Differential Geometry in Physics

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These notes were developed as a supplement to a course on Differential Geometry at the advanced undergraduate, first year graduate level, which by the author has taught for several years. There are many excellent good texts in Differential Geometry but very few have an early introduction to differential forms and their applications to Physics. It is the purpose of these notes to bridge some of these gaps and thus help the student get a more profound understanding of the concepts involved. When appropriate, the notes also correlate classical equations to the more elegant but less intuitive modern formulation of the subject.

These notes should be accessible to students who have completed a traditional training in Advanced Calculus, Linear Algebra, and differential Equations. Students who master the entirety of this material will have gained enough background to begin a formal study of the General Theory of relativity

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Chapter 1

1.1 Tangent Vectors

1.1 Definition Euclidean n-space $\mathbf{R}^n$ is defined as the set of ordered n-tuples $\mathbf{p} = (p^1, \ldots, p^n)$, where $p^i \in \mathbf{R}$, for each $i = 1, \ldots, n$.

Given any two n-tuples $\mathbf{p} = (p^1, \ldots, p^n)$, $\mathbf{q} = (q^1, \ldots, q^n)$ and any real number c, we define two operations:

$$\begin{aligned} \mathbf{p} + \mathbf{q} &= (p^1 + q^1, \ldots, p^n + q^n) \\ c\mathbf{p} &= (cp^1, \ldots, cp^n) \end{aligned} \tag{1.1}$$

With the sum and the scalar multiplication of ordered n-tuples defined this way, Euclidean space acquires the structure of a vector space of n dimensions[1].

1.2 Definition Let x^i be the real valued functions in $\mathbf{R}^n$ such that $x^i(\mathbf{p}) = p^i$ for any point $\mathbf{p} = (p^1, \ldots, p^n)$. The functions x^i are then called the natural *coordinates* of the the point $\mathbf{p}$. When the dimension of the space $n = 3$, we often write: $x^1 = x$, $x^2 = y$ and $x^3 = z$.

1.3 Definition A real valued function in $\mathbf{R}^n$ is of class C^r if all the partial derivatives of the function up to order r exist and are continuous. The space of infinitely differentiable (smooth) functions will be denoted by $C^\infty(\mathbf{R}^n)$.

In advanced calculus, vectors are usually regarded as arrows characterized by a direction and a length. Vectors as thus considered as independent of their location in space. Because of physical and mathematical reasons, it is advantageous to introduce a notion of vectors which does depend on location. For example, if the vector is to represent a force acting on a rigid body, then the resulting equations of motion will obviously depend on the point at which the force is applied.

In a later chapter we will also consider vectors on spaces which are curved. In these cases the position of the vectors is crucial. for instance, a unit vector pointing north at the earth's equator, is not at all the same as a unit vector pointing north at the tropic of Capricorn. This example should help motivate the following definition.

1.4 Definition A **tangent vector** X_p in $\mathbf{R}^n$, is an ordered pair $(\mathbf{X}, \mathbf{p})$. We may regard $\mathbf{X}$ as an ordinary advanced calculus vector and $\mathbf{p}$ is the position vector of the foot the arrow.

[1] In these notes we will use the following index conventions.
Indices such as i, j, k, l, m, n, run from 1 to n
Indices such as μ, ν, ρ, σ, run from 0 to n
Indices such as $\alpha, \beta, \gamma, \delta$, run from 1 to 2.

The collection of all tangent vectors at a point $\mathbf{p} \in \mathbf{R}^n$ is called the **tangent space** at $\mathbf{p}$ and will be denoted by $T_p(\mathbf{R}^n)$. Given two tangent vectors X_p, Y_p and a constant c, we can define new tangent vectors at $\mathbf{p}$ by $(X+Y)_p = X_p + Y_p$ and $(cX)_p = cX_p$. With this definition, it is easy to see that for each point $\mathbf{p}$, the corresponding tangent space $T_p(\mathbf{R}^n)$ at that point has the structure of a vector space. On the other hand, there is no natural way to add two tangent vectors at different points.

Let U be a open subset of $\mathbf{R}^n$. The set $T(U)$ consisting of the union of all tangent vectors at all points in U is called the **tangent bundle**. This object is not a vector space, but as we will see later it has much more structure than just a set.

1.5 Definition A vector field X in $U \in \mathbf{R}^n$ is a smooth function from U to $T(U)$.

We may think of a vector field as a smooth assignment of a tangent vector X_p to each point in in U. Given any two vector fields X and Y and any smooth function f, we can define new vector fields $X+Y$ and fX by

$$\begin{aligned} (X+Y)_p &= X_p + Y_p \\ (fX)_p &= fX_p \end{aligned} \tag{1.2}$$

Remark Since the space of smooth functions is not a field but only a ring, the operations above give the space of vector fields the structure of a ring module. The subscript notation X_p to indicate the location of a tangent vector is some times cumbersome. At the risk of introducing some confusion, we will drop the subscript to denote a tangent vector. Hopefully, it will e clear from the context, whether we are referring to a vector or to a vector field. At the risk of introducing some confusion, we

Vector fields are essential objects in physical applications. If we consider the flow of a fluid in a region, the velocity vector field indicates the speed and direction of the flow of the fluid at that point. Other examples of vector fields in classical physics are the electric, magnetic and gravitational fields.

1.6 Definition Let X_p be a tangent vector in an open neighborhood U of a point $\mathbf{p} \in \mathbf{R}^n$ and let f be a C^∞ function in U. The directional derivative of f at the point $\mathbf{p}$, in the direction of X_p is defined by

$$\nabla_X(f)(p) = \quad f(p) \cdot \mathbf{X}(p), \tag{1.3}$$

where $\quad f(p)$ is the gradient of the function f at the point $\mathbf{p}$. The notation

$$X_p(f) = \nabla_X(f)(p)$$

is also often used in these notes. We may think of a tangent vector at point as an operator on the space of smooth functions in a neighborhood of the point. The operator assigns to a function, the directional derivative of the function in the direction of the vector. It is easy to generalize the notion of directional derivatives to vector fields by defining $X(f)(p) = X_p(f)$.

1.7 Proposition If $f, g \in C^\infty \mathbf{R}^n$, $a, b \in \mathbf{R}$, and X is a vector field, then

$$\begin{aligned} X(af+bg) &= aX(f) + bX(g) \\ X(fg) &= fX(g) + gX(f) \end{aligned} \tag{1.4}$$

The proof of this proposition follows from fundamental properties of the gradient, and it is found in any advanced calculus text.

Any quantity in Euclidean space which satisfies relations 1.4 is a called a **linear derivation** on the space of smooth functions. The word linear here is used in the usual sense of a linear operator in linear algebra, and the word derivation means that the operator satisfies Leibnitz rule.

The proof of the following proposition is slightly beyond the scope of this course, but the proposition is important because it characterizes vector fields in a coordinate independent manner.

1.8 Proposition Any linear derivation on $C^\infty(\mathbf{R}^n)$ is a vector field.

This result allows us to identify vector fields with linear derivations. This step is a big departure from the usual concept of a "calculus" vector. To a differential geometer, a vector is a linear operator whose inputs are functions. At each point, the output of the operator is the directional derivative of the function in the direction of X.

Let $\mathbf{p} \in U$ be a point and let x^i be the coordinate functions in U. Suppose that $X_p = (\mathbf{X}, \mathbf{p})$, where the components of the Euclidean vector $\mathbf{X}$ are $a^1, \ldots, a^n$. Then, for any function f, the tangent vector X_p operates on f according to the formula

$$X_p(f) = \sum_{i=1}^{n} a^i (\frac{\partial f}{\partial x^i})(p). \tag{1.5}$$

It is therefore natural to identify the tangent vector X_p with the differential operator

$$\begin{aligned} X_p &= \sum_{i=1}^{n} a^i (\frac{\partial}{\partial x^i})(p) \\ X_p &= a^1 (\frac{\partial}{\partial x^1})_p + \ldots + a^n (\frac{\partial}{\partial x^n})_p . \end{aligned} \tag{1.6}$$

Notation: We will be using Einstein's convention to suppress the summation symbol whenever an expression contains a repeated index. Thus, for example, the equation above could be simply written

$$X_p = a^i (\frac{\partial}{\partial x^i})_p . \tag{1.7}$$

This equation implies that the action of the vector X_p acts on the coordinate functions x^i yields the components a^i of the vector. In elementary treatments, vectors are often identified with the components of the vector and this may cause some confusion.

The difference between a tangent vector and a vector field is that in the latter case, the coefficients a^i are smooth functions of x^i. The quantities

$$(\frac{\partial}{\partial x^1})_p, \ldots, (\frac{\partial}{\partial x^n})_p ,$$

form a basis for the tangent space $T_p(\mathbf{R}^n)$ at the point $\mathbf{p}$, and any tangent vector can be written as a linear combination of these basis vectors. The quantities a^i are called the **contravariant** components of the tangent vector. Thus, for example, the Euclidean vector in $\mathbf{R}^3$

$$\mathbf{X} = 3\mathbf{i} + 4\mathbf{j} - 3\mathbf{k}$$

located at a point $\mathbf{p}$, would correspond to the tangent vector

$$X_p = 3(\frac{\partial}{\partial x})_p + 4(\frac{\partial}{\partial y})_p - 3(\frac{\partial}{\partial z})_p ,$$

1.2 Curves in $\mathbf{R}^3$

1.9 Definition A **curve** $\alpha(t)$ in $\mathbf{R}^3$ is a C^∞ map from an open subset of $\mathbf{R}$ into $\mathbf{R}^3$. The curve assigns to each value of a parameter $t \in \mathbf{R}$, a point $(x^1(t), x^2(t), x^2(t))$ in $\mathbf{R}^3$

$$\begin{aligned} U \in \mathbf{R} &\stackrel{\alpha}{\longmapsto} \mathbf{R}^3 \\ t &\longmapsto \alpha(t) = (x^1(t), x^2(t), x^2(t)) \end{aligned}$$

One may think of the parameter t as representing time, and the curve α as representing the trajectory of a moving point particle.

1.10 Example Let

$$\alpha(t) = (a_1 t + b_1, a_2 t + b_2, a_3 t + b_3).$$

This equation represents a straight line passing through the point $\mathbf{p} = (b_1, b_2, b_3)$, in the direction of the vector $\mathbf{v} = (a_1, a_2, a_3)$.

1.11 Example Let

$$\alpha(t) = (a \cos \omega t, a \sin \omega t, bt).$$

This curve is called a circular helix. Geometrically, we may view the curve as the path described by the hypothenuse of a triangle with slope b, which is wrapped around a circular cylinder of radius a. The projection of the helix onto the xy-plane is a circle and the curves rises at a constant rate in the z-direction.

Occasionally we will revert to the position vector notation

$$\mathbf{x}(t) = (x^1(t), x^2(t), x^3(t)) \tag{1.8}$$

which is more prevalent in vector calculus and elementary physics textbooks. Of course, what this notation really means is

$$x^i(t) = (x^i \circ \alpha)(t), \tag{1.9}$$

where x^i are the coordinate slot functions in an open set in $\mathbf{R}^3$.

1.12 Definition The derivative $\alpha'(t)$ of the curve is called the **velocity** vector and the second derivative $\alpha''(t)$ is called the **acceleration**. The length $v = ||\alpha'(t)||$ of the velocity vector is called the speed of the curve. The components of the velocity vector are simply given by

$$\mathbf{V}(t) = \frac{d\mathbf{x}}{dt} = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt} \right), \tag{1.10}$$

and the speed is

$$v = \sqrt{\left(\frac{dx^1}{dt} \right)^2 + \left(\frac{dx^2}{dt} \right)^2 + \left(\frac{dx^3}{dt} \right)^2} \tag{1.11}$$

The differential of $d\mathbf{x}$ of the classical position vector given by

$$d\mathbf{x} = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt} \right) dt \tag{1.12}$$

is called an **infinitesimal tangent vector**, and the norm $||d\mathbf{x}||$ of the infinitesimal tangent vector is called the differential of arclength ds. Clearly we have

$$ds = ||d\mathbf{x}|| = v dt \tag{1.13}$$

As we will see later in this text, the notion of infinitesimal objects needs to be treated in a more rigorous mathematical setting. At the same time, we must not discard the great intuitive value of this notion as envisioned by the masters who invented of Calculus; even at the risk of some possible confusion! Thus, whereas in the more strict sense of modern differential geometry, the velocity vector is really a tangent vector and hence it should be viewed as a linear derivation on the space of functions, it is helpful to regard $d\mathbf{x}$ as a traditional vector which, at the infinitesimal level, gives a linear approximation to the curve.

If f is any smooth function on $\mathbf{R}^3$, we formally define $\alpha'(t)$ in local coordinates by the formula

$$\alpha'(t)(f)\mid_{\alpha(t)}= \frac{d}{dt}(f\circ\alpha)\mid_t . \tag{1.14}$$

The modern notation is more precise, since it takes into account that the velocity has a vector part as well as point of application. Given a point on the curve, the velocity of the curve acting on a function, yields the directional derivative of that function in the direction tangential to the curve at the point in question.

The diagram below provides a more geometrical interpretation of the the velocity vector formula (1.14). The map $\alpha(t)$ from $\mathbf{R}$ to $\mathbf{R}^3$ induces a map α_* from the tangent space of $\mathbf{R}$ to the tangent space of $\mathbf{R}^3$. The image $\alpha_*(\frac{d}{dt})$ in $T\mathbf{R}^3$ of the tangent vector $\frac{d}{dt}$ is what we call $\alpha'(t)$

$$\alpha_*(\frac{d}{dt}) = \alpha'(t).$$

Since $\alpha'(t)$ is a tangent vector in $\mathbf{R}^3$, it acts on functions in $\mathbf{R}^3$. The action of $\alpha'(t)$ on a function f on $\mathbf{R}^3$ is the same as the action of $\frac{d}{dt}$ on the composition $f\circ\alpha$. In particular, if we apply $\alpha'(t)$ to the coordinate functions x^i, we get the components of the the tangent vector, as illustrated

$$\begin{array}{ccccc} \frac{d}{dt}\in T\mathbf{R} & \stackrel{\alpha_*}{\longmapsto} & T\mathbf{R}^3 \ni \alpha'(t) & & \\ \downarrow & & \downarrow & & \\ \mathbf{R} & \stackrel{\alpha}{\longmapsto} & \mathbf{R}^3 & \stackrel{x^i}{\longmapsto} & \mathbf{R} \end{array}$$

$$\alpha'(t)(x^i)\mid_{\alpha(t)}= \frac{d}{dt}(x^i\circ\alpha)\mid_t . \tag{1.15}$$

The map α_* on the tangent spaces induced by the curve α is called the **push-forward**. Many authors use the notation $d\alpha$ to denote the push-forward, but we prefer to avoid this notation because most students fresh out of advanced calculus have not yet been introduced to the interpretation of the differential as a linear isomorphism on tangent spaces.

1.13 Definition

If $t = t(s)$ is a smooth, real valued function and $\alpha(t)$ is a curve in $\mathbf{R}^3$, we say that the curve $\beta(s) = \alpha(t(s))$ is a **reparametrization** of α

A common reparametrization of curve is obtained by using the arclength as the parameter. Using this reparametrization is quite natural, since we know from basic physics that the rate of change of the arclength is what we call speed

$$v = \frac{ds}{dt} = ||\alpha'(t)||. \tag{1.16}$$

The arc length is obtained by integrating the above formula

$$s = \int ||\alpha'(t)||\, dt = \int \sqrt{\left(\frac{dx^1}{dt}\right)^2 + \left(\frac{dx^2}{dt}\right)^2 + \left(\frac{dx^3}{dt}\right)^2}\, dt \tag{1.17}$$

In practice it is typically difficult to actually find an explicit arclength parametrization of a curve since not only does one have calculate the integral, but also one needs to be able to find the inverse function t in terms of s. On the other hand, from a theoretical point of view, arclength parametrizations are ideal since any curve so parametrized, has unit speed. The proof of this fact is a simple application of the chain rule and the inverse function theorem.

$$\beta'(s) \quad = \quad [\alpha(t(s))]'$$

$$
\begin{aligned}
&= \alpha'(t(s))t'(s) \\
&= \alpha'(t(s))\frac{1}{s'(t(s))} \\
&= \frac{\alpha'(t(s))}{||\alpha'(t(s))||},
\end{aligned}
$$

and any vector divided by its length is a unit vector. Leibnitz notation makes this even more self evident

$$
\begin{aligned}
\frac{d\mathbf{x}}{ds} &= \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \frac{\frac{d\mathbf{x}}{dt}}{\frac{ds}{dt}} \\
&= \frac{\frac{d\mathbf{x}}{dt}}{||\frac{d\mathbf{x}}{dt}||}
\end{aligned}
$$

1.14 Example Let $\alpha(t) = (a\cos\omega t, a\sin\omega t, bt)$. Then

$$\mathbf{V}(t) = (-a\omega\sin\omega t, a\omega\cos\omega t, b),$$

$$
\begin{aligned}
s(t) &= \int_0^t \sqrt{(-a\omega\sin\omega u)^2 + (a\omega\cos\omega u)^2 + b^2}\, du \\
&= \int_0^t \sqrt{a^2\omega^2 + b^2}\, du \\
&= ct, \quad \text{where,} \quad c = \sqrt{a^2\omega^2 + b^2}.
\end{aligned}
$$

The helix of unit speed is then given by

$$\beta(s) = (a\cos\frac{\omega s}{c}, a\sin\frac{\omega s}{c}, b\frac{\omega s}{c}).$$

Frenet Frames

Let $\beta(s)$ be a curve parametrized by arc length and let T(s) be the vector

$$T(s) = \beta'(s). \tag{1.18}$$

The vector $T(s)$ is tangential to the curve and it has unit length. Hereafter, we will call T the unit **Tangent** vector. Differentiating the relation

$$T \cdot T = 1, \tag{1.19}$$

we get

$$2T \cdot T' = 0, \tag{1.20}$$

so we conclude that the vector T' is orthogonal to T. Let N be a unit vector orthogonal to T, and let κ be the scalar such that

$$T'(s) = \kappa N(s). \tag{1.21}$$

We call N the unit **normal** to the curve, and κ the **curvature**. Taking the length of both sides of last equation, and recalling that N has unit length, we deduce that

$$\kappa = ||T'(s)|| \tag{1.22}$$

It makes sense to call κ the curvature, since if T is a unit vector, then $T'(s)$ is not zero only if the direction of T is changing. The rate of change of the direction of the tangent vector is precisely what one would expect to measure how much a curve is curving. In particular, it $T' = 0$ at a particular point, we expect that at that point, the curve is locally well approximated by a straight line.

We now introduce a third vector

$$B = T \times N, \tag{1.23}$$

which we will call the **binormal** vector. The triplet of vectors (T, N, B) forms an orthonormal set; that is,

$$\begin{aligned} T \cdot T = N \cdot N = B \cdot B = 1 \\ T \cdot N = T \cdot B = N \cdot B = 0. \end{aligned} \tag{1.24}$$

If we differentiate the relation $B \cdot B = 1$, we find that $B \cdot B' = 0$, hence B' is orthogonal to B. Furthermore, differentiating the equation $T \cdot B = 0$, we get

$$B' \cdot T + B \cdot T' = 0.$$

rewriting the last equation

$$B' \cdot T = -T' \cdot B = -\kappa N \cdot B = 0,$$

we also conclude that B' must also be orthogonal to T. This can only happen if B' is orthogonal to the TB-plane, so B' must be proportional to N. In other words, we must have

$$B'(s) = -\tau N(s) \tag{1.25}$$

for some quantity τ, which we will call the **torsion**. The torsion is similar to the curvature in the sense that it measures the rate of change of the binormal. Since the binormal also has unit length, the only way one can have a non-zero derivative is if B is changing directions. The quantity B' then measures the rate of change in the up and down direction of an observer which is moving with the curve always facing forward in the direction of the tangent vector. The binormal B is something like the flag in the back of sand dune buggy.

The set of basis vectors $\{T, N, B\}$ is called the **Frenet Frame** or the **repere mobile** (moving frame). The advantage of this basis over the fixed $(\mathbf{i},\mathbf{j},\mathbf{k})$ basis is that the Frenet frame is naturally adapted to the curve. It propagates along with the curve with the tangent vector always pointing in the direction of motion, whereas, the normal and binormal vectors point towards the directions in which the curve is tending to curve. In particular, a complete description of how the curve is curving can be obtained by calculating the rate of change of the frame in terms of the frame itself.

1.15 Theorem Let $\beta(s)$ be a unit speed curve with curvature κ and torsion τ. Then

$$\begin{array}{lcccc} T' & = & & \kappa N & \\ N' & = & -\kappa T & & \tau B \\ B' & = & & -\tau B & \end{array} . \tag{1.26}$$

Proof: We only need to establish the equation for N'. Differentiating the equation $N \cdot N = 1$, we get $2N \cdot N' = 0$, so N' is orthogonal to N. Hence, N' must be a linear combination of T and B.

$$N' = aT + bB.$$

Taking the dot product of last equation with T and B respectively, we see that

$$a = N' \cdot T, \text{ and } \quad b = N' \cdot B.$$

On the other hand, differentiating the equations $N \cdot T = 0$, and $N \cdot B = 0$, we find that

$$\begin{aligned} N' \cdot T &= -N \cdot T' = -N \cdot (\kappa N) = -\kappa \\ N' \cdot B &= -N \cdot B' = -N \cdot (-\tau N) = \tau. \end{aligned}$$

We conclude that $a = -\kappa$, $b = \tau$, and thus

$$N' = -\kappa T + \tau B.$$

The Frenet frame equations (1.26) can also be written in matrix form as shown below.

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{1.27}$$

The group theoretic significance of this matrix formulation is quite important and we will come back to this later when we talk about general orthonormal frames. At this time, perhaps it suffices to point out that the appearance of an antisymmetric matrix in the Frenet equations is not at all coincidental.

The following theorem provides a computational method to calculate the curvature and torsion directly from the equation of a given unit speed curve.

1.16 Proposition Let $\beta(s)$ be a unit speed curve with curvature $\kappa > 0$ and torsion τ. Then

$$\begin{aligned} \kappa &= \|\beta''(s)\| \\ \tau &= \frac{\beta' \cdot [\beta'' \times \beta''']}{\beta'' \cdot \beta''} \end{aligned} \tag{1.28}$$

Proof: If $\beta(s)$ is a unit speed curve, we have $\beta'(s) = T$. Then

$$\begin{aligned} T' = \beta''(s) &= \kappa N, \\ \beta'' \cdot \beta'' &= (\kappa N) \cdot (\kappa N), \\ \beta'' \cdot \beta'' &= \kappa^2 \\ \kappa^2 &= \|\beta''\|^2 \end{aligned}$$

$$\begin{aligned} \beta'''(s) &= \kappa' N + \kappa N' \\ &= \kappa' N + \kappa(-\kappa T + \tau B) \\ &= \kappa' N + -\kappa^2 T + \kappa\tau B. \end{aligned}$$

$$\begin{aligned} \beta' \cdot [\beta'' \times \beta'''] &= T \cdot [\kappa N \times (\kappa' N + -\kappa^2 T + \kappa\tau B)] \\ &= T \cdot [\kappa^3 B + \kappa^2 \tau T] \\ &= \kappa^2 \tau \\ \tau &= \frac{\beta' \cdot [\beta'' \times \beta''']}{\kappa^2} \\ &= \frac{\beta' \cdot [\beta'' \times \beta''']}{\beta'' \cdot \beta''} \end{aligned}$$

1.17 Example Consider a circle of radius r whose equation is given by

$$\alpha(t) = (r \cos t, r \sin t, 0).$$

Then,

$$\begin{aligned}
\alpha'(t) &= (-r\sin t, r\cos t, 0) \\
\|\alpha'(t)\| &= \sqrt{(-r\sin t)^2 + (r\cos t)^2 + 0^2} \\
&= \sqrt{r^2(\sin^2 t + \cos^2 t)} \\
&= r.
\end{aligned}$$

Therefore $ds/dt = r$ and $s = rt$, which we recognize as the formula for the length of an arc of circle of radius t, subtended by a central angle whose measure is t radians. We conclude that

$$\beta(s) = (-r\sin\frac{s}{r}, r\cos\frac{s}{r}, 0)$$

is a unit speed reparametrization. The curvature of the circle can now be easily computed

$$\begin{aligned}
T &= \beta'(s) = (-\cos\frac{s}{r}, -\sin\frac{s}{r}, 0) \\
T' &= (\frac{1}{r}\sin\frac{s}{r}, -\frac{1}{r}\cos\frac{s}{r}, 0) \\
\kappa &= \|\beta''\| = \|T'\| \\
&= \sqrt{\frac{1}{r^2}\sin^2\frac{s}{r} + \frac{1}{r^2}\cos^2\frac{s}{r} + 0^2} \\
&= \sqrt{\frac{1}{r^2}(\sin^2\frac{s}{r} + \cos^2\frac{s}{r})} \\
&= \frac{1}{r}
\end{aligned}$$

This is a very simple but important example. The fact that for a circle of radius r the curvature is $\kappa = 1/r$ could not be more intuitive. A small circle has large curvature and a large circle has small curvature. As the radius of the circle approaches infinity, the circle locally looks more and more like a straight line, and the curvature approaches to 0. If one were walking along a great circle on a very large sphere (like the earth) one would be perceive the space to be locally flat.

1.18 Proposition Let $\alpha(t)$ be a curve of velocity $\mathbf{V}$, acceleration A, speed v and curvature κ, then

$$\begin{aligned}
\mathbf{V} &= vT, \\
\mathbf{A} &= \frac{dv}{dt}T + v^2\kappa N.
\end{aligned} \tag{1.29}$$

Proof: Let $s(t)$ be the arclength and let $\beta(s)$ be a unit speed reparametrization. Then $\alpha(t) = \beta(s(t))$ and by the chain rule

$$\begin{aligned}
\mathbf{V} &= \alpha'(t) \\
&= \beta'(s(t))s'(t) \\
&= vT
\end{aligned}$$

$$\begin{aligned}
\mathbf{A} &= \alpha''(t) \\
&= \frac{dv}{dt}T + vT'(s(t))s'(t) \\
&= \frac{dv}{dt}T + v(\kappa N)v \\
&= \frac{dv}{dt}T + v^2\kappa N
\end{aligned}$$

Equation 1.29 is important in physics. The equation states that a particle moving along a curve in space feels a component of acceleration along the direction of motion whenever there is a change of speed, and a centripetal acceleration in the direction of the normal whenever it changes direction. The **centripetal acceleration** and any point is

$$a = v^3\kappa = \frac{v^2}{r}$$

where r is the radius of a circle which has maximal tangential contact with the curve at the point in question. This tangential circle is called the **osculating circle**. The osculating circle can be envisioned by a limiting process similar to that of the tangent to a curve in differential calculus. Let $\mathbf{p}$ be point on the curve, and let $\mathbf{q}_1$ and $\mathbf{q}_2$ two nearby points. The three points determine a circle uniquely. This circle is a "secant" approximation to the tangent circle. As the points $\mathbf{q}_1$ and $\mathbf{q}_2$ approach the point $\mathbf{p}$, the "secant" circle approaches the osculating circle. The osculating circle always lies in the the TN-plane, which by analogy, is called the osculating plane.

1.19 Example (Helix)

$$\begin{aligned}
\beta(s) &= (a\cos\frac{\omega s}{c}, a\sin\frac{\omega s}{c}, \frac{bs}{c}), \quad \text{where } c = \sqrt{a^2\omega^2+b^2} \\
\beta'(s) &= (-\frac{a\omega}{c}\sin\frac{\omega s}{c}, \frac{a\omega}{c}\cos\frac{\omega s}{c}, \frac{b}{c}) \\
\beta''(s) &= (-\frac{a\omega^2}{c^2}\cos\frac{\omega^2 s}{c}, -\frac{a\omega^2}{c^2}\sin\frac{\omega s}{c}, 0) \\
beta'''(s) &= (-\frac{a\omega^3}{c^3}\cos\frac{\omega^2 s}{c}, -\frac{a\omega^3}{c^3}\sin\frac{\omega s}{c}, 0) \\
\kappa^2 &= \beta''\cdot\beta'' \\
&= \frac{a^2\omega^4}{c^4} \\
\kappa &= \pm\frac{a\omega^2}{c^2} \\
\tau &= \frac{(\beta'\beta''\beta''')}{\beta''\cdot\beta''} \\
&= \frac{b}{c}\begin{bmatrix} -\frac{a\omega^2}{c^2}\cos\frac{\omega s}{c} & -\frac{a\omega^2}{c^2}\sin\frac{\omega s}{c} \\ \frac{a\omega^3}{c^2}\sin\frac{\omega s}{c} & -\frac{a\omega^3}{c^2}\cos\frac{\omega s}{c}\end{bmatrix}\frac{c^4}{a^2\omega^4}. \\
&= \frac{b}{c}\frac{a^2\omega^5}{c^5}\frac{c^4}{a^2\omega^4}
\end{aligned}$$

Simplifying the last expression and substituting the value of c, we get

$$\begin{aligned}
\tau &= \frac{b\omega}{a^2\omega^2+b^2} \\
\kappa &= \pm\frac{a\omega^2}{a^2\omega^2+b^2}
\end{aligned}$$

Notice that if $b = 0$ the helix collapses to a circle in the xy-plane. In this case the formulas above reduce to $\kappa = 1/a$ and $\tau = 0$. The ratio $\kappa/\tau = a\omega/b$ is particularly simple. Any curve where $\kappa/\tau = constant$ is called a helix, of which the circular helix is a special case.

1.20 Example (Plane curves) Let $\alpha(t) = (x(t), y(t), 0)$. Then

$$\alpha' = (x', y', 0)$$

$$\begin{aligned}
\alpha'' &= (x'', y'', 0) \\
\alpha''' &= (x''', y''', 0) \\
\kappa &= \frac{||\alpha' \times \alpha''||}{||\alpha'||^3} \\
&= \frac{| x'y'' - y'x'' |}{(x'^2 + y'^2)^{3/2}} \\
\tau &= 0
\end{aligned}$$

1.21 Example (Cornu Spiral) Let $\beta(s) = (x(s), y(s), 0)$, where

$$\begin{aligned}
x(s) &= \int_0^s \cos \frac{t^2}{2c^2}\, dt \\
y(s) &= \int_0^s \sin \frac{t^2}{2c^2}\, dt.
\end{aligned} \tag{1.30}$$

Then, using the fundamental theorem of calculus, we have

$$\beta'(s) = (\cos \frac{s^2}{2c^2}, \sin \frac{t^2}{2c^2}, 0),$$

Since $||\beta' = v = 1||$, the curve is of unit speed, and s is indeed the arc length. The curvature is of the Cornu spiral is given by

$$\begin{aligned}
\kappa &= | x'y'' - y'x'' | = (\beta' \cdot \beta')^{1/2} \\
&= || - \frac{s}{c^2} \sin \frac{t^2}{2c^2}, \frac{s}{c^2} \cos \frac{t^2}{2c^2}, 0|| \\
&= \frac{s}{c^2}.
\end{aligned}$$

The integrals (1.30) defining the coordinates of the Cornu spiral are the classical **Frenel Integrals**. These functions, as well as the spiral itself arise in the computation of the diffraction pattern of a coherent beam of light by a straight edge.

In cases where the given curve $\alpha(t)$ is not of unit speed, the following proposition provides formulas to compute the curvature and torsion in terms of α

1.22 Proposition If $\alpha(t)$ is a regular curve in $\mathbf{R}^3$, then

$$\kappa^2 = \frac{||\alpha' \times \alpha''||^2}{||\alpha'||^6} \tag{1.31}$$

$$\tau = \frac{(\alpha'\alpha''\alpha''')}{||\alpha' \times \alpha''||^2}, \tag{1.32}$$

where $(\alpha'\alpha''\alpha''')$ is the triple vector product $[\alpha \times' \alpha''] \cdot \alpha'''$.

Proof:

$$\begin{aligned}
\alpha' &= vT \\
\alpha'' &= v'T + v^2\kappa N \\
\alpha''' &= (v^2\kappa)N'((s(t))s'(t) + \ldots \\
&= v^3\kappa N' + \ldots \\
&= v^3\kappa\tau B + \ldots
\end{aligned}$$

The other terms are unimportant here because as we will see $\alpha' \times \alpha''$ is proportional to B

$$\begin{aligned}
\alpha' \times \alpha'' &= v^3\kappa(T \times N) = v^3\kappa B \\
||\alpha' \times \alpha''|| &= v^3\kappa \\
\kappa &= \frac{||\alpha' \times \alpha''||}{v^3} \\
(\alpha' \times \alpha'') \cdot \alpha''' &= v^6\kappa^2\tau \\
\tau &= \frac{(\alpha'\alpha''\alpha''')}{v^6\kappa^2} \\
&= \frac{(\alpha'\alpha''\alpha''')}{||\alpha' \times \alpha''||^2}
\end{aligned}$$

1.3 Fundamental Theorem of Curves

Some geometrical insight into the significance of the curvature and torsion can be gained by considering the Taylor series expansion of an arbitrary unit speed curve $\beta(s)$ about $s = 0$

$$\beta(s) = \beta(0) + \beta'(0)s + \frac{\beta''(0)}{2!}s^2 + \frac{\beta'''(0)}{3!}s^3 + \ldots \tag{1.33}$$

Since we are assuming that s is an arclength parameter,

$$\begin{aligned}
\beta'(0) &= T(0) = T_0 \\
\beta''(0) &= (\kappa N)(0) = \kappa_0 N_0 \\
\beta'''(0) &= (-\kappa^2 T + \kappa' N + \kappa\tau B)(0) = -\kappa_0^2 T_0 + \kappa_0' N_0 + \kappa_0\tau_0 B_0
\end{aligned}$$

Keeping only the lowest terms in the components of T, N, and B, we get the first order Frenet approximation to the curve

$$\beta(s) \doteq \beta(0) + T_0 s + \frac{1}{2}\kappa_0 N_0 s^2 + \frac{1}{6}\kappa_0\tau_0 B_0 s^3. \tag{1.34}$$

The first two terms represent the linear approximation to the curve. The first three terms approximate the curve by a parabola which lies in the osculating plane (TN-plane). If $\kappa_0 = 0$, then locally the curve looks like a straight line. If $\tau_0 = 0$, then locally the curve is a plane curve which lies on the osculating plane. In this sense, the curvature measures the deviation of the curve from being a straight line and the torsion (also called the second curvature) measures the deviation of the curve from being a plane curve.

1.23 Theorem (Fundamental Theorem of Curves) Let $\kappa(s)$ and $\tau(s)$, $(s > 0)$ be any two analytic functions. Then there exists a unique curve (unique up to its position in $\mathbf{R}^3$) for which s is the arclength, $\kappa(s)$ its curvature and $\tau(s)$ its torsion.
Proof: Pick a point in $\mathbf{R}^3$. By an appropriate affine transformation, we may assume that this point is the origin. Pick any orthogonal frame $\{T, NB\}$. The curve is then determined uniquely by its Taylor expansion in the Frenet frame as in equation (1.34).

1.24 Remark It is possible to prove the theorem just assuming that $\kappa(s)$ and $\tau(s)$ are continuous. The proof however, becomes much harder and we refer the reader to other standard texts for the proof.

1.25 Proposition A curve with $\kappa = 0$ is part of a straight line.
we leave the proof as an exercise.

1.26 Proposition A curve $\alpha(t)$ with $\tau = 0$ is a plane curve.
Proof: If $\tau = 0$, then $(\alpha'\alpha''\alpha''') = 0$. This means that the three vectors α', α'', and α''' are linearly dependent and hence there exist functions $a_1(s)$,$a_2(s)$ and $a_3(s)$ such that

$$a_3\alpha''' + a_2\alpha'' + a_1\alpha' = 0.$$

This linear homogeneous equation will have a solution of the form

$$\alpha = \mathbf{c}_1\alpha_1 + \mathbf{c}_2\alpha_2 + \mathbf{c}_3, \quad c_i = \text{constant vectors.}$$

This curve lies in the plane

$$(\mathbf{x} - \mathbf{c}_3) \cdot \mathbf{n} = 0, \quad \text{where} \quad \mathbf{n} = \mathbf{c}_1 \times \mathbf{c}_2$$