Chapter 8
Nonlinear Systems

Section 8.1

1 (a). For $y'' + ty = \sin y', y(0) = 0, y'(0) = 1,$ let $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. Thus,

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ -ty + \sin y' \end{bmatrix} = \begin{bmatrix} y_2 \\ -ty_1 + \sin y_2 \end{bmatrix}, \ \mathbf{y}(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

1 (b). From part (a), $\mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} f_1(t,y_1,y_2) \\ f_2(t,y_1,y_2) \end{bmatrix} = \begin{bmatrix} y_2 \\ -ty_1 + \sin y_2 \end{bmatrix}$. Therefore, the requested partial derivatives are $\frac{\partial f_1}{\partial y_1} = 0,\ \frac{\partial f_1}{\partial y_2} = 1,\ \frac{\partial f_2}{\partial y_1} = -t,\ \frac{\partial f_2}{\partial y_2} = \cos y_2$.

1 (c). There are no points in 3-dimensional space where the hypotheses of Theorem 8.1 fail to be satisfied.

2 (a). For $y'' + (y')^3 + y^{1/3} = \tan(t/2), y(1) = 1, y'(1) = -2$, let $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. Thus,

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ \tan(t/2) - y_1^{1/3} - y_2^3 \end{bmatrix}, \ \mathbf{y}(1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

2 (b). For $\mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} f_1(t,y_1,y_2) \\ f_2(t,y_1,y_2) \end{bmatrix}$, the requested partial derivatives are $\frac{\partial f_1}{\partial y_1} = 0,\ \frac{\partial f_1}{\partial y_2} = 1,\ \frac{\partial f_2}{\partial y_1} = -\frac{1}{3}y_1^{-2/3},\ \frac{\partial f_2}{\partial y_2} = -3y_2^2$.

2 (c). The hypotheses of Theorem 8.1 are not satisfied at $t = \pm(2n+1)\pi/2$ and $y_1 = 0$.

3 (a). For $y'' + t^{-1}(1 + y + 2y')^{-1} = t^{-1}e^{-t}, y(2) = 2, y'(2) = 1,$ let

$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. Thus,

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ -t^{-1}(1 + y + 2y')^{-1} + t^{-1}e^{-t} \end{bmatrix} = \begin{bmatrix} y_2 \\ -t^{-1}(1 + y_1 + 2y_2)^{-1} + t^{-1}e^{-t} \end{bmatrix},$$

$$\mathbf{y}(2) = \begin{bmatrix} y_1(2) \\ y_2(2) \end{bmatrix} = \begin{bmatrix} y(2) \\ y'(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

3 (b). From part (a), $\mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} f_1(t,y_1,y_2) \\ f_2(t,y_1,y_2) \end{bmatrix} = \begin{bmatrix} y_2 \\ -t^{-1}(1 + y_1 + 2y_2)^{-1} + t^{-1}e^{-t} \end{bmatrix}$.

Therefore, the requested partial derivatives are

$$\frac{\partial f_1}{\partial y_1} = 0,\ \frac{\partial f_1}{\partial y_2} = 1,\ \frac{\partial f_2}{\partial y_1} = t^{-1}(1 + y_1 + 2y_2)^{-2},\ \frac{\partial f_2}{\partial y_2} = 2t^{-1}(1 + y_1 + 2y_2)^{-2}.$$

3 (c). The hypotheses of Theorem 8.1 are satisfied everywhere except on the planes $t=0$ and $1+y_1+2y_2=0$.

4 (a). For $y'''+\cos(ty')=t(y'')^2, y(0)=1, y'(0)=1, y''(0)=-2$, let

$$\mathbf{y}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}=\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}. \text{ Thus, } \mathbf{y}'=\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix}=\begin{bmatrix} y_2 \\ y_3 \\ -\cos(ty_2)+y_3^2 \end{bmatrix},\ \mathbf{y}(0)=\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

4 (b). For $\mathbf{f}(t,\mathbf{y})=\begin{bmatrix} f_1(t,y_1,y_2,y_3) \\ f_2(t,y_1,y_2,y_3) \\ f_3(t,y_1,y_2,y_3) \end{bmatrix}$, the requested partial derivatives are

$$\frac{\partial f_1}{\partial y_1}=0,\ \frac{\partial f_1}{\partial y_2}=1,\ \frac{\partial f_1}{\partial y_3}=0,\ \frac{\partial f_2}{\partial y_1}=0,\ \frac{\partial f_2}{\partial y_2}=0,\frac{\partial f_2}{\partial y_3}=1,$$
$$\frac{\partial f_3}{\partial y_1}=0,\ \frac{\partial f_3}{\partial y_2}=t\sin(ty_2),\frac{\partial f_3}{\partial y_3}=2ty_3 .$$

4 (c). The hypotheses of Theorem 8.1 are satisfied in all of $ty_1y_2y_3$ - space.

5 (a). For $y'''+2t^{1/3}(y-2)^{-1}(y''+2)^{-1}=0, y(0)=0, y'(0)=2, y''(0)=2$, let

$$\mathbf{y}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}=\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}. \text{ Thus,}$$

$$\mathbf{y}'=\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix}=\begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix}=\begin{bmatrix} y' \\ y'' \\ -2t^{1/3}(y-2)^{-1}(y''+2)^{-1} \end{bmatrix}=\begin{bmatrix} y_2 \\ y_3 \\ -2t^{1/3}(y_1-2)^{-1}(y_3+2)^{-1} \end{bmatrix},$$

$$\mathbf{y}(0)=\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix}=\begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix}=\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

5 (b). From part (a), $\mathbf{f}(t,\mathbf{y})=\begin{bmatrix} f_1(t,y_1,y_2,y_3) \\ f_2(t,y_1,y_2,y_3) \\ f_3(t,y_1,y_2,y_3) \end{bmatrix}=\begin{bmatrix} y_2 \\ y_3 \\ -2t^{1/3}(y_1-2)^{-1}(y_2+2)^{-1} \end{bmatrix}$.

Therefore, the requested partial derivatives are

$$\frac{\partial f_1}{\partial y_1}=0,\ \frac{\partial f_1}{\partial y_2}=1,\ \frac{\partial f_1}{\partial y_3}=0$$
$$\frac{\partial f_2}{\partial y_1}=0,\ \frac{\partial f_2}{\partial y_2}=0,\ \frac{\partial f_2}{\partial y_3}=1 .$$
$$\frac{\partial f_3}{\partial y_1}=2t^{1/3}(y_1-2)^{-2}(y_3+2)^{-1},\ \frac{\partial f_3}{\partial y_2}=0,\ \frac{\partial f_3}{\partial y_3}=2t^{1/3}(y_1-2)^{-1}(y_3+2)^{-2}$$

5 (c). The hypotheses of Theorem 8.1 are satisfied everywhere except on the "hyperplanes" $y_1=2$ and $y_3=-2$.

6. Since $y_2'=t\cos^2(y_2)-3y_1+t^4$, it follows that the scalar problem is $y''=t\cos^2(y')-3y+t^4$, $y(2)=1, y'(2)=-1$.

7. Since $y_2'=y_2\tan y_1+e^{y_2}$, it follows that the scalar problem is $y''=y'\tan y+e^{y'}$, $y(0)=0, y'(0)=1$.

8. Since $y_3' = y_1 y_2 + y_3^2$, it follows that the scalar problem is $y''' = yy' + (y'')^2$, $y(-1) = -1, y'(-1) = 2, y''(-1) = -4$.

9. Since $y_3' = (y_2 y_3 + t^2)^{1/2}$, it follows that the scalar problem is $y''' = (y'y'' + t^2)^{1/2}$, $y(1) = 1, y'(1) = 1/2, y''(1) = 3$.

11. Laplace transforms cannot be productively used because the equation is nonlinear.

14 (a). Let $a = \pi/(2\delta)$. Then $\tan ax = ax + (1/3)a^3x^3 + (2/15)a^5x^5 + \cdots$. Retaining the first term of the Maclaurin series in equation (7), we have $mx'' + (2k\delta/\pi)\tan(\pi x/2\delta) \approx mx'' + (2k\delta/\pi)(\pi x/2\delta) = mx'' + kx$.

14 (b). As in part (a), retaining the first two terms of the Maclaurin series in equation (7) results in equation (8).

14 (c). Equation (7) becomes $\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ -(2k\delta/m\pi)\tan(\pi y_1/2\delta) \end{bmatrix}$.

Equation (8) becomes $\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ -(k/m)(y_1 + (\pi^2/12\delta^2)y_1^3) \end{bmatrix}$.

14 (d). The system version of equation (7) satisfies the hypotheses of Theorem 8.1 everywhere except along $y_1 = \pm(2n+1)\pi/2$. The system version of equation (8) satisfies the hypotheses of Theorem 8.1 everywhere in ty_1y_2 - space

15 (a). Adding equations 3 and 4, we obtain $\frac{dc}{dt} + \frac{de}{dt} = 0$. Thus, using the linearity of differentiation, $\frac{d(c+e)}{dt} = 0$ and hence, $c(t) + e(t) \equiv e_0$ is a constant function.

15 (b). Substituting $e(t) = e_0 - c(t)$ in equations 1 and 3, we find $\frac{da}{dt} = -k_1 e_0 a(t) + k_1 c(t)a(t) + k_1' c(t)$ and $\frac{dc}{dt} = k_1 e_0 a(t) - k_1 c(t)a(t) - (k_1' + k_2)c(t)$.

15 (c). The hypotheses of Theorem 8.1 are satisfied for all points in (t,a,c) - space.

16 (a). At the instant shown in the figure,

$$V_{\text{sub}} = (2/3)\pi R^3 + \int_0^{y(t)} \pi r^2 dy = (2/3)\pi R^3 + \int_0^{y(t)} \pi(R^2 - y^2)\,dy$$

$$= (2/3)\pi R^3 + \pi[R^2 y(t) - (1/3)(y(t))^3].$$

16 (b). Equation (10) is physically relevant as long as $-R \le y(t) \le R$.

Section 8.2

1. For

$$x' = x(-1+y)$$
$$y' = y(1-x),$$

we see that $x' = 0$ if (a) $x = 0$ or (b) $y = 1$. In Case (a), we have $y' = 0$ only if $y = 0$, yielding the equilibrium point $(x,y) = (0,0)$. In Case (b), we have $y' = 0$ only if $x = 1$, yielding the equilibrium point $(x,y) = (1,1)$.

2. For

$$x' = y(x+3)$$
$$y' = (x-1)(y-2),$$

we see that $x'=0$ if (a) $x=-3$ or (b) $y=0$. In Case (a), we have $y'=0$ only if $y=2$, yielding the equilibrium point $(x,y)=(-3,2)$. In Case (b), we have $y'=0$ only if $x=1$, yielding the equilibrium point $(x,y)=(1,0)$.

3. For

$$x' = (x-2)(y+1)$$
$$y' = x^2-4x+3,$$

we see that $x'=0$ if (a) $x=2$ or (b) $y=-1$. In Case (a), we cannot have $y'=0$. In Case (b), we have $y'=0$ only if $x=1$ or $x=3$, yielding the equilibrium points $(x,y)=(1,-1)$ and $(x,y)=(3,-1)$.

4. For

$$x' = (x-1)(y+1)$$
$$y' = (x-2)y,$$

we see that $x'=0$ if (a) $x=1$ or (b) $y=-1$. In Case (a), we have $y'=0$ only if $y=0$, yielding the equilibrium point $(x,y)=(1,0)$. In Case (b), we have $y'=0$ only if $x=2$, yielding the equilibrium point $(x,y)=(2,-1)$.

5. For

$$x' = x(x-2y)$$
$$y' = y(3x-y),$$

we see that $x'=0$ if (a) $x=0$ or (b) $x=2y$. In Case (a), we have $y'=0$ only if $y=0$, yielding the equilibrium point $(x,y)=(0,0)$. In Case (b), we have $y'=0$ only if $y=0$, yielding the same equilibrium point as in Case (a), $(x,y)=(0,0)$.

6. For

$$x' = y(y-x)$$
$$y' = x(x+2y),$$

we see that $x'=0$ if (a) $\mathrm{y}=0$ or (b) $y=x$. In Case (a), we have $y'=0$ only if $x=0$, yielding the equilibrium point $(x,y)=(0,0)$. In Case (b), we have $y'=0$ only if $x=0$, yielding the same equilibrium point $(x,y)=(0,0)$.

7. For

$$x' = x^2+y^2-8$$
$$y' = x^2-y^2,$$

we see that $y'=0$ if $x^2=y^2$. Using this requirement in the first equation, we see that $x'=0$ requires $2x^2-8=0$ or $x=\pm 2$. Since $y=\pm x$, we find 4 equilibrium points, $(2,2),(2,-2),(-2,-2),$ and $(-2,2)$.

8. For

$$x' = x^2+2y^2-3$$
$$y' = 2x^2+y^2-3,$$

we see that $x'=0$ if $x^2=3-2y^2$. In this event, we have $y'=0$ only if $2(3-2y^2)+y^2-3=0$. Solving for y we obtain $y=\pm 1$. Then, since $x^2=3-2y^2$, we see that $x=\pm 1$ for each choice of y. The equilibrium points are
$(x,y)=(1,1),(-1,1),(1,-1),(-1,-1)$.

9. For

$$x' = y - 1$$
$$y' = x(y + x)$$
$$z' = y(2 - z),$$

we see that $x' = 0$ requires $y = 1$. Using this requirement in the second equation, we see that $y' = 0$ requires $x(1 + x) = 0$. Thus, we need in Case (a) $x = 0$ or in Case (b), $x = -1$. Finally, $z' = 0$ requires $z = 2$ since y is nonzero. We obtain 2 equilibrium points, $(x,y,z) = (0,1,2)$ and $(x,y,z) = (-1,1,2)$.

10. For

$$x' = z^2 - 1$$
$$y' = z(1 - 2x + y)$$
$$z' = -(1 - x - y)^2,$$

we see that $x' = 0$ requires $z = \pm 1$. Using this requirement in the second equation, we see that $y' = 0$ requires $1 - 2x + y = 0$ while $z' = 0$ requires $1 - x - y = 0$. Satisfying $y' = 0$ and $z' = 0$ therefore requires $x = 2/3$ and $y = 1/3$. Combining this requirement with $z = \pm 1$, we obtain 2 equilibrium points,
$(x,y,z) = (2/3, 1/3, 1)$ and $(x,y,z) = (2/3, 1/3, -1)$.

11. Making the substitution $y_1 = y$ and $y_2 = y'$ the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = -y_1 - y_1^3$$

Since $y_2' = -y_1(1 + y_1^2)$, we cannot have $y_2' = 0$ unless $y_1 = 0$. Similarly, from the first equation, $y_1' = 0$ requires $y_2 = 0$. Thus, the only equilibrium point is $(y_1, y_2) = (y, y') = (0,0)$.

12. Making the substitution $y_1 = y$ and $y_2 = y'$ the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = 1 - e^{y_1} y_2 - \sin^2(\pi y_1)$$

Thus, the equilibrium points are $(y_1, y_2) = (y, y') = (n + 0.5, 0), n = 0, \pm 1, \pm 2, \ldots$.

13. Making the substitution $y_1 = y$ and $y_2 = y'$ the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = 1 - y_1^2 - 2(1 + y_1^4)^{-1} y_2$$

From the first equation, $y_1' = 0$ requires $y_2 = 0$. Thus, in the second equation, $y_2' = 0$ requires $1 - y_1^2 = 0$ or $y_1 = \pm 1$. There are two equilibrium points
$(y_1, y_2) = (y, y') = (1,0)$ and $(y_1, y_2) = (y, y') = (-1,0)$.

14. Making the substitution $y_1 = y$, $y_2 = y'$, and $y_3 = y''$ the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = 1 + y_3 - 2\sin y_1$$

Thus, the equilibrium points are

$$(y_1, y_2, y_3) = ((\pi/6) + 2n\pi, 0, 0) \text{ and}$$
$$(y_1, y_2, y_3) = ((5\pi/6) + 2n\pi, 0, 0), n = 0, \pm 1, \pm 2, \ldots.$$

15. Making the substitution $y_1 = y, y_2 = y'$ and $y_3 = y''$, the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = y_2^2 + (y_1^2 - 4)(2 + y_2^2)^{-1}.$$

From the first equation, $y_1' = 0$ requires $y_2 = 0$ while (by the second equation) $y_2' = 0$ requires $y_3 = 0$. Having these requirements, the third equation tells us that $y_3' = 0$ only if $y_1 = \pm 2$. Hence, There are two equilibrium points $(y_1, y_2, y_3) = (y, y', y'') = (2,0,0)$ and $(y_1, y_2, y_3) = (y, y', y'') = (-2,0,0)$.

16. Since (0,0) is an equilibrium point, we know $\beta = 0$ and $\delta = 0$. Similarly, since (2,1) is an equilibrium point, we know $2\alpha + 2 = 0$ and $\gamma - 6 = 0$. Thus, $\alpha = -1$ and $\gamma = 6$.

17. Since (1,1) is an equilibrium point, we know $\alpha + \beta + 2 = 0$ and $\gamma + \delta - 1 = 0$. Similarly, since (2,0) is an equilibrium point, we know $2\alpha + 2 = 0$ and $2\gamma - 1 = 0$. Thus, $\alpha = -1$ and $\gamma = 1/2$. Using the equations derived from the equilibrium point (1,1), we have $-1 + \beta + 2 = 0$ and $(1/2) + \delta - 1 = 0$. Therefore, $\beta = -1$ and $\delta = 1/2$.

18. The slope of a phase plane trajectory is given by $y'/x' = g(x,y)/f(x,y)$, see equation (9). As given, $g(2,1)/f(2,1) = 1$ and $g(1,-1)/f(1,-1) = 0$. Therefore, $g(1,-1) = 0$ and so $\beta = 2$. Knowing $\beta = 2$ and $g(2,1)/f(2,1) = 1$, we obtain $(3+\beta)/(2+\alpha) = 1$ or $5/(2+\alpha) = 1$. Thus, we obtain $\alpha = 3$.

19. The slope of a phase plane trajectory is given by $y'/x' = g(x,y)/f(x,y)$, see equation (9). As given, $g(1,1)/f(1,1) = 0$ and $g(1,-1)/f(1,-1) = 4$. Therefore, $g(1,1) = 0$ and so $2 + \gamma = 0$ or $\gamma = -2$. Knowing $\gamma = -2$ and $g(1,-1)/f(1,-1) = 4$, we obtain $(2-\gamma)/(\alpha - \beta + 1) = 4$ or $1/(\alpha - \beta + 1) = 1$. Finally, since there is a vertical tangent at $(0,-1)$ we know $f(0,-1) = 0$, and thus $-\beta + 1 = 0$. Using $\beta = 1$ along with the prior equation $1/(\alpha - \beta + 1) = 1$, we obtain $\alpha = 1$.

20. The slope of a phase plane trajectory is given by $y'/x' = g(x,y)/f(x,y)$, see equation (9). As given, $g(1,2)/f(1,2) = 1/6$ and thus $1/6 = g(1,2)/f(1,2) = (-1+0.5)/(5-2^n)$. Solving for n, we obtain $n = 3$.

21. Making the substitution $y_1 = y$ and $y_2 = y'$ the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = y_2 - 2y_1^2 + \alpha.$$

Since $(y_1, y_2) = (2,0)$ is an equilibrium point, it follows that $2y_1^2 = 8 = \alpha$.

22 (a). $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$

22 (b). $\mathbf{v} = 15\mathbf{i} + \mathbf{j}$

22 (a). $\mathbf{v} = -\mathbf{j}$

24. For $A = \begin{bmatrix} -9 & 1 \\ 1 & -9 \end{bmatrix}$, the eigenvalues are $\lambda_1 = -10$ and $\lambda_2 = -8$ with corresponding eigenvectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution is

$\mathbf{y}(t) = c_1 e^{-10t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-8t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and hence all solution points are attracted to the origin. Thus, the direction field corresponding to the given matrix is C.

25. For $A=\begin{bmatrix}-1 & -3\\ -3 & -1\end{bmatrix}$, the eigenvalues are $\lambda_1=-4$ and $\lambda_2=2$ with corresponding eigenvectors $\mathbf{u}_1=\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{u}_2=\begin{bmatrix}1\\-1\end{bmatrix}$. The general solution is
$\mathbf{y}(t)=c_1e^{-4t}\begin{bmatrix}1\\1\end{bmatrix}+c_2e^{2t}\begin{bmatrix}1\\-1\end{bmatrix}$ and hence solution points that begin on the line $y=x$ are attracted to the origin whereas those that begin on the line $y=-x$ are repelled away from the origin. Thus, the direction field corresponding to the given matrix is B.

26. For $A=\begin{bmatrix}-4 & 6\\ 6 & -4\end{bmatrix}$, the eigenvalues are $\lambda_1=-10$ and $\lambda_2=2$ with corresponding eigenvectors $\mathbf{u}_1=\begin{bmatrix}1\\-1\end{bmatrix}$ and $\mathbf{u}_2=\begin{bmatrix}1\\1\end{bmatrix}$. The general solution is
$\mathbf{y}(t)=c_1e^{-10t}\begin{bmatrix}1\\-1\end{bmatrix}+c_2e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$ and hence solution points that begin on the line $y=x$ are repelled away from the origin whereas those that begin on the line $y=-x$ are attracted to the origin. Thus, the direction field corresponding to the given matrix is D.

27. For $A=\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}$, the eigenvalues are $\lambda_1=6$ and $\lambda_2=2$ with corresponding eigenvectors $\mathbf{u}_1=\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{u}_2=\begin{bmatrix}1\\-1\end{bmatrix}$. The general solution is
$\mathbf{y}(t)=c_1e^{6t}\begin{bmatrix}1\\1\end{bmatrix}+c_2e^{2t}\begin{bmatrix}1\\-1\end{bmatrix}$ and hence solution points that begin on the line $y=x$ are repelled away from the origin as are those that begin on the line $y=-x$. Thus, the direction field corresponding to the given matrix is A.

28. The phase plane point $(\alpha,0)$ is an equilibrium point when α is a root of $f(y)=0$.

29 (a). Making the substitution $y_1=y$ and $y_2=y'$ the scalar equation can be expressed as the system

$$y_1'=y_2$$
$$y_2'=-y_1-y_1^3\ .$$

The nullclines are the lines $y_1=0$ and $y_2=0$. The only equilibrium point is the point $(0,0)$.

30 (a). Making the substitution $y_1=y$ and $y_2=y'$ the scalar equation can be expressed as the system

$$y_1'=y_2$$
$$y_2'=-y_1(1-y_1^2)\ .$$

The nullclines are the lines $y_1=0, y_1=\pm1$, and $y_2=0$. The equilibrium points are $(0,0),(-1,0),(1,0)$.

31 (a). Making the substitution $y_1 = y$ and $y_2 = y'$ the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = 1 - 2\sin^2 y_1 .$$

The nullclines are the lines $y_1 = \pm(\pi/4) + n\pi, n = 0, \pm 1, \pm 2, \ldots$ and the line $y_2 = 0$ The equilibrium points are $(\pm(\pi/4) + n\pi, 0), n = 0, \pm 1, \pm 2, \ldots$

32 (a). The nullclines are the lines $y = 3x - 2$ and $y = x$. These lines intersect at the point $(1,1)$ yielding the only equilibrium point.

33 (a). The nullclines are the lines $y = 2 - x$ and $y = x$. These lines intersect at the point $(1,1)$ yielding the only equilibrium point.

34 (a). The nullclines are the lines $y = 2x - 2$ and $y = 4 - x$ where $f = 0$ and the line $y = (1/2)x$ where $g = 0$. The lines $f = 0$ and $g = 0$ intersect at the points $(4/3, 2/3)$ and $(8/3, 4/3)$ yielding the only equilibrium points.

35 (a). The nullclines are the lines $y = 2x - 6$ and $y = x$, where $f = 0$ and the line $y = -x$, where $g = 0$. The lines $f = 0$ and $g = 0$ intersect at the points $(0,0)$ and $(2,-2)$ yielding the only equilibrium points.

36 (a). The nullclines are the curves $y = 1 - x^2$ and $y = -1 + x^2$. These curves intersect at the equilibrium points $(-1,0)$ and $(1,0)$.

Section 8.3

1 (a). Given $x'' + 4x = 0$, multiply by x' to obtain $x'x'' + 4x'x = 0$. Integrating, we obtain $0.5(x')^2 + 2x^2 = C$.

1 (b). The equation $x'' + 4x = 0$ can be expressed as $\begin{aligned} x' &= y \\ y' &= -4x \end{aligned}$. With this notation, the conserved quantity found in part (a) is $0.5y^2 + 2x^2 = C$. The graph passes through the point $(x,y) = (1,1)$ when $C = 2.5$.

1 (c). At $(1,1)$, the velocity vector is $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} = \mathbf{i} - 4\mathbf{j}$. The velocity vector is tangent to the graph and indicates that the graph is traversed in the clockwise direction as t increases.

2 (a). Given $x'' - (x+1) = 0$, multiply by x' to obtain $x'x'' - x'(x+1) = 0$. Integrating, we obtain $(x')^2 - (x+1)^2 = C$.

2 (b). The equation $x'' - (x+1) = 0$ can be expressed as $\begin{aligned} x' &= y \\ y' &= x + 1 \end{aligned}$. With this notation, the conserved quantity found in part (a) is $y^2 - (x+1)^2 = C$. The graph passes through the point $(x,y) = (1,1)$ when $C = -3$.

2 (c). At $(1,1)$, the velocity vector is $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} = \mathbf{i} + 2\mathbf{j}$. The velocity vector indicates that the solution point moves upward and to the right along the right branch of the hyperbola as t increases.

3 (a). Given $x'' + x^3 = 0$, multiply by x' to obtain $x'x'' + x'x^3 = 0$. Integrating, we obtain $0.5(x')^2 + 0.25x^4 = C$.

3 (b). The equation $x'' + x^3 = 0$ can be expressed as $\begin{array}{l} x' = y \\ y' = -x^3. \end{array}$ With this notation, the conserved quantity found in part (a) is $0.5y^2 + 0.25x^4 = C$. The graph passes through the point $(x,y) = (1,1)$ when $C = 0.75$.

3 (c). At $(1,1)$, the velocity vector is $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} = \mathbf{i} - \mathbf{j}$. The velocity vector is tangent to the graph and indicates that the graph is traversed in the clockwise direction as *t* increases.

4 (a). Given $x'' - (x^3 + \pi \sin \pi x) = 0$, multiply by x' to obtain $x'x'' - x'(x^3 + \pi \sin \pi x) = 0$. Integrating, we obtain $2(x')^2 - (x^4 - 4\cos \pi x) = C$.

4 (b). The equation $x'' - (x^3 + \pi \sin \pi x) = 0$ can be expressed as $\begin{array}{l} x' = y \\ y' = x^3 + \pi \sin \pi x. \end{array}$ With this notation, the conserved quantity found in part (a) is $2y^2 - (x^4 - 4\cos \pi x) = C$.
The graph passes through the point $(x,y) = (1,1)$ when $C = -3$.

4 (c). At $(1,1)$, the velocity vector is $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} = \mathbf{i} + \mathbf{j}$. The velocity vector indicates that the solution point moves upward and to the right along the right branch of the graph as *t* increases.

5 (a). Given $x'' + x^2 = 0$, multiply by x' to obtain $x'x'' + x'x^2 = 0$. Integrating, we obtain $0.5(x')^2 + (1/3)x^3 = C$.

5 (b). The equation $x'' + x^2 = 0$ can be expressed as $\begin{array}{l} x' = y \\ y' = -x^2. \end{array}$ With this notation, the conserved quantity found in part (a) is $0.5y^2 + (1/3)x^3 = C$. The graph passes through the point $(x,y) = (1,1)$ when $C = 5/6$.

5 (c). At $(1,1)$, the velocity vector is $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} = \mathbf{i} - \mathbf{j}$. The velocity vector is tangent to the graph and indicates that the solution point moves "down the graph" as *t* increases.

6 (a). Given $x'' + x/(1+x^2) = 0$, multiply by x' to obtain $x'x'' + x'x/(1+x^2) = 0$. Integrating, we obtain $(x')^2 + \ln(1+x^2) = C$.

6 (b). The equation $x'' + x/(1+x^2) = 0$ can be expressed as $\begin{array}{l} x' = y \\ y' = -x/(1+x^2). \end{array}$ With this notation, the conserved quantity found in part (a) is $y^2 + \ln(1+x^2) = C$.
The graph passes through the point $(x,y) = (1,1)$ when $C = 1 + \ln 2$.

6 (c). At $(1,1)$, the velocity vector is $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} = \mathbf{i} - 0.5\mathbf{j}$. The velocity vector indicates that the solution point moves clockwise along the curve as *t* increases.

7. Rewriting the conservation law in terms of x and x', we have $(x')^2 + x^2 \cos x = C$. Differentiating with respect to *t*, we obtain $2x'x'' + 2x'x\cos x - x^2 x' \sin x = 0$ or $x'(2x'' + 2x\cos x - x^2 \sin x) = 0$. Therefore, the differential equation is $x'' + x\cos x - 0.5x^2 \sin x = 0$.

8. Rewriting the conservation law in terms of x and x', we have $(x')^2 - e^{-x^2} = C$. Differentiating with respect to *t*, we obtain $2x'x'' - (e^{-x^2})(-2xx') = 0$. Therefore, the differential equation is $x'' + xe^{-x^2} = 0$.

9 (a). The equation $x'' + x + x^3 = 0$ can be expressed as $\begin{array}{l} x' = y \\ y' = -x - x^3. \end{array}$ The nullclines are the lines defined by $y = 0$ and $-x(1+x^2) = 0$; the lines $y = 0$ and $x = 0$. Thus, the only equilibrium point is the point $(x,y) = (0,0)$.

9 (b). The velocity vector has the form $\mathbf{v}(x,y) = y\mathbf{i} - (x + x^3)\mathbf{j}$. Thus, we obtain $\mathbf{v}(1,1) = \mathbf{i} - 2\mathbf{j}$, $\mathbf{v}(1,-1) = -\mathbf{i} - 2\mathbf{j}$, $\mathbf{v}(-1,1) = \mathbf{i} + 2\mathbf{j}$, and $\mathbf{v}(-1,-1) = -\mathbf{i} + 2\mathbf{j}$.

9 (c). Multiplying by x', the equation becomes $x'x'' + x'(x + x^3) = 0$. Integrating, we obtain $0.5(x')^2 + 0.5x^2 + 0.25x^4 = C$ or $2y^2 + 2x^2 + x^4 = C_1$. The graph of the conserved quantity passes through the point $(1,1)$ when $C_1 = 5$. The graph passes through the other three points and is consistent with the sketch in part (b).

10. Since $x'' + \alpha x = 0$ it follows that $0.5(x')^2 + 0.5\alpha x^2 = C_1$ and hence $\alpha x^2 + y^2 = C$.

10 (a). Figure A is a circle of radius 2 and thus $\alpha = 1$ and $x^2 + y^2 = 4$.
Figure B is a hyperbola with asymptotes $y = \pm x$. Since (0, 2) is on the graph, we see that $\alpha = -1$ and $y^2 - x^2 = 4$.
Figure C shows horizontal lines, $y = \pm 2$. Thus, $\alpha = 0$.

10 (b). The solution point in Figure A travels clockwise around the circle. Solution points in Figure B move to the right on the upper branch and to the left on the lower branch. Solutions points in Figure C move to the right on the upper line and to the left on the lower line.

11. In analogy with Exercise 9, multiply the equation $y''' + f(y') = 0$ by y'', obtaining $y''y''' + y''f(y') = 0$. Integrating, we find $0.5y'' + F(y') = C$ where $F(u)$ is an antiderivative of $f(u)$. Thus, the differential equation has a conservation law given by $0.5(y'')^2 + F(y') = C$.

12. (a) From the definition of $E(t)$, it follows that $\dfrac{dE}{dt} = mx'x'' + kxx' = (mx'' + kx)x'$. From the differential equation, $mx'' + \gamma x' + kx = 0$ and hence $mx'' + kx = -\gamma x'$. Therefore, $\dfrac{dE}{dt} = (-\gamma x')x' \le 0$.
(b) Energy is not conserved. On t-intervals where $x'(t) \neq 0$, $E(t)$ is a decreasing function of t and energy is being lost.

13 (a). For the system

$$x' = 2x$$
$$y' = -2y$$

we have $f(x,y) = 2x$ and $g(x,y) = -2y$. Thus, $f_x = 2$ and $g_y = -2$. Since $f_x = -g_y$, the system is Hamiltonian.

13 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = 2y$. Integrating with respect to x, we obtain $H(x,y) = 2xy + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = 2x + p'(y) = f(x,y) = 2x$. Therefore, $p'(y) = 0$ and hence $p(y) = C$ is a constant function. Dropping the constant, we obtain a Hamiltonian function, $H(x,y) = 2xy$.

13 (c). From part (b), the phase-plane trajectories are defined by $2xy = C$. If a phase-plane trajectory passes through the point $(1,1)$, then $C = 2$ and the trajectory is given by $xy = 1$.

14 (a). For the system

$$x' = 2xy$$
$$y' = -y^2$$

we have $f(x,y) = 2xy$ and $g(x,y) = -y^2$. Thus, $f_x = 2y$ and $g_y = -2y$. Since $f_x = -g_y$, the system is Hamiltonian.

14 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = y^2$. Integrating with respect to x, we obtain $H(x,y) = xy^2 + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = 2xy + p'(y) = f(x,y) = 2xy$.

Therefore, $p'(y) = 0$ and hence $p(y) = C$ is a constant function. Dropping the constant, we obtain a Hamiltonian function, $H(x,y) = xy^2$.

14 (c). From part (b), the phase-plane trajectories are defined by $xy^2 = C$. If a phase-plane trajectory passes through the point $(1,1)$, then $C = 1$ and the trajectory is given by $xy^2 = 1$.

15 (a). For the system

$$x' = x - x^2 + 1$$
$$y' = -y + 2xy + 4x$$

we have $f(x,y) = x - x^2 + 1$ and $g(x,y) = -y + 2xy + 4x$. Thus, $f_x = 1 - 2x$ and $g_y = -1 + 2x$. Since $f_x = -g_y$, the system is Hamiltonian.

15 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = y - 2xy - 4x$. Integrating with respect to *x*, we obtain $H(x,y) = xy - x^2y - 2x^2 + p(y)$. Differentiating with respect to *y* in order to determine $p(y)$, we find $H_y(x,y) = x - x^2 + p'(y) = f(x,y) = x - x^2 + 1$. Therefore, $p'(y) = 1$ and hence $p(y) = y + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = xy - x^2y - 2x^2 + y$.

15 (c). From part (b), the phase-plane trajectories are defined by $xy - x^2y - 2x^2 + y = C$. If a phase-plane trajectory passes through the point $(1,1)$, then $C = -1$ and the trajectory is given by $xy - x^2y - 2x^2 + y + 1 = 0$.

16 (a). For the system

$$x' = -8y$$
$$y' = 2x$$

we have $f(x,y) = -8$ and $g(x,y) = 2x$. Thus, $f_x = 0$ and $g_y = 0$. Since $f_x = -g_y$, the system is Hamiltonian.

16 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_y(x,y) = f(x,y) = -8y$. Integrating with respect to *y*, we obtain $H(x,y) = -4y^2 + q(x)$. Differentiating with respect to *x* in order to determine $q(x)$, we find $H_x(x,y) = q'(x) = -2x$. Therefore, $q(x) = -x^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = -x^2 - 4y^2$.

16 (c). From part (b), the phase-plane trajectories are defined by $-x^2 - 4y^2 = C$. If a phase-plane trajectory passes through the point $(1,1)$, then $C = -5$ and the trajectory is given by $x^2 + 4y^2 = 5$.

17 (a). For the system

$$x' = 2y\cos x$$
$$y' = y^2 \sin x$$

we have $f(x,y) = 2y\cos x$ and $g(x,y) = y^2 \sin x$. Thus, $f_x = -2y\sin x$ and $g_y = 2y\sin x$. Since $f_x = -g_y$, the system is Hamiltonian.

17 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = -y^2 \sin x$. Integrating with respect to *x*, we obtain $H(x,y) = y^2 \cos x + p(y)$. Differentiating with respect to *y* in order to determine $p(y)$, we find $H_y(x,y) = 2y\cos x + p'(y) = f(x,y) = 2y\cos x$. Therefore, $p'(y) = 0$ and hence $p(y) = C$ is a constant function. Dropping the constant, we obtain a Hamiltonian function, $H(x,y) = y^2 \cos x$.

17 (c). From part (b), the phase-plane trajectories are defined by $y^2\cos x = C$. If a phase-plane trajectory passes through the point $(1,1)$, then $C = \cos 1$ and the trajectory is given by $y^2\cos x = \cos 1$.

18 (a). For the system

$$x' = 2y - x + 3$$
$$y' = y + 4x^3 - 2x$$

we have $f_x = -1$ and $g_y = 1$. Since $f_x = -g_y$, the system is Hamiltonian.

18 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_y(x,y) = f(x,y) = 2y - x + 3$. Integrating with respect to y, we obtain $H(x,y) = y^2 - xy - 3y + q(x)$. Differentiating with respect to x in order to determine $q(x)$, we find $H_x(x,y) = -y + q'(x) = -y - 4x^3 + 2x$. Therefore, $q(x) = -x^4 + x^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = y^2 - xy + 3y - x^4 + x^2$.

18 (c). If a phase-plane trajectory $H(x,y) = C$ passes through the point $(1,1)$, then the trajectory is given by $y^2 - xy + 3y - x^4 + x^2 = 8$.

19 (a). For the system

$$x' = -2y$$
$$y' = 3x^2$$

we have $f(x,y) = -2y$ and $g(x,y) = 3x^2$. Thus, $f_x = 0$ and $g_y = 0$. Since $f_x = -g_y$, the system is Hamiltonian.

19 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = -3x^2$. Integrating with respect to x, we obtain $H(x,y) = -x^3 + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = p'(y) = f(x,y) = -2y$. Therefore, $p'(y) = -2y$ and hence $p(y) = -y^2 + C$ is a constant function. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = -x^3 - y^2$.

19 (c). From part (b), the phase-plane trajectories are defined by $-x^3 - y^2 = C$. If a phase-plane trajectory passes through the point $(1,1)$, then $C = -2$ and the trajectory is given by $x^3 + y^2 = 2$.

20 (a). For the system

$$x' = xe^{xy}$$
$$y' = -2x - ye^{xy}$$

we have $f_x = e^{xy} + xye^{xy}$ and $g_y = -e^{xy} - xye^{xy}$. Since $f_x = -g_y$, the system is Hamiltonian.

20 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_y(x,y) = f(x,y) = xe^{xy}$. Integrating with respect to y, we obtain $H(x,y) = e^{xy} + q(x)$. Differentiating with respect to x in order to determine $q(x)$, we find $H_x(x,y) = ye^{xy} + q'(x) = 2x + ye^{xy}$. Therefore, $q(x) = x^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = e^{xy} + x^2$.

20 (c). If a phase-plane trajectory $H(x,y) = C$ passes through the point $(1,1)$, then the trajectory is given by $e^{xy} + x^2 = 1 + e$.

21. Consider the system

$$x' = x^3 + 3\sin(2x+3y)$$
$$y' = -3x^2y - 2\sin(2x+3y)\ .$$

Calculating the partial derivatives, we have $f_x = 3x^2 + 6\cos(2x+3y)$ and $g_y = -3x^2 - 6\cos(2x+3y)$. Since $f_x = -g_y$, the system is Hamiltonian.
Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = 3x^2y + 2\sin(2x+3y)$. Integrating with respect to x, we obtain $H(x,y) = x^3y - \cos(2x+3y) + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = x^3 + 3\sin(2x+3y) + p'(y) = f(x,y) = x^3 + 3\sin(2x+3y)$. Therefore, $p'(y) = 0$ and hence $p(y) = C$ is a constant function. We obtain a Hamiltonian function, $H(x,y) = x^3y - \cos(2x+3y)$.

22. Consider the system

$$x' = e^{xy} + y^3$$
$$y' = -e^{xy} - x^3\ .$$

Calculating the partial derivatives, we have $f_x = ye^{xy}$ and $g_y = -xe^{xy}$. Since $f_x \neq -g_y$, the system is not Hamiltonian.

23. Consider the system

$$x' = -\sin(2xy) - x$$
$$y' = \sin(2xy) + y\ .$$

Calculating the partial derivatives, we have $f_x = -2y\cos(2xy) - 1$ and $g_y = 2x\cos(2xy) + 1$. Since $f_x \neq -g_y$, the system is not Hamiltonian.

24. Consider the system

$$x' = -3x^2 + xe^y$$
$$y' = 6xy + 3x - e^y\ .$$

Calculating the partial derivatives, we have $f_x = -6x + e^y$ and $g_y = 6x - e^y$. Since $f_x = -g_y$, the system is Hamiltonian. Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = -6xy - 3x + e^y$. Integrating with respect to x, we obtain $H(x,y) = -3x^2y - (3/2)x^2 + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = -3x^2 + p'(y) = f(x,y) = -3x^2 + xe^y$. Therefore, $p'(y) = xe^y$ and hence $p(y) = xe^y + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = -3x^2y - (3/2)x^2 + xe^y$.

25. Consider the system

$$x' = y$$
$$y' = x - x^2\ .$$

Calculating the partial derivatives, we have $f_x = 0$ and $g_y = 0$. Since $f_x = -g_y$, the system is Hamiltonian.
Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = x^2 - x$. Integrating with respect to x, we obtain $H(x,y) = (1/6)(2x^3 - 3x^2) + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = p'(y) = f(x,y) = y$. Therefore, $p'(y) = y$ and hence $p(y) = 0.5y^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = (1/6)(2x^3 - 3x^2 + 3y^2)$.

26. Consider the system

$$x' = x + 2y$$
$$y' = x^3 - 2x + y \ .$$

Calculating the partial derivatives, we have $f_x = 1$ and $g_y = 1$. Since $f_x \neq -g_y$, the system is not Hamiltonian.

27. Consider the system

$$x' = f(y)$$
$$y' = g(x) \ .$$

Calculating the partial derivatives, we have $\partial_x[f(y)] = 0$ and $\partial_y[g(x)] = 0$. Since $\partial_x[f(y)] = -\partial_y[g(x)]$, the system is Hamiltonian.

Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x)$. Integrating with respect to x, we obtain $H(x,y) = -G(x) + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = p'(y) = f(y)$. Therefore, $p(y) = F(y) + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = F(y) - G(x)$.

28. Consider the system

$$x' = f(y) + 2y$$
$$y' = g(x) + 6x \ .$$

Calculating the partial derivatives, we have $\partial_x[f(y) + 2y] = 0$ and $\partial_y[g(x) + 6x] = 0$. Since $\partial_x[f(y) + 2y] = -\partial_y[g(x) + 6x]$, the system is Hamiltonian. Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x) - 6x$. Integrating with respect to x, we obtain $H(x,y) = -G(x) - 3x^2 + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = p'(y) = f(y) + 2y$. Therefore, $p(y) = F(y) + y^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = -G(x) - 3x^2 + F(y) + y^2$.

29. Consider the system

$$x' = 3f(y) - 2xy$$
$$y' = g(x) + y^2 + 1 \ .$$

Calculating the partial derivatives, we have $\partial_x[3f(y) - 2xy] = -2y$ and $\partial_y[g(x) + y^2 + 1] = 2y$. Since $\partial_x[3f(y) - 2xy] = -\partial_y[g(x) + y^2 + 1]$, the system is Hamiltonian.

Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x) - y^2 - 1$. Integrating with respect to x, we obtain $H(x,y) = -G(x) - y^2 x - x + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = -2yx + p'(y) = 3f(y) - 2xy$. Therefore, $p(y) = 3F(y) + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = 3F(y) - G(x) - y^2 x - x$.

30. Consider the system

$$x' = f(x - y) + 2y$$
$$y' = f(x - y) \ .$$

Calculating the partial derivatives, we have $\partial_x[f(x - y) + 2y] = f'(x - y)$ and $\partial_y[f(x - y)] = -f'(x - y)$. Since $\partial_x[f(x - y) + 2y] = -\partial_y[f(x - y)]$, the system is Hamiltonian. Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -f(x - y)$. Integrating with respect to x, we obtain $H(x,y) = -F(x - y) + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = f(x - y) + p'(y) = f(x - y) + 2y$.

Therefore, $p(y) = y^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = -F(x-y) + y^2$.

31. Consider the composition $K(x(t),y(t))$. Differentiating with respect to t, we obtain $\frac{d}{dt}K(x(t),y(t)) = \frac{\partial K}{\partial x}\frac{dx}{dt} + \frac{\partial K}{\partial y}\frac{dy}{dt} = -(\mu g)f + (\mu f)g = 0$. Therefore, $K(x(t),y(t))$ is a conserved quantity.

Section 8.4

1 (a). All points lying within the ellipse E having semi-major axis ε and semi-minor axis $\varepsilon/2$ lie within the circle of radius ε. Likewise, all points lying within the circle of radius $\varepsilon/2$ lie within the ellipse E. Therefore, given $\varepsilon > 0$, choose $\delta = \varepsilon/2$.

1 (b). The origin is not an asymptotically stable equilibrium point since the solution points remain on an ellipse and do not approach the origin as $t \to \infty$.

2. The origin is an unstable equilibrium point. Any solution point starting near the origin will follow a branch of the hyperbola and will eventually exit any circle centered at the origin.

3 (a). Making the substitution $y = x'$, the scalar equation $x'' + \gamma x' + x = 0$ can be expressed as the system

$$x' = y$$
$$y' = -x - \gamma y.$$

The origin is the only equilibrium point for this system.

3 (b). We analyze stability by appealing to Theorem 8.3. The system in part (a) has the form $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix}$. The characteristic polynomial for A is $p(\lambda) = \lambda^2 + \gamma\lambda + 1$ and thus the eigenvalues of A are $\lambda_1 = 0.5\left(-\gamma - \sqrt{\gamma^2 - 4}\right)$ and $\lambda_2 = 0.5\left(-\gamma + \sqrt{\gamma^2 - 4}\right)$. When $\gamma^2 - 4 \ge 0$, we see that $\lambda_1 \le \lambda_2$. Thus, if $2 \le \gamma$, then $\lambda_1 \le \lambda_2 < 0$ which shows the origin is asymptotically stable. On the other hand, if $\gamma \le -2$, then $0 < \lambda_1 \le \lambda_2$ which shows the origin is an unstable equilibrium point. For $-2 < \gamma < 2$, the eigenvalues are complex with nonzero imaginary parts. For $-2 < \gamma < 0$, the real parts of λ_1 and λ_2 are positive, which shows the origin is an unstable equilibrium point. Likewise, for $0 < \gamma < 2$, the origin is an asymptotically stable equilibrium point. When $\gamma = 0$, the origin is a stable (but not asymptotically stable) equilibrium point.

4. For the system $\mathbf{y}' = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$. Thus, by Theorem 8.3, the origin is an unstable equilibrium point.

5. For the system $\mathbf{y}' = \begin{bmatrix} 5 & -14 \\ 3 & -8 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. Thus, by Theorem 8.3, the origin is an asymptotically stable equilibrium point.

6. For the system $\mathbf{y}' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = 2i$ and $\lambda_2 = -2i$. Thus, by Theorem 8.3, the origin is a stable equilibrium point but not an asymptotically stable equilibrium point.

7. For the system $\mathbf{y}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Thus, by Theorem 8.3, the origin is an unstable equilibrium point.

8. For the system $\mathbf{y}' = \begin{bmatrix} -7 & -3 \\ 5 & 1 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -4$ and $\lambda_2 = -2$. Thus, by Theorem 8.3, the origin is an asymptotically stable equilibrium point.

9. For the system $\mathbf{y}' = \begin{bmatrix} 9 & 5 \\ -7 & -3 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$. Thus, by Theorem 8.3, the origin is an unstable equilibrium point.

10. For the system $\mathbf{y}' = \begin{bmatrix} -3 & -5 \\ 2 & -1 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$. Thus, by Theorem 8.3, the origin is an asymptotically stable equilibrium point.

11. For the system $\mathbf{y}' = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$. Thus, by Theorem 8.3, the origin is an unstable equilibrium point.

12. For the system $\mathbf{y}' = \begin{bmatrix} -13 & -8 \\ 15 & 9 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$. Thus, by Theorem 8.3, the origin is an asymptotically stable equilibrium point.

13. For the system $\mathbf{y}' = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$. Thus, by Theorem 8.3, the origin is a stable (but not asymptotically stable) equilibrium point.

14. For the system $\mathbf{y}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. Thus, by Theorem 8.3, the origin is an asymptotically stable equilibrium point.

15. For the system $\mathbf{y}' = \begin{bmatrix} -3 & 3 \\ 1 & -5 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -6$ and $\lambda_2 = -2$. Thus, by Theorem 8.3, the origin is an asymptotically stable equilibrium point.

16. Eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$. Since one of the eigenvalues is real and positive, the origin is an unstable equilibrium point.

17. Eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. Since the eigenvalues are real and positive, the origin is an unstable equilibrium point.

18. Eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -2$. Since the eigenvalues are real and negative, the origin is an asymptotically stable equilibrium point.

19. Eigenvalues are $\lambda_1 = 1 - 2i$ and $\lambda_2 = 1 + 2i$. Since the eigenvalues are complex with positive real parts, the origin is an unstable equilibrium point.

20. Eigenvalues are $\lambda_1 = -2i$ and $\lambda_2 = 2i$. Since the eigenvalues are purely imaginary, the origin is a stable equilibrium point but it is not an asymptotically stable equilibrium point.

21. Eigenvalues are $\lambda_1 = -2 - 2i$ and $\lambda_2 = -2 + 2i$. Since the eigenvalues are complex with negative real parts, the origin is an asymptotically stable equilibrium point.

22. Eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$. Since one of the eigenvalues is real and positive, the origin is an unstable equilibrium point.

23. Eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -3$. Since the eigenvalues are real and negative, the origin is an asymptotically stable equilibrium point.

24 (a). Solving $\mathbf{0} = A\mathbf{y}_e + \mathbf{g}_0$, it follows that $\mathbf{y}_e = -A^{-1}\mathbf{g}_0$ is the unique equilibrium point.

24 (b). Let $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$. Then, $\mathbf{z}' = \mathbf{y}' = A\mathbf{y} + \mathbf{g}_0 = A\mathbf{y} - A\mathbf{y}_e = A\mathbf{z}$. Theorem 8.3 can be applied to the new system $\mathbf{z}' = A\mathbf{z}$.

25. For the system $\mathbf{y}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, the unique equilibrium point is

$\mathbf{y}_e = -A^{-1}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = -(1/3)\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$. With the change of variable $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$ the system becomes $(\mathbf{z} + \mathbf{y}_e)' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}(\mathbf{z} + \mathbf{y}_e) + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$ or $\mathbf{z}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{z} + \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{y}_e + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

This last system reduces to the homogeneous system $\mathbf{z}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{z}$. The coefficient matrix has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$. By Theorem 8.3, the origin is an asymptotically stable equilibrium point of $\mathbf{z}' = A\mathbf{z}$ and therefore, $\mathbf{y}_e$ is an asymptotically stable equilibrium point of the nonhomogeneous system $\mathbf{y}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

26. For the system $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, the unique equilibrium point is $\mathbf{y}_e = -A^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. With the change of variable $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$ the system reduces to the homogeneous system $\mathbf{z}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{z}$. The coefficient matrix has eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$. By Theorem 8.3, the origin is a stable but not an asymptotically stable equilibrium point of $\mathbf{z}' = A\mathbf{z}$. Therefore, $\mathbf{y}_e$ is a stable but not an asymptotically stable equilibrium point of the nonhomogeneous system.

27. For the system $\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, the unique equilibrium point is

$\mathbf{y}_e = -A^{-1}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. With the change of variable $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$ the system becomes $(\mathbf{z} + \mathbf{y}_e)' = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}(\mathbf{z} + \mathbf{y}_e) + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ or $\mathbf{z}' = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}\mathbf{y}_e + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. This last system reduces to the homogeneous system $\mathbf{z}' = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}\mathbf{z}$. The coefficient matrix has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$. By Theorem 8.3, the origin is an unstable equilibrium point of $\mathbf{z}' = A\mathbf{z}$ and therefore, $\mathbf{y}_e$ is an unstable equilibrium point of the nonhomogeneous system $\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

28. For the system $\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ -10 & 5 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, the unique equilibrium point is $\mathbf{y}_e = -A^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/5 \\ -8/5 \end{bmatrix}$.
With the change of variable $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$ the system reduces to the homogeneous system

$\mathbf{z}' = \begin{bmatrix} -1 & 1 \\ -10 & 5 \end{bmatrix}\mathbf{z}$. The coefficient matrix has eigenvalues $\lambda_1 = 2+i$ and $\lambda_2 = 2-i$. By Theorem 8.3, the origin is an unstable equilibrium point of $\mathbf{z}' = A\mathbf{z}$. Therefore, $\mathbf{y}_e$ is an unstable equilibrium point of the nonhomogeneous system.

29. For the system $\mathbf{y}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -1, \lambda_2 = 2,$ and $\lambda_3 = 3$. Thus, by the discussion following Theorem 8.3, the origin is an unstable equilibrium point.

30. For the system $\mathbf{y}' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, the unique equilibrium point is $\mathbf{y}_e = -A^{-1}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$. With the change of variable $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$ the system reduces to the homogeneous system $\mathbf{z}' = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}\mathbf{z}$. The coefficient matrix has eigenvalues $\lambda_1 = 1, \lambda_1 = -1,$ and $\lambda_3 = -1$. By Theorem 8.3, the origin is an unstable equilibrium point of $\mathbf{z}' = A\mathbf{z}$. Therefore, $\mathbf{y}_e$ is an unstable equilibrium point of the nonhomogeneous system.

31. For the system $\mathbf{y}' = \begin{bmatrix} -3 & -5 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -2+3i, \lambda_2 = -2-3i, \lambda_3 = 2i,$ and $\lambda_4 = -2i$. Thus, by the discussion following Theorem 8.3, the origin is a stable (but not asymptotically stable) equilibrium point.

32. For the system $\mathbf{y}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, unique equilibrium point is given by

$\mathbf{y}_e = -A^{-1}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. With the change of variables $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$, the system reduces to the homogeneous system $\mathbf{z}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\mathbf{z}$. The coefficient matrix has eigenvalues $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1,$ and $\lambda_4 = 1$. Thus, by the discussion following Theorem 8.3, the origin is an unstable equilibrium point.

34 (a). Since the coefficient matrix A is real and symmetric, it has real eigenvalues and a full set of eigenvectors.

34 (b). From the discussion following Theorem 8.3, the equilibrium point $\mathbf{y}_e = \mathbf{0}$ is isolated if and only if $\det[A] \neq 0$. Now, $\det[A] = 1 - \alpha^2$ and therefore, $\mathbf{y}_e = \mathbf{0}$ is an isolated equilibrium point if and only if $\alpha \neq \pm 1$.

34 (c). When $\alpha = 1$ the equilibrium points lie on the line $y = x$. When $\alpha = -1$ the equilibrium points lie on the line $y = -x$.

34 (d). No, since the eigenvalues of A are real and not purely imaginary; see Theorem 8.3.

34 (e). The eigenvalues of A are $\lambda_1 = -1 + \alpha$, and $\lambda_2 = -1 - \alpha$. By part (b), if $\mathbf{y}_e = \mathbf{0}$ is an isolated equilibrium point, then $\alpha \neq \pm 1$. Clearly, both eigenvalues are negative when $-1 < \alpha < 1$ whereas one of the eigenvalues is positive when $|\alpha| > 1$.

35. Since $\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, it follows that $1 + 2a_{12} = 2$ and $a_{21} + 2a_{22} = 4$. From the first equation, we have $a_{12} = 1/2$. Since $\mathbf{y} = \mathbf{0}$ is not an isolated equilibrium point, it follows that $\det[A] = 0$. Thus, $a_{22} - a_{12}a_{21} = 0$ or $a_{22} - (1/2)a_{21} = 0$. This last equation, together with the prior equation $a_{21} + 2a_{22} = 4$ tells us that $a_{21} = 2$ and $a_{22} = 1$. Thus, $A = \begin{bmatrix} 1 & 1/2 \\ 2 & 1 \end{bmatrix}$.

Section 8.5

1 (a). For the system

$$x' = x^2 + y^2 - 32$$
$$y' = y - x\,,$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ and $\mathbf{y}_e = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$.

1 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} 2x & 2y \\ -1 & 1 \end{bmatrix}$.

Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} 8 & 8 \\ -1 & 1 \end{bmatrix}\mathbf{z}$

and (ii) $\mathbf{z}' = \begin{bmatrix} -8 & -8 \\ -1 & 1 \end{bmatrix}\mathbf{z}$.

1 (c). In case (i), the eigenvalues are $\lambda_1 = 2.438\ldots$ and $\lambda_2 = 6.561\ldots$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$. For case (ii), the eigenvalues are $\lambda_1 = -8.815\ldots$ and $\lambda_2 = 1.815\ldots$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$.

2 (a). For the system

$$x' = x^2 + 9y^2 - 9$$
$$y' = x\,,$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{y}_e = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

2 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} 2x & 18y \\ 1 & 0 \end{bmatrix}$.

Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} 0 & 18 \\ 1 & 0 \end{bmatrix}\mathbf{z}$ and (ii) $\mathbf{z}' = \begin{bmatrix} 0 & -18 \\ 1 & 0 \end{bmatrix}\mathbf{z}$

2 (c). In case (i), the eigenvalues are $\lambda_1 = 4.242\ldots$ and $\lambda_2 = -4.242\ldots$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$. For case (ii), the eigenvalues are $\pm 3\sqrt{2}\,i$ and thus nothing can be inferred about the stability of the nonlinear system.

3 (a). For the system

$$x' = 1 - x^2$$
$$y' = x^2 + y^2 - 2,$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{y}_e = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{y}_e = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, and $\mathbf{y}_e = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

3 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} -2x & 0 \\ 2x & 2y \end{bmatrix}$.

Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} -2 & 0 \\ 2 & 2 \end{bmatrix}\mathbf{z}$,

(ii) $\mathbf{z}' = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix}\mathbf{z}$, (iii) $\mathbf{z}' = \begin{bmatrix} 2 & 0 \\ -2 & -2 \end{bmatrix}\mathbf{z}$, and (iv) $\mathbf{z}' = \begin{bmatrix} -2 & 0 \\ 2 & -2 \end{bmatrix}\mathbf{z}$.

3 (c). In cases (i) – (iii), $\lambda = 2$ is an eigenvalue and thus the nonlinear system is unstable at each of the corresponding equilibrium points $\mathbf{y}_e$. For case (iv), the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -2$ and thus the nonlinear system is asymptotically stable at the corresponding equilibrium point $\mathbf{y}_e$.

4 (a). For the system

$$x' = x - y - 1$$
$$y' = x^2 - y^2 + 1,$$

the equilibrium point is $\mathbf{y}_e = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

4 (b). At the equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} 1 & -1 \\ 2x & -2y \end{bmatrix}$.

Thus, the linearized system is $\mathbf{z}' = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\mathbf{z}$.

4 (c). The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$ and thus the nonlinear system is unstable at the equilibrium point $\mathbf{y}_e$.

5 (a). For the system

$$x' = (x-2)(y-3)$$
$$y' = (x+2y)(y-1),$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{y}_e = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\mathbf{y}_e = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$.

5 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} y-3 & x-2 \\ y-1 & x+4y-2 \end{bmatrix}$. Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} -4 & 0 \\ -2 & -4 \end{bmatrix}\mathbf{z}$, (ii) $\mathbf{z}' = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}\mathbf{z}$, and (iii) $\mathbf{z}' = \begin{bmatrix} 0 & -8 \\ 2 & 4 \end{bmatrix}\mathbf{z}$.

5 (c). In case (i), the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -4$ and thus the nonlinear system is asymptotically stable at the corresponding equilibrium point $\mathbf{y}_e$. For case (ii), the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 4$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$. In case (iii), the eigenvalues are $\lambda_1 = 2 + 2\sqrt{3}\,i$ and $\lambda_2 = 2 - 2\sqrt{3}\,i$. Thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$.

6 (a). For the system

$$x' = (x-y)(y+1)$$
$$y' = (x+2)(y-4),$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \mathbf{y}_e = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, and $\mathbf{y}_e = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

6 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} y+1 & x-2y-1 \\ y-4 & x+2 \end{bmatrix}$. Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} -1 & 1 \\ -6 & 0 \end{bmatrix}\mathbf{z}$, (ii) $\mathbf{z}' = \begin{bmatrix} 5 & -5 \\ 0 & 6 \end{bmatrix}\mathbf{z}$, and (iii) $\mathbf{z}' = \begin{bmatrix} 0 & -1 \\ -5 & 0 \end{bmatrix}\mathbf{z}$.

6 (c). In case (i), the eigenvalues are $-0.5 \pm 0.5i\sqrt{23}$ and thus the nonlinear system is asymptotically stable at the corresponding equilibrium point $\mathbf{y}_e$. For case (ii), the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 6$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$. In case (iii), the eigenvalues are $\pm\sqrt{5}$. Thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$.

7 (a). For the system

$$x' = (x-2y)(y+4)$$
$$y' = 2x - y,$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{y}_e = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$.

7 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} y+4 & x-4y-8 \\ 2 & -1 \end{bmatrix}$. Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} 4 & -8 \\ 2 & -1 \end{bmatrix}\mathbf{z}$, and (ii) $\mathbf{z}' = \begin{bmatrix} 0 & 6 \\ 2 & -1 \end{bmatrix}\mathbf{z}$.

7 (c). In case (i), the eigenvalues are $\lambda_1 = 0.5(3+\sqrt{39}\,i)$ and $\lambda_2 = 0.5(3-\sqrt{39}\,i)$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$. For case (ii), the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 3$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$.

8 (a). For the system

$$x' = xy - 1$$
$$y' = (x+4y)(x-1)\,,$$

the equilibrium point is $\mathbf{y}_e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

8 (b). At the equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} y & x \\ 2x+4y-1 & 4(x-1) \end{bmatrix}$. Thus, the linearized system is $\mathbf{z}' = \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}\mathbf{z}$.

8 (c). The eigenvalues are $0.5\left(1\pm\sqrt{21}\right)$ and thus the nonlinear system is unstable at the equilibrium point $\mathbf{y}_e$.

9 (a). For the system

$$x' = y^2 - x$$
$$y' = x^2 - y\,,$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{y}_e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

9 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} -1 & 2y \\ 2x & -1 \end{bmatrix}$.

Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{z}$,

and (ii) $\mathbf{z}' = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\mathbf{z}$.

9 (c). In case (i), the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -1$ and thus the nonlinear system is asymptotically stable at the corresponding equilibrium point $\mathbf{y}_e$. For case (ii), the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 1$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$.

10. At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} (1/2)[1-x-(1/2)y] & -(1/4)x \\ -(1/12)y & (1/4)[1-(1/3)x-(4/3)y] \end{bmatrix}$. Thus, the linearized systems are: (i) at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{z}' = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}\mathbf{z}$, (ii) at $\begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$, $\mathbf{z}' = \begin{bmatrix} 1/8 & 0 \\ -1/8 & -1/4 \end{bmatrix}\mathbf{z}$,

(iii) at $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{z}' = \begin{bmatrix} -1/2 & -1/2 \\ 0 & 1/12 \end{bmatrix}\mathbf{z}$. Thus, in all three of these cases, the system is unstable at the corresponding equilibrium point.

11 (c). By Taylor's theorem, $f(z) = f(0) + f'(0)z + f''(\gamma)z^2/2$ where γ is between z and 0. For $f(z) = \sin z$, we have $\sin z_1 - z_1 = (-\sin\gamma)z_1^2/2$ where γ is between z_1 and 0. Now, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = |z_1 - \sin z_1|/\sqrt{z_1^2 + z_2^2} \le |z_1 - \sin z_1|/|z_1|$. So, by the remarks above, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| \le |z_1^2/2|/|z_1| = |z_1|/2$. Hence, since $|z_1|/2$ goes to 0 as $\mathbf{z}$ goes to $\mathbf{0}$, the system is almost linear at both equilibrium points.

12 (a). For the given system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$, the coefficient matrix A is $A = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_2^2 \\ 0 \end{bmatrix}$.

12 (b). $\|\mathbf{g}(\mathbf{z})\| = z_2^2$, or using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = r^2\sin^2\theta$.

12 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = r^2\sin^2\theta/r = r\sin^2\theta$. Thus, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

12 (d). The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$. Thus, by Theorem 8.4, $\mathbf{z} = \mathbf{0}$ is an unstable equilibrium point.

13 (a). For the system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$,

$$z_1' = 5z_1 - 14z_2 + z_1z_2$$
$$z_2' = 3z_1 - 8z_2 + z_1^2 + z_2^2 ,$$

the coefficient matrix A is given by $A = \begin{bmatrix} 5 & -14 \\ 3 & -8 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_1z_2 \\ z_1^2 + z_2^2 \end{bmatrix}$.

13 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = \sqrt{(z_1z_2)^2 + (z_1^2 + z_2^2)^2} = \sqrt{(r^2\cos\theta\sin\theta)^2 + (r^2)^2}$ or $\|\mathbf{g}(\mathbf{z})\| = \sqrt{r^4(\cos^2\theta\sin^2\theta + 1)}$. (Also note that $\|\mathbf{z}\| = r$.)

13 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = \sqrt{r^4(\cos^2\theta\sin^2\theta + 1)}/r \le r^2\sqrt{2}/r = r\sqrt{2}$. Thus, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

13 (d). The eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = -1$. Thus, by Theorem 8.4, $\mathbf{z} = \mathbf{0}$ is an asymptotically stable equilibrium point.

14 (a). For the given system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$, the coefficient matrix A is $A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_1^2 + z_2^2 \\ (z_1^2 + z_2^2)^{1/3} \end{bmatrix}$.

14 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = r^{2/3}\sqrt{1 + r^{8/3}}$.

14 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = r^{2/3}\sqrt{1 + r^{8/3}}/r = \sqrt{1 + r^{8/3}}/r^{1/3}$. Thus, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\|$ does not exist as $\|\mathbf{z}\| \to 0$. The system is not almost linear at $\mathbf{z} = \mathbf{0}$.

15 (a). For the system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$,

$$z_1' = -z_1 + 3z_2 + z_2 \cos\sqrt{z_1^2 + z_2^2}$$
$$z_2' = -z_1 - 5z_2 + z_1 \cos\sqrt{z_1^2 + z_2^2}\ ,$$

the coefficient matrix A is given by $A = \begin{bmatrix} -1 & 3 \\ -1 & -5 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_2 \cos\sqrt{z_1^2 + z_2^2} \\ z_1 \cos\sqrt{z_1^2 + z_2^2} \end{bmatrix}$.

15 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\left\|\mathbf{g}(\mathbf{z})\right\| = \sqrt{(z_1^2 + z_2^2)\cos^2\sqrt{z_1^2 + z_2^2}} = \sqrt{r^2\cos^2 r}$ or $\left\|\mathbf{g}(\mathbf{z})\right\| = r\left|\cos r\right|$. (Also note that $\left\|\mathbf{z}\right\| = r$.)

15 (c). From part (b), $\left\|\mathbf{g}(\mathbf{z})\right\| / \left\|\mathbf{z}\right\| = r\left|\cos r\right| / r = \left|\cos r\right|$. Thus, $\left\|\mathbf{g}(\mathbf{z})\right\| / \left\|\mathbf{z}\right\| \to 1$ as $\left\|\mathbf{z}\right\| \to 0$. Therefore, the system is not an almost linear system.

16 (a). For the given system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$, the coefficient matrix A is $A = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_1 z_2 \cos z_2 \\ z_1 z_2 \sin z_2 \end{bmatrix}$.

16 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\left\|\mathbf{g}(\mathbf{z})\right\| = r^2\left|\cos\theta\sin\theta\right|$.

16 (c). From part (b), $\left\|\mathbf{g}(\mathbf{z})\right\| / \left\|\mathbf{z}\right\| = r^2\left|\sin\theta\cos\theta\right| / r \le r$. Thus, $\left\|\mathbf{g}(\mathbf{z})\right\| / \left\|\mathbf{z}\right\| \to 0$ as $\left\|\mathbf{z}\right\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

16 (d). The eigenvalues of A are $\lambda_1 = -4$ and $\lambda_2 = -1$. Thus, by Theorem 8.4, $\mathbf{z} = \mathbf{0}$ is an asymptotically stable equilibrium point.

17 (a). For the system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$,

$$z_1' = 2z_2 + z_2^2$$
$$z_2' = -2z_1 + z_1 z_2\ ,$$

the coefficient matrix A is given by $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_2^2 \\ z_1 z_2 \end{bmatrix}$.

17 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\left\|\mathbf{g}(\mathbf{z})\right\| = \sqrt{(z_1 z_2)^2 + z_2^4} = \sqrt{(r^2\cos\theta\sin\theta)^2 + r^4\sin^4\theta}$ or $\left\|\mathbf{g}(\mathbf{z})\right\| = \sqrt{r^4\sin^2\theta(\cos^2\theta + \sin^2\theta)} = r^2\left|\sin\theta\right|$. (Also note that $\left\|\mathbf{z}\right\| = r$.)

17 (c). From part (b), $\left\|\mathbf{g}(\mathbf{z})\right\| / \left\|\mathbf{z}\right\| = r^2\left|\sin\theta\right| / r = r\left|\sin\theta\right|$. Thus, $\left\|\mathbf{g}(\mathbf{z})\right\| / \left\|\mathbf{z}\right\| \to 0$ as $\left\|\mathbf{z}\right\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

(d) The eigenvalues of A are $\lambda_1 = -2i$ and $\lambda_2 = 2i$. No conclusion can be drawn from Theorem 8.4 relative to the stability of $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$.

18 (a). For the given system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$, the coefficient matrix A is $A = \begin{bmatrix} -3 & -5 \\ 2 & -1 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_1 e^{-\sqrt{z_1^2 + z_2^2}} \\ z_2 e^{-\sqrt{z_1^2 + z_2^2}} \end{bmatrix}$.

18 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\left\|\mathbf{g}(\mathbf{z})\right\| = re^{-r}$.

18 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = e^{-r}$. Thus, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| \to 1$ as $\|\mathbf{z}\| \to 0$; the system is not almost linear at $\mathbf{z} = \mathbf{0}$.

19 (a). For the system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$,

$$z_1' = 9z_1 + 5z_2 + z_1 z_2$$
$$z_2' = -7z_1 - 3z_2 + z_1^2 ,$$

the coefficient matrix A is given by $A = \begin{bmatrix} 9 & 5 \\ -7 & -3 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_1 z_2 \\ z_1^2 \end{bmatrix}$.

19 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain

$\|\mathbf{g}(\mathbf{z})\| = \sqrt{(z_1 z_2)^2 + z_1^4} = \sqrt{(r^2\cos\theta\sin\theta)^2 + r^4\cos^4\theta}$ or

$\|\mathbf{g}(\mathbf{z})\| = \sqrt{r^4\cos^2\theta(\cos^2\theta + \sin^2\theta)} = r^2|\cos\theta|$. (Also note that $\|\mathbf{z}\| = r$.)

19 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = r^2|\cos\theta|/r = r|\cos\theta|$. Thus, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

(d) The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 4$. Thus, by Theorem 8.4, $\mathbf{z} = \mathbf{0}$ is an unstable equilibrium point of the system.

20 (a). For the given system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$, the coefficient matrix A is $A = \begin{bmatrix} 2 & 2 \\ -5 & -2 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} 0 \\ z_1^2 \end{bmatrix}$.

20 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = r^2\cos^2\theta$.

20 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = r\cos^2\theta$. Thus, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

20 (d). The eigenvalues of A are $\lambda_1 = i\sqrt{6}$ and $\lambda_2 = -i\sqrt{6}$. Thus, no conclusions can be drawn by using Theorem 8.4.

21 (a). The system

$$x' = -x + xy + y$$
$$y' = x - xy - 2y$$

can be expressed as $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$ where the coefficient matrix A is given by $A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_1 z_2 \\ -z_1 z_2 \end{bmatrix}$. Since A is invertible, the solutions of $A\mathbf{z} + \mathbf{g}(\mathbf{z}) = \mathbf{0}$ are vectors $\mathbf{z}_e$ such that $\mathbf{0} = -A^{-1}\mathbf{g}(\mathbf{z}_e)$ and therefore, we need $\mathbf{g}(\mathbf{z}_e) = \mathbf{0}$. Clearly, the only solution of $\mathbf{g}(\mathbf{z}) = \mathbf{0}$ is $\mathbf{z}_e = \mathbf{0}$.

21 (b). The linearized system is $\mathbf{z}' = A\mathbf{z}$ and we find that A has eigenvalues $\lambda_1 = -2.618\ldots$ and $\lambda_2 = -0.382\ldots$ we see that $\mathbf{z} = \mathbf{0}$ is an asymptotically stable equilibrium point of $\mathbf{z}' = A\mathbf{z}$.

21 (c). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = \sqrt{2(z_1 z_2)^2} = \sqrt{2r^4\cos^2\theta\sin^2\theta} = \sqrt{2}\, r^2|\cos\theta\sin\theta|$. (Also note that $\|\mathbf{z}\| = r$.) Therefore, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = \sqrt{2}\, r^2|\cos\theta\sin\theta|/r = \sqrt{2}\, r|\cos\theta|$. Thus, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

21 (d). By Theorem 8.4, $\mathbf{z} = \mathbf{0}$ is an asymptotically stable equilibrium point of the original system.

22 (a). The system has the form

$$x' = y$$

$$y' = 1 - (1+x)^{3/2} .$$

22 (c). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -(3/2)(1+x)^{1/2} & 0 \end{bmatrix}$. Thus, at $\mathbf{z} = \mathbf{0}$, $A = \begin{bmatrix} 0 & 1 \\ -3/2 & 0 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = i\sqrt{3/2}$ and $\lambda_2 = -i\sqrt{3/2}$ and hence the linearized system is stable but not asymptotically stable at $\mathbf{z} = \mathbf{0}$.

22 (d). Theorem 8.4 does not provide any information about the stability of the nonlinear system since the eigenvalues of the linearized system $\mathbf{z}' = A\mathbf{z}$ are purely imaginary.

23 (a). Multiplying by x' we obtain $x'x'' = x'[1-(1+x)^{3/2}]$. Integrating, we obtain $0.5(x')^2 = x - 0.4(1+x)^{5/2}$. Therefore, with $y = x'$ we have $y^2 = 2x - 0.8(1+x)^{5/2} + C$.

24 (a). At the equilibrium point (0, 0), the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Since A is not invertible, Theorem 8.4 does not apply.

24 (b). Let $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. For the given system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$, $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} -z_1^{2/3} \\ 2z_2^{1/3} \end{bmatrix}$. Using polar coordinates, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = \sqrt{r^{-2/3}\cos^{4/3}\theta + 4r^{-4/3}\sin^{2/3}\theta}$. Thus, the limit of $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\|$ does not exist as $\|\mathbf{z}\| \to 0$; The system is not almost linear at (0, 0).

27. In this case, $a_{11} = 0, a_{12} = 1, a_{21} = -1, a_{22} = 0, g_1 = \alpha r^3\cos\theta$, and $g_2 = \alpha r^3\sin\theta$. Thus, $h(r) = \alpha r^2$ and we obtain the system

$$r' = \alpha r^3$$

$$\theta' = -1 .$$

Solving, $r(t) = (C_1 - 2\alpha t)^{-1/2}$ and $\theta(t) = -t + C_2$. Hence, $x = (C_1 - 2\alpha t)^{-1/2}\cos(-t + C_2)$ and $y = (C_1 - 2\alpha t)^{-1/2}\sin(-t + C_2)$.

28. So, $a_{11} = 1, a_{12} = 0, a_{21} = 0, a_{22} = 1, g_1 = r^2\cos\theta$, and $g_2 = r^2\sin\theta$. Thus, $h(r) = r$ and we obtain the initial value problem

$$r' = r + r^2 , \quad r(0) = 1$$

$$\theta' = 0 , \quad \theta(0) = \sqrt{3} .$$

The solution is $r = (2/3)e^t/[1-(2/3)e^t]$, $\theta = \pi/3$. However, the denominator in the expression for *r*, $1-(2/3)e^t$, vanishes at $3/2 = e^t$. Solving for *t*, we have $t = \ln 1.5 = 0.405\ldots$. Thus, the solution does not exist at $t = 1$.

29. So, $a_{11}=0, a_{12}=1, a_{21}=-1, a_{22}=0, g_1=-r\cos\theta\ln r^2$, and $g_2=-r\sin\theta\ln r^2$. Thus, $h(r)=-\ln r^2$ and we obtain the initial value problem

$$r'=-2r\ln r\,,\ r(0)=1$$
$$\theta'=1\,,\ \theta(0)=\pi/4\,.$$

The general solution is $r=C_1\exp(e^{-2t})$, $\theta=t+C_2$. Imposing the initial conditions we arrive at $r=\exp(e^{-2t}-1)$, $\theta=t+\pi/4$. Hence, at $t=1$, we find $x=\exp(e^{-2}-1)\cos(1+\pi/4)\approx -0.0896\ldots$ and $y=\exp(e^{-2}-1)\sin(1+\pi/4)\approx 0.411\ldots$

Section 8.6

1 (a). Since the eigenvalues are real and have opposite signs, $\mathbf{y}=\mathbf{0}$ is an unstable saddle point.

1 (d). We have $\Psi(t)=[e^{\lambda_1 t}\mathbf{x}_1, e^{\lambda_2 t}\mathbf{x}_2]=\begin{bmatrix} e^{2t} & e^{-t} \\ e^{2t} & -e^{-t}\end{bmatrix}$ and $\Psi'(t)=\begin{bmatrix} 2e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t}\end{bmatrix}$.

Therefore, $A=\Psi'(t)\Psi^{-1}(t)=\begin{bmatrix} 2e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t}\end{bmatrix}\begin{bmatrix} 0.5e^{-2t} & 0.5e^{-2t} \\ 0.5e^{t} & -0.5e^{t}\end{bmatrix}=\begin{bmatrix} 0.5 & 1.5 \\ 1.5 & 0.5\end{bmatrix}$.

2 (a). Since the eigenvalues are real and positive, $\mathbf{y}=\mathbf{0}$ is an unstable node.

2 (d). We have $\Psi(t)=[e^{\lambda_1 t}\mathbf{x}_1, e^{\lambda_2 t}\mathbf{x}_2]=\begin{bmatrix} e^{t} & 2e^{2t} \\ 2e^{t} & -e^{2t}\end{bmatrix}$ and $\Psi'(t)=\begin{bmatrix} e^{t} & 4e^{2t} \\ 2e^{t} & -2e^{2t}\end{bmatrix}$.

Therefore, $A=\Psi'(t)\Psi^{-1}(t)=\begin{bmatrix} 9/5 & -2/5 \\ -2/5 & 6/5\end{bmatrix}$.

3 (a). Since both eigenvalues are real and positive, $\mathbf{y}=\mathbf{0}$ is an unstable improper node.

3 (d). We have $\Psi(t)=[e^{\lambda_1 t}\mathbf{x}_1, e^{\lambda_2 t}\mathbf{x}_2]=\begin{bmatrix} 2e^{2t} & 0 \\ 0 & 2e^{t}\end{bmatrix}$ and $\Psi'(t)=\begin{bmatrix} 4e^{2t} & 0 \\ 0 & 2e^{t}\end{bmatrix}$.

Therefore, $A=\Psi'(t)\Psi^{-1}(t)=\begin{bmatrix} 4e^{2t} & 0 \\ 0 & 2e^{t}\end{bmatrix}\begin{bmatrix} 0.5e^{-2t} & 0 \\ 0 & 0.5e^{-t}\end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0 & 1\end{bmatrix}$.

4 (a). Since the eigenvalues are real and negative, $\mathbf{y}=\mathbf{0}$ is an asymptotically stable node.

4 (d). We have $\Psi(t)=[e^{\lambda_1 t}\mathbf{x}_1, e^{\lambda_2 t}\mathbf{x}_2]=\begin{bmatrix} e^{-2t} & e^{-t} \\ 0 & e^{-t}\end{bmatrix}$ and $\Psi'(t)=\begin{bmatrix} -2e^{-2t} & -e^{-t} \\ 0 & -e^{-t}\end{bmatrix}$.

Therefore, $A=\Psi'(t)\Psi^{-1}(t)=\begin{bmatrix} -2 & 1 \\ 0 & -1\end{bmatrix}$.

5 (a). Since the eigenvalues are real and have opposite signs, $\mathbf{y}=\mathbf{0}$ is an unstable saddle point.

5 (d). We have $\Psi(t)=[e^{\lambda_1 t}\mathbf{x}_1, e^{\lambda_2 t}\mathbf{x}_2]=\begin{bmatrix} e^{t} & 2e^{-t} \\ 0 & e^{-t}\end{bmatrix}$ and $\Psi'(t)=\begin{bmatrix} e^{t} & -2e^{-t} \\ 0 & -e^{-t}\end{bmatrix}$.

Therefore, $A=\Psi'(t)\Psi^{-1}(t)=\begin{bmatrix} e^{t} & -2e^{-t} \\ 0 & -e^{-t}\end{bmatrix}\begin{bmatrix} e^{-t} & -2e^{-t} \\ 0 & e^{t}\end{bmatrix}=\begin{bmatrix} 1 & -4 \\ 0 & -1\end{bmatrix}$.

6 (a). For $A=\begin{bmatrix} 1 & -6 \\ 1 & -4\end{bmatrix}$, the eigenvalues are $\lambda_1=-1$ and $\lambda_2=-2$.

6 (b). Since the eigenvalues are real and negative, $\mathbf{y}=\mathbf{0}$ is an asymptotically stable improper node.

7 (a). For $A = \begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$.

7 (b). Since the eigenvalues are real and positive, $\mathbf{y} = \mathbf{0}$ is an unstable improper node.

8 (a). For $A = \begin{bmatrix} -6 & 14 \\ -2 & 5 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -2$.

8 (b). Since the eigenvalues have opposite sign, $\mathbf{y} = \mathbf{0}$ is an unstable saddle point.

9 (a). For $A = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 3i$ and $\lambda_2 = -3i$.

9 (b). Since the eigenvalues are complex with zero real part, $\mathbf{y} = \mathbf{0}$ is a stable, but not asymptotically stable, center.

10 (a). For $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$, the eigenvalues are $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$.

10 (b). Since the eigenvalues are complex with negative real part, $\mathbf{y} = \mathbf{0}$ is an asymptotically stable spiral point.

11 (a). For $A = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}$, the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -2$.

11 (b). Since the eigenvalues are real and negative, $\mathbf{y} = \mathbf{0}$ is an asymptotically stable improper node.

12 (a). For $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$.

12 (b). Since the eigenvalues are complex with positive real part, $\mathbf{y} = \mathbf{0}$ is an unstable spiral point.

13 (a). For $A = \begin{bmatrix} -2 & -4 \\ 5 & 2 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 4i$ and $\lambda_2 = -4i$.

13 (b). Since the eigenvalues are complex with zero real part, $\mathbf{y} = \mathbf{0}$ is a stable, but not asymptotically stable, center.

14 (a). For $A = \begin{bmatrix} 7 & -24 \\ 2 & -7 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$.

14 (b). Since the eigenvalues are real with opposite sigen, $\mathbf{y} = \mathbf{0}$ is an unstable saddle point.

15 (a). For $A = \begin{bmatrix} -1 & 8 \\ -1 & 5 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 3$.

15 (b). Since the eigenvalues are real and positive, $\mathbf{y} = \mathbf{0}$ is an unstable improper node.

16 (a). For $A = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$, the eigenvalues are $\lambda_1 = -2 + i$ and $\lambda_2 = -2 - i$.

16 (b). Since the eigenvalues are complex with negative real part, $\mathbf{y} = \mathbf{0}$ is an asymptotically stable spiral point.

17 (a). For $A = \begin{bmatrix} 2 & 4 \\ -4 & -6 \end{bmatrix}$, the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -2$.

17 (b). Since the eigenvalues are real and negative and A is not a multiple of the identity, $\mathbf{y} = \mathbf{0}$ is an asymptotically stable improper node.

18 (a). For $A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 3$.

18 (b). Since the eigenvalues are real and positive and A is a multiple of the identity, $\mathbf{y}=\mathbf{0}$ is an unstable proper node.

19 (a). For $A=\begin{bmatrix}1 & 2\\ -8 & 1\end{bmatrix}$, the eigenvalues are $\lambda_1=1+4i$ and $\lambda_2=1-4i$.

19 (b). Since the eigenvalues are complex with positive real part, $\mathbf{y}=\mathbf{0}$ is an unstable spiral point.

20 (a). For $A=\begin{bmatrix}-1 & -2\\ 2 & 3\end{bmatrix}$, the eigenvalues are $\lambda_1=1$ and $\lambda_2=1$.

20 (b). Since the eigenvalues are real and positive and A is not a multiple of the identity, $\mathbf{y}=\mathbf{0}$ is an unstable improper node.

21 (a). For $A_1=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}$, the eigenvalues are $\lambda_1=-3$ and $\lambda_2=-1$. Since the eigenvalues are real and negative, $\mathbf{y}=\mathbf{0}$ is an asymptotically stable equilibrium point. Therefore, A_1 corresponds to Direction Field 2.

21 (b). For $A_2=\begin{bmatrix}1 & 2\\ -2 & -1\end{bmatrix}$, the eigenvalues are $\lambda_1=-\sqrt{3}i$ and $\lambda_2=\sqrt{3}i$. Since the eigenvalues are complex with zero real part, $\mathbf{y}=\mathbf{0}$ is a stable, but not asymptotically stable, center. Therefore, A_2 corresponds to Direction Field 4.

21 (c). For $A_3=\begin{bmatrix}2 & 1\\ -1 & -2\end{bmatrix}$, the eigenvalues are $\lambda_1=-\sqrt{3}$ and $\lambda_2=\sqrt{3}$. Since the eigenvalues are real and have opposite sign, $\mathbf{y}=\mathbf{0}$ is an unstable saddle point. Therefore, A_3 corresponds to Direction Field 1.

21 (d). For $A_4=\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix}$, the eigenvalues are $\lambda_1=1-2i$ and $\lambda_2=1+2i$. Since the eigenvalues are complex with positive real part, $\mathbf{y}=\mathbf{0}$ is an unstable spiral point. Therefore, A_4 corresponds to Direction Field 3.

22. For a center, eigenvalues are purely imaginary. Therefore, $\alpha=-2$.

23. Consider $A=\begin{bmatrix}-4 & \alpha\\ -2 & 2\end{bmatrix}$. The characteristic polynomial is $p(\lambda)=\lambda^2+2\lambda+(2\alpha-8)$. Thus, the eigenvalues are $\lambda=-1\pm\sqrt{9-2\alpha}$. In order to have an asymptotically stable spiral point at $\mathbf{y}=\mathbf{0}$, we need complex eigenvalues with negative real parts. Thus, we need $9-2\alpha<0$ or $9/2<\alpha$.

24. Note that $\lambda_1=-2$ and $\lambda_2=-2$ no matter the value of α. Thus, $\mathbf{y}=\mathbf{0}$ is always an asymptotically stable equilibrium point; it will be a proper node if $\alpha=0$.

25. Consider $A=\begin{bmatrix}4 & -2\\ \alpha & -4\end{bmatrix}$. The characteristic polynomial is $p(\lambda)=\lambda^2+(2\alpha-16)$. Thus, the eigenvalues are $\lambda=\pm\sqrt{16-2\alpha}$. In order to have a saddle point at $\mathbf{y}=\mathbf{0}$, we need real eigenvalues with opposite signs. Thus, we need $16-2\alpha>0$ or $\alpha<8$.

26. Consider the nonhomogeneous system $\mathbf{y}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. The system has a unique equilibrium point given by $\mathbf{y}_e = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Making the substitution $\mathbf{z} = \mathbf{y} - \mathbf{y}_e$, we obtain $\mathbf{z}' = \begin{bmatrix} 1 & -4 \\ -1 & 1 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Therefore, $\mathbf{z} = \mathbf{0}$ is an unstable spiral point and consequently, $\mathbf{y} = \mathbf{y}_e$ is an unstable spiral point of the original system.

27. Consider the nonhomogeneous system $\mathbf{y}' = \begin{bmatrix} 6 & 5 \\ -7 & -6 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 4 \\ -6 \end{bmatrix}$. The system has a unique equilibrium point given by $\mathbf{y}_e = -\begin{bmatrix} 6 & 5 \\ -7 & -6 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} -6 & -5 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \end{bmatrix}$. Making the substitution $\mathbf{z} = \mathbf{y} - \mathbf{y}_e$, we obtain $\mathbf{z}' = \begin{bmatrix} 6 & 5 \\ -7 & -6 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = -1$ and $\lambda_2 = 1$. Therefore, $\mathbf{z} = \mathbf{0}$ is an unstable saddle point and consequently, $\mathbf{y} = \mathbf{y}_e$ is an unstable saddle point of the original system.

28. Consider the nonhomogeneous system $\mathbf{y}' = \begin{bmatrix} 5 & -14 \\ 3 & -8 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The system has a unique equilibrium point given by $\mathbf{y}_e = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$. Making the substitution $\mathbf{z} = \mathbf{y} - \mathbf{y}_e$, we obtain $\mathbf{z}' = \begin{bmatrix} 5 & -14 \\ 3 & -8 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = -2$ and $\lambda_2 = -1$. Therefore, $\mathbf{z} = \mathbf{0}$ is an asymptotically stable improper node and consequently, $\mathbf{y} = \mathbf{y}_e$ is an asymptotically stable improper node of the original system.

29. Consider the nonhomogeneous system $\mathbf{y}' = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2 \\ -4 \end{bmatrix}$. The system has a unique equilibrium point given by $\mathbf{y}_e = -\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Making the substitution $\mathbf{z} = \mathbf{y} - \mathbf{y}_e$, we obtain $\mathbf{z}' = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = -1$ and $\lambda_2 = 2$. Therefore, $\mathbf{z} = \mathbf{0}$ is an unstable saddle point and consequently, $\mathbf{y} = \mathbf{y}_e$ is an unstable saddle point of the original system.

30 (a). The characteristic equation is $\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$. The origin is a center if the roots are purely imaginary. That is, if $a_{11} + a_{22} = 0$ and $a_{11}a_{22} - a_{12}a_{21} < 0$.

30 (b). Note that $f(x,y) = a_{11}x + a_{12}y$ and $g(x,y) = a_{21}x + a_{22}y$. Thus, $f_x = a_{11}$ and $g_y = a_{22}$. By part (a), $f_x = -g_y$ and hence the system is Hamiltonian.

30 (c). The converse is not true since the system can be Hamiltonian even though $a_{11}a_{22} - a_{12}a_{21} = 0$.

32 (a). The eigenvalues of the coefficient matrix $A = \begin{bmatrix} -2 & 1 \\ 5 & 2 \end{bmatrix}$ are $\lambda_1 = 3$ and $\lambda_2 = -3$.

32 (b). Since the eigenvalues are real with opposite sign, $\mathbf{y} = \mathbf{0}$ is an (unstable) saddle point.

32 (c). Since the system is Hamiltonian, we know that $H_y(x,y) = -2x + y$. Therefore, $H(x,y) = -2xy + 0.5y^2 + q(x)$. We determine $q(x)$ by differentiating $H(x,y)$ with respect to *x*, finding $H_x(x,y) = -2y + q'(x) = -5x - 2y$. Thus, $q'(x) = -5x$ and so $q(x) = -2.5x^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = -2.5x^2 - 2xy + 0.5y^2$. The conservation law for the system is $H(x,y) = C$.

33 (a). The eigenvalues of the coefficient matrix $A = \begin{bmatrix} 1 & 3 \\ -3 & -1 \end{bmatrix}$ are $\lambda_1 = -2\sqrt{2}i$ and $\lambda_2 = -2\sqrt{2}i$.

33 (b). Since the eigenvalues are complex with zero real part, $\mathbf{y} = \mathbf{0}$ is a stable, but not asymptotically stable, center.

33 (c). Since the system is Hamiltonian, we know that $H_y(x,y) = x + 3y$. Therefore, $H(x,y) = xy + 1.5y^2 + q(x)$. We determine $q(x)$ by differentiating $H(x,y)$ with respect to *x*, finding $-3x - y = -H_x(x,y) = -y - q'(x)$. Thus, $q'(x) = 3x$ and so $q(x) = 1.5x^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = xy + 1.5(x^2 + y^2)$. The conservation law for the system is $H(x,y) = C$.

34 (a). The eigenvalues of the coefficient matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$ are $\lambda_1 = 2$ and $\lambda_2 = -2$.

34 (b). Since the eigenvalues are real with opposite sign, $\mathbf{y} = \mathbf{0}$ is an (unstable) saddle point.

34 (c). Since the system is Hamiltonian, we know that $H_y(x,y) = 2x + y$. Therefore, $H(x,y) = 2xy + 0.5y^2 + q(x)$. We determine $q(x)$ by differentiating $H(x,y)$ with respect to *x*, finding $H_x(x,y) = 2y + q'(x) = 2y$. Thus, $q'(x) = 0$ and so $q(x) = C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = 2xy + 0.5y^2$. The conservation law for the system is $H(x,y) = C$.

Section 8.7

1 (a). Consider the system

$$x' = x - x^2 - xy$$
$$y' = y - 3y^2 - 0.5xy\ .$$

If $y = 0$, then all direction field filaments on the positive *x*-axis point towards $x = 1$. Thus, *x* approaches an equilibrium value of $x_e = 1$ as *t* increases. Similarly, if $x = 0$, then *y* approaches an equilibrium value of $y_e = 1/3$ as *t* increases.
In each case, the presence of the *xy* term causes the derivative to decrease. Therefore, the presence of the other species is harmful in each case.

1 (b). Rewriting the system as

$$x' = x(1 - x - y)$$
$$y' = y(1 - 3y - 0.5x)\ ,$$

we see that $x' = 0$ if (i) $x = 0$ or (ii) $1 - x - y = 0$. In case (i), $y' = 0$ if $y = 0$ or $y = 1/3$. Thus, two equilibrium points are $(x,y) = (0,0)$ and $(x,y) = (0,1/3)$. In case (ii), $y' = 0$ if $y = 0$ (and hence, $x = 1$) or if $1 - 3y - 0.5x = 0$ (and hence $x + y = 1$ and $0.5x + 3y = 1$). Thus, case (ii) leads us to two more equilibrium points $(x,y) = (1,0)$ and $(x,y) = (0.8,0.2)$.

1 (c). At the equilibrium point $\mathbf{z}=\mathbf{0}$, the linearized system takes the form $\mathbf{z}'=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1=1$ and $\lambda_2=1$. Since, $\mathbf{z}=\mathbf{0}$ is an unstable proper node of the linearized system, the original system is also unstable at $\mathbf{y}=\mathbf{0}$.

2 (a). Consider the system

$$x'=-x-x^2$$
$$y'=-y+xy\ .$$

If $y=0$, then x approaches an equilibrium value of $x_e=0$ as t increases. If $x=0$, then y approaches an equilibrium value of $y_e=0$ as t increases.

The presence of y is a matter of indifference to x. The presence of x is beneficial to y.

2 (b). The only equilibrium point in the first quadrant is $(x,y)=(0,0)$.

2 (c). At the equilibrium point $\mathbf{z}=\mathbf{0}$, the linearized system takes the form $\mathbf{z}'=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1=-1$ and $\lambda_2=-1$. Since, $\mathbf{z}=\mathbf{0}$ is an asymptotically stable proper node of the linearized system, the original system is also asymptotically stable at $\mathbf{y}=\mathbf{0}$.

3 (a). Consider the system

$$x'=x-x^2-xy$$
$$y'=-y-y^2+xy\ .$$

If $y=0$, then all direction field filaments on the positive x-axis point towards $x=1$. Thus, x approaches an equilibrium value of $x_e=1$ as t increases. Similarly, if $x=0$, then y approaches an equilibrium value of $y_e=0$ as t increases. The presence of the xy term in the first equation causes the derivative to decrease. Therefore, the presence of y is harmful to x. On the other hand, the presence of the xy term in the second equation causes the derivative to increase. Therefore, the presence of x is beneficial to y.

3 (b). Rewriting the system as

$$x'=x(1-x-y)$$
$$y'=-y(1+y-x)\ ,$$

we see that $x'=0$ if (i) $x=0$ or (ii) $1-x-y=0$. In case (i), $y'=0$ if $y=0$ or $y=-1$. The latter possibility has been excluded and thus case (i) leads to a single equilibrium point, $(x,y)=(0,0)$. In case (ii), $y'=0$ if $y=0$ (and hence, $x=1$) or if $1+y-x=0$ (and hence $x+y=1$ and $x-y=1$). This second set of equations also has solution $x=1$ and $y=0$. Thus, case (ii) leads us to one more equilibrium point $(x,y)=(1,0)$.

3 (c). At the equilibrium point $\mathbf{z}=\mathbf{0}$, the linearized system takes the form $\mathbf{z}'=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1=-1$ and $\lambda_2=1$. Since, $\mathbf{z}=\mathbf{0}$ is an unstable saddle point of the linearized system, the original system is also unstable at $\mathbf{y}=\mathbf{0}$.

4 (a). Consider the system

$$x' = x - x^2 + xy$$
$$y' = y - y^2 + xy \ .$$

If $y = 0$, then x approaches an equilibrium value of $x_e = 1$ as t increases. If $x = 0$, then y approaches an equilibrium value of $y_e = 1$ as t increases.
In both cases, the presence of one species is beneficial to the other species.

4 (b). The only equilibrium points in the first quadrant are $(x,y) = (0,0)$, $(x,y) = (0,1)$, and $(x,y) = (1,0)$.

4 (c). At the equilibrium point $\mathbf{z} = \mathbf{0}$, the linearized system takes the form $\mathbf{z}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = 1$ and $\lambda_2 = 1$. Since, $\mathbf{z} = \mathbf{0}$ is an unstable proper node of the linearized system, the original system is also unstable at $\mathbf{y} = \mathbf{0}$.

5 (a). When $y = 0$, the assumed model reduces to $x' = r_1(1 + \alpha_1 x)x$. In this case, we see from the figure, that $\ln x(t) = 0.5t + \ln x(0)$. Differentiating, we obtain $\dfrac{x'(t)}{x(t)} = 0.5$ or $x' = 0.5x$. Thus, $\alpha_1 = 0$ and $r_1 = 0.5$. Similarly, when $x = 0$, the model reduces to $y' = r_2(1 + \alpha_2 y)y$. In this case, we see from the figure, that $\ln y(t) = -t + \ln y(0)$. Differentiating, we obtain $\dfrac{y'(t)}{y(t)} = -1$ or $y' = -y$. Thus, $\alpha_2 = 0$ and $r_2 = -1$. So far, we have deduced that the assumptions of the population model imply it has the form

$$x' = 0.5(1 + \beta_1 y)x$$
$$y' = -(1 + \beta_2 x)y \ .$$

Knowing the equilibrium point $(x_e, y_e) = (2,3)$, allows us to determine the last remaining model parameters, β_1 and β_2. In particular, we know from the first equation that $0.5(1 + 3\beta_1)2 = 0$ while the second equation gives $-(1 + 2\beta_2)3 = 0$. Consequently, $\beta_1 = -1/3$ and $\beta_2 = -1/2$.

5 (b). From part (a), the model is given by

$$x' = (1/2)x - (1/6)xy$$
$$y' = -y + (1/2)xy \ .$$

The presence of y causes x' to decrease and hence y is harmful to x. The presence of x causes y' to increase and hence x is beneficial to y.

6 (a). Consider the system

$$x' = r(1 - \alpha x - \beta y)x + \mu x$$
$$y' = r(1 - \alpha y - \beta x)y \ .$$

The equilibrium points are $(x,y) = (0,0)$, $(x,y) = (0,\alpha^{-1})$, $(x,y) = (\alpha^{-1}(1 + \mu r^{-1}),0)$, and $(x,y) = \delta^{-1}(\alpha(1 + \mu r^{-1}) - \beta, \alpha - \beta(1 + \mu r^{-1}))$ where $\delta = \alpha^2 - \beta^2$.

6 (b). If μ is chosen large enough so that $\beta(1 + \mu r^{-1}) > \alpha$ then we see from part (a) that the "coexisting species" equilibrium point is moved into the fourth quadrant and is therefore physically irrelevant.

6 (c). At $\mathbf{z}=\mathbf{0}$, the linearized system has the form $\mathbf{z}'=\begin{bmatrix} r+\mu & 0 \\ 0 & r \end{bmatrix}\mathbf{z}$. The point $\mathbf{z}=\mathbf{0}$ is an unstable improper node. At the equilibrium point $\mathbf{z}=\begin{bmatrix} 0 \\ 1/\alpha \end{bmatrix}$, the linearized system is $\mathbf{z}'=\begin{bmatrix} r(1+\mu r^{-1}-\beta\alpha^{-1}) & 0 \\ -r\beta\alpha^{-1} & -r \end{bmatrix}\mathbf{z}$. The eigenvalues are $\lambda_1=-r$ and $\lambda_2=r(1+\mu r^{-1}-\beta\alpha^{-1})$. Since the eigenvalues have opposite sign, the equilibrium point is an unstable saddle point. The equilibrium point $(x,y)=(\alpha^{-1}(1+\mu r^{-1}),0)$ is an asymptotically stable improper node since the eigenvalues of the linearized system are negative and different:
$\lambda_1=-r(1+\mu r^{-1})$ and $\lambda_2=r[1-\beta\mu(\alpha r)^{-1}-\beta\alpha^{-1}]$.

6 (d). For the nonlinear system, $(0,0)$ and $(0,\alpha^{-1})$ are unstable equilibrium points. The equilirium point $(x,y)=(\alpha^{-1}(1+\mu r^{-1}),0)$ is stable.

6 (e). It appears that the *y* species will be driven to extinction with the *x* species approaching the limiting value $\alpha^{-1}(1+\mu r^{-1})$.

7 (a). Consider the system

$$x'=r(1-\alpha x-\beta y)x$$
$$y'=r(1-\alpha y-\beta x)y-\mu y\ .$$

We see that $x'=0$ if (i) $x=0$ or (ii) $1-\alpha x-\beta y=0$. In case (i), $y'=0$ if $y=0$ or $y=(r-\mu)/(\alpha r)$. Thus case (i) leads to two equilibrium points, $(x,y)=(0,0)$ and $(x,y)=(0,(r-\mu)/(\alpha r))$. In case (ii), $y'=0$ if $y=0$ or if $1-(\mu/r)-\alpha y-\beta x=0$. Thus case (ii) leads to two equilibrium points, $(x,y)=(1/\alpha,0)$ and
$(x,y)=(\delta^{-1}[\alpha-\beta(1-\mu r^{-1})],\delta^{-1}[-\beta+\alpha(1-\mu r^{-1})])$ where $\delta=\alpha^2-\beta^2$.

7 (b). If $\mu>r$, then $1-\mu r^{-1}<0$. In this case, we see from part (a) that the only physically relevant equilibrium points are $(x,y)=(0,0)$ and $(x,y)=(1/\alpha,0)$.

7 (c). At $\mathbf{z}=\mathbf{0}$, the linearized system has the form $\mathbf{z}'=\begin{bmatrix} r & 0 \\ 0 & r-\mu \end{bmatrix}\mathbf{z}$. Since we are assuming $\mu>r$, the point $\mathbf{z}=\mathbf{0}$ is an unstable saddle point. At the equilibrium point $\mathbf{z}=\begin{bmatrix} 1/\alpha \\ 0 \end{bmatrix}$, the linearized system is $\mathbf{z}'=\begin{bmatrix} -r & -r\beta\alpha^{-1} \\ 0 & r-\mu-r\beta\alpha^{-1} \end{bmatrix}\mathbf{z}$. The eigenvalues are $\lambda_1=-r$ and $\lambda_2=r-\mu-r\beta\alpha^{-1}$.
Since both eigenvalues are negative, the equilibrium point is an asymptotically stable improper node.

7 (d). For the nonlinear system, $(0,0)$ is unstable and $(\alpha^{-1},0)$ is stable.

7 (e). If $\mu>r$, it appears that the *y* species will be driven to extinction with the *x* species approaching the limiting value α^{-1}.

8. The strategy of nurturing the desirable species leads to an equilibrium *x*-population of $\alpha^{-1}(1+\mu r^{-1})$. This is greater than the equilibrium *x*-population of α^{-1} that results from harvesting the undesirable species.

9. Consider the population model

$$x' = \pm a_1 x \pm b_1 x^2 \pm c_1 xy \pm d_1 xz$$
$$y' = \pm a_2 y \pm b_2 y^2 \pm c_2 xy \pm d_2 yz$$
$$z' = \pm a_3 z \pm c_3 xz \pm d_3 yz\ .$$

Since x and y are mutually competitive, we need to choose a negative sign for c_1 and c_2 (the presence of x reduces the growth rate y' and similarly the presence of y reduces the growth rate x'). The same argument applies to the signs of d_1 and d_2 since the predator is harmful to x and to y. The presence of the prey is beneficial to the predator z and thus we need to choose a positive sign for c_3 and d_3.
So far, we have deduced

$$x' = \pm a_1 x \pm b_1 x^2 - c_1 xy - d_1 xz$$
$$y' = \pm a_2 y \pm b_2 y^2 - c_2 xy - d_2 yz$$
$$z' = \pm a_3 z + c_3 xz + d_3 yz\ .$$

We also know that, in the absence of the other two species, x and y each evolve towards a nonzero equilibrium value. Thus, from the first equation, we know the term $\pm a_1 x \pm b_1 x^2 = x(\pm a_1 \pm b_1 x)$ has a positive zero, as does the corresponding term in the second equation, $\pm a_2 y \pm b_2 y^2 = y(\pm a_2 \pm b_2 y)$. From this fact, we infer that a_1 and b_1 have opposite signs, as do a_2 and b_2. The general solution of an equation of the form $u' = au + bu^2$ is $u = Ae^{-at} + Bt^2 + Ct + D$. If a is negative, then $u(t) \to \infty$ as $t \to \infty$. Hence, there cannot be a nonzero equilibrium solution when a is negative. Applying this observation to the equations $x' = \pm a_1 x \pm b_1 x^2$ and $y' = \pm a_2 y \pm b_2 y^2$, we deduce that a_1 and a_2 are positive and b_1 and b_2 are negative. Likewise, in order that z decrease to zero in the absence of x and y, we need to have a_3 negative. In summary, we arrive at the following model which will support the observations:

$$x' = a_1 x - b_1 x^2 - c_1 xy - d_1 xz$$
$$y' = a_2 y - b_2 y^2 - c_2 xy - d_2 yz$$
$$z' = -a_3 z + c_3 xz + d_3 yz\ .$$

10 (a). Consider the system

$$s' = -\alpha si + \gamma r$$
$$i' = \alpha si - \beta i$$
$$r' = \beta i - \gamma r\ .$$

Summing these three equations, we obtain $s'(t) + i'(t) + r'(t) = 0$. Hence, $s(t) + i(t) + r(t)$ is constant, say $s(t) + i(t) + r(t) = N$ where N denotes the size of the population.

10 (b). If those who recover are permanently immunized, then

$$s' = -\alpha si$$
$$i' = \alpha si - \beta i$$
$$r' = \beta i\ .$$

As in part (a), we can sum these equations and again conclude that $s(t) + i(t) + r(t) = N$.

10 (c). If some infected members perish, then

$$s' = -\alpha si$$
$$i' = \alpha si - \beta i$$
$$r' = \beta i - \gamma r\ .$$

In this case, $s'(t) + i'(t) + r'(t) = -\gamma r(t)$. Thus, the population is not constant but rather is decreasing.

11 (a). Consider the system

$$s' = -\alpha si + \gamma r$$
$$i' = \alpha si - \beta i$$
$$r' = \beta i - \gamma r\ .$$

Using the fact, from Exercise 10, that $s + i + r = N$, we obtain a reduced system,

$$s' = -\alpha si + \gamma(N - i - s)$$
$$i' = \alpha si - \beta i\ .$$

11 (b). For the given values, $\alpha = \beta = \gamma = 1$ and $N = 9$, the reduced system has the form

$$s' = -si + (9 - i - s)$$
$$i' = si - i\ .$$

Rewriting this system slightly,

$$s' = -si + 9 - i - s$$
$$i' = i(s - 1)\ .$$

We see that $i' = 0$ if (i) $i = 0$ or (ii) $s = 1$. In case (i), $s' = 0$ if $s = 9$. Thus case (i) leads to the equilibrium point $(s,i) = (9,0)$. In case (ii), $s' = 0$ if $i = 4$. Thus case (ii) leads to the equilibrium point $(s,i) = (1,4)$.

11 (c). At $\mathbf{z} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$, the linearized system has the form $\mathbf{z}' = \begin{bmatrix} -1 & -10 \\ 0 & 8 \end{bmatrix}\mathbf{z}$. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 8$. This equilibrium point is an unstable saddle point. At $\mathbf{z} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, the linearized system has the form $\mathbf{z}' = \begin{bmatrix} -5 & -2 \\ 4 & 0 \end{bmatrix}\mathbf{z}$. The eigenvalues are $\lambda_1 = (-5 - i\sqrt{7})/2$ and $(-5 + i\sqrt{7})/2$. This equilibrium point is an asymptotically stable spiral point.

11 (d). $(9,0)$ is an unstable equilibrium point while $(1,4)$ is stable.