Chapter 1
Introduction to Differential Equations

Section 1.1

1. This D.E. is of order two because the highest derivative in the equation is y''.

2. Order is 1.

3. This D.E. is of order one because the highest derivative in the equation is y'. (Note: $(y')^3 \neq y''')$

4. Order is 3.

5 (a). $y = Ce^{t^2}$. Differentiating gives us $y' = Ce^{t^2} \cdot 2t = 2ty$. Therefore, $y' - 2ty = 0$ for any value of C.

5 (b). Substituting into the differential equation yields $y(1) = Ce^{1^2} = Ce$. Using the initial condition, $y(1) = 2 = Ce$. Solving for C, we find $C = 2e^{-1}$.

6. $y''' = 2$. $\quad y'' = 2t + c_1,\ y^1 = t^2 + c_1 t + c_2,\ y = \dfrac{t^3}{3} + c_1\dfrac{t^2}{2} + c_2 t + c_3$.

Order = 3 $\qquad$ 3 arbitrary constants

7 (a). $y = C_1 \sin 2t + C_2 \cos 2t$. Differentiating gives us $y' = 2C_1 \cos 2t - 2C_2 \sin 2t$ and $y'' = -4C_1 \sin 2t - 4C_2 \cos 2t = -4(C_1 \sin 2t + C_2 \cos 2t) = -4y$. Therefore, $y'' + 4y = -4y + 4y = 0$ and thus $y(t) = C_1 \sin 2t + C_2 \cos 2t$ is a solution of the D.E. $y'' + 4y = 0$.

7 (b). $y\left(\frac{\pi}{4}\right) = C_1(1) + C_2(0) = C_1 = 3$ and $y'\left(\frac{\pi}{4}\right) = 2C_1(0) - 2C_2(1) = -2C_2 = -2 \Rightarrow C_2 = 1$.

8. $y = 2e^{-4t}$. $\quad y' + ky = -8e^{-4t} + 2ke^{-4t} = 2(k-4)e^{-4t} = 0$

$\therefore\ k = 4.\quad y(0) = 2 = y_0. \quad \therefore\ k = 4, y_0 = 2.$

9. $y = ct^{-1}$. Differentiating gives us $y' = -ct^{-2}$. Thus $y' + y^2 = -ct^{-2} + c^2 t^{-2} = (c^2 - c)t^{-2} = 0$. Solving this for c, we find that $c^2 - c = c(c-1) = 0$. Therefore, $c = 0,1$.

10. $y = -e^{-t} + \sin t \qquad y' + y = g(t),\ y(0) = y_0. \qquad y' = e^{-t} + \cos t$

$y' + y = e^{-t} + \cos t - e^{-t} + \sin t = g \quad \therefore \quad g(t) = \cos t + \sin t,\ y(0) = -1 = y_0$

11. $y = t^r$. Differentiating gives us $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Thus $t^2y'' - 2ty' + 2y = r(r-1)t^r - 2rt^r + 2t^r = [r(r-1) - 2r + 2]t^r = 0$. Solving this for r, we find that $r(r-1) - 2r + 2 = r^2 - 3r + 2 = (r-2)(r-1) = 0$. Therefore, $r = 1,2$.

12. $y = c_1e^{2t} + c_2e^{-2t}$. $y' = 2c_1e^{2t} - 2c_2e^{-2t}$, $y'' = 4c_1e^{2t} + 4c_2e^{-2t} = 4y$
$\therefore\ y'' - 4y = 0.$

13. From (12), $y = C_1e^{2t} + C_2e^{-2t}$, which we differentiate to get $y' = 2C_1e^{2t} - 2C_2e^{-2t}$. Using the initial conditions, $y(0) = 2$ and $y'(0) = 0$, we have two equations containing C_1 and C_2: $C_1 + C_2 = 2$ and $2C_1 - 2C_2 = 0$. Solving these simultaneous equations gives us $C_1 = C_2 = 1$. Thus, the solution to the initial value problem is $y = e^{2t} + e^{-2t} = 2\cosh(2t)$.

14. $y(0) = c_1 + c_2 = 1,\ 2c_1 - 2c_2 = 2 \ \therefore\ c_1 = 1,\ c_2 = 0 \quad y(t) = e^{2t}$.

15. From (12), $y(t) = C_1e^{2t} + C_2e^{-2t}$. Using the initial condition $y(0) = 3$, we find that $C_1 + C_2 = 3$. From the initial condition $\lim_{t\to\infty} y(t) = 0$ and the equation for $y(t)$ given to us in (12), we can conclude that $C_1 = 0$ (if $C_1 \neq 0$, then $\lim_{t\to\infty} = \pm\infty$). Therefore, $C_2 = 3$ and $y(t) = 3e^{-2t}$.

16. $c_1 + c_2 = 10 \quad \lim_{t\to-\infty} y(t) = 0 \ \Rightarrow\ c_2 = 0 \ \therefore\ c_1 = 10\ \&\ y(t) = 10e^{2t}$.

17. From the graph, we can see that $y' = -1$ and that $y(1) = 1$. Thus $m = y' - 1 = -1 - 1 = -2$ and $y_0 = y(1) = 1$.

18. $y' = mt \ \Rightarrow\ y = \frac{m}{2}t^2 + c$. From graph, $y = -1$ only at $t = 0 \ \therefore\ t_0 = 0$.
Also $c = -1$. From graph $y(1) = -0.5 \ \therefore\ -\frac{1}{2} = \frac{m}{2} - 1 \ \Rightarrow\ m = 1$.

19. We know that this is a freefall problem, so we can begin with the generic equation for freefall situations: $y(t) = -\frac{g}{2}t^2 + v_0t + y_0$. The object is released from rest, so $v_0 = 0$. The impact time corresponds to the time at which $y = 0$, so we are left with the following equation for the impact time t: $0 = -\frac{g}{2}t^2 + y_0$. Solving this for t yields $t = \sqrt{\frac{2y_0}{g}}$. For the velocity at the time of impact: $v = y' = -gt + v_0 = -gt = -\sqrt{2gy_0}$.

20. $x'' = a \quad x' = at + v_0,\ v_0 = x_0 = 0 \Rightarrow x = \frac{at^2}{2} + 0$.
$88 = a(8) \Rightarrow a = 11\ \ ft/\sec^2$. At $t = 8,\ x = 11\left(\frac{64}{2}\right) = 352\,ft$.

21. $a = y'' = 32 - \varepsilon \sin\left(\frac{\pi t}{4}\right)$. Integrating gives us $y' = -32t - \frac{4}{\pi}\varepsilon \cos\left(\frac{\pi t}{4}\right) + C$. The object is dropped from rest, so $y'(0) = 0 = -\frac{4}{\pi}\varepsilon + C$. Solving for C yields $C = \frac{4}{\pi}\varepsilon$, and putting this value back into the equation for y' and simplifying gives us $y' = -32t + \frac{4}{\pi}\varepsilon\left(1 - \cos\left(\frac{\pi t}{4}\right)\right)$.

Integrating again gives us $y = -16t^2 + \frac{4}{\pi}\varepsilon t - \left(\frac{4}{\pi}\right)^2 \varepsilon \sin\left(\frac{\pi t}{4}\right) + C'$. Since the object is dropped from a height of 252 ft. (at $t = 0$), $y(0) = C' = 252$ and thus

$y = -16t^2 + \frac{4}{\pi}\varepsilon t - \left(\frac{4}{\pi}\right)^2 \varepsilon \sin\left(\frac{\pi t}{4}\right) + 252$. Finally, since $y(4) = 0$,

$y(4) = 0 = -16 \cdot 4^2 + \frac{4\varepsilon}{\pi} \cdot 4 - \left(\frac{4}{\pi}\right)^2 \varepsilon \sin(\pi) + 252$. Solving for ε yields $\varepsilon = \frac{\pi}{4}$.

Section 1.2

1 (a). The equation is autonomous because y' depends only on y.

1 (b). Setting $y' = 0$, we have $0 = -y + 1$. Solving this for y yields the equilibrium solution: $y = 1$.

2 (a). not autonomous

2 (b). no equilibrium solutions, isoclines are $t = \text{constant}$.

3 (a). The equation is autonomous because y' depends only on y.

3 (b). Setting $y' = 0$, we have $0 = \sin y$. Solving this for y yields the equilibrium solutions: $y = \pm n\pi$.

4 (a). autonomous

4 (b). $y(y-1) = 0, \quad y = 0,\ 1$.

5 (a). The equation is autonomous because y' does not depend explicitly on t.

5 (b). There are no equilibrium solutions because there are no points at which $y' = 0$.

6 (a). not autonomous

6 (b). $y = 0$ is equilibrium solution, isoclines are hyperbolas.

7 (a). $c = -1$: Setting $c = -1$ gives us $-y + 1 = -1$ which, solved for y, reads $y = 2$. This is the isocline for $c = -1$.

$c = 0$: Setting $c = 0$ gives us $-y + 1 = 0$ which, solved for y, reads $y = 1$. This is the isocline for $c = 0$.

$c = 1$: Setting $c = 1$ gives us $-y + 1 = 1$ which, solved for y, reads $y = 0$, the isocline for $c = 1$.

8 (a). $-y+t=-1 \Rightarrow y=t+1$

$-y+t=0 \Rightarrow y=t$

$-y+t=1 \Rightarrow y=t-1$

9 (a). $c=-1$: Setting $c=-1$ gives us $y^2-t^2=-1$ which can be simplified to $t^2-y^2=1$ (a hyperbola). This is the isocline for $c=-1$.

$c=0$: Setting $c=0$ gives us $y^2-t^2=0$ which can be simplified to $y=\pm t$. This is the isocline for $c=0$.

$c=1$: Setting $c=1$ gives us $y^2-t^2=1$ (a hyperbola). This is the isocline for $c=1$.

10. $f(0)=f(2)=0 \quad y'=y(2-y)$

$y'>0$ for $0<y<2$, $y'<0$ for $-\infty<y<0$ and $2<y<\infty$.

11. One example that would fit these criteria is $y'=-(y-1)^2$. For this autonomous D.E., $y'=0$ at $y=1$ and $y'<0$ for $-\infty<y<1$ and $1<y<\infty$.

12. $y'=1$.

13. One example that would fit these criteria is $y'=\sin(2\pi y)$. For this autonomous D.E., $y'=0$ at $y=\frac{n}{2}$.

14. c.

15. f.

16. a.

17. b.

18. d.

19. e.