Chapter 2
First Order Linear Differential Equations

Section 2.1

1. This equation is linear because it can be written in the form $y' + p(t)y = g(t)$. It is nonhomogeneous because when it is put in this form, $g(t) \neq 0$.

2. nonlinear

3. This equation is nonlinear because it cannot be written in the form $y' + p(t)y = g(t)$.

4. nonlinear

5. This equation is nonlinear because it cannot be written in the form $y' + p(t)y = g(t)$.

6. linear, homogeneous

7. This equation is nonlinear because it can be written in the form $y' + p(t)y = g(t)$.

8. nonlinear

9. This equation is linear because it cannot be written in the form $y' + p(t)y = g(t)$. It is nonhomogeneous because when it is put in this form, $g(t) \neq 0$.

10. linear, homogeneous

11 (a). Theorem 2.1 guarantees a unique solution for the interval $(-\infty,\infty)$, since $\frac{t}{t^2+1}$ and $\sin(t)$ are both continuous for all t and -2 is on this interval.

11 (b). Theorem 2.1 guarantees a unique solution for the interval $(-\infty,\infty)$, since $\frac{t}{t^2+1}$ and $\sin(t)$ are both continuous for all t and 0 is on this interval.

11 (c). Theorem 2.1 guarantees a unique solution for the interval $(-\infty,\infty)$, since $\frac{t}{t^2+1}$ and $\sin(t)$ are both continuous for all t and π is on this interval.

12 (a). $2 < t < \infty$

12 (b). $-2 < t < 2$

12 (c). $-2 < t < 2$

12 (d). $-\infty < t < -2$

13 (a). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(3,\infty)$, the largest interval that includes $t = 5$.

13 (b). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(-2,2)$, the largest interval that includes $t = -\frac{3}{2}$.

13 (c). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(-2,2)$, the largest interval that includes $t = 0$.

13 (d). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(-\infty,-2)$, the largest interval that includes $t = -5$.

13 (e). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(-2,2)$, the largest interval that includes $t = \frac{3}{2}$.

14. $\dfrac{\ln\left|t + t^{-1}\right|}{t-2} = \dfrac{\ln\left|\frac{t^2+1}{t}\right|}{t-2}$ undefined at $t = 0,2$.

14 (a). $2 < t < \infty$.

14 (b). $0 < t < 2$.

14 (c). $-\infty < t < 0$.

14 (d). $-\infty < t < 0$.

15. $y(t) = 3e^{t^2}$. Differentiating gives us $y' = 3e^{t^2}(2t) = 2ty$. Substituting these values into the given equation yields $2ty + p(t)y = 0$. Solving this for $p(t)$, we find that $p(t) = -2t$. Putting $t = 0$ into the equation for y gives us $y_0 = 3$.

16(a). $y = Ct^r \qquad y' = Crt^{r-1} \qquad 2ty' - 6y = 0$

$\therefore\ 2Crt^r - 6Ct^r = (2r-6)Ct^r = 0 \ \Rightarrow\ (2r-6)y = 0 \ \Rightarrow\ 2r - 6 = 0 \ \Rightarrow\ r = 3$

$y(-2) = C(-2)^r = 8 \ \Rightarrow\ C \neq 0 \quad \therefore\ C(-2)^3 = 8 \Rightarrow C = -1$

16 (b). $-\infty < t < 0$ since $p(t) = \dfrac{-3}{t}$

16 (c). $y(t) = -t^3,\ -\infty < t < \infty$.

17. $y(t) = 0$ satisfies all of these conditions.

Section 2.2

1 (a). First, we will integrate $p(t) = 3$ to find $P(t) = 3t$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{-3t}$.

1 (b). $y(0) = C = -3$. Therefore, the solution to the initial value problem is $y = -3e^{-3t}$.

2 (a). $y' - \frac{1}{2}y = 0 \qquad (e^{-t/2}y)' = 0,\ y = Ce^{t/2}$.

2 (b). $y(-1) = Ce^{-1/2} = 2,\ C = 2e^{1/2} \qquad y(t) = 2e^{(t+1)/2}$

3 (a). We can rewrite this equation into the conventional form: $y' - 2ty = 0$. Then we will integrate $p(t) = -2t$ to find $P(t) = -t^2$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{t^2}$.

3 (b). $y(1) = Ce = 3$. Solving for C yields $C = 3e^{-1}$. Therefore, the solution to the initial value problem is $y(t) = 3e^{-1}e^{t^2} = 3e^{(t^2-1)}$.

4 (a). $ty' - 4y = 0 \ \Rightarrow\ y' - \frac{4}{t}y = 0. \qquad \int -\frac{4}{t}dt = -4\ln|t| = -\ln(t^4) \quad \therefore\ \mu = \frac{1}{t^4}$

$\frac{1}{t^4}y' - \frac{4}{t^5}y = (t^{-4}y)' = 0 \qquad y = Ct^4$.

4 (b). $y(1) = C = 1 \quad \therefore\ y(t) = t^4$.

5 (a). We can rewrite this equation into the conventional form: $y' + \frac{4}{t}y = 0$. Then we will integrate $p(t) = \frac{4}{t}$ to find $P(t) = 4\ln|t| = \ln t^4$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{-\ln t^4} = Ce^{\ln t^{-4}} = Ct^{-4}$.

5 (b). $y(1) = C = 1$. Therefore, the solution to the initial value problem is $y(t) = t^{-4}$.

6 (a). $\mu = \exp(t - \cos t) \quad \therefore\ y(t) = Ce^{-(t-\cos t)}$.

6 (b). $y\left(\frac{\pi}{2}\right) = Ce^{-\pi/2} = 1 \qquad C = e^{\pi/2} \qquad y = e^{\pi/2}e^{-(t-\cos t)} = e^{\pi/2 - t + \cos t}$.

7 (a). First, we will integrate $p(t) = -2\cos(2t)$ to find $P(t) = -\sin(2t)$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{\sin(2t)}$.

7 (b). $y(\pi) = C = -2$. Therefore, the solution to the initial value problem is $y(t) = -2e^{\sin(2t)}$.

8 (a). $((t^2+1)y)' = 0 \qquad y = \frac{C}{t^2+1}$.

8 (b). $y(0) = C = 3 \quad \therefore \quad y(t) = \dfrac{3}{t^2+1}$.

9 (a). We can rewrite this equation into the conventional form: $y' - 3(t^2+1)y = 0$. Then we will integrate $p(t) = -3(t^2+1)$ to find $P(t) = -t^3 - 3t$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{t^3+3t}$.

9 (b). $y(1) = Ce^4 = 4$. Solving for C yields $C = 4e^{-4}$. Therefore, the solution to the initial value problem is $y(t) = 4e^{t^3+3t-4}$.

10 (a). $y' + e^{-t}y = 0 \quad \therefore \quad \int e^{-t}dt = -e^{-t} \qquad (-e^{e^{-t}}y)' = 0 \qquad y = Ce^{e^{-t}}$.

10 (b). $y(0) = Ce^1 = 2 \qquad C = 2e^{-1} \qquad y(t) = 2e^{e^{-t}-1}$.

11 (a). #2

11 (b). #3

11 (c). #1

12. $y(t) = y_0e^{-\alpha t} \qquad 4 = y_0e^{-\alpha},\ 1 = y_0e^{-3\alpha}$ Divide: $4 = e^{2\alpha} \ \Rightarrow\ \alpha = \frac{1}{2}\ln 4 = \ln 2$ and $y_0 = e^{3\alpha} = e^{\frac{3}{2}\ln 4} = e^{\ln(8)} = 8. \quad \therefore \quad y(t) = 8e^{-(\ln 2)t}$.

13. First, we should put the equation into our conventional form: $y' - \dfrac{\alpha}{t}y = 0$. Integrating $p(t) = -\dfrac{\alpha}{t}$ gives us $P(t) = -\alpha\ln|t| = \ln|t^{-\alpha}|$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{-\ln|t^{-\alpha}|} = Ce^{\ln|t^{\alpha}|} = Ct^{\alpha}$. Using the general solution and the point $(2,1)$, we can solve for C in terms of α: $y(2) = 1 = C\cdot 2^{\alpha}$; $C = 2^{-\alpha}$. We can then substitute this value for C into the general solution at the point $(4,4)$: $y(4) = 4 = 2^{-\alpha}\cdot 4^{\alpha} = 4^{-\alpha/2}\cdot 4^{\alpha} = 4^{\alpha/2}$. Setting the exponents equal to each other yields $1 = \dfrac{\alpha}{2};\alpha = 2$. Finally, solving for y_0, $y_0 = y(1) = 2^{-2}\cdot 1^2 = \dfrac{1}{4}$.

14. $z' = 2z,\ z = y+2 \quad \therefore \quad z(0) = -1+2 = 1 \ \Rightarrow\ z = e^{2t} = y+2 \quad \therefore \quad y = -2+e^{2t}$

15. Putting this equation into a form more like #14, we have $y' = -2ty + 6t = -2t(y-3)$. We will then let $z = y-3$ (and $z' = y'$, accordingly). Substituting into our modified original equation yields an equation for $z(t)$: $z' = -2tz$, or put in a more conventional form, $z' + 2tz = 0$. Using the same substitution for the initial condition yields $z(0) = 4 - 3 = 1$. Integrating $p(t) = 2t$ gives us $P(t) = t^2$. The general solution is then $z(t) = Ce^{-t^2}$. Our initial condition requires that $C = 1$,

so the solution for $z(t)$ is $z(t)=e^{-t^2}$. In terms of $y(t)$, this solution reads $y-3=e^{-t^2}$. Solved for $y(t)$, this solution is $y(t)=e^{-t^2}+3$.

16 (a). $\dfrac{dB}{dc}=-kB,\ B(0)=-A^*$

16 (b). $B(c)=-A^*e^{-kc}=A(c)-A^* \quad \therefore \quad A(c)=A^*(1-e^{-kc})$ No. $A(c)\ \uparrow A^*$ as $c\uparrow\infty$

16 (c). $0.95A^*=A^*(1-e^{-kc}) \ \Rightarrow\ -0.05=-e^{-kc} \ \Rightarrow\ -kc=\ln(1/20)=-\ln(20)$

$\therefore\ c_{0.95}=\dfrac{1}{k}\ln(20)$.

17. Solving the equation $y'+cy=0$ with our method yields the general solution $y(t)=y_0e^{-ct}$. Looking at the graph, we can see that $y(0)=2=y_0$ and $y(-0.4)=3=y_0e^{-c(-0.4)}=2e^{0.4c}$. Solving for c gives us $c=\dfrac{5}{2}\ln\left(\dfrac{3}{2}\right)\approx 1.01$.

18. $y'=Ce^{-ct} \qquad y(1)=Ce^{-c}=y_0 \ \Rightarrow\ C=y_0e^{c} \quad \therefore\ y=y_0e^{-c(t-1)}$

$y(1)=y_0=-1 \qquad y(0.3)\approx-\dfrac{1}{2} \quad \therefore\ -\dfrac{1}{2}=-e^{-c(-0.7)}=0.7c\approx\ln\left(\dfrac{1}{2}\right)$

$c\approx-\dfrac{1}{0.7}\ln(2)=-0.990 \quad \therefore\ c=-1$.

19 (a). The general solution to this D.E. is $y(t)=y_0e^{-t}$, which can be rewritten as $\ln(y)=-t+c$. Thus, this D.E. corresponds to graph #2 with $y_0=y(0)=e^{\ln(y(0))}=e^2$.

19 (b). The general solution to this D.E. is $y(t)=y_0e^{t\sin 4t}$, which can be rewritten as $\ln(y)=t\sin 4t+c$. Thus, this D.E. corresponds to graph #1 with $y_0=y(0)=e^{\ln(y(0))}=1$.

19 (c). The general solution to this D.E. is $y(t)=y_0e^{-t^2/2}$, which can be rewritten as $\ln(y)=-\dfrac{t^2}{2}+c$. Thus, this D.E. corresponds to graph #4 with $y_0=y(0)=e^{\ln(y(0))}=e$.

19 (d). The general solution to this D.E. is $y(t)=y_0e^{t-\sin 4t}$, which can be rewritten as $\ln(y)=t-\sin 4t+c$. Thus, this D.E. corresponds to graph #3 with $y_0=y(0)=e^{\ln(y(0))}=1$.

20. $\ln y(t)=\dfrac{3-1}{4-0}t+1=\dfrac{t}{2}+1 \quad \therefore\ p(t)=\dfrac{d}{dt}\ln(y(t))=\dfrac{1}{2} \qquad y_0=e$.

21 (a). Integrating $p(t)=t^n$ gives us $P(t)=\dfrac{t^{n+1}}{n+1}$. Thus the solution to this initial value problem is

$y(t)=y_0e^{-t^{n+1}/n+1}$ which can be rewritten as $\ln y=\ln y_0-\dfrac{t^{n+1}}{n+1}$.

Substituting values from the table gives us the necessary equations to solve for y_0 and n. First, $-\frac{1}{4} = \ln y_0 - \frac{1}{n+1}$ and $-4 = \ln y_0 - \frac{2^{n+1}}{n+1}$ can be combined to solve for n:

$4 - \frac{1}{4} = \frac{15}{4} = \frac{2^{n+1}-1}{n+1}$, so $n = 3$. $-\frac{1}{4} = \ln y_0 - \frac{1}{4}$ by substitution, and therefore $y_0 = 1$.

21 (b). $y(t) = y_0 e^{-t^{n+1}/_{n+1}} = 1 \cdot e^{-t^4/_4} \Rightarrow y(-1) = e^{-\frac{1}{4}}$.

Section 2.3

1. For this D.E., $p(t) = 2$. Integrating gives us $P(t) = 2t$. An integrating factor is, then, $\mu(t) = e^{2t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{2t}y' + 2e^{2t}y = (e^{2t}y)' = e^{2t}$. Integrating both sides yields $e^{2t}y = \frac{1}{2}e^{2t} + C$. Therefore, the general solution is $y(t) = \frac{1}{2} + Ce^{-2t}$.

2. $y' + 2y = e^{-t} \Rightarrow (e^{2t}y)' = e^t \Rightarrow e^{2t}y = e^t + C \Rightarrow y = e^{-t} + Ce^{-2t}$.

3. For this D.E., $p(t) = 2$. Integrating gives us $P(t) = 2t$. An integrating factor is, then, $\mu(t) = e^{2t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{2t}y' + 2e^{2t}y = (e^{2t}y)' = 1$. Integrating both sides yields $e^{2t}y = t + C$. Therefore, the general solution is $y(t) = te^{-2t} + Ce^{-2t}$.

4. $y' + 2ty = t \Rightarrow (e^{t^2}y)' = te^{t^2} \Rightarrow e^{t^2}y = \frac{1}{2}e^{t^2} + C \Rightarrow y = \frac{1}{2} + Ce^{-t^2}$.

5. Putting this equation into the conventional form gives us $y' + \frac{2}{t}y = t$. For this D.E., $p(t) = \frac{2}{t}$. Integrating gives us $P(t) = 2\ln t$. An integrating factor is, then, $\mu(t) = e^{\ln t^2} = t^2$. Multiplying the D.E. by $\mu(t)$, we obtain $t^2y' + 2ty = (t^2y)' = t^3$. Integrating both sides yields $t^2y = \frac{1}{4}t^4 + C$. Therefore, the general solution is $y(t) = \frac{1}{4}t^2 + Ct^{-2}$.

6. $(t^2+4)y' + 2ty = t^2(t^2+4) \Rightarrow y' + \frac{2t}{t^2+4}y = t^2$, $\mu = e^{\ln(t^2+4)} = t^2 + 4$

   $\therefore ((t^2+4)y)' = t^2(t^2+4) = t^4 + 4t^2 \Rightarrow (t^2+4)y = \frac{t^5}{5} + \frac{4t^3}{3} + C \qquad y = \frac{t^5/_5 + 4t^3/_3 + C}{(t^2+4)}$.

7. For this D.E., $p(t) = 1$. Integrating gives us $P(t) = t$. An integrating factor is, then, $\mu(t) = e^t$. Multiplying the D.E. by $\mu(t)$, we obtain $e^ty' + e^ty = (e^ty)' = te^t$. Integrating both sides yields $e^ty = te^t - e^t + C$. Therefore, the general solution is $y(t) = t - 1 + Ce^{-t}$.

8. $y' + 2y = \cos 3t \;\Rightarrow\; (e^{2t}y)' = e^{2t}\cos 3t$

$u = e^{2t} \qquad dv = \cos 3t dt$

$du = 2e^{2t}dt \qquad v = \frac{1}{3}\sin 3t \qquad \int e^{2t}\cos 3t dt = \frac{e^{2t}}{3}\sin 3t - \frac{2}{3}\int e^{2t}\sin 3t dt$

$u = e^{2t} \qquad dv = \sin 3t dt$

$du = 2e^{2t}dt \qquad v = -\frac{1}{3}\cos 3t \qquad \int e^{2t}\sin 3t dt = -\frac{e^{2t}}{3}\cos 3t + \frac{2}{3}\int e^{2t}\cos 3t dt$

$\therefore\; I = \frac{e^{2t}}{3}\sin 3t - \frac{2}{3}\left\{-\frac{e^{2t}}{3}\cos 3t + \frac{2}{3}I\right\} \;\Rightarrow\; I(1+\frac{4}{9}) = \frac{e^{2t}}{3}(\sin 3t + 2\cos 3t)$

$\therefore\; I = \frac{3}{13}e^{2t}(\sin 3t + 2\cos 3t)$

$\therefore\; e^{2t}y = \frac{3}{13}e^{2t}(\sin 3t + 2\cos 3t) + C \;\Rightarrow\; y = \frac{3}{13}(\sin 3t + 2\cos 3t) + Ce^{-2t}$

9. For this D.E., $p(t) = -3$. Integrating gives us $P(t) = -3t$. An integrating factor is, then, $\mu(t) = e^{-3t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{-3t}y' - 3e^{-3t}y = (e^{-3t}y)' = 6e^{-3t}$. Integrating both sides yields $e^{-3t}y = -2e^{-3t} + C$. Solving for y gives us $y = -2 + Ce^{3t}$, and with our initial condition, $y(0) = 1 = -2 + C$. Solving for C yields $C = 3$, and thus our final solution is $y = -2 + 3e^{3t}$.

10. $y' - 2y = e^{3t},\; y(0) = 3. \quad (e^{-2t}y)' = e^{t} \;\Rightarrow\; e^{-2t}y = e^{t} + C \;\Rightarrow\; y = e^{3t} + Ce^{2t}$

$y(0) = 1 + C = 3 \;\Rightarrow\; C = 2,\; y = e^{3t} + 2e^{2t}$.

11. Putting this D.E. in the conventional form, we have $y' + \frac{3}{2}y = \frac{1}{2}e^{t}$. For this D.E., $p(t) = \frac{3}{2}$. Integrating gives us $P(t) = \frac{3}{2}t$. An integrating factor is, then, $\mu(t) = e^{\frac{3}{2}t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{\frac{3}{2}t}y' + \frac{3}{2}e^{\frac{3}{2}t}y = (e^{\frac{3}{2}t}y)' = \frac{1}{2}e^{\frac{5}{2}t}$. Integrating both sides yields $e^{\frac{3}{2}t}y = \frac{1}{5}e^{\frac{5}{2}t} + C$. Solving for y gives us $y = \frac{1}{5}e^{t} + Ce^{-\frac{3}{2}t}$, and with our initial condition, $y(0) = 0 = \frac{1}{5} + C$. Solving for C yields $C = -\frac{1}{5}$, and thus our final solution is $y = \frac{1}{5}e^{t} - \frac{1}{5}e^{-\frac{3}{2}t}$.

12. $y' + y = 1 + 2e^{-t}\cos(2t),\ y(\pi/2) = 0 \quad \therefore\ (e^t y)' = e^t + 2\cos 2t$

$e^t y = e^t + \sin 2t + C \ \Rightarrow\ y = 1 + e^{-t}\sin 2t + Ce^{-t}$

$y(\pi/2) = 1 + Ce^{-\pi/2} = 0 \ \Rightarrow\ C = -e^{\pi/2};\ y = 1 + e^{-t}\sin 2t - e^{-(t-\pi/2)}$.

13. Putting this D.E. in the conventional form, we have $y' + \frac{\cos(t)}{2}y = -\frac{3}{2}\cos(t)$. For this D.E., $p(t) = \frac{\cos(t)}{2}$. Integrating gives us $P(t) = \frac{\sin(t)}{2}$. An integrating factor is, then, $\mu(t) = e^{\frac{\sin(t)}{2}}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{\frac{\sin(t)}{2}}y' + \frac{\cos(t)}{2}e^{\frac{\sin(t)}{2}}y = (e^{\frac{\sin(t)}{2}}y)' = -\frac{3\cos(t)}{2}e^{\frac{\sin(t)}{2}}$. Integrating both sides yields $e^{\frac{\sin(t)}{2}}y = -3e^{\frac{\sin(t)}{2}} + C$. Solving for y gives us $y = -3 + Ce^{-\frac{\sin(t)}{2}}$, and with our initial condition, $y(0) = -4 = -3 + C$. Solving for C yields $C = -1$, and thus our final solution is $y = -3 - e^{-\frac{\sin(t)}{2}}$.

14. $y' + 2y = e^{-t} + t + 1,\ y(-1) = e,\ (e^{2t}y)' = e^t + te^{2t} + e^{2t}$

$ye^{2t} = e^t + \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + \frac{1}{2}e^{2t} + C \ \Rightarrow\ y = e^{-t} + \frac{t}{2} + \frac{1}{4} + Ce^{-2t}$

$y(-1) = e - \frac{1}{2} + \frac{1}{4} + Ce^2 = e \ \Rightarrow\ C = \frac{1}{4}e^{-2}$

$\therefore\ y = e^{-t} + \frac{t}{2} + \frac{1}{4} + \frac{1}{4}e^{-2(t+1)}$.

15. Putting this D.E. in the conventional form, we have $y' + \frac{3}{t}y = 1 + \frac{1}{t}$. For this D.E., $p(t) = \frac{3}{t}$. Integrating gives us $P(t) = 3\ln(t)$. An integrating factor is, then, $\mu(t) = e^{3\ln(t)} = e^{\ln(t^3)} = t^3$. Multiplying the D.E. by $\mu(t)$, we obtain $t^3y' + 3t^2y = (t^3y)' = t^3 + t^2$. Integrating both sides yields $t^3y = \frac{1}{4}t^4 + \frac{1}{3}t^3 + C$. Solving for y gives us $y = \frac{t}{4} + \frac{1}{3} + Ct^{-3}$, and with our initial condition, $y(-1) = \frac{1}{3} = -\frac{1}{4} + \frac{1}{3} - C$. Solving for C yields $C = -\frac{1}{4}$, and thus our final solution is $y = \frac{t}{4} + \frac{1}{3} - \frac{1}{4}t^{-3}$. The t-interval on which this solution exists is $-\infty < t < 0$.

16. $y' + \frac{4}{t}y = \alpha t,\ \mu = t^4$

$t^4 y' + 4t^3 y = \alpha t^5 = (t^4 y)' \Rightarrow t^4 y = \alpha \frac{t^6}{6} + C \Rightarrow y = \frac{\alpha t^2}{6} + Ct^{-4}$

$y(1) = -\frac{1}{3} = \frac{\alpha}{6} + C \Rightarrow C = -\frac{1}{3} - \frac{\alpha}{6} \equiv 0 \Rightarrow \alpha = -2,\ y = -\frac{t^2}{3}.$

17. Multiplying both sides of the equation by the integrating factor, $\mu(t) = e^{2t}$, we have $e^{2t}y = e^{2t}(Ce^{-2t} + t + 1) = e^{2t}(t+1) + C$. Differentiating gives us $(e^{2t}y)' = e^{2t}(1) + 2e^{2t}(t+1) = e^{2t}(2t+3)$. Therefore, $(e^{2t}y)' = (\mu(t)y)' = \mu(t)\cdot g(t) = e^{2t}(2t+3) \Rightarrow g(t) = 2t+3$ and $\mu(t) = e^{2t} = e^{P(t)} \Rightarrow P(t) = 2t \Rightarrow p(t) = 2$.

18. $2tCe^{t^2} + pCe^{t^2} = 0 \Rightarrow p(t) = -2t$. Substituting, $(Ce^{t^2} + 2)' - 2t(Ce^{t^2} + 2) = -4t \Rightarrow g(t) = -4t$.

19. Multiplying both sides of the equation by the integrating factor, $\mu(t) = t$, we have $ty = t(Ct^{-1} + 1) = t + C$. Differentiating gives us $(ty)' = 1$. Therefore, $(ty)' = (\mu(t)y)' = \mu(t)\cdot g(t) = 1 = (t)(t^{-1}) \Rightarrow g(t) = t^{-1}$ and $\mu(t) = t = e^{P(t)} \Rightarrow P(t) = \ln t \Rightarrow p(t) = \frac{1}{t} = t^{-1}$.

20. $(e^{-t} + t - 1)' + (e^{-t} + t - 1) = t \Rightarrow g(t) = t,\ y_0 = 0$.

21. $y(t) = -2e^{-t} + e^t + \sin t \Rightarrow y_0 = y(0) = -2 + 1 + 0 = -1$.

If $y(t) = -2e^{-t} + e^t + \sin t$, then $y' = 2e^{-t} + e^t + \cos t$.

Substituting in $y' + y = g(t)$, $(2e^{-t} + e^t + \cos t) + (-2e^{-t} + e^t + \sin t) = 2e^t + \cos t + \sin t = g(t)$.

22. $y' + (1 + \cos t)y = 1 + \cos t,\ y(0) = 3,\ \mu = e^{t+\sin t}$.

$(e^{t+\sin t}y)' = (1+\cos t)e^{t+\sin t} = (e^{t+\sin t})' \Rightarrow e^{t+\sin t}y = e^{t+\sin t} + C \Rightarrow y = 1 + Ce^{-(t+\sin t)}$.

$y(0) = 1 + C = 3 \Rightarrow C = 2 \ \therefore y = 1 + 2e^{-(t+\sin t)}$ and $\lim_{t\to\infty} y(t) = 1$.

23. Putting this D.E. in the conventional form, we have $y' + 2y = e^{-t} - 2$. For this D.E., $p(t) = 2$. An integrating factor is, then, $\mu(t) = e^{2t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{2t}y' + 2e^{2t}y = (e^{2t}y)' = e^t - 2e^{2t}$. Integrating both sides yields $e^{2t}y = e^t - e^{2t} + C$. Solving for y gives us $y = e^{-t} - 1 + Ce^{-2t}$, and with our initial condition, $y(0) = -2 = 1 - 1 + C$. Solving for C yields $C = -2$, and thus our final solution is $y = e^{-t} - 1 - 2e^{-2t}$. Therefore, $\lim_{t\to\infty} y(t) = -1$.

24. On [1,2]:

$y'+\frac{1}{t}y=3t,\ y(1)=1$. An integrating factor is $\mu(t)=t$. Multiplying the D.E. by $\mu(t)$, we obtain $(ty)'=3t^2\Rightarrow ty=t^3+C\Rightarrow y=t^2+Ct^{-1},\ y(1)=1+C=1\Rightarrow C=0$. Therefore, the solution for $1\le t\le 2$ is $y=t^2$ and $y(2)=4$.

On [2,3]:

$y'+\frac{1}{t}y=0,\ y(2)=4$. An integrating factor is $\mu(t)=t$. Multiplying the D.E. by $\mu(t)$, we obtain $(ty)'=0\Rightarrow ty=C\Rightarrow y=Ct^{-1},\ y(2)=\frac{C}{2}=4\Rightarrow C=8$. Therefore, the solution for $2\le t\le 3$ is $y=\frac{8}{t}$.

25. On $[0,\pi]$:

$y'+(\sin t)y=\sin t,\ y(0)=3$. An integrating factor is $\mu(t)=e^{-\cos t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{-\cos t}y'+e^{-\cos t}(\sin t)y=(e^{-\cos t}y)'=(\sin t)e^{-\cos t}$. Integrating both sides yields $e^{-\cos t}y=e^{-\cos t}+C$. Solving for y gives us $y=1+Ce^{\cos t}$, and with our initial condition, $y(0)=3=1+Ce\Rightarrow C=2e^{-1}$. Therefore, the solution for $0\le t\le\pi$ is $y=1+2e^{\cos t-1}$ and $y(\pi)=1+2e^{-2}$.

On $[\pi,2\pi]$:

$y'+(\sin t)y=-\sin t,\ y(\pi)=1+2e^{-2}$. Multiplying the D.E. by $\mu(t)=e^{-\cos t}$, we obtain $e^{-\cos t}y'+e^{-\cos t}(\sin t)y=(e^{-\cos t}y)'=(-\sin t)e^{-\cos t}$. Integrating both sides yields $e^{-\cos t}y=-e^{-\cos t}+C$. Solving for y gives us $y=-1+Ce^{\cos t}$, and with our initial condition, $y(\pi)=1+2e^{-2}=-1+Ce^{-1}\Rightarrow C=2e^{1}+2e^{-1}$. Therefore, the solution for $\pi\le t\le 2\pi$ is $y=-1+2e^{\cos t+1}+2e^{\cos t-1}$.

26. On [0,1]: $y'=2,\ y(0)=1$.

$y=2t+C,\ y(0)=C=1\Rightarrow C=1$.

Therefore, the solution for $0\le t\le 1$ is $y=2t+1$ and $y(1)=3$.

On [1,2]: $y'+\frac{1}{t}y=2,\ y(1)=3$. An integrating factor is $\mu(t)=t$. Multiplying the D.E. by $\mu(t)$, we obtain $(ty)'=2t\Rightarrow ty=t^2+C\Rightarrow y=t+Ct^{-1},\ y(1)=1+C=3\Rightarrow C=2$. Therefore, the solution for $1\le t\le 2$ is $y=t+\frac{2}{t}$.

27. On [0,1]:

$y' + (2t-1)y = 0,\ y(0) = 3$. An integrating factor is $\mu(t) = e^{t^2-t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{t^2-t}y' + e^{t^2-t}(2t-1)y = (e^{t^2-t}y)' = 0$. Integrating both sides yields $e^{t^2-t}y = C$. Solving for y gives us $y = Ce^{t-t^2}$, and with our initial condition, $y(0) = 3 = C$. Therefore, the solution for $0 \le t \le 1$ is $y = 3e^{t-t^2}$ and $y(1) = 3$.

On [1,3]:

$y' + (0)y = y' = 0,\ y(1) = 3$. Integrating gives us $y = C = 3$. Therefore, the solution for $1 \le t \le 3$ is $y = 3$ and $y(3) = 3$.

On [3,4]:

$y' + \left(-\frac{1}{t}\right)y = 0,\ y(3) = 3$. An integrating factor is $\mu(t) = e^{-\ln t} = \frac{1}{t}$. Multiplying the D.E. by $\mu(t)$, we obtain $\frac{1}{t}y' - \frac{1}{t^2}y = (\frac{1}{t}y)' = 0$. Integrating both sides yields $\frac{1}{t}y = C$. Solving for y gives us $y = Ct$, and with our initial condition, $y(3) = 3 = C(3) \Rightarrow C = 1$. Therefore, the solution for $3 \le t \le 4$ is $y = t$.

28. $y(t) = t\{Si(t) - Si(1) + 3\}$

Section 2.4

1. $P(t) = A_0e^{rt} = 5000e^{.05t}$. Thus, $P(30) = 5000e^{.05\cdot 30} = 22408.45$.

2. $P_2(t) = (1+\frac{r}{2})^{2t}A_0 \qquad P_2(30) = (1.025)^{60}\cdot 5000$

$\therefore\ \ln P_2(30) = 60\ln(1.025) + \ln 5000 = 9.999 \qquad P_2(30) \approx 21999$

3 (a). $P_1(t) = (1+r)^tA_0 = (1.06)^tA_0$. Setting $P_1(t) = 2A_0$ yields $2 = 1.06^t$, and solving for t gives us $t \approx 11.9$ years.

3 (b). $P_2(t) = (1+\frac{r}{2})^{2t}A_0 = (1.03)^{2t}A_0$. Setting $P_2(t) = 2A_0$ yields $2 = 1.03^{2t}$, and solving for t gives us $t \approx 11.72$ years.

3 (c). $P(t) = A_0e^{rt} = A_0e^{.06t}$. Setting $P(t) = 2A_0$ yields $2 = e^{.06t}$, and solving for t gives us $t \approx 11.55$ years.

4. With $r = .05 \qquad P(t) = e^{.05t}A_0 \qquad P(10) = e^{0.5}A_0$

With unknown r, $P(8) = e^{r8}A_0 = e^{0.5}A_0$

$\therefore\ 8r = 0.5 \ \Rightarrow\ r = 1/16 \approx 0.0625 \quad (6.25\%)$

5 (a). $P_B{}' = (0.04 + 0.004t)P_B$; $P_B(0) = A_0$.

5 (b). $P_B = A_0 e^{.04t + .002t^2}$. This can be verified easily through differentiation.

5 (c). For Plan A, $P_A(t) = A_0 e^{.06t}$. To find the time t at which Plan B "catches up" with Plan A, let us set $P_A(t) = P_B(t)$: $A_0 e^{.06t} = A_0 e^{.04t+.002t^2}$. Dividing by A_0 and taking the natural logarithm of both sides yields $.06t = .04t + .002t^2$, and solving for t gives us $t = 0$ (the time of the initial investment) and $t = 10$ years (the time at which Plan B "catches up").

6. After 4 yrs, $P(4) = 1000e^{.05(4)}$, $\quad P(10) = 1000e^{.05(4)+.07(6)} = 1000e^{.2+.42} = 1858.93$

7. We can simplify this problem by considering the two deposits separately and then adding the principals of each deposit together at a time of twelve years. We have, then, $1000e^{12r} + 1000e^{6r} = 4000$. Introducing a new variable $x \equiv e^{6r}$, we have $x^2 + x - 4 = 0$. Solving this with the quadratic formula yields one positive value of x: $x \approx 1.5616 = e^{6r}$. Solving for r yields $r \approx 0.0743$.

8. $11{,}000{,}000 = 10{,}000{,}000e^{5k}$. Solving for k yields $k = \frac{1}{5}\ln\left(\frac{11}{10}\right)$.

$P(30) = 10{,}000{,}000e^{\frac{1}{5}\ln\left(\frac{11}{10}\right)(30)} = 10{,}000{,}000e^{\ln\left(\frac{11}{10}\right)^6} = 17{,}715{,}610$.

9. $2 = e^{kt}$, and thus $t = \dfrac{\ln 2}{k} = 5\dfrac{\ln 2}{\ln\frac{11}{10}} \approx 36.36$ days.

10. $1.3 = e^{2k} \;\Rightarrow\; k = \frac{1}{2}\ln(1.3)$. $\quad 3 = e^{kt} \;\Rightarrow\; t = \dfrac{\ln 3}{k} = \dfrac{2\ln(3)}{\ln(1.3)} \approx 8.375$ wks.

11. $80{,}000 = 100{,}000e^{6k}$. Solving for k yields $k = \frac{1}{6}\ln(.8)$. Using this value for k, we have $(80{,}000 + 50{,}000)e^{\ln(0.8)} = 130{,}000 \cdot 0.8 = 104{,}000$.

12 (a). $P' = kP + M,\; P(0) = P_0 \qquad P' - kP = M,\; (e^{-kt}P)' = Me^{-kt}$

$$e^{-kt}P = -\frac{M}{k}e^{-kt} + C \;\Rightarrow\; P = -\frac{M}{k} + Ce^{kt},\; P_0 = -\frac{M}{k} + C$$

$$\therefore\; P(t) = -\frac{M}{k} + (P_0 + \frac{M}{k})e^{kt}$$

12 (b). $P_0 = -\dfrac{M}{k}$. $\quad P_0$ and P must be nonnegative $\Rightarrow -\dfrac{M}{k} \ge 0$. $\quad$ If net immigration rate $M > 0$, net growth rate $k < 0$ and vice versa.

12 (c). Set $kP + M = 0 \;\Rightarrow\; P = -\dfrac{M}{k}$. $\quad P(t) = P_0 = -\dfrac{M}{k}$ in this case.

13 (a). For Strategy I, we have $M_I = kP_0$. For Strategy II, we have $M_{II} = P_0(e^k - 1)$.

13 (b). The net profit for each strategy would equal $(M)(\text{profit}/\text{fish})$, and so the profit for Strategy I is, then: $\Pr_I = 500{,}000(.3172)(.75) = 118{,}950$, and the profit for Strategy II is: $\Pr_{II} = 500{,}000(e^{.3172} - 1)(0.6) \approx 111{,}983$. Strategy I would be more profitable for the farm.

14 (a). $P_1(1) = -\dfrac{M}{k} + (P_0 + \dfrac{M}{k})e^k,\ P_1(2) = P_1(1)e^k = -\dfrac{M}{k}e^k + (P_0 + \dfrac{M}{k})e^{2k}$

$P_2(1) = P_0 e^k,\ P_2(2) = -\dfrac{M}{k} + (P_0 e^k + \dfrac{M}{k})e^k$

14 (b). $P_1(2) - P_2(2) = -\dfrac{M}{k}e^k + P_0 e^{2k} + \dfrac{M}{k}e^{2k} + \dfrac{M}{k} - P_0 e^{2k} - \dfrac{M}{k}e^k = \dfrac{M}{k}(e^{2k} - 2e^k + 1)$

$= \dfrac{M}{k}(e^k - 1)^2$. Since $M > 0$, $P_1(2) > P_2(2)$ if $k > 0$ and $P_1(2) < P_2(2)$ if $k < 0$.

14 (c). If $k > 0$, introduce the immigrants as early as possible. If $k < 0$, introduce as late as possible.

15 (a). From the general solution of the radioactive decay equation, $Q(t) = Ce^{-kt}$, we can use the data given to find C and k. $Q(1) = Ce^{-k} = 100$ and $Q(4) = Ce^{-4k} = 30$, so combining these equations, we find that $e^{-3k} = \dfrac{3}{10}$ and therefore, $k = \dfrac{1}{3}\ln\left(\dfrac{10}{3}\right) \approx 0.4013$. Using this value of k with the $t = 1$ data, we find that $C = Q_0 = 149.4\,mg$. $C = Q_0$, since the exponential falls off the expression for Q at $t = 0$.

15 (b). $\tau = \dfrac{\ln 2}{k} \approx 1.727$ months.

15 (c). $0.01 = e^{-kt}$. Solving for t, we have $t = -\dfrac{\ln(0.01)}{k} \approx 11.475$ months.

16 (a). $\tau = \dfrac{\ln 2}{k} = 5730 \ \Rightarrow\ k = \dfrac{\ln 2}{5730}.\qquad 0.3 = e^{-kt} \ \Rightarrow\ t = \dfrac{-\ln(0.3)}{k}$

$t = \ln(\tfrac{10}{3}) \cdot \dfrac{\tau}{\ln 2} = \left(\dfrac{\ln(\frac{10}{3})}{\ln 2}\right)\tau \approx 9953$ yr.

16 (b). From (a) $t = \dfrac{\ln(\frac{10}{3})}{\ln 2}\tau \quad \therefore \quad \dfrac{\ln(\frac{10}{3})}{\ln 2}(\tau - 30) \le t \le \dfrac{\ln(\frac{10}{3})}{\ln 2}(\tau + 30)$

or $9901 \le t \le 10005$ yrs.

16 (c). $\dfrac{Q(60{,}000)}{Q(0)} = e^{-60{,}000k} = e^{-60{,}000\left(\frac{\ln 2}{5730}\right)} \approx 2.83\left(10^{-5}\right)$.

17. $Q' = -kQ + M$. Writing this D.E. in the conventional form, we have $Q' + kQ = M$. For this D.E., $p(t) = k$ and $P(t) = kt$, which yields an integrating factor of $\mu(t) = e^{kt}$. Thus, $e^{kt}Q' + ke^{kt}Q = (e^{kt}Q)' = e^{kt}M$. Integrating both sides gives us $e^{kt}Q = e^{kt}\frac{M}{k} + C$. Solving for Q, we have $Q = \frac{M}{k} + Ce^{-kt}$. $Q_0 = \frac{M}{k} + C$, so our equation for Q in terms of Q_0 now reads $Q(t) = \frac{M}{k} + \left(Q_0 - \frac{M}{k}\right)e^{-kt} = 50e^{-kt} + \frac{M}{k}\left(1 - e^{-kt}\right)$. Setting $Q(2) = 100$ and substituting $k = \frac{\ln 2}{\tau} = \frac{\ln 2}{3} \approx 0.231$, we have $100 = 50e^{-2k} + \frac{M}{k}\left(1 - e^{-2k}\right) = 31.5 + \frac{M}{.231}(0.37)$. Solving for M, we find $M = 42.78$ (mg/yr.).

18. $\tau = \frac{\ln 2}{k} = 8$ days. $Q(t) = Q_0 e^{-kt} = Q_0 e^{-\ln 2 \frac{t}{\tau}}$

$30 = Q_0 e^{-\frac{3}{8}\ln 2} \Rightarrow Q_0 = 30e^{\frac{3}{8}\ln 2} \approx 38.9\mu\text{g}$

19. $0.99Q_0 = Q_0 e^{-kt}$. Solving for t in terms of k, we have

$t = \frac{1}{k}\ln\left(\frac{100}{99}\right) = \frac{\tau}{\ln 2}\ln\left(\frac{100}{99}\right) = 4 \cdot 10^9 \cdot 0.0145 \approx 0.058 \cdot 10^9$=58 million years.

20. Contact angle is $180 - 30 + 45 = 195°$ or $\theta_2 - \theta_1 = 3.403$ rad.

$\therefore\ T_2 = e^{0.3(3.403)}(100) \approx 277.6$ lb.

21. The contact angle, $\theta_2 - \theta_1 = 2\pi + 2\pi + \pi = 5\pi$. $T_2 = e^{0.1(\theta_2 - \theta_1)}(100 \cdot 9.8) = e^{0.1 \cdot 5\pi}(980) \approx 4714$ N.

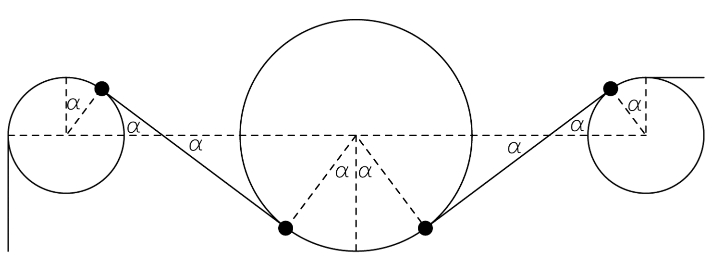
22. Contact $\angle$: $90° + \alpha + \alpha + 90° = 240°$ where $\sin\alpha = \frac{a}{2a} = \frac{1}{2} \Rightarrow \alpha = 30°$

$\theta_2 - \theta_1 = \frac{4}{3}\pi$ for T_3 and $\frac{2\pi}{3}$ for T_2

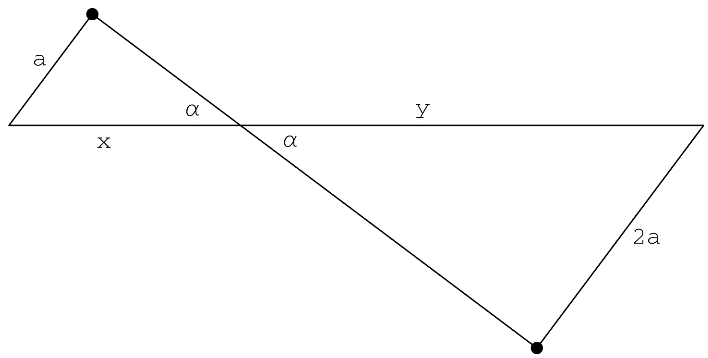
$T_2 = 100e^{.2(2\pi/3)} \approx 152$ lb.

$T_3 = 100e^{.2(4\pi/3)} \approx 231$ lb.

23. The angle, α, is marked at various places on the diagram below. A right angle occurs at each of the dots.



To determine the angle α, part of the diagram is shown here with the radii of the circles marked.



$\sin\alpha = \frac{a}{x}$ and $\sin\alpha = \frac{2a}{y} \Rightarrow y = 2x$.

In the text, we are given that $x + y = 5a$.

Therefore, $x + 2x = 3x = 5a \Rightarrow x = \frac{5a}{3} \therefore \sin\alpha = \frac{a}{x} = \frac{a}{\frac{5a}{3}} = .6 \Rightarrow \alpha \approx .6435 \ \ radians$.

The corresponding contact angles and belt tensions are:

For T_2: $\frac{\pi}{2} + \alpha \approx 2.214 \ \ radians \Rightarrow T_2 = T_1 e^{\mu(angle)} = 100e^{(.2)(2.214)} \approx 155.7\ lb.$

For T_3: $(\frac{\pi}{2} + \alpha) + 2\alpha = \frac{\pi}{2} + 3\alpha \approx 3.501 \ \ radians$

$\Rightarrow T_3 = T_1 e^{\mu(angle)} = 100e^{(.2)(3.501)} \approx 201.4\ lb.$

For T_4: $(\frac{\pi}{2} + 3\alpha) + \alpha = \frac{\pi}{2} + 4\alpha \approx 4.145 \ \ radians$

$\Rightarrow T_4 = T_1 e^{\mu(angle)} = 100e^{(.2)(4.415)} \approx 229.1\ lb.$

24. Contact $\angle$: $2\pi + 2\pi + 2\pi + \frac{\pi}{3} = \frac{19\pi}{3}$

$F = \frac{1}{3} e^{0.4(19\pi/3)} \approx 953.5$ lb.

Section 2.5

1 (a). To begin, $Q(0) = 0$ and $Q' = (0.2)(3) - \frac{Q}{100}(3)$. Putting the second equation in the conventional form, we have $Q' + 0.03Q = 0.6$. Multiplying both sides of this equation by the integrating factor $\mu(t) = e^{0.03t}$ gives us $(e^{0.03t}Q)' = 0.6e^{0.03t}$. Integrating both sides yields $e^{0.03t}Q = 0.6 \cdot \frac{100}{3} e^{0.03t} + C = 20e^{0.03t} + C$. Solving for Q, we have $Q = 20 + Ce^{-0.03t}$.

$Q(0) = 0 = 20 + C$, so $C = -20$. With this value for C, our final equation for Q is $Q = 20(1 - e^{-0.03t})$. Thus, $Q(10) = 20(1 - e^{-0.3}) \approx 5.18\text{lb}$.

1 (b). $\lim_{t \to \infty} Q(t) = 20\text{lb}$ and the limiting concentration is 0.2lb/gal.

2. $V = 100(70)(20) = 140{,}000 m^3$. $\quad Q' = 0 - \dfrac{Q}{v} r \;\Rightarrow\; Q = Q_0 e^{-\frac{r}{v}t}$

$$0.01Q_0 = Q_0 e^{-\frac{r}{v}30} \;\Rightarrow\; -\frac{r}{v} = \frac{1}{30}\ln(0.01) \;\Rightarrow\; r = \frac{v}{30}\ln(100).$$

$$r = \frac{140{,}000}{30}\ln(100) \approx 21{,}491 \ \text{m}^3/\text{min}. \qquad \frac{r}{v} = \frac{1}{30}\ln(100) = 0.1535 \quad (\approx 15.4\%).$$

3 (a). To begin, $Q(0) = 5$ and $Q' = 0.25r - \dfrac{Q}{200}r$. Putting the second equation in the conventional form, we have $Q' + 0.005rQ = 0.25r$. Multiplying both sides of this equation by the integrating factor $\mu(t) = e^{0.005rt}$ gives us $(e^{0.005rt}Q)' = 0.25re^{0.005rt}$. Integrating both sides yields $e^{0.005rt}Q = 0.25(200)e^{0.005rt} + C = 50e^{0.005rt} + C$. Solving for Q, we have $Q = 50 + Ce^{-0.005rt}$. $Q(0) = 5 = 50 + C$, so $C = -45$. With this value for C, our equation for Q now reads $Q = 50 - 45e^{-0.005rt}$. We know that $Q(20) = 30 = 50 - 45e^{-\frac{20}{200}r}$, and solving for r yields

$$r = \ln\left(\frac{(50-30)}{45}\right)(-10) = 10\ln\left(\frac{9}{4}\right) \approx 8.11\text{gal/min}.$$

3 (b). This would be impossible, since $Q(t) < 50\text{lb}$ for all $0 \le t < \infty$.

4 (a). $Q' = (10te^{-t/50})(100) - \dfrac{Q}{5000}(100) \qquad Q(0) = 0$

$$Q' = -\frac{1}{50}Q + 1000te^{-t/50} \;\Rightarrow\; (Qe^{t/50})' = 1000t$$

$$Qe^{t/50} = 500t^2 + C \;\Rightarrow\; Q = 500t^2e^{-t/50} + Ce^{-t/50}. \quad Q(0) = C = 0. \quad \therefore\; Q(t) = 500t^2e^{-t/50} \text{ oz.}$$

4 (b). $Q' = 500\left(2t - \frac{t^2}{50}\right)e^{-t/50} = 0 \;\Rightarrow\; t^2 = 100t \;\Rightarrow\; t = 100$ min.,

$$\frac{Q(100)}{5000} = \frac{500(100)^2}{5000}e^{-2} = 1000e^{-2} \approx 135.3 \ \text{oz}/\text{gal}$$

4 (c). Plot $c(t)$ vs t. Yes.

5 (a). To begin, $Q(0) = 10$, $V(0) = 100$, and $V(t) = 100 + t$. Since the tank has a capacity of 700 gallons, $100 + t = 700$. Solving for t yields $t = 600$ minutes.

5 (b). $Q' = (0.5)(3) - \frac{Q}{100+t}(2)$. Putting this in the conventional form, we have $Q' + \frac{2}{100+t}Q = \frac{3}{2}$.

Multiplying both sides of the equation by the integrating factor $\mu(t) = e^{2\ln(100+t)} = (100+t)^2$ gives us $((100+t)^2 Q)' = \frac{3}{2}(100+t)^2$. Integrating both sides yields $(100+t)^2 Q = \frac{(100+t)^3}{2} + C$, and solving for Q, we have $Q = \frac{100+t}{2} + \frac{C}{(100+t)^2}$. $Q(0) = 10 = 50 + \frac{C}{100^2}$, and solving for C yields $C = -40(100)^2 = -400{,}000$.

Substituting this value of C back into our equation for Q gives us our final equation for Q, $Q(t) = \frac{100+t}{2} - \frac{400{,}000}{(100+t)^2}$. $V(t) = 400$ at $t = 300$, so $Q(300) = \frac{400}{2} - \frac{400{,}000}{(400)^2} = 197.5$ lb. The concentration, then, is $\frac{197.5}{400}$ lb/gal.

5 (c). $Q(600) = \frac{700}{2} - \frac{400{,}000}{(700)^2} \approx 349.2$ lb. The concentration, then, is $\frac{349.2}{700} \approx .4988$ lb/gal.

6 (a). $Q' = \alpha \frac{Q}{500}(15) - \frac{Q}{500}(15)$

6 (b). $Q(180) = 0.01Q_0 \qquad Q' = \frac{-(1-\alpha)}{500}(15)Q \qquad Q = Q_0 e^{-.03(1-\alpha)t}$

$.01 = e^{-.03(1-\alpha)(180)} \;\Rightarrow\; e^{-5.4(1-\alpha)} = .01$

$5.4(1-\alpha) = \ln(100) \;\Rightarrow\; 1-\alpha = 0.8528 \;\Rightarrow\; \alpha = 0.1472$.

7 (a). $Q_A(0) = 1000$, $Q_B(0) = 0$, $Q_A{}' = 0 - 1000\left(\frac{Q_A}{500{,}000}\right)$, and

$Q_B{}' = 1000\left(\frac{Q_A}{500{,}000}\right) - 1000\left(\frac{Q_B}{200{,}000}\right)$.

7 (b). Putting the equation for $Q_A{}'$ into the conventional form, we have $Q_A{}' = -\frac{1}{500}Q_A$. Thus, $Q_A = 1000e^{-\frac{t}{500}}$. Putting the equation for $Q_B{}'$ into the conventional form, we have $Q_B{}' + \frac{1}{200}Q_B = 2e^{-\frac{t}{500}}$. Multiplying both sides by the integrating factor $\mu(t) = e^{\frac{t}{200}}$ yields $(Q_B e^{\frac{t}{200}})' = 2e^{t\left(\frac{1}{200} - \frac{1}{500}\right)} = 2e^{\frac{3t}{1000}}$. Integrating both sides gives us $Q_B e^{\frac{t}{200}} = \frac{2000}{3}e^{\frac{3t}{1000}} + C$, and

solving for Q_B, $Q_B = \frac{2000}{3}e^{-\frac{t}{500}} + Ce^{-\frac{t}{200}}$. $Q_B(0) = 0 = \frac{2000}{3} + C$, so $C = -\frac{2000}{3}$. Substituting this value back into our equation, we have $Q_B = \left(\frac{2000}{3}\right)\left(e^{-\frac{t}{500}} - e^{-\frac{t}{200}}\right)$

7 (c). Setting $Q_B{}' = 0$, we have $0 = \left(\frac{2000}{3}\right)\left(-\frac{1}{500}e^{-\frac{t}{500}} + \frac{1}{200}e^{-\frac{t}{200}}\right)$. Since $e^{-\frac{t}{500}+\frac{t}{200}} = \frac{500}{200}$, $\frac{3}{1000}t = \ln\left(\frac{5}{2}\right)$, and thus $t = \frac{1000}{3}\ln\left(\frac{5}{2}\right) \approx 305.4$ hours.

7 (d). Here, we want to determine t_A such that $Q_A(t_A) = \frac{1}{2}$ lb and t_B such that $Q_B(t) \le 0.2$ lb where $t \le t_B$. This can be solved via plotting: $t_A \approx 3800$ hours and $t_B \approx 4056$ hours. Therefore, $t \approx 4056$ hours.

8 (a). $r_i = r_0 = 3 + \sin t \;\Rightarrow\; V = \text{constant}$.

8 (b). Expect $\lim_{t\to\infty} Q(t) = .5(200) = 100$ lb.

The tank is being "flushed out", albeit in a pulsating manner.

8 (c). $Q' = .5(3+\sin t) - \frac{Q}{200}(3+\sin t),\; Q(0) = 10$

$Q' + \frac{3+\sin t}{200}Q = \frac{1}{2}(3+\sin t) \;\Rightarrow\; (Qe^{(3t-\cos t)/200})' = \frac{1}{2}(3+\sin t)e^{(3t-\cos t)/200}$

$Qe^{(3t-\cos t)/200} = 100e^{3t-\cos t} + C \;\Rightarrow\; Q = 100 + Ce^{-(3t-\cos t)/200}$

$Q(0) = 10 = 100 + Ce^{1/200} \;\Rightarrow\; C = -90e^{-1/200} \Rightarrow\; Q(t) = 100 - 90e^{-(3t-\cos t+1)/200}$.

8(d). $\lim_{t\to\infty} e^{-(3t-\cos t+1)/200} = 0 \;\Rightarrow\; \lim_{t\to\infty} Q(t) = 100$ lb.

9. $f(t) = 3 + \sin t$. Therefore, $\tau = \int_0^t (3+\sin s)ds = [3s - \cos s]_0^t = 3t - \cos t + 1$. Now, $\frac{dQ}{d\tau} = 0.5 - \frac{1}{200}Q$ and $Q(0) = 10$. Putting the first equation into the conventional form, we have $\frac{dQ}{d\tau} + \frac{1}{200}Q = 0.5$, and multiplying both sides by the integrating factor $\mu(t) = e^{\frac{\tau}{200}}$ gives us $\left(e^{\frac{\tau}{200}}Q\right)' = 0.5e^{\frac{\tau}{200}}$. Integrating both sides yields $e^{\frac{\tau}{200}}Q = 100e^{\frac{\tau}{200}} + C$, and solving for Q, $Q = 100 + Ce^{-\frac{\tau}{200}}$. Now, $Q(\tau = 0) = 10 = 100 + C$, and therefore, $C = -90$.

Substituting this back into our equation for Q yields $Q = 100 - 90e^{-\frac{\tau}{200}}$, which in terms of t reads $Q = 100 - 90e^{-\frac{3t - \cos t + 1}{200}}$.

10 (a). No limit since we do not expect concentration to stabilize.

10 (b). $Q' = .2(1 + \sin t)(3) - \dfrac{Q}{200}(3),\ Q(0) = 10$

10 (c). $Q' + \dfrac{3}{200}Q = 0.6(1 + \sin t) \qquad \left(e^{3/200\,t}Q\right)' = 0.6e^{3/200\,t}(1 + \sin t)$.

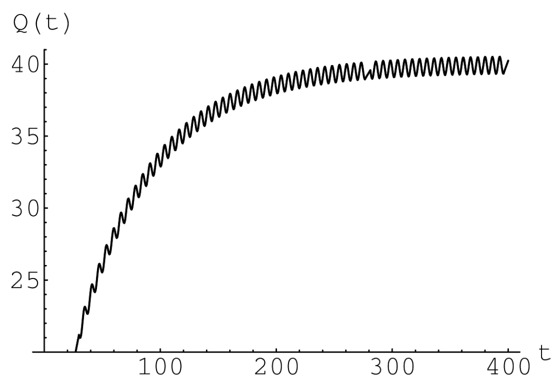
$$\int e^{at}\sin t\,dt = e^{at}\frac{(-\cos t + a\sin t)}{(1 + a^2)} \qquad e^{3/200\,t}Q = 0.6\left\{\frac{200}{3}e^{3/200\,t} + \frac{e^{3/200\,t}(-\cos t + \frac{3}{200}\sin t)}{1 + (3/200)^2}\right\} + C$$

$$Q(t) = 0.6\left\{\frac{200}{3} + \frac{(-\cos t + \frac{3}{200}\sin t)}{1 + (3/200)^2}\right\} + Ce^{-3/200\,t} = 40 + \frac{-0.6\cos t + 0.009\sin t}{1.000225} + Ce^{-3/200\,t}$$

$$Q(0) = 10 = 40 - \frac{0.6}{1.000225} + C \ \Rightarrow\ C = -30 + \frac{0.6}{1.000225}$$

$$Q(t) = 40 - 30e^{-3/200\,t} + \left(\frac{0.6(e^{-3/200\,t} - \cos t) + 0.009\sin t)}{1.000225}\right)$$

10 (d).



11 (a). First, $Q = Q_0e^{-kt}$ for the radioactive material. To find k from the half-life of the material, $\frac{1}{2}Q_0 = Q_0e^{-18k}$. Solving for k, we have $k = \dfrac{\ln 2}{18}$. Thus for the decay of the radioactive material alone, we have $Q(t) = 5e^{-\frac{\ln 2}{18}t}$ with t measured in hours. Now, for the lake, we know that Q varies both with decay and with the water flow. Accordingly, we will begin with the relationship $Q(t + \Delta t) - Q(t) \approx -kQ(t)\Delta t - \dfrac{Q(t)}{V}r\Delta t$.

Using a form of the definition of the derivative and solving for Q', we have

$Q' = -\left(k+\frac{r}{V}\right)Q = -\left(\frac{\ln 2}{18}+\frac{60{,}000}{1{,}200{,}000}\right)Q \approx -0.0885Q$. We know that $Q_0 = Q(0) = 5$ lb, so our final equation for Q reads $Q(t) = 5e^{-0.0885t}$.

11 (b). Here, $(0.0001)(5) = 5e^{-0.0885t}$. Thus, $t = 104.07$ hours.

12. $\theta' = k(S-\theta)$, $S = 72$, $\theta(0) = 350$, $\theta(10) = 290$

$$\theta' + k\theta = kS \;\Rightarrow\; (e^{kt}\theta)' = ke^{kt}S \;\Rightarrow\; e^{kt}\theta = e^{kt}S + C \;\Rightarrow\; \theta = S + Ce^{-kt}$$

$$\theta(0) = \theta_0 = S + C \;\Rightarrow\; C = \theta_0 - S \;\Rightarrow\; \theta = S + (\theta_0 - S)e^{-kt}$$

$$290 = 72 + (350-72)e^{-k(10)} \;\Rightarrow\; 218 = 278e^{-10k},\; 10k = \ln\left(\frac{278}{218}\right)$$

$$k = \frac{1}{10}\ln\left(\frac{278}{218}\right);\; 120 = 72 + (350-72)e^{-kt} \;\Rightarrow\; e^{-kt} = \frac{48}{278}$$

$$t = -\frac{1}{k}\ln\left(\frac{48}{278}\right) = \frac{10\ln\left(\frac{278}{48}\right)}{\ln\left(\frac{278}{218}\right)} = \frac{10(1.756)}{0.243} \approx 72.2 \text{ min.}$$

13. To begin, $\theta = S + (\theta_0 - S)e^{-kt}$. With our substitutions for the time the food was in the oven, this equation reads $120 = 350 + (40-350)e^{-10k}$. Solving for k, we have

$k = -\frac{1}{10}\ln\left(\frac{350-120}{350-40}\right) \approx .02985$. The temperature of the food after 20 minutes in the oven is, then, $\theta(20) = 350 + (40-350)e^{-20k} = 350 - (310)(0.550) \approx 179.5$ degrees. Finally, the food is cooled at room temperature, so $\theta(t) = 110 = 72 + (179.5-72)e^{-0.02985t}$. Solving for t yields

$$t \approx -\frac{1}{0.02985}\ln\left(\frac{110-72}{179.5-72}\right) \approx 34.8 \text{ minutes.}$$

14. $\theta = S + (\theta_0 - S)e^{-kt}$; $170 = 212 + (72-212)e^{-k5}$

$$k = \frac{1}{5}\ln\left(\frac{140}{42}\right) = \frac{1}{5}\ln\left(\frac{10}{3}\right) \text{ min.}^{-1}$$

$$P' = r\left(1 - \frac{Q(t)}{140}\right)P,\quad P = P_0\exp\left\{r\left(t - \frac{1}{140}\int_0^t \theta(s)\,ds\right)\right\}$$

$$\theta(t) = 212 + (72-212)e^{-kt} = 212 - 140e^{-kt}$$

$$\int_0^t \theta\,ds = 212t - \frac{140}{k}\left(1 - e^{-kt}\right)$$

$$\therefore\quad 0.01 = \exp\left\{r\left(10 - \frac{1}{140}\left[2120 - \frac{140}{k}\left(1 - e^{-10k}\right)\right]\right)\right\}$$

$$\frac{1}{100}=\exp\left\{r\left(10-\frac{212}{14}+\frac{1}{k}\left(1-e^{-10k}\right)\right)\right\}$$

$$-\ln(100)=r\left(10-15.143+\frac{5}{\ln\left(\frac{10}{3}\right)}(1-.09)\right)=r(-5.143+3.78)$$

$$-4.6052\approx r(-1.363)\ \Rightarrow\ r\approx 3.379\ \text{min}^{-1}$$

15. For the first cup, $\theta_1=72+(34-72)e^{-kt}$. Thus, with the proper substitutions, $53=72-38e^{-kt_1}$. e^{-kt_1}, then, is equal to $\frac{19}{38}$. For the second cup, $\theta_2=34+(72-34)e^{-kt}$. With the proper substitutions, we have $53=34+38e^{-kt_2}$. e^{-kt_2}, then, is equal to $\frac{19}{38}$. Thus, the two times are equal.

16. $\theta=S+(\theta_0-S)e^{-kt}$ For casserole, $45=72+(40-72)e^{-k2}$

$$-27=-32e^{-k2},\ \ k=\frac{1}{2}\ln\left(\frac{32}{27}\right)$$

$$S(t)=72+228\left(1-e^{-\alpha t}\right)\qquad S(2)=150=72+228\left(1-e^{-\alpha 2}\right)$$

$$1-e^{-2\alpha}=\frac{78}{228}\ \Rightarrow\ e^{-2\alpha}=\frac{150}{228}\ \Rightarrow\ \alpha=\frac{1}{2}\ln\left(\frac{228}{150}\right)$$

$$\theta'=k(S(t)-\theta)\ \Rightarrow\ \theta'+k\theta=kS(t)\ \Rightarrow\ (e^{kt}\theta)'=ke^{kt}S(t)$$

$$=ke^{kt}(72+228-228e^{-\alpha t})=ke^{kt}(300-228e^{-\alpha t})$$

$$e^{kt}\theta=300e^{kt}-\frac{228k}{k-\alpha}e^{(k-\alpha)t}+C\ \Rightarrow\ \theta=300-\frac{228k}{k-\alpha}e^{-\alpha t}+Ce^{-kt}$$

$$\theta(0)=45=300-\frac{228k}{k-\alpha}+C\ \Rightarrow\ C=\frac{228k}{k-\alpha}-255$$

$$\theta(8)=300-\frac{228k}{k-\alpha}e^{-8\alpha}+\left(\frac{228k}{k-\alpha}-255\right)e^{-8k}$$

$$e^{-8\alpha}=\left(e^{-2\alpha}\right)^4=\left(\frac{150}{228}\right)^4\approx .1873,\ \ e^{-8k}=\left(e^{-2k}\right)^4=\left(\frac{27}{32}\right)^4=.5068$$

$$k=\frac{1}{2}\ln\left(\frac{32}{27}\right)\approx 0.08495\qquad \alpha=\frac{1}{2}\ln\left(\frac{228}{150}\right)\approx 0.2094$$

$$k-\alpha\approx -0.1244\qquad \frac{k}{k-\alpha}=-0.682843$$

$$\theta(8)=300-228(-0.682843)(.1873379)+(228(-.682843)-255)(.5068216)$$

$$=300+29.166-208.14565=121.02^\circ$$