Chapter 6
First Order Linear Systems

Section 6.1

1. $2A(t)-3tB(t)=2\begin{bmatrix} t-1 & t^2 \\ 2 & 2t+1 \end{bmatrix}-3t\begin{bmatrix} t & -1 \\ 0 & t+2 \end{bmatrix}=\begin{bmatrix} 2t-2 & 2t^2 \\ 4 & 4t+2 \end{bmatrix}-\begin{bmatrix} 3t^2 & -3t \\ 0 & 3t^2+6t \end{bmatrix}$

$=\begin{bmatrix} 2t-2-3t^2 & 2t^2+3t \\ 4 & 2-2t-3t^2 \end{bmatrix}$

2. $A(t)B(t)-B(t)A(t)=\begin{bmatrix} 2 & 2t^2+t+2 \\ -4 & -2 \end{bmatrix}$

3. $A(t)\mathbf{c}(t)=\begin{bmatrix} t-1 & t^2 \\ 2 & 2t+1 \end{bmatrix}\begin{bmatrix} t+1 \\ -1 \end{bmatrix}=\begin{bmatrix} (t-1)(t+1)+t^2(-1) \\ 2(t+1)+(2t+1)(-1) \end{bmatrix}=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

4. $\det[tA(t)]=-t^3-t^2$

5. There are two natural ways to do this problem. We can form the matrix $A(t)B(t)$ and then calculate $\det[A(t)B(t)]$. Alternatively, we can separately calculate $\det[A(t)]$ and $\det[B(t)]$ and use the fact that $\det[A(t)B(t)]=\det[A(t)]\det[B(t)]$.
Taking the latter course, $\det[A(t)]=(t-1)(2t+1)-2t^2=-(t+1)$, and $\det[B(t)]=t(t+2)=t^2+2t$. Thus, $\det[A(t)B(t)]=-(t+1)(t^2+2t)=-(t^3+3t^2+2t)$.

6. $\det[A(t)]=2t+1$ and so the matrix $A(t)$ is invertible for every value t except $t=-\frac{1}{2}$. The inverse of $A(t)$ is given by $A^{-1}(t)=\frac{1}{2t+1}\begin{bmatrix} t+1 & -t \\ -t & t+1 \end{bmatrix}$, $t\neq-\frac{1}{2}$.

7. As noted in Example 2, a square matrix is invertible if and only if its determinant is nonzero. Now, $\det[A(t)]=t(t-3)-4=t^2-3t-4=(t-4)(t+1)$ and so the matrix $A(t)$ is invertible for every value t except $t=4$ and $t=-1$. The inverse of $A(t)$ is given by $A^{-1}(t)=\frac{1}{(t-4)(t+1)}\begin{bmatrix} t-3 & -2 \\ -2 & t \end{bmatrix}$, $t\neq 4, t\neq -1$.

8. $\det[A(t)] = 2\sin t\cos t = \sin 2t$ and so the matrix $A(t)$ is invertible for every value t except $2t = n\pi \Rightarrow t = \frac{n\pi}{2},\ n = 0, \pm 1, \pm 2, \pm 3, \ldots$. The inverse of $A(t)$ is given by

$$A^{-1}(t) = \frac{1}{2\sin t\cos t}\begin{bmatrix} \cos t & \cos t \\ -\sin t & \sin t \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\csc t & \frac{1}{2}\csc t \\ -\frac{1}{2}\sec t & \frac{1}{2}\sec t \end{bmatrix},\quad t \neq \frac{n\pi}{2},\ n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

9. In this case, $\det[A(t)] = e^t e^{4t} - e^{3t}e^{2t} = e^{5t} - e^{5t}$ and so $\det[A(t)]$ is zero for every value of t. Hence, the given matrix $A(t)$ is never invertible.

10. $$\lim_{t\to 0} A(t) = \lim_{t\to 0}\begin{bmatrix} \frac{\sin t}{t} & t\cos t & \frac{3}{t+1} \\ e^{3t} & \sec t & \frac{2t}{t^2 - 1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}.$$

11. $$\lim_{t\to 0} A(t) = \lim_{t\to 0}\begin{bmatrix} te^{-t} & \tan t \\ t^2 - 2 & e^{\sin t} \end{bmatrix} = \begin{bmatrix} \lim_{t\to 0} te^{-t} & \lim_{t\to 0}\tan t \\ \lim_{t\to 0}[t^2 - 2] & \lim_{t\to 0} e^{\sin t} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}.$$

12. Differentiating $A(t)$ component wise, we have $A'(t) = \begin{bmatrix} \cos t & 3 \\ 2t & 0 \end{bmatrix}$ and

$A''(t) = \begin{bmatrix} -\sin t & 0 \\ 2 & 0 \end{bmatrix}$. $A(t), A'(t)$ and $A''(t)$ are defined for $-\infty < t < \infty$.

13. Differentiating $A(t)$ component wise, we have $A'(t) = \begin{bmatrix} 0 & t^{-1} \\ -0.5(1-t)^{-1/2} & 3e^{3t} \end{bmatrix}$ and

$A''(t) = \begin{bmatrix} 0 & -t^{-2} \\ -0.25(1-t)^{-3/2} & 9e^{3t} \end{bmatrix}$. $A(t)$ is defined for $-\infty < t < 0$ and $0 < t \le 1$.

$A'(t)$ and $A''(t)$ are defined for $-\infty < t < 0$ and $0 < t < 1$.

14. $P(t) = \begin{bmatrix} t^2 & 3 \\ \sin t & t \end{bmatrix}$ and $\mathbf{g}(t) = \begin{bmatrix} \sec t \\ -5 \end{bmatrix}$.

15. $$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} t^{-1}y_1 + (t^2+1)y_2 + t \\ 4y_1 + t^{-1}y_2 + 8t\ln t \end{bmatrix} = \begin{bmatrix} t^{-1}y_1 + (t^2+1)y_2 \\ 4y_1 + t^{-1}y_2 \end{bmatrix} + \begin{bmatrix} t \\ 8t\ln t \end{bmatrix} =$$

$\begin{bmatrix} t^{-1} & t^2+1 \\ 4 & t^{-1} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} t \\ 8t\ln t \end{bmatrix}$. Therefore, $P(t) = \begin{bmatrix} t^{-1} & t^2+1 \\ 4 & t^{-1} \end{bmatrix}$ and $\mathbf{g}(t) = \begin{bmatrix} t \\ 8t\ln t \end{bmatrix}$.

16. Let $A'(t) = \begin{bmatrix} 2t & 1 \\ \cos t & 3t^2 \end{bmatrix}$. Integrating component wise, we find

$$A(t) = \begin{bmatrix} t^2 + C_{11} & t + C_{12} \\ \sin t + C_{21} & t^3 + C_{22} \end{bmatrix}.$$

Since $A(0)=\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}=\begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$, we obtain $A(t)=\begin{bmatrix} t^2+2 & t+5 \\ \sin t+1 & t^3-2 \end{bmatrix}$.

17. Let $A'(t)=\begin{bmatrix} t^{-1} & 4t \\ 5 & 3t^2 \end{bmatrix}$. Integrating component wise, we find $A(t)=\begin{bmatrix} \ln|t|+C_{11} & 2t^2+C_{12} \\ 5t+C_{21} & t^3+C_{22} \end{bmatrix}$. Since $A(1)=\begin{bmatrix} C_{11} & 2+C_{12} \\ 5+C_{21} & 1+C_{22} \end{bmatrix}=\begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$, we obtain $A(t)=\begin{bmatrix} \ln|t|+2 & 2t^2+3 \\ 5t-4 & t^3-3 \end{bmatrix}$.

18. Let $A''(t)=\begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}$. Integrating component wise, we find

$$A'(t)=\begin{bmatrix} t+C_{11} & \dfrac{t^2}{2}+C_{12} \\ C_{21} & C_{22} \end{bmatrix} \Rightarrow A(t)=\begin{bmatrix} \dfrac{t^2}{2}+C_{11}t+D_{11} & \dfrac{t^3}{6}+C_{12}t+D_{12} \\ C_{21}t+D_{21} & C_{22}t+D_{22} \end{bmatrix}.$$

Since $A(0)=\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ and $A(1)=\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$, we obtain $A(t)=\begin{bmatrix} \dfrac{t^2}{2}-\dfrac{5}{2}t+1 & \dfrac{t^3}{6}+\dfrac{5}{6}t+1 \\ -2 & 2t+1 \end{bmatrix}$.

19. Integrating component wise, we obtain

$$\int_0^t B(s)\,ds=\begin{bmatrix} \int_0^t 2s\,ds & \int_0^t \cos s\,ds & \int_0^t 2\,ds \\ \int_0^t 5\,ds & \int_0^t (s+1)^{-1}\,ds & \int_0^t 3s^2\,ds \end{bmatrix}=\begin{bmatrix} t^2 & \sin t & 2t \\ 5t & \ln|t+1| & t^3 \end{bmatrix}.$$

20. Integrating component wise, we obtain $\int_0^t B(s)\,ds=\begin{bmatrix} e^t-1 & 3t^2 \\ \dfrac{\sin 2\pi t}{2\pi} & \dfrac{1-\cos 2\pi t}{2\pi} \end{bmatrix}$.

21 (a). One example is $A=\begin{bmatrix} 1 & t \\ t^2 & 0 \end{bmatrix}$.

22. One example is $A=\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$.

Section 6.2

1. The given problem can be written as $\mathbf{y}'(t)=P(t)\mathbf{y}(t)+\mathbf{g}(t),\quad \mathbf{y}(3)=\mathbf{y}_0$ where $P(t)=\begin{bmatrix} t^{-1} & \tan t \\ \ln|t| & e^t \end{bmatrix}$, $\mathbf{g}(t)=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{y}_0=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The coefficient functions $p_{11}(t)=t^{-1}$ and $p_{21}(t)=\ln|t|$ are discontinuous at $t=0$. The coefficient function $p_{12}(t)=\tan t$ has discontinuities at $\pm\pi/2, \pm 3\pi/2, \ldots$. The largest interval containing $t_0=3$ but containing no discontinuities of any coefficient function is the interval $\pi/2<t<3\pi/2$.

2. In standard form, the problem is $\mathbf{y}' = \begin{bmatrix} 1 & \tan t \\ t^2-2 & 4 \end{bmatrix}\mathbf{y} + \begin{bmatrix} (t+1)^{-2} \\ 0 \end{bmatrix}, \mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\tan t$ is discontinuous at $t = \pm\pi/2$ and $(t+1)^{-2}$ is discontinuous at $t = -1$. The largest interval containing $t_0 = 0$ but containing no discontinuities of any coefficient function is the interval $-1 < t < \pi/2$.

3. In standard form, the problem is $\mathbf{y}' = \begin{bmatrix} (\cos t)/t^2 & 1/t^2 \\ 2 & 4t \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1/t^2 \\ \sec t \end{bmatrix}, \mathbf{y}(1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

The only discontinuities of $p_{11}(t)$ and $p_{12}(t)$ are at $t = 0$, while $g_2(t)$ is discontinuous at $t = \pm\pi/2, \pm 3\pi/2, \ldots$. The largest interval containing $t_0 = 1$ but containing no discontinuities of any coefficient function is the interval $0 < t < \pi/2$.

4. In standard form, the problem is $\mathbf{y}' = \begin{bmatrix} \dfrac{3t}{t+2} & \dfrac{5}{t+2} \\ \dfrac{2}{t-2} & \dfrac{4t}{t-2} \end{bmatrix}\mathbf{y}, \mathbf{y}(1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

The largest interval containing $t_0 = 1$ but containing no discontinuities of any coefficient function is the interval $-2 < t < 2$.

5. Differentiating, $y_1' = 5c_1e^{5t} + 3c_2e^{3t}$ and $y_2' = 5c_1e^{5t} - 3c_2e^{3t}$. Calculating the right hand sides, $4y_1 + y_2 = 4(c_1e^{5t} + c_2e^{3t}) + (c_1e^{5t} - c_2e^{3t}) = 5c_1e^{5t} + 3c_2e^{3t} = y_1'$ and $y_1 + 4y_2 = (c_1e^{5t} + c_2e^{3t}) + 4(c_1e^{5t} - c_2e^{3t}) = 5c_1e^{5t} - 3c_2e^{3t} = y_2'$.

7 (a). $\mathbf{y}' = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\mathbf{y}$

7 (b). $\mathbf{y} = c_1e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

8 (a). $\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{y}$

8 (b). $\mathbf{y} = c_1\begin{bmatrix} e^t\cos t \\ -e^t\sin t \end{bmatrix} + c_2\begin{bmatrix} e^t\sin t \\ e^t\cos t \end{bmatrix}$

9. For $\mathbf{y} = c_1e^{2t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we have $\mathbf{y}' = 2c_1e^{2t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 3c_2e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. On the other hand, $A\mathbf{y} = A\left(c_1e^{2t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = c_1e^{2t}A\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2e^{3t}A\begin{bmatrix} 1 \\ -1 \end{bmatrix} =$ $c_1e^{2t}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1e^{2t}\begin{bmatrix} 4 \\ -2 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 3 \\ -3 \end{bmatrix}$. Thus, $\mathbf{y}' = A\mathbf{y}$ for every choice of c_1 and c_2.

In order to solve the initial value problem, we first note that

$\mathbf{y}(0)=c_1\begin{bmatrix}2\\-1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}2&1\\-1&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$. Thus, solving $\begin{bmatrix}2&1\\-1&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}4\\-3\end{bmatrix}$, we obtain $c_1=1$ and $c_2=2$. Therefore, $\mathbf{y}(t)=e^{2t}\begin{bmatrix}2\\-1\end{bmatrix}+2e^{3t}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}2e^{2t}+2e^{3t}\\-e^{2t}-2e^{3t}\end{bmatrix}$ is the solution of the given initial value problem.

10. For $\mathbf{y}'=c_1e^{5t}\begin{bmatrix}5\\5\end{bmatrix}+c_2e^{-t}\begin{bmatrix}1\\-2\end{bmatrix}$, $A\mathbf{y}=c_1e^{5t}\begin{bmatrix}3+2\\4+1\end{bmatrix}+c_2e^{-t}\begin{bmatrix}-3+4\\-4+2\end{bmatrix}$

Solving $\begin{bmatrix}1&-1\\1&2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}-1\\8\end{bmatrix}$, we obtain $c_1=2$ and $c_2=3$. Therefore, $\mathbf{y}(t)=\begin{bmatrix}2e^{5t}-3e^{-t}\\2e^{5t}+6e^{-t}\end{bmatrix}$ is the solution of the given initial value problem.

11. Let $\mathbf{Y}(t)=\begin{bmatrix}y(t)\\y'(t)\end{bmatrix}$. Calculating $\mathbf{Y}'(t)$, we find

$\mathbf{Y}'(t)=\begin{bmatrix}y'(t)\\y''(t)\end{bmatrix}=\begin{bmatrix}y'(t)\\-t^2y'(t)-4y(t)+\sin t\end{bmatrix}=\begin{bmatrix}0&1\\-4&-t^2\end{bmatrix}\begin{bmatrix}y(t)\\y'(t)\end{bmatrix}+\begin{bmatrix}0\\\sin t\end{bmatrix}$. Therefore, the scalar equation can be written as $\mathbf{Y}'=P(t)\mathbf{Y}+\mathbf{G}(t)$ where $P(t)=\begin{bmatrix}0&1\\-4&-t^2\end{bmatrix}$ and $\mathbf{G}(t)=\begin{bmatrix}0\\\sin t\end{bmatrix}$.

12. Let $\mathbf{Y}(t)=\begin{bmatrix}y(t)\\y'(t)\end{bmatrix}$. Calculating $\mathbf{Y}'(t)$, we find

$\mathbf{Y}'(t)=\begin{bmatrix}y'(t)\\y''(t)\end{bmatrix}=\begin{bmatrix}0&1\\-\sqrt{t}\sec t&3t\sec t\end{bmatrix}\begin{bmatrix}y(t)\\y'(t)\end{bmatrix}+\begin{bmatrix}0\\(t^2+1)\sec t\end{bmatrix}$. Therefore, the scalar equation can be written as $\mathbf{Y}'=P(t)\mathbf{Y}+\mathbf{G}(t)$ where $P(t)=\begin{bmatrix}0&1\\-\sqrt{t}\sec t&3t\sec t\end{bmatrix}$ and $\mathbf{G}(t)=\begin{bmatrix}0\\(t^2+1)\sec t\end{bmatrix}$.

13. Let $\mathbf{Y}(t)=\begin{bmatrix}y(t)\\y'(t)\\y''(t)\end{bmatrix}$. We solve for y''' by multiplying the equation by e^{-t} and find

$\mathbf{Y}'(t)=\begin{bmatrix}y'(t)\\y''(t)\\-5e^{-t}y''(t)-e^{-t}t^{-1}y'(t)-(e^{-t}\tan t)y(t)+e^{-t}\end{bmatrix}$. Expressing $\mathbf{Y}'(t)$ in matrix terms, we have $\mathbf{Y}'(t)=\begin{bmatrix}0&1&0\\0&0&1\\-e^{-t}\tan t&-e^{-t}t^{-1}&-5e^{-t}\end{bmatrix}\begin{bmatrix}y(t)\\y'(t)\\y''(t)\end{bmatrix}+\begin{bmatrix}0\\0\\e^{-t}\end{bmatrix}$. Therefore, the scalar equation can be written as $\mathbf{Y}'=P(t)\mathbf{Y}+\mathbf{G}(t)$ where

$$P(t)=\begin{bmatrix}0&1&0\\0&0&1\\-e^{-t}\tan t&-e^{-t}t^{-1}&-5e^{-t}\end{bmatrix}\text{ and }\mathbf{G}(t)=\begin{bmatrix}0\\0\\e^{-t}\end{bmatrix}.$$

14. Let $\mathbf{Y}(t)=\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}$. $\mathbf{Y}'(t)=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -\cos t & 2 \end{bmatrix}\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}$. Therefore, the scalar equation can be written as $\mathbf{Y}'=P(t)\mathbf{Y}+\mathbf{G}(t)$ where $P(t)=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -\cos t & 2 \end{bmatrix}$ and $\mathbf{G}(t)=\begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}$.

15. Let $\mathbf{Y}(t)=\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$ so that $\mathbf{Y}'(t)=\begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix}$. We are given that

$$\mathbf{Y}'(t)=\begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}+\begin{bmatrix} 0 \\ 2\cos 2t \end{bmatrix}=\begin{bmatrix} y'(t) \\ -3y(t)+2y'(t)+2\cos 2t \end{bmatrix}.$$

Therefore, equating components of the vector $\mathbf{Y}'(t)$, we see that the scalar equation is $y''=-3y+2y'+2\cos 2t, \quad y(-1)=1, y'(-1)=4$.

16. $y'''-4y''+2y=e^{3t}, \quad y(0)=1, y'(0)=-2, \ y''(0)=3$.

17. Let $\mathbf{Y}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}=\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ y'''(t) \end{bmatrix}$ so that $\mathbf{Y}'(t)=\begin{bmatrix} y'(t) \\ y''(t) \\ y'''(t) \\ y^{(4)}(t) \end{bmatrix}$. We are given that

$$\mathbf{Y}'(t)=\begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_2+y_3\sin(y_1)+y_3^2 \end{bmatrix}=\begin{bmatrix} y' \\ y'' \\ y''' \\ y'+y''\sin(y)+(y'')^2 \end{bmatrix}.$$

Equating components of the vector $\mathbf{Y}'(t)$, we see that the scalar equation is $y^{(4)}=y'+y''\sin(y)+(y'')^2, \quad y(1)=0, y'(1)=0, y''(1)=-1, y'''(1)=2$.

18. Making the indicated change of variables, the system of differential equations is

$$\begin{aligned} Y_2' &= Y_2+Y_3+tY_4 \\ Y_4' &= 2tY_1+Y_2+Y_4 \end{aligned}$$

. Therefore, the system can be expressed in the form $\mathbf{Y}'=P(t)\mathbf{Y}+\mathbf{G}(t)$ where

$$P(t)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & t \\ 0 & 0 & 0 & 1 \\ 2t & 1 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{G}(t)=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

19. Making the indicated change of variables, the system of differential equations is

$$\begin{aligned} Y_2' &= t^{-1}Y_2+4Y_1-tY_3+(\sin t)Y_4+e^{2t} \\ Y_4' &= Y_1-5Y_4 \end{aligned}.$$

Therefore, the system can be expressed in the form $\mathbf{Y}' = P(t)\mathbf{Y} + \mathbf{G}(t)$ where

$$P(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & t^{-1} & -t & \sin t \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -5 \end{bmatrix} \text{ and } \mathbf{G}(t) = \begin{bmatrix} 0 \\ e^{2t} \\ 0 \\ 0 \end{bmatrix}.$$

20. Making the indicated change of variables, the system of differential equations is

$$\begin{aligned} Y_2' &= 4Y_1 + 7Y_2 - 8Y_3 + 6Y_4 + t^2 \\ Y_4' &= 3Y_1 - 6Y_2 + 2Y_3 + 5Y_4 - \sin t \end{aligned}$$

. Therefore, the system can be expressed in the form $\mathbf{Y}' = P(t)\mathbf{Y} + \mathbf{G}(t)$ where

$$P(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & 7 & -8 & 6 \\ 0 & 0 & 0 & 1 \\ 3 & -6 & 2 & 5 \end{bmatrix} \text{ and } \mathbf{G}(t) = \begin{bmatrix} 0 \\ t^2 \\ 0 \\ -\sin t \end{bmatrix}.$$

21. Making the indicated change of variables, the system of differential equations is

$$\begin{aligned} 15Y_3 + 9Y_2 + 3Y_2' &= 12Y_1 - 6Y_4 + 3t^2 \\ Y_4 + 5Y_1 - Y_4' &= 2Y_3 - 6Y_2 + t \end{aligned}.$$

Writing this system in standard form,

$$\begin{aligned} Y_2' &= 4Y_1 - 3Y_2 - 5Y_3 - 2Y_4 + t^2 \\ Y_4' &= 5Y_1 + 6Y_2 - 2Y_3 + Y_4 - t \end{aligned}.$$

Therefore, the system can be expressed in the form $\mathbf{Y}' = P(t)\mathbf{Y} + \mathbf{G}(t)$ where

$$P(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & -3 & -5 & -2 \\ 0 & 0 & 0 & 1 \\ 5 & 6 & -2 & 1 \end{bmatrix} \text{ and } \mathbf{G}(t) = \begin{bmatrix} 0 \\ t^2 \\ 0 \\ -t \end{bmatrix}.$$

Section 6.3

1 (a). In matrix terms, the system has the form $\mathbf{y}' = A\mathbf{y}$ where

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ or } \mathbf{y}' = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix} \mathbf{y}.$$

1 (b). We have

$\mathbf{y}' = \begin{bmatrix} 6e^{3t} \\ 9e^{3t} \end{bmatrix}$. Calculating $A\mathbf{y}$, we obtain $\begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix} \begin{bmatrix} 2e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 18e^{3t} - 12e^{3t} \\ 30e^{3t} - 21e^{3t} \end{bmatrix}$ and

therefore, $A\mathbf{y} = \begin{bmatrix} 6e^{3t} \\ 9e^{3t} \end{bmatrix}$, showing that the function $\mathbf{y}(t)$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

2 (a). $\mathbf{y}' = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix} \mathbf{y}$.

3 (a). In matrix terms, the system has the form $\mathbf{y}' = A\mathbf{y}$ where

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ or } \mathbf{y}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\mathbf{y}.$$

3 (b). We have

$\mathbf{y}' = \begin{bmatrix} 2e^t \cos 2t - 4e^t \sin 2t \\ -e^t \sin 2t - 2e^t \cos 2t \end{bmatrix}$. Calculating $A\mathbf{y}$, we obtain

$\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2e^t \cos 2t \\ -e^t \sin 2t \end{bmatrix} = \begin{bmatrix} 2e^t \cos 2t - 4e^t \sin 2t \\ -2e^t \cos 2t - e^t \sin 2t \end{bmatrix}$. Therefore, the function $\mathbf{y}(t)$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

4 (a). $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ \frac{2}{t^2} & -\frac{2}{t} \end{bmatrix}\mathbf{y}.$

5 (a). In matrix terms, the system has the form $\mathbf{y}' = A\mathbf{y}$ where

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -6 & -3 & 1 \\ -8 & -2 & 4 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ or } \mathbf{y}' = \begin{bmatrix} 0 & 1 & 1 \\ -6 & -3 & 1 \\ -8 & -2 & 4 \end{bmatrix}\mathbf{y}.$$

5 (b). We have

$\mathbf{y}' = \begin{bmatrix} e^t \\ -e^t \\ 2e^t \end{bmatrix}$. Calculating $A\mathbf{y}$, we obtain

$\begin{bmatrix} 0 & 1 & 1 \\ -6 & -3 & 1 \\ -8 & -2 & 4 \end{bmatrix}\begin{bmatrix} e^t \\ -e^t \\ 2e^t \end{bmatrix} = \begin{bmatrix} -e^t + 2e^t \\ -6e^t + 3e^t + 2e^t \\ -8e^t + 2e^t + 8e^t \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t \\ 2e^t \end{bmatrix}$ and therefore the function $\mathbf{y}(t)$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

6 (a). $\mathbf{y}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\mathbf{y}.$

7 (a). $\mathbf{y}_1' = \begin{bmatrix} 6e^{3t} \\ 9e^{3t} \end{bmatrix}$ and also $A\mathbf{y}_1 = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix}\begin{bmatrix} 2e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 18e^{3t} - 12e^{3t} \\ 30e^{3t} - 21e^{3t} \end{bmatrix} = \mathbf{y}_1'.$

Similarly for $\mathbf{y}_2'$.

7 (b). The Wronskian $W(t)$ is given by

$W(t) = \det[\Psi(t)]$ where $\Psi(t) = \begin{bmatrix} 2e^{3t} & 2e^{-t} \\ 3e^{3t} & 5e^{-t} \end{bmatrix}$. Thus, $W(t) = 10e^{2t} - 6e^{2t} = 4e^{2t}$.

7 (c). $\mathbf{y}(t) = \begin{bmatrix} 2e^{3t} & 2e^{-t} \\ 3e^{3t} & 5e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

7 (d). Given the general solution in part (c), $\mathbf{y}(0)=\begin{bmatrix}2 & 2\\ 3 & 5\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$. Solving, we find

$\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}3/4\\ -1/4\end{bmatrix}$. Therefore, the solution of the initial value problem is

$$\mathbf{y}(t)=(3/4)\begin{bmatrix}2e^{3t}\\ 3e^{3t}\end{bmatrix}-(1/4)\begin{bmatrix}2e^{-t}\\ 5e^{-t}\end{bmatrix}=(1/4)\begin{bmatrix}6e^{3t}-2e^{-t}\\ 9e^{3t}-5e^{-t}\end{bmatrix}.$$

8 (b). The Wronskian $W(t)$ is given by

$W(t)=\det[\Psi(t)]$ where $\Psi(t)=\begin{bmatrix}2e^{3t}-4e^{-t} & 4e^{3t}+2e^{-t}\\ 3e^{3t}-10e^{-t} & 6e^{3t}+5e^{-t}\end{bmatrix}$. Thus, $W(t)=20e^{2t}\neq 0$.

8 (c). $\mathbf{y}(t)=\begin{bmatrix}2e^{3t}-4e^{-t} & 4e^{3t}+2e^{-t}\\ 3e^{3t}-10e^{-t} & 6e^{3t}+5e^{-t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$

8 (d). Given the general solution in part (c), $\mathbf{y}(0)=\begin{bmatrix}-2 & 6\\ -7 & 11\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}$. Solving, we find

$\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}-3/10\\ -1/10\end{bmatrix}$. Therefore, the solution of the initial value problem is

$$\mathbf{y}(t)=(-3/10)\begin{bmatrix}2e^{3t}-4e^{-t}\\ 3e^{3t}-10e^{-t}\end{bmatrix}-(1/10)\begin{bmatrix}4e^{3t}+2e^{-t}\\ 6e^{3t}+5e^{-t}\end{bmatrix}=\begin{bmatrix}-e^{3t}+e^{-t}\\ -\frac{3}{2}e^{3t}+\frac{5}{2}e^{-t}\end{bmatrix}.$$

9 (a). $\mathbf{y}_1'=\begin{bmatrix}-e^{-t}\\ 2e^{-t}\end{bmatrix}$ and also $A\mathbf{y}_1=\begin{bmatrix}3 & 2\\ -4 & -3\end{bmatrix}\begin{bmatrix}e^{-t}\\ -2e^{-t}\end{bmatrix}=\begin{bmatrix}3e^{-t}-4e^{-t}\\ -4e^{-t}+6e^{-t}\end{bmatrix}=\mathbf{y}_1'$.

Similarly for $\mathbf{y}_2'$.

9 (b). The Wronskian $W(t)$ is given by

$W(t)=\det[\Psi(t)]$ where $\Psi(t)=\begin{bmatrix}e^{-t} & -3e^{-t}\\ -2e^{-t} & 6e^{-t}\end{bmatrix}$. Thus, $W(t)=6e^{-2t}-6e^{-2t}\equiv 0$

and therefore, the given set of solutions is not a fundamental set of solutions.

10 (b). The Wronskian $W(t)$ is given by

$W(t)=\det[\Psi(t)]$ where $\Psi(t)=\begin{bmatrix}-5e^{-2t}\cos 3t & -5e^{-2t}\sin 3t\\ e^{-2t}(\cos 3t-3\sin 3t) & e^{-2t}(3\cos 3t+\sin 3t)\end{bmatrix}$. Thus,

$W(t)=-15e^{-4t}\neq 0$.

10 (c). $\mathbf{y}(t)=\begin{bmatrix}-5e^{-2t}\cos 3t & -5e^{-2t}\sin 3t\\ e^{-2t}(\cos 3t-3\sin 3t) & e^{-2t}(3\cos 3t+\sin 3t)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$

10 (d). Given the general solution in part (c), $\mathbf{y}(0)=\begin{bmatrix}-5 & 0\\ 1 & 3\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}5\\ 2\end{bmatrix}$. Solving, we find $\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}-1\\ 1\end{bmatrix}$.

Therefore, the solution of the initial value problem is $\mathbf{y}(t)=\begin{bmatrix}5e^{-2t}(\cos 3t-\sin 3t)\\ e^{-2t}(2\cos 3t+4\sin 3t)\end{bmatrix}$.

11 (a). $\mathbf{y}_1'=\begin{bmatrix}e^{t}\\ -2e^{t}\end{bmatrix}$ and also $A\mathbf{y}_1=\begin{bmatrix}-3 & -2\\ 4 & 3\end{bmatrix}\begin{bmatrix}e^{t}\\ -2e^{t}\end{bmatrix}=\begin{bmatrix}-3e^{t}+4e^{t}\\ 4e^{t}-6e^{t}\end{bmatrix}=\mathbf{y}_1'$.

Similarly for $\mathbf{y}_2'$.

11 (b). The Wronskian $W(t)$ is given by

$$W(t) = \det[\Psi(t)] \text{ where } \Psi(t) = \begin{bmatrix} e^t & e^{-t} \\ -2e^t & -e^{-t} \end{bmatrix}. \text{ Thus, } W(t) = -1 + 2 = 1.$$

11 (c). $\mathbf{y}(t) = \begin{bmatrix} e^t & e^{-t} \\ -2e^t & -e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

11 (d). Given the general solution in part (c), $\mathbf{y}(1) = \begin{bmatrix} e & e^{-1} \\ -2e & -e^{-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Solving, we find

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2e^{-1} \\ -e \end{bmatrix}$. Therefore, the solution of the initial value problem is

$$\mathbf{y}(t) = 2e^{-1} \begin{bmatrix} e^t \\ -2e^t \end{bmatrix} - e \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} 2e^{t-1} - e^{1-t} \\ -4e^{t-1} + e^{1-t} \end{bmatrix}.$$

12 (b). The Wronskian $W(t)$ is given by

$$W(t) = \det[\Psi(t)] \text{ where } \Psi(t) = \begin{bmatrix} 1 & e^{3t} \\ 1 & -2e^{3t} \end{bmatrix}. \text{ Thus, } W(t) = -3e^{-3t} \neq 0.$$

12 (c). $\mathbf{y}(t) = \begin{bmatrix} 1 & e^{3t} \\ 1 & -2e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

12 (d). Given the general solution in part (c), $\mathbf{y}(-1) = \begin{bmatrix} 1 & e^{-3} \\ 1 & -2e^{-3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$. Solving, we find

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{e^3}{3} \begin{bmatrix} 0 \\ 6 \end{bmatrix}$. Therefore, the solution of the initial value problem is $\mathbf{y}(t) = \begin{bmatrix} -2e^{3(t+1)} \\ 4e^{3(t+1)} \end{bmatrix}$.

13 (a).

$$\mathbf{y}_1' = \begin{bmatrix} 2t-2 \\ 2 \end{bmatrix} \text{ and also } A\mathbf{y}_1 = \begin{bmatrix} 2t^{-2} & 1 - 2t^{-1} + 2t^{-2} \\ -2t^{-2} & 2t^{-1} - 2t^{-2} \end{bmatrix} \begin{bmatrix} t^2 - 2t \\ 2t \end{bmatrix}.$$

$$= \begin{bmatrix} (2 - 4t^{-1}) + (2t - 4 + 4t^{-1}) \\ (-2 + 4t^{-1}) + (4 - 4t^{-1}) \end{bmatrix} = \mathbf{y}_1'$$

Similarly for $\mathbf{y}_2'$.

13 (b). The Wronskian $W(t)$ is given by

$$W(t) = \det[\Psi(t)] \text{ where } \Psi(t) = \begin{bmatrix} t^2 - 2t & t-1 \\ 2t & 1 \end{bmatrix}. \text{ Thus, } W(t) = -t^2.$$

13 (c). $\mathbf{y}(t) = \begin{bmatrix} t^2 - 2t & t-1 \\ 2t & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

13 (d). Given the general solution in part (c), $\mathbf{y}(2) = \begin{bmatrix} 0 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Solving, we find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Therefore, the solution of the initial value problem is $\mathbf{y}(t) = \begin{bmatrix} t^2 - 2t \\ 2t \end{bmatrix} - 2 \begin{bmatrix} t-1 \\ 1 \end{bmatrix} = \begin{bmatrix} t^2 - 4t + 2 \\ 2t - 2 \end{bmatrix}$.

14 (b). The Wronskian $W(t)$ is given by

$$W(t)=\det[\Psi(t)] \text{ where } \Psi(t)=\begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & 2e^t\cos 2t & 2e^t\sin 2t \\ 0 & -e^t\sin 2t & e^t\cos 2t \end{bmatrix}. \text{ Thus, } W(t)=2.$$

14 (c). $\mathbf{y}(t)=\begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & 2e^t\cos 2t & 2e^t\sin 2t \\ 0 & -e^t\sin 2t & e^t\cos 2t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$

14 (d). Given the general solution on part (c), $\mathbf{y}(0)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}.$

Solving, we find $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$. Therefore, the solution of the initial value problem

is $\mathbf{y}(t)=\begin{bmatrix} 3e^{-2t} \\ 4e^t(\cos 2t-\sin 2t) \\ 2e^t(-\cos 2t+\sin 2t) \end{bmatrix}.$

15 (a). $\mathbf{y}_1'=\begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix}$ and also $A\mathbf{y}_1=\begin{bmatrix} -21 & -10 & 2 \\ 22 & 11 & -2 \\ -110 & -50 & 11 \end{bmatrix}\begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix}=\begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix}=\mathbf{y}_1'.$

Similarly for $\mathbf{y}_2'$ and $\mathbf{y}_3'$.

15 (b). The Wronskian $W(t)$ is given by

$$W(t)=\det[\Psi(t)] \text{ where } \Psi(t)=\begin{bmatrix} 5e^t & e^t & e^{-t} \\ -11e^t & 0 & -e^{-t} \\ 0 & 11e^t & 5e^{-t} \end{bmatrix}. \text{ Thus, } W(t)=-11e^t.$$

15 (c). $\mathbf{y}(t)=\begin{bmatrix} 5e^t & e^t & e^{-t} \\ -11e^t & 0 & -e^{-t} \\ 0 & 11e^t & 5e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$

15 (d). Given the general solution on part (c), $\mathbf{y}(0)=\begin{bmatrix} 5 & 1 & 1 \\ -11 & 0 & -1 \\ 0 & 11 & 5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 3 \\ -10 \\ -16 \end{bmatrix}.$

Solving, we find $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Therefore, the solution of the initial value problem

is $\mathbf{y}(t)=\begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix}-\begin{bmatrix} e^t \\ 0 \\ 11e^t \end{bmatrix}-\begin{bmatrix} e^{-t} \\ -e^{-t} \\ 5e^{-t} \end{bmatrix}=\begin{bmatrix} 4e^t-e^{-t} \\ -11e^t+e^{-t} \\ -11e^t-5e^{-t} \end{bmatrix}.$

16 (a). $W(t)=\det\begin{bmatrix}5e^{-t} & e^{t}\\ -7e^{-t} & -e^{t}\end{bmatrix}=2$

16 (b). The trace of A is equal to $6-6=0$.

16 (c). For $t_0=-1, W(t_0)e^{\int_{t_0}^{t} tr[P(s)]ds}=2e^{\int_{-1}^{t}0ds}=2$.

17 (a). $W(t)=\det\begin{bmatrix}5e^{2t} & e^{4t}\\ -7e^{2t} & -e^{4t}\end{bmatrix}=2e^{6t}$

17 (b). The trace of $A=\begin{bmatrix}9 & 5\\ -7 & -3\end{bmatrix}$ is equal to $9+(-3)=6$.

17 (c). For $t_0=0, W(t_0)e^{\int_{t_0}^{t} tr[P(s)]ds}=2e^{\int_0^t 6ds}=2e^{6t}$.

18 (a). $W(t)=\det\begin{bmatrix}-1 & e^{t}\\ t^{-1} & 0\end{bmatrix}=-t^{-1}e^{t}$

18 (b). The trace of A is equal to $1-t^{-1}$.

18 (c). For $t_0=1, W(t_0)e^{\int_{t_0}^{t} tr[P(s)]ds}=-e\left(e^{\int_1^t(1-s^{-1})ds}\right)=-ee^{s-\ln s|_1^t}=-ee^{t-\ln t-1}=-e^{t-\ln t}=-t^{-1}e^{t}$.

19 (a). $W(t)=\det\begin{bmatrix}2e^{t} & 0 & e^{4t}\\ -e^{t} & -e^{-t} & e^{4t}\\ -e^{t} & e^{-t} & e^{4t}\end{bmatrix}=e^{t}e^{-t}e^{4t}\det\begin{bmatrix}2 & 0 & 1\\ -1 & -1 & 1\\ -1 & 1 & 1\end{bmatrix}=-6e^{4t}$

19 (b). The trace of $A=\begin{bmatrix}2 & 1 & 1\\ 1 & 1 & 2\\ 1 & 2 & 1\end{bmatrix}$ is equal to $2+1+1=4$.

19 (c). For $t_0=0, W(t_0)e^{\int_{t_0}^{t} tr[P(s)]ds}=-6e^{\int_0^t 4ds}=-6e^{4t}$.

20 (a). $W(t)=\det\begin{bmatrix}5 & 2e^{3t}\\ 1 & e^{3t}\end{bmatrix}=3e^{3t}\neq 0$.

20 (b). $3e^{3t}=3e^{\int_0^t tr[A]ds}\Rightarrow tr[A]=3$.

20 (c). $\psi=\begin{bmatrix}5 & 2e^{3t}\\ 1 & e^{3t}\end{bmatrix}\Rightarrow \psi'=\begin{bmatrix}0 & 6e^{3t}\\ 0 & 3e^{3t}\end{bmatrix}=A\begin{bmatrix}5 & 2e^{3t}\\ 1 & e^{3t}\end{bmatrix}$.

20 (d). $A=\begin{bmatrix}0 & 6e^{3t}\\ 0 & 3e^{3t}\end{bmatrix}\cdot\dfrac{1}{3e^{3t}}\begin{bmatrix}e^{3t} & -2e^{3t}\\ -1 & 5\end{bmatrix}=\begin{bmatrix}-2 & 10\\ -1 & 5\end{bmatrix}$.

The results are consistent since $tr[A]=-2+5=3$

21. If $W(t)$ is constant, then by Abel's Theorem, the function $e^{\int_{t_0}^{t} tr[P(s)]ds}$ must also be constant. Therefore, $g(t)=\int_{t_0}^{t} tr[P(s)]ds$ must be constant and hence the derivative of $g(t)$ is identically zero. However, by the fundamental theorem of calculus, $g'(t)=tr[P(t)]$ and hence the trace of $P(t)$ must be zero. Since the trace is equal to $3+\alpha$ we conclude that $\alpha=-3$.

Section 6.4

1 (a). Let $F(t) = [\mathbf{f}_1(t), \mathbf{f}_2(t)] = \begin{bmatrix} t & t^2 \\ 1 & 2 \end{bmatrix}$. Then, $\det[F(t)] = 2t - t^2$.

1 (b). No, because we do not know whether the functions $\mathbf{f}_1(t)$ and $\mathbf{f}_2(t)$ form a fundamental set of solutions for a linear system.

1 (c). Yes. At $t = 1$, the determinant is $2 - 1 = 1 \neq 0$. Therefore, $[\mathbf{f}_1(t), \mathbf{f}_2(t)]\mathbf{k} = \mathbf{0} \Rightarrow \mathbf{k} = \mathbf{0}$.

2 (a). $\det[F(t)] = t^2 e^t - t\sin t$.

2 (b). No

2 (c). Yes. At $t = 1$, the determinant is e - $\sin 1 \neq 0$.

3 (a). Let $F(t) = [\mathbf{f}_1(t), \mathbf{f}_2(t)] = \begin{bmatrix} te^t & \sin^2 t \\ t-1 & 2 \end{bmatrix}$. Then, $\det[F(t)] = 2te^t - (t-1)\sin^2 t$.

3 (b). No, because we do not know whether the functions $\mathbf{f}_1(t)$ and $\mathbf{f}_2(t)$ form a fundamental set of solutions for a linear system.

3 (c). Yes. At $t = 1$, the determinant is $2e \neq 0$.

4. $k_1\begin{bmatrix} t \\ 1 \end{bmatrix} + k_2\begin{bmatrix} t^2 \\ 1 \end{bmatrix} = \begin{bmatrix} t & t^2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; det = $t - t^2 \neq 0$ at $t = 2$ for example $\Rightarrow \mathbf{k} = \mathbf{0}$. Therefore, the given set of functions is linearly independent.

5. We need to solve the equation $k_1\begin{bmatrix} e^t \\ 1 \end{bmatrix} + k_2\begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} k_1e^t + k_2e^{-t} \\ k_1 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This vector equation requires $k_1e^t + k_2e^{-t} \equiv 0$ and $k_1 \equiv 0$. By the second equation, $k_1 = 0$ and hence, using this fact in the first equation, $k_2e^{-t} \equiv 0$. Multiplying this identity by the nonzero function e^t, we see that $k_2 = 0$ as well. Hence, the only way to satisfy $k_1\begin{bmatrix} e^t \\ 1 \end{bmatrix} + k_2\begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is to choose $k_1 = k_2 = 0$. This means the given set of functions is linearly independent.

6. $k_1\begin{bmatrix} e^t \\ 1 \end{bmatrix} + k_2\begin{bmatrix} e^{-t} \\ 1 \end{bmatrix} + k_3\begin{bmatrix} \sinh t \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; Let $k_1 = 1,\ k_2 = -1,\ k_3 = -2$. The given set of functions is linearly dependent.

7. We need to solve the equation $k_1\begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ t^2 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} k_1 \\ k_1t + k_2 \\ k_2t^2 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The first component of this vector identity cannot be satisfied unless $k_1 = 0$ and the third component cannot be satisfied unless $k_2 = 0$. Hence, the only way to satisfy the identity $k_1\begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ t^2 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is to choose $k_1 = k_2 = 0$. This means the given set of functions is linearly independent.

8. $k_1\begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ t^2 \end{bmatrix} + k_3\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; Let $k_1 = 0,\ k_2 = 0,\ k_3 = 1$. The given set of functions is linearly dependent.

9. We need to solve the equation $k_1\begin{bmatrix}1\\t\\0\end{bmatrix}+k_2\begin{bmatrix}0\\1\\t^2\end{bmatrix}+k_3\begin{bmatrix}0\\0\\1\end{bmatrix}\equiv\begin{bmatrix}0\\0\\0\end{bmatrix}$ or $\begin{bmatrix}k_1\\k_1t+k_2\\k_2t^2+k_3\end{bmatrix}\equiv\begin{bmatrix}0\\0\\0\end{bmatrix}$. The first component of this vector identity cannot be satisfied unless $k_1=0$ and therefore the second component requires $k_2=0$. Given that k_1 and k_2 must both be zero, the third component then requires that $k_3=0$. Hence, the only way to satisfy the identity $k_1\begin{bmatrix}1\\t\\0\end{bmatrix}+k_2\begin{bmatrix}0\\1\\t^2\end{bmatrix}+k_3\begin{bmatrix}0\\0\\1\end{bmatrix}\equiv\begin{bmatrix}0\\0\\0\end{bmatrix}$ is to choose $k_1=k_2=k_3=0$. This means the given set of functions is linearly independent.

10. $k_1\begin{bmatrix}1\\\sin^2 t\\0\end{bmatrix}+k_2\begin{bmatrix}0\\2-2\cos^2 t\\-2\end{bmatrix}+k_3\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$; Let $k_1=1,\ k_2=-\dfrac{1}{2},\ k_3=-1$. The given set of functions is linearly dependent.

11 (a). Let $F(t)=\begin{bmatrix}e^t & t^2\\0 & t\end{bmatrix}$. Then, $\det[F(t)]=te^t$.

11 (b). Since the determinant is zero at $t=0$, $F(t)$ cannot be a fundamental matrix for a linear system defined on an interval containing $t=0$.

11 (c). A fundamental matrix $\Psi(t)$ satisfies the matrix differential equation $\Psi'=P(t)\Psi$. Given that $\Psi(t)=\begin{bmatrix}e^t & t^2\\0 & t\end{bmatrix}$, we know that $\Psi'(t)=\begin{bmatrix}e^t & 2t\\0 & 1\end{bmatrix}$. Therefore, the equation $\Psi'=P(t)\Psi$ implies that $\begin{bmatrix}e^t & 2t\\0 & 1\end{bmatrix}=P(t)\begin{bmatrix}e^t & t^2\\0 & t\end{bmatrix}$. Postmultiplying by Ψ^{-1}, we see that $\Psi'\Psi^{-1}=P(t)$. Therefore, $1/(te^t)\begin{bmatrix}e^t & 2t\\0 & 1\end{bmatrix}\begin{bmatrix}t & -t^2\\0 & e^t\end{bmatrix}=P(t)$ and so $P(t)=1/(te^t)\begin{bmatrix}te^t & (2t-t^2)e^t\\0 & e^t\end{bmatrix}$. Canceling the nonzero term e^t we have $P(t)=t^{-1}\begin{bmatrix}t & (2t-t^2)\\0 & 1\end{bmatrix}$.

12 (a). Let $F(t)=\begin{bmatrix}t^2 & 2t\\0 & 1\end{bmatrix}$. Then, $\det[F(t)]=t^2$.

12 (b). Since the determinant is zero at $t=0$, $F(t)$ cannot be a fundamental matrix for a linear system defined on an interval containing $t=0$.

12 (c). A fundamental matrix $\Psi(t)$ satisfies the matrix differential equation $\Psi'=P(t)\Psi$. Given that $\Psi(t)=\begin{bmatrix}t^2 & 2t\\0 & 1\end{bmatrix}$, we know that $\Psi'(t)=\begin{bmatrix}2t & 2\\0 & 0\end{bmatrix}$. Therefore, the equation $\Psi'=P(t)\Psi$ implies that $\begin{bmatrix}2t & 2\\0 & 0\end{bmatrix}=P(t)\begin{bmatrix}t^2 & 2t\\0 & 1\end{bmatrix}$. Postmultiplying by Ψ^{-1}, we see that $\Psi'\Psi^{-1}=P(t)$. Therefore, $1/(t^2)\begin{bmatrix}2t & 2\\0 & 0\end{bmatrix}\begin{bmatrix}1 & -2t\\0 & t^2\end{bmatrix}=P(t)$ and so $P(t)=\begin{bmatrix}2t^{-1} & -2\\0 & 0\end{bmatrix}$ which is continuous on $(-\infty,0)$ and $(0,\infty)$.

13 (a). We first show that $\Psi' = P(t)\Psi$. Now, $\Psi'(t) = \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}$ whereas

$P(t)\Psi(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} = \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}$. Thus, since $\Psi' = P(t)\Psi$, we know that $\Psi(t)$ is a solution matrix. To show $\Psi(t)$ is a fundamental matrix, we need to verify that $\det[\Psi(t)] \neq 0$. Since $\det[\Psi(t)] \equiv -2$, we know $\Psi(t)$ is a fundamental matrix.

13 (b). $\hat{\Psi}(t) = \begin{bmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t - e^{-t} & e^t + e^{-t} \\ e^t + e^{-t} & e^t - e^{-t} \end{bmatrix}$. Thus, we need a matrix $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$\frac{1}{2}\begin{bmatrix} e^t - e^{-t} & e^t + e^{-t} \\ e^t + e^{-t} & e^t - e^{-t} \end{bmatrix} = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Expanding the right-hand side of this matrix equation, we arrive at the requirement

$\frac{1}{2}\begin{bmatrix} e^t - e^{-t} & e^t + e^{-t} \\ e^t + e^{-t} & e^t - e^{-t} \end{bmatrix} = \begin{bmatrix} ae^t + ce^{-t} & be^t + de^{-t} \\ ae^t - ce^{-t} & be^t - de^{-t} \end{bmatrix}$. Comparing entries, we see that

$a = 1/2, c = -1/2, b = 1/2,$ and $d = 1/2$. Thus, $C = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$.

13 (c). $\det[C] = 1/2$ and thus, $\hat{\Psi}(t)$ is a fundamental matrix.

14 (a). Since $\det[\Psi(t)] \neq 0$, we know $\Psi(t)$ is a fundamental matrix.

14 (b). $\hat{\Psi}(t) = \begin{bmatrix} 2e^t - e^{-t} & e^t + 3e^{-t} \\ 2e^t + e^{-t} & e^t - 3e^{-t} \end{bmatrix} = \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$. Thus, $C = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$.

14 (c). $\det[C] = -7$ and thus, $\hat{\Psi}(t)$ is a fundamental matrix.

15 (a). We first show that $\hat{\Psi} = P(t)\Psi$. Now, $\Psi'(t) = \begin{bmatrix} e^t & -2e^{-2t} \\ 0 & 6e^{-2t} \end{bmatrix}$ whereas

$P(t)\Psi(t) = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^t & -2e^{-2t} \\ 0 & 6e^{-2t} \end{bmatrix}$. Thus, since $\Psi' = P(t)\Psi$, we know that $\Psi(t)$ is a solution matrix. To show $\Psi(t)$ is a fundamental matrix, we need to verify that $\det[\Psi(t)] \neq 0$. Now $\det[\Psi(t)] = -3e^{-t}$ and thus is never zero for any value t. Therefore, $\Psi(t)$ is a fundamental matrix.

15 (b). $\hat{\Psi}(t) = \begin{bmatrix} 2e^{-2t} & 0 \\ -6e^{-2t} & 0 \end{bmatrix}$ and so we need a matrix $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$\begin{bmatrix} 2e^{-2t} & 0 \\ -6e^{-2t} & 0 \end{bmatrix} = \begin{bmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Expanding the right-hand side of this matrix equation, we arrive at the requirement

$\begin{bmatrix} 2e^{-2t} & 0 \\ -6e^{-2t} & 0 \end{bmatrix} = \begin{bmatrix} ae^t + ce^{-2t} & be^t + de^{-2t} \\ -3ce^{-2t} & -3de^{-2t} \end{bmatrix}$. Comparing entries in the second column, we see that $d = 0$ and $b = 0$. Comparing entries in the first column, we see $c = 2$ and $a = 0$. Thus,

$C = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$.

15 (c). $\det[C]=0$ and thus, $\widehat{\Psi}(t)$ is a solution matrix but not a fundamental matrix.

16 (a). Since $\det[\Psi(t)]=-6e^{2t}\neq 0$, we know $\Psi(t)$ is a fundamental matrix.

16 (b). $\widehat{\Psi}(t)=\begin{bmatrix} e^t+e^{-t} & 4e^{2t} & e^t+4e^{2t} \\ -2e^{-t} & e^{2t} & e^{2t} \\ 0 & 3e^{2t} & 3e^{2t}\end{bmatrix}=\begin{bmatrix} e^t & e^{-t} & 4e^{2t} \\ 0 & -2e^{-t} & e^{2t} \\ 0 & 0 & 3e^{2t}\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}$. Thus, $C=\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}$.

16 (c). $\det[C]=1$ and thus, $\widehat{\Psi}(t)$ is a fundamental matrix.

17. For $\Psi(t)=\begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t}\end{bmatrix}$, we need a matrix C such that $\widehat{\Psi}(t)=\Psi(t)C$ where $\widehat{\Psi}(0)=I$. This requirement means that $I=\widehat{\Psi}(0)=\Psi(0)C$.
Equivalently, C is the inverse of $\Psi(0)=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$. Thus, $C=\Psi(0)^{-1}=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$.

18. For $\Psi(t)=\begin{bmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t}\end{bmatrix}$, we need a matrix C such that $\widehat{\Psi}(t)=\Psi(t)C$ where $\widehat{\Psi}(0)=I$. This requirement means that $I=\widehat{\Psi}(0)=\Psi(0)C$.
Equivalently, C is the inverse of $\Psi(0)=\begin{bmatrix}1 & 1\\ 0 & -3\end{bmatrix}$. Thus, $C=\Psi(0)^{-1}=-\frac{1}{3}\begin{bmatrix}-3 & -1\\ 0 & 1\end{bmatrix}$.

Section 6.5

1 (a). $A\mathbf{x}_1=\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}2\\ -2\end{bmatrix}=2\mathbf{x}_1$. Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1=2$. Similarly, $A\mathbf{x}_2=\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}\begin{bmatrix}-2\\ 1\end{bmatrix}=\begin{bmatrix}-6\\ 3\end{bmatrix}=3\mathbf{x}_2$. Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue $\lambda_2=3$.

1 (b). Solutions are $\mathbf{y}_1(t)=e^{2t}\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{3t}\begin{bmatrix}-2\\ 1\end{bmatrix}$.

1 (c). The Wronskian is $W(t)=\det\begin{bmatrix}e^{2t} & -2e^{3t}\\ -e^{2t} & e^{3t}\end{bmatrix}=e^{5t}-2e^{5t}=-e^{5t}$. Since $W(t)$ is nonzero for any value t, the two solutions form a fundamental set of solutions.

2 (a). $A\mathbf{x}_1=\begin{bmatrix}7 & -3\\ 16 & -7\end{bmatrix}\begin{bmatrix}3\\ 8\end{bmatrix}=\begin{bmatrix}-3\\ -8\end{bmatrix}=-1\mathbf{x}_1$. Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1=-1$. Similarly, $A\mathbf{x}_2=\begin{bmatrix}7 & -3\\ 16 & -7\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}=1\mathbf{x}_2$. Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue $\lambda_2=1$.

2 (b). Solutions are $\mathbf{y}_1(t)=e^{-t}\begin{bmatrix}3\\ 8\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{t}\begin{bmatrix}1\\ 2\end{bmatrix}$.

2 (c). The Wronskian is $W(t) = \det\begin{bmatrix} 3e^{-t} & e^{t} \\ 8e^{-t} & 2e^{t} \end{bmatrix} = -2 \neq 0$. Since $W(t)$ is nonzero for any value t, the two solutions form a fundamental set of solutions.

3 (a). The vector $\mathbf{x}_1 = \mathbf{0}$ cannot be an eigenvector since an eigenvector must be nonzero. Considering the other vector, $A\mathbf{x}_2 = \begin{bmatrix} 11 & 5 \\ -22 & -10 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{x}_2$. Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 1$.

3 (b). The solution is $\mathbf{y}_2(t) = e^{t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

4 (a). $A\mathbf{x}_1 = \begin{bmatrix} -5 & 2 \\ -18 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1\mathbf{x}_1$, $A\mathbf{x}_2 = \begin{bmatrix} -5 & 2 \\ -18 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$. Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 1$, but $\mathbf{x}_2$ is not an eigenvector.

4 (b). The solution is $\mathbf{y}_1(t) = e^{t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

5 (a). $A\mathbf{x}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\mathbf{x}_1$. Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = -1$. Similarly, $A\mathbf{x}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \mathbf{x}_2$. Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 1$.

5 (b). Solutions are $\mathbf{y}_1(t) = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{t}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

5 (c). The Wronskian is $W(t) = \det\begin{bmatrix} e^{-t} & 2e^{t} \\ -e^{-t} & 2e^{t} \end{bmatrix} = 2 + 2 = 4$. Since $W(t)$ is nonzero for any value t, the two solutions form a fundamental set of solutions.

6 (a). $A\mathbf{x}_1 = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} = 4\mathbf{x}_1$. Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 4$. Similarly, $A\mathbf{x}_2 = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\mathbf{x}_2$. Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 0$.

6 (b). Solutions are $\mathbf{y}_1(t) = e^{4t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{0}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

6 (c). The Wronskian is $W(t) = \det\begin{bmatrix} e^{4t} & 1 \\ -2e^{4t} & 2 \end{bmatrix} = 4e^{4t} \neq 0$. Since $W(t)$ is nonzero for any value t, the two solutions form a fundamental set of solutions.

7. For $A=\begin{bmatrix}-4 & 3\\ -4 & 4\end{bmatrix}$ the equation $(A-2I)\mathbf{x}=\mathbf{0}$ has the form $\begin{bmatrix}-6 & 3\\ -4 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Elementary row operations $[-(1/3)R_1$ then $R_2+2R_1]$ can be used to row reduce the system to $\begin{bmatrix}2 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ or $2x_1=x_2$. Thus an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ 2x_1\end{bmatrix}=x_1\begin{bmatrix}1\\ 2\end{bmatrix}, x_1\neq 0$.

8. For $A=\begin{bmatrix}5 & 3\\ -4 & -3\end{bmatrix}$ the equation $(A+I)\mathbf{x}=\mathbf{0}$ has the form $\begin{bmatrix}6 & 3\\ -4 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Thus an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ -2x_1\end{bmatrix}=x_1\begin{bmatrix}1\\ -2\end{bmatrix}, x_1\neq 0$.

9. For $A=\begin{bmatrix}1 & 1\\ -4 & 6\end{bmatrix}$ the equation $(A-5I)\mathbf{x}=\mathbf{0}$ has the form $\begin{bmatrix}-4 & 1\\ -4 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Elementary row operations $[-(1/4)R_1$ then $R_2+4R_1]$ can be used to row reduce the system to $\begin{bmatrix}1 & -1/4\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ or $4x_1=x_2$. Thus an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ 4x_1\end{bmatrix}=x_1\begin{bmatrix}1\\ 4\end{bmatrix}, x_1\neq 0$.

10. For $A=\begin{bmatrix}1 & -7 & 3\\ -1 & -1 & 1\\ 4 & -4 & 0\end{bmatrix}$ the equation $(A+4I)\mathbf{x}=\mathbf{0}$ has the form $\begin{bmatrix}5 & -7 & 3\\ -1 & 3 & 1\\ 4 & -4 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$. Thus an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}2x_2\\ x_2\\ -x_2\end{bmatrix}=x_2\begin{bmatrix}2\\ 1\\ -1\end{bmatrix}, x_2\neq 0$.

11. For $A=\begin{bmatrix}3 & 1 & 1\\ -1 & 1 & -1\\ 2 & 1 & 2\end{bmatrix}$ the equation $(A-2I)\mathbf{x}=\mathbf{0}$ has the form $\begin{bmatrix}1 & 1 & 1\\ -1 & -1 & -1\\ 2 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$. Elementary row operations $(R_2+R_1$, then $R_3-2R_1)$ followed by R_1+R_3 and $(-R_3)$ can be used to row reduce the system to $\begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$, or $\begin{matrix}x_1=x_3\\ x_2=-2x_3\end{matrix}$. Thus an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}x_3\\ -2x_3\\ x_3\end{bmatrix}=x_3\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}, x_3\neq 0$.

12. For $A=\begin{bmatrix}1 & 3 & 1\\ 2 & 1 & 2\\ 4 & 3 & -2\end{bmatrix}$ the equation $(A-5I)\mathbf{x}=\mathbf{0}$ has the form $\begin{bmatrix}-4 & 3 & 1\\ 2 & -4 & 2\\ 4 & 3 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$.

Thus an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_1\\x_1\end{bmatrix}=x_1\begin{bmatrix}1\\1\\1\end{bmatrix}, x_1\neq 0$.

13. For $A=\begin{bmatrix}-2&3&1\\-8&13&5\\11&-17&-6\end{bmatrix}$ the equation $A\mathbf{x}=\mathbf{0}$ has the form

$\begin{bmatrix}-2&3&1\\-8&13&5\\11&-17&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. Since the arithmetic looks forbidding, we turned to MATLAB and used the RREF command. MATLAB says the system is equivalent to $\begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, or

$x_1=-x_3$
$x_2=-x_3$. Thus, an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-x_3\\-x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}-1\\-1\\1\end{bmatrix}, x_3\neq 0$.

14. The characteristic polynomial is $p(\lambda)=\begin{vmatrix}-5-\lambda&1\\0&4-\lambda\end{vmatrix}$, or

$p(\lambda)=(\lambda+5)(\lambda-4)$. Thus, the eigenvalues are $\lambda_1=-5$ and $\lambda_2=4$.

15. For $A=\begin{bmatrix}8&0\\3&2\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\begin{vmatrix}8-\lambda&0\\3&2-\lambda\end{vmatrix}$, or

$p(\lambda)=(8-\lambda)(2-\lambda)$. Thus, the eigenvalues are $\lambda_1=8$ and $\lambda_2=2$.

16. The characteristic polynomial is $p(\lambda)=\begin{vmatrix}3-\lambda&-3\\-6&6-\lambda\end{vmatrix}$, or

$p(\lambda)=(\lambda)(\lambda-9)$. Thus, the eigenvalues are $\lambda_1=0$ and $\lambda_2=9$.

17. For $A=\begin{bmatrix}5&2\\4&3\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\begin{vmatrix}5-\lambda&2\\4&3-\lambda\end{vmatrix}$, or

$p(\lambda)=(5-\lambda)(3-\lambda)-8=\lambda^2-8\lambda+7=(\lambda-7)(\lambda-1)$. Thus, the eigenvalues are $\lambda_1=7$ and $\lambda_2=1$.

18. The characteristic polynomial is

$$p(\lambda)=\begin{vmatrix}5-\lambda&0&0\\0&1-\lambda&3\\0&2&2-\lambda\end{vmatrix}$$

or $p(\lambda)=(5-\lambda)(\lambda^2-3\lambda-4)=(5-\lambda)(\lambda-4)(\lambda+1)=-\lambda^3+8\lambda^2-11\lambda-20$ and the eigenvalues are $\lambda_1=-1, \lambda_2=4$, and $\lambda_3=5$.

19. For $A=\begin{bmatrix}-2 & 3 & 1\\ -8 & 13 & 5\\ 11 & -17 & -6\end{bmatrix}$, the characteristic polynomial is

$p(\lambda)=\begin{vmatrix}-2-\lambda & 3 & 1\\ -8 & 13-\lambda & 5\\ 11 & -17 & -6-\lambda\end{vmatrix}$. Given the arithmetic required to find the characteristic polynomial, it is advisable to use a computer routine such as poly(A) from MATLAB. However, it is possible to find $p(\lambda)$ by hand:

$$\begin{vmatrix}-2-\lambda & 3 & 1\\ -8 & 13-\lambda & 5\\ 11 & -17 & -6-\lambda\end{vmatrix}=(-2-\lambda)\begin{vmatrix}13-\lambda & 5\\ -17 & -6-\lambda\end{vmatrix}-3\begin{vmatrix}-8 & 5\\ 11 & -6-\lambda\end{vmatrix}+\begin{vmatrix}-8 & 13-\lambda\\ 11 & -17\end{vmatrix}$$

or $p(\lambda)=(-2-\lambda)(\lambda^2-7\lambda+7)-3(8\lambda-7)+(11\lambda-7)=-\lambda^3+5\lambda^2-6\lambda$.

Thus, $p(\lambda)=-\lambda(\lambda^2-5\lambda+6)=-\lambda(\lambda-3)(\lambda-2)$ and hence the eigenvalues are $\lambda_1=0, \lambda_2=3$, and $\lambda_3=2$.

20. The characteristic polynomial is

$p(\lambda)=\begin{vmatrix}1-\lambda & -7 & 3\\ -1 & -1-\lambda & 1\\ 4 & -4 & -\lambda\end{vmatrix}$

or $p(\lambda)=-\lambda(\lambda^2-16)=(-\lambda)(\lambda-4)(\lambda+4)=-\lambda^3+16\lambda$ and the eigenvalues are $\lambda_1=0, \lambda_2=4$, and $\lambda_3=-4$.

21. The eigenvalues are $\lambda_1=-2$ and $\lambda_2=2$ with corresponding eigenvectors

$\mathbf{x}_1=\begin{bmatrix}3\\2\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\2\end{bmatrix}$. A fundamental set of solutions consists of the functions

$\mathbf{y}_1(t)=e^{-2t}\begin{bmatrix}3\\2\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{2t}\begin{bmatrix}1\\2\end{bmatrix}$. Therefore, the general solution is

$\mathbf{y}(t)=c_1e^{-2t}\begin{bmatrix}3\\2\end{bmatrix}+c_2e^{2t}\begin{bmatrix}1\\2\end{bmatrix}$. The solution of the initial value problem is $\mathbf{y}(t)=e^{-2t}\begin{bmatrix}3\\2\end{bmatrix}+e^{2t}\begin{bmatrix}1\\2\end{bmatrix}$.

22. The eigenvalues are $\lambda_1=-1$ and $\lambda_2=3$ with corresponding eigenvectors

$\mathbf{x}_1=\begin{bmatrix}1\\-2\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}3\\-2\end{bmatrix}$. A fundamental set of solutions consists of the functions

$\mathbf{y}_1(t)=e^{-t}\begin{bmatrix}1\\-2\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{3t}\begin{bmatrix}3\\-2\end{bmatrix}$. Therefore, the general solution is

$\mathbf{y}(t)=c_1e^{-t}\begin{bmatrix}1\\-2\end{bmatrix}+c_2e^{3t}\begin{bmatrix}3\\-2\end{bmatrix}$. The solution of the initial value problem is

$\mathbf{y}(t)=\begin{bmatrix}3e^{3(t-1)}-e^{-(t-1)}\\ -2e^{3(t-1)}+2e^{-(t-1)}\end{bmatrix}$.

23. The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Therefore, the general solution is $\mathbf{y}(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The solution of the initial value problem is $\mathbf{y}(t) = 3e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

24. The eigenvalues are $\lambda_1 = -0.11$ and $\lambda_2 = -0.05$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{-0.11t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{-0.05t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

25. The eigenvalues are $\lambda_1 = 1, \lambda_2 = 2,$ and $\lambda_3 = 3$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{t}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{3t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. The solution of the initial value problem is $\mathbf{y}(t) = e^{t}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2e^{3t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

26. The eigenvalues are $\lambda_1 = -2, \lambda_2 = 1,$ and $\lambda_3 = 4$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{-2t}\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{t}\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{4t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The solution of the initial value problem is $\mathbf{y}(t) = e^{-2t}\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} + e^{t}\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + e^{4t}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

27. The eigenvalues are $\lambda_1 = -2, \lambda_2 = 2$, and $\lambda_3 = 4$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{-2t}\begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{4t}\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$.

28. The eigenvalues are $\lambda_1 = 1, \lambda_2 = 3$, and $\lambda_3 = 5$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{3t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{5t}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

29. The eigenvalues are $\lambda_1 = -1, \lambda_2 = 1$, and $\lambda_3 = 2$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{-t}\begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{t}\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{2t}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$.

30. The eigenvalues are $\lambda_1 = -2, \lambda_2 = 1$, and $\lambda_3 = 2$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -5 \\ -6 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{-2t}\begin{bmatrix} 1 \\ -5 \\ -6 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{t}\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{2t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

31. We need to have $\begin{bmatrix} 2 & x \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for some value λ. Therefore, equating vectors, it follows that we $2 - x = \lambda$ and $1 + 5 = -\lambda$. This requires $\lambda = -6$ and $x = 8$.

32. We need to have $\begin{bmatrix} x & y \\ 2x & -y \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Therefore, it follows that $-x + y = -1$ and $-2x$ - y $= 1$. This requires $x = 0,\ y = -1$.

39 (a). The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$. Corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution is $\mathbf{y}(t) = c_1 e^{-3\tau}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-\tau}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

39 (b). The solution of the initial value problem is $\mathbf{y}(t) = -\frac{Q_0}{2}e^{-3\tau}\begin{bmatrix}1\\-1\end{bmatrix} + 3\frac{Q_0}{2}e^{-\tau}\begin{bmatrix}1\\1\end{bmatrix}$.

Therefore, $Q_1(t) = \frac{Q_0}{2}\left(-e^{-3\tau} + 3e^{-\tau}\right)$ and $Q_2(t) = \frac{Q_0}{2}\left(e^{-3\tau} + 3e^{-\tau}\right)$.

Note that $0 < Q_1(\tau) < Q_2(\tau)$. Therefore, we need τ such that $\frac{Q_0}{2}(e^{-3\tau} + 3e^{-\tau}) < .01Q_0 \Rightarrow (e^{-3\tau} + 3e^{-\tau}) < .02$. Graphically, we find that a value $\tau \approx 5.011$ will suffice. Since $t = (V/r)\tau = 50\tau$, we obtain a value of $t \approx 250.55$ sec or $t \approx 4.18$ min.

40 (a). The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = \lambda_3 = -4$. Corresponding eigenvectors are

$\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$. The general solution is

$$\mathbf{Q}(t) = c_1e^{-\tau}\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2e^{-4\tau}\begin{bmatrix}1\\-1\\0\end{bmatrix} + c_3e^{-4\tau}\begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

40 (b). The solution of the initial value problem is $\mathbf{y}(t) = 2Q_0e^{-\tau}\begin{bmatrix}1\\1\\1\end{bmatrix} - Q_0e^{-4\tau}\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

Therefore, $\mathbf{Q}(t) = Q_0\begin{bmatrix}2e^{-\frac{r}{V}t} - e^{-4\frac{r}{V}t}\\ 2e^{-\frac{r}{V}t}\\ 2e^{-\frac{r}{V}t} + e^{-4\frac{r}{V}t}\end{bmatrix}$.

Section 6.6

1. For $A = \begin{bmatrix}2 & 1\\-1 & 2\end{bmatrix}$, the characteristic polynomial is $p(\lambda) = \begin{vmatrix}2-\lambda & 1\\-1 & 2-\lambda\end{vmatrix} = \lambda^2 - 4\lambda + 5$.

Therefore, the eigenvalues are $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$. We find an eigenvector $\mathbf{x}_1$ by solving $(A - \lambda_1 I)\mathbf{x} = \mathbf{0}$ or

$\begin{bmatrix}2-(2+i) & 1\\-1 & 2-(2+i)\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$. This equation reduces to $\begin{bmatrix}-i & 1\\-1 & -i\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$. The elementary row operation $R_2 + iR_1$ reduces the system to $\begin{bmatrix}-i & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ or $-ix_1 + x_2 = 0$.

Thus, an eigenvector is $\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\ix_1\end{bmatrix} = x_1\begin{bmatrix}1\\i\end{bmatrix}$, $x_1 \neq 0$. Since the eigenvalues and eigenvectors occur in conjugate pairs, the eigenpairs are

$\lambda_1 = 2 + i$, $\mathbf{x}_1 = \begin{bmatrix}1\\i\end{bmatrix}$ and $\lambda_2 = 2 - i$, $\mathbf{x}_2 = \begin{bmatrix}1\\-i\end{bmatrix}$.

2. The characteristic polynomial is $p(\lambda)=\begin{vmatrix}-\lambda & -9\\ 1 & -\lambda\end{vmatrix}=\lambda^2+9$. Therefore, the eigenvalues are $\lambda_1=3i$ and $\lambda_2=-3i$. We find an eigenvector $\mathbf{x}_1$ by solving $(A-\lambda_1 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}-3i & -9\\ 1 & -3i\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Thus, an eigenvector is $\mathbf{x}_1=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}3ix_2\\ x_2\end{bmatrix}=x_2\begin{bmatrix}3i\\ 1\end{bmatrix}$, $x_2\neq 0$. Since the eigenvalues and eigenvectors occur in conjugate pairs, the eigenpairs are

$$\lambda_1=3i,\ \mathbf{x}_1=\begin{bmatrix}3i\\ 1\end{bmatrix} \text{ and } \lambda_2=-3i,\ \mathbf{x}_2=\begin{bmatrix}-3i\\ 1\end{bmatrix}.$$

3. For $A=\begin{bmatrix}6 & -13\\ 1 & 0\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\begin{vmatrix}6-\lambda & -13\\ 1 & -\lambda\end{vmatrix}=\lambda^2-6\lambda+13$. Therefore, the eigenvalues are $\lambda_1=3+2i$ and $\lambda_2=3-2i$. We find an eigenvector $\mathbf{x}_1$ by solving $(A-\lambda_1 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}6-(3+2i) & -13\\ 1 & -(3+2i)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. This equation reduces to $\begin{bmatrix}3-2i & -13\\ 1 & -3-2i\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. The elementary row operations $R_1\leftrightarrow R_2$, then $R_2-(3-2i)R_1$ reduces the system to $\begin{bmatrix}1 & -(3+2i)\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ or $x_1-(3+2i)x_2=0$. Thus, an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}(3+2i)x_2\\ x_2\end{bmatrix}=x_2\begin{bmatrix}3+2i\\ 1\end{bmatrix}$, $x_2\neq 0$. Choosing $x_2=1$, we obtain the eigenvector $\mathbf{x}_1=\begin{bmatrix}3+2i\\ 1\end{bmatrix}$. Since the eigenvalues and eigenvectors occur in conjugate pairs, the eigenpairs are

$$\lambda_1=3+2i,\ \mathbf{x}_1=\begin{bmatrix}3+2i\\ 1\end{bmatrix} \text{ and } \lambda_2=3-2i,\ \mathbf{x}_2=\begin{bmatrix}3-2i\\ 1\end{bmatrix}.$$

4. The characteristic polynomial is $p(\lambda)=\begin{vmatrix}3-\lambda & 1\\ -2 & 1-\lambda\end{vmatrix}=(\lambda-2)^2+1$. Therefore, the eigenvalues are $\lambda_1=2+i$ and $\lambda_2=2-i$. We find an eigenvector $\mathbf{x}_1$ by solving $(A-\lambda_1 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}1-i & 1\\ -2 & -1-i\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Thus, an eigenvector is $\mathbf{x}_1=\begin{bmatrix}1\\ -1+i\end{bmatrix}$. Since the eigenvalues and eigenvectors occur in conjugate pairs, the eigenpairs are

$$\lambda_1=2+i,\ \mathbf{x}_1=\begin{bmatrix}1\\ -1+i\end{bmatrix} \text{ and } \lambda_2=2-i,\ \mathbf{x}_2=\begin{bmatrix}1\\ -1-i\end{bmatrix}.$$

5. Using the EIG command in MATLAB, we find eigenvalues $\lambda_1=1, \lambda_2=1+i$, and $\lambda_3=1-i$. For each eigenvalue λ, we use the RREF command in MATLAB to solve $(A-\lambda I)\mathbf{x}=\mathbf{0}$, finding $\mathbf{x}_1=\begin{bmatrix}-3\\ 0\\ 1\end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix}3+2i\\ 1\\ -1-i\end{bmatrix}$, and $\mathbf{x}_3=\begin{bmatrix}3-2i\\ 1\\ -1+i\end{bmatrix}$.

Note that another possible eigenvector for λ_2 is $\mathbf{x}_2 = (-1+i)\begin{bmatrix} 3+2i \\ 1 \\ -1-i \end{bmatrix} = \begin{bmatrix} -5+i \\ -1+i \\ 2 \end{bmatrix}$.

6. The eigenvalues are $\lambda_1 = 2, \lambda_2 = 2+3i,$ and $\lambda_3 = 2-3i$. The corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -4-i \\ 3i \\ 1+i \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} -4+i \\ -3i \\ 1-i \end{bmatrix}$.

7. As in Example 1, $\mathbf{y}(t) = e^{(4+2i)t}\begin{bmatrix} 4 \\ -1+i \end{bmatrix} = e^{4t}(\cos 2t + i\sin 2t)\begin{bmatrix} 4 \\ -1+i \end{bmatrix}$ is one solution of $\mathbf{y}' = A\mathbf{y}$. Expanding and collecting real and imaginary parts, we obtain $\mathbf{y}(t) = e^{4t}\begin{bmatrix} 4\cos 2t \\ -\cos 2t - \sin 2t \end{bmatrix} + ie^{4t}\begin{bmatrix} 4\sin 2t \\ \cos 2t - \sin 2t \end{bmatrix}$. Thus, a fundamental set of solutions can be formed from $\mathbf{y}_1(t) = e^{4t}\begin{bmatrix} 4\cos 2t \\ -\cos 2t - \sin 2t \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{4t}\begin{bmatrix} 4\sin 2t \\ \cos 2t - \sin 2t \end{bmatrix}$.

8. $\mathbf{y}(t) = e^{it}\begin{bmatrix} -2+i \\ 5 \end{bmatrix} = (\cos t + i\sin t)\begin{bmatrix} -2+i \\ 5 \end{bmatrix}$ is one solution of $\mathbf{y}' = A\mathbf{y}$. Expanding and collecting real and imaginary parts, we obtain $\mathbf{y}(t) = \begin{bmatrix} -2\cos t - \sin t \\ 5\cos t \end{bmatrix} + i\begin{bmatrix} \cos t - 2\sin t \\ 5\sin t \end{bmatrix}$. Thus, a fundamental set of solutions can be formed from $\mathbf{y}_1(t) = \begin{bmatrix} -2\cos t - \sin t \\ 5\cos t \end{bmatrix}$ and $\mathbf{y}_2(t) = \begin{bmatrix} \cos t - 2\sin t \\ 5\sin t \end{bmatrix}$.

9. As in Example 1, $\mathbf{y}(t) = e^{2it}\begin{bmatrix} -1-i \\ 1 \end{bmatrix} = (\cos 2t + i\sin 2t)\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ is one solution of $\mathbf{y}' = A\mathbf{y}$. Expanding and collecting real and imaginary parts, we obtain $\mathbf{y}(t) = \begin{bmatrix} -\cos 2t + \sin 2t \\ \cos 2t \end{bmatrix} + i\begin{bmatrix} -\cos 2t - \sin 2t \\ \sin 2t \end{bmatrix}$. Thus, a fundamental set of solutions can be formed from $\mathbf{y}_1(t) = \begin{bmatrix} -\cos 2t + \sin 2t \\ \cos 2t \end{bmatrix}$ and $\mathbf{y}_2(t) = \begin{bmatrix} -\cos 2t - \sin 2t \\ \sin 2t \end{bmatrix}$.

10. $\mathbf{y}(t) = e^{t}(\cos t + i\sin t)\begin{bmatrix} -1+i \\ i \end{bmatrix}$ is one solution of $\mathbf{y}' = A\mathbf{y}$. Expanding and collecting real and imaginary parts, we obtain $\mathbf{y}(t) = e^{t}\begin{bmatrix} -\cos t - \sin t \\ -\sin t \end{bmatrix} + ie^{t}\begin{bmatrix} \cos t - \sin t \\ \cos t \end{bmatrix}$. Thus, a fundamental set of solutions can be formed from $\mathbf{y}_1(t) = e^{t}\begin{bmatrix} -\cos t - \sin t \\ -\sin t \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{t}\begin{bmatrix} \cos t - \sin t \\ \cos t \end{bmatrix}$.

11. As in Example 1, $\mathbf{y}(t)=e^{(2+3i)t}\begin{bmatrix}-5+3i\\3+3i\\2\end{bmatrix}=e^{2t}(\cos 3t+i\sin 3t)\begin{bmatrix}-5+3i\\3+3i\\2\end{bmatrix}$ is one solution of $\mathbf{y}'=A\mathbf{y}$. Expanding and collecting real and imaginary parts, we obtain $\mathbf{y}(t)=e^{2t}\begin{bmatrix}-5\cos 3t-3\sin 3t\\3\cos 3t-3\sin 3t\\2\cos 3t\end{bmatrix}+ie^{2t}\begin{bmatrix}3\cos 3t-5\sin 3t\\3\cos 3t+3\sin 3t\\2\sin 3t\end{bmatrix}$. Thus, two linearly independent solutions are $\mathbf{y}_1(t)=e^{2t}\begin{bmatrix}-5\cos 3t-3\sin 3t\\3\cos 3t-3\sin 3t\\2\cos 3t\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{2t}\begin{bmatrix}3\cos 3t-5\sin 3t\\3\cos 3t+3\sin 3t\\2\sin 3t\end{bmatrix}$. The third solution needed to complete the fundamental set is obtained from the real eigenvalue $\lambda=2$, $\mathbf{y}_3(t)=e^{2t}\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

12. $e^t(\cos 5t+i\sin 5t)\begin{bmatrix}i\\1\\0\\0\end{bmatrix}=e^t\begin{bmatrix}-\sin 5t\\\cos 5t\\0\\0\end{bmatrix}+ie^t\begin{bmatrix}\cos 5t\\\sin 5t\\0\\0\end{bmatrix}$. Also,
$e^t(\cos 2t+i\sin 2t)\begin{bmatrix}0\\0\\i\\1\end{bmatrix}=e^t\begin{bmatrix}0\\0\\-\sin 2t\\\cos 2t\end{bmatrix}+ie^t\begin{bmatrix}0\\0\\\cos 2t\\\sin 2t\end{bmatrix}$ Thus, a fundamental set of solutions can be formed from $e^t\begin{bmatrix}-\sin 5t\\\cos 5t\\0\\0\end{bmatrix}$, $e^t\begin{bmatrix}\cos 5t\\\sin 5t\\0\\0\end{bmatrix}$, $e^t\begin{bmatrix}0\\0\\-\sin 2t\\\cos 2t\end{bmatrix}$, $e^t\begin{bmatrix}0\\0\\\cos 2t\\\sin 2t\end{bmatrix}$.

13. Proceeding as in Exercises 7-12, we find the general solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}(t)=c_1e^{2t}\begin{bmatrix}\cos t\\-\sin t\end{bmatrix}+c_2e^{2t}\begin{bmatrix}\sin t\\\cos t\end{bmatrix}$. Imposing the initial condition, $\mathbf{y}(0)=c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}4\\7\end{bmatrix}$, we obtain the solution $\mathbf{y}(t)=e^{2t}\begin{bmatrix}4\cos t+7\sin t\\-4\sin t+7\cos t\end{bmatrix}$.

14. Proceeding as in Exercises 7-12, we find the general solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}(t)=c_1\begin{bmatrix}3\sin 3t\\\cos 3t\end{bmatrix}+c_2\begin{bmatrix}3\cos 3t\\\sin 3t\end{bmatrix}$. Imposing the initial condition, $\mathbf{y}(0)=c_1\begin{bmatrix}0\\1\end{bmatrix}+c_2\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}6\\2\end{bmatrix}$, we obtain the solution $\mathbf{y}(t)=\begin{bmatrix}-6\sin 3t+6\cos 3t\\2\cos 3t+2\sin 3t\end{bmatrix}$.

15. Proceeding as in Exercises 7-12, we find the general solution of $\mathbf{y}' = A\mathbf{y}$ is
$\mathbf{y}(t) = c_1 e^{3t}\begin{bmatrix} 3\cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 2\cos 2t + 3\sin 2t \\ \sin 2t \end{bmatrix}$. Imposing the initial condition,
$\mathbf{y}(0) = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, we find $c_1 = 3$ and $c_2 = -4$ and the solution is
$\mathbf{y}(t) = e^{3t}\begin{bmatrix} \cos 2t - 18\sin 2t \\ 3\cos 2t - 4\sin 2t \end{bmatrix}$.

16. Proceeding as in Exercises 7-12, we find the general solution of $\mathbf{y}' = A\mathbf{y}$ is
$\mathbf{y}(t) = c_1 e^{2t}\begin{bmatrix} \cos t \\ -\cos t - \sin t \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} \sin t \\ \cos t - \sin t \end{bmatrix}$. Imposing the initial condition,
$\mathbf{y}(0) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$, we obtain the solution $\mathbf{y}(t) = e^{2t}\begin{bmatrix} 8\cos t + 14\sin t \\ 6\cos t - 22\sin t \end{bmatrix}$.

17. Proceeding as in Exercises 7-12, we find the general solution of $\mathbf{y}' = A\mathbf{y}$ is
$\mathbf{y}(t) = c_1 e^{t}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 3\cos t - 2\sin t \\ \cos t \\ -\cos t + \sin t \end{bmatrix} + c_3 e^{t}\begin{bmatrix} 2\cos t + 3\sin t \\ \sin t \\ -\cos t - \sin t \end{bmatrix}$.
Imposing the initial condition, $\mathbf{y}(0) = c_1\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$, we find
$c_1 = -9$, $c_2 = 1$, and $c_3 = -12$, and the solution $\mathbf{y}(t) = e^{t}\begin{bmatrix} 27 - 21\cos t - 38\sin t \\ \cos t - 12\sin t \\ -9 + 11\cos t + 13\sin t \end{bmatrix}$.

18. Proceeding as in Exercises 7-12, we find the general solution of $\mathbf{y}' = A\mathbf{y}$ is
$\mathbf{y}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -4\cos 3t + \sin 3t \\ -3\sin 3t \\ \cos 3t - \sin 3t \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} -\cos 3t - 4\sin 3t \\ 3\cos 3t \\ \cos 3t + \sin 3t \end{bmatrix}$.
Imposing the initial condition, $\mathbf{y}(0) = c_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 4 \end{bmatrix}$, we find
$c_1 = -2$, $c_2 = -1$, and $c_3 = 3$, and the solution $\mathbf{y}(t) = e^{2t}\begin{bmatrix} -2 + \cos 3t - 13\sin 3t \\ 9\cos 3t + 3\sin 3t \\ 2 + 2\cos 3t + 4\sin 3t \end{bmatrix}$.

22. The eigenvalues of A are $\lambda = \left(-1 \pm \sqrt{9 + 12\mu}\right)/2$. If $9 + 12\mu < 0$, $\lambda = -\frac{1}{2} \pm i\beta$ $(\beta \neq 0)$, therefore distinct and $y(t) \to 0$. If $0 < 9 + 12\mu < 1$, , the eigenvalues are distinct, real and negative. Therefore, $-\infty < 9 + 12\mu < 1$ $-\infty < \mu < -\frac{2}{3}$.

23. The eigenvalues of A are $\lambda = \left(-5 \pm \sqrt{1+4\mu}\right)/2$. In order that both components of $\mathbf{y}(t)$ go to zero as $t \to \infty$, we need each of these (real) eigenvalues to be negative. Therefore, we need $\left(-5 + \sqrt{1+4\mu}\right)/2 < 0$ or $\sqrt{1+4\mu} < 5$. This inequality holds if and only if $1+4\mu < 25$ or $-\infty < \mu < 6$.

24. The eigenvalues of A are $\lambda = \left(-2 \pm \sqrt{16-4\mu^2}\right)/2 = -1 \pm \sqrt{4-\mu^2}$. Require $-\infty < 4-\mu^2 < 1 \Rightarrow 3 < \mu^2 < \infty$. Therefore, $-\infty < \mu < -\sqrt{3}$ and $\sqrt{3} < \mu < \infty$.

25. The eigenvalues of A are $\lambda = -1 \pm \sqrt{4+\mu^2}$. In order that both components of $\mathbf{y}(t)$ go to zero as $t \to \infty$, we need each of these (real) eigenvalues to be negative. Therefore, we need $-1+\sqrt{4+\mu^2} < 0$ or $\sqrt{4+\mu^2} < 1$. This inequality cannot hold for any real value of μ.

26 (a). $\dfrac{d}{dt}\mathbf{v} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{v} \Rightarrow \mathbf{v}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

26 (b). $\mathbf{v}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \mathbf{v}(t) = \begin{bmatrix} \cos t + 2\sin t \\ -\sin t + 2\cos t \end{bmatrix}$. $\mathbf{r}(t) = \begin{bmatrix} \sin t - 2\cos t \\ \cos t + 2\sin t \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$.

$\mathbf{r}(0) = \begin{bmatrix} -2+d_1 \\ 1+d_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{r}(t) = \begin{bmatrix} \sin t - 2\cos t + 4 \\ \cos t + 2\sin t \end{bmatrix}$. $\mathbf{v}\left(\dfrac{3\pi}{2}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathbf{r}\left(\dfrac{3\pi}{2}\right) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

27 (d). If the charged particle is launched with initial velocity parallel to the magnetic field, it will move with constant velocity.

28 (b). The eigenpairs are $-\dfrac{\gamma}{m} + \lambda_1, \mathbf{x}_1$ and $-\dfrac{\gamma}{m} + \lambda_2, \mathbf{x}_2$ and $-\dfrac{\gamma}{m} + \lambda_3, \mathbf{x}_3$.

The corresponding fundamental matrix is $e^{-\frac{\gamma}{m}t}\psi(t)$.

Section 6.7

1 (a). For $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda-2)^2$. The eigenvalue $\lambda_1 = 2$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-2I)\mathbf{x} = \mathbf{0}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1 = 2$ have the form $\mathbf{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_1 \neq 0$. The geometric multiplicity of $\lambda_1 = 2$ is 1.

1 (b). We find a generalized eigenvector corresponding to $\lambda_1 = 2$ by solving the equation $(A-2I)\mathbf{x} = \mathbf{x}_1$ where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ 1 \end{bmatrix}$ where x_1 is arbitrary. Choosing $x_1 = 0$, we obtain the generalized eigenvector $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus, we have solutions $\mathbf{y}_1(t) = e^{2t}\mathbf{x}_1$ and, as in equation (6), $\mathbf{y}_2(t) = te^{2t}\mathbf{x}_1 + e^{2t}\mathbf{x}_2$. A fundamental matrix is $\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$.

1 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition, $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{2t}\begin{bmatrix} 1-t \\ -1 \end{bmatrix}$.

2 (a). The characteristic polynomial is $p(\lambda) = (3-\lambda)^2$. The eigenvalue $\lambda_1 = 3$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-3I)\mathbf{x} = \mathbf{0}$. Therefore, all the eigenvectors corresponding to $\lambda_1 = 3$ have the form $\mathbf{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_1 \neq 0$. The geometric multiplicity of $\lambda_1 = 3$ is 1.

2 (b). $\mathbf{y}_1(t) = e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y}_2(t) = te^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{3t}\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} = e^{3t}\begin{bmatrix} t \\ \frac{1}{2} \end{bmatrix}$. A fundamental matrix is $\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} e^{3t} & te^{3t} \\ 0 & \frac{1}{2}e^{3t} \end{bmatrix}$.

2 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition, $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{3t}\begin{bmatrix} 2t+4 \\ 1 \end{bmatrix}$.

3 (a). For $A = \begin{bmatrix} 6 & 0 \\ 2 & 6 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda-6)^2$. The eigenvalue $\lambda_1 = 6$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-6I)\mathbf{x} = \mathbf{0}$ or $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1 = 6$ have the form $\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x_2 \neq 0$. The geometric multiplicity of $\lambda_1 = 6$ is 1.

3 (b). We find a generalized eigenvector corresponding to $\lambda_1 = 6$ by solving the equation $(A-6I)\mathbf{x} = \mathbf{x}_1$ where $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The solution is $\mathbf{x} = \begin{bmatrix} 0.5 \\ x_2 \end{bmatrix}$ where x_2 is arbitrary. Choosing $x_2 = 0$, we obtain the generalized eigenvector $\mathbf{x}_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$. Thus, we have solutions $\mathbf{y}_1(t) = e^{6t}\mathbf{x}_1$ and, as in equation (6), $\mathbf{y}_2(t) = te^{6t}\mathbf{x}_1 + e^{6t}\mathbf{x}_2$. A fundamental matrix is $\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} 0 & 0.5e^{6t} \\ e^{6t} & te^{6t} \end{bmatrix}$.

3 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition requires $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 0 & 0.5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{6t}\begin{bmatrix} -2 \\ -4t \end{bmatrix}$.

4 (a). The characteristic polynomial is $p(\lambda) = (3-\lambda)^2$. The eigenvalue $\lambda_1 = 3$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-3I)\mathbf{x} = \mathbf{0}$. Therefore, all the eigenvectors corresponding to $\lambda_1 = 3$ have the form $\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x_2 \neq 0$. The geometric multiplicity of $\lambda_1 = 3$ is 1.

4 (b). $\mathbf{y}_1(t) = e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{y}_2(t) = te^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{3t}\begin{bmatrix} 1 \\ t \end{bmatrix}$. A fundamental matrix is $\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} 0 & e^{3t} \\ e^{3t} & te^{3t} \end{bmatrix}$.

4 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition, $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{3t}\begin{bmatrix} 2 \\ 2t-3 \end{bmatrix}$.

5 (a). For $A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda - 3)^2$. The eigenvalue $\lambda_1 = 3$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-3I)\mathbf{x} = \mathbf{0}$ or $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1 = 3$ have the form $\mathbf{x} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $x_1 \neq 0$. The geometric multiplicity of $\lambda_1 = 3$ is 1.

5 (b). We find a generalized eigenvector corresponding to $\lambda_1 = 3$ by solving the equation $(A-3I)\mathbf{x} = \mathbf{x}_1$ where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The solution is $\mathbf{x} = \begin{bmatrix} .5x_2 + .5 \\ x_2 \end{bmatrix}$ where x_2 is arbitrary. Choosing $x_2 = 0$, we obtain the generalized eigenvector $\mathbf{x}_2 = \begin{bmatrix} .5 \\ 0 \end{bmatrix}$. Thus, we have solutions $\mathbf{y}_1(t) = e^{3t}\mathbf{x}_1$ and, as in equation (6), $\mathbf{y}_2(t) = te^{3t}\mathbf{x}_1 + e^{3t}\mathbf{x}_2$. A fundamental matrix is $\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} e^{3t} & (t+.5)e^{3t} \\ 2e^{3t} & 2te^{3t} \end{bmatrix}$.

5 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition requires $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 1 & .5 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} .5 \\ 1 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{3t}\begin{bmatrix} t+1 \\ 2t+1 \end{bmatrix}$.

6 (a). The characteristic polynomial is $p(\lambda) = (3-\lambda)^2$. The eigenvalue $\lambda_1 = 3$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-3I)\mathbf{x} = \mathbf{0}$. Use $\mathbf{x}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$. The geometric multiplicity of $\lambda_1 = 3$ is 1.

6 (b). $\mathbf{y}_1(t) = e^{3t}\begin{bmatrix} 6 \\ -1 \end{bmatrix}$ and $\mathbf{y}_2(t) = te^{3t}\begin{bmatrix} 6 \\ -1 \end{bmatrix} + e^{3t}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = e^{3t}\begin{bmatrix} 6t-1 \\ -t \end{bmatrix}$. A fundamental matrix is $\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} 6e^{3t} & (6t-1)e^{3t} \\ -e^{3t} & -te^{3t} \end{bmatrix}$.

6 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition, $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} -2 \\ -12 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{3t}\begin{bmatrix} -72t \\ 12t+2 \end{bmatrix}$.

7 (a). For $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda-2)^2$. The eigenvalue $\lambda_1 = 2$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-2I)\mathbf{x} = \mathbf{0}$ or $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1 = 2$ have the form $\mathbf{x} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $x_1 \neq 0$. The geometric multiplicity of $\lambda_1 = 2$ is 1.

7 (b). We find a generalized eigenvector corresponding to $\lambda_1 = 2$ by solving the equation $(A-2I)\mathbf{x} = \mathbf{x}_1$ where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The solution is $\mathbf{x} = \begin{bmatrix} -1-x_2 \\ x_2 \end{bmatrix}$ where x_2 is arbitrary. Choosing $x_2 = 0$, we obtain the generalized eigenvector $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Thus, we have solutions $\mathbf{y}_1(t) = e^{2t}\mathbf{x}_1$ and, as in equation (6), $\mathbf{y}_2(t) = te^{2t}\mathbf{x}_1 + e^{2t}\mathbf{x}_2$. A fundamental matrix is $\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} e^{2t} & (t-1)e^{2t} \\ -e^{2t} & -te^{2t} \end{bmatrix}$.

7 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition requires $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{2t}\begin{bmatrix} 4-3t \\ 3t-1 \end{bmatrix}$.

8 (a). The characteristic polynomial is $p(\lambda)=(\lambda-5)^2$. The eigenvalue $\lambda_1=5$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-5I)\mathbf{x}=\mathbf{0}$. Use $\mathbf{x}_1=\begin{bmatrix}1\\-1\end{bmatrix}$. The geometric multiplicity of $\lambda_1=5$ is 1.

8 (b). $\mathbf{y}_1(t)=e^{5t}\begin{bmatrix}1\\-1\end{bmatrix}$ and $\mathbf{y}_2(t)=te^{5t}\begin{bmatrix}1\\-1\end{bmatrix}+e^{5t}\begin{bmatrix}1\\0\end{bmatrix}=e^{5t}\begin{bmatrix}t+1\\-t\end{bmatrix}$. A fundamental matrix is

$$\Psi(t)=[\mathbf{y}_1(t),\mathbf{y}_2(t)]=\begin{bmatrix}e^{5t} & (t+1)e^{5t}\\-e^{5t} & -te^{5t}\end{bmatrix}.$$

8 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition, $\Psi(0)\mathbf{c}=\mathbf{y}_0$. We find $\begin{bmatrix}1 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}4\\-4\end{bmatrix}$ or $\mathbf{c}=\begin{bmatrix}4\\0\end{bmatrix}$. Thus, the solution of the initial value problem is

$$\mathbf{y}(t)=e^{5t}\begin{bmatrix}4\\-4\end{bmatrix}.$$

9 (a). For $A=\begin{bmatrix}2 & 1 & 0\\0 & 2 & 1\\0 & 0 & 2\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=(\lambda-2)^3$. The eigenvalue $\lambda_1=2$ has algebraic multiplicity 3. Corresponding eigenvectors are obtained by solving $(A-2I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1=2$ have the form $\mathbf{x}=\begin{bmatrix}x_1\\0\\0\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\end{bmatrix}$, $x_1\neq 0$. The geometric multiplicity of $\lambda_1=2$ is 1.

9 (b). For $A=\begin{bmatrix}2 & 1 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=(\lambda-2)^3$. The eigenvalue $\lambda_1=2$ has algebraic multiplicity 3. Corresponding eigenvectors are obtained by solving $(A-2I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1=2$ have the form $\mathbf{x}=\begin{bmatrix}x_1\\0\\x_3\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$, where $\mathbf{x}$ is nonzero. The geometric multiplicity of $\lambda_1=2$ is 2.

13. For $A=\begin{bmatrix}5 & 0 & 0\\1 & 5 & 0\\1 & 1 & 5\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=(\lambda-5)^3$. The eigenvalue $\lambda_1=5$ has algebraic multiplicity 3.

Corresponding eigenvectors are obtained by solving $(A-5I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}0&0&0\\1&0&0\\1&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Therefore, all the eigenvectors corresponding to $\lambda_1=5$ have the form

$\mathbf{x}=\begin{bmatrix}0\\0\\x_3\end{bmatrix}=x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$, $x_3\neq 0$. The geometric multiplicity of $\lambda_1=5$ is 1, so A does not have a full set of eigenvectors.

14. The characteristic polynomial is $p(\lambda)=(\lambda-5)^3$. The eigenvalue $\lambda_1=5$ has algebraic multiplicity 3. Corresponding eigenvectors are obtained by solving $(A-5I)\mathbf{x}=\mathbf{0}$. Use $\mathbf{x}_1=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix}0\\0\\1\end{bmatrix}$. The geometric multiplicity of $\lambda_1=5$ is 2, so A does not have a full set of eigenvectors.

15. For $A=\begin{bmatrix}5&0&1\\0&5&0\\0&0&5\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=(\lambda-5)^3$. The eigenvalue $\lambda_1=5$ has algebraic multiplicity 3. Corresponding eigenvectors are obtained by solving $(A-5I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1=5$ have the form $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\0\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}$, where $\mathbf{x}$ is nonzero. The geometric multiplicity of $\lambda_1=5$ is 2, so A does not have a full set of eigenvectors.

16. The characteristic polynomial is $p(\lambda)=(\lambda-5)^3$. The eigenvalue $\lambda_1=5$ has algebraic multiplicity 3. Corresponding eigenvectors are obtained by solving $(A-5I)\mathbf{x}=0\mathbf{x}=\mathbf{0}$. Use $\mathbf{x}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{x}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$. The geometric multiplicity of $\lambda_1=5$ is 3, so A does have a full set of eigenvectors.

17. For $A=\begin{bmatrix}2&0&0&0\\1&2&0&0\\0&0&3&0\\0&0&1&3\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=(\lambda-2)^2(\lambda-3)^2$. The eigenvalue $\lambda_1=2$ has algebraic multiplicity 2 as does $\lambda_2=3$.

Corresponding eigenvectors for $\lambda_1 = 2$ have the form $\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $x_2 \neq 0$. Therefore, the geometric multiplicity of $\lambda_1 = 2$ is 1. Similarly, eigenvectors corresponding to $\lambda_2 = 3$ have the form $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $x_4 \neq 0$ and so $\lambda_2 = 3$ has geometric multiplicity 1. *A* does not have a full set of eigenvectors.

18. The characteristic polynomial is $p(\lambda) = (2-\lambda)^4$. The eigenvalue $\lambda_1 = 2$ has algebraic multiplicity 4. Corresponding eigenvectors are obtained by solving $(A-2I)\mathbf{x} = \mathbf{0}$. Use $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. The geometric multiplicity of $\lambda = 2$ is 3, so *A* does not have a full set of eigenvectors.

19. For $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda-2)^3(\lambda-3)$. The eigenvalue $\lambda_1 = 2$ has algebraic multiplicity 3 while $\lambda_2 = 3$ has algebraic multiplicity 1. Corresponding eigenvectors for $\lambda_1 = 2$ have the form $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$. Therefore, the geometric multiplicity of $\lambda_1 = 2$ is 3. Eigenvectors corresponding to $\lambda_2 = 3$ have the form $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $x_4 \neq 0$ and so $\lambda_2 = 3$ has geometric multiplicity 1. Since every eigenvalue of *A* has geometric multiplicity equal to its algebraic multiplicity, *A* has a full set of eigenvectors.

20. *A* must have $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ as two distinct eigenvalues. Therefore, *A* cannot have a repeated eigenvalue and cannot be defective.

21. In order for *A* to be symmetric, $a_{12} = x$ must be the same as $a_{21} = 9$. Thus, $x = 9$. Similarly, $a_{23} = y$ must equal $a_{32} = 4$.

22. $x = 6$, $y = 1$.

23. In order for A to be symmetric, $a_{13} = x^2 - 1$ must be the same as $a_{31} = 0$. Thus, we can have either $x = 1$ or $x = -1$. Similarly, $a_{21} = 2/y$ must equal $a_{12} = 1$. Hence, $y = 2$.

24. In order for A to be Hermitian, $\overline{a}_{12} = x - 3i = 9 - 3i \Rightarrow x = 9$ and $\overline{a}_{23} = 2 - yi = 2 + 5i \Rightarrow y = -5$.

25. In order for A to be Hermitian, $a_{11} = 2 + xi$ must be the same as $\overline{a}_{11} = 2 - xi$. Thus, we need $x = 0$. Similarly, $a_{21} = 1 + yi$ must equal $\overline{a}_{12} = 1 - 2i$. Hence, $y = 2$. These choices are consistent with the remaining undetermined entries, a_{22} and a_{23}.

26 (a). $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, for example.

26 (b). $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, for example.

30. The equation $(A - 2I)\mathbf{x} = \mathbf{v}_1$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Choose $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

The equation $(A - 2I)\mathbf{x} = \mathbf{v}_2$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Choose $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. A fundamental set of solutions can be formed from $\mathbf{y}_1(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{2t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{2t}\begin{bmatrix} \frac{t^2}{2} \\ t \\ 1 \end{bmatrix}$.

31. For $A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 3 & 4 \end{bmatrix}$, the equation $(A - 4I)\mathbf{x} = \mathbf{v}_1$ is $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The solution is $\mathbf{x} = \begin{bmatrix} 0 \\ 1/3 \\ x_3 \end{bmatrix}$ where x_3 is arbitrary. Choosing $x_3 = 0$ we have $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1/3 \\ 0 \end{bmatrix}$.

The equation $(A - 4I)\mathbf{x} = \mathbf{v}_2$ is $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ 0 \end{bmatrix}$. The solution is $\mathbf{x} = \begin{bmatrix} 1/6 \\ -1/18 \\ x_3 \end{bmatrix}$ where x_3 is arbitrary. Choosing $x_3 = 0$ we have $\mathbf{v}_3 = \begin{bmatrix} 1/6 \\ -1/18 \\ 0 \end{bmatrix}$. By equation (12), a fundamental set of solutions can be formed from $\mathbf{y}_1(t) = e^{4t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{4t}(\mathbf{v}_2 + t\mathbf{v}_1) = e^{4t}\begin{bmatrix} 0 \\ 1/3 \\ t \end{bmatrix}$, and

$$\mathbf{y}_3(t) = e^{4t}(\mathbf{v}_3 + t\mathbf{v}_2 + 0.5t^2\mathbf{v}_1) = \frac{e^{4t}}{18}\begin{bmatrix} 3 \\ -1 + 6t \\ 9t^2 \end{bmatrix}.$$

32. The equation $(A-I)\mathbf{x}=\mathbf{v}_1$ is $\begin{bmatrix} 2 & -8 & -10 \\ -2 & 6 & 8 \\ 2 & -6 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Choose $\mathbf{v}_2=\begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$.

The equation $(A-I)\mathbf{x}=\mathbf{v}_2$ is $\begin{bmatrix} 2 & -8 & -10 \\ -2 & 6 & 8 \\ 2 & -6 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$. Choose $\mathbf{v}_3=\begin{bmatrix} -\frac{3}{4} \\ -\frac{1}{4} \\ 0 \end{bmatrix}$. A fundamental set of solutions can be formed from $\mathbf{y}_1(t)=e^t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{y}_2(t)=e^t\begin{bmatrix} \frac{1}{2}+t \\ -t \\ t \end{bmatrix}$, and $\mathbf{y}_3(t)=e^t\begin{bmatrix} -\frac{3}{4}+\frac{t}{2}+\frac{t^2}{2} \\ -\frac{1}{4}-\frac{t^2}{2} \\ \frac{t^2}{2} \end{bmatrix}$.

33. For $A=\begin{bmatrix} -6 & -8 & 22 \\ 2 & 4 & -4 \\ -2 & -2 & 8 \end{bmatrix}$, the equation $(A-2I)\mathbf{x}=\mathbf{v}_1$ is given by $\begin{bmatrix} -8 & -8 & 22 \\ 2 & 2 & -4 \\ -2 & -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. A convenient solution is $\mathbf{v}_2=\begin{bmatrix} -1.5 \\ 0 \\ -0.5 \end{bmatrix}$. The equation $(A-2I)\mathbf{x}=\mathbf{v}_2$ is $\begin{bmatrix} -8 & -8 & 22 \\ 2 & 2 & -4 \\ -2 & -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -1.5 \\ 0 \\ -0.5 \end{bmatrix}$. One solution is $\mathbf{v}_3=\begin{bmatrix} -0.5 \\ 0 \\ -0.25 \end{bmatrix}$. A fundamental set consists of

$\mathbf{y}_1(t)=e^{2t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{y}_2(t)=e^{2t}(\mathbf{v}_2+t\mathbf{v}_1)=e^{2t}\begin{bmatrix} t-1.5 \\ -t \\ -0.5 \end{bmatrix}$, and

$\mathbf{y}_3(t)=e^{2t}(\mathbf{v}_3+t\mathbf{v}_2+0.5t^2\mathbf{v}_1)=\dfrac{e^{2t}}{4}\begin{bmatrix} -2-6t+2t^2 \\ -2t^2 \\ -1-2t \end{bmatrix}$.

36 (a). Two linearly independent solutions are $\mathbf{x}_1=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

36 (b). Choose $\mathbf{x}_3=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

36 (c). $\mathbf{Q}(t)=c_1e^{-4t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}+c_2e^{-4t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}+c_3e^{-t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Section 6.8

1 (a). For $A=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2+4\lambda+3=(\lambda+3)(\lambda+1)$. The eigenvalues are $\lambda_1=-3$ and $\lambda_2=-1$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$. Thus, the complementary solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}_C=\begin{bmatrix}e^{-3t} & e^{-t}\\ -e^{-3t} & e^{-t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$.

1 (b). Inserting the suggested trial form $\mathbf{y}_P=\begin{bmatrix}a_1\\ a_2\end{bmatrix}$ into the nonhomogeneous equation leads to $\mathbf{y}_P'=A\mathbf{y}_P+\mathbf{g}(t)$ or $\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}$. Solving this system, we obtain $\mathbf{y}_P=\begin{bmatrix}1\\ 1\end{bmatrix}$.

1 (c). The general solution of the nonhomogeneous problem is $\mathbf{y}_C+\mathbf{y}_P=\begin{bmatrix}e^{-3t} & e^{-t}\\ -e^{-3t} & e^{-t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}$.

1 (d). Imposing the initial condition, $\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}3\\ 1\end{bmatrix}$. Solving, we find $c_1=1$ and $c_2=1$. Thus, $\mathbf{y}(t)=\begin{bmatrix}e^{-3t}+e^{-t}+1\\ -e^{-3t}+e^{-t}+1\end{bmatrix}$ is the unique solution of the given initial value problem.

2 (a). For $A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-4\lambda+3=(\lambda-1)(\lambda-3)$. The eigenvalues are $\lambda_1=1$ and $\lambda_2=3$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$. Thus, the complementary solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}_C=\begin{bmatrix}e^{t} & e^{3t}\\ -e^{t} & e^{3t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$.

2 (b). Inserting the suggested trial form $\mathbf{y}_P=e^{-t}\begin{bmatrix}a_1\\ a_2\end{bmatrix}$ into the nonhomogeneous equation and solving the system, we obtain $\mathbf{y}_P=e^{-t}\begin{bmatrix}-\frac{3}{8}\\ \frac{1}{8}\end{bmatrix}$.

2 (c). The general solution of the nonhomogeneous problem is $\mathbf{y}_C+\mathbf{y}_P=\begin{bmatrix}e^{t} & e^{3t}\\ -e^{t} & e^{3t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+e^{-t}\begin{bmatrix}-\frac{3}{8}\\ \frac{1}{8}\end{bmatrix}$.

2 (d). Imposing the initial condition, $\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\begin{bmatrix}-\frac{3}{8}\\ \frac{1}{8}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Solving, we find $c_1=\dfrac{1}{4}$ and $c_2=\dfrac{1}{8}$. Thus, $\mathbf{y}(t)=\begin{bmatrix}\frac{1}{4}e^{t}+\frac{1}{8}e^{3t}-\frac{3}{8}e^{-t}\\ -\frac{1}{4}e^{t}+\frac{1}{8}e^{3t}+\frac{1}{8}e^{-t}\end{bmatrix}$ is the unique solution of the given initial value problem.

3 (a). For $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-1=(\lambda+1)(\lambda-1)$. The eigenvalues are $\lambda_1=-1$ and $\lambda_2=1$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$. Thus, the complementary solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}_C=\begin{bmatrix}e^{-t} & e^{t}\\ -e^{-t} & e^{t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$.

3 (b). Inserting the suggested trial form $\mathbf{y}_P=t\begin{bmatrix}a_1\\ a_2\end{bmatrix}+\begin{bmatrix}b_1\\ b_2\end{bmatrix}$ into the nonhomogeneous equation leads to $\mathbf{y}_P'=A\mathbf{y}_P+\mathbf{g}(t)$ or $\begin{bmatrix}a_1\\ a_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}ta_1+b_1\\ ta_2+b_2\end{bmatrix}+\begin{bmatrix}t\\ -1\end{bmatrix}$. Solving this system, we obtain $\mathbf{y}_P=\begin{bmatrix}0\\ -t\end{bmatrix}$.

3 (c). The general solution of the nonhomogeneous problem is $\mathbf{y}_C+\mathbf{y}_P=\begin{bmatrix}e^{-t} & e^{t}\\ -e^{-t} & e^{t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\begin{bmatrix}0\\ -t\end{bmatrix}$.

3 (d). Imposing the initial condition, $\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}2\\ -1\end{bmatrix}$. Solving, we find $c_1=1.5$ and $c_2=0.5$. Thus, $\mathbf{y}(t)=0.5\begin{bmatrix}3e^{-t}+e^{t}\\ -3e^{-t}+e^{t}-t\end{bmatrix}$ is the unique solution of the given initial value problem.

4 (a). For $A=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-1$. The eigenvalues are $\lambda_1=-1$ and $\lambda_2=1$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$. Thus, the complementary solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}_C=\begin{bmatrix}e^{-t} & e^{t}\\ e^{-t} & -e^{t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$.

4 (b). Inserting the suggested trial form $\mathbf{y}_P=e^{2t}\begin{bmatrix}a_1\\ a_2\end{bmatrix}+t\begin{bmatrix}b_1\\ b_2\end{bmatrix}+\begin{bmatrix}c_1\\ c_2\end{bmatrix}$ into the nonhomogeneous equation and solving the system, we obtain $\mathbf{y}_P=\begin{bmatrix}-\frac{1}{3}e^{2t}-1\\ \frac{2}{3}e^{2t}+t\end{bmatrix}$.

4 (c). The general solution of the nonhomogeneous problem is

$$\mathbf{y}_C+\mathbf{y}_P=\begin{bmatrix}e^{-t} & e^{t}\\ e^{-t} & -e^{t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\begin{bmatrix}-\frac{1}{3}e^{2t}-1\\ \frac{2}{3}e^{2t}+t\end{bmatrix}.$$

4 (d). Imposing the initial condition, $\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\begin{bmatrix}-\frac{4}{3}\\ \frac{2}{3}\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}$. Solving, we find $c_1=\dfrac{5}{6}$ and $c_2=\dfrac{1}{2}$. Thus, $\mathbf{y}(t)=\begin{bmatrix}\frac{5}{6}e^{-t}+\frac{1}{2}e^{t}-\frac{1}{3}e^{2t}-1\\ \frac{5}{6}e^{-t}-\frac{1}{2}e^{t}+\frac{2}{3}e^{2t}+t\end{bmatrix}$ is the unique solution of the given initial value problem.

5 (a). For $A=\begin{bmatrix}-3 & -2\\ 4 & 3\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-1=(\lambda+1)(\lambda-1)$. The eigenvalues are $\lambda_1=-1$ and $\lambda_2=1$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ -2\end{bmatrix}$. Thus, the complementary solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}_C=\begin{bmatrix}e^{-t} & e^{t}\\ -e^{-t} & -2e^{t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$.

5 (b). Inserting the suggested trial form $\mathbf{y}_P=\sin t\begin{bmatrix}a_1\\ a_2\end{bmatrix}+\cos t\begin{bmatrix}b_1\\ b_2\end{bmatrix}$ into the nonhomogeneous equation leads to $\mathbf{y}_P'=A\mathbf{y}_P+\mathbf{g}(t)$ or $\begin{bmatrix}a_1\cos t-b_1\sin t\\ a_2\cos t-b_2\sin t\end{bmatrix}=\begin{bmatrix}-3 & -2\\ 4 & 3\end{bmatrix}\begin{bmatrix}a_1\sin t+b_1\cos t\\ a_2\sin t+b_2\cos t\end{bmatrix}+\begin{bmatrix}\sin t\\ 0\end{bmatrix}$. Solving this system, we obtain $\mathbf{y}_P=0.5\begin{bmatrix}3\sin t-\cos t\\ -4\sin t\end{bmatrix}$.

5 (c). The general solution of the nonhomogeneous problem is

$$\mathbf{y}_C+\mathbf{y}_P=\begin{bmatrix}e^{-t} & e^{t}\\ -e^{-t} & -2e^{t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+0.5\begin{bmatrix}3\sin t-\cos t\\ -4\sin t\end{bmatrix}.$$

5 (d). Imposing the initial condition, $\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\begin{bmatrix}-.5\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Solving, we find $c_1=1$ and $c_2=-0.5$. Thus, $\mathbf{y}(t)=0.5\begin{bmatrix}2e^{-t}-e^{t}+3\sin t-\cos t\\ -2e^{-t}+2e^{t}-4\sin t\end{bmatrix}$ is the unique solution of the given initial value problem.

6 (a). For $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-2\lambda$. The eigenvalues are $\lambda_1=0$ and $\lambda_2=2$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$. Thus, the complementary solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}_C=\begin{bmatrix}1 & e^{2t}\\ -1 & e^{2t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$.

7. Given $\mathbf{y}(t)=\begin{bmatrix}1+\sin 2t\\ e^{t}+\cos 2t\end{bmatrix}$ it follows that $\mathbf{y}_0=\mathbf{y}(\pi/2)=\begin{bmatrix}1+\sin\pi\\ e^{\pi/2}+\cos\pi\end{bmatrix}=\begin{bmatrix}1\\ e^{\pi/2}-1\end{bmatrix}$. Inserting $\mathbf{y}(t)$ into the differential equation, we see that $\mathbf{y}'(t)=A\mathbf{y}(t)+\mathbf{g}(t)$ and thus $\begin{bmatrix}2\cos 2t\\ e^{t}-2\sin 2t\end{bmatrix}=\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}\begin{bmatrix}1+\sin 2t\\ e^{t}+\cos 2t\end{bmatrix}+\mathbf{g}(t)$. Solving for $\mathbf{g}(t)$, we obtain $\mathbf{g}(t)=\begin{bmatrix}-2e^{t}\\ e^{t}+2\end{bmatrix}$.

8. Given $\mathbf{y}(t)=\begin{bmatrix}t+\alpha\\ t^2+\beta\end{bmatrix}$ it follows that $\mathbf{y}(1)=\begin{bmatrix}1+\alpha\\ 1+\beta\end{bmatrix}=\begin{bmatrix}2\\ -1\end{bmatrix}\Rightarrow\alpha=1,\ \beta=-2$. Inserting $\mathbf{y}(t)$ into the differential equation, we see that $\mathbf{y}'(t)=A\mathbf{y}(t)+\mathbf{g}(t)$ and thus $\mathbf{y}'=\begin{bmatrix}1\\ 2t\end{bmatrix}=\begin{bmatrix}1 & t\\ t^2 & 1\end{bmatrix}\begin{bmatrix}t+1\\ t^2-2\end{bmatrix}+\mathbf{g}(t)$. Solving for $\mathbf{g}(t)$, we obtain $\mathbf{g}(t)=\begin{bmatrix}-t^3+t\\ -t^3-2t^2+2t+2\end{bmatrix}$.

9. Following the hint, we form $[\mathbf{y}_1', \mathbf{y}_2'] = P(t)[\mathbf{y}_1, \mathbf{y}_2] + [\mathbf{g}_1(t), \mathbf{g}_2(t)]$ which has the form $\begin{bmatrix} 0 & e^t \\ -e^{-t} & 0 \end{bmatrix} = P(t)\begin{bmatrix} 1 & e^t \\ e^{-t} & -1 \end{bmatrix} + \begin{bmatrix} -2 & e^t \\ 0 & -1 \end{bmatrix}$. Solving for $P(t)$, we have

$$P(t) = \begin{bmatrix} 2 & 0 \\ -e^{-t} & 1 \end{bmatrix}\begin{bmatrix} 1 & e^t \\ e^{-t} & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 \\ -e^{-t} & 1 \end{bmatrix}(-1/2)\begin{bmatrix} -1 & -e^t \\ -e^{-t} & 1 \end{bmatrix} = \begin{bmatrix} 1 & e^t \\ 0 & -1 \end{bmatrix}.$$

10. If A^{-1} exists, $\mathbf{y}_2 = -A^{-1}\mathbf{b}$ is the unique solution.
If A^{-1} does not exist, the matrix equation $A\mathbf{y} = -\mathbf{b}$ will either have no solution or a non-unique solution. Therefore, either no equilibrium solution or a non-unique equilibrium solution.

11. An equilibrium solution of $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$ is a constant solution. Therefore, since $\mathbf{y}' = \mathbf{0}$ we need $A\mathbf{y}_e = -\mathbf{b}$. For $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ we see that $\mathbf{y}_e = -A^{-1}\mathbf{b} = -\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$.

12. A^{-1} exists and $\mathbf{y}_e = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

13. As noted in the solution of Exercise 11, we need $A\mathbf{y}_e = -\mathbf{b}$. For $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ we see that $\mathbf{y}_e = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where a is arbitrary.

14. A^{-1} exists and $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{y}_e = \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix} \Rightarrow \mathbf{y}_e = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$.

15. As noted in the solution of Exercise 11, we need $A\mathbf{y}_e = -\mathbf{b}$. For $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ we see that $\mathbf{y}_e = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + a\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ where a is arbitrary.

16. The characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda$. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
Thus, one fundamental matrix is $\hat{\Psi}(t) = \begin{bmatrix} 1 & e^{2t} \\ 1 & -e^{2t} \end{bmatrix}$. Set

$$\Psi = \hat{\Psi}C.\ \Psi(1) = I = \hat{\Psi}(1)C.\ \therefore C = \hat{\Psi}(1)^{-1} = \begin{bmatrix} 1 & e^2 \\ 1 & -e^2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}e^{-2} & -\frac{1}{2}e^{-2} \end{bmatrix}.$$

$$\Psi(t) = \begin{bmatrix} 1 & e^{2t} \\ 1 & -e^{2t} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}e^{-2} & -\frac{1}{2}e^{-2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(1 + e^{2(t-1)}\right) & \frac{1}{2}\left(1 - e^{2(t-1)}\right) \\ \frac{1}{2}\left(1 - e^{2(t-1)}\right) & \frac{1}{2}\left(1 + e^{2(t-1)}\right) \end{bmatrix}.$$

17. For $A=\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2+4=(\lambda+2i)(\lambda-2i)$. The eigenvalues are $\lambda_1=-2i$ and $\lambda_2=2i$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}-1\\ i\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}-1\\ -i\end{bmatrix}$. Converting to real solutions, we have $\mathbf{y}(t)=e^{-2it}\begin{bmatrix}-1\\ i\end{bmatrix}=(\cos 2t-i\sin 2t)\begin{bmatrix}-1\\ i\end{bmatrix}$. Therefore, a fundamental set of solutions is $\mathbf{y}_1(t)=\begin{bmatrix}-\cos 2t\\ \sin 2t\end{bmatrix}$ and $\mathbf{y}_2(t)=\begin{bmatrix}\sin 2t\\ \cos 2t\end{bmatrix}$.

Thus, one fundamental matrix is $\Psi(t)=\begin{bmatrix}-\cos 2t & \sin 2t\\ \sin 2t & \cos 2t\end{bmatrix}$. The solution of the given initial value problem has the form $\hat{\Psi}(t)=\Psi(t)C=\begin{bmatrix}-\cos 2t & \sin 2t\\ \sin 2t & \cos 2t\end{bmatrix}C$ where C is a (2×2) matrix chosen so that $\hat{\Psi}(\pi/4)=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$. Imposing this condition, we have $\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}=\Psi(\pi/4)C=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}C$. Solving for C, we obtain $C=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}$ and thus $\hat{\Psi}(t)=\begin{bmatrix}-\cos 2t & \sin 2t\\ \sin 2t & \cos 2t\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}$ or $\hat{\Psi}(t)=\begin{bmatrix}\sin 2t & -\cos 2t-\sin 2t\\ \cos 2t & \sin 2t-\cos 2t\end{bmatrix}$.

18. The characteristic polynomial is $p(\lambda)=\lambda^2-2\lambda$. The eigenvalues are $\lambda_1=0$ and $\lambda_2=2$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$.

Thus, one fundamental matrix is $\hat{\Psi}(t)=\begin{bmatrix}1 & e^{2t}\\ 1 & -e^{2t}\end{bmatrix}$. Set

$\Psi=\hat{\Psi}C$. $\Psi(0)=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}=\hat{\Psi}(0)C$. $\therefore C=\hat{\Psi}(0)^{-1}\Psi(0)=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}=\begin{bmatrix}\frac{3}{2} & \frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2}\end{bmatrix}$.

$$\Psi(t)=\begin{bmatrix}1 & e^{2t}\\ 1 & -e^{2t}\end{bmatrix}\begin{bmatrix}\frac{3}{2} & \frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{3}{2}-\frac{1}{2}e^{2t} & \frac{1}{2}-\frac{1}{2}e^{2t}\\ \frac{3}{2}+\frac{1}{2}e^{2t} & \frac{1}{2}+\frac{1}{2}e^{2t}\end{bmatrix}.$$

19. For $A=\begin{bmatrix}3 & -4\\ 2 & -3\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-1=(\lambda+1)(\lambda-1)$. The eigenvalues are $\lambda_1=-1$ and $\lambda_2=1$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}2\\ 1\end{bmatrix}$. Therefore, a fundamental set of solutions is $\mathbf{y}_1(t)=e^{-t}\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{t}\begin{bmatrix}2\\ 1\end{bmatrix}$. Thus, one fundamental matrix is $\Psi(t)=\begin{bmatrix}e^{-t} & 2e^{t}\\ e^{-t} & e^{t}\end{bmatrix}$.

The solution of the given initial value problem has the form $\hat{\Psi}(t) = \Psi(t)C = \begin{bmatrix} e^{-t} & 2e^{t} \\ e^{-t} & e^{t} \end{bmatrix} C$ where C is a (2×2) matrix chosen so that $\hat{\Psi}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Imposing this condition, we have $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \Psi(0)C = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} C$. Solving for C, we obtain $C = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ and thus $\hat{\Psi}(t) = \begin{bmatrix} e^{-t} & 2e^{t} \\ e^{-t} & e^{t} \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$ or $\hat{\Psi}(t) = \begin{bmatrix} -e^{-t} + 2e^{t} & 2e^{-t} - 2e^{t} \\ -e^{-t} + e^{t} & 2e^{-t} - e^{t} \end{bmatrix}$.

20. The characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda + 5$. The eigenvalues are $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$, with corresponding eigenvectors $\mathbf{x} = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$. Then

$$\mathbf{y}(t) = e^{t}(\cos 2t + i \sin 2t)\begin{bmatrix} -2i \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{t}\sin 2t \\ e^{t}\cos 2t \end{bmatrix} + i\begin{bmatrix} -2e^{t}\cos 2t \\ e^{t}\sin 2t \end{bmatrix}.$$

Thus, one fundamental matrix is $\hat{\Psi}(t) = \begin{bmatrix} 2e^{t}\sin 2t & -2e^{t}\cos 2t \\ e^{t}\cos 2t & e^{t}\sin 2t \end{bmatrix}$. Set

$$\Psi = \hat{\Psi}C.\ \Psi\left(\frac{\pi}{4}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2e^{\frac{\pi}{4}} & 0 \\ 0 & e^{\frac{\pi}{4}} \end{bmatrix} C.\ \therefore C = \begin{bmatrix} \frac{1}{2}e^{-\frac{\pi}{4}} & 0 \\ 0 & e^{-\frac{\pi}{4}} \end{bmatrix}.$$

$$\Psi(t) = \begin{bmatrix} e^{\left(t-\frac{\pi}{4}\right)}\sin 2t & -2e^{\left(t-\frac{\pi}{4}\right)}\cos 2t \\ \frac{1}{2}e^{\left(t-\frac{\pi}{4}\right)}\cos 2t & e^{\left(t-\frac{\pi}{4}\right)}\sin 2t \end{bmatrix}.$$

21. For $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda = \lambda(\lambda - 2)$. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore, a fundamental set of solutions is $\mathbf{y}_1(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. A fundamental matrix is $\Psi(t) = \begin{bmatrix} 1 & e^{2t} \\ -1 & e^{2t} \end{bmatrix}$ and therefore, $\Psi^{-1}(s) = 0.5e^{-2s}\begin{bmatrix} e^{2s} & -e^{2s} \\ 1 & 1 \end{bmatrix} = 0.5\begin{bmatrix} 1 & -1 \\ e^{-2s} & e^{-2s} \end{bmatrix}$. From equation (11), the solution is $\mathbf{y}(t) = \Psi(t)\Psi^{-1}(t_0)\mathbf{y}_0 + \Psi(t)\int_{t_0}^{t}\Psi^{-1}(s)\mathbf{g}(s)\,ds$. Since $\mathbf{y}_0 = \mathbf{0}$ and $t_0 = 0$, we have

$$\mathbf{y}(t) = \Psi(t)\int_0^t \Psi^{-1}(s)\mathbf{g}(s)\,ds = \begin{bmatrix} 1 & e^{2t} \\ -1 & e^{2t} \end{bmatrix}\int_0^t 0.5\begin{bmatrix} e^{2s} \\ 1 \end{bmatrix} ds = \begin{bmatrix} 1 & e^{2t} \\ -1 & e^{2t} \end{bmatrix} 0.25\begin{bmatrix} e^{2t} - 1 \\ 2t \end{bmatrix}$$

$$= 0.25\begin{bmatrix} e^{2t} - 1 + 2te^{2t} \\ -(e^{2t} - 1) + 2te^{2t} \end{bmatrix}.$$

22. For $A=\begin{bmatrix}9 & -4\\ 15 & -7\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-2\lambda-3=(\lambda-3)(\lambda+1)$. The eigenvalues are $\lambda_1=-1$ and $\lambda_2=3$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}2\\5\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}2\\3\end{bmatrix}$.

A fundamental matrix is $\Psi(t)=\begin{bmatrix}2e^{-t} & 2e^{3t}\\ 5e^{-t} & 3e^{3t}\end{bmatrix}$ and therefore,

$$\Psi^{-1}(s)=-\frac{1}{4}e^{-2s}\begin{bmatrix}3e^{3s} & -2e^{3s}\\ -5e^{-s} & 2e^{-s}\end{bmatrix}=\begin{bmatrix}-\frac{3}{4}e^{s} & \frac{1}{2}e^{s}\\ \frac{5}{4}e^{-3s} & -\frac{1}{2}e^{-3s}\end{bmatrix}.$$

$$\int_0^t\Psi^{-1}(s)\mathbf{g}(s)\,ds=\int_0^t\begin{bmatrix}-\frac{3}{4}e^{2s}\\ \frac{5}{4}e^{-2s}\end{bmatrix}ds=\begin{bmatrix}-\frac{3}{8}\left(e^{2t}-1\right)\\ -\frac{5}{8}\left(e^{-2t}-1\right)\end{bmatrix}.$$

$$\Psi(t)\int_0^t\Psi^{-1}(s)\mathbf{g}(s)\,ds=\begin{bmatrix}-\frac{3}{4}(e^{t}-e^{-t})-\frac{5}{4}(e^{t}-e^{3t})\\ -\frac{15}{8}(e^{t}-e^{-t})-\frac{15}{8}(e^{t}-e^{3t})\end{bmatrix}=\begin{bmatrix}\frac{3}{4}e^{-t}-2e^{t}+\frac{5}{4}e^{3t}\\ \frac{15}{8}e^{-t}-\frac{15}{4}e^{t}+\frac{15}{8}e^{3t}\end{bmatrix}.$$

Then, $\mathbf{y}(t)=\Psi(t)\mathbf{y}_0+\Psi(t)\int_0^t\Psi^{-1}(s)\mathbf{g}(s)\,ds$, $\mathbf{y}(0)=\Psi(0)\mathbf{y}_0+\mathbf{0}=\begin{bmatrix}2\\5\end{bmatrix}\Rightarrow$

$\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}2 & 2\\ 5 & 3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}\Rightarrow\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$. Therefore,

$$\mathbf{y}(t)=\begin{bmatrix}2e^{-t}\\5e^{-t}\end{bmatrix}+\begin{bmatrix}\frac{3}{4}e^{-t}-2e^{t}+\frac{5}{4}e^{3t}\\ \frac{15}{8}e^{-t}-\frac{15}{4}e^{t}+\frac{15}{8}e^{3t}\end{bmatrix}=\begin{bmatrix}\frac{11}{4}e^{-t}-2e^{t}+\frac{5}{4}e^{3t}\\ \frac{55}{8}e^{-t}-\frac{15}{4}e^{t}+\frac{15}{8}e^{3t}\end{bmatrix}.$$

23. For $A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2+1=(\lambda-i)(\lambda+i)$. The eigenvalues are $\lambda_1=i$ and $\lambda_2=-i$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\i\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\-i\end{bmatrix}$.

$\mathbf{x}_1=\begin{bmatrix}-1\\i\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}-1\\-i\end{bmatrix}$. Converting to real solutions, we have $\mathbf{y}(t)=e^{it}\begin{bmatrix}1\\i\end{bmatrix}=(\cos t+i\sin t)\begin{bmatrix}1\\i\end{bmatrix}$.

Therefore, a fundamental set of solutions is $\mathbf{y}_1(t)=\begin{bmatrix}\cos t\\-\sin t\end{bmatrix}$ and $\mathbf{y}_2(t)=\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$. A fundamental matrix is $\Psi(t)=\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}$. We have $\Psi^{-1}(s)=\begin{bmatrix}\cos s & -\sin s\\ \sin s & \cos s\end{bmatrix}$.

From equation (11), the solution is $\mathbf{y}(t)=\Psi(t)\Psi^{-1}(t_0)\mathbf{y}_0+\Psi(t)\int_{t_0}^t\Psi^{-1}(s)\mathbf{g}(s)\,ds$. Since $\mathbf{y}_0=\begin{bmatrix}0\\1\end{bmatrix}$, $\mathbf{g}(s)=\begin{bmatrix}2\\1\end{bmatrix}$, and $t_0=0$, we have

$$\mathbf{y}(t)=\Psi(t)\Psi^{-1}(t_0)\mathbf{y}_0+\Psi(t)\int_0^t\Psi^{-1}(s)\mathbf{g}(s)\,ds=\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}+\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\int_0^t\begin{bmatrix}2\cos s-\sin s\\ 2\sin s+\cos s\end{bmatrix}ds$$

$$=\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}+\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\begin{bmatrix}2\sin t+\cos t-1\\ -2\cos t+\sin t+2\end{bmatrix}=\begin{bmatrix}1-\cos t+3\sin t\\ -2+3\cos t+\sin t\end{bmatrix}.$$

24. For $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=(1-\lambda)^2$. The eigenvalue is $\lambda=1$, with corresponding eigenvector $\mathbf{x}=\begin{bmatrix}1\\ 0\end{bmatrix}$. Then

$\mathbf{y}_1=\begin{bmatrix}e^t\\ 0\end{bmatrix}$. Let $\mathbf{y}_2=e^t\left(t\mathbf{v}_1+\mathbf{v}_2\right)$. $\mathbf{v}_1=\begin{bmatrix}1\\ 0\end{bmatrix}$, $\mathbf{v}_2=\begin{bmatrix}0\\ 1\end{bmatrix}\Rightarrow \mathbf{y}_2=e^t\begin{bmatrix}t\\ 1\end{bmatrix}$.

A fundamental matrix is $\Psi(t)=\begin{bmatrix}e^t & te^t\\ 0 & e^t\end{bmatrix}$ and therefore,

$\Psi^{-1}(s)=e^{-2s}\begin{bmatrix}e^s & -se^s\\ 0 & e^s\end{bmatrix}=\begin{bmatrix}e^{-s} & -se^{-s}\\ 0 & e^{-s}\end{bmatrix}$. $\int_0^t \Psi^{-1}(s)\mathbf{g}(s)\,ds=\int_0^t\begin{bmatrix}e^{-s}(1-s)\\ e^{-s}\end{bmatrix}ds=\begin{bmatrix}te^{-t}\\ 1-e^{-t}\end{bmatrix}$.

Since $\mathbf{y}(0)=\begin{bmatrix}0\\ 0\end{bmatrix}$, $\mathbf{y}(t)=\Psi(t)\int_0^t \Psi^{-1}(s)\mathbf{g}(s)\,ds=\begin{bmatrix}t+te^t-t\\ e^t-1\end{bmatrix}=\begin{bmatrix}te^t\\ e^t-1\end{bmatrix}$.

25 (a). $\mathbf{Q}(t)$ is an equilibrium solution if $\frac{r}{V}\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\mathbf{Q}+\begin{bmatrix}cr\\ 0\end{bmatrix}=\mathbf{0}$. Solving for $\mathbf{Q}$, we obtain

$\mathbf{Q}_e(t)=\frac{Cv}{3}\begin{bmatrix}2\\ 1\end{bmatrix}$.

25 (b). The characteristic polynomial is $p(\lambda)=\lambda^2+(4r/V)\lambda+3r^2/V^2=(\lambda+3r/V)(\lambda+r/V)$. The eigenvalues are $\lambda_1=-3r/V$ and $\lambda_2=-r/V$, with corresponding eigenvectors

$\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$. Therefore, a fundamental set of solutions is

$\mathbf{y}_1(t)=e^{-3rt/V}\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{-rt/V}\begin{bmatrix}1\\ 1\end{bmatrix}$. The complementary solution is

$\mathbf{Q}_C(t)=\begin{bmatrix}e^{-3rt/V} & e^{-rt/V}\\ -e^{-3rt/V} & e^{-rt/V}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$.

25 (c). Finding a constant particular solution is equivalent to finding an equilibrium solution, as in part (a).

25 (d). The general solution is $\mathbf{Q}(t)=\mathbf{Q}_C(t)+\mathbf{Q}_e(t)=\begin{bmatrix}e^{-3rt/V} & e^{-rt/V}\\ -e^{-3rt/V} & e^{-rt/V}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\frac{cV}{3}\begin{bmatrix}2\\ 1\end{bmatrix}$. Imposing the initial condition leads to $\mathbf{Q}(0)=\mathbf{0}$ or $\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\frac{cV}{3}\begin{bmatrix}2\\ 1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. The solution of the initial value problem is $\mathbf{Q}(t)=\frac{cV}{6}\begin{bmatrix}4-e^{-3rt/V}-3e^{-rt/V}\\ 2+e^{-3rt/V}-3e^{-rt/V}\end{bmatrix}$.

25 (e). $\frac{1}{V}\lim_{t\to\infty}\mathbf{Q}(t)=\frac{c}{3}\begin{bmatrix}2\\ 1\end{bmatrix}$.

26 (a). $-V_s + \frac{1}{2}I_1' + I_1 + 2(I_1 - I_2) = 0,\ 2(I_2 - I_1) + I_2 + \frac{1}{2}I_2' = 0$. Therefore,

$$\frac{d}{dt}\begin{bmatrix} I_1' \\ I_2' \end{bmatrix} = \begin{bmatrix} -6I_1 + 4I_2 + 2V_s \\ 4I_1 - 6I_2 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} 2V_s \\ 0 \end{bmatrix},\ t > 0,\ \begin{bmatrix} I_1(0) \\ I_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

26 (b). The characteristic polynomial is $p(\lambda) = \lambda^2 + 12\lambda + 20 = (\lambda + 10)(\lambda + 2)$. The eigenvalues are $\lambda_1 = -10$ and $\lambda_2 = -2$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore, a fundamental matrix is $\Psi(t) = \begin{bmatrix} e^{-10t} & e^{-2t} \\ -e^{-10t} & e^{-2t} \end{bmatrix}$.

26 (c). $\mathbf{I}(t) = \Psi(t)\int_0^t \Psi^{-1}(s)\begin{bmatrix} 2V_s \\ 0 \end{bmatrix} ds$ since $\mathbf{I}(0) = \mathbf{0}$. $\Psi^{-1}(s) = \dfrac{1}{2e^{-12s}}\begin{bmatrix} e^{-2s} & -e^{-2s} \\ e^{-10s} & e^{-10s} \end{bmatrix} = \dfrac{1}{2}\begin{bmatrix} e^{10s} & -e^{10s} \\ e^{2s} & e^{2s} \end{bmatrix}$.

With $V_s(t) = 1,\ t > 0,\ \int_0^t \Psi^{-1}(s)\begin{bmatrix} 2V_s \\ 0 \end{bmatrix} ds = \int_0^t \begin{bmatrix} e^{10s} \\ e^{2s} \end{bmatrix} = \begin{bmatrix} \frac{1}{10}(e^{10t} - 1) \\ \frac{1}{2}(e^{2t} - 1) \end{bmatrix}$, Therefore,

$$\mathbf{I}(t) = \begin{bmatrix} \frac{1}{10}(1 - e^{-10t}) + \frac{1}{2}(1 - e^{-2t}) \\ -\frac{1}{10}(1 - e^{-10t}) + \frac{1}{2}(1 - e^{-2t}) \end{bmatrix} = \begin{bmatrix} -\frac{1}{10}e^{-10t} - \frac{1}{2}e^{-2t} + \frac{3}{5} \\ \frac{1}{10}e^{-10t} - \frac{1}{2}e^{-2t} + \frac{2}{5} \end{bmatrix}.$$

27 (a). In the vector system $\mathbf{v}' = -\mathbf{v} + (\mathbf{v} \times \mathbf{k}) + \mathbf{f}$, the term $\mathbf{v} \times \mathbf{k}$ is given by $(v_x\mathbf{i} + v_y\mathbf{j}) \times \mathbf{k} = -v_x\mathbf{j} + v_y\mathbf{i}$. Therefore, the system is $\begin{bmatrix} v_x' \\ v_y' \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \end{bmatrix}$.

27 (b). For $\mathbf{f} = 0.5\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$, we seek a constant solution $\mathbf{v}$; that is, an equilibrium solution. Thus, we need to solve, if possible, $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix} + 0.5\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This system does indeed have a solution, namely $\mathbf{v}_e = 0.25\begin{bmatrix} 1 + \sqrt{3} \\ -1 + \sqrt{3} \end{bmatrix}$. If we choose the initial velocity equal to the "equilibrium velocity," $\mathbf{v}_e$, then the particle will move at that constant velocity.

28 (b). The characteristic polynomial is $p(\lambda) = -\lambda(\lambda^2 + \omega_c^2)$. The eigenvalues are $\lambda_1 = 0,\ \lambda_2 = i\omega_c,\ \lambda_3 = -i\omega_c$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$. Therefore,

A fundamental matrix is $\Psi(t) = \begin{bmatrix} 0 & \cos\omega_c t & \sin\omega_c t \\ 0 & -\sin\omega_c t & \cos\omega_c t \\ 1 & 0 & 0 \end{bmatrix}$.

28 (c). $\Phi(t)=\Psi(t)\Psi^{-1}(0)=\begin{bmatrix}0 & \cos\omega_c t & \sin\omega_c t\\ 0 & -\sin\omega_c t & \cos\omega_c t\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}^{-1}=$

$$\begin{bmatrix}0 & \cos\omega_c t & \sin\omega_c t\\ 0 & -\sin\omega_c t & \cos\omega_c t\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}=\begin{bmatrix}\cos\omega_c t & \sin\omega_c t & 0\\ -\sin\omega_c t & \cos\omega_c t & 0\\ 0 & 0 & 1\end{bmatrix}.$$

28 (d). From equation 11, $\mathbf{v}(t)=\Phi(t)\mathbf{v}_0+\Phi(t)\int_0^t\Phi^{-1}(s)\mathbf{g}(s)\,ds$, using $\Phi(t)$ as a fundamental matrix and noting that $\Phi^{-1}(0)=\mathbf{I}$. Therefore, $\mathbf{v}(t)=\Phi(t)\mathbf{v}_0+\mathbf{f}(t)$.

28 (e). $\mathbf{r}(t)=\int_0^\tau\mathbf{v}(t)dt=\left[\int_0^\tau\Phi(t)dt\right]\mathbf{v}_0+\int_0^\tau\mathbf{f}(t)dt=\hat{\mathbf{r}}$. $\therefore\left[\int_0^\tau\Phi(t)dt\right]\mathbf{v}_0=\hat{\mathbf{r}}-\int_0^\tau\mathbf{f}(t)dt$

$$\int_0^\tau\Phi(t)dt=\begin{bmatrix}\omega_c^{-1}\sin\omega_c t & \omega_c^{-1}(1-\cos\omega_c t) & 0\\ -\omega_c^{-1}(1-\cos\omega_c t) & \omega_c^{-1}\sin\omega_c t & 0\\ 0 & 0 & \tau\end{bmatrix}.$$

$$D=\det\left\{\int_0^\tau\Phi(t)dt\right\}=\frac{2\tau}{\omega_c^2}(1-\cos\omega_c\tau)=\frac{4\tau}{\omega_c^2}\sin^2\left(\frac{\omega_c\tau}{2}\right)\ \therefore\frac{\omega_c\tau}{2}\neq n\pi\Rightarrow\tau\neq\frac{2n\pi}{\omega_c}.$$

Section 6.9

1 (a). For $\mathbf{y}'=P(t)\mathbf{y}+\mathbf{g}(t),\ \mathbf{y}(t_0)=\mathbf{y}_0$, Euler's method has the form $\mathbf{y}_{n+1}=\mathbf{y}_n+h[P(t_n)\mathbf{y}_n+\mathbf{g}(t_n)]$.

For $P(t)=\begin{bmatrix}1&2\\2&3\end{bmatrix}$, $\mathbf{g}(t)=\begin{bmatrix}1\\1\end{bmatrix}$, $\mathbf{y}_0=\begin{bmatrix}-1\\1\end{bmatrix}$, and $t_0=0$ the iteration is

$$\mathbf{y}_{n+1}=\mathbf{y}_n+h\left[\begin{bmatrix}1&2\\2&3\end{bmatrix}\mathbf{y}_n+\begin{bmatrix}1\\1\end{bmatrix}\right],\ \mathbf{y}_0=\begin{bmatrix}-1\\1\end{bmatrix}.$$

1 (b). In general, $t_k=t_0+kh, k=0,1,\ldots$. Since $t_0=0$, we have $t_k=kh, k=0,1,\ldots$. In general, $h=(b-a)/N$. So, for $a=0, b=1$, and $h=0.01$, we obtain $N=1/h=100$.

2 (a). $\mathbf{y}_{n+1}=\mathbf{y}_n+h\left[\begin{bmatrix}1&t_n\\2+t_n&2\end{bmatrix}\mathbf{y}_n+\begin{bmatrix}1\\t_n\end{bmatrix}\right],\ \mathbf{y}_0=\begin{bmatrix}2\\1\end{bmatrix}.$

2 (b). $t_k=1+kh, k=0,1,\ldots$. $N=.5/h=50$.

3 (a). For $\mathbf{y}'=P(t)\mathbf{y}+\mathbf{g}(t),\ \mathbf{y}(t_0)=\mathbf{y}_0$, Euler's method has the form $\mathbf{y}_{n+1}=\mathbf{y}_n+h[P(t_n)\mathbf{y}_n+\mathbf{g}(t_n)]$.

For $P(t)=\begin{bmatrix}-t^2&t\\2-t&0\end{bmatrix}$, $\mathbf{g}(t)=\begin{bmatrix}1\\t\end{bmatrix}$, $\mathbf{y}_0=\begin{bmatrix}2\\0\end{bmatrix}$, and $t_0=1$ the iteration is

$$\mathbf{y}_{n+1}=\mathbf{y}_n+h\left[\begin{bmatrix}-t_n^2&t_n\\2-t_n&0\end{bmatrix}\mathbf{y}_n+\begin{bmatrix}1\\t_n\end{bmatrix}\right],\ \mathbf{y}_0=\begin{bmatrix}2\\0\end{bmatrix}.$$

3 (b). In general, $t_k=t_0+kh, k=0,1,\ldots$. Since $t_0=1$, we have $t_k=1+kh, k=0,1,\ldots$. In general, $h=(b-a)/N$. So, for $a=1, b=4$, and $h=0.01$, we obtain $N=3/h=300$.

4 (a). $\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}\mathbf{y}_n + \begin{bmatrix} 0 \\ 2 \\ t_n \end{bmatrix}\right],\ \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

4 (b). $t_k = -1 + kh, k = 0,1,\ldots.\quad N = 1/h = 100.$

5 (a). For $\mathbf{y}' = P(t)\mathbf{y} + \mathbf{g}(t),\ \mathbf{y}(t_0) = \mathbf{y}_0$, Euler's method has the form $\mathbf{y}_{n+1} = \mathbf{y}_n + h[P(t_n)\mathbf{y}_n + \mathbf{g}(t_n)]$.

For $P(t) = \begin{bmatrix} t^{-1} & \sin t \\ 1-t & 1 \end{bmatrix}, \mathbf{g}(t) = \begin{bmatrix} 0 \\ t^2 \end{bmatrix}, \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $t_0 = 1$ the iteration is

$\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\begin{bmatrix} t_n^{-1} & \sin t_n \\ 1-t_n & 1 \end{bmatrix}\mathbf{y}_n + \begin{bmatrix} 0 \\ t_n^2 \end{bmatrix}\right],\ \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

5 (b). In general, $t_k = t_0 + kh, k = 0,1,\ldots$. Since $t_0 = 1$, we have $t_k = 1 + kh, k = 0,1,\ldots$. In general, $h = (b-a)/N$. So, for $a = 1, b = 6$, and $h = 0.01$, we obtain $N = 5/h = 500$.

6. $\mathbf{y}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} -0.98 \\ 1.02 \end{bmatrix}$ and

$\mathbf{y}_2 = \begin{bmatrix} -0.98 \\ 1.02 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -0.98 \\ 1.02 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} -0.9594 \\ 1.041 \end{bmatrix}$

7. The iteration has the form $\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\begin{bmatrix} 1 & t_n \\ 2+t_n & 2 \end{bmatrix}\mathbf{y}_n + \begin{bmatrix} 1 \\ t_n \end{bmatrix}\right],\ \mathbf{y}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ where

$t_0 = 1$ and $t_1 = 1.01$. Therefore, $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0.01\begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 2.04 \\ 1.09 \end{bmatrix}$ and

$\mathbf{y}_2 = \begin{bmatrix} 2.04 \\ 1.09 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & 1.01 \\ 3.01 & 2 \end{bmatrix}\begin{bmatrix} 2.04 \\ 1.09 \end{bmatrix} + \begin{bmatrix} 1 \\ 1.01 \end{bmatrix}\right] = \begin{bmatrix} 2.04 \\ 1.09 \end{bmatrix} + 0.01\begin{bmatrix} 4.1409 \\ 9.3304 \end{bmatrix}$

$= \begin{bmatrix} 2.081409 \\ 1.183304 \end{bmatrix}.$

8. $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0.01\left[\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 1.99 \\ 0.03 \end{bmatrix}$ and

$\mathbf{y}_2 = \begin{bmatrix} 1.99 \\ 0.03 \end{bmatrix} + 0.01\left[\begin{bmatrix} -(1.01)^2 & 1.01 \\ 2-1.01 & 0 \end{bmatrix}\begin{bmatrix} 1.99 \\ 0.03 \end{bmatrix} + \begin{bmatrix} 1 \\ 1.01 \end{bmatrix}\right] = \begin{bmatrix} 1.9800030 \\ 0.059801 \end{bmatrix}$

9. The iteration has the form $\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}\mathbf{y}_n + \begin{bmatrix} 0 \\ 2 \\ t_n \end{bmatrix}\right],\ \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ where

$t_0 = -1$ and $t_1 = -0.99$. Therefore,

$\mathbf{y}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0.01\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.03 \\ 0.99 \end{bmatrix}$

and

$$\mathbf{y}_2 = \begin{bmatrix} 0.01 \\ 0.03 \\ 0.99 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0.01 \\ 0.03 \\ 0.99 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -0.99 \end{bmatrix}\right] = \begin{bmatrix} 0.01 \\ 0.03 \\ 0.99 \end{bmatrix} + 0.01\begin{bmatrix} 1 \\ 3.08 \\ -0.92 \end{bmatrix}$$

$$= \begin{bmatrix} 0.02 \\ 0.0608 \\ 0.9808 \end{bmatrix}.$$

10. $\mathbf{y}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & \sin(1) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$ and

$$\mathbf{y}_2 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} + 0.01\left[\begin{bmatrix} \frac{1}{1.01} & \sin(1.01) \\ 1-1.01 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0.01 \end{bmatrix} + \begin{bmatrix} 0 \\ (1.01)^2 \end{bmatrix}\right] = \begin{bmatrix} 0.00008468318 \\ 0.020301 \end{bmatrix}$$

11 (a). Let $\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. With this,

$$\mathbf{z}'(t) = \begin{bmatrix} z_1'(t) \\ z_2'(t) \end{bmatrix} = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} z_2(t) \\ -z_1(t) + t^{3/2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ t^{3/2} \end{bmatrix}, \mathbf{z}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

11 (b). Thus, the iteration has the form $\mathbf{z}_{n+1} = \mathbf{z}_n + h\left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{z}_n + \begin{bmatrix} 0 \\ t_n^{3/2} \end{bmatrix}\right]$, $\mathbf{z}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where $t_0 = 0$ and $t_1 = 0.01$.

11 (c). Therefore, $\mathbf{z}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.01\left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.01\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.01 \end{bmatrix}$ and

$$\mathbf{z}_2 = \begin{bmatrix} 1 \\ -0.01 \end{bmatrix} + 0.01\left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -0.01 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix}\right] = \begin{bmatrix} 1 \\ -0.01 \end{bmatrix} + 0.01\begin{bmatrix} -.01 \\ -.999 \end{bmatrix}$$

$$= \begin{bmatrix} .9999 \\ -.01999 \end{bmatrix}.$$

12 (a). $\mathbf{z}'(t) = \begin{bmatrix} z_1'(t) \\ z_2'(t) \end{bmatrix} = \begin{bmatrix} z_2(t) \\ -t^2 z_1(t) - z_2(t) + 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -t^2 & -1 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \mathbf{z}(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

12 (b). $\mathbf{z}_{n+1} = \mathbf{z}_n + h\left[\begin{bmatrix} 0 & 1 \\ -t^2 & -1 \end{bmatrix}\mathbf{z}_n + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right]$, $\mathbf{z}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $t_0 = 1$.

12 (c). $\mathbf{z}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.01\left[\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} 1.01 \\ 1 \end{bmatrix}$ and

$$\mathbf{z}_2 = \begin{bmatrix} 1.01 \\ 1 \end{bmatrix} + 0.01\left[\begin{bmatrix} 0 & 1 \\ -(1.01)^2 & -1 \end{bmatrix}\begin{bmatrix} 1.01 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} 1.02 \\ 0.999696699 \end{bmatrix}$$

13 (a). Let $\mathbf{z}=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}=\begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$. With this,

$$\mathbf{z}'=\begin{bmatrix} z_1' \\ z_2' \\ z_3' \end{bmatrix}=\begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix}=\begin{bmatrix} z_2 \\ z_3 \\ -2z_2-tz_1+t+1 \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -t & -2 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ t+1 \end{bmatrix}, \mathbf{z}(0)=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

13 (b). Thus, the iteration has the form $\mathbf{z}_{n+1}=\mathbf{z}_n+h\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -t_n & -2 & 0 \end{bmatrix}\mathbf{z}_n+\begin{bmatrix} 0 \\ 0 \\ t_n+1 \end{bmatrix}\right]$, $\mathbf{z}_0=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ where $t_0=0$ and $t_1=0.01$.

13 (c). Therefore, $\mathbf{z}_1=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}+0.01\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}+0.01\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}=\begin{bmatrix} 0.99 \\ -1 \\ 0.03 \end{bmatrix}$ and

$$\mathbf{z}_2=\begin{bmatrix} 0.99 \\ -1 \\ 0.03 \end{bmatrix}+0.01\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.01 & -2 & 0 \end{bmatrix}\begin{bmatrix} 0.99 \\ -1 \\ 0.03 \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ 1.01 \end{bmatrix}\right]=\begin{bmatrix} 0.99 \\ -1 \\ 0.03 \end{bmatrix}+0.01\begin{bmatrix} -1 \\ 0.03 \\ 3.0001 \end{bmatrix}$$

$$=\begin{bmatrix} 0.98 \\ -0.9997 \\ 0.060001 \end{bmatrix}.$$

14 (a). $\mathbf{z}'(t)=\begin{bmatrix} z_1'(t) \\ z_2'(t) \end{bmatrix}=\begin{bmatrix} z_2(t) \\ -e^{-t}z_1(t)-z_2(t)+2 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -e^{-t} & -1 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}+\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \mathbf{z}(0)=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

14 (b). $\mathbf{z}_{n+1}=\mathbf{z}_n+h\left[\begin{bmatrix} 0 & 1 \\ -e^{-t} & -1 \end{bmatrix}\mathbf{z}_n+\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right]$, $\mathbf{z}_0=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ where $t_0=0$.

14 (c). $\mathbf{z}_1=\begin{bmatrix} -1 \\ 1 \end{bmatrix}+0.01\left[\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}+\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right]=\begin{bmatrix} -0.99 \\ 1.02 \end{bmatrix}$ and

$$\mathbf{z}_2=\begin{bmatrix} -0.99 \\ 1.02 \end{bmatrix}+0.01\left[\begin{bmatrix} 0 & 1 \\ -e^{-0.01} & -1 \end{bmatrix}\begin{bmatrix} -0.99 \\ 1.02 \end{bmatrix}+\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right]=\begin{bmatrix} -0.9798 \\ 1.039601493 \end{bmatrix}$$

18. Actual error: $\mathbf{y}(1)-\overline{\mathbf{y}}_{200}=\begin{bmatrix} -0.00807508729 \\ 0.0759433736... \end{bmatrix}$

Estimated error: $\overline{\mathbf{y}}_{200}-\mathbf{y}_{100}=\begin{bmatrix} -0.0086591617... \\ 0.0764878206... \end{bmatrix}$

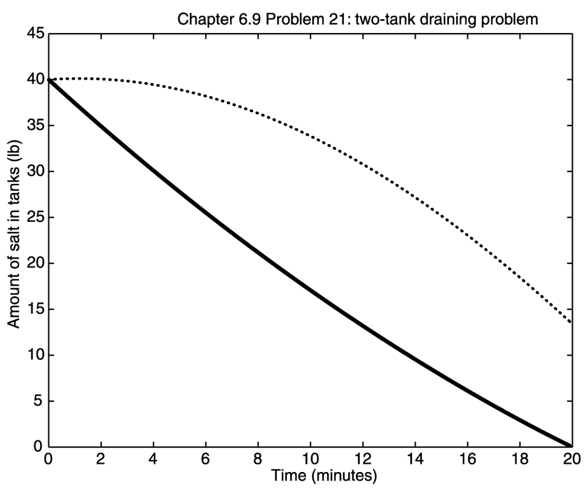
20. Actual error: $\mathbf{y}(1)-\overline{\mathbf{y}}_{200}=\begin{bmatrix} 0.0027112167... \\ -0.0027112167... \end{bmatrix}$

Estimated error: $\overline{\mathbf{y}}_{200}-\mathbf{y}_{100}=\begin{bmatrix} 0.0027202379... \\ -0.0027202379... \end{bmatrix}$

21 (a). $\dfrac{dQ_1}{dt} = -15\dfrac{Q_1}{V_1} + \dfrac{5}{V_2}Q_2 = \dfrac{-15}{200-10t}Q_1 + \dfrac{5}{500-20t}Q_2$

$\dfrac{dQ_2}{dt} = \dfrac{15}{200-10t}Q_1 - \dfrac{35}{500-20t}Q_2$

21 (b).
```
t=0:.01:19.9;
Q1(1)=40;Q2(1)=40;
h=0.01;
V1=200-10*t;
V2=500-20*t;
N=19.9/h;
for i=1:N
Q1(i+1)=Q1(i)+h*(-(15/V1(i))*Q1(i)+(5/V2(i))*Q2(i));
Q2(i+1)=Q2(i)+h*((15/V1(i))*Q1(i)-(35/V2(i))*Q2(i));
end
plot(t,Q1,t,Q2,':')
ylabel('Amount of salt in tanks (lb)')
xlabel('time (minutes)')
title('Chapter 6.9 Problem 21: two-tank draining problem')
```
21 (c).

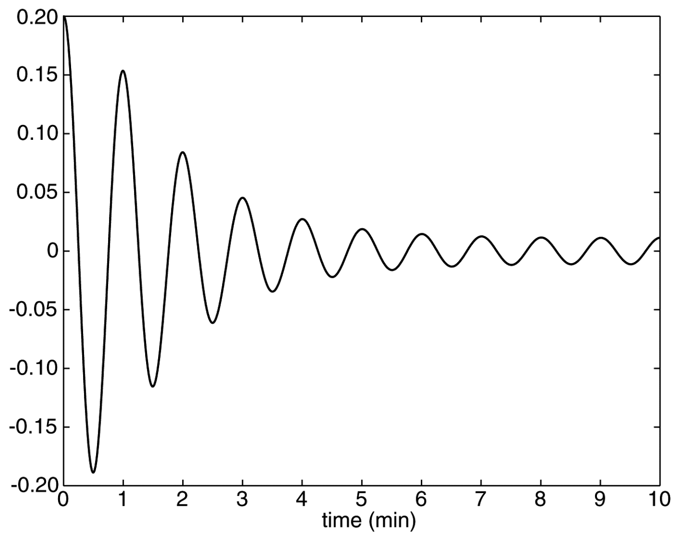


21 (d). The coefficients $\pm\dfrac{15}{200-10t}$ are not continuous at $t = 20$. Therefore, Existence-Uniqueness Theorem 6.1 does not apply to any interval containing $t = 20$.

22 (a). $my'' + \gamma y' + ky = 0,\ m = 1,\ \gamma = 2te^{-\frac{t}{2}},\ k = 4\pi^2,\ y(0) = \dfrac{1}{5}\mathit{meters},\ y'(0) = 0.$

$$\mathbf{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4\pi^2 & -2te^{-\frac{t}{2}} \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \mathbf{y}(0) = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}.$$

22 (b).
```
t=0:.005:10;
h=.005;
N=2000;
y1(1)=0.2;y2(1)=0;
gamma=2*t.*exp(-0.5*t);
k=4*(pi^2);
for i=1:N
y1(i+1)=y1(i)+h*y2(i);
y2(i+1)=y2(i)+h*(-k*y1(i)-gamma(i)*y2(i));
end
plot(t,y1)
```

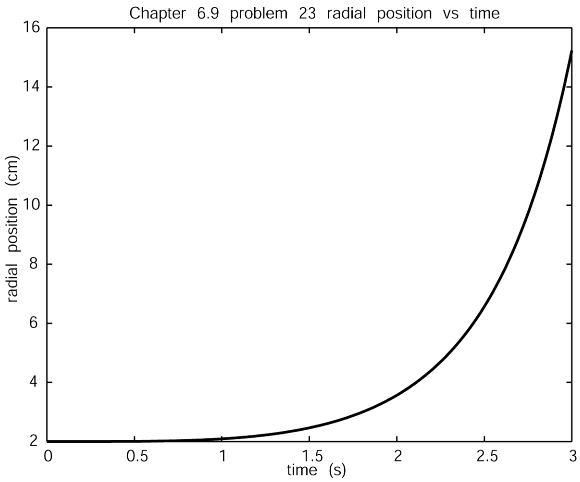


22 (c).

The amplitude of displacement decreases significantly during the time when damping is significant. As damping dimishes, the vibration amplitude seems to settle down to a constant value.

23 (b).
```
t=0:0.01:3;
h=0.01;
N=300;
y1(1)=2;y2(1)=0;
for i=1:N
y1(i+1)=y1(i)+h*y2(i);
y2(i+1)=y2(i)+h*(((pi/4)^2)*(t(i)^2)*y1(i)-0.5*y2(i));
end
plot(t,y1)
xlabel('time (s)');
ylabel('radial position (cm)');
title('Chapter 6.9 problem 23 radial position vs time')
y1(301)
```

23 (c).



$r(3) = 15.2268..cm$

Section 6.10

1. For $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ the characteristic polynomial is $p(\lambda) = \lambda^2 - \lambda - 2$. Eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$. Corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. As in Example 1, we can construct a diagonalizing matrix T from the eigenvectors of A, $T = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. The corresponding matrix of eigenvalues, $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, is such that $T^{-1}AT = D$.

2. The characteristic polynomial is $p(\lambda) = \lambda^2 - 1$. Eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$. Corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Therefore, $T = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

3. For $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ the characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda$. Eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$. Corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. As in Example 1, we can construct a diagonalizing matrix T from the eigenvectors of A, $T = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. The corresponding matrix of eigenvalues, $D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, is such that $T^{-1}AT = D$.

4. The characteristic polynomial is $p(\lambda)=\lambda^2-5\lambda$. Eigenvalues are $\lambda_1=0$ and $\lambda_2=5$. Corresponding eigenvectors are $\mathbf{x}_1=\begin{bmatrix}3\\-2\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\1\end{bmatrix}$. Therefore, $T=\begin{bmatrix}3&1\\-2&1\end{bmatrix}$ and $D=\begin{bmatrix}0&0\\0&5\end{bmatrix}$.

5. For $A=\begin{bmatrix}2&3\\3&2\end{bmatrix}$ the characteristic polynomial is $p(\lambda)=\lambda^2-4\lambda-5$. Eigenvalues are $\lambda_1=-1$ and $\lambda_2=5$. Corresponding eigenvectors are $\mathbf{x}_1=\begin{bmatrix}-1\\1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\1\end{bmatrix}$. As in Example 1, we can construct a diagonalizing matrix T from the eigenvectors of A, $T=\begin{bmatrix}-1&1\\1&1\end{bmatrix}$. The corresponding matrix of eigenvalues, $D=\begin{bmatrix}-1&0\\0&5\end{bmatrix}$, is such that $T^{-1}AT=D$.

6. The characteristic polynomial is $p(\lambda)=\lambda^2-2\lambda-3=(\lambda+1)(\lambda-3)$. Eigenvalues are $\lambda_1=-1$ and $\lambda_2=3$. Corresponding eigenvectors are $\mathbf{x}_1=\begin{bmatrix}1\\-1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\1\end{bmatrix}$. Therefore, $T=\begin{bmatrix}1&1\\-1&1\end{bmatrix}$ and $D=\begin{bmatrix}-1&0\\0&3\end{bmatrix}$.

7. For $A=\begin{bmatrix}2&0\\1&1\end{bmatrix}$ the characteristic polynomial is $p(\lambda)=\lambda^2-3\lambda+2$. Eigenvalues are $\lambda_1=1$ and $\lambda_2=2$. Corresponding eigenvectors are $\mathbf{x}_1=\begin{bmatrix}0\\1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\1\end{bmatrix}$. As in Example 1, we can construct a diagonalizing matrix T from the eigenvectors of A, $T=\begin{bmatrix}0&1\\1&1\end{bmatrix}$. The corresponding matrix of eigenvalues, $D=\begin{bmatrix}1&0\\0&2\end{bmatrix}$, is such that $T^{-1}AT=D$.

8. The characteristic polynomial is $p(\lambda)=\lambda^2-\lambda-6=(\lambda+2)(\lambda-3)$. Eigenvalues are $\lambda_1=-2$ and $\lambda_2=3$. Corresponding eigenvectors are $\mathbf{x}_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}2\\1\end{bmatrix}$. Therefore, $T=\begin{bmatrix}1&2\\0&1\end{bmatrix}$ and $D=\begin{bmatrix}-2&0\\0&3\end{bmatrix}$.

9. For $A=\begin{bmatrix}25&-8&30\\24&-7&30\\-12&4&-14\end{bmatrix}$, the eigenvalues are $\lambda_1=1$ and $\lambda_2=2$. From the characteristic polynomial given, it follows that λ_1 has algebraic multiplicity 2 and λ_2 has algebraic multiplicity 1.

In order to find the eigenvectors corresponding to λ_1, we solve $(A-\lambda_1 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix} 24 & -8 & 30 \\ 24 & -8 & 30 \\ -12 & 4 & -15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This system reduces to $\begin{bmatrix} 12 & -4 & 15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and hence eigenvectors corresponding to λ_1 all have the form

$\mathbf{x}=\begin{bmatrix} (4x_2-15x_3)/12 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} x_2/3 \\ x_2 \\ 0 \end{bmatrix}+\begin{bmatrix} -5x_3/4 \\ 0 \\ x_3 \end{bmatrix}$. Thus, we find two linearly independent eigenvectors, $\mathbf{x}_1=\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix} -5 \\ 0 \\ 4 \end{bmatrix}$ corresponding to λ_1 and therefore, λ_1 has geometric multiplicity 2. Since λ_2 has algebraic multiplicity 1, it also has geometric multiplicity 1. Thus, A is not defective (that is, A is diagonalizable). In order to find the eigenvectors corresponding to λ_2, we solve $(A-\lambda_2 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix} 23 & -8 & 30 \\ 24 & -9 & 30 \\ -12 & 4 & -16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solving this system, we obtain an eigenvector corresponding to λ_2, $\mathbf{x}=\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$. Therefore, if $T=\begin{bmatrix} 1 & -5 & 2 \\ 3 & 0 & 2 \\ 0 & 4 & -1 \end{bmatrix}$, and $D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then $T^{-1}AT=D$.

10. $\lambda_1=-1$ has algebraic multiplicity 1 and $\lambda_2=3$ has algebraic multiplicity 2. The corresponding eigenvectors are $\mathbf{x}=\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ for λ_1 and $\mathbf{x}_1=\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ for λ_2. Therefore, λ_1 has geometric multiplicity 1 and λ_2 has geometric multiplicity 2. A is diagonalizable and $T=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & -2 & 0 \end{bmatrix}$, and $D=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

11. For $A=\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$, the eigenvalues are $\lambda_1=1$ and $\lambda_2=2$. From the characteristic polynomial given, it follows that λ_1 has algebraic multiplicity 2 and λ_2 has algebraic multiplicity 1.

In order to find the eigenvectors corresponding to λ_1, we solve $(A-\lambda_1 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}0&0&1\\2&1&-3\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. This system reduces to $\begin{bmatrix}0&0&1\\2&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ and hence eigenvectors corresponding to λ_1 all have the form $\mathbf{x}=\begin{bmatrix}x_1\\-2x_1\\0\end{bmatrix}=x_1\begin{bmatrix}1\\-2\\0\end{bmatrix}$.

Thus, there is only one linearly independent eigenvector corresponding to λ_1. Therefore, λ_1 has geometric multiplicity 1 and consequently A is defective (not diagonalizable).

12. $\lambda_1=2$ has algebraic multiplicity 2 and $\lambda_2=3$ has algebraic multiplicity 1. The corresponding eigenvectors are $\mathbf{x}=\begin{bmatrix}1\\4\\1\end{bmatrix}$ for λ_1 and $\mathbf{x}=\begin{bmatrix}0\\1\\1\end{bmatrix}$ for λ_2. Therefore, λ_1 has geometric multiplicity 1 and λ_2 has geometric multiplicity1 and A is not diagonalizable.

13. For $A=\begin{bmatrix}4&-1&1\\10&-2&3\\1&0&1\end{bmatrix}$, the only eigenvalue is $\lambda_1=1$. From the characteristic polynomial given, it follows that λ_1 has algebraic multiplicity 3. In order to find the eigenvectors corresponding to λ_1, we solve $(A-\lambda_1 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}3&-1&1\\10&-3&3\\1&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. This system reduces to $\begin{bmatrix}1&0&0\\0&1&-1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ and hence eigenvectors corresponding to λ_1 all have the form $\mathbf{x}=\begin{bmatrix}0\\x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}0\\1\\1\end{bmatrix}$. Thus, there is only one linearly independent eigenvector corresponding to λ_1. Therefore, λ_1 has geometric multiplicity 1 and consequently A is defective (not diagonalizable).

14. All four matrices are diagonalizable.
Matrices (a) and (d) have distinct eigenvalues.
Matrix (b) is a real, symmetric matrix.
Matrix (c) is lower triangular and has distinct eigenvalues.

15. For $A=\begin{bmatrix}6&-6\\2&-1\end{bmatrix}$ the eigenvalues are $\lambda_1=2$ and $\lambda_2=3$ with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}3\\2\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}2\\1\end{bmatrix}$. Make the substitution $\mathbf{y}=T\mathbf{z}=\begin{bmatrix}3&2\\2&1\end{bmatrix}\mathbf{z}$ to obtain $T\mathbf{z}'=AT\mathbf{z}+\mathbf{g}(t)$.

Multiplying by T^{-1} gives $\mathbf{z}' = T^{-1}AT\mathbf{z} + T^{-1}\mathbf{g}(t)$ or

$\mathbf{z}' = D\mathbf{z} + T^{-1}\mathbf{g}(t) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 4+3e^t \\ 2+2e^t \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} e^t \\ 2 \end{bmatrix}$. Thus, the system uncouples into $\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} 2z_1 + e^t \\ 3z_2 + 2 \end{bmatrix}$. Solving these uncoupled first order equations, we obtain $\mathbf{z} = \begin{bmatrix} -e^t + c_1e^{2t} \\ -(2/3) + c_2e^{3t} \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the general solution

$$\mathbf{y} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -e^t + c_1e^{2t} \\ -(2/3) + c_2e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{2t} & 2e^{3t} \\ 2e^{2t} & e^{3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} 3e^t + 4/3 \\ 2e^t + 2/3 \end{bmatrix}.$$

16. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Make the substitution $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\mathbf{z}$.

$$\mathbf{z}' = D\mathbf{z} + T^{-1}\mathbf{g}(t) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} e^{2t} - 2e^t \\ -e^{2t} + e^t \end{bmatrix} = \begin{bmatrix} -e^t \\ e^{2t} \end{bmatrix}.$$

Solving these first order equations, we obtain $\mathbf{z} = \begin{bmatrix} -(1/2)e^t + c_1e^{-t} \\ te^{2t} + c_2e^{2t} \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the solution $\mathbf{y} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -(1/2)e^t + c_1e^{-t} \\ te^{2t} + c_2e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} & e^{2t} \\ -e^{-t} & -e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -e^t + te^{2t} \\ \frac{1}{2}e^t - te^{2t} \end{bmatrix}$.

17. For $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Make the substitution $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\mathbf{z}$ to obtain $T\mathbf{z}' = AT\mathbf{z} + \mathbf{g}(t)$.

Multiplying by T^{-1} gives $\mathbf{z}' = T^{-1}AT\mathbf{z} + T^{-1}\mathbf{g}(t)$ or

$$\mathbf{z}' = D\mathbf{z} + T^{-1}\mathbf{g}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} t \\ 3-t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} t-1 \\ 1 \end{bmatrix}.$$

Thus, the system uncouples into $\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} t-1 \\ 3z_2 + 1 \end{bmatrix}$. Solving these uncoupled first order equations, we obtain $\mathbf{z} = \begin{bmatrix} (1/2)t^2 - t + c_1 \\ -(1/3) + c_2e^{3t} \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the solution

$$\mathbf{y} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (1/2)t^2 - t + c_1 \\ -(1/3) + c_2e^{3t} \end{bmatrix} = \begin{bmatrix} 1 & e^{3t} \\ -1 & 2e^{3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} (1/2)t^2 - t - (1/3) \\ -(1/2)t^2 + t - (2/3) \end{bmatrix}.$$

18. The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 5$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Make the substitution $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{z}$.

$\mathbf{z}' = D\mathbf{z} + T^{-1}\mathbf{g}(t) = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\mathbf{z} + \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 4t+4 \\ -2t+1 \end{bmatrix} = \begin{bmatrix} 2t+1 \\ 2 \end{bmatrix}$.

Solving these first order equations, we obtain $\mathbf{z} = \begin{bmatrix} -t-1+c_1e^{2t} \\ -\frac{2}{5}+c_2e^{5t} \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the solution $\mathbf{y} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -t-1+c_1e^{2t} \\ -\frac{2}{5}+c_2e^{5t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} & e^{5t} \\ -e^{2t} & e^{5t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -2t-\frac{12}{5} \\ t+\frac{3}{5} \end{bmatrix}$.

19. For $A = \begin{bmatrix} -9 & -5 \\ 8 & 4 \end{bmatrix}$ the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -4$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Make the substitution $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 5 & 1 \\ -8 & -1 \end{bmatrix}\mathbf{z}$ to obtain $T\mathbf{z}'' = AT\mathbf{z}$.

Multiplying by T^{-1} gives $\mathbf{z}'' = T^{-1}AT\mathbf{z}$ or $\mathbf{z}'' = D\mathbf{z} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\mathbf{z}$. Thus, the system uncouples into $\begin{bmatrix} z_1'' \\ z_2'' \end{bmatrix} = \begin{bmatrix} -z_1 \\ -4z_2 \end{bmatrix}$.

Solving these uncoupled equations, we obtain $\mathbf{z} = \begin{bmatrix} c_1\cos t + d_1\sin t \\ c_2\cos 2t + d_2\sin 2t \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the solution

$$\mathbf{y} = \begin{bmatrix} 5 & 1 \\ -8 & -1 \end{bmatrix}\begin{bmatrix} c_1\cos t + d_1\sin t \\ c_2\cos 2t + d_2\sin 2t \end{bmatrix} = \begin{bmatrix} 5(c_1\cos t + d_1\sin t) + c_2\cos 2t + d_2\sin 2t \\ -8(c_1\cos t + d_1\sin t) - c_2\cos 2t - d_2\sin 2t \end{bmatrix}.$$

20. The eigenvalues are $\lambda_1 = -9$ and $\lambda_2 = -1$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 7 \\ -15 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Make the substitution $\mathbf{x} = T\mathbf{z} = \begin{bmatrix} 7 & 1 \\ -15 & -1 \end{bmatrix}\mathbf{z}$ to obtain $\mathbf{z}'' + \begin{bmatrix} -9 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{z} = \mathbf{0}$. Solving the equations, we obtain $\mathbf{z} = \begin{bmatrix} c_1e^{-3t} + c_2e^{3t} \\ k_1e^{-t} + k_2e^{t} \end{bmatrix}$. Finally, we obtain the solution $\mathbf{x} = \begin{bmatrix} 7 & 1 \\ -15 & -1 \end{bmatrix}\begin{bmatrix} c_1e^{-3t} + c_2e^{3t} \\ k_1e^{-t} + k_2e^{t} \end{bmatrix} = \begin{bmatrix} 7(c_1e^{-3t} + c_2e^{3t}) + k_1e^{-t} + k_2e^{t} \\ -15(c_1e^{-3t} + c_2e^{3t}) - (k_1e^{-t} + k_2e^{t}) \end{bmatrix}$.

21. For $A = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$ the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Make the substitution $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}\mathbf{z}$ to obtain $T\mathbf{z}'' = AT\mathbf{z}$.

Multiplying by T^{-1} gives $\mathbf{z}'' = T^{-1}AT\mathbf{z}$ or $\mathbf{z}'' = D\mathbf{z} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{z}$. Thus, the system uncouples into $\begin{bmatrix} z_1'' \\ z_2'' \end{bmatrix} = \begin{bmatrix} -z_1 \\ z_2 \end{bmatrix}$.

Solving these uncoupled equations, we obtain $\mathbf{z} = \begin{bmatrix} c_1\cos t + d_1\sin t \\ c_2e^{-t} + d_2e^{t} \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the solution $\mathbf{y} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} c_1\cos t + d_1\sin t \\ c_2e^{-t} + d_2e^{t} \end{bmatrix} = \begin{bmatrix} c_1\cos t + d_1\sin t + c_2e^{-t} + d_2e^{t} \\ -(c_1\cos t + d_1\sin t) - 3(c_2e^{-t} + d_2e^{t}) \end{bmatrix}$.

22. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 5$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Make the substitution $\mathbf{x} = T\mathbf{z} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\mathbf{z}$ to obtain $\mathbf{z}'' + \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}\mathbf{z} = \mathbf{0}$.

Solving the equations, we obtain $\mathbf{z} = \begin{bmatrix} c_1t + c_2 \\ k_1\cos(\sqrt{5}t) + k_2\sin(\sqrt{5}t) \end{bmatrix}$. Finally, we obtain the solution

$$\mathbf{x} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} c_1t + c_2 \\ k_1\cos(\sqrt{5}t) + k_2\sin(\sqrt{5}t) \end{bmatrix} = \begin{bmatrix} (c_1t + c_2) + 2\left[k_1\cos(\sqrt{5}t) + k_2\sin(\sqrt{5}t)\right] \\ -2(c_1t + c_2) + \left[k_1\cos(\sqrt{5}t) + k_2\sin(\sqrt{5}t)\right] \end{bmatrix}.$$

27 (a). For $A = \begin{bmatrix} 500 & -200 \\ -200 & 200 \end{bmatrix}$ the eigenvalues are $\lambda_1 = 100$ and $\lambda_2 = 600$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

27 (b). Make the substitution $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\mathbf{z}$ to obtain $T\mathbf{z}'' + AT\mathbf{z} = \mathbf{0}$. Multiplying by T^{-1} gives $\mathbf{z}'' + T^{-1}AT\mathbf{z} = \mathbf{0}$ or $\mathbf{z}'' + \begin{bmatrix} 100 & 0 \\ 0 & 600 \end{bmatrix}\mathbf{z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus, the system uncouples into $\begin{bmatrix} z_1'' + 100z_1 \\ z_2'' + 600z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The initial condition is $\mathbf{z}(0) = T^{-1}\mathbf{y}(0) = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & -0.2 \end{bmatrix}\begin{bmatrix} 0.1 \\ 0.15 \end{bmatrix} = \begin{bmatrix} 0.08 \\ 0.01 \end{bmatrix}$.

27 (c). Solving the uncoupled equations $\mathbf{z}'' + D\mathbf{z} = \mathbf{0}$, we obtain $\mathbf{z} = \begin{bmatrix} c_1\cos 10t + d_1\sin 10t \\ c_2\cos 10\sqrt{6}t + d_2\sin 10\sqrt{6}t \end{bmatrix}$.

Imposing the initial condition, we find $\mathbf{z} = \begin{bmatrix} 0.08\cos 10t \\ 0.01\cos 10\sqrt{6}t \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the solution of the initial value problem:

$$\mathbf{y} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0.08\cos 10t \\ 0.01\cos 10\sqrt{6}t \end{bmatrix} = \begin{bmatrix} 0.08\cos 10t + 0.02\cos\left(10\sqrt{6}t\right) \\ 0.16\cos 10t - 0.01\cos\left(10\sqrt{6}t\right) \end{bmatrix}.$$

Section 6.11

1 (a). We proceed as in Example 2. For $A = \begin{bmatrix} 5 & -4 \\ 5 & -4 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = \lambda^2 - \lambda$. Eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 1$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since A is diagonalizable, we obtain from equation (7) $e^{tA} = T\Lambda(t)T^{-1}$ where $T = [\mathbf{x}_1, \mathbf{x}_2] = \begin{bmatrix} 4 & 1 \\ 5 & 1 \end{bmatrix}$ and $\Lambda(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}$. Thus,

$$\Phi(t) = e^{tA} = \begin{bmatrix} 4 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} -4+5e^t & 4-4e^t \\ -5+5e^t & 5-4e^t \end{bmatrix}.$$

1 (b). The solution of $\mathbf{y}' = A\mathbf{y}, \mathbf{y}(-1) = \mathbf{y}_0$ is given by $\mathbf{y}(t) = e^{(t+1)A}\mathbf{y}_0$. Therefore,

$$\mathbf{y}(2) = e^{(2+1)A}\mathbf{y}_0 = e^{3A}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4+5e^3 & 4-4e^3 \\ -5+5e^3 & 5-4e^3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4+5e^3 \\ -5+5e^3 \end{bmatrix}.$$

2 (a). The characteristic polynomial is $p(\lambda) = (\lambda - 2)^2$. Eigenvalues are $\lambda_1 = \lambda_2 = 2$ with corresponding eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Therefore,

$$\mathbf{y}_1(t) = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}. \text{ Let } \mathbf{y}_2(t) = e^{2t}(t\xi + 7),\ \xi = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ (A - 2\mathbf{I})\eta = \xi \Rightarrow \eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \mathbf{y}_2(t) = e^{2t}\begin{bmatrix} t \\ 1 \end{bmatrix}.$$

$$\Psi(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} = \Phi(t) \text{ since } \Psi(0) = \mathbf{I}.$$

2 (b). $\mathbf{y}(2) = \Phi(1)\mathbf{y}(1) = \begin{bmatrix} e^2 & e^2 \\ 0 & e^2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3e^2 \\ 2e^2 \end{bmatrix}$.

3 (a). We proceed as in Example 2. For $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = \lambda^2 - 8\lambda + 7$. Eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 7$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. Since A is diagonalizable, we obtain from equation (7) $e^{tA} = T\Lambda(t)T^{-1}$ where $T = [\mathbf{x}_1, \mathbf{x}_2] = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}$ and $\Lambda(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & e^{7t} \end{bmatrix}$. Thus,

$$\Phi(t) = e^{tA} = \begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{7t} \end{bmatrix}\begin{bmatrix} 1/6 & -5/6 \\ 1/6 & 1/6 \end{bmatrix} = (1/6)\begin{bmatrix} e^t + 5e^{7t} & -5e^t + 5e^{7t} \\ -e^t + e^{7t} & 5e^t + e^{7t} \end{bmatrix}.$$

3 (b). The solution of $\mathbf{y}' = A\mathbf{y}, \mathbf{y}(0) = \mathbf{y}_0$ is given by $\mathbf{y}(t) = e^{tA}\mathbf{y}_0$. Therefore,

$$\mathbf{y}(-1) = e^{(-1)A}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1/6)\begin{bmatrix} e^{-1} + 5e^{-7} & -5e^{-1} + 5e^{-7} \\ -e^{-1} + e^{-7} & 5e^{-1} + e^{-7} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1/6)\begin{bmatrix} -4e^{-1} + 10e^{-7} \\ 4e^{-1} + 2e^{-7} \end{bmatrix}.$$

4 (a). The characteristic polynomial is $p(\lambda)=(1-\lambda)(2-\lambda)(-1-\lambda)$. Eigenvalues are $\lambda_1=-1,\ \lambda_2=1,\ \lambda_3=2$ with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\1\\-3\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \mathbf{x}_3=\begin{bmatrix}1\\1\\0\end{bmatrix}$.

Therefore, $\Psi(t)=\begin{bmatrix}e^{-t} & e^{t} & e^{2t}\\ e^{-t} & 0 & e^{2t}\\ -3e^{-t} & 0 & 0\end{bmatrix},\ \Psi(0)=\begin{bmatrix}1&1&1\\1&0&1\\-3&0&0\end{bmatrix}\Rightarrow\Psi^{-1}(0)=\begin{bmatrix}0&0&-\frac{1}{3}\\1&-1&0\\0&1&\frac{1}{3}\end{bmatrix}$

and $\Phi(t)=\begin{bmatrix}e^{-t} & e^{t} & e^{2t}\\ e^{-t} & 0 & e^{2t}\\ -3e^{-t} & 0 & 0\end{bmatrix}\begin{bmatrix}0&0&-\frac{1}{3}\\1&-1&0\\0&1&\frac{1}{3}\end{bmatrix}=\begin{bmatrix}e^{t} & -e^{t}+e^{2t} & \frac{1}{3}(-e^{-t}+e^{2t})\\ 0 & e^{2t} & \frac{1}{3}(-e^{-t}+e^{2t})\\ 0 & 0 & e^{-t}\end{bmatrix}$

4 (b). $\mathbf{y}(1)=\Phi(1)\mathbf{y}(0)=\Phi(1)\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}e^2\\e^2\\0\end{bmatrix}$.

5 (a). From Theorem 6.15, $\Phi(t,s)=\Psi(t)\Psi^{-1}(s)=\begin{bmatrix}t & t^2\\1 & 2t\end{bmatrix}\begin{bmatrix}2s^{-1} & -1\\-s^{-2} & s^{-1}\end{bmatrix}$, and thus

$\Phi(t,s)=\begin{bmatrix}2s^{-1}t-s^{-2}t^2 & -t+t^2s^{-1}\\ 2s^{-1}-2s^{-2}t & -1+2ts^{-1}\end{bmatrix}$; $\Phi(t,s)$ is not a function of $t-s$.

5 (b). From Theorem 6.15, $\mathbf{y}(3)=\Phi(3,1)\mathbf{y}(1)=\begin{bmatrix}6-9 & -3+9\\2-6 & -1+6\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-9\\-9\end{bmatrix}$.

6. $B=T^{-1}p(A)T=T^{-1}(2A^3-A+3I)T=2T^{-1}A^3T-T^{-1}AT+3T^{-1}T=2D^3-D+3I$.
Therefore, $B=\begin{bmatrix}2\lambda_1^3-\lambda_1+3 & 0\\ 0 & 2\lambda_2^3-\lambda_2+3\end{bmatrix}$.

9 (a). As we saw in equation (6), if $T^{-1}AT=D$ then $A^n=TD^nT^{-1}$. (For this present case, $T=\begin{bmatrix}3&2\\4&3\end{bmatrix}$ and $D=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$.) Since $A^n=T\begin{bmatrix}1&0\\0&-1\end{bmatrix}^nT^{-1}$ and since $\begin{bmatrix}1&0\\0&-1\end{bmatrix}^n=I$ when n is even, it follows that $A^n=I$.

9 (b). $A^n=TD^nT^{-1}=T\begin{bmatrix}1&0\\0&-1\end{bmatrix}^nT^{-1}=TDT^{-1}=A$ when n is odd.

9 (c). As in parts (a) and (b), we see that $A^{-n}=I$ when n is even and $A^{-n}=A$ when n is odd.

10. $A=T\begin{bmatrix}1&0\\0&-1\end{bmatrix}T^{-1}$. The four matrices are: $D=\begin{bmatrix}\pm1&0\\0&\pm i\end{bmatrix}$

$D_1=\begin{bmatrix}1&0\\0&i\end{bmatrix},\ D_2=\begin{bmatrix}-1&0\\0&i\end{bmatrix},\ D_3=\begin{bmatrix}1&0\\0&-i\end{bmatrix},\ D_4=\begin{bmatrix}-1&0\\0&-i\end{bmatrix}$.

11. For the given matrix, $A^{-1}=A$. Thus, if $B=A^{1/2}$, then $B^2=A=A^{-1}$ as requested. Exercise 10 asks for four different square roots of A and any one of these will serve as B.

12. $A^2+A^{\frac{1}{2}}=TBT^{-1}\Rightarrow B=T^{-1}A^2T+T^{-1}A^{\frac{1}{2}}T$. Since $A^2=I,\ B=I+D=\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}\pm1&0\\0&\pm i\end{bmatrix}$.

13. Since $A = TDT^{-1}$, it follows that $A^3 = TD^3T^{-1} = \begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix}\begin{bmatrix}-2 & 0\\ 0 & 2\end{bmatrix}^3\begin{bmatrix}-1 & 1\\ 2 & -1\end{bmatrix} = \begin{bmatrix}24 & -16\\ 32 & -24\end{bmatrix}$.

14. $f_1(A) = \cos(\pi A) = T\begin{bmatrix}\cos(\pi\lambda_1) & 0\\ 0 & \cos(\pi\lambda_2)\end{bmatrix}T^{-1}$. $T = \begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}$, $T^{-1} = \begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}$.

$\cos(\pi\lambda_1) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\cos(\pi\lambda_2) = \cos\left(\frac{\pi}{2}\right) = 0$. Therefore,

$f_1(A) = \cos(\pi A) = \begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix} = \begin{bmatrix}3\sqrt{2} & -\sqrt{2}\\ \frac{15}{2}\sqrt{2} & -\frac{5}{2}\sqrt{2}\end{bmatrix}$.

$\sin(\pi\lambda_1) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\sin(\pi\lambda_2) = \sin\left(\frac{\pi}{2}\right) = 1$. Therefore,

$f_2(A) = \sin(\pi A) = \begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix} = \begin{bmatrix}3\sqrt{2}-5 & -\sqrt{2}+2\\ \frac{15}{2}\sqrt{2}-15 & -\frac{5}{2}\sqrt{2}+6\end{bmatrix}$.

15. As we saw following Theorem 6.16, $\cos(tA) = T\begin{bmatrix}\cos\lambda_1 t & 0\\ 0 & \cos\lambda_2 t\end{bmatrix}T^{-1}$ when A is a (2×2) diagonalizable matrix with eigenvalues λ_1 and λ_2. Thus, with $t = \pi$ and the given eigenvalues, we have

$$\cos(\pi A) = T\begin{bmatrix}\cos(\pi/3) & 0\\ 0 & \cos(7\pi/3)\end{bmatrix}T^{-1} = T\begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix}T^{-1}$$
$$= (1/2)T\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}T^{-1} = (1/2)I.$$

Similarly, we find $\sin(\pi A) = (\sqrt{3}/2)I$.

16. Let $T = \begin{bmatrix}1 & 1\\ -2 & -1\end{bmatrix}$. Make the substitution $\mathbf{y} = T\mathbf{z}$. Premultiplying by $T^{-1} = \begin{bmatrix}-1 & -1\\ 2 & 1\end{bmatrix}$ gives $\mathbf{z}'' + D\mathbf{z} = T^{-1}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}-1\\ 2\end{bmatrix}$, $D = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$. The solution is $\mathbf{z}(t) = \begin{bmatrix}c_1e^{-t} + c_2e^{t} + 1\\ k_1\cos t + k_2\sin t + 2\end{bmatrix}$. Converting to the original variables, we obtain

$\mathbf{y}(t) = T\mathbf{z}(t) = \begin{bmatrix}1 & 1\\ -2 & -1\end{bmatrix}\begin{bmatrix}c_1e^{-t} + c_2e^{t} + 1\\ k_1\cos t + k_2\sin t + 2\end{bmatrix} = \begin{bmatrix}c_1e^{-t} + c_2e^{t} + k_1\cos t + k_2\sin t + 3\\ -2c_1e^{-t} - 2c_2e^{t} - k_1\cos t - k_2\sin t - 4\end{bmatrix}$.

17. Let $T = \begin{bmatrix}1 & 1\\ -2 & -1\end{bmatrix}$. Making the substitution $\mathbf{y} = T\mathbf{z}$, the system becomes $AT\mathbf{z}' + T\mathbf{z} = \begin{bmatrix}1\\ 1\end{bmatrix}$. Premultiplying by $T^{-1} = \begin{bmatrix}-1 & -1\\ 2 & 1\end{bmatrix}$ gives $T^{-1}AT\mathbf{z}' + \mathbf{z} = \begin{bmatrix}-2\\ 3\end{bmatrix}$ or $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\mathbf{z}' + \mathbf{z} = \begin{bmatrix}-2\\ 3\end{bmatrix}$. Thus, the system uncouples into $\begin{bmatrix}-z_1' + z_1\\ z_2' + z_2\end{bmatrix} = \begin{bmatrix}-2\\ 3\end{bmatrix}$. The solution is $\mathbf{z}(t) = \begin{bmatrix}c_1e^{t} - 2\\ c_2e^{-t} + 3\end{bmatrix}$. Converting to the original variables, we obtain $\mathbf{y} = T\mathbf{z} = \begin{bmatrix}1 & 1\\ -2 & -1\end{bmatrix}\begin{bmatrix}c_1e^{t} - 2\\ c_2e^{-t} + 3\end{bmatrix} = \begin{bmatrix}c_1e^{t} + c_2e^{-t} + 1\\ -2c_1e^{t} - c_2e^{-t} + 1\end{bmatrix}$.

18. Make the substitution $\mathbf{y} = T\mathbf{z}$. $\mathbf{z}'' + \mathbf{z}' + D\mathbf{z} = \mathbf{0}$.. The solution is

$$\mathbf{z}(t) = \begin{bmatrix} c_1 e^{\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)t} + c_2 e^{\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t} \\ k_1 e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) + k_2 e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{bmatrix}.$$ Converting to the original variables, we obtain

$$\mathbf{y}(t) = T\mathbf{z}(t) = \begin{bmatrix} c_1 e^{\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)t} + c_2 e^{\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t} + k_1 e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) + k_2 e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right) \\ -2c_1 e^{\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)t} - 2c_2 e^{\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t} - k_1 e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) - k_2 e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right) \end{bmatrix}.$$

19. Let $T = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$. Making the substitution $\mathbf{y} = T\mathbf{z}$, the system becomes $T\mathbf{z}'' + 2AT\mathbf{z}' = \mathbf{0}$.

Premultiplying by $T^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ gives $\mathbf{z}'' + 2T^{-1}AT\mathbf{z}' = \mathbf{0}$ or $\mathbf{z}'' + 2\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{z}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus, the system uncouples into $\begin{bmatrix} z_1'' - 2z_1' \\ z_2'' + 2z_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The solution is $\mathbf{z}(t) = \begin{bmatrix} c_1 + c_2 e^{2t} \\ d_1 + d_2 e^{-2t} \end{bmatrix}$. Converting to the original variables, we obtain

$$\mathbf{y}(t) = T\mathbf{z}(t) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} c_1 + c_2 e^{2t} \\ d_1 + d_2 e^{-2t} \end{bmatrix} = \begin{bmatrix} c_1 + c_2 e^{2t} + d_1 + d_2 e^{-2t} \\ -2(c_1 + c_2 e^{2t}) - (d_1 + d_2 e^{-2t}) \end{bmatrix}.$$

20 (a). $m_1 x_1'' = k_1(x_2 - x_1)$; $m_2 x_2'' = k_2(x_3 - x_2) - k_1(x_2 - x_1)$; $m_3 x_3'' = -k_2(x_3 - x_2)$. Therefore, $m_1 x_1'' + k_1(x_1 - x_2) = 0$; $m_2 x_2'' - k_1 x_1 + (k_1 + k_2)x_2 - k_2 x_3 = 0$; $m_3 x_3'' - k_2 x_2 + k_2 x_3 = 0$. The result follows.

20 (b). $K\mathbf{v}_0 = \mathbf{0}$, where $\mathbf{v}_0$ is any nonzero multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Therefore, $0, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenpair.

20 (c). Let $\mathbf{x} = f(t)\mathbf{v}_0$. $M\mathbf{x}'' + K\mathbf{x} = M(f''(t)\mathbf{v}_0) + Kf(t)\mathbf{v}_0 = \mathbf{0}$. Therefore, since $K(f(t)\mathbf{v}_0) = f(t)K\mathbf{v}_0 = \mathbf{0}$, $Mf''(t)\mathbf{v}_0 = \mathbf{0}$ *or* $m_j f''(t) = 0$, $j = 1,2,3 \Rightarrow f''(t) = 0$. Therefore, $f(t) = c_1 t + c_2$ and $\mathbf{x}(t) = (c_1 t + c_2)\mathbf{v}_0$. $\mathbf{x}(0) = c_2\mathbf{v}_0 = \mathbf{0} \Rightarrow c_2 = 0$, $\mathbf{x}(t) = c_1\mathbf{v}_0 = \mathbf{v}_0 \Rightarrow c_1 = 1$. Therefore, $\mathbf{x}(t) = t\mathbf{v}_0$. The system is executing motion at constant velocity $\mathbf{v}_0$. There is no relative motion; the three-mass system is translating like a rigid body.

21 (a). For this case, we have $A = M^{-1}K = \frac{k}{m}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. Using MATLAB, we find the eigenvalues of $B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ are $\gamma_1 = 0, \gamma_2 = 1,$ and $\gamma_3 = 3$ with corresponding eigenvectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$ and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Since $A = (k/m)B$, the eigenvalues of A are multiples of k/m times the eigenvalues of B while corresponding eigenvectors can be chosen to be the same as those of B.

21 (b). Making the substitution $\mathbf{x} = T\mathbf{z}$, the system becomes $T\mathbf{z}'' + AT\mathbf{z} = \mathbf{0}$. Premultiplying by gives $\mathbf{z}'' + T^{-1}AT\mathbf{z} = \mathbf{0}$ or $\mathbf{z}'' + \begin{bmatrix} 0 & 0 & 0 \\ 0 & km^{-1} & 0 \\ 0 & 0 & 3km^{-1} \end{bmatrix}\mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Thus, the system uncouples into $\begin{bmatrix} z_1'' \\ z_2'' + km^{-1}z_2 \\ z_3'' + 3km^{-1}z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The solution is $\mathbf{z}(t) = \begin{bmatrix} c_1 t + c_2 \\ d_1 \cos\omega t + d_2 \sin\omega t \\ e_1 \cos\sqrt{3}\omega t + e_2 \sin\sqrt{3}\omega t \end{bmatrix}$, where $\omega = \sqrt{km^{-1}}$.
Converting to the original variables, we obtain

$$\mathbf{x}(t) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} c_1 t + c_2 \\ d_1 \cos\omega t + d_2 \sin\omega t \\ e_1 \cos\sqrt{3}\omega t + e_2 \sin\sqrt{3}\omega t \end{bmatrix}$$

$$= \begin{bmatrix} c_1 t + c_2 + d_1 \cos\omega t + d_2 \sin\omega t + e_1 \cos\sqrt{3}\omega t + e_2 \sin\sqrt{3}\omega t \\ c_1 t + c_2 - 2(e_1 \cos\sqrt{3}\omega t + e_2 \sin\sqrt{3}\omega t) \\ c_1 t + c_2 - (d_1 \cos\omega t + d_2 \sin\omega t) + e_1 \cos\sqrt{3}\omega t + e_2 \sin\sqrt{3}\omega t \end{bmatrix}.$$