Chapter 4
Second Order Linear Differential Equations

Introduction

1 (a). The block's weight is $W_b = \pi(0.5)^2(2)(50) = 25\pi$ lb. Therefore, we have $25\pi = \pi(0.5)^2(Y)(62.4)$. Solving for Y yields $Y = 1.60256$ feet.

1 (b). $y_0 = -\frac{1}{4}$, $y_0' = 0$, and $\omega^2 = \frac{\rho_\ell g}{\rho L} = \frac{W}{W_b}\frac{g}{L} = \frac{62.4}{50}\frac{32}{2} = 19.968$. Therefore, $\omega = 4.469$, and thus our equation for $y(t)$ is $y(t) = -\frac{1}{4}\cos(4.469t)$.

1 (c). The maximum depth to which the block sinks would be $Y + |y_0| = 1.603 + .0.25 = 1.853$ feet.

2. $y(t) = \frac{y_0'}{\omega}\sin\omega t$. $Y + \frac{y_0'}{\omega} = 2 \Rightarrow y_0' = \omega(2 - Y) \approx 1.77596 \;\; ft.$

3 (a). We know that the frequency of oscillation is given by $\omega^2 = \frac{\rho_\ell g}{\rho L}$. We can see that the density for the second drum is greater than that of the first. Since all the other variables determining ω would be the same for both drums, $\rho_2 > \rho_1$ means that $\omega_2 < \omega_1$, which in turn means that $T_2 > T_1$. Thus the first drum bobs more rapidly.

3 (b). By the same rationale as in the first part, $L_2 > L_1$ means that $\omega_2 < \omega_1$, which in turn means that $T_2 > T_1$. Thus the first drum bobs more rapidly.

4 (a). $y_0 = 0, \; T = 2$

4 (b). $y_0' = \frac{1}{2} \;\; ft.\;\; y(t) = \frac{1}{2}\sin\omega t$, with $\omega \cdot 2 = 2\pi \Rightarrow \omega = \pi. \;\; \therefore \;\; y(t) = \frac{1}{2}\sin\pi t,$

4 (c). $\omega^2 = \pi^2 = \frac{W_l}{W_b}\frac{g}{L} \Rightarrow W_b = \frac{W_l}{\omega^2}\frac{g}{L}; \; V = \pi\left(\frac{3}{2}\right)^2 \cdot 5 = \frac{45\pi}{4} ft^3$

$$W_b = 62.4\left(\frac{45\pi}{4}\right)\frac{1}{\pi^2}\frac{32}{5} = 62.4 \cdot \frac{9}{\pi} \cdot 8 \approx 1430.1 \;\; lb.$$

5 (a). Using the model provided, $my'' = mg - \rho_\ell Vg$. We can rewrite this equation with m in terms of V and ρ: $\rho V y'' = \rho V g - \rho_\ell V g$. Simplifying this equation and solving for y'', we have $y'' = \left(\frac{\rho - \rho_\ell}{\rho}\right)g$. We need not restrict this equation to the motion of cylindrical objects.

5 (b). Using these initial conditions, antidifferentiation yields the general solution

$$y = \left(\frac{\rho - \rho_\ell}{\rho}\right) g \frac{t^2}{2} + y_0' t + y_0 .$$

6. $$\frac{\rho - \rho_\ell}{\rho} = \frac{30 - 62.4}{62.4} = -.51923 .$$

$$\therefore\ 0 = -.51923 \cdot 32 \cdot \frac{t^2}{2} + 0 + 99 \Rightarrow 8.30769 t^2 = 99 \Rightarrow t = 11.9166\ldots \text{ sec} .$$

Section 4.1

1. All the relevant functions are continuous everywhere, so Theorem 4.1 guarantees a unique solution for the interval $(-\infty, \infty)$.

2. $t_0 = \pi$. Since $g(t) = \tan t$, $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

3. Dividing the equation by e^t yields the functions $p(t) = 0$, $q(t) = \frac{1}{e^t (t^2 - 1)}$, and $g(t) = \frac{4}{te^t}$.

   These functions are discontinuous at the points ± 1 and 0, and since $t_0 = -2$, the largest t-interval on which Theorem 4.1 guarantees a unique solution is $(-\infty, -1)$.

4. $p(t) = \frac{\sin 2t}{t\left(t^2 - 9\right)}$, $q(t) = \frac{2}{t}$, $0 < t < 3$.

5. $y'' + y = 0$, $y(t_0) = y_0$, $y'(t_0) = y_0'$ for any t_0, y_0, y_0'.

6. $y'' + \frac{1}{t-3} y = 0$, $y(t_0) = y_0$, $y'(t_0) = y_0'$, $t_0 > 3$, any y_0, y_0' (not both 0).

7. $y'' + \frac{y'}{t-5} + \frac{y}{t+1} = 0$, $-1 < t_0 < 5$, $y(t_0) = y_0$, $y'(t_0) = y_0'$. y_0 and y_0' cannot both be zero.

8 (a). $y'' - \frac{y'}{t} + \frac{y}{t^2} = 0$, $t_0 = 1$. $0 < t < \infty$.

8 (b). $t^2(0) - t(1) + t = 0$.

8 (c). No.

9. Theorem 4.1 guarantees a unique solution on the interval $(-4, 4)$, so it is not possible for the limit to hold.

10. Theorem 4.1 guarantees a unique solution on the interval $(-\infty, 3)$, so it is not possible for the limit to hold.

11 (a). B

11 (b). D

11 (c). A

11 (d). C

Section 4.2

1 (a). $y_1'' = 4e^{2t} = 4y_1$ and $y_2'' = 8e^{-2t} = 4y_2$, so both equations are solutions.

1 (b). $W = \begin{vmatrix} e^{2t} & 2e^{-2t} \\ 2e^{2t} & -4e^{-2t} \end{vmatrix} = -8 \neq 0$, so yes, the functions do form a fundamental set of solutions.

1 (c). The general solution would be $y = c_1e^{2t} + c_2 \cdot 2e^{-2t}$. Differentiating yields $y' = 2c_1e^{2t} - 4c_2e^{-2t}$. $y(0) = c_1 + 2c_2 = 1$ and $y'(0) = 2c_1 - 4c_2 = -2$, and solving these two simultaneous equations yields $c_1 = 0$ and $c_2 = \frac{1}{2}$. Thus the unique solution for this initial value problem is $y(t) = e^{-2t}$.

2 (a). $y_1'' = 2e^t = y_1$ and $y_2'' = e^{-t+3} = y_2$, so both equations are solutions.

2 (b). $W = \begin{vmatrix} 2e^{t} & e^{-t+3} \\ 2e^{t} & -e^{-t+3} \end{vmatrix} = -4e^3 \neq 0$, so yes, the functions do form a fundamental set of solutions.

2 (c). The general solution would be $y = c_1 \cdot 2e^t + c_2 \cdot e^{-t+3}$. Differentiating yields $y' = 2c_1e^t - c_2e^{-t+3}$. $y(-1) = 2c_1e^{-1} + c_2e^4 = 1$ and $y'(-1) = 2c_1e^{-1} - c_2e^4 = 0$, and solving these two simultaneous equations yields $c_1 = \frac{e}{4}$ and $c_2 = \frac{1}{2e^4}$. Thus the unique solution for this initial value problem is $y(t) = \frac{e^{t+1}}{2} + \frac{e^{-t-1}}{2}$.

3 (a). y_1 is not a solution. y_2 is a solution.

4 (a). Both equations are solutions.

4 (b). $W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1 \neq 0$, so yes, the functions do form a fundamental set of solutions.

4 (c). The general solution would be $y = c_1 \cos t + c_2 \sin t$. Differentiating yields $y' = -c_1 \sin t + c_2 \cos t$. $y\left(\frac{\pi}{2}\right) = c_2 = 1$ and $y'\left(\frac{\pi}{2}\right) = -c_1 = 1 \Rightarrow c_1 = -1$. Thus the unique solution for this initial value problem is $y(t) = -\cos t + \sin t$.

5 (a). $y_1' = 2e^{2t} = 2y_1$ and $y_1'' = 4e^{2t} = 4y_1$. Thus $y_1'' - 4y_1' + 4y_1 = 4y_1 - 8y_1 + 4y_1 = 0$.

$y_2' = 2te^{2t} + e^{2t}$ and $y_2'' = 4te^{2t} + 2e^{2t} + 2e^{2t} = 4te^{2t} + 4e^{2t}$. Thus

$y_2'' - 4y_2' + 4y_2 = 4te^{2t} + 4e^{2t} - 8te^{2t} - 4e^{2t} + 4te^{2t} = 0$, and so both equations are solutions.

5 (b). $W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (2t+1)e^{2t} \end{vmatrix} = e^{4t} \neq 0$, so yes, the functions do form a fundamental set of solutions.

5 (c). The general solution would be $y = c_1 e^{2t} + c_2 te^{2t}$. Differentiating yields

$y' = 2c_1 e^{2t} + 2c_2 te^{2t} + c_2 e^{2t}$. $y(0) = c_1 = 2$ and $y'(0) = 2c_1 + c_2 = 0 \Rightarrow c_2 = -4$.

Thus the unique solution for this initial value problem is $y(t) = 2e^{2t} - 4te^{2t}$.

6 (a). Both equations are solutions.

6 (b). $W = \begin{vmatrix} 1 & e^{t/2} \\ 0 & \frac{1}{2}e^{t/2} \end{vmatrix} = \frac{1}{2}e^{t/2} \neq 0$, so yes, the functions do form a fundamental set of solutions.

6 (c). The general solution would be $y = c_1 + c_2 e^{t/2}$. Differentiating yields $y' = \frac{1}{2}c_2 e^{t/2}$.

$y(2) = c_1 + c_2 e = 0$ and $y'(2) = \frac{1}{2}c_2 e = 2 \Rightarrow c_2 = 4e^{-1}$ and $c_1 = -4$. Thus the unique solution for this initial value problem is $y(t) = -4 + 4e^{\frac{t-2}{2}}$.

7 (a). $y_1' = \frac{1}{2}\cos\left(\frac{t}{2}+\frac{\pi}{3}\right)$ and $y_1'' = -\frac{1}{4}\sin\left(\frac{t}{2}+\frac{\pi}{3}\right)$. Thus $4y_1'' + y_1 = -\sin\left(\frac{t}{2}+\frac{\pi}{3}\right) + \sin\left(\frac{t}{2}+\frac{\pi}{3}\right) = 0$.

$y_2' = \frac{1}{2}\cos\left(\frac{t}{2}-\frac{\pi}{3}\right)$ and $y_2'' = -\frac{1}{4}\sin\left(\frac{t}{2}-\frac{\pi}{3}\right)$. Thus $4y_2'' + y_2 = -\sin\left(\frac{t}{2}-\frac{\pi}{3}\right) + \sin\left(\frac{t}{2}-\frac{\pi}{3}\right) = 0$,

and so both equations are solutions.

7 (b). $W = \begin{vmatrix} \sin\left(\frac{t}{2}+\frac{\pi}{3}\right) & \sin\left(\frac{t}{2}-\frac{\pi}{3}\right) \\ \frac{1}{2}\cos\left(\frac{t}{2}+\frac{\pi}{3}\right) & \frac{1}{2}\cos\left(\frac{t}{2}-\frac{\pi}{3}\right) \end{vmatrix}$

$= \frac{1}{2}\left[\sin\left(\frac{t}{2}+\frac{\pi}{3}\right)\cos\left(\frac{t}{2}-\frac{\pi}{3}\right) - \sin\left(\frac{t}{2}-\frac{\pi}{3}\right)\cos\left(\frac{t}{2}+\frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\sin\left(\frac{t}{2}+\frac{\pi}{3}-\frac{t}{2}+\frac{\pi}{3}\right) = \frac{1}{2}\sin\left(\frac{2\pi}{3}\right) \neq 0$, so yes, the two equations do form a fundamental set of solutions.

7 (c). The general solution would be $y = c_1\sin\left(\frac{t}{2}+\frac{\pi}{3}\right) + c_2\sin\left(\frac{t}{2}-\frac{\pi}{3}\right)$. Differentiating yields

$y' = \frac{1}{2}c_1\cos\left(\frac{t}{2}+\frac{\pi}{3}\right) + \frac{1}{2}c_2\cos\left(\frac{t}{2}-\frac{\pi}{3}\right)$.

Using the first initial condition, $y(0) = c_1 \sin\left(\frac{\pi}{3}\right) + c_2 \sin\left(-\frac{\pi}{3}\right) = 0$, and so $c_1 = c_2$. From the second initial condition, $y'(0) = \frac{1}{2}c_1 \cos\left(\frac{\pi}{3}\right) + \frac{1}{2}c_2 \cos\left(-\frac{\pi}{3}\right) = \frac{1}{4}c_1 + \frac{1}{4}c_2 = 1$. Thus $c_1 = c_2 = 2$, and the unique solution is $y = 2\sin\left(\frac{t}{2} + \frac{\pi}{3}\right) + 2\sin\left(\frac{t}{2} - \frac{\pi}{3}\right) = 4\sin\left(\frac{t}{2}\right)\cos\left(\frac{\pi}{3}\right) = 2\sin\left(\frac{t}{2}\right)$.

8 (a). Both equations are solutions.

8 (b). $W = \begin{vmatrix} 2e^t & e^{2t} \\ 2e^t & 2e^{2t} \end{vmatrix} = 2e^{3t} \neq 0$, so yes, the functions do form a fundamental set of solutions.

8 (c). The general solution would be $y = c_1 \cdot 2e^t + c_2 \cdot e^{2t}$. Differentiating yields $y' = 2c_1e^t + 2c_2e^{2t}$. $y(-1) = 2c_1e^{-1} + c_2e^{-2} = 1$ and $y'(-1) = 2c_1e^{-1} + 2c_2e^{-2} = 0$, and solving these two simultaneous equations yields $c_1 = e$ and $c_2 = -e^2$. Thus the unique solution for this initial value problem is $y(t) = 2e^{t+1} - e^{2(t+1)}$.

9 (a). $y_1' = \frac{1}{t}$, $y_1'' = -\frac{1}{t^2}$, $y_2' = \frac{1}{t}$, and $y_2'' = -\frac{1}{t^2}$. Thus $ty_1'' + y_1' = ty_2'' + y_2' = 0$, and so both equations are solutions.

9 (b). $W = \begin{vmatrix} \ln t & \ln 3t \\ \frac{1}{t} & \frac{1}{t} \end{vmatrix} = \frac{1}{t}(\ln t - \ln 3t) = t^{-1}(\ln t - \ln 3 - \ln t) = -t^{-1}\ln 3 \neq 0$ on $(0,\infty)$, so yes, the two equations do form a fundamental set of solutions.

9 (c). The general solution would be $y = c_1 \ln t + c_2 \ln 3t = (c_1 + c_2)\ln t + c_2 \ln 3$. Differentiating yields $y' = \frac{c_1 + c_2}{t}$. From the first initial condition, $y(3) = (c_1 + c_2)\ln 3 + c_2 \ln 3 = 0$, so $c_1 + 2c_2 = 0$. From the second initial condition, $y'(3) = \frac{c_1 + c_2}{3} = 3$. Thus $c_1 = 18$ and $c_2 = -9$, and so the unique solution is $y = 18\ln t - 9\ln 3t,\ 0 < t < \infty$.

10 (a). Both equations are solutions.

10 (b). $W = \begin{vmatrix} \ln t & \ln 3 \\ \frac{1}{t} & 0 \end{vmatrix} = \frac{-\ln 3}{t} \neq 0$, so yes, the functions do form a fundamental set of solutions.

10 (c). The general solution would be $y = c_1 \cdot \ln t + c_2 \cdot \ln 3$. Differentiating yields $y' = \frac{c_1}{t} + 0$. $y(1) = 0 + c_2 \ln 3 = 0$ and $y'(1) = c_1 = 3$, and solving these two simultaneous equations yields $c_1 = 3$ and $c_2 = 0$. Thus the unique solution for this initial value problem is $y(t) = 3\ln t,\ 0 < t < \infty$.

11 (a). $y_1' = 3t^2$, $y_1'' = 6t$, $y_2' = t^{-2}$, and $y_2'' = -2t^{-3}$. Thus $t^2y_1'' - ty_1' - 3y_1 = t^2y_2'' - ty_2' - 3y_2 = 0$, and so both equations are solutions.

11 (b). $W = \begin{vmatrix} t^3 & -t^{-1} \\ 3t^2 & t^{-2} \end{vmatrix} = t + 3t = 4t \neq 0$ on $(-\infty,0)$, so yes, the two equations do form a fundamental set of solutions.

11 (c). The general solution would be $y = c_1t^3 - c_2t^{-1}$. Differentiating yields $y' = 3c_1t^2 + c_2t^{-2}$. From the first initial condition, $y(-1) = -c_1 + c_2 = 0$. From the second initial condition,

$y'(-1) = 3c_1 + c_2 = -2$. Thus $c_1 = c_2 = -\dfrac{1}{2}$, and so the unique solution is

$y = -\dfrac{1}{2}t^3 + \dfrac{1}{2}t^{-1}, \ -\infty < t < 0$.

12 (a). Both equations are solutions.

12 (b). $W = \begin{vmatrix} e^{-t} & 2e^{1-t} \\ -e^{-t} & -2e^{1-t} \end{vmatrix} = 0$, Therefore, the Wronskian calculation does not establish that y_1 and y_2 form a fundamental set.

13 (a). $y_1' = 1$, $y_1'' = 0$, $y_2' = -1$, and $y_2'' = 0$. Thus both equations are solutions.

13 (b). $W = \begin{vmatrix} t+1 & -t+2 \\ 1 & -1 \end{vmatrix} = -t - 1 + t - 2 = -3 \neq 0$, so yes, the two equations do form a fundamental set of solutions.

13 (c). The general solution would be $y = c_1(t+1) + c_2(2-t)$. Differentiating yields $y' = c_1 - c_2$. From the first initial condition, $y(1) = 2c_1 + c_2 = 4$. From the second initial condition,

$y'(1) = c_1 - c_2 = -1$. Thus $c_1 = 1$ and $c_2 = 2$, and so the unique solution is

$y = t + 1 + 4 - 2t = -t + 5$.

14 (a). Both equations are solutions.

14 (b). $W = \begin{vmatrix} \sin\pi t + \cos\pi t & \sin\pi t - \cos\pi t \\ \pi\cos\pi t - \pi\sin\pi t & \pi\cos\pi t + \pi\sin\pi t \end{vmatrix}$

$= 2\pi\sin\pi t\cos\pi t + \pi(\sin^2\pi t + \cos^2\pi t) - 2\pi\sin\pi t\cos\pi t + \pi(\sin^2\pi t + \cos^2\pi t)$,

$= 2\pi \neq 0$ so yes, the functions do form a fundamental set of solutions.

14 (c). The general solution would be $y = c_1(\sin\pi t + \cos\pi t) + c_2(\sin\pi t - \cos\pi t)$. Differentiating yields

$y' = \pi c_1(\cos\pi t - \sin\pi t) + \pi c_2(\cos\pi t + \sin\pi t)$.

$y\left(\dfrac{1}{2}\right) = c_1 + c_2 = 1$ and $y'\left(\dfrac{1}{2}\right) = -\pi c_1 + \pi c_2 = 0 \Rightarrow c_1 = c_2 = \tfrac{1}{2}$.

Thus the unique solution for this initial value problem is

$y = \frac{1}{2}(\sin\pi t + \cos\pi t) + \frac{1}{2}(\sin\pi t - \cos\pi t) = \sin\pi t$.

15 (a). y_1 is not a solution. y_2 is not a solution.

16 (b). $\bar{y} = \sin(2t)\cos\left(\frac{\pi}{4}\right) + \cos(2t)\sin\left(\frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}(2\cos 2t) + \frac{1}{\sqrt{2}}(\sin 2t)$.

Thus $c_1 = \frac{1}{2\sqrt{2}}$ and $c_2 = \frac{1}{\sqrt{2}}$.

17 (a). $\bar{y}' = 2 + 1 + \ln 3t = \ln 3t + 3,\ \bar{y}'' = \frac{1}{t}$. Thus $t^2\bar{y}'' - t\bar{y}' + \bar{y} = t - 3t - t\ln 3t + 2t + t\ln 3t = 0$.

17(b). $\bar{y} = 2t + t(\ln t + \ln 3) = t(2 + \ln 3) + t\ln t$. Thus $c_1 = 2 + \ln 3$ and $c_2 = 1$.

18 (b). $\bar{y} = 2\cosh\left(\frac{t}{2}\right) = e^{t/2} + e^{-t/2} = 1\cdot e^{-t/2} + \left(-\frac{1}{2}\right)\cdot\left(-2e^{t/2}\right)$.

Thus $c_1 = 1$ and $c_2 = -\frac{1}{2}$.

19. Substituting y_1 into the equation yields $9e^{3t} + 3\alpha e^{3t} + \beta e^{3t} = 0;\ 3\alpha + \beta = -9$. Substituting y_2 into the equation yields $9e^{-3t} - 3\alpha e^{-3t} + \beta e^{-3t} = 0;\ -3\alpha + \beta = -9$. Solving the two simultaneous equations gives us $\alpha = 0$ and $\beta = -9$.

20 (a). $y_1(t) = t, y_1'(t) = 1, y_1''(t) = 0,\ y_2(t) = e^t, y_2'(t) = y_2''(t) = e^t$.

$0 + p(t)\cdot 1 + q(t)\cdot t = 0,\ e^t + p(t)\cdot e^t + q(t)\cdot e^t = 0$

$p + qt = 0,\ e^t(1 + p + q) = 0 \Rightarrow p + q = -1$

$\therefore (t-1)q = 1 \Rightarrow q = \frac{1}{t-1},\ p = \frac{-t}{t-1}$

20 (b). Both p and q continuous on $(-\infty,1)$ *and* $(1,\infty)$.

20 (c). $W = \begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix} = (t-1)e^t \qquad W \neq 0 \ \text{on}\ (-\infty,1)\ \text{and}\ (1,\infty)$

20 (d). Yes, $W \neq 0$ on the two intervals on which p and q are both continuous.

21. From Abel's Theorem, we have $W(t) = W(1)e^{-\int_1^t s\,ds} \Rightarrow W(2) = 4e^{-\int_1^2 s\,ds} = 4e^{-(4-1)/2} = 4e^{-3/2}$.

22. Substitute, $4e^{2t} + 2\alpha e^{2t} + \beta e^{2t} = 0 \Rightarrow 2\alpha + \beta = -4$.

Also, $W = e^{-t} \Rightarrow p(t) = \alpha = 1$. $\therefore \alpha = 1, \beta = -6\ (y'' + y' - 6y = 0)$.

23. $p(t) = 0$. From Abel's Theorem, $W(t) = W(t_0)$, which is a constant. Therefore, $W(4) = -3$.

24. $y'' + py' + 3y = 0,\ W = W(t_0)e^{-\int_{t_0}^t p\,ds} = e^{-t^2}$.

$\therefore p(t) = \frac{d}{dt}\left(t^2\right) = 2t$.

Section 4.3

1 (a). $W(1)=\begin{vmatrix}2 & -1\\ 2 & -1\end{vmatrix}=0$, so the two solutions do not form a fundamental set.

1 (b). Since $-\frac{1}{2}y_1$ and y_2 satisfy identical initial conditions, we can conclude from Theorem 4.1 that $-\frac{1}{2}y_1(t)\equiv y_2(t)$. Therefore, $\frac{1}{2}y_1(t)+y_2(t)=0,\ -\infty<t<\infty$.

2 (a). $W(-2)=\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix}=1\neq 0$, so the two solutions do form a fundamental set.

2 (b). Yes (Theorem 4.7)

3 (a). $W(0)=\begin{vmatrix}0 & -1\\ 1 & 0\end{vmatrix}=1\neq 0$, so the two solutions do form a fundamental set.

3 (b). From Theorem 4.7, the two solutions do form a linearly independent set of functions on $-\infty<t<\infty$.

4 (a). $W(3)=\begin{vmatrix}0 & 1\\ 0 & 2\end{vmatrix}=0$, so the two solutions do not form a fundamental set.

4 (b). By Theorem 4.1, $y_1(t)\equiv 0$. Therefore, $1\cdot y_1(t)+0\cdot y_2(t)=0$ and $y_1(t), y_2(t)$ are linearly dependent.

5 (a). $y_1''-4y_1=4e^{2t}-4e^{2t}=0,\ y_2''-4y_2=4e^{-2t}-4e^{-2t}=0$, so yes, both are solutions to the differential equation.

5 (b). $y_1(1)=e^2,\ y_1'(1)=2e^2,\ y_2(1)=e^{-2},\ y_2'(1)=-2e^{-2}$.

5 (c). $W(1)=\begin{vmatrix}e^2 & e^{-2}\\ 2e^2 & -2e^{-2}\end{vmatrix}=-4\neq 0$, so the solutions do form a fundamental set.

6 (a). $4y_1''-y_1=4\left(\frac{1}{4}e^{t/2}\right)-e^{t/2}=0,\ 4y_2''-y_2=4\left(\frac{-1}{2}e^{-t/2}\right)-\left(-2e^{-t/2}\right)=0$,

6 (b). $y_1(-2)=e^{-1},\ y_1'(-2)=\frac{1}{2}\mathrm{e}^{-1},\ y_2(-2)=-2e,\ y_2'(-2)=e$.

6 (c). $W(-2)=\begin{vmatrix}e^{-1} & -2e\\ \frac{e^{-1}}{2} & e\end{vmatrix}=2\neq 0$, so the solutions do form a fundamental set.

7 (a). $y_1''+9y_1=-9\sin(3(t-1))+9\sin(3(t-1))=0$,

$y_2''+9y_2=-9(2\cos(3(t-1)))+9(2\cos(3(t-1)))=0$, so yes, both are solutions to the differential equation.

7 (b). $y_1(1) = 0,\ y_1{}'(1) = 3,\ y_2(1) = 2,\ y_2{}'(1) = 0$.

7 (c). $W(1) = \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} = -6 \neq 0$, so the solutions do form a fundamental set.

8 (a). $y_1' = e^{-2t}(-2\cos t - \sin t), y_1'' = e^{-2t}(3\cos t + 4\sin t)$

$y_1'' + 4y_1' + 5y_1 = e^{-2t}(3\cos t + 4\sin t - 8\cos t - 4\sin t + 5\cos t) = 0$

$y_2' = e^{-2t}(-2\sin t + \cos t), y_2'' = e^{-2t}(3\sin t - 4\cos t)$

$y_2'' + 4y_2' + 5y_2 = e^{-2t}(3\sin t - 4\cos t - 8\sin t + 4\cos t + 5\sin t) = 0$

8 (b). $y_1(0) = 1,\ y_1{}'(0) = -2,\ y_2(0) = 0,\ y_2{}'(0) = 1$.

8 (c). $W(0) = \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} = 1 \neq 0$, so the solutions do form a fundamental set.

9 (a). $y_1'' + 2y_1' - 3y_1 = 9e^{-3t} - 6e^{-3t} - 3e^{-3t} = 0,$

$y_2'' + 2y_2' - 3y_2 = 9e^{-3(t-2)} - 6e^{-3(t-2)} - 3e^{-3(t-2)} = 0$, so yes, both are solutions to the differential equation.

9 (b). $y_1(2) = e^{-6},\ y_1{}'(2) = -3e^{-6},\ y_2(2) = 1,\ y_2{}'(2) = -3$.

9 (c). $W(2) = \begin{vmatrix} e^{-6} & 1 \\ -3e^{-6} & -3 \end{vmatrix} = 0$, so the solutions do not form a fundamental set.

10 (a). $y_1'' - 6y_1' + 9y_1 = e^{3(t+2)}(9 - 18 + 9) = 0$

$y_2'' - 6y_2' + 9y_2 = e^{3(t+2)}(9t + 6 - 6(3t+1) + 9t)) = 0$

10 (b). $y_1(-2) = 1,\ y_1{}'(-2) = 3,\ y_2(-2) = -2,\ y_2{}'(-2) = -5$.

10 (c). $W(-2) = \begin{vmatrix} 1 & -2 \\ 3 & -5 \end{vmatrix} = 1 \neq 0$, so the solutions do form a fundamental set.

11. $\begin{bmatrix} \bar{y}_1 & \bar{y}_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}; \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 3 \neq 0$, so $\{\bar{y}_1 \quad \bar{y}_2\}$ is a fundamental set.

12. $\begin{bmatrix} \bar{y}_1 & \bar{y}_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}; \begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} = 0$, so $\{\bar{y}_1 \quad \bar{y}_2\}$ is not a fundamental set.

13. $\begin{bmatrix} \bar{y}_1 & \bar{y}_2 \end{bmatrix} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}; \begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$, so $\{\bar{y}_1 \quad \bar{y}_2\}$ is a fundamental set.

14. The set is linearly independent since one function is not a constant multiple of the other.

15. $f_2 = \ln t^2 = 2\ln t = 2f_1$, so the set is linearly dependent $(2f_1 - f_2 = 0)$.

16. The set is linearly independent since one function is not a constant multiple of the other.

17. The set is linearly independent since one function is not a constant multiple of the other.

18. Set $k_1 \cdot 2 + k_2 \cdot t + k_3 \cdot (-t^2) = 0$ and evaluate at $t = -1, 0, 1$.

$$\begin{bmatrix} 2 & -1 & -1 \\ 2 & 0 & 0 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad \begin{vmatrix} 2 & -1 & -1 \\ 2 & 0 & 0 \\ 2 & 1 & -1 \end{vmatrix} = -2 \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} = -4 \neq 0.$$

Therefore, $k_1 = k_2 = k_3 = 0$ and the set is linearly independent.

19. $f_1 = 2, f_2 = \sin^2 t, f_3 = 2\cos^2 t$. Therefore, $2f_2 + f_3 - f_1 = 0$ on $-3 < t < 2$, so the set is linearly dependent.

20. $f_1 = e^t, f_2 = 2e^{-t}, f_3 = \sinh t$. Therefore, $\frac{1}{2} f_1 - \frac{1}{4} f_2 - f_3 = 0$, so the set is linearly dependent.

21 (a). $f_1 = t, f_2 = 2t = 2f_1$, so the functions form a linearly dependent set.

21 (b). $f_1 = t, f_2 = -t = -f_1$, so the functions form a linearly dependent set.

21 (c). $f_1 = t, f_2 = t - 1$. These functions form a linearly independent set since one function is not a constant multiple of the other.

22 (a). If $f_1 = cf_2$, then $f_1 - cf_2 = 0$. Therefore, the set is linearly dependent.

22 (b). Assume $\{f_1, f_2\}$ is a linearly dependent set. Then

$k_1 f_1(t) + k_2 f_2(t) = 0$ with k_1 and k_2 not both zero. Assume that $k_1 \neq 0$. Then $f_1(t) = -\left(\frac{k_2}{k_1}\right) f_2(t)$ on the domain.

23 (a). If $f_2 = 3f_1 - 2f_3$, then $3f_1 - f_2 - 2f_3 = 0$ on the domain. Therefore, the set is linearly dependent.

23 (b). Assume $\{f_1, f_2, f_3\}$ is a linearly dependent set. Then

$k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) = 0$ with k_1, k_2, k_3 not all zero. Assume, without loss of generality, that $k_1 \neq 0$. Then $f_1(t) = -\left(\frac{k_2}{k_1}\right) f_2(t) - \left(\frac{k_3}{k_1}\right) f_3(t)$.

24. Any set of functions containing the zero function is linearly dependent.

Consider $\{0, f_2, f_3, \ldots, f_n\}$. Then $1 \cdot 0 + 0 \cdot f_2 + 0 \cdot f_3 + \ldots + 0 \cdot f_n = 0$.

25. Suppose that $f_3 = a_1 f_1 + a_2 f_2$ and $f_3 = b_1 f_1 + b_2 f_2$. Then $(a_1 - b_1) f_1 + (a_2 - b_2) f_2 = 0$. Since the functions are linearly independent, $a_1 - b_1 = a_2 - b_2 = 0$; $a_1 = b_1$, $a_2 = b_2$.

26. On $0 < t < \infty$, $f_2 = |t| = t = f_1$. $\therefore f_1 - f_2 = 0$ and $\{f_1, f_2\}$ is linearly dependent.

On $-\infty < t < \infty$, let $k_1 f_1 + k_2 f_2 = k_1 t + k_2 |t| = 0$ and evaluate at

$t = \pm 1$. Then $k_1 + k_2 = 0, -k_1 + k_2 = 0 \Rightarrow k_1 = k_2 = 0$ and this is a linearly independent set.

27. $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

28. $\begin{bmatrix} -1 & -3 \\ 2 & 1 \end{bmatrix}$

Section 4.4

1 (a). $\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1) = 0$. Thus $y = c_1 e^{-2t} + c_2 e^t$.

1 (b). $y(0) = c_1 + c_2 = 3,\ y'(0) = -2c_1 + c_2 = -3$. Solving these simultaneous equations yields $c_1 = 2$ and $c_2 = 1$. Thus the unique solution to the initial value problem is $y = 2e^{-2t} + e^t$.

1 (c). $\lim_{t\to-\infty} y = \infty$ and $\lim_{t\to\infty} y = \infty$.

2 (a). $\lambda^2 - \frac{1}{4} = 0 \Rightarrow \lambda = \pm\frac{1}{2}$. Thus $y = c_1 e^{-t/2} + c_2 e^{t/2}$.

2 (b). $y(2) = c_1 e^{-1} + c_2 e^{1} = 1,\ y'(2) = -\frac{1}{2} c_1 e^{-1} + \frac{1}{2} c_2 e^{1} = 0$. Therefore, $c_2 e = c_1 e^{-1} = \frac{1}{2} \Rightarrow c_1 = \frac{e}{2}$ and $c_2 = \frac{e^{-1}}{2}$. Thus the unique solution to the initial value problem is $y = \frac{1}{2} e^{-(t-2)/2} + \frac{1}{2} e^{(t-2)/2}$.

2 (c). $\lim_{t\to-\infty} y = \infty$ and $\lim_{t\to\infty} y = \infty$.

3 (a). $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$. Thus $y = c_1 e^{3t} + c_2 e^t$.

3 (b). $y(0) = c_1 + c_2 = -1,\ y'(0) = 3c_1 + c_2 = 1$. Solving these simultaneous equations yields $c_1 = 1$ and $c_2 = -2$. Thus the unique solution to the initial value problem is $y = e^{3t} - 2e^t$.

3 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = \infty$.

4 (a). $2\lambda^2 - 5\lambda + 2 = (2\lambda - 1)(\lambda - 2) = 0$. Thus $y = c_1 e^{t/2} + c_2 e^{2t}$.

4 (b). $y(0) = c_1 + c_2 = -1,\ y'(0) = \frac{1}{2} c_1 + 2c_2 = -5$. Solving these simultaneous equations yields $c_1 = 2$ and $c_2 = -3$. Thus the unique solution to the initial value problem is $y = 2e^{t/2} - 3e^{2t}$.

4 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = -\infty$.

5 (a). $\lambda^2 - 1 = (\lambda + 1)(\lambda - 1) = 0$. Thus $y = c_1 e^{-t} + c_2 e^t$.

5 (b). $y(0) = c_1 + c_2 = 1,\ y'(0) = -c_1 + c_2 = -1$. Solving these simultaneous equations yields $c_1 = 1$ and $c_2 = 0$. Thus the unique solution to the initial value problem is $y = e^{-t}$.

5 (c). $\lim_{t\to-\infty} y = \infty$ and $\lim_{t\to\infty} y = 0$.

6 (a). $\lambda^2 + 2\lambda = \lambda(\lambda + 2) = 0$. Thus $y = c_1 e^{-2t} + c_2$.

6 (b). $y(-1)=c_1e^2+c_2=0,\ y'(-1)=-2c_1e^2=2$. Therefore, $c_1=-e^{-2}$ and $c_2=1$, and $y=1-e^{-2(t-+1)}$.

6 (c). $\lim_{t\to-\infty} y=-\infty$ and $\lim_{t\to\infty} y=1$.

7 (a). $\lambda^2+5\lambda+6=(\lambda+2)(\lambda+3)=0$. Thus $y=c_1e^{-3t}+c_2e^{-2t}$.

7 (b). $y(0)=c_1+c_2=1,\ y'(0)=-3c_1-2c_2=-1$. Solving these simultaneous equations yields $c_1=-1$ and $c_2=2$. Thus the unique solution to the initial value problem is $y=-e^{-3t}+2e^{-2t}$.

7 (c). $\lim_{t\to-\infty} y=-\infty$ and $\lim_{t\to\infty} y=0$.

8 (a). $\lambda^2-5\lambda+6=(\lambda-2)(\lambda-3)=0$. Thus $y=c_1e^{2t}+c_2e^{3t}$.

8 (b). $c_1+c_2=1,\ 2c_1+3c_2=-1$. Therefore, $c_1=4$ and $c_2=-3$, and $y=4e^{2t}-3e^{3t}$.

8 (c). $\lim_{t\to-\infty} y=0$ and $\lim_{t\to\infty} y=-\infty$.

9 (a). $\lambda^2-4=(\lambda+2)(\lambda-2)=0$. Thus $y=c_1e^{-2t}+c_2e^{2t}$.

9 (b). $y(3)=c_1e^{-6}+c_2e^{6}=0,\ y'(3)=-2c_1e^{-6}+2c_2e^{6}=0$. Solving these simultaneous equations yields $c_1=0$ and $c_2=0$. Thus the unique solution to the initial value problem is $y=0$.

9 (c). $\lim_{t\to-\infty} y=0$ and $\lim_{t\to\infty} y=0$.

10 (a). $8\lambda^2-6\lambda+1=(4\lambda-1)(2\lambda-1)=0$. Thus $y=c_1e^{t/4}+c_2e^{t/2}$.

10 (b). $c_1e^{1/4}+c_2e^{1/2}=4,\ \frac{1}{4}c_1e^{1/4}+\frac{1}{2}c_2e^{1/2}=\frac{3}{2}$. Therefore, $c_1=2e^{-1/4}$ and $c_2=2e^{-1/2}$, and $y=2e^{(t-1)/4}+2e^{(t-1)/2}$.

10 (c). $\lim_{t\to-\infty} y=0$ and $\lim_{t\to\infty} y=\infty$.

11 (a). $2\lambda^2-3\lambda=(\lambda)(2\lambda-3)=0$. Thus $y=c_1+c_2e^{\frac{3}{2}t}$.

11 (b). $y(-2)=c_1+c_2e^{-3}=3,\ y'(-2)=\dfrac{3}{2}c_2e^{-3}=0$. Solving these simultaneous equations yields $c_1=3$ and $c_2=0$. Thus the unique solution to the initial value problem is $y=3$.

11 (c). $\lim_{t\to-\infty} y=3$ and $\lim_{t\to\infty} y=3$.

12 (a). $\lambda^2-6\lambda+8=(\lambda-2)(\lambda-4)=0$. Thus $y=c_1e^{2t}+c_2e^{4t}$.

12 (b). $c_1e^2+c_2e^4=2,\ 2c_1e^2+4c_2e^4=-8$. Therefore, $c_1=8e^{-2}$ and $c_2=-6e^{-4}$, and $y=8e^{2(t-1)}-6e^{4(t-1)}$.

12 (c). $\lim_{t\to-\infty} y=0$ and $\lim_{t\to\infty} y=-\infty$.

13 (a). $\lambda^2+4\lambda+2=0$. Thus $\lambda=\dfrac{-4\pm\sqrt{16-8}}{2}=-2\pm\sqrt{2}$ and $y=c_1e^{(-2-\sqrt{2})t}+c_2e^{(-2+\sqrt{2})t}$.

13 (b). $y(0)=c_1+c_2=0,\ y'(0)=(-2-\sqrt{2})c_1+(-2+\sqrt{2})c_2=4$. Solving these simultaneous equations yields $c_1=-\sqrt{2}$ and $c_2=\sqrt{2}$. Thus the unique solution to the initial value problem is $y=-\sqrt{2}e^{(-2-\sqrt{2})t}+\sqrt{2}e^{(-2+\sqrt{2})t}$.

13 (c). $\lim_{t\to-\infty} y=-\infty$ and $\lim_{t\to\infty} y=0$.

14 (a). $\lambda^2-4\lambda-1=0 \Rightarrow \lambda=\dfrac{4\pm\sqrt{16+4}}{2}=2\pm\sqrt{5}$. Thus $y=c_1e^{(2-\sqrt{5})t}+c_2e^{(2+\sqrt{5})t}$.

14 (b). $c_1+c_2=1,\ (2-\sqrt{5})c_1+(2+\sqrt{5})c_2=2+\sqrt{5}$. Therefore, $c_1=0$ and $c_2=1$, and $y=e^{(2+\sqrt{5})t}$.

14 (c). $\lim_{t\to-\infty} y=0$ and $\lim_{t\to\infty} y=\infty$.

15 (a). $2\lambda^2-1=0$. Thus $\lambda=\pm\dfrac{1}{\sqrt{2}}$ and $y=c_1e^{-t/\sqrt{2}}+c_2e^{t/\sqrt{2}}$.

15 (b). $y(0)=c_1+c_2=-2,\ y'(0)=-\dfrac{1}{\sqrt{2}}c_1+\dfrac{1}{\sqrt{2}}c_2=\sqrt{2}$. Solving these simultaneous equations yields $c_1=-2$ and $c_2=0$. Thus the unique solution to the initial value problem is $y=-2e^{-t/\sqrt{2}}$.

15 (c). $\lim_{t\to-\infty} y=-\infty$ and $\lim_{t\to\infty} y=0$.

16. Since $\lim_{t\to\infty} e^{-3t}=0,\ y(t)=c_1e^{-3t}+2,\ y(0)=c_1+2=1 \Rightarrow c_1=-1$. Therefore, $y(t)=2-e^{-3t},\ \lambda^2+\alpha\lambda+\beta=\lambda(\lambda+3) \Rightarrow \alpha=3, \beta=0$ and $y'(0)=y_0'=3$.

17 (a). $W=\begin{vmatrix} e^{-t} & ce^{\lambda_2 t} \\ -e^{-t} & \lambda_2 ce^{\lambda_2 t}\end{vmatrix}=(\lambda_2+1)ce^{(\lambda_2-1)t}=4e^{2t}$. Thus $\lambda_2=3$ and $c=1$. The second member of the fundamental set is then $y_2=e^{3t}$.

17 (b). $\lambda^2+\alpha\lambda+\beta=(\lambda+1)(\lambda-3)$. Therefore $\alpha=-2$ and $\beta=-3$.

17 (c). The general solution is $y=c_1e^{-t}+c_2e^{3t}$. Using the initial conditions, we have $y(0)=c_1+c_2=3,\ y'(0)=-c_1+3c_2=5$. Solving these simultaneous equations yields $c_1=1$ and $c_2=2$. The unique solution to the initial value problem is $y=e^{-t}+2e^{3t}$.

18 (a). $\lambda^2+2\lambda=0 \Rightarrow \lambda=0,-2$. Graph (C) since it is the only equation admitting a nonzero limit as $t\to\infty$.

18 (b). $6\lambda^2-5\lambda+1=(3\lambda-1)(2\lambda-1)=0 \Rightarrow \lambda=\dfrac{1}{3},\dfrac{1}{2}$. Graph (B)

18 (c). $\lambda^2-1=0 \Rightarrow \lambda=1,-1$. Graph (A).

19. Utilizing the hint given, we can make the substitution $u(t)=y'(t)$. The equation then becomes $u''-5u'+6u=0$.

The characteristic equation of this new differential equation is $\lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2) = 0$.

Thus $u(t) = c_1' e^{2t} + c_2' e^{3t} = y'(t)$. Antidifferentiation gives us

$y = \frac{1}{2}c_1' e^{2t} + \frac{1}{3}c_2' e^{3t} + c_3 = c_1 e^{2t} + c_2 e^{3t} + c_3$.

20 (a). $mx'' + kx' = 0,\ m\lambda^2 + k\lambda = m\lambda\left(\lambda + \frac{k}{m}\right) = 0 \Rightarrow \lambda = -\frac{k}{m}, 0$. Thus $x(t) = c_1 e^{-\frac{k}{m}t} + c_2$.

20 (b). $x' = -\frac{k}{m}c_1 e^{-\frac{k}{m}t} \quad \therefore c_1 + c_2 = x_0,\ -\frac{k}{m}c_1 = v_0 \Rightarrow c_1 = -\frac{m}{k}v_0, c_2 = x_0 + \frac{m}{k}v_0$.

$x(t) = -\frac{mv_0}{k}e^{-\frac{k}{m}t} + x_0 + \frac{mv_0}{k} = x_0 + \frac{mv_0}{k}(1 - e^{-\frac{k}{m}t})$.

20 (c). $\lim_{t\to\infty} x(t) = x_0 + \frac{mv_0}{k}$.

21 (a). $\lambda^2 - \Omega^2 = (\lambda + \Omega)(\lambda - \Omega) = 0$. Thus $r(t) = c_1 e^{-\Omega t} + c_2 e^{\Omega t}$. From the initial conditions,

$r(0) = c_1 + c_2 = r_0$ and $r'(0) = -\Omega c_1 + \Omega c_2 = r_0'$. Solving these simultaneous equations yields

$c_1 = \frac{1}{2}\left(r_0 - \Omega^{-1}r_0'\right)$ and $c_2 = \frac{1}{2}\left(r_0 + \Omega^{-1}r_0'\right)$. Thus the unique solution is

$r = \frac{1}{2}\left(r_0 - \Omega^{-1}r_0'\right)e^{-\Omega t} + \frac{1}{2}\left(r_0 + \Omega^{-1}r_0'\right)e^{\Omega t} = r_0\cosh(\Omega t) + \Omega^{-1}r_0'\sinh(\Omega t)$. When $r_0 = r_0' = 0$,

$r(t) = 0$ and the particle remains at rest at the pivot.

21 (b). $r_0 = 0,\ r_0' = \frac{1}{5},\ \ell = 3,\ \Omega = \frac{30(2\pi)}{60} = \pi$. We need to find t when $r(t) = \ell = 3$. With this condition,

we have $r(t) = 3 = \frac{1}{\pi}\cdot\frac{1}{5}\sinh(\pi t)$. Solved for t, $t = 1.44705$ seconds.

22 (a). $\lambda^2 + \frac{k}{m}\lambda - \Omega^2 = 0$. Thus $\lambda_\pm = \dfrac{-\frac{k}{m} \pm \sqrt{\left(k/m\right)^2 + 4\Omega^2}}{2}$ and the general solution to the differential

equation is $r = c_1 e^{\lambda_- t} + c_2 e^{\lambda_+ t}$. From the initial conditions, we have

$c_1 + c_2 = 0$ and $\lambda_- c_1 + \lambda_+ c_2 = r_0'$. Solving these simultaneous equations yields

$c_2 = -c_1 = \dfrac{r_0'}{\sqrt{\left(k/m\right)^2 + 4\Omega^2}}$, and thus $r(t) = \dfrac{r_0' e^{-\frac{k}{2m}t}\cdot 2\sinh\left[\frac{1}{2}\left(\sqrt{\left(k/m\right)^2 + 4\Omega^2}\right)t\right]}{\sqrt{\left(k/m\right)^2 + 4\Omega^2}}$

22 (b). $m \approx R^3, k \approx R^2 \quad \therefore \frac{k}{m} \approx \frac{1}{R}$ would decrease.

22 (c). $\Omega = 20\,{}^{rev}\!/_{min} = 20(2\pi)/60 = \dfrac{2\pi}{3}\,{}^{rad}\!/_{sec}$, $r_0' = 1$, $\dfrac{k}{m} = 4s^{-1}$

$$\sqrt{\left(\frac{k}{m}\right)^2 + 4\Omega^2} = \sqrt{16 + 4\left(\frac{4\pi^2}{9}\right)} = 5.79189,\ \frac{k}{2m} = 2,\ r_0' = 1.$$

$$\therefore r(t) = \frac{1 \cdot e^{-2t} \cdot 2\sinh[2.89594t]}{5.79189},\ r(2) = 1.03605cm.$$

23 (a). $r = c_2\left[e^{\lambda_+ t} - e^{\lambda_- t}\right]$ where $\lambda_+ > 0$ and $\lambda_- < 0$ (from question 22). For the positive limit,

$\lambda_+ = \dfrac{\Omega}{2} = \dfrac{1}{2}\left(-\dfrac{k}{m} + \sqrt{\left(k/m\right)^2 + 4\Omega^2}\right)$. Therefore

$\Omega + \dfrac{k}{m} = \sqrt{\left(k/m\right)^2 + 4\Omega^2}$, and $\Omega^2 + 2\dfrac{k}{m}\Omega + \left(\dfrac{k}{m}\right)^2 = \left(\dfrac{k}{m}\right)^2 + 4\Omega^2$. Then

$2\dfrac{k}{m}\Omega = 3\Omega^2$, and $\dfrac{k}{m} = \dfrac{3}{2}\Omega$.

23 (b). Using the relation reached in part (a), $\left(\dfrac{k}{m}\right)^2 + 4\Omega^2 = \dfrac{9}{4}\Omega^2 + 4\Omega^2 = \dfrac{25}{4}\Omega^2$. Therefore

$\sqrt{\left(k/m\right)^2 + 4\Omega^2} = \dfrac{5}{2}\Omega$ and $\lambda_- = \dfrac{1}{2}\left[-\dfrac{k}{m} - \sqrt{\left(k/m\right)^2 + 4\Omega^2}\right] = \dfrac{1}{2}\left[-\dfrac{3}{2}\Omega - \dfrac{5}{2}\Omega\right] = -2\Omega.$

$\lambda_+ = \dfrac{1}{2}\left[-\dfrac{3}{2}\Omega + \dfrac{5}{2}\Omega\right] = \dfrac{\Omega}{2}$, and so $r(t) = \dfrac{r_0'}{\frac{5}{2}\Omega}\left[e^{\frac{\Omega}{2}t} - e^{-2\Omega t}\right]$.

Section 4.5

1 (a). $\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$. Thus $y = c_1e^{-t} + c_2te^{-t}$.

1 (b). $y' = -c_1e^{-t} + c_2(1-t)e^{-t}$. From the initial conditions, we have

$c_1e^{-1} + c_2e^{-1} = 1$ and $-c_1e^{-1} + c_2 \cdot 0 = 0$, and thus $c_1 = 0$ and $c_2 = e$. The unique solution is then

$y(t) = te^{-(t-1)}$.

1 (c). $\lim\limits_{t\to-\infty} y = -\infty$ and $\lim\limits_{t\to\infty} y = 0$.

2 (a). $9\lambda^2 - 6\lambda + 1 = (3\lambda - 1)^2 = 0$. Thus $y = c_1e^{t/3} + c_2te^{t/3}$.

2 (b). $y' = \frac{1}{3}c_1e^{t/3} + c_2(1 + \frac{t}{3})e^{t/3}$. $c_1e + 3c_2e = -2$ and $\frac{1}{3}c_1e + 2c_2e = -\frac{5}{3} \Rightarrow c_1 = e^{-1}$ and $c_2 = -e^{-1}$. The unique solution is then $y(t) = e^{(t-3)/3} - te^{(t-3)/3} = (1-t)e^{(t-3)/3}$.

2 (c). $\lim\limits_{t\to-\infty} y = 0$ and $\lim\limits_{t\to\infty} y = -\infty$.

3 (a). $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$. Thus $y = c_1e^{-3t} + c_2te^{-3t}$.

3 (b). $y' = -3c_1e^{-3t} + c_2(1-3t)e^{-3t}$. From the initial conditions, we have

$c_1 = 2$ and $-3c_1 + c_2 = -2$, thus $c_2 = 4$. The unique solution is then $y(t) = (2+4t)e^{-3t}$.

3 (c). $\lim_{t\to-\infty} y = -\infty$ and $\lim_{t\to\infty} y = 0$.

4 (a). $25\lambda^2 + 20\lambda + 4 = (5\lambda+2)^2 = 0$. Thus $y = c_1e^{-\frac{2}{5}t} + c_2te^{-\frac{2}{5}t}$.

4 (b). $y' = \frac{-2}{5}c_1e^{-\frac{2}{5}t} + c_2(1-\frac{2t}{5})e^{-\frac{2}{5}t}$.

$c_1e^{-2} + 5c_2e^{-2} = 4e^{-2}$ and $\frac{-2}{5}c_1e^{-2} + (1-2)c_2e^{-2} = -\frac{3}{5}e^{-2} \Rightarrow c_1 = -1$ and $c_2 = 1$.

The unique solution is then $y(t) = (t-1)e^{-\frac{2}{5}t}$.

4 (c). $\lim_{t\to-\infty} y = -\infty$ and $\lim_{t\to\infty} y = 0$.

5 (a). $4\lambda^2 - 4\lambda + 1 = (2\lambda-1)^2 = 0$. Thus $y = c_1e^{\frac{t}{2}} + c_2te^{\frac{t}{2}}$.

5 (b). $y' = \frac{1}{2}c_1e^{\frac{t}{2}} + c_2(1+\frac{t}{2})e^{\frac{t}{2}}$. From the initial conditions, we have

$c_1e^{\frac{1}{2}} + c_2e^{\frac{1}{2}} = -4$ and $\frac{1}{2}c_1e^{\frac{1}{2}} + \frac{3}{2}c_2e^{\frac{1}{2}} = 0$, and thus $c_1 = -6e^{-\frac{1}{2}}$ and $c_2 = 2e^{-\frac{1}{2}}$. The unique

solution is then $y(t) = (-6+2t)e^{\frac{t-1}{2}}$.

5 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = \infty$.

6 (a). $\lambda^2 - 4\lambda + 4 = (\lambda-2)^2 = 0$. Thus $y = c_1e^{2t} + c_2te^{2t}$.

6 (b). $c_1e^{-2} - c_2e^{-2} = 2$ and $2c_1e^{-2} + (1-2)c_2e^{-2} = 1 \Rightarrow c_1 = -e^2$ and $c_2 = -3e^2$

The unique solution is then $y(t) = (-1-3t)e^{2(t+1)}$.

6 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = -\infty$.

7 (a). $16\lambda^2 - 8\lambda + 1 = (4\lambda-1)^2 = 0$. Thus $y = c_1e^{\frac{t}{4}} + c_2te^{\frac{t}{4}}$.

7 (b). From the initial conditions, we have $y(0) = c_1 = -4$ and $y_0' = \frac{1}{4}c_1 + c_2 = 3$. Thus $c_2 = 4$, and so

the unique solution is $y(t) = (-4+4t)e^{\frac{t}{4}}$.

7 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = \infty$.

8 (a). $\lambda^2 + 2\sqrt{2}\lambda + 2 = (\lambda+\sqrt{2})^2 = 0$. Thus $y = c_1e^{-\sqrt{2}t} + c_2te^{-\sqrt{2}t}$.

8 (b). $c_1 + 0 = 1$ and $-\sqrt{2}c_1 + (1-0)c_2 = 0 \Rightarrow c_1 = 1$ and $c_2 = \sqrt{2}$

The unique solution is then $y(t) = (1+\sqrt{2}t)e^{-\sqrt{2}t}$.

8 (c). $\lim_{t\to-\infty} y = -\infty$ and $\lim_{t\to\infty} y = 0$.

9 (a). $\lambda^2-5\lambda+6.25=\left(\lambda-\frac{5}{2}\right)^2=0$. Thus $y=c_1e^{\frac{5}{2}t}+c_2te^{\frac{5}{2}t}$.

9 (b). $y'=\frac{5}{2}c_1e^{\frac{5}{2}t}+c_2\left(1+\frac{5}{2}t\right)e^{\frac{5}{2}t}$. From the initial conditions, we have

$c_1e^{-5}-2c_2e^{-5}=0$ and $\frac{5}{2}c_1e^{-5}-4c_2e^{-5}=1$, and thus $c_1=2e^5$ and $c_2=e^5$. The unique solution is

then $y(t)=(2+t)e^{\frac{5}{2}(t+2)}$.

9 (c). $\lim_{t\to-\infty} y=0$ and $\lim_{t\to\infty} y=\infty$.

10 (a). $3\lambda^2+2\sqrt{3}\lambda+1=(\sqrt{3}\lambda+1)^2=0$. Thus $y=c_1e^{-\frac{1}{\sqrt{3}}t}+c_2te^{-\frac{1}{\sqrt{3}}t}$.

10 (b). $y'=\frac{-1}{\sqrt{3}}c_1e^{-\frac{1}{\sqrt{3}}t}+c_2(1-\frac{t}{\sqrt{3}})e^{-\frac{1}{\sqrt{3}}t}$. $c_1+0=2\sqrt{3}$ and $\frac{-1}{\sqrt{3}}c_1+(1-0)c_2=3\Rightarrow$ $c_1=2\sqrt{3}$ and $c_2=5$.

The unique solution is then $y(t)=(2\sqrt{3}+5t)e^{-\frac{1}{\sqrt{3}}t}$.

10 (c). $\lim_{t\to-\infty} y=-\infty$ and $\lim_{t\to\infty} y=0$.

11. $\lambda^2-2\alpha\lambda+\alpha^2=(\lambda-\alpha)^2=0$. Thus $y=c_1e^{\alpha t}+c_2te^{\alpha t}$. From the initial conditions and the graph provided, $y(0)=c_1=0$ and at the maximum, $y'=c_2(1+\alpha t)e^{\alpha t}=0$. Solving for the t coordinate at the maximum gives us $t_{\max}=-\frac{1}{\alpha}=2,$ and thus $\alpha=-\frac{1}{2}$.

Solving for the y coordinate at the maximum gives us $y_{\max}=c_2\left(-\frac{1}{\alpha}\right)e^{-1}=2c_2e^{-1}=8e^{-1}$, and

thus $c_2=4$. Finally, the equation for $y(t)$ is $y(t)=4te^{-\frac{t}{2}}$, and $\alpha=-\frac{1}{2},\ y_0=0,\ y_0'=4$.

12. $y''=0\ \therefore \alpha=0,\ y=y_0't+y_0,\ y(0)=y_0=2,\ y(4)=(y_0')(4)+2=0\Rightarrow y_0'=-\frac{1}{2}$.

Therefore, $y(t)=-\frac{1}{2}t+2,\ \alpha=0,\ y_0=2,\ y_0'=-\frac{1}{2}$.

13. $4\lambda^2+4\lambda+1=(2\lambda+1)^2=0$. Thus $y=c_1e^{-\frac{t}{2}}+c_2te^{-\frac{t}{2}}$. From the first point given, we have $y(1)=c_1e^{-\frac{1}{2}}+c_2e^{-\frac{1}{2}}=e^{-\frac{1}{2}}$. From the second, we have $y(2)=c_1e^{-1}+2c_2e^{-1}=0$. Solving these two simultaneous equations yields $c_1=2$ and $c_2=-1$. Finally, we have $y(t)=2e^{-\frac{t}{2}}-te^{-\frac{t}{2}}$, and differentiation gives us $y'=-e^{-\frac{t}{2}}-\frac{1}{2}(2-t)e^{-\frac{t}{2}}$. Thus $y(0)=2$ and $y'(0)=-2$.

14 (a). $y_2 = e^t\upsilon,\ y_2' = e^t\upsilon + e^t\upsilon',\ \ y_2'' = e^t v + 2e^t v' + e^t\upsilon''$. Therefore

$$t(\upsilon'' + 2\upsilon' + v) - (2t+1)(\upsilon' + v) + (t+1)\upsilon = 0 \Rightarrow$$

$$t\upsilon'' + (2t - 2t - 1)\upsilon' + (t - 2t - 1 + t + 1)v = t\upsilon'' - \upsilon' = 0;$$

$$u = v',\ u' - t^{-1}u = 0,\ \left(t^{-1}u\right)' = 0,\ u = ct \Rightarrow v = c\frac{t^2}{2}\ \ \therefore y_2 = t^2e^t$$

14 (b). $W = \begin{vmatrix} e^t & t^2e^t \\ e^t & (t^2+2t)e^t \end{vmatrix} = 2te^{2t} \neq 0$ on $(-\infty,0)$ and $(0,\infty)$.

14 (c). $p(t) = -\dfrac{(2t+1)}{t},\ q(t) = \dfrac{t+1}{t}$, continuous on $(-\infty,0)$ and $(0,\infty)$.

15 (a). $y_2 = t\upsilon,\ y_2' = \upsilon + t\upsilon',\ \ y_2'' = t\upsilon'' + 2\upsilon'$. Therefore

$t^2(t\upsilon'' + 2\upsilon') - t(\upsilon + t\upsilon') + t\upsilon = t^3\upsilon'' + t^2\upsilon' = t^2\left(tv'' + v'\right) = 0$, and so

$\left(t\upsilon'\right)' = 0$, which means that $\upsilon' = \dfrac{c}{t}$. Antidifferentiation yields $\upsilon = c\ln|t| + c'$, and thus

$y_2 = t\ln|t|$.

15 (b). $W = \begin{vmatrix} t & t\ln|t| \\ 1 & \ln|t|+1 \end{vmatrix} = t \neq 0$ on $(-\infty,0)$ and $(0,\infty)$.

15 (c). $p(t) = -\dfrac{1}{t},\ q(t) = \dfrac{1}{t^2}$, continuous on $(-\infty,0)$ and $(0,\infty)$.

16 (a). $y_2 = \upsilon\sin t,\ y_2' = \upsilon'\sin t + \upsilon\cos t,\ \ y_2'' = v''\sin t + 2v'\cos t - v\sin t$. Therefore

$$(\upsilon''\sin t + 2\upsilon'\cos t - v\sin t) - (2\cot t)(v'\sin t + v\cos t) + (1 + 2\cot^2 t)\sin v = 0 \Rightarrow$$

$$\upsilon''(\sin t) + \upsilon'(2\cos t - 2\cot t\sin t) + v(-\sin t - 2\cot t\cos t + \sin t + 2\cot^2 t\sin t) = 0$$

$$\Rightarrow v''\sin t = 0 \Rightarrow v'' = 0,\ \ v = c_1t + c_2\ \ \therefore y_2 = t\sin t.$$

16 (b). $W = \begin{vmatrix} \sin t & t\sin t \\ \cos t & \sin t + t\cos t \end{vmatrix} = \sin^2 t \neq 0$ on $(\mathrm{n}\pi,(\mathrm{n}+1)\pi)$, n = ...,-2,-1,0,1,2,....

16 (c). $p(t) = -2\cot t,\ q(t) = 1 + 2\cot^2 t$, continuous on same intervals.

17 (a). $y_2 = (t+1)^2\upsilon,\ y_2' = 2(t+1)\upsilon + (t+1)^2\upsilon',\ \ y_2'' = (t+1)^2\upsilon'' + 4(t+1)\upsilon' + 2\upsilon$. Therefore

$$(t+1)^4\upsilon'' + 4(t+1)^3\upsilon' + 2(t+1)^2\upsilon - 4\left[2(t+1)^2\upsilon + (t+1)^3\upsilon'\right] + 6(t+1)^2v$$

$= (t+1)^4\upsilon'' = 0$, and so $\upsilon'' = 0$. Antidifferentiation yields $\upsilon = c_1(t+1) + c_2$, and thus

$y_2 = (t+1)^3$.

17 (b). $W = \begin{vmatrix} (t+1)^2 & (t+1)^3 \\ 2(t+1) & 3(t+1)^2 \end{vmatrix} = (t+1)^4 \neq 0$ on $(-\infty,-1)$ and $(-1,\infty)$.

17 (c). $p(t) = -\dfrac{4}{t+1}$, $q(t) = \dfrac{6}{(t+1)^2}$, continuous on $(-\infty,-1)$ and $(-1,\infty)$.

18 (a). $y_2 = e^{-t^2}\upsilon,\ y_2' = -2te^{-t^2}\upsilon + e^{-t^2}\upsilon'$, .

$y_2'' = -2e^{-t^2}v + 4t^2e^{-t^2}v - 2te^{-t^2}v' - 2te^{-t^2}v' + e^{-t^2}\upsilon''$.

Therefore, $\upsilon'' - 4t\upsilon' + (-2+4t^2)v + 4t(-2tv + \upsilon') + (2+4t^2)\upsilon = 0 \Rightarrow v'' = 0$;

$\therefore v = c_1t + c_2,\ y_2 = te^{-t^2}$.

18 (b). $W = \begin{vmatrix} e^{-t^2} & te^{-t^2} \\ -2te^{-t^2} & e^{-t^2} - 2t^2e^{-t^2} \end{vmatrix} = e^{-2t^2} \neq 0$ on $(-\infty,\infty)$.

18 (c). $p(t) = 4t,\ q(t) = 2 + 4t^2$, continuous on $(-\infty,\infty)$.

19 (a). $y_2 = (t-2)^2\upsilon,\ y_2' = 2(t-2)\upsilon + (t-2)^2\upsilon',\ y_2'' = (t-2)^2\upsilon'' + 4(t-2)\upsilon' + 2\upsilon$. Therefore

$(t-2)^4\upsilon'' + 4(t-2)^3\upsilon' + 2(t-2)^2\upsilon + 2(t-2)^2\upsilon + (t-2)^3\upsilon' - 4(t-2)^2\upsilon = 0$, and so

$\upsilon'' + \dfrac{5}{(t-2)}\upsilon' = 0$. Thus $\left((t-2)^5\upsilon'\right)' = 0$, and antidifferentiation yields $\upsilon = -\dfrac{c_1}{4(t-2)^4} + c_2$,

and thus $y_2 = (t-2)^{-2}$.

19 (b). $W = \begin{vmatrix} (t-2)^2 & (t-2)^{-2} \\ 2(t-2) & -2(t-2)^{-3} \end{vmatrix} = -\dfrac{4}{(t-2)} \neq 0$ on $(-\infty,2)$ and $(2,\infty)$.

19 (c). $p(t) = \dfrac{1}{t-2}$, $q(t) = -\dfrac{4}{(t-2)^2}$, continuous on $(-\infty,2)$ and $(2,\infty)$.

20 (a). $y_2 = e^t\upsilon,\ y_2' = e^t\upsilon + e^t\upsilon',\ y_2'' = e^tv + 2e^tv' + e^t\upsilon''$. Therefore

$\upsilon'' + 2\upsilon' + v - \left(2 + \dfrac{n-1}{t}\right)(\upsilon' + v) + \left(1 + \dfrac{n-1}{t}\right)\upsilon = 0 \Rightarrow$

$\upsilon'' - \dfrac{n-1}{t}v' = 0 \Rightarrow \left(t^{-(n-1)}v'\right)' = 0 \Rightarrow v' = c_1t^{n-1} \Rightarrow v = \frac{c_1}{n}t^n$, $\therefore y_2 = t^ne^t$

20 (b). $W = \begin{vmatrix} e^t & t^ne^t \\ e^t & (nt^{n-1} + t^n)e^t \end{vmatrix} = nt^{n-1}e^{2t}$.

If $n = 1$, $W \neq 0$ on $(-\infty,\infty)$. If $n \geq 2$, $W \neq 0$ on $(-\infty,0),(0,\infty)$.

20 (c). $p(t) = -\left(2 + \dfrac{n-1}{t}\right)$, $q(t) = \left(1 + \dfrac{n-1}{t}\right)$, continuous on same intervals.

Section 4.6

1 (a). $2e^{i\pi/3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + i\sqrt{3}$

1 (b). $-2\sqrt{2}e^{-i\pi/4} = -2\sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right) = -2 + i2$

1 (c). $(2-i)e^{i3\pi/2} = (2-i)(-i) = -1 - i2$.

1 (d). $\frac{1}{2\sqrt{2}}e^{i7\pi/6} = \frac{1}{2\sqrt{2}}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}}$

1 (e). $\left(\sqrt{2}e^{i\pi/6}\right)^4 = 4e^{i2\pi/3} = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -2 + i2\sqrt{3}$

2 (a). $2\cos(\sqrt{2}t) + i2\sin(\sqrt{2}t)$

2 (b). $\frac{2}{\pi}e^{-2t}\cos(3t) - i\frac{2}{\pi}e^{-2t}\sin(3t)$

2 (c). $-\frac{1}{2}e^{2t}[\cos(t+\pi) + i\sin(t+\pi)] = \frac{1}{2}e^{2t}\cos t + i\frac{1}{2}e^{2t}\sin t$.

2 (d). $3\sqrt{3}e^{3t}\cos(3t) + i3\sqrt{3}e^{3t}\sin(3t)$

2 (e). $-\frac{\sqrt{2}}{4}(\cos 3\pi t + i\sin 3\pi t) = -\frac{\sqrt{2}}{4}\cos(3\pi t) - i\frac{\sqrt{2}}{4}\sin(3\pi t)$

3 (a). $\lambda^2 + 4 = 0$, and thus $\lambda = \pm i2$.

3 (b). $y = c_1\cos(2t) + c_2\sin(2t)$

3 (c). $y' = -2c_1\sin(2t) + 2c_2\cos(2t)$. Using the initial conditions, we have

$y(\pi/4) = c_2 = -2$ and $y'(\pi/4) = -2c_1 = 1$. Thus

$c_1 = -\frac{1}{2},\ c_2 = -2,$ and $y = -\frac{1}{2}\cos(2t) - 2\sin(2t)$.

4 (a). $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$.

4 (b). $y = c_1e^{-t}\cos t + c_2e^{-t}\sin t$

4 (c). $y' = -c_1e^{-t}\cos t - c_1e^{-t}\sin t - c_2e^{-t}\sin t + c_2e^{-t}\cos t$. $y(0) = c_1 = 3$ and $y'(0) = -c_1 + c_2 = -1$.

Thus $c_1 = 3,\ c_2 = 2,$ and $y = 3e^{-t}\cos t + 2e^{-t}\sin t$.

5 (a). $9\lambda^2 + 1 = 0$, and thus $\lambda = \pm i\frac{1}{3}$.

5 (b). $y = c_1\cos\left(\frac{t}{3}\right) + c_2\sin\left(\frac{t}{3}\right)$

5 (c). $y' = -\frac{1}{3}c_1 \sin\left(\frac{t}{3}\right) + \frac{1}{3}c_2 \cos\left(\frac{t}{3}\right)$. Using the initial conditions, we have

$$y(\pi/2) = \frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = 4 \text{ and } y'(\pi/2) = -\frac{1}{3}c_1\left(\frac{1}{2}\right) + \frac{1}{3}c_2\left(\frac{\sqrt{3}}{2}\right) = 0.$$

Solving these simultaneous equations gives us

$$c_1 = 2\sqrt{3},\ c_2 = 2, \text{ and } y = 2\sqrt{3}\cos\left(\frac{t}{3}\right) + 2\sin\left(\frac{t}{3}\right).$$

6 (a). $2\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{4-8}}{4} = \frac{1}{2} \pm i\frac{1}{2}$.

6 (b). $y = c_1 e^{\frac{t}{2}} \cos\left(\frac{t}{2}\right) + c_2 e^{\frac{t}{2}} \sin\left(\frac{t}{2}\right)$

6 (c). $y' = \frac{1}{2}c_1 e^{\frac{t}{2}} \cos\left(\frac{t}{2}\right) - \frac{1}{2}c_1 e^{\frac{t}{2}} \sin\left(\frac{t}{2}\right) + \frac{1}{2}c_2 e^{\frac{t}{2}} \sin\left(\frac{t}{2}\right) + \frac{1}{2}c_2 e^{\frac{t}{2}} \cos\left(\frac{t}{2}\right)$.

$y(-\pi) = -c_2 e^{-\frac{\pi}{2}} = 1$ and $y'(-\pi) = -\frac{1}{2}c_1 e^{-\frac{\pi}{2}}(-1) + \frac{1}{2}c_2 e^{-\frac{\pi}{2}}(-1) = -1$. Thus

$c_1 = -3e^{\frac{\pi}{2}}$, $c_2 = -e^{\frac{\pi}{2}}$, and $y = -3e^{\frac{t+\pi}{2}} \cos\left(\frac{t}{2}\right) - e^{\frac{t+\pi}{2}} \sin\left(\frac{t}{2}\right)$.

7 (a). $\lambda^2 + \lambda + 1 = 0$, and thus $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

7 (b). $y = c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_2 e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right)$

7 (c). $y' = -\frac{1}{2}c_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2}c_1 e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right)$

$$-\frac{1}{2}c_2 e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2}c_2 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2}t\right).$$

Using the initial conditions, we have $y(0) = c_1 = -2$ and $y'(0) = -\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = -2$.

Solving these simultaneous equations gives us

$$c_1 = -2,\ c_2 = -2\sqrt{3}, \text{ and } y = -2e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2}t\right) - 2\sqrt{3}e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right).$$

8 (a). $\lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$.

8 (b). $y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$

8 (c). $y' = -2c_1 e^{-2t} \cos t - c_1 e^{-2t} \sin t - 2c_2 e^{-2t} \sin t + c_2 e^{-2t} \cos t$.

$y\left(\frac{\pi}{2}\right) = c_2 e^{-\pi} = \frac{1}{2}$ and $y'\left(\frac{\pi}{2}\right) = -c_1 e^{-\pi} - 2c_2 e^{-\pi} = -2$. Thus

$c_1 = e^{\pi}$, $c_2 = \frac{1}{2}e^{\pi}$, and $y = e^{-2\left(t-\frac{\pi}{2}\right)}(\cos t + \frac{1}{2}\sin t)$.

9 (a). $9\lambda^2 + 6\lambda + 2 = 0$, and thus $\lambda = -\frac{1}{3} \pm i\frac{1}{3}$.

9 (b). $y = c_1 e^{-\frac{t}{3}} \cos\left(\frac{t}{3}\right) + c_2 e^{-\frac{t}{3}} \sin\left(\frac{t}{3}\right)$

9 (c). $y' = -\frac{1}{3}c_1 e^{-\frac{t}{3}} \cos\left(\frac{t}{3}\right) - \frac{1}{3}c_1 e^{-\frac{t}{3}} \sin\left(\frac{t}{3}\right) - \frac{1}{3}c_2 e^{-\frac{t}{3}} \sin\left(\frac{t}{3}\right) + \frac{1}{3}c_2 e^{-\frac{t}{3}} \cos\left(\frac{t}{3}\right)$. Using the initial conditions, we have $y(3\pi) = -c_1 e^{-\pi} \Rightarrow c_1 = 0$ and $y'(3\pi) = -\frac{1}{3}c_2 e^{-\pi} = \frac{1}{3}$. Solving these simultaneous equations gives us $c_1 = 0,\ c_2 = -e^{\pi}$, and $y = -e^{-\frac{t}{3}} e^{\pi} \sin\left(\frac{t}{3}\right)$.

10 (a). $\lambda^2 + 4\pi^2 = 0 \Rightarrow \lambda = \pm 2\pi i$.

10 (b). $y = c_1 \cos(2\pi t) + c_2 \sin(2\pi t)$

10 (c). $y' = -2\pi c_1 \sin(2\pi t) + 2\pi c_2 \cos(2\pi t)$.

$y(1) = c_1 = 2$ and $y'(1) = 2\pi c_2 = 1 \Rightarrow c_2 = \frac{1}{2\pi}$.

Thus $y = 2\cos(2\pi t) + \frac{1}{2\pi}\sin(2\pi t)$.

11 (a). $\lambda^2 - 2\sqrt{2}\lambda + 3 = 0$, and thus $\lambda = \sqrt{2} \pm i$.

11 (b). $y = c_1 e^{\sqrt{2}t} \cos(t) + c_2 e^{\sqrt{2}t} \sin(t)$

11 (c). $y' = \sqrt{2}c_1 e^{\sqrt{2}t} \cos(t) - c_1 e^{\sqrt{2}t} \sin(t) + \sqrt{2}c_2 e^{\sqrt{2}t} \sin(t) + c_2 e^{\sqrt{2}t} \cos(t)$. Using the initial conditions, we have $y(0) = c_1 = -\frac{1}{2}$ and $y'(0) = \sqrt{2}c_1 + c_2 = \sqrt{2}$.

Solving these simultaneous equations gives us

$c_1 = -\frac{1}{2},\ c_2 = \frac{3\sqrt{2}}{2}$, and $y = -\frac{1}{2}e^{\sqrt{2}t}\cos(t) + \frac{3\sqrt{2}}{2}e^{\sqrt{2}t}\sin(t)$.

12 (a). $9\lambda^2 + \pi^2 = 0 \Rightarrow \lambda = \pm\frac{\pi}{3}i$.

12 (b). $y = c_1 \cos\left(\frac{\pi}{3}t\right) + c_2 \sin\left(\frac{\pi}{3}t\right)$

12 (c). $y' = -\frac{\pi}{3}c_1 \sin\left(\frac{\pi}{3}t\right) + \frac{\pi}{3}c_2 \cos\left(\frac{\pi}{3}t\right)$.

$y(3) = -c_1 = 2$ and $y'(3) = -\frac{\pi}{3}c_2 = -\pi \Rightarrow c_1 = -2,\ c_2 = 3$.

Thus $y = -2\cos\left(\frac{\pi}{3}t\right) + 3\sin\left(\frac{\pi}{3}t\right)$

13. $\lambda = \pm i$, so the characteristic equation must be $\lambda^2 + 1 = 0$. Thus $a = 0$ and $b = 1$.
$y_0 = y(\pi/4) = \dfrac{1}{\sqrt{2}} - 1$ and $y_0' = y'(\pi/4) = \dfrac{1}{\sqrt{2}} + 1$.

14. $\lambda = \pm 2i$, so the characteristic equation must be $\lambda^2 + 4 = 0$. Thus $a = 0$ and $b = 4$.
$y_0 = y(\pi/4) = 2$ and $y_0' = y'(\pi/4) = -2$.

15. $\lambda = -2 \pm i$, so the characteristic equation must be $\lambda^2 + 4\lambda + 5 = 0$. Thus $a = 4$ and $b = 5$.
$y_0 = y(0) = 1$ and $y_0' = y'(0) = -2 - 1 = -3$.

16. $\lambda = 1 \pm 2i$, so the characteristic equation must be $(\lambda - 1)^2 + 4 = \lambda^2 - 2\lambda + 5 = 0$. Thus
$a = -2$ and $b = 5$. $y_0 = y(\pi/6) = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}$ and $y_0' = y'(\pi/6) = -\dfrac{1}{2} - \dfrac{3\sqrt{3}}{2}$.

17. $\lambda = \pm i\pi$, so the characteristic equation must be $\lambda^2 + \pi^2 = 0$. Thus $a = 0$ and $b = \pi^2$.
$y_0 = y(1/2) = -1$ and $y_0' = y'(1/2) = -\sqrt{3}\pi$.

18. $y = \sin t + \cos t$, so $\alpha = 0$ and $\beta = 1$. $R\cos\delta = 1$ and $R\sin\delta = 1$, so $R = \sqrt{2}$ and $\delta = \dfrac{\pi}{4}$. Thus
$y = \sqrt{2}\cos\left(t - \dfrac{\pi}{4}\right)$.

19. $y = \cos \pi t - \sin \pi t$, so $\alpha = 0$ and $\beta = \pi$. $R\cos\delta = 1$ and $R\sin\delta = -1$, so $R = \sqrt{2}$ and $\delta = \dfrac{7\pi}{4}$.
Thus $y = \sqrt{2}\cos\left(\pi t - \dfrac{7\pi}{4}\right)$.

20. $y = e^t \cos t + \sqrt{3}e^t \sin t$, so $\alpha = 1$ and $\beta = 1$. $R\cos\delta = 1$ and $R\sin\delta = \sqrt{3}$, so $R = 2$ and $\delta = \dfrac{\pi}{3}$.
Thus $y = \sqrt{3}e^t \cos\left(t - \dfrac{\pi}{3}\right)$.

21. $y = -e^{-t}\cos t + \sqrt{3}e^{-t}\sin t$, so $\alpha = -1$ and $\beta = 1$. $R\cos\delta = -1$ and $R\sin\delta = \sqrt{3}$, so
$R = 2$ and $\delta = \dfrac{2\pi}{3}$. Thus $y = 2e^{-t}\cos\left(t - \dfrac{2\pi}{3}\right)$.

22. $y = e^{-2t}\cos 2t - e^{-2t}\sin 2t$, so $\alpha = -2$ and $\beta = 2$. $R\cos\delta = 1$ and $R\sin\delta = -1$, so
$R = \sqrt{2}$ and $\delta = \dfrac{7\pi}{4}$. Thus $y = \sqrt{2}e^{-2t}\cos\left(2t - \dfrac{7\pi}{4}\right)$.

23. $y(t) = 2\cos\left(\dfrac{\pi}{2}t\right)$, $a = 0$, $b = \dfrac{\pi^2}{4}$, $y_0 = 2$, $y_0' = 0$.

24. $y(t) = \cos\left(\dfrac{3}{2}t - \dfrac{\pi}{8}\right)$, $a = 0$, $b = \dfrac{9}{4}$, $y_0 = \cos\left(\dfrac{\pi}{8}\right)$, $y_0' = \dfrac{3}{2}\sin\left(\dfrac{\pi}{8}\right)$.

25. $y(t)=\frac{1}{2}\cos\left(2t-\frac{5\pi}{6}\right),\ a=0,\ b=4,\ y_0=\frac{1}{2}\cos\frac{5\pi}{6},\ y_0'=\sin\frac{5\pi}{6}$.

26 (a). $\lambda^2+\mu\lambda+\omega^2=0$. $\lambda=\frac{-\mu\pm\sqrt{\mu^2-4\omega^2}}{2}=-\frac{\mu}{2}\pm\frac{i}{2}\sqrt{4\omega^2-\mu^2}$

$$y=c_1e^{-\frac{\mu}{2}t}\cos\left(\sqrt{\omega^2-\tfrac{\mu^2}{4}}\,t\right)+c_2e^{-\frac{\mu}{2}t}\sin\left(\sqrt{\omega^2-\tfrac{\mu^2}{4}}\,t\right).$$

26 (b). $y(0)=c_1=2,\ y'(0)=-\frac{\mu}{2}c_1+\sqrt{\omega^2-\frac{\mu^2}{4}}\,c_2=0\Rightarrow c_2=\frac{\mu}{\sqrt{\omega^2-\frac{\mu^2}{4}}}$.

$$y=e^{-\frac{\mu}{2}t}\left[2\cos\left(\sqrt{\omega^2-\tfrac{\mu^2}{4}}\,t\right)+\frac{\mu}{\sqrt{\omega^2-\frac{\mu^2}{4}}}\sin\left(\sqrt{\omega^2-\tfrac{\mu^2}{4}}\,t\right)\right].$$

26 (c). $\alpha=-\frac{\mu}{2},\ \beta=\sqrt{\omega^2-\frac{\mu^2}{4}}$.

$$R^2=4+\frac{\mu^2}{\omega^2-\frac{\mu^2}{4}}=\frac{4\omega^2}{\omega^2-\frac{\mu^2}{4}}\Rightarrow R=\frac{2\omega}{\sqrt{\omega^2-\frac{\mu^2}{4}}},\ \tan\delta=\frac{\mu}{2\sqrt{\omega^2-\frac{\mu^2}{4}}}=\frac{\mu}{\sqrt{4\omega^2-\mu^2}}.$$

27 (a). $e^{\alpha t}\cos\beta t=\frac{1}{2}e^{(\alpha+i\beta)t}+\frac{1}{2}e^{(\alpha-i\beta)t},\ e^{\alpha t}\sin\beta t=\frac{1}{2i}e^{(\alpha+i\beta)t}-\frac{1}{2i}e^{(\alpha-i\beta)t}$.

27 (b). $\left[e^{\alpha t}\cos\beta t\ ,\ e^{\alpha t}\sin\beta t\right]=\left[e^{(\alpha+i\beta)t}\ ,\ e^{(\alpha-i\beta)t}\right]\begin{bmatrix}\frac{1}{2} & \frac{1}{2i}\\ \frac{1}{2} & -\frac{1}{2i}\end{bmatrix}$.

27 (c). $\det A=-\frac{1}{2i}$, so $A^{-1}=-2i\begin{bmatrix}-\frac{1}{2i} & -\frac{1}{2i}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}=\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}$.

$\left[e^{(\alpha+i\beta)t}\ ,\ e^{(\alpha-i\beta)t}\right]=\left[e^{\alpha t}\cos\beta t\ ,\ e^{\alpha t}\sin\beta t\right]\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}$, so

$e^{(\alpha+i\beta)t}=e^{\alpha t}\cos\beta t+ie^{\alpha t}\sin\beta t$ and $e^{(\alpha-i\beta)t}=e^{\alpha t}\cos\beta t-ie^{\alpha t}\sin\beta t$.

28. From Abel's Theorem, $a=0\ \therefore\ y=c_1\cos 3t+c_2\sin 3t$.

29. $\lambda^2+4i\lambda+5=0$, and thus $\lambda=\frac{-4i\pm\sqrt{-16-20}}{2}=i,\ -5i$. Thus $y=c_1e^{it}+c_2e^{-5it}$.

Section 4.7

2 (a). $ky = mg,\ k = \dfrac{9.8(10)}{0.030} = 3266.67\ \ N/m$ or $3.2667\ \ N/mm$.

2 (b). $my'' + ky = 0,\ y'' + 326.67y = 0,\ y(0) = 0.07,\ y'(0) = 0$.

2 (c). $\omega = \sqrt{\frac{k}{m}} = 18.0739...,\ y = c_1 \cos \omega t + c_2 \sin \omega t$.

$y(0) = 0.07,\ c_2\omega = 0 \Rightarrow c_2 = 0\ \ \therefore y = 0.07\cos(18.0739t)$.

3. $my'' + ky = 0,\ y(0) = 0,\ y'(0) = 2$. The general solution to this differential equation is $y = c_1 \cos \omega t + c_2 \sin \omega t$, where $\omega = \sqrt{\dfrac{k}{m}}$. From the initial conditions, we have $y(0) = c_1 = 0$ and $y'(0) = \omega c_2 = 2$, and thus $c_2 = \dfrac{2}{\omega}$. The unique solution is then $y = \dfrac{2}{\omega}\sin \omega t$. We know that $y_{\max} = 0.2 = \dfrac{2}{\omega}$, so $\omega = 10 = \sqrt{\dfrac{k}{20}}$. Solving for k yields $k = 2000$ N/m.

4 (a). $T = 4\left(\frac{5}{4} - \frac{3}{4}\right) = 2\ \ s$.

4 (b). $f = \dfrac{1}{T} = \dfrac{1}{2}\ \ Hz,\ \omega = 2\pi f = \pi\ \ rad/s$.

4 (c). $R = 3$; the first maximum occurs at

$t = \dfrac{3}{4} - \left(\dfrac{5}{4} - \dfrac{3}{4}\right) = \dfrac{1}{4}\ \ \therefore \omega\left(\frac{1}{4}\right) - \delta = 0 \Rightarrow \delta = \dfrac{\pi}{4},\ y = 3\cos\left(\pi t - \frac{\pi}{4}\right)\ \ cm$.

4 (d). $y(0) = 3\cos\left(\frac{\pi}{4}\right) = 2.1213..cm,\ y'(0) = -3\pi\sin\left(-\frac{\pi}{4}\right) = \dfrac{3\pi}{\sqrt{2}} = 6.6643..cm/s$.

5 (a). $my'' + \gamma y' + ky = 0$ with $m = 10,\ \gamma = 7,\ k = 100,\ y(0) = 0.5,\ y'(0) = 1$. Thus with numerical constants the initial value problem reads $y'' + 0.7y' + 10y = 0,\ y(0) = 0.5,\ y'(0) = 1$.

5 (b). $\lambda^2 + 0.7\lambda + 10 = 0$, and thus $\lambda = -0.35 \pm i3.14285 = -\alpha \pm i\beta$. The general solution is then $y = c_1 e^{-0.35t}\cos 3.14285t + c_2 e^{-0.35t}\sin 3.14285t$. From the initial conditions, we have $y(0) = c_1 = 0.5$ and $y'(0) = -0.35c_1 + 3.14285c_2 = 1$. Solving these simultaneous equations yields $c_1 = 0.5$ and $c_2 = 0.37386$. Thus the unique solution to the initial value problem is $y = e^{-0.35t}(0.5\cos(3.14285t) + 0.37386\sin(3.14285t))$. $\lim_{t\to\infty} y(t) = 0$, which means that the damping dissipates the energy of the system, causing the motion to decrease.

6 (a). $my'' + \gamma y' + ky = 0,\ m\lambda^2 + \gamma\lambda + k = 0 \Rightarrow \lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$. Critical damping: $\gamma^2 = 4mk$.

6 (b). $y = c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t},\ y(0) = c_1 = 0,\ y'(0) = c_2 = 4.$

$y'(t) = c_2\left(1 - \frac{\gamma}{2m}t\right)e^{-\frac{\gamma}{2m}t},\ y'(t) = 0 \text{ when } t_m = \dfrac{2m}{\gamma}.$

$y(t_m) = c_2 \dfrac{2m}{\gamma} e^{-1} = \dfrac{1}{2} \quad \therefore 4\dfrac{2m}{\gamma} = \dfrac{e}{2} \Rightarrow \dfrac{m}{\gamma} = \dfrac{e}{16}.$

$m = 1\ slug,\ \gamma = 16e^{-1} \approx 5.886...lb \cdot \sec/ft,\ k = \dfrac{\gamma^2}{4m} = 8.6614...lb/ft.$

7 (a). $my'' + \gamma y' + ky = 0$ with $y(0) = y_0,\ y'(0) = 0$. $m\lambda^2 + \gamma\lambda + k = 0$, and thus $\lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$.

We can rewrite this as $\lambda_1 = \dfrac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m}$ and $\lambda_2 = \dfrac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m}$. The general solution to this initial value problem is $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$. From the initial conditions, we have $y(0) = c_1 + c_2 = y_0$ and $y'(0) = \lambda_1 c_1 + \lambda_2 c_2 = 0$. Solving these simultaneous equations for c_1 and c_2 gives us $c_1 = \dfrac{\lambda_2 y_0}{\lambda_2 - \lambda_1}$ and $c_1 = \dfrac{-\lambda_1 y_0}{\lambda_2 - \lambda_1}$, and thus the unique solution is

$$y = \frac{\left(\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}\right)}{\lambda_2 - \lambda_1} y_0.$$

7 (b).

$$\lambda_1\lambda_2 = \left(-\frac{\gamma}{2m} - \frac{\sqrt{\gamma^2 - 4mk}}{2m}\right)\left(-\frac{\gamma}{2m} + \frac{\sqrt{\gamma^2 - 4mk}}{2m}\right) = \frac{\gamma^2}{4m^2} - \frac{\gamma^2 - 4mk}{4m^2}$$

$\lambda_1\lambda_2 = \dfrac{4mk}{4m^2} = \dfrac{k}{m} \Rightarrow \lambda_2 = \dfrac{k/m}{\lambda_1} = -\dfrac{2k}{\gamma + \sqrt{\gamma^2 - 4mk}}$. Therefore, $\lim\limits_{\gamma\to\infty} \lambda_2 = 0$.

Then, since $\lambda_1 + \lambda_2 = 2\left(\dfrac{-\gamma}{2m}\right) = -\dfrac{\gamma}{m}, \lim\limits_{\gamma\to\infty} \lambda_1 = \lim\limits_{\gamma\to\infty}\left(-\dfrac{\gamma}{m} - \lambda_2\right) = -\infty.$

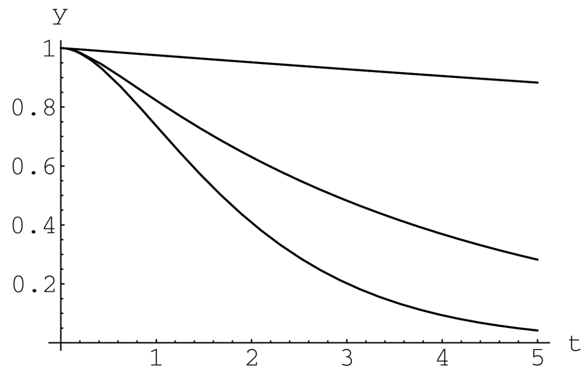
7 (c). $\lim\limits_{\gamma\to\infty} y(t) = \lim\limits_{\gamma\to\infty}\left[\left(\dfrac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1}\right) y_0\right] = y_0 \lim\limits_{\gamma\to\infty}\left[\left(\dfrac{\lambda_2 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} - \dfrac{\lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1}\right)\right]$

$= y_0\left[0 - \lim\limits_{\gamma\to\infty} \dfrac{e^{\lambda_2 t}}{\frac{\lambda_2}{\lambda_1} - 1}\right] = y_0\left[-\dfrac{1}{0 - 1}\right] = y_0.$ As damping increases, the motion becomes suppressed; the system "locks up" and tends to stay at its initial displacement.

8 (a). $y'' + \gamma y' + y = 0,\ y(0) = 1,\ y'(0) = 0,\ \lambda^2 + \gamma\lambda + 1 = 0 \Rightarrow \lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2},$

$\gamma_{crit}^2 - 4 = 0 \Rightarrow \gamma_{crit} = 2.$

8 (b).



The plots are consistent. For a fixed t, y is tending toward $y_0 = 1$ as γ increases.

9. For this problem, we must make $\mu = \dfrac{\gamma}{m}$ and $\dfrac{\rho_\ell g}{\rho L} = \dfrac{k}{m}$. The volume of the drum is

$V = \pi\left(\dfrac{5}{2}\right)^2 8 = 50\pi$ cubic feet. Therefore,

$$\frac{\rho_\ell}{\rho} = \frac{\text{weight of equiv. vol. of water}}{\text{weight of drum}} = \frac{50\pi(62.4)}{6000} = 1.634. \quad \frac{k}{m} = \frac{\rho_\ell}{\rho}\frac{g}{L} = 1.634 \cdot \left(\frac{32}{8}\right) = 6.535 \text{ s}^{-2}$$

and $\dfrac{\gamma}{m} = 0.1 \text{ s}^{-1}$, $m = 5\ kg$, $k = 32.675\ \text{N/m}$, $\gamma = 0.5\ \text{kg/s}$.

10. $my'' + \gamma y' + ky = 0,\ y(0) = 0,\ y'(0) = y_0',\ m\lambda^2 + \gamma\lambda + k = 0 \Rightarrow \lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$.

10 (a). Underdamped: $\lambda = -\alpha \pm i\beta,\ \alpha = \frac{\gamma}{2m},\ \beta = \dfrac{\sqrt{4mk - \gamma^2}}{2m}$

$$y = c_1 e^{-\alpha t}\cos\beta t + c_2 e^{-\alpha t}\sin\beta t,\ y(0) = c_1 = 0,\ y'(0) = \beta c_2 = y_0' \quad \therefore y(t) = \tfrac{y_0'}{\beta} e^{-\alpha t}\sin\beta t.$$

Critically damped:

$$y = c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t},\ y(0) = c_1 = 0,\ y'(0) = c_2 = y_0' \quad \therefore y(t) = y_0' t e^{-\frac{\gamma}{2m}t} = y_0' t e^{-\sqrt{\frac{k}{m}}t}.$$

Overdamped: $\lambda = -\dfrac{\gamma}{2m} \pm \dfrac{1}{2m}\sqrt{\gamma^2 - 4mk} = -\dfrac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}}$

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},\ c_1 + c_2 = 0,\ \lambda_1 c_1 + \lambda_2 c_2 = y_0' = (\lambda_2 - \lambda_1)c_2 \quad \therefore c_2 = \frac{y_0'}{(\lambda_2 - \lambda_1)} = -c_1.$$

$$y(t) = \frac{1}{\lambda_2 - \lambda_1}\left[e^{\lambda_1 t} - e^{\lambda_2 t}\right]y_0' = \frac{1}{2\sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}}} e^{-\frac{\gamma}{2m}t}\left[-e^{-\sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}}\ t} + e^{\sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}}\ t}\right]y_0'.$$

10 (b). As $\gamma \to \gamma_{crit}$, $e^{-\frac{\gamma}{2m}t} \to e^{-\sqrt{\frac{k}{m}}t}$ and use: $\displaystyle\lim_{x\downarrow 0}\frac{\sinh(xt)}{x} = t,\ \lim_{x\uparrow 0}\frac{\sin(xt)}{x} = t.$

Section 4.8

1 (a). $y_P' = 3,\ y_P'' = 0,\ 0 - 2(3) - 3(3t-1) = -9t - 6 + 3 = -9t - 3$.

1 (b). $\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$, and thus $y_C = c_1 e^{-t} + c_2 e^{3t}$.

1 (c). $y = c_1 e^{-t} + c_2 e^{3t} + 3t - 1$. From the initial conditions, we have
$y(0) = c_1 + c_2 - 1 = 1$ and $y'(0) = -c_1 + 3c_2 + 3 = 3$. Solving these simultaneous equations yields
$c_1 = \dfrac{3}{2}$ and $c_2 = \dfrac{1}{2}$, and so $y = \dfrac{3}{2}e^{-t} + \dfrac{1}{2}e^{3t} + 3t - 1$.

2 (b). $y_C = c_1 e^{-t} + c_2 e^{3t}$.

2 (c). $y = c_1 e^{-t} + c_2 e^{3t} - \dfrac{1}{3}e^{2t}$.
$y(0) = c_1 + c_2 - \frac{1}{3} = 1$ and $y'(0) = -c_1 + 3c_2 - \frac{2}{3} = 0 \Rightarrow c_1 = \dfrac{5}{6}$ and $c_2 = \dfrac{1}{2}$, and so
$y = \frac{5}{6}e^{-t} + \frac{1}{2}e^{3t} - \frac{1}{3}e^{2t}$.

3 (a). $y_P' = 8e^{4t},\ y_P'' = 32e^{4t},\ e^{4t}(32 - 8 - 2(2)) = 20e^{4t}$.

3 (b). $\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$, and thus $y_C = c_1 e^{-t} + c_2 e^{2t}$.

3 (c). $y = c_1 e^{-t} + c_2 e^{2t} + 2e^{4t}$. From the initial conditions, we have
$y(0) = c_1 + c_2 + 2 = 0$ and $y'(0) = -c_1 + 2c_2 + 8 = 1$. Solving these simultaneous equations yields
$c_1 = 1$ and $c_2 = -3$, and so $y = e^{-t} - 3e^{2t} + 2e^{4t}$.

4 (b). $y_C = c_1 e^{-t} + c_2 e^{2t}$.

4 (c). $y = c_1 e^{-t} + c_2 e^{2t} - 5$.
$y(-1) = c_1 e^{1} + c_2 e^{-2} - 5 = 0$ and $y'(-1) = -c_1 e^{1} + 2c_2 e^{-2} = 1 \Rightarrow c_1 = 3e^{-1}$ and $c_2 = 2e^{2}$, and so
$y = 3e^{-(t+1)} + 2e^{2(t+1)} - 5$.

5 (a). $2 + (2t - 2) = 2t$.

5 (b). $\lambda^2 + \lambda = \lambda(\lambda + 1) = 0$, and thus $y_C = c_1 e^{-t} + c_2$.

5 (c). $y = c_1 e^{-t} + c_2 + t^2 - 2t$. From the initial conditions, we have
$y(1) = c_1 e^{-1} + c_2 - 1 = 1$ and $y'(1) = -c_1 e^{-1} + 2 - 2 = -2$. Solving these simultaneous equations
yields $c_1 = 2e$ and $c_2 = 0$, and so $y = 2e^{-(t-1)} + t^2 - 2t$.

6 (b). $y_C = c_1 e^{-t} + c_2$.

6 (c). $y = c_1 e^{-t} + c_2 - 2te^{-t}$.
$y(0) = c_1 + c_2 = 2$ and $y'(0) = -c_1 - 2 = 2 \Rightarrow c_1 = -4$ and $c_2 = 6$, and so $y = -4e^{-t} + 6 - 2te^{-t}$.

7 (a). $y_P' = 2 - 2\sin 2t,\ y_P'' = -4\cos 2t,\ -4\cos 2t + (2t + \cos 2t) = 2t - 3\cos 2t$.

7 (b). $\lambda^2+1=0$, and thus $y_C=c_1\cos t+c_2\sin t$.

7 (c). $y=c_1\cos t+c_2\sin t+2t+\cos 2t$. From the initial conditions, we have $y(0)=c_1+1=0$ and $y'(0)=c_2+2=0$. Solving these simultaneous equations yields $c_1=-1$ and $c_2=-2$, and so $y=-\cos t-2\sin t+2t+\cos 2t$.

8 (b). $y_C=c_1\cos 2t+c_2\sin 2t$.

8 (c). $y=c_1\cos 2t+c_2\sin 2t+2e^{t-\pi}$.

$y(\pi)=c_1+2=2$ and $y'(\pi)=2c_2+2=0\Rightarrow c_1=0$ and $c_2=-1$, and so $y=-\sin 2t+2e^{t-\pi}$.

9 (a). $10-2(10(t+1))+10(t+1)^2=10-20t-20+10t^2+20t+10=10t^2$.

9 (b). $\lambda^2-2\lambda+2=0$, so $\lambda=1\pm i$, and thus $y_C=c_1e^t\cos t+c_2e^t\sin t$.

9 (c). $y=c_1e^t\cos t+c_2e^t\sin t+5(t+1)^2$. From the initial conditions, we have $y(0)=c_1+5=0$ and $y'(0)=c_1+c_2+10=0$. Solving these simultaneous equations yields $c_1=-5$ and $c_2=-5$, and so $y=-5e^t\cos t-5e^t\sin t+5(t+1)^2$.

10 (b). $y_C=c_1e^t\cos t+c_2e^t\sin t$.

10 (c). $y=c_1e^t\cos t+c_2e^t\sin t+2\cos t+\sin t$.

$y\left(\frac{\pi}{2}\right)=c_2e^{\frac{\pi}{2}}+1=1$ and $y'\left(\frac{\pi}{2}\right)=-c_1e^{\frac{\pi}{2}}+c_2e^{\frac{\pi}{2}}-2=0\Rightarrow c_1=-2e^{-\frac{\pi}{2}}$ and $c_2=0$, and so

$y=-2e^{t-\frac{\pi}{2}}\cos t+2\cos t+\sin t$.

11 (a). $y_P'=\frac{1}{2}(2t+t^2)e^t$, $y_P''=\frac{1}{2}(2+4t+t^2)e^t$, and $\left(\frac{t^2}{2}+2t+1\right)e^t-2\cdot\frac{1}{2}\left(2t+t^2\right)e^t+\frac{t^2}{2}e^t=e^t$.

11 (b). $\lambda^2-2\lambda+1=(\lambda-1)^2=0$, and thus $y_C=c_1e^t+c_2te^t$.

11 (c). $y=c_1e^t+c_2te^t+\frac{t^2}{2}e^t$. From the initial conditions, we have $y(0)=c_1=$ -2 and $y'(0)=c_1+c_2=2$. Solving these simultaneous equations yields $c_1=-2$ and $c_2=4$, and so $y=-2e^t+4te^t+\frac{t^2}{2}e^t$.

12 (b). $y_C=c_1e^t+c_2te^t$.

12 (c). $y=c_1e^t+c_2te^t+t^2+4t+10+\cos t$.

$y(0)=c_1+10+1=1$ and $y'(0)=c_1+c_2+4=3\Rightarrow c_1=-10$ and $c_2=9$, and so

$y=-10e^t+9te^t+t^2+4t+10+\cos t$.

13. First, $y_P=a_1u+a_2v$. Now we have

$$y_P''+p(t)y_P'+q(t)y_P=a_1\left(u''+p(t)u'+q(t)u\right)+a_2\left(v''+p(t)v'+q(t)v\right)=a_1g_1+a_2g_2$$

14. $e^t + 2t + \frac{1}{2} = \frac{1}{2}\left[2e^t + 1\right] + \frac{2}{3}[3t]$. Thus $y_P = \frac{1}{2}u_1 + \frac{2}{3}u_3$.

15. $4e^{-t} - 2 = 2\left[2e^{-t} - t - 1\right] + \frac{2}{3}[3t]$. Thus $y_P = 2u_2 + \frac{2}{3}u_3$.

16. $\cosh t = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \frac{1}{4}\left[2e^t + 1\right] + \frac{1}{4}\left[2e^{-t} - t - 1\right] + \frac{1}{12}[3t]$. Thus $y_P = \frac{1}{4}u_1 + \frac{1}{4}u_2 + \frac{1}{12}u_3$.

17. Differentiation gives us $y_P' = 2e^{2t} - 2t,\ y_P'' = 4e^{2t} - 2$. From the given differential equation, we have $g(t) = 4e^{2t} - 2 + 2e^{2t} - 2t - e^{2t} + t^2 = 5e^{2t} + t^2 - 2t - 2$.

18. Differentiation gives us $y_P' = 3 + \frac{1}{2}t^{-\frac{1}{2}},\ y_P'' = -\frac{1}{4}t^{-\frac{3}{2}}$. From the given differential equation, we have $g(t) = -\frac{1}{4}t^{-\frac{3}{2}} - 2\left(3 + \frac{1}{2}t^{-\frac{1}{2}}\right) = -6 - t^{-\frac{1}{2}} - \frac{1}{4}t^{-\frac{3}{2}}$.

19. Differentiation gives us $y_P' = 3,\ y_P'' = 0$. From the given differential equation, we have $g(t) = t \cdot 0 + e^t \cdot 3 + 2 \cdot 3t = 3e^t + 6t$.

20. Differentiation gives us $y_P' = \frac{1}{1+t},\ y_P'' = -\frac{1}{(1+t)^2}$. From the given differential equation, we have $g(t) = -\frac{1}{(1+t)^2} + \ln(1+t),\ t > -1$.

21. Differentiation gives us $y_P' = -\sin t,\ y_P'' = -\cos t$. From the given differential equation, we have $g(t) = -\cos t - \sin^2 t + 2|t|\cos t = \left(2|t| - 1\right)\cos t - \sin^2 t$.

22. $(\lambda - 1)(\lambda - 2) = \lambda^2 - 3\lambda + 2 = 0 \Rightarrow\ \alpha = -3,\ \beta = 2$. Differentiation gives us $y_P' = -4e^{-2t},\ y_P'' = 8e^{-2t}$, and so $g(t) = 8e^{-2t} - 3\left(-4e^{-2t}\right) + 2\left(2e^{-2t}\right) = 24e^{-2t}$.

23. $\lambda = -1, 0 \Rightarrow (\lambda + 1)\lambda = 0$, so $\lambda^2 + \lambda = 0$ and $\alpha = 1,\ \beta = 0$. Differentiation gives us $y_P' = 2t,\ y_P'' = 2$, and so $g(t) = 2 + 2t$.

24. $(\lambda - 1)^2 = \lambda^2 - 2\lambda + 1 = 0 \Rightarrow\ \alpha = -2,\ \beta = 1$. Differentiation gives us $y_P' = \left(t^2 + 2t\right)e^t,\ y_P'' = \left(t^2 + 4t + 2\right)e^t$, and so $g(t) = \left(t^2 + 4t + 2\right)e^t - 2\left(t^2 + 2t\right)e^t + t^2e^t = 2e^t$.

25. $\lambda = 1 \pm i \Rightarrow (\lambda - 1)^2 = -1$, so $\lambda^2 - 2\lambda + 2 = 0$ and $\alpha = -2,\ \beta = 2$. Differentiation gives us $y_P' = e^t + \cos t,\ y_P'' = e^t - \sin t$, and so
$g(t) = e^t - \sin t - 2\left[e^t + \cos t\right] + 2\left[e^t + \sin t\right] = e^t - 2\cos t + \sin t$.

26. $\lambda^2 + 4 = 0 \Rightarrow\ \alpha = 0,\ \beta = 4$. Differentiation gives us $y_P' = \cos t,\ y_P'' = -\sin t$, and so $g(t) = -\sin t + 4(-1 + \sin t) = 3\sin t - 4$.

Section 4.9

1 (a). $\lambda^2 - 4 = 0 \Rightarrow y_C = c_1 e^{-2t} + c_2 e^{2t}$

1 (b). $y_P = A_2 t^2 + A_1 t + A_0,\ y_P' = 2A_2 t + A_1,\ y_P'' = 2A_2.$

$$y_P'' - 4y_P = 2A_2 - 4(A_2 t^2 + A_1 t + A_0) = 4t^2 \Rightarrow A_0 = -\frac{1}{2}, A_1 = 0, A_2 = -1.$$

Therefore, $y_P = -t^2 - \frac{1}{2}$

1 (c). $y = c_1 e^{-2t} + c_2 e^{2t} - t^2 - \frac{1}{2}$

2 (a). $y_C = c_1 e^{-2t} + c_2 e^{2t}$

2 (b). $y_P = -\frac{1}{8}\sin 2t.$

2 (c). $y = c_1 e^{-2t} + c_2 e^{2t} - \frac{1}{8}\sin 2t.$

3 (a). $\lambda^2 + 1 = 0 \Rightarrow y_C = c_1 \cos t + c_2 \sin t$

3 (b). $y_P = Ae^t,\ y_P' = Ae^t,\ y_P'' = Ae^t.$

$y_P'' + y_P = 2Ae^t = 8e^t \Rightarrow A = 4.$ Therefore, $y_P = 4e^t$

3 (c). $y = c_1 \cos t + c_2 \sin t + 4e^t$

4 (a). $y_C = c_1 \cos t + c_2 \sin t$

4 (b). $y_P = -\frac{2}{5}e^t \cos t + \frac{1}{5}e^t \sin t.$

4 (c). $y = c_1 \cos t + c_2 \sin t - \frac{2}{5}e^t \cos t + \frac{1}{5}e^t \sin t.$

5 (a). $(\lambda - 2)^2 = 0 \Rightarrow y_C = c_1 e^{2t} + c_2 t e^{2t}$

5 (b). $y_P = At^2 e^{2t},\ y_P' = (2At^2 + 2At)e^{2t},\ y_P'' = (4At^2 + 8At + 2A)e^{2t}.$

$y_P'' - 4y_P' + 4y_P = e^{2t} \Rightarrow A = \frac{1}{2}.$ Therefore, $y_P = \frac{t^2}{2}e^{2t}$

5 (c). $y = c_1 e^{2t} + c_2 t e^{2t} + \frac{t^2}{2}e^{2t}$

6 (a). $y_C = c_1 e^{2t} + c_2 t e^{2t}$

6 (b). $y_P = \frac{1}{8}\cos 2t + 2.$

6 (c). $y = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{8}\cos 2t + 2.$

7 (a). $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm i \Rightarrow y_C = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

7 (b). $y_P = A_3 t^3 + A_2 t^2 + A_1 t + A_0,\ y_P' = 3A_3 t^2 + 2A_2 t + A_1,\ y_P'' = 6A_3 t + 2A_2$.

$y_P'' + 2y_P' + 2y_P = t^3 \Rightarrow A_0 = 0, A_1 = \frac{3}{2}, A_2 = -\frac{3}{2}, A_3 = \frac{1}{2}$.

Therefore, $y_P = \frac{1}{2}t^3 - \frac{3}{2}t^2 + \frac{3}{2}t$

7 (c). $y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + \frac{1}{2}t^3 - \frac{3}{2}t^2 + \frac{3}{2}t$

8 (a). $y_C = c_1 e^{\frac{t}{2}} + c_2 e^{2t}$

8 (b). $y_P = -te^t + e^t$.

8 (c). $y = c_1 e^{\frac{t}{2}} + c_2 e^{2t} - te^t + e^t$.

9 (a). $y_C = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

9 (b). $y_P = A_0 e^{-t} + A_1\cos t + A_2 \sin t,\ y_P' = -A_0 e^{-t} - A_1 \sin t + A_2\cos t,$

$y_P'' = A_0 e^{-t} - A_1\cos t - A_2 \sin t$. $y_P'' + 2y_P' + 2y_P = e^{-t} + \cos t \Rightarrow A_0 = 1, A_1 = \frac{1}{5}, A_2 = \frac{2}{5}$

Therefore, $y_P = e^{-t} + \frac{1}{5}\cos t + \frac{2}{5}\sin t$

9 (c). $y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + e^{-t} + \frac{1}{5}\cos t + \frac{2}{5}\sin t$

10 (a). $y_C = c_1 e^{-t} + c_2$

10 (b). $y_P = 2t^3 - 6t^2 + 12t$.

10 (c). $y = c_1 e^{-t} + c_2 + 2t^3 - 6t^2 + 12t$.

11 (a). $2\lambda^2 - 5\lambda + 2 = (2\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda = \frac{1}{2}, 2 \Rightarrow y_C = c_1 e^{\frac{t}{2}} + c_2 e^{2t}$

11 (b). $y_P = A_0 t e^{t/2}$. Substituting into the differential equation yields $y_P = 2te^{\frac{t}{2}}$.

11 (c). $y = c_1 e^{\frac{t}{2}} + c_2 e^{2t} + 2te^{\frac{t}{2}}$

12 (a). $y_C = c_1 e^{-t} + c_2$

12 (b). $y_P = -\frac{1}{2}\cos t + \frac{1}{2}\sin t$.

12 (c). $y = c_1 e^{-t} + c_2 - \frac{1}{2}\cos t + \frac{1}{2}\sin t$.

13 (a). $9\lambda^2 - 6\lambda + 1 = (3\lambda - 1)^2 = 0 \Rightarrow y_C = c_1 e^{\frac{t}{3}} + c_2 te^{\frac{t}{3}}$

13 (b). $y_P = (A_1t^3 + A_0t^2)e^{\frac{t}{3}}$. Substituting into the differential equation yields $y_P = \frac{1}{6}t^3e^{\frac{t}{3}}$

13 (c). $y = c_1e^{\frac{t}{3}} + c_2te^{\frac{t}{3}} + \frac{1}{6}t^3e^{\frac{t}{3}}$

14 (a). $y_C = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t$

14 (b). $y_P = t - \frac{5}{4} + \frac{e^{-t}}{2}$.

14 (c). $y = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t + t - \frac{5}{4} + \frac{e^{-t}}{2}$.

15 (a). $\lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = -2 \pm i \Rightarrow y_C = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t$

15 (b). $y_P = Ae^{-2t} + B_1\cos t + B_2\sin t$. Substituting into the differential equation yields

$$y_P = 2e^{-2t} + \frac{1}{8}\cos t + \frac{1}{8}\sin t.$$

15 (c). $y = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t + 2e^{-2t} + \frac{1}{8}\cos t + \frac{1}{8}\sin t$

16 (a). $y_C = c_1e^{-t} + c_2e^{3t}$

16 (b). $y_P = Ae^{-t}\cos t + Be^{-t}\cos t + C_2t^2 + C_1t + C_0 + t(D_1 + D_0)e^{3t}$

$$= Ae^{-t}\cos t + Be^{-t}\cos t + C_2t^2 + C_1t + C_0 + D_1t^2e^{3t} + D_0te^{3t}$$

17 (a). $y_C = c_1\cos 3t + c_2\sin 3t$

17 (b). $y_P = t\left(A_2t^2 + A_1t + A_0\right)\cos 3t + t\left(B_2t^2 + B_1t + B_0\right)\sin 3t + C\cos t + D\sin t$

$$= \left(A_2t^3 + A_1t^2 + A_0t\right)\cos 3t + \left(B_2t^3 + B_1t^2 + B_0t\right)\sin 3t + +C\cos t + D\sin t$$

18 (a). $y_C = c_1 + c_2e^t$

18 (b). $y_P = t(A_2t^2 + A_1t + A_0) + t(B_2t^2 + B_1t + B_0)e^t = A_2t^3 + A_1t^2 + A_0t + (B_2t^3 + B_1t^2 + B_0t)e^t$

19 (a). $\lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1 = 0$; $y_C = c_1e^t\cos t + c_2e^t\sin t$

19 (b). $y_P = Ae^{-t}\cos 2t + Be^{-t}\sin 2t + C_1t + C_0 + e^{-t}\left(D_1t + D_0\right)\cos t + e^{-t}\left(E_1t + E_0\right)\sin t$

20 (a). $y_C = c_1e^t + c_2e^{-t}$

20 (b). $y_P = Ate^t + Bte^{-t} + Ce^{2t} + De^{-2t}$

21 (a). $y_C = c_1\cos 2t + c_2\sin 2t$

21 (b). Using $\sin(2t) = 2\sin t\cos t$ and $\cos(2(2t)) = 2\cos^2 2t - 1$,

$\sin t\cos t + \cos^2 2t = \frac{1}{2}\sin 2t + \frac{1}{2} + \frac{1}{2}\cos 4t$. Therefore,

$$y_P = At\cos 2t + Bt\sin 2t + C + D\cos 4t + E\sin 4t$$

22 (a). $y_C = c_1 \sin 2t + c_2 \cos 2t$

22 (b). $y_P = Ae^{-2t} + B + Ce^{2t}$

23. $(\lambda+1)(\lambda-2) = \lambda^2 - \lambda - 2 = 0$, so $\alpha = -1$ and $\beta = -2$. $y'' - y' - 2y = 4t$, which leads to the general solution of $y = c_1 e^{-t} + c_2 e^{2t} - 2t + 1$.

24. $\lambda(\lambda+1) = \lambda^2 + \lambda = 0$, so $\alpha = 1$ and $\beta = 0$. $y'' + y' = t$, which leads to the general solution of $y = c_1 + c_2 e^{-t} + \frac{t^2}{2} - t$.

25. $(\lambda+2)(\lambda+2) = \lambda^2 + 4\lambda + 4 = 0$, so $\alpha = 4$ and $\beta = 4$. $y'' + 4y' + 4y = 5\sin t$, which leads to the general solution of $y = c_1 e^{-2t} + c_2 te^{-2t} - \frac{4}{5}\cos t + \frac{3}{5}\sin t$.

26. $\alpha = 0$ and $\beta = 1$, $y = c_1 \cos t + c_2 \sin t + t - \frac{1}{3}\sin 2t$.

27. $\lambda = -1 \pm i2$, so $(\lambda+1)^2 = -4$ and thus $\lambda^2 + 2\lambda + 5 = 0$. Therefore. $\alpha = 2$ and $\beta = 5$. $y'' + 2y' + 5y = 8e^{-t}$, which leads to the general solution of $y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + 2e^{-t}$.

28. Since $y_P = t(A_1 t + A_0) + B_0 te^{3t}$, we know that 1 and e^{3t} are solutions and $\lambda^2 - 3\lambda = 0$, and thus $\alpha = -3$ and $\beta = 0$.

29. Since $y_P = A_0 te^{2t} + B_0 te^{-2t} + C_1 t + C_0$, we know that e^{2t} and e^{-2t} are solutions of the homogeneous differential equation. This means that $\lambda = \pm 2$, so $(\lambda+2)(\lambda-2) = \lambda^2 - 4 = 0$, and thus $\alpha = 0$ and $\beta = -4$.

30. We know that $\cos 2t$ and $\sin 2t$ are solutions and $\lambda^2 + 4 = 0$, and thus $\alpha = 0$ and $\beta = 4$.

31 (a). Graph C

31 (b). Graph E

31 (c). Graph A

31 (d). Graph B

31 (e). Graph D

32. $W_b = 200\ lb$. The weight of an equivalent weight of water is $W_b = 8(62.4) = 499.2\ lb$. $m_b = \frac{200}{32}$. Note: $\frac{\rho_\ell}{p} = \frac{499.2}{200} = 2.496 \Rightarrow \omega^2 = 2.496\left(\frac{32}{2}\right) \Rightarrow \omega = 39.936$.

32 (a). $y'' + \omega^2 y = \frac{10}{m_b}\sin \omega t,\ y(0) = 0,\ y'(0) = 0$.

32 (b). $y_c = c_1\cos\omega t + c_2\sin\omega t,\ y_p = At\cos\omega t + Bt\sin\omega t$

$$y_p' = A\cos\omega t - \omega At\sin\omega t + B\sin\omega t + \omega Bt\cos\omega t = (A+\omega Bt)\cos\omega t + (B-\omega At)\sin\omega t$$

$$y_p'' = \omega B\cos\omega t - \omega(A+\omega Bt)\sin\omega t - \omega A\sin\omega t + \omega(B-\omega At)\cos\omega t$$

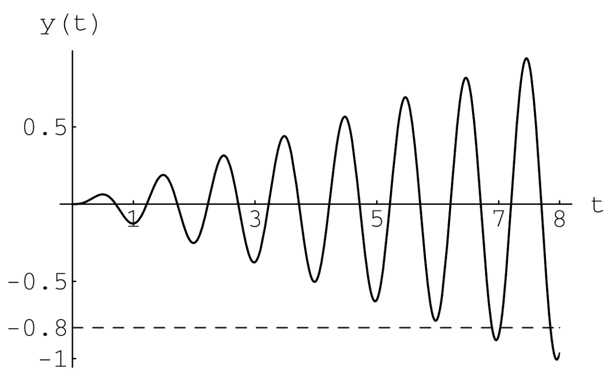
$$y_p'' + \omega^2 y_p = \omega B\cos\omega t - \omega A\sin\omega t - \omega A\sin\omega t + \omega B\cos\omega t = \frac{10}{m_b}\sin\omega t.$$

Therefore, $B=0,\ A = -\frac{10}{2\omega m_b} = -\frac{5}{\omega m_b}$ and

$$y = c_1\cos\omega t + c_2\sin\omega t - \frac{5}{\omega m_b}t\cos\omega t,\ y(0)=c_1=0,\ y'(0)=\omega c_2 - \frac{5}{\omega m_b}=0 \Rightarrow c_2 = \frac{5}{\omega^2 m_b}.$$

$$y = \frac{5}{\omega^2 m_b}\sin\omega t - \frac{5}{\omega m_b}t\cos\omega t,\ \omega \approx 6.3195\ \sec^{-1},\ m_b = 6.25\ \ slug.$$

32 (c). Before being put into motion, the block is floating with a depth Y submerged, where $62.4(4)Y = 200 \Rightarrow Y = 0.80128...ft$. Therefore, the model is valid if $-0.80128... \le y \le 2 - 0.80128$. From the graph, $y = -0.80128$ at $t \approx 7$ sec.



33. $y_P'' + p(t)y_P' + q(t)y_P = (u''+iv'') + p(t)(u'+iv') + q(t)(u+iv)$

$= (u'' + p(t)u' + q(t)u) + i(v'' + p(t)v' + q(t)v) = g_1(t) + ig_2(t)$. The real and imaginary parts must be equal on both sides of the equation, so

$u'' + p(t)u' + q(t)u = g_1(t)$ and $v'' + p(t)v' + q(t)v = g_2(t)$.

34 (a). $y'' - y = e^{i2t},\ y_p = Ae^{i2t},\ y_p' = i2Ae^{i2t},\ y_p'' = -4Ae^{i2t}$

$$-4Ae^{i2t} - Ae^{i2t} = e^{i2t} \Rightarrow A = -\frac{1}{5}.$$

34 (b). $Ae^{i2t} = -\frac{1}{5}(\cos 2t + i\sin 2t) \Rightarrow u = -\frac{1}{5}\cos 2t,\ v = -\frac{1}{5}\sin 2t.$

35 (a). $y_P' = iAe^{it},\ y_P'' = -Ae^{it}$. $-Ae^{it} + 2iAe^{it} + Ae^{it} = 2iAe^{it} = e^{it}$, so $A = \frac{1}{2i} = -\frac{i}{2}$ and $y_P = -\frac{i}{2}e^{it}$.

35 (b). $y_P = -\frac{i}{2}(\cos t + i\sin t) = \frac{1}{2}\sin t - \frac{i}{2}\cos t$. Thus $u = \frac{1}{2}\sin t$ and $v = -\frac{1}{2}\cos t$. For the real function,

$u'' + 2u' + u = -\frac{1}{2}\sin t + 2\left(\frac{1}{2}\cos t\right) + \frac{1}{2}\sin t = \cos t$. For the imaginary function,

$v'' + 2v' + v = \frac{1}{2}\cos t + 2\left(\frac{1}{2}\sin t\right) - \frac{1}{2}\cos t = \sin t$.

36 (a). $y_P' = iAe^{it}$, $y_P'' = -Ae^{it}$. $-Ae^{it} + 4Ae^{it} = e^{it} \Rightarrow A = \frac{1}{3}$ and $y_P = \frac{1}{3}e^{it}$.

36 (b). $y_P = \frac{1}{3}(\cos t + i\sin t) \Rightarrow u = \frac{1}{3}\cos t,\ v = \frac{1}{3}\sin t$.

37 (a). $y_P' = A(1 + i2t)e^{i2t}$, $y_P'' = (i2 + i2 - 4t)Ae^{i2t} = (-4t + i4)Ae^{i2t}$. $(-4t + i4)Ae^{i2t} + 4Ate^{i2t} = e^{i2t}$, so

$A = -\frac{i}{4}$ and $y_P = -\frac{i}{4}te^{i2t}$.

37 (b). $y_P = -\frac{i}{4}t(\cos 2t + i\sin 2t) = \frac{t}{4}\sin 2t + i\left(-\frac{t}{4}\cos 2t\right)$. Thus $u = \frac{t}{4}\sin 2t$ and $v = -\frac{t}{4}\cos 2t$. For the

real function, $u'' + 4u = \cos 2t - t\sin 2t + 4\left(\frac{t}{4}\sin 2t\right) = \cos 2t$. For the imaginary function,

$v'' + 4v = \sin 2t + t\cos 2t - 4\left(\frac{t}{4}\cos 2t\right) = \sin 2t$.

38 (a). $y_P' = -i2Ae^{-i2t}$, $y_P'' = -4Ae^{-i2t}$. $(-4 - i2)Ae^{-i2t} = e^{-i2t} \Rightarrow A = \frac{-1}{4 + i2} = \frac{-(4 - i2)}{20} = -\frac{1}{5} + i\frac{1}{10}$ and

$y_p = \left(-\frac{1}{5} + i\frac{1}{10}\right)e^{-i2t}$.

38 (b). $y_P = \left(-\frac{1}{5} + i\frac{1}{10}\right)(\cos 2t - i\sin 2t) \Rightarrow u = -\frac{1}{5}\cos 2t + \frac{1}{10}\sin 2t,\ v = \frac{1}{5}\sin 2t + \frac{1}{10}\cos 2t$.

39 (a). $y_P' = (1+i)Ae^{(1+i)t}$, $y_P'' = (1+i)^2Ae^{(1+i)t} = i2Ae^{(1+i)t}$. $i2Ae^{(1+i)t} + Ae^{(1+i)t} = e^{(1+i)t}$, so

$A = \frac{1}{1 + i2} = \frac{1 - i2}{5}$ and $y_P = \left(\frac{1}{5} - i\frac{2}{5}\right)e^{(1+i)t}$.

39 (b). $y_P = \left(\frac{1}{5} - i\frac{2}{5}\right)e^t(\cos t + i\sin t) = e^t\left(\frac{1}{5}\cos t + \frac{2}{5}\sin t\right) + ie^t\left(\frac{1}{5}\sin t - \frac{2}{5}\cos t\right)$. Thus

$u = e^t\left(\frac{1}{5}\cos t + \frac{2}{5}\sin t\right)$ and $v = e^t\left(\frac{1}{5}\sin t - \frac{2}{5}\cos t\right)$. For the real function,

$u'' + u = e^t\left(\frac{4}{5}\cos t - \frac{2}{5}\sin t\right) + e^t\left(\frac{1}{5}\cos t + \frac{2}{5}\sin t\right) = e^t\cos t$. For the imaginary function,

$v'' + v = e^t\left(\frac{2}{5}\cos t + \frac{4}{5}\sin t\right) + e^t\left(-\frac{2}{5}\cos t + \frac{1}{5}\sin t\right) = e^t\sin t$.

Section 4.10

1 (a). $y_C = c_1 \cos 2t + c_2 \sin 2t$

1 (b). $\begin{bmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2\sin 2t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2\cos 2t & -\sin 2t \\ 2\sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} 0 \\ 2\sin 2t \end{bmatrix} = \begin{bmatrix} -\sin^2 2t \\ \sin 2t \cos 2t \end{bmatrix}$.

Antidifferentiation gives us $u_1 = -\frac{t}{2} + \frac{1}{8}\sin 4t$ and $u_2 = \frac{\sin^2 2t}{4}$. Thus

$y_P = -\frac{t}{2}\cos 2t + \frac{1}{8}\sin 4t \cos 2t + \frac{1}{4}\sin^3 2t$ and

$y = c_1 \cos 2t + c_2 \sin 2t - \frac{t}{2}\cos 2t + \frac{1}{8}\sin 4t \cos 2t + \frac{1}{4}\sin^3 2t$.

1 (c). $y_P = At\sin 2t + Bt\cos 2t$, $y_P' = (A - 2Bt)\sin 2t + (B + 2At)\cos 2t$,

$y_P'' = (-4B - 4At)\sin 2t + (4A - 4Bt)\cos 2t$. $-4B\sin 2t + 4A\cos 2t = 2\sin 2t$, and thus

$A = 0,\ B = -\frac{1}{2}$, and $y_P = -\frac{1}{2}t\cos 2t$. Combining the particular solution with the

complementary solution gives us $y = C_1 \cos 2t + C_2 \sin 2t - \frac{t}{2}\cos 2t$.

To reconcile, $y_P = \frac{1}{8}\sin 4t \cos 2t + \frac{1}{4}\sin^3 2t = \frac{1}{4}\sin 2t \cos^2 2t + \frac{1}{4}\sin^3 2t = \frac{1}{4}\sin 2t$. Therefore, the

solution in (b) can be written $y = c_1 \cos 2t + (c_2 + \frac{1}{4})\sin 2t - \frac{t}{2}\cos 2t$.

2 (a). $y_C = c_1 \cos t + c_2 \sin t$

2 (b). $\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sec t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} 0 \\ \sec t \end{bmatrix} = \begin{bmatrix} -\frac{\sin t}{\cos t} \\ 1 \end{bmatrix}$. Antidifferentiation gives

us $u_1 = \ln|\cos t|$ and $u_2 = t$. Thus $y_P = (\cos t)\ln|\cos t| + t\sin t$ and

$y = c_1 \cos t + c_2 \sin t + (\cos t)\ln(\cos t) + t\sin t$, since $\cos t > 0$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

2 (c). Method of undetermined coefficients is not applicable.

3 (a). $y_C = c_1 e^t + c_2 t^2 e^t$

3 (b). $\begin{bmatrix} e^t & t^2 e^t \\ e^t & (t^2 + 2t)e^t \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{2te^{2t}} \begin{bmatrix} (t^2 + 2t)e^t & -t^2 e^t \\ -e^t & e^t \end{bmatrix} \begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t^2}{2}e^{-t} \\ \frac{1}{2}e^{-t} \end{bmatrix}$.

Antidifferentiation gives us $u_1 = \left(\frac{t^2}{2} + t + 1\right)e^{-t}$ and $u_2 = -\frac{1}{2}e^{-t}$. Thus

$y_P = \left(\frac{t^2}{2} + t + 1\right)e^{-t}e^t - \frac{1}{2}e^{-t}t^2 e^t = t + 1$ and $y = c_1 e^t + c_2 t^2 e^t + t + 1$.

3 (c). The method of undetermined coefficients is not applicable.

4 (a). $y_C = c_1 e^{-t} + c_2 e^t$

4 (b). $\begin{bmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{1+e^t} \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t & -e^t \\ e^{-t} & e^{-t} \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{1+e^t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\frac{e^t}{1+e^t} \\ \frac{1}{2}\frac{e^{-t}}{1+e^t} \end{bmatrix}$. Antidifferentiation gives us

$u_1 = -\frac{1}{2}\ln(1+\mathrm{e}^t)$ and $u_2 = -\frac{1}{2}e^{-t} + \frac{1}{2}\ln(1+e^{-t})$. Thus $y_P = -\frac{e^{-t}}{2}\ln(1+\mathrm{e}^t) - \frac{1}{2} + \frac{e^t}{2}\ln(1+e^{-t})$

and $y = c_1 e^{-t} + c_2 e^t - \frac{e^{-t}}{2}\ln(1+\mathrm{e}^t) - \frac{1}{2} + \frac{e^t}{2}\ln(1+e^{-t})$.

4 (c). The method of undetermined coefficients is not applicable.

5 (a). $y_C = c_1 e^{-t} + c_2 e^t$

5 (b). $\begin{bmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t & -e^t \\ e^{-t} & e^{-t} \end{bmatrix}\begin{bmatrix} 0 \\ e^t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}e^{2t} \\ \frac{1}{2} \end{bmatrix}$. Antidifferentiation gives us

$u_1 = -\frac{1}{4}\mathrm{e}^{2t}$ and $u_2 = \frac{t}{2}$. Thus $y_P = -\frac{1}{4}e^t + \frac{t}{2}e^t$ and $y = c_1 e^{-t} + c_2 e^t - \frac{1}{4}e^t + \frac{t}{2}e^t$.

5 (c). $y_P = Ate^t$, and differentiation gives us $y_P' = A(1+t)e^t$ and $y_P'' = A(2+t)e^t$. Then we have

$A(2+t)e^t - Ate^t = e^t$, and so $A = \frac{1}{2}$, $y_P = \frac{t}{2}e^t$, and $y = C_1 e^{-t} + C_2 e^t + \frac{t}{2}e^t$.

To reconcile, the solution in (b) can be written $y = c_1 e^{-t} + (c_2 - \tfrac{1}{4})e^t + \frac{t}{2}e^t$.

6 (a). $y_1 = t^2$. Use Reduction of Order to obtain $y_2(t)$. $y_2 = t^2 v$, $y_2' = 2tv + t^2 v'$, $y_2'' = 2v + 4tv' + t^2 v''$

Therefore,

$2v + 4tv' + t^2 v'' - 4v - 2tv' + 2v = 0 \Rightarrow t^2 v'' + 2tv' = (t^2 v')' \Rightarrow v' = \frac{k_1}{t^2} \Rightarrow v = -\frac{k_1}{t} + k_2$.

Using $v = t^{-1}$, $y_2 = t$, $y_c = c_1 t^2 + c_2 t$.

6 (b). $\begin{bmatrix} t^2 & t \\ 2t & 1 \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{t}{1+t^2} \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = -\frac{1}{t^2}\begin{bmatrix} 1 & -t \\ -2t & t^2 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{t}{1+t^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{1+t^2} \\ \frac{-t}{1+t^2} \end{bmatrix}$. Antidifferentiation gives us

$u_1 = \tan^{-1}\mathrm{t}$ and $u_2 = -\frac{1}{2}\ln(1+t^2)$. Thus $y_P = t^2\tan^{-1}\mathrm{t} - \frac{t}{2}\ln(1+t^2)$ and

$y = c_1 t^2 + c_2 t + t^2\tan^{-1}\mathrm{t} - \frac{t}{2}\ln(1+t^2)$.

6 (c). The method of undetermined coefficients is not applicable.

7 (a). $y_C = c_1 e^t + c_2 te^t$

7 (b). $\begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{e^{2t}}\begin{bmatrix} (t+1)e^t & -te^t \\ -e^t & e^t \end{bmatrix}\begin{bmatrix} 0 \\ e^t \end{bmatrix} = \begin{bmatrix} -t \\ 1 \end{bmatrix}$. Integrating gives us

$y_P = -\frac{t^2}{2}e^t + t^2e^t = \frac{t^2}{2}e^t$ and $y = c_1e^t + c_2te^t + \frac{t^2}{2}e^t$.

7 (c). $y_P = At^2e^t$, and differentiation gives us $y_P' = A\left(t^2+2t\right)e^t$ and $y_P'' = A\left(t^2+4t+2\right)e^t$. Then we have $A\left(t^2+4t+2\right)e^t - 2A\left(t^2+2t\right)e^t + At^2e^t = e^t$, and so

$A = \frac{1}{2}$, $y_P = \frac{t^2}{2}e^t$, and $y = c_1e^t + c_2te^t + \frac{t^2}{2}e^t$.

8 (a). $y_C = c_1\cos 6t + c_2\sin 6t$

8 (b). $\begin{bmatrix} \cos 6t & \sin 6t \\ -6\sin 6t & 6\cos 6t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \csc^3 6t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6\cos 6t & -\sin 6t \\ 6\sin 6t & \cos 6t \end{bmatrix}\begin{bmatrix} 0 \\ \csc^3 6t \end{bmatrix} = \begin{bmatrix} -\frac{\csc^3 6t}{6} \\ \frac{\cos 6t}{6\sin^3 6t} \end{bmatrix}$.

Antidifferentiation gives us $u_1 = \frac{1}{36}\cot(6\text{t})$ and $u_2 = -\frac{1}{72}\csc^2(6t)$. Thus

$y_P = \frac{1}{36}\cos(6\text{t})\cot(6\text{t}) - \frac{1}{72}\csc(6t)$ and $y = c_1\cos 6t + c_2\sin 6t + \frac{1}{36}\cos(6\text{t})\cot(6\text{t}) - \frac{1}{72}\csc(6t)$.

8 (c). The method of undetermined coefficients is not applicable.

9 (a). $y_C = c_1\sin t + c_2 t\sin t$

9 (b). $\begin{bmatrix} \sin t & t\sin t \\ \cos t & \sin t + t\cos t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t\sin t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{\sin^2 t}\begin{bmatrix} \sin t + t\cos t & -t\sin t \\ -\cos t & \sin t \end{bmatrix}\begin{bmatrix} 0 \\ t\sin t \end{bmatrix} = \begin{bmatrix} -t^2 \\ t \end{bmatrix}$.

Antidifferentiation gives us $u_1 = -\frac{t^3}{3}$, $u_2 = \frac{t^2}{2}$, $y_P = -\frac{t^3}{3}\sin t + \frac{t^3}{2}\sin t = \frac{t^3}{6}\sin t$, and

$y = c_1\sin t + c_2 t\sin t + \frac{t^3}{6}\sin t$.

9 (c). The method of undetermined coefficients is not applicable.

10 (a). $y_C = c_1 t + c_2 t\ln|t|$

10 (b). $\begin{bmatrix} t & t\ln|t| \\ 1 & \ln|t|+1 \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{t}\ln|t| \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{t}\begin{bmatrix} \ln|t|+1 & -t\ln|t| \\ -1 & t \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{t}\ln|t| \end{bmatrix} = \begin{bmatrix} -\frac{1}{t}\left(\ln|t|\right)^2 \\ \frac{1}{t}\ln|t| \end{bmatrix}$.

Antidifferentiation gives us $u_1 = -\frac{1}{3}\left(\ln|\text{t}|\right)^3$ and $u_2 = \frac{1}{2}\left(\ln|\text{t}|\right)^2$. Thus

$y_P = -\frac{t}{3}\left(\ln|\text{t}|\right)^3 + \frac{t}{2}\left(\ln|\text{t}|\right)^3 = \frac{t}{6}\left(\ln|\text{t}|\right)^3$ and $y = c_1 t + c_2 t\ln|t| + \frac{t}{6}\left(\ln|\text{t}|\right)^3$.

10 (c). The method of undetermined coefficients is not applicable.

11 (a). $y_C = c_1 t + c_2 e^t$

11 (b). $\begin{bmatrix} t & e^t \\ 1 & e^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ (t-1)e^t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \dfrac{1}{(t-1)e^t}\begin{bmatrix} e^t & -e^t \\ -1 & t \end{bmatrix}\begin{bmatrix} 0 \\ (t-1)e^t \end{bmatrix} = \begin{bmatrix} -e^t \\ t \end{bmatrix}$. Antidifferentiation gives us $u_1 = -e^t$, $u_2 = \dfrac{t^2}{2}$, $y_P = -te^t + \dfrac{t^2}{2}e^t$, and $y = c_1 t + c_2 e^t + \left(\dfrac{t^2}{2} - t\right)e^t$.

11 (c). The method of undetermined coefficients is not applicable.

12 (a). $y_C = c_1 e^{-t^2} + c_2 t e^{-t^2}$

12 (b). $\begin{bmatrix} e^{-t^2} & te^{-t^2} \\ -2te^{-t^2} & (1-2t^2)e^{-t^2} \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t^2 e^{-t^2} \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = e^{2t^2}\begin{bmatrix} (1-2t^2)e^{-t^2} & -te^{-t^2} \\ 2te^{-t^2} & e^{-t^2} \end{bmatrix}\begin{bmatrix} 0 \\ t^2 e^{-t^2} \end{bmatrix} = \begin{bmatrix} -t^3 \\ t^2 \end{bmatrix}$.

Antidifferentiation gives us $u_1 = -\dfrac{t^4}{4}$ and $u_2 = \dfrac{t^3}{3}$. Thus $y_P = -\dfrac{t^4}{4}e^{-t^2} + \dfrac{t^4}{3}e^{-t^2} = \dfrac{t^4}{12}e^{-t^2}$ and

$y = c_1 e^{-t^2} + c_2 t e^{-t^2} + \dfrac{t^4}{12}e^{-t^2}$.

12 (c). The method of undetermined coefficients is not applicable.

13 (a). $y_1 = (t-1)^2$, and using reduction of order we have $y_2 = (t-1)^2 v$. Differentiation yields $y_2' = 2(t-1)v + (t-1)^2 v'$ and $y_2'' = 2v + 4(t-1)v' + (t-1)^2 v''$. Then we have

$(t-1)^4 v'' + 4(t-1)^3 v' + 2(t-1)^2 v - 4(t-1)\left[2(t-1)v + (t-1)^2 v'\right] + 6(t-1)^2 v = 0$. Thus

$(t-1)^4 v'' = 0$, and antidifferentiation of $v'' = 0$ gives us $v = k_1(t-1) + k_2$. Then $y_2 = (t-1)^3$, and so $y_C = c_1(t-1)^2 + c_2(t-1)^3$

13 (b). $\begin{bmatrix} (t-1)^2 & (t-1)^3 \\ 2(t-1) & 3(t-1)^2 \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{t}{(t-1)^2} \end{bmatrix}$, so

$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \dfrac{1}{(t-1)^4}\begin{bmatrix} 3(t-1)^2 & -(t-1)^3 \\ -2(t-1) & (t-1)^2 \end{bmatrix}\begin{bmatrix} 0 \\ \dfrac{t}{(t-1)^2} \end{bmatrix} = \begin{bmatrix} -\dfrac{t}{(t-1)^3} \\ \dfrac{t}{(t-1)^4} \end{bmatrix}$. Antidifferentiation gives us

$u_1 = (t-1)^{-1} + \dfrac{1}{2}(t-1)^{-2}$, $u_2 = -\dfrac{1}{2}(t-1)^{-2} - \dfrac{1}{3}(t-1)^{-3}$, $y_P = (t-1) + \dfrac{1}{2} - \dfrac{1}{2}(t-1) - \dfrac{1}{3} = \dfrac{t}{2} - \dfrac{1}{3}$,

and $y = c_1(t-1)^2 + c_2(t-1)^3 + \dfrac{t}{2} - \dfrac{1}{3}$.

13 (c). The method of undetermined coefficients is not applicable.

14 (a). $y_1 = e^t$. Use Reduction of Order to obtain $y_2(t)$. $y_2 = e^t v,\ y_2' = e^t v + e^t v',\ y_2'' = e^t(v + 2v' + v'')$

Therefore, $v'' + 2v' + v - \left(2 + \frac{2}{t}\right)(v + v') + \left(1 + \frac{2}{t}\right)v = 0 \Rightarrow v'' - \frac{2}{t}v' = 0$

$\Rightarrow (t^{-2}v')' = 0 \Rightarrow t^{-2}v' = k_1 \Rightarrow v = k_1\dfrac{t^3}{3} + k_2$.

Using $y_2 = t^3 e^t$, $y_c = c_1 e^t + c_2 t^3 e^t$.

14 (b). $\begin{bmatrix} e^t & t^3 e^t \\ e^t & (t^3 + 3t^2)e^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \dfrac{e^{-2t}}{3t^2}\begin{bmatrix} (t^3 + 3t^2)e^t & -t^3 e^t \\ -e^t & e^t \end{bmatrix}\begin{bmatrix} 0 \\ e^t \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ \frac{1}{3t^2} \end{bmatrix}$.

Antidifferentiation gives us $u_1 = -\dfrac{t^2}{6}$ and $u_2 = -\dfrac{t^{-1}}{3}$. Thus $y_P = -\dfrac{t^2}{6}\mathrm{e}^t - \dfrac{t^2}{3}e^t = -\dfrac{t^2}{2}e^t$ and

$y = c_1 e^t + c_2 t^3 e^t - \dfrac{t^2}{2}e^t$.

14 (c). The method of undetermined coefficients is not applicable.

15. $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix} \Rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \dfrac{1}{W}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}\begin{bmatrix} 0 \\ g \end{bmatrix} = \begin{bmatrix} -\dfrac{y_2 g}{W} \\ \dfrac{y_1 g}{W} \end{bmatrix}$. Antidifferentiation yields

$u_1 = -\displaystyle\int_0^t \frac{y_2(\lambda)g(\lambda)}{W(\lambda)}d\lambda$ and $u_2 = \displaystyle\int_0^t \frac{y_1(\lambda)g(\lambda)}{W(\lambda)}d\lambda$, and so $y_P = \displaystyle\int_0^t \frac{\left[y_2(t)y_1(\lambda) - y_1(t)y_2(\lambda)\right]}{W(\lambda)}g(\lambda)d\lambda$

and $y = c_1 y_1 + c_2 y_2 + \displaystyle\int_0^t \frac{\left[y_2(t)y_1(\lambda) - y_1(t)y_2(\lambda)\right]}{W(\lambda)}g(\lambda)d\lambda$. $y(0) = c_1 y_1(0) + c_2 y_2(0) = y_0$ and

$y'(0) = c_1 y_1'(0) + c_2 y_2'(0) = y_0'$, since $y_P(0) = y_P'(0) = 0$.

16. For this problem, we have $y_1 = \cos 2t,\ y_2 = \sin 2t$, and $W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$. Then we have $y_P = \displaystyle\int_0^t \frac{\sin 2t \cos 2\lambda - \cos 2t \sin 2\lambda}{2}g(\lambda)d\lambda = \frac{1}{2}\int_0^t \sin(2(t-\lambda))g(\lambda)d\lambda$. Since $y = y_P$, $\alpha = 0,\ \beta = 4,\ y_0 = 0,\ y_0' = 0$.

17. For this problem, we have $y_1 = e^{-t},\ y_2 = e^t$, and $W = \begin{vmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{vmatrix} = 2$. Then we have

$y_P = \displaystyle\int_0^t \frac{\left[e^t e^{-\lambda} - e^{-t}e^{\lambda}\right]}{2}g(\lambda)d\lambda = \int_0^t \sinh(t-\lambda)g(\lambda)d\lambda$. Thus we can see that $y = e^{-t} + y_P$, and so $\alpha = 0,\ \beta = -1,\ y_0 = 1,\ y_0' = -1$.

18. For this problem, we have $y_1 = 1,\ y_2 = t$, and $W = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix} = 1$. Then we have

$y_P = \int_0^t [t-\lambda] g(\lambda)\, d\lambda$. Thus we can see that $y = t + y_P(t)$, and so $\alpha = 0,\ \beta = 0,\ y_0 = 0,\ y_0' = 1$.

Section 4.11

1 (a). $y_C = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$

1 (b). Case i: $\omega \neq \omega_0$. $y_P = A\cos\omega t + B\sin\omega t$, and differentiation yields

$y_P'' = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$. Then we have $\left(\omega_0{}^2 - \omega^2\right)A\cos\omega t + \left(\omega_0{}^2 - \omega^2\right)B\sin\omega t = F\cos\omega t$,

and thus $B = 0$ and $A = \dfrac{F}{\omega_0{}^2 - \omega^2}$. The particular solution is then $y_P = \dfrac{F}{\omega_0{}^2 - \omega^2}\cos\omega t$.

Case ii: : $\omega = \omega_0$. $y_P = At\cos\omega_0 t + Bt\sin\omega_0 t$, and differentiation yields

$y_P'' = \omega_0 B\cos\omega_0 t - \omega_0\left(A + \omega_0 Bt\right)\sin\omega_0 t - \omega_0 A\sin\omega_0 t + \omega_0\left(B - \omega_0 At\right)\cos\omega_0 t$

$= \left(2\omega_0 B - \omega_0{}^2 At\right)\cos\omega_0 t + \left(-2\omega_0 A - \omega_0{}^2 Bt\right)\sin\omega_0 t$. Then we have

$2\omega_0 B\cos\omega_0 t - 2\omega_0 A\sin\omega_0 t = F\cos\omega_0 t$, and thus $A = 0$ and $B = \dfrac{F}{2\omega_0}$. The particular solution is

then $y_P = \dfrac{F}{2\omega_0} t\sin\omega_0 t$.

2 (a). $ky = mg,\ k = \dfrac{10\cdot 9.8}{0.098} = 1000\,\mathrm{N}/\mathrm{m}$.

2 (b). $10y'' + 1000y = 20\cos(10t);\ \ y'' + 100y = 2\cos(10t),\ \ y(0) = 0,\ \ y'(0) = 0$.

$y_P = At\cos(10t) + Bt\sin(10t)$ and differentiation yields

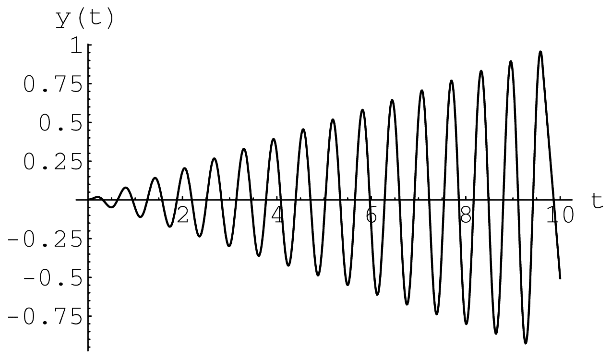
$y_P'' = (20B - 100At)\cos(10t) + (-20A - 100Bt)\sin(10t)$. Then we have

$y_p'' + 100y_p = 20B\cos(10t) - 20A\sin(10t) = 2\cos(10t) \Rightarrow B = \dfrac{1}{10},\ \ A = 0$, and

so $y_P = \dfrac{t}{10}\sin(10t)$, and $y = c_1\cos 10t + c_2\sin 10t + \dfrac{t}{10}\sin(10t)$. From the initial conditions, we

have $y(0) = c_1 = 0$ and $y'(0) = 10c_2 = 0$. Thus the unique solution is $y = \dfrac{t}{10}\sin(10t)$.

2 (c). There is no maximum excursion.



3 (a). $ky = mg,\ k = \dfrac{10 \cdot 9.8}{0.098} = 1000 \,{}^{\text{N}}\!/\!{}_{\text{m}}$.

3 (b). $10y'' + 1000y = 20e^{-t};\ y'' + 100y = 2e^{-t},\ y(0) = 0,\ y'(0) = 0$.

$y_P = Ae^{-t}$, and differentiation yields $y_P'' = Ae^{-t}$. Then we have $Ae^{-t} + 100Ae^{-t} = 2e^{-t}$, and so $A = \dfrac{2}{101}$, $y_P = \dfrac{2}{101}e^{-t}$, and $y = c_1 \cos 10t + c_2 \sin 10t + \dfrac{2}{101}e^{-t}$. From the initial conditions, we have $y(0) = c_1 + \dfrac{2}{101} = 0$ and $y'(0) = 10c_2 - \dfrac{2}{101} = 0$, and thus $c_1 = -\dfrac{2}{101}$ and $c_2 = \dfrac{1}{10}\left(\dfrac{2}{101}\right)$.

Thus the unique solution is $y = \dfrac{2}{101}\left(-\cos 10t + \dfrac{1}{10}\sin 10t + e^{-t}\right)$.

3 (c). $|y|_{\max} \approx 0.035\text{m}$.

4 (a). $ky = mg,\ k = \dfrac{10 \cdot 9.8}{0.098} = 1000 \,{}^{\text{N}}\!/\!{}_{\text{m}}$.

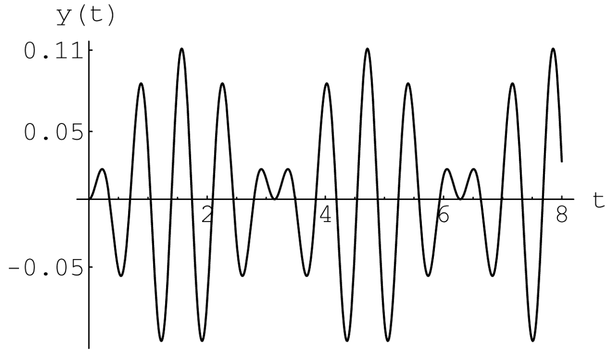
4 (b). $10y'' + 1000y = 20\cos(8t);\ y'' + 100y = 2\cos(8t),\ y(0) = 0,\ y'(0) = 0$.

$y_P = A\cos(8t) + B\sin(8t)$ and differentiation yields $y_P'' = -64A\cos(8t) - 64B\sin(8t)$. Then we have $y_p'' + 100y_p = 36A\cos(8t) + 36B\sin(8t) = 2\cos(8t) \Rightarrow A = \dfrac{1}{18},\ B = 0$, and

so $y_P = \dfrac{1}{18}\cos(8t)$, and $y = c_1 \cos 10t + c_2 \sin 10t + \dfrac{1}{18}\cos(8t)$. From the initial conditions, we have $y(0) = c_1 + \dfrac{1}{18} = 0$ and $y'(0) = 10c_2 = 0 \Rightarrow c_1 = -\dfrac{1}{18},\ c_2 = 0$. Thus the unique solution is

$y = -\dfrac{1}{18}\cos(10t) + \dfrac{1}{18}\cos(8t)$.

4 (c). $|y|_{\max} \approx 0.11\text{m}$.



5 (a). See 3 (a)

5 (b). On $0 \le t \le \pi$: $y'' + 100y = 2,\ y(0) = 0,\ y'(0) = 0$. $y_P = A$, and differentiation yields $y_P'' = 0$.

Then we have $0 + 100A = 2$, and so $A = \frac{1}{50}$, $y_P = \frac{1}{50}$, and $y = c_1 \cos 10t + c_2 \sin 10t + \frac{1}{50}$. From the initial conditions, we have $y(0) = c_1 + \frac{1}{50} = 0$ and $y'(0) = 10c_2 = 0$, and thus

$c_1 = -\frac{1}{50}$ and $c_2 = 0$. Thus the unique solution is $y = \frac{1}{50} - \frac{1}{50}\cos 10t$. At

$t = \pi,\ y(\pi) = \frac{1}{50} - \frac{1}{50}\cos(10\pi) = 0$ and $y'(\pi) = \frac{10}{50}\sin(10\pi) = 0$. Then we have

$y'' + 100y = 0,\ y(\pi) = 0,\ y'(\pi) = 0$ for $t > \pi$.

$y = c_1 \cos 10t + c_2 \sin 10t \Rightarrow y(\pi) = c_1 = 0,\ y'(\pi) = 10c_2 = 0$. Thus $y = 0$ for this region.

5 (c). $|y|_{\max} = \frac{2}{50} = 0.04\text{m}$.

6. $$y = 0.1\sin(\pi t)\sin(7\pi t) = 0.1\left[\frac{1}{2}(\cos(6\pi t) - \cos(8\pi t))\right] = 0.05[\cos(6\pi t) - \cos(8\pi t]$$

Therefore, $y_c = A\cos(6\pi t) + B\sin(6\pi t)$ and $\frac{k}{m} = (6\pi)^2$. If $y_p = A\cos(8\pi t) + B\sin(8\pi t)$, then

$-(8\pi)^2[A\cos(8\pi t) + B\sin(8\pi t)] + (6\pi)^2[A\cos(8\pi t) + B\sin(8\pi t)] = \frac{20}{m}\cos(8\pi t)$.

Therefore, $(-64 + 36)\pi^2 A = \frac{20}{m}$, $(-64 + 36)\pi^2 B = 0 \Rightarrow A = -\frac{1}{28\pi^2}\frac{20}{m} = -\frac{5}{7\pi^2 m}$

and $\frac{k}{m} = 36\pi^2$, $-\frac{5}{7\pi^2 m} = -0.05 \Rightarrow m = 1.447..kg,\ k = 36\pi^2 m = 514.2857..N/m$.

7 (a). $2y''+8y'+80y=20\cos 8t,\ y(0)=0,\ y'(0)=0$. For the complementary solution, we have $\lambda^2+4\lambda+40=0$, and so $\lambda=-2\pm i6$. Thus $y_C=c_1e^{-2t}\cos 6t+c_2e^{-2t}\sin 6t$. For the particular solution, we have $y_P=A\cos 8t+B\sin 8t$, and differentiation yields $y_P'=-8A\sin 8t+8B\cos 8t$ and $y_P''=-64A\cos 8t-64B\sin 8t$. Then we have

$-64A\cos 8t-64B\sin 8t+4(-8A\sin 8t+8B\cos 8t)+40(A\cos 8t+B\sin 8t)=10\cos 8t$. Solving for A and B yields $A=-\dfrac{3}{20}$ and $B=\dfrac{1}{5}$. Thus $y=c_1e^{-2t}\cos 6t+c_2e^{-2t}\sin 6t-\dfrac{3}{20}\cos 8t+\dfrac{1}{5}\sin 8t$. From the initial conditions, we have $y(0)=c_1-\dfrac{3}{20}=0$ and $y'(0)=-2c_1+6c_2+\dfrac{8}{5}=0$. Solving these simultaneous equations yields $c_1=\dfrac{3}{20}$ and $c_2=-\dfrac{13}{60}$, so

$y=\dfrac{3}{20}e^{-2t}\cos 6t-\dfrac{13}{60}e^{-2t}\sin 6t-\dfrac{3}{20}\cos 8t+\dfrac{1}{5}\sin 8t$.

7 (b). For sufficiently large t, $y(t)\approx-\dfrac{3}{20}\cos 8t+\dfrac{1}{5}\sin 8t$, and so the limit does not exist. This equation is called the steady state solution.

8 (a). $y_C=c_1e^{-2t}\cos 6t+c_2e^{-2t}\sin 6t$. For the particular solution, we have $y_P=Ae^{-t}$,

$y_p''+4y_p'+40y_p=10e^{-t}\Rightarrow A-4A+40A=10\Rightarrow A=\dfrac{10}{37}$. Thus

$y=c_1e^{-2t}\cos 6t+c_2e^{-2t}\sin 6t+\dfrac{10}{37}e^{-t}$. From the initial conditions, we have

$y(0)=c_1+\dfrac{10}{37}=0$ and $y'(0)=-2c_1+6c_2-\dfrac{10}{37}=0\Rightarrow c_1=-\dfrac{10}{37},\ c_2=-\dfrac{5}{111}$.

$y=-\dfrac{10}{37}e^{-2t}\cos 6t-\dfrac{5}{111}e^{-2t}\sin 6t+\dfrac{10}{37}e^{-t}$.

8 (b). $\lim\limits_{t\to\infty}y(t)=0$.

9 (a). $y_C=c_1e^{-2t}\cos 6t+c_2e^{-2t}\sin 6t$. For the particular solution, we have $y_P=A\cos 6t+B\sin 6t$, and differentiation yields $y_P'=-6A\sin 6t+6B\cos 6t$ and $y_P''=-36A\cos 6t-36B\sin 6t$. Then we have $-36A\cos 6t-36B\sin 6t+4(-6A\sin 6t+6B\cos 6t)+40(A\cos 6t+B\sin 6t)=10\sin 6t$. Solving for A and B yields $A=-\dfrac{30}{74}$ and $B=\dfrac{5}{74}$. Thus $y=c_1e^{-2t}\cos 6t+c_2e^{-2t}\sin 6t-\dfrac{30}{74}\cos 6t+\dfrac{5}{74}\sin 6t$.

From the initial conditions, we have $y(0)=c_1-\dfrac{30}{74}=0$ and $y'(0)=-2c_1+6c_2+\dfrac{30}{74}=0$.

Solving these simultaneous equations yields $c_1 = \frac{30}{74}$ and $c_2 = \frac{5}{74}$, so

$$y = \frac{30}{74}e^{-2t}\cos 6t + \frac{5}{74}e^{-2t}\sin 6t - \frac{30}{74}\cos 6t + \frac{5}{74}\sin 6t.$$

9 (b). For sufficiently large t, $y(t) \approx -\frac{30}{74}\cos 6t + \frac{5}{74}\sin 6t$, and so the limit does not exist. This equation is called the steady state solution.

10 (a). On $0 \le t \le \frac{\pi}{2}$:

$y_C = c_1 e^{-2t}\cos 6t + c_2 e^{-2t}\sin 6t$. For the particular solution, we have $y_P = \frac{1}{4}$. Thus

$y = c_1 e^{-2t}\cos 6t + c_2 e^{-2t}\sin 6t + \frac{1}{4}$. From the initial conditions, we have

$y(0) = c_1 + \frac{1}{4} = 0$ and $y'(0) = -2c_1 + 6c_2 = 0 \Rightarrow c_1 = -\frac{1}{4},\ c_2 = -\frac{1}{12}$.

$$y = -\frac{1}{4}e^{-2t}\cos 6t - \frac{1}{12}e^{-2t}\sin 6t + \frac{1}{4},\ y\left(\frac{\pi}{2}\right) = \frac{1}{4}e^{-\pi} + \frac{1}{4},\ y'\left(\frac{\pi}{2}\right) = -\frac{1}{2}e^{-\pi} + \frac{1}{2}e^{-\pi} = 0.$$

On $\frac{\pi}{2} < t < \infty$:

$y = d_1 e^{-2t}\cos 6t + d_2 e^{-2t}\sin 6t$. From the initial conditions, we have

$$y\left(\frac{\pi}{2}\right) = -d_1 e^{-\pi} = \frac{1}{4}\left(1 + e^{-\pi}\right) \text{ and } y'\left(\frac{\pi}{2}\right) = 2d_1 e^{-\pi} - 6d_2 e^{-\pi} = 0$$

$$\Rightarrow d_1 = -\frac{1}{4}\left(e^{\pi} + 1\right),\ d_2 = -\frac{1}{12}\left(e^{\pi} + 1\right).\ y = -\frac{1}{4}\left(e^{\pi} + 1\right)\left[e^{-2t}\cos 6t + \frac{1}{3}e^{-2t}\sin 6t\right].$$

10 (b). $\lim_{t\to\infty} y(t) = 0$.

11 (a). $y'' + 2\delta y' + \omega_0{}^2 y = F\cos\omega_0 t,\ y(0) = 0,\ y'(0) = 0$. $\lambda = \frac{-2\delta \pm \sqrt{4\delta^2 - 4\omega_0{}^2}}{2} = -\delta \pm i\sqrt{\omega_0{}^2 - \delta^2}$. Thus

$y_C = c_1 e^{-\delta t}\cos\left(t\sqrt{\omega_0{}^2 - \delta^2}\right) + c_2 e^{-\delta t}\sin\left(t\sqrt{\omega_0{}^2 - \delta^2}\right)$. $y_P = A\cos\omega_0 t + B\sin\omega_0 t$, and differentiation

yields $y_P' = -\omega_0 A\sin\omega_0 t + \omega_0 B\cos\omega_0 t$ and $y_P'' = -\omega_0{}^2 A\cos\omega_0 t - \omega_0{}^2 B\sin\omega_0 t$.

Then we have $y_P'' + 2\delta y_P' + \omega_0{}^2 y_P = 2\delta\left[-\omega_0 A\sin\omega_0 t + \omega_0 B\cos\omega_0 t\right] = F\cos\omega_0 t$. Solving for

A and B yields $A = 0$ and $B = \frac{F}{2\delta\omega_0}$. Thus

$$y = c_1 e^{-\delta t}\cos\left(t\sqrt{\omega_0{}^2 - \delta^2}\right) + c_2 e^{-\delta t}\sin\left(t\sqrt{\omega_0{}^2 - \delta^2}\right) + \frac{F}{2\delta\omega_0}\sin\omega_0 t.$$

From the initial conditions, we have $y(0)=c_1=0$ and $y'(0)=c_2\sqrt{\omega_0{}^2-\delta^2}+\dfrac{F}{2\delta}=0$. Thus

$c_1=0$ and $c_2=-\dfrac{F}{2\delta\sqrt{\omega_0{}^2-\delta^2}}$, and so $y=\dfrac{F}{2\delta}\left[\dfrac{\sin\omega_0 t}{\omega_0}-\dfrac{e^{-\delta t}\sin\left(t\sqrt{\omega_0{}^2-\delta^2}\right)}{\sqrt{\omega_0{}^2-\delta^2}}\right]$.

11 (b). First, let us rewrite $y=\dfrac{F}{2}\left[\dfrac{\sqrt{\omega_0{}^2-\delta^2}\sin\omega_0 t-\omega_0 e^{-\delta t}\sin\left(t\sqrt{\omega_0{}^2-\delta^2}\right)}{\delta\omega_0\sqrt{\omega_0{}^2-\delta^2}}\right]\equiv\dfrac{F}{2}\dfrac{N(\delta)}{D(\delta)}$. To use

L'Hopital's Rule to find the limit, we need

$\dfrac{dN}{d\delta}=\dfrac{1}{2}\left(\omega_0{}^2-\delta^2\right)^{-\frac{1}{2}}(-2\delta)\sin\omega_0 t+\omega_0 te^{-\delta t}\sin\left(t\sqrt{\omega_0{}^2-\delta^2}\right)$

$-\omega_0 e^{-\delta t}\cos\left(t\sqrt{\omega_0{}^2-\delta^2}\right)\cdot\dfrac{1}{2}\left(\omega_0{}^2-\delta^2\right)^{-\frac{1}{2}}(-2\delta)t$. Thus $\dfrac{dN}{d\delta}\to 0+\omega_0 t\sin\omega_0 t+0$ as $\delta\to 0$.

$\dfrac{dD}{d\delta}=\omega_0\sqrt{\omega_0{}^2-\delta^2}+\delta\omega_0\cdot\dfrac{1}{2}\left(\omega_0{}^2-\delta^2\right)^{-\frac{1}{2}}(-2\delta)$. Thus $\dfrac{dD}{d\delta}\to\omega_0{}^2$ as $\delta\to 0$. Therefore,

$\lim\limits_{\delta\to 0^+}\dfrac{F}{2}\dfrac{N(\delta)}{D(\delta)}=\dfrac{F}{2\omega_0}t\sin\omega_0 t$.

11 (c). For sufficiently large t, $y\approx\dfrac{F}{2\delta}\dfrac{\sin\omega_0 t}{\omega_0}$. Knowing m and k means that we know $\omega_0=\sqrt{\dfrac{k}{m}}$.

Therefore, by measuring the amplitude $\dfrac{F}{2\delta\omega_0}$ of the steady state solution and knowing $F=\dfrac{\overline{F}}{m}$, we can determine δ.

12 (a). $y''+2\delta y'+\omega_0{}^2 y=F\cos\omega_1 t,\ y(0)=0,\ y'(0)=0$.

$y_C=c_1e^{-\delta t}\cos\left(t\sqrt{\omega_0{}^2-\delta^2}\right)+c_2e^{-\delta t}\sin\left(t\sqrt{\omega_0{}^2-\delta^2}\right)$. $y_P=A\cos\omega_1 t+B\sin\omega_1 t$, and differentiation

yields $y_P'=-\omega_1 A\sin\omega_1 t+\omega_1 B\cos\omega_1 t$ and $y_P''=-\omega_1{}^2 A\cos\omega_1 t-\omega_1{}^2 B\sin\omega_1 t$.

Then we have

$y_P''+2\delta y_P'+\omega_0{}^2 y_P=\left(\omega_0{}^2-\omega_1{}^2\right)\left[A\cos\omega_1 t+B\sin\omega_1 t\right]+2\delta\left[-\omega_1 A\sin\omega_1 t+\omega_1 B\cos\omega_1 t\right]=F\cos\omega_1 t$.

Solving for A and B yields $A=\dfrac{\left(\omega_0{}^2-\omega_1{}^2\right)F}{\left(\omega_0{}^2-\omega_1{}^2\right)^2+\left(2\delta\omega_1\right)^2}$ and $B=\dfrac{2\delta\omega_1 F}{\left(\omega_0{}^2-\omega_1{}^2\right)^2+\left(2\delta\omega_1\right)^2}$. Thus

$y=c_1e^{-\delta t}\cos\left(t\sqrt{\omega_0{}^2-\delta^2}\right)+c_2e^{-\delta t}\sin\left(t\sqrt{\omega_0{}^2-\delta^2}\right)+A\cos\omega_1 t+B\sin\omega_1 t$. From the initial conditions,

we have $y(0)=c_1+A=0$ and $y'(0)=-\delta c_1+c_2\sqrt{\omega_0{}^2-\delta^2}+\omega_1 B=0$.

Thus $c_1 = -A$ and $c_2 = -\dfrac{\delta A + \omega_1 B}{\sqrt{\omega_0^{\ 2} - \delta^2}}$, and so

$$y = \frac{F}{\left(\omega_0^{\ 2} - \omega_1^{\ 2}\right)^2 + \left(2\delta\omega_1\right)^2}\left[\left(\omega_0^{\ 2} - \omega_1^{\ 2}\right)\cos\omega_1 t + 2\delta\omega_1 \sin\omega_1 t\right] -$$

$$\frac{Fe^{-\delta t}}{\left(\omega_0^{\ 2} - \omega_1^{\ 2}\right)^2 + \left(2\delta\omega_1\right)^2}\left\{\left(\omega_0^{\ 2} - \omega_1^{\ 2}\right)\cos\left(t\sqrt{\omega_0^{\ 2} - \delta^2}\right) + \left[\delta\left(\omega_0^{\ 2} - \omega_1^{\ 2}\right) + \omega_1(2\delta\omega_1)\right]\frac{\sin\left(t\sqrt{\omega_0^{\ 2} - \delta^2}\right)}{\sqrt{\omega_0^{\ 2} - \delta^2}}\right\}$$

12 (b). Using $\delta\left(\omega_0^{\ 2} - \omega_1^{\ 2}\right) + \omega_1(2\delta\omega_1) = \delta\left(\omega_0^{\ 2} + \omega_1^{\ 2}\right)$,

$$\lim_{\omega_1 \to \omega_0} y(t) = \frac{F}{\left(2\delta\omega_0\right)^2} 2\delta\omega_0 \sin\omega_0 t - \frac{Fe^{-\delta t}}{\left(2\delta\omega_0\right)^2}\left\{\left[2\delta\omega_0^{\ 2}\right]\frac{\sin\left(t\sqrt{\omega_0^{\ 2} - \delta^2}\right)}{\sqrt{\omega_0^{\ 2} - \delta^2}}\right\}$$

$$= \frac{F}{2\delta}\left\{\frac{\sin\omega_0 t}{\omega_0} - e^{-\delta t}\,\frac{\sin\left(t\sqrt{\omega_0^{\ 2} - \delta^2}\right)}{\sqrt{\omega_0^{\ 2} - \delta^2}}\right\}$$

12 (c).

$$\lim_{\delta \to 0^+} y(t) = \frac{F}{\left(\omega_0^{\ 2} - \omega_1^{\ 2}\right)^2}\left\{\left(\omega_0^{\ 2} - \omega_1^{\ 2}\right)\cos\omega_1 t\right\} - \frac{F}{\left(\omega_0^{\ 2} - \omega_1^{\ 2}\right)^2}\left\{\left(\omega_0^{\ 2} - \omega_1^{\ 2}\right)\cos\omega_0 t\right\}$$

$$= \frac{F}{\left(\omega_0^{\ 2} - \omega_1^{\ 2}\right)}\left\{\cos\omega_1 t - \cos\omega_0 t\right\} = \frac{F}{\left(\omega_1^{\ 2} - \omega_0^{\ 2}\right)}\left\{\cos\omega_0 t - \cos\omega_1 t\right\}.$$

13 (a). $mx'' = -kx - mg\cos\left(\dfrac{\pi}{4}\right)$, $x(0) = -10$, $x'(0) = 0$.

13 (b). $x'' + \dfrac{k}{m}x = -\dfrac{g}{\sqrt{2}}$. $\dfrac{k}{m} = \dfrac{150}{\left(150/32\right)} = 32\text{s}^{-2}$. Thus the complementary solution is

$x_C = c_1\cos\left(t\sqrt{32}\right) + c_2\sin\left(t\sqrt{32}\right)$. $x_P = A$, so $\dfrac{k}{m}A = -\dfrac{g}{\sqrt{2}}$, and so $A = -\dfrac{1}{\sqrt{2}}$ and

$x = c_1\cos\left(t\sqrt{32}\right) + c_2\sin\left(t\sqrt{32}\right) - \dfrac{1}{\sqrt{2}}$.

From the initial conditions, we have $x(0) = c_1 - \dfrac{1}{\sqrt{2}} = -10$ and $x'(0) = \sqrt{32}c_2 = 0$. Thus

$c_1 = \dfrac{1}{\sqrt{2}} - 10$, $c_2 = 0$, and $x = \left(\dfrac{1}{\sqrt{2}} - 10\right)\cos\left(t\sqrt{32}\right) - \dfrac{1}{\sqrt{2}}$.

Differentiation gives us $x'(t)=\sqrt{32}\left(10-\frac{1}{\sqrt{2}}\right)\sin\left(t\sqrt{32}\right)$. We need x' when $x=0$, so

$\cos\left(t\sqrt{32}\right)=\dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}-10}$. Solving for $\sin\left(t\sqrt{32}\right)=.9971$, and so $x'=52.416\,{}^{\text{ft}}\!/\!{}_{\text{s}}$.

13 (c). Letting x and y represent horizontal and vertical coordinates which have their origin at the mouth o the "cannon," we have $y''=-g,\ y(0)=0,\ y'(0)=\frac{v_0}{\sqrt{2}}$ and $x''=0,\ x(0)=0,\ x'(0)=\frac{v_0}{\sqrt{2}}$. For the y initial value problem, we have $y'=-gt+\frac{v_0}{\sqrt{2}}$ and antidifferentiation yields $y=-\frac{gt^2}{2}+\frac{v_0 t}{\sqrt{2}}$. Settin $y=0$ gives us $t=0,\ \frac{\sqrt{2}v_0}{g}$. Substituting the second value for t into the solution of the initial value problem for x gives us $x(t)=\frac{v_0}{\sqrt{2}}t=\frac{v_0}{\sqrt{2}}\sqrt{2}\frac{v_0}{g}=\frac{{v_0}^2}{g}=\frac{(52.416)^2}{32}=85.857\text{ft}.$

14 (b). (i). $\frac{v^2}{2}=-gy+C_1,\ \frac{30^2}{2}=0+C_1\Rightarrow v^2=-2gy+900,\ v(2)=(900-128)^{\frac{1}{2}}=\sqrt{772}\,ft/\sec$.

(ii). $\frac{v^2}{2}=-\frac{k}{2m}(y-2)^2-g(y-2)+C_2,\ C_2=\frac{772}{2}\Rightarrow v^2=-\frac{k}{m}(y-2)^2-2g(y-2)+772$

$v(3)=0\Rightarrow 0=-\frac{k}{m}-2g+772\Rightarrow\frac{k}{m}=772-64=708,\ m=\frac{\frac{1}{16}}{32}\Rightarrow k=\frac{708}{16(32)}=1.3828\ \ lb/ft$.

15. $V_s=L\frac{dI}{dt}+\frac{1}{C}\int_0^t I(\lambda)d\lambda$, so $\frac{dV_s}{dt}=LI''+\frac{1}{C}I,\ I(0)=0,\ I'(0)=0$, and therefore $I''+\frac{1}{4}I=\frac{dV_s}{dt}$.

$V_s=5\sin 3t,$ so $I''+\frac{1}{4}I=15\cos 3t$. Thus $I_C=c_1\cos\left(\frac{t}{2}\right)+c_2\sin\left(\frac{t}{2}\right)$ and $I_P=A\cos 3t+B\sin 3t$.

Differentiation gives us $I_P''=-9A\cos 3t-9B\sin 3t$, and then we have

$\left(-9+\frac{1}{4}\right)A\cos 3t+\left(-9+\frac{1}{4}\right)B\sin 3t=15\cos 3t$. Thus $B=0$ and $A=-\frac{12}{7}$, and so

$I_P=-\frac{12}{7}\cos 3t$ and $I=c_1\cos\left(\frac{t}{2}\right)+c_2\sin\left(\frac{t}{2}\right)-\frac{12}{7}\cos 3t$. From the initial conditions, we have

$I(0)=c_1-\frac{12}{7}=0$ and $I'(0)=\frac{1}{2}c_2=0$. Thus $c_1=\frac{12}{7}$ and $c_2=0$ and $I(t)=\frac{12}{7}\left(\cos\frac{t}{2}-\cos 3t\right)$.

16. $I'' + \frac{1}{4}I = \frac{dV_s}{dt}$. $V_s = 10te^{-t}$, so $I'' + \frac{1}{4}I = 10te^{-t}$. Thus $I_C = c_1\cos\left(\frac{t}{2}\right) + c_2\sin\left(\frac{t}{2}\right)$ and

$I_P = (At+B)e^{-t}$. Differentiation gives us $I_P'' = (At - 2A + B)e^{-t}$, and then we have

$At - 2A + B + \frac{1}{4}(At+B) = -10t + 10 \Rightarrow A = -8,\ B = -\frac{24}{5}$.

Thus, $I_P = \left(-8t - \frac{24}{5}\right)e^{-t}$ and $I = c_1\cos\left(\frac{t}{2}\right) + c_2\sin\left(\frac{t}{2}\right) + \left(-8t - \frac{24}{5}\right)e^{-t}$. From the initial conditions, we have $I(0) = c_1 - \frac{24}{5} = 0$ and $I'(0) = \frac{1}{2}c_2 - 8 + \frac{24}{5} = 0$. Thus $c_1 = \frac{24}{7}$ and $c_2 = \frac{32}{5}$ and

$I(t) = \frac{24}{5}\cos\left(\frac{t}{2}\right) + \frac{32}{5}\sin\left(\frac{t}{2}\right) - \left(8t + \frac{24}{5}\right)e^{-t}$.

17. $I_S = \frac{V}{R} + \frac{1}{L}\int_0^t V(\lambda)d\lambda + C\frac{dV}{dt}$, and then we have $CV'' + \frac{1}{R}V' + \frac{1}{L}V = \frac{dI_s}{dt}$, $V(0) = 0,\ V'(0) = 0$.

$R = 1\text{k}\Omega,\ L = 1\text{H},\ \text{C} = \frac{1}{2}\mu\text{F}$, and so $V'' + 2V' + 2V = 2\frac{dI_s}{dt}$. For this problem,

$I_s = 1 - e^{-t} \Rightarrow 2\frac{dI_s}{dt} = 2e^{-t}$. For the complementary solution, we have $\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$, and so $V_C = c_1e^{-t}\cos t + c_2e^{-t}\sin t$. $V_P = Ae^{-t}$, and substituting this into the original differential equation, we have $A - 2A + 2A = 2$. Thus $A = 2$, and so $V = c_1e^{-t}\cos t + c_2e^{-t}\sin t + 2e^{-t}$. From the initial conditions, we have $V(0) = c_1 + 2 = 0$ and $V'(0) = -c_1 + c_2 - 2 = 0$. Solving these simultaneous equations yields $c_1 = -2$ and $c_2 = 0$, and so $V = -2e^{-t}\cos t + 2e^{-t}$.

18. $V'' + 2V' + 2V = 2\frac{dI_s}{dt}$, $V(0) = 0,\ V'(0) = 0$. For this problem, $I_s = 5\sin t \Rightarrow 2\frac{dI_s}{dt} = 10\cos t$. For the complementary solution, $V_C = c_1e^{-t}\cos t + c_2e^{-t}\sin t$.

$V_P = A\cos t + B\sin t,\ V_P' = -A\sin t + B\cos t,\ V_P'' = -A\cos t - B\sin t$, and substituting this into the original differential equation, we have $(-2A + B)\sin t + (A + 2B)\cos t = 10\cos t \Rightarrow A = 2,\ B = 4$.

Thus, $V = c_1e^{-t}\cos t + c_2e^{-t}\sin t + 2\cos t + 4\sin t$. From the initial conditions, we have $V(0) = c_1 + 2 = 0$ and $V'(0) = -c_1 + c_2 + 4 = 0$. Solving these simultaneous equations yields $c_1 = -2$ and $c_2 = -6$, and so $V = -2e^{-t}\cos t - 6e^{-t}\sin t + 2\cos t + 4\sin t$.