Chapter 7
Laplace Transforms

Section 7.1

1. $\mathcal{L}\{1\} = \lim_{T\to\infty}\int_0^T 1\cdot e^{-st}dt = \lim_{T\to\infty}\left.\frac{-e^{-st}}{s}\right|_0^T = \lim_{T\to\infty}\frac{1}{s}\left(1-e^{-sT}\right) = \frac{1}{s},\ s>0$

2. $\mathcal{L}\{e^{3t}\} = \lim_{T\to\infty}\int_0^T e^{3t}\cdot e^{-st}dt = \lim_{T\to\infty}\int_0^T e^{-(s-3)t}dt = \lim_{T\to\infty}-\frac{1}{s-3}e^{-(s-3)t}\Big|_0^T = \lim_{T\to\infty}\frac{1}{s-3}\left(1-e^{-(s-3)T}\right)$

   $= \frac{1}{s-3},\ s>3.$

3. $\mathcal{L}\{te^{-t}\} = \lim_{T\to\infty}\int_0^T te^{-t}\cdot e^{-st}dt = \lim_{T\to\infty}\int_0^T te^{-(s+1)t}dt$. For integration by parts, we will use

   $u=t,\ du=dt,\ dv=e^{-(s+1)t}dt,$ and $v=-\frac{e^{-(s+1)t}}{s+1}$. Then we have

   $\lim_{T\to\infty}\int_0^T te^{-(s+1)t}dt = \lim_{T\to\infty}\left\{\left.\frac{-te^{-(s+1)t}}{s+1}\right|_0^T + \frac{1}{s+1}\int_0^T e^{-(s+1)t}dt\right\} = \lim_{T\to\infty}\left\{\frac{-Te^{-(s+1)T}}{s+1} + \frac{1}{(s+1)^2}\left(1-e^{-(s+1)T}\right)\right\}$

   $= \frac{1}{(s+1)^2},\ s>-1.$

4. $\mathcal{L}\{t-5\} = \lim_{T\to\infty}\int_0^T (t-5)\cdot e^{-st}dt$. For integration by parts, we will use

   $u=t-5,\ du=dt,\ dv=e^{-st}dt,$ and $v=-\frac{e^{-st}}{s}$. Then we have

   $\lim_{T\to\infty}\int_0^T (t-5)e^{-st}dt = \lim_{T\to\infty}\left\{\left.\frac{-(t-5)e^{-st}}{s}\right|_0^T + \frac{1}{s}\int_0^T e^{-st}dt\right\} = \lim_{T\to\infty}\left\{\frac{-(T-5)e^{-sT}+5}{s} - \left.\frac{e^{-st}}{s^2}\right|_0^T\right\}$

   $= \frac{1}{s^2}-\frac{5}{s},\ s>0.$

5. $\mathcal{L}\{f(t)\}$ does not exist because $\lim_{t\to\infty} te^{t\sqrt{t}}e^{-st} = \infty$ for all $s>0$.

6. $\mathcal{L}\{f(t)\}$ does not exist because $\lim_{t\to\infty} e^{(t-1)^2}e^{-st} = \infty$ for all $s>0$.

7. $\mathcal{L}\{|t-1|\} = \underbrace{\int_0^1 (1-t)e^{-st}dt}_{(1)} + \underbrace{\int_1^\infty (t-1)e^{-st}dt}_{(2)}$.

(1): $u = 1-t,\ du = -dt,\ dv = e^{-st}dt,\ v = -\frac{e^{-st}}{s}$. Then we have

$$(1) = -(1-t)\frac{e^{-st}}{s}\Big|_0^1 - \int_0^1 \frac{e^{-st}}{s}dt = \frac{1}{s} + \frac{1}{s^2}e^{-st}\Big|_0^1 = \frac{1}{s} + \frac{1}{s^2}\left(e^{-s}-1\right)$$

(2): $u = t-1,\ du = dt,\ dv = e^{-st}dt,\ v = -\frac{e^{-st}}{s}$. Then we have

$$(2) = -\frac{t-1}{s}e^{-st}\Big|_1^\infty + \frac{1}{s}\int_1^\infty e^{-st}dt = \frac{e^{-s}}{s^2}$$

$(1)+(2) = \frac{1}{s} - \frac{1}{s^2} + \frac{2}{s^2}e^{-s},\ s > 0$.

8. $\mathcal{L}\{(t-2)^2\} = \lim_{T\to\infty}\int_0^T (t-2)^2 \cdot e^{-st}dt$. Using

$u = (t-2)^2,\ du = 2(t-2)dt,\ dv = e^{-st}dt,$ and $v = -\frac{e^{-st}}{s},$ we have

$\lim_{T\to\infty}\int_0^T (t-2)^2 \cdot e^{-st}dt = \lim_{T\to\infty}\left\{\frac{-(t-2)^2e^{-st}}{s}\Big|_0^T + \frac{2}{s}\int_0^T (t-2)e^{-st}dt\right\} = \frac{4}{s} + \frac{2}{s}\lim_{T\to\infty}\int_0^T (t-2)e^{-st}dt$. Using

parts with $u = t-2,\ du = dt,\ dv = e^{-st}dt,$ and $v = -\frac{e^{-st}}{s}$, we have

$$\lim_{T\to\infty}\int_0^T (t-2)^2 \cdot e^{-st}dt = \frac{4}{s} + \frac{2}{s}\lim_{T\to\infty}\left\{\left(\frac{-(t-2)e^{-st}}{s} + \frac{1}{s^2}e^{-st}\right)\Big|_0^T\right\} = \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3},\ s > 0.$$

9. $\mathcal{L}\{f(t)\} = \lim_{T\to\infty}\int_1^T 1\cdot e^{-st}dt = \lim_{T\to\infty}\frac{-e^{-st}}{s}\Big|_1^T = \lim_{T\to\infty}\frac{1}{s}\left(e^{-s} - e^{-sT}\right) = \frac{e^{-s}}{s},\ s > 0$.

10. $\mathcal{L}\{f(t)\} =$

$$\lim_{T\to\infty}\int_1^T (t-1)\cdot e^{-st}dt = \lim_{T\to\infty}\left\{\frac{-(t-1)e^{-st}}{s}\Big|_1^T + \frac{1}{s}\int_1^T e^{-st}dt\right\} = \lim_{T\to\infty}\left\{-\frac{(T-1)e^{-sT}}{s} + \frac{e^{-s}-e^{-sT}}{s^2}\right\}$$

$= \frac{e^{-s}}{s^2},\ s > 0.$

11. $\mathcal{L}\{f(t)\} = \int_1^2 1\cdot e^{-st}dt = \frac{-e^{-st}}{s}\Big|_1^2 = \frac{1}{s}\left(e^{-s} - e^{-2s}\right),\ s \neq 0;\ = 1,\ s = 0$.

12. $\mathcal{L}\{f(t)\} = \int_1^2 (t-1)\cdot e^{-st}dt = \left.\frac{-(t-1)e^{-st}}{s}\right|_1^2 + \frac{1}{s}\int_1^2 e^{-st}dt = -\frac{e^{-2s}}{s} + \frac{1}{s^2}\left(e^{-s} - e^{-2s}\right),\ s \neq 0.$

13. $\int t^n e^{-st}dt$. $u = t^n$, $du = nt^{n-1}dt$, $dv = e^{-st}dt$, and $v = -\frac{e^{-st}}{s}$. Then we have

$$-\frac{t^n e^{-st}}{s} + \frac{n}{s}\int t^{n-1}e^{-st}dt,\ s > 0.$$

14 (a). Since $\lim_{t\to 0^+} t^n = 0$ and $\lim_{t\to 0^+} e^{-st} = 1$, $\lim_{t\to 0^+} t^n e^{-st} = 0$.

14 (b). Using L'Hopital's rule:

$$\lim_{t\to\infty} t^n e^{-st} = \lim_{t\to\infty}\frac{t^n}{e^{st}} = \lim_{t\to\infty}\frac{nt^{n-1}}{se^{st}} = \ldots = \lim_{t\to\infty}\frac{n!}{s^n e^{st}} = 0.$$

15 (a). Using 13 and 14,

$$\mathcal{L}\{t^n\} = \lim_{T\to\infty}\left\{\left.\frac{-t^n e^{-st}}{s}\right|_0^T + \frac{n}{s}\int_0^T t^{n-1}e^{-st}dt\right\} = 0 + \frac{n}{s}\lim_{T\to\infty}\int_0^T t^{n-1}e^{-st}dt = \frac{n}{s}\mathcal{L}\{t^{n-1}\},\ s > 0.$$

15 (b).

$$\mathcal{L}\{t^2\} = \frac{2}{s}\mathrm{L}\{t\} = \frac{2}{s^3},\ \mathcal{L}\{t^3\} = \frac{3}{s}\mathcal{L}\{t^2\} = \frac{3!}{s^4},\ \mathcal{L}\{t^4\} = \frac{4}{s}\mathcal{L}\{t^3\} = \frac{4!}{s^5},$$
$$\mathcal{L}\{t^5\} = \frac{5}{s}\mathcal{L}\{t^4\} = \frac{5!}{s^6},\ s > 0.$$

15 (c). $\mathcal{L}\{t^m\} = \frac{m!}{s^{m+1}},\ s > 0.$

16. $\mathcal{L}\{f(t)\} = \lim_{T\to\infty}\left\{e^{-st}\left.\left(\frac{-s\cos\omega t - \omega\sin\omega t}{s^2+\omega^2}\right)\right|_0^T\right\} = \frac{s}{s^2+\omega^2},\ s > 0.$

17. $\mathcal{L}\{f(t)\} = \lim_{T\to\infty}\left\{e^{-st}\left.\left(\frac{-s\sin\omega t - \omega\cos\omega t}{s^2+\omega^2}\right)\right|_0^T\right\} = \frac{\omega}{s^2+\omega^2},\ s > 0.$

18. $f(t) = \cos(\omega(t-2)) = \cos(\omega t)\cos(2\omega) + \sin(\omega t)\sin(2\omega)$. Using 16 and 17,

$$\mathcal{L}\{f(t)\} = \frac{s\cos(2\omega) + \omega\sin(2\omega)}{s^2+\omega^2},\ s > 0.$$

19. $f(t) = \sin(\omega(t-2)) = \sin\omega t\cos 2\omega - \cos\omega t\sin 2\omega$. Then we have

$$\mathcal{L}\{f(t)\} = \frac{\omega\cos 2\omega - s\sin 2\omega}{s^2+\omega^2},\ s > 0.$$

20. $\mathcal{L}\{f(t)\} = \lim_{T\to\infty}\int_0^T e^{-(s-3)t}\sin t\,dt = \lim_{T\to\infty}\left\{e^{-(s-3)t}\left.\left(\frac{-(s-3)\sin t - \cos t}{(s-3)^2+1}\right)\right|_0^T\right\} = \frac{1}{(s-3)^2+1},\ s > 3.$

21. $\mathcal{L}\{f(t)\} = \lim_{T\to\infty}\int_0^T e^{-(s+2)t}\cos 4t\,dt = \lim_{T\to\infty}\left\{ e^{-(s+2)t}\left(\frac{-(s+2)\cos 4t + 4\sin 4t}{(s+2)^2+16}\right)\Bigg|_0^T \right\}$

$= \frac{s+2}{(s+2)^2+16},\ s > -2.$

22. $\mathcal{L}\{2e^{-5t}\} = \frac{2}{s+5},\ s > -5$. $\mathcal{L}\{6t\} = \frac{6}{s^2},\ s > 0$. Then $\mathcal{L}\{r(t)\} = \frac{2}{s+5} + \frac{6}{s^2},\ s > 0$.

23. $\mathcal{L}\{5e^{-7t}\} = \frac{5}{s+7},\ s > -7$. $\mathcal{L}\{t\} = \frac{1}{s^2},\ s > 0$, and $\mathcal{L}\{2e^{2t}\} = \frac{2}{s-2},\ s > 2$. Then

$\mathcal{L}\{r(t)\} = \frac{5}{s+7} + \frac{1}{s^2} + \frac{2}{s-2},\ s > 2$.

24 (a). The function is discontinuous on $0 \le t < \infty$ because the one-sided limits do not exist at the vertical asymptotes.

24 (b). The function is not exponentially bounded on $0 \le t < \infty$.

25 (a). The function is continuous on $0 \le t < \infty$.

25 (b). The function is exponentially bounded on $0 \le t < \infty$. $|f(t)| \le e^t$, so we can take $M = 1,\ a = 1$.

26 (a). The function is continuous on $0 \le t < \infty$.

26 (b). The function is exponentially bounded on $0 \le t < \infty$.

If $f(t) = t^2 e^{-t}$, then $f'(t) = (2t - t^2)e^{-t} = 0 \Rightarrow t = 2$, is a maximum point, so we can take $M = f(2) = 4e^{-2},\ a = 0$.

27 (a). The function is continuous on $0 \le t < \infty$.

27 (b). The function is exponentially bounded on $0 \le t < \infty$, since $\cosh 2t \le \frac{e^{2t}+1}{2} \le e^{2t}$ on $0 \le t < \infty$, so we can take $M = 1,\ a = 2$.

28 (a). The function is piecewise continuous on $0 \le t < \infty$.

28 (b). Consider $g(t) = te^{-t},\ g'(t) = (1-t)e^{-t} = 0 \Rightarrow t = 1$. $\therefore t = 1$ is a maximum and $g(t) \le e^{-1} \Rightarrow t \le e^{-1}e^t,\ 0 \le t < \infty$. Since $[[t]] \le t,\ 0 \le t < \infty$, the function is exponentially bounded on $0 \le t < \infty$, taking $M = e^{-1},\ a = 1$.

29 (a). The function is piecewise continuous on $0 \le t < \infty$.

29 (b). $|f(t)| \le e^{2t}$, and so the function is exponentially bounded on $0 \le t < \infty$, taking $M = 1,\ a = 2$.

30 (a). The function is continuous on $0 \le t < \infty$.

30 (b). The function is not exponentially bounded on $0 \le t < \infty$ because

$$f(t) \ge \frac{e^{t^2}}{e^{2t}+e^{2t}} = \frac{1}{2}e^{t^2-2t} \text{ and } e^{t^2-2t} > e^{\frac{t^2}{2}},\ t > 4.$$

31 (a). The function is discontinuous and not piecewise continuous on $0 \le t < \infty$.

31 (b). The function is not exponentially bounded on $0 \le t < \infty$.

32. $\lim_{T\to\infty}\int_0^T \frac{1}{1+t^2}\,dt = \lim_{T\to\infty}\left(\tan^{-1} t\Big|_0^T\right) = \lim_{T\to\infty}\left(\tan^{-1} T\right) = \frac{\pi}{2}$, so the improper integral converges.

33. $\int_0^T \frac{t}{1+t^2}\,dt = \frac{1}{2}\ln\left(1+t^2\right)\Big|_0^T = \frac{1}{2}\ln\left(1+T^2\right)$. Since $\lim_{T\to\infty}\frac{1}{2}\ln\left(1+T^2\right) = \infty$, the improper integral diverges.

34. $\lim_{T\to\infty}\int_0^T e^{-t}\cos(-e^{-t})dt = \lim_{T\to\infty}\int_1^{e^{-T}} -\cos(u)du = \int_0^1 \cos u\,du = \sin u\Big|_0^1 = \sin(1)$, so the improper integral converges.

35. $\int_0^\infty te^{-t^2}dt = \lim_{T\to\infty}\int_0^T te^{-t^2}dt = \lim_{T\to\infty}\int_0^{T^2} e^{-u}\left(\frac{1}{2}du\right) = \lim_{T\to\infty}\frac{1}{2}\int_0^{T^2} e^{-u}du = \lim_{T\to\infty}\frac{1}{2}\left(1-e^{-T^2}\right) = \frac{1}{2}$, so the integral converges to $\frac{1}{2}$.

36. $f(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = 3e^{2t},\ t \ge 0$.

37. $f(t) = -2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = -2t + e^{-t},\ t \ge 0$.

38. $f(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = 2e^{-2t} + 2e^{2t} = 4\cosh(2t),\ t \ge 0$.

39. $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{t} - e^{-t} = 2\sinh t,\ t \ge 0$.

Section 7.2

1. $\mathcal{L}\{f(t)\} = 3\left(\frac{2}{s^3}\right) + \frac{2}{s^2} + \frac{1}{s} = \frac{6}{s^3} + \frac{2}{s^2} + \frac{1}{s},\ s > 0$

2. $\mathcal{L}\{f(t)\} = \frac{2}{s-1} + \frac{5}{s},\ s > 1$

3. $\mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{3}{s^2+9},\ s > 0$

4. $\mathcal{L}\{f(t)\} = e^{-s}\mathcal{L}\{e^{3t}\} = \frac{e^{-s}}{s-3},\ s > 3$

5. $\mathcal{L}\{f(t)\} = e^{-s}\mathcal{L}\{t^2\} = \dfrac{2e^{-s}}{s^3},\ s > 0$

6. $\mathcal{L}\{\sin^2 \omega t\} = \mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}\cos(2\omega t)\right\} = \dfrac{1}{2}\cdot\dfrac{1}{s} - \dfrac{1}{2}\cdot\dfrac{s}{s^2+4\omega^2} = \dfrac{1}{2}\left(\dfrac{1}{s} - \dfrac{s}{s^2+4\omega^2}\right),\ s > 0$

7. $\mathcal{L}\{f(t)\} = \mathcal{L}\{2t\}\big|_{s\to s+2} = \dfrac{2}{(s+2)^2},\ s > -2$

8. $\mathcal{L}\{\sin 3t \cos 3t\} = \mathcal{L}\left\{\frac{1}{2}\sin 6t\right\} = \dfrac{3}{s^2+36},\ s > 0$

9. $\mathcal{L}\{f(t)\} = \mathcal{L}\{2(t-2)h(t-2) + 4h(t-2)\}$

$= e^{-2s}\left[\mathcal{L}\{2t\} + \mathcal{L}\{4\}\right] = e^{-2s}\left[\dfrac{2}{s^2} + \dfrac{4}{s}\right],\ s > 0$

10. $\mathcal{L}\{e^{2t}\cos 3t\} = \mathcal{L}\{\cos 3t\}\big|_{s\to s-2} = \dfrac{s-2}{(s-2)^2+9},\ s > 2$

11. $\mathcal{L}\{f(t)\} = e^3\mathcal{L}\{e^{3(t-1)}h(t-1)\} = e^3 e^{-s}\mathcal{L}\{e^{3t}\} = \dfrac{e^{3-s}}{s-3},\ s > 3$

12. $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{t^2+3t+5\right\}\Big|_{s\to s-4} = \dfrac{2}{(s-4)^3} + \dfrac{3}{(s-4)^2} + \dfrac{5}{(s-4)},\ s > 4$

13. $\mathcal{L}^{-1}\{F(s)\} = 3 + \dfrac{24}{3!}t^3 = 3 + 4t^3,\ t \ge 0$

14. $\mathcal{L}^{-1}\{F(s)\} = 2\sin 5t + 4e^{3t},\ t \ge 0$

15. $\mathcal{L}^{-1}\{F(s)\} = 2e^{2t}\cos 3t,\ t \ge 0$

16. $\mathcal{L}^{-1}\{F(s)\} = 5e^{3t}\dfrac{t^3}{3!} = \dfrac{5}{6}e^{3t}t^3,\ t \ge 0$

17. $\mathcal{L}^{-1}\{F(s)\} = \sin\big(3(t-2)\big)h(t-2),\ t \ge 0$

18. $\mathcal{L}^{-1}\{F(s)\} = e^{9(t-2)}h(t-2),\ t \ge 0$

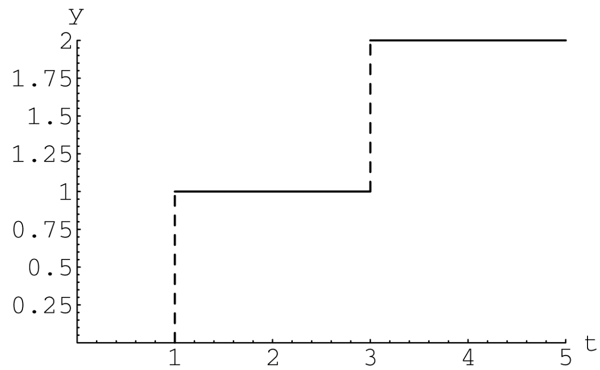
19. $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left(\dfrac{4(s-1)-2}{(s-1)^2+9}\right) = e^t\left(4\cos 3t - \dfrac{2}{3}\sin 3t\right),\ t \ge 0$

20. $\mathcal{L}^{-1}\{F(s)\} = \left[2\cos(4(t-3)) + \dfrac{7}{4}\sin(4(t-3))\right]h(t-3),\ t \ge 0$

21. $\mathcal{L}^{-1}\{F(s)\} = \dfrac{48}{4!}\left((t-3)^4 h(t-3) + 2(t-5)^4 h(t-5)\right)$

$= 2(t-3)^4 h(t-3) + 4(t-5)^4 h(t-5),\ t \ge 0$

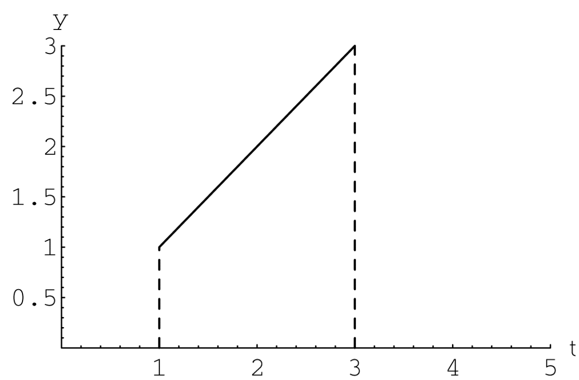
22. $\mathcal{L}\{f(t)\} = \dfrac{e^{-s} + e^{-3s}}{s}, \; s > 0$



23. $\mathcal{L}\{f(t)\} = e^{-2\pi s}\dfrac{1}{s^2+1}, \; s > 0$

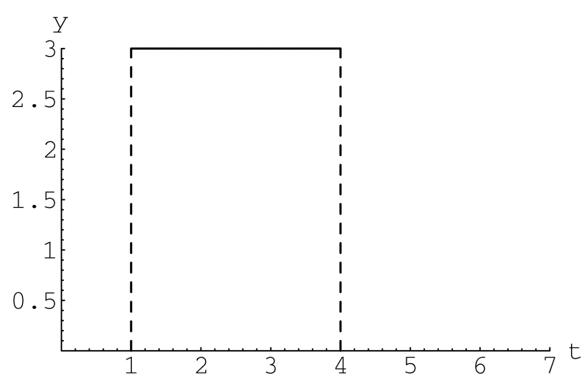
24. $f(t) = (t-1)h(t-1) + h(t-1) - (t-3)h(t-3) - 3h(t-3)$, and so

$$\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right), \; s \neq 0 \;\; (= 4, s = 0)$$



25. $\mathcal{L}\{f(t)\} = \dfrac{\left(1 - e^{-3s}\right)}{s}, \; s > 0$

26. $\mathcal{L}\{f(t)\} = \dfrac{3\left(e^{-s} - e^{-4s}\right)}{s}, \; s \neq 0 \;\; (= 9, s = 0)$

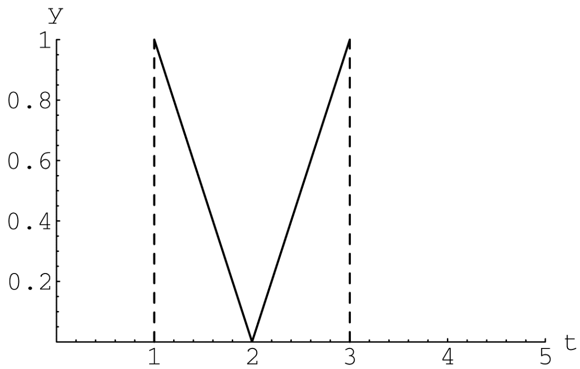


27. $f(t) = -(t-1)h(t-1) + h(t-1) + (t-3)h(t-3) + h(t-3)$, and so

$$\mathcal{L}\{f(t)\} = -\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-3s}}{s^2} + \frac{e^{-3s}}{s},\ s \neq 0\ (=0, s=0)$$

28. $f(t) = -(t-1)h(t-1) + h(t-1) + (t-2)h(t-2) + (t-2)h(t-2) - (t-3)h(t-3) - h(t-3)$, and

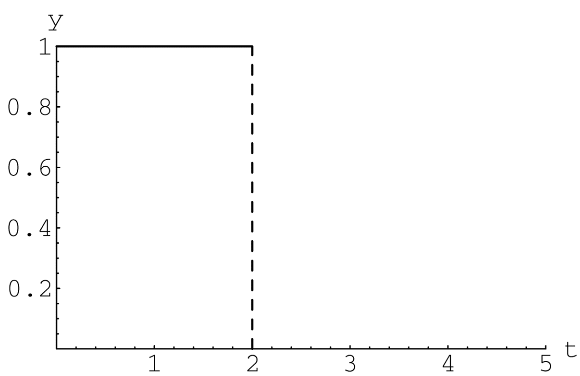
so $\mathcal{L}\{f(t)\} = -\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} + \frac{2e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s},\ s \neq 0\ (=1, s=0)$



29. $\mathcal{L}\{f(t)\} = \frac{e^{-s} - 2e^{-2s} + e^{-3s}}{s},\ s \neq 0\ (=0, s=0)$

30. $\mathcal{L}\{f(t)\} = \int_0^2 e^{-st}dt = -\frac{e^{-st}}{s}\Big|_0^2 = \frac{1+e^{-2s}}{s},\ s \neq 0\ (=2, s=0)$

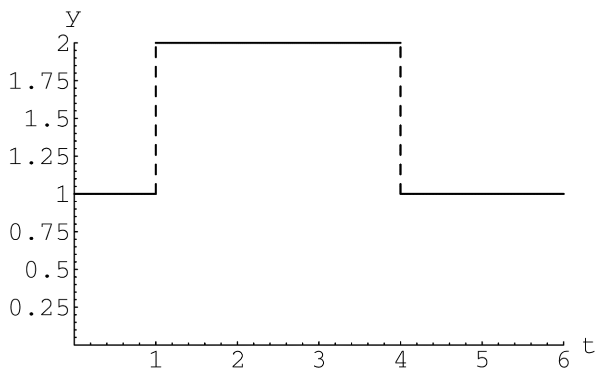
Note: $f(t) = 1 - h(t-2)$ for all t except $t = 2$.



31. $\mathcal{L}\{f(t)\} = \int_0^1 e^{-(s-2)t}dt = -\frac{e^{-(s+2)t}}{s+2}\Big|_0^1 = \frac{1-e^{-(s+2)}}{s+2},\ s \neq -2\ (=1, s=-2)$

32. $\mathcal{L}\{f(t)\} = \frac{1+e^{-s}-e^{-4s}}{s},\ s > 0$

Note: $f(t) = 1 + [h(t-1) - h(t-4)]$ for all t except $t = 4$.



33. $\mathcal{L}\{f(t)\} = -\int_0^2 e^{-st}\,dt + \int_3^\infty e^{-st}\,dt = \left.\frac{e^{-st}}{s}\right|_0^2 - \left.\frac{e^{-st}}{s}\right|_3^\infty = \frac{-1 + e^{-2s} + e^{-3s}}{s},\ s > 0$

34. $f(t) = (t-2)[h(t-2) - h(t-3)] + [h(t-3) - h(t-4)]$

$= (t-2)h(t-2) - [(t-3)+1]h(t-3) + [h(t-3) - h(t-4)]$

$= (t-2)h(t-2) - (t-3)h(t-3) - h(t-4)$ and $\mathcal{L}\{f(t)\} = \frac{e^{-2s} - e^{-3s}}{s^2} - \frac{e^{-4s}}{s},\ s > 0$

35. $f(t) = h(t-1) + h(t-2) - 2h(t-3)$ and $\mathcal{L}\{f(t)\} = \frac{e^{-s} + e^{-2s} - 2e^{-3s}}{s},\ s \neq 0;\ = 3,\ s = 0$

36. $f(t) = (t-1)[h(t-1) - h(t-2)] + (3-t)[h(t-2) - h(t-3)]$

$= (t-1)h(t-1) - [(t-2)+1]h(t-2) + [-(t-2)+1]h(t-2) + (t-3)h(t-3)$

and $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s^2} = \frac{e^{-s} - 2e^{-2s} + e^{-3s}}{s^2},\ s \neq 0\ (=1, s=0)$

37. $f(t) = (1-t)[h(t) - h(t-1)] + (2-t)[h(t-1) - h(t-2)] = 1 - t + h(t-1) + (t-2)h(t-2),\ t \geq 0$

and $\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2},\ s \neq 0;\ = 1,\ s = 0$

38. $F(s) = \frac{A_1}{s-3} + \frac{A_2}{s+1}$, and so $A_1(s+1) + A_2(s-3) = 12$ and $A_1 + A_2 = 0,\ A_1 - 3A_2 = 12$. Thus

$A_1 = 3,\ A_2 = -3$, and $F(s) = \frac{3}{s-3} - \frac{3}{s+1}$, and so $f(t) = -3e^{-t} + 3e^{3t},\ t \geq 0$.

39. $F(s) = \frac{A_1}{s} + \frac{A_2}{s+2}$, and so $A_1(s+2) + A_2 s = 4$ and $A_1 + A_2 = 0,\ 2A_1 = 4$. Thus

$A_1 = 2,\ A_2 = -2$, and $F(s) = \frac{2}{s} - \frac{2}{s+2}$, and so $f(t) = 2 - 2e^{-2t},\ t \geq 0$.

40. $F(s)=24e^{-5s}\left(\dfrac{A_1}{s-3}+\dfrac{A_2}{s+3}\right)$, and so $A_1(s+3)+A_2(s-3)=1$ and $A_1+A_2=0,\ 3A_1-3A_2=1$.

Thus $A_1=\dfrac{1}{6},\ A_2=-\dfrac{1}{6}$, and $F(s)=4e^{-5s}\left(\dfrac{1}{s-3}-\dfrac{1}{s+3}\right)$, and so

$f(t)=4\left[e^{3(t-5)}-e^{-3(t-5)}\right]h(t-5),\ t\geq 0$.

From (6), Table 7.1: $\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left(\dfrac{3}{s^2-9}\right)=\sinh 3t \Rightarrow f(t)=8\sinh(3(t-5))h(t-5),\ t\geq 0$.

Answers agree since $\sinh(3(t-5))=\dfrac{e^{3(t-5)}-e^{-3(t-5)}}{2}$.

41. $F(s)=\left[\dfrac{A_1}{(s-2)}+\dfrac{A_2}{(s-3)}\right]10e^{-s}$ and so $A_1(s-3)+A_2(s-2)=1$ and $A_1+A_2=0,\ -3A_1-2A_2=1$.

Thus $A_1=-1,\ A_2=1$, and $F(s)=10e^{-s}\left[\dfrac{-1}{s-2}+\dfrac{1}{s-3}\right]$, and so

$f(t)=10\left(e^{3(t-1)}-e^{2(t-1)}\right)h(t-1),\ t\geq 0$.

42. $g(t)=12\left[h(t-1)-h(t-3)\right]$, and so $sY(s)-y(0)+4Y=\dfrac{12}{s}\left(e^{-s}-e^{-3s}\right)$. Therefore,

$Y=\dfrac{2}{s+4}+\dfrac{12}{s(s+4)}\left(e^{-s}-e^{-3s}\right)$, and so $\dfrac{1}{s(s+4)}=\dfrac{A_1}{s}+\dfrac{A_2}{s+4}$. Thus

$A_1(s+4)+A_2s=1$ and $A_1+A_2=0,\ 4A_1=1$. Solving these simultaneous equations yields

$A_1=\dfrac{1}{4},\ A_2=-\dfrac{1}{4}$, and so $Y=\dfrac{2}{s+4}+3\left(e^{-s}-e^{-3s}\right)\left[\dfrac{1}{s}-\dfrac{1}{s+4}\right]$. Thus

$y(t)=2e^{-4t}+3\left[h(t-1)-h(t-3)\right]-3\left[e^{-4(t-1)}h(t-1)-e^{-4(t-3)}h(t-3)\right],\ t\geq 0$.

43. $g(t)=e^{3t}h(t-4)=e^{12}e^{3(t-4)}h(t-4)$, and so $sY-1-Y=e^{12}\dfrac{e^{-4s}}{s-3}$. Therefore,

$Y=\dfrac{1}{s-1}+\dfrac{e^{12}e^{-4s}}{(s-1)(s-3)}$, and so $\dfrac{1}{(s-1)(s-3)}=\dfrac{A_1}{s-1}+\dfrac{A_2}{s-3}$. Thus

$A_1(s-3)+A_2(s-1)=1$ and $A_1+A_2=0,\ -3A_1-A_2=1$. Solving these simultaneous equations

yields $A_1=-\dfrac{1}{2},\ A_2=\dfrac{1}{2}$, and so $Y=\dfrac{1}{s-1}+\dfrac{e^{12}}{2}e^{-4s}\left[\dfrac{-1}{s-1}+\dfrac{1}{s-3}\right]$. Thus

$y(t)=e^{t}+\dfrac{1}{2}e^{12}\left(-e^{t-4}+e^{3(t-4)}\right)h(t-4),\ t\geq 0$.

44. $s^2Y - sy(0) - y'(0) - 4Y = \frac{1}{s-3}$. Therefore, $Y = \frac{1}{(s-3)(s-2)(s+2)}$, and so

$\frac{1}{(s-3)(s+2)(s-2)} = \frac{A_1}{s-3} + \frac{A_2}{s+2} + \frac{A_3}{s-2}$. Thus,

$A_1 = \left.\frac{1}{(s-2)(s+2)}\right|_{s=3} = \frac{1}{5},\ A_2 = \left.\frac{1}{(s-3)(s-2)}\right|_{s=-2} = \frac{1}{20},\ A_3 = \left.\frac{1}{(s-3)(s+2)}\right|_{s=2} = -\frac{1}{4}$. Therefore,

$y(t) = \frac{1}{5}e^{3t} + \frac{1}{20}e^{-2t} - \frac{1}{4}e^{2t},\ t \geq 0$.

45. $s^2Y - s(0) - 1 - 2(sY - 0) - 8Y = \frac{1}{s-1}$. Therefore,

$Y(s^2 - 2s - 8) - 1 = \frac{1}{s-1} \Rightarrow Y(s^2 - 2s - 8) = \frac{1}{s-1} + 1 = \frac{s}{s-1}$, which means that

$Y = \frac{s}{(s-1)(s+2)(s-4)}$, and so $\frac{s}{(s-1)(s+2)(s-4)} = \frac{A_1}{s-1} + \frac{A_2}{s+2} + \frac{A_3}{s-4}$. Thus

$A_1(s^2 - 2s - 8) + A_2(s^2 - 5s + 4) + A_3(s^2 + s - 2) = s$, and so $A_1 = -\frac{1}{9},\ A_2 = -\frac{1}{9},\ A_3 = \frac{2}{9}$.

Finally, we have $Y = \frac{1}{9}\left[\frac{-1}{s-1} + \frac{-1}{s+2} + \frac{2}{s-4}\right]$, and so $y(t) = -\frac{1}{9}e^{t} - \frac{1}{9}e^{-2t} + \frac{2}{9}e^{4t},\ t \geq 0$.

46 (a). $\frac{d}{ds}F(s) = \int_0^\infty \frac{d}{ds}(e^{-st})f(t)dt = \int_0^\infty -te^{-st}f(t)dt = -\int_0^\infty e^{-st}(tf(t))dt = -\mathcal{L}\{tf(t)\}$.

46 (b). $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{t\sin\omega t}{2\omega}\right\} = -\frac{1}{2\omega}\frac{d}{ds}\mathcal{L}\{\sin\omega t\} = -\frac{1}{2\omega}\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right)$

$= -\frac{1}{2}(-1)(s^2+\omega^2)^{-2}(2s) = \frac{s}{(s^2+\omega^2)^2},\ s > 0$.

47. $\mathcal{L}\left(\int_0^t\left[\int_0^\lambda f(\sigma)d\sigma\right]d\lambda\right) = \frac{1}{s}\mathcal{L}\left(\int_0^t f(\sigma)d\sigma\right) = \frac{1}{s^2}F(s),\ s > \max\{a,0\}$

48. $\mathcal{L}\left\{\int_2^t f(\lambda)d\lambda\right\} = \mathcal{L}\left\{\int_0^t f(\lambda)d\lambda - \int_0^2 f(\lambda)d\lambda\right\} = \frac{1}{s}F(s) - \mathcal{L}\{3\} = \frac{1}{s}[F(s) - 3],\ s > \max\{a,0\}$

49 (a). $f(t) = h(t)h(3-t) = \begin{cases} 1,\ 0 \leq t \leq 3 \\ 0,\ 3 < t \end{cases}$, and $g(t) = h(t) - h(t-3) = \begin{cases} 1,\ 0 \leq t < 3 \\ 0,\ 3 \leq t \end{cases}$, and so the two functions are equal for all $t \neq 3$ and hence not identical.

49 (b). $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} = \int_0^3 e^{-st}dt = \frac{1-e^{-3s}}{s},\ s \neq 0;\ = 3,\ s = 0$, and so the transformed functions are identical.

Section 7.3

1. $F(s)=\dfrac{A}{s-1}+\dfrac{B_2}{(s-2)^2}+\dfrac{B_1}{s-2}$

2. $F(s)=\dfrac{A_3}{(s-1)^3}+\dfrac{A_2}{(s-1)^2}+\dfrac{A_1}{s-1}+\dfrac{B_2}{(s-2)^2}+\dfrac{B_1}{s-2}$

3. $F(s)=\dfrac{A_2}{s^2}+\dfrac{A_1}{s}+\dfrac{Bs+C}{(s+1)^2+9}$

4. $F(s)=\dfrac{A}{s-2}+\dfrac{Bs+C}{s^2+16}$

5. $F(s)=\dfrac{A_2}{(s-3)^2}+\dfrac{A_1}{s-3}+\dfrac{B_2}{(s+3)^2}+\dfrac{B_1}{s+3}$

6. $F(s)=\dfrac{A_2}{(s+4)^2}+\dfrac{A_1}{s+4}+\dfrac{B_2s+C_2}{(s^2+1)^2}+\dfrac{B_1s+C_1}{s^2+1}$

7. $F(s)=\dfrac{Bs+C}{(s+4)^2+1}+\dfrac{Ds+E}{(s+3)^2+4}$

8. $F(s)=\dfrac{A_2}{(s-2)^2}+\dfrac{A_1}{s-2}+\dfrac{B_2s+C_2}{((s+4)^2+1)^2}+\dfrac{B_1s+C_1}{(s+4)^2+1}$

9. $f(t)=2e^{3t},\ t\ge 0$

10. $f(t)=\dfrac{1}{2}t^2e^{-t},\ t\ge 0$

11. $F(s)=4\left(\dfrac{s}{s^2+9}\right)+\dfrac{5}{3}\left(\dfrac{3}{s^2+9}\right)$, and so $f(t)=4\cos 3t+\dfrac{5}{3}\sin 3t,\ t\ge 0$.

12. $F(s)=\dfrac{A}{s-1}+\dfrac{B}{s-2}$. $A=\left.\dfrac{2s-3}{s-2}\right|_{s=1}=1$ and $B=\left.\dfrac{2s-3}{s-1}\right|_{s=2}=1$. Therefore, $f(t)=e^t+e^{2t},\ t\ge 0$.

13. $F(s)=\dfrac{A}{s+3}+\dfrac{B}{s+1}$. $A=\left.\dfrac{3s+7}{s+1}\right|_{s=-3}=\dfrac{-2}{-2}=1$ and $B=\left.\dfrac{3s+7}{s+3}\right|_{s=-1}=\dfrac{4}{2}=2$. Thus

$F(s)=\dfrac{1}{s+3}+\dfrac{2}{s+1}$ and so $f(t)=e^{-3t}+2e^{-t},\ t\ge 0$.

14. $F(s)=\dfrac{A}{s}+\dfrac{Bs+C}{s^2+1}=\dfrac{As^2+A+Bs^2+Cs}{s(s^2+1)}$. Then we have $A+B=4,\ C=1,\ A=1,\ B=3$.

Therefore, $F(s)=\dfrac{1}{s}+\dfrac{3s}{s^2+1}+\dfrac{1}{s^2+1}$ and so $f(t)=1+3\cos t+\sin t,\ t\ge 0$.

15. $F(s)=\dfrac{A}{s}+\dfrac{Bs+C}{s^2+4}$. Then we have $A+B=3,\ C=1,$ and $4A=8$, so $A=2,\ B=1,\ C=1$. Thus

$F(s)=\dfrac{2}{s}+\dfrac{s+1}{s^2+4}$ and so $f(t)=2+\cos 2t+\dfrac{1}{2}\sin 2t,\ t\ge 0$.

16. $F(s)=\dfrac{B_2s+C_2}{\left(s^2+4\right)^2}+\dfrac{B_1s+C_1}{\left(s^2+4\right)}$. Then we have $B_1=0,\ C_1=1,\ B_2=6,\ C_2=4$. Therefore,

$F(s)=\dfrac{6s+4}{\left(s^2+4\right)^2}+\dfrac{1}{s^2+4}$ and so

$f(t)=6\left(\dfrac{t}{4}\sin 2t\right)+4\left(\dfrac{1}{16}[\sin 2t-2t\cos 2t]\right)+\dfrac{1}{2}\sin 2t=\dfrac{3}{2}t\sin 2t+\dfrac{3}{4}\sin 2t-\dfrac{1}{2}t\cos 2t,\ t\ge 0$.

17. $F(s)=\dfrac{s}{(s-1)^3}=\dfrac{(s-1)+1}{(s-1)^3}=\dfrac{1}{(s-1)^2}+\dfrac{1}{(s-1)^3}$, so $f(t)=te^t+\dfrac{1}{2}t^2e^t,\ t\ge 0$.

18. $sY-3+2Y=26\left(\dfrac{3}{s^2+9}\right)$, and thus $Y=\dfrac{3}{s+2}+26\dfrac{3}{(s+2)\left(s^2+9\right)}$.

$\dfrac{1}{(s+2)\left(s^2+9\right)}=\dfrac{A}{s+2}+\dfrac{Bs+C}{s^2+9}$, and so $A=\left.\dfrac{s}{s^2+9}\right|_{s=-2}=\dfrac{1}{13}$.

$\dfrac{1}{(s+2)(s^2+9)}-\dfrac{1/13}{s+2}=\dfrac{1-1/13(s^2+9)}{(s+2)(s^2+9)}=\dfrac{-1/13(s^2-4)}{(s+2)(s^2+9)}=\dfrac{-1/13(s-2)}{(s^2+9)}=\dfrac{Bs+C}{s^2+9}$. Then,

$B=-\dfrac{1}{13},\ C=\dfrac{2}{13}$, and so

$Y=\dfrac{3}{s+2}+26\cdot 3\left(\dfrac{1/13}{s+2}-\dfrac{1}{13}\cdot\dfrac{s}{s^2+9}+\dfrac{2}{13}\cdot\dfrac{1}{s^2+9}\right)=\dfrac{9}{s+2}-6\left(\dfrac{s}{s^2+9}\right)+4\left(\dfrac{3}{s^2+9}\right)$. Finally, we have $y(t)=9e^{-2t}-6\cos 3t+4\sin 3t$.

19. $sY-1-3Y=13\left(\dfrac{s}{s^2+4}\right)$, and thus $Y=\dfrac{1}{s-3}+13\dfrac{s}{(s-3)\left(s^2+4\right)}$.

$\dfrac{s}{(s-3)\left(s^2+4\right)}=\dfrac{A}{s-3}+\dfrac{Bs+C}{s^2+4}$, and so $A=\left.\dfrac{s}{s^2+4}\right|_{s=3}=\dfrac{3}{13}$. Setting $s=0$ gives us

$0=-\dfrac{1}{13}+\dfrac{C}{4},\ C=\dfrac{4}{13}$. Setting $s=1$ gives us $\dfrac{1}{-2\cdot 5}=-\dfrac{1}{2}\left(\dfrac{3}{13}\right)+\dfrac{B+\frac{4}{13}}{5}$. Solving for B yields

$B=-\dfrac{3}{13}$, and so $Y=\dfrac{1}{s-3}+13\left(\dfrac{\frac{3}{13}}{s-3}+\dfrac{\frac{-3}{13}s+\frac{4}{13}}{s^2+4}\right)=\dfrac{4}{s-3}-3\left(\dfrac{s}{s^2+4}\right)+2\left(\dfrac{2}{s^2+4}\right)$. Finally, we have $y(t)=4e^{3t}-3\cos 2t+2\sin 2t$.

20. $sY-3+2Y=\frac{4}{s^2}$, and thus $Y=\frac{3}{s+2}+\frac{4}{s^2(s+2)}=\frac{3s^2+4}{s^2(s+2)}=\frac{A}{s^2}+\frac{B}{s}+\frac{C}{s+2}$, and so

$C=\left.\frac{3s^2+4}{s^2}\right|_{s=-2}=4$, $A=\left.\frac{3s^2+4}{s+2}\right|_{s=0}=2$. Setting $s=1$ gives us $\frac{7}{3}=2+B+\frac{4}{3}\Rightarrow B=-1$.

Therefore, $Y=\frac{2}{s^2}-\frac{1}{s}+\frac{4}{s+2}$. Finally, we have $y(t)=4e^{-2t}+2t-1$.

21. $sY-1-3Y=\frac{1}{s-3}$, so $Y=\frac{1}{s-3}+\frac{1}{(s-3)^2}$ and thus $y(t)=e^{3t}+te^{3t}$.

22. $s^2Y-s(1)-2+3(sY-1)+2Y=\frac{6}{s+1}$, and thus $Y=\frac{s+5}{(s+1)(s+2)}+\frac{6}{(s+1)^2(s+2)}$.

$\frac{s^2+6s+11}{(s+1)^2(s+2)}=\frac{A}{s+2}+\frac{B}{s+1}+\frac{C}{(s+1)^2}$, and so

$A=\left.\frac{s^2+6s+11}{(s+1)^2}\right|_{s=-2}=3$, $C=\left.\frac{s^2+6s+11}{s+2}\right|_{s=-1}=6$. Setting $s=0$ gives us

$\frac{11}{2}=\frac{3}{2}+B+6\Rightarrow B=-2$. Therefore, $Y=\frac{3}{s+2}-\frac{2}{s+1}+\frac{6}{(s+1)^2}$. Finally, we have

$y(t)=3e^{-2t}-2e^{-t}+6te^{-t}$.

23. $s^2Y-s(2)-6+4Y=\frac{8}{s^2}$, so $Y=\frac{2s+6}{s^2+4}+\frac{8}{s^2(s^2+4)}$.

If $\frac{8}{s^2(s^2+4)}=\frac{A}{s^2}+\frac{B}{s}+\frac{Cs+D}{s^2+4}$, then $\frac{A(s^2+4)+Bs(s^2+4)+Cs^3+Ds^2}{s^2(s^2+4)}=\frac{8}{s^2(s^2+4)}$, and so

we have $B+C=0,\ A+D=0,\ 4B=0,\ 4A=8$, which means that $B=C=0,\ A=2,\ D=-2$.

Then $Y=2\left(\frac{s}{s^2+4}\right)+3\left(\frac{2}{s^2+4}\right)+\frac{2}{s^2}-\left(\frac{2}{s^2+4}\right)$ and $y(t)=2\cos 2t+2\sin 2t+2t$.

24. $s^2Y-s(1)-1+4Y=\frac{s}{s^2+4}$, so $Y=\frac{s+1}{s^2+4}+\frac{s}{(s^2+4)^2}$. Therefore,

$y(t)=\cos 2t+\frac{1}{2}\sin 2t+\frac{t}{4}\sin 2t$.

25. $s^2Y-s(1)-0+4Y=\frac{2}{s^2+4}$, so $Y=\frac{s}{s^2+4}+\frac{2}{(s^2+4)^2}$. Therefore,

$y(t)=\cos 2t+\frac{2}{2\cdot 8}(\sin 2t-2t\cos 2t)=\cos 2t+\frac{1}{8}\sin 2t-\frac{t}{4}\cos 2t$.

26. $s^2Y - s(0) - 0 - 2(sY - 0) + Y = \frac{1}{s-2}$, and thus $Y = \frac{1}{(s-2)(s-1)^2} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$, and

so $A = \left.\frac{1}{(s-1)^2}\right|_{s=2} = 1$, $C = \left.\frac{1}{s-2}\right|_{s=1} = -1$. Setting $s = 0$ gives us $-\frac{1}{2} = -\frac{1}{2} - B - 1 \Rightarrow B = -1$.

Therefore, $Y = \frac{1}{s-2} - \frac{1}{s-1} - \frac{1}{(s-1)^2}$. Finally, we have $y(t) = e^{2t} - e^t - te^t$.

27. $s^2Y - 1 + 2sY - 0 + Y = \frac{1}{s+1}$, so $Y = \frac{1}{(s+1)^2} + \frac{1}{(s+1)^3}$. Therefore, $y(t) = te^{-t} + \frac{t^2}{2}e^{-t}$.

28. $g(t) = 6[h(t) - h(t-\pi)]$, and then we have $s^2Y - s(1) - 3 + 9Y = \frac{6}{s}\left(1 - e^{-\pi s}\right)$, so

$Y = \frac{s+3}{s^2+9} + \frac{6}{s\left(s^2+9\right)}\left(1 - e^{-\pi s}\right)$. $\frac{6}{s\left(s^2+9\right)} = \frac{A}{s} + \frac{Bs+C}{s^2+9} = \frac{As^2 + 9A + Bs^2 + Cs}{s\left(s^2+9\right)}$. Thus

$A + B = 0,\ C = 0,\ A = \frac{2}{3},\ B = -\frac{2}{3}$, and so $Y = \frac{s}{s^2+9} + \frac{3}{s^2+9} + \left(\frac{2/3}{s} - \frac{2}{3}\cdot\frac{s}{s^2+9}\right)\left(1 - e^{-\pi s}\right)$. Then,

$$y(t) = \cos 3t + \sin 3t + \frac{2}{3}(1 - \cos 3t) - \frac{2}{3}(1 - \cos 3(t-\pi))h(t-\pi)$$
$$= \cos 3t + \sin 3t + \frac{2}{3}(1 - \cos 3t) - \frac{2}{3}(1 + \cos 3t)h(t-\pi).$$

29. $g(t) = t[1 - h(t-2)] = t - (t-2)h(t-2) - 2h(t-2)$, and then we have

$s^2Y - s(1) - 0 + Y = \frac{1}{s^2} - \frac{1}{s^2}e^{-2s} - \frac{2}{s}e^{-2s}$, so $Y = \frac{s}{s^2+1} + \frac{1}{s^2\left(s^2+1\right)}\left(1 - e^{-2s}\right) - \frac{2}{s\left(s^2+1\right)}e^{-2s}$.

$\frac{1}{s^2\left(s^2+1\right)} = \frac{1}{s^2} - \frac{1}{s^2+1}$ and $\frac{-2}{s\left(s^2+1\right)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{As^2 + A + Bs^2 + Cs}{s\left(s^2+1\right)}$. Thus

$A + B = 0,\ C = 0,\ A = -2 \ \Rightarrow\ B = 2$, and so

$Y = \frac{s}{s^2+1} + \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)\left(1 - e^{-2s}\right) + \left(-\frac{2}{s} + \frac{2s}{s^2+1}\right)e^{-2s}$. The inverse Laplace transform yields

$$y(t) = \cos t + t - (t-2)h(t-2) - \sin t + \sin(t-2)h(t-2) - 2h(t-2) + 2\cos(t-2)h(t-2)$$
$$= \cos t - \sin t + t + [-(t-2) + \sin(t-2) - 2 + 2\cos(t-2)]h(t-2)$$
$$= \cos t - \sin t + t + [-t + \sin(t-2) + 2\cos(t-2)]h(t-2).$$

30. $s^2Y - sy_0 - y_0' + \alpha(sY - y_0) + \beta Y = 0 \Rightarrow Y = \frac{sy_0 + \left(y_0' + \alpha y_0\right)}{s^2 + \alpha s + \beta} = \frac{2s-1}{s^2+s+2}$, so

$\alpha = 1,\ \beta = 2,\ y_0 = 2,\ y_0' + \alpha y_0 = y_0' + 1(2) = -1 \Rightarrow y_0' = -3$.

31. $\frac{sy_0 + \left(y_0' + \alpha y_0\right)}{s^2 + \alpha s + \beta} = \frac{3}{s^2-4}$, so $\alpha = 0,\ \beta = -4,\ y_0 = 0,\ y_0' = 3$.

32. $\dfrac{sy_0+(y_0'+\alpha y_0)}{s^2+\alpha s+\beta}=\dfrac{s}{(s+1)^2}$, so $\alpha=2,\ \beta=1,\ y_0=1,\ y_0'=-2$.

Section 7.4

1. $T=4.\ 3\int_0^2 e^{-st}dt-3\int_2^4 e^{-st}dt=3\left[\left.\dfrac{e^{-st}}{-s}\right|_0^2-\left.\dfrac{e^{-st}}{-s}\right|_2^4\right]=\dfrac{3}{s}\left[1-e^{-2s}+e^{-4s}-e^{-2s}\right]=\dfrac{3}{s}\left(1-e^{-2s}\right)^2.$

Therefore, $\mathcal{L}\{f\}=\dfrac{3}{s}\dfrac{\left(1-e^{-2s}\right)^2}{1-e^{-4s}}=\dfrac{3}{s}\left(\dfrac{1-e^{-2s}}{1+e^{-2s}}\right).$

2. $T=2.\ 3\int_0^1 e^{-st}dt+\int_1^2 e^{-st}dt=3\left.\dfrac{e^{-st}}{-s}\right|_0^1+\left.\dfrac{e^{-st}}{-s}\right|_1^2=\dfrac{1}{s}\left(3-3e^{-s}+e^{-s}-e^{-2s}\right)=\dfrac{1}{s}\left(3-2e^{-s}-e^{-2s}\right).$

Therefore, $\mathcal{L}\{f\}=\dfrac{3-2e^{-s}-e^{-2s}}{s(1-e^{-2s})}=\dfrac{-3+e^{-s}}{s(1-e^{-s})}.$

3. $T=4.\ -3\int_0^2 e^{-st}dt+2\int_2^4 e^{-st}dt=\left.\dfrac{3}{s}e^{-st}\right|_0^2-\left.\dfrac{2}{s}e^{-st}\right|_2^4=\dfrac{3}{s}\left(e^{-2s}-1\right)-\dfrac{2}{s}\left(e^{-4s}-e^{-2s}\right)$

$=\dfrac{1}{s}\left(-2e^{-4s}+5e^{-2s}-3\right)$. Therefore, $\mathcal{L}\{f\}=\dfrac{-2e^{-4s}+5e^{-2s}-3}{s\left(1-e^{-4s}\right)}=\dfrac{-3+2e^{-2s}}{s\left(1+e^{-2s}\right)}.$

4. $T=4.\ 2\int_0^1 e^{-st}dt+\int_1^3 e^{-st}dt=2\left.\dfrac{e^{-st}}{-s}\right|_0^1+\left.\dfrac{e^{-st}}{-s}\right|_1^3=\dfrac{1}{s}\left(2-2e^{-s}+e^{-s}-e^{-3s}\right)=\dfrac{1}{s}\left(2-e^{-s}-e^{-3s}\right).$

Therefore, $\mathcal{L}\{f\}=\dfrac{2-e^{-s}-e^{-3s}}{s(1-e^{-4s})}=\dfrac{2+e^{-s}+e^{-2s}}{s(1+e^{-2s})(1+e^{-s})}.$

5. $T=2.\ \int_0^1 te^{-st}dt+\int_1^2(2-t)e^{-st}dt=-\left.\dfrac{1}{s^2}(st+1)e^{-st}\right|_0^1+\left.\dfrac{1}{s^2}(st+1)e^{-st}\right|_1^2+\left.\dfrac{2}{-s}e^{-st}\right|_1^2$

$=-\dfrac{1}{s^2}\left[(s+1)e^{-s}-1\right]+\dfrac{1}{s^2}\left[(2s+1)e^{-2s}-(s+1)e^{-s}\right]+\dfrac{2}{s}\left[e^{-s}-e^{-2s}\right]=\dfrac{1}{s^2}\left(1-e^{-s}\right)^2$. Therefore,

$\mathcal{L}\{f\}=\dfrac{1}{s^2}\dfrac{\left(1-e^{-s}\right)^2}{1-e^{-2s}}=\dfrac{1-e^{-s}}{s^2\left(1+e^{-s}\right)}.$

6. $T=2.\ \int_1^2(t-1)e^{-st}dt=\int_0^1 ue^{-s(u+1)}du=e^{-s}\int_0^1 ue^{-su}du=\left.\dfrac{-e^{-s}}{s^2}(su+1)e^{-su}\right|_0^1=\dfrac{-e^{-s}}{s^2}\left[(s+1)e^{-s}-1\right].$

Therefore, $\mathcal{L}\{f\}=\dfrac{e^{-s}\left[1-(s+1)e^{-s}\right]}{s^2\left(1-e^{-2s}\right)}.$

7. $T=2.\ \int_0^1 (1-t^2)e^{-st}dt = \frac{1}{s^3}\left[s^2t^2+2st+2-s^2\right]e^{-st}\Big|_0^1$

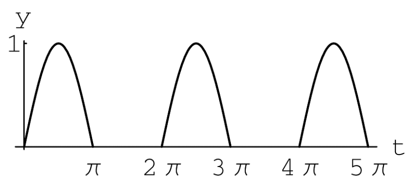
$= \frac{1}{s^3}\left[(s^2+2s+2-s^2)e^{-s}-2+s^2\right] = \frac{1}{s^3}\left[(2s+2)e^{-s}+s^2-2\right]$. Therefore,

$\mathcal{L}\{f\} = \frac{\left[(2s+2)e^{-s}+(s^2-2)\right]}{s^3(1-e^{-2s})}$.

8. $T=2.\ \int_0^2 te^{-st}dt = \frac{-1}{s^2}(st+1)e^{-st}\Big|_0^2 = \frac{-1}{s^2}\left[(2s+1)e^{-2s}-1\right]$. Therefore, $\mathcal{L}\{f\} = \frac{1-(2s+1)e^{-2s}}{s^2(1-e^{-2s})}$.

9. $T=\frac{\pi}{2},\ \mathcal{L}\{f\} = \frac{2+2e^{-\frac{\pi}{2}s}}{(s^2+4)\left(1-e^{-\frac{\pi}{2}s}\right)}$

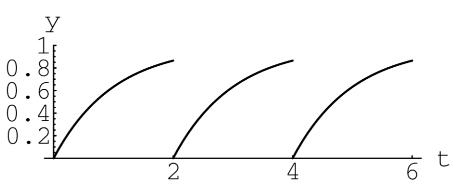
10. $T=2\pi,\ \mathcal{L}\{f\} = \frac{1+e^{-\pi s}}{(s^2+1)(1-e^{-2\pi s})} = \frac{1}{(s^2+1)(1-e^{-\pi s})}$



11. $T=1,\ \int_0^1 e^{-t}e^{-st}dt = -\frac{e^{-(s+1)t}}{s+1}\Big|_0^1 = \frac{1}{s+1}\left(1-e^{-(s+1)}\right),\ \mathcal{L}\{f\} = \frac{1-e^{-(s+1)}}{(s+1)(1-e^{-s})}$.

12. $T=2,\ \int_0^2 (1-e^{-t})e^{-st}dt = \frac{e^{-st}}{-s}\Big|_0^2 + \frac{e^{-(s+1)t}}{s+1}\Big|_0^2 = \frac{1}{s}\left(1-e^{-2s}\right) - \frac{1}{s+1}\left(1-e^{-2(s+1)}\right)$

$\mathcal{L}\{f\} = \frac{1}{s} - \frac{1-e^{-2(s+1)}}{(s+1)(1-e^{-2s})}$.

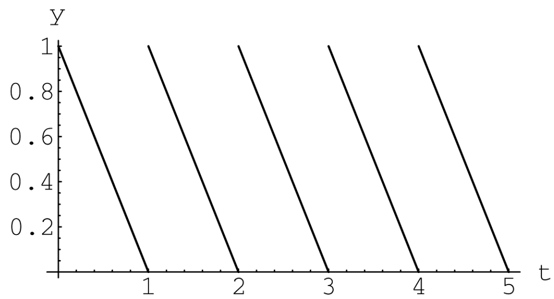


13. $\mathcal{L}^{-1}\left(\frac{e^{-\alpha s}}{s(1-e^{-\alpha s})}\right) = \mathcal{L}^{-1}\left(\frac{e^{-\alpha s}}{s}\left(1+e^{-\alpha s}+e^{-2\alpha s}+e^{-3\alpha s}+\ldots\right)\right)$

$= \mathcal{L}^{-1}\left(\frac{e^{-\alpha s}}{s}+\frac{e^{-2\alpha s}}{s}+\frac{e^{-3\alpha s}}{s}+\ldots\right) = h(t-\alpha)+h(t-2\alpha)+\ldots,$

14. $F(s)=\frac{1}{s}-\frac{1}{s^2}-\frac{e^{-s}}{s(1-e^{-s})}$

and so $f(t)=1-t+h(t-1)+h(t-2)+h(t-3)+...=1-t+\sum_{n=1}^{\infty}h(t-n)$.

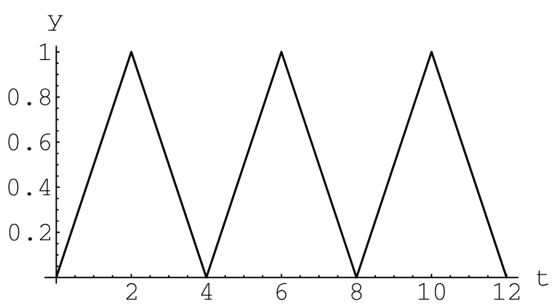


15. $F(s)=\frac{3}{s^2}-\frac{3e^{-2s}}{s(1-e^{-2s})}$ and so $f(t)=3[t-h(t-2)-h(t-4)-h(t-6)...]=3\left[t-\sum_{n=1}^{\infty}h(t-2n)\right]$.

16. $F(s)=\frac{1}{2s^2}-\frac{1}{s^2}\frac{e^{-2s}}{1+e^{-2s}}=\frac{1}{2s^2}-\frac{e^{-2s}}{s^2}\left(1-e^{-2s}+e^{-4s}-e^{-6s}+e^{-8s}+...\right)=\frac{1}{2s^2}+\sum_{n=1}^{\infty}(-1)^n\frac{e^{-2ns}}{s^2}$

and so $f(t)=\frac{t}{2}-(t-2)h(t-2)+(t-4)h(t-4)-(t-6)h(t-6)+...$

$=\frac{t}{2}+\sum_{n=1}^{\infty}(-1)^n(t-2n)h(t-2n)$.



17 (a). $\frac{dq}{dt}=rC_i(t)-\frac{r}{v}q$. With q in kg and t in days, we have $\frac{dq}{dt}+\frac{5(10^6)}{50(10^6)}q=\frac{5(10^6)}{10^6}C_i(t)$, where

$C_i(t)=\begin{cases}1, & 0\le t<\frac{1}{2}\\ 0, & \frac{1}{2}\le t<1\end{cases}$, $C_i(t+1)=C_i(t)$. Then $\frac{dq}{dt}+\frac{1}{10}q=5C_i(t),\ q(0)=0$.

17 (b). $sQ+0.1Q=5\left\{\frac{\int_0^{0.5}e^{-st}dt}{1-e^{-s}}\right\}=\frac{5}{s}\left(\frac{1-e^{-\frac{s}{2}}}{1-e^{-s}}\right)=\frac{5}{s\left(1+e^{-\frac{s}{2}}\right)}$. Then

$Q=\frac{1}{s(s+0.1)}\cdot\frac{5}{1+e^{-\frac{s}{2}}}=50\left(\frac{1}{s}-\frac{1}{s+0.1}\right)\frac{1}{1+e^{-\frac{s}{2}}}$.

17 (c). $Q = 50\left(\frac{1}{s} - \frac{1}{s+0.1}\right)\left(1 - e^{-\frac{s}{2}} + e^{-s} - e^{-\frac{3s}{2}} + e^{-2s} + \ldots\right)$. Noting that $\mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s+0.1}\right) = \left(1 - e^{-0.1t}\right)$,

we have $q(t) = 50\left[1 - e^{-0.1t} - \left(1 - e^{-0.1\left(t-\frac{1}{2}\right)}\right)h\left(t - \frac{1}{2}\right) + \left(1 - e^{-0.1(t-1)}\right)h(t-1) - \ldots\right]$. Thus for

$$1 \le t < 2,\ q(t) = \begin{cases} 50\left[1 - e^{-0.1t} + e^{-0.1\left(t-\frac{1}{2}\right)} - e^{-0.1(t-1)}\right], & 1 \le t < \frac{3}{2} \\ 50\left[-e^{-0.1t} + e^{-0.1\left(t-\frac{1}{2}\right)} - e^{-0.1(t-1)}\right], & \frac{3}{2} \le t < 2 \end{cases}.$$

18 (a). $ms^2X(s) = \frac{f_0\int_0^{\frac{T}{2}} e^{-st}dt}{1 - e^{-st}} = \frac{f_0}{s}\left(\frac{1 - e^{-s\frac{T}{2}}}{1 - e^{-sT}}\right) \Rightarrow X(s) = \frac{f_0}{m}\cdot\frac{1}{s^3}\cdot\frac{1}{1 + e^{-s\frac{T}{2}}}$ and

$$V(s) = sX(s) = \frac{f_0}{m}\cdot\frac{1}{s^2}\cdot\frac{1}{1 + e^{-s\frac{T}{2}}} = \frac{f_0}{m}\cdot\frac{1}{s^2}\left(1 - e^{-s\frac{T}{2}} + e^{-sT} - e^{-s\frac{3T}{2}} + \ldots\right)$$

$$v(t) = \frac{f_0}{m}\left[t - \left(t - \frac{T}{2}\right)h\left(t - \frac{T}{2}\right) + (t-T)h(t-T) - \left(t - \frac{3T}{2}\right)h\left(t - \frac{3T}{2}\right) + \ldots\right]$$

$$= \frac{f_0}{m}\sum_{n=0}^{\infty}(-1)^n\left(t - \frac{nT}{2}\right)h\left(t - \frac{nT}{2}\right).$$

Similarly, $x(t) = \frac{f_0}{2m}\left[t^2 - \left(t - \frac{T}{2}\right)^2 h\left(t - \frac{T}{2}\right) + (t-T)^2 h(t-T) - \left(t - \frac{3T}{2}\right)^2 h\left(t - \frac{3T}{2}\right) + \ldots\right]$

$$= \frac{f_0}{2m}\sum_{n=0}^{\infty}(-1)^n\left(t - \frac{nT}{2}\right)^2 h\left(t - \frac{nT}{2}\right).$$

18 (b). $m = 1,\ f_0 = 1,\ T = 1,\ t = \frac{5}{4} \Rightarrow v\left(\frac{5}{4}\right) = \left[t - (t - ½) + (t-1)\right]\Big|_{t=\frac{5}{4}} = (t - ½)\Big|_{t=\frac{5}{4}} = \frac{3}{4}\ m/s$ and

$$x\left(\frac{5}{4}\right) = \frac{1}{2}\left[t^2 - (t - ½)^2 + (t-1)^2\right]\Big|_{t=\frac{5}{4}} = \frac{1}{2}\left(\frac{25}{16} - \frac{9}{16} + \frac{1}{16}\right) = \frac{17}{32}\ m.$$

19. We know that $ay_i'' + by_i' + cy_i = f_i(t),\ i = 1,\ 2$. Therefore,

$$a\left(c_1y_1 + c_2y_2\right)'' + b\left(c_1y_1 + c_2y_2\right)' + c\left(c_1y_1 + c_2y_2\right) = c_1\left[ay_1'' + by_1' + cy_1\right] + c_2\left[ay_2'' + by_2' + cy_2\right]$$

$$= c_1f_1 + c_2f_2.$$

20 (a). $\mathcal{L}\{ay'' + by' + cy\} = \mathcal{L}\{f\}$. Since

$y(0) = 0,\ y'(0) = 0,\ \left(as^2 + bs + c\right)Y = F \Rightarrow \Phi(s) = \frac{1}{as^2 + bs + c} = \frac{1}{2s^2 + 5s + 2}$. Comparing:

$a = 2,\ b = 5,\ c = 2$.

20 (b). If $f(t)=e^{-t}$, $F(s)=\dfrac{1}{s+1}$, $Y(s)=\dfrac{1}{\left(2s^2+5s+2\right)(s+1)}$. Since

$2s^2+5s+2=2(s+2)(s+½)$, $Y(s)=\dfrac{1}{2}\left[\dfrac{1}{(s+2)(s+1)(s+½)}\right]=\dfrac{1}{2}\left[\dfrac{A}{s+2}+\dfrac{B}{s+1}+\dfrac{C}{s+½}\right]$. Then

$A=\left.\dfrac{1}{(s+1)(s+½)}\right|_{s=-2}=\dfrac{2}{3}$, $B=\left.\dfrac{1}{(s+2)(s+½)}\right|_{s=-1}=-2$, $C=\left.\dfrac{1}{(s+2)(s+1)}\right|_{s=-½}=\dfrac{4}{3}$ and

$y(t)=\dfrac{1}{3}e^{-2t}-e^{-t}+\dfrac{2}{3}e^{-\frac{t}{2}}$.

21. $f(t)=t$, $F(s)=\dfrac{1}{s^2}$, $y(t)=2\left(e^{-t}-1\right)+t\left(e^{-t}+1\right)$,

$Y(s)=\dfrac{2}{s+1}-\dfrac{2}{s}+\dfrac{1}{(s+1)^2}+\dfrac{1}{s^2}=\dfrac{-2\left(s^2+s\right)+2s^2+2s+1}{s^2(s+1)^2}=\dfrac{1}{s^2(s+1)^2}$. Since $Y(s)=\Phi(s)F(s)$,

$\Phi(s)=\dfrac{1}{(s+1)^2}$.

22. From 21, $\Phi(s)=\dfrac{1}{(s+1)^2}$. If $f(t)=h(t)$, $F(s)=\dfrac{1}{s}$ and $Y(s)=\dfrac{1}{s(s+1)^2}=\dfrac{A}{s}+\dfrac{B}{s+1}+\dfrac{C}{(s+1)^2}$

$A=\left.\dfrac{1}{(s+1)^2}\right|_{s=0}=1$, $C=\left.\dfrac{1}{s}\right|_{s=-1}=-1$. Setting $s=1$, $\dfrac{1}{4}=1+\dfrac{B}{2}-\dfrac{1}{4}\Rightarrow B=-1$. Therefore,

$Y(s)=\dfrac{1}{s}-\dfrac{1}{s+1}-\dfrac{1}{(s+1)^2}\Rightarrow y(t)=1-e^{-t}-te^{-t}$.

23 (a). $s^2Y+4Y=F$, so $\Phi=\dfrac{1}{\left(s^2+4\right)}$.

23 (b). $F=\dfrac{2}{s^3}$, so $Y=\dfrac{2}{s^3\left(s^2+4\right)}$.

24 (a). $s^2Y+sY+Y=F$, so $\Phi=\dfrac{1}{\left(s^2+s+1\right)}$.

24 (b). $\int_0^2 e^{-st}f(t)dt=\int_0^1 e^{-st}dt-\int_1^2 e^{-st}dt=\left.\dfrac{e^{-st}}{-s}\right|_0^1-\left.\dfrac{e^{-st}}{-s}\right|_1^2=\dfrac{1}{s}(1-e^{-s})+\dfrac{1}{s}(e^{-2s}-e^{-s})=\dfrac{1}{s}(1-e^{-s})^2$.

Therefore, $F=\dfrac{(1-e^{-s})^2}{s(1-e^{-2s})}=\dfrac{(1-e^{-s})}{s(1+e^{-s})}$, so $Y=\dfrac{(1-e^{-s})}{s(1+e^{-s})(s^2+s+1)}$.

25 (a). $s^2Y+4sY+4Y=F$, so $\Phi=\dfrac{1}{\left(s^2+4s+4\right)}=\dfrac{1}{(s+2)^2}$.

25 (b). $\int_0^1 te^{-st}dt = -\frac{1}{s^2}(st+1)e^{-st}\Big|_0^1 = \frac{1}{s^2}\left[1-(s+1)e^{-s}\right]$, so $F = \frac{1-(s+1)e^{-s}}{s^2\left(1-e^{-s}\right)}$ and $Y = \frac{1-(s+1)e^{-s}}{s^2\left(1-e^{-s}\right)(s+2)^2}$.

26 (a). $s^3Y - 4Y = F$, so $\Phi = \frac{1}{\left(s^3-4\right)}$.

26 (b). $F = \frac{1}{s-1} + \frac{1}{s^2} = \frac{s^2+s-1}{s^2(s-1)}$, so $Y = \frac{s^2+s-1}{s^2(s-1)(s^3-4)}$.

27 (a). $s^3Y + 4sY = F$, so $\Phi = \frac{1}{\left(s^3+4s\right)}$.

27 (b). $F = \frac{s}{s^2+4}$, so $Y = \frac{s}{\left(s^2+4\right)\left(s^3+4s\right)} = \frac{1}{\left(s^2+4\right)^2}$.

28. $y'' + by' + cy = f \Rightarrow s^2Y - sy(0) - y'(0) + b(sY - y(0)) + cY = F$. Therefore,

$(s^2+bs+c)Y - sy_0 - y_0' - by_0 = F \Rightarrow Y = \frac{sy_0 + y_0' + by_0}{s^2+bs+c} + \frac{F(s)}{s^2+bs+c}$. If

$f(t) = h(t),\ F(s) = \frac{1}{s} \Rightarrow Y(s) = \frac{s^2y_0 + s(y_0' + by_0) + 1}{s^3 + bs^2 + cs} = \frac{s^2+2s+1}{s^3+3s^2+2s}$. Therefore,

$b = 3,\ c = 2,\ y_0 = 1,\ y_0' + by_0 = y_0' + 3 = 2 \Rightarrow y_0' = -1$.

29. $Y = \frac{sy_0 + y_0' + by_0}{s^2+bs+c} + \frac{F(s)}{s^2+bs+c}$. If $f(t) = e^{-t},\ F(s) = \frac{1}{s+1}$ and

$\frac{(s+1)\left(sy_0 + y_0' + by_0\right) + 1}{(s+1)\left(s^2+bs+c\right)} = \frac{s^2+s+1}{(s+1)\left(s^2+4\right)}$. Therefore,

$b = 0,\ c = 4$, and $(s+1)\left(sy_0 + y_0'\right) + 1 = s^2y_0 + sy_0' + sy_0 + y_0' = s^2 + s + 1$. Finally, $y_0 = 1,\ y_0' = 0$.

Section 7.5

1. $$\mathcal{L}\left\{\begin{bmatrix}\cos t \\ t \\ te^t\end{bmatrix}\right\} = \begin{bmatrix}\frac{s}{s^2+1} \\ \frac{1}{s^2} \\ \frac{1}{(s-1)^2}\end{bmatrix}$$

2. $$\mathcal{L}\left\{\frac{d}{dt}\begin{bmatrix}e^{-t}\cos 2t \\ 0 \\ t+e^t\end{bmatrix}\right\} = s\mathcal{L}\left\{\frac{d}{dt}\begin{bmatrix}e^{-t}\cos 2t \\ 0 \\ t+e^t\end{bmatrix}\right\} - \begin{bmatrix}1 \\ 0 \\ 1\end{bmatrix} = s\begin{bmatrix}\frac{s+1}{(s+1)^2+4} \\ 0 \\ \frac{1}{s^2} + \frac{1}{s-1}\end{bmatrix} - \begin{bmatrix}1 \\ 0 \\ 1\end{bmatrix} = \begin{bmatrix}\frac{s(s+1)}{(s+1)^2+4} - 1 \\ 0 \\ \frac{1}{s} + \frac{s}{s-1} - 1\end{bmatrix}$$

3. $$\mathcal{L}\left\{\begin{bmatrix} 2t-h(t-2) \\ 2h(t-2) \end{bmatrix}\right\} = \begin{bmatrix} \frac{2}{s^2}-\frac{e^{-2s}}{s} \\ \frac{2e^{-2s}}{s} \end{bmatrix}$$

4. $$\mathcal{L}\left\{\int_0^t \begin{bmatrix} 1 \\ \lambda \\ e^{-\lambda} \end{bmatrix} d\lambda\right\} = \frac{1}{s}\mathcal{L}\left\{\begin{bmatrix} 1 \\ t \\ e^{-t} \end{bmatrix}\right\} = \begin{bmatrix} \frac{1}{s^2} \\ \frac{1}{s^3} \\ \frac{1}{s(s+1)} \end{bmatrix}$$

5. $$\mathcal{L}\left\{\begin{bmatrix} \sin(t-1)h(t-1) \\ e^{t-1}-2t \end{bmatrix}\right\} = \begin{bmatrix} \frac{e^{-s}}{s^2+1} \\ \frac{e^{-1}}{s-1}-\frac{2}{s^2} \end{bmatrix}$$

6. $$\mathcal{L}^{-1}\left\{\begin{bmatrix} 1/s \\ \frac{2}{s^2+2s+2} \\ \frac{1}{s^2+s} \end{bmatrix}\right\}; \ \frac{2}{(s+1)^2+1} = \mathcal{L}\{[2e^{-t}\sin t]\}; \ \frac{1}{s(s+1)} = \frac{1}{s}-\frac{1}{s+1}. \text{ Therefore, } \begin{bmatrix} 1 \\ 2e^{-t}\sin t \\ 1-e^{-t} \end{bmatrix}.$$

7. $$\mathcal{L}^{-1}\left\{\begin{bmatrix} e^{-s}\left(\frac{1}{s}-\frac{1}{s^2+1}\right) \\ \frac{2e^{-s}}{s^2+1} \end{bmatrix}\right\} = \begin{bmatrix} (1-\sin(t-1))h(t-1) \\ 2\sin(t-1)h(t-1) \end{bmatrix}$$

8. $$\mathcal{L}^{-1}\{\mathbf{Y}(s)\} = \begin{bmatrix} t^3-e^{2t}+2\sin t \\ 2t^3+3\sin t \\ t^3-2e^{2t}+\sin t \end{bmatrix}$$

9. $s\mathbf{Y}-\begin{bmatrix} 5 \\ 6 \end{bmatrix} = A\mathbf{Y}$, so $\begin{bmatrix} s-5 & 4 \\ -5 & s+4 \end{bmatrix}\mathbf{Y} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$. Thus $\mathbf{Y} = \frac{1}{s^2-s}\begin{bmatrix} s+4 & -4 \\ 5 & s-5 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{5s-4}{s^2-s} \\ \frac{6s-5}{s^2-s} \end{bmatrix}$, and

since $\frac{5s-4}{s^2-s} = \frac{A}{s}+\frac{B}{s-1} \Rightarrow A=4,\ B=1$ and $\frac{6s-5}{s^2-s} = \frac{5}{s}+\frac{1}{s-1}$, we have $\mathbf{y}(t) = \begin{bmatrix} 4+e^t \\ 5+e^t \end{bmatrix}$.

10. $s\mathbf{Y}-\mathbf{0} = A\mathbf{Y}+\begin{bmatrix} 0 \\ 1/s \end{bmatrix} = \begin{bmatrix} s-5 & 4 \\ -5 & s+4 \end{bmatrix}\mathbf{Y}+\begin{bmatrix} 0 \\ 1/s \end{bmatrix} \Rightarrow \mathbf{Y} = \frac{1}{s^2-s}\begin{bmatrix} s+4 & -4 \\ 5 & s-5 \end{bmatrix}\begin{bmatrix} 0 \\ 1/s \end{bmatrix} = \begin{bmatrix} \frac{-4}{s^2(s-1)} \\ \frac{s+4}{s^2(s-1)} \end{bmatrix}$, and

since $\frac{-4}{s^2(s-1)} = \frac{A}{s^2}+\frac{B}{s}+\frac{C}{s-1} \Rightarrow A=4,\ B=4,\ C=-4$. then $\frac{-4}{s^2(s-1)} = \frac{4}{s^2}+\frac{4}{s}-\frac{4}{s-1}$

and $\dfrac{s-5}{s^2(s-1)}=\dfrac{-5}{s^2(s-1)}+\dfrac{1}{s(s-1)}=\dfrac{5}{s^2}+\dfrac{5}{s}-\dfrac{5}{s-1}+\dfrac{1}{s-1}-\dfrac{1}{s}=\dfrac{5}{s^2}+\dfrac{4}{s}-\dfrac{4}{s-1}$. Therefore,

$$\mathbf{Y}(s)=\begin{bmatrix}\dfrac{4}{s^2}+\dfrac{4}{s}-\dfrac{4}{s-1}\\ \dfrac{5}{s^2}+\dfrac{4}{s}-\dfrac{4}{s-1}\end{bmatrix}\Rightarrow \mathbf{y}(t)=\begin{bmatrix}4t+4-4e^t\\ 5t+4-4e^t\end{bmatrix}.$$

11. $s\mathbf{Y}=A\mathbf{Y}+\begin{bmatrix}\dfrac{1}{s^2}\\ \dfrac{1}{s}\end{bmatrix}$, so $\begin{bmatrix}s-5 & 4\\ -3 & s+2\end{bmatrix}\mathbf{Y}=\begin{bmatrix}\dfrac{1}{s^2}\\ \dfrac{1}{s}\end{bmatrix}$. Thus

$$\mathbf{Y}=\frac{1}{s^2-3s+2}\begin{bmatrix}s+2 & -4\\ 3 & s-5\end{bmatrix}\begin{bmatrix}\dfrac{1}{s^2}\\ \dfrac{1}{s}\end{bmatrix}=\frac{1}{s^2(s-1)(s-2)}\begin{bmatrix}-3s+2\\ s^2-5s+3\end{bmatrix}.$$

$Y_1=\dfrac{-3s+2}{s^2(s-1)(s-2)}=\dfrac{A}{s^2}+\dfrac{B}{s}+\dfrac{C}{s-1}+\dfrac{D}{s-2}\Rightarrow A=1,\ B=0,\ C=1,\ D=-1$, so $y_1=t+e^t-e^{2t}$.

$Y_2=\dfrac{s^2-5s+3}{s^2(s-1)(s-2)}=\dfrac{A}{s^2}+\dfrac{B}{s}+\dfrac{C}{s-1}+\dfrac{D}{s-2}\Rightarrow A=\dfrac{3}{2},\ B=-\dfrac{1}{4},\ C=1,\ D=-\dfrac{3}{4}$, so

$y_2=\dfrac{3}{2}t-\dfrac{1}{4}+e^t-\dfrac{3}{4}e^{2t}$. Finally, we have $\mathbf{y}(t)=\begin{bmatrix}t+e^t-e^{2t}\\ \dfrac{3}{2}t-\dfrac{1}{4}+e^t-\dfrac{3}{4}e^{2t}\end{bmatrix}$.

12. From 11, $\mathbf{Y}=\dfrac{1}{(s-1)(s-2)}\begin{bmatrix}s+2 & -4\\ 3 & s-5\end{bmatrix}\begin{bmatrix}3\\ 2\end{bmatrix}=\dfrac{1}{(s-1)(s-2)}\begin{bmatrix}3s-2\\ 2s-1\end{bmatrix}$.

$Y_1=\dfrac{3s-2}{(s-1)(s-2)}=\dfrac{A}{s-1}+\dfrac{B}{s-2}\Rightarrow A=-1,\ B=4$, so $y_1=-e^t+4e^{2t}$.

$Y_2=\dfrac{2s-1}{(s-1)(s-2)}=\dfrac{A}{s-1}+\dfrac{B}{s-2}\Rightarrow A=-1,\ B=3$, so $y_2=-e^t+3e^{2t}$. Finally, we have

$$\mathbf{y}(t)=\begin{bmatrix}-e^t+4e^{2t}\\ -e^t+3e^{2t}\end{bmatrix}.$$

13. $s\mathbf{Y}=A\mathbf{Y}+\begin{bmatrix}2\\ 0\end{bmatrix}$, so $\begin{bmatrix}s-1 & -4\\ 1 & s-1\end{bmatrix}\mathbf{Y}=\begin{bmatrix}2\\ 0\end{bmatrix}$. Thus $\mathbf{Y}=\dfrac{1}{(s-1)^2+4}\begin{bmatrix}s-1 & 4\\ -1 & s-1\end{bmatrix}\begin{bmatrix}2\\ 0\end{bmatrix}$.

$Y_1=\dfrac{2(s-1)}{(s-1)^2+4}$, so $y_1=2e^t\cos 2t$. $Y_2=\dfrac{-2}{(s-1)^2+4}$, so $y_2=-e^t\sin 2t$. Finally, we have

$$\mathbf{y}(t)=\begin{bmatrix}2e^t\cos 2t\\ -e^t\sin 2t\end{bmatrix}.$$

14. $s\mathbf{Y}-\begin{bmatrix}3\\0\end{bmatrix}=A\mathbf{Y}+\begin{bmatrix}0\\ \frac{3}{s-1}\end{bmatrix}$, so $\begin{bmatrix}s-1 & -4\\ 1 & s-1\end{bmatrix}\mathbf{Y}=\begin{bmatrix}0\\ \frac{3}{s-1}\end{bmatrix}$. Thus $\mathbf{Y}=\frac{1}{(s-1)^2+4}\begin{bmatrix}s-1 & 4\\ -1 & s-1\end{bmatrix}\begin{bmatrix}0\\ \frac{3}{s-1}\end{bmatrix}$.

$Y_1=\frac{3(s-1)+\frac{12}{s-1}}{(s-1)^2+4}=\frac{3(s-1)}{(s-1)^2+4}+\frac{12}{(s-1)\left[(s-1)^2+4\right]}$. For

$\frac{12}{(s-1)\left[(s-1)^2+4\right]}=\frac{A}{s-1}+\frac{B(s-1)+C}{(s-1)^2+4}$, $A=3,\ B=-3,\ C=0$. Therefore,

$Y_1=\frac{3(s-1)}{(s-1)^2+4}+\frac{3}{s-1}-\frac{3(s-1)}{(s-1)^2+4}=\frac{3}{s-1}$, so $y_1=3e^t$. $Y_2=0$, so $y_2=0$. Finally, we have

$\mathbf{y}(t)=\begin{bmatrix}3e^t\\0\end{bmatrix}$.

15. Letting $t=\tau+1;\ \tau=t-1$, we have $\frac{d\mathbf{y}}{d\tau}=\begin{bmatrix}6 & -3\\ 8 & -5\end{bmatrix}\mathbf{y}$, $\mathbf{y}\big|_{\tau=0}=\begin{bmatrix}5\\10\end{bmatrix}$. Then $s\mathbf{Y}=A\mathbf{Y}+\begin{bmatrix}5\\10\end{bmatrix}$, so

$\begin{bmatrix}s-6 & 3\\ -8 & s+5\end{bmatrix}\mathbf{Y}=\begin{bmatrix}5\\10\end{bmatrix}$. Thus $\mathbf{Y}=\frac{1}{s^2-s-6}\begin{bmatrix}s+5 & -3\\ 8 & s-6\end{bmatrix}\begin{bmatrix}5\\10\end{bmatrix}$.

$Y_1=\frac{5s-5}{(s-3)(s+2)}=\frac{A}{s+2}+\frac{B}{s-3}\Rightarrow A=3,\ B=2$, so $y_1=3e^{-2\tau}+2e^{3\tau}$.

$Y_2=\frac{10s-20}{(s-3)(s+2)}=\frac{A}{s+2}+\frac{B}{s-3}\Rightarrow A=8,\ B=2$, so $y_2=8e^{-2\tau}+2e^{3\tau}$.Finally, we have

$\mathbf{y}(t)=\begin{bmatrix}3e^{-2(t-1)}+2e^{3(t-1)}\\ 8e^{-2(t-1)}+2e^{3(t-1)}\end{bmatrix}$.

16. $s^2\mathbf{Y}-s\begin{bmatrix}1\\0\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}=A\mathbf{Y}$, so $\begin{bmatrix}s^2+3 & 2\\ -4 & s^2-3\end{bmatrix}\mathbf{Y}=\begin{bmatrix}s\\1\end{bmatrix}$. Thus $\mathbf{Y}=\frac{1}{s^4-1}\begin{bmatrix}s^2-3 & -2\\ 4 & s^2+3\end{bmatrix}\begin{bmatrix}s\\1\end{bmatrix}$.

$Y_1=\frac{s^3-3s-2}{(s-1)(s+1)(s^2+1)}=\frac{A}{s+1}+\frac{B}{s-1}+\frac{Cs+D}{s^2+1}\Rightarrow A=0,\ B=-1,\ C=2,\ D=1$, so

$y_1=-e^t+2\cos t+\sin t$.

$Y_2=\frac{s^2+4s+3}{(s-1)(s+1)(s^2+1)}=\frac{s+3}{(s-1)(s^2+1)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+1}\Rightarrow A=2,\ B=-2,\ C=-1$, so

$y_2=2e^t-2\cos t-\sin t$. Finally, we have $\mathbf{y}(t)=\begin{bmatrix}-e^t+2\cos t+\sin t\\ 2e^t-2\cos t-\sin t\end{bmatrix}$.

17. $s^2\mathbf{Y} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\mathbf{Y} + \begin{bmatrix} \frac{1}{s^2} \\ \frac{1}{s} \end{bmatrix}$, so $\begin{bmatrix} s^2-1 & 1 \\ -1 & s^2+1 \end{bmatrix}\mathbf{Y} = \begin{bmatrix} \frac{1}{s^2} \\ \frac{1}{s} \end{bmatrix}$. Thus $\mathbf{Y} = \frac{1}{s^4}\begin{bmatrix} s^2+1 & -1 \\ 1 & s^2-1 \end{bmatrix}\begin{bmatrix} \frac{1}{s^2} \\ \frac{1}{s} \end{bmatrix}$.

$Y_1 = \frac{1}{s^4}\left(1 + \frac{1}{s^2} - \frac{1}{s}\right)$, so $y_1 = \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{4!}t^4 = \frac{t^5}{120} - \frac{t^4}{24} + \frac{t^3}{6}$. $Y_2 = \frac{1}{s^4}\left(\frac{1}{s^2} + s - \frac{1}{s}\right)$, so

$y_2 = \frac{1}{5!}t^5 + \frac{1}{2!}t^2 - \frac{1}{4!}t^4 = \frac{t^5}{120} - \frac{t^4}{24} + \frac{t^2}{2}$. Finally, we have

$$\mathbf{y}(t) = \begin{bmatrix} \frac{t^5}{120} - \frac{t^4}{24} + \frac{t^3}{6} \\ \frac{t^5}{120} - \frac{t^4}{24} + \frac{t^2}{2} \end{bmatrix} = \frac{1}{120}\begin{bmatrix} t^5 - 5t^4 + 20t^3 \\ t^5 - 5t^4 + 60t^2 \end{bmatrix}.$$

18. $s^2\mathbf{Y} - s\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\mathbf{Y} + \begin{bmatrix} \frac{2}{s} \\ \frac{1}{s} \end{bmatrix}$, so $\begin{bmatrix} s^2-1 & 1 \\ -1 & s^2+1 \end{bmatrix}\mathbf{Y} = \begin{bmatrix} \frac{2}{s} \\ \frac{1}{s} + s \end{bmatrix}$. Thus

$\mathbf{Y} = \frac{1}{s^4}\begin{bmatrix} s^2+1 & -1 \\ 1 & s^2-1 \end{bmatrix}\begin{bmatrix} \frac{2}{s} \\ \frac{1}{s} + s \end{bmatrix}$. $Y_1 = \frac{1}{s^4}\left(2s + \frac{2}{s} - \frac{1}{s} - s\right) = \frac{1}{s^3} + \frac{1}{s^5}$, so $y_1 = \frac{t^4}{4!} + \frac{t^2}{2!} = \frac{t^4}{24} + \frac{t^2}{2}$.

$Y_2 = \frac{1}{s^4}\left(\frac{2}{s} + s - \frac{1}{s} + s^3 - s\right) = \frac{1}{s} + \frac{1}{s^5}$, so $y_2 = 1 + \frac{t^4}{24}$. Finally, we have $\mathbf{y}(t) = \begin{bmatrix} \frac{t^2}{2} + \frac{t^4}{24} \\ 1 + \frac{t^4}{24} \end{bmatrix}$.

19. $s\mathbf{Y} - \mathbf{y}(0) = A\mathbf{Y}$, so $\begin{bmatrix} s-6 & -5 & 0 \\ 7 & s+6 & 0 \\ 0 & 0 & s+2 \end{bmatrix}\mathbf{Y} = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}$. Thus

$$\mathbf{Y} = \begin{bmatrix} \frac{s+6}{s^2-1} & \frac{5}{s^2-1} & 0 \\ \frac{-7}{s^2-1} & \frac{s-6}{s^2-1} & 0 \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix}\begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{2s-8}{s^2-1} \\ \frac{-4s+10}{s^2-1} \\ \frac{-1}{s+2} \end{bmatrix}.$$

$Y_1 = \frac{2s-8}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1} \Rightarrow A = 5,\ B = -3$, so $y_1 = 5e^{-t} - 3e^t$.

$Y_2 = \frac{-4s+10}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1} \Rightarrow A = -7,\ B = 3$, so $y_2 = -7e^{-t} + 3e^t$.

$Y_3 = \dfrac{-1}{s+2}$, so $y_3 = -e^{-2t}$. Finally, we have $\mathbf{y}(t) = \begin{bmatrix} 5e^{-t} - 3e^{t} \\ -7e^{-t} + 3e^{t} \\ -e^{-2t} \end{bmatrix}$.

20. $s\mathbf{Y} = A\mathbf{Y} + \begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{s} \\ -\frac{2}{s^2} \end{bmatrix}$, so $\begin{bmatrix} s-1 & 0 & 0 \\ 0 & s+1 & -1 \\ 0 & 0 & s-2 \end{bmatrix}\mathbf{Y} = \begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{s} \\ -\frac{2}{s^2} \end{bmatrix}$. Thus

$\mathbf{Y} = \begin{bmatrix} \frac{1}{s-1} & 0 & 0 \\ 0 & \frac{1}{s+1} & \frac{1}{(s+1)(s-2)} \\ 0 & 0 & \frac{1}{s-2} \end{bmatrix}\begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{s} \\ -\frac{2}{s^2} \end{bmatrix}$. $Y_1 = \dfrac{1}{(s-1)^2}$, so $y_1 = te^{t}$.

$Y_2 = \dfrac{s(s-2)-2}{s^2(s+1)(s-2)} = \dfrac{A}{s^2} + \dfrac{B}{s} + \dfrac{C}{s+1} + \dfrac{D}{s-2} \Rightarrow A = 1,\ B = \dfrac{1}{2},\ C = -\dfrac{1}{3},\ D = -\dfrac{1}{6}$, so

$y_2 = t + \dfrac{1}{2} - \dfrac{1}{3}e^{-t} - \dfrac{1}{6}e^{2t}$.

$Y_3 = \dfrac{-2}{s^2(s-2)} = \dfrac{A}{s^2} + \dfrac{B}{s} + \dfrac{C}{s-2} \Rightarrow A = 1,\ B = \dfrac{1}{2},\ C = -\dfrac{1}{2}$, so $y_3 = t + \dfrac{1}{2} - \dfrac{1}{2}e^{2t}$. Finally, we have

$$\mathbf{y}(t) = \begin{bmatrix} te^{t} \\ t + \frac{1}{2} - \frac{1}{3}e^{-t} - \frac{1}{6}e^{2t} \\ t + \frac{1}{2} - \frac{1}{2}e^{2t} \end{bmatrix}.$$

21 (a). $s^2 - 9s + 18 = (s-3)(s-6) = 0 \Rightarrow \lambda = 3,\ 6$.

21 (b). $s\mathbf{Y} - \mathbf{y}(0) = A\mathbf{Y} \Rightarrow \mathbf{Y} = (sI - A)^{-1}\mathbf{y}_0$. Then $-A^{-1} = \dfrac{1}{18}\begin{bmatrix} -2 & -1 \\ 4 & -7 \end{bmatrix} \Rightarrow A^{-1} = \dfrac{1}{18}\begin{bmatrix} 2 & 1 \\ -4 & 7 \end{bmatrix}$, and

$\det A^{-1} = \left(\dfrac{1}{18}\right)^2 \cdot 18 = \dfrac{1}{18}$. Thus $A = \left(A^{-1}\right)^{-1} = \dfrac{18}{18}\begin{bmatrix} 7 & -1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 4 & 2 \end{bmatrix}$.

22 (a). $s\mathbf{Y}_1 = A\mathbf{Y}_1 + \mathbf{G}(s),\ s\mathbf{Y}_2 = A\mathbf{Y}_2 + \mathbf{Y}_1 \Rightarrow \mathbf{Y}_1 = (sI - A)^{-1}\mathbf{G},\ \mathbf{Y}_2 = (sI - A)^{-1}\mathbf{Y}_1$ and

$\mathbf{Y}_2 = (sI - A)^{-2}\mathbf{G}(s)\ \therefore \Omega(s) = (sI - A)^{-2}$.

22 (b). $(sI-A)=\begin{bmatrix} s-1 & 1 \\ -1 & s+1 \end{bmatrix} \Rightarrow (sI-A)^{-1}=\frac{1}{s^2}\begin{bmatrix} s+1 & -1 \\ 1 & s-1 \end{bmatrix}$ and

$$\Omega(s)=(sI-A)^{-2}=\frac{1}{s^4}\begin{bmatrix} s+1 & -1 \\ 1 & s-1 \end{bmatrix}\begin{bmatrix} s+1 & -1 \\ 1 & s-1 \end{bmatrix}=\frac{1}{s^4}\begin{bmatrix} s^2+2s & -2s \\ 2s & s^2-2s \end{bmatrix}.$$

$$\mathbf{G}(s)=\begin{bmatrix} \frac{1}{s} \\ \frac{1}{s^2} \end{bmatrix} \Rightarrow \mathbf{Y}_2(s)=\frac{1}{s^4}\begin{bmatrix} s+2-\frac{2}{s} \\ 2+1-\frac{2}{s} \end{bmatrix}=\begin{bmatrix} -\frac{2}{s^5}+\frac{2}{s^4}+\frac{1}{s^3} \\ -\frac{2}{s^5}+\frac{3}{s^4} \end{bmatrix} \text{Therefore,}$$

$$\mathbf{Y}_2(t)=\begin{bmatrix} -\frac{2t^4}{4!}+\frac{2t^3}{3!}+\frac{t^2}{2!} \\ -\frac{2t^4}{4!}+\frac{3t^3}{3!} \end{bmatrix}=\begin{bmatrix} -\frac{t^4}{12}+\frac{t^3}{3}+\frac{t^2}{2} \\ -\frac{t^4}{12}+\frac{t^3}{2} \end{bmatrix}.$$

23. $\mathbf{Y}_1=\frac{1}{s-3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{Y}_2=\begin{bmatrix} \frac{3}{s+2}-\frac{3}{s-3} \\ \frac{8}{s+2}-\frac{3}{s-3} \end{bmatrix}=\begin{bmatrix} \frac{-15}{(s+2)(s-3)} \\ \frac{5s-30}{(s+2)(s-3)} \end{bmatrix}$. Therefore,

$$\begin{bmatrix} \frac{1}{s-3} & \frac{-15}{(s+2)(s-3)} \\ \frac{1}{s-3} & \frac{5s-30}{(s+2)(s-3)} \end{bmatrix}=\left(sI-A\right)^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix}, \text{ and so } \begin{bmatrix} \frac{-1}{3} & \frac{5}{2} \\ \frac{-1}{3} & 5 \end{bmatrix}=-A^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix}. \text{ Thus}$$

$$A=-\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} \frac{-1}{3} & \frac{5}{2} \\ \frac{-1}{3} & 5 \end{bmatrix}^{-1}=\frac{6}{5}\begin{bmatrix} 5 & \frac{-5}{2} \\ \frac{20}{3} & \frac{-25}{6} \end{bmatrix}=\begin{bmatrix} 6 & -3 \\ 8 & -5 \end{bmatrix}.$$

24. $\mathbf{Y}_1=\begin{bmatrix} \frac{1}{(s+2)^2} \\ \frac{1}{s+2} \end{bmatrix}$, $\mathbf{Y}_2=\begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix}$, $\mathbf{G}(s)=\begin{bmatrix} \frac{2}{s} \\ 0 \end{bmatrix}$. Therefore, $\begin{bmatrix} \frac{1}{(s+2)^2} & \frac{1}{s} \\ \frac{1}{s+2} & 0 \end{bmatrix}=\left(sI-A\right)^{-1}\begin{bmatrix} 0 & 1+\frac{2}{s} \\ 1 & 0 \end{bmatrix}$, and so

$$(sI-A)=\begin{bmatrix} 0 & \frac{s+2}{s} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{(s+2)^2} & \frac{1}{s} \\ \frac{1}{s+2} & 0 \end{bmatrix}^{-1}=-s(s+2)\begin{bmatrix} 0 & \frac{s+2}{s} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\frac{1}{s} \\ -\frac{1}{s+2} & \frac{1}{(s+2)^2} \end{bmatrix}$$

$$=-s(s+2)\begin{bmatrix} -\frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & -\frac{1}{s} \end{bmatrix}=\begin{bmatrix} s+2 & -1 \\ 0 & s+2 \end{bmatrix}=sI-\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}; \; A=\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}.$$

25 (a). $-V_1(s)+R_1I_1+LsI_1+R_2(I_1-I_2)=0$, $R_2(I_2-I_1)+\frac{1}{Cs}I_2+R_3I_2+V_2(s)=0$.

$$\begin{bmatrix} R_1+R_2+sL & -R_2 \\ -R_2 & R_2+R_3+\frac{1}{Cs} \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}=\begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}, \text{ so}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}=\frac{1}{(R_1+R_2+sL)\left(R_2+R_3+\frac{1}{Cs}\right)-R_2{}^2}\begin{bmatrix} R_2+R_3+\frac{1}{Cs} & R_2 \\ R_2 & R_1+R_2+sL \end{bmatrix}\begin{bmatrix} V_1 \\ -V_2 \end{bmatrix}.$$

25 (b). $$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix}=\frac{1}{(2+s)\left(2+\frac{1}{s}\right)-1}\begin{bmatrix} 2+\frac{1}{s} & 1 \\ 1 & 2+s \end{bmatrix}\begin{bmatrix} \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2} \end{bmatrix}=\frac{(s+1)^{-2}}{4+2s+\frac{2}{s}}\begin{bmatrix} \frac{s+1}{s} \\ -(s+1) \end{bmatrix}=\frac{s}{2(s+1)^4}\begin{bmatrix} \frac{s+1}{s} \\ -(s+1) \end{bmatrix}.$$

$I_1(s)=\frac{1}{2(s+1)^3}\Rightarrow i_1(t)=\frac{t^2}{4}e^{-t}$ and

$I_2(s)=\frac{-s}{2(s+1)^3}=\frac{-(s+1)+1}{2(s+1)^3}=\frac{-1}{2(s+1)^2}+\frac{1}{2(s+1)^3}\Rightarrow i_2(t)=-\frac{t}{2}e^{-t}+\frac{t^2}{4}e^{-t}$.

Section 7.6

1. First, let $\sigma=t-\lambda$, and differentiation yields $d\sigma=-d\lambda$. Then,

$$f*g=\int_0^t f(t-\lambda)g(\lambda)d\lambda=\int_t^0 f(\sigma)g(t-\sigma)(-d\sigma)=\int_0^t g(t-\sigma)f(\sigma)d\sigma=g*f.$$

2 (a). $f*g=\int_0^t h(t-\lambda)h(\lambda)d\lambda=\int_0^t 1d\lambda=t$

2 (b). $F=G=\frac{1}{s}$, so $f*g=\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)=t$

3 (a). $f*g=\int_0^t (t-\lambda)\lambda^2 d\lambda=\left(t\frac{\lambda^3}{3}-\frac{\lambda^4}{4}\right)\Big|_0^t=t^4\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{t^4}{12}$

3 (b). $F=\frac{1}{s^2}$, $G=\frac{2}{s^3}$, so $f*g=\mathcal{L}^{-1}\left\{\frac{2}{s^5}\right\}=\frac{2t^4}{4!}=\frac{t^4}{12}$

4 (a). $f*g=\int_0^t e^{(t-\lambda)}e^{-2\lambda}d\lambda=\left(e^t\frac{e^{-3\lambda}}{-3}\right)\Big|_0^t=\frac{e^t}{3}\left(1-e^{-3t}\right)=\frac{e^t-e^{-2t}}{3}$

4 (b). $F=\frac{1}{s-1}$, $G=\frac{1}{s+2}$, so $f*g=\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{1}{s-1}-\frac{1}{s+2}\right)\right\}=\frac{1}{3}\left(e^t-e^{-2t}\right)$

5 (a). $f*g=\int_0^t (t-\lambda)\sin\lambda d\lambda = -t\cos\lambda\Big|_0^t - \int_0^t \lambda\sin\lambda d\lambda$

$= -t(\cos t-1)-(-\lambda\cos\lambda+\sin\lambda)\Big|_0^t = -t\cos t + t + t\cos t - \sin t = t-\sin t$

5 (b). $F=\dfrac{1}{s^2},\ G=\dfrac{1}{s^2+1}$, so $f*g=\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s^2+1)}\right\}=\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}-\dfrac{1}{s^2+1}\right\}=t-\sin t$

6 (a). $f*g=\int_0^t \sin(t-\lambda)\cos\lambda d\lambda = \dfrac{1}{2}\int_0^t[\sin t+\sin(t-2\lambda)]d\lambda = \dfrac{1}{2}\left[t\sin t+\int_t^{-t}(\sin\sigma)\left(-\dfrac{1}{2}d\sigma\right)\right]$

$=\dfrac{1}{2}\left[t\sin t+\dfrac{1}{2}\cos\sigma\Big|_{-t}^{t}\right]=\dfrac{t}{2}\sin t$, where $\sigma=t-2\lambda$.

6 (b). $F=\dfrac{1}{s^2+1},\ G=\dfrac{s}{s^2+1}$, so $f*g=\mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}=\dfrac{t}{2}\sin t$

7 (a). $$f*g=\int_0^t (t-\lambda)[h(\lambda)-h(\lambda-1)]d\lambda = \begin{cases} \int_0^t (t-\lambda)d\lambda = -\dfrac{(t-\lambda)^2}{2}\Big|_0^t = \dfrac{t^2}{2}, & 0\le t\le 1 \\ \int_0^1 (t-\lambda)d\lambda = -\dfrac{(t-\lambda)^2}{2}\Big|_0^1 = \dfrac{t^2}{2}-\dfrac{(t-1)^2}{2}, & 1\le t<\infty \end{cases}$$

7 (b). $F=\dfrac{1}{s^2},\ G=\dfrac{(1-e^{-s})}{s}$, so $f*g=\mathcal{L}^{-1}\left\{\dfrac{1}{s^3}-\dfrac{e^{-s}}{s^3}\right\}=\dfrac{t^2}{2}-\dfrac{(t-1)^2}{2}h(t-1)$

8. $$P*\mathbf{y}=\mathcal{L}^{-1}\left\{\begin{bmatrix}\dfrac{1}{s} & \dfrac{1}{s-1}\\ 0 & \dfrac{1}{s^2}\end{bmatrix}\begin{bmatrix}\dfrac{1}{s}\\ \dfrac{1}{s+1}\end{bmatrix}\right\}=\mathcal{L}^{-1}\left\{\begin{bmatrix}\dfrac{1}{s^2}+\dfrac{1}{(s-1)(s+1)}\\ \dfrac{1}{s^2(s+1)}\end{bmatrix}\right\}.$$

$\dfrac{1}{(s-1)(s+1)}=\dfrac{1}{2}\left(\dfrac{1}{s-1}-\dfrac{1}{s+1}\right);\ \dfrac{1}{s^2(s+1)}=\dfrac{A}{s^2}+\dfrac{B}{s}+\dfrac{C}{s+1}\Rightarrow A=1,\ B=-1,\ C=1$. Therefore,

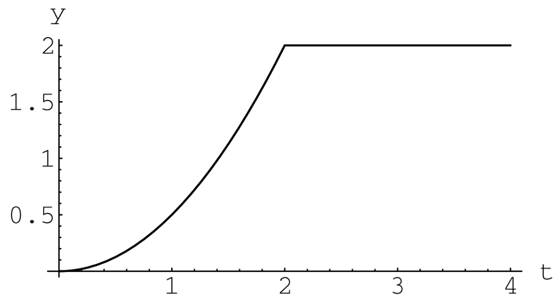
$$P*\mathbf{y}=\begin{bmatrix}t+\dfrac{1}{2}e^{t}-\dfrac{1}{2}e^{-t}\\ t-1+e^{-t}\end{bmatrix}$$

9. $t*\begin{bmatrix}t\\ \cos t\end{bmatrix}=\mathcal{L}^{-1}\left\{\begin{bmatrix}\dfrac{1}{s^4}\\ \dfrac{s}{s^2(s^2+1)}\end{bmatrix}\right\}$. $\mathcal{L}^{-1}\left\{\dfrac{1}{s^4}\right\}=\dfrac{t^3}{6}$. Then we have

$\dfrac{1}{s(s^2+1)}=\dfrac{A}{s}+\dfrac{Bs+C}{s^2+1}\Rightarrow A=1,\ B=-1,\ C=0$. $\mathcal{L}^{-1}\left\{\dfrac{1}{s}-\dfrac{s}{s^2+1}\right\}=1-\cos t$, so $t*\begin{bmatrix} t \\ \cos t \end{bmatrix} = \begin{bmatrix} \dfrac{t^3}{6} \\ 1-\cos t \end{bmatrix}$.

10. $g(t) = th(t) - (t-2)h(t-2) - 2h(t-2),\ F(s) = \dfrac{1}{s},\ G(s) = \dfrac{1}{s^2} - \dfrac{e^{-2s}}{s^2} - \dfrac{2e^{-2s}}{s},$

so $FG = \dfrac{1}{s^3} - \dfrac{e^{-2s}}{s^3} - \dfrac{2e^{-2s}}{s^2}$. Then $f*g = \dfrac{t^2}{2} - \dfrac{(t-2)^2}{2}h(t-2) - 2(t-2)h(t-2)$.

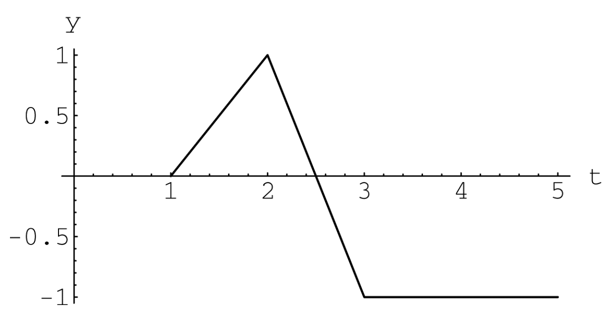


11. $F = G = \dfrac{e^{-s} - e^{-2s}}{s}$, so $FG = \dfrac{e^{-2s} - 2e^{-3s} + e^{-4s}}{s^2}$. Then

$f*g = (t-2)h(t-2) - 2(t-3)h(t-3) + (t-4)h(t-4)$.

12. $F(s) = \dfrac{1-e^{-s}}{s},\ G(s) = \dfrac{e^{-s} - 2e^{-2s}}{s}$, so $FG = \dfrac{e^{-s} - 3e^{-2s} + 2e^{-3s}}{s^2}$. Then

$f*g = (t-1)h(t-1) - 3(t-2)h(t-2) + 2(t-3)h(t-3)$.



13. $\mathcal{L}\{t*t*t\} = \left(\dfrac{1}{s^2}\right)^3 = \dfrac{1}{s^6}$, so $t*t*t = \dfrac{t^5}{5!} = \dfrac{t^5}{120}$.

14. $\mathcal{L}\{h(t)*e^{-t}*e^{-2t}\} = \dfrac{1}{s(s+1)(s+2)} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{s+2}$. Thus $A = \dfrac{1}{2},\ B = -1,\ C = \dfrac{1}{2}$ and

$h(t)*e^{-t}*e^{-2t} = \dfrac{1}{2} - e^{-t} + \dfrac{1}{2}e^{-2t}$.

15. $\mathcal{L}\{t * e^{-t} * e^{t}\} = \dfrac{1}{s^2(s+1)(s-1)} = \dfrac{A}{s^2} + \dfrac{B}{s} + \dfrac{C}{s-1} + \dfrac{D}{s+1}$. Thus $A=-1,\ B=0,\ C=\dfrac{1}{2},\ D=-\dfrac{1}{2}$

and $t * e^{-t} * e^{t} = \mathcal{L}^{-1}\left\{\dfrac{-1}{s^2} + \dfrac{\frac{1}{2}}{s-1} - \dfrac{\frac{1}{2}}{s+1}\right\} = -t + \dfrac{1}{2}\left(e^{t} - e^{-t}\right)$.

16. $\mathcal{L}\{h(t) * h(t) * \ldots * h(t)\} = \dfrac{1}{s^n}$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s^n}\right\} = \dfrac{t^{n-1}}{(n-1)!}$.

Therefore, $\dfrac{t^{n-1}}{(n-1)!} = Ct^8 \Rightarrow n = 9,\ C = \dfrac{1}{8!}$.

17. $\mathcal{L}\{\underbrace{e^{-t} * e^{-t} * \ldots * e^{-t}}_{\text{n times}}\} = \dfrac{1}{(s+1)^n}$, $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^n}\right\} = \dfrac{t^{n-1}}{(n-1)!}e^{-t} = Ct^4e^{\alpha t}$. Thus

$n = 5,\ C = \dfrac{1}{4!},\ \alpha = -1.$

18. $\int_0^t \sin(t-\lambda)y(\lambda)d\lambda = t^2 = \sin t * y$. Therefore,

$\dfrac{2}{s^3} = \dfrac{1}{s^2+1}Y \Rightarrow Y = \dfrac{2\left(s^2+1\right)}{s^3} = \dfrac{2}{s} + \dfrac{2}{s^3} \Rightarrow y(t) = 2 + t^2.$

19. $t^2e^{-t} = \int_0^t \cos(t-\lambda)y(\lambda)d\lambda = \cos t * y$. Therefore,

$\dfrac{2}{(s+1)^3} = \dfrac{s}{s^2+1}Y \Rightarrow Y = \dfrac{2\left(s^2+1\right)}{s(s+1)^3} = \dfrac{A}{s} + \dfrac{B}{s+1} + \dfrac{C}{(s+1)^2} + \dfrac{D}{(s+1)^3}$. Thus

$A = 2,\ B = -2,\ C = 0,\ D = -4$, and so we have $Y = \dfrac{2}{s} - \dfrac{2}{s+1} - \dfrac{4}{(s+1)^3}$. Finally,

$y(t) = 2 - 2e^{-t} - 2t^2e^{-t}$.

20. $y(t) - \int_0^t e^{t-\lambda}y(\lambda)d\lambda = t \Rightarrow y - e^{t} * y = t,\ Y - \dfrac{1}{s-1}Y = \dfrac{1}{s^2}$. Therefore,

$\dfrac{1}{s^2} = \dfrac{s-2}{s-1}Y \Rightarrow Y = \dfrac{s-1}{s^2(s-2)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-2}$. Thus $A = -\dfrac{1}{4},\ B = \dfrac{1}{2},\ C = \dfrac{1}{4}$, and so we have

$Y = \dfrac{-1/4}{s} + \dfrac{1/2}{s^2} + \dfrac{1/4}{s-2}$. Finally, $y(t) = -\dfrac{1}{4} + \dfrac{t}{2} + \dfrac{1}{4}e^{2t}$.

21. $\int_0^t y(t-\lambda)y(\lambda)d\lambda = 6t^3 = y * y \Rightarrow Y^2 = (6)\dfrac{3!}{s^4} = \dfrac{36}{s^4} \Rightarrow Y = \pm\dfrac{6}{s^2}$ and $y(t) = \pm 6t$.

22. $\frac{1}{s^2}Y = \frac{2}{s^3} - \frac{2}{(s+1)^3} \Rightarrow Y = \frac{2}{s} - \frac{2s^2}{(s+1)^3}$.

$s^2 = (s^2+2s+1)-(2s+1) = (s+1)^2-(2s+2-1) = (s+1)^2-2(s+1)+1$. Therefore,

$Y = \frac{2}{s} - 2\left(\frac{1}{s+1} - \frac{2}{(s+1)^2} + \frac{1}{(s+1)^3}\right)$, so $y(t) = 2-2\left(e^{-t}-2te^{-t}+\frac{t^2}{2}e^{-t}\right) = 2-2\left(1-2t+\frac{t^2}{2}\right)e^{-t}$.

23. $sY - 0 + \frac{1}{s+2}Y = \frac{1}{s} \Rightarrow Y = \frac{1}{s}\left(\frac{s+2}{(s+1)^2}\right) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$. Thus $A=2,\ B=-2, C=-1$, and

$Y = \frac{2}{s} - \frac{2}{s+1} - \frac{1}{(s+1)^2}$, so $y(t) = 2-2e^{-t}-te^{-t}$.

24. $s\mathbf{Y} - \begin{bmatrix}1\\2\end{bmatrix} = \frac{1}{s}\mathbf{Y} \Rightarrow \left(s-\frac{1}{s}\right)\mathbf{Y} = \begin{bmatrix}1\\2\end{bmatrix} \Rightarrow \mathbf{Y} = \frac{s}{s^2-1}\begin{bmatrix}1\\2\end{bmatrix} = \frac{1}{2}\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\begin{bmatrix}1\\2\end{bmatrix}$, so

$\mathbf{y}(t) = \frac{1}{2}\left(e^{t}+e^{-t}\right)\begin{bmatrix}1\\2\end{bmatrix} = \cosh t\begin{bmatrix}1\\2\end{bmatrix}$.

25. $sY - 1 = \frac{1}{s^2}\cdot\frac{1}{s^2} \Rightarrow Y = \frac{1}{s} + \frac{1}{s^5} \Rightarrow y(t) = 1 + \frac{t^4}{4!} = 1 + \frac{t^4}{24}$.

26. $sY - (-1) - Y = \frac{1}{s^2}\cdot\frac{1}{s-1} \Rightarrow (s-1)Y = -1 + \frac{1}{s^2(s-1)} \Rightarrow Y = \frac{-1}{s-1} + \frac{1}{s^2(s-1)^2}$.

$\frac{1}{s^2(s-1)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} \Rightarrow A=2,\ B=1,\ C=-2,\ D=1$. Thus,

$Y = -\frac{1}{s-1} + \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2} = \frac{2}{s} + \frac{1}{s^2} - \frac{3}{s-1} + \frac{1}{(s-1)^2}$, so $y(t) = 2+t-3e^{t}+te^{t}$.

Section 7.7

1 (a). $\int_0^3 \left(1+e^{-t}\right)\delta(t-2)\,dt = 1+e^{-2}$

1 (b). $\int_{-2}^1 \left(1+e^{-t}\right)\delta(t-2)\,dt = 0$ since $t=2$ lies outside the integration interval.

1 (c). $\int_{-1}^2 \begin{bmatrix}\cos 2t\\ te^{-t}\end{bmatrix}\delta(t)\,dt = \begin{bmatrix}1\\0\end{bmatrix}$

1 (d). $\int_{-3}^2 \left(e^{2t}+t\right)\begin{bmatrix}\delta(t+2)\\ \delta(t-1)\\ \delta(t-3)\end{bmatrix}dt = \begin{bmatrix}e^{-4}-2\\ e^2+1\\ 0\end{bmatrix}$

2. From Equation 7b, $f*\delta = f(t)$.

3. $\int_0^1 \sin^2[\pi(t-t_0)]\delta\left(t-\frac{1}{2}\right)dt = \sin^2\left[\pi\left(\frac{1}{2}-t_0\right)\right] = \frac{3}{4} \Rightarrow \left|\sin\left[\pi\left(\frac{1}{2}-t_0\right)\right]\right| = \frac{\sqrt{3}}{2}$

One possible t_0: $\pi\left(\frac{1}{2}-t_0\right) = \frac{\pi}{3} \Rightarrow \frac{1}{2}-t_0 = \frac{1}{3} \Rightarrow t_0 = \frac{1}{6}$.

4. $\int_1^5 t^n \delta(t-2)dt = 2^n = 8 \Rightarrow n = 3$.

5. $f(t) = 1 - h(1-t) = h(t-1)$ for all t except t=1.

6. $g(t) = \int_0^t h(\lambda-1)d\lambda = \begin{cases} 0, & t<1 \\ t-1, & t \ge 1 \end{cases}$. Therefore, $g(t) = (t-1)h(t-1)$.

7. $k = h(2-t) - h(1-t) = h(t-1) - h(t-2)$ for all t except t=1, 2.

8. $g(t) = \int_0^t e^{\alpha t}\delta(t-t_0)dt = \begin{cases} 0, & 0 \le t \le t_0 \\ e^{\alpha t_0}, & t_0 < t < \infty \end{cases}$. Therefore, $t_0 = 2,\ e^{2\alpha} = e^{-2} \Rightarrow \alpha = -1$.

9 (a). $\left(e^{-t}y\right)' = e^{-t} \Rightarrow e^{-t}y = -e^{-t} + C \Rightarrow y = -1 + Ce^t$. From the initial condition, we have $y(0) = 0 = -1 + C$. Thus C=1 and $y = -1 + e^t$.

9 (b). $s\Phi - \Phi = 1 \Rightarrow \Phi = \frac{1}{s-1}$. Therefore, $\phi(t) = e^t$.

9 (c). $\phi * g = \int_0^t e^{(t-\lambda)}h(\lambda)d\lambda = \int_0^t e^{t-\lambda}d\lambda = e^t\left(-e^{-\lambda}\right)\Big|_0^t = -1 + e^t$

10 (a). $\left(e^{-t}y\right)' = 1 \Rightarrow e^{-t}y = t + C \Rightarrow y = te^t + Ce^t$. From the initial condition, we have $y(0) = 0 = C$. Thus $y = te^t$.

10 (b). From 9b, $\phi(t) = e^t$.

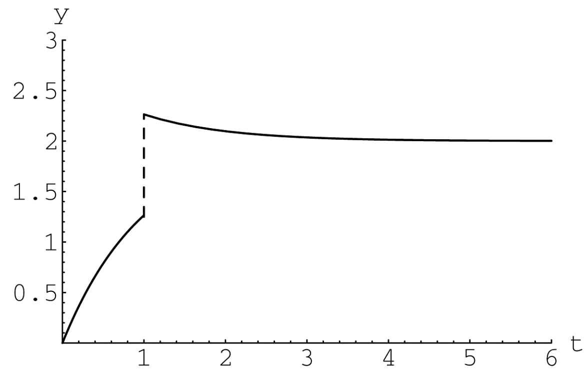
10 (c). $\phi * g = \int_0^t e^{(t-\lambda)}e^{\lambda}d\lambda = e^t \int_0^t d\lambda = te^t$.

11 (a). $\left(e^{-t}y\right)' = te^{-t} \Rightarrow e^{-t}y = -te^{-t} - e^{-t} + C \Rightarrow y = -(t+1) + Ce^t$. From the initial condition, we have $y(0) = 0 = -1 + C$. Thus C=1 and $y = e^t - (t+1)$.

11 (b). $s\Phi - \Phi = 1 \Rightarrow \Phi = \frac{1}{s-1}$. Therefore, $\phi(t) = e^t$.

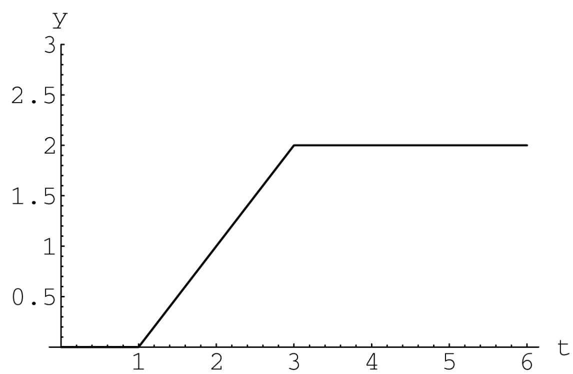
11 (c). $\phi * g = \int_0^t e^{(t-\lambda)}\lambda d\lambda = e^t\left(-\lambda e^{-\lambda} - e^{-\lambda}\right)\Big|_0^t = e^t - t - 1$

12. $sY + Y = \frac{2}{s} + e^{-s} \Rightarrow Y = \frac{2}{s(s+1)} + \frac{e^{-s}}{s+1},\ \frac{2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow A = 2,\ B = -2$. Therefore, $y(t) = 2 - 2e^{-t} + e^{-(t-1)}h(t-1)$.



13. $sY + Y = e^{-s} - e^{-2s} \Rightarrow Y = \frac{e^{-s}}{s+1} - \frac{e^{-2s}}{s+1}$. Therefore, $y(t) = e^{-(t-1)}h(t-1) - e^{-(t-2)}h(t-2)$.

14. $s^2Y = e^{-s} - e^{-3s} \Rightarrow Y = \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2}$. Therefore, $y(t) = (t-1)h(t-1) - (t-3)h(t-3)$.

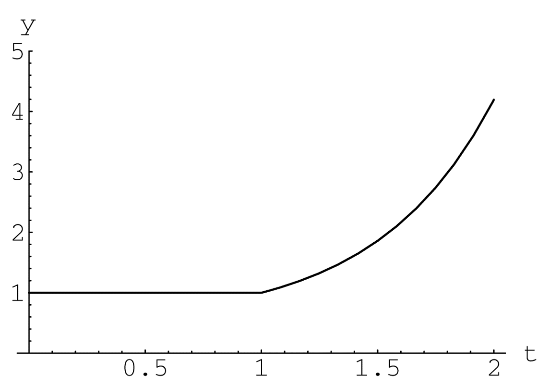


15. $\left(s^2 + 4\pi^2\right)Y = 2\pi e^{-2s} \Rightarrow Y = \frac{2\pi}{s^2 + 4\pi^2}e^{-2s}$. Therefore, $y(t) = \sin\left(2\pi(t-2)\right)h(t-2)$.

16. $s^2Y - s - 2(sY-1) = e^{-s} \Rightarrow Y = \frac{s-2}{s(s-2)} + \frac{e^{-s}}{s(s-2)} = \frac{1}{s} + \frac{e^{-s}}{s(s-2)}$.

$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} \Rightarrow A = -\frac{1}{2},\ B = \frac{1}{2}$. Therefore, $Y = \frac{1}{s} + e^{-s}\left(-\frac{1}{2s} + \frac{1}{s(s-2)}\right)$ and

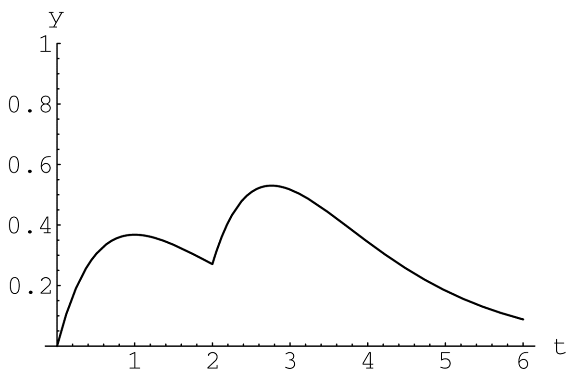
$y(t) = 1 - \frac{1}{2}h(t-1) + \frac{1}{2}e^{2(t-1)}h(t-1)$.



17. $\left(s^2+2s+2\right)Y = e^{-s} \Rightarrow Y = e^{-s}\dfrac{1}{(s+1)^2+1}$. Therefore, $y(t) = e^{-(t-1)}\sin(t-1)h(t-1)$.

18. $s^2Y - 1 + 2sY + Y = e^{-2s} \Rightarrow \left(s^2+2s+1\right)Y = 1 + e^{-2s} \Rightarrow Y = \dfrac{1}{(s+1)^2} + \dfrac{e^{-2s}}{(s+1)^2}$. Therefore,

$y(t) = te^{-t} + (t-2)e^{-(t-2)}h(t-2)$.



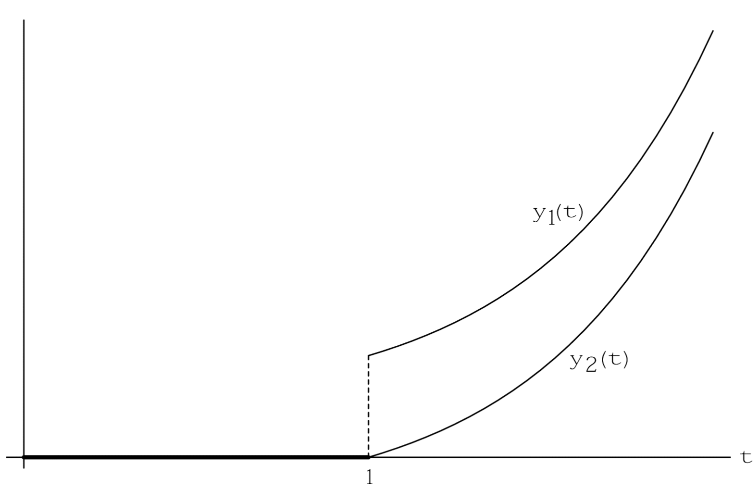
19. $s\mathbf{Y} = A\mathbf{Y} + e^{-s}\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow \mathbf{Y} = e^{-s}(sI-A)^{-1}\begin{bmatrix}1\\0\end{bmatrix}$. $sI - A = \begin{bmatrix}s-1 & -1\\ -1 & s-1\end{bmatrix}$, so

$(sI-A)^{-1} = \dfrac{1}{s^2-2s}\begin{bmatrix}s-1 & 1\\ 1 & s-1\end{bmatrix}$. Then $\mathbf{Y} = \dfrac{e^{-2s}}{s(s-2)}\begin{bmatrix}s-1 & 1\\ 1 & s-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \dfrac{e^{-2s}}{s(s-2)}\begin{bmatrix}s-1\\1\end{bmatrix}$.

$\dfrac{s-1}{s(s-2)} = \dfrac{1}{s} + \dfrac{1}{s(s-2)}$, $\dfrac{1}{s(s-2)} = \dfrac{-\frac{1}{2}}{s} + \dfrac{\frac{1}{2}}{s-2} \Rightarrow \dfrac{s-1}{s(s-2)} = \dfrac{\frac{1}{2}}{s} + \dfrac{\frac{1}{2}}{s-2}$, so

$y_1(t) = \dfrac{1}{2}\left(1+e^{2(t-1)}\right)h(t-1)$ and $y_2(t) = \dfrac{1}{2}\left(-1+e^{2(t-1)}\right)h(t-1)$. Finally, we have

$$\mathbf{y}(t) = \begin{bmatrix}\frac{1}{2}\left(1+e^{2(t-1)}\right)h(t-1)\\ \frac{1}{2}\left(-1+e^{2(t-1)}\right)h(t-1)\end{bmatrix}.$$



20. $s\mathbf{Y} = A\mathbf{Y} + \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix} - e^{-s}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{Y} = (sI - A)^{-1}\left(\begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix} - e^{-s}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. $sI - A = \begin{bmatrix} s-2 & -1 \\ 0 & s-1 \end{bmatrix}$, so

$(sI - A)^{-1} = \dfrac{1}{(s-1)(s-2)}\begin{bmatrix} s-1 & 1 \\ 0 & s-2 \end{bmatrix}$. Then

$$\mathbf{Y} = \frac{1}{(s-1)(s-2)}\begin{bmatrix} \frac{1}{s} - e^{-s}(s-1) \\ \frac{s-2}{s} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s(s-1)(s-2)} - \dfrac{e^{-s}}{(s-2)} \\ \dfrac{1}{s(s-1)} \end{bmatrix}.$$

$\dfrac{1}{s(s-1)(s-2)} = \dfrac{A}{s} + \dfrac{B}{s-1} + \dfrac{C}{s-2} \Rightarrow A = \dfrac{1}{2},\ B = -1,\ C = \dfrac{1}{2};\ \dfrac{1}{s(s-1)} = -\dfrac{1}{s} + \dfrac{1}{s-1}$, so

$y_1(t) = \dfrac{1}{2} - e^t + \dfrac{1}{2}e^{2t} - e^{2(t-1)}h(t-1)$ and $y_2(t) = -1 + e^t$.