7

Fitting classes and injectors

7.1 A non-injective Fitting class

After B. Fischer, W. Gaschütz, and B. Hartley's result about the injective character of the Fitting classes of soluble groups (Theorem 2.4.26), and bearing in mind the extension of the projective theory to the general universe of finite groups, it seemed to be reasonable to think about the validity of Theorem 2.4.26 outside the soluble realm. It was conjectured then that if $\mathfrak{F}$ is an arbitrary Fitting class and G is a finite group, then $\mathrm{Inj}_{\mathfrak{F}}(G) \neq \emptyset$. In the eighties of the last century, a big effort of some mathematicians was addressed to find methods to obtain injectors for Fitting classes in all finite groups. These efforts were successful for a big number of Fitting classes and they will be presented in Section 7.2. In this atmosphere, the construction of E. Salomon [Sal] of an example of a non-injective Fitting class caused a deep shock.

Salomon's construction, never published, is based in a pull-back construction of induced extensions due to F. Gross and L. G. Kovács (see Section 1.1). The aim of this section is to present the Salomon's example in full detail.

We begin with a quick insight to the group $A = \mathrm{Aut}\big(\mathrm{Alt}(6)\big)$. Let D denote the normal subgroup of inner automorphisms $D \cong \mathrm{Alt}(6)$ of A. It is well-known that the quotient group A/D is isomorphic to an elementary abelian 2-group of order 4 and A does not split over D, i.e. there is no complement of D in A (see [Suz82]).

If u is an involution of $\mathrm{Sym}(6)$, the symmetric group of degree 6, then $\langle u\rangle$ is a complement of $\mathrm{Alt}(6)$ in $\mathrm{Sym}(6)$ and the element u acts on $\mathrm{Alt}(6)$ as an outer automorphism.

Likewise, $\mathrm{Alt}(6) \cong \mathrm{PSL}(2,9)$ but $\mathrm{Sym}(6) \not\cong \mathrm{PGL}(2,9)$ (see [Hup67, pages 183 and 184]). There exist elements of order 2 in $\mathrm{PGL}(2,9)$ which are not in $\mathrm{PSL}(2,9)$ (for instance the coclass of the matrix $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ in the quotient group $\mathrm{GL}(2,9)/\mathrm{Z}\big(\mathrm{GL}(2,9)\big) \cong \mathrm{PGL}(2,9)$). If v is one of these involutions, then $\langle v\rangle$ is a complement of $\mathrm{PSL}(2,9)$ in $\mathrm{PGL}(2,9)$ and the element v acts on $\mathrm{Alt}(6) \cong \mathrm{PSL}(2,9)$ as an outer automorphism.

The subgroup $B = D\langle u\rangle \cong \mathrm{Sym}(6)$ and the subgroup $C = D\langle v\rangle \cong \mathrm{PGL}(2,9)$ are normal subgroups of A of index 2. Clearly $A = BC$ and $B \cap C = D$.

Let S be a non-abelian simple group. If x is an involution in S, define the group homomorphism

$$\alpha_1 \colon B \longrightarrow S \quad \text{such that } \mathrm{Ker}(\alpha_1) = D,\ B^{\alpha_1} = \langle x\rangle,$$

Put $|S \colon \mathrm{Im}(\alpha_1)| = |S|/2 = n_1$, and consider the right transversal

$$\mathcal{T}_1 = \{s_1 = 1, s_2, \dots, s_{n_1}\},$$

of $\mathrm{Im}(\alpha_1)$ in S and the transitive action

$$\rho_1 \colon S \longrightarrow \mathrm{Sym}(n_1)$$

on the set of indices $\mathcal{I}_1 = \{1, \dots, n_1\}$. For each $i \in \mathcal{I}_1$ and each $s \in S$, $s_i s = x_{i,s} s_j$, for some $x_{i,s} \in \mathrm{Im}(\alpha_1)$ and $i^{s^{\rho_1}} = j$. Write $P_S = S^{\rho_1} \leq \mathrm{Sym}(n_1)$ and consider the monomorphism (see Lemma 1.1.26)

$$\lambda_1 = \lambda_{\mathcal{T}_1} \colon S \longrightarrow \mathrm{Im}(\alpha_1) \wr_{\rho_1} P_S,$$

defined by $s^{\lambda_1} = (x_{1,s}, \dots, x_{n_1,s})s^{\rho_1}$, for any $x \in S$, and the epimorphism

$$\bar{\alpha}_1 \colon W_1 = B \wr_{\rho_1} P_S \longrightarrow \mathrm{Im}(\alpha_1) \wr_{\rho_1} P_S$$

defined by $\big((b_1, \dots, b_{n_1})\tau\big)^{\bar{\alpha}_1} = (b_1^{\alpha_1}, \dots, b_{n_1}^{\alpha_1})\tau$, for $b_1, \dots, b_{n_1} \in B$ and $\tau \in P_S$. Write $M_1 = \mathrm{Ker}(\bar{\alpha}_1) = D^{n_1} \cong \mathrm{Alt}(6)^{n_1}$.

Construct the induced extension G_1, defined by α_1 (see Definition 1.1.27),

$$E\lambda_1 \colon 1 \longrightarrow M_1 \longrightarrow G_1 \xrightarrow{\sigma_1} S \longrightarrow 1$$

Recall that

$$G_1 = \{w \in W_1 : w^{\bar{\alpha}_1} = s^{\lambda_1} \quad \text{for some } s \in S\},$$

and

$$\sigma_1 \colon G_1 \longrightarrow S \quad \text{defined by } w^{\sigma_1} = s, \text{ where } w^{\bar{\alpha}_1} = s^{\lambda_1}.$$

The following diagram is commutative:

$$\begin{tikzcd}
E\lambda_1 \colon 1 \arrow[r] & M_1 \arrow[r] \arrow[d, "\mathrm{id}"] & G_1 \arrow[r, "\sigma_1"] \arrow[d] & S \arrow[r] \arrow[d, "\lambda_1"] & 1 \\
E \colon 1 \arrow[r] & M_1 \arrow[r] & W_1 \arrow[r, "\bar{\alpha}_1"] & \mathrm{Im}(\alpha_1) \wr_{\rho_1} P_s \arrow[r] & 1
\end{tikzcd}$$

Then, applying Theorem 1.1.35, G_1 splits over M_1, since B splits over D.

For the group C we repeat the previous arguments to construct a similar group G_2. Let T be a non-abelian simple group. If y is an involution in T, define the group homomorphism

$$\alpha_2 \colon C \longrightarrow T \quad \text{such that } \mathrm{Ker}(\alpha_2) = D,\, C^{\alpha_2} = \langle y \rangle.$$

Put $|T : \mathrm{Im}(\alpha_2)| = |T|/2 = n_2$, and consider the right transversal

$$\mathcal{T}_2 = \{t_1 = 1, t_2, \dots, t_{n_2}\}$$

of $\mathrm{Im}(\alpha_2)$ in T and the transitive action

$$\rho_2 \colon T \longrightarrow \mathrm{Sym}(n_2)$$

on the set of indices $\mathcal{I}_2 = \{1, \dots, n_2\}$. For each $i \in \mathcal{I}_2$ and each $t \in T$, $t_i t = y_{i,t} t_j$, for some $y_{i,t} \in \mathrm{Im}(\alpha_2)$ and $i^{t^{\rho_2}} = j$.

With the obvious changes of notation, construct the induced extension defined by α_2 as in Definition 1.1.27. Then, for $G_2 = \{w \in W_2 = C \wr_{\rho_2} P_T : w^{\bar{\alpha}_2} = t^{\lambda_2} \text{ for some } t \in T\}$ and $\sigma_2 \colon G_2 \longrightarrow T$ defined as above, we also have that the following diagram is commutative

$$\begin{tikzcd}
E\lambda_2\colon 1 \arrow[r] & M_2 \arrow[r] \arrow[d, "\mathrm{id}"] & G_2 \arrow[r, "\sigma_2"] \arrow[d] & T \arrow[r] \arrow[d, "\lambda_2"] & 1 \\
E_2\colon 1 \arrow[r] & M_2 \arrow[r] & W_2 \arrow[r, "\bar{\alpha}_2"] & \mathrm{Im}(\alpha_2) \wr_{\rho_2} P_T \arrow[r] & 1
\end{tikzcd}$$

Then, again by Theorem 1.1.35, G_2 splits over M_2 since C splits over D.

Finally, consider the homomorphism $\alpha \colon A \longrightarrow S \times T$ such that $b^{\alpha} = (b^{\alpha_1}, 1)$, $c^{\alpha} = (1, c^{\alpha_2})$ for any $b \in B$, $c \in C$. Then, $\mathrm{Ker}(\alpha) = D$ and $\mathrm{Im}(\alpha) = \mathrm{Im}(\alpha_1) \times \mathrm{Im}(\alpha_2)$. Put $|S \times T : \mathrm{Im}(\alpha)| = \frac{|S|}{2}\frac{|T|}{2} = n_1 n_2$, and consider the right transversal of $\mathrm{Im}(\alpha)$ in $S \times T$

$$\begin{aligned}
\mathcal{T} &= \mathcal{T}_1 \times \mathcal{T}_2 \\
&= \{(s_1, t_1) = (1,1), (s_1, t_2), \dots, (s_1, t_{n_2}), (s_2, t_1), (s_2, t_2), \dots, (s_{n_1}, t_{n_2})\}.
\end{aligned}$$

The transitive action $\rho \colon S \times T \longrightarrow \mathrm{Sym}(n_1 n_2)$ on the set of indices $\mathcal{I} = \mathcal{I}_1 \times \mathcal{I}_2 = \{(1,1), \dots, (n_1, n_2)\}$ (lexicographically ordered) gives $P = (S \times T)^{\rho} = P_S \times P_T$.

Consider the monomorphism

$$\lambda = \lambda_{\mathcal{T}} \colon S \times T \longrightarrow \mathrm{Im}(\alpha) \wr_{\rho} P,$$

defined by

$$(s,t)^{\lambda} = \big((x_{1,s}, y_{1,t}), (x_{1,s}, y_{2,t}), \dots, (x_{n_1,s}, y_{n_2,t})\big)(s,t)^{\rho}$$

for any $s \in S$, $t \in T$, the epimorphism

$$\bar{\alpha}\colon W = A \wr_\rho P \longrightarrow \mathrm{Im}(\alpha) \wr_\rho P$$

defined by

$$\big((a_{(1,1)}, a_{(1,2)}, \dots, a_{(n_1,n_2)})\tau\big)^{\bar{\alpha}} = (a^\alpha_{(1,1)}, a^\alpha_{(1,2)}, \dots, a^\alpha_{(n_1,n_2)})\tau$$

for $a_{(1,1)}, a_{(1,2)}, \dots, a_{(n_1,n_2)} \in A$ and $\tau \in P$, and write $M = \mathrm{Ker}(\bar{\alpha}) = D^\natural = D^{n_1 n_2} \cong \mathrm{Alt}(6)^{n_1 n_2}$.

Construct the induced extension defined by the homomorphism $\alpha\colon A \longrightarrow S \times T$:

$$\begin{tikzcd}
E\lambda\colon 1 \arrow[r] & M \arrow[r] \arrow[d, "\mathrm{id}"] & G \arrow[r, "\sigma"] \arrow[d] & S \times T \arrow[r] \arrow[d, "\lambda"] & 1 \\
E\colon 1 \arrow[r] & M \arrow[r] & W \arrow[r, "\bar{\alpha}"] & \mathrm{Im}(\alpha) \wr_p (P_S \times P_T) \arrow[r] & 1
\end{tikzcd}$$

Then,

$$G = \{w \in W = A \wr_\rho P : w^{\bar{\alpha}} = (s,t)^\lambda \quad \text{for some } (s,t) \in S \times T\}$$

and $\sigma\colon G \longrightarrow S \times T$ defined by $w^\sigma = (s,t)$ such that $w^{\bar{\alpha}} = (s,t)^\lambda$, for all $w \in G$. Now applying Theorem 1.1.35, the group G does not split over M, since A does not split over D.

Every element $w \in W$ can be written uniquely as

$$w = (a_{(1,1)}, \dots, a_{(n_1,n_2)})(\tau_1, \tau_2)$$

where $a_{(1,1)}, a_{(1,2)}, \dots, a_{(n_1,n_2)} \in A$ for all $(i,j) \in \mathcal{I}$, $\tau_1 \in P_S$ and $\tau_2 \in P_T$. If $w \in G$, and $w^{\bar{\alpha}} = (s,t)^\lambda$, then

$$\begin{aligned}
w^{\bar{\alpha}} &= (a^\alpha_{(1,1)}, \dots, a^\alpha_{(n_1,n_2)})(\tau_1, \tau_2) \\
&= w^{\sigma\lambda} \\
&= \big((x_{1,s}, y_{1,t}), (x_{1,s}, y_{2,t}), \dots, (x_{n_1,s}, y_{n_2,t})\big)(s,t)^\rho
\end{aligned}$$

and $a^\alpha_{(i,j)} = (x_{i,s}, y_{j,t})$, for all $(i,j) \in \mathcal{I}$, $s^{\rho_1} = \tau_1$ and $t^{\rho_2} = \tau_2$.

Proposition 7.1.1. *The group W possesses subgroups $W_{(1)}$ and $W_{(2)}$ which are isomorphic to W_1 and W_2, respectively.*

Proof. Let $W_{(1)}$ be the subset of all elements w in W such that

1. $a_{(i,1)} = a_{(i,2)} = \dots = a_{(i,n_2)}$, for all $i = 1, \dots, n_1$,
2. $a_{(i,j)} \in B$, for all $(i,j) \in \mathcal{I}$, and
3. $\tau_2 = 1$.

Then $W_{(1)}$ is a subgroup of W and the map $\psi_1\colon W_1 \longrightarrow W_{(1)}$ such that $\big((b_1,\dots,b_{n_1})\tau\big)^{\psi_1}$ is the element $w \in W_{(1)}$ such that

1. $a_{(i,1)} = a_{(i,2)} = \cdots = a_{(i,n_2)} = b_i$, for all $i = 1,\dots,n_1$,
2. $\tau_1 = \tau$ and $\tau_2 = 1$,

is a group isomorphism. Put $M_{(1)} = M_1^{\psi_1}$.

A similar argument and construction holds for W_2. □

Proposition 7.1.2. *The group G possesses two subgroups which are isomorphic to G_1 and G_2, respectively.*

Proof. Consider the subgroup $G_{(1)} = W_{(1)} \cap G$ and note that

$$G_{(1)} = \{x \in W_{(1)} : x^{\bar\alpha} = (s,1)^\lambda \text{ for some } s \in S\}.$$

Note that the kernel of the group epimorphism

$$\sigma_{(1)} = \sigma\pi_1\colon G_{(1)} \longrightarrow S,$$

where $\pi_1\colon S \times T \longrightarrow S$ is the canonical projection, is $M_{(1)} = M_1^{\psi_1}$, as in Proposition 7.1.1. Define the group homomorphism

$$\beta_1 = \iota_{(1)}\psi_1^{-1}\colon G_{(1)} \longrightarrow W_1,$$

where $\iota_{(1)}\colon G_{(1)} \longrightarrow W_{(1)}$ is the canonical inclusion and ψ_1 as in Proposition 7.1.1.

Consider an element $x = (a_{(1,1)},\dots,a_{(n_1,n_2)})(\tau_1,1) \in G_{(1)}$. Then, if $x^{\bar\alpha} = (s,1)^\lambda$, we have that $s^{\rho_1} = \tau_1$ and $a_{(i,j)}^{\alpha} = (x_{i,s},1) \in S\times 1$, for all $i = 1,\dots,n_1$, i.e. $a_{(i,j)} \in B$ and $a_{(i,j)}^{\alpha_1} = x_{i,s}$, for all $i = 1,\dots,n_1$. Observe that

$$x^{\bar\alpha} = (s,1)^\lambda = \big((x_{1,s},1),(x_{1,s},1)\dots,(x_{n_1,s},1)\big)(s^{\rho_1},1),$$

and

$$\begin{aligned} x^{\beta_1\bar\alpha_1} &= x^{\iota_{(1)}\psi_1^{-1}\bar\alpha_1} = x^{\psi_1^{-1}\bar\alpha_1} = \big((a_{(1,1)},\dots,a_{(n_1,1)})\tau_1\big)^{\bar\alpha_1} = \\ &= (a_{(1,1)}^{\alpha_1},\dots,a_{(n_1,1)}^{\alpha_1})\tau_1 = (x_{1,s},\dots,x_{n_1,s})s^{\rho_1} = \\ &= s^{\lambda_1} = (s,1)^{\pi_1\lambda_1} = x^{\sigma\pi_1\lambda_1} = x^{\sigma_{(1)}\lambda_1}. \end{aligned}$$

Then the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & M_1 & \longrightarrow & G_{(1)} & \xrightarrow{\sigma_{(1)}} & S & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle \beta_1} & & \big\downarrow{\scriptstyle \lambda_1} & & \\ 1 & \longrightarrow & M_1 & \longrightarrow & W_1 & \xrightarrow{\bar\alpha_1} & \mathrm{Im}(\alpha_1) \wr_{\rho_1} P_s & \longrightarrow & 1 \end{array}$$

is commutative.

By the universal property, Theorem 1.1.23 (2), we have that $G_{(1)}$ is isomorphic to G_1.

Analogously we can proceed with G_2 and it appears a subgroup $G_{(2)}$ in $W_{(2)}$ which is isomorphic to G_2. □

Let S and T be two non-abelian simple groups. Recall that the class $\mathfrak{F} = \mathrm{D}_0(S, T, 1)$ composed by the trivial group and all groups which are direct products of the form

$$S_1 \times \cdots \times S_n \times T_1 \times \cdots \times T_m,$$

where $S_i \cong S$, $T_j \cong T$, $1 \leq i \leq n$, $1 \leq j \leq m$, for some positive integers n and m, is a Fitting formation (see Lemma 2.2.3).

Theorem 7.1.3. *Let S and T be two non-abelian simple groups. Suppose that S and T satisfy the three following conditions:*

1. *no subgroup of S is isomorphic to T,*
2. *no subgroup of T is isomorphic to S, and*
3. *either S or T are isomorphic to no subgroup of a direct product of copies of the alternating group* $\mathrm{Alt}(6)$ *of degree 6.*

Consider the Fitting formation $\mathfrak{F} = \mathrm{D}_0(S, T, 1)$. Then the group G, constructed above, has no $\mathfrak{F}$-injectors.

Proof. The group G possesses two subgroups, $\tilde{S}$ and $\tilde{T}$, which are isomorphic to S and T, respectively. Write $G/M = (H_1/M) \times (H_2/M)$, with $H_1/M \cong S$ and $H_2/M \cong T$. Observe that $\tilde{S}M/M \cong \tilde{S}/(\tilde{S} \cap M) = \tilde{S}$, since $\tilde{S} \cap M = 1$, by condition 3. If $(H_1/M) \cap (\tilde{S}M/M) = 1$, then the group $G/H_1 \cong T$ would have a subgroup isomorphic to S, and this is not possible by Condition 2. Hence $H_1 = \tilde{S}M$. A similar argument with $\tilde{T}$ and H_2 leads to $H_2 = \tilde{T}M$. Both H_1 and H_2 are maximal normal subgroups of G.

We observe that $\mathrm{Max}_{\mathfrak{F}}(\tilde{S}M) = \{U : UM = \tilde{S}M, U \cong S\}$. If $U \in \mathrm{Max}_{\mathfrak{F}}(\tilde{S}M)$, then $U \cap M = 1$ by condition 3. Since $U \in \mathfrak{F}$ and $UM \leq \tilde{S}M$, we have that $U \cong S$ and $UM = \tilde{S}M$.

Similarly $\mathrm{Max}_{\mathfrak{F}}(\tilde{T}M) = \{V : VM = \tilde{T}M, V \cong T\}$.

Suppose that X is an $\mathfrak{F}$-injector of G. Then, the subgroup $X \cap \tilde{S}M = R_1$ is $\mathfrak{F}$-maximal in $\tilde{S}M$. Hence $R_1 \cong S$. Likewise, $X \cap \tilde{T}M = R_2 \cong T$. Hence $R_1 \times R_2$ is a normal subgroup of X and $R_1 \times R_2 \cong S \times T$. Moreover, $(R_1 \times R_2) \cap M = 1$. Since $|G| = |M||S \times T| = |M||R_1 \times R_2|$, we conclude that $R_1 \times R_2$ is a complement of M in G, i.e. G splits over M. But this is not true. Therefore the group G has no $\mathfrak{F}$-injectors and $\mathfrak{F}$ is a non-injective Fitting class. □

Remark 7.1.4. The simple groups $S = \mathrm{Alt}(7)$ and $T = \mathrm{PSL}(2, 11)$ satisfy the above conditions 1, 2, and 3.

7.2 Injective Fitting classes

We have proved in Corollary 2.4.28 that every Fitting class $\mathfrak{F}$ is injective in the universe $\mathfrak{F}\mathfrak{S}$. In fact, in the attempt of investigating classes of groups, larger than the soluble one, in which there exist $\mathfrak{F}$-injectors for a particular Fitting class $\mathfrak{F}$, the first remarkable contribution comes from A. Mann in [Man71]. There, following some ideas due to B. Fischer and E. C. Dade (see [DH92, page 623]), it is proved that in every $\mathfrak{N}$-constrained group G, there exists a single conjugacy class of $\mathfrak{N}$-injectors and each $\mathfrak{N}$-injector is an $\mathfrak{N}$- maximal subgroup containing the Fitting subgroup. A group G is said to be $\mathfrak{N}$-*constrained* if $\mathrm{C}_G\big(\mathrm{F}(G)\big) \leq \mathrm{F}(G)$. It is well-known that every soluble group is $\mathfrak{N}$-constrained (see [DH92, A, 10.6]).

In [BL79] D. Blessenohl and H. Laue proved that the class $\mathfrak{Q}$ of all quasinilpotent groups is an injective Fitting class in $\mathfrak{E}$. In fact they prove something more (see [DH92, IX, 4.15]).

Theorem 7.2.1 (D. Blessenohl and H. Laue). *Every finite group G has a single conjugacy class of $\mathfrak{Q}$-injectors, and this consists of those $\mathfrak{Q}$-maximal subgroups of G containing $\mathrm{F}^*(G)$.*

In the decade of the eighties of the last century there was a considerable amount of contributions to obtain more injective Fitting classes. P. Förster proved the existence of a certain non-empty characteristic conjugacy class of $\mathfrak{N}$-injectors in every finite group in [För85a]. Later M. J. Iranzo and F. Pérez-Monasor obtained the existence of injectors in all finite groups with respect to various Fitting classes, including a new type of $\mathfrak{N}$-injectors. Their investigations, together with M. Torres, gave light to a "test" to prove the injectivity of a number of Fitting classes. Some of the most interesting results obtained from this test have been published recently by M. J. Iranzo, J. Lafuente, and F. Pérez-Monasor. Their achievements illuminate the validity of a L. A. Shemetkov conjecture saying that any Fitting class composed of soluble groups is injective.

We present here some of the fruits of these investigations.

Proposition 7.2.2. *Let $\mathfrak{F}$ be a Fitting class and G be a group.*

1. *A perfect comonolithic subnormal subgroup E of G is an $\mathfrak{F}$-component of G if and only of $EG_{\mathfrak{F}}/G_{\mathfrak{F}}$ is a component of $G/G_{\mathfrak{F}}$.*
2. *If E is an $\mathfrak{F}$-component of G, the $\mathfrak{F}$-maximal subgroups of E containing $E_{\mathfrak{F}}$ are $\mathfrak{F}$-injectors of E.*

Proof. 1. Let E be a perfect comonolithic subnormal subgroup of a group G. Suppose that E is an $\mathfrak{F}$-component of G. Then $\mathrm{N}(E)$ is a subnormal $\mathfrak{F}$-subgroup of G, i.e. $\mathrm{N}(E) \leq G_{\mathfrak{F}}$. Therefore $EG_{\mathfrak{F}}/G_{\mathfrak{F}}$ is isomorphic to a quotient group of $E/\mathrm{N}(E)$, and then $EG_{\mathfrak{F}}/G_{\mathfrak{F}}$ is a quasisimple subnormal subgroup of $G/G_{\mathfrak{F}}$. Conversely, if $EG_{\mathfrak{F}}/G_{\mathfrak{F}}$ is a component of $G/G_{\mathfrak{F}}$, then $E/(E \cap G_{\mathfrak{F}})$ is a quasisimple group. Since E is subnormal in G, $E_{\mathfrak{F}} = E \cap G_{\mathfrak{F}}$

by Remark 2.4.4. If $E \in \mathfrak{F}$, then E is contained in $G_{\mathfrak{F}}$, contrary to supposition. Hence $E_{\mathfrak{F}} \leq \mathrm{Cosoc}(E)$. Moreover, $\mathrm{Cosoc}(E)/E_{\mathfrak{F}} = \mathrm{Z}(E/E_{\mathfrak{F}})$. Therefore $\mathrm{N}(E) = [E, \mathrm{Cosoc}(E)] \leq E_{\mathfrak{F}}$. Hence $\mathrm{N}(E) \in \mathfrak{F}$.

2. Suppose E is an $\mathfrak{F}$-component of G and V is an $\mathfrak{F}$-maximal subgroup of E such that $E_{\mathfrak{F}} \leq V$. Since $\mathrm{N}(E) \leq E_{\mathfrak{F}} \leq \mathrm{Cosoc}(E)$ and $\mathrm{Cosoc}(E)/\mathrm{N}(E)$ is abelian, $E_{\mathfrak{F}}$ is the $\mathfrak{F}$-injector of $\mathrm{Cosoc}(G)$. Moreover, $V \cap \mathrm{Cosoc}(E)$ is normal in $\mathrm{Cosoc}(E)$ and then is a subnormal $\mathfrak{F}$-subgroup of E. Hence $V \cap \mathrm{Cosoc}(E) = E_{\mathfrak{F}}$ and V is an $\mathfrak{F}$-injector of E. □

Proposition 7.2.3. *Let K be a subnormal subgroup of a group G. If E is an $\mathfrak{F}$-component of G such that E is not contained in K, we have that $[K, E] \leq \mathrm{N}(E)$.*

Proof. Denote $M = \mathrm{Cosoc}(E)$. By Theorem 2.2.19, the subgroup K normalises E. Therefore K normalises M. Clearly K is subnormal in KE and KM is normal in KE. Since $K \cap E$ is subnormal in the comonolithic group E and $E \nleq K$, we have that $K \cap E \leq M$. Therefore

$$[K, E] \leq [KM, E] \leq KM \cap E = M(K \cap E) \leq M.$$

Hence

$$[K, E, E] = [E, K, E] \leq [M, E] = \mathrm{N}(E)$$

and the Three-Subgroups Lemma (see [KS04, 1.5.6]) yields that $[E, K] = [E, E, K] \leq \mathrm{N}(E)$. □

Now we are ready to state and prove the result of Iranzo, Pérez-Monasor, and Torres.

Theorem 7.2.4 ([IPMT90]). *Let $\mathfrak{F}$ be a Fitting class and G a group. Let $\{E_1, \dots, E_n\}$ be a set of $\mathfrak{F}$-components of G which is invariant by conjugation of the elements of G. For each $i = 1, \dots, n$, let J_i be an $\mathfrak{F}$-injector of E_i. Consider the subgroup $J = \langle J_1, \dots, J_n \rangle$.*

Then $\mathrm{Inj}_{\mathfrak{F}}\big(\mathrm{N}_G(J)\big) \subseteq \mathrm{Inj}_{\mathfrak{F}}(G)$.

Proof. Note that, by Proposition 7.2.2 (2) and Proposition 7.2.3, J is a normal product $J = J_1 \cdots J_n$, and therefore $J \in \mathfrak{F}$. Let H be an $\mathfrak{F}$-injector of $\mathrm{N}_G(J)$. We have to prove that for any subnormal subgroup S of G, the subgroup $H \cap S$ is $\mathfrak{F}$-maximal in S. To do that we consider an $\mathfrak{F}$-subgroup K of S such that $H \cap S \leq K$ and argue that $H \cap S = K$.

We may assume without loss of generality that the $\mathfrak{F}$-components $E_1, \dots, E_m$ are those contained in S, for $m \leq n$, and the other ones are not in S. This implies that $\{E_1, \dots, E_m\}$ is a set of $\mathfrak{F}$-components of S which is invariant by conjugation of the elements of S.

Observe that $J \leq \mathrm{N}_G(J)_{\mathfrak{F}} \leq H$. Therefore, for any $i = 1, \dots, m$, we have that

$$J_i \leq J \cap E_i \leq H \cap E_i \leq H \cap S \cap E_i \leq K \cap E_i \in \mathfrak{F},$$

since $K \cap E_i$ is subnormal in K. Therefore

$$J_i = J \cap E_i = H \cap E_i = K \cap E_i,$$

since $J_i \in \mathrm{Max}_{\mathfrak{F}}(E_i)$, $i = 1, \dots, m$.

Observe that if $x \in K$, for every $i \in \{1, \dots, m\}$, there exists an index $j \in \{1, \dots, m\}$ such that

$$J_i^x = (J \cap E_i)^x = K \cap E_i^x = K \cap E_j = J_j.$$

Choose now $j \in \{m+1, \dots, n\}$. Applying Proposition 7.2.3, it can be deduced that $[J_j, S] \leq [E_j, S] \leq \mathrm{N}(E_j) \leq J_j$. This is to say that S normalises J_j for every $j \in \{m+1, \dots, n\}$. Therefore

$$K \leq \mathrm{N}_S(J_1 \dots J_m) \leq \mathrm{N}_S(J).$$

Hence $H \cap S \leq K \leq \mathrm{N}_S(J)$ and then $H \cap S = H \cap \mathrm{N}_S(J)$.

The subgroup $\mathrm{N}_S(J)$ is subnormal in $\mathrm{N}_G(J)$. Since $H \in \mathrm{Inj}_{\mathfrak{F}}\bigl(\mathrm{N}_G(J)\bigr)$, we have that $H \cap S \in \mathrm{Max}_{\mathfrak{F}}(\mathrm{N}_S(J))$. This implies that $H \cap S = K$, as desired. $\square$

Theorem 7.2.4 is a crucial result when proving the injectivity of a Fitting class by inductive arguments: with the above notation, if $\mathrm{Inj}_{\mathfrak{F}}\bigl(\mathrm{N}_G(J)\bigr) \neq \emptyset$, then the group G possesses $\mathfrak{F}$-injectors. Equipped with this theorem we can obtain several results of M. J. Iranzo, J. Lafuente, and F. Pérez-Monasor in [ILPM03] and [ILPM04], which go much further on the theorems about the existence of injectors.

Lemma 7.2.5 (see [ILPM03]). *Let G be a group and $\mathfrak{m}$ a preboundary of perfect groups. Set $\mathfrak{B} = \mathrm{Fit}\bigl(\mathrm{Cosoc}(Z) : Z \in \mathfrak{m}\bigr)$.*

1. *If $X, Y \in \mathrm{b}_{\mathfrak{m}}(G)$, then*
   a) *$\mathrm{Cosoc}(X) = X_{\mathfrak{B}}$, $[X, Y] \leq X \cap Y$ and $(XY)_{\mathfrak{B}} = X_{\mathfrak{B}} Y_{\mathfrak{B}}$,*
   b) *$X \neq Y$ if and only if $XG_{\mathfrak{B}}/G_{\mathfrak{B}} \neq YG_{\mathfrak{B}}/G_{\mathfrak{B}}$.*
2. *Suppose that $\mathrm{b}_{\mathfrak{m}}(G) = \{X_1, \dots, X_n\} \neq \emptyset$ and write $E = \mathrm{E}_{\mathfrak{m}}(G)$; then*
   a) *$E = X_1 \dots X_n$ and $E_{\mathfrak{B}} = (X_1)_{\mathfrak{B}} \dots (X_n)_{\mathfrak{B}}$,*
   b) *$E/E_{\mathfrak{B}} \cong X_1/(X_1)_{\mathfrak{B}} \times \cdots \times X_n/(X_n)_{\mathfrak{B}}$ is a direct product of non-abelian simple groups.*

Proof. 1a. By definition of $\mathfrak{B}$, we have that $\mathrm{Cosoc}(X) \in \mathfrak{B}$. Assume that $X \in \mathfrak{B}$. Then $X \in \mathrm{s}_n\bigl(\mathrm{Cosoc}(Z) : Z \in \mathfrak{m}\bigr)$, by [DH92, XI, 4.14]. But this is not possible since $\mathfrak{m}$ is subnormally independent. Therefore $\mathrm{Cosoc}(X) = X_{\mathfrak{B}}$.

Trivially, if $X = Y$, then $[X, Y] \leq X \cap Y$. Suppose that $X \neq Y$. Observe that, since $\mathfrak{m}$ is subnormally independent, we have that $X \nleq Y$ and $Y \nleq X$. By Theorem 2.2.19, Y normalises X and X normalises Y. Hence $[X, Y] \leq X \cap Y$.

If $X \neq Y$, then $X \cap Y \leq \mathrm{Cosoc}(X) \cap \mathrm{Cosoc}(Y) = X_{\mathfrak{B}} \cap Y_{\mathfrak{B}}$. Moreover,

$$XY_{\mathfrak{B}} \cap YX_{\mathfrak{B}} = (X \cap YX_{\mathfrak{B}})Y_{\mathfrak{B}} = (X \cap Y)X_{\mathfrak{B}}Y_{\mathfrak{B}} = X_{\mathfrak{B}}Y_{\mathfrak{B}}$$

and then

$$XY/X_{\mathfrak{B}}Y_{\mathfrak{B}} = XY_{\mathfrak{B}}/X_{\mathfrak{B}}Y_{\mathfrak{B}} \times YX_{\mathfrak{B}}/X_{\mathfrak{B}}Y_{\mathfrak{B}}$$

is a direct product of non-abelian simple groups. Since $(XY)_{\mathfrak{B}}/X_{\mathfrak{B}}Y_{\mathfrak{B}} \leq \mathrm{Z}(XY/X_{\mathfrak{B}}Y_{\mathfrak{B}})$ by [DH92, IX, 1.1], we conclude that $(XY)_{\mathfrak{B}} = X_{\mathfrak{B}}Y_{\mathfrak{B}}$.

1b. Observe that $XG_{\mathfrak{B}}/G_{\mathfrak{B}} \cong X/(X \cap G_{\mathfrak{B}}) = X/X_{\mathfrak{B}}$ is a non-abelian simple group. Suppose that $X \neq Y$ and $XG_{\mathfrak{B}}/G_{\mathfrak{B}} = YG_{\mathfrak{B}}/G_{\mathfrak{B}}$. Notice that $[X,Y] \leq X \cap Y \in \mathfrak{B}$, and then, $XG_{\mathfrak{B}}/G_{\mathfrak{B}} = (XG_{\mathfrak{B}}/G_{\mathfrak{B}})' = [XG_{\mathfrak{B}}/G_{\mathfrak{B}}, YG_{\mathfrak{B}}/G_{\mathfrak{B}}] = [X,Y]G_{\mathfrak{B}}/G_{\mathfrak{B}} = 1$. This is a contradiction.

Part 2 follows immediately from 1. □

Lemma 7.2.6 (M. J. Iranzo, J. Lafuente, and F. Pérez-Monasor, unpublished). *Let $\mathfrak{F}$ be a Fitting class and $\mathfrak{n}$ a subclass of $\bar{\mathrm{b}}(\mathfrak{F})$. Then*

$$\mathrm{Fit}(\mathfrak{F}, \mathfrak{n}) = \mathfrak{F} \cdot \mathrm{Fit}\,\mathfrak{n} = \big(G \in \mathfrak{E} : G = G_{\mathfrak{F}}\,\mathrm{E}_{\mathfrak{n}}(G)\big).$$

Proof. Let G be a group. If $X \in b_{\mathfrak{n}}(G)$, then clearly $\mathrm{Cosoc}(X) = X_{\mathfrak{F}}$.

Write $\mathfrak{X} = \big(G \in \mathfrak{E} : G = G_{\mathfrak{F}}\,\mathrm{E}_{\mathfrak{n}}(G)\big)$ and $\mathfrak{Y} = \mathrm{Fit}\,\mathfrak{n}$. For each group G, the subgroup $\mathrm{E}_{\mathfrak{n}}(G)$ is in $\mathrm{Fit}\,\mathfrak{n}$, i.e. $\mathrm{E}_{\mathfrak{n}}(G) \leq G_{\mathfrak{Y}}$. Therefore $\mathfrak{X} \subseteq \mathfrak{F} \cdot \mathrm{Fit}\,\mathfrak{n} \subseteq \mathrm{Fit}(\mathfrak{F}, \mathfrak{n})$. Let us prove that $\mathfrak{X}$ is a Fitting class.

If $G \in \mathfrak{X}$, then $G/G_{\mathfrak{F}} \cong \mathrm{E}_{\mathfrak{n}}(G)/\mathrm{E}_{\mathfrak{n}}(G)_{\mathfrak{F}}$ is a direct product of non-abelian simple groups by Lemma 7.2.5 (2b). Let N be a normal subgroup of G. Then $b_{\mathfrak{n}}(N) \subseteq b_{\mathfrak{n}}(G)$. Thus, if $b_{\mathfrak{n}}(N) = \{X_1, \dots, X_r\}$, then

$$NG_{\mathfrak{F}}/G_{\mathfrak{F}} = X_1G_{\mathfrak{F}}/G_{\mathfrak{F}} \times \cdots \times X_rG_{\mathfrak{F}}/G_{\mathfrak{F}}$$

and then $N = N \cap NG_{\mathfrak{F}} = N \cap X_1 \dots X_rG_{\mathfrak{F}} = N \cap \mathrm{E}_{\mathfrak{n}}(N)G_{\mathfrak{F}} = \mathrm{E}_{\mathfrak{n}}(N)N_{\mathfrak{F}} \in \mathfrak{X}$.

If N and M are normal subgroups of a group $G = NM$ and $N, M \in \mathfrak{X}$, then $G = NM = N_{\mathfrak{F}}\,\mathrm{E}_{\mathfrak{n}}(N)M_{\mathfrak{F}}\,\mathrm{E}_{\mathfrak{n}}(M) \leq G_{\mathfrak{F}}\,\mathrm{E}_{\mathfrak{n}}(G)$. Hence $G \in \mathfrak{X}$.

Therefore $\mathfrak{X}$ is a Fitting class. It is clear that $\mathfrak{F}$ and $\mathfrak{n}$ are contained in $\mathfrak{X}$. Hence $\mathfrak{X} = \mathrm{Fit}(\mathfrak{F}, \mathfrak{n})$. □

Lemma 7.2.7. *Let $\mathfrak{T}$ be a Fitting class such that $\mathfrak{T} = \mathfrak{T}\mathfrak{S}$. Consider $\mathfrak{F} = \mathfrak{T}^{\mathrm{b}} = \mathrm{Fit}\big(\mathrm{Cosoc}(X) : X \in \mathrm{b}(\mathfrak{T})\big)$. Then $\mathrm{b}(\mathfrak{T}) = \bar{\mathrm{b}}(\mathfrak{T}) \subseteq \bar{\mathrm{b}}(\mathfrak{F})$.*

Proof. Let G be a group in $\mathrm{b}(\mathfrak{T})$. Then G is a comonolithic perfect group and $\mathrm{Cosoc}(G) \in \mathfrak{F}$. If $G \in \mathfrak{F}$, then $G \in \mathrm{s}_n\big(\mathrm{Cosoc}(X) : X \in \mathrm{b}(\mathfrak{T})\big)$ by [DH92, XI, 4.14]. This is to say that there exists a group $X \in \mathrm{b}(\mathfrak{T})$ such that G is a proper subnormal subgroup of X. In particular $G \in \mathfrak{T}$, and this contradicts our assumption. Hence $G \in \bar{\mathrm{b}}(\mathfrak{F})$. □

Theorem 7.2.8. *Let $\mathfrak{T}$ be a class of groups. The following statements are equivalent:*

1. *$\mathfrak{T}$ is a Fitting class such that $\mathfrak{T} = \mathfrak{T}\mathfrak{S}$.*
2. *$\mathfrak{T} = (G \in \mathfrak{E} : G_{\mathfrak{X}} \in \mathfrak{F})$ for a pair of Fitting classes $\mathfrak{X}$ and $\mathfrak{F}$ such that $\mathfrak{F} = \mathfrak{X} \cap \mathfrak{F}\mathfrak{A}$.*

In this case, for each group G, we have $G_{\mathfrak{T}} = \mathrm{C}_G(G_{\mathfrak{X}}/G_{\mathfrak{F}})$.

Proof. 1 implies 2. Set $\mathfrak{m} = \mathrm{b}(\mathfrak{T})$, and consider the Fitting classes $\mathfrak{F} = \mathfrak{T}^{\mathrm{b}}$ and $\mathfrak{X} = \mathrm{Fit}\,\mathfrak{m}$. Clearly $\mathfrak{F} \subseteq \mathfrak{X} \cap \mathfrak{T}$. Since $\mathfrak{T} = \mathfrak{T}\mathfrak{S}$, we have that $\mathfrak{m} = \bar{\mathrm{b}}(\mathfrak{T}) \subseteq \bar{\mathrm{b}}(\mathfrak{F})$, by the above lemma. Then we can apply Lemma 7.2.6 and conclude that

$$\mathfrak{X} = \mathrm{Fit}(\mathfrak{F}, \mathfrak{m}) = \bigl(G \in \mathfrak{E} : G = G_{\mathfrak{F}}\,\mathrm{E}_{\mathfrak{m}}(G)\bigr).$$

If $G \in \mathfrak{X} \cap \mathfrak{F}\mathfrak{A}$, then $G/G_{\mathfrak{F}} \cong \mathrm{E}_{\mathfrak{m}}(G)/\bigl(\mathrm{E}_{\mathfrak{m}}(G) \cap G_{\mathfrak{F}}\bigr)$ and this group is abelian and a direct product of non-abelian simple groups, by Lemma 7.2.5 (2b). Hence $G \in \mathfrak{F}$, and then $\mathfrak{F} = \mathfrak{X} \cap \mathfrak{F}\mathfrak{A}$.

Set $\mathfrak{H} = (G \in \mathfrak{E} : G_{\mathfrak{X}} \in \mathfrak{F})$. If a group $G \in \mathfrak{H} \setminus \mathfrak{T}$, there exists a subnormal subgroup N of G such that $N \in \mathfrak{m}$. Thus $N \leq G_{\mathfrak{X}} \in \mathfrak{F} \subseteq \mathfrak{T}$, and this is a contradiction. Hence $\mathfrak{H} \subseteq \mathfrak{T}$. Conversely if G is a group in $\mathfrak{T}$ and $N = G_{\mathfrak{X}}$, then $N = N_{\mathfrak{F}} E_{\mathfrak{m}}(N)$. But since $\mathfrak{T}$ is a Fitting class, $\mathrm{E}_{\mathfrak{m}}(G) = 1 = \mathrm{E}_{\mathfrak{m}}(N)$. Then $N \in \mathfrak{F}$. Therefore $G \in \mathfrak{H}$. Hence $\mathfrak{H} = \mathfrak{T}$.

2 implies 1. We see that, under these hypotheses, the class $\mathfrak{T}$ is a Fitting class. Let N be a normal subgroup of a $\mathfrak{T}$-group G. Clearly $N_{\mathfrak{X}} \leq G_{\mathfrak{X}} \in \mathfrak{F}$, and then $N \in \mathfrak{T}$. Consider now a group $G = NM$ such that N and M are normal $\mathfrak{T}$-subgroups of G. Then $N_{\mathfrak{X}}$, $M_{\mathfrak{X}} \in \mathfrak{F}$ and the subgroup $F = N_{\mathfrak{X}} M_{\mathfrak{X}} \in \mathfrak{F}$. By [DH92, IX, 1.1], we have that $G_{\mathfrak{X}}/F \leq \mathrm{Z}(G/F)$, and then $G_{\mathfrak{X}} \in \mathfrak{X} \cap \mathfrak{F}\mathfrak{A} = \mathfrak{F}$. Therefore $G \in \mathfrak{T}$. Thus, $\mathfrak{T}$ is a Fitting class.

Suppose that N is a normal $\mathfrak{T}$-subgroup of a group G, such that $G/N \in \mathfrak{A}$. Then $N_{\mathfrak{X}} \in \mathfrak{F}$. Since $G_{\mathfrak{X}}/N_{\mathfrak{X}} = G_{\mathfrak{X}}/(N \cap G_{\mathfrak{X}}) \cong NG_{\mathfrak{X}}/N \in \mathfrak{A}$, we have that $G_{\mathfrak{X}} \in \mathfrak{X} \cap \mathfrak{F}\mathfrak{A} = \mathfrak{F}$. Therefore $G \in \mathfrak{T}$. This implies that $\mathfrak{T} = \mathfrak{T}\mathfrak{S}$.

Finally, observe that in this situation $\mathfrak{F} = \mathfrak{X} \cap \mathfrak{T}$. Therefore $G_{\mathfrak{F}} = G_{\mathfrak{T}} \cap G_{\mathfrak{X}}$. Thus $G_{\mathfrak{T}} \leq \mathrm{C}_G(G_{\mathfrak{X}}/G_{\mathfrak{F}}) = C$. Obviously $(C \cap G_{\mathfrak{X}})/G_{\mathfrak{F}}$ is an abelian group and then $C_{\mathfrak{X}} = C \cap G_{\mathfrak{X}} \in \mathfrak{F}$, since $\mathfrak{F} = \mathfrak{X} \cap \mathfrak{F}\mathfrak{A}$. Therefore $C \in \mathfrak{T}$ and $C = G_{\mathfrak{T}}$. □

Corollary 7.2.9. *Let $\mathfrak{T}$ be a Fitting class such that $\mathfrak{T} = \mathfrak{T}\mathfrak{S}$. Then*

$$\mathrm{Fit}\bigl(\mathrm{b}(\mathfrak{T})\bigr) \cap \mathfrak{T} = \mathfrak{T}^{\mathrm{b}}.$$

Proof. Set $\mathfrak{m} = \mathrm{b}(\mathfrak{T})$ and consider again the Fitting classes $\mathfrak{F} = \mathfrak{T}^{\mathrm{b}}$ and $\mathfrak{X} = \mathrm{Fit}\,\mathfrak{m}$. By the above arguments, if a group G is in $\mathfrak{X} \cap \mathfrak{T}$, then $G = G_{\mathfrak{F}}\,\mathrm{E}_{\mathfrak{m}}(G) \in \mathfrak{T}$. Hence $\mathrm{E}_{\mathfrak{m}}(G) \in \mathfrak{T}$, and this implies that $\mathrm{E}_{\mathfrak{m}}(G) = 1$. Thus $G \in \mathfrak{F}$. Therefore $\mathfrak{X} \cap \mathfrak{T} = \mathfrak{F}$. □

The following proposition is motivated by a result due to W. Gaschütz (see [DH92, X, 3.14]).

Proposition 7.2.10. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be two Fitting classes in the same Lockett section such that $\mathfrak{F} \subseteq \mathfrak{G}$. For each group G denote*

$$\psi\colon G_{\mathfrak{G}}/G_{\mathfrak{F}} \longrightarrow (G_{\mathfrak{G}}G')/(G_{\mathfrak{F}}G')$$

the natural epimorphism. If p is a prime divisor of $|\mathrm{Ker}(\psi)|$, then $\mathfrak{G}\mathfrak{S}_p \neq \mathfrak{G}$.

Proof. Observe that $\mathrm{Ker}(\psi) = (G_{\mathfrak{G}}/G_{\mathfrak{F}}) \cap (G/G_{\mathfrak{F}})'$. Let p be a prime divisor of $|\mathrm{Ker}(\psi)|$ and suppose that $\mathfrak{G}\mathfrak{S}_p = \mathfrak{G}$. If $P/G_{\mathfrak{F}}$ is a Sylow p-subgroup of $G/G_{\mathfrak{F}}$, then $P \in \mathfrak{F}\mathfrak{S}_p \subseteq \mathfrak{G}\mathfrak{S}_p = \mathfrak{G}$. Since $\mathfrak{F}$ and $\mathfrak{G}$ are in the same Lockett section and $\mathfrak{F} \subseteq \mathfrak{G}$, the groups $P/P_{\mathfrak{F}}$ and $G_{\mathfrak{G}}/G_{\mathfrak{F}}$ are abelian, by [DH92, X, 1.21]. Thus $P' \leq P_{\mathfrak{F}}$ and $P \cap G_{\mathfrak{G}}$ is a normal subgroup of $G_{\mathfrak{G}}$. Hence $P' \cap G_{\mathfrak{G}} \in \mathfrak{F}$ and $P' \cap G_{\mathfrak{G}}$ is subnormal in $G_{\mathfrak{G}}$. Therefore $P' \cap G_{\mathfrak{G}} \leq (G_{\mathfrak{G}})_{\mathfrak{F}} = G_{\mathfrak{F}}$. Then $(P/G_{\mathfrak{F}})' \cap (G_{\mathfrak{G}}/G_{\mathfrak{F}}) = 1$. By [DH92, X, 1.21] again, $G_{\mathfrak{G}}/G_{\mathfrak{F}} \leq \mathrm{Z}(G/G_{\mathfrak{F}})$ and then

$$(P/G_{\mathfrak{F}}) \cap (G/G_{\mathfrak{F}})' \cap (G_{\mathfrak{G}}/G_{\mathfrak{F}}) \leq (P/G_{\mathfrak{F}}) \cap (G/G_{\mathfrak{F}})' \cap \mathrm{Z}(G/G_{\mathfrak{F}}) \leq (P/G_{\mathfrak{F}})'$$

by [Hup67, IV, 2.2]. Thus, $(P/G_{\mathfrak{F}}) \cap (G/G_{\mathfrak{F}})' \cap (G_{\mathfrak{G}}/G_{\mathfrak{F}}) = 1$ and this contradicts the choice of P. $\square$

Lemma 7.2.11. *Let $\mathfrak{X}$ be a Fitting class such that $\mathfrak{X}\mathfrak{S} = \mathfrak{X}$. Then*

$$\mathfrak{X}^{\mathrm{b}} \subseteq \mathfrak{X}_* \subseteq \mathfrak{X} = \mathfrak{X}^*.$$

Proof. By [DH92, X, 1.8], we have that $\mathfrak{X} = \mathfrak{X}^*$. If $X \in \mathrm{b}(\mathfrak{X})$, then X is perfect. By Proposition 7.2.10, $X_{\mathfrak{X}} = X_{\mathfrak{X}_*}$. Then $\mathrm{Cosoc}(X) \in \mathfrak{X}_*$ and $\mathfrak{X}^{\mathrm{b}} \subseteq \mathfrak{X}_*$. $\square$

Theorem 7.2.12 (see [ILPM04]). *Let $\mathfrak{X}$ be a Fitting class such that $\mathfrak{X}\mathfrak{S} = \mathfrak{X}$. The correspondence $\mathfrak{F} \longrightarrow \mathfrak{F} \cdot \mathrm{Fit}\big(\mathrm{b}(\mathfrak{X})\big)$, for every Fitting class $\mathfrak{F} \in \mathrm{Sec}(\mathfrak{X}^{\mathrm{b}}, \mathfrak{X})$, defines a bijection*

$$\mathrm{Sec}(\mathfrak{X}^{\mathrm{b}}, \mathfrak{X}) \longrightarrow \mathrm{Sec}\left(\mathrm{Fit}\big(\mathrm{b}(\mathfrak{X})\big), \mathfrak{X} \cdot \mathrm{Fit}\big(\mathrm{b}(\mathfrak{X})\big)\right)$$

whose inverse is defined by $\mathfrak{G} \longrightarrow \mathfrak{G} \cap \mathfrak{X}$, for every $\mathfrak{G} \in \mathrm{Sec}\left(\mathrm{Fit}\big(\mathrm{b}(\mathfrak{X})\big), \mathfrak{X} \cdot \mathrm{Fit}\big(\mathrm{b}(\mathfrak{X})\big)\right)$.

Moreover, the restriction of this bijection to the Lockett section $\mathrm{Locksec}(\mathfrak{X})$ *gives a bijection*

$$\mathrm{Locksec}(\mathfrak{X}) \longrightarrow \mathrm{Locksec}\left(\mathfrak{X} \cdot \mathrm{Fit}\big(\mathrm{b}(\mathfrak{X})\big)\right).$$

Proof. Set $\mathfrak{m} = \mathrm{b}(\mathfrak{X})$, $\mathfrak{M} = \mathrm{Fit}\,\mathfrak{m}$, $\mathfrak{B} = \mathfrak{X}^{\mathrm{b}}$ and $\mathfrak{R} = \mathfrak{X} \cdot \mathfrak{M}$.

If $\mathfrak{F} \in \mathrm{Sec}(\mathfrak{B}, \mathfrak{X})$, then $\mathfrak{F} \cdot \mathfrak{M}$ is a Fitting class by [DH92, XI, 4.7] and Lemma 7.2.6. Obviously $\mathfrak{F} \cdot \mathfrak{M} \in \mathrm{Sec}(\mathfrak{M}, \mathfrak{R})$ and $\mathfrak{F} \subseteq \mathfrak{F} \cdot \mathfrak{M} \cap \mathfrak{X}$. Let G be a group in $\mathfrak{F} \cdot \mathfrak{M} \cap \mathfrak{X}$. Then $G_{\mathfrak{M}} \in \mathfrak{M} \cap \mathfrak{X} = \mathfrak{B}$, by Corollary 7.2.9. Hence $G = G_{\mathfrak{F}} G_{\mathfrak{M}} \in \mathfrak{F}$. Thus, $\mathfrak{F} = \mathfrak{F} \cdot \mathfrak{M} \cap \mathfrak{X}$.

On the other hand, if $\mathfrak{G} \in \mathrm{Sec}(\mathfrak{M}, \mathfrak{R})$, then $\mathfrak{X} \cap \mathfrak{G} \in \mathrm{Sec}(\mathfrak{B}, \mathfrak{X})$ by Corollary 7.2.9 and $(\mathfrak{X} \cap \mathfrak{G}) \cdot \mathfrak{M} \subseteq \mathfrak{G}$. Let G be a group in $\mathfrak{G}$. Then $G_{\mathfrak{X}} = G_{\mathfrak{X} \cap \mathfrak{G}}$ and, since $\mathfrak{G} \subseteq \mathfrak{R}$, we have that

$$G = G_{\mathfrak{X}} G_{\mathfrak{M}} = G_{\mathfrak{X} \cap \mathfrak{G}} G_{\mathfrak{M}} \in (\mathfrak{X} \cap \mathfrak{G}) \cdot \mathfrak{M}$$

and then $\mathfrak{G} = (\mathfrak{X} \cap \mathfrak{G}) \cdot \mathfrak{M}$.

Hence it only remains to prove the properties of the second bijection. We have to prove that $\mathfrak{R}$ is a Lockett class and $\mathfrak{R}_* = \mathfrak{T}_* \cdot \mathfrak{M}$.

If G and H are groups, then it is clear that $\mathrm{E}_{\mathfrak{m}}(G \times H) = \mathrm{E}_{\mathfrak{m}}(G) \times \mathrm{E}_{\mathfrak{m}}(H)$. Since $\mathfrak{T}$ is a Lockett class, by Theorem 7.2.11, we also have that $(G \times H)_{\mathfrak{T}} = G_{\mathfrak{T}} \times H_{\mathfrak{T}}$. Hence

$$(G \times H)_{\mathfrak{R}} = (G \times H)_{\mathfrak{T}} \mathrm{E}_{\mathfrak{m}}(G \times H) = G_{\mathfrak{T}} \mathrm{E}_{\mathfrak{m}}(G) \times H_{\mathfrak{T}} \mathrm{E}_{\mathfrak{m}}(H) = G_{\mathfrak{R}} \times H_{\mathfrak{R}},$$

and $\mathfrak{R}$ is a Lockett class.

Let $\mathfrak{s}(\mathfrak{R})$ denote the largest Fitting subclass of $\mathfrak{R}$ which has a generating system of perfect groups. Then $\mathfrak{M} \subseteq \mathfrak{s}(\mathfrak{R}) \subseteq \mathfrak{R}_*$. Hence $\mathfrak{T}_* \cdot \mathfrak{M} \subseteq \mathfrak{R}_*$. On the other hand, for an arbitrary group G, we have that

$$[G_{\mathfrak{R}}, G] = [G_{\mathfrak{T}} G_{\mathfrak{M}}, G] = [G_{\mathfrak{T}}, G][G_{\mathfrak{M}}, G] \leq G_{\mathfrak{T}_*} G_{\mathfrak{M}},$$

by [DH92, X, 1.3]. Hence $\mathfrak{T}_* \cdot \mathfrak{M} \in \mathrm{Locksec}(\mathfrak{R})$ by [DH92, X, 1.21]. Therefore $\mathfrak{T}_* \cdot \mathfrak{M} = \mathfrak{R}_*$ and we conclude the proof. □

Lemma 7.2.13. *Let $\mathfrak{T}$ be a Fitting class such that $\mathfrak{T} = \mathfrak{T}\mathfrak{S}$.*

1. *Set $\mathfrak{M} = \mathrm{Fit}\big(\mathrm{b}(\mathfrak{T})\big)$. If U is an $\mathfrak{M}$-subgroup of a group G containing $G_{\mathfrak{M}}$, then U is a subgroup of $G_{\mathfrak{M}} G_{\mathfrak{T}}$.*
2. *The class $\mathfrak{T} \cdot \mathrm{Fit}\big(\mathrm{b}(\mathfrak{T})\big)$ is a normal Fitting class.*

Proof. Denote $\mathfrak{m} = \mathrm{b}(\mathfrak{T})$ and $\mathfrak{B} = \mathfrak{T}^{\mathrm{b}}$.

1. We can assume that $G \notin \mathfrak{T}$ and then $\mathrm{b}_{\mathfrak{m}}(G) = \{X_1, \dots, X_n\}$ is a non-empty set and $\mathrm{E}_{\mathfrak{m}}(G) = X_1 \cdots X_n \leq G_{\mathfrak{M}} \leq U$. Hence $\mathrm{b}_{\mathfrak{m}}(U) = \{X_1, \dots, X_n, \dots, X_t\}$, for $n \leq t$, and $\mathrm{E}_{\mathfrak{m}}(U) = \mathrm{E}_{\mathfrak{m}}(G)L$, for $L = X_{n+1} \cdots X_t$. As in the proof of Theorem 7.2.8, $G_{\mathfrak{M}} = G_{\mathfrak{B}} \mathrm{E}_{\mathfrak{m}}(G)$ and $U = U_{\mathfrak{B}} \mathrm{E}_{\mathfrak{m}}(U)$.

Since $X_i \not\leq U_{\mathfrak{B}}$ for each index i, we have that $[U_{\mathfrak{B}}, X_i] \leq U_{\mathfrak{B}} \cap X_i \leq (X_i)_{\mathfrak{B}}$. Thus

$$[\mathrm{E}_{\mathfrak{m}}(G), U_{\mathfrak{B}}] = [X_1, U_{\mathfrak{B}}] \cdots [X_n, U_{\mathfrak{B}}] \leq (X_1)_{\mathfrak{B}} \cdots (X_n)_{\mathfrak{B}} = \mathrm{E}_{\mathfrak{m}}(G)_{\mathfrak{B}},$$

by Lemma 7.2.5 (2a). Analogously, by Lemma 7.2.5 (1a), $[X_i, L] \leq X_i \cap L \leq (X_i)_{\mathfrak{B}}$, for each i. Hence $[\mathrm{E}_{\mathfrak{m}}(G), L] \leq \mathrm{E}_{\mathfrak{m}}(G)_{\mathfrak{B}}$. Therefore

$$[G_{\mathfrak{M}}, U_{\mathfrak{B}} L] = [G_{\mathfrak{B}} \mathrm{E}_{\mathfrak{m}}(G), U_{\mathfrak{B}} L] \leq G_{\mathfrak{B}} [E_{\mathfrak{m}}(G), U_{\mathfrak{B}}][\mathrm{E}_{\mathfrak{m}}(G), L] \leq G_{\mathfrak{B}}.$$

By Theorem 7.2.8, $U_{\mathfrak{B}} L \leq G_{\mathfrak{T}}$ and $U = U_{\mathfrak{B}} \mathrm{E}_{\mathfrak{m}}(G) L \leq E_{\mathfrak{m}}(G) G_{\mathfrak{T}} = G_{\mathfrak{M}} G_{\mathfrak{T}}$.

2. To see that the class $\mathfrak{R} = \mathfrak{T} \cdot \mathfrak{M}$ is a normal Fitting class consider a group G and suppose that U is an $\mathfrak{R}$-subgroup such that $G_{\mathfrak{R}} \leq U \leq G$. By Statement 1, $U_{\mathfrak{M}} \leq G_{\mathfrak{M}} G_{\mathfrak{T}} = G_{\mathfrak{R}}$. On the other hand, using the arguments of the proof of Statement 1, $[\mathrm{E}_{\mathfrak{m}}(G), U_{\mathfrak{T}}] \leq U_{\mathfrak{T}} \cap \mathrm{E}_{\mathfrak{m}}(G) \leq \mathrm{E}_{\mathfrak{m}}(G)_{\mathfrak{B}}$. Then

$$[G_{\mathfrak{M}}, U_{\mathfrak{T}}] = [G_{\mathfrak{B}} \mathrm{E}_{\mathfrak{m}}(G), U_{\mathfrak{T}}] \leq G_{\mathfrak{B}} [E_{\mathfrak{m}}(G), U_{\mathfrak{T}}] \leq G_{\mathfrak{B}} \mathrm{E}_{\mathfrak{m}}(G)_{\mathfrak{B}} \leq G_{\mathfrak{B}}.$$

Hence $U_{\mathfrak{T}} \leq \mathrm{C}_G(G_{\mathfrak{M}}/G_{\mathfrak{B}}) = G_{\mathfrak{T}}$, by Theorem 7.2.8. Thus, $U = U_{\mathfrak{M}} U_{\mathfrak{T}} \leq G_{\mathfrak{R}}$ and $U = G_{\mathfrak{R}}$. □

Lemma 7.2.14. *If* $\mathfrak{X}$ *is a Fitting class such that* $\mathfrak{X} = \mathfrak{X}\mathfrak{S}$*,* X *is a group in* $\mathrm{b}(\mathfrak{X})$ *and* $\mathfrak{F} \in \mathrm{Locksec}(\mathfrak{X})$*, then* $X_{\mathfrak{F}}$ *is not* $\mathfrak{F}$*-maximal in* X*.*

Proof. If $\mathfrak{F} \in \mathrm{Locksec}(\mathfrak{X})$, then, in particular, $\mathfrak{X}^{\mathrm{b}} \subseteq \mathfrak{F} \subseteq \mathfrak{X}$ by Lemma 7.2.11. Moreover $\mathrm{b}(\mathfrak{X}) \subseteq \mathrm{b}(\mathfrak{F})$ by [DH92, XI, 4.7]. Since $X \in \mathrm{b}(\mathfrak{X})$, then $\mathrm{Cosoc}(X) = X_{\mathfrak{F}}$. Suppose that $X_{\mathfrak{F}}$ is $\mathfrak{F}$-maximal in X. Consider a soluble subgroup $Y/X_{\mathfrak{F}}$ of $X/X_{\mathfrak{F}}$. Then $Y \in \mathfrak{X}\mathfrak{S} = \mathfrak{X}$, and by maximality of $X_{\mathfrak{F}}$ in X, we have that $X_{\mathfrak{F}} = Y_{\mathfrak{F}}$. Since $\mathfrak{F} \in \mathrm{Locksec}(\mathfrak{X})$, the quotient $Y/X_{\mathfrak{F}}$ is abelian, by [DH92, X, 1.21]. Then $X/X_{\mathfrak{F}}$ is soluble, and this is a contradiction. ☐

Theorem 7.2.15 (see [ILPM04]). *Let* $\mathfrak{X}$ *be a Fitting class such that* $\mathfrak{X} = \mathfrak{X}\mathfrak{S}$*. If* $\mathfrak{H} \in \mathrm{Sec}\big(\mathfrak{X}_*, \mathfrak{X} \cdot \mathrm{Fit}\big(\mathrm{b}(\mathfrak{X})\big)\big)$*, then*

1. *$\mathfrak{H}$ is an injective Fitting class;*
2. *$\mathfrak{H}$ is a normal Fitting class if and only if* $\mathfrak{H} \in \mathrm{Locksec}\big(\mathfrak{X} \cdot \mathrm{Fit}\big(\mathrm{b}(\mathfrak{X})\big)\big)$*.*

Proof. 1. Write $\mathfrak{m} = \mathrm{b}(\mathfrak{X})$, $\mathfrak{F} = \mathfrak{X} \cap \mathfrak{H}$ and $\mathfrak{G} = \mathfrak{F} \cdot \mathrm{Fit}\,\mathfrak{m}$. If $H \in \mathfrak{H}$, then $H = H_{\mathfrak{X}}\,\mathrm{E}_{\mathfrak{m}}(H)$, by Lemma 7.2.6, since $\mathfrak{H} \subseteq \mathfrak{X} \cdot \mathrm{Fit}\,\mathfrak{m}$. Thus, $H_{\mathfrak{X}} \in \mathfrak{H} \cap \mathfrak{X} = \mathfrak{F}$. Hence $H = H_{\mathfrak{F}}\,\mathrm{E}_{\mathfrak{m}}(H) \in \mathfrak{F} \cdot \mathrm{Fit}\,\mathfrak{m} = \mathfrak{G}$. Hence $\mathfrak{H} \subseteq \mathfrak{G}$.

To see that $\mathfrak{H}$ is injective, let G be a group and let us prove that G possesses $\mathfrak{H}$-injectors. If $\mathrm{b}_{\mathfrak{m}}(G) = \emptyset$, then $G \in \mathfrak{X}$. Hence $G_{\mathfrak{F}} = G_{\mathfrak{H}}$. Since $\mathfrak{F} \in \mathrm{Locksec}(\mathfrak{X})$ by Theorem 7.2.12, the quotient $G/G_{\mathfrak{H}}$ is abelian. Therefore $G_{\mathfrak{H}}$ is a normal $\mathfrak{H}$-injector of G.

Assume that $\mathrm{b}_{\mathfrak{m}}(G) \neq \emptyset$. Since $G_{\mathfrak{H}}$ is a normal subgroup of G we can assume that $\mathrm{b}_{\mathfrak{m}}(G_{\mathfrak{H}}) = \{X_1, \dots, X_r\}$ and $\mathrm{b}_{\mathfrak{m}}(G) = \{X_1, \dots, X_n\}$, for $r \leq n$. If $r = n$, then $G_{\mathfrak{H}} = G_{\mathfrak{F}}\,\mathrm{E}_{\mathfrak{m}}(G_{\mathfrak{H}}) = G_{\mathfrak{F}}\,\mathrm{E}_{\mathfrak{m}}(G) = G_{\mathfrak{G}}$. By Theorem 7.2.12, $\mathfrak{G} \in \mathrm{Locksec}\big(\mathfrak{X} \cdot \mathrm{Fit}\big(\mathrm{b}(\mathfrak{X})\big)\big)$. Since, by Lemma 7.2.13, $\mathfrak{X} \cdot \mathrm{Fit}\big(\mathrm{b}(\mathfrak{X})\big)$ is a normal Fitting class, we deduce that so is $\mathfrak{G}$, by [DH92, X, 3.3]. Therefore $G_{\mathfrak{G}}$ is $\mathfrak{G}$-injector of G and $G_{\mathfrak{H}}$ is $\mathfrak{H}$-injector of G.

Now assume that $r < n$. Fix an index $i \in \{r+1, \dots, n\}$. Clearly, X_i is a perfect comonolithic group such that $X_i \notin \mathfrak{H}$. In addition, $\mathrm{Cosoc}(X_i) \in \mathfrak{H}$, by virtue of Lemma 7.2.11. In particular, X_i is an $\mathfrak{H}$-component of G, By Proposition 7.2.2, X_i possesses $\mathfrak{H}$-injectors. Consider $H = H_{r+1} \cdots H_n$, with $H_i \in \mathrm{Inj}_{\mathfrak{H}}(X_i)$ (note that H_i normalises H_j, $i, j \in \{r+1, \dots, n\}$, by Lemma 7.2.3). By induction on the order of G, if $\mathrm{N}_G(H)$ is a proper subgroup of G, then $\mathrm{N}_G(H)$ possesses $\mathfrak{H}$-injectors. Then G possesses $\mathfrak{H}$-injectors by Theorem 7.2.4. Therefore we can suppose that H is a normal subgroup of G. Then H_i is a normal subgroup of X_i and then $H_i = \mathrm{Cosoc}(X_i) = (X_i)_{\mathfrak{H}}$. Thus $(X_i)_{\mathfrak{H}}$ is an $\mathfrak{F}$-maximal subgroup of X_i, which contradicts Lemma 7.2.14.

2. It is shown in Theorem 7.2.12 that $\mathfrak{X} \cdot \mathrm{Fit}\,\mathfrak{m}$ is a Lockett class. Moreover, by Lemma 7.2.13, it is a normal Fitting class. If $\mathfrak{H} \in \mathrm{Locksec}(\mathfrak{X} \cdot \mathrm{Fit}\,\mathfrak{m})$, then $\mathfrak{H}$ is also a normal Fitting class by [DH92, X, 3.3]. For the converse, consider $\mathfrak{H} \notin \mathrm{Locksec}(\mathfrak{X} \cdot \mathrm{Fit}\,\mathfrak{m})$. Observe that $(\mathfrak{X} \cdot \mathrm{Fit}\,\mathfrak{m})_* = \mathfrak{X}_* \cdot \mathrm{Fit}\,\mathfrak{m}$, by Theorem 7.2.12 and then $\mathrm{Fit}\,\mathfrak{m} \not\subseteq \mathfrak{H}$. Let X be a group in $\mathfrak{m} \setminus \mathfrak{H}$. Then X is a perfect and comonolithic group and $\mathrm{Cosoc}(X) \in \mathfrak{H} \cap \mathfrak{X} = \mathfrak{F}$. Hence $X_{\mathfrak{F}} = \mathrm{Cosoc}(X)$. Since

$\mathfrak{T}_*$ is contained in $\mathfrak{F}$, it follows that $\mathfrak{F} \in \mathrm{Locksec}(\mathfrak{T})$. By Lemma 7.2.14, $X_{\mathfrak{F}}$ is not $\mathfrak{F}$-maximal in X. Therefore $\mathfrak{H}$ is not a normal Fitting class. □

Corollary 7.2.16 (see [ILPM04]). *If $\mathfrak{F}$ is a Fitting class in* $\mathrm{Locksec}(\mathfrak{S})$, *then $\mathfrak{F}$ is injective.*

Proof. If $\mathfrak{F} \in \mathrm{Sec}\big(\mathfrak{S}_*, \mathfrak{S} \cdot \mathrm{Fit}\big(\mathrm{b}(\mathfrak{S})\big)\big) = \mathrm{Sec}\big(\mathfrak{S}_*, \mathfrak{S}^* \cdot \mathrm{Fit}\big(\mathrm{b}(\mathfrak{S})\big)\big)$, then $\mathfrak{F}$ is an injective Fitting class. In particular if $\mathfrak{F} \in \mathrm{Locksec}(\mathfrak{S}) = \{\mathfrak{F} : \mathfrak{S}_* \subseteq \mathfrak{F} \subseteq \mathfrak{S} = \mathfrak{S}^*\}$, then $\mathfrak{F}$ is injective. □

Remarks 7.2.17. The example of a non-injective Fitting class in Section 7.1 affords counterexamples to possible extensions of Theorem 7.2.15:

1. Fitting classes $\mathfrak{H} \in \mathrm{Sec}\big(\mathfrak{T}^{\mathrm{b}}, \mathrm{Fit}\big(\mathrm{b}(\mathfrak{T})\big)\big)$ need not be injective;
2. if $\mathfrak{T} = \mathfrak{T}\mathfrak{S}$, then $\mathrm{Fit}\big(\mathrm{b}(\mathfrak{T})\big)$ need not be injective;
3. Fitting classes $\mathfrak{H} \in \mathrm{Sec}\big(\mathfrak{T}^{\mathrm{b}}, \mathrm{Fit}\big(\mathrm{b}(\mathfrak{T})\big)\big)$ need not be normal. There are normal Fitting classes which does not belong to $\mathrm{Sec}\big(\mathfrak{T}^{\mathrm{b}}, \mathrm{Fit}\big(\mathrm{b}(\mathfrak{T})\big)\big)$.

Proof. Let S and T be non-abelian simple groups such that $\mathrm{D}_0(S, T, 1)$ is a non-injective Fitting class.

1. Let R be a non-abelian simple group and consider the regular wreath product $W = (S \times T) \wr R$. Then W is a perfect comonolithic group (see [DH92, A, 18.8]). Hence $\mathfrak{m} = (W)$ is a preboundary and $\mathfrak{T} = \mathrm{h}(\mathfrak{m})$ is a Fitting class such that $\mathfrak{T} = \mathfrak{T}\mathfrak{S}$ by Theorem 2.4.12 (3). Note that $\mathfrak{T}^{\mathrm{b}} = \mathrm{Fit}\big(\mathrm{Cosoc}(W)\big) = \mathrm{D}_0(S, T)$ is not injective.

2. If $\mathfrak{m} = (S, T, 1)$ and $\mathfrak{T} = \mathrm{h}(\mathfrak{m})$, then $\mathfrak{T} = \mathfrak{T}\mathfrak{S}$ and $\mathrm{Fit}\big(\mathrm{b}(\mathfrak{T})\big) = \mathrm{D}_0(S, T, 1)$ is a non-injective Fitting class.

3. Let $\mathfrak{D}$ denote the class of all direct products of non-abelian simple groups. Let E and F be any two non-abelian simple groups. The regular wreath product $W = E \wr F$ is a perfect comonolithic group. Set $\mathfrak{m} = (W)$, $\mathfrak{T} = \mathrm{h}(\mathfrak{m})$ and $\mathfrak{H} = \mathfrak{S}_*\mathfrak{D}$. Then $\mathfrak{T}^{\mathrm{b}} = \mathrm{D}_0(E, 1) \subseteq \mathfrak{H}$. Moreover, $\mathfrak{H}$ is the smallest normal Fitting class, by [DH92, X, 3.27], and then $\mathfrak{H} \subseteq \mathfrak{T} \cdot \mathrm{Fit}\big(\mathrm{b}(\mathfrak{T})\big)$ by Lemma 7.2.13. If R is a non-abelian simple group, $R \not\cong F$, then the regular wreath product $G = E \wr R \in \mathfrak{T}$. The base subgroup is $E^{\natural} = G_{\mathfrak{H}}$ and $G/G_{\mathfrak{H}} \cong R$ is non-abelian. Therefore $\mathfrak{T}_* \not\subseteq \mathfrak{H}$, by [DH92, X, 1.2]. Clearly $\mathrm{Fit}\big(\mathrm{b}(\mathfrak{T})\big) = \mathrm{Fit}(W) \not\subseteq \mathfrak{H}$. Note that $\mathfrak{T}^{\mathrm{b}}$ is not normal. □

Corollary 7.2.18. *If $\mathfrak{F}$ is a Fitting class such that $\mathfrak{F}\mathfrak{S} = \mathfrak{F}$, then $\mathfrak{F}$ is injective. In particular, the class $\mathfrak{S}$ of all soluble groups is injective.*

Corollary 7.2.19. *A group G possesses a single conjugacy class of $\mathfrak{S}$-injectors if and only if G is soluble.*

Proof. Applying Theorem 2.4.26, only the necessity of the condition is in doubt. Assume that a group G possesses a single conjugacy class of $\mathfrak{S}$-injectors. Let p and q be two different primes dividing the order of $E_{\mathfrak{S}}(G)$

and let P and Q be a Sylow p-subgroup and a Sylow q-subgroup of $E_{\mathfrak{S}}(G)$ respectively. Applying Proposition 7.2.2 (2) and Theorem 7.2.4, there exist $\mathfrak{S}$-injectors V and W of G such that $P \leq V$ and $Q \leq W$. Since V and W are conjugate in G and $E_{\mathfrak{S}}(G)$ is normal in G, it follows that $V \cap E_{\mathfrak{S}}(G)$ contains a Sylow q-subgroup of $E_{\mathfrak{S}}(G)$ for each prime q dividing $|E_{\mathfrak{S}}(G)|$. Therefore $E_{\mathfrak{S}}(G)$ is contained in V and so $E_{\mathfrak{S}}(G) = 1$. This yields that G is soluble. □

Theorem 7.2.20. *Let $\mathfrak{X}$ be a class of quasisimple groups and consider the class*

$$\mathfrak{K}(\mathfrak{X}) = (G : \text{every component of } G \text{ is in } \mathfrak{X}).$$

Then $\mathfrak{K}(\mathfrak{X})$ is an injective Fitting class.

Proof. Let $\mathfrak{X}$ be a class of quasisimple groups and denote $\mathfrak{K} = \mathfrak{K}(\mathfrak{X})$. We first prove that $\mathfrak{K}$ is a Fitting class.

If $G \in \mathfrak{K}$ and N is a normal subgroup of G, then every component of N is a component of G. Hence every component of N is in $\mathfrak{X}$ and then $N \in \mathfrak{K}$.

Suppose that a group G is product $G = NM$, where N and M are normal $\mathfrak{K}$-subgroups of G. Let E be a component of G. Assume that E is not contained in M and E is not contained in N. Applying Proposition 7.2.3, it follows that E centralises MN. Hence E is central in G. This is a contradiction. Therefore either E is contained in M or E is contained in N. Hence E belongs to $\mathfrak{X}$. It implies that $G \in \mathfrak{K}$.

Let E be a component of a group $G \in \mathfrak{K}\mathfrak{S}$. Then $E \in \mathfrak{K}\mathfrak{S}$. Since E is perfect, it follows that $E \in \mathfrak{K}$. Hence $\mathfrak{K} = \mathfrak{K}\mathfrak{S}$ and therefore $\mathfrak{K}$ is injective by Corollary 7.2.18. □

Let $\mathfrak{K}$ be a Fitting class as in Theorem 7.2.20. By Proposition 2.4.6 (5) and Proposition 2.4.6 (2), $\mathfrak{F} \diamond \mathfrak{K} \diamond \mathfrak{S} = \mathfrak{F} \diamond \mathfrak{K}$ for each Fitting class $\mathfrak{F}$. Hence we have the following:

Corollary 7.2.21. *Let $\mathfrak{X}$ be a class of quasisimple groups and consider the class $\mathfrak{K} = \mathfrak{K}(\mathfrak{X})$ as in Theorem 7.2.20. Then $\mathfrak{F} \diamond \mathfrak{K}$ is an injective Fitting class for any Fitting class $\mathfrak{F}$.*

Note that [För87, 2.5(b)] is a consequence of the above corollary.

In the following, we describe another injective Fitting class, the class of all $\mathfrak{F}$-constrained groups.

Proposition 7.2.22. *Let $\mathfrak{F}$ be a Fitting class. In a group G, the following statements are equivalent:*

1. $\mathrm{C}_G(G_{\mathfrak{F}}) \leq G_{\mathfrak{F}}$,
2. $\mathrm{F}^*(G) \in \mathfrak{F}$.

Proof. 1 implies 2. Suppose that E is a component of G such that $E \not\leq G_{\mathfrak{F}}$. Then $[G_{\mathfrak{F}}, E] = 1$, by Proposition 7.2.3. Therefore $E \leq \mathrm{C}_G(G_{\mathfrak{F}}) \leq G_{\mathfrak{F}}$. This contradiction yields $\mathrm{E}(G) \leq G_{\mathfrak{F}}$.

Denote $\pi = \operatorname{char}\mathfrak{F}$. Applying Proposition 2.2.22 (2) we have that $\mathrm{F}^*(G) = \mathrm{F}(G)\,\mathrm{E}(G) = \mathrm{O}_{\pi'}\big(\mathrm{F}(G)\big)\,\mathrm{O}_{\pi}\big(\mathrm{F}(G)\big)\,\mathrm{E}(G)$. On the other hand, the normal $\mathfrak{F}$-subgroup $\mathrm{O}_{\pi'}\big(\mathrm{F}(G)\big) \cap G_{\mathfrak{F}}$ is a nilpotent π'-group. Hence $\mathrm{O}_{\pi'}\big(\mathrm{F}(G)\big) \cap G_{\mathfrak{F}} = 1$ and then $\mathrm{O}_{\pi'}\big(\mathrm{F}(G)\big) \le \mathrm{C}_G(G_{\mathfrak{F}}) \le G_{\mathfrak{F}}$. Therefore $\mathrm{O}_{\pi'}\big(\mathrm{F}(G)\big) = 1$ and $\mathrm{F}(G) = \mathrm{O}_{\pi}\big(\mathrm{F}(G)\big) \in \mathfrak{F}$. Then $\mathrm{F}^*(G) \in \mathfrak{F}$.

2 implies 1. Since $\mathrm{F}^*(G) \in \mathfrak{F}$, it follows that $\mathrm{F}^*(G) \le G_{\mathfrak{F}}$. Thus, by Proposition 2.2.22 (4),

$$\mathrm{C}_G(G_{\mathfrak{F}}) \le \mathrm{C}_G\big(\mathrm{F}^*(G)\big) \le \mathrm{F}^*(G) \le G_{\mathfrak{F}}. \qquad \square$$

Corollary 7.2.23. *Let $\mathfrak{F}$ be a Fitting class. Let G be a group such that $\mathrm{C}_G(G_{\mathfrak{F}}) \le G_{\mathfrak{F}}$. Then for any subnormal subgroup S of G, we have that $\mathrm{C}_S(S_{\mathfrak{F}}) \le S_{\mathfrak{F}}$.*

Corollary 7.2.24 ([IPM86]). *Let $\mathfrak{F}$ be a Fitting class and $\pi = \operatorname{char}\mathfrak{F}$. For any group G, write $\bar{G} = G/\mathrm{O}_{\pi'}(G)$ and adopt the "bar convention:" if $H \le G$, then $\bar{H} = H\,\mathrm{O}_{\pi'}(G)/\mathrm{O}_{\pi'}(G)$.*

The following statements are pairwise equivalent:

1. $\mathrm{C}_{\bar{G}}(\bar{G}_{\mathfrak{F}}) \le \bar{G}_{\mathfrak{F}}$,
2. $\mathrm{E}(\bar{G}) \in \mathfrak{F}$,
3. $\mathrm{F}^*(\bar{G}) \in \mathfrak{F}$.

Definition 7.2.25. *For a Fitting class $\mathfrak{F}$, a group G is said to be* $\mathfrak{F}$-constrained *if G satisfies one condition of Corollary 7.2.24.*

Note that every group is $\mathfrak{Q}$-constrained by Proposition 2.2.22 (4) and a group G is $\mathfrak{N}$-constrained if $\mathrm{C}_G(F(G)) \le F(G)$.

Corollary 7.2.26. *Let $\mathfrak{F}$ be a Fitting class. The class of all $\mathfrak{F}$-constrained groups is an injective Fitting class.*

Proof. Let $\mathfrak{X}$ be the class of all quasisimple $\mathfrak{F}$-groups and consider the Fitting class $\mathfrak{K} = \mathfrak{K}(\mathfrak{X})$. A group G is $\mathfrak{F}$-constrained if and only if $\mathrm{E}\big(G/\mathrm{O}_{\pi'}(G)\big) \in \mathfrak{F}$. This is equivalent to say that every component of the group $G/\mathrm{O}_{\pi'}(G) \in \mathfrak{X}$. This happens if and only if $G/\mathrm{O}_{\pi}(G) \in \mathfrak{K}$, or, in other words, if and only if $G \in \mathfrak{E}_{\pi'} \diamond \mathfrak{K}$. Therefore the class of all $\mathfrak{F}$-constrained groups is the Fitting class $\mathfrak{E}_{\pi'} \diamond \mathfrak{K}$. By Corollary 7.2.21, is an injective Fitting class. $\square$

Recall that the first result of existence and conjugacy of $\mathfrak{N}$-injectors in a universe larger that the soluble groups is due to Mann working on $\mathfrak{N}$-constrained groups [Man71]. Theorem 7.2.1 proves that every group, i.e. every $\mathfrak{Q}$-constrained group, possesses a unique conjugacy class of $\mathfrak{Q}$-injectors. Thus it seems that for every Fitting class $\mathfrak{F}$, the property of being an $\mathfrak{F}$-constrained group is closely related to the conjugacy of $\mathfrak{F}$-injectors. In general the equivalence does not hold as we observed in Corollary 7.2.19 inasmuch as the class $\mathfrak{S}$ of all soluble groups is properly contained in the class of all $\mathfrak{S}$-constrained groups (which is the same as the class of all $\mathfrak{N}$-constrained groups). For Fitting classes $\mathfrak{F}$ such that $\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{Q}$, we have the following result.

Proposition 7.2.27 ([IPM86]). *Let $\mathfrak{F}$ be a Fitting class such that $\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{Q}$.*

If G is an $\mathfrak{F}$-constrained group, then

1. *G possesses a single conjugacy class of $\mathfrak{F}$-injectors, and*
2. *the $\mathfrak{F}$-injectors and the $\mathfrak{Q}$-injectors of G coincide.*

Conversely, if G is a group such that the $\mathfrak{Q}$-injectors are in $\mathfrak{F}$, then G is an $\mathfrak{F}$-constrained group.

Proof. Let G be an $\mathfrak{F}$-constrained group. Then, since $\operatorname{char}\mathfrak{F} = \mathbb{P}$, we have that $\mathrm{F}^*(G) = G_{\mathfrak{F}}$, by Corollary 7.2.24. Let V be an $\mathfrak{Q}$-injector of G. Then V is an $\mathfrak{Q}$-maximal subgroup containing $\mathrm{F}^*(G)$ [BL79]. Observe that, since $\mathrm{F}^*(G) \leq V_{\mathfrak{F}}$, we have that

$$\mathrm{C}_V(V_{\mathfrak{F}}) \leq \mathrm{C}_V\big(\mathrm{F}^*(G)\big) \leq \mathrm{F}^*(G) \leq V_{\mathfrak{F}},$$

and V is an $\mathfrak{F}$-constrained group. Thus $V = \mathrm{F}^*(V) = V_{\mathfrak{F}}$ and V is an $\mathfrak{F}$-maximal subgroup of G.

If S is a subnormal subgroup of G, then $V \cap S$ is an $\mathfrak{Q}$-injector of S. Since $\mathfrak{F}$ is contained in $\mathfrak{Q}$, we have that $V \cap S$ is $\mathfrak{F}$-maximal in S.

In order to obtain the conjugacy of all $\mathfrak{F}$-injectors of G, it is enough to prove that each $\mathfrak{F}$-injector of G is an $\mathfrak{Q}$-injector of G. Let H be an $\mathfrak{F}$-injector of G, then H is an $\mathfrak{F}$-maximal subgroup of G containing $G_{\mathfrak{F}} = \mathrm{F}^*(G)$. Hence H is an $\mathfrak{Q}$-subgroup of G containing $\mathrm{F}^*(G)$ and there exists a $\mathfrak{Q}$-injector V of G such that $H \leq V$. By the previous arguments, $V = H$.

The converse is obvious. ◻

Lemma 7.2.28. *Let $\mathfrak{H}$ and $\mathfrak{F}$ be Fitting classes and let G be a group such that*

$$\mathrm{C}_G(G_{\mathfrak{H}\diamond\mathfrak{F}}/G_{\mathfrak{H}}) \leq G_{\mathfrak{H}\diamond\mathfrak{F}}.$$

Let J be subgroup of G containing $G_{\mathfrak{H}\diamond\mathfrak{F}}$. Then

1. *$J \in \mathrm{Max}_{\mathfrak{H}\diamond\mathfrak{F}}(G)$ if and only if $J/G_{\mathfrak{H}} \in \mathrm{Max}_{\mathfrak{F}}(G/G_{\mathfrak{H}})$.*
2. *$J \in \mathrm{Inj}_{\mathfrak{H}\diamond\mathfrak{F}}(G)$ if and only if $J/G_{\mathfrak{H}} \in \mathrm{Inj}_{\mathfrak{F}}(G/G_{\mathfrak{H}})$.*

Proof. The condition $\mathrm{C}_G(G_{\mathfrak{H}\diamond\mathfrak{F}}/G_{\mathfrak{H}}) \leq G_{\mathfrak{H}\diamond\mathfrak{F}}$ is equivalent to $\mathrm{C}_{\bar{G}}(\bar{G}_{\mathfrak{F}}) \leq \bar{G}_{\mathfrak{F}}$ for the quotient group $\bar{G} = G/G_{\mathfrak{H}}$. Let S be a subnormal subgroup of G. By Corollary 7.2.23 we have that $\mathrm{C}_{\bar{S}}(\bar{S}_{\mathfrak{F}}) \leq \bar{S}_{\mathfrak{F}}$, for $\bar{S} = SG_{\mathfrak{H}}/G_{\mathfrak{H}}$. But, since $S_{\mathfrak{H}} = G_{\mathfrak{H}} \cap S$, we have that $\bar{S} \cong S/S_{\mathfrak{H}}$. Therefore, for any subnormal subgroup S of G, $\mathrm{C}_S(S_{\mathfrak{H}\diamond\mathfrak{F}}/S_{\mathfrak{H}}) \leq S_{\mathfrak{H}\diamond\mathfrak{F}}$.

Let K be a subgroup of G such that $G_{\mathfrak{H}\diamond\mathfrak{F}} \leq K$. Observe that $G_{\mathfrak{H}} \leq K$ implies that $G_{\mathfrak{H}} \leq K_{\mathfrak{H}} \cap G_{\mathfrak{H}\diamond\mathfrak{F}}$. On the other hand $K_{\mathfrak{H}} \cap G_{\mathfrak{H}\diamond\mathfrak{F}}$ is a normal $\mathfrak{H}$-subgroup of K and then of $G_{\mathfrak{H}\diamond\mathfrak{F}}$, i.e.

$$K_{\mathfrak{H}} \cap G_{\mathfrak{H}\diamond\mathfrak{F}} \leq (G_{\mathfrak{H}\diamond\mathfrak{F}})_{\mathfrak{H}} \leq G_{\mathfrak{H}}$$

and therefore $G_{\mathfrak{H}} = K_{\mathfrak{H}} \cap G_{\mathfrak{H}\diamond\mathfrak{F}}$. Thus $[K_{\mathfrak{H}}, G_{\mathfrak{H}\diamond\mathfrak{F}}] \leq G_{\mathfrak{H}}$. This implies that

$$K_{\mathfrak{H}} \leq \mathrm{C}_G(G_{\mathfrak{H}\diamond\mathfrak{F}})/G_{\mathfrak{H}} \leq G_{\mathfrak{H}\diamond\mathfrak{F}}$$

and then $G_{\mathfrak{H}} = K_{\mathfrak{H}}$.

Using this fact, the proof is a routine checking. □

Corollary 7.2.29. *Let $\mathfrak{F}$ be a Fitting class containing the class of all nilpotent groups $\mathfrak{N}$. Assume that every $\mathfrak{F}$-constrained group possesses $\mathfrak{F}$-injectors.*

Then, for every Fitting class $\mathfrak{H}$, the class $\mathfrak{H} \diamond \mathfrak{F}$ is injective.

Proof. We have to prove that $\mathrm{Inj}_{\mathfrak{H}\diamond\mathfrak{F}}(G) \neq \emptyset$ for every group G. Let G be a minimal counterexample. First we notice that a subgroup E is an $\mathfrak{H}\diamond\mathfrak{F}$-component of G such that $\mathrm{N}(E) \in \mathfrak{H}$ if and only if $EG_{\mathfrak{H}}/G_{\mathfrak{H}}$ is a component of $G/G_{\mathfrak{H}}$ such that $EG_{\mathfrak{H}}/G_{\mathfrak{H}} \notin \mathfrak{F}$.

Let $\mathcal{E} = \{E_1, \dots, E_n\}$ be the set of all $\mathfrak{H} \diamond \mathfrak{F}$-components of G such that $\mathrm{N}(E_i) \in \mathfrak{H}$ and suppose that $\mathcal{E} \neq \emptyset$. For $J_i \in \mathrm{Inj}_{\mathfrak{H}\diamond\mathfrak{F}}(E_i)$, $i = 1, \dots, n$, construct the product $J = J_1 \cdots J_n$. If $\mathrm{N}_G(J)$ is a proper subgroup of G, then $\mathrm{Inj}_{\mathfrak{H}\diamond\mathfrak{F}}\big(\mathrm{N}_G(J)\big) \neq \emptyset$, by minimality of G. Since the set $\mathcal{E}$ is invariant by conjugation of the elements of G, we can apply Theorem 7.2.4 and then $\mathrm{Inj}_{\mathfrak{H}\diamond\mathfrak{F}}(G) \neq \emptyset$. This contradicts our assumption. Therefore J is a normal subgroup of G and then each J_i is normal in E_i, for $i = 1, \dots, n$. This implies that $J_i \leq \mathrm{Cosoc}(E_i)$.

Let $P/(E_i)_{\mathfrak{H}}$ be a Sylow subgroup of $E_i/(E_i)_{\mathfrak{H}}$. Then $P \in \mathfrak{H} \diamond \mathfrak{F}$. Observe that, since $J_i/\mathrm{N}(E_i) \leq \mathrm{Z}\big(E_i/\mathrm{N}(E_i)\big)$, the subgroup P is normal in PJ_i. Then $PJ_i \in \mathfrak{H} \diamond \mathfrak{F}$. By maximality of J_i, we have that $P \leq J_i$. Since this happens for any Sylow subgroup of E_i, we have that $E_i \leq J_i$, which is a contradiction. Hence $\mathcal{E} = \emptyset$ and every component of $G/G_{\mathfrak{H}}$ is in $\mathfrak{F}$. Therefore $\mathrm{E}(G/G_{\mathfrak{H}}) \in \mathfrak{F}$. This implies that $G/G_{\mathfrak{H}}$ is $\mathfrak{F}$-constrained, i.e. $\mathrm{C}_G(G_{\mathfrak{H}\diamond\mathfrak{F}})/G_{\mathfrak{H}} \leq G_{\mathfrak{H}\diamond\mathfrak{F}}$ by Corollary 7.2.24 . By hypothesis, the group $G/G_{\mathfrak{H}}$, possesses $\mathfrak{F}$-injectors. By Lemma 7.2.28, the group G possesses $\mathfrak{H}\diamond\mathfrak{F}$-injectors. This is the final contradiction. □

Corollary 7.2.30 (M. J. Iranzo and F. Pérez-Monasor). *Let $\mathfrak{F}$ be a Fitting class such that $\mathfrak{N} \subseteq \mathfrak{F} \subseteq \mathfrak{Q}$. Then, for every Fitting class $\mathfrak{H}$, the class $\mathfrak{H} \diamond \mathfrak{F}$ is injective.*

In particular, the class $\mathfrak{N}$ of all nilpotent groups is injective (P. Förster [För85a]).

Observe that $\mathfrak{E}_{\pi'}\mathfrak{N}_\pi = \mathfrak{E}_{\pi'}\mathfrak{N}$. This leads us to the following.

Corollary 7.2.31. *Let π be a set of prime numbers. The Fitting class $\mathfrak{E}_{\pi'}\mathfrak{N}_\pi$ is injective.*

In particular, for any prime p, the Fitting class $\mathfrak{E}_{p'}\mathfrak{S}_p$ of all p-nilpotent groups is injective.

Remark 7.2.32. Let p be a prime. We say that a group G is *p-constrained* if G is $\mathfrak{S}_p$-constrained group. M. J. Iranzo and M. Torres proved in [IT89] that a group G possesses a unique conjugacy class of p-nilpotent injectors if and only if G is p-constrained. Moreover, in this case,

$$\mathrm{Inj}_{\mathfrak{E}_{p'}\mathfrak{S}_p}(G) = \{\mathrm{O}_{p',p}(G)P : P \in \mathrm{Syl}_p(G)\},$$

and the p-nilpotent injectors of G are the p-nilpotent maximal subgroups of G containing $\mathrm{O}_{p',p}(G)$.

Theorem 7.2.33 ([IPM88]). *Every extensible saturated Fitting formation is injective.*

Proof. Assume the result is false and let G be counterexample of least order. Clearly $\pi = \operatorname{char}\mathfrak{F} = \pi(\mathfrak{F})$ and $\mathfrak{N}_\pi \subseteq \mathfrak{F} \subseteq \mathfrak{E}_\pi$ since $\mathfrak{F}$ is saturated.

Assume the result is false and let G be counterexample of least order. Since G possesses $\mathfrak{F}$-injectors if and only if $G/\mathrm{O}_{\pi'}(G)$ possesses $\mathfrak{F}$-injectors, it follows that $\mathrm{O}_{\pi'}(G) = 1$. Also, since $\mathfrak{F}$ is an extensible homomorph, G has $\mathfrak{F}$-injectors if and only if $G/G_\mathfrak{F}$ possesses $\mathfrak{F}$-injectors. Therefore $G_\mathfrak{F} = 1$.

Consider, as in Theorem 7.2.4, the set $\mathcal{E} = \{E_1, \dots, E_n\}$ of all $\mathfrak{F}$-components of G and suppose that $\mathcal{E} \neq \emptyset$. Observe that, since $G_\mathfrak{F} = 1$, the $\mathfrak{F}$-components of G are just the components. Let $i = 1, \dots, n$. Then every $\mathfrak{F}$-maximal subgroup J_i of E_i containing the $\mathfrak{F}$-radical of E_i is an $\mathfrak{F}$-injector of E_i by Proposition 7.2.2 (2). Consider the subgroup $J = \langle J_1, \dots, J_n\rangle$. By Theorem 7.2.4, we have that J is normal in G. Moreover, J is an $\mathfrak{F}$-group. Hence J is contained in $G_\mathfrak{F}$ and then $J_i = 1$. This implies that $E_i \in \mathfrak{E}_{\pi'}$ and, since E_i is subnormal in G, we obtain that $E_i = 1$. Then $\mathrm{E}(G) = 1$ and $\mathrm{F}^*(G) = \mathrm{F}(G) = \mathrm{O}_\pi\big(\mathrm{F}(G)\big) \times \mathrm{O}_{\pi'}\big(\mathrm{F}(G)\big)$. But $\mathrm{O}_\pi\big(\mathrm{F}(G)\big) \leq G_\mathfrak{F} = 1$ and $\mathrm{O}_{\pi'}\big(\mathrm{F}(G)\big) \leq \mathrm{O}_{\pi'}(G) = 1$. Hence $\mathrm{F}^*(G) = 1$. This contradiction proves the theorem. ☐

It is not difficult to prove that every extensible saturated Fitting formation $\mathfrak{F}$ is of the form

$$\mathfrak{F} = \big(G : \text{all composition factors of } G \text{ belong to } \mathfrak{F} \cap \mathfrak{J}\big).$$

The most popular extensible saturated Fitting formations are the class $\mathfrak{E}_\pi$, π a set of primes, and the class $\mathfrak{S}$ of all soluble groups.

Applying the above result, every finite group possesses $\mathfrak{E}_\pi$-injectors. In general, if V is an $\mathfrak{E}_\pi$-injector of a group G, then V is a maximal π-subgroup of G containing $\mathrm{O}_\pi(G)$; but $|G : V|$ need not to be a π'-number. If G possesses Hall π-subgroups, in particular if G is soluble, then the $\mathfrak{E}_\pi$-injectors of G are the Hall π-subgroups of G.

Concluding Remarks 7.2.34. There are many other injective Fitting classes closely related to the ones presented in the section. For instance, for each prime p, let us consider the class $\mathfrak{E}_{p^*p}$, the *p^*p-groups*, defined by H. Bender (see [HB82b]). This is the class composed by all groups G factorising as $G = N\,\mathrm{C}^*_G(P)$ for any normal subgroup N and any $P \in \mathrm{Syl}_p(N)$, where $\mathrm{C}^*_G(P)$

is the largest normal subgroup of $\mathrm{N}_G(P)$ acting nilpotently on P. A group $G \in \mathfrak{E}_{p^*p}$ such that $\mathrm{O}^p(G) = G$ is said to be a *p^*-group* and the class of all p^*-groups is denoted by $\mathfrak{E}_{p^*}$. The class $\mathfrak{E}_{p^*p}$ is an injective Fitting class and, in fact, any Fitting class $\mathfrak{F}$ such that $\mathfrak{E}_{p^*p} \subseteq \mathfrak{F} \subseteq \mathfrak{E}_{p^*}\mathfrak{S}_p$ is injective (see [IT89]).

Other examples of injective Fitting classes are the class $\mathfrak{E}_{p'}\mathfrak{Q}$ of all p-quasinilpotent groups and the class $\mathfrak{O}^p = \big(G : G/\mathrm{C}_G\big(\mathrm{O}_p(G)\big) \in \mathfrak{S}_p\big)$ (see [MP92]). These classes satisfy the following chain

$$\mathfrak{E}_{p'}\mathfrak{Q} \subset \mathfrak{E}_{p^*p} \subset \mathfrak{E}_{p^*}\mathfrak{S}_p \subset \mathfrak{O}^p$$

where all containments are strict.

Finally let us mention the contribution of M. J. Iranzo, J. Medina, and F. Pérez-Monasor in [IMPM01] that, using that the class $\mathfrak{E}_\pi$ is injective, proves that the class of all p-decomposable groups is an injective Fitting class.

Bearing in mind Salomon's example in Section 7.1 and the results of the present section, the following question arises:

Open question 7.2.35. *Is it possible to characterise the injective Fitting classes?*

7.3 Supersoluble Fitting classes

It is well-known that the product of two supersoluble normal subgroups of a group need not to be supersoluble. In other words, the class $\mathfrak{U}$ of all supersoluble groups is not a Fitting class, although $\mathfrak{U}$ is closed for subnormal subgroups. This failure is the starting point of two fruitful lines of research.

1. Obviously the direct product of supersoluble subgroups is always supersoluble; hence the study of different types of products, with extra conditions, such that those special products of supersoluble subgroups give a new supersoluble subgroup makes sense; following these ideas a considerable amount of papers has been published in the last years dealing with totally permutable products, mutually permutable products, ... (see, for instance, [AS89], [BBPR96a])

2. On the other hand we can analyse the properties of supersoluble Fitting classes, i.e. those Fitting classes contained in the class $\mathfrak{U}$ of all supersoluble groups. This investigation was encouraged by the excellent results obtained in metanilpotent Fitting classes due to T. O. Hawkes, T. R. Berger, R. A. Bryce, and J. Cossey (see [DH92, XI, Section 2]).

The question of the existence of Fitting classes composed of supersoluble groups was settled by M. Menth in [Men95b]. In this paper he presented a family of supersoluble non-nilpotent Fitting classes. These Fitting classes are constructed via Dark's method (see [DH92, IX, Section 5]). Terminology and notation are mainly taken from [DH92, IX, Sections 5 and 6] and the papers of Menth [Men94, Men95b, Men95a, Men96].

Following Dark's strategy, we start with a identification of the universe of groups to consider. Let p be a prime such that $p \equiv 1 \pmod 3$, and n a primitive 3rd root of unity in the field $\mathrm{GF}(p)$. The universe to consider will be the class $\mathfrak{S}_p\mathfrak{S}_3$.

Now the ingredients are:

1. *The key section* $\kappa(G)$ of a group $G \in \mathfrak{S}_p\mathfrak{S}_3$ is $\kappa(G) = \mathrm{O}^p(G)$.
2. *The associated class* $\mathfrak{X}$. Consider the groups
$$T = \langle a, b : a^p = b^p = [a, b, a, a] = [a, b, a, b] = [a, b, b, b] = 1\rangle$$
and
$$V = \langle T, s : s^3 = 1, a^s = a^n, b^s = b^n\rangle.$$
These groups have the following properties:
 a) $|T| = p^5$, $T' = \mathrm{Z}_2(T)$ and the factors of the central series are $T/T' \cong C_p \times C_p$, $T'/\mathrm{Z}(T) \cong C_p$, and $\mathrm{Z}(T) \cong C_p \times C_p$;
 b) $\mathrm{Z}(V) = \mathrm{Z}(T)$ and the conjugation by s induces on T/T' the power automorphism $x \longmapsto x^n$, on $T'/\mathrm{Z}(T)$ the power automorphism $x \longmapsto x^{n^2}$, and centralises $\mathrm{Z}(T)$;
 c) every extension of T by an elementary abelian 3-group is supersoluble; in particular V is supersoluble.

 Let $\mathfrak{V}_0$ be the class of all finite groups G which can be factorised as $G = XY$ where
 a) $X = \mathrm{O}_p(G)$ is a central product of copies T_i of T (the empty product, i.e. the case $\mathrm{O}_p(G) = 1$, is admitted);
 b) $Y \in \mathrm{Syl}_3(G)$ and for every index i, we have that $Y/\mathrm{C}_Y(T_i) \cong C_3$ and $[T_i]\big(Y/\mathrm{C}_Y(T_i)\big) \cong V$.
3. *The class* $\mathfrak{V} = D^p(\mathfrak{V}_0) = \big(G \in \mathfrak{S}_p\mathfrak{S}_3 : \kappa(G) \in \mathfrak{V}_0\big)$.

The following result is due to Menth. We quote it here without proof.

Theorem 7.3.1 ([Men95b, 4.2]). *The class* $\mathfrak{V} = \mathrm{Fit}(V)$ *is the Fitting class generated by* V. *If* $G \in \mathfrak{V}$ *and write* $P = \mathrm{O}_p(G)$, $V_0 = \mathrm{O}^p(G)$, *and* $C = \mathrm{O}_3\big(\mathrm{Z}_\infty(V_0)\big)$, *then*

1. G *is supersoluble;*
2. $\mathrm{F}(G) = PC$ *and* $G/\mathrm{F}(G)$ *is an elementary abelian 3-group;*
3. $G = \mathrm{C}_P(Y)V_0$ *for every Sylow 3-subgroup* Y *of* G;
4. $\mathrm{Soc}(G) \leq \mathrm{Z}(G)$.

 Moreover, $\mathfrak{V}$ *is a Lockett class ([Men94, 2.2]).*

This supersoluble Fitting class is contained in $\mathfrak{S}_p\mathfrak{S}_3$. The above construction can be generalised to include examples of supersoluble Fitting classes in $\mathfrak{S}_p\mathfrak{S}_q$ for other odd primes q. In [Tra98], G. Traustason gives an example of a supersoluble Fitting class in $\mathfrak{S}_p\mathfrak{S}_2$. This class is also constructed following Dark's strategy.

In contrast with metanilpotent Fitting classes, supersoluble Fitting classes are extremely restricted in additional closure properties. This is also proved by M. Menth in [Men95a]. In this section we will present the most relevant results of this paper.

Lemma 7.3.2. *Let G be a supersoluble group. Then,* $\mathrm{Fit}(G)$ *is supersoluble if and only if* $\mathrm{Fit}(G) \subseteq \mathrm{lform}(G)$.

Proof. Denote $\mathfrak{G} = \mathrm{lform}(G)$. Since G is supersoluble, $\mathfrak{G} \subseteq \mathfrak{U}$. Hence $\mathrm{Fit}(G)$ is a supersoluble Fitting class.

For the converse, observe that since G is supersoluble, the quotient group $G/\mathrm{O}_{p',p}(G)$ is an abelian group of exponent $e(p)$ dividing $p-1$ for each prime p by [DH92, IV, 3.4 (f)]. Applying Theorem 3.1.11, the saturated formation $\mathfrak{G}$ is locally defined by the formation function f, where $f(p) = \mathrm{form}\big(G/\mathrm{O}_{p',p}(G)\big)$, if p divides $|G|$, and $f(p) = \emptyset$ if p does not divide $|G|$. It is rather easy to see that $f(p) = \mathfrak{A}\big(e(p)\big)$, where $\mathfrak{A}(m)$ denotes the class of all abelian groups of exponent dividing m. Since $f(p)$ is subgroup-closed for all primes p, the formation $\mathfrak{G} = \mathrm{LF}(f)$ is subgroup-closed by [DH92, IV, 3.14]. Hence the class $\mathrm{Fit}(G) \cap \mathfrak{G}$ is S_n-closed.

Let X be a group which is the product of two normal subgroups N_1, N_2 of X such that $N_1, N_2 \in \mathrm{Fit}(G) \cap \mathfrak{G}$. For each prime p, we have that $X/\mathrm{O}_{p',p}(X)$ is the normal product of $N_1\mathrm{O}_{p',p}(X)/\mathrm{O}_{p',p}(X)$ and $N_2\mathrm{O}_{p',p}(X)/\mathrm{O}_{p',p}(X)$. Since $X \in \mathrm{Fit}(G)$, then X is supersoluble and so $X/\mathrm{O}_{p',p}(X)$ is abelian by [DH92, IV, 3.4 (f)]. Moreover, for $i = 1, 2$, we have that

$$N_i\mathrm{O}_{p',p}(X)/\mathrm{O}_{p',p}(X) \cong N_i/\mathrm{O}_{p',p}(N_i) \in \mathfrak{A}\big(e(p)\big),$$

since $N_i \in \mathrm{LF}(f)$. Hence $X/\mathrm{O}_{p',p}(X) \in \mathfrak{A}\big(e(p)\big)$. Hence $X \in \mathfrak{G}$. This is to say that the class $\mathrm{Fit}(G) \cap \mathfrak{G}$ is N_0-closed.

Therefore $\mathrm{Fit}(G) \cap \mathfrak{G}$ is a Fitting class containing G. Thus, $\mathrm{Fit}(G) \subseteq \mathrm{lform}(G)$. ☐

Lemma 7.3.3. *Let X be a group such that the regular wreath product $W = X \wr C$ is a supersoluble group for some non-trivial group C. Then X is nilpotent.*

Proof. Suppose that the result is false and let X be a counterexample of minimal order. Then X is a non-nilpotent group and the regular wreath product $W = X \wr C$ is a supersoluble group for some non-trivial group C. Denote by $X^\natural$ the base of group of W. If Y is a subgroup of X, denote by $Y^\natural$ the corresponding subgroup of $X^\natural$. Let N be a minimal normal subgroup of X. Then $(X/N) \wr C \cong W/N^\natural$ by [DH92, A, 18.2(d)]. Moreover $(X/N) \wr C$ is supersoluble. By minimality of X, we have that X/N is nilpotent. Since X is non-nilpotent, it follows that $X \in \mathrm{b}(\mathfrak{N})$ and so X is a primitive group. Since X is a supersoluble non-nilpotent primitive group, then X possesses a unique minimal normal subgroup Y which is a cyclic group of prime order, q say, and $\mathrm{Z}(X) = 1$. Then $Y^\natural$ is a minimal normal subgroup of W by [DH92, A, 18.5(a)]), and W is primitive by [DH92, A, 18.5(b)]. In particular, the order of the minimal normal subgroup of W is a prime. Note that the order of $Y^\natural$ is $q^{|C|}$. This contradiction proves the lemma. $\square$

Theorem 7.3.4. *Let $\mathfrak{F}$ be a supersoluble Fitting class. Assume that X is a group and p is a prime such that the regular wreath product $X \wr C_p \in \mathfrak{F}$. Then X is a p-group.*

Proof. Set $G = X \wr C_p \in \mathfrak{F}$. We can assume, without loss of generality that $\mathfrak{F} = \operatorname{Fit}(G)$. By Lemma 7.3.2, $\mathfrak{F} \subseteq \operatorname{lform}(G)$. We can apply now some results due to P. Hauck (see [DH92, X, 2.9 and 2.10]) to deduce that $X \wr P \in \mathfrak{F} \subseteq \operatorname{lform}(G)$, for every p-group P.

Suppose further that X is not a p-group. Then there exists a prime divisor $q \neq p$ of $|X|$. Since $X \wr C_p$ is supersoluble, it follows that X is nilpotent by Lemma 7.3.3. Therefore $X = \mathrm{O}_{q',q}(X)$.

Applying Theorem 3.1.11, $\operatorname{lform}(G) = \operatorname{LF}(f)$ is locally defined by the formation function f, where $f(r) = \operatorname{form}\big(G/\operatorname{O}_{r',r}(G)\big)$, if r divides $|G|$, and $f(r) = \emptyset$ if r does not divide $|G|$. Then $P \in f(q)$ for all p-groups P. Hence $\mathfrak{S}_p \subseteq f(q) = \operatorname{form}\big(G/\operatorname{O}_{q',q}(G)\big)$.

Observe that for every natural number e, the class $\mathfrak{S}_p^{(e)} = \big(G \in \mathfrak{S}_p : \exp(G) \leq p^e\big)$ is a subformation of $\mathfrak{S}_p$. Hence $\operatorname{form}\big(G/\operatorname{O}_{q',q}(G)\big)$ has infinitely many subformations, and this contradicts the theorem of R. M. Bryant, R. A. Bryce, and B. Hartley ([DH92, VII, 1.6]). $\square$

Fitting classes with the property of Theorem 7.3.4 are called *abstoßend* by P. Hauck. This term is translated into English as *repellent* (see [DH92, X, 2, Exercise 4]).

Proposition 7.3.5. *Let $\mathfrak{F}$ be a Fitting class of soluble groups. Suppose that the group G is a semidirect non-direct product $G = [N]A$ of the normal subgroup N by a q-subgroup A, q a prime. Suppose that A induces the automorphism group A^* on N and consider the semidirect product $G^* = [N]A^*$. Then $G \in \mathfrak{F}$ if and only if $G^* \in \mathfrak{F}$.*

Proof. First observe that $A^* \cong A/\operatorname{C}_A(N)$ and $C = \operatorname{C}_A(N)$ is a normal subgroup of G. Thus, the group $G^* \cong G/C$ is an epimorphic image of G. Moreover, since the semidirect product is non-direct, $C \neq A$.

Suppose that $G \in \mathfrak{F}$. Then $\mathfrak{S}_q \subseteq \mathfrak{F}$, by [DH92, IX, 1.9], and $G/N \cong A \in \mathfrak{F}$. Moreover $N \cap C = 1$ and $G/NC \cong A^*$ is nilpotent. By Lemma 2.4.2, the $G^* \cong G/C \in \mathfrak{F}$.

The same arguments show that G is in $\mathfrak{F}$ if $G^* \cong G/C \in \mathfrak{F}$. $\square$

Proposition 7.3.6. *Let $\mathfrak{F}$ be a Fitting class and suppose that G is an $\mathfrak{F}$-group such that G is the semidirect product $G = [N]\langle s\rangle$ where $N = N_1 \times \dots \times N_n$, N_i normal in G, $1 \leq i \leq n$. Let σ_i be the automorphism of N_i induced by conjugation of s. For each $i = 1, \dots, n$, consider a copy $\overline{N_i} \cong N_i$ and construct the semidirect product $H_i = [\overline{N_i} \times N_i]\langle s\rangle$, where s induces on $\overline{N_i}$ the automorphism σ_i^{-1}. Then $H_i \in \mathfrak{F}$.*

Proof. Without loss of generality, we can argue with the normal subgroup N_1. Consider the direct product $N^* = \overline{N_1} \times N_1 \times \cdots \times N_n$ and a cyclic group $\langle t \rangle$ such that $\langle s \rangle \cong \langle t \rangle$. Construct the semidirect product $G^* = [N^*](\langle s \rangle \times \langle t \rangle)$, where $\overline{N_1}$ and all factors N_i are normal in G^* and the operation of s and t on the N_i is as follows: s centralises $\overline{N_1}$ and acts on N_i in the same way as σ_i; t centralises N_1, operates on N_i in the same way as σ_i for $2 \leq i \leq n$ and on $\overline{N_1}$ as σ_1. Since $N_1 \in \mathfrak{F}$, we have that $\overline{N_1} \in \mathfrak{F}$. Therefore $\langle N^*, s \rangle \cong \langle N^*, t \rangle \cong \overline{N_1} \times G \in \mathfrak{F}$. Then $G^* \in \mathfrak{F}$. The normal subgroup $\langle N^*, st^{-1} \rangle$ of G^* is an $\mathfrak{F}$-group. Finally, observe that $H_1 \cong \langle \overline{N_1} \times N_1, st^{-1} \rangle$ and this is normal in $\langle N^*, st^{-1} \rangle$. Hence $H_1 \in \mathfrak{F}$. □

Remarks and notation 7.3.7. Let p and q be different primes, p odd, such that q divides $p - 1$. Let e and r be natural numbers.

1. Recall that $\mathrm{Aut}(C_{p^e}) \cong C_{p^{e-1}(p-1)}$ (see [DH92, A, 21.1]). Each natural number m, with $\gcd(m, p) = 1$ and $1 \leq m \leq p^e$ can be uniquely written in the form $m = tp + k$, for $0 \leq t \leq p^{e-1} - 1$ and $1 \leq k \leq p - 1$. The pair (t, k) uniquely determines the automorphism $\sigma(t, k)$ of the cyclic group $C_{p^e} = \langle x \rangle$ of order p^e, defined by $x^{\sigma(t,k)} = x^{tp+k} = x^m$.
2. Therefore there exists an automorphism $\alpha = \sigma(t, k)$ of C_{p^e} of order q. This means that $n = tp + k \neq 1$ is an integer such that $n^q \equiv 1 \pmod{p^e}$. Moreover any automorphism of C_{p^e} of order q is of the form α^t for $1 \leq t \leq q - 1$. If x is a generator of the cyclic group C_{p^e}, then $x^{\alpha^t} = x^{n^t}$.
3. Let X_r be the direct product of r copies of the cyclic group of order p^e. Construct the semidirect product $G_r = [X_r]C$ of X_r and a cyclic group $C = \langle s \rangle \cong C_q$ where s raises all elements of X_r to the same n-th power. If $\{x_1, \ldots, x_r\}$ is a set of r generators (a basis) of X_r, observe that all subgroups of the form $\langle x_i, s \rangle$, for $i = 1, \ldots, r$, are isomorphic to $\mathrm{E}(q|p^e)$ (see [DH92, B, 12.5]).

Lemma 7.3.8. *Consider the Fitting class,* $\mathrm{Fit}(G_r)$*, generated by the group* G_r*. For any natural number* k*, let* $H_k = [X_k]C$ *denote a group which is a semidirect product of the homocyclic abelian group* X_k *of exponent* p^e *and rank* $k \geq 1$ *by a cyclic group* $C = \langle \alpha \rangle$ *such that* α *is an automorphism of* X_k *of order* q *and* $\det(\alpha) = 1$*. Then* $H_k \in \mathrm{Fit}(G_r)$*.*

Proof. The prime q is a divisor of $p - 1$ and then $\gcd(q, p) = 1$. By [DH92, A, 11.6], X_k has a direct decomposition $X_k = X_{k(1)} \times \cdots \times X_{k(s)}$ into $\langle \alpha \rangle$-admissible subgroups $X_{k(i)}$ with the following properties for each $i = 1, \ldots, s$:

1. $X_{k(i)}$ is indecomposable as a $\langle \alpha \rangle$-module;
2. $Y_{k(i)} = X_{k(i)}/\Phi(X_{k(i)})$ is an irreducible $\mathrm{GF}(p)\langle \alpha \rangle$-module.

The finite field $\mathrm{GF}(p)$ contains a primitive q-th root of unity n. This implies that every irreducible representation of the cyclic group C_q over the field $\mathrm{GF}(p)$ is linear ([DH92, B, 8.9 (d)]). Therefore $Y_{k(i)} \cong C_p$ for each $i = 1, \ldots, s$. Therefore $X_{k(i)} \cong C_{p^e}$ for each $i = 1, \ldots, s$. This is to say that there exists a basis of X_k such that the action of α on X_k, according to this basis, can be written as a diagonal matrix $\operatorname{diag}(n^{\lambda_1}, \dots, n^{\lambda_{k-1}}, n^{\lambda})$, where $\lambda = -(\lambda_1 + \dots + \lambda_{k-1})$.

Consider the homocyclic group X_{k+r-1} of exponent p^e and rank $k+r-1$ and fix a basis $\{x_1, \dots, x_k, y_1, \dots, y_{r-1}\}$ of X_{k+r-1}. For each $j = 1, \dots, k-1$, consider the extension $L_j = [X_{k+r-1}]\langle \alpha_j \rangle$ of X_{k+r-1} such that $x_j^{\alpha_j} = x_j^{n^{\lambda_j}}$, $x_l^{\alpha_j} = x_l$, if $l \in \{1, \dots, k\} \setminus \{j\}$, and $y_s^{\alpha_j} = y_s^{n^{\lambda_j}}$, for $s = 1, \dots, r-1$. Consider also the extension $L_k = [X_{k+r-1}]\langle \alpha_k \rangle$ of X_{k+r-1} such that $x_k^{\alpha_k} = x_k^{n^{\lambda}}$, $x_l^{\alpha_k} = x_l$, if $l \in \{1, \dots, k-1\}$, and $y_s^{\alpha_k} = y_s^{n^{\lambda}}$, for $s = 1, \dots, r-1$.

In other words, the action of the automorphism α_j on X_{k+r-1}, in the fixed basis, can be written as a diagonal matrix

$$\alpha_j = \operatorname{diag}(\underbrace{1, \dots, 1}_{j-1}, n^{\lambda_j}, \underbrace{1, \dots, 1}_{k-j}, \underbrace{n^{\lambda_j}, \dots, n^{\lambda_j}}_{r-1}), \qquad \text{if } 1 \le j \le k-1,$$

and

$$\alpha_k = \operatorname{diag}(\underbrace{1, \dots, 1}_{k-1}, \underbrace{n^{\lambda}, \dots, n^{\lambda}}_{r}).$$

Hence, for all $j = 1, \dots, k$, we have that $L_j \cong G_r \times X_{k-1}$ and therefore $L_j \in \operatorname{Fit}(G_r)$.

Set $L = [X_{k+r-1}]\langle \alpha_1, \dots, \alpha_k \rangle$. Clearly L is a normal product of $L_1, \dots, L_k$. Hence $L \in \operatorname{Fit}(G_r)$. Consider the product

$$\alpha = \prod_{j=1}^{k} \alpha_j = \operatorname{diag}(n^{\lambda_1}, n^{\lambda_2}, \dots, n^{\lambda_{k-1}}, n^{\lambda}, \underbrace{1, \dots, 1}_{r-1})$$

and the normal subgroup $L_0 = [X_{k+r-1}]\langle \alpha \rangle$ of L. Identify $X_k = \langle x_1, \dots, x_k \rangle$ and observe that the subgroup $\langle X_k, \alpha \rangle$ is isomorphic to H_k and $L_0 \cong H_k \times X_{r-1}$. Therefore H_k is isomorphic to a subnormal subgroup of L. Hence $H_k \in \operatorname{Fit}(G_r)$. ◻

Lemma 7.3.9. *Let α be any nontrivial automorphism of X_r of order a power of q and write $G = [X_r]\langle \alpha \rangle$. Then $G_q \in \operatorname{Fit}(G)$.*

Proof. If the order of α is q^m and $m > 1$, then the order of $\alpha^{q^{m-1}}$ is q. Since $\langle X_r, \alpha^{q^{m-1}} \rangle$ is normal in G, then $\langle X_r, \alpha^{q^{m-1}} \rangle \in \operatorname{Fit}(G)$. Therefore we can assume that the order of α is q. As in Lemma 7.3.8, there exists a basis $\{x_1, \dots, x_r\}$ of X_r such that the matrix of α with respect to this basis is diagonal and $\alpha = \operatorname{diag}(n^{\lambda_1}, \dots, n^{\lambda_r})$. Since $\alpha \neq \mathrm{id}$, not all λ_i are equal to 0. Without loss of generality we can assume that $\lambda_1 = 1$. As a consequence of Proposition 7.3.6, the class $\operatorname{Fit}(G)$ contains the group $E_1 = [X_q]\langle \beta_1 \rangle$ which is an extension of X_q by the automorphism β_1 such that in a fixed basis of X_q has a diagonal matrix expression as follows: $\beta_1 = \operatorname{diag}(n, n^{-1}, 1, \dots, 1)$. Clearly, this group is isomorphic to $E_2 = [X_q]\langle \beta_2 \rangle$, where the automorphism β_2 in the fixed basis of X_q has a diagonal matrix expression $\beta_2 = \mathrm{diag}(1, n^2, n^{-2}, 1, \dots, 1)$. Hence E_2 belongs to $\mathrm{Fit}(G)$. Therefore the class $\mathrm{Fit}(G)$ contains the extensions of X_q by the automorphisms β_j, for $j = 1, \dots, q-1$ such that in the fixed basis have diagonal matrix expressions as follows:

$$\begin{aligned} \beta_1 &= \mathrm{diag}(n, n^{-1}, 1, \dots, 1) \\ \beta_2 &= \mathrm{diag}(1, n^2, n^{-2}, 1, \dots, 1) \\ &\dots \\ \beta_{q-1} &= \mathrm{diag}(1, \dots, 1, n^{q-1}, n) \end{aligned}$$

Thus $\mathrm{Fit}(G)$ contains the extension of X_q by the automorphism

$$\beta = \prod_{i=1}^{q-1} \beta_i = \mathrm{diag}(n, \dots, n)$$

and then $G_q = [X_q]\langle\beta\rangle \in \mathrm{Fit}(G)$. □

Lemma 7.3.10. *Let X be a homocyclic group of exponent p^e and let $G = [X]Q$ be a semidirect non-direct product of X and a q-group Q.*

1. *If $q \geq 3$, then $C_{p^e} \wr C_q \in \mathrm{Fit}(G)$.*
2. *If $q = 2$, then $\mathrm{Fit}(G)$ contains the extension of X_4 by $\langle\alpha, \beta\rangle$, where α and β are automorphisms of X_4, i.e. members of the group $\mathrm{GL}(4, \mathbb{Z}/p^e\mathbb{Z})$, such that in a fixed basis $\{x_1, x_2, x_3, x_4\}$ of X_4 have matrix expressions*

$$\alpha = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} -1 &&& \\ & -1 && \\ && 1 & \\ &&& 1 \end{pmatrix}$$

3. *In both cases 1 and 2 the Fitting class $\mathrm{Fit}(G)$ is not supersoluble.*

Proof. By Proposition 7.3.5, we can assume that Q is a group of automorphisms of X. Since the semidirect product is non-direct, there exists an element $s \in Q$ which is a non-trivial automorphism of X of order a power of q. It is clear that $[X]\langle s\rangle$ is subnormal in G and then $H = [X]\langle s\rangle \in \mathrm{Fit}(G)$.

By Lemma 7.3.9, we have that $\mathrm{Fit}(G_q) \subseteq \mathrm{Fit}([X]\langle s\rangle) \subseteq \mathrm{Fit}(G)$. By Lemma 7.3.8, the class $\mathrm{Fit}(G)$ contains all extensions of a homocyclic group X of exponent p^e by $\alpha \in \mathrm{Aut}(X)$ of order q such that $\det\alpha = 1$.

1. Suppose that q is odd. Observe that the regular wreath product $C_{p^e} \wr C_q$ is isomorphic to a extension of the homocyclic group X_q of exponent p^e and rank q by an automorphism α of order q whose action on X_q has matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

whose determinant is $(-1)^{q-1} = 1$. Hence $C_{p^e} \wr C_q \in \mathrm{Fit}(G)$.

2. Since α and β have both order 2 and determinant 1, the extensions $\langle X_4, \alpha\rangle$ and $\langle X_4, \beta\rangle$ are in $\mathrm{Fit}(G)$. The group $\langle \alpha, \beta\rangle$ is isomorphic to a dihedral group of order 8. Therefore the extension $H = [X_4]\langle \alpha, \beta\rangle$ is a subnormal product of $\langle X_4, \alpha\rangle$ and $\langle X_4, \beta\rangle$ and then $H \in \mathrm{Fit}(G)$.

3. In Case 1, the Fitting class $\mathrm{Fit}(G)$ is not supersoluble by Theorem 7.3.4.

In Case 2, suppose that the group H is supersoluble and consider the Frattini quotient $Y_4 = X_4/\Phi(X_4)$. The group $H^* = [Y_4]\langle \alpha, \beta\rangle$ is an epimorphic image of H and then H^* is supersoluble. Denote $Y_4 = \langle y_1, y_2, y_3, y_4\rangle$, where $y_i = x_i\Phi(X_4)$, for $i = 1, 2, 3, 4$. Now the respective actions of α and β on the 4-dimensional $\mathrm{GF}(p)$-vector space Y_4 have the same matrix representation, but now considered in $\mathrm{GL}(4, p)$. Let N be a minimal normal subgroup of H^* contained in Y_4. Since H^* is supersoluble, the group N is cyclic, $N = \langle y\rangle$ say. This is to say that y is an eigenvector for α and for β. Since y is an eigenvector for β, then either $y = x_1^{n_1}x_2^{n_2}$ or $y = x_3^{n_3}x_4^{n_4}$. But then y is not an eigenvector for α. Hence H is not supersoluble and $\mathrm{Fit}(G)$ is not a supersoluble Fitting class. □

Theorem 7.3.11. *If $\mathfrak{F}$ is a supersoluble Fitting class, then every metabelian $\mathfrak{F}$-group is nilpotent.*

Proof. Assume that the result is not true and let G be a metabelian non-nilpotent $\mathfrak{F}$-group of minimal order. Note that $N = G'$ is abelian. For every element $x \notin N$, $N\langle x\rangle$ is a metanilpotent normal subgroup of G. If $N\langle x\rangle$ were a proper subgroup of G for each element $x \in G$, then G would be nilpotent. This would contradict the choice of G. Therefore $G = N\langle x\rangle$, for some element $x \notin N$. By the same argument, we can assume that x is a q-element for some prime q. Clearly N is not a q-group and $G = \mathrm{O}_{q'}(N)Q$ for some $Q \in \mathrm{Syl}_q(G)$ such that $x \in Q$. The subgroup $G_0 = \mathrm{O}_{q'}(N)\langle x\rangle$ is subnormal in G. Hence $G_0 \in \mathfrak{F}$. If G_0 were nilpotent, then $G = NG_0$ would be a product of two subnormal nilpotent subgroups and therefore G would be nilpotent, contrary to supposition. The minimal choice of G implies that $G = G_0$, i. e., we can assume that N is a q'-group. We also may suppose that x is of order q. For a prime p with $p \neq q$, the subgroup $\mathrm{O}_p(N)$ is normal in G. If $\mathrm{O}_p(N)\langle x\rangle$ is nilpotent, then x centralises $\mathrm{O}_p(N)$. In this case $G = N^*\langle x\rangle \times \mathrm{O}_p(N)$, where N^* is the Hall p'-subgroup of N. By minimality of G, $N^*\langle x\rangle$ is nilpotent. Thus G is nilpotent, and this contradicts our choice of G. Hence, we can assume that $N = N_1 \times \cdots \times N_n$, where $N_i \in \mathrm{Syl}_{p_i}(N)$, for all primes p_i dividing $|N|$, and x induces on each N_i a non-trivial automorphism σ_i. Since x does not centralise N_1, it follows that x does not centralise some chief factor of G below N_1. This implies that q divides $p_1 - 1$ since G is supersoluble.

Consider the semidirect product $H = [P]C$, where $P = N_0 \times N_1$, with $N_0 \cong N_1$, and $C = \langle x \rangle$. Suppose that x induces on N_1 the automorphism σ_1 and on N_0 the automorphism σ_1^{-1}. By Proposition 7.3.6, we have that $H \in \mathfrak{F}$ and H is non-nilpotent.

By [DH92, A, 11.6], we have that N_0 has a direct decomposition $N_0 = A_{1(0)} \times \cdots \times A_{k(0)}$ with the following properties for each $i = 1, \dots, k$:

1. $A_{i(0)}$ is indecomposable as a C-module;
2. $A_{i(0)}/\Phi(A_{i(0)})$ is an irreducible $\mathrm{GF}(p_1)C$-module;
3. $A_{i(0)}$ is homocyclic.

Note that $A_{i(0)}/\Phi(A_{i(0)})$ is a faithful C-module and so its dimension is 1 because q divides $p_1 - 1$ ([DH92, B, 8.9 (d)]). Therefore $A_{i(0)} \cong C_{p_1^e}$ for each $i = 1, \dots, k$. Moreover x induces on each $A_{i(0)}$ an automorphism σ_1^{-1}. Analogously $N_1 = A_{1(1)} \times \cdots \times A_{k(1)}$, $A_{i(1)} \cong C_{p_1^e}$ for each $i = 1, \dots, k$ and x induces on each $A_{i(1)}$ the automorphism σ_1. By Lemma 7.3.6, we have that $[A_{1(0)} \times A_{1(1)}]C \in \mathfrak{F}$. Hence Lemma 7.3.10 implies that $\mathfrak{F}$ is not supersoluble. This contradiction proves the theorem. □

Theorem 7.3.12. *Let $\mathfrak{F}$ be a supersoluble non-nilpotent Fitting class. Then $\mathfrak{F}$ is not closed with respect to any of the operators* Q, S, *and* E_Φ.

Proof. Assume that $\mathfrak{F}$ is a Q-closed non-nilpotent supersoluble Fitting class. Let H be a supersoluble non-nilpotent $\mathfrak{F}$-group of minimal order. Then H/N is nilpotent $\mathfrak{F}$-group for every minimal normal subgroup N of H. Consequently $H \in \mathrm{b}(\mathfrak{N})$ and so H is a primitive group. Then, by Theorem 1, $N = \mathrm{Soc}(H)$ is a minimal normal subgroup of H and $N = \mathrm{C}_H(N)$ and N is cyclic of prime order. In particular, H is metabelian. This contradicts Theorem 7.3.11. Therefore the class $\mathfrak{F}$ is not Q-closed.

Suppose that $\mathfrak{F}$ is an E_Φ-closed supersoluble non-nilpotent Fitting class. Since $\mathfrak{F}$ is composed of metanilpotent groups we can apply the theorem [DH92, XI, 2.16] to conclude that $\mathfrak{F}$ is s-closed. Applying Theorem 2.5.2, $\mathfrak{F}$ is a saturated formation. In particular, $\mathfrak{F}$ is Q-closed. This contradiction proves that $\mathfrak{F}$ is not E_Φ-closed. Note that $\mathfrak{F}$ cannot be subgroup-closed either. □

Recall that a *Fischer class* is a Fitting class $\mathfrak{F}$ satisfying the following property: if G is a group in $\mathfrak{F}$ and H/K is a normal nilpotent subgroup of G/K for some normal subgroup K of G, it follows that $H \in \mathfrak{F}$. These classes were originally introduced by Fischer in the soluble universe. If $\mathfrak{F}$ is a Fischer class of soluble groups, then the $\mathfrak{F}$-injectors of a soluble group are exactly the *Fischer $\mathfrak{F}$-subgroups*, which are the natural duals of Gaschütz's covering subgroups (see [DH92, IX, Section 3]).

Corollary 7.3.13. *Let $\mathfrak{F}$ be a supersoluble non-nilpotent Fitting class. Then $\mathfrak{F}$ is not a Fischer class.*

Proof. Assume that $\mathfrak{F}$ is a Fischer class. We shall prove that $\mathfrak{F}$ is subgroup-closed. Suppose that this is not true and let G be a group of minimal order such that $G \in \mathfrak{F}$ but $M \notin \mathfrak{F}$ for some subgroup M of G. Among the subgroups of G which are not in $\mathfrak{F}$, we choose M of maximal order. Clearly M is a maximal subgroup of G. If G' is contained in M, then M is normal and so $M \in \mathfrak{F}$, contrary to supposition. Consequently, $G = MG'$. Since, by [DH92, VII, 2.2], M has prime index, it follows that $M/M \cap G'$ is a cyclic group of prime order. Note that G' is nilpotent and $M \cap G'$ has prime index in G'. This implies that $M \cap G'$ is normal in G'. Therefore $M \cap G'$ is normal in G. Since $\mathfrak{F}$ is a Fischer class, we have that $M \in \mathfrak{F}$, contrary to the choice of M. Then $\mathfrak{F}$ is subgroup-closed. This contradicts Theorem 7.3.12. Consequently, $\mathfrak{F}$ is not a Fischer class.

Since metanilpotent R_0-closed Fitting classes need not be Q-closed, the exclusion of the R_0-closure cannot be argued in the same way. What Menth shows is that the supersoluble Fitting class $\mathfrak{V}$ introduced at the beginning of the section is not R_0-closed.

Theorem 7.3.14. *The class $\mathfrak{V}$ is not R_0-closed.*

Proof. We will use the notation introduced at the beginning of the section. Let us consider the direct product $W = V \times V^{\varphi}$ of two copies of V. The diagonal subgroup $D = \{(x, x^{\varphi}) : x \in V\}$ of W is isomorphic to V. The subgroups $A = \{(x, 1) : x \in T'\}$ and $B = \{(1, x^{\varphi}) : x \in T'\}$ are normal in W and $A \cap B = (1, 1)$. Observe that the subgroup $G = \langle A, D\rangle$ is a semidirect product $G = [A]D = [B]D$ and $G/A \cong G/B \cong V \in \mathfrak{V}$. Next we see that $G \notin \mathfrak{V}$.

The element (s, s^{φ}) is a 3-element and then $(s, s^{\varphi}) \in \mathrm{O}^p(G)$. Hence the commutator $[(a, a^{\varphi}), (s, s^{\varphi})] = (a, a^{\varphi})^{n-1} \in \mathrm{O}^p(G)$ and also $(b, b^{\varphi})^{n-1} \in \mathrm{O}^p(G)$ for the generators a, b of T. Therefore D is contained in $\mathrm{O}^p(G)$. There exists an element $t \in T' \setminus \mathrm{Z}(T)$ such that $t^s = t^{n^2}$. Hence $[(t, 1), (s, s^{\varphi})] = ([t, s], 1) = (t^{n^2-1}, 1) \in \mathrm{O}^p(G)$. Since n is a primitive cube root of unity in $\mathrm{GF}(p)$, we have that p divides $n^3 - 1$ but $\gcd(p, n^2 - 1) = 1$. Therefore $(t, 1) \in \mathrm{O}^p(G)$. Then $[(t, 1), (a, a^{\varphi})] = ([t, a], 1)$ and $[(t, 1), (b, b^{\varphi})] = ([t, b], 1)$ are in $\mathrm{O}^p(G)$. Then $A \leq \mathrm{O}^p(G)$. Therefore $G = \mathrm{O}^p(G)$ and the group G is p-perfect.

Observe that the subgroup $\mathrm{Z}(T) \times \mathrm{Z}(T)^{\varphi}$ is a subgroup of $\mathrm{Z}\big(\mathrm{O}_p(G)\big)$ of order p^4. If we suppose that $G \in \mathfrak{V}$, then $G \in \mathfrak{V}_0$ and then $\mathrm{O}_p(G)$ is a central product of copies of T. Since $|\mathrm{O}_p(G)| = p^8$, we need exactly two copies of T, T_1, T_2 say, such that $|T_1 \cap T_2| = p^2$. Therefore $\mathrm{Z}(T_1) = \mathrm{Z}(T_2) = \mathrm{Z}\big(\mathrm{O}_p(G)\big)$ has order p^2. This contradicts the previous observation. Hence $G \notin \mathfrak{V}$. We conclude then that $\mathfrak{V}$ is not R_0-closed. $\square$

Let $\mathfrak{F}$ be a Fitting class of soluble groups. If π is a set of primes, $\mathfrak{F}$ is said to be *Hall-π-closed* provided that whenever H is a Hall π-subgroup of G and $G \in \mathfrak{F}$, then $H \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be *Hall-closed* if it is Hall-π-closed for all sets of primes π.

Theorem 7.3.15. *Every metanilpotent Lockett class is Hall-closed.*

Proof. Assume that the result is false and let $\mathfrak{F}$ be a metanilpotent Fitting class that is not Hall-closed. There exists a set π of primes and a group $G \in \mathfrak{F}$ such that G has a Hall π-subgroup $H \notin \mathfrak{F}$. Set $F = \mathrm{F}(G)$, and let $p_1, \dots, p_n$ be the prime divisors of $|F|$. Then F is the direct product of its Sylow p_i-subgroups P_i, $1 \leq i \leq n$, and G/F is nilpotent. Having numbered the primes suitably, there is an integer k ($1 \leq k \leq n$) such that $p_1, \dots, p_k$ are elements of π. Note that $k < n$ because otherwise H would be subnormal in G. Then $P = H \cap F = P_1 \cdots P_k$. The quotient H/P is isomorphic to a subgroup of G/F and therefore nilpotent. Hence H/P is generated by cyclic subgroups $\langle x_i P\rangle$. At least one of the subgroups $\langle P, x_i\rangle$ is not an $\mathfrak{F}$-group. Let us choose $H^* = \langle P, x\rangle$ such that $|H^*|$ is of minimal order. Then $H^*_{\mathfrak{F}}$ is a normal maximal subgroup of H^*. Now we replace G by $G^* = \langle F, x\rangle$, because $G^* \in \mathfrak{F}$ and H^* is a Hall π-subgroup of G^*. Set $Q = P_{k+1} \cdots P_n$. We define a direct product $D = \langle P, x_1\rangle \times \langle Q, x_2\rangle$, where $\langle P, x_1\rangle$ is a copy of H^* and $\langle Q, x_2\rangle$ is a copy of $Q\langle x\rangle$. Then $K = PQ\langle x_1x_2\rangle$ is a normal subgroup of D isomorphic to G^*. Hence K is contained in $D_{\mathfrak{F}} = \langle P, x_1\rangle_{\mathfrak{F}} \times \langle Q, x_2\rangle_{\mathfrak{F}}$. Since $|\langle P, x_1\rangle : \langle P, x_1\rangle_{\mathfrak{F}}| = P$ and $|\langle Q, x_2\rangle : \langle Q, x_2\rangle_{\mathfrak{F}}| = p$, it follows that $|D : D_{\mathfrak{F}}| = p^2$. However $|D : K| = p$. This contradiction proves the theorem. $\square$

Not every supersoluble Fitting class is a Lockett class ([Men96, Example 1]). In the following we shall prove that every supersoluble Fitting class is contained in a supersoluble Lockett class.

Theorem 7.3.16. *Every supersoluble Fitting class is contained in a supersoluble Lockett class.*

Proof. Assume that $\mathfrak{F}$ is a supersoluble Fitting class. If $G \in \mathfrak{F}^*$, then $D = \{(g, g^{-1}) : g \in G\}$ is a subgroup of $(G \times G)_{\mathfrak{F}}$ by [DH92, X, 1.5, 1.9]. Therefore D is supersoluble. Since G is an epimorphic image of D, it follows that G is supersoluble. Therefore $\mathfrak{F}^*$ is a supersoluble Lockett class. $\square$

7.4 Fitting sets, Fitting sets pairs, and outer Fitting sets pairs

This section has two main themes. The first is connected with Fitting sets and injectors. The second subject under investigation is the localised theory of Fitting pairs and outer Fitting pairs developed in [AJBBPR00].

As mentioned in Section 2.4, the theory of Fitting classes has been enriched by the introduction of Fitting sets by W. Anderson in [And75]. Recall that a subgroup H of a group G is an injector of G if H is an $\mathcal{F}$-injector of G for some Fitting set $\mathcal{F}$ of G. One the most important motivating questions in the theory of Fitting sets is to determine which subgroups are injectors. Some results in this direction are presented in [DH92, VIII, Section 3]. There Doerk and Hawkes proposed the problem of describing injectors of soluble groups without explicit use of the concept of a Fitting set.

This problem is complicated by the general nature of injectors: there are likely to be many Fitting sets for a given group, often leading to different sets of injectors. For example, the set of injectors of a soluble group includes all its normal subgroups, all its Hall subgroups, and all its maximal subgroups [DH92, VIII, 3.5]. An injector A of a finite soluble group B must have rather strong properties that can be described without direct reference to Fitting sets: $A \cap K$ must be a CAP subgroup of K and pronormal (see [DH92, Section I, 6]) in B for each normal subgroup K of B [DH92, VIII, 2.14]. However, these properties are inadequate to characterise injectors [DH92, Exercise 2, p. 553]. We present here the best attempt to accomplish that task. This characterisation, unpublished at the moment of writing this, was communicated privately by its authors, R. Dark and A. Feldman ([DF]), to us.

If G is a group, denote by $\mathrm{Inj}(G)$ the set of all injectors of G. The following result is a very useful characterisation of this set. Recall that if H is a subgroup of G then

$$\mathrm{s}_n\, H^G = \{S \leq G : S \text{ is a subnormal subgroup of } H^g, \text{ for some } g \in G\}.$$

Lemma 7.4.1 ([DH92, VIII, 3.3]). *Let G be a soluble group and H a subgroup of G. Then any two of the following statements are equivalent*

1. *$H \in \mathrm{Inj}(G)$*
2. *$\mathrm{s}_n\, H^G$ is a Fitting set of G.*
3. *$\mathrm{s}_n\, H^G$ is the smallest Fitting set of G which contains H.*

Lemma 7.4.2. *Suppose S and T are pronormal subgroups of a soluble group G and x, $y \in G$. If S and T are subnormal in $\langle S,T\rangle$ and S^x and T^y are subnormal in $\langle S^x,T^y\rangle$, then there exists $z \in G$ with $S^x = S^z$ and $T^y = T^z$.*

Proof. Let Σ be a Hall system of G which reduces into $\langle S,T\rangle$. Applying [DH92, I, 6.3], S and T are normal in $\langle S,T\rangle = ST$. By [DH92, I, 4.21], Σ reduces into both S and T. Analogously, S^x and T^y are normal in $\langle S^x,T^y\rangle = S^xT^y$. Then by [DH92, I, 6.11], $S^xT^y = (ST)^z$ for some $z \in G$. This implies that Σ^z, which reduces into $(ST)^z$, reduces into the subnormal subgroups S^x and S^z and T^y and T^z of that group. But the pronormality of S and T then implies, by [DH92, I, 6.6], that $S^x = S^z$ and $T^y = T^z$, as claimed. □

Now we prove a result that will supply the inductive step in our eventual characterisation of injectors:

Theorem 7.4.3. *Let G be a soluble group and suppose H is a subgroup of G and M is a normal subgroup of G. Assume that the following condition holds:*

Whenever S is a subnormal subgroup of H, $g \in G$, $S^g \leq HM$ and $S_1 = H \cap S^g M$ is subnormal in H, then S_1 and S^g are conjugate in $J = \langle S_1, S^g \rangle$. (7.1)

Then

1. *if S is a subnormal subgroup of H, then S is pronormal in $\mathrm{N}_G(SM)$ and*
2. *if $HM \in \mathrm{Inj}(G)$, then $H \in \mathrm{Inj}(G)$.*

Proof. 1. Let g be an element of $\mathrm{N}_G(SM)$, so that $S^g M = (SM)^g = SM$. Note that if S is subnormal in H, then SM is subnormal in HM, and therefore $S_1 = H \cap S^g M = H \cap SM = S(H \cap M)$ is subnormal in H. Applying (7.1) with $g = 1$ yields S and $S_1 = S(H \cap M)$ are conjugate. Now, by order considerations, $S = S_1$. By (7.1) then, S and S^g are conjugate in $\langle S, S^g \rangle$; i.e. S is pronormal in $\mathrm{N}_G(SM)$.

2. Suppose that S and T are subnormal subgroups of H and $a, b \in G$ with S^a and T^b normal in $S^a T^b$. By Lemma 7.4.1, it suffices to find an element w such that $S^a T^b$ is subnormal in H^w. Now SM and TM are subnormal subgroups of HM and $S^a M$ and $T^b M$ are normal in $Y = S^a T^b M = S^a M T^b M$, and because $HM \in \mathrm{Inj}(G)$, there exists $c \in G$ such that Y is subnormal in $(HM)^c = H^c M$. Let $H_0 = H^c$ and $S_0 = S^c$. Note that condition (7.1) still holds when H is replaced by the conjugate H_0. Replacing S and g by S_0 and $c^{-1}a$ we have S_0 is subnormal in H_0, $S_0^g = S^a \leq H_0 M$, and $S_0^g M = S^a M$ is normal in Y which is subnormal in $H_0 M$. Hence $S_0^g M$ is subnormal in $H_0 M$, and $S_1 = H_0 \cap S_0^g M$ is subnormal in H_0. Then by (7.1), S_1 and S^a are conjugate in $\langle S_1, S^a \rangle \leq S^a M \leq Y$. Similarly, $T_1 = H_0 \cap T^b M$ is subnormal in H_0, and T^b is conjugate in Y to T_1; hence there are elements $x, y \in Y$ such that $S_1^x = S^a$ and $T_1^y = T^b$.

Now $S^a M = H_0 M \cap S_0^g M = (H_0 \cap S_0^g M)M = S_1 M$ and then $Y \leq \mathrm{N}_G(S^a M) = \mathrm{N}_G(S_1 M)$, and if follows from Assertion 1 that S_1 is pronormal in Y. Similarly, T_1 is pronormal in Y. We also have that S_1 and T_1 are subnormal in $\langle S_1, T_1 \rangle$ and S_1^x, T_1^y normal in $S_1^x T_1^y$. By Lemma 7.4.2, there exists $z \in Y$ with $S_1^x = S_1^z$ and $T_1^y = T_1^z$. Hence $S^a T^b = S_1^x T_1^y = (S_1 T_1)^z$ is subnormal in $H_0^z = H^{cz}$, so setting $w = cz$ yields our result. □

Now we are ready to prove that two properties that do not involve Fitting sets are equivalent to that of being an injector. Not surprisingly, conjugation, which is crucial to the definition of Fitting set and normality (and therefore indirectly, subnormality) play an important role in these properties. In particular, for convenience we introduce the following definition:

Definition 7.4.4. *If H and X are subgroups of a soluble group G and $g \in G$, we say H is (X, g)*-pronormal *if $H \cap X$ and $H^g \cap X$ are conjugate in $J = \langle H \cap X, H^g \cap X \rangle$.*

Note that H is a pronormal subgroup of G if and only if H is (G,g)-pronormal for all $g \in G$.

We now can prove:

Theorem 7.4.5 (R. Dark and A. Feldman). *Let G be a soluble group, and suppose that H is a subgroup of G. Then any two of the following conditions are equivalent:*

1. *H is an injector of G;*
2. *whenever $H \le K \le G$, $g \in G$, and X and $X^{g^{-1}}$ are subnormal subgroups of K, then H is (X,g)-pronormal;*
3. *whenever M/N is a chief factor of G which is not covered by H, S is a subnormal subgroup of H such that $H \cap N \le S$, $g \in G$, and $S^g \le HM$ with $S_1 = H \cap S^g M$ subnormal in H, then S_1 and S^g are conjugate in $J = \langle S_1, S^g\rangle$.*

Proof. 1 implies 2. Suppose that H is an $\mathcal{F}$-injector of G for some Fitting set $\mathcal{F}$ of G. Then, with K and X as in 2 and J as in the definition of (X,g)-pronormal, H is an $\mathcal{F}_K$-injector of K by [DH92, VIII, 2.13], and then $H \cap X$ is an $\mathcal{F}_X$-injector of X by [DH92, VIII, 2.6], and hence $H \cap X$ is an $\mathcal{F}_J$-injector of J by [DH92, VIII, 2.13] again. Similarly, H^g is an $\mathcal{F}_{K^g}$-injector of K^g, and X is subnormal in K^g by hypothesis, and then $H^g \cap X$ is an $\mathcal{F}_X$-injector of X, and $H^g \cap X$ is an $\mathcal{F}_J$-injector of J. Thus by Theorem 2.4.26, $H \cap X$ and $H^g \cap X$ are conjugate in J, establishing 2.

2 implies 3. First observe that, in these hypotheses, we certainly have that H avoids M/N. With $X = M$, we see that $H \cap M$ and $H^g \cap M$ are conjugate in $J = \langle H \cap M, H^g \cap M\rangle$, and then $(H \cap M)N$ and $(H^g \cap M)N$ are conjugate in JN. But $JN/N \le M/N$, which is abelian, and it follows that $(H \cap M)N = (H^g \cap M)N$. This holds for all $g \in G$ because $X = M$ is normal in G, and then $(H \cap M)N$ is normal in G. Since H does not cover the chief factor M/N of G, we have that $(H \cap M)N < M$. Then $(H \cap M)N = N$, establishing the result.

Assume the hypotheses of 3 and take $X = S^g M$. Then X is subnormal in $H^g M$ and $X^{g^{-1}}$ is subnormal in HM. Also, $X \le HM$, and $X = HM \cap S^g M = (H \cap S^g M)M = S_1 M$ is subnormal in HM. Moreover, $H \cap X = S_1$ by definition, and $H^g \cap X = H^g \cap S^g M = S^g(H^g \cap M)$, which equals $S^g(H^g \cap N)$ inasmuch as H^g avoids M/N. But $H \cap N \le S$ by hypothesis, and then $H^g \cap N \le S^g$, and $H^g \cap X = S^g$. Thus 2 yields that S_1 and S^g are conjugate in $\langle S_1, S^g\rangle$, as claimed.

To see that 3 implies 1, we pass through an intermediate Step 4.

4. *Whenever M/N is a chief factor of G which is not covered by H, and such that $\mathrm{Core}_G(H) \le N < M \le \langle H^G\rangle$, and S is a subnormal subgroup of H such that $H \cap N \le S$, $g \in G$, and $S^g \le HM$ with $S_1 = H \cap S^g M$ subnormal in H, then S_1 and S^g are conjugate in $J = \langle S_1, S^g\rangle$.*

It is clear that 3 implies 4. Hence we have to prove that 4 implies 1.

Note first that if $C = \mathrm{Core}_G(H)$, it is easy to see that if 4 holds for $H \le G$, then 4 also holds for $H/C \le G/C$. Moreover, if $H/C \in \mathrm{Inj}(G/C)$, then $H \in \mathrm{Inj}(G)$ by [DH92, VIII, 2.17]. Thus it suffices to prove that if 4 holds for H/C in G/C, then $H/C \in \mathrm{Inj}(G/C)$, and we may assume that $C = 1$, i.e. H is core-free in G.

We proceed by induction on the index $|\langle H^G\rangle : H|$. If $|\langle H^G\rangle : H| = 1$, then $H = 1$ inasmuch as H is core-free. In this case H is obviously an injector of G. Hence we may assume that $|\langle H^G\rangle : H| > 1$. Let M_1 be a minimal normal subgroup of G such that $M_1 \le \langle H^G\rangle$. Since H is core-free, H does not cover M_1. We see next that because 4 holds for H, it also holds for HM_1.

Suppose that M/N is a chief factor of G which $1 < M_1 \le \mathrm{Core}_G(HM_1) \le N < M \le \langle (HM_1)^G\rangle = \langle H^G\rangle$ and M/N is not covered by HM_1.

Now suppose that $g \in G$ and $\bar{S}$ is a subnormal subgroup of HM_1 such that $HM_1 \cap N \le \bar{S}$, $g \in G$, and $\bar{S}^g \le (HM_1)M$ with $\bar{S}_1 = HM_1 \cap \bar{S}^g M$ subnormal in HM_1,

Consider $S = H \cap \bar{S}$. Then S is subnormal in H. Since $M_1 \le HM_1 \cap N \le \bar{S}$, then $\bar{S} = HM_1 \cap \bar{S} = (H \cap \bar{S})M_1 = SM_1$, and then $\bar{S}^g M = S^g M$. Observe also that $H \cap N = H \cap HM_1 \cap N \le H \cap \bar{S} = S$ and $S^g \le \bar{S}^g \le HM$. Finally, it is clear that $S_1 = H \cap S^g M = H \cap (HM_1 \cap \bar{S}^g M) = H \cap \bar{S}_1$ is subnormal in H.

Thus the hypotheses of 4 hold, implying S_1 and S^g are conjugate in $J = \langle S_1, S^g\rangle$. Moreover, $\bar{S} = SM_1$, and $\bar{S}^g = S^g M_1$, and $\bar{S}_1 = HM_1 \cap \bar{S}^g M = (H \cap S^g M)M_1 = S_1 M_1$, and $\bar{J} = \langle \bar{S}_1, \bar{S}^g\rangle = JM_1$. Hence $\bar{S}_1$ and $\bar{S}^g$ are conjugate in $\bar{J}$.

Observe that $|\langle H^G\rangle| = |\langle (HM_1)^G\rangle : HM_1| < |\langle H^G\rangle : H|$. Thus the induction hypothesis implies that $HM_1 \in \mathrm{Inj}(G)$. To complete the proof, we apply Theorem 7.4.3 (2) with $M = M_1$. With $N = 1$, and by 4 applied to the chief factor M_1/N, Condition (7.1) of Theorem 7.4.3 holds. Thus, Theorem 7.4.3 (2) shows that $H \in \mathrm{Inj}(G)$. ☐

Corollary 7.4.6. *Let G be a soluble group. Suppose that H is an injector of G and M a normal subgroup of G. Then $H \cap M$ is pronormal in G.*

Applying [DH92, VIII, 3.5], a maximal subgroup of a group is always an injector. Hence, in particular, in a soluble group the intersection of a maximal subgroup and a normal subgroup is pronormal in the group.

By [DH92, VIII, 3.8] every normally embedded subgroup of a soluble group is an injector. In the following we give a proof of this fact using Theorem 7.4.5.

Corollary 7.4.7. *Suppose H is a normally embedded subgroup of a soluble group G. Then $H \in \mathrm{Inj}(G)$.*

Proof. Assume that H is normally embedded in G, $H \le K \le G$, and $X, X^{g^{-1}}$ are subnormal in K for some $g \in G$. We shall show that H is (X, g)-pronormal.

First we show that $H \cap X$ and $H^g \cap X$ are locally conjugate in X. For an arbitrary prime, p, let $P \in \mathrm{Syl}_p(H)$; let $P_1 \in \mathrm{Syl}_p(K)$ such that $P_1 \cap H = P$. Because X is subnormal in K, $P_1 \cap X \in \mathrm{Syl}_p(X)$, by [DH92, I, 4.21]. Also, $H \cap X$ is subnormal in H, and $P \cap X = P \cap (H \cap X) \in \mathrm{Syl}_p(H \cap X)$. Now H normally embedded in G implies $P \in \mathrm{Syl}_p(\langle P^G \rangle)$, and $P \le \langle P^G \rangle \cap P_1 \le \langle P^G \rangle$, and then $P = \langle P^G \rangle \cap P_1$. Because $\langle P^G \rangle \cap X$ is normal in X, $(P_1 \cap X) \cap (\langle P^G \rangle \cap X) \in \mathrm{Syl}_p(\langle P^G \rangle \cap X)$. But $(\langle P^G \rangle \cap X) \cap (P_1 \cap X) = (\langle P^G \rangle \cap P_1) \cap X = P \cap X \in \mathrm{Syl}_p(H \cap X)$. Hence any Sylow p-subgroup of $H \cap X$ is a Sylow p-subgroup of $\langle P^G \rangle \cap X$. By similar arguments, $P^g \in \mathrm{Syl}_p(H^g)$ implies $P^g \cap X \in \mathrm{Syl}_p\big(\langle (P^g)^G \rangle \cap X\big) = \mathrm{Syl}_p(\langle P^G \rangle \cap X)$ and $P^g \cap X \in \mathrm{Syl}_p(H^g \cap X)$. Thus we have Sylow p-subgroups of $H \cap X$ and $H^g \cap X$ that are Sylow p-subgroups of the same subgroup of X, and they are conjugate in X, as desired.

Now note that $\langle P^G \rangle \cap X$ is normal in X, and since this works for all primes p, $H \cap X$ and $H^g \cap X$ are normally embedded in X. Thus $H \cap X$ and $H^g \cap X$ are locally pronormal [DH92, I, 7.13] and therefore pronormal [DH92, I, 6.14] in X. Thus $H \cap X$ and $H^g \cap X$ are locally conjugate and locally pronormal subgroups in X, and they are conjugate in X [DH92, I, 6.16]. Finally, the pronormality of $H \cap X$ in X implies that $H \cap X$ and $H^g \cap X$ are conjugate in their join; i.e. H is (X, g)-pronormal, establishing the result. $\square$

Let $\mathfrak{F}$ be a Fitting class. Blessenohl and Gaschütz [BG70] introduced the notion of $\mathfrak{F}$-Fitting pair which turns out to be useful for the construction of normal Fitting classes in the Lockett section of $\mathfrak{F}$.

We need to deal with arbitrary (possibly infinite) groups. Hence if we denote a group by G, we are assuming that the group G is finite. Otherwise, we put $\mathbf{G}$.

Definition 7.4.8. *If N and M are groups, an* embedding *is a group monomorphism $\nu\colon N \longrightarrow M$.*

If N^ν is a normal subgroup of M, then ν is said to be a normal embedding*.*

Definition 7.4.9 ([BG70]). *Let $\mathfrak{F}$ be a Fitting class. An* $\mathfrak{F}$-Fitting pair *is a pair $(d, \mathbf{A})$ which consists of a group $\mathbf{A}$ and a family $\big(d_U \in \mathrm{Hom}(U, \mathbf{A}) : U \in \mathfrak{F}\big)$ such that for each normal embedding $\nu\colon U \longrightarrow V \in \mathfrak{F}$, the assertion $d_U = \nu d_V$ holds.*

It can be proved that in this case $\{(g)^{d_G} : g \in G, G \in \mathfrak{F}\}$ is an abelian subgroup of $\mathbf{A}$ ([DH92, IX, 2.12 (b)]). Hence, without loss of generality, we may assume that $\mathbf{A}$ is abelian.

In the same paper, Blessenohl and Gaschütz gave examples of Fitting pairs and proved the following result, which remains valid in the general finite universe.

Proposition 7.4.10 (see [DH92, IX, 2.11]). *Let $\mathfrak{F}$ be a Fitting class and let $(d, \mathbf{A})$ be an $\mathfrak{F}$-Fitting pair. Then the class $\mathfrak{R} = \mathrm{Ker}(d, \mathbf{A})$ of all groups $G \in \mathfrak{F}$ such that $G^{d_G} = 1$ is a normal Fitting class such that $\mathfrak{F}_* \subseteq \mathfrak{R} \subseteq \mathfrak{F}$.*

Lausch [Lau73] showed that every non-trivial normal Fitting class in the soluble universe can be described as the kernel of a Fitting pair. He also described a universal $\mathfrak{F}$-Fitting pair, leading to the so-called Lausch group. He carried out the construction for the case $\mathfrak{F} = \mathfrak{S}$, but as Bryce and Cossey pointed out in [BC75], Lausch's method applies to an arbitrary Fitting class (see [DH92, X, Section 4] for details).

J. Pense, in his Dissertation [Pen87], generalised the concept of an $\mathfrak{F}$-Fitting pair to that of outer $\mathfrak{F}$-Fitting pair.

Definition 7.4.11 (see [Pen88]). *Let $\mathfrak{F}$ be a Fitting class. An* outer $\mathfrak{F}$-Fitting pair *is a pair $(d, \mathbf{A})$ which consists of a group $\mathbf{A}$ and a family $\bigl(d_U \in \operatorname{Hom}(U, \mathbf{A}) : U \in \mathfrak{F}\bigr)$ such that for each normal embedding $\nu\colon U \longrightarrow V \in \mathfrak{F}$, there exists an inner automorphism α of $\mathbf{A}$ such that $d_U\alpha = \nu d_V$.*

Obviously, if $\mathbf{A}$ is an abelian group, then an outer $\mathfrak{F}$-Fitting pair is just an $\mathfrak{F}$-Fitting pair.

Pense extended the definition of a Fitting set to an infinite group by requiring it to mean a set of *finite* subgroups closed under conjugation and under the usual operations of taking normal subgroups and forming finite normal products. He also introduced the concept of $\mathcal{F}$-Fitting sets pair $(d, \mathbf{A})$, where $\mathbf{A}$ is an abelian group, to develop a local version of the Lausch group in certain type of groups ([Pen87]).

Definition 7.4.12. *If N and M are finite subgroups of $\mathbf{G}$, a* $\mathbf{G}$-embedding *is a group monomorphism $\nu\colon N \longrightarrow M$ which is the restriction to N of an inner automorphism of $\mathbf{G}$.*

If N^ν is a normal subgroup of M, then ν is said to be a normal $\mathbf{G}$-embedding.

Definition 7.4.13. *Let $\mathcal{F}$ be a Fitting set of a group $\mathbf{G}$. An* $\mathcal{F}$-Fitting sets pair relative to $\mathbf{G}$ *is a pair $(d, \mathbf{A})$ which consists of a group $\mathbf{A}$ and a family $\bigl(d_U \in \operatorname{Hom}(U, \mathbf{A}) : U \in \mathcal{F}\bigr)$ such that for each normal $\mathbf{G}$-embedding $\nu\colon U \longrightarrow V \in \mathcal{F}$, the assertion $d_U = \nu d_V$ holds.*

Note that, in our definition of $\mathcal{F}$-Fitting sets pair, we do not require that $\mathbf{A}$ is an abelian group. An outer $\mathcal{F}$-Fitting sets pair is defined as follows:

Definition 7.4.14 ([AJBBPR00]). *Let $\mathcal{F}$ be a Fitting set of a group $\mathbf{G}$. An* outer $\mathcal{F}$-Fitting sets pair relative to $\mathbf{G}$ *is a pair $(d, \mathbf{A})$ which consists of a group $\mathbf{A}$ and a family $\bigl(d_U \in \operatorname{Hom}(U, \mathbf{A}) : U \in \mathcal{F}\bigr)$ such that for each normal $\mathbf{G}$-embedding $\nu\colon U \longrightarrow V \in \mathcal{F}$, there exists an inner automorphism α of $\mathbf{A}$ such that $d_U\alpha = \nu d_V$.*

If $\mathfrak{F}$ is a Fitting class, then $\operatorname{Tr}_{\mathfrak{F}}(\mathbf{G})$ is a Fitting set of the group $\mathbf{G}$, and if $(d, \mathbf{A})$ is an (outer) $\mathfrak{F}$-Fitting pair, then the pair $(d, \mathbf{A})$, for $\bigl(d_U \in \operatorname{Hom}(U, \mathbf{A}) : U \in \operatorname{Tr}_{\mathfrak{F}}(\mathbf{G})\bigr)$, is an (outer) $\operatorname{Tr}_{\mathfrak{F}}(\mathbf{G})$-Fitting sets pair relative to $\mathbf{G}$.

Definition 7.4.15. *Two outer $\mathcal{F}$-Fitting sets pairs $(d_i, \mathbf{A}_i)$, $i = 1, 2$, are* equivalent *if there exists an isomorphism $\sigma \colon \mathbf{A}_1 \longrightarrow \mathbf{A}_2$, such that for each $U \in \mathcal{F}$, there exists $\alpha_U \in \mathrm{Inn}(\mathbf{A}_2)$ such that $d_{2U} = d_{1U}\sigma\alpha_U$.*

In [AJBBPR00], P. Arroyo-Jordá, A. Ballester-Bolinches, and M. D. Pérez-Ramos made a complete study of outer Fitting sets pairs. In the sequel, we will present the main results of this paper.

To begin with, we point out that there are some differences between Fitting pairs and Fitting sets pairs. We shall show two of them.

Remarks 7.4.16. 1. In Definition 7.4.9 of Fitting pair, the group $\mathbf{A}$ can be assumed abelian without loss of generality. This is not true for Fitting sets pairs in general.

Let G be the alternating group of degree 5, $G = \mathrm{Alt}(5)$, and $\mathcal{F}$ the trace in G of the Fitting class $\mathfrak{F} = \mathfrak{S}_3\mathfrak{S}_5\mathfrak{S}_2$. In other words, the Fitting set $\mathcal{F}$ is composed of all subgroups of G of prime-power order, and the normalisers of the Sylow 5- and 3-subgroups. Consider the symmetric group $S = \mathrm{Sym}(3)$ of degree 3. If X is a subgroup of prime-power order of G, then put $d_X : X \longrightarrow S$ to be the trivial homomorphism: $x^{d_X} = 1$ for all $x \in X$. If $P \in \mathrm{Syl}_3(G)$ and $N_3 = \mathrm{N}_G(P)$, then put $d_{N_3} \colon N_3 \longrightarrow S$ to be a homomorphism such that $P = \mathrm{Ker}(d_{N_3})$ and $\mathrm{Im}(d_{N_3}) = \langle (12) \rangle$. If $Q \in \mathrm{Syl}_5(G)$ and $N_5 = \mathrm{N}_G(Q)$, then put $d_{N_5} \colon N_5 \longrightarrow S$ to be a homomorphism such that $Q = \mathrm{Ker}(d_{N_5})$ and $\mathrm{Im}(d_{N_5}) = \langle (23) \rangle$.

The pair $(\{d_H : H \in \mathcal{F}\}, S)$ is an $\mathcal{F}$-Fitting sets pair relative to G.

Observe that S is not abelian and $S = \langle h^{d_H} : H \in \mathcal{F}, h \in H \rangle$.

2. Pense [Pen87, Kollollar 3.30] shows that if (d, A) is a outer Fitting pair with A finite, then it is equivalent to a Fitting pair. This is not true for outer Fitting sets pairs.

Let $Q = \langle x, y : x^4 = 1, x^2 = y^2, x^y = x^{-1} \rangle$ be a quaternion group of order 8 and fix a subgroup $C = \langle x \rangle$ of order 4 of Q. The set of all subgroups of C is a Fitting set $\mathcal{F}$ of Q. The inclusion $\iota \colon C \longrightarrow Q$ induces a family of monomorphisms between the members of $\mathcal{F}$ and Q. The pair (ι, Q) is an outer $\mathcal{F}$-Fitting sets pair relative to Q. The inner automorphism α_y of Q induced by y gives a normal Q-embedding of $\nu \colon C \longrightarrow C$ such that $x^\nu = x^{-1}$ and $\iota\alpha_y = \nu\iota$.

If (ι, Q) were equivalent to a $\mathcal{F}$-Fitting sets pair (d, A), there would exist an isomorphism $\psi \colon Q \longrightarrow A$ such that for each subgroup T of C there would exist $\alpha_T \in \mathrm{Inn}(A)$ such that $d_T = \iota_T\psi\alpha_T$. Since $d_C = \nu d_C$, we have that $x^2 \in \mathrm{Ker}(d_C)$. But $\iota_C\psi\alpha_C$ is a monomorphism and therefore $d_C \neq \iota_C\psi\alpha_C$. Thus (ι, Q) cannot be equivalent to an $\mathcal{F}$-Fitting sets pair (d, A).

The following result is the “Fitting sets” version of [Pen87, Satz 3.2].

Theorem 7.4.17. *Let $(d, \mathbf{A})$ be an outer $\mathcal{F}$-Fitting sets pair relative to $\mathbf{G}$ and let $\mathcal{H}$ be a Fitting set of $\mathbf{A}$.*

1. *The collection* $\mathcal{H}d^{-1} = \{U \in \mathcal{F} : U^{d_U} \in \mathcal{H}\}$ *of finite subgroups of* $\mathbf{G}$ *is a Fitting set of* $\mathbf{G}$.
2. *If* $U \in \mathcal{F}$, *then* $U_{\mathcal{H}d^{-1}} = \big((U^{d_U})_{\mathcal{H}}\big)^{d_U^{-1}}$.

Proof. 1. If N is a normal subgroup of $U \in \mathcal{H}d^{-1}$, then N^{d_N} is conjugate in $\mathbf{A}$ to the normal subgroup N^{d_U} of $U^{d_U} \in \mathcal{H}$. Thus $N \in \mathcal{H}d^{-1}$.

Assume that N_1 and N_2 are subgroups of $\mathbf{G}$ which are normal in $T = N_1N_2$ and $N_i \in \mathcal{H}d^{-1}$, for $i = 1, 2$. Then $T^{d_T} = N_1^{d_T}N_2^{d_T}$ and $N_i^{d_T}$ is normal in T^{d_T}, for $i = 1, 2$. Moreover, $N_i^{d_T}$ is conjugate in $\mathbf{A}$ to $N_i^{d_{N_i}}$, for $i = 1, 2$. Therefore $T \in \mathcal{H}d^{-1}$.

2. Let $C = \big((U^{d_U})_{\mathcal{H}}\big)^{d_U^{-1}}$. By Statement 1, C is a normal $\mathcal{H}d^{-1}$-subgroup of U. If M is a normal subgroup of U, with $M \in \mathcal{H}d^{-1}$, then $M^{d_M} \in \mathcal{H}$ and it is conjugate in $\mathbf{A}$ to M^{d_U}. Hence $M^{d_U} \leq (U^{d_U})_{\mathcal{H}}$ and then $M \leq C$. □

Definition 7.4.18. *For an outer* $\mathcal{F}$*-Fitting sets pair relative to* $\mathbf{G}$, $(d, \mathbf{A})$, *and a homomorphism* $\varphi\colon \mathbf{A} \longrightarrow \mathbf{B}$, *we define the* induced outer $\mathcal{F}$-Fitting pair relative to $\mathbf{G}$, $(d\varphi, \mathbf{B})$, *by* $(d\varphi)_T = d_T\varphi$, *for every* $T \in \mathcal{F}$.

The next theorem provides a criterion for the Fitting sets constructed by means of outer Fitting sets pairs to be injective.

Theorem 7.4.19 ([AJBBPR00]). *Let* $\mathbf{G}$ *be a group and denote by* $\mathcal{E}_{\mathbf{G}}$ *the Fitting set composed of all finite subgroups of* $\mathbf{G}$. *Let* $(d, \mathbf{A})$ *be an outer* $\mathcal{E}_{\mathbf{G}}$*-Fitting sets pair relative to* $\mathbf{G}$. *Suppose that* $\mathcal{F}$ *is a Fitting set of* $\mathbf{A}$ *and the pair* $(d, \mathbf{A})$ *satisfies the following condition:*

For each $\mathbf{G}$*-embedding* $\nu\colon V \longrightarrow U$, *for* $U, V \in E_{\mathbf{G}}$ *such that* $U_{\mathcal{F}d^{-1}} \leq V^{\nu}$, *there exists* $\eta \in \mathrm{Inn}(A)$ *such that* $\nu d_U = d_V\eta$. (7.2)

Let $X \in E_{\mathbf{G}}$. *If the group* X^{d_X} *possesses a single conjugacy class of* $\mathcal{F}$*-injectors, then* X *also possesses a single conjugacy class of* $\mathcal{F}d^{-1}$*-injectors.*

Proof. Let X be a subgroup of $\mathbf{G}$ and assume that T is an $\mathcal{F}$-injector of X^{d_X}. Denote by $U = T^{d_X^{-1}}$. We shall see that U is an $\mathcal{F}d^{-1}$-injector of X. Since T is an $\mathcal{F}$-injector of X^{d_X}, it follows that $(X^{d_X})_{\mathcal{F}}$ is a subgroup of T. Hence $X_{\mathcal{F}d^{-1}} = ((X^{d_X})_{\mathfrak{F}})^{d_X^{-1}}$ by Theorem 7.4.17 (2) and it is contained in U. By property (7.2) there exists $a \in \mathbf{A}$ such that $(U^{d_U})^a = U^{d_X}$. Since $T = U^{d_X} \in \mathcal{F}$ it follows that $U \in \mathcal{F}d^{-1}$.

Let N be a subnormal subgroup of X and suppose that $U \cap N \leq W \leq N$, where $W \in \mathcal{F}d^{-1}$. Since N is a subnormal subgroup of X, it holds that $N_{\mathcal{F}d^{-1}} = N \cap X_{\mathcal{F}d^{-1}} \leq N \cap U \leq W$. By (7.2), the subgroup W^{d_N} is conjugate in $\mathbf{A}$ to W^{d_W} which is in $\mathcal{F}$. On the other hand, since $(d, \mathbf{A})$ is an outer $\mathcal{E}_{\mathbf{G}}$-Fitting sets pair relative to $\mathbf{G}$, there exists $\theta \in \mathrm{Inn}(A)$ such that d_N is $d_X\theta$ restricted to N. Hence W^{d_N} is conjugate in $\mathbf{A}$ to W^{d_X}. Consequently $W^{d_X} \in \mathcal{F}$. Now $\mathrm{Ker}(d_X) \leq X_{\mathfrak{F}d^{-1}} \leq U$. Hence $(U \cap N)^{d_X} = T \cap N^{d_X}$ which is contained in $W^{d_X} \leq N^{d_X}$. Since T is an $\mathcal{F}$-injector of X^{d_X} and $W^{d_X} \in \mathcal{F}$, it follows that $T \cap N^{d_X} = W^{d_X}$. Therefore $W \leq U$ and $U \cap N = W$. This means that U is an $\mathcal{F}d^{-1}$-injector of X.

Suppose now that X^{d_X} has a single conjugacy class of $\mathcal{F}$-injectors. Let U and $\tilde{U}$ be two $\mathcal{F}d^{-1}$-injectors of X. A straightforward proof using analogous arguments provides that U^{d_X} and $\tilde{U}^{d_X}$ are $\mathcal{F}$-injectors of X^{d_X}. By hypothesis, there exists $x \in X$ such that $U^{d_X} = (\tilde{U}^x)^{d_X}$. Since $\operatorname{Ker}(d_X) \leq U \cap \tilde{U}$, it follows that $U = \tilde{U}^x$. □

The rest of the section is devoted to construct injective Fitting sets using outer Fitting sets pairs. We shall give some examples of outer Fitting sets pairs which are local versions of the outer Fitting pairs constructed in [Pen88, Sections 4 and 5]. These local constructions provide further information and show that Fitting sets pairs are worth investigating.

Our first example leads to a p-supersoluble Fitting set, p a prime, in every group. This Fitting set is dominant in the set of all p-constrained groups (see Definition 2.4.29).

Example 7.4.20. Let G be a group and let J be a simple group. Suppose that n_G is the largest natural number such that $|J|^{n_G}$ divides $|G|$. Denote by $\mathrm{D}_J(n_G)$ the direct product of n_G copies of J. If $n_G = 0$, we agree that $\mathrm{D}_J(n_G) = 1$. Let $\mathrm{A}_J(n_G) = \operatorname{Aut}\bigl(\mathrm{D}_J(n_G)\bigr)$ and $\mathrm{O}_J(n_G) = \operatorname{Out}\bigl(\mathrm{D}_J(n_G)\bigr)$. It is known that

1. if J is non-abelian, then $\mathrm{A}_J(n_G)$ is isomorphic to the natural wreath product

$$\mathrm{A}_J(n_G) \cong \operatorname{Aut}(J) \wr_{\mathrm{nat}} \operatorname{Sym}(n_G) \quad \text{and} \quad \mathrm{O}_J(n_G) \cong \operatorname{Out}(J) \wr_{\mathrm{nat}} \operatorname{Sym}(n_G).$$

2. if $J \cong C_p$, for a prime p, then

$$\mathrm{A}_J(n_G) \cong \mathrm{GL}(p, n_G).$$

Also let $\mathbf{D}_J$ be the restricted direct product of countably infinitely many copies of J and let $\mathbf{A}_J = \operatorname{Aut}^0(\mathbf{D}_J)$ be the group of all automorphisms of $\mathbf{D}_J$ with finite support Denote $\mathbf{O}_J$ the group of outer automorphisms of $\mathbf{D}_J$ with finite support.

Let $\mathfrak{F}$ and $\mathfrak{G}$ be two Fitting classes such that $\mathfrak{G} \subseteq \mathfrak{F}$.

1. ([Pen88, Theorem II]) For any group G and any chief series Γ of G through $G_{\mathfrak{F}}$ and $G_{\mathfrak{G}}$, let $\mathbf{D}_J(\Gamma, \mathfrak{F}/\mathfrak{G})$ be the direct product of all the J-chief factors of Γ between $G_{\mathfrak{F}}$ and $G_{\mathfrak{G}}$, taken in the order of occurrence in Γ. We consider this group as the subgroup of $\mathbf{D}_J$ consisting of the first direct components of $\mathbf{D}_J$. The group G operates on every such $\mathbf{D}_J(\Gamma, G_{\mathfrak{F}}/G_{\mathfrak{G}})$ and by identical continuation also on $\mathbf{D}_J$. This action defines a homomorphism

$$d_G^{J,\mathfrak{F}/\mathfrak{G}} \colon G \longrightarrow \mathbf{A}_J.$$

Then the pair $(d^{J,\mathfrak{F}/\mathfrak{G}}, \mathrm{A}_J)$ is an outer $\mathfrak{E}$-Fitting pair. This is called the *chief factor product Fitting pair*.

The construction is dependent on the inherent choices only within equivalence of outer Fitting pairs.

2. ([AJBBPR00, Ex. IV]) Let G be a group. Let $\mathcal{E}_G$ denote the Fitting set composed of all subgroups of G. For each $T \in \mathcal{E}_G$, i.e. for each subgroup T of G, we consider a chief series Γ_T of T through $T_{\mathfrak{F}}$ and $T_{\mathfrak{G}}$. Let $\mathrm{D}_J(\Gamma_T)$ be the direct product of all the J-chief factors of T taken in the order of occurrence in Γ_T. We consider this group as the subgroup of $\mathrm{D}_J(n_G)$ consisting of the first direct components of $\mathrm{D}_J(n_G)$. T acts by conjugacy on $\mathrm{D}_J(\Gamma_T)$ and in trivial way on the rest of components of $\mathrm{D}_J(n_G)$. This action defines a homomorphism

$$d_T^{J,\mathfrak{F}/\mathfrak{G}} : T \longrightarrow \mathrm{A}_J(n_G).$$

Then the pair $\left(d^{J,\mathfrak{F}/\mathfrak{G}}, \mathrm{A}_J(n_G)\right)$ is an outer $\mathcal{E}_G$-Fitting sets pair relative to G. This is called the *chief factor product Fitting sets pair* relative to G.

The construction is dependent on the inherent choices only within equivalence of outer Fitting sets pairs.

Remark 7.4.21. With the above notation, if $\mathcal{F}$ is a Fitting set of $\mathbf{A}_J$, then $\mathfrak{F} = \mathcal{F}d^{-1}$ is a Fitting class defined by the chief factor product Fitting pair by [Pen87, Satz 3.2]. Then $\mathrm{Tr}_{\mathfrak{F}}(G)$ is the Fitting set of G defined by the chief factor product Fitting sets pair relative to G (see Theorem 7.4.17).

There exist Fitting sets associated with chief factor product Fitting sets pairs which cannot be obtained in this way.

Let G be a group and p a prime dividing $|G|$. Following the notation the above example, we take $J = C_p$, the cyclic group of order p, $\mathfrak{F} = \mathfrak{E}$ the class of all finite groups, and $\mathfrak{G} = (1)$, the trivial class. Let n_G be the natural number such that p^{n_G} is the order of a Sylow p-subgroup of G. Then $\mathrm{D}_J(n_G)$ is an elementary abelian p-group of order p^{n_G} and $\mathrm{A}_J(n_G) = \mathrm{GL}(n_G, p)$. Denote by $\left(d, \mathrm{GL}(n_G, p)\right)$ the chief factor product Fitting sets pair relative to G of Example 7.4.20 (2), that is $d = d^{C_p,\mathfrak{E}/(1)}$.

Let $\mathcal{F} = \left\{U \leq \mathrm{GL}(n_G, p) \colon U \leq \mathrm{Z}\left(\mathrm{GL}(n_G, p)\right)\right\}$. Since $\mathrm{Z}\left(\mathrm{GL}(n_G, p)\right)$ is a normal subgroup of $\mathrm{GL}(n_G, p)$, it is clear that $\mathcal{F}$ is a Fitting set of $\mathrm{GL}(n_G, p)$. By Theorem 7.4.17 we have that $\mathcal{F}_Z = \mathcal{F}d^{-1}$ is a Fitting set of G.

It is proved in [AJBBPR00, Ex. VI]) that there exist groups G for which $\mathcal{F}_Z$ is not the trace in G of any Fitting class. In particular, $\mathcal{F}_Z$ is not the trace in G of the Fitting class obtained by the inverse image of a Fitting set of $\mathbf{A}_{C_p}$ through the chief factor product Fitting pair.

We study the Fitting set $\mathcal{F}_Z$ in a group G. We assume that $n_G \neq 0$. For any subgroup $B \leq G$, write p^{n_B} the order of a Sylow p-subgroup of B. If $x \in B$, then

$$x^{d_B} = \begin{pmatrix} M(x) & 0 \\ 0 & I_{n_G - n_B} \end{pmatrix},$$

where $M(x) \in \mathrm{GL}(n_B, p)$ is the matrix of the action of x on the p-chief factors of a fixed chief series of B.

If $B \in \mathcal{F}_Z$, then $x^{d_B} = \lambda I_{n_G}$, for some non-zero scalar λ of $\mathrm{GF}(p)$. Hence, the p-chief factors of B are simple and all of them are B-isomorphic. In particular, $\mathrm{Ker}(d_B) = \mathrm{O}_{p',p}(B)$. Moreover, $B/\mathrm{Ker}(d_B)$ is a subgroup of $\mathrm{Z}\left(\mathrm{GL}(n_G, p)\right)$

and then it is isomorphic to a cyclic group of order dividing $p - 1$. If B does not contain any Sylow p-subgroup of G, then B is p-nilpotent; that is, $B = \mathrm{Ker}(d_B)$.

Note that all p-nilpotent subgroups of G are in $\mathcal{F}_Z$. If H is a subgroup of G, the order of a Sylow p-subgroup of H is denoted by $|H|_p$.

Lemma 7.4.22. *Let H be a subgroup of G. Assume that H is a p-soluble group of p-length at most 1. Then:*

1. *$H_{\mathcal{F}_Z}$ is the unique $\mathcal{F}_Z$-maximal subgroup of H containing $\mathrm{O}_{p',p}(H)$; in particular $H_{\mathcal{F}_Z}$ is the unique $\mathcal{F}_Z$-injector of H.*
2. *If $|H|_p < p^{n_G}$, then $H_{\mathcal{F}_Z} = \mathrm{O}_{p',p}(H)$.*
3. *If $|H|_p = p^{n_G}$, then $H_{\mathcal{F}_Z}$ is the set of all $m \in H$ such that m has scalar action on the direct product of the p-chief factors of H in a chief series of H.*

Proof. 1. Let M be an $\mathcal{F}_Z$-subgroup of H containing $\mathrm{O}_{p',p}(H)$. We claim that M is normal in H, so that the conclusion is clear.

Since the p-length of H is smaller than or equal to 1, then $M/\mathrm{O}_{p',p}(H)$ is a p'-group. Consequently the p-chief factors of H are completely reducible $\mathrm{GF}(p)M$-modules. Hence the direct product of the p-chief factors of H in a chief series of H, viewed as a $\mathrm{GF}(p)M$-module in the natural way, is $\mathrm{GF}(p)$ M-isomorphic to the direct product of the p-chief factors of M in a chief series of M. Since $M \in \mathcal{F}_Z$, then M has scalar action on the above mentioned direct product of the p-chief factors of H. Therefore $[M, H] \leq \mathrm{O}_{p',p}(H) \leq M$. In particular M is normal in H.

2. If $|H|_p < p^{n_G}$, it is clear that $H_{\mathcal{F}_Z}$ is p-nilpotent and then $H_{\mathcal{F}_Z} = \mathrm{O}_{p',p}(H)$.

3. Assume now that $|H|_p = p^{n_G}$. Denote by S the set of all $m \in H$ such that m has scalar action on the direct product of the p-chief factors of H in a chief series of H. It is clear that S is a normal subgroup of H containing $\mathrm{O}_{p',p}(H)$. Note that the p-chief factors of H are completely reducible as $\mathrm{GF}(p)H_{\mathcal{F}_Z}$-modules and also as $\mathrm{GF}(p)S$-modules because $H_{\mathcal{F}_Z}$ and S are normal subgroups of H. Moreover, since $|H|_p = p^{n_G}$ we can easily deduce that $S \in \mathcal{F}_Z$ and also that $S = H_{\mathcal{F}_Z}$. ◻

Recall that the class $\mathfrak{E}_{p'}\mathfrak{S}_p$ of all p-nilpotent groups is injective, and a group G possesses a unique conjugacy class of $\mathfrak{E}_{p'}\mathfrak{S}_p$-injectors if and only if G is p-constrained (see Corollary 7.2.31 and Remark 7.2.32). Moreover, in this case,

$$\mathrm{Inj}_{\mathfrak{E}_{p'}\mathfrak{S}_p}(G) = \{\mathrm{O}_{p',p}(G)P : P \in \mathrm{Syl}_p(G)\},$$

and the p-nilpotent injectors of G are the p-nilpotent maximal subgroups of G containing $\mathrm{O}_{p',p}(G)$.

Lemma 7.4.23. *Let H be a p-constrained subgroup of G such that $|H|_p = p^{n_G}$. Suppose that M is an $\mathcal{F}_Z$-maximal subgroup of H containing $\mathrm{O}_{p',p}(H)$.*

1. *There exists a p-nilpotent injector I of H such that $I = \mathrm{O}_{p',p}(M)$.*
2. *Moreover, M is the $\mathcal{F}_Z$-radical of $\mathrm{N}_H(I)$ and is the set of all elements $m \in \mathrm{N}_H(I)$ such that m has scalar action on the direct product of the p-chief factors of $\mathrm{N}_H(I)$ in a chief series of $\mathrm{N}_H(I)$.*

Proof. Suppose that $|M|_p < p^{n_G}$. In this case since $M \in \mathcal{F}_Z$, we have that M is a p-nilpotent group and then M is contained in a p-nilpotent injector, X say, of H, because $\mathrm{O}_{p',p}(H) \leq M$. But clearly $X \in \mathcal{F}_Z$, which implies $X = M$. In particular $M = \mathrm{O}_{p'}(H)H_p$ for some $H_p \in \mathrm{Syl}_p(H)$, which is a contradiction. Consequently there exists a Sylow p-subgroup H_p of H such that $\mathrm{O}_{p',p}(H)H_p \leq \mathrm{O}_{p',p}(M)$. But $I = \mathrm{O}_{p',p}(H)H_p$ is a p-nilpotent injector of H, which implies that $I = \mathrm{O}_{p',p}(M)$.

Observe that $I \leq M \leq \mathrm{N}_H(I)$ and $I = \mathrm{O}_{p',p}\big(\mathrm{N}_H(I)\big)$. Since $\mathrm{N}_H(I)$ is a p-soluble group of p-length at most 1, the conclusion follows from Lemma 7.4.22. □

Theorem 7.4.24. *Let H be a p-constrained subgroup of G. Then H has a unique conjugacy class of $\mathcal{F}_Z$-injectors. Moreover, the $\mathcal{F}_Z$-injectors of H are exactly the $\mathcal{F}_Z$-maximal subgroups of H containing $\mathrm{O}_{p',p}(H)$, or equivalently, the $\mathcal{F}_Z$-radical of H.*

Moreover, we have:

1. *If $|H|_p < p^{n_G}$, then the $\mathcal{F}_Z$-injectors of H are exactly the p-nilpotent injectors of H.*
2. *If $|H|_p = p^{n_G}$, then the set of $\mathcal{F}_Z$-injectors of H is exactly*

$$\mathrm{Inj}_{\mathcal{F}_Z}(G) = \big\{ \big(\mathrm{N}_H(I)\big)_{\mathcal{F}_Z} : I \in \mathrm{Inj}_{\mathfrak{E}_{p'}\mathfrak{S}_p}(H) \big\}$$

In particular, the $\mathcal{F}_Z$-injectors of H are the subgroups composed of all elements $m \in \mathrm{N}_H(I)$ such that m has scalar action on the direct product of the p-chief factors of $\mathrm{N}_H(I)$ in a chief series of $\mathrm{N}_H(I)$, where I is a p-nilpotent injector of H.

Proof. Note that if $|H|_p < p^{n_G}$, then the $\mathcal{F}_Z$-subgroups of H are exactly the p-nilpotent subgroups. On the other hand, if $|H|_p = p^{n_G}$, it is clear by Lemma 7.4.23 that the set of $\mathcal{F}_Z$-maximal subgroups of H containing $\mathrm{O}_{p',p}(H)$ is exactly the set $\big\{ \big(\mathrm{N}_H(I)\big)_{\mathcal{F}_Z} : I \in \mathrm{Inj}_{\mathfrak{E}_{p'}\mathfrak{S}_p}(H) \big\}$ which is a conjugacy class of subgroups of H. Since $\mathrm{O}_{p',p}(H) \leq H_{\mathcal{F}_Z}$, we deduce that this set also coincides with the set of all $\mathcal{F}_Z$-maximal subgroups of H containing $H_{\mathcal{F}_Z}$.

Therefore the Fitting set $\mathcal{F}_Z$ is dominant in the set $\mathcal{X} = \{H \leq G : H \text{ is } p\text{-constrained}\}$. □

J. Pense ([Pen87, 4.14]) presented a type of Fitting classes, constructed by means of Fitting pairs, with respect to which every finite group has a unique conjugacy class of injectors. An improved version of this result is presented in [Pen90c]. We shall show in the sequel that Pense's result is actually a particular case of a more general one.

Definition 7.4.25. *Let G be a group and let S be a perfect comonolithic group whose head is isomorphic to a simple group J. Let L be the subgroup generated by all subnormal subgroups of G isomorphic to S*

$$L = \langle T : T \text{ is subnormal in } G \text{ and } T \cong S\rangle$$

and let

$$M = \langle \mathrm{Cosoc}(T) : T \text{ is subnormal in } G \text{ and } T \cong S\rangle$$

(which is a normal subgroup of L by Theorem 2.2.19). The factor group L/M is called the S-head-section of G.

By Theorem 2.2.19, $L = T_1 \cdots T_m$, where all T_i are normal subgroups of L and $T_i \cong S$. Note that if S a perfect comonolithic subnormal subgroup of a group which is the join of two subnormal subgroups S_1 and S_2, then either S is contained in S_1 or S is contained in S_2 ([Wie39]). This implies that $T_i \cap M = \mathrm{Cosoc}(T_i)$ and then $T_iM/M \cong J$. Hence L/M is a group in the Fitting class $\mathrm{Fit}(J)$ generated by J, i.e. L/M is isomorphic to a direct product of copies of J, by Example 2.2.3 (1).

Example 7.4.26. Let G be a group and let S be a perfect comonolithic group whose head is isomorphic to a simple group J. Let $\mathrm{D}_J(n_G)$, $\mathrm{A}_J(n_G)$, $\mathfrak{F}$, and $\mathfrak{G}$ be as in Example 7.4.20.

1. ([Pen88, Theorem III]) For any group G fix an embedding of the S-head-section of $G_{\mathfrak{F}}/G_{\mathfrak{G}}$ as the first components of $\mathbf{D}_J$. Then G operates on $\mathbf{D}_J$ via this embedding, and therefore we have a homomorphism

$$H_G^{S,\mathfrak{F}/\mathfrak{G}} : G \longrightarrow \mathbf{A}_J.$$

The pair $(H^{S,\mathfrak{F}/\mathfrak{G}}, \mathbf{A}_J)$ is an outer $\mathfrak{E}$-Fitting pair.

2. ([AJBBPR00, Ex. V])

For each subgroup T of the group G, we fix an embedding of the S-head-section of $T_{\mathfrak{F}}/T_{\mathfrak{G}}$ as the first components of $\mathrm{D}_J(n_G)$. Then T operates on $\mathrm{D}_J(n_G)$ via this embedding, and therefore we have a homomorphism

$$h_T^{S,\mathfrak{F}/\mathfrak{G}} : T \longrightarrow \mathrm{A}_J(n_G).$$

Denote by $\mathcal{E}_G$ the Fitting set of all subgroups of G. Thus the pair

$$\big(h^{S,\mathfrak{F}/\mathfrak{G}}, \mathrm{A}_J(n_G)\big)$$

is an outer $\mathcal{E}_G$-Fitting sets pair relative to G.

Let S be a perfect comonolithic group whose head is isomorphic to a non-abelian simple group J. Consider the Fitting classes $\mathfrak{F} = \mathfrak{E}$, the class of all finite groups, and $\mathfrak{G} = (1)$, the trivial class. Write $H^{S,\mathfrak{F}/\mathfrak{G}} = H^S$. Then it appears the outer $\mathfrak{E}$-Fitting pair, $(H^S, \mathbf{A}_J)$ say. Consider the projection from

$\mathbf{A}_J$ to $\mathbf{O}_J$ and let $(\tilde{H}^S, \mathbf{O}_J)$ be the induced outer Fitting pair from the pair $(H^S, \mathbf{A}_J)$.

Analogously, if we consider the projection from $\mathrm{A}_J(n_G)$ onto $\mathrm{O}_J(n_G) = \mathrm{Out}\big(\mathrm{D}_J(n_G)\big)$ and let $\big(\tilde{h}^S, \mathrm{O}_J(n_G)\big)$ be the induced outer Fitting sets pair relative to G from the pair $\big(h^S, \mathrm{A}_J(n_G)\big)$.

Theorem 7.4.27. *With the notation introduced above, let $\mathcal{F}$ be a Fitting set of $\mathrm{O}_J(n_G)$ all whose elements are subgroups of the base group of $\mathrm{O}_J(n_G)$ and let $\mathcal{T} = \mathcal{F}(\tilde{h}^S)^{-1}$ be the Fitting set corresponding to the pair $\big(\tilde{h}^S, \mathrm{O}_J(n_G)\big)$.*

If $\mathrm{Out}(J)$ is soluble, then each subgroup of G has exactly a conjugacy class of $\mathcal{T}$-injectors.

Proof. Note that for every subgroup B of $\mathrm{O}_J(n_G)$, the $\mathcal{F}$-injectors of $B \cap \mathrm{Out}(J)^\natural$, where $\mathrm{Out}(J)^\natural$ is the base group of $\mathrm{O}_J(n_G)$, are exactly the $\mathcal{F}$-injectors of B. Therefore each subgroup of $\mathrm{O}_J(n_G)$ possesses a single conjugacy class of $\mathcal{F}$-injectors by Theorem 2.4.26. Then it is enough to show that the pair $\big(\tilde{h}^S, \mathrm{O}_J(n_G)\big)$ satisfies the property (7.2) of Theorem 7.4.19.

Write $f = \tilde{h}^S$. Let $\nu\colon V \longrightarrow U$ be a G-embedding between subgroups U and V of G such that $U_\mathcal{T} \le V^\nu$. We consider L_U/M_U and L_{V^ν}/M_{V^ν} the S-head-section of U and V^ν respectively. It is clear that L_U/M_U is the S-head-section of L_U and so $L_U^{f_{L_U}} = 1 \in \mathcal{F}$. Then $L_U \in \mathcal{F}f^{-1} = \mathcal{T}$ and $L_U \le U_\mathcal{T}$ and so also $L_U \le V^\nu$. This implies that $L_U \le L_{V^\nu}$. Now suppose that there exists a subnormal subgroup X of V^ν such that $X \cong S$ and X is not subnormal in U. Then, for any subnormal subgroup T of U such that $T \cong S$, we have that X and T are normal in XT, by Theorem 2.2.19, and then $[X, T] \le \mathrm{Cosoc}(T)$. Hence $[X, L_U] \le M_U$. Therefore $X \le \mathrm{C}_{V^\nu}\big(L_U/M_U\big) \le \mathrm{C}_U\big(L_U/M_U\big)$. Since $\mathrm{C}_U(L_U/M_U) \le \mathrm{Ker}(f_U) \le U_\mathcal{T} \le V^\nu$, it follows that X is subnormal in $\mathrm{C}_U(L_U/M_U)$ and also is in U, contrary to supposition.

Therefore the S-head-section of V^ν coincides with the S-head-section of U and then it is conjugate to the S-section of V. By construction of the Fitting sets pair, it follows that there exists $\eta \in \mathrm{Inn}\big(\mathrm{O}_J(n_G)\big)$, such that $\nu f_U = f_V \eta$. ☐

Now we deduce the aforesaid result of J. Pense.

Theorem 7.4.28 ([Pen90c]). *Let S be a perfect comonolithic group with head J. Consider the outer Fitting pair $(\tilde{H}^S, \mathbf{O}_J)$. Let $\mathcal{F}$ be a Fitting set in the base group of $\mathbf{O}_J$ and let $\mathfrak{F} = \mathcal{F}(\tilde{H}^S)^{-1}$ be the corresponding Fitting class. If the outer automorphism group of J is soluble, then every finite group has exactly a conjugacy class of $\mathfrak{F}$-injectors.*

Proof. First of all, note that $\mathbf{A}_J = \lim_{n\to\infty}\big(\mathrm{Aut}(\overbrace{J \times \cdots \times J}^{(n \text{ copies})})\big)$ and so $\mathbf{A}_J$ is the (restricted, natural) wreath product $\lim_{n\to\infty}\big(\mathrm{Aut}(J) \wr_{\mathrm{nat}} S_n\big)$ with base group $\mathrm{Aut}(J)^\natural$. Then $\mathbf{O}_J$ is $\mathbf{A}_J/\mathrm{Inn}(J)^\natural$ with base group $\mathrm{Out}(J)^\natural$.

For each group G we consider $\mathrm{O}_J(n_G)$ as a subgroup of $\mathbf{O}_J$. With respect to the outer $\mathcal{E}_G$-Fitting sets pair relative to G, $\big(\tilde{h}^S, \mathrm{O}_J(n_G)\big)$ and for each subgroup T of G, we have

$$(t)^{\tilde{h}^S_T} = (t)^{\tilde{H}^S_T} \in \mathrm{O}_J(n_G) \leq \mathbf{O}_J \qquad \text{for every } t \in T.$$

Therefore it follows that $\mathrm{Tr}_G(\mathfrak{F}) = \big(\mathrm{Tr}_{\mathrm{O}_J(n_G)}(\mathcal{F})\big)(\tilde{H}^S)^{-1}$. Applying Theorem 7.4.27, G has a conjugacy class of $\mathfrak{F}$-injectors. $\square$

Recall finally Schreier's conjecture, whose validity has been proved using the classification of finite simple groups, which states that the group $\mathrm{Out}(J)$, of all outer automorphisms of a non-abelian simple group J, is always soluble (see [KS04, page 151]).