# 4

# Normalisers and prefrattini subgroups

The aim of this chapter is to obtain information about the structure of a finite group through the study of $\mathfrak{H}$-normalisers and subgroups of prefrattini type.

In the soluble universe, after the introduction of saturated formations and covering subgroups by W. Gaschütz, R. W. Carter, and T. O. Hawkes introduced in [CH67] a conjugacy class of subgroups associated to saturated formations $\mathfrak{F}$ of full characteristic, the $\mathfrak{F}$-normalisers, defined in terms of a local definition of $\mathfrak{F}$, which generalised Hall's system normalisers. The Carter-Hawkes's $\mathfrak{F}$-normalisers keep all essential properties of system normalisers and, in the case of the saturated formation $\mathfrak{N}$ of the nilpotent groups, the $\mathfrak{N}$-normalisers of a group are exactly Hall's system normalisers.

In this context, and having in mind the known characterisation of $\mathfrak{F}$-normalisers by means of $\mathfrak{F}$-critical subgroups, it is natural to think about $\mathfrak{H}$-normalisers associated with Schunck classes $\mathfrak{H}$ for which the existence of $\mathfrak{H}$-critical subgroups is assured in each soluble group not in $\mathfrak{H}$. A. Mann [Man70] chose this characterisaton as his starting point and was able to extend introduced the normaliser concept to certain Schunck classes following this arithmetic-free way.

Concerning the prefrattini subgroups, we said in Sections 1.3 and 1.4 that the classical prefrattini subgroups of soluble groups were introduced by W. Gaschütz ([Gas62]). A prefrattini subgroup is defined by W. Gaschütz as an intersection of complements of the crowns of the group. They form a characteristic conjugacy class of subgroups which cover the Frattini chief factors and avoid the complemented ones. Gaschütz's original prefrattini subgroups have been widely investigated and variously generalised. The first extension is due to T. O. Hawkes ([Haw67]). He introduced the idea of obtaining analogues to Gaschütz's prefrattini subgroups, associated with a saturated formation $\mathfrak{F}$, by taking intersections of certain maximal subgroups defined in terms of $\mathfrak{F}$ into which a Hall system of the group reduces. Note that Hawkes restricts the set of maximal subgroups considered to the set of $\mathfrak{F}$-abnormal maximal subgroups. He observed that all of the relevant properties of the original idea were kept and, furthermore, he presented an original new theorem of factorisation of the $\mathfrak{F}$-normaliser and the new prefrattini subgroup associated to the same Hall system.

The extension of this theory to Schunck classes, still in the soluble realm, was done by P. Förster in [För83].

Another generalisation of the Gaschütz work in the soluble universe is due to H. Kurzweil [Kur89]. He introduced the $H$-prefrattini subgroups of a soluble group $G$, where $H$ is a subgroup of $G$. The $H$-prefrattini subgroups are conjugate in $G$ and they have the cover-avoidance property; if $H = 1$ they coincide with the classical prefrattini subgroups of Gaschütz and if $\mathfrak{F}$ is a saturated formation and $H$ is an $\mathfrak{F}$-normaliser of $G$ the $H$-prefrattini subgroups are those described by Hawkes.

The first attempts to develop a theory of prefrattini subgroups outside the soluble universe appeared in the papers of A. A. Klimowicz in [Kli77] and A. Brandis in [Bra88]. Both defined some types of prefrattini subgroups in $\pi$-soluble groups. They manage to adapt the arithmetical methods of soluble groups to the complements of crowns of $p$-chief factors, for $p \in \pi$, of $\pi$-soluble groups. Also the extension of prefrattini subgroups to a class of non finite groups with a suitable Sylow structure, made by M. J. Tomkinson in [Tom75], has to be mentioned.

All these types of prefrattini subgroups keep the original properties of Gaschütz: they form a conjugacy class of subgroups, they are preserved by epimorphic images and they avoid some chief factors, exactly those associated to the crowns whose complements are used in their definition, and cover the rest. Moreover, some other papers (see [Cha72, Mak70, Mak73]) analysed their excellent permutability properties, following the example of the theorem of factorisation of Hawkes.

At the beginning of the decade of the eighties of the past twentieth century, when the classification of simple groups was almost accomplished, H. Wielandt proposed, as a main aim after the classification, to progress in the universe of non-necessarily soluble groups trying to extend the magnificent results obtained in the soluble realm. As we have mentioned in Section 2.3, R. P. Erickson, P. Förster and P. Schmid answered this Wielandt's challenge analysing the projective classes in the non-soluble universe. It seems natural to progress in that direction and think about normalisers and prefrattini subgroups in the general finite universe. This was the starting point A. Ballester-Bolinches' Ph. Doctoral Thesis at the Universitat de València in 1989 [BB89b].

This chapter has two main themes which are organised in three sections. The first two sections are devoted to study the theory of normalisers of finite, non-necessarily soluble, groups. The second subject under investigation is the theory of prefrattini subgroups outside the soluble universe. This is presented in Section 4.3.

## 4.1 $\mathfrak{H}$-normalisers

Obviously the definition of $\mathfrak{H}$-normalisers in the general universe has to be motivated by the characterisation of $\mathfrak{H}$-normalisers of soluble groups by chains of $\mathfrak{H}$-critical subgroups.

In this section, $\mathfrak{H}$ will be a Schunck class of the form $\mathfrak{H} = \mathrm{E}_{\Phi}\mathfrak{F}$, for some formation $\mathfrak{F}$. Thus, by Theorem 2.3.24, the existence of $\mathfrak{H}$-critical subgroups is assured in every group which does not belong to $\mathfrak{H}$.

Here we present the extension of the theory of $\mathfrak{H}$-normalisers to general non-necessarily soluble groups done by A. Ballester-Bolinches in his Ph. Doctoral Thesis [BB89b] and published in [BB89a]. Previous ways of extending the soluble theory had been looked at. J. Beidleman and B. Brewster [BB74] studied normalisers associated to saturated formations in the $\pi$-soluble universe, $\pi$ a set of primes, and L. A. Shemetkov [She76] introduced normalisers associated to saturated formations in the general universe of all finite groups by means of critical supplements of the residual.

The definition of $\mathfrak{H}$-normaliser presented here is obviously motivated by the most abstract characterisation of the classical $\mathfrak{H}$-normalisers.

**Definition 4.1.1.** *Let $G$ be a group. A subgroup $D$ of $G$ is said to be an* $\mathfrak{H}$-normaliser *of $G$ if either $D = G$ or there exists a chain of subgroups*

$$D = H_n \leq H_{n-1} \leq \cdots \leq H_1 \leq H_0 = G \tag{4.1}$$

*such that $H_i$ is $\mathfrak{H}$-critical subgroup of $H_{i-1}$, for each $i \in \{1, \dots, n\}$, and $H_n$ contains no $\mathfrak{H}$-critical subgroup.*

The condition on $H_n$ is equivalent to say that $D \in \mathfrak{H}$. Moreover $D = G$ if and only if $G \in \mathfrak{H}$.

The non-empty set of all $\mathfrak{H}$-normalisers of $G$ will be denoted by $\mathrm{Nor}_{\mathfrak{H}}(G)$.

If we restrict ourselves to the universe of soluble groups, this definition is equivalent to the classical ones of R. W. Carter and T. O. Hawkes in [CH67] and A. Mann in [Man70] (see [DH92, V, 3.8]).

In this section, we analyse the main properties of $\mathfrak{H}$-normalisers, primarily motivated by their behaviour in the soluble universe. In particular, we study their relationship with systems of maximal subgroups and projectors.

Each $\mathfrak{H}$-normaliser of a soluble group is associated with a particular Hall system of the group ([Man70]). Obviously this is no longer true in the general case. But bearing in mind the relationship between systems of maximal subgroups and Hall systems (see Theorem 1.4.17 and Corollary 1.4.18), it seems natural to wonder about the relationship between $\mathfrak{H}$-normalisers and systems of maximal subgroups.

Assume that $D$ is an $\mathfrak{H}$-normaliser of a group $G$ constructed by the chain

$$D = H_n \leq H_{n-1} \leq \cdots \leq H_1 \leq H_0 = G \tag{4.2}$$

such that $H_i$ is $\mathfrak{H}$-critical subgroup of $H_{i-1}$, for each $i \in \{1, \dots, n\}$, and $H_n$ contains no $\mathfrak{H}$-critical subgroup. Let $\mathbf{X}(D)$ be a system of maximal subgroups of $D$. Applying several times Theorem 1.4.14, we can obtain a system of maximal subgroups $\mathbf{X}$ of $G$ such that there exist systems of maximal subgroups $\mathbf{X}_i$ of $H_i$, for $i = 0, 1, \dots, n$, with $\mathbf{X}_0 = \mathbf{X}$, $\mathbf{X}_n = \mathbf{X}(D)$ and for each $i$, $H_i \in \mathbf{X}_{i-1}$ and $(\mathbf{X}_{i-1})_{H_i} = \{H_i \cap S : S \in \mathbf{X}_{i-1}, S \neq H_i\} \subseteq \mathbf{X}_i$. This motivates the following definition.

**Definition 4.1.2.** *Let $D$ be an $\mathfrak{H}$-normaliser of a group $G$ constructed by a chain* (4.2) *and let $\mathbf{X}$ be a system of maximal subgroups of $G$ such that there exist systems of maximal subgroups $\mathbf{X}_i$ of $H_i$, $i = 0, 1, \dots, n$, with $\mathbf{X}_0 = \mathbf{X}$, $\mathbf{X}_n = \mathbf{X}(D)$ and for each $i$, $H_i \in \mathbf{X}_{i-1}$ and $(\mathbf{X}_{i-1})_{H_i} = \{H_i \cap S : S \in \mathbf{X}_{i-1}, S \neq H_i\} \subseteq \mathbf{X}_i$. We will say that $D$ is an $\mathfrak{H}$-*normaliser of $G$ associated with $\mathbf{X}$.

By the previous paragraph, every $\mathfrak{H}$-normaliser is associated with some system of maximal subgroups. Next we see that every system of maximal subgroups has an associated $\mathfrak{H}$-normaliser.

**Proposition 4.1.3.** *Given a system of maximal subgroups $\mathbf{X}$ of a group $G$, there exists an $\mathfrak{H}$-normaliser of $G$ associated with $\mathbf{X}$.*

*Proof.* We argue by induction on the order of $G$. We can assume that $G \notin \mathfrak{H}$. Let $M$ be an $\mathfrak{H}$-critical maximal subgroup of $G$ such that $M \in \mathbf{X}$. By Corollary 1.4.16, there exists a system of maximal subgroups $\mathbf{Y}$ of $M$, such that $\mathbf{X}_M \subseteq \mathbf{Y}$. By induction, there exists an $\mathfrak{H}$-normaliser $D$ of $M$ associated with $\mathbf{Y}$. Then $D$ is an $\mathfrak{H}$-normaliser of $G$ associated with $\mathbf{X}$. ☐

*Remarks 4.1.4.* 1. An $\mathfrak{H}$-normaliser can be associated with some different systems of maximal subgroups. Consider the symmetric group of order 5, $G = \mathrm{Sym}(5)$, and $\mathfrak{H} = \mathfrak{N}$ the class of nilpotent groups. Write $D = \langle (12), (45) \rangle$. The subgroups $M_1 = D\langle (123) \rangle$ and $M_2 = D\langle (345) \rangle$ are $\mathfrak{N}$-critical maximal subgroups of $G$ and $\mathbf{X}_1 = \{M_1, \mathrm{Alt}(5)\}$ and $\mathbf{X}_2 = \{M_2, \mathrm{Alt}(5)\}$ are systems of maximal subgroups of $G$. Observe that $D$ is an $\mathfrak{N}$-normaliser of $G$ associated with $\mathbf{X}_1$ and $\mathbf{X}_2$.

2. Given a system of maximal subgroups $\mathbf{X}$ of a group $G$, there is not a unique $\mathfrak{H}$-normaliser of $G$ associated with $\mathbf{X}$. In the soluble group

$$G = \langle a, b : a^9 = b^2 = 1, a^b = a^{-1} \rangle,$$

the Hall system $\Sigma = \{G, \langle a \rangle, \langle b \rangle\}$ reduces into the $\mathfrak{N}$-critical subgroup $M = \langle a^3, b \rangle$ and then the $\mathfrak{N}$-normalisers $D_1 = \langle b \rangle$ and $D_2 = \langle a^3 b \rangle$ are associated with the system of maximal subgroups defined by $\Sigma$: $\mathbf{X}(\Sigma) = \{\langle a \rangle, \langle a^3, b \rangle\}$.

For a non-soluble example, consider the Example of 1 and observe that $D_1 = \langle (12), (45) \rangle$, $D_2 = \langle (13), (45) \rangle$ and $D_3 = \langle (23), (45) \rangle$ are $\mathfrak{N}$-normalisers associated with $\mathbf{X}_1$.

One of the basic properties of $\mathfrak{H}$-normalisers of soluble groups is that they are preserved by epimorphic images (see [DH92, V, 3.2]). This is also true in the general case.

**Proposition 4.1.5.** *Let $G$ be a group. Let $N$ be a normal subgroup of $G$. If $D$ is an $\mathfrak{H}$-normaliser of $G$ associated with a system of maximal subgroups $\mathbf{X}$, then $DN/N$ is an $\mathfrak{H}$-normaliser of $G/N$ associated with $\mathbf{X}/N$.*

*In particular, the $\mathfrak{H}$-normalisers of a group are preserved under epimorphic images.*

*Proof.* We argue by induction on the order of $G$. Suppose first that $N$ is a minimal normal subgroup of $G$. If $G \in \mathfrak{H}$, $D = G$ and there is nothing to prove. If $G \notin \mathfrak{H}$, then $G$ has an $\mathfrak{H}$-critical subgroup $M \in \mathbf{X}$ such that $D$ is an $\mathfrak{H}$-normaliser of $M$ associated with a system of maximal subgroups $\mathbf{Y}$ of $M$ and $\mathbf{X}_M \subseteq \mathbf{Y}$. If $N$ is contained in $M$, then $DN/N$ is, applying induction, an $\mathfrak{H}$-normaliser of $M/N$ associated with the system of maximal subgroups $\mathbf{Y}/N$ of $M/N$. Since $\mathbf{X}/N_{M/N} = \mathbf{X}_M/N$ is contained in $\mathbf{Y}/N$ and $M/N$ is $\mathfrak{H}$-critical in $G/N$ by Lemma 2.3.23, it follows that $DN/N$ is an $\mathfrak{H}$-normaliser of $G/N$ associated with $\mathbf{X}/N$. Suppose that $G = MN$. By induction, $D(M \cap N)/(M \cap N)$ is an $\mathfrak{H}$-normaliser of $M/(M \cap N)$ associated with $\mathbf{Y}/(M \cap N)$. Therefore, by virtue of the canonical isomorphism between $G/N$ and $M/(M \cap N)$, it follows that $DN/N$ is an $\mathfrak{H}$-normaliser of $G/N$ associated with $\mathbf{X}/N$ (note that the image of $\mathbf{X}/N = \{YN/N : Y \in \mathbf{X}_M\}$ under the above isomorphism is just $\mathbf{Y}/(M \cap N)$).

Assume now that $N$ is not a minimal normal subgroup of $G$ and let $A$ be a minimal normal subgroup of $G$ contained in $N$. Then, by induction, $DA/A$ is an $\mathfrak{H}$-normaliser of $G/A$ associated with $\mathbf{X}/A$ and $(DN/A)\big/(N/A)$ is $\mathfrak{H}$-normaliser of $(G/A)\big/(N/A)$ associated with $(\mathbf{X}/A)\big/(N/A)$. Consequently, $DN/N$ is an $\mathfrak{H}$-normaliser of $G/N$ associated with $\mathbf{X}/N$.

The proof of the proposition is now complete.

It is well-known that $\mathfrak{H}$-normalisers of soluble groups cover the $\mathfrak{H}$-central chief factors and avoid the $\mathfrak{H}$-eccentric ones (see [DH92, V, 3.3]). The cover-avoidance property is a typical property of the soluble universe that we cannot expect to be satisfied in the general one.

We present here some results to show partial aspects of the cover-avoidance property of $\mathfrak{H}$-normalisers in the general universe.

**Lemma 4.1.6.** *Let $M$ be an $\mathfrak{H}$-critical subgroup of a group $G$. If $H/K$ is an $\mathfrak{H}$-central chief factor of $G$, then $M$ covers $H/K$ and $[H/K] * G \cong [(H \cap M)/(K \cap M)] * M$. In particular $(H \cap M)/(K \cap M)$ is an $\mathfrak{H}$-central chief factor of $M$.*

*Proof.* If $M$ does not cover $H/K$, then $K = H \cap \mathrm{Core}_G(M)$ and $M$ supplements $H/K$. Moreover $H \,\mathrm{Core}_G(M)/\mathrm{Core}_G(M)$ is the socle of the monolithic primitive group $G/\mathrm{Core}_G(M)$. Since $H \,\mathrm{Core}_G(M)/\mathrm{Core}_G(M) \cong_G H/K$, then

$G/\operatorname{Core}_G(M) \cong [H/K] * G \in \mathfrak{H}$, contrary to the $\mathfrak{H}$-abnormality of $M$ in $G$. Hence $M$ covers $H/K$. Since $H/K$ is $\mathfrak{H}$-central in $G$, then $\mathrm{C}_G(H/K)$ is not contained in $\operatorname{Core}_G(M)$ and therefore $G = M\,\mathrm{C}_G(H/K)$. Now the result follows from [DH92, A, 13.9]. □

**Corollary 4.1.7.** *Let $D$ be an $\mathfrak{H}$-normaliser of a group $G$. If $H/K$ is an $\mathfrak{H}$-central chief factor of $G$, then $D$ covers $H/K$ and $(H \cap D)/(K \cap D)$ is an $\mathfrak{H}$-central chief factor of $D$. Moreover, $\operatorname{Aut}_G(H/K) \cong \operatorname{Aut}_D\big((H \cap D)/(K \cap D)\big)$.*

**Proposition 4.1.8.** *Let $D$ be an $\mathfrak{H}$-normaliser of a group $G$. If $H/K$ is a supplemented chief factor of $G$ covered by $D$, then $[H/K]*G \cong [(H \cap D)/(K \cap D)] * D \in \mathfrak{H}$.*

*Proof.* If $D = G$ the result is clear. Suppose that $D$ is an $\mathfrak{H}$-critical subgroup of $G$. Since $H/K$ is avoided by $\Phi(G)$ and covered by $D$, then $(H \cap D)/(K \cap D)$ is a chief factor of $D$, $\operatorname{Aut}_G(H/K) \cong \operatorname{Aut}_D\big((H \cap D)/(K \cap D)\big)$ and $[H/K]*G \cong [(H \cap D)/(K \cap D)] * D$, by Statements (1), (2), and (3) of Proposition 1.4.11. Thus, if $H/K$ is non-abelian, then $[H/K] * G$ is isomorphic to a quotient group of $D$ and therefore $[H/K] * G \in \mathfrak{H}$. If $H/K$ is abelian, then $H/K$ it is complemented by a maximal subgroup $M$ of $G$. By Proposition 1.4.11 (4), we have that $M \cap D$ is a maximal subgroup of $D$, and $(H \cap D)/(K \cap D)$ is a chief factor of $D$ complemented by $M \cap D$. Since $D \in \mathfrak{H}$, the primitive group associated with $(H \cap D)/(K \cap D)$ is isomorphic to a quotient group of $D$ and therefore $[(H \cap D)/(K \cap D)] * D \in \mathfrak{H}$.

In the general case, we consider the chain (4.2) of subgroups of $G$. If $H/K$ is a supplemented chief factor of $G$ covered by $D$, then $H/K$ is covered by $H_1$ and avoided by $\Phi(G)$. By Proposition 1.4.11, $(H \cap H_1)/(K \cap H_1)$ is a supplemented chief factor of $H_1$. Now, since $D$ is an $\mathfrak{H}$-normaliser of $H_1$, then $[(H \cap H_1)/(K \cap H_1)] * H_1 \cong [(H \cap D)/(K \cap D)] * D$ by induction. Since clearly $[(H \cap H_1)/(K \cap H_1)] * H_1 \cong [H/K] * G$, we deduce that $[H/K] * G \cong [(H \cap D/(K \cap D)] * D \in \mathfrak{H}$. □

**Corollary 4.1.9.** *Let $D$ be an $\mathfrak{H}$-normaliser of a group $G$. Then, among all supplemented chief factors of $G$, $D$ covers exactly the $\mathfrak{H}$-central ones.*

We show next that nothing can be said about the $\mathfrak{H}$-eccentric chief factors of $G$.

*Example 4.1.10.* Let $S$ be the alternating group of degree 5. Consider the class $\mathfrak{F} = \big(G : S \notin \mathrm{Q}(G)\big)$. Then $\mathrm{b}(\mathfrak{F}) = \big(S\big)$. Hence $\mathfrak{F}$ is a saturated formation by Example 2.3.21. Let $E$ be the maximal Frattini extension of $S$ with 3-elementary abelian kernel (see [DH92, Appendix $\beta$] for details). The group $E$ possesses a 3-elementary abelian normal subgroup $N$ such that $N \leq \Phi(E)$, and $E/N \cong S$. Let $M$ be a maximal subgroup of $E$, such that $M/N \cong \mathrm{Alt}(4)$. Then $M$ is $\mathfrak{F}$-critical in $E$ and, since $M$ is soluble, and then $M \in \mathfrak{F}$, we have that $M$ is an $\mathfrak{F}$-normaliser of $E$. Observe also that if a minimal normal subgroup $K$ of $E$ in $N$ is $\mathfrak{F}$-central in $E$, then $K \leq \mathrm{Z}(E)$. Recall that $N \cong \mathrm{A}_3(S)$, the

3-Frattini module, and we can think of $N$ as an GF(3)[$S$]-module. If we denote $S(N)$ the socle of such module, we have that $\mathrm{Ker}\big(S \,\mathrm{on}\, S(N)\big) = \mathrm{O}_{3',3}(S) = 1$, by a theorem of R. Griess and P. Schmid [GS78]. Therefore there exists an $\mathfrak{F}$-eccentric minimal normal subgroup $K$ of $E$, such that $K \leq N$. It is clear that $M$ covers $K$.

Note that the group $E$ has at least three conjugacy classes of $\mathfrak{F}$-normalisers. Moreover, none of these $\mathfrak{F}$-normalisers has the cover-avoidance property in $E$.

**Lemma 4.1.11.** *Let $G$ be a group. Consider a system of maximal subgroups $\mathbf{X}$ of $G$ and an $\mathfrak{H}$-normaliser $D$ of $G$ associated with $\mathbf{X}$. Then, for any monolithic $\mathfrak{H}$-abnormal maximal subgroup $H \in \mathbf{X}$, we have that $D$ is contained in $H$.*

*Proof.* We prove the assertion by induction on $|G|$. Let $H$ be a monolithic $\mathfrak{H}$-abnormal maximal subgroup in $\mathbf{X}$. Assume that $G$ has an $\mathfrak{H}$-central minimal normal subgroup, $N$ say. By Corollary 4.1.7, $N$ is contained in $D \cap H$. Moreover, applying Proposition 4.1.5, $D/N$ is an $\mathfrak{H}$-normaliser of $G$ associated with $\mathbf{X}/N$. By induction, $D/N \leq H/N$ and then $D \leq H$. Thus, we can assume that every minimal normal subgroup of $G$ is $\mathfrak{H}$-eccentric in $G$. If $N$ is contained in $H$, then, again by Proposition 4.1.5 and induction, we have that $D \leq DN \leq H$. Therefore we assume that $\mathrm{Core}_G(H) = 1$ and $G$ is a monolithic primitive group. There exists a unique minimal normal subgroup $N$ of $G$. Observe that $\mathrm{F}'(G) = N$ and so $H$ is $\mathfrak{H}$-critical in $G$. Since $H \in \mathbf{X}$, we have that $D$ is contained in $H$ by construction of $D$. ◻

**Lemma 4.1.12.** *If a maximal subgroup $M$ of a group $G$ contains an $\mathfrak{H}$-normaliser of $G$, then $M$ is $\mathfrak{H}$-abnormal in $G$.*

*Proof.* Suppose that $D$ is an $\mathfrak{H}$-normaliser of the group $G$ and $D$ is contained in the maximal subgroup $M$ of $G$. If $H/K$ is a chief factor supplemented by $M$ and $H/K$ is $\mathfrak{H}$-central in $G$, then $D$ covers $H/K$, by Corollary 4.1.9, and so does $M$, a contradiction. Hence $H/K$ is $\mathfrak{H}$-eccentric in $G$ and $M$ is $\mathfrak{H}$-abnormal in $G$. ◻

The previous lemmas allow us to discover the relationship between $\mathfrak{H}$-normalisers and monolithic maximal subgroups. The corresponding result in the soluble universe is in [DH92, V, 3.4].

**Corollary 4.1.13.** *Let $M$ be a monolithic maximal subgroup of a group $G$. Then $M$ is $\mathfrak{H}$-abnormal in $G$ if and only if $M$ contains an $\mathfrak{H}$-normaliser of $G$.*

It is not true in general that an $\mathfrak{H}$-abnormal maximal subgroup $M$ of a group $G$ contains an $\mathfrak{H}$-normaliser of $G$.

*Example 4.1.14.* Consider the saturated formation $\mathfrak{F}$ composed of all $S$-perfect groups, for $S \cong \mathrm{Alt}(5)$, the alternating group of degree 5 as in Example 4.1.10. Let $G$ be the direct product $G = S_1 \times S_2$ of two copies $S_1, S_2$ of $S$. Clearly each core-free maximal subgroup is $\mathfrak{F}$-abnormal in $G$. Suppose, arguing by contradiction, that $U$ is a core-free maximal subgroup of $G$ and there exists $E \in \mathrm{Nor}_{\mathfrak{F}}(G)$ such that $E$ is contained in $U$. Let $M$ be an $\mathfrak{F}$-critical maximal subgroup of $G$ such that $E$ is contained in $M$ and $E$ is an $\mathfrak{F}$-normaliser of $M$. Since $M$ is monolithic, we can assume that $S_1 = \mathrm{Core}_G(M)$. Therefore $M = S_1 \times (M \cap S_2)$. It is clear that $M \cap S_2 \neq 1$. Let $N$ be a minimal normal subgroup of $M$ contained in $M \cap S_2$. Since $N$ is a supplemented $\mathfrak{F}$-central chief factor of $M$, then $N$ is covered by $E$ by virtue of Corollary 4.1.9. Consequently, $N \leq M \cap S_2 \cap U = 1$. This contradiction yields that no core-free maximal subgroup of $G$ contains an $\mathfrak{F}$-normaliser of $G$.

The fundamental connection between $\mathfrak{H}$-normalisers and $\mathfrak{H}$-projectors of a soluble group is that every $\mathfrak{H}$-projector contains an $\mathfrak{H}$-normaliser (see [Man70, Theorem 9] and [DH92, V, 4.1]). This is no longer true in the general case: any Sylow 5-subgroup of $G = \mathrm{Alt}(5)$, the alternating group of degree 5, is an $\mathfrak{N}$-projector of $G$ and contains no $\mathfrak{N}$-normaliser of $G$.

However we can prove some interesting results that confirm the close relation between $\mathfrak{H}$-normalisers and $\mathfrak{H}$-projectors, especially when saturated formations $\mathfrak{H}$ are considered.

**Definitions 4.1.15.** *Let $G$ be a group.*

1. *A maximal subgroup $M$ of $G$ is said to be* $\mathfrak{H}$-crucial *in $G$ if $M$ is $\mathfrak{H}$-abnormal and $M/\mathrm{Core}_G(M) \in \mathfrak{H}$.*
2. *If $G \notin \mathfrak{H}$, an $\mathfrak{H}$-normaliser $D$ of $G$ is said to be* $\mathfrak{H}$-crucial *in $G$ if there exists a chain of subgroups*

$$D = H_n \leq H_{n-1} \leq \cdots \leq H_1 \leq H_0 = G \tag{4.3}$$

*such that $H_i$ is $\mathfrak{H}$-crucial $\mathfrak{H}$-critical subgroup of $H_{i-1}$, for each $i \in \{1, \dots, n\}$, and $H_n$ contains no $\mathfrak{H}$-critical subgroup.*

**Proposition 4.1.16.** *If $D$ is an $\mathfrak{H}$-crucial $\mathfrak{H}$-normaliser of a group $G$, then $D$ is an $\mathfrak{H}$-projector of $G$.*

*Proof.* Clearly $G \notin \mathfrak{H}$. Suppose first that $D$ is maximal in $G$. Then we have that $D/\mathrm{Core}_G(D)$ is an $\mathfrak{H}$-maximal subgroup of the group $G/\mathrm{Core}_G(D)$ and $G/\mathrm{Core}_G(D)$ is a primitive group in the boundary of $\mathfrak{H}$. Since $D/\mathrm{Core}_G(D)$ is an $\mathfrak{H}$-projector of $G/\mathrm{Core}_G(D)$, then $D$ is an $\mathfrak{H}$-projector of $G$ by Proposition 2.3.14.

Suppose that $D$ is not maximal in $G$, and let $M$ be an $\mathfrak{H}$-crucial $\mathfrak{H}$-critical subgroup of $G$ such that $D$ is an $\mathfrak{H}$-crucial $\mathfrak{H}$-normaliser of $M$. By induction, $D$ is an $\mathfrak{H}$-projector of $M$. By Proposition 2.3.14, $D$ is an $\mathfrak{H}$-projector of $G$. ◻

**Lemma 4.1.17.** *Let $G$ be a group and $E$ an $\mathfrak{H}$-maximal subgroup of $G$ such that $G = E\,\mathrm{F}(G)$, then $E$ is an $\mathfrak{H}$-normaliser of $G$.*

*Proof.* We proceed by induction on $|G|$. If $E = G$, there is nothing to prove. We can assume that $G \notin \mathfrak{H}$ and $E$ is then a proper subgroup of $G$. Let $M$ be a maximal subgroup of $G$ containing $E$. Since $M = E\,\mathrm{F}(M)$ and $E$ is $\mathfrak{H}$-maximal in $M$, then $E$ is an $\mathfrak{H}$-normaliser of $M$, by induction. Applying Proposition 2.3.17, $E$ is an $\mathfrak{H}$-projector of $G$ and then $M$ is $\mathfrak{H}$-critical in $G$. Therefore $E$ is an $\mathfrak{H}$-normaliser of $G$. □

Let $\mathfrak{F}$ be a saturated formation. It is known that in a soluble group in $\mathfrak{N}\mathfrak{F}$, the $\mathfrak{F}$-projectors and the $\mathfrak{F}$-normalisers coincide (see [DH92, V, 4.2]). The above lemma allows us to extend this result to Schunck classes in the general universe.

**Theorem 4.1.18.** *If $G$ is a group in $\mathfrak{N}\mathfrak{H}$, then the $\mathfrak{H}$-projectors and the $\mathfrak{H}$-normalisers of $G$ coincide.*

*Proof.* We prove by induction on the order of $G$ that the $\mathfrak{H}$-normalisers of $G$ are $\mathfrak{H}$-crucial in $G$. If $G \in \mathfrak{H}$, the result is trivial. Thus, we can assume that $G \notin \mathfrak{H}$. Let $M$ be an $\mathfrak{H}$-critical subgroup of $G$. Then $G = M\,\mathrm{F}(G)$ and $M \cap \mathrm{F}(G)$ is contained in $\mathrm{Core}_G(M)$ because $\mathrm{F}(G)/\Phi(G)$ is abelian. Hence $M/\mathrm{Core}_G(M)$ is a quotient group of $M/\big(M \cap \mathrm{F}(G)\big) \cong G/\mathrm{F}(G)$, and then $M/\mathrm{Core}_G(M) \in \mathfrak{H}$. Therefore $M$ is $\mathfrak{H}$-crucial in $G$. If $D \in \mathrm{Nor}_{\mathfrak{H}}(G)$, then there exists an $\mathfrak{H}$-critical subgroup $M$ of $G$ such that $D \in \mathrm{Nor}_{\mathfrak{H}}(M)$. Since $M \in \mathfrak{N}\mathfrak{H}$, we have that $D$ is an $\mathfrak{H}$-crucial $\mathfrak{H}$-normaliser of $M$ by induction. Therefore $D$ is an $\mathfrak{H}$-crucial $\mathfrak{H}$-normaliser of $G$.

Therefore we can apply Proposition 4.1.22 to conclude that each $\mathfrak{H}$-normaliser of $G$ is an $\mathfrak{H}$-projector of $G$.

Now, let $E$ be an $\mathfrak{H}$-projector of $G$. Since $G \in \mathfrak{N}\mathfrak{H}$, it follows that $G = E\,\mathrm{F}(G)$. By Lemma 4.1.17, $E$ is an $\mathfrak{H}$-normaliser of $G$. □

The previous result can be used to show that, for saturated formations $\mathfrak{F}$, the $\mathfrak{F}$-normalisers of groups with soluble $\mathfrak{F}$-residual can be described in terms of projectors. The corresponding result for soluble groups appears in [DH92, V, 4.3].

**Theorem 4.1.19.** *1. Let $\mathfrak{F}$ be a formation and $\mathfrak{H} = \mathrm{E}_{\Phi}\mathfrak{F}$. Then, for any group $G$, if $D$ is an $\mathfrak{N}\mathfrak{F}$-normaliser of $G$, the $\mathfrak{H}$-projectors of $D$ are $\mathfrak{H}$-normalisers of $G$.*

*2. Let $\mathfrak{F}$ be a saturated formation and let $G$ be a group such that the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is a soluble group of nilpotent length $r$. We construct the chain of subgroups*

$$D_r \leq D_{r-1} \leq D_{r-2} \leq \cdots \leq D_1 \leq D_0 = G$$

*where $D_i$ is an $\mathfrak{N}^{r-i}\mathfrak{F}$-projector of $D_{i-1}$, for $i \in \{1, \ldots, r\}$. Then $D_r$ is an $\mathfrak{F}$-normaliser of $G$.*

*Proof.* 1. By Corollary 3.3.9, $\mathfrak{N}\mathfrak{F}$ is a saturated formation. Moreover, $\mathfrak{H}$ is contained in $\mathfrak{N}\mathfrak{F}$.

If $G \in \mathfrak{N}\mathfrak{F}$, then $G \in \mathfrak{N}\mathfrak{H}$ and so $\operatorname{Proj}_{\mathfrak{H}}(G) = \operatorname{Nor}_{\mathfrak{H}}(G)$ by Theorem 4.1.18. Thus we can assume that $G \notin \mathfrak{N}\mathfrak{F}$. Let $D$ be an $\mathfrak{N}\mathfrak{F}$-normaliser of $G$. Then there exists a chain of subgroups (4.2), such that $H_{i-1}$ is an $\mathfrak{N}\mathfrak{F}$-critical subgroup of $H_i$, for each index $i$. Since $\mathfrak{H} \subseteq \mathfrak{N}\mathfrak{F}$, every $\mathfrak{H}$-normaliser of $D$ is an $\mathfrak{H}$-normaliser of $G$. Since $D \in \mathfrak{N}\mathfrak{F} \subseteq \mathfrak{N}\mathfrak{H}$, we have that $\operatorname{Proj}_{\mathfrak{H}}(D) = \operatorname{Nor}_{\mathfrak{H}}(D)$ by Theorem 4.1.18. Hence each $\mathfrak{H}$-projector of $D$ is an $\mathfrak{H}$-normaliser of $G$.

2. Let $\mathfrak{F}$ be a saturated formation and let $G$ be a group whose $\mathfrak{F}$-residual, $G^{\mathfrak{F}}$, is a soluble group of nilpotent length $r$. This is to say that $G \in \mathfrak{N}^r\mathfrak{F}$. We construct the chain of subgroups

$$D_{r-1} \leq D_{r-2} \leq \cdots \leq D_1 \leq D_0 = G$$

where $D_i$ is an $\mathfrak{N}^{r-i}\mathfrak{F}$-projector of $D_{i-1}$, for $i \in \{1, \dots, r-1\}$. Since $G \in \mathfrak{N}(\mathfrak{N}^{r-1}\mathfrak{F})$, then the $\mathfrak{N}^{r-1}\mathfrak{F}$-projectors and the $\mathfrak{N}^{r-1}\mathfrak{F}$-normalisers of $G$ coincide by Theorem 4.1.18. Therefore $D_1$ is an $\mathfrak{N}^{r-1}\mathfrak{F}$-normaliser of $G$. By Statement 1, the $\mathfrak{N}^{r-2}\mathfrak{F}$-projectors of $D_1$ are $\mathfrak{N}^{r-2}\mathfrak{F}$-normalisers of $G$. Thus, $D_2$ is an $\mathfrak{N}^{r-2}\mathfrak{F}$-normaliser of $G$. Repeating this argument, we obtain that $D_{r-1}$ is an $\mathfrak{N}\mathfrak{F}$-normaliser of $G$. Hence, every $\mathfrak{F}$-projector of $D_{r-1}$ is an $\mathfrak{F}$-normaliser of $G$ by Statement 1. Consequently $D_r$ is an $\mathfrak{F}$-normaliser of $G$. □

The next result yields a sufficient condition for a subgroup of a group in $\mathfrak{N}\mathfrak{H}$ to contain an $\mathfrak{H}$-normaliser.

**Theorem 4.1.20.** *Let $G$ be a group in $\mathfrak{N}\mathfrak{H}$ and $E$ a subgroup of $G$ that covers all $\mathfrak{H}$-central chief factors of a given chief series of $G$. Then $E$ contains an $\mathfrak{H}$-normaliser of $G$.*

*Proof.* We argue by induction on the order of $G$. Clearly we can assume that $G \notin \mathfrak{H}$ and that $E$ is a proper subgroup of $G$. If $M$ is a maximal subgroup of $G$ such that $E \leq M$, then $M$ is an $\mathfrak{H}$-abnormal subgroup of $G$ and $G = M\operatorname{F}(G)$ because $E$ covers the section $G/\operatorname{F}(G)$. This is to say that $M$ is $\mathfrak{H}$-critical in $G$. Moreover $M$ is has the cover-avoidance property and the intersections of $M$ with all normal subgroups of a chief series of $G$ give a chief series of $M$. If $H/K$ is a chief factor of $G$ in that series covered by $M$, then $(M \cap H)/(M \cap K)$ is a chief factor of $M$ such that $[H/K] * G \cong [(M \cap H)/(M \cap K)] * M$ by Proposition 1.4.11. Consequently, $E$ covers all $\mathfrak{H}$-central chief factors of a chief series of $M$. By induction, $E$ contains an $\mathfrak{H}$-normaliser of $M$ which is an $\mathfrak{H}$-normaliser of $G$. □

We end this section with the analysis of the relation between the $\mathfrak{F}$-normalisers and the $\mathfrak{F}$-hypercentre, $\mathfrak{F}$ a saturated formation.

Recall that a normal subgroup $N$ of a group $G$ is said to be *$\mathfrak{F}$-hypercentral in $G$* if every chief factor of $G$ below $N$ is $\mathfrak{F}$-central in $G$. The product of $\mathfrak{F}$-hypercentral normal subgroups of a group is again an $\mathfrak{F}$-hypercentral normal subgroup of the group (see [DH92, IV, 6.4]). Thus every group $G$ possesses a unique maximal normal $\mathfrak{F}$-hypercentral subgroup called the $\mathfrak{F}$-*hypercentre* of $G$ and denoted by $\mathrm{Z}_{\mathfrak{F}}(G)$.

Let $G$ be a group. By Corollary 4.1.7, the $\mathfrak{F}$-hypercentre of $G$ is contained in every $\mathfrak{F}$-normaliser of $G$. Therefore $\mathrm{Z}_{\mathfrak{F}}(G)$ is contained in $\mathrm{Core}_G(D)$, for every $\mathfrak{H}$-normaliser $D$ of $G$. However, the equality does not hold in general.

*Example 4.1.21.* Consider $E$ and $\mathfrak{F}$ as in Example 4.1.10. By [GS78, Example 1 (b)], $\mathrm{Z}_{\mathfrak{F}}(E) = 1$. If $M$ is a maximal subgroup of $E$ such that $M/N \cong \mathrm{Alt}(4)$, then $M$ is an $\mathfrak{F}$-normaliser of $E$ and $\mathrm{Core}_E(M) = N \neq 1$.

In the next section, we shall see that the equality holds in groups with soluble $\mathfrak{F}$-residual.

Next we describe the $\mathfrak{F}$-hypercentre of a group in terms of the $\mathfrak{F}$-residual of the group and an $\mathfrak{F}$-normaliser. A similar description appears in [DH92, IV, 6.14] for $\mathfrak{F}$-maximal subgroups supplementing the $\mathfrak{F}$-residual. Note that, in general, the $\mathfrak{F}$-normalisers are not $\mathfrak{F}$-maximal subgroups.

**Proposition 4.1.22.** *Let $\mathfrak{F}$ be a saturated formation. If $D$ is an $\mathfrak{F}$-normaliser of a group $G$, then $\mathrm{Z}_{\mathfrak{F}}(G) = \mathrm{C}_D(G^{\mathfrak{F}})$.*

*Proof.* Applying [DH92, IV, 6.10]), we have that $[G^{\mathfrak{F}}, \mathrm{Z}_{\mathfrak{F}}(G)] = 1$. Therefore $\mathrm{Z}_{\mathfrak{F}}(G)$ is contained in $\mathrm{C}_D(G^{\mathfrak{F}})$. Next we prove that $\mathrm{C}_D(G^{\mathfrak{F}})$ is an $\mathfrak{F}$-hypercentral normal subgroup of $G$. Since $G = DG^{\mathfrak{F}}$, the $\mathrm{C}_D(G^{\mathfrak{F}})$ is normal in $G$. Let $H/K$ be a chief factor of $G$ below $\mathrm{C}_D(G^{\mathfrak{F}})$. Then $G^{\mathfrak{F}} \leq \mathrm{C}_G(H/K)$. This implies that $G = D\,\mathrm{C}_G(H/K)$. Consequently $H/K$ is a chief factor of $D$ by [DH92, A, 13.9]). Since $D \in \mathfrak{F}$, the chief factor $H/K$ is $\mathfrak{F}$-central in $D$ and then in $G$ by [DH92, A, 13.9]). Consequently $\mathrm{C}_D(G^{\mathfrak{F}})$ is an $\mathfrak{F}$-hypercentral normal subgroup of $G$ and hence it is contained in $\mathrm{Z}_{\mathfrak{F}}(G)$. □

## 4.2 Normalisers of groups with soluble residual

In this section we assume that $\mathfrak{F}$ is a saturated formation. Most of the properties of $\mathfrak{F}$-normalisers of soluble groups, such as conjugacy, cover-avoidance property, relation with $\mathfrak{F}$-projectors, do not hold in the general case (see examples of the previous section). However $\mathfrak{F}$-normalisers of groups $G$ in which the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is soluble (i.e. groups in the class $\mathfrak{S}\mathfrak{F}$) do really satisfy these classical properties. The purpose of the section is to give a full account of these results. We remark that no use of the corresponding results for soluble groups occurs in our arguments.

The following elementary result will be used frequently in the section. Let $M$ be an $\mathfrak{F}$-abnormal maximal subgroup of a group $G$. Then $G = MG^{\mathfrak{F}}$. Assume, in addition, that $G^{\mathfrak{F}}$ is soluble. Then every chief factor of $G$ supplemented by $M$ is abelian. In particular, $M$ is a maximal subgroup of $G$ of type 1.

Our starting point is a result of P. Schmid which proves that the $\mathfrak{F}$-projectors of a group with soluble $\mathfrak{F}$-residual form a conjugacy class of subgroups.

**Theorem 4.2.1 ([Sch74]).** *Let $\mathfrak{F}$ be a saturated formation. Let $G$ be group whose $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is soluble. Then $\mathrm{Proj}_{\mathfrak{F}}(G)$ is a conjugacy class of subgroups of $G$.*

*Proof.* We argue by induction on $|G|$. Obviously we can assume that $G^{\mathfrak{F}} \neq 1$. Let $N$ be a minimal normal subgroup of $G$ such that $N \leq G^{\mathfrak{F}}$ and suppose that $E$ and $D$ are $\mathfrak{F}$-projectors of $G$. By induction, $X = EN = D^g N$ for some $g \in G$. Since $N$ is abelian, we have that $E$ and $D^g$ are $\mathfrak{F}$-projectors of $X$, by Lemma 4.1.17 and Theorem 4.1.18. If $X$ is a proper subgroup of $G$, then $E$ and $D^g$ are conjugate in $X$ by induction. Thus we can assume that $G = EN$, for every minimal normal subgroup $N$ which is contained in $G^{\mathfrak{F}}$. Since $G/N \cong E/(E \cap N) \in \mathrm{Q}\mathfrak{F} = \mathfrak{F}$, we have that $N = G^{\mathfrak{F}}$. This is to say that $G^{\mathfrak{F}}$ is an abelian minimal normal subgroup of $G$ and every $\mathfrak{F}$-projector of $G$ is a maximal subgroup of $G$. Let $p$ be the prime dividing $|G|$. Let $F$ be the canonical local definition of $\mathfrak{F} = \mathrm{LF}(F)$, and consider the $F(p)$-residual $T = G^{F(p)}$ of $G$. Clearly $T$ contains $N$. Since $G/N \in \mathfrak{E}_{p'}F(p)$ (see [DH92, IV, 3.2]), it follows that $T/N$ is a $p'$-group. Moreover, since $F$ is full, we have that $\mathrm{O}^p(T) = T$. Hence, for any $E \in \mathrm{Proj}_{\mathfrak{F}}(G)$, we have that $T = N(T \cap E)$ and $T \cap E$ is a Hall $p'$-subgroup of $T$. By the Schur-Zassenhaus theorem [Hup67, I, 18.1 and 18.2], the Hall $p'$-subgroups of $T$ are a conjugacy class of subgroups of $T$. If $T \cap E$ is normal in $G$, then $T \cap E = \mathrm{O}^p(T) = T$. This is a contradiction. Hence $E = \mathrm{N}_G(T \cap E)$ and then $\mathrm{Proj}_{\mathfrak{F}}(G)$ is a conjugacy class of subgroups of $G$. □

Assume that $G$ is a group with soluble $\mathfrak{F}$-residual, $\mathfrak{F}$ a saturated formation. Then $\mathrm{Proj}_{\mathfrak{F}}(G) = \mathrm{Cov}_{\mathfrak{F}}(G)$. This can be proved by reducing the problem to the case $G \in \mathrm{b}(\mathfrak{F})$ (note that if $E$ is an $\mathfrak{F}$-projector of $G$, then $E$ is an $\mathfrak{F}$-projector of $EN$ for every minimal normal subgroup $N$ of $G$ by Proposition 2.3.16). In such case, the equality is obviously true because $G$ is a primitive group of type 1 (see [DH92, III, 3.9]).

We show next that in groups with soluble $\mathfrak{F}$-residual, the $\mathfrak{F}$-normalisers can be joined to the group by means of some special chains.

**Lemma 4.2.2.** *Let $G$ be a group whose $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is soluble. If $D$ is an $\mathfrak{F}$-normaliser of $G$, there exists a chain of subgroups*

$$D = H_n \leq H_{n-1} \leq \cdots \leq H_1 \leq H_0 = G \tag{4.4}$$

*such that $H_i$ is $\mathfrak{H}$-critical maximal subgroup of $H_{i-1}$ of type 1, for each $i \in \{1, \dots, n\}$, and $H_n$ contains no $\mathfrak{F}$-critical subgroup.*

*Proof.* We prove the assertion by induction on $|G|$. We can assume that $G \notin \mathfrak{F}$. If $M$ is an $\mathfrak{F}$-critical subgroup of $G$ containing $D$ as $\mathfrak{F}$-normaliser, then $M$ is a maximal subgroup of type 1. Moreover $M^{\mathfrak{F}} \leq G^{\mathfrak{F}}$ by Proposition 2.2.8 (3). Hence $M^{\mathfrak{F}}$ is soluble. By induction, $D$ can be joined to $M$ by means of a chain of $\mathfrak{F}$-critical maximal subgroups of type 1. This completes the proof the lemma. □

**Lemma 4.2.3 (see [Ezq86]).** *If $M$ is a maximal subgroup of a group $G$ which supplements the Fitting subgroup $\mathrm{F}(G)$, then every subgroup with the cover-avoidance property in $M$ is a subgroup with the cover-avoidance property in $G$.*

*Proof.* Let $D$ be a subgroup with the cover-avoidance property in $M$. Let $H/K$ be a chief factor of $G$ covered by $M$. Observe that $G = M\,\mathrm{F}(G) = M\,\mathrm{C}_G(H/K)$. Then $(H \cap M)/(K \cap M)$ is a chief factor of $M$. If $D$ covers $(H\cap M)/(K\cap M)$, then $H\cap M = (K\cap M)(H\cap D)$. Since $H = K(H\cap M)$, we have that $H = K(H \cap D)$ and $D$ covers $H/K$. If $D$ avoids $(H\cap M)/(K\cap M)$, then $D\cap H \leq K$ and $D$ avoids $H/K$. Finally $D$ avoids all chief factors avoided by $M$. □

**Theorem 4.2.4.** *Let $G$ be a group whose $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is soluble. If $D$ is an $\mathfrak{F}$-normaliser of $G$, then $D$ covers all the $\mathfrak{F}$-central chief factors of $G$ and avoids all the $\mathfrak{F}$-eccentric ones.*

*Proof.* We use induction on the order of $G$ to prove that $\mathfrak{F}$-normalisers are subgroups with the cover-avoidance property in $G$. Let $D \neq G$ be an $\mathfrak{F}$-normaliser of $G$ and suppose that $D$ is maximal in $G$. If $H/K$ is a non-abelian chief factor of $G$, then $D$ covers $H/K$ since $D$ is of type 1. If $H/K$ is abelian and $D$ does not cover $H/K$, then $G = DH$ and $K \leq D$. In the group $G/K$, the minimal normal subgroup $H/K$ is abelian and complemented by the maximal subgroup $D/K$. Then $D$ avoids $H/K$.

If $D$ is not maximal in $G$, there exists an $\mathfrak{F}$-critical maximal subgroup $M$ of $G$ such that $D \in \mathrm{Nor}_{\mathfrak{F}}(M)$. By induction, $D$ has the cover-avoidance property in $M$. Since $M$ supplements $\mathrm{F}(G)$, $D$ has the cover-avoidance property in $G$ by Lemma 4.2.3.

If $H/K$ is an $\mathfrak{F}$-central chief factor of $G$, then, by Corollary 4.1.7, $D$ covers $H/K$. Suppose that $H/K$ is an $\mathfrak{F}$-eccentric chief factor of $G$ which is covered by $D$. Suppose that $D$ is defined by a chain (4.4) as in Lemma 4.2.2. Observe that $G = H_1\,\mathrm{F}(G) = H_1\,\mathrm{C}_G(H/K)$ and $H_1$ covers $H/K$. Hence, $(H\cap H_1)/(K\cap H_1)$ is a chief factor of $H_1$ such that $\mathrm{Aut}_G(H/K) \cong \mathrm{Aut}_{H_1}\big((H \cap H_1)/(K \cap H_1)\big)$. By repeating the argument we obtain that $(H\cap D)/(K\cap D)$ is an $\mathfrak{F}$-eccentric chief factor of $D$. Since $D \in \mathfrak{F}$, all chief factors of $D$ are $\mathfrak{F}$-central. This contradiction yields that $H/K$ is avoided by $D$. □

Combining Corollary 4.1.7 and Theorem 4.2.4, a chief series of an $\mathfrak{F}$-normaliser $D$ of a group $G$ with soluble $\mathfrak{F}$-residual can be obtained by intersecting $D$ with the members of a given chief series of $G$.

Our next result partially extends a result of J. D. Gillam (see [DH92, V, 3.3]) on the cover-avoidance property of $\mathfrak{F}$-normalisers. We wonder whether the cover-avoidance property characterises the $\mathfrak{F}$-normalisers of groups whose $\mathfrak{F}$-residual is soluble. The answer in general is negative even in soluble groups (see an example in [DH92, page 401]). Gillam's result characterises the $\mathfrak{F}$-normaliser of a soluble group associated with a particular Hall system by the cover-avoidance property together with the permutability with the Hall system. Obviously this is not possible in our context. However, Theorem 4.1.20 allows us to show that the characterisation of the $\mathfrak{F}$-normalisers by the cover-avoidance property, holds in groups whose $\mathfrak{F}$-residual is nilpotent.

**Corollary 4.2.5.** *If $\mathfrak{F}$ is a saturated formation and $G$ is a group in $\mathfrak{N}\mathfrak{F}$, then, for a subgroup $D$ of $G$, the following sentences are equivalent:*

1. *$D$ is an $\mathfrak{F}$-normaliser of $G$,*
2. *$D$ covers the $\mathfrak{F}$-central chief factors of $G$ and avoids the $\mathfrak{F}$-eccentric ones.*

We have seen in Example 4.1.21 that, in general, the $\mathfrak{F}$-hypercentre of a group $G$ is not the core in $G$ of an $\mathfrak{F}$-normaliser of $G$. The equality in groups with soluble $\mathfrak{F}$-residual follows from the cover-avoidance property of the $\mathfrak{F}$-normalisers.

**Proposition 4.2.6.** *Let $G$ be a group such that the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is a soluble group. If $D$ is an $\mathfrak{F}$-normaliser of $G$, then $\mathrm{Z}_{\mathfrak{F}}(G) = \mathrm{Core}_G(D)$.*

*Proof.* If $\mathrm{Z}_{\mathfrak{F}}(G) = 1$, the core of any $\mathfrak{F}$-normaliser is trivial by Theorem 4.2.4. If $\mathrm{Z}_{\mathfrak{F}}(G)$ is non-trivial, the group $G/\mathrm{Z}_{\mathfrak{F}}(G)$ has trivial $\mathfrak{F}$-hypercentre and the quotient $D\,\mathrm{Z}_{\mathfrak{F}}(G)/\mathrm{Z}_{\mathfrak{F}}(G)$ is an $\mathfrak{F}$-normaliser of $G/\mathrm{Z}_{\mathfrak{F}}(G)$ by Proposition 4.1.5. Consequently $\mathrm{Core}_G(D) \leq \mathrm{Z}_{\mathfrak{F}}(G)$. □

Our next major objective is to show that the connections between $\mathfrak{F}$-normalisers and $\mathfrak{F}$-projectors of groups with soluble $\mathfrak{F}$-residual are similar to the ones of the soluble case. In particular every $\mathfrak{F}$-normaliser is contained in an $\mathfrak{F}$-projector. Since, by Theorem 4.2.1, the $\mathfrak{F}$-projectors of groups in $\mathfrak{S}\mathfrak{F}$ form a conjugacy class of subgroups, every $\mathfrak{F}$-projector contains an $\mathfrak{F}$-normaliser.

**Theorem 4.2.7.** *Let $\mathfrak{F}$ be a saturated formation. If $G \in \mathfrak{N}\mathfrak{F}$ and $H$ is a subgroup of $G$ such that $G = H\,\mathrm{F}(G)$, then each $\mathfrak{F}$-projector of $H$ is of the form $H \cap E$, for some $\mathfrak{F}$-projector $E$ of $G$.*

*Proof.* Clearly we can assume that $\mathrm{F}(G) \neq 1$, $G \neq H$, and $G \notin \mathfrak{F}$. Moreover, arguing by induction on the order of $G$, we can assume that $H$ is a maximal subgroup of $G$. Since $H/\big(H \cap \mathrm{F}(G)\big) \in \mathfrak{F}$, each $\mathfrak{F}$-projector $D$ of $H$ satisfies $H = D\big(H \cap \mathrm{F}(G)\big)$. Then $G = D\,\mathrm{F}(G)$. If $E$ is an $\mathfrak{F}$-maximal subgroup of $G$ such that $D \leq E$, then $E \in \mathrm{Proj}_{\mathfrak{F}}(G)$ by Proposition 2.3.17. It is rather easy to show that $D$ and $E \cap H$ cover and avoid the same chief factors of a given chief series of $G$. Consequently $D = E \cap H$. □

**Theorem 4.2.8.** *Let $\mathfrak{F}$ be a saturated formation. If $G$ is a group whose $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is soluble, and $H$ is a subgroup of $G$ such that $G = H\,\mathrm{F}(G)$, then there exist an $\mathfrak{F}$-projector $A$ of $H$ and an $\mathfrak{F}$-projector $E$ of $G$ such that $A = H \cap E$.*

*Proof.* By Theorem 4.2.7, we can assume that $G \notin \mathfrak{N}\mathfrak{F}$. The quotient group $\bar{G} = G/\operatorname{F}(G)$ has soluble non-trivial $\mathfrak{F}$-residual $\bar{G}^{\mathfrak{F}} = G^{\mathfrak{F}}\operatorname{F}(G)/\operatorname{F}(G)$, Since $\bar{G}^{\mathfrak{F}} \neq 1$, we can consider a chief factor of $G$ of the form $\bar{G}^{\mathfrak{F}}/\bar{K}$. Since $\mathfrak{F}$ is saturated, then $\bar{G}^{\mathfrak{F}}/\bar{K}$ is a complemented abelian chief factor of $\bar{G}$. Let $M/\operatorname{F}(G)$ be a complement of $\bar{G}^{\mathfrak{F}}/\bar{K}$ in $\bar{G}$. Then $M$ is an $\mathfrak{F}$-crucial maximal subgroup of $G$. If $N/\operatorname{Core}_G(M) = \operatorname{Soc}\bigl(G/\operatorname{Core}_G(M)\bigr)$, then $H$ covers $N/\operatorname{Core}_G(M)$ and $(N \cap H)/\bigl(\operatorname{Core}_G(M) \cap H\bigr)$ is an $\mathfrak{F}$-eccentric chief factor of $H$. Moreover, $H = (N \cap H)(M \cap H)$ and $(N \cap H)/\bigl(\operatorname{Core}_G(M) \cap H\bigr)$ is an abelian chief factor of $H$. Consequently $M \cap H$ is an $\mathfrak{F}$-crucial maximal subgroup of $H$. On the other hand, $M = (M \cap H)\operatorname{F}(M)$ and so $M^{\mathfrak{F}}\operatorname{F}(M) = (M \cap H)^{\mathfrak{F}}\operatorname{F}(M)$ by Proposition 2.2.8 (2). Analogously $G^{\mathfrak{F}}\operatorname{F}(G) = H^{\mathfrak{F}}\operatorname{F}(G)$. This implies that $M^{\mathfrak{F}}$ is soluble. By induction, there exist $A \in \operatorname{Proj}_{\mathfrak{F}}(M \cap H)$ and $E \in \operatorname{Proj}_{\mathfrak{F}}(M)$ such that $A = H \cap E \cap M = H \cap E$. By Proposition 2.3.16, the $\mathfrak{F}$-projectors of any $\mathfrak{F}$-crucial monolithic maximal subgroup of a group are $\mathfrak{F}$-projectors of the group. Since $M \cap H$ is $\mathfrak{F}$-crucial in $H$, we have that $A$ is an $\mathfrak{F}$-projector of $H$, and since $M$ is $\mathfrak{F}$-crucial in $G$, then $E$ is an $\mathfrak{F}$-projector of $G$. ◻

**Theorem 4.2.9.** *Let $\mathfrak{F}$ be a saturated formation. Let $G$ be a group whose $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is soluble. Then each $\mathfrak{F}$-normaliser of $G$ is contained in an $\mathfrak{F}$-projector of $G$ and each $\mathfrak{F}$-projector contains an $\mathfrak{F}$-normaliser.*

*Proof.* We argue by induction of the order of $G$. We can assume that $G \notin \mathfrak{F}$. Let $D$ be an $\mathfrak{F}$-normaliser of $G$. There exists an $\mathfrak{F}$-critical subgroup $M$ of $G$ such that $D \in \operatorname{Nor}_{\mathfrak{F}}(M)$. Since $M^{\mathfrak{F}}$ is soluble, there exists an $\mathfrak{F}$-projector $A$ of $M$ such that $D$ is contained in $A$. Since $M$ is critical in $G$, we can apply Theorem 4.2.8 to conclude that there exist $B \in \operatorname{Proj}_{\mathfrak{F}}(M)$ and $E \in \operatorname{Proj}_{\mathfrak{F}}(G)$ such that $B = M \cap E$. By Theorem 4.2.1, the subgroups $A$ and $B$ are conjugate in $M$. Hence there exists an element $x \in M$ such that $A = B^x$. Thus, $A = M \cap E^x$ and $D$ is contained in $E^x$ which is an $\mathfrak{F}$-projector of $G$.

By Theorem 4.2.1, the $\mathfrak{F}$-projectors of $G$ form a conjugacy class of subgroups. Hence, every $\mathfrak{F}$-projector contains an $\mathfrak{F}$-normaliser. ◻

Assume that $\mathfrak{F}$ is a saturated formation. Let $G$ be a group whose $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is soluble. If $\Sigma$ is a Hall system of $G^{\mathfrak{F}}$, then we denote $\operatorname{N}_G(\Sigma) = \bigcap\{\operatorname{N}_G(H) : H \in \Sigma\}$. Sometimes $\operatorname{N}_G(\Sigma)$ is said to be the *absolute system normaliser in $G$ of $\Sigma$*.

In [Yen70], it is proved that if $G$ is a soluble group, then the $\mathfrak{F}$-projectors of $T$ are $\mathfrak{F}$-normalisers of $G$. Our next objective is to show that this result holds not only in soluble groups but also in groups whose $\mathfrak{F}$-residual is soluble. As a consequence we will obtain the conjugacy of $\mathfrak{F}$-normalisers in such groups.

In general, if $N$ is a soluble normal subgroup of a group $G$ and $\Sigma$ is a Hall system of $N$, then $\Sigma^g$ is also a Hall system of $N$, for all $g \in G$. Since Hall systems of a soluble group are conjugate, there exists an element $x \in N$ such that $\Sigma^g = \Sigma^x$. Hence, by the Frattini argument, we have that $G = \operatorname{N}_G(\Sigma)N$. Then $\operatorname{N}_G(\Sigma) \cap N$ is a system normaliser of $N$. Hence $\operatorname{N}_G(\Sigma) \cap N$ is nilpotent by [DH92, I, 5.4] and $\operatorname{N}_G(\Sigma)/\operatorname{N}_N(\Sigma)$ is isomorphic to $G/N$. If, in addition,

$N$ contains $G^{\mathfrak{F}}$, it follows that $G/N \in \mathfrak{F}$ and so $\mathrm{N}_G(\Sigma)$ belongs to $\mathfrak{N}\mathfrak{F}$. In that case, $\mathrm{N}_G(\Sigma)^{\mathfrak{F}}$ is contained in $\mathrm{N}_N(\Sigma)$ and so $\Sigma$ reduces into $\mathrm{N}_G(\Sigma)^{\mathfrak{F}}$.

The next lemma will be used in subsequent proofs.

**Lemma 4.2.10.** *Let $G$ be a group whose $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is soluble. Consider a Hall system $\Sigma$ of $G^{\mathfrak{F}}$ and write $T = \mathrm{N}_G(\Sigma)$. If $N$ is a normal subgroup of $G$, then $TN/N = \mathrm{N}_{G/N}(\Sigma N/N)$.*

*Therefore, if $E$ is an $\mathfrak{F}$-projector of $T$, then $EN/N$ is an $\mathfrak{F}$-projector of $\mathrm{N}_{G/N}(\Sigma N/N)$.*

*Proof.* We argue by induction on the order of $G$. Clearly $TN/N$ is contained in $\mathrm{N}_{G/N}(\Sigma N/N)$. Assume that $N$ is a minimal normal subgroup of $G$.

Suppose that $N \cap G^{\mathfrak{F}} = 1$. Note that $G$ acts transitively by conjugation on the set of Hall systems of $G^{\mathfrak{F}}N/N$. Hence $|G/N : \mathrm{N}_{G/N}(\Sigma N/N)|$ is the number of Hall systems of $G^{\mathfrak{F}}N/N$. Moreover, by the same argument, the number of Hall systems of $G^{\mathfrak{F}}$ is $|G : T|$. Hence $|G/N : \mathrm{N}_{G/N}(\Sigma N/N)| = |G : T|$. Now $|G : TN| \leq |G : T| = |G/N : \mathrm{N}_{G/N}(\Sigma N/N)| \leq |G/N : TN/N|$. This implies that $TN/N = \mathrm{N}_{G/N}(\Sigma N/N)$.

Assume now that $N \leq G^{\mathfrak{F}} = R$. Since system normalisers are preserved under epimorphisms by [DH92, I, 5.8], we have that $\mathrm{N}_{R/N}(\Sigma N/N) = \mathrm{N}_R(\Sigma)N/N$. Hence, since $G = RT$, we have that $|G/N : \mathrm{N}_{G/N}(\Sigma N/N)| = |R/N : \mathrm{N}_{R/N}(\Sigma N/N)| = |R : \mathrm{N}_R(\Sigma)N| = |R : (T \cap R)N| = |R : R \cap TN| = |G : TN| = |G/N : TN/N|$ and then $TN/N = \mathrm{N}_{G/N}(\Sigma N/N)$.

If $N$ is not a minimal normal subgroup of $G$ and $A$ is a minimal normal subgroup of $G$ contained in $N$, it follows that $TA/A = \mathrm{N}_{G/A}(\Sigma A/A)$. By induction, $(TN/A)/(N/A) = \mathrm{N}_{(G/A)/(N/A)}\big((\Sigma N/A)/(N/A)\big)$. Then $TN/N = \mathrm{N}_{G/N}(\Sigma N/N)$. ◻

The following result is also useful.

**Proposition 4.2.11 ([Hal37]).** *Let $G$ be a soluble group and $N$ a normal subgroup of $G$. Let $\Sigma^*$ be a Hall system of $N$ such that $\Sigma^* = \Sigma \cap N$ for some Hall system $\Sigma$ of $G$. Put $M = \mathrm{N}_G(\Sigma^*)$. We have*

1. *$\mathrm{N}_G(\Sigma)$ is contained in $M$,*
2. *$\Sigma_1 = \Sigma \cap M$ is a Hall system of $M$, and*
3. *$\mathrm{N}_M(\Sigma_1) = \mathrm{N}_G(\Sigma)$.*

*Proof.* 1. For any Hall subgroup $H^*$ of $N$ in $\Sigma^*$, there exists a Hall subgroup $H$ of $G$ in $\Sigma$ such that $H^* = H \cap N$. If $x \in \mathrm{N}_G(\Sigma)$, then $H^{*x} = (H \cap N)^x = H^x \cap N = H \cap N = H^*$, since $N$ is normal in $G$. Then $x \in \mathrm{N}_G(\Sigma^*)$. Hence $\mathrm{N}_G(\Sigma) \leq \mathrm{N}_G(\Sigma^*) = M$.

2. Let $p$ be any prime dividing the order of $G$, $H$ the Hall $p'$-subgroup of $N$ in $\Sigma^*$ and $P$ the Sylow $p$-subgroup of $G$ in $\Sigma$. There exists a Hall $p'$-subgroup $S$ of $G$ in $\Sigma$ such that $S \cap N = H$. Since $S$ normalises $H$ and $G = PS$, it follows that $T = \mathrm{N}_G(H) \cap P$ is a Sylow $p$-subgroup of $\mathrm{N}_G(H)$. Moreover, for any prime $q \neq p$, $P$ is contained in the Hall $q'$-subgroup $S_q$ of $G$ in $\Sigma$.

Hence, $T \leq S_q$. The subgroup $S_q \cap N$ is the Hall $q'$-subgroup of $N$ in $\Sigma^*$ and $S_q \cap N$ is normal in $S_q$. Therefore $T$ normalises $S_q \cap N$. This means that $T \leq \mathrm{N}_G(\Sigma^*) = M$ and $T = M \cap P$.

For two different primes $p_i$, $i = 1, 2$, dividing the order of $G$ consider the corresponding Sylow subgroups $P_i \in \mathrm{Syl}_{p_i}(G)$ of $G$ in $\Sigma$ and $T_i = P_i \cap M$, $i = 1, 2$. Note that $P_1P_2$ is a subgroup of $G$ and $\langle T_1, T_2 \rangle$ is contained in $P_1P_2 \cap M$. Hence, $\langle T_1, T_2 \rangle$ is a $\{p_1, p_2\}$-subgroup and so $\langle T_1, T_2 \rangle = T_1T_2$. Therefore $\Sigma \cap M = \Sigma_1$ is a Hall system of $M$.

3. Clearly, $\Sigma^{*g}$ is a Hall system of $N$, for all $g \in G$. Therefore, there exists $x \in N$, such that $\Sigma^{*g} = \Sigma^{*x}$. The Frattini argument implies that $G = MN$. Therefore, if $P \in \mathrm{Syl}_p(G) \cap \Sigma$, then $(P \cap M)N/N = PN/N \in \mathrm{Syl}_p(G/N)$. Hence $(P \cap M)(P \cap N) = P \cap (P \cap M)N = P \cap PN = P$.

If $x \in \mathrm{N}_G(\Sigma)$, then $x \in M$ and, for any Sylow subgroup $P \in \Sigma$, we have that $(P \cap M)^x = (P \cap M)$. Hence $\mathrm{N}_G(\Sigma) \leq \mathrm{N}_M(\Sigma_1)$. Conversely, if $x \in \mathrm{N}_M(\Sigma_1)$, for any Sylow subgroup $P \in \Sigma$, we have that $x \in \mathrm{N}_G(P \cap M)$ and $x \in M \leq \mathrm{N}_G(P \cap N)$. Hence $x \in \mathrm{N}_G(P)$. Consequently $\mathrm{N}_M(\Sigma_1) \leq \mathrm{N}_G(\Sigma)$ and the equality holds. ☐

**Lemma 4.2.12.** *Let $G$ be a group with a soluble normal subgroup $H$ such that $G^{\mathfrak{F}} \leq H$. Let $\Sigma$ be a Hall system of $H$. Denote $R = \mathrm{N}_G(\Sigma)$. Then each $\mathfrak{F}$-projector of $R$ is contained in an $\mathfrak{F}$-projector of* $\mathrm{N}_G(\Sigma \cap G^{\mathfrak{F}})$.

*Proof.* Assume that the result is not true and let $G$ be a minimal counterexample. Let $H$ be a normal subgroup of $G$ of minimal index $|H : G^{\mathfrak{F}}|$ among all normal subgroups for which the assertion does not hold. Let $H/K$ a chief factor of $G$ such that $G^{\mathfrak{F}} \leq K$. Note that $\Sigma \cap K$ is a Hall system of $K$ and denote $B = \mathrm{N}_G(\Sigma \cap K)$. Since the lemma is true for $G$, $K$, and $\Sigma \cap K$, we have that each $\mathfrak{F}$-projector of $B$ is contained in an $\mathfrak{F}$-projector of $\mathrm{N}_G(\Sigma \cap G^{\mathfrak{F}})$. By Proposition 4.2.11 (2), we have that $\Sigma^* = \Sigma \cap (H \cap B)$ is a Hall system of $H \cap B = \mathrm{N}_H(\Sigma \cap K)$. On the other hand, since $G = \mathrm{N}_G(\Sigma \cap K)K = BH$, and then $B/(B \cap H) \cong G/H \in \mathfrak{F}$, the subgroups $B$, $H \cap B$, and the Hall system $\Sigma^*$ satisfy the hypotheses of the lemma. If $B$ is a proper subgroup of $G$, each $\mathfrak{F}$-projector of $Q = \mathrm{N}_B(\Sigma^*)$ is contained in an $\mathfrak{F}$-projector of $\mathrm{N}_B(\Sigma^* \cap B^{\mathfrak{F}})$. Note that $\mathrm{N}_{H \cap B}(\Sigma^*) = \mathrm{N}_H(\Sigma)$ by Proposition 4.2.11 (3). Moreover $\mathrm{N}_G(\Sigma) \leq Q$. Since $G = H \, \mathrm{N}_G(\Sigma)$, we have that $B = (H \cap B) \, \mathrm{N}_G(\Sigma)$. Consequently $Q = \mathrm{N}_G(\Sigma)(Q \cap H \cap B) = \mathrm{N}_G(\Sigma) \, \mathrm{N}_{H \cap B}(\Sigma^*) = \mathrm{N}_G(\Sigma) = R$. This contradiction yields $B = G$. In other words, every Sylow subgroup of $K$ is normal in $G$. In particular, $G \in \mathfrak{N}\mathfrak{F}$. Suppose that $p$ is the prime divisor of the order of $H/K$. If $P$ is the Sylow $p$-subgroup of $H$ in $\Sigma$, we have that $H = PK$ and $R = \mathrm{N}_G(\Sigma) = \mathrm{N}_G(P)$. Let $E$ be an $\mathfrak{F}$-projector of $R$, then $G = HR = KR = K(ER^{\mathfrak{F}}) = EK = E \, \mathrm{F}(G)$ because $R^{\mathfrak{F}}$ is contained in $G^{\mathfrak{F}}$. By Theorem 4.2.7, $E$ is contained in an $\mathfrak{F}$-projector of $G = \mathrm{N}_G(\Sigma \cap G^{\mathfrak{F}})$. This is the final contradiction. ☐

**Theorem 4.2.13.** *Let $G$ be a group whose $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is soluble. Consider a Hall system $\Sigma$ of $G^{\mathfrak{F}}$ and denote $T = \mathrm{N}_G(\Sigma)$. Suppose that $M$ is an*

*$\mathfrak{F}$-abnormal maximal subgroup of $G$. If $\Sigma$ reduces into $M \cap G^{\mathfrak{F}}$, then there exists an $\mathfrak{F}$-projector of $T$ contained in an $\mathfrak{F}$-projector of $\mathrm{N}_M(\Sigma \cap M^{\mathfrak{F}})$.*

*Proof.* We split the proof in two steps.

1. *There exists an $\mathfrak{F}$-projector of $T$ contained in $M$.*

We use induction on the order of $G$. Note that, by [DH92, I, 4.17a] and Lemma 4.2.10, the hypotheses of the lemma hold in $G/\operatorname{Core}_G(M)$ and $M/\operatorname{Core}_G(M)$. If $\operatorname{Core}_G(M)$ is non-trivial, then, by induction, there exists an $\mathfrak{F}$-projector of $T\operatorname{Core}_G(M)/\operatorname{Core}_G(M)$, $D/\operatorname{Core}_G(M)$ say, contained in $M/\operatorname{Core}_G(M)$. We know that the $\mathfrak{F}$-residual of $T\operatorname{Core}_G(M)/\operatorname{Core}_G(M)$ is nilpotent and therefore the $\mathfrak{F}$-projectors of $T\operatorname{Core}_G(M)/\operatorname{Core}_G(M)$ are conjugate by Theorem 4.2.1. If $E$ is an $\mathfrak{F}$-projector of $T$, then there exists $g \in T$ such that $D = E^g\operatorname{Core}_G(M)$. Hence $E^g$ is an $\mathfrak{F}$-projector of $T$ contained in $M$. Assume now that $\operatorname{Core}_G(M) = 1$. Since $M$ is $\mathfrak{F}$-abnormal in $G$, the group $G$ is a primitive group of type 1 and $G = MN$, where $N$ is the minimal normal subgroup of $G$. Clearly we can assume that $G$ is not an $\mathfrak{F}$-group. Then $N \leq G^{\mathfrak{F}}$ and, by Proposition 2.2.8 (3), $M \cap G^{\mathfrak{F}} = M^{\mathfrak{F}}$. If $M^{\mathfrak{F}} = 1$, then $M$ is an $\mathfrak{F}$-group and $\mathrm{N}_M(\Sigma \cap M^{\mathfrak{F}}) = M$. Then $M$ is an $\mathfrak{F}$-projector of $G$ and $G \in \mathfrak{N}\mathfrak{F}$. In this case $G = T$ and our claim is true. Suppose that $M^{\mathfrak{F}} \neq 1$. We see that in this case $T$ is contained in $M$. Consider an element $am \in T$, with $a \neq 1, a \in N$, and $m \in M$. If $p$ is the prime divisor of $|N| = |G^{\mathfrak{F}} : M^{\mathfrak{F}}|$ and $S^p$ is the Hall $p'$-subgroup of $G^{\mathfrak{F}}$ in $\Sigma$, then $(S^p)^{am} = S^p$. Moreover, $S^p \leq M^{\mathfrak{F}} \leq M$ and then $(S^p)^a \leq M$. If $x \in S^p$, then $[x, a] \in M \cap N = 1$. Consequently $a$ centralises $S^p$ and $N \leq \mathrm{Z}(G^{\mathfrak{F}})$ by [DH92, I, 5.5]. Thus $G^{\mathfrak{F}}$ is contained in $\mathrm{C}_G(N)$ which is equal to $N$ by Theorem 1. Hence $M^{\mathfrak{F}} \leq N \cap M = 1$. This contradiction shows that $a = 1$ and $T$ is contained in $M$.

2. *Conclusion.*

Let $D$ be an $\mathfrak{F}$-projector of $T$ contained in $M$. Since $T^{\mathfrak{F}} \leq G^{\mathfrak{F}}$, we have that $G = TG^{\mathfrak{F}} = DG^{\mathfrak{F}}$. Put $R = M \cap G^{\mathfrak{F}}$; by hypothesis $\Sigma \cap R$ is a Hall system of $R$. Then we have that $D \leq \mathrm{N}_M(\Sigma) \leq \mathrm{N}_M(\Sigma \cap R) = D\big(G^{\mathfrak{F}} \cap \mathrm{N}_M(\Sigma \cap R)\big) = D\,\mathrm{N}_R(\Sigma \cap R)$. Since system normalisers of soluble groups are nilpotent, it follows that $\mathrm{N}_R(\Sigma \cap R)$ is a nilpotent normal subgroup of $\mathrm{N}_M(\Sigma \cap R)$. Hence $\mathrm{N}_M(\Sigma \cap R) \in \mathfrak{N}\mathfrak{F}$ and $D$ supplements the Fitting subgroup of $\mathrm{N}_M(\Sigma \cap R)$. By Theorem 4.2.8, $D$ is contained in an $\mathfrak{F}$-projector $E$ of $\mathrm{N}_M(\Sigma \cap R)$. Since $M^{\mathfrak{F}} \leq R$ and $R$ is soluble, we can apply Lemma 4.2.12 to $M$, $R$, and $\Sigma \cap R$ and deduce that each $\mathfrak{F}$-projector of $\mathrm{N}_M(\Sigma \cap R)$ is contained in an $\mathfrak{F}$-projector of $\mathrm{N}_M(\Sigma \cap M^{\mathfrak{F}})$. Therefore $E$, and then $D$, is contained in an $\mathfrak{F}$-projector of $\mathrm{N}_M(\Sigma \cap M^{\mathfrak{F}})$. □

**Theorem 4.2.14.** *Let $G$ be a group whose $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is soluble. Consider a Hall system $\Sigma$ of $G^{\mathfrak{F}}$ and denote $T = \mathrm{N}_G(\Sigma)$. If $D$ is an $\mathfrak{F}$-projector of $T$, then $D$ covers all $\mathfrak{F}$-central chief factors of $G$ and avoids the $\mathfrak{F}$-eccentric ones.*

*Proof.* By Lemma 4.2.10, it is enough to prove that $D$ covers the $\mathfrak{F}$-central minimal normal subgroups of $G$ and avoids the $\mathfrak{F}$-eccentric ones. Let $N$ be a $\mathfrak{F}$-central minimal normal subgroup of $G$. Then $N \leq \mathrm{C}_G(G^{\mathfrak{F}})$. It implies that $N$ is contained in $T$ and $G = DG^{\mathfrak{F}} = D\,\mathrm{C}_G(N)$. Hence $N$ is a minimal normal subgroup of $ND$ and $[N]*(ND) \cong [N]*G \in \mathfrak{F}$. Since $\mathfrak{F}$ is a saturated formation and $ND/N \in \mathfrak{F}$, we have that $ND \in \mathfrak{F}$. Since $D$ is $\mathfrak{F}$-maximal in $T$, we have that $N \leq D$. Suppose now that $N$ is $\mathfrak{F}$-eccentric in $G$. Then $N \leq G^{\mathfrak{F}}$ and $N$ is abelian. If $D$ does not avoid $N$, then $N \cap D \neq 1$. By [DH92, I, 5.5], we deduce that $N \leq \mathrm{Z}(G^{\mathfrak{F}})$, and then $N$ is $\mathfrak{F}$-central in $G$, contrary to supposition. Therefore $N$ is avoided by $D$. □

Now we can give a characterisation of the $\mathfrak{F}$-normalisers of a group $G$ whose $\mathfrak{F}$-residual is soluble in terms of the $\mathfrak{F}$-projectors of the absolute system normalisers of the Hall systems of $G^{\mathfrak{F}}$.

**Theorem 4.2.15.** *Let $G$ be a group whose $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is soluble. For every Hall system $\Sigma$ of $G^{\mathfrak{F}}$, every $\mathfrak{F}$-projector of $\mathrm{N}_G(\Sigma)$ is an $\mathfrak{F}$-normaliser of $G$. Thus*

$$\mathrm{Nor}_{\mathfrak{F}}(G) = \bigcup\bigl\{E \in \mathrm{Proj}_{\mathfrak{F}}\bigl(\mathrm{N}_G(\Sigma)\bigr) : \Sigma \textit{ is a Hall system of } G^{\mathfrak{F}}\bigr\},$$

*and $\mathrm{Nor}_{\mathfrak{F}}(G)$ is a conjugacy class of subgroups of $G$.*

*Proof.* We can assume that $G$ is not an $\mathfrak{F}$-group. Let $\Sigma$ be a Hall system of $G^{\mathfrak{F}}$ and let $M$ be an $\mathfrak{F}$-critical subgroup of $G$ such that $\Sigma$ reduces into $M \cap G^{\mathfrak{F}}$. By Theorem 4.2.13 there exists an $\mathfrak{F}$-projector $D$ of $\mathrm{N}_G(\Sigma)$ contained in an $\mathfrak{F}$-projector $D^*$ of $\mathrm{N}_M(\Sigma \cap M^{\mathfrak{F}})$. Arguing by induction, $D^*$ is an $\mathfrak{F}$-normaliser of $M$, and then of $G$. Applying Theorem 4.2.4, for $D^*$, and Theorem 4.2.14, for $D$, we have that both cover simultaneously all $\mathfrak{F}$-central chief factors of $G$ and avoid the $\mathfrak{F}$-eccentric ones. Therefore $D$ and $D^*$ have the same order and $D = D^*$. Since $\mathrm{N}_G(\Sigma) \in \mathfrak{N}\mathfrak{F}$, the $\mathfrak{F}$-projectors of $\mathrm{N}_G(\Sigma)$ are a conjugacy class of subgroups by Theorem 4.2.1. Therefore, every $\mathfrak{F}$-projector of $\mathrm{N}_G(\Sigma)$ is an $\mathfrak{F}$-normaliser of $G$.

Conversely, if $D$ is an $\mathfrak{F}$-normaliser of $G$ and $D \neq G$, then $D$ is an $\mathfrak{F}$-normaliser of an $\mathfrak{F}$-critical subgroup $M$ of $G$. By induction, there exists a Hall system $\Sigma^*$ of $M^{\mathfrak{F}}$ such that $D \in \mathrm{Proj}_{\mathfrak{F}}\bigl(\mathrm{N}_M(\Sigma^*)\bigr)$. Since, by Proposition 2.2.8 (3), $M^{\mathfrak{F}}$ is contained in $G^{\mathfrak{F}}$, we can find a Hall system $\Sigma$ of $G^{\mathfrak{F}}$ which reduces into $M \cap G^{\mathfrak{F}}$ and $\Sigma \cap M^{\mathfrak{F}} = \Sigma^*$ by [DH92, I, 4.16]. Applying Theorem 4.2.13, $\mathrm{N}_M(\Sigma^*)$ contains an $\mathfrak{F}$-projector of $\mathrm{N}_G(\Sigma)$. Since $\mathrm{Proj}_{\mathfrak{F}}\bigl(\mathrm{N}_M(\Sigma^*)\bigr)$ is a conjugacy class of subgroups of $\mathrm{N}_M(\Sigma^*)$, it follows that there exists an $\mathfrak{F}$-projector $E$ of $\mathrm{N}_G(\Sigma^g)$, for some $g \in G$, contained in $D$. Thus, $D$ is an $\mathfrak{F}$-projector of $\mathrm{N}_G(\Sigma^g)$ by Theorem 4.2.4 and Theorem 4.2.14. Consequently,

$$\bigcup\bigl\{E \in \mathrm{Proj}_{\mathfrak{F}}\bigl(\mathrm{N}_G(\Sigma)\bigr) : \Sigma \text{ is a Hall system of } G^{\mathfrak{F}}\bigr\} = \mathrm{Nor}_{\mathfrak{F}}(G). \qquad \square$$

**Corollary 4.2.16.** *Let $G$ be a group whose $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is soluble. If $H$ is an $\mathfrak{F}$-projector of $G$ complementing $G^{\mathfrak{F}}$ in $G$, then $H$ normalises some Sylow $p$-subgroup of $G^{\mathfrak{F}}$, for each prime $p$ dividing the order of $G^{\mathfrak{F}}$.*

*Proof.* By Theorem 4.2.9, $H$ contains an $\mathfrak{F}$-normaliser of $G$. Since in this case both complement $G^{\mathfrak{F}}$, then $H$ is an $\mathfrak{F}$-normaliser of $G$. By Theorem 4.2.15, there exists a Hall system $\Sigma$ of $G^{\mathfrak{F}}$ such that $H \leq \mathrm{N}_G(\Sigma)$. This means that $H$ normalises every Sylow subgroup of $G^{\mathfrak{F}}$ in $\Sigma$. $\square$

The following useful splitting theorem is a generalisation of a theorem due to G. Higman on complementation of abelian normal subgroups. The corresponding result for finite soluble groups was obtained by R. W. Carter and T. O. Hawkes (see [CH67] and [DH92, IV, 5.18]).

**Theorem 4.2.17.** *Let $\mathfrak{F}$ be a saturated formation and let $G$ be group whose $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ is abelian. Then $G^{\mathfrak{F}}$ is complemented in $G$ and two any complements are conjugate in $G$. The complements are the $\mathfrak{F}$-normalisers of $G$.*

*Proof.* First we prove that an $\mathfrak{F}$-normaliser of $G$ is a complement of $G^{\mathfrak{F}}$. Suppose that this is not true and let $G$ be a minimal counterexample. Put $R = G^{\mathfrak{F}}$. Then there exists $D \in \mathrm{Nor}_{\mathfrak{F}}(G)$ such that $D \cap R \neq 1$. Observe that, since $R$ is abelian and $G = RD$, the subgroup $R \cap D$ is normal in $G$.

Assume that there exists an $\mathfrak{F}$-eccentric minimal normal subgroup $N$ of $G$ such that $N \leq R$. The quotient $DN/N$ is an $\mathfrak{F}$-normaliser of $G/N$ and $R/N = (G/N)^{\mathfrak{F}}$. By minimality of $G$, we have that $R \cap D = N$. But then $D$ covers $N$ and $N$ has to be $\mathfrak{F}$-central in $G$ by Theorem 4.2.4. This is a contradiction. Hence every minimal normal subgroup of $G$ below $R$ is $\mathfrak{F}$-central in $G$. Then, if $N$ is any minimal normal subgroup of $G$ below $R$, we have that $N \leq D$ and, by minimality of $G$, $R \cap D = N$. Consequently, $N$ is the unique minimal normal subgroup of $G$ below $R$.

Let $M$ be an $\mathfrak{F}$-critical subgroup of $G$ such that $D \in \mathrm{Nor}_{\mathfrak{F}}(M)$. Since $M^{\mathfrak{F}}$ is contained in $R$, we have that $M^{\mathfrak{F}}$ is an abelian normal subgroup of $G$. If $M^{\mathfrak{F}} \neq 1$, then $N$ is contained in $M^{\mathfrak{F}}$ and, by minimality of $G$, we have that $M^{\mathfrak{F}} \cap D = 1$. This is a contradiction. Hence $M \in \mathfrak{F}$ and then $M = D$. This implies that $R/N$ is chief factor of $G$ complemented by $D$. Let $p$ be the prime dividing the order of $N$. Then $R$ is an abelian $p$-group. Suppose that $F$ is the integrated and full local definition of $\mathfrak{F}$. Then $F(p) \neq \emptyset$ and $R \leq G^{F(p)}$. Observe that $\mathfrak{F}$ is contained in $\mathfrak{E}_{p'}F(p)$ and that $G^{F(p)}/R$ is therefore a $p'$-group. Thus $R \in \mathrm{Syl}_p(G^{F(p)})$. By the Schur-Zassenhaus Theorem [Hup67, I, 18.1 and 18.2], there exists a complement $Q$ of $R$ in $G^{F(p)}$. Observe that $R/N$ is a chief factor avoided by $D$. Therefore $R/N$ is $\mathfrak{F}$-eccentric in $G$. Consequently $G/\mathrm{C}_G(R/N) \notin F(p)$, and $G^{F(p)}$ is not contained in $\mathrm{C}_G(R/N)$. Consider the $p'$-group $Q$ acting on the normal $p$-subgroup $R$ by conjugation. Then $R = [R, Q] \times \mathrm{C}_R(Q)$ by [DH92, A, 12.5]. Observe that both $\mathrm{C}_R(Q) = \mathrm{C}_R(QR) = \mathrm{C}_R(G^{F(p)})$ and $[R, Q] = [R, QR] = [R, G^{F(p)}]$ are normal subgroups of $G$. Since $N$ is the unique minimal normal subgroup of $G$ below $R$, then either $\mathrm{C}_R(Q) = 1$ or $[R, Q] = 1$. Since $N$ is $\mathfrak{F}$-central in $G$, we have that $N \leq \mathrm{C}_R(G^{F(p)}) = \mathrm{C}_R(Q)$. Consequently, $G^{F(p)} = QR \leq \mathrm{C}_G(R) \leq \mathrm{C}_G(R/N)$, contrary to supposition. Therefore each $\mathfrak{F}$-normaliser complements $G^{\mathfrak{F}}$ in $G$.

Consider now a subgroup $H$ of $G$ such that $G = HG^{\mathfrak{F}}$ and $H \cap G^{\mathfrak{F}} = 1$. Since every chief factor of $G$ below $G^{\mathfrak{F}}$ is $\mathfrak{F}$-eccentric, the subgroup $H$ covers all $\mathfrak{F}$-central chief factors of a chief series of $G$ through $G^{\mathfrak{F}}$. By Theorem 4.1.20, there exists $D \in \mathrm{Nor}_{\mathfrak{F}}(G)$ such that $D \leq H$. Therefore $D = H \in \mathrm{Nor}_{\mathfrak{F}}(G)$.

Finally, by Theorem 4.2.15, $\mathrm{Nor}_{\mathfrak{F}}(G)$ is a conjugacy class of subgroups of $G$. Hence the complements of $G^{\mathfrak{F}}$ are the $\mathfrak{F}$-normalisers of $G$ and they are conjugate. □

A consequence of Theorem 4.2.17 is the following result due to P. Schmid.

**Corollary 4.2.18 ([Sch74]).** *For every group $G$, we have that*

$$G^{\mathfrak{F}} \cap \mathrm{Z}_{\mathfrak{F}}(G) \leq (G^{\mathfrak{F}})' \cap \mathrm{Z}(G^{\mathfrak{F}}).$$

*Proof.* Theorem 4.2.17, applied to the group $G/(G^{\mathfrak{F}})'$, leads to $\mathrm{Z}_{\mathfrak{F}}(G) \cap G^{\mathfrak{F}} \leq (G^{\mathfrak{F}})'$. By [DH92, IV, 6.10]), we have that $[G^{\mathfrak{F}}, \mathrm{Z}_{\mathfrak{F}}(G)] = 1$. Therefore $G^{\mathfrak{F}} \cap \mathrm{Z}_{\mathfrak{F}}(G) \leq (G^{\mathfrak{F}})' \cap \mathrm{Z}(G^{\mathfrak{F}})$. □

Next, we use Corollary 4.2.18 to give a short proof of a well-known result of L. A. Shemetkov ([She72]).

**Theorem 4.2.19.** *Let $G$ be a group such that for some prime $p$, the Sylow $p$-subgroups of $G^{\mathfrak{F}}$ are abelian. Then every chief factor of $G$ below $G^{\mathfrak{F}}$ whose order is divisible by $p$ is an $\mathfrak{F}$-eccentric chief factor of $G$.*

*Proof.* Suppose that the theorem is false and let $G$ be a minimal counterexample. Then $G^{\mathfrak{F}} \neq 1$. Let $N$ be a minimal normal subgroup of $G$ such that $N \leq G^{\mathfrak{F}}$. From minimality of $G$, every chief factor of $G$ between $N$ and $G^{\mathfrak{F}}$ whose order is divisible by $p$ is $\mathfrak{F}$-eccentric, the prime $p$ divides $|N|$ and $N$ is an $\mathfrak{F}$-central chief factor of $G$. Then $N \leq G^{\mathfrak{F}} \cap \mathrm{Z}_{\mathfrak{F}}(G) \leq (G^{\mathfrak{F}})' \cap \mathrm{Z}(G^{\mathfrak{F}})$ by Corollary 4.2.18. Let $P$ be a Sylow $p$-subgroup of $G^{\mathfrak{F}}$. Since $P$ is abelian, we have that $N \leq (G^{\mathfrak{F}})' \cap \mathrm{Z}(G^{\mathfrak{F}}) \cap P = 1$ by Taunt's Theorem (see [Hup67, VI, 14.3]). This contradiction concludes the proof. □

We round the section off with another interesting splitting theorem.

**Theorem 4.2.20.** *Let $G$ be a group such that every chief factor of $G$ below $G^{\mathfrak{F}}$ is $\mathfrak{F}$-eccentric. Assume that $G^{\mathfrak{F}}$ is $p$-nilpotent for every prime $p$ in $\pi = \pi(|G : G^{\mathfrak{F}}|)$, Then*

1. *(P. Schmid, [Sch74]) $G^{\mathfrak{F}}$ is complemented in $G$ and any two complements are conjugate;*
2. *(A. Ballester-Bolinches, [BB89a]) the complements of $G^{\mathfrak{F}}$ in $G$ are the $(\mathfrak{F} \cap \mathfrak{S}_{\pi})$-normalisers of $G$.*

*Proof.* First we note that the class $\mathfrak{L} = \mathfrak{F} \cap \mathfrak{S}_{\pi}$ is a saturated formation and $G^{\mathfrak{L}} = G^{\mathfrak{F}}$.

We argue by induction on the order of $G$. Consider $N = \mathrm{O}^{\pi}(G^{\mathfrak{F}})$ and suppose that $N \neq 1$. The quotient group $G^{\mathfrak{F}}/N = (G/N)^{\mathfrak{F}}$ is a nilpotent $\pi$-group.

By induction, $G^{\mathfrak{F}}/N$ is complemented in $G/N$ and any two complements are conjugate. If $L/N$ is a complement of $G^{\mathfrak{F}}/N$ in $G/N$, then $N$ is a normal Hall $\pi'$-subgroup of $L$. By the Schur-Zassenhaus Theorem [Hup67, I, 18.1 and 18.2], there exists a Hall $\pi$-subgroup $H$ of $L$ and two Hall $\pi$-subgroups of $L$ are conjugate in $L$. Observe that $H \cap G^{\mathfrak{F}} = 1$ and then $G^{\mathfrak{F}}$ is complemented in $G$. Moreover if $A$ and $B$ are two complements of $G^{\mathfrak{F}}$ in $G$, then $AN/N$ and $BN/N$ are conjugate in $G/N$. Without loss of generality we can assume that $AN = BN$. Since $A$ and $B$ are Hall $\pi$-subgroups of $AN$ and $N$ is a normal Hall $\pi'$-subgroup of $AN$, it follows that $A$ and $B$ are conjugate by the Schur-Zassenhaus Theorem. If $E$ is an $\mathfrak{L}$-normaliser of $G$, then $EN/N$ is an $\mathfrak{L}$-normaliser of $G/N$ by Proposition 4.1.5. By induction, $E \cap G^{\mathfrak{F}} \leq N$. Since $E$ is a $\pi$-group and $N$ is a $\pi'$-group, we have that $E \cap G^{\mathfrak{F}} = 1$ and $E$ complements $G^{\mathfrak{F}}$ in $G$.

Therefore we can assume that $N = 1$, i.e. $G^{\mathfrak{F}}$ is a nilpotent $\pi$-group, and $G$ is a $\pi$-group in $\mathfrak{N}\mathfrak{F}$. Here the $\mathfrak{L}$-normalisers and the $\mathfrak{F}$-normalisers of $G$ coincide. Since every chief factor of $G$ below $G^{\mathfrak{F}}$ is $\mathfrak{F}$-eccentric in $G$, if $D$ is an $\mathfrak{F}$-normaliser of $G$, then $D \cap G^{\mathfrak{F}} = 1$, by Corollary 4.2.5, and $D$ is a complement of $G^{\mathfrak{F}}$ in $G$. Any complement $E$ of $G^{\mathfrak{F}}$ is an $\mathfrak{F}$-group. By Lemma 4.1.17, $E$ is contained in an $\mathfrak{F}$-normaliser. Hence $E$ is an $\mathfrak{F}$-normaliser of $G$. Thus, the complements of $G^{\mathfrak{F}}$ in $G$ are the $\mathfrak{F}$-normalisers of $G$, and they are conjugate, by Theorem 4.2.15. ☐

**Postscript**

K. Doerk (see [DH92, V, 3.18]) used the $\mathfrak{F}$-normalisers to show that a saturated formation $\mathfrak{F}$ has a unique upper bound for all local definitions, that is, a maximal local definition, in the soluble universe. In fact, he proved that the formation function $g$ given by

$$g(p) = \big(G : \text{the } \mathfrak{F}\text{- normalisers of } G \text{ are in } F(p)\big),$$

for all primes $p$, is the maximal local definition of $\mathfrak{F}$.

As we have seen in Chapter 3, the situation in the general finite universe is not so clear cut. However, it is possible to use the $\mathfrak{F}$-normalisers of finite, non-necessarily soluble, groups to give necessary and sufficient conditions for a saturated formation $\mathfrak{F}$ to have a maximal local definition ([BB89a], [BB91]).

## 4.3 Subgroups of prefrattini type

The introduction of systems of maximal subgroups in [BBE91] made possible the extension of prefrattini subgroups to finite, non-necessarily soluble, groups. Later, in [BBE95], we introduced the concept of a *weakly solid* (or simply *w-solid*) *set of maximal subgroups* following some ideas due to M. J. Tomkinson [Tom75]. Equipped with these new notions, we were able to present a common generalisation of all prefrattini subgroups of the literature. These new subgroups enjoy most of the properties of the soluble case, for instance they are preserved by epimorphic images and enjoy excellent factorisation properties. Unfortunately, we cannot expect to keep cover-avoidance property and conjugacy. In fact, conjugacy characterises solubility, and conjugacy and cover-avoidance property are equivalent in some sense (see Corollary 4.3.14). In fact we can repeat here the comment said in the introduction of Section 1.4: we lose the arithmetical properties, but we find deep relations between maximal subgroups which are general to all finite groups.

We present here a distillation of the preceding concepts. Observe, for instance, that the definition of system of maximal subgroups given in [BBE91] is different, but equivalent, to the one in Section 1.4. In fact this presentation allows us to speak of a particular subgroup of prefrattini type, which is defined by the intersection of all maximal subgroups in a subsystem of maximal subgroups. This point of view is new since all precedents of prefrattini subgroups in the past were families of subgroups of the group. To recover this classical idea of a set of prefrattini subgroups, we include the concept of w-solid set as a union-set of subsystems of maximal subgroups.

**Definitions 4.3.1.** *Let $\mathbf{X}$ be a (possibly empty) set of monolithic maximal subgroups of a group $G$.*

1. *We will say that $\mathbf{X}$ is a* weakly solid (w-solid) set of maximal subgroups *of $G$ if*

   *for any $U$, $S \in \mathbf{X}$ such that $\operatorname{Core}_G(U) \neq \operatorname{Core}_G(S)$ and both complement the same abelian chief factor $H/K$ of $G$, then $M = (U \cap S)H \in \mathbf{X}$.* (4.5)
2. *$\mathbf{X}$ is said to be* solid *if it satisfies (4.5) and whenever a chief factor is $\mathbf{X}$-supplemented in $G$, then all its monolithic supplements are in $\mathbf{X}$.*

Next we give a varied selection of examples of w-solid and solid sets.

*Examples 4.3.2.* 1. The set $\operatorname{Max}^*(G)$, of all monolithic maximal subgroups of a group $G$, is solid.

2. Consider a subgroup $L$ of a group $G$; the set $\mathbf{X}_L$ of all monolithic maximal subgroups of $G$ containing $L$ is w-solid.

3. Given a w-solid (respectively solid) set $\mathbf{X}$ of maximal subgroups of a group $G$ and a class $\mathfrak{H}$ of groups, then the set $\mathbf{X}^a_{\mathfrak{H}}$ of all $\mathfrak{H}$-abnormal subgroups in $\mathbf{X}$ and the set $\mathbf{X}^n_{\mathfrak{H}}$ of all $\mathfrak{H}$-normal subgroups in $\mathbf{X}$ are w-solid (respectively solid) as well.

If $\mathbf{X}$ is a system of maximal subgroups, then $\mathbf{X}^a_{\mathfrak{H}}$ and $\mathbf{X}^n_{\mathfrak{H}}$ are subsystems of maximal subgroups.

Let $M$ be a monolithic maximal subgroup of $G$. Recall that the normal index of $M$ in $G$, defined by W. E. Deskins in [Des59] and denoted by $\eta(G, M)$, is indeed $\eta(G, M) = \bigl|\operatorname{Soc}\bigl(G/\operatorname{Core}_G(M)\bigr)\bigr|$.

3. The following families of monolithic maximal subgroups of a group $G$ are w-solid:

a) Fixed a prime $p$, the set $\mathbf{X}_p$ of all monolithic maximal subgroups $M$ of $G$ such that $|G : M|$ is a $p$-power. In fact, if $G$ is $p$-soluble, then $\mathbf{X}_p$ is indeed solid. However this is not true in the non-soluble case; in $G = \mathrm{Alt}(5)$ the set $\mathbf{X}_5$ is composed of all maximal subgroups isomorphic to $\mathrm{Alt}(4)$ and clearly it is not solid.

b) Fixed a set of primes $\pi$, the set $\mathbf{X}^\pi$ of all monolithic maximal subgroups $M$ of $G$ such that $|G : M|$ is a $\pi'$-number.

c) the set of all monolithic maximal subgroups of $G$ of composite index in $G$.

d) the set of all monolithic maximal subgroups $M$ of the group $G$ such that $\eta(G, M) \neq |G : M|$.

If $G$ is a group, the set $\mathcal{S}(G)$ composed of all systems of maximal subgroups of $G$ is non-empty by Theorem 1.4.7. If $\mathbf{X}$ is a w-solid set of maximal subgroups of $G$ and $\mathbf{Y} \in \mathcal{S}(G)$, then $\mathbf{X} \cap \mathbf{Y}$ is a subsystem of maximal subgroups of $G$. Applying Theorem 1.4.7, we have that $\mathbf{X} = \bigcup\{\mathbf{X} \cap \mathbf{Y} : \mathbf{Y} \in \mathcal{S}(G)\}$.

**Definitions 4.3.3.** *1. Let $G$ be a group. Let $\mathbf{X}$ be a non-empty subsystem of maximal subgroups of $G$. Define*

$$\mathrm{W}(G, \mathbf{X}) = \bigcap\{M : M \in \mathbf{X}\}.$$

*For convenience, we define* $\mathrm{W}(G, \emptyset) = G$.
*We will say that $W$ is a* subgroup of prefrattini type *of $G$ if $W = \mathrm{W}(G, \mathbf{X})$ for some subsystem $\mathbf{X}$ of maximal subgroups of $G$.*

*2. If $\mathbf{X}$ be a w-solid set of maximal subgroups of $G$, we say that*

$$\mathrm{Pref}_{\mathbf{X}}(G) = \{\mathrm{W}(G, \mathbf{X} \cap \mathbf{Y}) : \mathbf{Y} \in \mathcal{S}(G),\ \mathbf{X} \cap \mathbf{Y} \neq \emptyset\}$$

*is the set of all $\mathbf{X}$-prefrattini subgroups of $G$.*

We show in the following that the known prefrattini subgroups are associated with w-solid sets of maximal subgroups.

*Examples 4.3.4.* 1. The $\mathrm{Max}^*(G)$-prefrattini subgroups are simply called *prefrattini subgroups* of $G$. We write

$$\mathrm{Pref}(G) = \{\mathrm{W}(G, \mathbf{X}) : \mathbf{X} \in \mathcal{S}(G)\}.$$

In other words, a prefrattini subgroup of a group $G$ is a subgroup of the form $\mathrm{W}(G, \mathbf{X})$, where $\mathbf{X}$ is a system of maximal subgroups of $G$. If $G$ is a soluble group, we can apply Corollary 1.4.18 and conclude that the prefrattini subgroups of $G$ are those introduced by W. Gaschütz in [Gas62] which originated this theory.

2. Let $\mathfrak{H}$ be a Schunck class. The $\mathrm{Max}^*(G)^a_{\mathfrak{H}}$-prefrattini subgroups of a group $G$ are the $\mathfrak{H}$-prefrattini subgroups defined in [BBE91]. If $G$ is soluble, they are the $\mathfrak{H}$-prefrattini subgroups studied by P. Förster in [För83] and, if $\mathfrak{H}$ is a saturated formation, the $\mathrm{Max}^*(G)^a_{\mathfrak{H}}$-prefrattini subgroups of $G$ are the ones introduced by T. O. Hawkes in [Haw67].

3. If $G$ is a soluble group, then $\mathrm{Pref}_{\mathbf{X}_L}(G)$ is the set of all $L$-prefrattini subgroups introduced by H. Kurzweil in [Kur89].

4. The $\mathbf{X}_p$-prefrattini subgroups of a $p$-soluble group are the $p$-prefrattini subgroups studied by A. Brandis in [Bra88].

*Notation 4.3.5.* If $\mathfrak{H}$ is a Schunck class, $G$ is a group, and $\mathbf{X}$ is a system of maximal subgroups of $G$, we denote

$$\mathrm{W}(G, \mathfrak{H}, \mathbf{X}) = \mathrm{W}(G, \mathbf{X}^a_{\mathfrak{H}}),$$

and say that $\mathrm{W}(G, \mathfrak{H}, \mathbf{X})$ is the $\mathfrak{H}$-prefrattini subgroup of $G$ associated with $\mathbf{X}$. We write

$$\mathrm{Pref}_{\mathfrak{H}}(G) = \{\mathrm{W}(G, \mathfrak{H}, \mathbf{X}) : \mathbf{X} \in \mathcal{S}(G)\}$$

for the set of all $\mathfrak{H}$-prefrattini subgroups of $G$.

**Theorem 4.3.6.** *Consider a group $G$, $\mathbf{X}$ a subsystem of maximal subgroups of $G$ and $W = \mathrm{W}(G, \mathbf{X})$. Then*

$$W = \bigcap \{\mathrm{T}(G, \mathbf{X}, F) : F \textit{ is an } \mathbf{X}\textit{-supplemented chief factor of } G\}.$$

*Moreover $W$ has the following properties.*

1. *Let $1 = G_0 < G_1 < \cdots < G_n = G$ be a chief series of $G$; write $\mathcal{I} = \{i : 1 \leq i \leq n$ such that $G_i/G_{i-1}$ is $\mathbf{X}$-supplemented$\}$; then, if $\mathcal{I}$ is non-empty,*
$$W = \bigcap_{i \in \mathcal{I}} \{S_i : S_i \textit{ is an } \mathbf{X}\textit{-supplement of } G_i/G_{i-1}\}.$$
2. *If $N$ is a normal subgroup of $G$, then $WN/N = \mathrm{W}(G/N, \mathbf{X}/N)$.*

*Proof.* Applying Proposition 1.3.11, we can deduce that

$$W = \bigcap \{\mathrm{T}(G, \mathbf{X}, F) : F \text{ is an } \mathbf{X}\text{-supplemented chief factor of } G\}.$$

Now Assertion 1 follows from Theorem 1.2.36 and Theorem 1.3.8.

In proving Assertion 2, suppose first that $N$ is a minimal normal subgroup of $G$ and let $1 = G_0 < G_1 = N < \cdots < G_n = G$ be a chief series of $G$. Clearly we can assume that $\mathbf{X}$ is non-empty. Then $\mathcal{I} = \{i : 1 \leq i \leq n \text{ such that } G_i/G_{i-1} \text{ is } \mathbf{X}\text{-supplemented}\}$ is non-empty and $W = \bigcap_{i \in \mathcal{I}} \{S_i : S_i \text{ is an } \mathbf{X}\text{-supplement of } G/G_{i-1}\}$ by Statement 1. If $N$ is an $\mathbf{X}$-Frattini, then $N$ is contained in $S_i$ for all $i \in \mathcal{I}$ and then $W/N = \mathrm{W}(G/N, \mathbf{X}/N)$. Otherwise, $N$ is contained in $S_i$ for all $i \in \mathcal{I} \setminus \{1\}$ and $G = NS_1$. The case $\mathcal{I} = \{1\}$ leads to $W = S_1$ and $\mathbf{X}/N = \emptyset$. Then $G = WN$ and $WN/N = \mathrm{W}(G/N, \mathbf{X}/N)$. Suppose that $\mathcal{I} \setminus \{1\} \neq \emptyset$. Then $WN = \bigcap_{i \in \mathcal{I} \setminus \{1\}} S_i$ and then $WN/N = \mathrm{W}(G/N, \mathbf{X}/N)$. Therefore Assertion 2 holds when $N$ is a minimal normal subgroup of $G$.

A familiar inductive argument proves the validity of Statement 2 for any normal subgroup $N$ of $G$. $\square$

*Remark 4.3.7.* Theorem 4.3.6 does not hold when $\mathbf{X}$ is simply a JH-solid set (see Example 1.3.10). This is the reason why we introduce the prefrattini subgroups associated with subsystems of maximal subgroups and not with JH-solid sets of maximal subgroups.

All classical examples of prefrattini subgroups in the soluble universe, including Kurzweil's, enjoy the conjugacy and the cover-avoidance property. Now we prove that, roughly speaking, it can be said that conjugacy and cover-avoidance property of soluble chief factors are equivalent properties for subgroups of prefrattini type. In fact, conjugacy of prefrattini subgroups characterises solubility. The consideration of primitive non-soluble groups, whose core-free maximal subgroups are neither conjugate nor CAP-subgroups, causes that in the general non-soluble universe these properties fail.

**Proposition 4.3.8.** *Let $G$ be a group and $\mathbf{X}$ a subsystem of maximal subgroups of $G$. Put $W = \mathrm{W}(G,\mathbf{X})$. Let $H/K$ be a chief factor of $G$.*

1. *If $H/K$ is $\mathbf{X}$-Frattini, then $\mathrm{W}(G,\mathbf{X})$ covers $H/K$.*
2. *If $H/K$ possesses $\mathbf{X}$-complement in $G$, then $\mathrm{W}(G,\mathbf{X})$ avoids $H/K$.*

*Proof.* Assume that $H/K$ is an $\mathbf{X}$-Frattini chief factor of $G$. Then $H/K \leq MK/K$ for all $M \in \mathbf{X}$. Hence,

$$H/K \leq \bigcap\{MK/K : M \in \mathbf{X}\} = \mathrm{W}(G/K, \mathbf{X}/K) = WK/K,$$

by Proposition 4.3.6, and $\mathrm{W}(G,\mathbf{X})$ covers $H/K$.

If a maximal subgroup $M$ of $G$ belongs to $\mathbf{X}$, then $W \leq M$. Hence, if $M$ complements $H/K$, $W$ avoids $H/K$. □

**Corollary 4.3.9.** *Let $G$ be a group, $\mathbf{X}$ a solid set of maximal subgroups of $G$ and $H/K$ an abelian chief factor of $G$. Then*

1. *$H/K$ is either covered or avoided by all $W \in \mathrm{Pref}_{\mathbf{X}}(G)$;*
2. *$H/K$ is covered by some $W \in \mathrm{Pref}_{\mathbf{X}}(G)$ if and only if $H/K$ is an $\mathbf{X}$-Frattini chief factor of $G$.*

The above result justifies the following definition.

**Definition 4.3.10.** *Let $G$ be a group and $\mathbf{X}$ a w-solid set of maximal subgroups of $G$. We say that* $\mathrm{Pref}_{\mathbf{X}}(G)$ satisfies ACAP *if whenever $F$ is an abelian chief factor of $G$,*

1. *then $F$ is either covered or avoided by all $W \in \mathrm{Pref}_{\mathbf{X}}(G)$, and*
2. *$F$ is covered by some $W \in \mathrm{Pref}_{\mathbf{X}}(G)$ if and only if $F$ is an $\mathbf{X}$-Frattini chief factor of $G$.*

Clearly if $\mathrm{Pref}_{\mathbf{X}}(G)$ satisfies ACAP, any $W \in \mathrm{Pref}_{\mathbf{X}}(G)$ covers all abelian $\mathbf{X}$-Frattini chief factors of $G$ and avoids all abelian $\mathbf{X}$-complemented.

By the above corollary, if $\mathbf{X}$ is a solid set of maximal subgroups of a group $G$, then $\mathrm{Pref}_{\mathbf{X}}(G)$ satisfies ACAP. We give some more examples.

*Examples 4.3.11.* 1. By Lemma 1.5 of [Kur89], if $L$ is a subgroup of a soluble group $G$, the set $\mathrm{Pref}_{\mathfrak{X}_L}(G)$ of all $L$-prefrattini subgroups of $G$ satisfies ACAP (note that $\mathbf{X}_L$ is w-solid, but not solid in general).

2. Let $G$ be the group as in Example 1.3.10. We consider the set $\mathbf{X} = \{\langle a, z\rangle, \langle b, z\rangle, \langle ab, z\rangle, \langle a^2b, z\rangle\}$. Then $\mathbf{X}$ is a subsystem of maximal subgroups of $G$ and $\mathrm{W}(G, \mathbf{X}) = \langle z\rangle$. We consider the system $\mathbf{Y}$ of maximal subgroups defined by the Hall system $\Sigma = \{N, \langle abz\rangle\}$ (see Theorem 1.4.17). Then $\mathrm{W}(G, \mathbf{X} \cap \mathbf{Y}) = \langle ab, z\rangle$. It is clear that the $\mathbf{X}$-prefrattini subgroups of $G$ do not satisfy ACAP.

**Proposition 4.3.12.** *Let $G$ be a group, and let $\mathbf{X}$ be a w-solid set of maximal subgroups of $G$. Assume that $\mathrm{Pref}_{\mathbf{X}}(G)$ satisfies ACAP. Let $\mathbf{X}_1$, $\mathbf{X}_2$ be two systems of maximal subgroups of $G$ and $H/K$ an abelian chief factor of $G$. Then, there exists an $\mathbf{X}$-complement of $H/K$ in $\mathbf{X}_1$ if and only if there exists an $\mathbf{X}$-complement of $H/K$ in $\mathbf{X}_2$.*

*Proof.* Put $\{i, j\} = \{1, 2\}$. Suppose that $M_i$ is an $\mathbf{X}$-complement of $H/K$ in $\mathbf{X}_i$ but for all maximal subgroups $S \in \mathbf{X} \cap \mathbf{X}_j$ such that $K \leq S$, we have $H \leq S$. Denote by $W_k$ the $(\mathbf{X} \cap \mathbf{X}_k)$-prefrattini subgroup of $G$, $k = 1, 2$. Applying Theorem 4.3.6, $W_i \leq M_i$. Then $K = W_iK \cap H$. Since $\mathrm{Pref}_{\mathbf{X}}(G)$ satisfies ACAP, we have $K = W_jK \cap H$. However $W_jK/K$ is the $\mathbf{X}/K$-prefrattini subgroup of $G/K$ associated with $\mathbf{X}_j/K$ by Theorem 4.3.6 (2). Then $W_jK/K = \bigcap\{S/K : S \in \mathbf{X} \cap \mathbf{X}_j, K \leq S\}$. Our assumption implies $H/K \leq W_jK/K$. This contradiction proves that $H/K$ has an $\mathbf{X}$-complement in $\mathbf{X}_j$. ☐

**Theorem 4.3.13.** *Let $\mathbf{X}$ be a w-solid set of maximal subgroups of group $G$. For $\mathbf{Y} = \mathbf{X}^n_{\mathfrak{S}}$, the set of all $\mathfrak{S}$-normal maximal subgroups in $\mathbf{X}$, the following statements are equivalent:*

1. $\mathrm{Pref}_{\mathbf{Y}}(G)$ *satisfies ACAP;*
2. $\mathrm{Pref}_{\mathbf{Y}}(G)$ *is a set of conjugate subgroups of subgroups of $G$.*

*Proof.* 1 implies 2. Assume that Assertion 2 does not hold and choose for $G$ a counterexample of least order. If $H$ is any non-trivial normal subgroup of $G$, then $\mathbf{X}/H$ is w-solid set of maximal subgroups of $G/H$ and $(\mathbf{X}/H)^n_{\mathfrak{S}} = \mathbf{Y}/H$. It is clear that $\mathrm{Pref}_{\mathbf{Y}/H}(G/H)$ satisfies ACAP. Hence the minimal choice of $G$ implies that $\mathrm{Pref}_{\mathbf{Y}/H}(G/H)$ is a set of conjugate subgroups of $G/H$.

Let $N$ be a minimal normal subgroup of $G$. If $N$ is $\mathbf{Y}$-Frattini, then $N$ is covered by every $\mathbf{Y}$-prefrattini subgroup of $G$ by Theorem 4.3.6. In that case, the Since the theorem holds in $G/N$, $\mathrm{Pref}_{\mathbf{Y}}(G)$ is a conjugacy class of subgroups of $G$, contrary to supposition. Hence $N$ is $\mathbf{Y}$-supplemented in $G$. In particular $N$ is $\mathfrak{S}$-central in $G$ and therefore $N$ is abelian. Let $M \in \mathbf{Y}$ such that $G = MN$ and $M \cap N = 1$, and let $\mathbf{S}$ be a system of maximal subgroups of $G$ such that $M \in \mathbf{S}$ (Theorem 1.4.7). Denote by $A$ the $(\mathbf{Y} \cap \mathbf{S})$-prefrattini subgroup of $G$. Then $A \leq M$ by Theorem 4.3.6. Since by hypothesis $\mathrm{Pref}_{\mathbf{Y}}(G)$

is not a set of conjugate subgroups of $G$, there exists a system $\mathbf{S}_0$ of maximal subgroups of $G$ such that $A_0 = \mathrm{W}(G, \mathbf{Y} \cap \mathbf{S}_0)$ and $A$ are not conjugate in $G$.

Let $\varphi$ be the isomorphism between $G/N$ and $M$. We have $(\mathbf{X}/N)^n_{\mathfrak{S}} = \mathbf{Y}/N$ and $(\mathbf{Y}/N)^\varphi = (\mathbf{X} \cap M)^n_{\mathfrak{S}} = \mathbf{Y} \cap M$ by Lemma 1.2.23. Denote $C = \mathrm{Core}_G(M) = \mathrm{C}_M(N)$. Suppose that $C \neq 1$. Since the theorem holds in $G/C$, there exists $x \in G$ such that $A_0^x C = AC \leq M$. Without loss of generality we can assume that $x = 1$. In particular $A_0 \leq M$. Then $AN \cap M = A$ and $A_0 N \cap M = A_0$ are $(\mathbf{X} \cap M)^n_{\mathfrak{S}}$-prefrattini subgroups of $M$. The minimal choice of $G$ implies that $A$ and $A_0$ are conjugate in $M$. This contradiction leads to $C = 1$. Since $M$ is $\mathfrak{S}$-normal in $G$, we have $G$ is a primitive soluble group. By Corollary 1.4.18, there exists $g \in G$ such that $\mathbf{S}_0^g = \mathbf{S}$. If $U \in \mathbf{Y} \cap \mathbf{S}$, then $U$ complements the chief factor $\mathrm{Soc}\bigl(G/\mathrm{Core}_G(U)\bigr)$. By Proposition 4.3.12, there exists $V \in \mathbf{Y} \cap \mathbf{S}_0$ such that $V$ complements $\mathrm{Soc}\bigl(G/\mathrm{Core}_G(U)\bigr)$. Since $G/\mathrm{Core}_G(U)$ is a soluble primitive group, $\mathrm{Core}_G(U) = \mathrm{Core}_G(V)$ and $U$ and $V$ are conjugate in $G$ by Theorem 1.1.10. This implies that $\mathbf{Y} \cap \mathbf{S} = \mathbf{Y} \cap \mathbf{S}_0^g = (\mathbf{Y} \cap \mathbf{S}_0)^g$. Applying Theorem 4.3.6, $A = \bigcap\{U : U \in \mathbf{Y} \cap \mathbf{S}\} = \bigcap\{U : U \in (\mathbf{Y} \cap \mathbf{S}_0)^g\} = \bigcap\{V^g : V \in \mathbf{Y} \cap \mathbf{S}_0\} = A_0^g$. This contradiction proves the implication.

2 implies 1. Note that all non-abelian chief factors of $G$ are $\mathbf{Y}$-Frattini. This means that $\mathbf{Y}$-prefrattini subgroups are conjugate CAP-subgroups indeed. □

**Corollary 4.3.14.** *Let* $\mathbf{X}$ *be a w-solid set of maximal subgroups of a soluble group* $G$. *The following statements are equivalent:*

1. $\mathrm{Pref}_{\mathbf{X}}(G)$ *is a set of conjugate subgroups of* $G$, *and*
2. *every* $W \in \mathrm{Pref}_{\mathbf{X}}(G)$ *is a CAP-subgroup of* $G$ *which covers all* $\mathbf{X}$*-Frattini chief factors of* $G$ *and avoids the* $\mathbf{X}$*-complemented ones.*

In general the prefrattini subgroups of a group are not conjugate: in any non-abelian simple group the prefrattini subgroups are the maximal subgroups. We prove next that the solubility of a group is characterised by the conjugacy of its prefrattini subgroups.

**Theorem 4.3.15.** *A group* $G$ *is soluble if and only if the set* $\mathrm{Pref}(G)$ *of all prefrattini subgroups is a conjugacy class of subgroups of* $G$.

*Proof.* If $G$ is a soluble group, then the conjugation of the prefrattini subgroups of $G$ follows directly from Theorem 4.3.6 and Corollary 1.4.18.

Conversely, assume that $G$ is a group such that the set $\mathrm{Pref}(G)$ of all prefrattini subgroups of $G$ is a conjugacy class of subgroups of $G$. We prove that $G$ is soluble by induction on the order of $G$. By Theorem 4.3.6 (2), we have that, for every normal subgroup $N$ of $G$, the set $\mathrm{Pref}(G/N)$ of all prefrattini subgroups of $G/N$ is a conjugacy class of subgroups of $G/N$. Therefore $G/N$ is soluble for each minimal normal subgroup $N$ of $G$ and $G$ is a monolithic primitive group. Suppose that $G$ is not soluble. Then $N = \mathrm{Soc}(G)$ is not abelian. Let $W/N$ be a prefrattini subgroup of $G/N$ associated with an arbitrary system of maximal subgroups $\mathbf{X}^*$ of $G/N$. Let $P_1$ be a non-trivial Sylow $p_1$-subgroup of $N$, for some prime $p_1$; there exists a maximal subgroup $M$ of $G$ such that $\mathrm{N}_G(P_1) \leq M$. Clearly $\mathrm{Core}_G(M) = 1$. The set $\mathbf{X}_1 = \{H \leq G : N \leq H, H/N \in \mathbf{X}^*\} \cup \{M\}$ is a system of maximal subgroups of $G$. Applying Theorem 4.3.6, $W \cap M$ is the prefrattini subgroup of $G$ associated with $\mathbf{X}_1$. Let $P_2$ be a non-trivial Sylow $p_2$-subgroup of $N$, for a prime $p_2$ such that $p_1 \neq p_2$. This is always possible since $N$ is non-abelian. Consider now a maximal subgroup $S$ of $G$ such that $\mathrm{N}_G(P_2) \leq S$ and the system of maximal subgroups $\mathbf{X}_2 = \{H \leq G : N \leq H, H/N \in \mathbf{X}^*\} \cup \{S\}$ of $G$. As above, we have that $W \cap S = \mathrm{W}(G, \mathbf{X}_2)$. Consequently $W \cap M$ and $W \cap S$ are conjugate in $G$. This implies that $W \cap M$ contains a Sylow $p_2$-subgroup of $N$. Since $p_2$ is arbitrary, we have that $W \cap M$ contains a Sylow $p$-subgroup of $N$ for any prime $p$ dividing the order of $N$. This implies that $N \leq M$, which is a contradiction. Hence $G$ is soluble. □

Finally in this section, we touch on the question of the description of the core and the normal closure of subgroups of prefrattini type. For solid sets $\mathbf{X}$ of maximal subgroups, the core of the $\mathbf{X}$-prefrattini subgroups is the $\mathbf{X}$-Frattini subgroup defined in Definition 1.2.18 (1).

**Proposition 4.3.16.** *If $\mathbf{X}$ is a solid set of maximal subgroups of a group $G$ and $W$ is an $\mathbf{X}$-prefrattini subgroup of a group $G$, then* $\mathrm{Core}_G(W) = \Phi_{\mathbf{X}}(G)$.

*Proof.* Let $\mathbf{Y}$ be a system of maximal subgroups of $G$. Consider $W = \mathrm{W}(G, \mathbf{X} \cap \mathbf{Y})$. Since $\mathbf{X}$ is solid, we have that $\Phi_{\mathbf{X}}(G) = \bigcap\{\mathrm{Core}_G(M) : M \in \mathbf{X} \cap \mathbf{Y}\} = \mathrm{Core}_G(W)$.

The classical Frattini subgroup of a group $G$, $\Phi(G)$, is clearly the $\mathrm{Max}^*(G)$-Frattini subgroup of $G$. The $\mathrm{Max}^*(G)^a_{\mathfrak{N}}$-Frattini subgroup is denoted by $\mathrm{L}(G)$ in [Bec64]. H. Bechtell also denotes the $\mathrm{Max}(G)^n_{\mathfrak{N}}$-Frattini subgroup by $\mathrm{R}(G)$. Following his notation, if $\mathfrak{H}$ is a Schunck class and $G$ is a group, we denote

$$\mathrm{L}_{\mathfrak{H}}(G) = \bigcap\{M : M \text{ is } \mathfrak{H}\text{-abnormal monolithic maximal subgroup of } G\}$$

the $\mathrm{Max}^*(G)^a_{\mathfrak{H}}$-Frattini subgroup of $G$, and similarly

$$\mathrm{R}_{\mathfrak{H}}(G) = \bigcap\{M : M \text{ is } \mathfrak{H}\text{-normal monolithic maximal subgroup of } G\}$$

the $\mathrm{Max}^*(G)^n_{\mathfrak{H}}$-Frattini subgroup of $G$.

**Theorem 4.3.17.** *Let $\mathfrak{F}$ be a saturated formation and let $\mathbf{X}$ be a system of maximal subgroups of a group $G$, then*

$$\mathrm{Core}_G\big(\mathrm{W}(G, \mathbf{X}^a_{\mathfrak{F}})\big) = \mathrm{Z}_{\mathfrak{F}}\big(G \bmod \Phi(G)\big) = \mathrm{L}_{\mathfrak{F}}(G).$$

*Proof.* Denote $W = \mathrm{W}(G, \mathbf{X}^a_{\mathfrak{F}})$. Applying Theorem 4.3.6, $\mathrm{L}_{\mathfrak{F}}(G) \leq W$. Hence $\mathrm{L}_{\mathfrak{F}}(G)$ is contained in $\mathrm{Core}_G(W)$. Conversely, if $S$ is an $\mathfrak{F}$-abnormal monolithic maximal subgroup of $G$ in $\mathbf{X}$, then we have $W \leq S$ and $\mathrm{Core}_G(W) \leq$

$\mathrm{Core}_G(S)$. Then $\mathrm{Core}_G(W)$ is contained in every $\mathfrak{F}$-abnormal monolithic maximal subgroup of $G$. Hence $\mathrm{Core}_G(W) \leq \mathrm{L}_{\mathfrak{F}}(G)$.

To prove that $\mathrm{Z}_{\mathfrak{F}}\big(G/\Phi(G)\big) = \mathrm{L}_{\mathfrak{F}}(G)/\Phi(G)$ suppose first that $\Phi(G) = 1$. Since every chief factor of $G$ below $\mathrm{Z}_{\mathfrak{F}}(G)$ is $\mathfrak{F}$-central in $G$, it follows that $\mathrm{Z}_{\mathfrak{F}}(G) \leq \mathrm{L}_{\mathfrak{F}}(G)$. To prove the converse observe that if $\Phi(G) = 1$, then $\mathrm{L}_{\mathfrak{F}}(G) \cap G^{\mathfrak{F}} = 1$. Assume not and let $N$ be a minimal normal subgroup of $G$ such that $N \leq \mathrm{L}_{\mathfrak{F}}(G) \cap G^{\mathfrak{F}}$. Since $\Phi(G) = 1$, it follows that $N$ is supplemented in $G$ by a monolithic $\mathfrak{F}$-normal maximal subgroup $M$. Hence $G^{\mathfrak{F}} \leq M$. This contradiction leads to $\mathrm{L}_{\mathfrak{F}}(G) \cap G^{\mathfrak{F}} = 1$. Consider a chief factor $H/K$ of $G$ such that $H \leq \mathrm{L}_{\mathfrak{F}}(G)$. Since $G^{\mathfrak{F}} \cap \mathrm{L}_{\mathfrak{F}}(G) = 1$, then $HG^{\mathfrak{F}}/KG^{\mathfrak{F}}$ is a chief factor of $G$ which is $G$-isomorphic to $H/K$. This means that $H/K$ is $\mathfrak{F}$-central in $G$. Therefore $\mathrm{L}_{\mathfrak{F}}(G) \leq \mathrm{Z}_{\mathfrak{F}}(G)$ and equality holds.

If $\Phi(G) \neq 1$, then consider the quotient group $G^* = G/\Phi(G)$. Since $\Phi(G^*) = 1$, we obtain the required equality. ☐

**Proposition 4.3.18.** *Let $G$ be a group. If $\mathfrak{F}$ is a saturated formation and* $\mathbf{X}$ *is a system of maximal subgroups of $G$, then*

$$\mathrm{Core}_G\big(\mathrm{W}(G, \mathbf{X}^n_{\mathfrak{F}})\big) = \mathrm{R}_{\mathfrak{F}}(G) = \Phi(G \bmod G^{\mathfrak{F}}).$$

*Proof.* First notice that $G^{\mathfrak{F}}$ is contained in every $\mathfrak{F}$-normal maximal subgroup of $G$ and if $G \in \mathfrak{F}$, then every maximal subgroup of $G$ is $\mathfrak{F}$-normal. Therefore, $\mathrm{R}_{\mathfrak{F}}(G)/G^{\mathfrak{F}} = \mathrm{R}_{\mathfrak{F}}(G/G^{\mathfrak{F}}) = \Phi(G/G^{\mathfrak{F}})$. Since $G^{\mathfrak{F}}$ is contained in $\mathrm{W}(G, \mathbf{X}^n_{\mathfrak{F}})$, we have $\mathrm{W}(G, \mathbf{X}^n_{\mathfrak{F}})/G^{\mathfrak{F}} = \mathrm{W}(G/G^{\mathfrak{F}}, \mathbf{X}^n_{\mathfrak{F}}/G^{\mathfrak{F}})$ by Theorem 4.3.6 (2) and so $\mathrm{Core}_G\big(\mathrm{W}(G, \mathbf{X}^n_{\mathfrak{F}})\big)/G^{\mathfrak{F}} = \Phi(G/G^{\mathfrak{F}})$. ☐

**Definition 4.3.19.** *Let $G$ be a group and suppose that* $\mathbf{X}$ *is a solid set of maximal subgroups of $G$. A normal subgroup $N$ of $G$ is said to be*

1. *an* $\mathbf{X}$-profrattini normal subgroup of $G$ *if either $N = 1$ or every chief factor of $G$ of the form $N/K$ is an* $\mathbf{X}$*-Frattini chief factor of $G$, and*
2. *an* $\mathbf{X}$-parafrattini normal subgroup of $G$ *if either $N = 1$ or every chief factor of $G$ of the form $N/K$ is a non-*$\mathbf{X}$*-complemented chief factor of $G$, that is, no maximal subgroup in* $\mathbf{X}$ *is a complement of $N/K$ in $G$.*

*For* $\mathbf{X} = \mathrm{Max}^*(G)$*, the solid set of all monolithic maximal subgroups of $G$, we say simply profrattini and parafrattini.*

*Examples and remarks 4.3.20.* 1. If $N$ is an $\mathbf{X}$-profrattini normal subgroup of $G$, then $N$ is an $\mathbf{X}$-parafrattini normal subgroup of $G$. The converse does not hold in general. It is enough to consider a non-abelian simple group $S$. It is clear that $S$ is $\mathbf{X}$-parafrattini for all solid sets $\mathbf{X}$ of maximal subgroups of $S$. However $S$ is not $\mathbf{X}$-profrattini.

If $N$ is soluble, $N$ is $\mathbf{X}$-profrattini if and only if $N$ is $\mathbf{X}$-parafrattini.

2. If $\mathfrak{F}$ is a totally nonsaturated formation (see [BBE91]), then $G^{\mathfrak{F}}$ is a profrattini normal subgroup of $G$ for every group $G$.

3. If $\mathbf{X}$ is a solid set of maximal subgroups of a group $G$, a quasinilpotent normal subgroup $N$ of $G$ is $\mathbf{X}$-profrattini if and only if $N \leq \Phi_{\mathbf{X}}(G)$.

*Proof.* Assume that $N$ is a quasinilpotent $\mathbf{X}$-profrattini normal subgroup of $G$ but $N \not\leq \Phi_{\mathbf{X}}(G)$. Then there exists a maximal subgroup $U$ of $G$ such that $K \leq U$, $U \in \mathbf{X}$ and $G = UN$. We have that $G/\operatorname{Core}_G(U) = \big(N\operatorname{Core}_G(U)/\operatorname{Core}_G(U)\big)\big(U/\operatorname{Core}_G(U)\big)$ and $N\operatorname{Core}_G(U)/\operatorname{Core}_G(U)$ is quasinilpotent. Therefore

$$N\operatorname{Core}_G(U)/\operatorname{Core}_G(U) = \mathrm{F}^*\big(G/\operatorname{Core}_G(U)\big) = \operatorname{Soc}\big(G/\operatorname{Core}_G(U)\big).$$

But this contradicts $N$ being $\mathbf{X}$-profrattini. Hence $N \leq \Phi_{\mathbf{X}}(G)$. The converse holds trivially. □

**Theorem 4.3.21.** *Let $G$ be a group and suppose that $\mathbf{X}$ is a solid set of maximal subgroups of $G$.*

1. *If $N$, $M$ are both $\mathbf{X}$-profrattini normal subgroups of $G$, then $NM$ is an $\mathbf{X}$-profrattini normal subgroup of $G$.*
2. *If $N$, $M$ are both $\mathbf{X}$-parafrattini normal subgroups of $G$, then $NM$ is an $\mathbf{X}$-parafrattini normal subgroup of $G$.*

*Proof.* Let $(NM)/K$ be a chief factor of $G$. The normal subgroups $KM$ and $KN$ lie between $K$ and $NM$. If $K = KN = KM$, then $NM \leq K$, which is impossible. Hence, either $NM = NK$ or $NM = MK$. Suppose that $NM = NK$ (the other case is analogous). By Lemma 1.2.16, if $S$ supplements (respectively, complements) $NM/K = NK/K$, then $S$ also supplements (respectively, complements) the chief factor $N/(N \cap K)$. If $N$ is a $\mathbf{X}$-profrattini (respectively, $\mathbf{X}$-parafrattini) normal subgroup of $G$, then $S \notin \mathbf{X}$. Hence $MN$ is also $\mathbf{X}$-profrattini (respectively, $\mathbf{X}$-parafrattini) normal subgroup of $G$. □

*Remark 4.3.22.* Let $G$ be a group and $\mathbf{X}$ be a solid set of maximal subgroups of $G$. Suppose that $N$ is a normal subgroup of $G$ satisfying the property that either $N = 1$ or every chief factor $N/K$ of $G$ is $\mathbf{X}$-complemented in $G$. If $M$ is a normal subgroup of $G$ with the same property, then $MN$ does not have this property in general. For instance, consider $G = A \times B$ where $A = \langle a : a^4 = 1\rangle$, $B = \langle b : b^2 = 1\rangle$, and $\mathbf{X} = \operatorname{Max}^*(G)$. Then $B$ and $D = \langle a^2 b\rangle$ are two complemented minimal normal subgroups of $G$. However $BD/B$ is a Frattini chief factor of $G$.

**Definitions 4.3.23.** *Let $G$ be a group and $\mathbf{X}$ be a solid set of maximal subgroups of $G$.*

1. *The* $\mathbf{X}$-profrattini subgroup of $G$ *is the normal subgroup*

$$\operatorname{Pro}_{\mathbf{X}}(G) = \langle N : N \textit{ is an } \mathbf{X}\textit{-profrattini normal subgroup of } G\rangle.$$

2. *The* $\mathbf{X}$-parafrattini subgroup of $G$ *is the normal subgroup*

$$\operatorname{Para}_{\mathbf{X}}(G) = \langle N : N \textit{ is an } \mathbf{X}\textit{-parafrattini normal subgroup of } G\rangle.$$

*For* $\mathbf{X} = \mathrm{Max}^*(G)$*, the solid set of all monolithic maximal subgroups of* $G$*, we write simply* $\mathrm{Pro}(G)$ *and* $\mathrm{Para}(G)$*.*

It is clear that $\mathrm{Pro}_{\mathbf{X}}(G) \leq \mathrm{Para}_{\mathbf{X}}(G)$. If $\mathbf{X}$ is a solid set of maximal subgroups of $G$ composed of maximal subgroups of type 1, then $\mathrm{Pro}_{\mathbf{X}}(G) = \mathrm{Para}_{\mathbf{X}}(G)$. In particular, the equality holds when $G$ is soluble. There are non-soluble groups such that $\mathrm{Pro}_{\mathbf{X}}(G) = \mathrm{Para}_{\mathbf{X}}(G)$. Consider a prime $p$ and a cyclic group $Z$ of order $p^2$. Let $G = S \wr Z$ be the regular wreath product of $S$ with $Z$, where $S$ is a non-abelian simple group. Then $\mathrm{Pro}(G) = \mathrm{Para}(G)$ is the unique maximal normal subgroup of $G$.

It is clear that for each normal subgroup $\mathrm{Para}_{\mathbf{X}}(G) < N$ (respectively, $\mathrm{Pro}_{\mathbf{X}}(G) < N$) there is at least one $G$-chief factor $N/K$ which is $\mathbf{X}$-supplemented (respectively, $\mathbf{X}$-complemented) in $G$. We can say much more than this.

**Proposition 4.3.24.** *Let* $G$ *be a primitive group of type 2 which splits over* $\mathrm{Soc}(G) = N$ *by a maximal subgroup* $S$ *of* $G$*. Then* $\mathrm{Soc}(S)$ *is non-abelian.*

*Proof.* Let $A$ be an abelian minimal normal subgroup of $S$. Then $A$ is an elementary abelian $p$-group for some prime $p$. Since $S \leq \mathrm{N}_G(A)$, then $\mathrm{N}_G(A) = S$ since proper containment leads to a contradiction that $A$ is normal in $G$, by maximality of $S$ in $G$. Hence $N \cap \mathrm{C}_G(A) = 1$. If $p$ divides $|N|$, a contradiction arises since $A$ would be contained in a Sylow $p$-subgroup $P = [T]A$ of $NA$ with $T = P \cap N$. Hence, $T \cap \mathrm{Z}(P) \neq 1$ and there exists an element $x \in \mathrm{C}_N(A)$ such that $x \neq 1$. This is not possible. Consequently $p$ does not divide $|N|$. Let $q$ be a prime dividing $|N|$. By [Gor80, 6.2.2], there exists a unique $A$-invariant Sylow $q$-subgroup $Q$ of $N$. For any element $s \in S$, $Q^s$ is also $A$-invariant. Consequently, $Q = Q^s$ and $S \leq \mathrm{N}_G(Q)$. Since $N \cap S = 1$, $Q$ is not contained in $S$ and so $G = QS = NS$. This implies $N = Q$, a contradiction. $\square$

**Corollary 4.3.25.** *Denote by* $\mathfrak{K}$ *the class of all groups* $G$ *such that every chief factor of* $G$ *is complemented in* $G$ *by a maximal subgroup of* $G$*. Then* $\mathfrak{K}$ *is composed of soluble groups.*

*Proof.* Suppose that $\mathfrak{K}$ is not contained in $\mathfrak{S}$ and consider a group of minimal order $G \in \mathfrak{K} \setminus \mathfrak{S}$. Then $G \in \mathrm{b}(\mathfrak{S})$ and $G$ is a primitive group of type 2. By hypothesis, $N = \mathrm{Soc}(G)$ is a non-abelian minimal normal subgroup which is complemented in $G$ by a core-free maximal soluble subgroup $S$ of $G$. But $\mathrm{Soc}(S)$ abelian contradicts Proposition 4.3.24. $\square$

**Proposition 4.3.26.** *Let* $G$ *be a group and let* $\mathbf{X}$ *be a solid set of maximal subgroups of* $G$*.*

1. *Denote by* $\mathcal{N}$ *the set of all normal subgroups* $N$ *of* $G$ *satisfying the property that every chief factor of* $G$ *between* $N$ *and* $G$ *is* $\mathbf{X}$*-supplemented in* $G$*. If* $N, M \in \mathcal{N}$*, then* $N \cap M \in \mathcal{N}$*.*

2. *Denote by $\mathcal{K}$ the set of all normal subgroups $N$ of $G$ satisfying the property that every chief factor of $G$ between $N$ and $G$ is $\mathbf{X}$-complemented in $G$. If $N$, $M \in \mathcal{K}$, then $N \cap M \in \mathcal{K}$.*

*Proof.* Consider a chief series of $G$ from $M$ to $M \cap N$.

$$N \cap M \leq \cdots \leq M. \tag{4.6}$$

1. Consider a chief factor $H/K$ of $G$ in (4.6). Then $HN/KN$ is a chief factor of $G$ between $N$ and $G$. Since $N \in \mathcal{N}$, it follows that $HN/KN$ is $\mathbf{X}$-supplemented in $G$ by $S \in \mathbf{X}$, say. This means that $G = S(HN)$ and $KN \leq S \cap NH$. Hence $G = SH$ and $K \leq S \cap H$. Hence $H/K$ is $\mathbf{X}$-supplemented in $G$ by $S$. Therefore Assertion 1 follows from Theorem 1.2.36.

2. Note that by Corollary 4.3.25, the groups $G/N$ and $G/M$ are soluble. Then $G/(N \cap M)$ is soluble. Therefore all chief factors in (4.6) are abelian.

The Assertion 2 now follows by applying the same arguments as those used in the proof of Statement 1 replacing "supplemented" by "complemented." □

**Corollary 4.3.27.** *Let $G$ be a group and $\mathbf{X}$ a solid set of maximal subgroups of $G$. Then*

1. $\mathrm{Pro}_{\mathbf{X}}(G) = \bigcap\{N : N \in \mathcal{N}\} \in \mathcal{N}$ *and every chief factor of $G$ between* $\mathrm{Pro}_{\mathbf{X}}(G)$ *and $G$ is $\mathbf{X}$-supplemented in $G$;*
2. $\mathrm{Para}_{\mathbf{X}}(G) = \bigcap\{N : N \in \mathcal{K}\} \in \mathcal{K}$ *and every chief factor of $G$ between* $\mathrm{Para}_{\mathbf{X}}(G)$ *and $G$ is $\mathbf{X}$-complemented in $G$.*

*Proof.* 1. Denote $K = \bigcap\{N : N \in \mathcal{N}\}$. By Proposition 4.3.26, $K \in \mathcal{N}$. If $K/L$ is an $\mathbf{X}$-supplemented chief factor of $G$, then $L \in \mathcal{N}$ by Theorem 1.2.34 and this is not possible. Therefore every chief factor of $G$ of the form $K/L$ is $\mathbf{X}$-Frattini. Hence $K \leq \mathrm{Pro}_{\mathbf{X}}(G)$. Assume that $K < \mathrm{Pro}_{\mathbf{X}}(G)$. Let $\mathrm{Pro}_{\mathbf{X}}(G)/N$ be a chief factor of $G$ such that $K \leq N$. Then $\mathrm{Pro}_{\mathbf{X}}(G)/N$ should be $\mathbf{X}$-Frattini. This contradicts Proposition 4.3.26.

The proof for 2 is analogous. □

**Corollary 4.3.28.** *If $\mathbf{X}$ is a solid set of maximal subgroups of a group $G$, then $G/\mathrm{Para}_{\mathbf{X}}(G)$ is a soluble group.*

*Proof.* Note that $G/\mathrm{Para}_{\mathbf{X}}(G) \in \mathfrak{K}$. Apply now Corollary 4.3.25. □

It is clear from the above result that $G^{\mathfrak{S}}$, the soluble residual of $G$, is contained in $\mathrm{Para}_{\mathbf{X}}(G)$.

**Corollary 4.3.29.** *Let $\mathbf{X}$ be a solid set of maximal subgroups of a group $G$. Then* $\mathrm{Para}_{\mathbf{X}}(G) = \mathrm{Pro}_{\mathbf{X}}(G)G^{\mathfrak{S}}$.

*Proof.* It is clear that $\mathrm{Pro}_{\mathbf{X}}(G)G^{\mathfrak{S}} \leq \mathrm{Para}_{\mathbf{X}}(G)$. Suppose there exists a chief factor $F = \mathrm{Para}_{\mathbf{X}}(G)/N$ of $G$ with $\mathrm{Pro}_{\mathbf{X}}(G)G^{\mathfrak{S}} \leq N$. By definition of $\mathrm{Para}_{\mathbf{X}}(G)$, the chief factor $F$ is non-$\mathbf{X}$-complemented in $G$. On the other hand, $F$ is abelian and $\mathbf{X}$-supplemented in $G$ because $\mathrm{Pro}_{\mathbf{X}}(G)G^{\mathfrak{S}} \leq N$. Such $F$ cannot exist. Hence $\mathrm{Para}_{\mathbf{X}}(G) = \mathrm{Pro}_{\mathbf{X}}(G)G^{\mathfrak{S}}$. □

**Theorem 4.3.30.** *Let $G$ be a group and let $\mathbf{X}$ be a solid set of maximal subgroups of $G$. Then $N$ is an $\mathbf{X}$-parafrattini normal subgroup of $G$ if and only if $N = \langle N \cap W^g : g \in G \rangle$ for each $W \in \mathrm{Pref}_{\mathbf{X}}(G)$.*

*Proof.* Suppose that $N = \langle N \cap W^g : g \in G \rangle$ for each $W \in \mathrm{Pref}_{\mathbf{X}}(G)$. Let $N/K$ be a chief factor of $G$. Assume that $N/K$ is $\mathbf{X}$-complemented in $G$. Then there exists a maximal subgroup $M \in \mathbf{X}$ of $G$ such that $G = MN$ and $N \cap M = K$. If $W$ is an $\mathbf{X}$-prefrattini subgroup of $G$ such that $W \leq M$, it follows that $W \cap N \leq M \cap N = K$. Hence $N = \langle N \cap W^g : g \in G \rangle \leq K$,contrary to supposition. Therefore $N/K$ is non-$\mathbf{X}$-complemented in $G$. Hence $N$ is $\mathbf{X}$-parafrattini.

Conversely, assume that $N$ is an $\mathbf{X}$-parafrattini normal subgroup of $G$. We may suppose that $N \neq 1$. Let $W \in \mathrm{Pref}_{\mathbf{X}}(G)$ and $L = \langle N \cap W^g : g \in G \rangle$. Suppose $L < N$. Let $N/H$ be a chief factor of $G$ such that $L \leq H$. Since $N$ is $\mathbf{X}$-parafrattini, we have that $N/H$ is non-$\mathbf{X}$-complemented in $G$. Note that $W \cap N \leq L \leq H$. Hence $W$ avoids $N/H$. This implies that $N/H$ is $\mathbf{X}$-supplemented. Let $\mathbf{S}$ be the system of maximal subgroups of $G$ such that $W = \mathrm{W}(G, \mathbf{X} \cap \mathbf{S})$ and $M$ be an $\mathbf{X}$-supplement of $N/H$ in $G$ such that $M \in \mathbf{S}$. Consider a chief series of $G$ passing through $H$ and $N$. Let $S_1, \dots, S_r$ be the $\mathbf{X}$-supplements of the chief factors of $G$ above $N$ such that $S_i \in \mathbf{S}$ $(1 \leq i \leq r)$. Then $WN/N = \bigcap_{i=1}^r S_i/N$ and $WH/H = \bigcap_{i=1}^r (S_i/H) \cap (M/H)$ by Theorem 4.3.6. Therefore $WH = \bigcap_{i=1}^r (S_i \cap M) = WN \cap M = W(M \cap N)$. Since $W \cap N \cap M = W \cap N = W \cap H$, it follows that $|H| = |M \cap N|$ and so $H = M \cap N$. Hence $M$ is an $\mathbf{X}$-complement of $N/H$ in $G$. This contradicts our assumption. Consequently $N = \langle N \cap W^g : g \in G \rangle$ for each $W \in \mathrm{Pref}_{\mathbf{X}}(G)$. □

The following result describes the normal closure of an $\mathbf{X}$-prefrattini subgroup.

**Corollary 4.3.31.** *Let $\mathbf{X}$ be a solid set of maximal subgroups of group $G$. If $W \in \mathrm{Pref}_{\mathbf{X}}(G)$, we have that $\langle W^G \rangle = \langle W^g : g \in G \rangle = \mathrm{Para}_{\mathbf{X}}(G)$.*

*Proof.* Write $P = \mathrm{Para}_{\mathbf{X}}(G)$. Each abelian chief factor of $G$ which is $\mathbf{X}$-complemented in $G$ is avoided by every $\mathbf{X}$-prefrattini subgroup of $G$ by Corollary 4.3.9. Since every chief factor $H/K$ such that $P \leq K < H \leq G$ is abelian and $\mathbf{X}$-complemented in $G$, it follows that $W \leq \mathrm{Para}_{\mathbf{X}}(G)$ for all $W \in \mathrm{Pref}_{\mathbf{X}}(G)$. From Theorem 4.3.30, $\langle W^G \rangle = P$. □

In [Haw67] an elegant theorem of factorisation of prefrattini subgroups of soluble groups is proved. There T. O. Hawkes makes a strong use of the cover-avoidance property. Here we present a similar factorisation in the general non-soluble universe but, obviously, with no use of the cover-avoidance property.

**Theorem 4.3.32.** *Let $G$ be a group and let $\mathfrak{H}$ be a Schunck class of the form $\mathfrak{H} = \mathrm{E}_{\Phi}\,\mathfrak{F}$, for some formation $\mathfrak{F}$. Consider a system of maximal subgroups $\mathbf{X}$ of $G$. Then, if $\mathbf{Y}$ is a w-solid set of maximal subgroups of $G$, we have*

$$\mathrm{W}\big(G, (\mathbf{X} \cap \mathbf{Y})^a_{\mathfrak{H}}\big) = D\,\mathrm{W}(G, \mathbf{X} \cap \mathbf{Y}),$$

*where $D$ is an $\mathfrak{H}$-normaliser of $G$ associated with $\mathbf{X}$.*

*Proof.* We argue by induction on the order of $G$. Obviously we can suppose that $\Phi(G) = 1$. Denote $W^* = \mathrm{W}\big(G, (\mathbf{X} \cap \mathbf{Y})^a_{\mathfrak{H}}\big)$ and $W = \mathrm{W}\big(G, \mathbf{X} \cap \mathbf{Y}\big)$. By Theorem 4.3.6, $W^*$ is contained in $W$.

By Lemma 4.1.11, we know that $D$ is contained in every $\mathfrak{H}$-abnormal maximal subgroup of $G$ in $\mathbf{X}$. Hence $\langle D, W\rangle \leq W^*$. If $G \in \mathfrak{H}$, then $G = D$ and $(\mathbf{X} \cap \mathbf{Y})^a_{\mathfrak{H}} = \emptyset$. Thus, $W^* = G = D$. Therefore we may assume that $G \notin \mathfrak{H}$. Consider an $\mathfrak{H}$-critical maximal subgroup $M$ of $G$ in $\mathbf{X}$ such that $D$ is an $\mathfrak{H}$-normaliser of $M$ associated with a system of maximal subgroups $\mathbf{X}(M)$ such that $\mathbf{X}_M \subseteq \mathbf{X}(M)$. Then $M$ supplements a minimal normal subgroup $N$ of $G$. If $G$ is a simple group, then every maximal subgroup of $G$ is $\mathfrak{H}$-abnormal and then $W^* = W$ and the theorem is true in this case. Hence we can assume that $N$ is a proper subgroup of $G$ and $N \cap M \neq M$.

If $N$ is an $(\mathbf{X} \cap \mathbf{Y})$-Frattini minimal normal subgroup, then $N \leq W \leq W^*$ and the assertion follows by induction. Hence we may suppose that $M \in \mathbf{X} \cap \mathbf{Y}$.

Moreover, arguing as in Lemma 1.2.23, we have that $\mathbf{Y}_M/(M \cap N) = \{(S \cap M)/(M \cap N) : N \leq S \in \mathbf{Y}\}$ is a w-solid set of maximal subgroups of $M/(M \cap N)$. Thus $\mathbf{Y}_M = \{S \cap M : N \leq S \in \mathbf{Y}\}$ is a w-solid set of maximal subgroups of $M$. By induction,

$$\mathrm{W}\big(M, (\mathbf{X}(M) \cap \mathbf{Y}_M)^a_{\mathfrak{H}}\big) = D\,\mathrm{W}\big(M, \mathbf{X}(M) \cap \mathbf{Y}_M\big). \tag{4.7}$$

Consider a chief series $\Gamma$ of $G$ through $N$. Then, by Theorem 4.3.6 (1), we have that $W = M \cap S_{i_1} \cap \dots \cap S_{i_r}$, where the $S_{i_j}$ are $(\mathbf{X} \cap \mathbf{Y})$-supplements of chief factors in $\Gamma$ over $N$. Observe that $\Gamma \cap M$ gives a piece of chief series of $M$ over $N \cap M$. Moreover, again by Theorem 4.3.6 (1),

$$W\big(M, \mathbf{X}(M) \cap \mathbf{Y}_M\big)(N \cap M)/(N \cap M) = \bigcap_{j=1}^{r} (M \cap S_{i_j})/(N \cap M)$$

and then

$$W = W\big(M, \mathbf{X}(M) \cap \mathbf{Y}_M\big)(N \cap M).$$

Similarly,

$$W^* = W\big(M, (\mathbf{X}(M) \cap \mathbf{Y}_M)^a_{\mathfrak{H}}\big)(N \cap M).$$

Hence, by taking the product with $N \cap M$ in both sides of the equality (4.7) we obtain the required factorisation. □

Motivated by [Tom75, Theorem 5.3], we present the following factorisation involving $\mathfrak{H}$-normal maximal subgroups.

**Theorem 4.3.33.** *Let $G$ be a group and let $\mathfrak{F}$ be a saturated formation. Consider a system of maximal subgroups $\mathbf{X}$ of $G$. Then if $\mathbf{Y}$ is a w-solid set of maximal subgroups of $G$ we have that*

$$\mathrm{W}\big(G, (\mathbf{X} \cap \mathbf{Y})^n_{\mathfrak{F}}\big) = \mathrm{W}(G, \mathbf{X} \cap \mathbf{Y})G^{\mathfrak{F}}.$$

*Proof.* Since $G^{\mathfrak{F}}$ is contained in $M$, for every $\mathfrak{F}$-normal maximal subgroup $M$ of $G$, it follows that $G^{\mathfrak{F}} \leq \mathrm{W}\big(G, (\mathbf{X} \cap \mathbf{Y})^{n}_{\mathfrak{F}}\big)$. Since $G/G^{\mathfrak{F}} \in \mathfrak{F}$, it is clear that $(\mathbf{X} \cap \mathbf{Y})^{n}_{\mathfrak{F}}/G^{\mathfrak{F}} = (\mathbf{X} \cap \mathbf{Y})/G^{\mathfrak{F}}$. Therefore $\mathrm{W}\big(G, (\mathbf{X} \cap \mathbf{Y})^{n}_{\mathfrak{F}}\big)/G^{\mathfrak{F}} = \mathrm{W}\big(G/G^{\mathfrak{F}}, (\mathbf{X} \cap \mathbf{Y})^{n}_{\mathfrak{F}}/G^{\mathfrak{F}}\big) = \mathrm{W}(G, \mathbf{X} \cap \mathbf{Y})G^{\mathfrak{F}}/G^{\mathfrak{F}}$ by Theorem 4.3.6 (2). □

**Corollary 4.3.34.** *Let $G$ be a group and let $\mathfrak{F}$ be a saturated formation. Consider a system of maximal subgroups $\mathbf{X}$ of $G$. Then if $\mathbf{Y}$ is a w-solid set of maximal subgroups of $G$ we have that*

$$G = \mathrm{W}\big(G, (\mathbf{X} \cap \mathbf{Y})^{n}_{\mathfrak{F}}\big) W\big(G, (\mathbf{X} \cap \mathbf{Y})^{a}_{\mathfrak{F}}\big).$$

*Proof.* Just notice that if $D$ is an $\mathfrak{F}$-normaliser of $G$, then $G = DG^{\mathfrak{F}}$. Now apply the factorisations presented in Theorem 4.3.32 and Theorem 4.3.33. □

The theory of prefrattini subgroups was continued by X. Soler-Escrivà in her Ph. Doctoral Thesis at the Universidad Pública de Navarra, [SE02]. Her work is another example of the progress produced by using non-arithmetical properties, even in soluble groups. In its place all relations between maximal subgroups of a group and maximal subgroups of its critical subgroups are used thoroughly (see [ESE05]). This leads to the existence and properties of some distributive lattices, generated by three types of pairwise permutable subgroups, namely hypercentrally embedded subgroups (see [CM98]), $\mathfrak{F}$-normalisers, and subgroups of prefrattini type (see [ESE]).