Chapter 3
First Order Nonlinear Differential Equations

Section 3.1

1 (a). Solving for y', we have $y' = \frac{1}{3}(1 - 2t\cos y)$. Thus, $f(t,y) = \frac{1}{3}(1 - 2t\cos y)$.

1 (b). $\frac{\partial f}{\partial y} = \frac{1}{3}(0 + 2t\sin y) = \frac{2}{3}t\sin y$. f and $\frac{\partial f}{\partial y}$ are continuous in the entire ty plane.

1 (c). The largest open rectangle is the entire ty plane, since f and $\frac{\partial f}{\partial y}$ are continuous in the entire ty plane.

2 (a). $f(t,y) = \frac{1}{3t}(1 - 2\cos y)$.

2 (b). $\frac{\partial f}{\partial y} = \frac{2}{3t}\sin y$. f and $\frac{\partial f}{\partial y}$ are continuous when $t < 0,\ t > 0$.

2 (c). $R = \{(t,y): t > 0, -\infty < y < \infty\}$.

3 (a). Solving for y', we have $y' = -\frac{2t}{1+y^2}$. Thus, $f(t,y) = -\frac{2t}{1+y^2}$.

3 (b). $\frac{\partial f}{\partial y} = (-2t)(-1)(1+y^2)^{-2}(2y) = \frac{4ty}{(1+y^2)^2}$. f and $\frac{\partial f}{\partial y}$ are continuous in the entire ty plane.

3 (c). The largest open rectangle is the entire ty plane, since f and $\frac{\partial f}{\partial y}$ are continuous in the entire ty plane.

4 (a). $f(t,y) = \frac{-2t}{1+y^3}$.

4 (b). $\frac{\partial f}{\partial y} = \frac{6ty^2}{(1+y^3)^2}$. f and $\frac{\partial f}{\partial y}$ are continuous everywhere in the ty-plane except on the line $y = -1$.

4 (c). $R = \{(t,y): -\infty < t < \infty, y > -1\}$.

5 (a). Solving for y', we have $y' = \tan t - ty^{\frac{1}{3}}$. Thus, $f(t,y) = \tan t - ty^{\frac{1}{3}}$.

5 (b). $\dfrac{\partial f}{\partial y} = -\dfrac{1}{3} t y^{-\frac{2}{3}}$. f and $\dfrac{\partial f}{\partial y}$ are continuous except on the lines $t = \left(n + \dfrac{1}{2}\right)\pi$ (where n is an integer) and $y = 0$.

5 (c). The largest open rectangle is $R = \left\{(t,y): -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}, 0 < y < \infty\right\}$.

6 (a). $f(t,y) = \dfrac{t^2 - e^{-y}}{y^2 - 9}$.

6 (b). $\dfrac{\partial f}{\partial y} = \dfrac{(y^2 + 2y - 9)e^{-y} - 2t^2 y}{(y^2 - 9)^2}$. f and $\dfrac{\partial f}{\partial y}$ are continuous everywhere in the ty-plane except $y = \pm 3$.

6 (c). $R = \{(t,y): -\infty < t < \infty, -3 < y < 3\}$.

7 (a). Solving for y', we have $y' = \dfrac{2 + \tan t}{\cos y}$. Thus, $f(t,y) = \dfrac{2 + \tan t}{\cos y}$.

7 (b). $\dfrac{\partial f}{\partial y} = (2 + \tan t)(-1)(\cos y)^{-2}(-\sin y) = (2 + \tan t)\sec y \tan y$. f and $\dfrac{\partial f}{\partial y}$ are continuous except on the lines $t = \left(n + \dfrac{1}{2}\right)\pi$ (where n is an integer) and $y = \left(m + \dfrac{1}{2}\right)\pi$ (where m is an integer).

7 (c). The largest open rectangle is $R = \left\{(t,y): -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}, -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}\right\}$.

8 (a). $f(t,y) = \dfrac{2 + \tan y}{\cos 2t}$.

8 (b). $\dfrac{\partial f}{\partial y} = \dfrac{\sec^2 y}{\cos 2t}$. f and $\dfrac{\partial f}{\partial y}$ are continuous except where $\tan y$ is not defined and $\cos 2t = 0$, or where $y = \left(n + \frac{1}{2}\right)\pi$, $n = \ldots,-2,-1,0,1,2,\ldots$, and $t = \left(m + \frac{1}{2}\right)\frac{\pi}{2}$, $m = \ldots,-2,-1,0,1,2,\ldots$

8 (c). $R = \left\{(t,y): \dfrac{3\pi}{4} < t < \dfrac{5\pi}{4}, -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}\right\}$.

9. One possible example is $y' = \dfrac{1}{t(t-4)(y+1)(y-2)}$ with $(t_0, y_0) = (2,0)$.

10 (a). $f(t,y) = \dfrac{y^2}{t^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{2y}{t^2}$. f and $\dfrac{\partial f}{\partial y}$ are continuous except where $t = 0$.

$R = \{(t,y): 0 < t < \infty, -\infty < y < \infty\}$.

10 (b). No contradiction. If the hypotheses are not satisfied, "bad things need not happen".

11. $\bar{y}(t) = \dfrac{2}{\sqrt{1 - (t-1)}}$, so $\bar{y}(0) = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

12. $\bar{y}(t) = (4 + (t - t_0))^{\frac{3}{2}}$, so $\bar{y}(0) = (4 - t_0)^{\frac{3}{2}} = 1 \Rightarrow t_0 = 3$.

13 (a). $z_1(t) = y(t+2)$, so $z_1(-5) = y(-3) = 2$.

13 (b). $z_2(t) = y(t-2)$, so $z_2(3) = y(1) = 0$.

Section 3.2

1 (a). Antidifferentiation gives us $\frac{y^2}{2} + \cos t = C$. From the initial condition, we have $\frac{(-2)^2}{2} + \cos\frac{\pi}{2} = C = 2$. Then we have $y^2 = 4 - 2\cos t$, $y = -\sqrt{4 - 2\cos t}$.

1 (b). $-\infty < t < \infty$

2 (a). $y^2 y' = 1$, so $\frac{y^3}{3} - t = C$. From the initial condition, we have $\frac{8}{3} - 1 = \frac{5}{3} = C$. Then we have $y^3 = 3t + 5 \Rightarrow y = (3t+5)^{\frac{1}{3}}$.

2 (b). $-\infty < t < \infty$

3 (a). $(y+1)y' + 1 = 0$, so $\frac{y^2}{2} + y + t = C$. From the initial condition, we have $0 + 0 + 1 = C$. Then we have $\frac{y^2}{2} + y + t = 1 \Rightarrow y^2 + 2y + 2(t-1) = 0$, $y = \frac{-2 \pm \sqrt{4 - 8(t-1)}}{2}$. Since $y(1) = 0$, we only want the plus sign. Finally, $y = \frac{-2 + \sqrt{4 - 8(t-1)}}{2} = -1 + \sqrt{3 - 2t}$.

3 (b). $-\infty < t \le \frac{3}{2}$

4 (a). $y^{-2} y' - 2t = 0$, so $-y^{-1} - t^2 = C$. From the initial condition, we have $1 - 0 = C$. Then we have $-y^{-1} = t^2 + 1 \Rightarrow y = \frac{-1}{1 + t^2}$.

4 (b). $-\infty < t < \infty$

5 (a). $y^{-3} y' - t = 0$, so $\frac{y^{-2}}{-2} - \frac{t^2}{2} = C$. From the initial condition, we have $C = -\frac{1}{8}$. Then we have $y^{-2} + t^2 = \frac{1}{4}$, $y = \frac{1}{\sqrt{1/4 - t^2}} = \frac{2}{\sqrt{1 - 4t^2}}$.

5 (b). $-\frac{1}{2} < t < \frac{1}{2}$

6 (a). $e^{-y}y' + (t - \sin t) = 0$, so $-e^{-y} + \left(\frac{t^2}{2} + \cos t\right) = C$. From the initial condition, we have $-1 + 1 = 0 = C$. Then we have $e^{-y} = \frac{t^2}{2} + \cos t \Rightarrow y = -\ln\left(\frac{t^2}{2} + \cos t\right)$.

6 (b). $-\infty < t < \infty$

7 (a). $\frac{1}{1+y^2}y' - 1 = 0$, so $\tan^{-1} y - t = C$. From the initial condition, we have $C = -\frac{\pi}{2}$. Then we have $\tan^{-1} y = t - \frac{\pi}{2}$, $y = \tan\left(t - \frac{\pi}{2}\right)$.

7 (b). $0 < t < \pi$

8 (a). $(\cos y)y' + t^{-2} = 0$, so $\sin y - t^{-1} = C$. From the initial condition, we have $0 - (-1) = 1 = C$. Then we have $\sin y = 1 + t^{-1} \Rightarrow y = \sin^{-1}\left(1 + t^{-1}\right)$.

8 (b). $-\infty < t < -\frac{1}{2}$

9 (a). $\frac{1}{1-y^2}y' - t = 0$.

By partial fractions, $\frac{1}{1-y^2} = \frac{-1}{y^2-1} = \frac{-1}{(y-1)(y+1)} = \frac{-\frac{1}{2}}{y-1} + \frac{\frac{1}{2}}{y+1}$, and so $\frac{1}{2}\ln\left|\frac{y+1}{y-1}\right| - \frac{t^2}{2} = C$.

From the initial condition, we have $\frac{1}{2}\ln 3 = C$. Then we have

$\ln\left|\frac{y+1}{y-1}\right| - t^2 = \ln 3 \Rightarrow \ln\left|\frac{1}{3}\left(\frac{y+1}{y-1}\right)\right| = t^2$, and solving for y yields $y = \frac{3e^{t^2} - 1}{3e^{t^2} + 1}$.

9 (b). $-\infty < t < \infty$

10 (a). $3y^2y' + 2t - 1 = 0$, so $y^3 + t^2 - t = C$. From the initial condition, we have $-1 + 1 - (-1) = 1 = C$.

Then we have $y^3 = 1 + t - t^2 \Rightarrow y = \left(1 + t - t^2\right)^{\frac{1}{3}}$.

10 (b). $-\infty < t < \infty$

11 (a). $e^y y' - e^t = 0$, so $e^y - e^t = C$. From the initial condition, we have $C = e - 1$. Then we have $e^y - e^t = e - 1$, $y = \ln\left(e^t + e - 1\right)$.

11 (b). $-\infty < t < \infty$

12 (a). $yy' - t = 0$, so $\frac{y^2}{2} - \frac{t^2}{2} = C$. From the initial condition, we have $2 - 0 = C$. Then we have $\frac{y^2}{2} - \frac{t^2}{2} = 2 \Rightarrow y = -\sqrt{4 + t^2}$.

12 (b). $-\infty < t < \infty$

13 (a). $\sec^2 y(y') + e^{-t} = 0$, so $\tan y - e^{-t} = C$. From the initial condition, we have $C = 1 - 1 = 0$. Then we have $\tan y = e^{-t}$, $y = \tan^{-1}(e^{-t})$.

13 (b). $-\infty < t < \infty$

14 (a). $(2y - \sin y)(y') + (t - \sin t) = 0$, so $y^2 + \cos y + \frac{t^2}{2} + \cos t = C$. From the initial condition, we have $0 + 1 + 0 + 1 = 2 = C$. Then we have $y^2 + \cos y = 2 - \frac{t^2}{2} - \cos t$. There is no explicit solution.

15 (a). $(y+1)e^y y' + (t-2) = 0$, so $ye^y + \frac{(t-2)^2}{2} = C$. From the initial condition, we have $C = 2e^2 + \frac{1}{2}$.

Then we have $ye^y = 2e^2 + \frac{1}{2} - \frac{(t-2)^2}{2}$. There is no explicit solution.

16. $y = (4+t)^{-\frac{1}{2}}$, so $y' = -\frac{1}{2}(4+t)^{-\frac{3}{2}} = -\frac{1}{2}y^3 \Rightarrow y' + \frac{1}{2}y^3 = 0$, $y(0) = 4^{-\frac{1}{2}} = \frac{1}{2}$. Therefore, $\alpha = \frac{1}{2}$, $n = 3$, $y_0 = \frac{1}{2}$.

17. $y = \frac{6}{(5+t^4)}$, so $y' = 6(-1)(5+t^4)^{-2}(4t^3) = \frac{-24t^3}{(5+t^4)^2} = -24t^3\left(\frac{y}{6}\right)^2 = -\frac{2}{3}t^3y^2$. Then we have $y' + \frac{2}{3}t^3y^2 = 0$, so $\alpha = \frac{2}{3}$, $n = 3$, $y_0 = \frac{6}{5+1} = 1$.

18. $y^3 + t^2 + \sin y = 4 \Rightarrow 3y^2y' + 2t + (\cos y)y' = 0 \Rightarrow (3y^2 + \cos y)y' + 2t = 0$.

When $t = 2$, $y_0^3 + 4 + \sin y_0 = 4 \Rightarrow y_0^3 + \sin y_0 = 0 \Rightarrow y_0 = 0 \Rightarrow y(2) = 0$.

19. First, $y'e^y + ye^y y' + 2t = \cos t$. Then $(1+y)e^y y' + (2t - \cos t) = 0$. At $t_0 = 0$, we have $y_0 e^{y_0} + 0 = 0$, so $y_0 = 0$, and thus $y(0) = 0$.

20. $y^{-2}y' = 2 \Rightarrow -y^{-1} = 2t + C$, $-y_0^{-1} = C \Rightarrow -y^{-1} = 2t - y_0^{-1} \Rightarrow y^{-1} = y_0^{-1} - 2t \Rightarrow y = \frac{1}{y_0^{-1} - 2t}$.

Require $y_0^{-1} - 2(4) = 0 \Rightarrow y_0 = \frac{1}{8}$.

21 (a). $\left(\frac{K}{S} + 1\right)S' + \alpha = 0$, so $K\ln S + S + \alpha t = C$. From the initial condition, we have $K\ln S_0 + S_0 = C$, so $K\ln S + S = -\alpha t + K\ln S_0 + S_0$.

21 (b). When $t=0$, $S(0)=S_0=1$, so $C=K\cdot 0+1=1$. Then we have $K\ln S+S=-\alpha t+1$. From the other conditions, we have $K\ln\left(\frac{3}{4}\right)+\frac{3}{4}=-\alpha+1\Rightarrow\left(\ln\frac{3}{4}\right)K+\alpha=\frac{1}{4}$ and $K\ln\left(\frac{1}{8}\right)+\frac{1}{8}=-6\alpha+1\Rightarrow\left(\ln\frac{1}{8}\right)K+6\alpha=\frac{7}{8}$. Solving these simultaneous equations yields $K\approx 1.769$ and $\alpha\approx 0.759$.

21 (c). $K\ln\left(\frac{1}{50}\right)+\frac{1}{50}=-\alpha t+1$, so $1.769(-3.912)+0.02=-0.759t+1$. Solving for t yields $t\approx 10.41$.

22. $y'=1+(y+1)^2$. Let $u=y+1$, $u'=1+u^2$, $\frac{1}{(1+u^2)}u'=1\Rightarrow\tan^{-1}(u)=t+C$.

Then, $y(0)=0\Rightarrow u(0)=1$, $\frac{\pi}{4}=0+C\Rightarrow\tan^{-1}(u)=t+\frac{\pi}{4}\Rightarrow u=y+1=\tan\left(t+\frac{\pi}{4}\right)$.

Therefore, $y=\tan\left(t+\frac{\pi}{4}\right)-1$, $-\frac{3\pi}{4}<t<\frac{\pi}{4}$.

23. $y'=t\left((y+2)^2+1\right)$. Letting $u=y+2$, we have $u'=t\left(u^2+1\right)$, so $\frac{1}{u^2+1}u'=t$. Then $\tan^{-1}u=\frac{t^2}{2}+C$. From the initial condition, we have $y(0)=-3$ and $u(0)=-1$, so $-\frac{\pi}{4}=0+C$, $C=-\frac{\pi}{4}$, and $\tan^{-1}u=\frac{t^2}{2}-\frac{\pi}{4}$. In terms of y, this reads $y=-2+\tan\left(\frac{t^2}{2}-\frac{\pi}{4}\right)$.

Setting $-\frac{\pi}{2}<\frac{t^2}{2}-\frac{\pi}{4}<\frac{\pi}{2}$ and simplifying, we have

$-\frac{\pi}{2}<t^2<\frac{3\pi}{2}\Rightarrow|t|<\sqrt{\frac{3\pi}{2}}\Rightarrow-\sqrt{\frac{3\pi}{2}}<t<\sqrt{\frac{3\pi}{2}}$.

24. $y'=(y+1)^2\sin t$. $\frac{y'}{(y+1)^2}=\sin t\Rightarrow\frac{-1}{y+1}=-\cos t+C$.

Then, $y(0)=0\Rightarrow-1=-1+C\Rightarrow C=0\Rightarrow\frac{-1}{y+1}=-\cos t$.

Therefore, $y+1=\sec t\Rightarrow y=\sec t-1$.

25. $Q^{-3}Q'+k=0$, so $\frac{Q^{-2}}{-2}+kt=C'$ and $Q^{-2}=2kt-C$. From the implicit initial condition, we have $Q_0^{\ -2}=-C$, so $Q^{-2}=2kt+Q_0^{\ -2}$. Solved for Q, we have $Q(t)=\frac{1}{\sqrt{2kt+Q_0^{\ -2}}}=\frac{Q_0}{\sqrt{1+2kQ_0^{\ 2}t}}$.

Thus $\frac{1}{2}Q_0 = \frac{Q_0}{\sqrt{1+2kQ_0^2\tau}}$, where τ is the half-life of the reactant. Therefore,

$2 = \sqrt{1+2kQ_0^2\tau}$, which, solved for τ, gives $\tau = \frac{3}{2kQ_0^2}$. Thus the half-life depends upon Q_0.

26. $Q' = -kQ^2,\ Q(0) = Q_0;\ Q^{-2}Q' = -k \Rightarrow -Q^{-1} = -kt + C,\ C = -Q_0^{-1}$. Therefore,

$Q^{-1} = kt + Q_0^{-1} \Rightarrow Q = \frac{1}{kt + Q_0^{-1}} = \frac{Q_0}{1+kQ_0 t},\ Q(10) = 0.4Q_0$. Then,

$0.4Q_0 = \frac{Q_0}{1+kQ_0(10)} \Rightarrow 0.4 + 4kQ_0 = 1 \Rightarrow kQ_0 = 0.15$ and $Q = \frac{Q_0}{1+.15t}$.

Set $Q = 0.25Q_0$. Then, $0.25 = \frac{1}{1+.15t} \Rightarrow t = 20$ min.

27 (a). The equation is nonlinear and separable. $\frac{1}{|y|}y' - 1 = 0$.

27 (b). $|y| = \begin{cases} y, & y \ge 0 \\ -y, & y < 0 \end{cases}$. Thus $\int \frac{dy}{|y|} = \begin{cases} \ln y, & y > 0 \\ -\ln y, & y < 0 \end{cases} \Rightarrow y(t) = \begin{cases} y(0)e^t, & y > 0 \\ y(0)e^{-t}, & y < 0 \end{cases}$.

Since $y(0) = 1 > 0$, the solution $y(t) = e^t$ of $y' = |y|,\ y(0) = 1$ will be identical to that of $y' = y,\ y(0) = 1$ as long as $y(t) = e^t \ge 0$. This is true for all t, however, and so the two solution curves agree.

27 (c). If $y(0) = -1 < 0$, then the solution of $y' = |y|,\ y(0) = -1$, is $y(t) = -e^{-t}$, but the solution of $y' = y,\ y(0) = -1$, is $y(t) = -e^t$.

28. $y' = -y^2$ is graph c. $y' = y^3$ is graph a. $y' = y(4-y)$ is graph b.

29. This is a translation three units to the right of graph (a) in problem 28.

30. Yes. $\frac{1}{f(y)}y' - 1 = 0$.

31. $y'\sin y + y(\cos y)y' - 3 = 0 \Rightarrow y' = \frac{3}{\sin y + y\cos y} = f(y)$. The solution of

$y' = f(y)$ is $H(y) = t + C$, where $H(y) = \int \frac{1}{f(y)}dy$. The solution of $y' = t^2 f(y)$ is therefore

$H(y) = \frac{t^3}{3} + C_2$. From the initial condition, we know that $H(0) = 1 + C \Rightarrow C = H(0) - 1$, and

$H(0) = \frac{1}{3} + C_2$. Thus $C_2 = C + \frac{2}{3}$. Now we have $H(y) = t + C$ and $y\sin y - 3t + 3 = 0$, so

$y\sin y = 3(t-1) \Rightarrow \frac{1}{3}y\sin y = t + (-1) \Rightarrow H(y) = \frac{1}{3}y\sin y$, and $C = -1$.

Therefore, $C_2 = C + \frac{2}{3} = -1 + \frac{2}{3} = -\frac{1}{3}$ and the implicit solution of the initial value problem is

$\frac{y \sin y}{3} = \frac{t^3}{3} - \frac{1}{3} \Rightarrow y \sin y - t^3 + 1 = 0$.

Section 3.3

1. $M = 3t^2 - 2,\ N = y,\ M_y = N_t = 0$, so the equation is exact. $\frac{\partial H}{\partial y} = y \Rightarrow H = \frac{y^2}{2} + g(t)$ and

$\frac{\partial H}{\partial t} = g' = 3t^2 - 2 \Rightarrow g(t) = t^3 - 2t + C \Rightarrow \frac{y^2}{2} + t^3 - 2t = C$.

From the initial condition, we have $\frac{(-2)^2}{2} - 1 - 2(-1) = 2 - 1 + 2 = 3 = C$, and thus

$\frac{y^2}{2} + t^3 - 2t = 3 \Rightarrow y = -\sqrt{6 + 4t - 2t^3}$ (the minus sign can be checked by the initial condition).

2. $M = y + t^3,\ N = t + y^3,\ M_y = N_t = 1$, so the equation is exact.

$\frac{\partial H}{\partial t} = M = y + t^3 \Rightarrow H = yt + \frac{t^4}{4} + h(y)$ and $\frac{\partial H}{\partial y} = t + \frac{dh}{dy} = N = t + y^3 \Rightarrow \frac{dh}{dy} = y^3 \Rightarrow h = \frac{y^4}{4}$.

Therefore, $yt + \frac{t^4}{4} + \frac{y^4}{4} = C,\ y(0) = -2 \Rightarrow C = 4$ and $\frac{y^4}{4} + yt + \frac{t^4}{4} = 4 \Rightarrow y^4 + 4yt + t^4 = 16$.

3. The equation is separable, and therefore it is exact. $\left(y^2+1\right)^{-1} y' = 3t^2 + 1$ gives us

$\tan^{-1} y = t^3 + t + C$, and from the initial condition we have $\frac{\pi}{4} = C$. Thus $y = \tan\left(t^3 + t + \frac{\pi}{4}\right)$.

4. $M = 3t^2 y,\ N = 6t + y^3,\ M_y = 3t^2,\ Nt = 6$. Therefore, the differential equation is not exact.

5. $M = e^{t+y} + 3t^2,\ N = e^{t+y} + 2y,\ M_y = N_t = e^{t+y}$, so the equation is exact.

$\frac{\partial H}{\partial t} = M = e^{t+y} + 3t^2 \Rightarrow H = e^{t+y} + t^3 + h(y)$ and $\frac{\partial H}{\partial y} = e^{t+y} + \frac{dh}{dy} = N = e^{t+y} + 2y \Rightarrow \frac{dh}{dy} = 2y$,

and so $h = y^2 + C$. From the initial condition, we have $1 + 0 + 0 = C$, and thus

$e^{t+y} + t^3 + y^2 = 1$.

6. $M = y\cos(ty) + 1,\ N = t\cos(ty) + 2ye^{y^2},\ M_y = N_t = \cos(ty) - ty\sin(ty)$, so the equation is

exact. $\frac{\partial H}{\partial t} = M = y\cos(ty) + 1 \Rightarrow H = \sin(ty) + t + h(y)$

and $\frac{\partial H}{\partial y} = t\cos(ty) + \frac{dh}{dy} = N = t\cos(ty) + 2ye^{y^2} \Rightarrow \frac{dh}{dy} = 2ye^{y^2}$, and so $h = e^{y^2}$. From the initial condition, we have $0 + \pi + 1 = C$, and thus $\sin(ty) + t + e^{y^2} = \pi + 1$.

7. $M_y = \cos(t+y) - y\sin(t+y) + 1,\ N_t = \cos(t+y) - y\sin(t+y) + 1$, so the equation is exact.

$\frac{\partial H}{\partial t} = M = y\cos(t+y) + y + t \Rightarrow H = y\sin(t+y) + yt + \frac{t^2}{2} + h(y)$ and

$\frac{\partial H}{\partial y} = \sin(t+y) + y\cos(t+y) + t + \frac{dh}{dy} = \sin(t+y) + y\cos(t+y) + t + y$. Thus

$\frac{dh}{dy} = y \Rightarrow h = \frac{y^2}{2} + C$, and from the initial condition, we have

$(-1)\sin(1-1) + (-1)(1) + \frac{1}{2} + \frac{1}{2} = 0 = C$, and thus $y\sin(t+y) + yt + \frac{t^2}{2} + \frac{y^2}{2} = 0$.

8. $M = \alpha t^3 y^n,\ N = t^m y^2,\ M_y = n\alpha t^3 y^{n-1},\ N_t = mt^{m-1}y^2$. Therefore,

$m - 1 = 3 \Rightarrow m = 4,\ n - 1 = 2 \Rightarrow n = 3,\ 3\alpha = 4 \Rightarrow \alpha = \frac{4}{3}$.

9. $N = t^2 + y^2\sin t,\ N_t = M_y = 2t + y^2\cos t$. Thus $M = 2ty + \frac{y^3}{3}\cos t + m(t)$.

10. $M = t^2 + y^2\sin t,\ M_y = 2y\sin t = N_t \Rightarrow N = -2y\cos t + n(y)$.

11. $0 + 1 + {y_0}^2 = 5 \Rightarrow y_0 = \pm 2$. $3t^2y + t^3y' + e^t + 2yy' = 0$, so $(t^3 + 2y)y' + (3t^2y + e^t) = 0$ and thus $M = 3t^2y + e^t$ and $N = t^3 + 2y$.

12. $y = -t - \sqrt{4 - t^2} \Rightarrow y(0) = -2 = y_0$. Also, $N_t = a = M_y \Rightarrow$ exact.

Then, $H_y = y + at \Rightarrow H = \frac{y^2}{2} + aty + g(t),\ H_t = at + g' = ay + bt \Rightarrow g' = bt \Rightarrow g = \frac{bt^2}{2}$.

Therefore,

$$H = \frac{y^2}{2} + aty + \frac{bt^2}{2} = C \Rightarrow y^2 + 2aty + (bt^2 - 2C) = 0$$

$$\Rightarrow y = \frac{-2at \pm \sqrt{4a^2t^2 - 4(bt^2 - 2C)}}{2} = -at \pm \sqrt{a^2t^2 - bt^2 + 2C} = -at \pm \sqrt{2C - (b - a^2)t^2}$$

Choose the negative and $a = 1,\ 2C = 4,\ b - a^2 = 1 \Rightarrow b = 2$.

13. $\frac{2y}{y^2+1}y' + \frac{1}{t+1} = 0 \Rightarrow \ln\left(y^2+1\right) + \ln(t+1) = C$. From the initial condition, we have $\ln 2 = C$.

Thus $\left(y^2+1\right)(t+1) = 2$, and solving for y yields $y = \sqrt{\frac{1-t}{1+t}}$.

14. One example: $(y+2t)+(2y+t)y'=0,\ M=y+2t,\ N=2y+t$

$\dfrac{\partial H}{\partial t}=M=y+2t \Rightarrow H=yt+t^2+h(y),\ \dfrac{\partial H}{\partial y}=t+\dfrac{dh}{dy}=2y+t$. Therefore,

$\dfrac{dh}{dy}=2y \Rightarrow h=y^2 \Rightarrow yt+t^2+y^2=C$.

Section 3.4

1 (a). The equation is both separable and exact.

1 (b). (i) $y'=y(2-y) \Rightarrow y'-2y=-y^2 \Rightarrow\ 1-n=-1=m,\ v=y^{-1} \Rightarrow y=v^{-1}$, thus $y'=-v^2v'$ and $-v^{-2}v'=2v^{-1}-v^{-2}$ or $v'+2v=1,\ v(0)=1$.

(ii) $\left(e^{2t}v\right)'=e^{2t} \Rightarrow e^{2t}v=\dfrac{1}{2}e^{2t}+C$ or $v=\dfrac{1}{2}+Ce^{-2t}$. From the initial condition,

$\dfrac{1}{2}+C=1 \Rightarrow C=\dfrac{1}{2}$, and so $v=\dfrac{1}{2}\left(1+e^{-2t}\right)$.

(iii) $y=v^{-1}=\dfrac{2}{1+e^{-2t}}$.

1 (c). $-\infty<t<\infty$

2 (a). The equation is both separable and exact.

2 (b). (i) $y'=2ty-2ty^2 \Rightarrow\ 1-2=-1=m,\ v=y^{-1} \Rightarrow y=v^{-1}$, thus $-v^{-2}v'=2tv^{-1}-2tv^{-2}$ or $v'+2tv=2t,\ v(0)=-1$.

(ii) $\left(e^{t^2}v\right)'=2te^{t^2} \Rightarrow e^{t^2}v=e^{t^2}+C$ or $v=1+Ce^{-t^2}$. From the initial condition,

$1+C=-1 \Rightarrow C=-2$, and so $v=1-2e^{-t^2}$.

(iii) $y=v^{-1}=\dfrac{1}{1-2e^{-t^2}}$.

2 (c). $-\sqrt{\ln 2}<t<\sqrt{\ln 2}$

3 (a). The equation is neither separable nor exact.

3 (b). (i) $m=1-n=-1,\ v=y^{-1} \Rightarrow y=v^{-1}$, thus $y'=-v^{-2}v'=-v^{-1}+e^tv^{-2} \Rightarrow v'=v-e^t$ or $v'-v=-e^t,\ v(-1)=-1$.

(ii) $\left(e^{-t}v\right)'=-1 \Rightarrow e^{-t}v=-t+C$ or $v=-te^t+Ce^t$. From the initial condition,

$e^{-1}+Ce^{-1}=-1 \Rightarrow C=-(1+e)$, and so $v=e^t(-t-1-e)=-(t+1)e^t-e^{t+1}$.

(iii) $y=v^{-1}=\dfrac{-1}{(t+1)e^t+e^{t+1}}$.

3 (c). $-(1+e) < t < \infty$

4 (a). The equation is both separable and exact.

4 (b). (i) $1-n=2=m,\ v=y^2 \Rightarrow y=v^{\frac{1}{2}}$, thus $y'=\frac{1}{2}v^{-\frac{1}{2}}v'$ and $\frac{1}{2}v^{-\frac{1}{2}}v'=v^{\frac{1}{2}}+v^{-\frac{1}{2}}$ or

$v'=2v+2,\ v(0)=1$.

(ii) $\left(e^{-2t}v\right)'=2e^{-2t} \Rightarrow e^{-2t}v=-e^{-2t}+C$ or $v=-1+Ce^{2t}$. From the initial condition,

$-1+C=1 \Rightarrow C=2$, and so $v=-1+2e^{2t}$.

(iii) $y=-\sqrt{-1+2e^{2t}}$.

4 (c). $-\frac{1}{2}\ln 2 < t < \infty$

5 (a). The equation is neither separable nor exact.

5 (b). (i) $m=1-n=3,\ v=y^3 \Rightarrow y=v^{\frac{1}{3}}$, thus $y'=\frac{1}{3}v^{-\frac{2}{3}}v'$. Then $t\cdot\frac{1}{3}v^{-\frac{2}{3}}v'+v^{\frac{1}{3}}=t^3v^{-\frac{2}{3}}$, and so

$tv'+3v=3t^3,\ v(1)=1$.

(ii) $\left(t^3v\right)'=3t^5 \Rightarrow t^3v=\frac{t^6}{2}+C$ or $v=\frac{1}{2}t^3+\frac{C}{t^3}$. From the initial condition, $\frac{1}{2}+C=1 \Rightarrow C=\frac{1}{2}$,

and so $v=\frac{1}{2}\left(t^3+t^{-3}\right)$.

(iii) $y=v^{\frac{1}{3}}=\left(\frac{1}{2}\left(t^3+t^{-3}\right)\right)^{\frac{1}{3}}$.

5 (c). $0 < t < \infty$

6 (a). The equation is neither separable nor exact.

6 (b). (i) $m=1-n=\frac{2}{3},\ v=y^{\frac{2}{3}} \Rightarrow y=v^{\frac{3}{2}}$, thus $y'=\frac{3}{2}v^{\frac{1}{2}}v'$. Then $\frac{3}{2}v^{\frac{1}{2}}v'-v^{\frac{3}{2}}=tv^{\frac{1}{2}}$, and so

$v'-\frac{2}{3}v=\frac{2}{3}t,\ v(0)=4$.

(ii) $\left(e^{-\frac{2}{3}t}v\right)'=\frac{2}{3}te^{-\frac{2}{3}t} \Rightarrow e^{-\frac{2}{3}t}v=\frac{2}{3}\left(-\frac{3}{2}te^{-\frac{2}{3}t}-\frac{9}{4}e^{-\frac{2}{3}t}\right)+C$ or $v=-t-\frac{3}{2}+Ce^{\frac{2}{3}t}$. From the initial

condition, $-\frac{3}{2}+C=4 \Rightarrow C=\frac{11}{2}$, and so $v=-\left(t+\frac{3}{2}\right)+\frac{11}{2}e^{\frac{2}{3}t}$.

(iii) $y=-\left(\frac{11}{2}e^{\frac{2}{3}t}-\left(t+\frac{3}{2}\right)\right)^{\frac{3}{2}}$.

6 (c). $-\infty < t < \infty$

7. First, let $u = e^y$. Then $y = \ln u$ and $y' = \frac{u'}{u}$. Therefore, $\frac{u'}{u} = 2t^{-1} + \frac{1}{u} \Rightarrow u' - \frac{2}{t}u = 1$ which gives us $\frac{1}{t^2}u' - \frac{2}{t^3}u = \frac{1}{t^2}$. Then we have $\left(t^{-2}u\right)' = t^{-2} \Rightarrow t^{-2}u = -t^{-1} + C$. Solving for u gives us $u = -t + Ct^2$. From the initial condition, we have $y(1) = 0 \Rightarrow u(1) = 1$, and so $u = -t + 2t^2 \Rightarrow y = \ln\left(2t^2 - t\right),\ t > \frac{1}{2}$.

8. First, let $u = y + 1,\ u' = -u + tu^{-1},\ 1 - n = 3$. Therefore,
$v = u^3,\ u = v^{\frac{1}{3}},\ u' = \frac{1}{3}v^{-\frac{2}{3}}v' \Rightarrow \frac{1}{3}v^{-\frac{2}{3}}v' + v^{\frac{1}{3}} = tv^{-\frac{2}{3}}$. Then,
$v' + 3v = 3t,\ y(0) = 1 \Rightarrow u(0) = 2 \Rightarrow v(0) = 8$ and
$v = Ce^{-3t} + at + b,\ a + 3(at + b) = 3t \Rightarrow a = 1,\ b = -\frac{1}{3}$. Therefore,
$v = Ce^{-3t} + t - \frac{1}{3},\ v(0) = C - \frac{1}{3} = 8 \Rightarrow C = \frac{25}{3}$. Then,
$v = \frac{25}{3}e^{-3t} + t - \frac{1}{3},\ y = u - 1 = v^{\frac{1}{3}} - 1 = \left(\frac{25}{3}e^{-3t} + t - \frac{1}{3}\right)^{\frac{1}{3}} - 1,\ -\infty < t < \infty$.

9. $y_0 = 3$ by substitution. Differentiating yields
$$y' = \frac{-3e^{-t}}{1 - 3t} + 3e^{-t}\left(\frac{-1}{(1 - 3t)^2}\right)(-3) = -\frac{3}{(1 - 3t)e^t} + e^t\left(\frac{9}{(1 - 3t)^2 e^{2t}}\right) = -y + e^t y^2.$$
Thus $q(t) = e^t$.

Section 3.5

1. $1 - n = -1,\ v = P^{-1}, P = v^{-1}$. Thus $-v^{-2}v' - rv^{-1} = -\frac{r}{P_e}v^{-2}$, or $v' + rv = \frac{r}{P_e},\ v(0) = P_0^{-1}$. Then $v = Ce^{-rt} + \frac{1}{P_e},\ v(0) = \frac{1}{P_0} = C + \frac{1}{P_e}$. Solving for C yields $C = P_0^{-1} - P_e^{-1}$, so we have $v = P^{-1} = \left(P_0^{-1} - P_e^{-1}\right)e^{-rt} + P_e^{-1}$. Thus $P = \frac{1}{P_0^{-1}e^{-rt} - P_e^{-1}\left(e^{-rt} - 1\right)} = \frac{P_0 P_e}{P_0 - \left(P_0 - P_e\right)e^{-rt}}$.

2. Since P is measured in millions, $P_0 = 0.1,\ r = 0.1,\ P_e = 3$. Therefore,
$P = \frac{0.1(3)}{0.1 - (0.1 - 3)e^{-0.1t}},\ 0.9P_e = 2.7 \Rightarrow 2.7 = \frac{0.3}{0.1 + 2.9e^{-0.1t}} \Rightarrow e^{-.1t} \approx .003831417$
$\Rightarrow t \approx 55.65$ *years*.

3 (a). Setting $r\left(1-\frac{P}{P_e}\right)P+M=0$, we have $-\frac{P^2}{P_e}+P+\frac{M}{r}=0$. Then

$P^2-P_eP-P_e\frac{M}{r}=0\Rightarrow P=\frac{P_e\pm\sqrt{P_e^{\,2}+4P_e M/r}}{2}$. This makes sense; migration would alter the equilibrium state.

3 (b). $(2P-1)^2=1+4x\Rightarrow\left(P-\frac{1}{2}\right)^2=x+\frac{1}{4}$, where $x=\frac{M}{r}$. This is a parabola with vertex $\left(-\frac{1}{4},\frac{1}{2}\right)$. For $x>0$, there is one nonnegative equilibrium solution. Two such solutions exist for $-\frac{1}{4}<x\le 0$.

3 (c). When $x=-\frac{1}{4}$, the two nonnegative equilibrium solutions coalesce into a single equilibrium value. There are no equilibrium solutions for $x<-\frac{1}{4}$. This makes sense, since if the migration out of the colony is too large relative to reproduction, equilibrium could not be achieved.

4. Equilibrium values at 4 and -2. $P=\frac{P_e}{2}\left(1\pm\sqrt{1+\frac{4M}{P_e r}}\right)\Rightarrow 4+(-2)=P_e\Rightarrow P_e=2$.

$4-(-2)=6=P_e\sqrt{1+\frac{4M}{P_e r}}$ or $3\sqrt{1+\frac{2M}{r}}=8=\frac{2M}{r}\Rightarrow\frac{M}{r}=4$.

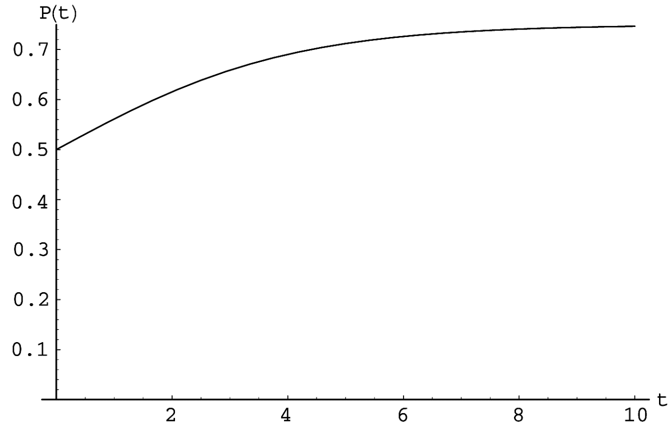
5. $P'=(1-P)P-\frac{3}{16},\ P(0)=\frac{1}{2}$. Then $P'=-P^2+P-\frac{3}{16}=-\left(P-\frac{3}{4}\right)\left(P-\frac{1}{4}\right)$. Then we have

$\frac{1}{\left(P-\frac{1}{4}\right)\left(P-\frac{3}{4}\right)}P'+1=0$, and by partial fractions, we have $\left(\frac{-2}{P-\frac{1}{4}}+\frac{2}{P-\frac{3}{4}}\right)P'+1=0$. Then

$\int\left[\frac{-2}{\left(P-\frac{1}{4}\right)}+\frac{2}{\left(P-\frac{3}{4}\right)}\right]dP+t=C$, and so $2\ln\left|\frac{P-\frac{3}{4}}{P-\frac{1}{4}}\right|+t=C$.

From the initial condition, we have $C=0$, so

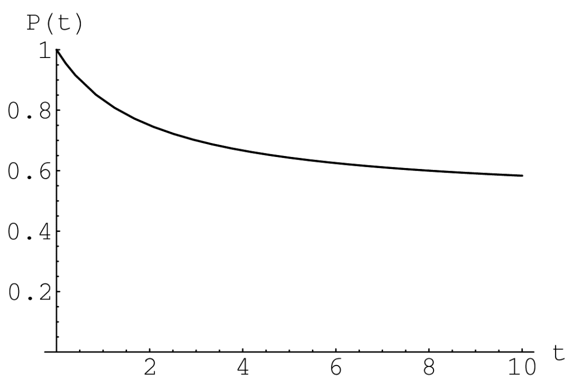
$\left|\frac{P-\frac{3}{4}}{P-\frac{1}{4}}\right|=e^{-\frac{t}{2}}$. But $\frac{1}{4}<P<\frac{3}{4}\Rightarrow\left|P-\tfrac{3}{4}\right|=-\left(P-\tfrac{3}{4}\right)\Rightarrow\frac{\frac{3}{4}-P}{P-\frac{1}{4}}=e^{-\frac{t}{2}}$, and then $P(t)=\frac{\frac{3}{4}+\frac{1}{4}e^{-\frac{t}{2}}}{1+e^{-\frac{t}{2}}}$.



6. $P' = (1-P)P - \frac{1}{4},\ P_0 = 1 \Rightarrow P' = -\left(P^2 - P + \frac{1}{4}\right) = -\left(P - \frac{1}{2}\right)^2$ which is separable.

$$\frac{1}{\left(P-\frac{1}{2}\right)^2}P' + 1 = 0 \Rightarrow -\left(P - \frac{1}{2}\right)^{-1} + t = C,\ -\left(1 - \frac{1}{2}\right)^{-1} + 0 = -2 = C$$

Therefore, $\left(P - \frac{1}{2}\right)^{-1} = 2 + t \Rightarrow P = \frac{1}{2} + \frac{1}{2+t} = \frac{2 + \frac{t}{2}}{2+t}$.

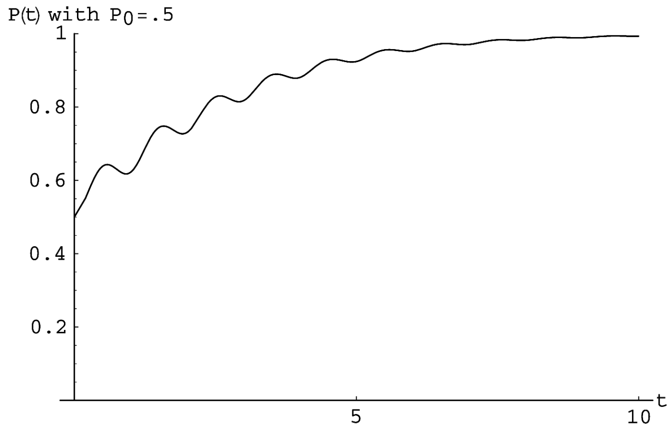


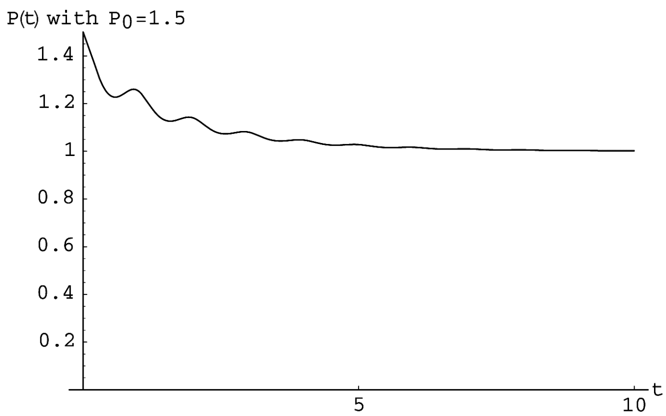
7. $P' = .5\left(1 + 2\sin(2\pi t)\right)(1-P)P,\ P(0) = P_0$. Following the derivation in the chapter with $r(t)$,

we have $R(t) = 0.5\int_0^t \left(1 + 2\sin(2\pi s)\right)ds = \frac{1}{2}\left(s - \frac{1}{\pi}\cos(2\pi s)\right)\Big|_0^t$

$= \frac{1}{2}\left(t - \frac{1}{\pi}\cos(2\pi t) + \frac{1}{\pi}\right) = \frac{1}{2}\left(t + \frac{1}{\pi}\left[1 - \cos(2\pi t)\right]\right)$. Therefore $P = \frac{P_0}{P_0 - \left(P_0 - 1\right)e^{-R(t)}}$ with

$R(t) = \frac{1}{2}\left(t + \frac{1}{\pi}\left[1 - \cos(2\pi t)\right]\right)$.





8. $\tau = \int_0^t r(s)ds \Rightarrow \dfrac{dP}{d\tau} = (1-P)P$. The solution procedure in the text leads to

$P(\tau) = \dfrac{P_0}{P_0 - (P_0 - 1)e^{-\tau}}$. Substitute $\tau = \dfrac{1}{2}\left(t + \dfrac{1}{\pi}[1 - \cos(2\pi t)]\right)$.

9. $P' = r(1-P)P = rP - rP^2 \Rightarrow v = P^{-1},\ P = v^{-1},\ -v^{-2}v' = rv^{-1} - rv^{-2} \Rightarrow v' + rv = r,\ v(0) = P_0^{-1}$.

Letting $R(t) = \int_0^t r(s)ds$, we have $\left(e^R v\right)' = re^R \Rightarrow e^R v = e^R + C \Rightarrow v = 1 + Ce^{-R}$. $v(0) = P_0^{-1}$, so

$C = P_0^{-1} - 1$ and thus $v = 1 + \left(P_0^{-1} - 1\right)e^{-R}$. Finally, $P = v^{-1} = \dfrac{1}{1 + \left(P_0^{-1} - 1\right)e^{-R}} = \dfrac{P_0}{P_0 - \left(P_0 - 1\right)e^{-R}}$

with $R(t) = \tau = \dfrac{1}{2}\left(t + \dfrac{1}{\pi}[1 - \cos(2\pi t)]\right)$.

10 (a). $P' = k(N-P)P$ with N and P in units of 100,000 and t in months. $N = 5,\ k = 2e^{-t} - 1$.

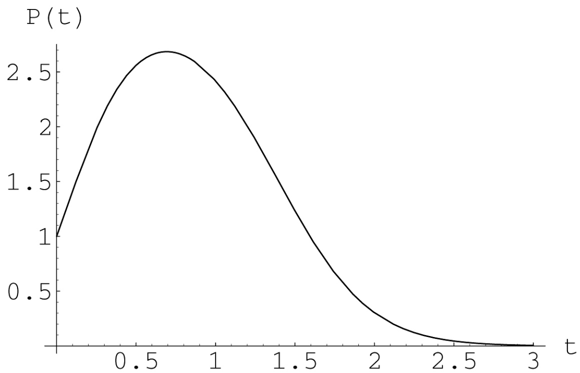
$\tau = \int_0^t (2e^{-s} - 1)ds = \left(-2e^{-s} - s\right)\Big|_0^t = \left(-2e^{-t} - t + 2\right) = -t + 2(1 - e^{-t})$.

$\dfrac{dP}{d\tau} = (5-P)P$ which is separable.

$\dfrac{1}{(5-P)P} = \dfrac{A}{P} + \dfrac{B}{5-P},\ A = \dfrac{1}{5},\ B = \dfrac{1}{5} \Rightarrow \dfrac{1}{5}(\ln P - \ln|5-P|) = \tau + C = \dfrac{1}{5}\ln\left|\dfrac{P}{5-P}\right|$

From the initial condition, $\frac{1}{5}\ln\left(\frac{1}{4}\right)=0+C \Rightarrow \ln\left|\frac{P}{5-P}\right|=5\tau+\ln\left(\frac{1}{4}\right) \Rightarrow \frac{P}{5-P}=\frac{1}{4}e^{5\tau}$

Therefore, $P=\frac{5e^{5\tau}}{4+e^{5\tau}}$, $\tau=-t+2(1-e^{-t})$.



10 (b). From the plot, $P_{\max}\approx 2.7$ (270,000).

10 (c). From the plot, $t\approx 1.8$ *months*.

11 (a). $(A-B)'=-kAB+kAB=0$, $A(t)-B(t)=A(0)-B(0)=5-2=3$ moles.

11 (b). $B=A-3$, $A'=-kA(A-3)=k(3-A)A$, $A(0)=5$.

11 (c). $A(1)=4$, $A'=3k\left(1-\frac{A}{3}\right)A$. Using equation (5), $A(t)=\frac{5\cdot 3}{5-(5-3)e^{-3kt}}$. Thus $A(t)=\frac{15}{5-2e^{-3kt}}$.

We know that $A(1)=4$, so $\frac{15}{5-2e^{-3k}}=4$. Solving for e^{-3k} yields $e^{-3k}=\frac{5}{8}$. Thus

$A(4)=\frac{15}{5-2\left(\frac{5}{8}\right)^4}=3.195$ moles. $B=A-3=0.195$ moles.

Section 3.6

1. With $v_0=0$, $v=-\frac{mg}{k}\left(1-e^{-\frac{k}{m}t}\right)$. Setting $v=-\frac{1}{2}\frac{mg}{k}$ gives us $1-e^{-\frac{k}{m}t}=\frac{1}{2}$. Thus $e^{-\frac{k}{m}t}=\frac{1}{2}$,

$\frac{k}{m}t=\ln 2$, $t=\frac{m}{k}\ln 2$.

2. $mv'=-mg+\kappa v^2$, $v(0)=0 \Rightarrow v'=-g+\frac{\kappa}{m}v^2=\frac{\kappa}{m}\left(v^2-\frac{mg}{\kappa}\right)\frac{v'}{v^2-\frac{mg}{\kappa}}=\frac{\kappa}{m}$

$$\frac{1}{v^2-\frac{mg}{\kappa}}=\frac{A}{v-\sqrt{\frac{mg}{\kappa}}}+\frac{B}{v+\sqrt{\frac{mg}{\kappa}}}\Rightarrow A=\frac{1}{2\sqrt{\frac{mg}{\kappa}}},\ B=-\frac{1}{2\sqrt{\frac{mg}{\kappa}}}.$$ Therefore,

$$\frac{1}{2\sqrt{\frac{mg}{\kappa}}}\ln\left|\frac{v-\sqrt{\frac{mg}{\kappa}}}{v+\sqrt{\frac{mg}{\kappa}}}\right|=\frac{\kappa}{m}t+C,\ v(0)=0\Rightarrow C=0 \text{ and } -\sqrt{\frac{mg}{\kappa}}<v\le 0.$$ Then,

$$\left|\frac{v-\sqrt{\frac{mg}{\kappa}}}{v+\sqrt{\frac{mg}{\kappa}}}\right|=\frac{\sqrt{\frac{mg}{\kappa}}-v}{\sqrt{\frac{mg}{\kappa}}+v}=e^{2\sqrt{\frac{\kappa g}{m}}t}\Rightarrow v=-\sqrt{\frac{mg}{\kappa}}\left(\frac{1-e^{-2\sqrt{\frac{\kappa g}{m}}t}}{1+e^{-2\sqrt{\frac{\kappa g}{m}}t}}\right)=-\sqrt{\frac{mg}{\kappa}}\tanh\left(\sqrt{\frac{\kappa g}{m}}t\right).$$

3. $10\ mi/hr=10\left(\frac{5280}{3600}\right)=14.67$ ft/sec. Then $14.67=\sqrt{\frac{200}{\kappa}}\Rightarrow\kappa\approx .929\frac{\text{lb}\cdot\text{sec}^2}{\text{ft}^2}$.

4 (a). $m\frac{dv}{dt}+kv=0\Rightarrow v(t)=v_0e^{-\frac{k}{m}t},\ m=\frac{3000}{32}\,slug$

$$\frac{v(4)}{v_0}=\frac{50}{220}=e^{-k\cdot\frac{32}{3000}\cdot 4}\Rightarrow\ln\left(\frac{22}{5}\right)=\frac{128}{3000}k.\ \text{Then, } k=\frac{3000}{128}\ln\left(\frac{22}{5}\right)=34.725\ \ lb\cdot\sec/ft.$$

4 (b). $d=\int_0^4 v(t)dt=v_0\int_0^4 e^{-\frac{k}{m}t}dt=v_0\left(-\frac{m}{k}e^{-\frac{k}{m}t}\right)\Big|_0^4=\frac{mv_0}{k}\left(1-e^{-\frac{4k}{m}}\right)$

$$=\frac{3000}{32}\left(220\cdot\frac{5280}{3600}\right)\left(\frac{1}{34.725}\right)\left(\frac{170}{220}\right)\approx 673\ \ ft.$$

5. $mv'+\kappa v^2=0\Rightarrow\frac{v'}{v^2}=-\frac{\kappa}{m}\Rightarrow -v^{-1}=-\frac{\kappa}{m}t+C,\ C=-v_0^{-1}$. Then we have

$v^{-1}=\frac{\kappa}{m}t+v_0^{-1}\Rightarrow v=\frac{v_0}{1+\frac{\kappa}{m}v_0t}$. From the condition provided, we have

$$\frac{v(4)}{v_0}=\frac{50}{220}=\frac{1}{1+4\frac{\kappa}{m}v_0}\Rightarrow 4\frac{\kappa}{m}v_0=\frac{1-\frac{5}{22}}{\frac{5}{22}}=\frac{17}{5}.$$ Solving for κ yields

$$\kappa=\frac{17}{5}\frac{m}{4v_0}=\frac{17}{5}\frac{3000}{32}\cdot\frac{1}{4}\div\left(220\left(\frac{5280}{3600}\right)\right)\approx .247\frac{\text{lb}\cdot\text{sec}^2}{\text{ft}^2}.$$

For the distance traveled, $d=\int_0^4 v(t)dt=v_0\int_0^4\frac{dt}{1+\frac{\kappa v_0}{m}t}=v_0\int_0^4\frac{dt}{1+\frac{17}{20}t}=v_0\left(\frac{20}{17}\right)\ln\left(1+\frac{17}{20}t\right)\Big|_0^4$

$$=220\left(\frac{5280}{3600}\right)\left(\frac{20}{17}\right)\ln\left(1+\frac{17}{5}\right)=562.4\ \text{ft}.$$

6. $mv' + kv = -mg,\ v(0) = v_0 \Rightarrow v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}$. Set

$v = 0$: $\frac{mg}{k} = \frac{mg}{k}\left(1 + \frac{kv_0}{mg}\right)e^{-\frac{k}{m}t_m} \Rightarrow \frac{k}{m}t_m = \ln\left(1 + \frac{kv_0}{mg}\right) \Rightarrow t_m = \frac{m}{k}\ln\left(1 + \frac{kv_0}{mg}\right)$.

7. $h = \int_0^{t_m} v(t)dt = \int_0^{t_m}\left[-\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}\right]dt = \left[-\frac{mg}{k}t - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}\right]_0^{t_m}$

$= -\frac{mg}{k}t_m + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(1 - e^{-\frac{k}{m}t_m}\right)$.

8. $mv' = -mg \Rightarrow v' = -g,\ v(0) = v_0$. Therefore, $v(t) = v_0 - gt$ and $y = -g\frac{t^2}{2} + v_0 t,\ t_m = \frac{v_0}{g}$. The

impact time is given by $-g\frac{t_i^2}{2} + v_0 t_i = 0 \Rightarrow -\frac{g}{2}t_i + v_0 = 0 \Rightarrow t_i = \frac{2v_0}{g} = 2t_m$.

9 (a). $v' = -g,\ v_0 = 0 \Rightarrow v = -gt = y' \Rightarrow y = -\frac{1}{2}gt^2 + y_0$. We want to find the time t at which y=7.

Thus $7 = -\frac{32}{2}t^2 + 555$, and solving for t yields $t \approx 5.852$ sec. At that time,

$v = -32(5.852) \approx -187.3$ft/sec.

9 (b). $mv' + kv = -mg \Rightarrow v' + \frac{kv}{m} = -g,\ v_0 = 0$. Thus $\left(ve^{\frac{k}{m}t}\right)' = -ge^{\frac{k}{m}t} \Rightarrow ve^{\frac{k}{m}t} = -\frac{mg}{k}e^{\frac{k}{m}t} + C$. From

the initial condition, we have $C = \frac{mg}{k}$, and so

$v = -\frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) \Rightarrow y = y_0 + \int_0^t v(s)ds = y_0 - \frac{mg}{k}\left(s + \frac{m}{k}e^{-\frac{k}{m}s}\right)\Big|_0^t = y_0 - \frac{mg}{k}\left(t + \frac{m}{k}\left(e^{-\frac{k}{m}t} - 1\right)\right)$.

$m = \frac{5\frac{1}{8}/16}{32} = \frac{41}{8 \cdot 16 \cdot 32}$ slug, so $\frac{m}{k} = \frac{41}{8(16)(32)(0.0018)} \approx 5.56098\ \text{sec}^{-1}$.

$mg = \frac{41}{8(16)} \approx 0.3203125$ lb, and so solving for t yields

$7 = 555 - 177.95139\left(t - 5.56098\left[1 - e^{\frac{-t}{5.56098}}\right]\right) \Rightarrow t = 7.08513$ sec. Substitution gives us

$v = \frac{-0.3203125}{0.0018}\left[1 - e^{\frac{-7.08513}{5.56098}}\right] \approx -128.18$ ft/sec.

10. $mg = 180\ \ lb$. For $0 \le t \le 10,\ v' = -g,\ v(0) = 0$.

For $10 < t \le 14,\ mv' + kv = -mg,\ y(14) = 0$.

For $mg = 200,\ \dfrac{200}{\text{k}} = 10\dfrac{5280}{3600} \Rightarrow k = \dfrac{3600(200)}{5280(10)} = 13.63636364$.

10 (a). $v = -gt$ At $t = 10,\ v = -320\ \ ft/\sec$.

10 (b). Solve $v' + \dfrac{k}{m}v = -g,\ v(0) = -320$, for $v(4)$.

$$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t} \Rightarrow v(4) = -\frac{180}{13.\overline{63}} + \left(-320 + \frac{180}{13.\overline{63}}\right)e^{-\frac{13.\overline{63}(32)}{180}(4)}$$

$= -13.2 - 306.8(.000061469) = -13.219\ \ ft/\sec$ (basically the terminal velocity).

10 (c). $$h = -\int_0^4 v(t)dt = \left(\frac{mg}{k}t - \left[v_0 + \frac{mg}{k}\right]\left(-\frac{m}{k}\right)e^{-\frac{k}{m}t}\right)\Bigg|_0^4 = \frac{mg}{k}(4) + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(e^{-\frac{4k}{m}} - 1\right)$$

$$= \frac{4mg}{k} - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(1 - e^{-\frac{4k}{m}}\right) = \frac{4(180)}{13.\overline{63}} - \frac{180}{32(13.\overline{63})}\left(-320 + \frac{180}{13.\overline{63}}\right)\left(1 - e^{-\frac{4(13.\overline{63})32}{180}}\right)$$

$= 52.8 - 0.4125(-306.8)(0.99994) = 179.347\ \ ft$.

10 (d). $h_{balloon} = h + \dfrac{1}{2}g(10)^2 = 179.347 + 1600 = 1779.347\ \ ft$.

11. For the first situation, $mv_1' + kv_1 = 0,\ v_1 = v_0 e^{-\frac{k}{m}t},\ m = \dfrac{3000}{32},\ k = 25$. Then

$$\frac{50}{220} = e^{\frac{-25\cdot 32}{3000}t_1} \Rightarrow t_1 = \frac{3000}{25(32)}\ln\frac{22}{5} \approx 5.556\ \text{sec.}$$

For the second situation, $mv_2' + k(\tanh t)v_2 = 0,\ v_2' + \dfrac{k}{m}\tanh t(v_2) = 0$. This is a first order linear equation. Letting $\mu = e^{\frac{k}{m}\ln(\cosh t)} = (\cosh t)^{\frac{k}{m}}$, we have

$$\left(v_2(\cosh t)^{\frac{k}{m}}\right)' = 0 \Rightarrow v_2 = C(\cosh t)^{-\frac{k}{m}}.$$

From the initial condition, we have $\cosh(0) = 1 \Rightarrow C = v_0$. Then

$$\frac{v_2}{v_0} = (\cosh t)^{-\frac{k}{m}} \Rightarrow \cosh t_2 = \left(\frac{v_0}{v_2}\right)^{\frac{m}{k}} = \left(\frac{220}{50}\right)^{\frac{3000}{32\cdot 25}}.$$

$\ln(\cosh t_2) = 3.75\ln\left(\dfrac{22}{5}\right) \approx 5.55602 \Rightarrow \cosh t_2 \approx 258.79$, so $t_2 \approx \cosh^{-1}(258.79) \approx 6.249$ sec.

This would be expected, since the size of the drag coefficient would be less for the second situation. Comparing the two values gives us $t_1 \approx 0.89 t_2$. These values do not seem appreciably different. However, it can be shown that this difference in stopping time leads to a difference in stopping distance of approximately 110 ft. If this distance is important for a certain situation, then the idealization is not reasonable.

Section 3.7

1. $\frac{dv}{dt} = -\frac{k}{m}x^2 v \Rightarrow v\frac{dv}{dx} = -\frac{k}{m}x^2 v \Rightarrow \frac{dv}{dx} = -\frac{k}{m}x^2 \Rightarrow v = -\frac{k}{m}\frac{x^3}{3} + C$. When $x = 0$, $v = v_0$.

   Therefore, $v_0 = C$, and so $v = -\frac{k}{m}\frac{x^3}{3} + v_0$ and $x_f{}^3 = 3\frac{m}{k}v_0 \Rightarrow x_f = \left(3\frac{m}{k}v_0\right)^{\frac{1}{3}}$.

2. $mv\frac{dv}{dx} = -kxv^2 \Rightarrow \frac{dv}{dx} = -\frac{k}{m}xv \Rightarrow \frac{dv}{dx} + \frac{k}{m}xv = 0$ (first order linear).

   $\frac{d}{dx}\left(e^{\frac{k}{m}\frac{x^2}{2}}v\right) = 0 \Rightarrow v = Ce^{-\frac{kx^2}{2m}}$, $C = v_0 \Rightarrow v = v_0 e^{-\frac{kx^2}{2m}}$. Since $v > 0$, $0 \le x < \infty$, $x_f = \infty$.

3. $mv\frac{dv}{dx} = -ke^{-x} \Rightarrow v\frac{dv}{dx} + \frac{k}{m}e^{-x} = 0 \Rightarrow \frac{v^2}{2} - \frac{k}{m}e^{-x} = C$. Then $C = \frac{v_0{}^2}{2} - \frac{k}{m}$, and so

   $v^2 = 2\left[\frac{v_0{}^2}{2} - \frac{k}{m} + \frac{k}{m}e^{-x}\right] \Rightarrow v = \left[v_0{}^2 - 2\frac{k}{m}\left(1 - e^{-x}\right)\right]^{\frac{1}{2}}$. If $v_0{}^2 \ge \frac{2k}{m}$, then v>0 for all nonnegative x and $x_f = \infty$. If $v_0{}^2 < \frac{2k}{m}$, then we have $v_0{}^2 = \frac{2k}{m}\left(1 - e^{-x_f}\right)$, which, solved for x_f, yields

   $x_f = -\ln\left(1 - \frac{mv_0{}^2}{2k}\right)$.

4. $mv\frac{dv}{dx} = -\frac{kv}{1+x} \Rightarrow \frac{dv}{dx} = -\frac{k}{m}\left(\frac{1}{1+x}\right) \Rightarrow v = -\frac{k}{m}\ln(1+x) + C$, $v_0 = C$. Therefore,

   $v = v_0 - \frac{k}{m}\ln(1+x)$ and $\frac{mv_0}{k} = \ln(1 + x_f) \Rightarrow x_f = e^{mv_0/k} - 1$.

5. $m\frac{dv}{dt} + kv^2 = 0$, $v(0) = v_0$, $x(0) = 0$. We want to find v when x=d.

   $mv\frac{dv}{dx} + kv^2 = 0 \Rightarrow \frac{dv}{dx} + \frac{k}{m}v = 0 \Rightarrow v = Ce^{-\frac{k}{m}x}$. From the initial condition, $v = v_0 e^{-\frac{k}{m}x}$, and so at x=d, $v = v_0 e^{-\frac{k}{m}d}$.

6. $m\dfrac{dv}{dt} = -mg - kv^2,\ v(0) = v_0,\ x(0) = 0 \Rightarrow mv\dfrac{dv}{dy} = -mg - kv^2 \Rightarrow \dfrac{dv}{dy} = -\dfrac{k}{m}v - gv^{-1}$

$\Rightarrow \dfrac{dv}{dy} + \dfrac{k}{m}v = -gv^{-1}$ (Bernoulli).

$1-n=2,\ u=v^2 \Rightarrow v = u^{\frac{1}{2}},\ \dfrac{dv}{dy} = \dfrac{1}{2}u^{-\frac{1}{2}}\dfrac{du}{dy} \Rightarrow \dfrac{1}{2}u^{-\frac{1}{2}}\dfrac{du}{dy} + \dfrac{k}{m}u^{\frac{1}{2}} = -gu^{-\frac{1}{2}}$. Therefore,

$\dfrac{du}{dy} + \dfrac{2k}{m}u = -2g,\ u = v_0^2$ when $y=0$. $\dfrac{d}{dy}\left(e^{\frac{2ky}{m}}u\right) = -2ge^{\frac{2ky}{m}} \Rightarrow e^{\frac{2ky}{m}}u = -\dfrac{mg}{k}e^{\frac{2ky}{m}} + C,\ C = v_0^2 + \dfrac{mg}{k}$

Therefore, $u = -\dfrac{mg}{k} + \left(v_0^2 + \dfrac{mg}{k}\right)e^{-\frac{2ky}{m}} = v^2 \Rightarrow v = \left[-\dfrac{mg}{k} + \left(v_0^2 + \dfrac{mg}{k}\right)e^{-\frac{2ky}{m}}\right]^{\frac{1}{2}}$. This equation is valid for $0 \le y \le h$, where $h =$ maximum height.

$$-\frac{mg}{k} + \left(v_0^2 + \frac{mg}{k}\right)e^{-\frac{2k}{m}h} = 0 \Rightarrow -\tfrac{2k}{m}h = \ln\left[\frac{\frac{mg}{k}}{v_0^2 + \frac{mg}{k}}\right] \Rightarrow h = \frac{m}{2k}\ln\left[1 + \frac{kv_0^2}{mg}\right].$$

7. With x measured as shown and $v = \dfrac{dx}{dt}$, we have $-m\dfrac{dv}{dt} = F\cos\theta$. Defining $\cos\theta = \dfrac{x}{\left(x^2+h^2\right)^{\frac{1}{2}}}$, we have $-mv\dfrac{dv}{dx} = \dfrac{Fx}{\left(x^2+h^2\right)^{\frac{1}{2}}} \Rightarrow -m\dfrac{v^2}{2} = F\left(x^2+h^2\right)^{\frac{1}{2}} + C$. We know that $v=0$ when $x=D$, so $C = -F\left(D^2+h^2\right)^{\frac{1}{2}}$. Then we have $v^2 = \dfrac{2}{m}\left(F\left(D^2+h^2\right)^{\frac{1}{2}} - F\left(x^2+h^2\right)^{\frac{1}{2}}\right)$.

When $x = \dfrac{D}{3},\ v = -\left(\dfrac{2F}{m}\left(\sqrt{D^2+h^2} - \sqrt{\dfrac{D^2}{9}+h^2}\right)\right)^{\frac{1}{2}}$.

8. $P = Fv,\ m\dfrac{dv}{dt} = F = \dfrac{P}{v} \Rightarrow mv\dfrac{dv}{dx} = \dfrac{P}{v} \Rightarrow v^2\dfrac{dv}{dx} = \dfrac{P}{m} \Rightarrow \dfrac{v^3}{3} = \dfrac{P}{m}x + C$

$\dfrac{v_1^3}{3} = \dfrac{P}{m}x_1 + C \Rightarrow C = \dfrac{v_1^3}{3} - \dfrac{P}{m}x_1,\ \dfrac{v_2^3}{3} = \dfrac{P}{m}x_2 + \dfrac{v_1^3}{3} - \dfrac{P}{m}x_1$. Therefore,

$x_2 - x_1 = \dfrac{m}{P}\left(\dfrac{v_2^3}{3} - \dfrac{v_1^3}{3}\right) = \dfrac{m}{3P}\left(v_2^3 - v_1^3\right),\ m = \dfrac{3000}{32},\ P = 200(550)\,ft\cdot lb/\sec$

$v_2 = 50\dfrac{5280}{3600}\,ft/\sec,\ v_1 = \dfrac{2}{5}v_2 \Rightarrow \Delta x = \dfrac{3000}{32}\cdot\dfrac{1}{3}\cdot\dfrac{1}{200(550)}\left(\dfrac{50(5280)}{3600}\right)^3\left(1-\left(\dfrac{2}{3}\right)^3\right)$

$\Rightarrow \Delta x = 112.04(.936) \approx 104.87\,ft$.

9 (a). $mv\dfrac{dv}{dx} + \kappa_0 xv^2 = 0,\ v = v_0$ when $x = 0$.

9 (b). $\frac{dv}{dx}+\frac{\kappa_0}{m}xv=0 \Rightarrow \left(e^{\frac{\kappa_0 x^2}{2m}}v\right)'=0 \Rightarrow v=v_0 e^{-\frac{\kappa_0 x^2}{2m}}$. Setting $x=d$ and $v=0.01v_0$, we have

$0.01v_0=v_0 e^{-\frac{\kappa_0 d^2}{2m}} \Rightarrow \frac{\kappa_0 d^2}{2m}=\ln 100$. Solving for κ_0 yields $\kappa_0=\frac{2m}{d^2}\ln 100$.

10 (a). $mv\frac{dv}{dr}=-\frac{GmM_e}{r^2}+\kappa v^2 \Rightarrow \frac{dv}{dr}=\frac{\kappa}{m}v-\frac{GM_e}{r^2}v^{-1},\ v=0$ when $r=R_e+h$.

10 (b). Bernoulli equation: $1-n=-1 \Rightarrow n=2,\ u=v^2 \Rightarrow v=u^{\frac{1}{2}} \Rightarrow \frac{dv}{dr}=\frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dr}=\frac{\kappa}{m}u^{\frac{1}{2}}-\frac{GM_e}{r^2}u^{-\frac{1}{2}}$

$\Rightarrow \frac{du}{dr}=\frac{2\kappa}{m}u-\frac{2GM_e}{r^2}$. Therefore,

$$\left(e^{-\frac{2\kappa}{m}r}u\right)'=2GM_e\frac{e^{-\frac{2\kappa}{m}r}}{r^2} \Rightarrow e^{-\frac{2\kappa}{m}(R_e+h)}u\Big|_{r=R_e+h}-e^{-\frac{2\kappa}{m}(R_e)}u\Big|_{r=R_e}-2GM_e\int_{R_e}^{R_e+h}\frac{e^{-\frac{2\kappa}{m}r}}{r^2}dr$$

$$=0-e^{-\frac{2\kappa}{m}(R_e)}u\Big|_{r=R_e}-2GM_e\int_{R_e}^{R_e+h}\frac{e^{-\frac{2\kappa}{m}r}}{r^2}dr.$$

Since $u=v^2,\ v=\frac{dr}{dt}<0,\ v_{impact}=-e^{\frac{\kappa}{m}(R_e)}\left[2GM_e\int_{R_e}^{R_e+h}\frac{e^{-\frac{2\kappa}{m}r}}{r^2}dr\right]^{\frac{1}{2}}$. Let

$r=R_e+s$. Then $v_{impact}=-\left[2GM_e\int_0^h\frac{e^{-\frac{2\kappa}{m}s}}{(R_e+s)^2}ds\right]^{\frac{1}{2}}$.

11. $m\frac{dv}{dt}=-\frac{GM_e m}{r^2} \Rightarrow v\frac{dv}{dr}=-\frac{GM_e}{r^2},\ v=v_0$ when $r=R_e$. Thus $\frac{v^2}{2}=\frac{G}{r}M_e+C$, and from our initial condition, $\frac{{v_0}^2}{2}=\frac{G}{R_e}M_e+C \Rightarrow \frac{v^2}{2}=\frac{{v_0}^2}{2}+GM_e\left(\frac{1}{r}-\frac{1}{R_e}\right)$. Since $v=0$ when $r=R_e+h$,

${v_0}^2=2GM_e\left(\frac{1}{R_e}-\frac{1}{R_e+h}\right)$. Thus

$$v_0=\left[2GM_e\left(\frac{1}{R_e}-\frac{1}{R_e+h}\right)\right]^{\frac{1}{2}}=\left[\frac{2(6.673)(10^{-11})(5.976)(10^{24})}{10^6}\left(\frac{1}{6.371}-\frac{1}{6.591}\right)\right]^{\frac{1}{2}}\approx 2044\text{ m/sec}.$$

12 (a). $m\ell^2\theta''=-mg\ell\sin\theta=m\ell^2\frac{d\omega}{dt} \Rightarrow m\ell^2\omega\frac{d\omega}{d\theta}=-mg\ell\sin\theta$

$m\ell^2\omega\frac{d\omega}{d\theta}=-mg\ell\sin\theta$ and $\omega=-\omega_0$ when $\theta=\pi$.

12 (b). $m\ell^2\frac{\omega^2}{2}=mg\ell\cos\theta+C,\ m\ell^2\frac{{\omega_0}^2}{2}=-mg\ell+C \Rightarrow m\ell^2\frac{\omega^2}{2}-mg\ell\cos\theta=m\ell^2\frac{{\omega_0}^2}{2}+mg\ell$.

12 (c). When $\theta = 0$, $m\ell^2 \frac{\omega^2}{2} - mg\ell = m\ell^2 \frac{\omega_0{}^2}{2} + mg\ell \Rightarrow \omega^2 = \omega_0^2 + 2mg\ell\left(\frac{2}{m\ell^2}\right) = \omega_0^2 + \frac{4g}{\ell}$

$\Rightarrow \omega = \sqrt{\omega_0^2 + \frac{4g}{\ell}}$.

13. $m\ell^2 \frac{\omega^2}{2} = mg\ell\cos\theta + C$, $\omega = \omega_0$ when $\theta = 0$. Therefore, $C = m\ell^2 \frac{\omega_0{}^2}{2} - mg\ell$, and so

$m\ell^2 \frac{\omega^2}{2} = mg\ell\cos\theta + m\ell^2 \frac{\omega_0{}^2}{2} - mg\ell$. We know that $\omega = 0$ when $\theta = \frac{3\pi}{4}$, so

$-\frac{mg\ell}{\sqrt{2}} + \frac{m\ell^2\omega_0{}^2}{2} - mg\ell = 0 \Rightarrow \omega_0{}^2 = \frac{2}{m\ell^2} mg\ell\left(1 + \frac{1}{\sqrt{2}}\right) = \frac{g}{\ell}\left(2 + \sqrt{2}\right)$. Thus

$\omega_0 = \sqrt{\frac{g}{\ell}\left(2 + \sqrt{2}\right)} = \sqrt{16\left(2 + \sqrt{2}\right)} \approx 7.391$ rad/sec.

Section 3.8

Note: for exercises 1-5, h=0.1

1 (a). $y = t^2 - t + C$, $y(1) = C = 0 \Rightarrow y = t^2 - t$

1 (b). $y_{k+1} = y_k + h(2t_k - 1)$

1 (c). $y_1 = 0.1$, $y_2 = 0.22$, $y_3 = 0.36$

1 (d). $y(1.1) = 0.11$, $y(1.2) = 0.24$, $y(1.3) = 0.39$

2 (a). $y = Ce^{-t}$, $y(0) = C = 1 \Rightarrow y = e^{-t}$

2 (b). $y_{k+1} = y_k - hy_k$

2 (c). $y_1 = 0.9$, $y_2 = 0.81$, $y_3 = 0.729$

2 (d). $y(0.1) = 0.90484$, $y(0.2) = 0.81873$, $y(0.3) = 0.74082$

3 (a). $y = Ce^{-\frac{t^2}{2}}$, $y(0) = C = 1 \Rightarrow y = e^{-\frac{t^2}{2}}$

3 (b). $y_{k+1} = y_k - h(t_k y_k)$

3 (c). $y_1 = 1$, $y_2 = 0.99$, $y_3 = 0.9702$

3 (d). $y(0.1) = 0.99501$, $y(0.2) = 0.98020$, $y(0.3) = 0.955997$

4 (a). $y = Ce^{-t} + t - 1$, $y(0) = C - 1 = 0 \Rightarrow y = e^{-t} + t - 1$

4 (b). $y_{k+1} = y_k + h(-y_k + t_k)$

4 (c). $y_1 = 0$, $y_2 = 0.01$, $y_3 = 0.029$

4 (d). $y(0.1) = 0.0048374$, $y(0.2) = 0.01873075$, $y(0.3) = 0.040818$

5 (a). $y^{-2}y' = 1,\ -y^{-1} = t + C,\ C = -1 \Rightarrow y = \dfrac{1}{1-t}$

5 (b). $y_{k+1} = y_k + h\left(y_k{}^2\right)$

5 (c). $y_1 = 1.1,\ y_2 = 1.221,\ y_3 = 1.3700841$

5 (d). $y(0.1) = 1.111111,\ y(0.2) = 1.25,\ y(0.3) = 1.4285714$

6. $y_{k+1} = y_k + 0.1(\alpha t_k + \beta)$. From k=0, $t_0 = 0,\ y_0 = -1$.

For $k = 0,\ y_1 = y_0 + 0.1(\alpha t_0 + \beta) \Rightarrow -0.9 = -1 + .1(0 + \beta) \Rightarrow 0.1 = .1\beta \Rightarrow \beta = 1$.

For $k = 1,\ y_2 = y_1 + 0.1(\alpha t_1 + \beta) \Rightarrow -0.81 = -0.9 + .1(\alpha(0.1) + 1) \Rightarrow -0.01 = .01\alpha \Rightarrow \alpha = -1$.

7. $y_{k+1} = y_k + 0.1\left(y_k{}^n + \alpha\right)$. From k=0, $t_0 = 1,\ y_0 = 1$.

For $k = 0,\ y_1 = y_0 + .1\left(y_0{}^n + \alpha\right) \Rightarrow 0.9 = 1 + .1(1^n + \alpha) \Rightarrow (1^n + \alpha) = -1 \Rightarrow \alpha = -2$.

For $k = 1,\ y_2 = y_1 + .1\left(y_1{}^n - 2\right) \Rightarrow 0.781 = 0.9 + .1(.9^n - 2) \Rightarrow (.9^n - 2) = -1.19$

$\Rightarrow .9^n = .81 \Rightarrow n = 2$.

8 (a). (i)Euler's method will underestimate the exact solution.

(ii)Euler's method will overestimate the exact solution.

(iii)Euler's method will underestimate the exact solution.

(iv)Euler's method will overestimate the exact solution.

8 (b). Exercise 2: decreasing, concave up, underestimates

Exercise 3: decreasing, concave down, overestimates

Exercise 5: increasing, concave up, underestimates

8 (c). Euler's method should initially underestimate (when solution curves are concave up) and then tend to "catch up" (when solution curves become concave down).

9. $y_{k+1} = y_k + h\left(t_k y_k + \sin(2\pi t_k)\right),\ y_0 = 1,\ h = 0.01,\ k = 0,1,\ldots,99$.

10. $V(0) = 90,\ V(t) = 90 + 5t,\ V(T) = 100$ when $T = 2 \Rightarrow 0 \le t \le 2$

$$\frac{dQ}{dt} = 6(2 - \cos(\pi t)) - 1 \cdot \frac{Q}{90 + 5t},\ Q(0) = 0$$

$$Q_{k+1} = Q_k + h\left[6(2 - \cos(\pi t_k)) - \frac{Q_k}{90 + 5t_k}\right],\ Q_0 = 0,\ h = 0.01,\ k = 0,1,2,,\ldots,199$$

Result: $Q(2) = 23.7556\ldots lb$.

11. $P' = 0.1\left(1 - \dfrac{P}{3}\right)P + e^{-t},\ P(0) = \dfrac{1}{2}$. $P_{k+1} = P_k + h\left[0.1\left(1 - \dfrac{1}{3}P_k\right)P_k + e^{-t_k}\right],\ P_0 = 0.5$. With $h = 0.01,\ k = 0,1,\ldots,199,\ t_k = 0.01k$, $P(2) = 1.502477$ million.

12 (a). $y = Ce^t - 1,\ C = 1 \Rightarrow y = e^t - 1$.

12 (b). $y_{k+1} = y_k + h(y_k + 1),\ y_0 = 0$. For $y_k^{(1)},\ h = 0.02,\ k = 0,1,...49$

For $y_k^{(2)},\ h = 0.01,\ k = 0,1,...99$.

13 (a). $y' - \lambda y = 0,\ \left(e^{-\lambda t} y\right)' = C,\ y = Ce^{\lambda t},\ y(0) = C = y_0$. Thus $y = e^{\lambda t} y_0$.

13 (b). $y_{k+1} = y_k + h\lambda y_k = (1 + \lambda h) y_k$. Therefore

$y_1 = (1 + \lambda h) y_0,\ y_2 = (1 + \lambda h) y_1 = (1 + \lambda h)^2 y_0,\ y_n = (1 + \lambda h)^n y_0,$

13 (c). $y_n = \left(1 + \frac{\lambda t}{n}\right)^n y_0$. Since $\lim_{n \to \infty}\left(1 + \frac{a}{n}\right)^n = e^a$, the result follows.

14 (a). $y^{-2} y' = 1,\ -y^{-1} = t + C,\ C = -1,\ y = \frac{1}{1 - t},\ -\infty < t < 1$

14 (b). $y_{k+1} = y_k + h y_k^2,\ y_0 = 1,\ h = 0.1,\ k = 0,1,...11$

14 (c). Numerical solution becomes worse as $t_k \uparrow 1$. The numerical solution gives the mistaken impression that the interval of existence extends to $t \geq 1$.