Chapter 9
Numerical Methods

Section 9.1

Unless indicated by $\ldots$, all results are rounded to the places shown.

1 (a). Integrating $y' = 2t - 1$, we find $y = t^2 - t + C$. Imposing the initial condition $y(1) = 0$, we obtain $y = t^2 - t$.

1 (b). Since $f(t,y) = 2t - 1$, it follows that $f(t+h, y + hf(t,y)) = 2(t+h) - 1$. Therefore, Heun's method takes the form

$$y_{n+1} = y_n + (h/2)[(2t_n - 1) + (2t_{n+1} - 1)].$$

1 (c). As in part (b), we find the modified Euler's method takes the form

$$y_{n+1} = y_n + h(2(t_n + h/2) - 1).$$

1 (d).

1.0000	0
1.1000	0.1100
1.2000	0.2400
1.3000	0.3900

1 (e).

1.0000	0
1.1000	0.1100
1.2000	0.2400
1.3000	0.3900

1 (f).

1.0000	0
1.1000	0.1100
1.2000	0.2400
1.3000	0.3900

2 (a). Integrating $y' = -y$ and imposing the initial condition, we obtain $y = e^{-t}$.

2 (b). Heun's method takes the form $y_{n+1} = (1 - h + 0.5h^2)y_n$.

2 (c). The modified Euler's method takes the form $y_{n+1} = (1 - h + 0.5h^2)y_n$.

2 (d).

0.0000	1.0000
0.1000	0.9050
0.2000	0.8190
0.3000	0.7412

2 (e).

0.0000	1.0000
0.1000	0.9050
0.2000	0.8190
0.3000	0.7412

2 (f).

0.0000	1.0000
0.1000	0.9048
0.2000	0.8187
0.3000	0.7408

3 (a). Solving the separable equation $y' = -ty$, we find $y = Ce^{-t^2/2}$. Imposing the initial condition $y(0) = 1$, we obtain $y = e^{-t^2/2}$.

3 (b). Since $f(t, y) = -ty$, it follows that $f(t + h, y + hf(t, y)) = -(t + h)[y + h(-ty)]$. Therefore, Heun's method takes the form

$$y_{n+1} = y_n + (h/2)[-t_n y_n - t_{n+1}(y_n - ht_n y_n)].$$

3 (c). As in part (b), we find the modified Euler's method takes the form

$$y_{n+1} = y_n - h(t_n + 0.5h)(y_n - 0.5ht_n y_n).$$

3 (d).

0	1.0000
0.1000	0.9950
0.2000	0.9802
0.3000	0.9560

3 (e).

0	1.0000
0.1000	0.9950
0.2000	0.9801
0.3000	0.9559

3 (f).

0	1.0000
0.1000	0.9950
0.2000	0.9802
0.3000	0.9560

4 (a). Integrating $y' = -y + t$ and imposing the initial condition, we obtain $y = t - 1 + e^{-t}$.

4 (b). Heun's method takes the form $y_{n+1} = y_n + 0.5h[-y_n + t_n - (y_n + h(-y_n + t_n)) + t_{n+1}]$.

4 (c). The modified Euler's method takes the form $y_{n+1} = y_n + h[-(y_n + 0.5h(-y_n + t_n)) + t_n + 0.5h]$.

4 (d).

0.0000	0.0000
0.1000	0.0050
0.2000	0.0190
0.3000	0.0412

4 (e).

0.0000	0.0000
0.1000	0.0050
0.2000	0.0190
0.3000	0.0412

4 (f).

0.0000	0.0000
0.1000	0.0048
0.2000	0.0187
0.3000	0.0408

5 (a). Solving the separable equation $y^2y' + t = 0$, we find $y^3 = -1.5t^2 + C$. Imposing the initial condition $y(0) = 1$, we obtain $y^3 = -1.5t^2 + 1$ or $y = (1 - 1.5t^2)^{1/3}$.

5 (b). Since $f(t,y) = -ty^{-2}$, it follows that $f(t+h, y + hf(t,y)) = -(t+h)[y + h(-ty)]^{-2}$. Therefore, Heun's method takes the form

$$y_{n+1} = y_n + (h/2)[-t_n y_n^{-2} - t_{n+1}(y_n - ht_n y_n^{-2})^{-2}].$$

5 (c). As in part (b), we find the modified Euler's method takes the form

$$y_{n+1} = y_n - h(t_n + 0.5h)(y_n - 0.5ht_n y_n^{-2})^{-2}.$$

5 (d).

0	1.0000
0.1000	0.9950
0.2000	0.9796
0.3000	0.9529

5 (e).

0	1.0000
0.1000	0.9950
0.2000	0.9797
0.3000	0.9531

5 (f).

0	1.0000
0.1000	0.9950
0.2000	0.9796
0.3000	0.9528

6 (a). Solving the separable equation $y' = 1 + y^2$ and imposing the initial condition we obtain $y = \tan(t - \pi / 4)$.

6 (b).

T	E	H	IE
0.0000	-1.0000	-1.0000	-1.0000
0.0500	-0.9000	-0.9047	-0.9049
0.1000	-0.8095	-0.8177	-0.8179
0.1500	-0.7267	-0.7375	-0.7378
0.2000	-0.6503	-0.6630	-0.6634
0.2500	-0.5792	-0.5933	-0.5937
0.3000	-0.5124	-0.5276	-0.5280
0.3500	-0.4493	-0.4653	-0.4657
0.4000	-0.3892	-0.4058	-0.4062
0.4500	-0.3316	-0.3486	-0.3491
0.5000	-0.2761	-0.2934	-0.2940
0.5500	-0.2223	-0.2399	-0.2404
0.6000	-0.1698	-0.1876	-0.1881
0.6500	-0.1184	-0.1362	-0.1368
0.7000	-0.0677	-0.0856	-0.0862
0.7500	-0.0175	-0.0354	-0.0360
0.8000	0.0326	0.0147	0.0140
0.8500	0.0826	0.0648	0.0641
0.9000	0.1329	0.1152	0.1145
0.9500	0.1838	0.1662	0.1654
1.0000	0.2355	0.2181	0.2173

The errors at $t = 1$ are, respectively, 0.0176, 1.6495e-004, and 6.8239e-004.

7 (a). Solving the separable equation $yy' + t = 0$, we find $y^2 = -t^2 + C$. Imposing the initial condition $y(0) = 3$, we obtain $y^2 = -t^2 + 9$ or $y = (9 - t^2)^{1/2}$.

7 (b).

T	E	H	IE
0	3.0000	3.0000	3.0000
0.0500	3.0000	2.9996	2.9996
0.1000	2.9992	2.9983	2.9983
0.1500	2.9975	2.9962	2.9962
0.2000	2.9950	2.9933	2.9933
0.2500	2.9917	2.9896	2.9896
0.3000	2.9875	2.9850	2.9850
0.3500	2.9825	2.9795	2.9795
0.4000	2.9766	2.9732	2.9732
0.4500	2.9699	2.9661	2.9661
0.5000	2.9623	2.9580	2.9580
0.5500	2.9539	2.9492	2.9492
0.6000	2.9445	2.9394	2.9394
0.6500	2.9344	2.9287	2.9287
0.7000	2.9233	2.9172	2.9172
0.7500	2.9113	2.9047	2.9047
0.8000	2.8984	2.8914	2.8914
0.8500	2.8846	2.8771	2.8771
0.9000	2.8699	2.8618	2.8618
0.9500	2.8542	2.8456	2.8456
1.0000	2.8376	2.8284	2.8284

The errors at $t = 1$ are, respectively, -9.1466e-003, -6.9021e-007, and -1.3752e-005.

8 (a). Solving the equation $y' + 2y = 4$ and imposing the initial condition we obtain $y = 2 + e^{-2t}$.

8 (b).

T	E	H	IE
0	3.0000	3.0000	3.0000
0.0500	2.9000	2.9050	2.9050
0.1000	2.8100	2.8190	2.8190
0.1500	2.7290	2.7412	2.7412
0.2000	2.6561	2.6708	2.6708
0.2500	2.5905	2.6071	2.6071
0.3000	2.5314	2.5494	2.5494
0.3500	2.4783	2.4972	2.4972
0.4000	2.4305	2.4500	2.4500
0.4500	2.3874	2.4072	2.4072
0.5000	2.3487	2.3685	2.3685
0.5500	2.3138	2.3335	2.3335
0.6000	2.2824	2.3018	2.3018
0.6500	2.2542	2.2732	2.2732
0.7000	2.2288	2.2472	2.2472
0.7500	2.2059	2.2237	2.2237
0.8000	2.1853	2.2025	2.2025
0.8500	2.1668	2.1832	2.1832
0.9000	2.1501	2.1658	2.1658
0.9500	2.1351	2.1501	2.1501
1.0000	2.1216	2.1358	2.1358

The errors at $t = 1$ are, respectively, 1.3758e-002, 4.8717e-004, and 4.8717e-004.

9 (a). Solving the separable equation $y' + 2ty = 0$, we find $y = Ce^{-t^2}$. Imposing the initial condition $y(0) = 2$, we obtain $y = 2e^{-t^2}$.

9 (b).

T	E	H	IE
0	2.0000	2.0000	2.0000
0.0500	2.0000	1.9950	1.9950
0.1000	1.9900	1.9801	1.9801
0.1500	1.9701	1.9555	1.9554
0.2000	1.9405	1.9216	1.9215
0.2500	1.9017	1.8788	1.8787
0.3000	1.8542	1.8278	1.8277
0.3500	1.7986	1.7694	1.7692
0.4000	1.7356	1.7043	1.7040
0.4500	1.6662	1.6334	1.6330
0.5000	1.5912	1.5576	1.5572
0.5500	1.5117	1.4780	1.4775
0.6000	1.4285	1.3955	1.3949
0.6500	1.3428	1.3110	1.3103
0.7000	1.2555	1.2255	1.2247
0.7500	1.1676	1.1398	1.1390
0.8000	1.0801	1.0549	1.0541
0.8500	0.9937	0.9715	0.9706
0.9000	0.9092	0.8902	0.8893
0.9500	0.8274	0.8116	0.8107
1.0000	0.7488	0.7364	0.7354

The errors at $t = 1$ are, respectively, -1.3009e-002, -6.0218e-004, and 3.3293e-004.

10. The iteration is Euler's method, with $t_0 = 2, T = 1$, and $f(t,y) = y + t^2 y^3$.

11. Since $t_n = 1 + nh,\ h = 0.05,\ n = 0,1,\ldots,99$, it follows that $t_0 = 1$ and $N - 1 = 99$. Thus, $N = 100$, and $T = t_N = 1 + Nh = 1 + 100h = 1 + (100)(0.05) = 5$. From the form of the iteration, it must be Heun's method. Therefore, $f(t,y) = ty^2 + 1$.

12. The iteration is the modified Euler's method, with $t_0 = 0, T = 2$, and $f(t,y) = t\sin^2 y$.

13. Since $t_n = 2 + nh,\ h = 0.01,\ n = 0,1,\ldots,99$, it follows that $t_0 = 2$ and $N - 1 = 99$. Thus, $N = 100$, and $T = t_N = 2 + Nh = 2 + 100h = 2 + (100)(0.01) = 1$. From the form of the iteration, it must be Euler's method. Therefore, $f(t,y) = y/(t^2 + y^2)$.

14. The iteration is the modified Euler's method, with $t_0 = -1, T = 10$, and $f(t,y) = \sin(t + y)$.

16. (a) The initial value problem is $Q'(t) = 6(2 - \cos\pi t) - \dfrac{Q(t)}{V(t)}$, $Q(0) = 0$ where $V(t) = 90 + 5t$.

(c) The tank contains 100 gallons when $t = 2$ minutes. As estimated by Heun's method, $Q(2) = 23.7538\ldots$ pounds.

17 (a). From Exercise 16, part (a), the problem to be solved is
$Q' = 12 - 6\cos \pi t - Q/(90+5t),\ Q(0)=0,\ 0 \le t \le 2$.

17 (b). Using the modified Euler's method with $h = 0.05$, we obtain

t	$Q(t)$
0	0
0.0500	0.3008
0.1000	0.6089
0.1500	0.9313
0.2000	1.2749
0.2500	1.6460
0.3000	2.0501
0.3500	2.4922
0.4000	2.9759
0.4500	3.5041
0.5000	4.0785
0.5500	4.6997
0.6000	5.3670
0.6500	6.0787
0.7000	6.8320
0.7500	7.6230
0.8000	8.4469
0.8500	9.2979
0.9000	10.1700
0.9500	11.0561
1.0000	11.9491
1.0500	12.8416
1.1000	13.7264
1.1500	14.5961
1.2000	15.4441
1.2500	16.2640
1.3000	17.0502
1.3500	17.7979
1.4000	18.5033
1.4500	19.1636
1.5000	19.7772
1.5500	20.3434
1.6000	20.8628
1.6500	21.3372
1.7000	21.7695
1.7500	22.1635
1.8000	22.5241
1.8500	22.8569
1.9000	23.1681
1.9500	23.4647
2.0000	23.7538

18. The Heun's method estimate is $P(2) = 1.5005$ million individuals.

19. Using the modified Euler's method, we estimate $P(2) = 1.5003$ million individuals.

20 (a). The results are listed below. The columns headed $H1$, $H2$, and $H3$ are the results obtained using step sizes $h = 0.05, h = 0.025,$ and $h = 0.0125$ respectively.

t	$H1$	$H2$	$H3$	$True$
0.0000	1.0000	1.0000	1.0000	1.0000
0.0500	1.0526	1.0526	1.0526	1.0526
0.1000	1.1109	1.1111	1.1111	1.1111
0.1500	1.1762	1.1764	1.1765	1.1765
0.2000	1.2495	1.2499	1.2500	1.2500
0.2500	1.3326	1.3332	1.3333	1.3333
0.3000	1.4275	1.4283	1.4285	1.4286
0.3500	1.5370	1.5381	1.5384	1.5385
0.4000	1.6645	1.6661	1.6665	1.6667
0.4500	1.8151	1.8174	1.8180	1.8182
0.5000	1.9954	1.9988	1.9997	2.0000
0.5500	2.2153	2.2204	2.2218	2.2222
0.6000	2.4894	2.4972	2.4993	2.5000
0.6500	2.8402	2.8527	2.8560	2.8571
0.7000	3.3049	3.3257	3.3314	3.3333
0.7500	3.9488	3.9860	3.9964	4.0000
0.8000	4.8975	4.9714	4.9925	5.0000
0.8500	6.4264	6.5969	6.6480	6.6667
0.9000	9.2615	9.7669	9.9353	10.0000
0.9500	15.9962	18.4267	19.5053	20.0000

The error ratios are denoted, respectively, by $R1$ and $R2$ where
$R1 = (H1 - True)/(H2 - True)$ and $R2 = (H2 - True)/(H3 - True)$

t	$R1$	$R2$
0.0500	3.8916	3.9471
0.1000	3.8879	3.9454
0.1500	3.8838	3.9436
0.2000	3.8791	3.9414
0.2500	3.8737	3.9390
0.3000	3.8674	3.9362
0.3500	3.8601	3.9329
0.4000	3.8513	3.9291
0.4500	3.8408	3.9244
0.5000	3.8279	3.9187
0.5500	3.8117	3.9117
0.6000	3.7909	3.9026
0.6500	3.7634	3.8907
0.7000	3.7255	3.8743
0.7500	3.6707	3.8503
0.8000	3.5860	3.8126
0.8500	3.4428	3.7459
0.9000	3.1683	3.6039
0.9500	2.5450	3.1800

20 (b). An error monitor is $y(t^*) - \hat{y}_{2n} = (\hat{y}_{2n} - y_n)/3$.

20 (c). The column headed *est* gives the estimated error using the error monitor from part (b). The column headed *true* gives the actual error. The error monitor used step sizes of h = 0.025 and h = 0.0125.

t	*est*	*true*
0.0500	4.4179e-006	4.4972e-006
0.1000	1.0382e-005	1.0574e-005
0.1500	1.8466e-005	1.8820e-005
0.2000	2.9497e-005	3.0084e-005
0.2500	4.4686e-005	4.5614e-005
0.3000	6.5850e-005	6.7281e-005
0.3500	9.5774e-005	9.7965e-005
0.4000	1.3886e-004	1.4222e-004
0.4500	2.0228e-004	2.0751e-004
0.5000	2.9820e-004	3.0650e-004
0.5500	4.4818e-004	4.6178e-004
0.6000	6.9264e-004	7.1587e-004
0.6500	1.1126e-003	1.1547e-003
0.7000	1.8854e-003	1.9679e-003
0.7500	3.4446e-003	3.6255e-003
0.8000	7.0273e-003	7.4956e-003
0.8500	1.7050e-002	1.8628e-002
0.9000	5.6137e-002	6.4677e-002
0.9500	3.5951e-001	4.9473e-001

Section 9.2

Unless indicated by $\ldots$, all results are rounded to the places shown.

1 (a). From the given equation $y' = -y + 2$, we know $y'(t) = -y(t) + 2$, $y''(t) = -y'(t)$, $y'''(t) = -y''(t)$, and $y^{(4)}(t) = -y'''(t)$. We also know that $y(0) = 1$. Therefore, $y'(0) = -y(0) + 2 = -1 + 2 = 1$, $y''(0) = -y'(0) = -1$, $y'''(0) = -y''(0) = 1$, and $y^{(4)}(0) = -y'''(0) = -1$. Thus,

$$P_4(t) = 1 + t - (1/2)t^2 + (1/6)t^3 - (1/24)t^4.$$

2 (a). From the given equation $y' = 2ty$, we know, $y'(0) = 0$, $y''(0) = 2$, $y'''(0) = 0$, and $y^{(4)}(0) = 12$. Thus,

$$P_4(t) = 1 + t^2 + (1/2)t^4.$$

3 (a). From the given equation $y' = ty^2$, we know $y'(t) = ty^2(t)$, $y''(t) = y^2(t) + 2ty(t)y'(t)$, $y'''(t) = 4y(t)y'(t) + 2ty'(t)y'(t) + 2ty(t)y''(t)$, and $y^{(4)}(t) = 6y'(t)y'(t) + 6y(t)y''(t) + 6ty'(t)y''(t) + 2ty(t)y'''(t)$. We also know that $y(0) = 1$. Therefore, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 0$, and $y^{(4)}(0) = 6$. Thus,

$$P_4(t) = 1 + (1/2)t^2 + (1/4)t^4.$$

4 (a). From the given equation $y' = t^2 + y$, we know, $y'(0) = 1$, $y''(0) = 1$, $y'''(0) = 3$, and $y^{(4)}(0) = 3$. Thus,

$$P_4(t) = 1 + t + (1/2)t^2 + (1/2)t^3 + (1/8)t^4.$$

5 (a). From the given equation $y' = y^{1/2}$, we know $y'(t) = y^{1/2}(t)$, $y''(t) = (1/2)y^{-1/2}(t)y'(t)$, $y'''(t) = -(1/4)y^{-3/2}(t)y'(t) + (1/2)y^{-1/2}(t)y''(t)$, and $y^{(4)}(t) = (3/8)y^{-5/2}(t)y'(t)y'(t) - (3/4)y^{-3/2}(t)y''(t)y'(t) + (1/2)y^{-1/2}(t)y'''(t)$. We also know that $y(0) = 1$. Therefore, $y'(0) = 1$, $y''(0) = 1/2$, $y'''(0) = -(1/4) + (1/2)(1/2) = 0$, and $y^{(4)}(0) = (3/8) - (3/4)(1/2) = 0$. Thus,

$$P_4(t) = 1 + t + (1/4)t^2.$$

6 (a). From the given equation $y' = ty^{-1}$, we know, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 0$, and $y^{(4)}(0) = -3$. Thus,

$$P_4(t) = 1 + (1/2)t^2 - (1/8)t^4.$$

7 (a). From the given equation $y' = y + \sin t$, we know $y'(t) = y(t) + \sin t$, $y''(t) = y'(t) + \cos t$, $y'''(t) = y''(t) - \sin t$, and $y^{(4)}(t) = y'''(t) - \cos t$. We also know that $y(0) = 1$. Therefore, $y'(0) = 1$, $y''(0) = 1 + 1 = 2$, $y'''(0) = 2 - 0 = 2$, and $y^{(4)}(0) = 2 - 1 = 1$. Thus,

$$P_4(t) = 1 + t + t^2 + (1/3)t^3 + (1/24)t^4.$$

8 (a). From the given equation $y' = y^{3/4}$, we know, $y'(0) = 1$, $y''(0) = 3/4$, $y'''(0) = 3/8$, and $y^{(4)}(0) = 3/32$. Thus,

$$P_4(t) = 1 + t + (3/8)t^2 + (1/16)t^3 + (1/256)t^4.$$

9 (a). From the given equation $y' = 1 + y^2$, we know $y'(t) = 1 + y^2(t)$, $y''(t) = 2y(t)y'(t)$, $y'''(t) = 2y'(t)y'(t) + 2y(t)y''(t)$, and $y^{(4)}(t) = 6y''(t)y'(t) + 2y(t)y''(t)$. We also know that $y(0) = 1$. Therefore, $y'(0) = 2$, $y''(0) = (2)(1)(2) = 4$, $y'''(0) = (2)(2)(2) + (2)(1)(4) = 16$, and $y^{(4)}(0) = (6)(4)(2) + (2)(1)(16) = 80$. Thus,

$$P_4(t) = 1 + 2t + 2t^2 + (8/3)t^3 + (10/3)t^4.$$

10 (a). From the given equation $y' = -4t^3y$, we know, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 0$, and $y^{(4)}(0) = -24$. Thus,

$$P_4(t) = 1 - t^4.$$

11 (a). From the given equation $y'' = 3y' - 2y$, we know $y''(t) = 3y'(t) - 2y(t)$, $y'''(t) = 3y''(t) - 2y'(t)$, $y^{(4)}(t) = 3y'''(t) - 2y''(t)$, and $y^{(5)}(t) = 3y^{(4)}(t) - 2y'''(t)$. We also know that $y(0) = 1$ and $y'(0) = 0$. Therefore, $y''(0) = (3)(0) - (2)(1) = -2$, $y'''(0) = (3)(-2) - (2)(0) = -6$, $y^{(4)}(0) = (3)(-6) - (2)(-2) = -14$, and $y^{(5)}(0) = (3)(-14) - (2)(-6) = -30$. Thus,

$$P_5(t) = 1 - t^2 - t^3 - (7/12)t^4 - (1/4)t^5.$$

12 (a). From the given equation $y'' - y' = 0$, we know, $y''(1) = 2$, $y'''(1) = 2$, $y^{(4)}(1) = 2$, and $y^{(5)}(1) = 2$. Thus,

$$P_5(t) = 1 + 2(t-1) + (t-1)^2 + (1/3)(t-1)^3 + (1/12)(t-1)^4 + (1/60)(t-1)^5.$$

13 (a). From the given equation $y''' = y'$, we know $y'''(t) = y'(t)$, $y^{(4)}(t) = y''(t)$, and $y^{(5)}(t) = y'''(t)$. We also know that $y(0) = 1, y'(0) = 2$ and $y''(0) = 0$. Therefore, $y'''(0) = 2$, $y^{(4)}(0) = 0$, $y^{(5)}(0) = 2$. Thus,

$$P_5(t) = 1 + 2t + (1/3)t^3 + (1/60)t^5.$$

14 (a). From the given equation $y'' + y + y^3 = 0$, we know, $y''(0) = -2$, $y'''(0) = 0$, $y^{(4)}(0) = 8$, and $y^{(5)}(0) = 0$. Thus,

$$P_5(t) = 1 - t^2 + (1/3)t^4 .$$

15. The function $q(h) = \sin 2h$ has a Maclaurin expansion given by $\sin 2h = 2h - (1/6)(2h)^3 + \cdots$. Therefore, $q(h) = O(h)$.

16. $q(h) = O(h)$.

17. The function $q(h) = 1 - \cos h$ has a Maclaurin expansion given by $1 - \cos h = 1 - (1 - (1/2)h^2 + (1/24)h^4 + \cdots = (1/2)h^2 + \cdots$. Therefore, $q(h) = O(h^2)$.

18. The function $q(h) = e^h - (1 + h)$ has a Maclaurin expansion given by $e^h - (1 + h) = [1 + h + (1/2)h^2 + \cdots] - (1 + h) = (1/2)h^2 + \cdots$. Therefore, $q(h) = O(h^2)$.

20 (a).

t	ts-1	ts-2	ts-3
0.0000	1.0000	1.0000	1.0000
0.0500	1.0000	1.0006	1.0006
0.1000	1.0013	1.0025	1.0025
0.1500	1.0037	1.0056	1.0056
0.2000	1.0075	1.0100	1.0100
0.2500	1.0125	1.0156	1.0156
0.3000	1.0187	1.0224	1.0224
0.3500	1.0261	1.0304	1.0304
0.4000	1.0348	1.0396	1.0396
0.4500	1.0446	1.0500	1.0500
0.5000	1.0556	1.0616	1.0616
0.5500	1.0677	1.0743	1.0742
0.6000	1.0810	1.0881	1.0881
0.6500	1.0955	1.1030	1.1030
0.7000	1.1110	1.1190	1.1190
0.7500	1.1276	1.1360	1.1360
0.8000	1.1452	1.1541	1.1541
0.8500	1.1638	1.1732	1.1731
0.9000	1.1835	1.1932	1.1932
0.9500	1.2041	1.2142	1.2142
1.0000	1.2256	1.2361	1.2361

20 (b). At $t = 1$, the errors are (respectively): -1.0441e-002, 5.5071e-005, and -1.0500e-006.

21 (a).

t	*ts-1*	*ts-2*	*ts-3*
0.0000	-1.0000	-1.0000	-1.0000
0.0500	-1.0000	-0.9975	-0.9975
0.1000	-0.9950	-0.9901	-0.9901
0.1500	-0.9851	-0.9779	-0.9780
0.2000	-0.9705	-0.9614	-0.9615
0.2500	-0.9517	-0.9409	-0.9412
0.3000	-0.9291	-0.9171	-0.9174
0.3500	-0.9032	-0.8905	-0.8908
0.4000	-0.8746	-0.8616	-0.8620
0.4500	-0.8440	-0.8311	-0.8316
0.5000	-0.8120	-0.7994	-0.8000
0.5500	-0.7790	-0.7672	-0.7677
0.6000	-0.7456	-0.7347	-0.7353
0.6500	-0.7123	-0.7024	-0.7030
0.7000	-0.6793	-0.6705	-0.6711
0.7500	-0.6470	-0.6394	-0.6400
0.8000	-0.6156	-0.6092	-0.6098
0.8500	-0.5853	-0.5800	-0.5806
0.9000	-0.5562	-0.5520	-0.5525
0.9500	-0.5283	-0.5252	-0.5256
1.0000	-0.5018	-0.4996	-0.5000

21 (b). At $t = 1$, the errors are (respectively): -1.8055e-003, 4.0475e-004, and -6.8372e-006

22 (a).

t	*ts*-1	*ts*-2	*ts*-3
0.0000	1.0000	1.0000	1.0000
0.0500	1.0250	1.0247	1.0247
0.1000	1.0494	1.0488	1.0488
0.1500	1.0732	1.0724	1.0724
0.2000	1.0965	1.0954	1.0954
0.2500	1.1193	1.1180	1.1180
0.3000	1.1416	1.1401	1.1402
0.3500	1.1635	1.1619	1.1619
0.4000	1.1850	1.1832	1.1832
0.4500	1.2061	1.2041	1.2042
0.5000	1.2269	1.2247	1.2247
0.5500	1.2472	1.2449	1.2450
0.6000	1.2673	1.2649	1.2649
0.6500	1.2870	1.2845	1.2845
0.7000	1.3064	1.3038	1.3038
0.7500	1.3256	1.3228	1.3229
0.8000	1.3444	1.3416	1.3416
0.8500	1.3630	1.3601	1.3601
0.9000	1.3814	1.3783	1.3784
0.9500	1.3995	1.3964	1.3964
1.0000	1.4173	1.4142	1.4142

22 (b). At $t = 1$, the errors are (respectively): 3.1075e-003, -5.7087e-005, and 1.3615e-006.

23 (a).

t	*ts-1*	*ts-2*	*ts-3*
0.0000	0.0000	0.0000	0.0000
0.0500	0.0500	0.0488	0.0488
0.1000	0.0977	0.0955	0.0956
0.1500	0.1436	0.1405	0.1407
0.2000	0.1880	0.1841	0.1844
0.2500	0.2311	0.2266	0.2269
0.3000	0.2733	0.2682	0.2686
0.3500	0.3146	0.3091	0.3095
0.4000	0.3553	0.3494	0.3498
0.4500	0.3955	0.3892	0.3897
0.5000	0.4354	0.4288	0.4293
0.5500	0.4751	0.4682	0.4687
0.6000	0.5146	0.5075	0.5080
0.6500	0.5541	0.5468	0.5473
0.7000	0.5937	0.5862	0.5868
0.7500	0.6335	0.6258	0.6264
0.8000	0.6736	0.6657	0.6664
0.8500	0.7139	0.7060	0.7067
0.9000	0.7547	0.7467	0.7475
0.9500	0.7960	0.7879	0.7888
1.0000	0.8379	0.8298	0.8307

23 (b). At $t = 1$, the errors are (respectively): 7.2979e-003, -8.2708e-004, and 3.0263e-005.

24. We find $E_1 = -1.0499 \times 10^{-6}$ and $E_2 = -1.2939 \times 10^{-7}$. The error ratio is 0.12323 while 1/8 is equal to 0.125. Thus, the error ratio is close to 1/8.

25. We find $E_1 = -6.8372 \times 10^{-6}$ and $E_2 = -8.4649 \times 10^{-7}$. The error ratio is 0.12381 while 1/8 is equal to 0.125. Thus, the error ratio is close to 1/8.

26. We find $E_1 = -1.3615 \times 10^{-6}$ and $E_2 = -1.6598 \times 10^{-7}$. The error ratio is 0.12191 while 1/8 is equal to 0.125. Thus, the error ratio is close to 1/8.

27. We find $E_1 = 3.0263 \times 10^{-5}$ and $E_2 = 3.6501 \times 10^{-6}$. The error ratio is 0.12061 while 1/8 is equal to 0.125. Thus, the error ratio is fairly close to 1/8.

Section 9.3

Unless indicated by $\ldots$, all results are rounded to the places shown.

1 (a). For the given initial value problem $y' = -y + 2$, $y(0) = 1$, we have

$$K_1 = f(t_0, y_0) = f(0,1) = 1$$
$$K_2 = f(t_0 + h/2, y_0 + (h/2)K_1) = f(0.05, 1 + 0.05(1)) = 0.95$$
$$K_3 = f(t_0 + h, y_0 - hK_1 + 2hK_2) = 0.91$$
$$y_1 = y_0 + h(K_1 + 4K_2 + K_3)/6 = 1 + (0.1)(1 + 3.8 + 0.91)/6 = 1.095166\ldots$$

1 (b). As in (a), we find $K_1 = 1, K_2 = 0.95, K_3 = 0.9525, K_4 = 0.90475$ and thus $y_1 = 1.0951625$.

1 (c). A *k*th order Runge-Kutta method will give the exact solution if the solution is a polynomial of degree *k*. In this case, the Runge-Kutta method will not give the exact solution.

2 (a). For the given initial value problem $y' = 2ty$, $y(0) = 1$, we have $y_1 = 1.01006667$

2 (b). $y_1 = 1.01005017$.

2 (c). Neither Runge-Kutta method will give the exact solution.

3 (a). For the given initial value problem $y' = ty^2$, $y(0) = 1$, we have
$K_1 = f(t_0, y_0) = f(0,1) = 0$
$K_2 = f(t_0 + h/2, y_0 + (h/2)K_1) = f(0.05, 1 + 0.05(0)) = 0.05$
$K_3 = f(t_0 + h, y_0 - hK_1 + 2hK_2) = 0.1020\ldots$
$y_1 = y_0 + h(K_1 + 4K_2 + K_3)/6 = 1 + (0.1)(0 + 0.05 + 0.1020)/6 = 1.0050335\ldots$

3 (b). As in (a), we find $K_1 = 0, K_2 = 0.05, K_3 = 0.0503\ldots, K_4 = 0.1010\ldots$ and thus $y_1 = 1.0050251\ldots$.

3 (c). A *k*th order Runge-Kutta method will give the exact solution if the solution is a polynomial of degree *k*. In this case, the Runge-Kutta method will not give the exact solution.

4 (a). For the given initial value problem $y' = t^2 + y$, $y(0) = 1$, we have $y_1 = 1.10550833$

4 (b). $y_1 = 1.10551271$.

4 (c). Neither Runge-Kutta method will give the exact solution.

5 (a). For the given initial value problem $y' = \sqrt{y}$, $y(0) = 1$, we have
$K_1 = f(t_0, y_0) = f(0,1) = 1$
$K_2 = f(t_0 + h/2, y_0 + (h/2)K_1) = f(0.05, 1 + 0.05(1)) = 1.0246\ldots$
$K_3 = f(t_0 + h, y_0 - hK_1 + 2hK_2) = 1.0511\ldots$
$y_1 = y_0 + h(K_1 + 4K_2 + K_3)/6 = 1.1024990\ldots$

5 (b). As in (a), we find $K_1 = 1, K_2 = 1.0246\ldots, K_3 = 1.0252\ldots, K_4 = 1.0500\ldots$ and thus $y_1 = 1.1024999\ldots$.

5 (c). A *k*th order Runge-Kutta method will give the exact solution if the solution is a polynomial of degree *k*. In this case, since the solution is a quadratic polynomial, both Runge-Kutta methods will give the exact solution.

6 (a). For the given initial value problem $y' = t/y$, $y(0) = 1$, we have $y_1 = 1.00498350$

6 (b). $y_1 = 1.00498757$.

6 (c). Neither Runge-Kutta method will give the exact solution.

7 (a). For the given initial value problem $y' = y + \sin t$, $y(0) = 1$, we have
$K_1 = f(t_0, y_0) = f(0,1) = 1$
$K_2 = f(t_0 + h/2, y_0 + (h/2)K_1) = f(0.05, 1 + 0.05(1)) = 1.0999\ldots$
$K_3 = f(t_0 + h, y_0 - hK_1 + 2hK_2) = 1.2198\ldots$
$y_1 = y_0 + h(K_1 + 4K_2 + K_3)/6 = 1.110329\ldots$

7 (b). As in (a), we find $K_1 = 1, K_2 = 1.0999\ldots, K_3 = 1.1049\ldots, K_4 = 1.2103\ldots$ and thus $y_1 = 1.110337\ldots$.

7 (c). A *k*th order Runge-Kutta method will give the exact solution if the solution is a polynomial of degree *k*. In this case, a Runge-Kutta method will not give the exact solution.

8 (a). For the given initial value problem $y' = y^{3/4}$, $y(0) = 1$, we have $y_1 = 1.10381059$

8 (b). $y_1 = 1.10381285$.

8 (c). The 4*th* order Runge-Kutta method will give the exact solution, but not the 3*rd* order.

9 (a). For the given initial value problem $y' = 1 + y^2$, $y(0) = 1$, we have

$K_1 = f(t_0, y_0) = f(0,1) = 2$

$K_2 = f(t_0 + h/2, y_0 + (h/2)K_1) = f(0.05, 1 + 0.05(2)) = 2.21$

$K_3 = f(t_0 + h, y_0 - hK_1 + 2hK_2) = 2.5425\ldots$

$y_1 = y_0 + h(K_1 + 4K_2 + K_3)/6 = 1.2230427\ldots$

9 (b). As in (a), we find $K_1 = 2, K_2 = 2.21, K_3 = 2.2332\ldots, K_4 = 2.4965\ldots$ and thus $y_1 = 1.2230489\ldots$.

9 (c). A kth order Runge-Kutta method will give the exact solution if the solution is a polynomial of degree k. In this case, a Runge-Kutta method will not give the exact solution.

10 (a). For the given initial value problem $y' = -4t^3 y$, $y(0) = 1$, we have $y_1 = 0.99990001$

10 (b). $y_1 = 0.99990000$.

10 (c). Neither Runge-Kutta method will give the exact solution.

11. Rewriting the given initial value problem, $y'' + ty' + y$, $y(0) = 1, y'(0) = -1$, as a first order system, we have

$$\begin{aligned} y_1' &= y_2, \quad y_1(0) = 1 \\ y_2' &= -ty_2 - y_1, \quad y_2(0) = -1, \end{aligned} \quad \text{or} \quad \mathbf{y}' = \mathbf{f}(t,\mathbf{y}), \mathbf{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} y_2 \\ -ty_2 - y_1 \end{bmatrix}.$$

Therefore,

$$\mathbf{K}_1 = \mathbf{f}(t_0, \mathbf{y}_0) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\mathbf{K}_2 = \mathbf{f}(t_0 + h/2, \mathbf{y}_0 + (h/2)\mathbf{K}_1) = \begin{bmatrix} -1.05 \\ -0.8975 \end{bmatrix}$$

$$\mathbf{K}_3 = \mathbf{f}(t_0 + h/2, \mathbf{y}_0 + (h/2)\mathbf{K}_2) = \begin{bmatrix} -1.0448\ldots \\ -0.8952\ldots \end{bmatrix}$$

$$\mathbf{K}_4 = \mathbf{f}(t_0 + h, \mathbf{y}_0 + h\mathbf{K}_3) = \begin{bmatrix} -1.0895\ldots \\ -0.7865\ldots \end{bmatrix}$$

$$\mathbf{y}_1 = \mathbf{y}_0 + h(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4)/6 = \begin{bmatrix} 0.895345\ldots \\ -1.089534\ldots \end{bmatrix}.$$

12. $\mathbf{y}_1 = \begin{bmatrix} 1.194834\ldots \\ 1.895042\ldots \end{bmatrix}$

13. For the given initial value problem, $\mathbf{y}' = \begin{bmatrix} 0 & t \\ e^t & 0 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1 \\ t \end{bmatrix}, \mathbf{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we have

$\mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} 1 + ty_2 \\ t + e^t y_1 \end{bmatrix}$. Therefore,

$$\mathbf{K}_1 = \mathbf{f}(t_0, \mathbf{y}_0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{K}_2 = \mathbf{f}(t_0 + h/2, \mathbf{y}_0 + (h/2)\mathbf{K}_1) = \begin{bmatrix} 1.055 \\ 2.2051\ldots \end{bmatrix}$$

$$\mathbf{K}_3 = \mathbf{f}(t_0 + h/2, \mathbf{y}_0 + (h/2)\mathbf{K}_2) = \begin{bmatrix} 1.0555\ldots \\ 2.2079\ldots \end{bmatrix}$$

$$\mathbf{K}_4 = \mathbf{f}(t_0 + h, \mathbf{y}_0 + h\mathbf{K}_3) = \begin{bmatrix} 1.1220\ldots \\ 2.4269\ldots \end{bmatrix}$$

$$\mathbf{y}_1 = \mathbf{y}_0 + h(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4)/6 = \begin{bmatrix} 2.105718\ldots \\ 1.220886\ldots \end{bmatrix}.$$

14. $\mathbf{y}_1 = \begin{bmatrix} -0.900625\ldots \\ 0.809968\ldots \end{bmatrix}$

15. Rewriting the given initial value problem, $y''' = ty$, $y(0) = 1, y'(0) = 0, y''(0) = -1$, as a first order system, we have

$$\begin{aligned} y_1' &= y_2, \quad y_1(0) = 1 \\ y_2' &= y_3, \quad y_2(0) = 0 \\ y_3' &= ty_1, \quad y_3(0) = -1, \end{aligned} \quad \text{or} \quad \mathbf{y}' = \mathbf{f}(t,\mathbf{y}), \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} y_2 \\ y_3 \\ ty_1 \end{bmatrix}.$$

Therefore,

$$\mathbf{K}_1 = \mathbf{f}(t_0, \mathbf{y}_0) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{K}_2 = \mathbf{f}(t_0 + h/2, \mathbf{y}_0 + (h/2)\mathbf{K}_1) = \begin{bmatrix} -0.05 \\ -1.0 \\ 0.05 \end{bmatrix}$$

$$\mathbf{K}_3 = \mathbf{f}(t_0 + h/2, \mathbf{y}_0 + (h/2)\mathbf{K}_2) = \begin{bmatrix} -0.05 \\ -0.9975 \\ 0.0498\ldots \end{bmatrix}$$

$$\mathbf{K}_4 = \mathbf{f}(t_0 + h, \mathbf{y}_0 + h\mathbf{K}_3) = \begin{bmatrix} -0.0997\ldots \\ -0.9950\ldots \\ 0.0995 \end{bmatrix}$$

$$\mathbf{y}_1 = \mathbf{y}_0 + h(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4)/6 = \begin{bmatrix} 0.995004\ldots \\ -0.099833\ldots \\ -0.995012\ldots \end{bmatrix}.$$

16. $\mathbf{y}_1 = \begin{bmatrix} 1.199637\ldots \\ 1.988834\ldots \\ 0.114991\ldots \end{bmatrix}$

19 (a). For the given initial value problem $y' = t/(1+y)$, $y(0) = 1$ and for the step size $h = 0.05$, we obtain $y_{20} = 1.2360679786\ldots$ as our estimate of $y(1)$.

19 (b). The actual value of the solution is $y(1) = 1.2360679749\ldots$.

20 (a). For the given initial value problem $y' = 2ty^2$, $y(0) = -1$ and for the step size $h = 0.05$, we obtain $y_{20} = -0.5000000409\ldots$ as our estimate of $y(1)$.

20 (b). The actual value of the solution is $y(1) = -0.5$.

21 (a). For the given initial value problem $y' = 1/(2y)$, $y(0) = 1$ and for the step size $h = 0.05$, we obtain $y_{20} = 1.4142135632\ldots$ as our estimate of $y(1)$.

21 (b). The actual value of the solution is $y_{20} = 1.4142135623\ldots$.

22 (a). For the given initial value problem $y' = (1+y^2)/(1+t)$, $y(0) = 0$ and for the step size $h = 0.05$, we obtain $y_{20} = 0.83064092\ldots$ as our estimate of $y(1)$.

22 (b). The actual value of the solution is $y(1) = 0.83064087\ldots$.

23 (a). Rewriting the given initial value problem, $y'' + 2y' + 2y = -2$, $y(0) = 0, y'(0) = 1$, as a first order system, we have

$$\begin{aligned} y_1' &= y_2, \quad y_1(0) = 0 \\ y_2' &= -2y_2 - 2y_1 - 2, \quad y_2(0) = 1, \end{aligned} \quad \text{or} \quad \mathbf{y}' = \mathbf{f}(t,\mathbf{y}), \mathbf{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} y_2 \\ -2y_2 - 2y_1 - 2 \end{bmatrix}.$$

23 (b). Using the step size $h = 0.1$, we obtain $\mathbf{y}_{20} = \begin{bmatrix} -0.810202\ldots \\ -0.425496\ldots \end{bmatrix}$ as our estimate to the solution value $\mathbf{y}(2) = \begin{bmatrix} -0.810199\ldots \\ -0.425499\ldots \end{bmatrix}$.

24 (b). Using the step size $h = 0.1$, we obtain $\mathbf{y}_{10} = \begin{bmatrix} 0.829662\ldots \\ 0.383398\ldots \end{bmatrix}$ as our estimate to the solution value $\mathbf{y}(1) = \begin{bmatrix} 0.829660\ldots \\ 0.383400\ldots \end{bmatrix}$.

25 (a). Rewriting the given initial value problem, $t^2y'' - ty' + y = t^2$, $y(1) = 2, y'(1) = 2$, as a first order system, we have

$$\begin{aligned} y_1' &= y_2, \quad y_1(1) = 2 \\ y_2' &= (ty_2 - y_1 + t^2)/t^2, \quad y_2(1) = 2, \end{aligned} \quad \text{or}$$

$$\mathbf{y}' = \mathbf{f}(t,\mathbf{y}), \mathbf{y}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} y_2 \\ (ty_2 - y_1 + t^2)/t^2 \end{bmatrix}.$$

25 (b). Using the step size $h = 0.1$, we obtain $\mathbf{y}_{10} = \begin{bmatrix} 4.6137054\ldots \\ 3.3068527\ldots \end{bmatrix}$ as our estimate to the solution value $\mathbf{y}_{10} = \begin{bmatrix} 4.6137056\ldots \\ 3.3068528\ldots \end{bmatrix}$.