Classes of Finite Groups

Mathematics and Its Applications

Managing Editor:

M. HAZEWINKEL

Centre for Mathematics and Computer Science, Amsterdam, The Netherlands

Volume 584

Classes of Finite Groups

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Springer

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN-10 1-4020-4718-5 (HB)
ISBN-13 978-1-4020-4718-3 (HB)
ISBN-10 1-4020-4719-3 (e-book)
ISBN-13 978-1-4020-4719-0 (e-book)

Published by Springer,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

www.springer.com

Printed on acid-free paper

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Printed in the Netherlands.

For the ones we love:
Fran, Isabel, Eneko

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Preface

. . . [El caballero andante] ha de saber las matemáticas,
porque a cada paso se le ofrecerá tener necesidad dellas; . . .

Miguel de Cervantes Saavedra
Segunda parte del ingenioso caballero
don Quijote de la Mancha, chapter 18

In the sixties and seventies of the last century, in parallel to the tremendous effort to classify the simple groups, a large number a papers created a beautiful and comprehensive view of finite soluble groups. In 1980, when the classification was almost completed, Helmut Wielandt proposed giving priority after the classification to the extension of these brilliant results of the theory of finite soluble groups to the more ambitious universe of all finite groups.

Almost at the same time Klaus Doerk and Trevor Hawkes started to write a volume gathering, ordering, and systematising the rich stuff of soluble groups. This encyclopedic work took more than ten years to accomplish. The publication of *Finite soluble groups* (De Gruyter, 1992) is a crucial milestone in the history of the development of the theory of classes of finite soluble groups. In fact lots of separate pieces of the manuscript, generously distributed by the authors to all interested specialists, had a strong influence on the research of the area even before the publication of the volume.

In the last decade, the Doerk-Hawkes' book has been one of the most powerful tools for undertaking Wielandt's task. The consequence is an impressive flourishing of ideas, methods and results illuminating the structure of finite groups. Furthermore, this process has produced a new arithmetic-free approach to understand some aspects of the soluble case.

We believe that there is already a lot of work published in this area and consequently there is a need for a detailed account of the theory of classes of

groups in the general finite universe. The present book represents an attempt to meet this need.

Our main objective in this book is to present the latest achievements and investigations continuing the Doerk-Hawkes book to enlarge and adapt the methods of the soluble case to classes of finite non-necessarily soluble, according to Wielandt's proposal.

The contents of the book are organised in seven chapters. Chapter 1 begins with primitive groups and crowns. These concepts are central to our approach. It continues with the study of solid sets and systems of maximal subgroups. They are, together with the generalised Jordan-Hölder theorem, the ingredients combined to introduce the prefrattini subgroups in Chapter 4. Chapter 2 contains definitions, and elementary and basic results on classes of groups. Chapter 3 deals with partially saturated formations. A unified extension of the theorems of Gaschütz-Lubeseder-Schmid and Baer on the local character of the saturated and solubly saturated formations is presented there. Normalisers associated with Schunck classes $\mathfrak{H}$ of the form $\mathrm{E}_{\Phi}\,\mathfrak{F}$ for some formation $\mathfrak{F}$ and prefrattini subgroups associated to arbitrary Schunck classes are studied in Chapter 4, whereas Chapter 5 is devoted to presenting an alternative approach to a theory of projectors and covering subgroups in arbitrary finite groups resembling the corresponding theory in finite soluble groups. It is based on Salomon's Dissertation *Strukturerhaltende Untergruppen, Schunckklassen und extreme Klassen endlicher Gruppen*, Johannes Gutenberg-Universität, Mainz, 1987. Subnormal subgroups associated to formations are the main theme of Chapter 6. This concept was introduced by Hawkes in 1969 in the soluble universe and it turns out to be very useful in the study of the structure of finite groups. The last chapter contains some of the recent developments of the theory of Fitting classes, focusing our attention on injective Fitting classes and supersoluble Fitting classes. In particular, a detailed account of Salomon's unpublished example of a non-injective Fitting class is included.

To end this preface, we would like to pay a tribute to the figure of Professor Klaus Doerk, recently deceased. Without Doerk and his research team's collaboration, this book would have never ever come to be.

Acknowledgements

We would like to conclude by expressing our deepest gratitude to Ramón Esteban-Romero for his patient work with our manuscripts. His knowledge on this book's issues as well as his master skills on the use of TEX made him the best helper for the meticulous task of editing this book.

We would also like to thank Homer Bechtell, John Cossey, Arny Feldman, María Jesús Iranzo, Paz Jiménez-Seral, Carmen Lacasa-Esteban, Julio Lafuente, Inmaculada Lizasoain, María del Carmen Pedraza-Aguilera, Tatiana Pedraza and Francisco Pérez-Monasor for their valious collaborations, as well as to the *Ministerio de Educación y Ciencia* (Spanish Government)

and *FEDER* (European Union) for their financial support via the grants MTM2004-08219-C02-01 and MTM2004-08219-C02-02.

To conclude, we must thank Springer for converting this project into a reality and for their continuous patience and help while writing this book.

Torres-Torres, *A. Ballester-Bolinches*
Pamplona, *Luis M. Ezquerro*
January, 2006