Chapter 5
Higher Order Linear Differential Equations

Section 5.1

1-5 Verify by substituting into differential equation.

6. Discontinuities for the relevant functions exist at $t = -3,-1,3$.

$y''' - \frac{1}{t^2 - 9} y'' + \ln(t+1) y' + \cos t y = 0$; Initial condition at $t = 0$. $-1 < t < 3$.

7. Discontinuities for the relevant functions exist at $t = -1$ and $n\pi + \frac{\pi}{2}$. Since $t_0 = 0$, the largest interval on which Theorem 5.1 guarantees a unique solution is $-1 < t < \frac{\pi}{2}$.

8. Discontinuities for the relevant functions exist at $t = \pm 4$ and $\pm\frac{\pi}{2}$. $\left(t^2 - 16\right) y^{(4)} + 2y'' + t^2 y = \sec t$; Initial condition at $t = 3$. $\frac{\pi}{2} < t < 4$.

9. There are no discontinuities for the relevant functions and $t_0 = 0$. Thus the largest interval on whic Theorem 5.1 guarantees a unique solution is $-\infty < t < \infty$.

10. $y^{(4)} - 5y'' + 4y = 0$; $\lambda^4 - 5\lambda^2 + 4 = \left(\lambda^2 - 1\right)\left(\lambda^2 - 4\right) = 0$ $\lambda = -1,\ 1,\ -2,\ 2$.

11. $\lambda^3 - \lambda = \lambda(\lambda - 1)(\lambda + 1) = 0$, so $\lambda = 0,\ \pm 1$.

12. $y''' - 2y'' - y' + 2y = 0$; $\lambda^3 - 2\lambda^2 - \lambda + 2 = \lambda^2(\lambda - 2) - (\lambda - 2) = (\lambda - 2)(\lambda^2 - 1) = 0$
$\lambda = -1,\ 1,\ 2$.

13. $\lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2 = (\lambda - 1)^2(\lambda + 1)^2 = 0$, so $\lambda = \pm 1$.

Section 5.2

1 (a). $y''' = 0$, and antidifferentiation yields $y'' = c_1$, $y' = c_1 t + c_2$, and $y = \frac{c_1}{2} t^2 + c_2 t + c_3$.

1 (b). $W = \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$, and thus the three functions form a fundamental set of solutions.

2. $W = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 1(2) = 2 \neq 0$.

3. $W = \begin{vmatrix} 1 & t & e^{-t} \\ 0 & 1 & -e^{-t} \\ 0 & 0 & e^{-t} \end{vmatrix} = e^{-t} \neq 0$, and thus the three functions form a fundamental set of solutions.

4. $W = \begin{vmatrix} 1 & t & \cos t & \sin t \\ 0 & 1 & -\sin t & \cos t \\ 0 & 0 & -\cos t & -\sin t \\ 0 & 0 & \sin t & -\cos t \end{vmatrix} = 1 \cdot 1 \cdot \left(\cos^2 t + \sin^2 t\right) = 1 \neq 0$.

5. $W = \begin{vmatrix} 1 & t & t^{-1} \\ 0 & 1 & -t^{-2} \\ 0 & 0 & 2t^{-3} \end{vmatrix} = 2t^{-3} \neq 0,\ t > 0$, and thus the three functions form a fundamental set of solutions.

6. $W = \begin{vmatrix} 1 & \ln t & t^2 \\ 0 & t^{-1} & 2t \\ 0 & -t^{-2} & 2 \end{vmatrix} = 1 \cdot (3t^{-1}) = 3t^{-1} \neq 0,\ t > 0$.

7. $y = c_1 + c_2 e^t + c_3 e^{-t}$, and differentiation yields $y' = c_2 e^t - c_3 e^{-t}$ and $y'' = c_2 e^t + c_3 e^{-t}$. From the initial conditions, we have $y(0) = c_1 + c_2 + c_3 = 3$, $y'(0) = c_2 - c_3 = -3$, and $y''(0) = c_2 + c_3 = 1$. Solving these simultaneous equations yields $c_1 = 2,\ c_2 = -1,$ and $c_3 = 2$, and so the unique solution is $y = 2 - e^t + 2e^{-t}$.

8. $y = c_1 + c_2 t + c_3 e^{-t}$, $y(1) = c_1 + c_2 + c_3 e^{-1} = 4$, $y'(1) = c_2 - c_3 e^{-1} = 3$, $y''(1) = c_3 e^{-1} = 0$
$\therefore\ c_3 = 0,\ c_2 = 3,\ c_1 = 1$ and $y(t) = 1 + 3t$.

9. $y = c_1 + c_2 t + c_3 \cos t + c_4 \sin t$, and differentiation yields $y' = c_2 - c_3 \sin t + c_4 \cos t$, $y'' = -c_3 \cos t - c_4 \sin t$, and $y''' = c_3 \sin t - c_4 \cos t$. From the initial conditions, we have $y\left(\frac{\pi}{2}\right) = c_1 + c_2 \frac{\pi}{2} + c_4 = 2 + \pi$, $y'\left(\frac{\pi}{2}\right) = c_2 - c_3 = 3$, $y''\left(\frac{\pi}{2}\right) = -c_4 = -3$, and $y'''\left(\frac{\pi}{2}\right) = c_3 = 1$.
Solving these simultaneous equations yields $c_1 = -1 - \pi,\ c_2 = 4,\ c_3 = 1,$ and $c_4 = 3$, and so the unique solution is $y = -1 - \pi + 4t + \cos t + 3\sin t$.

10. $y = c_1 + c_2 t + c_3 t^{-1}$, $y' = c_2 - c_3 t^{-2}$, $y'' = 2c_3 t^{-3}$, $y(2) = c_1 + 2c_2 + \frac{1}{2}c_3 = -1$, $y'(2) = c_2 - \frac{1}{4}c_3 = \frac{3}{2}$,
$y''(2) = \frac{2}{8}c_3 = -\frac{1}{2} \Rightarrow c_3 = -2$, $c_2 = \frac{3}{2} - \frac{1}{2} = 1$, $c_1 + 2(1) + \frac{1}{2}(-2) = -1 \Rightarrow c_1 = -2$
$y = -2 + t - 2t^{-1}$.

11. $y = c_1 + c_2 \ln t + c_3 t^2$, and differentiation yields $y' = c_2 t^{-1} + 2c_3 t$ and $y'' = -c_2 t^{-2} + 2c_3$. From the initial conditions, we have $y(1) = c_1 + c_3 = 1$, $y'(1) = c_2 + 2c_3 = 2$, and $y''(1) = -c_2 + 2c_3 = -6$. Solving these simultaneous equations yields $c_1 = 2,\ c_2 = 4,$ and $c_3 = -1$, and so the unique solution is $y = 2 + 4\ln t - t$.

12. $y''' - y' = 0;\ \ p_{n-1}(t) = p_2(t) = 0$. Abel's theorem predicts $W(t) = W(t_0)$.
If $t_0 = -1$, then $W(t) = W(-1) = \text{constant}$. From exercise 2, $W(t) = 2$.

13. $p_{n-1}(t) = p_2(t) = 1$, and so Abel's Theorem predicts $W(t) = W(0)e^{-t}$ with $t_0 = 0$. From Exercise 3, $W(t) = e^{-t} = W(0)e^{-t}$ since $W(0) = 1$.

14. $y^{(4)} + y'' = 0;\ \ p_{n-1}(t) = p_3(t) = 0$. If $t_0 = 1$, Abel's theorem predicts $W(t) = W(1) = \text{constant}$. From exercise 4, $W(t) = 1$.

15. $p_{n-1}(t) = p_2(t) = \dfrac{3}{t}$, and so Abel's Theorem predicts $W(t) = W(1)e^{-\int_1^t \frac{3}{s}ds} = W(1)e^{\{-3[\ln t - \ln 1]\}} = \dfrac{W(1)}{t^3}$ with $t_0 = 1$. From Exercise 5, $W(t) = 2t^{-3} = W(1)t^{-3}$ since $W(1) = 2$.

16. $t^2 y''' + ty'' - y' = 0;\ \ p_{n-1}(t) = p_2(t) = \dfrac{1}{t}$. With $t_0 = 2$, Abel's theorem predicts

$$W(t) = W(2)\exp\left\{-\int_2^t \frac{1}{s}ds\right\} = W(2)\exp\{-\ln t + \ln 2\} = W(2)\exp\left\{\ln\left(\frac{2}{t}\right)\right\} = 2\frac{W(2)}{t}.$$

From exercise 6, $W(t) = 3t^{-1}\ \ \therefore\ W(2) = \dfrac{3}{2}$ and $W(t) = 2\dfrac{W(2)}{t}$.

17. $p_{n-1}(t) = p_2(t) = -3$. $W(t) = W(1)e^{-\int_1^t (-3)ds} = e^{3(t-1)}$.

18. $p_{n-1}(t) = p_2(t) = \sin t$. $\quad W(t) = W(1)\exp\left\{-\int_1^t \sin s\,ds\right\} = 0$ since $W(1) = 0$.

19. $p_{n-1}(t) = p_2(t) = \dfrac{1}{t}$. $W(t) = W(1)e^{-\int_1^t s^{-1}ds} = 3t^{-1},\ t > 0$.

20. $p_{n-1}(t) = p_2(t) = 0$. $\therefore\ W(t) = W(1) = 3$.

21. $u'' - u = 0$, so $\lambda^2 - 1 = (\lambda + 1)(\lambda - 1) = 0$. Then we have $u = c_1 e^{-t} + c_2 e^{t} = y'$. Antidifferentiation yields $y = -c_1 e^{-t} + c_2 e^{t} + c_3 \equiv Ae^{-t} + Be^{t} + C(1)$.

22 (a). $u = y';\ \ u'' + u' = 0,\ \lambda^2 + \lambda = 0\ \Rightarrow\ \lambda = 0, -1 \qquad u = y' = c_1 + c_2 e^{-t}$

$\therefore\ \ y = c_1 t - c_2 e^{-t} + c_3\ \equiv\ c_1 t + c_2 e^{-t} + c_3 \cdot 1$

22 (b). $v = y'' \qquad v' + v = 0\ \Rightarrow\ v = y'' = k_1 e^{-t}\ \Rightarrow\ y' = -k_1 e^{-t} + k_2\ \Rightarrow\ y = k_1 e^{-t} + k_2 t + k_3$.

23. $v'' + v = 0$, so we have $v = c_1\cos t + c_2\sin t = y''$. Antidifferentiation twice yields $y = -c_1\cos t - c_2\sin t + c_3 t + c_4 \equiv A\cos t + B\sin t + Ct + D$.

24 (a). $W = \begin{vmatrix} 1 & t^2 & t^4 \\ 0 & 2t & 4t^3 \\ 0 & 2 & 12t^2 \end{vmatrix} = 1\cdot\left[24t^3 - 8t^3\right] = 16t^3$. Note that $W(0) = 0$ but $W(t)$ is not identically zero on $(-1,\ 1)$. If $y''' + p_2(t)y'' + p_1(t)y = 0$ were to have $1,\ t^2,\ t^4$ solutions with $p_2(t),\ p_1(t)$ continuous on $(-1,\ 1)$, we would contradict Abel's theorem. $\therefore$ No.

24 (b). If $y = 1$, then $y''' + p_2 y'' + p_1 y' = 0$ is satisfied for any $p_1,\ p_2$.

If $y = t^2,\ y' = 2t,\ y'' = 2 \Rightarrow 0 + 2p_2 + 2tp_1 = 0 \Rightarrow p_2 + tp_1 = 0$.

If $y = t^4,\ y' = 4t^3,\ y'' = 12t^2,\ y''' = 24t \Rightarrow 24t + 12t^2 p_2 + 4t^3 p_1 = 0 \Rightarrow 6 + 3tp_2 + t^2 p_1 = 0$.

Therefore, $\begin{bmatrix} t & 1 \\ t^2 & 3t \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix} \Rightarrow \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \dfrac{1}{2t^2}\begin{bmatrix} 3t & -1 \\ -t^2 & t \end{bmatrix}\begin{bmatrix} 0 \\ -6 \end{bmatrix} = \begin{bmatrix} 3/t^2 \\ -3/t \end{bmatrix}$.

Both functions fail to be continuous at $t = 0$.

25 (a). Differentiation yields $y' = c_2e^t - c_3e^{-t}$ and $y'' = c_2e^t + c_3e^{-t}$. From the initial conditions, we have $y(0) = c_1 + c_2 + c_3 = \alpha,\ y'(0) = c_2 - c_3 = \beta$, and $y''(0) = c_2 + c_3 = 4$. Solving these simultaneous equations in terms of α and β yields $c_1 = \alpha - 4,\ c_2 = 2 + \dfrac{1}{2}\beta$, and $c_3 = 2 - \dfrac{1}{2}\beta$. Then we have $y(t) = (\alpha - 4) + \left(2 + \dfrac{1}{2}\beta\right)e^t + \left(2 - \dfrac{1}{2}\beta\right)e^{-t}$. Since the third term goes to zero as t gets large, we must set $\alpha = 4$ and $\beta = -4$ so that the first two terms also become zero (for all t).

25 (b). y will be bounded on $0 \le t < \infty$ if $\beta = -4$ (α can be arbitrary). No choice of α and β will produce a solution that is bounded on $-\infty < t < \infty$ since $2 + \dfrac{1}{2}\beta$ and $2 - \dfrac{1}{2}\beta$ cannot simultaneously be zero.

Section 5.3

1. Antidifferentiation yields $y = c_1 + c_2 t + c_3 t^2$. Since $t_0 = 0$, we have $y_1 = 1,\ y_2 = t$, and $y_3 = \dfrac{t^2}{2}$ from the initial conditions provided.

2. $y = c_1 + c_2 t + c_3 t^2,\ \ y' = c_2 + 2c_3 t,\ \ y'' = 2c_3$

$t_0 = 1$: y_1: $c_1 + c_2 + c_3 = 1,\ c_2 + 2c_3 = 0,\ 2c_3 = 0 \quad \therefore\ c_1 = 1,\ c_2 = c_3 = 0,\ y_1(t) = 1$.

y_2: $c_1 + c_2 + c_3 = 0,\ c_2 + 2c_3 = 1,\ 2c_3 = 0 \ \Rightarrow\ c_3 = 0,\ c_2 = -1,\ c_1 = -1,\ y_2(t) = t - 1,$

y_3: $c_1+c_2+c_3=0,\ c_2+2c_3=0,\ 2c_3=1 \ \Rightarrow\ c_3=\frac{1}{2},\ c_2=-1,\ c_1=\frac{1}{2} \qquad y_3(t)=\frac{1}{2}(t-1)^2$

$y_1(t)=1,\ y_2(t)=t-1,\ y_3(t)=\frac{1}{2}(t-1)^2.$

3. Since $t_0=0$, we have $y_1(0)=c_1+c_3=1,\ y_1'(0)=c_2-c_3=0,$ and $y_1''=c_3=0$ from the initial conditions provided. Thus $c_1=1$ and $c_2=c_3=0$, and $y_1(t)=1$. Similarly, we have $y_2(0)=c_1+c_3=0,\ y_2'(0)=c_2-c_3=1,$ and $y_2''=c_3=0$ from the initial conditions provided. Thus $c_2=1$ and $c_1=c_3=0$, and $y_2(t)=t$. Finally, we have $y_3(0)=c_1+c_3=0,\ y_3'(0)=c_2-c_3=0,$ and $y_3''=c_3=1$ from the initial conditions provided. Thus $c_1=-1$ and $c_2=c_3=1$, and $y_3(t)=-1+t+e^{-t}$.

4. $y=c_1+c_2t+c_3e^{-t},\ y'=c_2-c_3e^{-t},\ y''=c_3e^{-t}$

$t_0=1$: $c_1+c_2+c_3e^{-1}=1,\ c_2-c_3e^{-1}=0,\ c_3e^{-1}=0 \ \Rightarrow\ c_1=1,\ c_2=c_3=0,\ y_1(t)=1$

y_2: $c_1+c_2+c_3e^{-1}=0,\ c_2-c_3e^{-1}=1,\ c_3e^{-1}=0,\ c_3=0,\ c_2=1,\ c_1=-1,\ y_2(t)=t-1$

y_3: $c_1+c_2+c_3e^{-1}=0,\ c_2-c_3e^{-1}=0,\ c_3e^{-1}=1 \ \Rightarrow\ c_3=e,\ c_2=1,\ c_1=-2,\ y_3(t)=-2+t+e^{-(t-1)}$

$y_1(t)=1,\ y_2(t)=t-1,\ y_3(t)=-2+t+e^{-(t-1)}.$

5 (a). $\{\cosh t,\ 1-\sinh t,\ 2+\sinh t\}$ is a solution set.

5 (b). $\cosh t=0\cdot 1+\frac{1}{2}e^{t}+\frac{1}{2}e^{-t},\ 1-\sinh t=1-\frac{1}{2}e^{t}+\frac{1}{2}e^{-t},$ and $2+\sinh t=2+\frac{1}{2}e^{t}-\frac{1}{2}e^{-t}$. Thus

$$A=\begin{bmatrix}0 & 1 & 2\\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\end{bmatrix}.$$

5 (c). $\det A=0\begin{vmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}-1\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}+2\begin{vmatrix}\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{vmatrix}=\frac{3}{2}\neq 0$, so the three functions form a fundamental set.

6 (a). $\{1-2t,\ t+2,\ e^{-(t+2)}\}$ is a solution set.

6 (b). $1-2t=1\cdot 1-2\cdot t+0\cdot e^{-t},\ t+2=2\cdot 1+1\cdot t+0\cdot e^{-t},\ e^{-(t+2)}=0\cdot 1+0\cdot t+e^{-2}\cdot e^{-t}$

$$A=\begin{bmatrix}1 & 2 & 0\\ -2 & 1 & 0\\ 0 & 0 & e^{-2}\end{bmatrix}$$

6 (c). $\det A=e^{-2}(5)=5e^{-2}\neq 0 \quad \therefore$ fundamental set.

7 (a). $\bar{y}_1 = 1+t$ and $\bar{y}_2 = \dfrac{t+1}{t}$ are solutions. However, $\bar{y}_3 = (t+1)^{-1}$ is not a solution and so

$\left\{1+t,\ \dfrac{t+1}{t},\ (t+1)^{-1}\right\}$ is not a solution set.

8 (a). $\{2t^2-1,\ 3,\ \ln(t^3)\}$ is a solution set; $\ln(t^3) = 3\ln t$.

8 (b). $2t^2-1 = -1\cdot 1 + 0\cdot \ln t + 2\cdot t^2,\ \ 3 = 3\cdot 1 + 0\cdot \ln t + 0\cdot t^2,\ \ 3\ln t = 0\cdot 1 + 3\cdot \ln t + 0\cdot t^2$

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 3 \\ 2 & 0 & 0 \end{bmatrix}$$

8 (c). $\det A = -3\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} = 18 \neq 0 \quad \therefore$ fundamental set.

9. Setting $c_1\cdot 1 + c_2 t + c_3 t^2 = 0$ and evaluating at $t = -1,\ 0,\ 1$ we have $c_1 - c_2 + c_3 = 0,\ c_1 = 0,$ and $c_1 + c_2 + c_3 = 0$. Thus $c_1 = c_2 = c_3 = 0$, and the three functions are linearly independent on the interval.

10. $c_1\cdot 1 + c_2\cdot(1+t) + c_3(1+t+t^2) = 0 \quad \therefore \ (c_1+c_2+c_3)\cdot 1 + (c_2+c_3)\cdot t + c_3\cdot t^2 = 0$

The argument of 9 leads to $c_1+c_2+c_3 = 0,\ c_2+c_3 = 0,\ c_3 = 0 \ \Rightarrow\ c_1 = c_2 = c_3 = 0$

$\therefore$ linearly independent on $-\infty < t < \infty$.

11. Setting $c_1\cos^2 t + c_2 2\cos 2t + c_3 2\sin^2 t = 0$ and using the identity $\cos^2 t - \sin^2 t = \cos 2t$, we have $c_1\cos^2 t + 2c_2(\cos^2 t - \sin^2 t) + 2c_3\sin^2 t = (c_1+2c_2)\cos^2 t + (-2c_2+2c_3)\sin^2 t = 0$. Taking $c_3 = 1,\ c_2 = 1,$ and $c_1 = -2$ to be one nontrivial solution, we can conclude that the three functions are linearly dependent on the interval.

12. $c_1(t^2+2t) + c_2(\alpha t+1) + c_3(t+\alpha) = c_1 t^2 + (2c_1 + \alpha c_2 + c_3)t + (c_2 + \alpha c_3) = 0$

$\therefore\ c_1 = 0,\ \alpha c_2 + c_3 = 0,\ c_2 + \alpha c_3 = 0$ or $\begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}\begin{bmatrix} c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $\det = \alpha^2 - 1$

$\therefore$ linearly dependent on $-\infty < t < \infty$ if $\alpha = \pm 1$ and linearly independent on $-\infty < t < \infty$ otherwise. (nontrivial $c_2,\ c_3$ if $\alpha = \pm 1$).

13. On $0 \le t < \infty$, $t|t| + 1 = t^2 + 1$. Then we have $c_1(t^2+1) + c_2(t^2-1) + c_3 t = (c_1+c_2)t^2 + c_3 t + (c_1 - c_2) = 0$. Thus $c_1 = c_2 = c_3 = 0$, and so the three functions are linearly independent on the interval.

14. On $-\infty < t \le 0$, $t|t| + 1 = -t^2 + 1 \Rightarrow 1(t|t|+1) + 1(t^2-1) + 0(t) = 0$, and so the three functions are linearly dependent on the interval.

15. Since the three functions are linearly independent on half of the interval (see 13), the functions are linearly independent on the entire interval.

Section 5.4

1 (a). $\lambda^3 - 4\lambda = 0$

1 (b). $\lambda^3 - 4\lambda = \lambda(\lambda+2)(\lambda-2) = 0$. Thus $\lambda = 0, \pm 2$.

1 (c). $y = c_1 + c_2 e^{-2t} + c_3 e^{2t}$, since the roots are distinct.

2 (a). $\lambda^3 + \lambda^2 - \lambda - 1$

2 (b). $\lambda^2(\lambda+1) - (\lambda+1) = (\lambda^2-1)(\lambda+1) = 0 \qquad \lambda = -1, -1, 1$

2 (c). $y = c_1 e^{-t} + c_2 te^{-t} + c_3 e^t; \quad W = \begin{vmatrix} e^{-t} & te^{-t} & e^t \\ -e^{-t} & (1-t)e^{-t} & e^t \\ e^{-t} & (-2+t)e^{-t} & e^t \end{vmatrix} =$

$$e^{-t}[3-2t] - te^{-t}[-2] + e^t[2-t-1+t]e^{-2t} = e^{-t}[3-2t+2t+1] = 4e^{-t} \neq 0$$

3 (a). $\lambda^3 + \lambda^2 + 4\lambda + 4 = 0$

3 (b). $\lambda^3 + \lambda^2 + 4\lambda + 4 = \lambda^2(\lambda+1) + 4(\lambda+1) = (\lambda^2+4)(\lambda+1) = 0$. Thus $\lambda = -1, \pm i2$.

3 (c). $y = c_1 e^{-t} + c_2 \cos 2t + c_3 \sin 2t$, since $W = \begin{vmatrix} e^{-t} & \cos 2t & \sin 2t \\ -e^{-t} & -2\sin 2t & 2\cos 2t \\ e^{-t} & -4\cos 2t & -4\sin 2t \end{vmatrix} = 10e^{-t} \neq 0$.

4 (a). $16\lambda^4 - 8\lambda^2 + 1$

4 (b). $(4\lambda^2 - 1)^2 = 0; \qquad \lambda = -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

4 (c). $y = c_1 e^{-t/2} + c_2 te^{-t/2} + c_3 e^{t/2} + c_4 te^{t/2} \qquad W = \begin{vmatrix} e^{-t/2} & te^{-t/2} & e^{t/2} & te^{t/2} \\ -\frac{1}{2}e^{-t/2} & \left(1-\frac{t}{2}\right)e^{-t/2} & \frac{1}{2}e^{t/2} & \left(1+\frac{t}{2}\right)e^{t/2} \\ \frac{1}{4}e^{-t/2} & \left(-1+\frac{t}{4}\right)e^{-t/2} & \frac{1}{4}e^{t/2} & \left(1+\frac{t}{4}\right)e^{t/2} \\ -\frac{1}{8}e^{-t/2} & \left(\frac{3}{4}-\frac{t}{8}\right)e^{-t/2} & \frac{1}{8}e^{t/2} & \left(\frac{3}{4}+\frac{t}{8}\right)e^{t/2} \end{vmatrix}$

$$W(0) = \begin{vmatrix} 1 & 0 & 1 & 0 \\ -\frac{1}{2} & 1 & \frac{1}{2} & 1 \\ \frac{1}{4} & -1 & \frac{1}{4} & 1 \\ -\frac{1}{8} & \frac{3}{4} & \frac{1}{8} & \frac{3}{4} \end{vmatrix} \to \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & \frac{3}{4} & 0 & \frac{3}{4} \end{vmatrix} \to \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -\frac{3}{4} & 0 \end{vmatrix} = 1 \cdot 1\left(\frac{3}{2}\right) \neq 0.$$

5 (a). $16\lambda^4 + 8\lambda^2 + 1 = 0$

5 (b). $16\lambda^4 + 8\lambda^2 + 1 = (4\lambda^2+1)^2 = 0$. Thus $\lambda = \pm\frac{i}{2}, \pm\frac{i}{2}$.

5 (c). $y = c_1\cos\frac{t}{2} + c_2 t\cos\frac{t}{2} + c_3\sin\frac{t}{2} + c_4 t\sin\frac{t}{2}$. To verify this,

$$W = \begin{vmatrix} \cos\left(\frac{t}{2}\right) & t\cos\left(\frac{t}{2}\right) & \sin\left(\frac{t}{2}\right) & t\sin\left(\frac{t}{2}\right) \\ -\frac{1}{2}\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right)-\frac{t}{2}\sin\left(\frac{t}{2}\right) & \frac{1}{2}\cos\left(\frac{t}{2}\right) & \sin\left(\frac{t}{2}\right)+\frac{t}{2}\cos\left(\frac{t}{2}\right) \\ -\frac{1}{4}\cos\left(\frac{t}{2}\right) & -\sin\left(\frac{t}{2}\right)-\frac{t}{4}\cos\left(\frac{t}{2}\right) & -\frac{1}{4}\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right)-\frac{t}{4}\sin\left(\frac{t}{2}\right) \\ \frac{1}{8}\sin\left(\frac{t}{2}\right) & -\frac{3}{4}\cos\left(\frac{t}{2}\right)+\frac{t}{8}\sin\left(\frac{t}{2}\right) & -\frac{1}{8}\cos\left(\frac{t}{2}\right) & -\frac{3}{4}\sin\left(\frac{t}{2}\right)-\frac{t}{8}\cos\left(\frac{t}{2}\right) \end{vmatrix}.$$

$$W(0) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & 0 & 1 \\ 0 & -\frac{3}{4} & -\frac{1}{8} & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & -\frac{3}{4} & -\frac{1}{8} \end{vmatrix} = -\frac{1}{4} \neq 0.$$

6 (a). $\lambda^3 - 1$

6 (b). $\lambda^3 = e^{i2k\pi} \;\Rightarrow\; \lambda_k = e^{i2k\pi/3},\ k = 0,\ 1,\ 2;\quad \lambda = 1,\ e^{i2\pi/3},\ e^{i4\pi/3} = 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

6 (c). $y = c_1 e^t + c_2 e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_3 e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$

7 (a). $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

7 (b). $\lambda^3 - 2\lambda^2 - \lambda + 2 = \lambda^2(\lambda-2) - (\lambda-2) = (\lambda+1)(\lambda-1)(\lambda-2)$. Thus $\lambda = 2, \pm 1$.

7 (c). $y = c_1 e^{-t} + c_2 e^t + c_3 e^{2t}$, since the roots are distinct.

8 (a). $\lambda^4 + 16$

8 (b). $\lambda^4 = -16 = 16e^{i(\pi+2k\pi)} \;\Rightarrow\; \lambda_k = 2e^{i\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)},\ k = 0,\ 1,\ 2,\ 3;\quad \lambda_1 = 2e^{i\frac{\pi}{4}} = \sqrt{2} + i\sqrt{2},\ \lambda_2 =$ $2e^{i\frac{3\pi}{4}} = -\sqrt{2} + i\sqrt{2},\ \lambda_3 = 2e^{i\frac{5\pi}{4}} = -\sqrt{2} - i\sqrt{2},\ \lambda_4 = 2e^{i\frac{7\pi}{4}} = \sqrt{2} - i\sqrt{2}\quad \therefore\ \lambda = \pm\sqrt{2} \pm i\sqrt{2}$

$y = c_1 e^{-\sqrt{2}t}\cos\left(\sqrt{2}t\right) + c_2 e^{-\sqrt{2}t}\sin\left(\sqrt{2}t\right) + c_3 e^{\sqrt{2}t}\cos\left(\sqrt{2}t\right) + c_4 e^{\sqrt{2}t}\sin\left(\sqrt{2}t\right).$

9 (a). $\lambda^3 + 4\lambda = 0$

9 (b). $\lambda^3 + 4\lambda = \lambda\left(\lambda^2 + 4\right) = 0$. Thus $\lambda = 0, \pm i2$.

9 (c). $y = c_1 + c_2\cos 2t + c_3\sin 2t$

9 (d). Differentiation gives us $y' = -2c_2\sin 2t + 2c_3\cos 2t$ and $y'' = -4c_2\cos 2t - 4c_3\sin 2t$. From the initial conditions, we have $y(0) = c_1 + c_2 = 1$, $y'(0) = 2c_3 = 6$, and $y''(0) = -4c_2 = 4$. Thus $c_1 = 2$, $c_2 = -1$, and $c_3 = 3$, and so $y = 2 - \cos 2t + 3\sin 2t$

10 (a). $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda+1)^3$

10 (b). $\lambda = -1, -1, -1$

10 (c). $y = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t}$

10 (d). $y(0) = 0,\ y'(0) = 1,\ y''(0) = 0$

$y' = -c_1 e^{-t} + c_2(1-t)e^{-t} + c_3\left(2t - t^2\right)e^{-t},\ y'' = c_1 e^{-t} + c_2(-2+t)e^{-t} + c_3\left(t^2 - 4t + 2\right)e^{-t}$

$y(0) = c_1 = 0,\ y'(0) = c_2 = 1,\ y''(0) = c_2(-2) + c_3(2) = 0 \Rightarrow c_3 = c_2 = 1 \qquad y = \left(t + t^2\right)e^{-t}$

11. $\lambda^2\left(\lambda^2 + 9\right) = \lambda^4 + 9\lambda^2 = 0$. Thus the differential equation is $y^{(4)} + 9y'' = 0$, and so $a_3 = 0,\ a_2 = 9,\ a_1 = 0$, and $a_0 = 0$.

12. $y = c_1 \cos t + c_2 \sin t + c_3 \cos 2t + c_4 \sin 2t; \quad \left(\lambda^2 + 1\right)\left(\lambda^2 + 4\right) = \lambda^4 + 5\lambda^2 + 4 = 0$

$y^{(4)} + 5y'' + 4y = 0; \quad a_3 = 0,\ a_2 = 5,\ a_1 = 0,\ a_0 = 4$.

13. $(\lambda - 1)^2(\lambda + 1)^2 = \left(\lambda^2 - 1\right)^2 = \lambda^4 - 2\lambda^2 + 1 = 0$. Thus the differential equation is $y^{(4)} - 2y'' + y = 0$, an so $a_3 = 0,\ a_2 = -2,\ a_1 = 0$, and $a_0 = 1$.

14.

$y = c_1 e^{-t}\sin t + c_2 e^{-t}\cos t + c_3 e^{t}\sin t + c_4 e^{t}\cos t; \quad \lambda = -1 \pm i1,\ 1 + i1$.

$(\lambda + 1 - i1)(\lambda + 1 + i1) = (\lambda + 1)^2 + 1 = \lambda^2 + 2\lambda + 2; \quad (\lambda - 1 - i1)(\lambda - 1 + i1) = (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2$

$\left(\lambda^2 + 2\lambda + 2\right)\left(\lambda^2 - 2\lambda + 2\right) = \left(\lambda^2 + 2\right)^2 - 4\lambda^2 = \lambda^4 + 4\lambda^2 + 4 - 4\lambda^2 = \lambda^4 + 4 = 0$

$y^{(4)} + 4y = 0; \quad a_3 = 0,\ a_2 = 0,\ a_1 = 0,\ a_0 = 4$.

15. $(\lambda - 1)^4 = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = 0$. Thus the differential equation is $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$, and so $a_3 = -4,\ a_2 = 6,\ a_1 = -4$, and $a_0 = 1$.

16 (a). $n = 5$

16 (b). $\{1,\ t,\ e^t,\ \cos t,\ \sin t\}$

17 (a). $n = 5$

17 (b). $\{e^t,\ e^t\cos 2t,\ e^t \sin 2t,\ e^{-t}\cos 2t,\ e^{-t}\sin 2t\}$

18 (a). $n = 8$

18 (b). $\{\sin t,\ \cos t,\ t\sin t,\ t\cos t,\ t^2\sin t,\ t^2\cos t,\ e^t\sin t,\ e^t\cos t\}$

19 (a). $n = 7$

19 (b). $\{\sin t,\ \cos t,\ t\sin t,\ t\cos t,\ e^t,\ te^t,\ t^2e^t\}$

20 (a). $n = 4$

20 (b). $\{1,\ t,\ t^2, e^{2t}\}$

21. $n = 1,\ a = 1,\ y(t) = Ce^{-t}$

22. $n=1,\ a=\pm 2;\quad y' \pm 2y = 0,\ y(t) = Ce^{\mp t}$

23. $n=4,\ a=0,\ y(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3$

24. $n=2,\ a=4;\quad y'' + 4y = 0,\ y = c_1\cos 2t + c_2 \sin 2t.$

25. Three values for λ must be 3 and $-\frac{3}{2} \pm i\frac{3\sqrt{3}}{2}$. Using these values to reach the characteristic equation gives us $(\lambda-3)(\lambda^2+3\lambda+9) = \lambda^3 + 3\lambda^2 + 9\lambda - 3\lambda^2 - 9\lambda - 27 = \lambda^3 - 27$. Thus $n = 3$ and $a = -27$.

Section 5.5

1 (a). $\lambda^3 - \lambda = \lambda(\lambda+1)(\lambda-1) = 0$. Thus $y_C = c_1 e^{-t} + c_2 + c_3 e^t$.

1 (b). $y_P = Ae^{2t}$, and substituting this into the original differential equation yields $(8A-2A)e^{2t} = e^{2t}$. Thus $A = \frac{1}{6}$ and so $y_P = \frac{1}{6}e^{2t}$.

1 (c). $y = c_1 e^{-t} + c_2 + c_3 e^t + \frac{1}{6}e^{2t}$

2 (a). $y_c = c_1 e^{-t} + c_2 + c_3 e^t$

2 (b). $y_p = At + B\cos 2t + C\sin 2t,\ y_p{}' = A - 2B\sin 2t + 2C\cos 2t,\ y'' =$

$-4B\cos 2t - 4C\sin 2t,\ y_p{}''' = 8B\sin 2t - 8C\cos 2t$

$\therefore\ 8B\sin 2t - 8C\cos 2t - A + 2B\sin 2t - 2C\cos 2t = 4 + 2\cos 2t$

$10B = 0,\ -10C = 2,\ -A = 4 \quad \therefore\ A = -4,\ B = 0,\ C = -\frac{1}{5};\qquad y_p = -4t - \frac{1}{5}\sin 2t$

2 (c). $y = c_1 e^{-t} + c_2 + c_3 e^t - 4t - \frac{1}{5}\sin 2t$

3 (a). $\lambda^3 - \lambda = \lambda(\lambda+1)(\lambda-1) = 0$. Thus $y_C = c_1 e^{-t} + c_2 + c_3 e^t$.

3 (b). $y_P = t(At+B) = At^2 + Bt$. Differentiation yields $y_P' = 2At + B,\ y_P'' = 2A,$ and $y_P''' = 0$, and substituting into the original differential equation yields $0 - 2At - B = 4t$. Thus $A = -2$ and $B = 0$ and so $y_P = -2t^2$.

3 (c). $y = c_1 e^{-t} + c_2 + c_3 e^t - 2t^2$

4 (a). $y_c = c_1 e^{-t} + c_2 + c_3 e^t$

4 (b). $y_p = Ate^t,\ y_p' = A(t+1)e^t,\ y_p'' = A(t+2)e^t,\ y_p''' = A(t+3)e^t;$

$y_p''' - y_p' = A[t+3-t-1]e^t = -4e^t \ \Rightarrow\ 2A = -4$ or $A = -2 \quad \therefore\ y_p = -2te^t$

4 (c). $y = c_1e^{-t} + c_2 + c_3e^t - 2te^t$

5 (a). $\lambda^3 + \lambda^2 = \lambda^2(\lambda+1) = 0$. Thus $y_C = c_1 + c_2t + c_3e^{-t}$.

5 (b). $y_P = Ate^{-t}$. Differentiation yields $y_P' = A(1-t)e^{-t}$, $y_P'' = A(t-2)e^{-t}$, and $y_P''' = A(-t+3)e^{-t}$, and substituting into the original differential equation yields $A(-t+3+t-2)e^{-t} = 6e^{-t}$. Thus $A = 6$ and so $y_P = 6te^{-t}$.

5 (c). $y = c_1 + c_2t + c_3e^{-t} + 6te^{-t}$

6 (a). $\lambda^3 - \lambda^2 = 0,\ \lambda = 0,\ 0,\ 1;\qquad y_c = c_1 + c_2t + c_3e^t$

6 (b). $y_p = Ae^{-2t},\ y_p' = -2Ae^{-2t},\ y_p'' = 4Ae^{-2t},\ y_p''' = -8Ae^{-2t}\quad \therefore\ (-8A - 4A)e^{-2t} = 4e^{-2t}$

$A = -\frac{1}{3},\ y_p = -\frac{1}{3}e^{-2t}$

6 (c). $y = c_1 + c_2t + c_3e^t - \frac{1}{3}e^{-2t}$

7 (a). $\lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda-1)^2 = 0$. Thus $y_C = c_1 + c_2e^t + c_3te^t$.

7 (b). $y_P = t(At+B) + Ct^2e^t = At^2 + Bt + Ct^2e^t$. Differentiation yields

$y_P' = 2At + B + C\left(t^2+2t\right)e^t$, $y_P'' = 2A + C\left(t^2+4t+2\right)e^t$, and $y_P''' = C\left(t^2+6t+6\right)e^t$, and substituting into the original differential equation yields

$C\left(t^2+6t+6\right)e^t - 2\left[2A + C\left(t^2+4t+2\right)e^t\right] + 2At + B + C\left(t^2+2t\right)e^t = t + 4e^t$. Thus

$A = \frac{1}{2}$, $B = 2$, and $C = 2$ and so $y_P = \frac{1}{2}t^2 + 2t + 2t^2e^t$.

7 (c). $y = c_1 + c_2e^t + c_3te^t + \frac{1}{2}t^2 + 2t + 2t^2e^t$

8 (a). $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda-1)^3 = 0;\qquad y_c = c_1e^t + c_2te^t + c_3t^2e^t$

8 (b). $y_p = At^3e^t,\ y_p' = A\left(t^3+3t^2\right)e^t,\ y_p'' = A\left(t^3+6t^2+6t\right)e^t,\ y_p''' = A\left(t^3+9t^2+18t+6\right)e^t$

$\therefore\ A\left[t^3+9t^2+18t+6-3\left(t^3+6t^2+6t\right)+3\left(t^3+3t^2\right)-t^3\right]e^t = 12e^t \ \Rightarrow\ A\cdot 6 = 12;\qquad y_p = 2t^3e^t$

8 (c). $y = c_1e^t + c_2te^t + c_3t^2e^t + 2t^3e^t$

9 (a). $\lambda^3 - 1 = 0$. Thus $\lambda = 1,\ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, and so $y_C = c_1e^t + c_2e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_3e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)$.

9 (b). $y_P = Ate^t$. Differentiation yields $y_P' = A(t+1)e^t$, $y_P'' = A(t+2)e^t$, and $y_P''' = A(t+3)e^t$, and substituting into the original differential equation yields $A(t+3-t)e^t = e^t$. Thus $A = \frac{1}{3}$ and so $y_P = \frac{1}{3}te^t$.

9 (c). $y_C = c_1 e^t + c_2 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_3 e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{3}te^t$

10 (a). $\lambda^3 + 1 = 0,\ \lambda^3 = -1 = e^{i(\pi+2k\pi)},\ \lambda_k = e^{i(\pi+2k\pi)/3},\ \lambda_1 = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2},$

$\lambda_2 = -1,\ \lambda_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2};\quad y_c = c_1 e^{-t} + c_2 e^{t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_3 e^{t/2} \sin\left(\frac{\sqrt{3}}{2}t\right)$

10 (b). $y_p = Ae^t + B\cos t + C\sin t,\ y_p' = Ae^t - B\sin t + C\cos t,\ y_p'' = Ae^t - B\cos t - C\sin t,$

$y_p''' = Ae^t + B\sin t - C\cos t \quad \therefore\ Ae^t + B\sin t - C\cos t + Ae^t + B\cos t + C\sin t =$

$= e^t + \cos t \quad \therefore\ 2A = 1,\ B + C = 0,\ -C + B = 1,\ A = \frac{1}{2},\ B = \frac{1}{2},\ C = -\frac{1}{2},$

$y_p = \frac{1}{2}e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t$

10 (c). $y = c_1 e^{-t} + c_2 e^{t/2} \cos\left(\frac{\sqrt{3}}{2}t\right) + c_3 e^{t/2} \sin\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{2}e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t$

11 (a). $\lambda^3 - 4\lambda^2 + 4\lambda = \lambda(\lambda-2)^2 = 0$. Thus $y_C = c_1 + c_2 e^{2t} + c_3 te^{2t}$.

11 (b). $y_P = t\left(A_3t^3 + A_2t^2 + A_1t + A_0\right) + t^2\left(B_2t^2 + B_1t + B_0\right)e^{2t}$

$= A_3t^4 + A_2t^3 + A_1t^2 + A_0t + \left(B_2t^4 + B_1t^3 + B_0t^2\right)e^{2t}$

12 (a). $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda-1)^3 = 0 \qquad y_c = c_1e^t + c_2te^t + c_3t^2e^t$

12 (b). $y_p = At^3e^t + Be^t\cos 3t + Ce^t\sin 3t + D$

13(a). $\lambda^4 - 16 = (\lambda^2+4)(\lambda-2)(\lambda+2) = 0,\ \lambda = \pm 2,\ \pm i2$, and so

$y_C = c_1e^{-2t} + c_2e^{2t} + c_3\cos 2t + c_4\sin 2t$.

13 (b). $y_P = t\left(A_1t + A_0\right)e^{2t} + t\left(B_1t + B_0\right)e^{-2t} + t\left(C_1t + C_0\right)\cos 2t + t\left(D_1t + D_0\right)\sin 2t$

14 (a). $\lambda^4 + 8\lambda^2 + 16 = \left(\lambda^2 + 4\right)^2 = 0 \quad \therefore\ \lambda = \pm i2,\ \pm i2$

$y_c = c_1\cos 2t + c_2\sin 2t + c_3t\cos 2t + c_4t\sin 2t$

14 (b). $y_p = t^2\left(A_1t + A_0\right)\cos 2t + t^2\left(B_1t + B_0\right)\sin 2t$

15 (a). $\lambda^4 - 1 = \left(\lambda^2+1\right)\left(\lambda+1\right)\left(\lambda-1\right) = 0$. Thus $y_C = c_1e^{-t} + c_2e^t + c_3\cos t + c_4\sin t$.

15 (b). $y_P = t\left(A_1t + A_0\right)e^{-t} + t\left(C_1t + C_0\right)\cos t + t\left(D_1t + D_0\right)\sin t$.

16. $y = c_1 + c_2t + c_3e^{2t} + 4\sin 2t;\quad \lambda^2(\lambda-2) = \lambda^3 - 2\lambda^2 = 0;\quad y''' - 2y'' = g(t)$

$\therefore\ a = -2,\ b = 0,\ c = 0.\quad y_p = 4\sin 2t,\ y_p' = 8\cos 2t,\ y_p'' = -16\sin 2t,\ y_p''' =$

$-32\cos 2t$.

Substitute: $-32\cos 2t - 2(-16\sin 2t) = g(t)$

$\therefore\ a=-2,\ b=0,\ c=0,\ g(t) = -32(\cos 2t - \sin 2t).$

17. $(\lambda^2+4)(\lambda-1) = \lambda^3 - \lambda^2 + 4\lambda - 4 = 0$. Therefore, $g(t) = y''' - y'' + 4y' - 4y$. $y_P = t^2$, and differentiation yields $y_P' = 2t,\ y_P'' = 2,$ and $y_P''' = 0$. Then we have

$g(t) = 0 - 2 + 4(2t) - 4t^2 = -4t^2 + 8t - 2$ and $a=-1,\ b=4,\ c=-4$.

18. $y = c_1 + c_2 t + c_3 t^2 - 2t^3;\quad \lambda^3 = 0,\ y''' = g(t);\quad y_p = -2t^3,\ y_p' = -6t^2,$

$y_p'' = -12t,\ y_p''' = -12;\quad a=b=c=0,\ g(t) = -12$

19. $y = c_1 + c_2 t + c_3 t^3 + t^4$, and so $1,\ t,\ t^3$ are solutions of the homogeneous equation.

$0+0+0+c\cdot 1 = 0$, so c $=0$. $0+0+bt\cdot 1 = 0$, so $b=0$. $t^3\cdot 6 + at^2(6t) = 0$, so $a=-1$.

$t^3y''' - t^2y'' = g(t)$ and $y_P = t^4$, so $g(t) = t^3\cdot 24t - t^2\cdot 12t^2 = 12t^4$.

20. $y = c_1 t + c_2 t^2 + c_3 t^4 + 2\ln t;\quad t,\ t^2,\ t^4$ are solutions of homogeneous equation.

$0+0+bt+ct=0\quad \therefore\ b+c=0,\ 0+at^2(2)+bt(2t)+c(t^2) = 0\quad \therefore\ 2a+2b+c=0$

$(t^4)' = 4t^3,\ (t^4)'' = 12t^2,\ (t^4)''' = 24t.\qquad t^3(24t) + at^2(12t^2) + bt(4t^3) + c(t^4) = 0$

$24 + 12a + 4b + c = 0\quad \therefore\ c=-b\ \Rightarrow\ 2a+b=0$ and $24+12a+3b=0$

$\therefore\ b=-2a\ \Rightarrow\ 24+12a-6a=0\ \Rightarrow\ a=-4,\ b=8,\ c=-8$

$t^3y''' - 4t^2y'' + 8ty' - 8y = g(t)$

$(2\ln t)' = \dfrac{2}{t},\ (2\ln t)'' = -\dfrac{2}{t^2},\ (2\ln t)''' = \dfrac{4}{t^3}$

$t^3\left(\dfrac{4}{t^3}\right) - 4t^2\left(-\dfrac{2}{t^2}\right) + 8t\left(\dfrac{2}{t}\right) - 16\ln t = g\ \Rightarrow\ g = 28 - 16\ln t$

$a=-4,\ b=8,\ c=-8,\ g(t) = 28 - 16\ln(t),\ t>0.$

21 (a). The three solutions can be verified by substitution.

21 (b). $\begin{bmatrix} t & t^2 & t^4 \\ 1 & 2t & 4t^3 \\ 0 & 2 & 12t^2 \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t^{-3}g \end{bmatrix}$ where $y_P = tu_1 + t^2u_2 + t^4u_3$. $\det = t[24t^3 - 8t^3] - 1[12t^4 - 2t^4] = 6t^4$.

Thus $u_1' = \begin{vmatrix} 0 & t^2 & t^4 \\ 0 & 2t & 4t^3 \\ t^{-3}g & 2 & 12t^2 \end{vmatrix}\cdot\dfrac{1}{6t^4} = \dfrac{1}{6}t^{-7}g[2t^5] = \dfrac{1}{3}t^{-2}g$,

$u_2' = \begin{vmatrix} t & 0 & t^4 \\ 1 & 0 & 4t^3 \\ 0 & t^{-3}g & 12t^2 \end{vmatrix}\cdot\dfrac{1}{6t^4} = -\dfrac{1}{6}t^{-7}g[3t^4] = -\dfrac{1}{2}t^{-3}g$, and $u_3' = \begin{vmatrix} t & t^2 & 0 \\ 1 & 2t & 0 \\ 0 & 2 & t^{-3}g \end{vmatrix} \cdot \dfrac{1}{6t^4} = \dfrac{1}{6}t^{-7}g\left[t^2\right] = \dfrac{1}{6}t^{-5}g$. $g = 2t^{\frac{1}{2}}$. Making this substitution,

antidifferentiating the three u' equations yields $u_1 = -\dfrac{4}{3}t^{-\frac{1}{2}}$, $u_2 = \dfrac{2}{3}t^{-\frac{3}{2}}$, and $u_3 = -\dfrac{2}{21}t^{-\frac{7}{2}}$. Thus

$y_P = -\dfrac{4}{3}t^{-\frac{1}{2}}(t) + \dfrac{2}{3}t^{-\frac{3}{2}}\left(t^2\right) - \dfrac{2}{21}t^{-\frac{7}{2}}\left(t^4\right) = -\dfrac{16}{21}t^{\frac{1}{2}}$. Finally, we have the general solution:

$y = c_1 t + c_2 t^2 + c_3 t^4 - \dfrac{16}{21}t^{\frac{1}{2}}$.

22. $g = 2t;\quad u_1' = \dfrac{2}{3}t^{-1} \;\Rightarrow\; u_1 = \dfrac{2}{3}\ln t;\quad u_2' = -t^{-2} \;\Rightarrow\; u_2 = t^{-1};$

$u_3' = \dfrac{1}{3}t^{-4} \;\Rightarrow\; u_3 = -\dfrac{1}{9}t^{-3};\quad y_p = \dfrac{2}{3}t\ln t + t - \dfrac{1}{9}t$

$y = c_1't + c_2't^2 + c_3't^4 + \dfrac{2}{3}t\ln t + t - \dfrac{1}{9}t \;=\; c_1 t + c_2 t^2 + c_3 t^4 + \dfrac{2}{3}t\ln t$

23. $g = t^3 + 2t^2 + 1$, and so $u_1' = \dfrac{1}{3}t^{-2}\left(t^3 + 2t^2 + 1\right) = \dfrac{1}{3}t + \dfrac{2}{3} + \dfrac{1}{3}t^{-2}$, and antidifferentiation yields

$u_1 = \dfrac{t^2}{6} + \dfrac{2}{3}t - \dfrac{1}{3}t^{-1}$. $u_2' = -\dfrac{1}{2}t^{-3}\left(t^3 + 2t^2 + 1\right) = -\dfrac{1}{2} - t^{-1} - \dfrac{1}{2}t^{-3}$, and antidifferentiation yields

$u_2 = -\dfrac{t}{2} - \ln t + \dfrac{1}{4}t^{-2}$. $u_3' = \dfrac{1}{6}t^{-5}\left(t^3 + 2t^2 + 1\right) = \dfrac{1}{6}t^{-2} + \dfrac{1}{3}t^{-3} + \dfrac{1}{6}t^{-5}$, and antidifferentiation yields

$u_3 = -\dfrac{1}{6}t^{-1} - \dfrac{1}{6}t^{-2} - \dfrac{1}{24}t^{-4}$. Thus

$y_P = t\left(\dfrac{t^2}{6} + \dfrac{2}{3}t - \dfrac{1}{3}t^{-1}\right) + t^2\left(-\dfrac{t}{2} - \ln t + \dfrac{1}{4}t^{-2}\right) + t^4\left(-\dfrac{1}{6}t^{-1} - \dfrac{1}{6}t^{-2} - \dfrac{1}{24}t^{-4}\right)$

$= -\dfrac{1}{2}t^3 + \dfrac{1}{2}t^2 - \dfrac{1}{8} - t^2\ln t$. Finally, we have the general solution:

$y = c_1 t + c_2 t^2 + c_3 t^4 - \dfrac{1}{2}t^3 - \dfrac{1}{8} - t^2\ln t$.