Chapter 10
Series Solutions of Linear Differential Equations

Section 10.1

1. Consider the power series $\sum_{n=0}^{\infty} \frac{t^n}{2^n}$. Applying the ratio test at an arbitrary value of t, $t \neq 0$, we obtain $\lim_{n\to\infty} \left| \frac{2^n t^{n+1}}{2^{n+1} t^n} \right| = \lim_{n\to\infty} \left| \frac{t}{2} \right| = \left| \frac{t}{2} \right|$. The limiting ratio is less than 1 if $|t| < 2$. Therefore, the radius of convergence is $R = 2$.

2. $\lim_{n\to\infty} \left| \frac{t^{n+1} n^2}{t^n (n+1)^2} \right| = \lim_{n\to\infty} \left| \frac{t}{\left(1+\frac{1}{n}\right)^2} \right| = |t|$. Therefore, the radius of convergence is $R = 1$.

3. Consider the power series $\sum_{n=0}^{\infty} (t-2)^n$. Applying the ratio test at an arbitrary value of t, $t \neq 2$, we obtain $\lim_{n\to\infty} \left| \frac{(t-2)^{n+1}}{(t-2)^n} \right| = \lim_{n\to\infty} |t-2| = |t-2|$. The limiting ratio is less than 1 if $|t-2| < 1$. Therefore, the radius of convergence is $R = 1$.

4. $\lim_{n\to\infty} \left| \frac{(3t-1)^{n+1}}{(3t-1)^n} \right| = |3t-1| < 1 \Rightarrow -1 < 3t-1 < 1 \Rightarrow 0 < t < \frac{2}{3}$. Therefore, the radius of convergence is $R = \frac{1}{3}$.

5. Consider the power series $\sum_{n=0}^{\infty} \frac{(t-1)^n}{n!}$. Applying the ratio test at an arbitrary value of t, $t \neq 1$, we obtain $\lim_{n\to\infty} \left| \frac{n!(t-1)^{n+1}}{(n+1)!(t-1)^n} \right| = \lim_{n\to\infty} \left| \frac{t-1}{n+1} \right| = 0$. The limiting ratio is less than 1 for all t, $t \neq 1$. Therefore, the radius of convergence is $R = \infty$.

6. $\lim_{n\to\infty} \left| \frac{(n+1)!(t-1)^{n+1}}{n!(t-1)^n} \right| = \lim_{n\to\infty} |(n+1)(t-1)| = \infty$, $t \neq 1$. Therefore, the radius of convergence is $R = 0$.

7. Consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n}$. Applying the ratio test at an arbitrary value of t, $t \neq 0$, we obtain $\lim_{n\to\infty} \left| \frac{nt^{n+1}}{(n+1)t^n} \right| = \lim_{n\to\infty} \left| \frac{nt}{n+1} \right| = |t|$. The limiting ratio is less than 1 if $|t| < 1$. Therefore, the radius of convergence is $R = 1$.

8. $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(t-3)^{n+1}4^n}{(-1)^n(t-3)^n 4^{n+1}}\right| = \left|\frac{t-3}{4}\right| < 1 \Rightarrow -4 < t-3 < 4 \Rightarrow -1 < t < 7$. Therefore, the radius of convergence is $R = 4$.

9. Consider the power series $\sum_{n=1}^{\infty}(\ln n)(t+2)^n$. Applying the ratio test at an arbitrary value of t, $t \neq -2$, we obtain $\lim_{n\to\infty}\left|\frac{(\ln(n+1))(t+2)^{n+1}}{(\ln n)(t+2)^n}\right| = \lim_{n\to\infty}\left|\frac{(\ln(n+1))(t+2)}{\ln n}\right| = \left|t+2\right|\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} = \left|t+2\right|$. (The last limit can be found using L'Hôpital's Rule.) The limiting ratio is less than 1 if $\left|t+2\right| < 1$. Therefore, the radius of convergence is $R = 1$.

10. $\lim_{n\to\infty}\left|\frac{(n+1)^3(t-1)^{n+1}}{n^3(t-1)^n}\right| = \left|t-1\right| < 1 \Rightarrow -1 < t-1 < 1 \Rightarrow 0 < t < 2$. Therefore, the radius of convergence is $R = 1$.

11. Consider the power series $\sum_{n=1}^{\infty}\frac{\sqrt{n}\,(t-4)^n}{2^n}$. Applying the ratio test at an arbitrary value of t, $t \neq 4$, we obtain $\lim_{n\to\infty}\left|\frac{2^n\sqrt{n+1}\,(t-4)^{n+1}}{2^{n+1}\sqrt{n}\,(t-4)^n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}\,(t-4)}{2\sqrt{n}}\right| = \left|\frac{t-4}{2}\right|$. The limiting ratio is less than 1 if $\left|t-4\right| < 2$. Therefore, the radius of convergence is $R = 2$.

12. $\lim_{n\to\infty}\left|\frac{(t-2)^{n+1}\arctan(n)}{(t-2)^n\arctan(n+1)}\right| = \left|t-2\right| < 1 \Rightarrow -1 < t-2 < 1 \Rightarrow 1 < t < 3$ $\left(\text{recall } \lim_{n\to\infty}\arctan(n) = \frac{\pi}{2}\right)$. Therefore, the radius of convergence is $R = 1$.

13. Applying the ratio test, we see the power series for $f(t)$ and $g(t)$ both have radius of convergence $R = 1$. Therefore, each series converges in the interval $-1 < t < 1$.
(a) $f(t) = 1 + t + t^2 + t^3 + t^4 + t^5 + \cdots$
$g(t) = 0 + t + 4t^2 + 9t^3 + 16t^4 + 25t^5 + \cdots$
(b) $f(t) + g(t) = 1 + 2t + 5t^2 + 10t^3 + 17t^4 + 26t^5 + \cdots$
(c) $f(t) - g(t) = 1 - 3t^2 - 8t^3 - 15t^4 - 24t^5 - \cdots$
(d) $f'(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + \cdots$
(e) $f''(t) = 2 + 6t + 12t^2 + 20t^3 + 30t^4 + 42t^5 + \cdots$

14. Applying the ratio test, we see the power series for $f(t)$ and $g(t)$ both have radius of convergence $R = 1$. Therefore, each series converges in the interval $-1 < t < 1$.
(a) $f(t) = t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + 6t^6 + \cdots$
$g(t) = -t + 2t^2 - 3t^3 + 4t^4 - 5t^5 + 6t^6 - \cdots$
(b) $f(t) + g(t) = 4t^2 + 8t^4 + 12t^6 + 16t^8 + 20t^{10} + \cdots$
(c) $f(t) - g(t) = 2t + 6t^3 + 10t^5 + 14t^7 + 18t^9 + 22t^{11} + \cdots$
(d) $f'(t) = 1 + 4t + 9t^2 + 16t^3 + 25t^4 + 36t^5 + \cdots$
(e) $f''(t) = 4 + 18t + 48t^2 + 100t^3 + 180t^4 + 294t^5 + \cdots$

15. Applying the ratio test, we see the power series for $f(t)$ has radius of convergence $R = 1/2$ while the series for $g(t)$ has radius of convergence $R = 1$. Therefore, each series converges in the interval $|t-1| < 1/2$, or $1/2 < t < 3/2$.

(a) $f(t) = 1 - 2(t-1) + 4(t-1)^2 - 8(t-1)^3 + 16(t-1)^4 - 32(t-1)^5 + \cdots$
$g(t) = 1 + (t-1) + (t-1)^2 + (t-1)^3 + (t-1)^4 + (t-1)^5 + \cdots$

(b) $f(t) + g(t) = 2 - (t-1) + 5(t-1)^2 - 7(t-1)^3 + 17(t-1)^4 - 31(t-1)^5 + \cdots$

(c) $f(t) - g(t) = -3(t-1) + 3(t-1)^2 - 9(t-1)^3 + 15(t-1)^4 - 33(t-1)^5 + \cdots$

(d) $f'(t) = -2 + 8(t-1) - 24(t-1)^2 + 64(t-1)^3 - 160(t-1)^4 + 384(t-1)^5 \cdots$

(e) $f''(t) = 8 - 48(t-1) + 192(t-1)^2 - 640(t-1)^3 + 1920(t-1)^4 - 5376(t-1)^5 \cdots$

16. Applying the ratio test, we see the power series for $f(t)$ is 1/2 and $g(t)$ is 1. Therefore, $R = \frac{1}{2}$.

(a) $f(t) = 1 + 2(t+1) + 4(t+1)^2 + 8(t+1)^3 + 16(t+1)^4 + 32(t+1)^5 + \cdots$
$g(t) = (t+1) + 2(t+1)^2 + 3(t+1)^3 + 4(t+1)^4 + 5(t+1)^5 + 6(t+1)^6 + \cdots$

(b) $f(t) + g(t) = 1 + 3(t+1) + 6(t+1)^2 + 11(t+1)^3 + 20(t+1)^4 + 37(t+1)^5 + \cdots$

(c) $f(t) - g(t) = 1 + (t+1) + 2(t+1)^2 + 5(t+1)^3 + 12(t+1)^4 + 27(t+1)^5 + \cdots$

(d) $f'(t) = 2 + 8(t+1) + 24(t+1)^2 + 64(t+1)^3 + 160(t+1)^4 + 384(t+1)^5 + \cdots$

(e) $f''(t) = 8 + 48(t+1) + 192(t+1)^2 + 640(t+1)^3 + 1920(t+1)^4 + 5376(t+1)^5 + \cdots$

17. Consider the power series $\sum_{n=0}^{\infty} 2^n t^{n+2}$. Make the change of index $k = n+2$. With this change, the lower limit of $n = 0$ transforms to $k = 2$ while the upper limit remains at ∞. Thus, the power series can be rewritten as $\sum_{k=2}^{\infty} 2^{k-2} t^k$. Finally, changing to the original summation index, n, we obtain $\sum_{n=2}^{\infty} 2^{n-2} t^n$.

18. Make the change of index $k = n+3$. The power series can be rewritten as $\sum_{k=3}^{\infty} (k-2)(k-1)t^k$. Finally, changing to the original summation index, n, we obtain $\sum_{n=3}^{\infty} (n-2)(n-1)t^n$.

19. Consider the power series $\sum_{n=0}^{\infty} a_n t^{n+2}$. Make the change of index $k = n+2$. With this change, the lower limit of $n = 0$ transforms to $k = 2$ while the upper limit remains at ∞. Thus, the power series can be rewritten as $\sum_{k=2}^{\infty} a_{k-2} t^k$. Finally, changing to the original summation index, n, we obtain $\sum_{n=2}^{\infty} a_{n-2} t^n$.

20. Make the change of index $k = n-1$. The power series can be rewritten as $\sum_{k=0}^{\infty} (k+1)a_{k+1} t^k$. Finally, changing to the original summation index, n, we obtain $\sum_{n=0}^{\infty} (n+1)a_{n+1} t^n$.

21. Consider the power series $\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$. Make the change of index $k=n-2$. With this change, the lower limit of $n=2$ transforms to $k=0$ while the upper limit remains at ∞. Thus, the power series can be rewritten as $\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}t^k$. Finally, changing to the original summation index, n, we obtain $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$.

22. Make the change of index $k=n+3$. The power series can be rewritten as $\sum_{k=3}^{\infty}(-1)^{k-3}a_{k-3}t^k$. Finally, changing to the original summation index, n, we obtain $\sum_{n=3}^{\infty}(-1)^{n-3}a_{n-3}t^n$.

23. Consider the power series $\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)a_n t^{n+2}$. Make the change of index $k=n+2$. With this change, the lower limit of $n=0$ transforms to $k=2$ while the upper limit remains at ∞. Thus, the power series can be rewritten as $\sum_{k=2}^{\infty}(-1)^{k-1}(k-1)a_{k-2}t^k$. Finally, changing to the original summation index, n, we obtain $\sum_{n=2}^{\infty}(-1)^{n-1}(n-1)a_{n-2}t^n$.

24. Let $f(t)=t^2(t-\sin t)$. $t-\sin t=-\sum_{n=1}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}$. Therefore, $f(t)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}t^{2n+3}}{(2n+1)!}$.
$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(2n+1)!(t)^{2n+5}}{(-1)^{n+1}(2n+3)!(t)^{2n+3}}\right|=0$. Thus, the radius of convergence is $R=\infty$.

25. Let $f(t)=1-\cos 3t$. From the Maclaurin series for $\cos u$ we have $\cos u=\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n}}{(2n)!}$.
Therefore, $\cos 3t=1-\frac{9t^2}{2!}+\frac{81t^4}{4!}-\frac{729t^6}{6!}+\cdots$. Hence,
$f(t)=\frac{9t^2}{2!}-\frac{81t^4}{4!}+\frac{729t^6}{6!}-\cdots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(3t)^{2n}}{(2n)!}$. We calculate the radius of convergence by using the ratio test. For an arbitrary value of t, $t\neq 0$, we have
$\lim_{n\to\infty}\left|\frac{(2n)!(3t)^{2n+2}}{(2n+2)!(3t)^{2n}}\right|=\lim_{n\to\infty}\left|\frac{9t^2}{(2n+2)(2n+1)}\right|=0$. Thus, the radius of convergence is $R=\infty$.

26. Let $f(t)=\frac{1}{1+2t}=\frac{1}{1-(-2t)}$. $\frac{1}{1-(-2t)}=\sum_{n=0}^{\infty}(-2t)^n=\sum_{n=0}^{\infty}(-2)^n t^n$. $\lim_{n\to\infty}\left|\frac{(-2t)^{n+1}}{(-2t)^n}\right|=2|t|<1$.
Thus, the radius of convergence is $R=\frac{1}{2}$.

27. Let $f(t)=1/(1-t^2)$. From the Maclaurin series for $1/(1-u)$ we have $\frac{1}{1-u}=\sum_{n=0}^{\infty}u^n$. Therefore, $\frac{1}{1-t^2}=1+t^2+t^4+t^6+\cdots$. Hence, $f(t)=\sum_{n=0}^{\infty}t^{2n}$.

We calculate the radius of convergence by using the ratio test. For an arbitrary value of t, $t \neq 0$, we have $\lim_{n\to\infty}\left|\dfrac{t^{2n+2}}{t^{2n}}\right| = \lim_{n\to\infty}\left|t^2\right| = t^2$. Thus, the radius of convergence is $R = 1$.

28 (a). $e^t = \sum_{n=0}^{\infty}\dfrac{t^n}{n!} = 1 + t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + \dfrac{t^4}{4!} + \dfrac{t^5}{5!} + ...$

$e^{-t} = \sum_{n=0}^{\infty}\dfrac{(-t)^n}{n!} = 1 - t + \dfrac{t^2}{2!} - \dfrac{t^3}{3!} + \dfrac{t^4}{4!} - \dfrac{t^5}{5!} + ...$

28 (b). $\sinh(t) = \dfrac{1}{2}\left\{\left(1 + t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + \dfrac{t^4}{4!} + \dfrac{t^5}{5!} + ...\right) - \left(1 - t + \dfrac{t^2}{2!} - \dfrac{t^3}{3!} + \dfrac{t^4}{4!} - \dfrac{t^5}{5!} + ...\right)\right\} = t + \dfrac{t^3}{3!} + \dfrac{t^5}{5!} + ...$

$\cosh(t) = \dfrac{1}{2}\left\{\left(1 + t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + \dfrac{t^4}{4!} + \dfrac{t^5}{5!} + ...\right) + \left(1 - t + \dfrac{t^2}{2!} - \dfrac{t^3}{3!} + \dfrac{t^4}{4!} - \dfrac{t^5}{5!} + ...\right)\right\} = 1 + \dfrac{t^2}{2!} + \dfrac{t^4}{4!} + ...$

29 (a). Consider the differential equation $y'' - \omega^2 y = 0$ and assume there is solution of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$.

Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \omega^2 \sum_{n=0}^{\infty} a_n t^n = 0$.

Making the change of index $k = n-2$ in the series for $y''(t)$, we obtain

$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - \omega^2\sum_{n=0}^{\infty} a_n t^n = 0$, or $\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - \omega^2 a_n]t^n = 0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2} = \dfrac{\omega^2 a_n}{(n+2)(n+1)}, n = 0,1,\ldots$

29 (b). The recurrence relation in part (a) leads us to

$a_2 = \omega^2 a_0 / 2,\ a_4 = \omega^2 a_2 / 12 = \omega^4 a_0 / 24,\ a_6 = \omega^2 a_4 / 30 = \omega^6 a_0 / 720,\ \ldots$

$a_3 = \omega^2 a_1 / 6,\ a_5 = \omega^2 a_3 / 20 = \omega^4 a_1 / 120,\ a_7 = \omega^2 a_5 / 42 = \omega^6 a_1 / 5040,\ \ldots$

Thus, $y(t) = a_0[1 + \dfrac{(\omega t)^2}{2} + \dfrac{(\omega t)^4}{24} + \dfrac{(\omega t)^6}{720} + \cdots] + \dfrac{a_1}{\omega}[\omega t + \dfrac{(\omega t)^3}{6} + \dfrac{(\omega t)^5}{120} + \dfrac{(\omega t)^7}{5040} + \cdots]$.

By Exercise 28, $y_1(t) = \cosh \omega t$ and $y_2(t) = \sinh \omega t$.

30 (a). $y(t) = \int_0^t \sum_{n=1}^{\infty} n\lambda^{n-1} d\lambda + C = \sum_{n=1}^{\infty} t^n + C,\ y(0) = C = 1 \Rightarrow y(t) = 1 + \sum_{n=1}^{\infty} t^n = \sum_{n=0}^{\infty} t^n$.

30 (b). $R = 1$.

30 (c). $y(t) = \dfrac{1}{1-t}$.

31 (a). Consider the function given by $y'(t) = \sum_{n=0}^{\infty}\dfrac{(t-1)^n}{n!}$, $y(1) = 1$. Integrating the series termwise, we obtain $y(t) = C + \sum_{n=0}^{\infty}\dfrac{(t-1)^{n+1}}{(n+1)!}$. Imposing the condition $y(1) = 1$, it follows that $C = 1$.

Adjusting the index of summation, we can write $y(t) = 1 + \sum_{n=1}^{\infty}\dfrac{(t-1)^n}{n!} = \sum_{n=0}^{\infty}\dfrac{(t-1)^n}{n!}$.

31 (b). Applying the ratio test, $\lim_{n\to\infty}\left|\dfrac{n!(t-1)^{n+1}}{(n+1)!(t-1)^n}\right| = \lim_{n\to\infty}\left|\dfrac{t-1}{n+1}\right| = 0$. Therefore, the radius of convergence is $R=\infty$.

31 (c). From the power series (7a), we see that $y(t)=e^{t-1}$.

32 (a). $y'(t) = -1 + \int_0^t \sum_{n=0}^{\infty}(-1)^n \dfrac{\lambda^n}{n!}\,d\lambda = -1 + \sum_{n=0}^{\infty}(-1)^n \dfrac{t^{n+1}}{(n+1)!} = -1 + \sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{t^n}{n!} = -\left\{1 + \sum_{n=1}^{\infty}(-1)^n \dfrac{t^n}{n!}\right\}$

$y' = -\sum_{n=0}^{\infty}(-1)^n \dfrac{t^n}{n!}$. Then, $y(t) = -\sum_{n=0}^{\infty}(-1)^n \dfrac{t^{n+1}}{(n+1)!} + 1 = 1 + \sum_{n=0}^{\infty}(-1)^{n+1}\dfrac{t^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}(-1)^n \dfrac{t^n}{n!}$.

32 (b). $R=\infty$.

32 (c). $y(t)=e^{-t}$.

33 (a). Consider the function given by $y'(t) = \sum_{n=2}^{\infty}(-1)^n \dfrac{(t-1)^n}{n!}$, $y(1)=0$. Integrating the series termwise, we obtain $y(t) = C + \sum_{n=2}^{\infty}(-1)^n \dfrac{(t-1)^{n+1}}{(n+1)!}$. Imposing the condition $y(1)=0$, it follows that $C=0$. Adjusting the index of summation, we can write

$y(t) = \sum_{n=3}^{\infty}(-1)^{n+1}\dfrac{(t-1)^n}{n!} = -\sum_{n=3}^{\infty}(-1)^n \dfrac{(t-1)^n}{n!}$.

33 (b). Applying the ratio test, $\lim_{n\to\infty}\left|\dfrac{(-1)^{n+1}n!(t-1)^{n+1}}{(-1)^n(n+1)!(t-1)^n}\right| = \lim_{n\to\infty}\left|\dfrac{t-1}{n+1}\right| = 0$. Therefore, the radius of convergence is $R=\infty$.

33 (c). From the power series (7a), we see that $\sum_{n=0}^{\infty}(-1)^n \dfrac{(t-1)^n}{n!} = e^{-(t-1)}$. Thus,

$1 - \dfrac{(t-1)}{1!} + \dfrac{(t-1)^2}{2!} + \sum_{n=3}^{\infty}(-1)^n \dfrac{(t-1)^n}{n!} = e^{-(t-1)}$. Or, using the results of part (a),

$1 - \dfrac{(t-1)}{1!} + \dfrac{(t-1)^2}{2!} - e^{-(t-1)} = y(t)$.

34 (a). $y(t) = \int_0^t \sum_{n=0}^{\infty}(-1)^n s^{2n}\,ds = \sum_{n=0}^{\infty}(-1)^n \dfrac{t^{2n+1}}{2n+1}$.

34 (b). $\lim_{n\to\infty}\left|\dfrac{(-1)^{n+1}t^{2n+3}(2n+1)}{(-1)^n t^{2n+1}(2n+3)}\right| = \left|t^2\right| < 1 \Rightarrow R=1$.

34 (c). $y(t)=\tan^{-1}(t)$.

35 (a). Consider the function $y(t)$ where $\int_0^t y(s)\,ds = \sum_{n=1}^{\infty}\dfrac{t^n}{n}$. Differentiating both sides, we obtain

$y(t) = \sum_{n=1}^{\infty} t^{n-1}$. Adjusting the index of summation, we can write $y(t) = \sum_{n=0}^{\infty} t^n$.

35 (b). Applying the ratio test, $\lim_{n\to\infty}\left|\dfrac{t^{n+1}}{t^n}\right| = |t|$. Therefore, the radius of convergence is $R=1$.

35 (c). From the power series (7d), we see that $y(t) = \sum_{n=0}^{\infty} t^n = \dfrac{1}{1-t}$.

36. Assume there is solution of the form $y(t)=\sum_{n=0}^{\infty}a_n t^n$. Differentiating, we obtain

$y'(t)=\sum_{n=1}^{\infty}na_n t^{n-1}$ and $y''(t)=\sum_{n=2}^{\infty}n(n-1)a_n t^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$, $ty'=\sum_{n=0}^{\infty}na_n t^n$.

Therefore, $\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}-(n+1)a_n]t^n=0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2}=\dfrac{(n+1)a_n}{(n+2)(n+1)}=\dfrac{a_n}{n+2}$ The recurrence leads us to

$a_2=\dfrac{a_0}{2}$, $a_3=\dfrac{a_1}{3}$, $a_4=\dfrac{a_2}{4}=\dfrac{a_0}{8}$, $a_5=\dfrac{a_3}{5}=\dfrac{a_1}{15}$

Therefore, $y(t)=a_0\left\{1+\dfrac{t^2}{2}+\dfrac{t^4}{8}+...\right\}+a_1\left\{t+\dfrac{t^3}{3}+\dfrac{t^5}{15}+...\right\}$, $y(0)=a_0=1$, $y'(0)=a_1=-1$.

Finally, $y(t)=\left\{1+\dfrac{t^2}{2}+\dfrac{t^4}{8}+...\right\}-\left\{t+\dfrac{t^3}{3}+\dfrac{t^5}{15}+...\right\}$.

37. Consider the initial value problem $y''+ty'-2y=0$, $y(0)=0, y'(0)=1$ and assume there is solution of the form $y(t)=\sum_{n=0}^{\infty}a_n t^n$. Differentiating, we obtain

$y'(t)=\sum_{n=1}^{\infty}na_n t^{n-1}$ and $y''(t)=\sum_{n=2}^{\infty}n(n-1)a_n t^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty}n(n-1)a_n t^{n-2}+t\sum_{n=1}^{\infty}na_n t^{n-1}-2\sum_{n=0}^{\infty}a_n t^n=0$. Making the change of index $k=n-2$ in the series for $y''(t)$, we obtain $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n+\sum_{n=1}^{\infty}na_n t^n-2\sum_{n=0}^{\infty}a_n t^n=0$, or

$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+(n-2)a_n]t^n=0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2}=\dfrac{-(n-2)a_n}{(n+2)(n+1)}$, $n=0,1,\ldots$ The recurrence leads us to

$a_2=2a_0/2=a_0$, $a_4=0a_2/12=0$, $a_6=-2a_4/30=0$, …

$a_3=a_1/6$, $a_5=-a_3/20=-a_1/120$, $a_7=-3a_5/42=a_1/1680$, …

Imposing the initial conditions, we have $a_0=0$ and $a_1=1$. Thus,

$y(t)=t+\dfrac{t^3}{6}-\dfrac{t^5}{120}+\dfrac{t^7}{1680}+\cdots$.

38. Assume there is solution of the form $y(t)=\sum_{n=0}^{\infty}a_n t^n$. Differentiating, we obtain

$y'(t)=\sum_{n=1}^{\infty}na_n t^{n-1}$ and $y''(t)=\sum_{n=2}^{\infty}n(n-1)a_n t^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$, $ty=\sum_{n=0}^{\infty}a_n t^{n+1}=\sum_{n=1}^{\infty}a_{n-1}t^n$

Therefore, $2a_2+\sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2}+a_{n-1}]t^n=0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2}=\dfrac{-a_{n-1}}{(n+2)(n+1)}$, $n=1,2,...$

The recurrence leads us to

$$a_3 = \frac{-a_0}{3\cdot 2},\ a_4 = \frac{-a_1}{4\cdot 3},\ a_5 = \frac{-a_2}{5\cdot 4} = 0$$

Therefore, $y(t) = a_0\left\{1 - \frac{t^3}{6} + ...\right\} + a_1\left\{t - \frac{t^4}{12} + ...\right\},\ a_0 = 1,\ a_1 = 2$.

Finally, $y(t) = \left\{1 - \frac{t^3}{6} + ...\right\} + 2\left\{t - \frac{t^4}{12} + ...\right\}$.

39. Consider the initial value problem $y'' + (1+t)y' + y = 0$, $y(0) = -1, y'(0) = 1$ and assume there is solution of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + (1+t)\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0$ or $\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty}(1+n)a_n t^n = 0$. Making the change of index $k = n-2$ in the series for $y''(t)$ and $k = n-1$ in the series for $y'(t)$, we obtain

$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + \sum_{n=0}^{\infty}(1+n)a_n t^n = 0$, or

$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + (n+1)a_n]t^n = 0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2} = \frac{-(n+1)a_{n+1} - (n+1)a_n}{(n+2)(n+1)} = \frac{-a_{n+1} - a_n}{n+2}$. The recurrence leads us to

$$a_2 = -(a_0 + a_1)/2,\ a_3 = -(a_2 + a_1)/3,\ a_4 = -(a_3 + a_2)/4 = 0,\ a_5 = -(a_4 + a_3)/5.$$

Imposing the initial conditions, we have $a_0 = -1$ and $a_1 = 1$. Thus,

$a_2 = 0,\ a_3 = -1/3,\ a_4 = 1/12,\ a_5 = 1/20$ and so we find

$$y(t) = -1 + t - \frac{1}{3}t^3 + \frac{1}{12}t^4 + \frac{1}{20}t^5 + \cdots.$$

40. Assume there is solution of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=0}^{\infty}(n+1)a_{n+1}t^n$ and $y''(t) = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$. Inserting these series into the differential equation, we have $\sum_{n=0}^{\infty}\left\{(n+2)(n+1)a_{n+2} - 5(n+1)a_{n+1} + 6a_n\right\}t^n = 0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2} = \frac{5(n+1)a_{n+1} - 6a_n}{(n+2)(n+1)}$, $n = 0,1,2,....$ The recurrence leads us to

$$a_2 = \frac{5a_1 - 6a_0}{2} = \frac{5(2) - 6(1)}{2} = 2,\ a_3 = \frac{5(2)a_2 - 6a_1}{3\cdot 2} = \frac{10(2) - 6(2)}{6} = \frac{4}{3}.$$

$$a_4 = \frac{5(3)a_3 - 6a_2}{4\cdot 3} = \frac{15(4/3) - 6(2)}{12} = \frac{2}{3},\ a_5 = \frac{5(4)a_4 - 6a_3}{5\cdot 4} = \frac{20(2/3) - 6(4/3)}{20} = \frac{4}{15}$$

Therrefore, $y(t) = 1 + 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4 + \frac{4}{15}t^5 + \cdots$.

41. Consider the initial value problem $y'' - 2y' + y = 0$, $y(0) = 0, y'(0) = 2$ and assume there is solution of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 2\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0$. Making the change of index $k = n-2$ in the series for $y''(t)$ and $k = n-1$ in the series for $y'(t)$, we obtain
$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - 2\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + \sum_{n=0}^{\infty} a_n t^n = 0$, or
$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n]t^n = 0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2} = \frac{2(n+1)a_{n+1} - a_n}{(n+2)(n+1)}$. The recurrence leads us to
$a_2 = (2a_1 - a_0)/2$, $a_3 = (4a_2 - a_1)/6$, $a_4 = (6a_3 - a_2)/12$, $a_5 = (8a_4 - a_3)/20$.
Imposing the initial conditions, we have $a_0 = 0$ and $a_1 = 2$. Thus,
$a_2 = 2$, $a_3 = 1$, $a_4 = 1/3$, $a_5 = 1/12$ and so we find $y(t) = 2t + 2t^2 + t^3 + \frac{1}{3}t^4 + \frac{1}{12}t^5 + \cdots$.

Section 10.2

1. Consider the differential equation $y'' + (\sec t)y' + t(t^2 - 4)^{-1}y = 0$. The coefficient function $p(t) = \sec t$ is not analytic at odd integer multiples of $\pi/2$. Thus, in the interval $-10 < t < 10$, $p(t)$ is not analytic at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$. Similarly, the coefficient function $q(t) = t(t^2 - 4)^{-1}$ is not analytic at $t = \pm 2$. These 8 points are the only singular points in $-10 < t < 10$.
2. The function $p(t) = t^{\frac{2}{3}}$ is not analytic at $t = 0$. The function $q(t) = \sin t$ is analytic everywhere. Therefore, $t = 0$ is the only singular point in $-10 < t < 10$.
3. Consider the differential equation $(1 - t^2)y'' + ty' + (\csc t)y = 0$. Putting the differential equation into the form of equation (1), we see that the coefficient function $p(t) = t(1 - t^2)^{-1}$ is not analytic at $t = \pm 1$. Similarly, the coefficient function $q(t) = (\csc t)(1 - t^2)^{-1}$ is not analytic at integer multiples of π or at $t = \pm 1$. Thus, in the interval $-10 < t < 10$, the singular points are given by $t = 0, \pm 1, \pm\pi, \pm 2\pi, \pm 3\pi$.
4. The function $p(t) = \frac{e^t}{\sin 2t}$ is not analytic at $t = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi, \pm\frac{5\pi}{2}, \pm 3\pi$. The function $q(t) = \frac{t}{(25 - t^2)\sin 2t}$ is also not analytic at $t = \pm 5$. Therefore, $t = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi, \pm\frac{5\pi}{2}, \pm 3\pi, \pm 5$ are the singular points in $-10 < t < 10$.

5. Consider the differential equation $(1+\ln|t|)y''+y'+(1+t^2)y=0$. Putting the differential equation into the form of equation (1), we see that the coefficient function $p(t)=(1+\ln|t|)^{-1}$ is not analytic at $t=0$ or at $t=\pm e^{-1}$. Similarly, the coefficient function $q(t)=(1+t^2)(1+\ln|t|)^{-1}$ is not analytic $t=0$ or at $t=\pm e^{-1}$. These three points are the only singular points in the interval $-10<t<10$.

6. The function $p(t)=\dfrac{t}{1+|t|}$ is not analytic at $t=0$. The function $q(t)=\tan t$ is not analytic at $t=\ \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2},\ \pm\dfrac{5\pi}{2},\ldots$. Therefore, $t=\ 0, \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2},\ \pm\dfrac{5\pi}{2}$ are the singular points in $-10<t<10$.

7. Consider the differential equation $y''+(1+2t)^{-1}y'+t(1-t^2)^{-1}y=0$. Since the coefficient functions are rational functions, each is analytic with a radius of convergence R equal to the distance from $t_0=0$ to its nearest singularity; see Figure 10.2. The only singularity of $p(t)=(1+2t)^{-1}$ is $t=-1/2$ while the only singularities of $q(t)=t(1-t^2)^{-1}$ are $t=\pm 1$. Thus, the radius of convergence of the series for $p(t)$ is $R=1/2$ while the series for $q(t)$ has radius of convergence $R=1$. The given initial value problem is guaranteed to have a unique solution that is analytic in the interval $-1/2<t<1/2$.

8. $p(t)=4(1-9t^2)^{-1}$ and $q(t)=t(1-9t^2)^{-1}$ are not analytic at $t=\pm 1/3$. Thus, for $t_0=1$, $R=\dfrac{2}{3}$.

9. Consider the differential equation $y''+(4-3t)^{-1}y'+3t(5+30t)^{-1}y=0$. Since the coefficient functions are rational functions, each is analytic with a radius of convergence R equal to the distance from $t_0=-1$ to its nearest singularity; see Figure 10.2. The only singularity of $p(t)=(4-3t)^{-1}$ is $t=4/3$ while the only singularity of $q(t)=3t(5+30t)^{-1}$ is $t=-1/6$. Thus, the radius of convergence of the series for $p(t)$ is $R=|-1-(4/3)|=7/3$ while the series for $q(t)$ has radius of convergence $R=|-1-(-1/6)|=5/6$. The given initial value problem is guaranteed to have a unique solution that is analytic in the interval $-5/6<t+1<5/6$.

10. $p(t)=(1+4t^2)^{-1}$ is not analytic at $t=\pm\dfrac{i}{2}$ and $q(t)=t(4+t)^{-1}$ is not analytic at $t=-4$. Thus, for $t_0=0$, $R=\dfrac{1}{2}$.

11. Consider the differential equation $y''+(1+3(t-2))^{-1}y'+(\sin t)y=0$. The coefficient function $p(t)=(3t-5)^{-1}$ is a rational function and is analytic with a radius of convergence R equal to the distance from $t_0=2$ to its nearest singularity; see Figure 10.2. The only singularity of $p(t)=(3t-5)^{-1}$ is $t=5/3$. The other coefficient function, $q(t)=\sin t$, is analytic everywhere with an infinite radius of convergence. The radius of convergence of the series for $p(t)$ is $R=|2-(5/3)|=1/3$. Therefore, the given initial value problem is guaranteed to have a unique solution that is analytic in the interval $-1/3<t-2<1/3$.

12. $p(t)=(t+3)(1+t^2)^{-1}$ is not analytic at $t=\pm i$ and $q(t)=t^2$ is analytic everywhere. Thus, for $t_0=1$, $R=\sqrt{2}$.

13 (a). Consider the differential equation $y''+ty'+y=0$. Let the solution be given by $y(t)=\sum_{n=0}^{\infty}a_n t^n$.

Differentiating, we obtain $y'(t)=\sum_{n=1}^{\infty}na_n t^{n-1}$ and $y''(t)=\sum_{n=2}^{\infty}n(n-1)a_n t^{n-2}$.

Inserting these series into the differential equation, we have

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + t\sum_{n=1}^{\infty} na_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0 \text{ or } \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=1}^{\infty} na_n t^n + \sum_{n=0}^{\infty} a_n t^n = 0.$$

Adjusting the indices, we obtain $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=1}^{\infty} na_n t^n + \sum_{n=0}^{\infty} a_n t^n = 0$ or

$2a_2 + a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_n]t^n = 0$. Consequently, the recurrence relation is given by $a_2 = -a_0/2$ and $a_{n+2} = -a_n/(n+2)$, $n = 1, 2, \ldots$.

13 (b). The recurrence leads us to

$a_2 = -a_0/2$, $a_4 = -a_2/4 = a_0/8$, ...

$a_3 = -a_1/3$, $a_5 = -a_3/5 = a_1/15$, ...

Thus, the general solution is

$$y(t) = a_0[1 - \frac{t^2}{2} + \frac{t^4}{8} - \cdots] + a_1[t - \frac{t^3}{3} + \frac{t^5}{15} - \cdots] = y_1(t) + y_2(t).$$

13 (c). Since the coefficient functions are analytic for $-\infty < t < \infty$, the series converges for $-\infty < t < \infty$.

13 (d). The coefficient function $p(t) = t$ is odd and the coefficient function $q(t) = 1$ is even. Therefore, Theorem 10.2 guarantees that the given equation has even solutions and odd solutions.

14 (a). $\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + 2na_n + 3a_n]t^n = 0$. Consequently, the recurrence relation is given by

$$a_{n+2} = \frac{-(2n+3)a_n}{(n+2)(n+1)},\ n = 0, 1, 2, \ldots.$$

14 (b). The recurrence leads us to

$a_2 = -3a_0/2$, $a_3 = -5a_1/6$, $a_4 = -7a_2/12 = 7a_0/8$, $a_5 = -9a_3/20 = 3a_1/8$ ...

$a_3 = -a_1/3$, $a_5 = -a_3/5 = a_1/15$, ...

Thus, the general solution is

$$y(t) = a_0[1 - \frac{3t^2}{2} + \frac{7t^4}{8} - \cdots] + a_1[t - \frac{5t^3}{6} + \frac{3t^5}{8} - \cdots].$$

14 (c). Since the coefficient functions are analytic for $-\infty < t < \infty$, $R = \infty$.

14 (d). $p(t) = 2t$ is odd and $q(t) = 3$ is even. Therefore, Theorem 10.2 guarantees that the given equation has even solutions and odd solutions.

15 (a). Consider the differential equation $(1+t^2)y'' + ty' + 2y = 0$. Let the solution be given by $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} na_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$.

Inserting these series into the differential equation, we have

$(1+t^2)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + t\sum_{n=1}^{\infty} na_n t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^n = 0$ or

$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^n + \sum_{n=1}^{\infty} na_n t^n + 2\sum_{n=0}^{\infty} a_n t^n = 0$. Adjusting the indices, we obtain

$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=2}^{\infty} n(n-1)a_n t^n + \sum_{n=1}^{\infty} na_n t^n + 2\sum_{n=0}^{\infty} a_n t^n = 0$. Consequently, the recurrence relation is given by $a_2 = -a_0, a_3 = -a_1/2$, and $a_{n+2} = -(n^2+2)a_n/[(n+2)(n+1)]$, $n = 2, 3, \ldots$.

15 (b). The recurrence leads us to

$a_2 = -a_0$, $a_4 = -a_2/2 = a_0/2$,…

$a_3 = -a_1/2$, $a_5 = -11a_3/20 = 11a_1/40$,…

Thus, the general solution is

$$y(t) = a_0[1 - t^2 + \frac{t^4}{2} - \cdots] + a_1[t - \frac{t^3}{2} + \frac{11t^5}{40} - \cdots] = y_1(t) + y_2(t).$$

15 (c). The coefficient functions $p(t) = t(1+t^2)^{-1}$ and $q(t) = 2(1+t^2)^{-1}$ fail to be analytic at $t = \pm i$. Therefore, the radius of convergence for each coefficient function is $R = 1$. Consequently, Theorem 10.1 guarantees that the power series solution converges in the interval $-1 < t < 1$.

15 (d). The coefficient function $p(t) = t(1+t^2)^{-1}$ is odd and the coefficient function $q(t) = 2(1+t^2)^{-1}$ is even. Therefore, Theorem 10.2 guarantees that the given equation has even solutions and odd solutions.

16 (a). $\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - 5(n+1)a_{n+1} + 6a_n]t^n = 0$. Consequently, the recurrence relation is given by $a_{n+2} = \dfrac{5(n+1)a_{n+1} - 6a_n}{(n+2)(n+1)}$, $n = 0,1,2,\ldots$.

16 (b). The recurrence leads us to

$a_2 = (5a_1 - 6a_0)/2 = 5a_1/2 - 3a_0$, $a_3 = (5(2)a_2 - 6a_1)/(3\cdot 2) = 19a_1/6 - 5a_0$

Thus, the general solution is

$$y(t) = a_0[1 - 3t^2 - 5t^3 - \cdots] + a_1[t + \frac{5t^2}{2} + \frac{19t^3}{6} + \cdots].$$

16 (c). Since the coefficient functions are analytic for $-\infty < t < \infty$, $R = \infty$.

16 (d). $p(t) = -5$ and $q(t) = 6$ are both even. Therefore, Theorem 10.2 does not apply.

17 (a). Consider the differential equation $y'' - 4y' + 4y = 0$. Let the solution be given by $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} na_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$.

Inserting these series into the differential equation, we have

$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 4\sum_{n=1}^{\infty} na_n t^{n-1} + 4\sum_{n=0}^{\infty} a_n t^n = 0$. Adjusting the indices, we obtain

$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - 4\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + 4\sum_{n=0}^{\infty} a_n t^n = 0$. Consequently, the recurrence relation is given by $a_{n+2} = [4(n+1)a_{n+1} - 4a_n]/[(n+2)(n+1)]$, $n = 0,1,\ldots$.

17 (b). The recurrence leads us to

$a_2 = 2a_1 - 2a_0$, $a_3 = (8a_2 - 4a_1)/6 = (16a_1 - 16a_0 - 4a_1)/6 = 2a_1 - (8/3)a_0$,…

Thus, the general solution is

$$y(t) = a_0[1 - 2t^2 - \frac{8t^3}{3} + \cdots] + a_1[t + 2t^2 + 2t^3 \cdots] = y_1(t) + y_2(t).$$

17 (c). The coefficient functions are constant and hence analytic everywhere. Consequently, Theorem 10.1 guarantees that the power series solution converges in the interval $-\infty < t < \infty$.

17 (d). The coefficient function $p(t) = -4$ is even and hence Theorem 10.2 does not apply.

18 (a). $\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+(n+1)na_{n+1}+a_n]t^n=0$. Consequently, the recurrence relation is given by $a_{n+2}=\dfrac{-[(n+1)na_{n+1}+a_n]}{(n+2)(n+1)}$.

18 (b). The recurrence leads us to

$$a_2=\frac{-a_0}{2}, a_3=\frac{-[(2)(1)a_2+a_1]}{3\cdot 2}=\frac{a_0}{6}-\frac{a_1}{6},\ a_4=\frac{-[(3)(2)a_3+a_2]}{4\cdot 3}=-\frac{a_0}{8}+\frac{a_1}{12}$$

Thus, the general solution is

$$y(t)=a_0[1-\frac{t^2}{2}-\frac{t^3}{6}-\cdots]+a_1[t-\frac{t^3}{6}+\frac{t^4}{12}+\cdots].$$

18 (c). $q(t)=\dfrac{1}{1+t}$ is not analytic at $t=-1$, $R=1$.

18 (d). $q(t)=\dfrac{1}{1+t}$ is neither even nor odd. Therefore, Theorem 10.2 does not apply.

19 (a). Consider the differential equation $(3+t)y''+3ty'+y=0$. Let the solution be given by $y(t)=\sum_{n=0}^{\infty}a_nt^n$. Differentiating, we obtain $y'(t)=\sum_{n=1}^{\infty}na_nt^{n-1}$ and $y''(t)=\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}$.

Inserting these series into the differential equation, we have

$(3+t)\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}+3t\sum_{n=1}^{\infty}na_nt^{n-1}+\sum_{n=0}^{\infty}a_nt^n=0$ or

$3\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}+\sum_{n=2}^{\infty}n(n-1)a_nt^{n-1}+3\sum_{n=1}^{\infty}na_nt^n+\sum_{n=0}^{\infty}a_nt^n=0$. Adjusting the indices, we obtain

$3\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n+\sum_{n=1}^{\infty}(n+1)na_{n+1}t^n+3\sum_{n=1}^{\infty}na_nt^n+\sum_{n=0}^{\infty}a_nt^n=0$. Consequently, the recurrence relation is given by

$a_2=-a_0/6$ and $a_{n+2}=-[n(n+1)a_{n+1}+(3n+1)a_n]/[3(n+2)(n+1)],\ n=1,2,\ldots$.

19 (b). The recurrence leads us to

$a_2=-a_0/6$, $a_3=-(2a_2+4a_1)/18=-(-2a_0/6+4a_1)/18=(a_0-12a_1)/54$, …

Thus, the general solution is

$$y(t)=a_0[1-\frac{t^2}{6}+\frac{t^3}{54}+\cdots]+a_1[t-\frac{2t^3}{9}+\cdots]=y_1(t)+y_2(t).$$

19 (c). The coefficient functions $p(t)=3t(3+t)^{-1}$ and $q(t)=(3+t)^{-1}$ fail to be analytic at $t=-3$. Therefore, the radius of convergence for each coefficient function is $R=3$. Consequently, Theorem 10.1 guarantees that the power series solution converges in the interval $-3<t<3$.

19 (d). The coefficient function $p(t)=3t(3+t)^{-1}$ is neither even nor odd. Therefore, Theorem 10.2 does not apply.

20 (a). $\sum_{n=0}^{\infty}[2(n+2)(n+1)a_{n+2}+n(n-1)a_n+4a_n]t^n=0$. Consequently, the recurrence relation is given by $a_{n+2}=\dfrac{-[n(n-1)+4]a_n}{2(n+2)(n+1)}$.

20 (b). The recurrence leads us to

$$a_2 = -a_0, a_3 = -\frac{a_1}{3},\ a_4 = \frac{a_0}{4},\ a_5 = \frac{a_1}{12}$$

Thus, the general solution is

$$y(t) = a_0[1 - t^2 + \frac{t^4}{4} - \cdots] + a_1[t - \frac{t^3}{3} + \frac{t^5}{12} + \cdots].$$

20 (c). $R = \sqrt{2}$.

20 (d). $p(t) = 0$ can be considered odd and $q(t) = \dfrac{4}{t^2+2}$ is even. Therefore, Theorem 10.2 guarantees that the given equation has even solutions and odd solutions.

21 (a). Consider the differential equation $y'' + t^2 y = 0$. Let the solution be given by $y(t) = \sum_{n=0}^{\infty} a_n t^n$.

Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + t^2 \sum_{n=0}^{\infty} a_n t^n = 0$ or $\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^{n+2} = 0$. Adjusting the indices, we obtain $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n + \sum_{n=2}^{\infty} a_{n-2} t^n = 0$. Consequently, the recurrence relation is given by $a_2 = 0, a_3 = 0,$ and $a_{n+2} = -a_{n-2}/[(n+2)(n+1)],\ n = 2, 3, \ldots$.

21 (b). The recurrence leads us to

$$a_2 = 0, a_3 = 0, a_4 = -a_0/12, a_5 = -a_1/20, \ldots$$

Thus, the general solution is

$$y(t) = a_0[1 - \frac{t^4}{12} + \cdots] + a_1[t - \frac{t^5}{20} + \cdots] = y_1(t) + y_2(t).$$

21 (c). The coefficient functions are polynomials and hence analytic everywhere. Consequently, Theorem 10.1 guarantees that the power series solution converges in the interval $-\infty < t < \infty$.

21 (d). The coefficient function $p(t) = 0$ can be considered an odd function while $q(t) = t^2$ is clearly an even function. Therefore, Theorem 10.2 guarantees that the given equation has even solutions and odd solutions.

22 (a). $\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + n a_n + a_n](t-1)^n = 0$. Consequently, the recurrence relation is given by

$$a_{n+2} = \frac{-(n+1)a_n}{(n+2)(n+1)} = \frac{-a_n}{n+2}, n = 0,1,2,....$$

22 (b). The recurrence leads us to

$$a_2 = -\frac{a_0}{2},\ a_3 = -\frac{a_1}{3},\ a_4 = -\frac{a_2}{4} = \frac{a_0}{8},\ a_5 = -\frac{a_3}{5} = \frac{a_1}{15}$$

Thus, the general solution is

$$y(t) = a_0[1 - \frac{(t-1)^2}{2} + \frac{(t-1)^4}{8} + \cdots] + a_1[(t-1) - \frac{(t-1)^3}{3} + \frac{(t-1)^5}{15} + \cdots].$$

22 (c). The coefficient functions are analytic everywhere. Consequently, $R = \infty$.

23 (a). Consider the differential equation $y'' + y = 0$. Let the solution be given by $y(z) = \sum_{n=0}^{\infty} a_n z^n$ where $z = t - 1$. Differentiating, we obtain $y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ and $y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \sum_{n=0}^{\infty} a_n z^n = 0$. Adjusting the indices, we obtain $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n + \sum_{n=0}^{\infty} a_n z^n = 0$. Consequently, the recurrence relation is given by $a_{n+2} = -a_n / [(n+2)(n+1)],\ n = 0, 1, \ldots$.

23 (b). The recurrence leads us to

$a_2 = -a_0 / 2, a_4 = -a_2 / 12 = a_0 / 24, \ldots$

$a_3 = -a_1 / 6, a_5 = -a_3 / 20 = a_1 / 120, \ldots$

Thus, the general solution is

$$y(t) = a_0 [1 - \frac{(t-1)^2}{2} + \frac{(t-1)^4}{24} + \cdots] + a_1 [(t-1) - \frac{(t-1)^3}{6} + \frac{(t-1)^5}{120} + \cdots].$$

23 (c). The coefficient functions are constants and hence analytic everywhere. Consequently, Theorem 10.1 guarantees that the power series solution converges in the interval $-\infty < t - 1 < \infty$.

24 (a). $\sum_{n=0}^{\infty} [(n+1) n a_{n+1} - (n+2)(n+1) a_{n+2} + (n+1) a_{n+1} + a_n](t-1)^n = 0$. Consequently, the recurrence relation is given by $a_{n+2} = \dfrac{(n+1)^2 a_{n+1} + a_n}{(n+2)(n+1)}, n = 0,1,2,\ldots.$

24 (b). The recurrence leads us to

$$a_2 = \frac{a_1 + a_0}{2} = \frac{a_1}{2} + \frac{a_0}{2},\ a_3 = \frac{4a_2 + a_1}{3 \cdot 2} = \frac{a_1}{2} + \frac{a_0}{3}$$

Thus, the general solution is

$$y(t) = a_0 [1 + \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} + \cdots] + a_1 [(t-1) - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{2} + \cdots].$$

24 (c). $p(t) = q(t) = \dfrac{1}{t-2}$ are not analytic at $t = 2$. Consequently, $R = 1$.

25 (a). Consider the differential equation $y'' + y' + (t-2) y = 0$ or $y'' + y' + [(t-1) - 1] y = 0$. Let the solution be given by $y(z) = \sum_{n=0}^{\infty} a_n z^n$ where $z = t - 1$. Differentiating, we obtain $y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ and $y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \sum_{n=1}^{\infty} n a_n z^{n-1} + \sum_{n=0}^{\infty} a_n z^{n+1} - \sum_{n=0}^{\infty} a_n z^n = 0$. Adjusting the indices, we obtain $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n + \sum_{n=1}^{\infty} a_{n-1} z^n - \sum_{n=0}^{\infty} a_n z^n = 0$.

Consequently, the recurrence relation is given by

$a_2 = (a_0 - a_1)/2$ and $a_{n+2} = -[(n+1) a_{n+1} - a_n + a_{n-1}] / [(n+2)(n+1)],\ n = 1, 2, \ldots$.

25 (b). The recurrence leads us to

$$a_3 = -(2a_2 - a_1 + a_0)/6 = -(a_0 - a_1)/3, \ldots$$

Thus, the general solution is

$$y(t) = a_0[1 + \frac{(t-1)^2}{2} - \frac{(t-1)^3}{3} + \cdots] + a_1[(t-1) - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} + \cdots].$$

25 (c). The coefficient functions are polynomials and hence analytic everywhere. Consequently, Theorem 10.1 guarantees that the power series solution converges in the interval $-\infty < t-1 < \infty$.

26. $a_{n+2} = \dfrac{(n^2 - \mu^2)a_n}{(n+2)(n+1)}, n = 0,1,2,...$

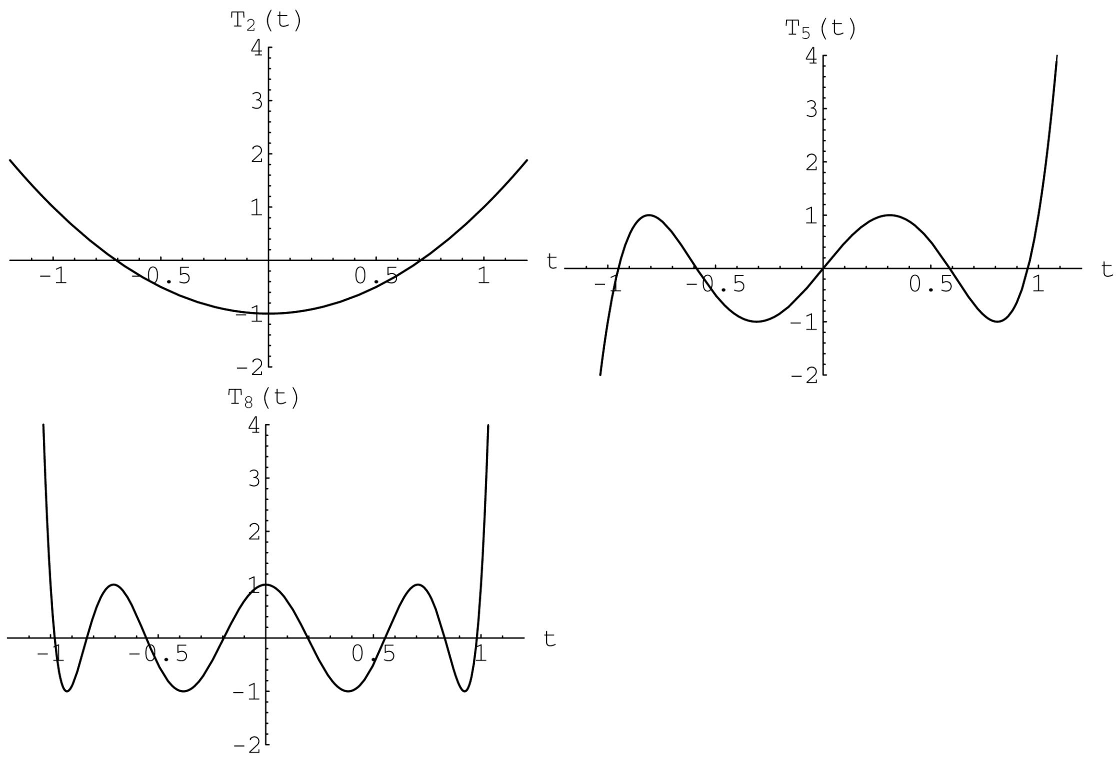
For $\mu = 5$, $a_3 = -4a_1$, $a_5 = \dfrac{16}{5}a_1$, $a_7 = a_9 = ... = 0$, $T_5(t) = a_1[t - 4t^3 + \frac{16}{5}t^5]$.

Set $T_5(1) = a_1[1 - 4 + \frac{16}{5}] = 1 \Rightarrow a_1 = 5$. Therefore, $T_5(t) = 16t^5 - 20t^3 + 5t$

For $\mu = 6$, $a_2 = -18a_0$, $a_4 = 48a_0$, $a_6 = -32a_0$, $T_6(t) = a_0[1 - 18t^2 + 48t^4 - 32t^6]$; $a_0 = -1$.

Therefore, $T_6(t) = 32t^6 - 48t^4 + 18t^2 - 1$

27 (c).



27 (d). $|T_N(t)| \le 1$ for $-1 < t < 1$. For $|t| \ge 1$, $\lim_{t \to \pm\infty} |T_N(t)| = \infty$.

28 (a). $\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \mu(\mu+1)a_n]t^n = 0$. Therefore the recurrence relation is $a_{n+2} = \dfrac{[n(n+1) - \mu(\mu+1)]a_n}{(n+2)(n+1)}, n = 0,1,2,...$

28 (b). When $\mu = N$, $a_{N+2} = a_{N+4} = a_{N+6} = ... = 0$. Therefore, if $\mu = 2M$, a polynomial solution of the form $a_0 + a_2t^2 + ... + a_{2M}t^{2M}$ exists, while if $\mu = 2M+1$, a polynomial solution of the form $a_1t + a_3t^3 + ... + a_{2M+1}t^{2M+1}$ exists.

28 (c). If $\mu = 0$ and $y = 1$, $(1-t^2)(0) - 2t(0) + 0(1) = 0$.
If $\mu = 1$ and $y = t$, $(1-t^2)(0) - 2t(1) + 1(2)(t) = 0$.

28 (d). If $\mu = 2$, $a_{n+2} = \dfrac{[n(n+1)-6]a_n}{(n+2)(n+1)} \Rightarrow P_2(t) = \dfrac{3}{2}t^2 - \dfrac{1}{2}$.
If $\mu = 3$, $a_{n+2} = \dfrac{[n(n+1)-12]a_n}{(n+2)(n+1)} \Rightarrow P_3(t) = \dfrac{5}{2}t^3 - \dfrac{3}{2}t$.
If $\mu = 4$, $a_{n+2} = \dfrac{[n(n+1)-20]a_n}{(n+2)(n+1)} \Rightarrow P_4(t) = \dfrac{35}{8}t^4 - \dfrac{15}{4}t^2 + \dfrac{3}{8}$.
If $\mu = 5$, $a_{n+2} = \dfrac{[n(n+1)-30]a_n}{(n+2)(n+1)} \Rightarrow P_5(t) = \dfrac{63}{8}t^5 - \dfrac{35}{4}t^3 + \dfrac{15}{8}t$.

29 (a). Consider the differential equation $y'' - 2ty' + 2\mu y = 0$. Let the solution be given by $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$.
Inserting these series into the differential equation, we have
$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 2\sum_{n=1}^{\infty} n a_n t^n + 2\mu \sum_{n=0}^{\infty} a_n t^n = 0$. Adjusting the indices, we obtain
$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n - 2\sum_{n=1}^{\infty} n a_n t^n + 2\mu \sum_{n=0}^{\infty} a_n t^n = 0$. Consequently, the recurrence relation is given by $a_2 = -\mu a_0$ and $a_{n+2} = (2n - 2\mu)a_n / [(n+2)(n+1)]$, $n = 1, 2, \ldots$.

29 (d). For $\mu = 2$, the even indexed coefficients a_n vanish when $n > 2$. From the recurrence relation, $H_2(t) = a_0 - 2a_0 t^2 = -a_0(2t^2 - 1)$. Choosing $a_0 = -2$ leads us to $H_2(t) = 4t^2 - 2$. For $\mu = 3$, the odd indexed coefficients a_n vanish when $n > 3$. From the recurrence relation, $H_3(t) = a_1 t - (2/3)a_1 t^3 = -a_1[(2/3)t^3 - t)$. Choosing $a_1 = -12$ leads us to $H_3(t) = 8t^3 - 12t$.
Similarly, $H_4(t) = 16t^4 - 48t^2 + 12$ and $H_5(t) = 32t^5 - 160t^3 + 120t$.

30 (a). Try $y(t) = \sum_{n=0}^{\infty} a_n t^n \Rightarrow \sum_{n=0}^{\infty} [(n+1)n a_{n+1} + (n+1)a_{n+1} - a_n]t^n = 0$.
$\Rightarrow a_{n+1} = \dfrac{a_n}{(n+1)^2} \Rightarrow y(t) = a_0 \sum_{n=0}^{\infty} \dfrac{t^n}{(n+1)^2}$. By the ratio test, $\lim_{n\to\infty} \left| \dfrac{t^{n+1}(n+1)^2}{t^n(n+2)^2} \right| = |t|$ and the series converges in $-1 < t < 1$.

30 (b). Try $y(t) = \sum_{n=0}^{\infty} a_n t^n \Rightarrow \sum_{n=0}^{\infty} [n(n-1)+1]a_n t^n = 0 \Rightarrow [n(n-1)+1]a_n = 0$.
The polynomial $x^2 - x + 1$ has roots $\dfrac{1 \pm \sqrt{1-4}}{2}$. Since there are no positive integer roots, the factor $[n(n-1)+1]$ is nonzero for all $n = 0,1,2,\ldots$ Therefore, $a_n = 0$, $n = 0,1,2,\ldots$ and $y(t) = 0$, The trivial solution results.

33. The coefficient function $p(t) = \sin t$ is odd and analytic everywhere. The coefficient function $q(t) = t^2$ is even and analytic everywhere. Thus, Theorem 10.2(b) applies. The differential equation has a general solution of the form (15).

34. No. $p(t) = \cos t$ is even; $q(t) = t$ is odd.

35. The coefficient function $p(t) = 0$ can be regarded as a function that is odd and analytic everywhere. The coefficient function $q(t) = t^2$ is even and analytic everywhere. Thus, Theorem 10.2(b) applies. The differential equation has a general solution of the form (15).

36. No. $p(t) = 1$ and $q(t) = t^2$ are both even.

37. The coefficient function $q(t) = t$ is odd. Thus, Theorem 10.2(b) does not apply.

38. No. $p(t) = e^t$ is neither even nor odd and $q(t) = 1$ is even.

39. Consider the differential equation $y'' + ay' + by = 0$. The coefficient function $p(t) = a$ can be regarded as an odd function if $a = 0$, but is even if a is nonzero. The coefficient function $q(t) = b$ is even. Both coefficient functions are analytic everywhere. Thus, Theorem 10.2(b) applies if $a = 0$ and b is arbitrary.

40 (a). $p(t) = 0,\ q(t) = \dfrac{1}{1+t^2}$. The denominator of $q(t)$ vanishes at $t = \pm i \Rightarrow R = 1$.

40 (b). $y(t) = \sum_{n=0}^{\infty} a_n t^n \Rightarrow \sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + n(n-1)a_n + a_n]t^n = 0$

$\Rightarrow r(n) = (n+2)(n+1),\ s(n) = n(n-1)+1$. Then $\lim_{n\to\infty}\left|\dfrac{a_{n+2}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{n(n-1)+1}{(n+2)(n+1)}\right| = 1$. Therefore, the series diverges for $\left|t^2\right| > 1 \Rightarrow |t| > 1$ by the Ratio Test.

40 (c). No contradiction. The unique solution of the initial value problem exists for $-\infty < t < \infty$, but its Maclaurin series has a radius of convergence $R = 1$.

Section 10.3

1 (a). $\lambda^2 + (-2\alpha + 1 - 1)\lambda + \alpha^2 = \lambda^2 - 2\alpha\lambda + \alpha^2 = 0$

1 (b). Using the technique in Section 4.5, the general solution is $y = c_1 t^\alpha + c_2 t^\alpha \ln t, t > 0$.

2. $W = \begin{vmatrix} t^\gamma \cos(\delta \ln t) & t^\gamma \sin(\delta \ln t) \\ t^{\gamma-1}[\gamma\cos(\delta\ln t) - \delta\sin(\delta\ln t)] & t^{\gamma-1}[\gamma\sin(\delta\ln t) + \delta\cos(\delta\ln t)] \end{vmatrix} = \delta t^{2\gamma-1} \neq 0$ in $0 < t < \infty$ since $\delta \neq 0$.

3. When put in standard form, the differential equation is $y'' - 4t^{-1}y' + 6t^{-2}y = 0$. Thus, $t_0 = 0$ is the only singular point. The characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$ which has roots $\lambda_1 = 2$ and $\lambda_2 = 3$. Hence, the general solution is $y = c_1 t^2 + c_2 t^3, t \neq 0$.

4. $t_0 = 0$. The characteristic equation is $\lambda^2 - \lambda - 6 = 0$ which has roots $\lambda_1 = -2$ and $\lambda_2 = 3$. Hence, the general solution is $y = c_1 t^{-2} + c_2 t^3, t \neq 0$.

5. When put in standard form, the differential equation is $y'' - 3t^{-1}y' + 4t^{-2}y = 0$. Thus, $t_0 = 0$ is the only singular point. The characteristic equation is $\lambda^2 - 4\lambda + 4 = 0$ which has roots $\lambda_1 = 2$ and $\lambda_2 = 2$. Hence, the general solution is $y = c_1 t^2 + c_2 t^2 \ln|t|, t \neq 0$.

6. $t_0 = 0$. The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$ which has roots $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Hence, the general solution is $y = c_1 t\cos(2\ln|t|) + c_2 t\sin(2\ln|t|), t \neq 0$.

7. When put in standard form, the differential equation is $y'' - 3t^{-1}y' + 29t^{-2}y = 0$. Thus, $t_0 = 0$ is the only singular point. The characteristic equation is $\lambda^2 - 4\lambda + 29 = 0$ which has roots $\lambda_1 = 2 + 5i$ and $\lambda_2 = 2 - 5i$. Hence, the general solution is $y = c_1 t^2 \cos(5\ln|t|) + c_2 t^2 \sin(5\ln|t|), t \neq 0$.

8. $t_0 = 0$. The characteristic equation is $\lambda^2 - 6\lambda + 9 = 0$ which has roots $\lambda_1 = \lambda_2 = 3$. Hence, the general solution is $y = c_1 t^3 + c_2 t^3 \ln|t|, t \neq 0$.

9. When put in standard form, the differential equation is $y'' + t^{-1}y' + 9t^{-2}y = 0$. Thus, $t_0 = 0$ is the only singular point. The characteristic equation is $\lambda^2 + 9 = 0$ which has roots $\lambda_1 = 3i$ and $\lambda_2 = -3i$. Hence, the general solution is $y = c_1\cos(3\ln|t|) + c_2\sin(3\ln|t|), t \neq 0$.

10. $t_0 = 0$. The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$ which has roots $\lambda_1 = \lambda_2 = -1$. Hence, the general solution is $y = c_1t^{-1} + c_2t^{-1}\ln|t|, t \neq 0$.

11. When put in standard form, the differential equation is $y'' + 3t^{-1}y' + 17t^{-2}y = 0$. Thus, $t_0 = 0$ is the only singular point. The characteristic equation is $\lambda^2 + 2\lambda + 17 = 0$ which has roots $\lambda_1 = -1 + 4i$ and $\lambda_2 = -1 - 4i$. Hence, the general solution is $y = c_1t^{-1}\cos(4\ln|t|) + c_2t^{-1}\sin(4\ln|t|), t \neq 0$.

12. $t_0 = 0$. The characteristic equation is $\lambda^2 + 10\lambda + 25 = 0$ which has roots $\lambda_1 = \lambda_2 = -5$. Hence, the general solution is $y = c_1t^{-5} + c_2t^{-5}\ln|t|, t \neq 0$.

13. Consider the differential equation $y'' + 5t^{-1}y' + 40t^{-2}y = 0$. We see that, $t_0 = 0$ is the only singular point. The characteristic equation is $\lambda^2 + 4\lambda + 40 = 0$ which has roots $\lambda_1 = -2 + 6i$ and $\lambda_2 = -2 - 6i$. Hence, the general solution is $y = c_1t^{-2}\cos(6\ln|t|) + c_2t^{-2}\sin(6\ln|t|), t \neq 0$.

14. $t_0 = 0$. The characteristic equation is $\lambda^2 - 3\lambda = 0$ which has roots $\lambda_1 = 0,\ \lambda_2 = 3$. Hence, the general solution is $y = c_1 + c_2t^3, t \neq 0$.

15. When put in standard form, the differential equation is $y'' - (t-1)^{-1}y' - 3(t-1)^{-2}y = 0$. Thus, $t_0 = 1$ is the only singular point. The characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$ which has roots $\lambda_1 = -3$ and $\lambda_2 = 1$. Hence, the general solution is $y = c_1(t-1)^3 + c_2(t-1)^{-1}, t \neq 1$.

16. $t_0 = 1$. The characteristic equation is $\lambda^2 + 2\lambda + 17 = 0$ which has roots $\lambda_1 = -1 + 4i,\ \lambda_2 = -1 - 4i$. Hence, the general solution is $y = c_1(t-1)^{-1}\cos(4\ln|t-1|) + c_2(t-1)^{-1}\sin(4\ln|t-1|), t \neq 1$.

17. When put in standard form, the differential equation is $y'' + 6(t+2)^{-1}y' + 6(t+2)^{-2}y = 0$. Thus, $t_0 = -2$ is the only singular point. The characteristic equation is $\lambda^2 + 5\lambda + 6 = 0$ which has roots $\lambda_1 = -3$ and $\lambda_2 = -2$. Hence, the general solution is $y = c_1(t+2)^{-3} + c_2(t+2)^{-2}, t \neq -2$.

18. $t_0 = 2$. The characteristic equation is $\lambda^2 + 4 = 0$ which has roots $\lambda_1 = 2i,\ \lambda_2 = -2i$. Hence, the general solution is $y = c_1\cos(2\ln|t-2|) + c_2\sin(2\ln|t-2|), t \neq 2$.

19. From the form of the general solution, $t_0 = -2$ and the characteristic equation has roots $\lambda_1 = 1$ and $\lambda_2 = -2$. Therefore, the characteristic equation is $\lambda^2 + \lambda - 2 = 0$. Matching the characteristic equation with the general form given in equation (3), we see that $\alpha - 1 = 1$ and $\beta = -2$. Thus, the differential equation is $(t+2)^2y'' + 2(t+2)y' - 2y = 0$.

20. $t_0 = 1,\ \lambda = 0,0.\ \therefore \lambda^2 = 0 \Rightarrow \alpha = 1,\ \beta = 0$.

21. From the form of the general solution, $t_0 = 0$ and the characteristic equation has roots $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$. Therefore, the characteristic equation is $\lambda^2 - 4\lambda + 5 = 0$. Matching the characteristic equation with the general form given in equation (3), we see that $\alpha - 1 = -4$ and $\beta = 5$. Thus, the differential equation is $t^2y'' - 3ty' + 5y = 0$.

22. The characteristic equation has roots $\lambda_1 = 2$ and $\lambda_2 = -1$. Therefore, the characteristic equation is $\lambda^2 - \lambda - 2 = 0 \Rightarrow \alpha = 0,\ \beta = -2$. Thus, the differential equation is $t^2y'' + ty' - y = g(t)$. We can determine the nonhomogenous term $g(t)$ by inserting the given particular solution $y_P(t) = 2t + 1$. Doing so, we obtain $t^2(0) + t(2) - 2(2t+1) = -2t - 2 = g(t)$.

23. From the form of the general solution, the characteristic equation has roots $\lambda_1 = 2$ and $\lambda_2 = 3$. Therefore, the characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$. Matching the characteristic equation with the general form given in equation (3), we see that $\alpha - 1 = -5$ and $\beta = 6$.
Thus, the differential equation is $t^2 y'' - 4ty' + 6y = g(t)$. We can determine the nonhomogenous term $g(t)$ by inserting the given particular solution $y_P(t) = \ln t$. Doing so, we obtain $t^2 y_P'' - 4ty_P' + 6y_P = g(t)$ or $t^2(-t^{-2}) - 4t(t^{-1}) + 6\ln t = g(t)$. Thus, $g(t) = -5 + 6\ln t$.

24. Under the change of variable $t = e^z$, the differential equation transforms into $Y''(z) - Y'(z) - 2Y(z) = 2$. The general solution is $Y(z) = c_1 e^{-z} + c_2 e^{2z} - 1 \Rightarrow y = c_1 t^{-1} + c_2 t^2 - 1$.

25. Under the change of variable $t = e^z$, the differential equation $t^2 y'' - ty' + y = t^{-1}$ transforms into $Y''(z) - 2Y'(z) + Y(z) = (e^z)^{-1}$ or $Y''(z) - 2Y'(z) + Y(z) = e^{-z}$. Solving this constant coefficient equation using the techniques of Chapter 4, we find the general solution $Y(z) = c_1 e^z + c_2 z e^z + 0.25e^{-z}$. Since $z = \ln t$, the solution can be converted to $y(t) = c_1 t + c_2 t \ln t + 0.25t^{-1}$.

26. Under the change of variable $t = e^z$, the differential equation transforms into $Y''(z) + 9Y(z) = 10e^z$.
The general solution is $Y(z) = c_1\cos(3z) + c_2\sin(3z) + e^z \Rightarrow y = c_1\cos(3\ln t) + c_2\sin(3\ln t) + t$.

27. Under the change of variable $t = e^z$, the differential equation $t^2 y'' - 6y = 10t^{-2} - 6$ transforms into $Y''(z) - Y'(z) - 6Y(z) = 10(e^z)^{-2} - 6$ or $Y''(z) - Y'(z) - 6Y(z) = 10e^{-2z} - 6$. Solving this constant coefficient equation using the techniques of Chapter 4, we find the general solution $Y(z) = c_1 e^{3z} + c_2 e^{-2z} - 2ze^{-2z} + 1$. Since $z = \ln t$, the solution can be converted to $y(t) = c_1 t^3 + c_2 t^{-2} - 2t^{-2}\ln t + 1$.

28. Under the change of variable $t = e^z$, the differential equation transforms into $Y''(z) - 5Y'(z) + 6Y(z) = 3z$. Therefore, $Y_c = c_1 e^{2z} + c_2 e^{3z}$, $Y_p = Az + B = \frac{1}{2}z + \frac{5}{12}$.
The general solution is $Y(z) = c_1 e^{2z} + c_2 e^{3z} + \frac{1}{2}z + \frac{5}{12} \Rightarrow y = c_1 t^2 + c_2 t^3 + \frac{1}{2}\ln t + \frac{5}{12}$.

29. Under the change of variable $t = e^z$, the differential equation $t^2 y'' + 8ty' + 10y = 36(t + t^{-1})$ transforms into $Y''(z) + 7Y'(z) + 10Y(z) = 36(e^z + e^{-z})$. Solving this constant coefficient equation using the techniques of Chapter 4, we find the general solution $Y(z) = c_1 e^{-5z} + c_2 e^{-2z} + 2e^z + 9e^{-z}$. Since $z = \ln t$, the solution can be converted to $y(t) = c_1 t^{-5} + c_2 t^{-2} + 2t + 9t^{-1}$.

30. The complementary solution is $y_C(t) = c_1 t^{-1} + c_2 t^3$. For a particular solution, use $y_P(t) = At + B$. Then, the general solution is $y(t) = c_1 t^{-1} + c_2 t^3 - 2t - 2$. Imposing the initial conditions, we obtain $y(1) = c_1 + c_2 - 2 - 2 = 1$ and $y'(1) = -c_1 + 3c_2 - 2 = 3$. Solving, we find the solution of the initial value problem is $y(t) = \frac{5}{2}t^{-1} + \frac{5}{2}t^3 - 2t - 2$. The interval of existence is $0 < t < \infty$.

31. Consider the initial value problem $t^2 y'' - 5ty' + 5y = 10, y(1) = 4, y'(1) = 6$. The complementary solution is $y_C(t) = c_1 t^5 + c_2 t$. By inspection, a particular solution is $y_P(t) = 2$. Thus, the general solution is $y(t) = c_1 t^5 + c_2 t + 2$. Imposing the initial conditions, we obtain $y(1) = c_1 + c_2 + 2 = 4$ and $y'(1) = 5c_1 + c_2 = 6$. Solving, we find the solution of the initial value problem is $y(t) = t^5 + t + 2$. The interval of existence is the entire t-axis.

32. The complementary solution is $y_C(t) = c_1 t^{-1} + c_2 t^{-1}\ln(-t)$. For a particular solution, use $y_P(t) = At + B$. Then, the general solution is $y_C(t) = c_1 t^{-1} + c_2 t^{-1}\ln(-t) + 2t + 9$. Imposing the initial conditions, we obtain $y(-1) = -c_1 - 2 + 9 = 1$ and $y'(-1) = -c_1 + c_2 + 2 = 0$. Solving, we find the solution of the initial value problem is $y(t) = 6t^{-1} + 4t^{-1}\ln(-t) + 2t + 9$. The interval of existence is $-\infty < t < 0$.

33. Consider the initial value problem $t^2 y'' + 3ty' + y = 2t^{-1}, y(1) = -2, y'(1) = 1$. The complementary solution is $y_C(t) = c_1 t^{-1} + c_2 t^{-1}\ln t$. Using the change of variable $t = e^z$ as in Example 2, we find a particular solution $y_P(t) = t^{-1}(\ln t)^2$. Thus, the general solution is $y(t) = c_1 t^{-1} + c_2 t^{-1}\ln t + t^{-1}(\ln t)^2$. Imposing the initial conditions, we obtain $y(1) = c_1 = -2$ and $y'(1) = -c_1 + c_2 = 1$. Solving, we find the solution of the initial value problem is $y(t) = -2t^{-1} - t^{-1}\ln t + t^{-1}(\ln t)^2$. The interval of existence is the positive t-axis.

34. $\dfrac{dy}{dt} = \dfrac{dy}{dz}\dfrac{dz}{dt} = \dfrac{1}{t}\dfrac{dy}{dz}$; $\dfrac{d^2y}{dt^2} = -\dfrac{1}{t^2}\dfrac{dy}{dz} + \dfrac{1}{t}\dfrac{d^2y}{dz^2}\dfrac{1}{t} = \dfrac{1}{t^2}\left(\dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}\right)$.

$\dfrac{d^3y}{dt^3} = -\dfrac{2}{t^3}\left(\dfrac{d^2y}{dz^2} - \dfrac{dy}{dz}\right) + \dfrac{1}{t^3}\left(\dfrac{d^3y}{dz^3} - \dfrac{d^2y}{dz^2}\right) = \dfrac{1}{t^3}\left(\dfrac{d^3y}{dz^3} - 3\dfrac{d^2y}{dz^2} + 2\dfrac{dy}{dz}\right)$. Therefore,

$$t^3 y''' + \alpha t^2 y'' + \beta t y' + \gamma y = \frac{d^3Y}{dz^3} - 3\frac{d^2Y}{dz^2} + 2\frac{dY}{dz} + \alpha\left(\frac{d^2Y}{dz^2} - \frac{dY}{dz}\right) + \beta\left(\frac{dY}{dz}\right) + \gamma Y = 0$$

$$\Rightarrow \frac{d^3Y}{dz^3} + (\alpha - 3)\frac{d^2Y}{dz^2} + (\beta - \alpha + 2)\frac{dY}{dz} + \gamma Y = 0.$$

35. Consider the differential equation $t^3 y''' + 3t^2 y'' - 3ty' = 0$. Assuming a solution of the form $y(t) = t^\lambda$, we obtain the characteristic equation $\lambda^3 - 4\lambda = 0$. The roots are $\lambda_1 = 0, \lambda_2 = 2$ and $\lambda_3 = -2$. The general solution is $y(t) = c_1 + c_2 t^2 + c_3 t^{-2},\ t \neq 0$.

36. $\alpha = 0,\ \beta = 1,\ \gamma = -1 \Rightarrow Y''' - 3Y'' + 3Y' - Y = 0$. The characteristic equation is $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0$. The roots are $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Therefore, $Y = c_1 e^z + c_2 z e^z + c_3 z^2 e^z \Rightarrow y = c_1 t + c_2 t\ln t + c_3 t(\ln t)^2$.

37. Consider the differential equation $t^3 y''' + 3t^2 y'' + ty' = 8t^2 + 12$. Using the change of variable $t = e^z$ as suggested in Exercise 34, the differential equation transforms to $Y'''(z) = 8e^{2z} + 12$. The general solution is $Y(z) = c_1 + c_2 z + c_3 z^2 + e^{2z} + 2z^3$. Using the fact that $z = \ln t$, the general solution becomes $y(t) = c_1 + c_2\ln t + c_3(\ln t)^2 + t^2 + 2(\ln t)^3,\ t > 0$.

38. $\alpha = 6,\ \beta = 7,\ \gamma = 1 \Rightarrow Y''' + 3Y'' + 3Y' + Y = 0$. The characteristic equation is $(\lambda + 1)^3 = 0$. The roots are $\lambda_1 = \lambda_2 = \lambda_3 = -1$. Therefore,

$Y_c = c_1 e^{-z} + c_2 z e^{-z} + c_3 z^2 e^{-z},\ Y_p = Az + B \Rightarrow Y = c_1 e^{-z} + c_2 z e^{-z} + c_3 z^2 e^{-z} + z - 1$

$\Rightarrow y = c_1 t^{-1} + c_2 t^{-1}\ln t + c_3 t^{-1}(\ln t)^2 + \ln t - 1$.

Section 10.4

1. When put in standard form, the differential equation is $y'' + t^{-1}(\cos t)y' + t^{-1}y = 0$. Thus, $t = 0$ is the only singular point. The coefficient functions are $p(t) = t^{-1}(\cos t)$ and $q(t) = t^{-1}$. Clearly $tp(t) = \cos t$ and $t^2 q(t) = t$ are analytic. Therefore, $t = 0$ is a regular singular point.

2. $p(t)=\dfrac{\sin t}{t^2}$ and $q(t)=\dfrac{1}{t^2}$. Since $tp(t)=\dfrac{\sin t}{t}=1-\dfrac{t^2}{3!}+\dfrac{t^4}{5!}-\dfrac{t^6}{7!}+\ldots$ and $t^2q(t)=1$ are both analytic at $t=0$, then $t=0$ is a regular singular point.

3. When put in standard form, the differential equation is $y''+(t+1)^{-1}y'+(t^2-1)^{-1}y=0$. Thus, $t=1$ and $t=-1$ are singular points. The coefficient functions are $p(t)=(t+1)^{-1}$ and $q(t)=(t^2-1)^{-1}$. Clearly $(t-1)p(t)=(t-1)(t+1)^{-1}$ and $(t-1)^2q(t)=(t-1)(t+1)^{-1}$ are analytic at $t=1$. Therefore, $t=1$ is a regular singular point. Similarly, $t=-1$ is also a regular singular point.

4. $p(t)=\dfrac{t+1}{(t^2-1)^2}=\dfrac{1}{(t-1)^2(t+1)}$ and $q(t)=\dfrac{1}{(t-1)^2(t+1)^2}$.

At $t=-1$, $(t+1)p(t)=\dfrac{1}{(t-1)^2}\to\dfrac{1}{4}$ and $(t+1)^2q(t)=\dfrac{1}{(t-1)^2}\to\dfrac{1}{4}$ as $t\to-1$. Therefore, $t=-1$ is a regular singular point.

At $t=1$, $\lim\limits_{t\to1}(t-1)p(t)=\lim\limits_{t\to1}\dfrac{1}{(t-1)(t+1)}$ does not exist.. Therefore, $t=1$ is an irregular singular point.

5. When put in standard form, the differential equation is $y''+t^{-2}(1-\cos t)y'+t^{-2}y=0$. Thus, $t=0$ is the only singular point. The coefficient functions are $p(t)=t^{-2}(1-\cos t)$ and $q(t)=t^{-2}$. Using a Maclaurin series, $tp(t)=t^{-1}(1-\cos t)=\dfrac{t}{2!}-\dfrac{t^3}{4!}+\dfrac{t^5}{6!}-\cdots$ is analytic at $t=0$ as is $t^2q(t)=1$. Therefore, $t=0$ is a regular singular point.

6. $p(t)=q(t)=\dfrac{1}{|t|}$. Since neither $tp(t)=\dfrac{t}{|t|}$ nor $t^2q(t)=\dfrac{t^2}{|t|}$ are analytic at $t=0$, there is an irregular singular point at $t=0$.

7. When put in standard form, the differential equation is $y''+(1-e^t)^{-1}y'+(1-e^t)^{-1}y=0$. Thus, $t=0$ is the only singular point. The coefficient functions are $p(t)=(1-e^t)^{-1}$ and $q(t)=(1-e^t)^{-1}$. Using a Maclaurin series,

$tp(t)=t(1-e^t)^{-1}=t\left(-t-\dfrac{t^2}{2!}-\dfrac{t^3}{3!}-\cdots\right)^{-1}=\left(-1-\dfrac{t}{2!}-\dfrac{t^2}{3!}-\cdots\right)^{-1}$ is analytic at $t=0$ as is $t^2q(t)$..

Therefore, $t=0$ is a regular singular point.

8. $p(t)=\dfrac{t+2}{(2-t)(2+t)}=\dfrac{-1}{(t-2)}$ and $q(t)=\dfrac{1}{(4-t^2)^2}=\dfrac{1}{(t-2)^2(t+2)^2}$.

At $t=-2$, $(t+2)p(t)=\dfrac{-(t+2)}{(t-2)}\to0$ and $(t+2)^2q(t)=\dfrac{1}{(t-2)^2}\to\dfrac{1}{16}$ as $t\to-2$. Therefore, $t=-2$ is a regular singular point.

At $t=2$, $(t-2)p(t)=-1$ and $(t-2)^2q(t)=\dfrac{1}{(t+2)^2}\to\dfrac{1}{16}$ as $t\to2$. Therefore, $t=2$ is a regular singular point.

9. When put in standard form, the differential equation is $y''+(1-t^2)^{-1/3}y'+(1-t^2)^{-1/3}ty=0$. Thus, $t=1$ and $t=-1$ are singular points. The coefficient functions are $p(t)=(1-t^2)^{-1/3}$ and $q(t)=t(1-t^2)^{-1/3}$. Neither of the functions $(t\pm1)p(t)$ or $(t\pm1)^2q(t)$ is analytic at $t=\pm1$. Therefore, $t=1$ is an irregular singular point as is $t=-1$.

10. $p(t)=1,\ q(t)=t^{\frac{1}{3}}$. Since $tp(t)=t$ is analytic at $t=0$, but $t^2q(t)=t^{\frac{7}{3}}$ is not, there is an irregular singular point at $t=0$.

11. For this problem, $p(t)=(\sin 2t)/P(t)$. Since we know there are singular points at $t=0$ and $t=\pm 1$, we know that $P(t)$ must be zero at those points. Since $tp(t)$ is analytic at $t=0$ and since $(\sin 2t)/t$ tends to 2 as $t\to 0$, it follows that t^2 is a factor of $P(t)$. Similarly, $(t-1)p(t)$ is *not* analytic at $t=1$ and thus $(t-1)^2$ must be a factor of $P(t)$. The same argument applies at $t=-1$ and thus $(t+1)^2$ must be a factor of $P(t)$. In summary, $P(t)=t^2(t-1)^2(t+1)^2=t^2(t^2-1)^2$.

12. $P(t)=1$.

13. For this problem, $p(t)=[tP(t)]^{-1}$. Since we know there are singular points at $t=\pm 1$, we know that $P(t)$ must be zero at $t=\pm 1$. Since $t^2q(t)=1/t$, it follows [without any assumptions on $P(t)$] that $t=0$ is an irregular singular point. Since, $(t-1)p(t)$ is *not* analytic at $t=1$ it follows that $(t-1)^2$ must be a factor of $P(t)$. The same argument applies at $t=-1$ and thus $(t+1)^2$ must be a factor of $P(t)$. In summary, $P(t)=(t-1)^2(t+1)^2=(t^2-1)^2$.

14(a). $t=0$ is a regular singular point if $n=1$.

14(b). $t=0$ is an irregular singular point if $n\ge 2$.

15. For this problem, $tp(t)=t/(\sin t)$ and $t^2q(t)=1/t^{n-2}$. Since $t/(\sin t)$ is analytic at $t=0$, it follows that $t=0$ is a regular singular point if $n=0,1,2$ and an irregular singular point if $n>2$.

16 (a). $tp(t)=-\dfrac{1}{2}$ and $t^2q(t)=\dfrac{t+1}{2}\to\dfrac{1}{2}$ as $t\to 0$. Thus, $t=0$ is a regular singular point.

16 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{\lambda+n}$ into the differential equation, we obtain

$[2\lambda(\lambda-1)-\lambda+1]a_0t^{\lambda}+\sum_{n=1}^{\infty}[(2(\lambda+n)(\lambda+n-1)-(\lambda+n)+1)a_n+a_{n-1}]t^{\lambda+n}=0$. Therefore, the indicial equation is $F(\lambda)=0$ where $F(\lambda)=2\lambda^2-3\lambda+1$. The roots of the indicial equation are $\lambda_1=\dfrac{1}{2}$ and $\lambda_2=1$.

16 (c). $a_n=\dfrac{-a_{n-1}}{F(\lambda+n)}=\dfrac{-a_{n-1}}{2(\lambda+n)^2-3(\lambda+n)+1},\ n=1,2,\ldots$

For $\lambda_2=1$, the recurrence relation is $a_n=\dfrac{-a_{n-1}}{2(1+n)^2-3(1+n)+1},\ n=1,2,\ldots.$

16 (d). $y(t)=a_0\left[t-\dfrac{t^2}{3}+\dfrac{t^3}{30}+\cdots\right]$.

17 (a). For this problem, $tp(t)=1$ and $t^2q(t)=(t-1)/4$. Thus, $t=0$ is a regular singular point.

17 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{\lambda+n}$ into the differential equation $4t^2y''+4ty'+(t-1)y=0$, we obtain $(4\lambda^2-1)a_0t^{\lambda}+\sum_{n=1}^{\infty}[(4(\lambda+n)^2-1)a_n+a_{n-1}]t^{\lambda+n}=0$. Therefore, the indicial equation is $F(\lambda)=0$ where $F(\lambda)=4\lambda^2-1$. The roots of the indicial equation are $\lambda_1=-1/2$ and $\lambda_2=1/2$.

17 (c). $a_n = \dfrac{-a_{n-1}}{F(\lambda+n)} = \dfrac{-a_{n-1}}{4(\lambda+n)^2-1}, n=1,2,\ldots$

For $\lambda = 1/2$, the recurrence relation is $a_n = -a_{n-1}/[4(n+0.5)^2-1], n=1,2,\ldots$.

17 (d). $y(t) = a_0[t^{1/2} - (1/8)t^{3/2} + (1/192)t^{5/2} - \cdots]$.

18 (a). $tp(t) = \dfrac{t}{16}$ and $t^2q(t) = \dfrac{3}{16}$. Both limits exist as $t \to 0$. Thus, $t = 0$ is a regular singular point.

18 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{\lambda+n}$ into the differential equation, we obtain

$[16\lambda(\lambda-1)+3]a_0t^{\lambda} + \sum_{n=1}^{\infty}[(16(\lambda+n)(\lambda+n-1)+3)a_n + (\lambda+n-1)a_{n-1}]t^{\lambda+n} = 0$. Therefore, the indicial equation is $F(\lambda) = 0$ where $F(\lambda) = 16\lambda^2 - 16\lambda + 3$. The roots of the indicial equation are $\lambda_1 = \dfrac{1}{4}$ and $\lambda_2 = \dfrac{3}{4}$.

18 (c). $a_n = \dfrac{-(\lambda+n-1)a_{n-1}}{F(\lambda+n)} = \dfrac{-(\lambda+n-1)a_{n-1}}{16(\lambda+n)(\lambda+n-1)+3}, n=1,2,\ldots$

For $\lambda_2 = \dfrac{3}{4}$, the recurrence relation is $a_n = \dfrac{-(3/4+n-1)a_{n-1}}{16(3/4+n)(3/4+n-1)+3}, n=1,2,\ldots$,.

18 (d). $y(t) = a_0\left[t^{\frac{3}{4}} - \dfrac{t^{\frac{7}{4}}}{32} + \dfrac{7t^{\frac{11}{4}}}{10240} + \cdots\right],\ t > 0$.

19 (a). For this problem, $tp(t) = 1$ and $t^2q(t) = t - 9$. Thus, $t = 0$ is a regular singular point.

19 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{\lambda+n}$ into the differential equation $t^2y'' + ty' + (t-9)y = 0$, we obtain $(\lambda^2-9)a_0t^{\lambda} + \sum_{n=1}^{\infty}[((\lambda+n)^2-9)a_n + a_{n-1}]t^{\lambda+n} = 0$. Therefore, the indicial equation is $F(\lambda) = 0$ where $F(\lambda) = \lambda^2 - 9$. The roots of the indicial equation are $\lambda_1 = -3$ and $\lambda_2 = 3$.

19 (c). $a_n = \dfrac{-a_{n-1}}{F(\lambda+n)} = \dfrac{-a_{n-1}}{(\lambda+n)^2-9}, n=1,2,\ldots$

For $\lambda = 3$, the recurrence relation is $a_n = -a_{n-1}/[(n+3)^2-9], n=1,2,\ldots$.

19 (d). $y(t) = a_0[t^3 - (1/7)t^4 + (1/112)t^5 - \cdots]$.

20 (a). $tp(t) = t + 2$ and $t^2q(t) = -t$. Both limits exist as $t \to 0$. Thus, $t = 0$ is a regular singular point.

20 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{\lambda+n}$ into the differential equation, we obtain

$[\lambda(\lambda-1)+2\lambda]a_0t^{\lambda-1} + \sum_{n=0}^{\infty}\{[(\lambda+n+1)(\lambda+n)+2(\lambda+n+1)]a_{n+1} + (\lambda+n-1)a_n\}t^{\lambda+n} = 0$.

Therefore, the indicial equation is $F(\lambda) = 0$ where $F(\lambda) = \lambda^2 + \lambda$. The roots of the indicial equation are $\lambda_1 = -1$ and $\lambda_2 = 0$.

20 (c). $a_{n+1} = \dfrac{-(\lambda+n-1)a_n}{(\lambda+n+2)(\lambda+n+1)}, n=0,1,2,\ldots$

For $\lambda_2 = 0$, the recurrence relation is $a_n = \dfrac{-(n-1)a_n}{(n+2)(n+1)}, n=0,1,2,\ldots$,.

20 (d). $y(t) = a_0\left[1 + \frac{t}{2}\right]$.

21 (a). For this problem, $tp(t) = 3$ and $t^2q(t) = 2t + 1$. Thus, $t = 0$ is a regular singular point.

21 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{\lambda+n}$ into the differential equation $t^2y'' + 3ty' + (2t+1)y = 0$,

we obtain $(\lambda^2 + 2\lambda + 1)a_0t^\lambda + \sum_{n=1}^{\infty}[((\lambda+n)^2 + 2(\lambda+n) + 1)a_n + 2a_{n-1}]t^{\lambda+n} = 0$. Therefore, the indicial equation is $F(\lambda) = 0$ where $F(\lambda) = \lambda^2 + 2\lambda + 1$. The roots of the indicial equation are $\lambda_1 = \lambda_2 = -1$.

21 (c). $a_n = \frac{-2a_{n-1}}{F(\lambda+n)} = \frac{-2a_{n-1}}{((\lambda+n)+1)^2}, n = 1, 2, \ldots$

For $\lambda = -1$, the recurrence relation is $a_n = -2a_{n-1}/n^2, n = 1, 2, \ldots$.

21 (d). $y(t) = a_0[t^{-1} - 2 + t - \cdots]$.

22 (a). Both limits exist as $t \to 0$. Thus, $t = 0$ is a regular singular point.

22 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{\lambda+n}$ into the differential equation, we obtain

$[\lambda(\lambda-1) - \lambda - 3]a_0t^\lambda + \sum_{n=1}^{\infty}\{[(\lambda+n)^2 - 2(\lambda+n) - 3)]a_n + (\lambda+n-1)a_{n-1}\}t^{\lambda+n} = 0$. Therefore, the indicial equation is $F(\lambda) = 0$ where $F(\lambda) = \lambda^2 - 2\lambda - 3$. The roots of the indicial equation are $\lambda_1 = -1$ and $\lambda_2 = 3$.

22 (c). $a_n = \frac{-(\lambda+n-1)a_{n-1}}{F(\lambda+n)} = \frac{-(\lambda+n-1)a_{n-1}}{(\lambda+n)^2 - 2(\lambda+n) - 3}, n = 1, 2, \ldots$

For $\lambda_2 = 3$, the recurrence relation is $a_n = \frac{-(n+2)a_{n-1}}{n(n+4)}, n = 1, 2, \ldots,$..

22 (d). $y(t) = a_0\left[t^3 - \frac{3t^4}{5} + \frac{t^5}{5} + \cdots\right]$.

23 (a). For this problem, $tp(t) = t - 2$ and $t^2q(t) = t$. Thus, $t = 0$ is a regular singular point.

23 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{\lambda+n}$ into the differential equation $ty'' + (t-2)y' + y = 0$, we

obtain $(\lambda^2 - 3\lambda)a_0t^{\lambda-1} + \sum_{n=0}^{\infty}(\lambda+n+1)[(\lambda+n-2)a_n + a_{n-1}]t^{\lambda+n} = 0$. Therefore, the indicial equation is $F(\lambda) = 0$ where $F(\lambda) = \lambda^2 - 3\lambda$. The roots of the indicial equation are $\lambda_1 = 0$ and $\lambda_2 = 3$.

23 (c). $a_{n+1} = \frac{-(\lambda+n+1)a_n}{F(\lambda+n)} = \frac{-(\lambda+n+1)a_n}{(\lambda+n+1)(\lambda+n-2)} = \frac{-a_n}{(\lambda+n-2)}, n = 0, 1, 2, \ldots$

For $\lambda = 3$, the recurrence relation is $a_n = -a_{n-1}/(n+1), n = 0, 1, \ldots$.

23 (d). $y(t) = a_0[t^3 - t^4 + (1/2)t^5 - \cdots]$.

24 (a). $tp(t) = -\frac{2\sin t}{t} \to -2$ as $t \to 0$ and $t^2q(t) = 2 + t \to 2$ as $t \to 0$. Thus, $t = 0$ is a regular singular point.

24 (b). $t^2y'' - 2\sin ty' + (2+t)y = \left[\lambda(\lambda-1)a_0t^\lambda + (\lambda+1)\lambda a_1t^{\lambda+1} + (\lambda+2)(\lambda+1)a_2t^{\lambda+2} + \cdots\right]$

$$-2\left[t - \frac{t^3}{3!} + ...\right]\left[\lambda a_0t^{\lambda-1} + (\lambda+1)a_1t^\lambda + (\lambda+2)a_2t^{\lambda+1} + \cdots\right] + (2+t)\left[a_0t^\lambda + a_1t^{\lambda+1} + a_2t^{\lambda+2} + ...\right] = 0.$$

For $t^\lambda: \lambda(\lambda-1)a_0 - 2\lambda a_0 + 2a_0 = (\lambda^2 - 3\lambda + 2)a_0 = (\lambda-1)(\lambda-2)a_0 = 0$.
For $t^{\lambda+1}: \lambda(\lambda+1)a_1 - 2(\lambda+1)a_1 + 2a_1 + a_0 = [(\lambda+1)(\lambda-2)+2]a_1 + a_0 = 0$.
For $t^{\lambda+2}: (\lambda+2)(\lambda+1)a_2 - 2(\lambda+2)a_2 + \frac{2}{3!}\lambda a_0 + 2a_2 + a_1 = 0$.
Therefore, the indicial equation is $F(\lambda) = (\lambda-1)(\lambda-2) = 0$. The roots of the indicial equation are $\lambda_1 = 1$ and $\lambda_2 = 2$.

24 (c). $y(t) = a_0[t^2 - \frac{t^3}{2} - \frac{t^4}{6} - \cdots]$

25 (a). For this problem, $tp(t) = 4$ and $t^2q(t) = te^t$. Thus, $t = 0$ is a regular singular point.

25 (b). Given the series $y = \sum_{n=0}^{\infty} a_nt^{\lambda+n}$, we have $ty'' = \lambda(\lambda-1)a_0t^{\lambda-1} + (\lambda+1)\lambda a_1t^\lambda + \cdots$,
$-4y' = \lambda a_0t^{\lambda-1} + (\lambda+1)a_1t^\lambda + \cdots$, and
$e^ty = [1 + t + (1/2!)t^2 + \cdots][a_0t^\lambda + a_1t^{\lambda+1} + \cdots] = a_0t^\lambda + (a_1+1)t^{\lambda+1} + \cdots$.
Therefore, substituting the series into the differential equation $ty'' - 4y' + e^ty = 0$, we obtain $\lambda(\lambda-5)a_0t^{\lambda-1} + [(\lambda+1)(\lambda-4) + a_0]t^\lambda + \cdots = 0$. Therefore, the indicial equation is $\lambda^2 - 5\lambda = 0$. The roots of the indicial equation are $\lambda_1 = 0$ and $\lambda_2 = 5$.

25 (c). $y(t) = a_0[t^5 - (1/6)t^6 - (5/84)t^7 - \cdots]$

26 (a). $tp(t) = -\frac{t}{\sin t} \to -1$ as $t \to 0$ and $t^2q(t) = \frac{t^2}{\sin t} \to 0$ as $t \to 0$. Thus, $t = 0$ is a regular singular point.

26 (b). $(\sin t)y'' - y' + y =$

$$\left[t - \frac{t^3}{3!} + \frac{t^5}{5!}...\right]\left[\lambda(\lambda-1)a_0t^{\lambda-2} + (\lambda+1)\lambda a_1t^{\lambda-1} + (\lambda+2)(\lambda+1)a_2t^\lambda + (\lambda+3)(\lambda+2)a_3t^{\lambda+1} + \cdots\right]$$

$$-\left[\lambda a_0t^{\lambda-1} + (\lambda+1)a_1t^\lambda + (\lambda+2)a_2t^{\lambda+1} + \cdots\right] + \left[a_0t^\lambda + a_1t^{\lambda+1} + a_2t^{\lambda+2} + \cdots\right] = 0..$$

For $t^{\lambda-1}: \lambda(\lambda-1)a_0 - \lambda a_0 = (\lambda^2 - 2\lambda)a_0 = \lambda(\lambda-2)a_0 = 0$.
For $t^\lambda: \lambda(\lambda+1)a_1 - (\lambda+1)a_1 + a_0 = (\lambda+1)(\lambda-1)a_1 + a_0 = 0$.
For $t^{\lambda+1}: (\lambda+2)(\lambda+1)a_2 + (\lambda+2)a_2 - \frac{1}{3!}\lambda(\lambda-1)a_0 + a_1 = (\lambda+2)^2a_2 + a_1 - \frac{1}{6}\lambda(\lambda-1)a_0 = 0$.
Therefore, the indicial equation is $F(\lambda) = \lambda(\lambda-2) = 0$. The roots of the indicial equation are $\lambda_1 = 0$ and $\lambda_2 = 2$.

26 (c). $y(t) = a_0[t^2 - \frac{t^3}{3} + \frac{t^4}{24} + \cdots]$

27 (a). For this problem, $tp(t) = t/(2 - 2e^t)$ and $t^2q(t) = t^2/(1 - e^t)$. Thus, $t = 0$ is a regular singular point.

27 (b). Given the series $y = \sum_{n=0}^{\infty} a_nt^{\lambda+n}$, we have
$(1 - e^t)y'' = -\lambda(\lambda-1)a_0t^{\lambda-1}[-0.5\lambda(\lambda-1)a_0 - (\lambda+1)\lambda a_1]t^\lambda + \cdots$,
$0.5y' = 0.5[\lambda a_0t^{\lambda-1} + (\lambda+1)a_1t^\lambda + \cdots]$.

Therefore, substituting the series into the differential equation $(1-e^t)y''+(1/2)y'+y=0$, we obtain $-\lambda(\lambda-1.5)a_0t^{\lambda-1}+[-(\lambda+1)(\lambda-0.5)a_1+0.5(-\lambda^2+\lambda+2)a_0]t^{\lambda}+\cdots=0$. Therefore, the indicial equation is $\lambda^2-1.5\lambda=0$. The roots of the indicial equation are $\lambda_1=0$ and $\lambda_2=1.5$.

27 (c). $y(t)=a_0[t^{3/2}+(1/2)t^{5/2}-(17/96)t^{7/2}+\cdots]$

Section 10.5

1 (a). When put in standard form, the differential equation is $y''-(2t)^{-1}(1+t)y'+t^{-1}y=0$. Therefore, $t=0$ is a regular singular point.

1 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{n+\lambda}$ into the differential equation, we obtain

$$(2\lambda^2-3\lambda)a_0t^{\lambda-1}+\sum_{n=0}^{\infty}[(\lambda+n+1)(2(\lambda+n)-1)a_{n+1}-(\lambda+n-2)a_n]t^{n+\lambda}=0.$$

Therefore, the exponents at the singularity are $\lambda_1=0$ and $\lambda_2=1.5$.

1 (c). The recurrence relation is $a_{n+1}=[(\lambda+n-2)a_n]/[(\lambda+n+1)(2\lambda+2n-1)]$, $n=0,1,\ldots$.

1 (d). For $\lambda_1=0$, $y=a_0[1+2t-t^2]$ is a polynomial solution.
For $\lambda_2=3/2$, $y=a_0[t^{3/2}-(1/10)t^{5/2}-(1/280)t^{7/2}-\cdots]$.

1 (e). Note that $tp(t)$ and $t^2q(t)$ are analytic everywhere. Thus, see equations (18)-(21), the second series found in part (d) converges for $0<t$.

2 (b). Substituting the series into the differential equation, we obtain

$$[2\lambda(\lambda-1)+5\lambda]a_0t^{\lambda-1}+[2\lambda(\lambda+1)+5(\lambda+1)]a_1t^{\lambda}+\sum_{n=1}^{\infty}[2(\lambda+n+1)(\lambda+n+5/2)a_{n+1}+3a_{n-1}]t^{n+\lambda}$$

$=0$. Therefore, $F(\lambda)=2\lambda(\lambda+3/2)\Rightarrow\lambda_1=-\frac{3}{2},\ \lambda_2=0$.

2 (c). The recurrence relation is $a_{n+1}=\dfrac{-3a_{n-1}}{2(\lambda+n+1)(\lambda+n+5/2)}$, $n=1,2,\ldots$ and $(\lambda+1)(2\lambda+5)a_1=0$

2 (d). For $\lambda_1=-\frac{3}{2}$, $y=a_0[t^{-3/2}-(3/2)t^{1/2}+(9/40)t^{5/2}+\cdots]$.
For $\lambda_2=0$, $y=a_0[1-(3/14)t^2+(9/616)t^4-\cdots]$.

2 (e). The series converges for $0<t$.

3 (a). When put in standard form, the differential equation is $y''-(3t)^{-1}y'+(3t^2)^{-1}(1+t)y=0$. Therefore, $t=0$ is a regular singular point.

3 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{n+\lambda}$ into the differential equation, we obtain

$$(3\lambda^2-4\lambda+1)a_0t^{\lambda}+\sum_{n=1}^{\infty}\{[3(\lambda+n)(\lambda+n-1)-\lambda-n+1]a_n+a_{n-1}\}t^{n+\lambda}=0.$$

Therefore, the exponents at the singularity are $\lambda_1=1/3$ and $\lambda_2=1$.

3 (c). The recurrence relation is $a_n=-a_{n-1}/[3(\lambda+n)(\lambda+n-1)-\lambda-n+1]$, $n=1,2,\ldots$.

3 (d). For $\lambda_1=1/3$, $y=a_0[t^{1/3}-t^{4/3}+(1/8)t^{7/3}+\cdots]$.
For $\lambda_2=1$, $y=a_0[t-(1/5)t^2+(1/80)t^3+\cdots]$.

3 (e). Note that $tp(t)$ and $t^2q(t)$ are analytic everywhere. Thus, see equations (18)-(21), the series found in part (d) converge for $0<t$.

4 (b). Substituting the series into the differential equation, we obtain

$$[6\lambda(\lambda-1)+\lambda+1]a_0t^{\lambda}+\sum_{n=1}^{\infty}\{[6(\lambda+n)(\lambda+n-1)+(\lambda+n)+1]a_n-a_{n-1}\}t^{n+\lambda}=0.$$ Therefore,

$$F(\lambda)=6\lambda^2-5\lambda+1\Rightarrow\lambda_1=\frac{1}{3},\ \lambda_2=\frac{1}{2}.$$

4 (c). The recurrence relation is $a_n=\dfrac{a_{n-1}}{6(\lambda+n)(\lambda+n-1)+(\lambda+n)+1}$, $n=1,2,...$

4 (d). For $\lambda_1=\dfrac{1}{3}$, $y=a_0[t^{1/3}+(1/5)t^{4/3}+(1/110)t^{7/3}+\cdots]$.

For $\lambda_2=\dfrac{1}{2}$, $y=a_0[t^{1/2}+(1/7)t^{3/2}+(1/182)t^{5/2}+\cdots]$.

4 (e). The series converges for $0<t$.

5 (a). When put in standard form, the differential equation is $y''-5t^{-1}y'+t^{-2}(9+t^2)y=0$. Therefore, $t=0$ is a regular singular point.

5 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{n+\lambda}$ into the differential equation, we obtain

$$(\lambda^2-6\lambda+9)a_0t^{\lambda}+[(\lambda+1)\lambda-5(\lambda+1)+9]a_1t^{\lambda+1}+$$
$$\sum_{n=2}^{\infty}\{[(\lambda+n)(\lambda+n-1)-5(\lambda+n)+9]a_n+a_{n-1}\}t^{n+\lambda}=0.$$

Therefore, the exponents at the singularity are $\lambda_1=\lambda_2=3$.

5 (c). The recurrence relation is $a_n=-a_{n-2}/(\lambda+n-3)^2$, $n=2,3,\ldots$.

5 (d). For $\lambda_1=3$, $y=a_0[t^3-(1/4)t^5+(1/64)t^7+\cdots]$.

5 (e). Note that $tp(t)$ and $t^2q(t)$ are analytic everywhere. Thus, see equations (18)-(21), the series found in part (d) converges for $0<t$.

6 (b). Substituting the series into the differential equation, we obtain

$$[4\lambda(\lambda-1)+8\lambda+1]a_nt^{\lambda}+\sum_{n=1}^{\infty}\left\{[4(\lambda+n)^2+4(\lambda+n)+1]a_n-2a_{n-1}\right\}t^{n+\lambda}=0.$$ Therefore,

$$F(\lambda)=4\lambda^2+4\lambda+1\Rightarrow\lambda_1=\lambda_2=-\frac{1}{2}.$$

6 (c). The recurrence relation is $a_n=\dfrac{2a_{n-1}}{(2(\lambda+n)+1)^2}$, $n=1,2,...$

6 (d). For $\lambda_1=-\dfrac{1}{2}$, $y=a_0[t^{-1/2}+(1/2)t^{1/2}+(1/8)t^{3/2}+\cdots]$.

6 (e). The series converges for $0<t$.

7 (a). When put in standard form, the differential equation is $y''-2t^{-1}y'+t^{-2}(2+t)y=0$. Therefore, $t=0$ is a regular singular point.

7 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{n+\lambda}$ into the differential equation, we obtain

$$(\lambda^2-3\lambda+2)a_0t^{\lambda}+\sum_{n=1}^{\infty}\{[(\lambda+n)^2-3(\lambda+n)+2]a_n+a_{n-1}\}t^{n+\lambda}=0.$$

Therefore, the exponents at the singularity are $\lambda_1=1$ and $\lambda_2=2$.

7 (c). The recurrence relation is $a_n=-a_{n-1}/[(\lambda+n-1)(\lambda+n-2)]$, $n=1,2,\ldots$.

7 (d). For $\lambda_2=2$, $y=a_0[t^2-(1/2)t^3+(1/12)t^4+\cdots]$.

7 (e). Note that $tp(t)$ and $t^2q(t)$ are analytic everywhere. Thus, see equations (18)-(21), the series found in part (d) converges for $0 < t$.

8 (b). Substituting the series into the differential equation, we obtain

$$[\lambda(\lambda-1)+4\lambda]a_0t^{\lambda} + [\lambda(\lambda+1)+4(\lambda+1)]a_1t^{\lambda+1} + \sum_{n=1}^{\infty}\{[(\lambda+n+1)(\lambda+n+4)]a_{n+1} - 2a_{n-1}\}t^{n+\lambda} = 0$$

Therefore, $F(\lambda) = \lambda^2 + 3\lambda \Rightarrow \lambda_1 = -3,\ \lambda_2 = 0$.

8 (c). The recurrence relation is $a_{n+1} = \dfrac{2a_{n-1}}{(\lambda+n+1)(\lambda+n+4)}$, $n = 1,2,...$ and $(\lambda+1)(\lambda+4)a_1 = 0$

8 (d). For $\lambda_2 = 0$, $y = a_0[1 + (1/5)t^2 + (1/70)t^4 + \cdots]$.

8 (e). The series converges for $0 < t$.

9 (a). When put in standard form, the differential equation is $y'' + t^{-1}y' - t^{-2}(1+t^2)y = 0$. Therefore, $t = 0$ is a regular singular point.

9 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{n+\lambda}$ into the differential equation, we obtain

$$(\lambda^2-1)a_0t^{\lambda} + [(\lambda+1)^2-1]a_1t^{\lambda+1} + \sum_{n=2}^{\infty}\{[(\lambda+n)^2-1]a_n - a_{n-2}\}t^{n+\lambda} = 0.$$

Therefore, the exponents at the singularity are $\lambda_1 = -1$ and $\lambda_2 = 1$.

9 (c). The recurrence relation is $a_n = a_{n-2}/[(\lambda+n)^2-1]$, $n = 2, 3, \ldots$.

9 (d). For $\lambda_2 = 1$, $y = a_0[t + (1/8)t^3 + (1/192)t^4 + \cdots]$.

9 (e). Note that $tp(t)$ and $t^2q(t)$ are analytic everywhere. Thus, see equations (18)-(21), the series found in part (d) converges for $0 < t$.

10 (b). Substituting the series into the differential equation, we obtain

$$[\lambda(\lambda-1)+5\lambda+4]a_0t^{\lambda} + [\lambda(\lambda+1)+5(\lambda+1)+4]a_1t^{\lambda+1}$$

$$+\sum_{n=2}^{\infty}\{[(\lambda+n)(\lambda+n+4)+4]a_n - a_{n-2}\}t^{n+\lambda} = 0.$$ Therefore, $F(\lambda) = \lambda^2 + 4\lambda + 4 \Rightarrow \lambda_1 = \lambda_2 = -2$.

10 (c). The recurrence relation is $a_n = \dfrac{a_{n-2}}{(\lambda+n+2)^2}$, $n = 2,3,...$ and $(\lambda+1)(\lambda+5)a_1 = 0$

10 (d). For $\lambda = -2$, $y = a_0[t^{-2} + (1/4) + (1/64)t^2 + \cdots]$.

10 (e). The series converges for $0 < t$.

11 (a). When put in standard form, the differential equation is $y'' + t^{-1}y' - t^{-2}(16+t)y = 0$. Therefore, $t = 0$ is a regular singular point.

11 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{n+\lambda}$ into the differential equation, we obtain

$$(\lambda^2-16)a_0t^{\lambda} + \sum_{n=1}^{\infty}\{[(\lambda+n)^2-16]a_n - a_{n-1}\}t^{n+\lambda} = 0.$$

Therefore, the exponents at the singularity are $\lambda_1 = -4$ and $\lambda_2 = 4$.

11 (c). The recurrence relation is $a_n = a_{n-1}/[(\lambda+n)^2-16]$, $n = 1, 2, \ldots$.

11 (d). For $\lambda_2 = 4$, $y = a_0[t^4 + (1/9)t^5 + (1/180)t^6 + \cdots]$.

11 (e). Note that $tp(t)$ and $t^2q(t)$ are analytic everywhere. Thus, see equations (18)-(21), the series found in part (d) converges for $0 < t$.

12 (b). Substituting the series into the differential equation, we obtain

$$\left[8\lambda^2-2\lambda-1\right]a_0t^\lambda+\sum_{n=1}^{\infty}\left\{[8(\lambda+n)^2-2(\lambda+n)-1]a_n+a_{n-1}\right\}t^{n+\lambda}=0.$$ Therefore,

$$F(\lambda)=8\lambda^2-2\lambda-1\Rightarrow\lambda_1=-\frac{1}{4},\ \lambda_2=\frac{1}{2}.$$

12 (c). The recurrence relation is $a_n=\dfrac{-a_{n-1}}{(4(\lambda+n)+1)(2(\lambda+n)-1)},\ n=1,2,...$

12 (d). For $\lambda_1=-\dfrac{1}{4}$, $y=a_0[t^{-1/4}-(1/2)t^{3/4}+(1/40)t^{7/4}+\cdots]$.

For $\lambda_2=\dfrac{1}{2}$, $y=a_0[t^{1/2}-(1/14)t^{3/2}+(1/616)t^{5/2}+\cdots]$..

12 (e). The series converges for $0<t$.

13 (a). When put in standard form, the differential equation is $y''-t^{-1}(t^2+1)^{-1}(1+t)y'+t^{-1}(t^2+1)^{-1}y=0$. Therefore, $t=0$ is a regular singular point and all other points are ordinary points.

13 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{n+\lambda}$ into the differential equation, we obtain

$$\sum_{n=1}^{\infty}(\lambda+n-1)(\lambda+n-2)a_{n-1}t^{n+\lambda}+\sum_{n=-1}^{\infty}(\lambda+n+1)(\lambda+n-1)a_{n+1}t^{n+\lambda}-\sum_{n=0}^{\infty}(\lambda+n-1)a_nt^{n+\lambda}=0$$

Therefore, indicial equation is $\lambda^2-2\lambda=0$. The exponents at the singularity are $\lambda_1=0$ and $\lambda_2=2$.

14 (a). $tp(t)=\dfrac{\sin 3t}{t}\to 3$ as $t\to 0$ and $t^2q(t)=\cos t\to 1$ as $t\to 0$. Thus, $t=0$ is a regular singular point.

14 (b). $t^2y''+\left(3t-\dfrac{(3t)^3}{3!}+...\right)y'+\left(1-\dfrac{t^2}{2!}+...\right)y=0$.

Therefore, indicial equation $(\lambda+1)^2=0\Rightarrow\lambda_1=\lambda_2=-1$.

15 (a). When put in standard form, the differential equation is $y''-(t^2-4)^{-2}y'+(t^2-4)^{-2}y=0$. Therefore, $t=2$ and $t=-2$ are irregular singular points. All other points are ordinary points.

16 (a). $tp(t)=\dfrac{1}{(1-t)^{\frac{1}{3}}}\to 1$ as $t\to 0$ and $t^2q(t)=-\dfrac{1}{(1-t)^{\frac{1}{3}}}\to -1$ as $t\to 0$. Thus, $t=0$ is a regular singular point.

Neither $(t-1)p(t)$ nor $(t\text{-}1)^2q(t)$ are analytical at $t=1$, so $t=1$ is an irregular singular point.

16 (b). $(1-t)^{\frac{1}{3}}=1-\dfrac{1}{3}t-\dfrac{1}{9}t^2+...\Rightarrow t^2\left(1-\dfrac{1}{3}t-\dfrac{1}{9}t^2+...\right)y''+ty'-y=0$.

Therefore, indicial equation $\lambda^2-1=0\Rightarrow\lambda_1=-1,\ \lambda_2=1$.

17 (a). We need to substitute the series $y=\sum_{n=0}^{\infty} a_n(t-1)^{n+\lambda}$ into the differential equation. Before doing so, let us make the change of variable $\tau=t-1$. We now substitute the series $y=\sum_{n=0}^{\infty} a_n\tau^{n+\lambda}$ into the transformed equation, $-\tau(\tau+2)y''-2(\tau+1)y'+\alpha(\alpha+1)y=0$, obtaining

$$-2\lambda^2 a_0\tau^{\lambda-1}+\sum_{n=0}^{\infty}\{[-(\lambda+n)^2-(\lambda+n)+\alpha(\alpha+1)]a_n-2(\lambda+n+1)^2 a_{n+1}\tau^{\lambda+n}=0.$$

Thus, the exponents at the singularity are $\lambda_1=\lambda_2=0$.

17 (b). For $\lambda=0$, the recurrence relation is $a_{n+1}=[-n^2-n+\alpha(\alpha+1)]a_n/[2(n+1)^2]$.

Thus, $y(t)=a_0\left[1+\dfrac{\alpha(\alpha+1)}{2}(t-1)+\dfrac{\alpha(\alpha+1)[-2+\alpha(\alpha+1)]}{16}(t-1)^2+\cdots\right].$

17 (c). When $\alpha=1$, $y(t)=a_0t$.

18 (a). $(1-t)^2=-(t-1)(t+1)=-(t-1)((t-1)+2)$, $t=(t-1)+1$. Let $\tau=t-1$. We now substitute the series into the transformed equation, $-\tau(\tau+2)y''-(\tau+1)y'+\alpha^2y=0$, obtaining

$$-[2\lambda(\lambda-1)+\lambda]a_0\tau^{\lambda-1}+\sum_{n=0}^{\infty}\{-[2(\lambda+n+1)(\lambda+n)+(\lambda+n+1)]a_{n+1}+[-(\lambda+n)^2+\alpha^2]a_n\}\tau^{\lambda+n}.$$

Thus, $F(\lambda)=2\lambda^2-\lambda=0$ and the exponents at the singularity are $\lambda_1=0$ and $\lambda_2=\dfrac{1}{2}$.

18 (b). For $\lambda_1=0$, the recurrence relation is $a_{n+1}=\dfrac{[-n^2+\alpha^2]a_n}{(n+1)(2n+1)}$.

and $y(t)=a_0\left[1+\alpha^2(t-1)+\dfrac{\alpha^2(\alpha^2-1)}{6}(t-1)^2+\cdots\right].$

For $\lambda_2=\dfrac{1}{2}$, the recurrence relation is $a_{n+1}=\dfrac{[-(n+1/2)^2+\alpha^2]a_n}{(n+3/2)(2n+2)}$.

and $y(t)=a_0\left[(t-1)^{\frac{1}{2}}+\dfrac{(\alpha^2-\frac{1}{4})}{3}(t-1)^{\frac{3}{2}}+\dfrac{(\alpha^2-\frac{1}{4})(\alpha^2-\frac{9}{4})}{30}(t-1)^{\frac{5}{2}}+\cdots\right],\ t-1>0.$

18 (c). By the Ratio Test, $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\dfrac{-(n+\lambda)^2+\alpha^2}{(n+\lambda+1)(2n+2\lambda+1)}\right|=\dfrac{1}{2}$

$\Rightarrow$ convergence for $\dfrac{1}{2}|\tau|<1$ or $|t-1|<2\ \ \therefore R=2$.

18 (d). When $\alpha=\dfrac{1}{2}$, one solution (with $\lambda=\dfrac{1}{2}$) reduces to $y(t)=a_0(t-1)^{\frac{1}{2}}$.

19 (a). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{n+\lambda}$ into the differential equation, we obtain

$$\lambda^2a_0t^{\lambda-1}+\sum_{n=0}^{\infty}\{(\lambda+n+1)^2a_{n+1}-(\lambda+n-\alpha)a_n\}t^{n+\lambda}=0.$$

19 (b). The recurrence relation is $a_{n+1}=(n-\alpha)a_n/(n+1)^2$. For $\alpha=5$, the solution is $y(t)=a_0[1-5t+5t^2-(5/3)t^3+(5/24)t^4-(1/120)t^5]$.

19 (c). $y(t)$ is neither an even nor an odd function. Theorem 10.2 does not apply.

20. The indicial equation is $\lambda(\lambda-1)+\alpha\lambda+\beta=\lambda^2+(\alpha-1)\lambda+\beta=0$. Since $\lambda_1=1,\ \lambda_2=2$, then $\lambda^2+(\alpha-1)\lambda+\beta=(\lambda-1)(\lambda-2)=\lambda^2-3\lambda+2\Rightarrow\alpha=-2,\ \beta=2$.

21. The indicial equation is $\lambda^2+(\alpha-1)\lambda+\beta=0$. In order to have $\lambda_1=1+2i$ and $\lambda_2=1-2i$, we need $(\lambda-\lambda_1)(\lambda-\lambda_2)=\lambda^2-(\lambda_1+\lambda_2)\lambda+\lambda_1\lambda_2=\lambda^2-2\lambda+5$. Therefore, $\alpha=-1$ and $\beta=5$.

22. The indicial equation is $\lambda(\lambda-1)+\alpha\lambda+2=0$ has $\lambda=2$ as a root. Therefore, $2(1)+2\alpha+2=0\Rightarrow\alpha=-2$. Therefore,

$$t^2y''-2ty'+(2+\beta t)y=\sum_{n=0}^{\infty}\{(\lambda+n)(\lambda+n-1)-2(\lambda+n)+2\}a_n\,t^{n+\lambda}+\beta\sum_{n=1}^{\infty}a_{n-1}\,t^{n+\lambda}=0$$

$$\Rightarrow[\lambda(\lambda-1)-2\lambda+2]a_0t^{\lambda}+\sum_{n=1}^{\infty}\{[(\lambda+n)^2-3(\lambda+n)+2]a_n+\beta a_{n-1}\}t^{n+\lambda}=0.$$

For $\lambda=2$, the recurrence relation becomes $\left[(n+2)^2-3(n+2)+2\right]a_n+\beta a_{n-1}=0,\ n=1,2,...$

Therefore, $\left[n^2+4n+4-3n-6+2\right]a_n+\beta a_{n-1}=(n^2+n)a_n+\beta a_{n-1}=0\Rightarrow\beta=-4$.

23. The indicial equation is $\lambda^2=0$ and the corresponding recurrence relation is $(n+1)^2a_{n+1}+\alpha na_n+\beta a_{n-1}=0$. Therefore, $\alpha=-1$ and $\beta=3$.

24 (a). $p(t)$ is odd and $q(t)$ is even, so we expect even and odd solutions.

24 (b). The indicial equation is $\lambda(\lambda-1)+\lambda-\upsilon^2=0$ or $F(\lambda)=\lambda^2-\upsilon^2\Rightarrow\lambda_1=-\upsilon,\ \lambda_2=\upsilon$.
For the Bessel equation, $\lambda(\lambda-1)+\lambda-\upsilon^2=0$ or $F(\lambda)=\lambda^2-\upsilon^2$.
The indicial equation and exponents at the singularity are the same for both equations.

24 (c). $[\lambda^2-\upsilon^2]a_0t^{\lambda}+[(\lambda+1)^2-\upsilon^2]a_1t^{\lambda-1}+\sum_{n=2}^{\infty}\{[(\lambda+n)^2-\upsilon^2]a_n-a_{n-2}\}t^{n+\lambda}=0$

$$\Rightarrow a_n=\frac{a_{n-2}}{(\lambda+n)^2-\upsilon^2},\ n=2,3,...$$

For Bessel's equation, $a_n=\dfrac{-a_{n-2}}{(\lambda+n)^2-\upsilon^2},\ n=2,3,...$ The minus sign creates a "term-to-term" change of sign in the series solution. This sign alteration is not present in the series solutions of the modified Bessel equation.