DEMIDEC RESOURCES

The impossible will take only a little bit longer*.

Mathematics Resource Part II of III: Geometry

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BY

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PROOFREADING

LEAH SLOAN

IN HER FOURTH YEAR WITH DEMIDEC

* A QUOTE OFFERED ENTERING FIRST-YEAR STUDENTS BY A NOBEL PRIZE WINNING CHEMIST AT A CERTAIN IVY LEAGUE UNIVERSITY, ATTRIBUTED TO THE ARMY CORPS OF ENGINEERS DURING WORLD WAR II: "THE DIFFICULT IS BUT THE WORK OF A MOMENT. THE IMPOSSIBLE WILL TAKE ONLY A LITTLE BIT LONGER." IT MAKES A GOOD DECATHLON MAXIM, TOO.

DEMIDEC

RESOURCES AND EXAMS

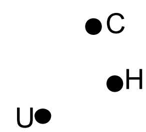
GEOMETRY

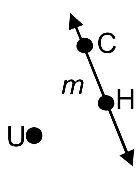
INTRODUCTION TO LINES, PLANES, AND ANGLES

Point **Line** **Ray** **Angle** **Vertex**

Geometry is a special type of mathematic that has fascinated man for centuries. Egyptian hieroglyphics, Greek sculpture, and the Roman arch all made use of specific shapes and their properties. Geometry is important not only as a tool in construction and as a component of the appearance of the natural world, but also as a branch of mathematics that requires us to make logical constructs to deduce what we know.

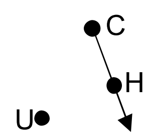
In the study of geometry, most terms are rigorously defined and used to convey specific conditions and characteristics. A few terms, however, exist primarily as concepts with no strict mathematical definition. The **point** is the first of these ideas. A point is represented on paper as a dot. It has no actual size; it simply represents a unique place. To the right are shown three points, named C, H, and U (capital letters are conventionally used to label and represent points in text).



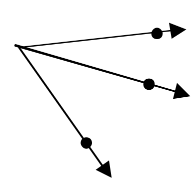


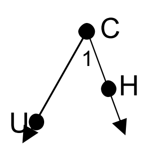
The next geometric idea is that of the **line**. A line is a one-dimensional object that extends infinitely in both directions. It contains points and is represented as a line on paper with arrows at both ends to indicate that it does indeed extend indefinitely. Lines are named either with two points that lie on the line or with a script letter. $\overleftrightarrow{CH}$, $\overleftrightarrow{HC}$, and *m* all refer to the same line in the diagram at left.

Similar to the line is the **ray**. Definitions for rays vary, but in general, a ray can be thought of as being similar to a line, but extending infinitely in only ONE direction. It has an endpoint and then extends infinitely in any one direction away from that endpoint. You might want to think of this endpoint as a "beginning-point." Rays are named similarly to lines, but their endpoints must be listed first. Because of this, $\overrightarrow{CH}$ and $\overrightarrow{HC}$ do not refer to the same ray. $\overrightarrow{CH}$ is shown at right. $\overrightarrow{HC}$ would point the other direction.



Last in our introduction to basic geometry is the **angle**. The definition for an angle remains pretty consistent from textbook to textbook; it is the figure formed by two rays with a common endpoint, known as the angle's **vertex**. An angle is named in one of three ways: (1) An angle can be named with the vertex point if it is the only angle with that vertex. (2) An angle can be named with a number that is written inside the angle. (3) Most commonly, an angle is named with three points, the center point representing the vertex. In the drawing at left, ∠1, ∠C, ∠UCH, and ∠HCU all refer to the same angle. In the drawing at right, ∠5 and ∠BAC refer to the same angle while ∠6 and ∠CAD refer to the same angle. ∠BAD then refers to an entirely different angle. Note that ∠A cannot be used to refer to an angle in the diagram at right because there are several angles that have vertex A. There would be no way of knowing which you meant.



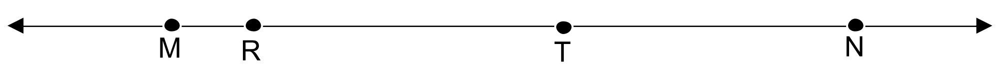


MORE ON LINES AND RAYS, AND A BIT ON PLANES

Distance **Ruler Postulate** **Between** **Collinear**
Line Segment **Midpoint** **Congruent** **Plane**
Coplanar

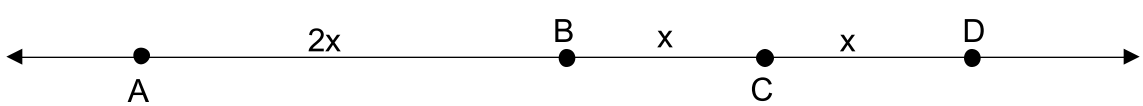
Now that we have defined and discussed lines, rays, angles, points, and vertices [plural of vertex], we can begin to set up a framework for geometry. Between any two points, there must be a positive **distance**. Even better, between any two points A and B on $\overleftrightarrow{AB}$, we can write the distance as AB or BA. The **Ruler Postulate** then states that we can set up a one-to-one correspondence between positive numbers and line distances. In case that sounds like Greek[1], it means that all distances can have numbers assigned to them, and we'll never run out of numbers in case we have a new distance that is so-and-so times as long, or only 75% as long, etc. Think of the longest distance you can. Maybe a million million miles? Well, now add one. A million million and one miles is indeed longer than a million million miles. You can keep doing this forever.

Next, we say that a point B is **between** two others A and C if all three points are **collinear** (lying on the same line) and $AB + BC = AC$. For example, given $\overleftrightarrow{MN}$ as shown, (remember that we could also call it $\overleftrightarrow{MR}, \overleftrightarrow{MT}, \overleftrightarrow{NT}$, or a whole slew of other names[2]),



We might be able to assign distances such that MR = 3, RT = 15, and TN = 13. Intuitively, RN would then equal 28 (because 15 + 13 = 28), and MT would equal 18 (because 3 + 15 = 18). (Can you find the distance MN?) Also note that R is between M and T, T is between R and N, T is between M and N, and R is between M and N. There are many true statements we could make concerning the betweenness properties on this line. In addition, M, R, T, and N are four collinear points.

> *Example*:
> Four points A, B, C, and D are collinear and lie on the line in that order. If B is the midpoint of $\overline{AD}$ and C is the midpoint of $\overline{BD}$, what is AD in terms of CD?
>
> *Solution*:
> Don't just try to think it through; draw it out. The drawing is somewhat similar to the one above with points M, R, T, and N. Since C is the midpoint of $\overline{BD}$, $BD = 2 \cdot CD$ and since B is the midpoint of $\overline{AD}$, $AD = 2 \cdot BD = 4 \cdot CD$.



Earlier, we came to a consensus[3] concerning what exactly lines and rays are. Now we explicitly define a new concept: the **line segment**. A line segment consists of two points on a line, along with all points between them. (Note that geometry is a very logical and tiered branch of mathematics; we had to define what it meant to be *between* before we could use that word in a definition.) The notation for line segments is similar to that for lines, but there are now no arrows over the ends of

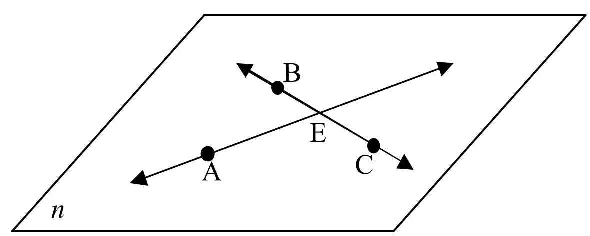
[1] As my high school physics teacher, Mr. Atman, pointed out to me, "the farther you go in your schooling, the more important dead Greek guys will become." It's true. We'll get to Pythagoras soon, and remember that we are studying Euclidean geometry.
[2] Finally, a time comes when name-calling is a good thing! It's amazing the things we get to do in mathematics. Go ahead and have fun with it; I promise the line won't mind.
[3] Okay, technically, only I came to the consensus. Pretend you had some input on it.

the bar. We can also say that the **midpoint** B of a line segment $\overline{AC}$ is the point that divides AC in half; in other words, the point where AB = BC. It should make sense that every line segment has a unique midpoint dividing the original segment into two smaller **congruent** segments. Two things in geometry are congruent if they are exactly the same size: two line segments are congruent if their lengths are equal, two angles are congruent if they have the same angle measure (more on that later), two figures are congruent if all their sides and angles are equal, etc. We write congruence almost like the "=" sign, but with a squiggly line on top. To say $\overline{AB}$ is congruent to $\overline{CD}$, we write $\overline{AB} \cong \overline{CD}$.

The final concept in this little section is the idea of the **plane**. Most people learn that a plane is a surface extending infinitely in two dimensions. The paper you are reading is a piece of a plane (provided it is not curled up and wrinkled). The geometric definition of a plane that many books offer is "a surface for which containment of two points A and B also implies containment of all points between A and B along $\overleftrightarrow{AB}$." Essentially, this is a convoluted but very precise way of saying that a plane must be completely flat and infinitely extended; otherwise, the line $\overleftrightarrow{AB}$ would extend beyond its edge or float above or below it.[4]

We have now discussed several basic terms, but there remain several key concepts concerning lines and planes that deserve emphasis. First is the idea that two distinct points determine a unique line. Think about this: if you take two points A and B anywhere in space, the one and only line through both A and B is $\overleftrightarrow{AB}$ (which we could also call $\overleftrightarrow{BA}$, of course). Second is the idea that two distinct lines intersect in at most one point. This is easy to conceptualize: to say that two lines are distinct is to say that they are not the same line, and two different lines must intersect each other once or not at all. Lastly, consider the idea that three non-collinear points determine a unique plane; related to that, also consider the concept that if two distinct lines intersect, there is exactly one plane containing both lines. Take three non-collinear points anywhere in space, and try to conceive a plane containing all three of them; there should only be one possible. (Do you understand why the points must be non-collinear? There are an infinite number of planes containing any given single line.) Similarly, if two distinct lines intersect, take the point of intersection, along with a distinct point from each line—this gives three non-collinear points, and a unique plane is determined!

Any plane can be named with either a script letter or three non-collinear points. Here, this plane could be called plane *n*, plane AEB, or even plane CEA. We could *not*, however, call it plane BEC because B, E, and C are collinear points and do not uniquely determine one plane. Note also now that $\overleftrightarrow{BC}$ and $\overleftrightarrow{AE}$ intersect in only one point, E. In this drawing[5], we say that A, B, C, and E are **coplanar** points because they all reside on the same plane. It should make sense to you that any three points must be coplanar, but four points are only sometimes coplanar. The fourth point may be "above" or "below" the plane created by the other three[6].

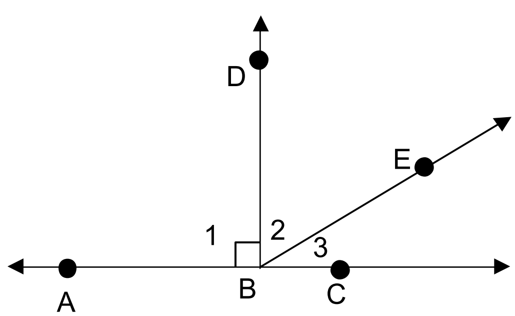


[4] Many formal definitions in mathematics seem very convoluted upon first (or even tenth) glance, but it's the fine points that make these definitions useful to mathematicians.
[5] My drawings, I'm afraid, are rather subpar. If something (as in this case) is supposed to have depth or more than two-dimensions, you'll just have to pretend. Pretend really hard.
[6] Why do you think that stools with four legs sometimes wobble while stools with three legs never do? It's because a stool with three legs can always have the ends of its legs match the plane of the floor while if four legs aren't all the same length, they won't be able to all stay coplanar on the floor. Cool, huh?
- Craig

EVEN MORE ON LINES, BUT FIRST A BIT ON ANGLES – A VERY *BIG* BIT

Protractor Postulate	**Straight Angle**	**Angle measure**	**Right Angle**
Acute Angle	**Obtuse Angle**	**Complementary**	**Supplementary**
Adjacent Angles	**Angle Bisector**	**Opposite Rays**	**Vertical Angles**
Perpendicular	**Parallel lines**	**Skew lines**	**Transversal**
Corresponding	**Alternate Interior**		

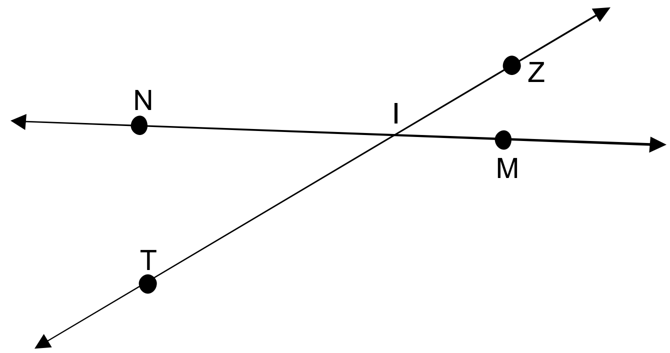
Just two pages ago, we discussed the Ruler Postulate, which says, in a nutshell, that arbitrary lengths can be assigned to line segments as long as they maintain correspondence with the positive numbers. Now, we discuss the **Protractor Postulate**, which states in a similar fashion that arbitrary measures can be assigned to *angles*, with 180° representing a **straight angle** and 360° equaling a full rotation.[7] The degree measure assigned to an angle is called the **angle measure**, and the measure of ∠1 is written "m∠1." While a straight angle is 180°, a **right angle** is an angle whose measure is 90°. Keep in mind that on diagrams, straight angles can be assumed when we see straight lines, but right angles conventionally cannot be assumed unless we see a little box in the angle. An **acute angle** is an angle whose measure is less than 90°, and an **obtuse angle** is an angle whose measure is between 90° and 180°. Two angles whose measures add to 90° are then said to be **complementary angles**, and two angles whose measures add to 180° are said to be **supplementary angles**. **Adjacent angles** are angles that share both a vertex and a ray. An **angle bisector** is a ray that divides an angle into two smaller congruent angles. This laundry list of terms is summed up in the picture below.



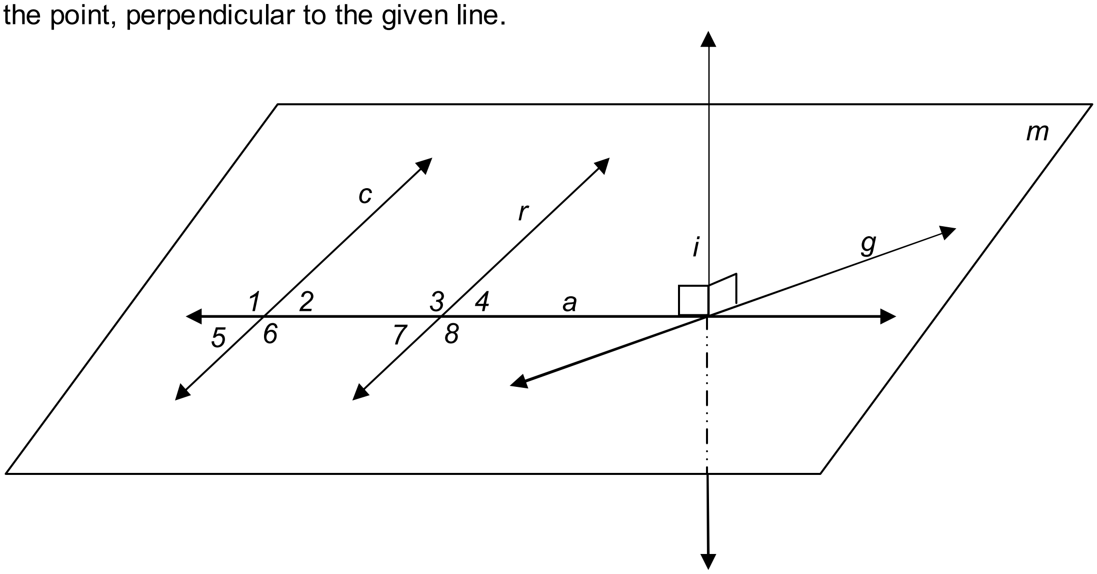
- ∠ABC is a straight angle. Notice it is flat.
- ∠ABD is a right angle. (We cannot assume a right angle on drawings, but the little box is a symbol indicating that ∠1 is a right angle. Whenever you see a box, feel confident you are dealing with a right angle.)
- ∠ABE is an obtuse angle—its measure is greater than 90°.
- ∠2 and ∠3 are acute angles—they measure less than 90°.
- ∠CBD is a right angle. (Because ∠1 is a right angle, ∠ABC is a straight angle, and 180-90=90.)
- ∠2 and ∠3 are complementary. Their measures add to 90° since ∠CBD is a right angle.
- ∠ABD and ∠CBD are supplementary. They combine to form straight angle ∠ABC, which measures 180°.
- ∠ABE and ∠3 are supplementary. They, too, combine to form straight angle ∠ABC, which measures 180°.
- ∠1 and ∠2 are adjacent because they share both a vertex and a ray, as are ∠ABE and ∠3, ∠1 and ∠DBC, and ∠2 and ∠3.
- ∠1 and ∠3 are not adjacent, however; they share vertex B but no common ray.
- $\overrightarrow{BD}$ is the angle bisector of ∠ABC because it divides the straight angle which measures 180° into two smaller congruent angles, each of which measures 90°.
- If m∠2 and m∠3 each happened to measure 45°, then $\overrightarrow{BE}$ would be the angle bisector of ∠DBC—splitting ∠DBC in half.

[7] As a student in geometry, I often wondered why 360° comprised a rotation. The most widely accepted theory is that it was arbitrarily defined at some point in mathematical history. This may have been because 360 is divisible by so many different numbers. Then again, it may well have been arbitrarily defined by a civilization that used a base-60 number system.

Now that the giant list of angle terms has been sorted through, there are just a couple more before we can finally move on. Suppose you started at a point and drew a ray in one direction, then drew another ray pointed in exactly the opposite direction. Two collinear rays such as these, with the same endpoint, are defined to be **opposite rays**, and two angles whose side rays are pairs of opposite rays are said to be **vertical angles**. For the drawing shown, we could say that $\overrightarrow{IT}$ and $\overrightarrow{IZ}$ are opposite rays, as are $\overrightarrow{IN}$ and $\overrightarrow{IM}$. There are then two pairs of vertical angles: the first is ∠NIZ and ∠MIT, and the second is ∠TIN and ∠MIZ. A critical theorem concerning vertical angles states that a pair of vertical angles is congruent.[8] Therefore, ∠NIT ≅ ∠MIZ and ∠NIZ ≅ ∠MIT.



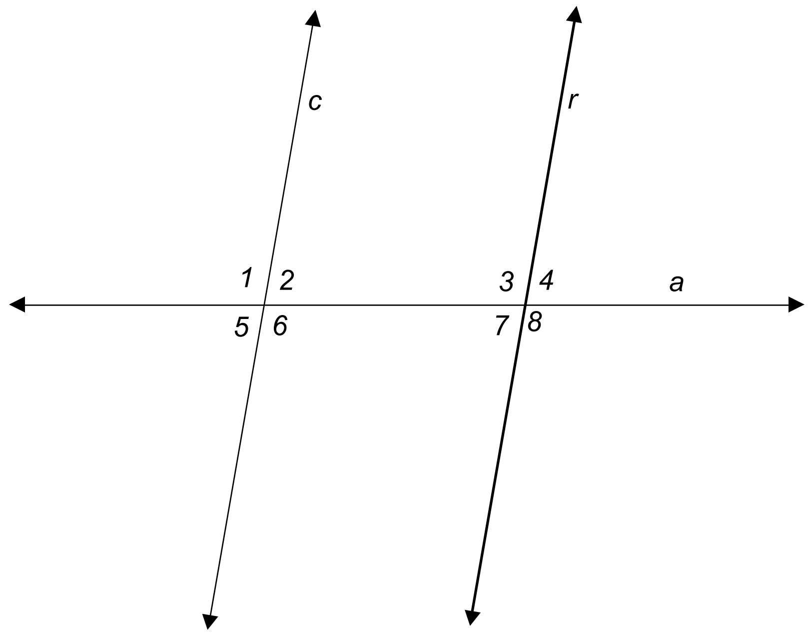
Two lines that intersect to form right angles are said to be **perpendicular** lines. Therefore, in the diagram on the previous page, $\overleftrightarrow{CB}$ and $\overleftrightarrow{BD}$ are perpendicular. In mathematical shorthand, we say $\overleftrightarrow{CB} \perp \overleftrightarrow{BD}$. Two lines that do not ever intersect are called either **parallel** if they are coplanar, or **skew** if they are not coplanar. To say that two lines *a* and *b* are parallel, we write $a \parallel b$. It is one of the most primary and fundamental tenets in geometry that given any line and any point not on that line, there exists exactly one line through the point, parallel to the previously given line. Analogously, it is also true that given the same circumstances, there exists exactly one line through the point, perpendicular to the given line.



In the diagram here—representing three dimensions—line *i* is skew to both line *c* and line *r*. Line *i* also intersects lines *a* and *g* and is perpendicular to both. A line is said to be perpendicular to a plane if it is perpendicular to all lines in that plane that intersect it through its foot (its foot being the point that intersects the plane). Line *i* is in this case perpendicular to plane *m*. Lines *c* and *r* appear to be parallel. If the distance between them remains constant no matter how far we travel along them, then they never intersect. Both lie in plane *m*, and thus they would be parallel lines. Line *a*, intersecting lines *c* and *r*, is called a **transversal**. A transversal is a line that intersects two coplanar lines in two points; thus, line *a* is also a transversal for lines *c* and *g*. There are two very useful theorems that would apply now if *c* and *r* are indeed parallel. The first is that when a transversal intersects two parallel lines, **corresponding** angles are congruent. The second is that when a

[8] In mathematics, an axiom or postulate is something that must be considered true without any type of proof. It lays the foundation. A theorem is then something that is proven true (either with axioms/postulates or with other theorems) to establish more mathematics. Can you prove the theorem "Vertical angles are congruent" ? Hint: it involves comparing supplementary angles.

transversal intersects two parallel lines, **alternate interior** angles are congruent. Rather than get bogged down by mathematical definition, consider these examples. In this diagram, there are four pairs of corresponding angles: ∠1 and ∠3, ∠2 and ∠4, ∠5 and ∠7, and ∠6 and ∠8. There are also two pairs of alternate interior angles: ∠2 and ∠7, along with ∠3 and ∠6. (∠5 and ∠4, along with the pair of ∠1 and ∠8, would be called alternate exterior angles.) The words “interior” and “exterior” refer to the angle placement in relation to the parallel lines, and the words “corresponding” and “alternate” refer to the angle placement in relation to the transversal.



Examples:
Assume *c* || *r* and
answer the following questions.

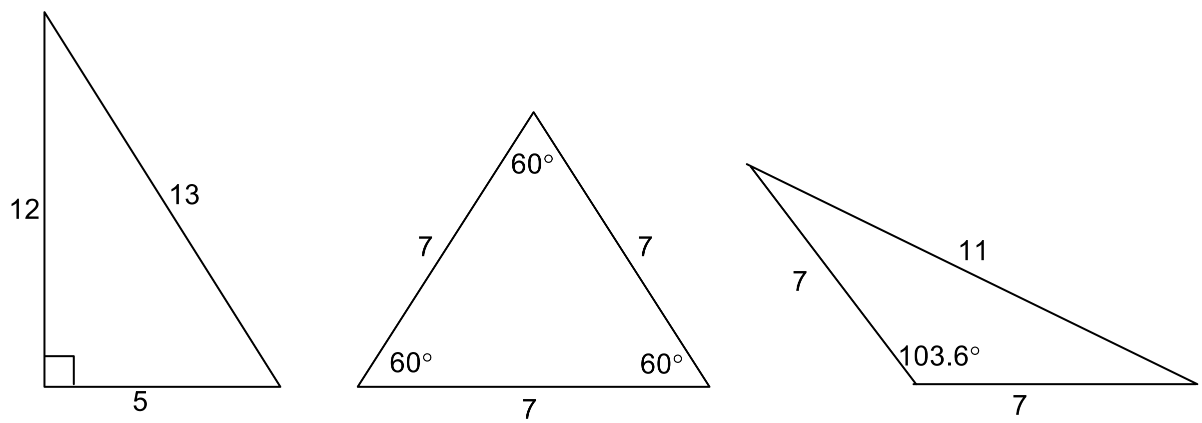
a) If m∠4=70°, then what is m∠7?
b) If m∠6=100°, then what is m∠8?
c) If m∠2=80°, then what is m∠7?
d) If m∠1=95°, then what is m∠4?
e) If m∠5=x°, then what is m∠3?
f) If *a*⊥*r*, is *a*⊥*c*?
g) Are lines *i* and *r* skew? Are lines *i* and *c*? Are lines *i* and *g*?

Solutions:

a) ∠4 ≅ ∠7 because they are vertical angles. Therefore, m∠7=70°
b) ∠6 ≅ ∠8 because they are corresponding angles. Therefore, m∠8=100°
c) ∠2 ≅ ∠7 because they are alternate interior angles. Therefore, m∠7=80°
d) ∠1 ≅ ∠3 because they are corresponding angles. Then ∠1 is supplementary to ∠4 because ∠3 is supplementary to ∠4. m∠4 = 180° - 95° = 85°
e) ∠5 ≅ ∠7 because they are corresponding angles. Then ∠7 is supplementary to ∠3 so m∠3 = 180-x.
f) If *a*⊥*r*, it means that ∠3, ∠4, ∠7, and ∠8 are all right angles. Thus, ∠1, ∠2, ∠5, and ∠6 are all right angles (corresponding and alternate interior angles) so *a*⊥*c.*
g) *i* and *r* are skew; *i* and *c* are skew. *i* and *g*, however, are not because they intersect. Lines are skew only if they are non-coplanar and do not intersect.

AN INTRODUCTION TO THE NUMERICAL PERSPECTIVE OF TRIANGLES

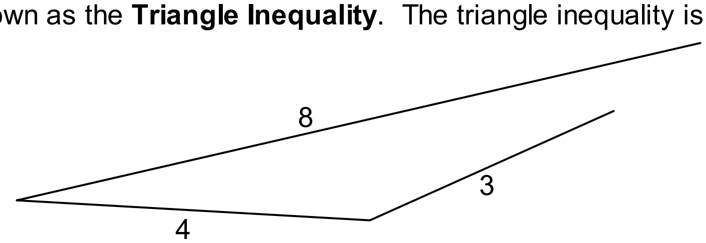
Right Triangle	**Acute Triangle**	**Obtuse Triangle**	**Scalene**
Isosceles	**Equilateral**	**Equiangular**	**Triangle Inequality**
Pythagorean Theorem	**Converse of the Pythagorean Theorem**		



There are three triangles shown above.[9] What can we say in describing them? The leftmost contains a right angle and two acute angles, with no two sides congruent. The triangle in the center contains three congruent acute angles and three congruent sides. The rightmost triangle contains an obtuse angle and two acute angles, along with two congruent sides. There are specific mathematical terms for each of these properties. The first set of three terms pertains to the largest angle within a triangle. If the largest angle of a triangle is 90°, the triangle is a **right triangle**. If the largest angle of a triangle is acute, we call it an **acute triangle**, and if the largest angle of a triangle is obtuse, we call it (logically) an **obtuse triangle**. The other set of terms concerns a triangle's sides. If a triangle has three sides with different lengths, the triangle is said to be **scalene**. If exactly two sides are congruent, we have an **isosceles** triangle, and if all three sides are congruent, the triangle is **equilateral**.[10] Lastly, the term **equiangular** applies to any polygon in which all angles are congruent. When observing triangles, it appears that any equilateral polygon must also be equiangular, but that is a true statement only in reference to triangles. For instance, a rectangle is equiangular but not always equilateral, and a star is equilateral but not equiangular.

> *Example:*
> Describe the three triangles above as specifically as possible.
>
> *Solution:*
> The triangle on the left is a right scalene triangle. The center triangle is an equilateral / equiangular and acute triangle. The triangle on the right is an obtuse isosceles triangle.

What else can be said about the three triangles above? Believe it or not, there are still other facts concerning the triangles that we have not yet uncovered. One, which many students learn very early (often even before taking geometry), is known as the **Triangle Inequality**. The triangle inequality is a theorem stating that any two side lengths of a triangle combined must be greater than the third side length. Some people understand the logic behind this theorem; if you'd like to, find three sticks (maybe toothpicks if you are willing to work with small objects), and cut them

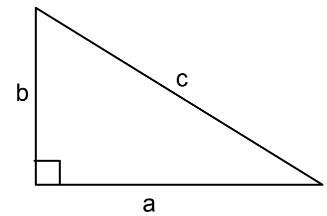


[9] I know that I should build from the ground up in mathematics (especially geometry), but I don't feel like I need to define "triangle" before using the word; *some* previous knowledge is expected. - Craig

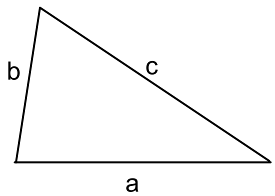
[10] The term equilateral applies not only to triangles, but to other polygons as well. (A polygon is any closed plane figure with many sides.) Any polygon can be said to be equilateral if all its sides are congruent.

into lengths of the ratio 3, 4, and 8. Now try to construct a triangle with the sticks. You will find the best you can do will be somewhat similar to what is pictured on the previous page. No matter how hard you try, the sticks with lengths 3 and 4 are not long enough to form two sides of a triangle. The side of 8 is just too long. This is the logic behind the triangle inequality. Two sides must be able to reach the ends of the third side.

Another item that is not completely evident in the triangle pictures but is still nevertheless important is the fact that the measures of the three angles in a triangle always sum to 180°. (This is a fact tested in great detail on standardized tests and one you probably learned years ago.)

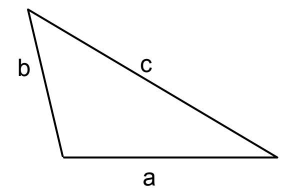


$a^2 + b^2 = c^2$



$a^2 + b^2 > c^2$

Lastly, concerning the three triangles previous, we can describe the triangles (or at least the right triangle) with the **Pythagorean Theorem**. Most students are familiar with the Pythagorean Theorem from previous math courses; many algebra courses even cover it. The Pythagorean Theorem states that in any right triangle, the sum of the squares of the legs[11] equals the square of the hypotenuse. If the legs are "a" and "b" and the hypotenuse is "c," then $a^2 + b^2 = c^2$. We can even check the Pythagorean Theorem with the right triangle on the previous page. 25 + 144 = 169 is a true equation, and thus the sides form a right triangle.[12] Most students are familiar with the Pythagorean Theorem but much less known is the **converse of the Pythagorean Theorem**: if $a^2 + b^2 > c^2$, then the triangle is acute, and if $a^2 + b^2 < c^2$, then the triangle is obtuse, where c is the longest side. (If $a^2 + b^2 = c^2$, then the triangle is right.)



$a^2 + b^2 < c^2$

Examples:

a) Find the hypotenuse of a right triangle with legs 44 and 117.
b) Find the unknown leg of a right triangle with hypotenuse 17 and one leg 8.
c) Find the altitude of an isosceles triangle with congruent sides 41 and base 18. Also find its area.
d) Find the area of an equilateral triangle with side "s".

Solutions:

a) We set a = 44 and b = 117 and use $c^2 = a^2 + b^2$ to obtain:

$c^2 = 44^2 + 117^2$

$c^2 = 15625$

$c = 125$

The hypotenuse has length 125.

b) We set c = 17 and b = 8 and use $a^2 = c^2 - b^2$ to obtain:

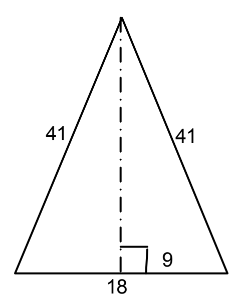
$a^2 = 17^2 - 8^2$

$a^2 = 225$

$a = 15$

The missing leg has length 15.

c) The altitude of a triangle forms a right angle with respect to its base. If the base is 18, the drawing looks like the one here. We now realize that we are looking for the unknown leg of a right triangle with hypotenuse 41 and leg 9. Using the same procedure



[11] Strictly speaking, we are dealing with the squares of the *lengths* of these various legs.

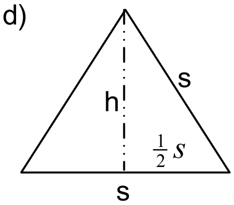
[12] Integer possibilities for right triangles are known as Pythagorean Triples. The most common Pythagorean Triples are 3, 4, 5 and 5, 12, 13. Lesser known Pythagorean Triples include 9, 40, 41 and 12, 35, 37. Pythagorean Triples have lots of nifty and spiffy properties; if you're curious (I promise being curious about math is nothing to be ashamed of!), consult any number theory book.

as in example (b), we find the other leg to be 40 units, which is the altitude of the isosceles triangle in question.
To find the area of this isosceles triangle, we dust off our memory from all the previous math that we've had and recall that the formula for the area of a triangle is $A=\frac{1}{2}bh$, where *b* and *h* represent the base and height of a triangle, respectively. This gives us:

$A=\frac{1}{2}bh$

$A=\frac{1}{2}\cdot 18\cdot 40=360$

The area is 360 square units.

d) 

This example is very similar to example (c). We use the Pythagorean Theorem after drawing the triangle in question and the missing height. We find the height:

$\left(\frac{1}{2}s\right)^2+h^2=s^2$

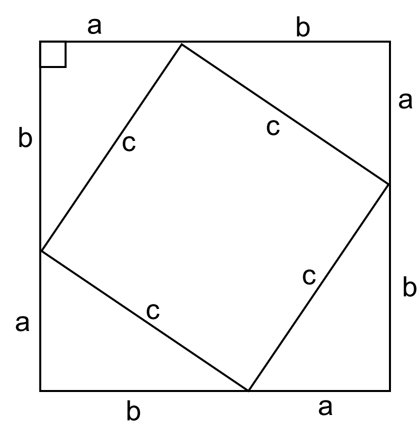
$h^2=s^2-\frac{1}{4}s^2=\frac{3}{4}s^2$

$h=\sqrt{\frac{3}{4}s^2}=\frac{s\sqrt{3}}{2}$. The area of a triangle, as was reviewed above, is $\frac{1}{2}bh$, so we can find the area:

$A=\frac{1}{2}bh=\frac{1}{2}\cdot s\cdot\frac{s\sqrt{3}}{2}=\frac{s^2\sqrt{3}}{4}$. If we have an equilateral triangle with side of s units, the area is $\frac{s^2\sqrt{3}}{4}$ square units. This is a fact that often rears its ugly head on tests, and it may be worth memorizing. If you find it difficult to memorize, then try to conceptualize this example to remember where the formula came from.

The Pythagorean Theorem is indeed a theorem, meaning that it has been proven mathematically (and many, many different proofs of it exist). Here, I will include a brief proof of the Pythagorean Theorem only because I find it fascinating and not for any competitive purpose. Feel free to skip to the next section if you are not interested.

In the drawing at right, four congruent right triangles have been laid, corner to corner. This forms an outer square with sides of length a+b and an inner square with sides of length c. To find the area of the inner square, we can take c^2, or we can take the area of the outer square and subtract the area of the four right triangles. Let's set these two possibilities equal to each other.



$(a+b)^2-4\cdot\left(\frac{1}{2}ab\right)=c^2$

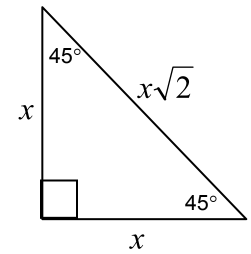
$\left(a^2+2ab+b^2\right)-2ab=c^2$

$a^2+b^2=c^2$

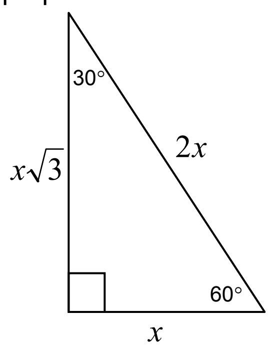
The Pythagorean Theorem is proven. Pretty nifty, huh?

A BRIEF CONTINUATION OF THE NUMERICAL PERSPECTIVE OF TRIANGLES

30-60-90 Triangle **45-45-90 Triangle**



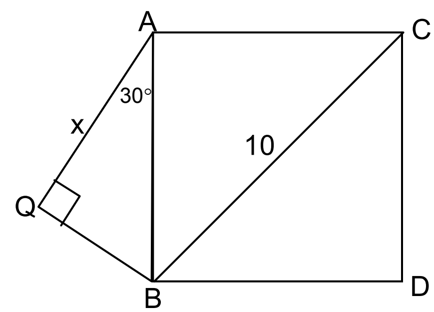
Anytime you encounter a right triangle, the Pythagorean Theorem will apply; however, there are two "special right triangles" that have additional properties as well. One is the isosceles right triangle, or the **45-45-90 triangle**, and the other is a right triangle in which the hypotenuse is twice the length of one of the legs, the **30-60-90 triangle**. Just as the names imply, a 45-45-90 triangle has two congruent 45° angles in addition to its right angle, and a 30-60-90 triangle has angles of 30° and 60° in addition to its right angle. What makes these triangles "special" is that the relationships between the sides are known and (relatively) easy to commit to memory.



Examples:

a) Prove the relationships of the 45-45-90 triangle and the 30-60-90 triangle.

b) Find *x* in the diagram below, given that ABDC is a square.



Solutions:

a) In an isosceles right triangle, we have two legs that are congruent. That means that for the Pythagorean Theorem, *a* and *b* are equal. If we set $a = b = x$, then we can solve for the hypotenuse.

$$c^2 = a^2 + b^2$$
$$c^2 = x^2 + x^2 = 2x^2$$
$$c = \sqrt{2x^2} = x\sqrt{2}$$

Now for the 30-60-90 triangle, we turn our attention to a triangle with angles of 30°, 60°, and 90°. The best place to find one is hidden inside the equilateral triangle. As in example (d) on the previous page, an altitude in an equilateral triangle creates two congruent 30-60-90 triangles. Since these two smaller triangles are the same size, we know the hypotenuse has twice the length of the shorter leg. If we say that the hypotenuse is $c = 2x$, and the shorter leg is $a = x$, then we can solve for b.

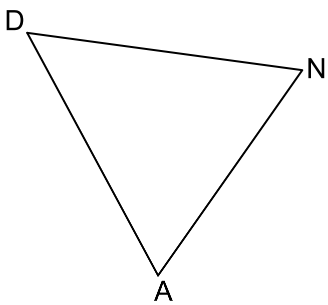
$$a^2 + b^2 = c^2$$
$$b^2 = c^2 - a^2$$
$$b^2 = (2x)^2 - x^2 = 4x^2 - x^2 = 3x^2$$
$$b = \sqrt{3x^2} = x\sqrt{3}$$

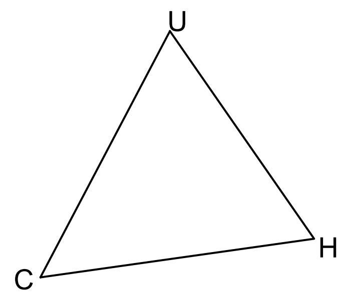
b) This resource has not had a chance yet to detail the specifics of special quadrilaterals. From a young age, though, most people seem to know that a square has four congruent sides and four right angles.[13][14] The diagonal of a square bisects the angles so that it divides the square into two congruent 45-45-90 triangles; the diagonal of the square is then the hypotenuse of those triangles. Thus, $AB = AC = CD = BD = \frac{10}{\sqrt{2}} = 5\sqrt{2}$. We know now the value of AB, so the value of $QA = AB \cdot \frac{\sqrt{3}}{2} = 5\sqrt{2} \cdot \frac{\sqrt{3}}{2} = \frac{5\sqrt{6}}{2}$.

AN INTRODUCTION TO THE ABSTRACT CONCEPTS OF TRIANGLES

SAS **SSS** **ASA** **AAS**
SSA—the ambiguous case

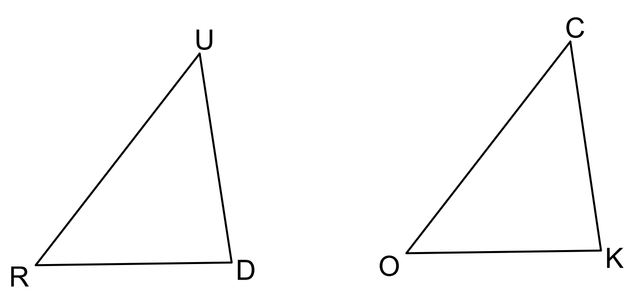
Very early on, geometric congruence was defined similarly to algebraic equivalence. If two line segments are congruent, their lengths are equal; if two angles are congruent, their angle measures are equal. What does it mean, then, to say that two *triangles* are congruent? It means, in short, exactly what one would intuitively expect: that all the corresponding parts of the triangles are congruent. To say that $\Delta DAN \cong \Delta CHU$ means that $\angle D \cong \angle C$, $\angle A \cong \angle H$, and $\angle N \cong \angle U$. It also means that $\overline{DA} \cong \overline{CH}$, $\overline{AN} \cong \overline{HU}$, and $\overline{DN} \cong \overline{CU}$. Would it be correct in this case to say that $\Delta DNA \cong \Delta HCU$? The answer is no. When a congruence between two polygons is written, it is written in an order so that corresponding parts of the two polygons are congruent. If we were to write $\Delta DNA \cong \Delta HCU$, that would imply $\overline{AN} \cong \overline{UC}$, which is not one of the congruencies listed earlier.





One of the core concepts and drills practiced in every high school geometry class is the proof. Usually written in a two-column form, the geometric proof is a series of logical statements proceeding from a list of givens to a desired conclusion, where each assertion is justified by a mathematical reason (a theorem, postulate, definition, or property in almost all cases). While some students enjoy the proof as a fun exercise in logic, most loathe it for its apparent pointlessness.[15] Decathletes are in luck, though, because the formal two-column proof in all likelihood will not be tested in the decathlon curriculum. The logic behind it, however, is still quite necessary in order to be successful.

Being able to prove the congruence of triangles is a difficult skill to master and takes up a majority of the year in many geometry courses. Just how many pieces of two triangles must be congruent before we can know for sure that the entire triangles are congruent? For example, in the two triangles here, if we wanted to prove $\Delta RUD \cong \Delta OCK$ and we knew only that $\overline{RU} \cong \overline{OC}$ and $\overline{UD} \cong \overline{CK}$, would we be able to conclude that the two triangles were



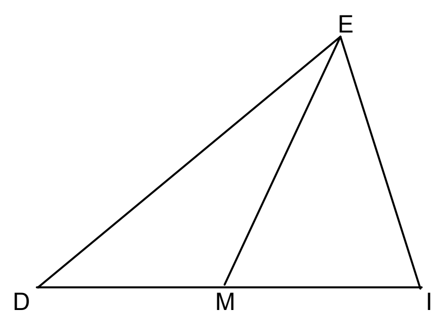
[13] If you didn't know this, then… well… you do now. ☺ - Craig
[14] I think I may have learned this on *Sesame Street*, now that you mention it. Oh, and these are two footnotes, 13 and 14, not footnote 1314. – Daniel
[15] It is true that the practice of proof develops logical reasoning skills (which *are* useful believe it or not), but quite frankly, no one in "the real world" will ask you to prove that a building is a rectangular prism if <blah blah blah>. My teacher even admitted it! Most people just learn proof because their geometry teacher tells them to.

congruent? If we could indeed conclude the triangle congruence, we would automatically know then that all of the corresponding angles were congruent, and that the third sides were congruent. Unfortunately, in this case, we would not be able to conclude the triangle congruence. We have only two congruent sides, and we would need also to know that either the included angles were congruent or the third sides were congruent. The conditions sufficient for proving triangle congruence are listed below; the logic behind these theorems is a bit too complicated to detail in this resource.

- **SSS** theorem – if two triangles have all three pairs of their corresponding sides congruent, then those two triangles are congruent
- **SAS** theorem – if two triangles have two pairs of corresponding sides congruent, along with the corresponding angle between those sides congruent, then those two triangles are congruent.
- **ASA** theorem – if two triangles have two pairs of corresponding angles congruent, along with the corresponding side between those angles congruent, then those two triangles are congruent.
- **AAS** theorem – if two triangles have two pairs of corresponding angles congruent, along with a corresponding, non-included side congruent, then those two triangles are congruent.
- ***SSA/ASS*** **ambiguous case** – if two triangles have two pairs of corresponding sides congruent, along with a corresponding angle congruent that is not between those two sides, we ***cannot*** conclude that those triangles are congruent.[16]

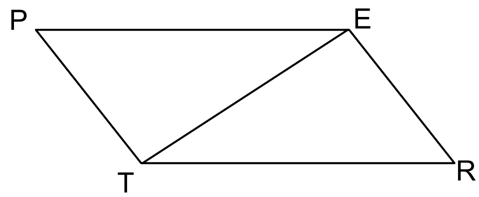
This will probably seem like old hash to anyone who has had a geometry course before and an overwhelming list of information for anyone new to geometry. If you fall in the latter category, take a breather to commit these to memory; you may also wish to take a day or two with a geometry book to practice some proof before continuing. Otherwise, these next examples might overwhelm you. For those of you who have had a geometry course before, let's look at a few brief examples, the first of which explains the ambiguous case. If these proofs do not seem to resemble anything you are used to, remember that the two-column proof is not the only valid type of proof. Here, I will simply offer the "paragraph proof."[17] In addition, you will do well to remember that in testing situations, figures are rarely drawn to scale.

Example:
Attempt to prove here that $\triangle DEM \cong \triangle DEI$ given only that $\overline{EM} \cong \overline{EI}$.



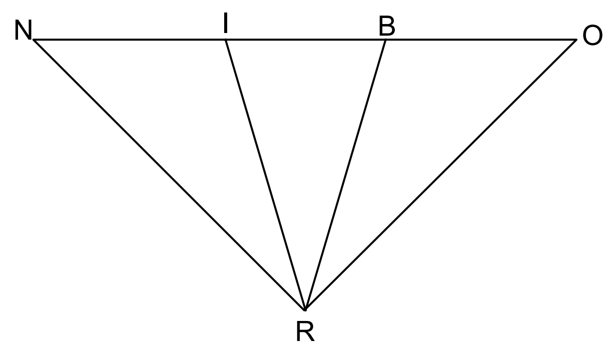
Solution:
We know trivially that $\angle D \cong \angle D$ and that $\overline{DE} \cong \overline{DE}$ by what is called the reflexive property.[18] We are also given that $\overline{EM} \cong \overline{EI}$. This means that we now have two congruent corresponding sides ($\overline{DE}$ & $\overline{DE}$) and ($\overline{EM}$ & $\overline{EI}$) along with a congruent corresponding angle that is not between the sides ($\angle D$). If it were possible, we could assert a triangle congruence by the SSA theorem, but the triangles are very obviously not congruent. In this case SSA is indeed ambiguous in that two obviously different triangles can be formed with $\angle D$, side $\overline{DE}$, and a third side equal in length to EM.

[16] Many geometry teachers either explicitly or implicitly hint that an easy way to remember the uselessness of the ambiguous case is its acronym's potential obscenity. Whatever works to remember.
[17] … frankly, because I think two-column proofs are too constrictive and sometimes restrict logic instead of letting it flow.
[18] It shouldn't be too mentally taxing that anything is congruent to itself.



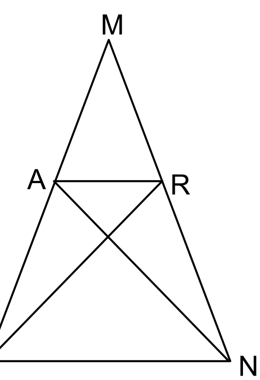
Example:
Prove that in this "shape" ΔPET ≅ ΔRTE, given that $\overleftrightarrow{PE} \parallel \overleftrightarrow{TR}$ and that $\overleftrightarrow{PT} \parallel \overleftrightarrow{ER}$.[19]

Solution:
If we consider that $\overleftrightarrow{PE} \parallel \overleftrightarrow{TR}$ and that $\overleftrightarrow{ET}$ is a transversal intersecting those parallel lines, we can note the alternate interior angle congruence ∠PET ≅ ∠RTE. If we then consider the other parallel lines $\overleftrightarrow{PT} \parallel \overleftrightarrow{ER}$ with the same transversal, we note another alternate interior angle congruence ∠PTE ≅ ∠RET. Now we use the reflexive property to say that $\overline{TE} \cong \overline{ET}$ (trivially, because they are obviously the same line segment!). This gives us the congruence of two corresponding angles and the included side, so we can conclude that ΔPET ≅ ΔRTE by the ASA theorem.



Example:
Prove $\overline{RB} \cong \overline{RI}$ given that $\overline{OB} \cong \overline{NI}$, $\overline{RO} \cong \overline{RN}$, and ∠O ≅ ∠N.

Solution:
The three given congruencies in this problem make proving a triangle congruence somewhat easier (even though triangle congruence is not our final goal in this proof). Because $\overline{OB} \cong \overline{NI}$, $\overline{RO} \cong \overline{RN}$, and ∠O ≅ ∠N, I can say that ΔRNI ≅ ΔROB by citing the SAS theorem. One of the key characteristics, then, of congruent shapes is that the parts of congruent shapes are congruent. In other words, all corresponding parts of congruent triangles are congruent. (Many books abbreviate this reason in two-column proofs as "CPCTC.") At any rate, we know that ΔRNI ≅ ΔROB so we know that $\overline{RB} \cong \overline{RI}$ because all corresponding parts of congruent triangles are congruent.



Example: In the drawing shown, $\overline{MT} \cong \overline{MN}$, $\overline{MA} \cong \overline{MR}$, ∠MAR ≅ ∠MRA, and $\overleftrightarrow{AR} \parallel \overleftrightarrow{TN}$. Prove that $\overline{AN} \cong \overline{RT}$.

Solution:
There are probably several equally valid ways of going about this proof. Here is one possibility. We are given that $\overleftrightarrow{AR} \parallel \overleftrightarrow{TN}$. We then have two transversals that intersect the parallel lines ($\overleftrightarrow{MN}$ and $\overleftrightarrow{MT}$) so we know that pairs of corresponding angles are congruent. That is, we know ∠MAR ≅ ∠MTN and ∠MRA ≅ ∠MNT. We are told that ∠MAR ≅ ∠MRA, so the transitive property tells us that ∠MTN ≅ ∠MNT.[20] We will need this particular angle congruency a bit later. Now we examine the other given

[19] To maintain the logical structure of geometry, I will avoid using the term "parallelogram" here because this resource has not yet defined the term. If you have had geometry before… then… I suppose you realize that this is a parallelogram and that we are proving that the diagonal of a parallelogram divides it into two congruent triangles. - Craig

[20] The transitive property as it pertains to geometry is very similar to the transitive property as it relates to algebra. In algebra, if x = y and y = z, then the transitive property tells us that x = z. For a geometric application, simply replace the = with ≅ and you have it. In this particular geometry problem, we've deduced a ≅ b and c ≅ d, and we were given a ≅ d, so our conclusion is b ≅ d.

information and see what else we can deduce. $\overline{MT} \cong \overline{MN}$ and $\overline{MA} \cong \overline{MR}$ so, subtracting the second pair of congruent sides from the first (a procedure guaranteed by the subtraction property of equality to produce two congruent line segments), we can get $\overline{AT} \cong \overline{RN}$. Furthermore, $\overline{TN} \cong \overline{NT}$ by the reflexive property, so we can prove ∆ATN ≅ ∆RNT by citing the SAS theorem ($\overline{AT} \cong \overline{RN}$, ∠MTN ≅ ∠MNT, and $\overline{TN} \cong \overline{NT}$). We can then say that $\overline{AN} \cong \overline{RT}$ because CPCTC. (The acronym was introduced in the previous example.)

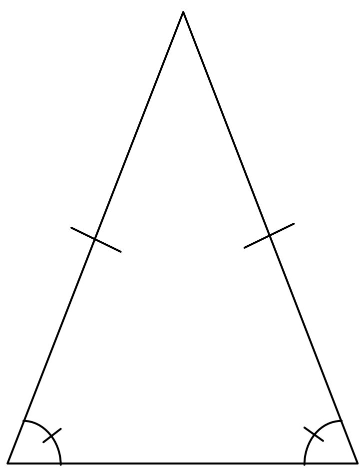
There are two things to note as we end these examples on triangle congruencies. The first is that none of these examples cited the SSS or AAS congruency theorems; the procedure for those, however, is essentially the same: the needed congruencies must first be established, and then the theorem can be cited. The second thing to note is that the last example was quite challenging. Avoid panicking if you found it difficult to follow; geometric proofs can become very complicated. Only with practice will you find yourself more comfortable with them.

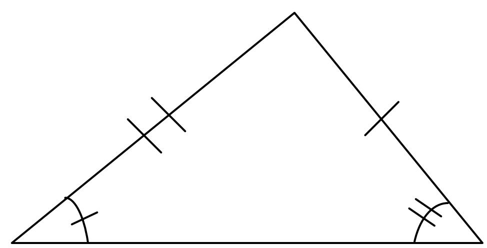
REVISITING THE NUMERICAL PERSPECTIVE ON TRIANGLES

Angle-Side Theorem **SAS similarity** **SSS similarity** **AA similarity**
Similar Triangles

Now that we have discussed the geometry of triangle congruence in detail, you may think it is time to move on to a new topic. Amazingly, and perhaps unfortunately, there is still more. One theorem that might have made some of the preceding proofs easier is the **Angle-Side Theorem**.[21]

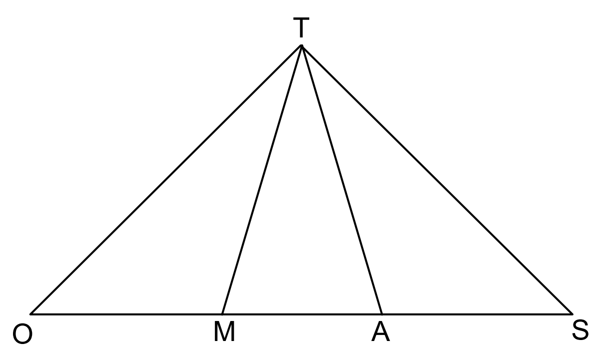
- The Angle-Side Theorem: If two sides of a triangle are congruent, then the angles opposite those sides are congruent. If two angles of a triangle are congruent, then the sides opposite those angles are congruent. If two sides are not congruent, then the angles opposite those sides are not congruent, and the larger angle is opposite the longer side. If two angles are not congruent, then the sides opposite those angles are not congruent, and the longer side is opposite the larger angle. In the drawing at left, the identical tick marks signify congruence between the appropriate parts. In the drawing at right, the side and angle with double tick marks are greater in length and measure than the side and angle with single tick marks.





The Angle-Side Theorem is quite versatile and can be used in a wide variety of proofs and calculations. Here are two examples.

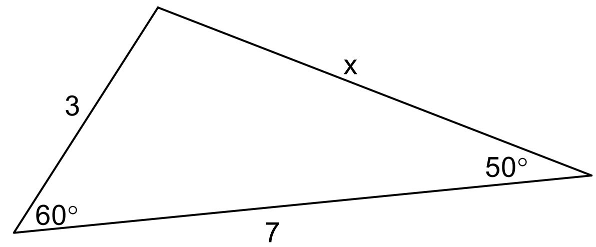
Example:
Prove $\overline{TM} \cong \overline{TA}$ given that $\overline{OM} \cong \overline{SA}$ and $\angle O \cong \angle S$.



Solution:
This proof is identical to an example given earlier except that there are now only two given statements and not three. Perhaps the Angle-Side Theorem can shed some light on the "missing" given. If we know now that $\angle O \cong \angle S$, we can cite the Angle-Side Theorem and know that $\overline{TO} \cong \overline{TS}$. With this congruence and the two congruencies given, we can assert $\Delta TOM \cong \Delta TSA$ by the SAS theorem. Then $\overline{TM} \cong \overline{TA}$ because CPCTC.

[21] Most books treat the parts of this "Angle-Side Theorem" as four separate theorems. I will lump them all together for two reasons. First, I think decathletes are bright and intelligent enough to understand all the parts of it at once. Second, I want this resource to be as concise as possible; since two-column proofs are absent from the official curriculum, the concepts are much more important than theorem distinctions.

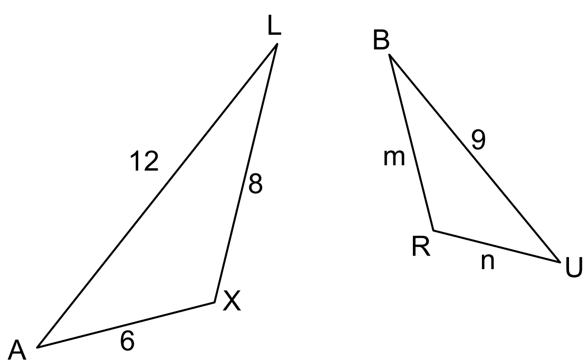
Example:
Make the most restrictive inequality possible for the length of this triangle's unknown side.[22]



Solution:
Because of the Triangle Inequality (stated two sections ago), we know that $x + 3 > 7$ and $3 + 7 > x$. Therefore, x must be greater than 4 and smaller than 10. Wait, though, there's more…let's apply the Angle-Side Theorem. The unlabeled angle must be 70° (because the three angles together must add to 180°), and the Angle-Side Theorem tells us that the comparative lengths of sides opposite non-congruent angles correspond to the sizes of those angles. So then, since 50° < 60° < 70°, the sides must satisfy the relationship $3 < x < 7$. Thus, the most restrictive statement we can make about x is $4 < x < 7$.

The last triangle topic we need to address now is the concept of **similar triangles**. What does it mean to say that two things are similar? In literature and English classes, it means that those two items share certain characteristics or traits. In geometry, the term "similar" takes on a more specific definition.

- Similar Polygons: Two shapes are *similar* if all of their corresponding angles are congruent and the ratios between corresponding sides are constant. We write triangle ABC similar to triangle DEF as $\triangle ABC \sim \triangle DEF$



Example:
Find the unknown sides m and n given that $\triangle LAX \sim \triangle BUR$.

Solution:
We know that the ratios between corresponding sides are equal, and we need only to set up a proportion between the two triangles' side lengths.

$$\frac{LA}{BU} = \frac{XL}{RB}$$

$$\frac{12}{9} = \frac{8}{m}$$

We also set up the other proportion necessary to find n:

$$\frac{LA}{BU} = \frac{XA}{RU}$$

$$\frac{12}{9} = \frac{6}{n}$$

With cross-multiplication in proportions, we get the final two equations $12m = 72$ and $12n = 52$ and the final answers are $m = 6; n = 4.5$

This is all very interesting, of course, but is there anything more to it? I'm afraid so. If you examined the term list at the beginning of this section, you saw some terms that seemed unsettlingly close to the theorems used to prove triangles congruent. The list is exactly what your intuition tells you. Earlier, the SSS, SAS, ASA, and AAS theorems were used to prove two triangles congruent, each

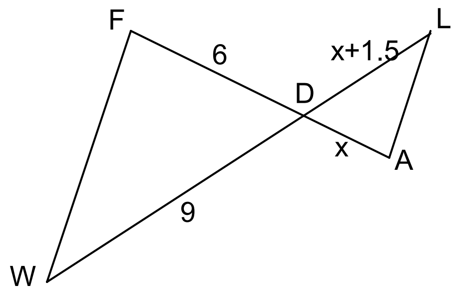
[22] With trigonometry, we could find the exact value of this missing side. With only geometric methods, though, our capabilities are a bit more limited. This example concerns the information available from geometry—sorry to all you knowledgeable trig experts out there.

theorem means that two triangles are identical in enough respects to declare the triangles wholly congruent. Proving triangle similarity is done in much the same manner. Triangles may exist with 0, 1, 2, or 3 angles congruent, and they may exist with 0, 1, 2, or 3 side lengths in proportion. How many corresponding angles must be congruent, and how many side lengths must be in proportion before we may assert that two triangles are necessarily similar? The triangle similarity theorems are listed below. Note if you are comparing them to the triangle congruence theorems that ASA and AAS have jointly been replaced with a single AA similarity theorem.

- **SSS similarity** theorem – if two triangles exist such that all three pairs of corresponding side lengths form a constant ratio, then the two triangles must be similar
- **SAS similarity** theorem – if two triangles exist such that two pairs of corresponding side lengths are form a constant ratio and the angles included between those sides are congruent, then the two triangles must be similar
- **AA similarity** theorem – if two triangles exist such that two pairs of corresponding angles are congruent, then the triangles must be similar[23]

Example:

Given $\overleftrightarrow{FW} \parallel \overleftrightarrow{LA}$, find x.



Solution:

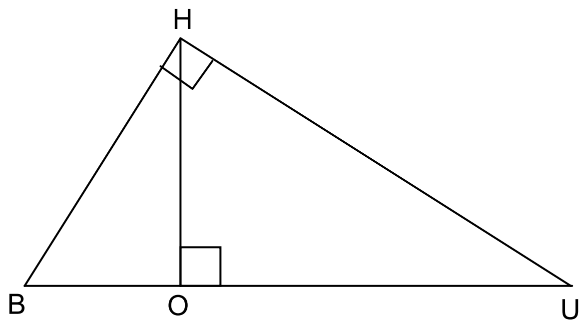
First we should try to prove triangle similarity. We are given that $\overleftrightarrow{FW} \parallel \overleftrightarrow{LA}$, and we observe that the parallel lines have two transversals, $\overleftrightarrow{LW}$ and $\overleftrightarrow{AF}$. With the two transversals, we then have two pairs of congruent alternate interior angles: ∠F ≅ ∠A and ∠W ≅ ∠L. Those two pairs of congruent angles are enough to cite the AA ~ theorem and say that △DFW ~ △DAL. We then set up a proportion between corresponding sides.

$$\frac{DF}{DA} = \frac{DW}{DL}$$

$$\frac{6}{x} = \frac{9}{x+1.5}$$

$$9x = 6(x+1.5)$$

$$x = 3$$



Example:

Prove the following statements given the diagram at left.

a) $HB^2 = OB \cdot BU$
b) $HU^2 = OU \cdot BU$
c) $HO^2 = OB \cdot OU$

[23] Remember that theorems can be proven from axioms, postulates, and other theorems. Given that AAA is a postulate (If all pairs of corresponding angles between two triangles are congruent, the triangles are similar.), could you prove the AA similarity theorem? Hint: It involves the number 180.

Solution:

a) We know that $\angle BHU \cong \angle BOH$ because they are both right angles. We also know that $\angle HBU \cong \angle OBH$ by the reflexive property. (It may be named differently, but it is still the same angle, and it must be congruent to itself.) With those two angle congruencies, we cite AA~ and say that $\Delta UHB \sim \Delta HOB$. We set up a proportion between corresponding sides and say that $\frac{HB}{BU} = \frac{OB}{BH}$. In any proportion, however, we can cross-multiply, and this gives us the seeked equation. $HB \cdot BH = OB \cdot BU$, or $HB^2 = OB \cdot BU$.

b) We know that $\angle HOU \cong \angle BHU$ because they are both right angles. We also know that $\angle OUH \cong \angle HUB$ by the reflexive property. With those two angle congruencies, we cite AA~ and say that $\Delta HOU \sim \Delta BHU$. We set up a proportion between corresponding sides and say that $\frac{HU}{OU} = \frac{BU}{HU}$. We then cross-multiply within our proportion: $HU \cdot HU = OU \cdot BU$, or $HU^2 = OU \cdot BU$.

c) In part (a), we successfully proved that $\Delta UHB \sim \Delta HOB$. If we rearrange the order of the lettering, we can equivalently say that $\Delta BHU \sim \Delta BOH$. In part (b), we successfully proved that $\Delta HOU \sim \Delta BHU$. We know that a pair of similar triangles must have all their angles congruent. Therefore, if two different triangles are similar to the same triangle, they must be similar to each other since all the corresponding angles are still congruent. Thus, saying both $\Delta BHU \sim \Delta BOH$ and $\Delta HOU \sim \Delta BHU$ means that $\Delta BOH \sim \Delta HOU$.[24] [The completion of this example is left as an exercise for the reader.]

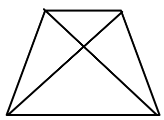
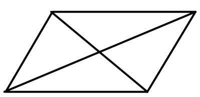
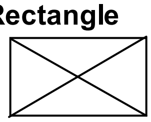
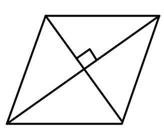
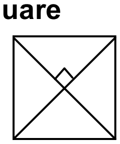
[These three theorems are known as the altitude-to-hypotenuse theorems and may be worth memorizing.]

[24] I suppose one might call this the transitive property of similarity if he or she were so inclined.

A PLETHORA OF PARALLELOGRAMS, AND THEY BROUGHT THEIR FRIENDS, AS WELL AS ONE UNINVITED GUEST

Parallelogram **Rectangle** **Square** **Rhombus** **Trapezoid**

Two examples so far in this resource have dealt with special quadrilaterals. (A quadrilateral is any polygon with four sides.) In one, we proved that the diagonal of a parallelogram creates two congruent triangles. In the other, we used the properties of 45-45-90 triangles on the two identical triangles created when a square is cut by its diagonal. The parallelogram and square are just two of a group of special quadrilaterals. The special quadrilaterals of concern to us, along with their definitions and a list of the major properties of each, is given below.[25]

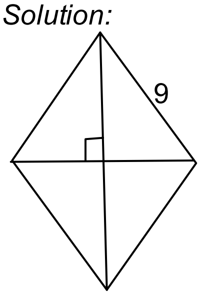
Shape	Mathematical Definition	Pertinent Information & Useful Properties
Trapezoid 	a quadrilateral with exactly one pair of parallel sides	The two parallel sides are known as the *bases*. A trapezoid may or may not be an *isosceles* trapezoid, in which the two non-base sides (called *legs*) are congruent. If a trapezoid is isosceles, then the pairs of base angles are congruent, as are the diagonals. In addition, sometimes a trapezoid with one right base angle is called a *right* trapezoid.
Parallelogram 	a quadrilateral with two pairs of parallel sides	Not only are both pairs of parallel sides parallel, they are also congruent. In addition, opposite angles are congruent, and consecutive angles are supplementary. Also, the parallelogram's diagonals bisect each other.
Rectangle 	a parallelogram containing at least one right angle	Not only is at least one angle a right angle, all four angles are right angles. All properties of parallelograms apply, and the diagonals are congruent in addition to bisecting each other.
Rhombus 	a parallelogram containing at least one pair of congruent adjacent sides	Not only is one pair of adjacent sides congruent, all four sides are congruent. All properties of parallelograms apply, and the diagonals are perpendicular bisectors of each other. In addition, the diagonals bisect the angles and form four congruent right triangles.
Square 	a parallelogram that is both a rectangle and a rhombus	All four angles are right angles, and all four sides are congruent. All the properties of both rectangles and rhombuses apply, and in addition the diagonals now form four congruent 45-45-90 right triangles (a.k.a. right-isosceles triangles).

For your benefit and reference, a checklist-style table drilling these properties is included in this year's math workbook. We won't repeat it here, but please be sure to fill it out using the information above, and consider referring back to it frequently as competition nears.

[25] The trapezoid is not listed in this year's official decathlon math outline. Nevertheless, it is still a quadrilateral with special properties, and no study of geometry is complete without it. We can just say that it appeared on this chart uninvited.

Example:
A rhombus has sides of length 9 and angles of measure 60°, 60°, 120°, and 120°. Find the lengths of its diagonals.

Solution:



When a picture is not given, it is usually a good idea to draw one labeled with what we know and what we need. We need to remember also that it is a property of rhombuses that the diagonals bisect the angles. So then, each of the four congruent right triangles is a 30-60-90 triangle with a hypotenuse of 9. A 30-60-90 triangle with a hypotenuse of 9 has a shorter leg of $\frac{9}{2}$ and a longer leg of $\frac{9\sqrt{3}}{2}$. Each of those legs measures half its respective diagonal so the diagonals must be 9 and $9\sqrt{3}$.

Example:
Classify each of the following statements as *true* or *false*.[26] Justify your answer.

a) All squares are rectangles.
b) Some rhombuses are squares.
c) Some rectangles are rhombuses.
d) No trapezoids are parallelograms.

Solution:

a) TRUE –Squares have four right angles and any parallelogram with at least one right angle qualifies as a rectangle.
b) TRUE – Squares are parallelograms with four congruent sides and any parallelogram with at least one pair of congruent adjacent sides qualifies as a rhombus. All squares are rhombuses; some rhombuses are squares.
c) TRUE – To be a rhombus, a parallelogram must have four congruent sides. It is possible for a rectangle to have four congruent sides (after all, some rectangles are squares) so it is true that some rectangles are rhombuses.
d) TRUE – A trapezoid is defined to have exactly one pair of parallel sides. A parallelogram has both pairs of sides parallel. It is obviously impossible for a quadrilateral to have both one pair of parallel sides and two pairs of parallel sides. Being a trapezoid and being a parallelogram are mutually exclusive conditions; no trapezoids are parallelograms.

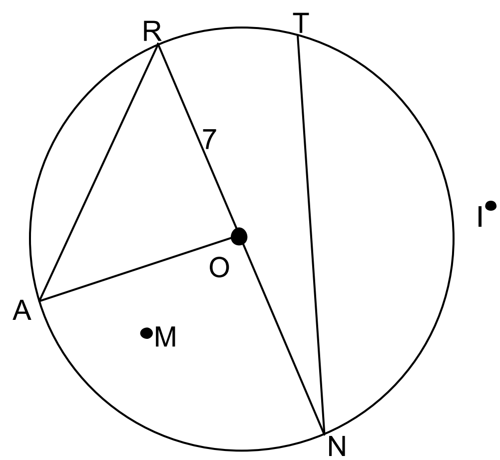
[26] For those of you who dealt with the inner workings of set theory last year (whether or not against your will), it may [or may not] make sense to you when I write that Squares $\subset$ Rectangles, Squares $\subset$ Rhombuses, and (Rectangles $\cap$ Rhombuses) = Squares.

TO EVERYTHING THERE IS A SEASON; TURN, TURN, TURN[27]

Center	**Radius**	**Chord**	**Diameter**
Interior	**Exterior**	**Tangent**	**Secant**
Point of Tangency			

We now turn our attention from things polygonal to something far more fun: circles. Circles appear everywhere in day-to-day life[28], from the tires on cars, to (as the first footnote says) hula hoops, to traffic roundabouts, to the shape of the earth. Besides, standing and "spinning in triangles" or "spinning in parallelograms" causes much less of a blissful, dizzy blur, and it makes you look even more ridiculous than spinning in circles would.

It's obvious what circles are: everyone has dealt with them since childhood. Unfortunately, the fact that circles are round is not a mathematical definition. The mathematical definition of "circle" is *the set of all coplanar points that are the same distance from a fixed point in the plane*[29]. That fixed point is called the **center**, and the uniform distance is called the circle's **radius**. The word "radius" also sometimes refers to a line segment that connects the center to a point on the circle (instead of referring to the distance itself). Circles are named by their center point. In the picture here, point O is the circle's center, and the circle has radius 7. We could also say that $\overline{AO}$, $\overline{RO}$, and $\overline{NO}$ are radii of circle O. (Remember, the radius can refer either to a segment connecting the center to a point on the circle or to the length of such a segment.)

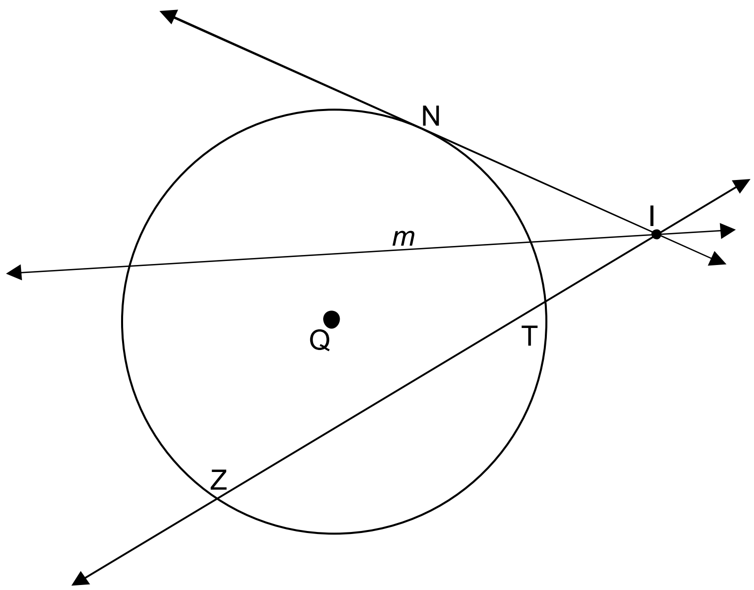


Example:
What are AO and ON in the diagram of the circle?

Solution:
In a circle, all radii must be congruent since the distance from the center of the circle to any point on the circle must be constant. Thus, AO = NO = 7.

What else do we have in our picture of a circle here? There are three line segments that connect two points on the circle: $\overline{AR}$, $\overline{RN}$, and $\overline{NT}$. There is a special term for such segments: a **chord** is any segment that connects two points on the circle. One of the three chords shown passes through the center (chord $\overline{RN}$) and any chord that passes through the center is, as most people probably know, called a **diameter**—chord $\overline{RN}$, then, is a diameter. Like "radius," the word "diameter" can refer to either the line segment itself, or the length of such a segment (so in this case the statements "the diameter is 14" and "$\overline{RN}$ is a diameter" are equally correct). In addition to the chords and radii, two other points are shown in the picture, points M and I. Point M rests comfortably inside the circle while point I is sadly neglected on the outside of the circle.[30] Point M is considered to be in the **interior** of the circle—the distance from M to the center is less than the radius. Point I, logically

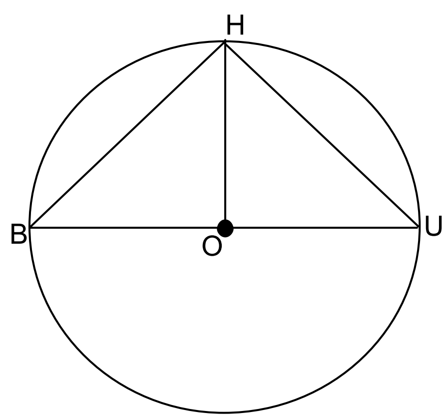
[27] This title is only marginally related to circles, and only in its last three words. Nevertheless, it's much more interesting than "Circles: They're Not Just for Hula Hoops Anymore." - Craig
[28] Those who are pretentious might use the word "ubiquitous." I just did.
[29] Oddly—as I review this resource on a different kind of plane returning prematurely from a coaches' clinic in Texas—I can look out the window and see circular green patches dotting the landscape below. Since nature abhors a vacuum but has no special preference for circles, Craig and I theorize that these actually result from the action of a central irrigation system—maybe a rotating sprinkler whose range is the radius? - Daniel
[30] poor point I



then, is considered to be in the **exterior** of the circle—its distance from the center exceeds the radius.

While it may seem that point I is receiving the short end of the stick and does not get to enjoy being in the inner circle, external points are far from insignificant. Any line passing through an external point of a circle can do one of three things. (1) not intersect the circle, (2) intersect the circle at exactly one point, or (3) intersect the circle at two distinct points. There has already been a veritable deluge of mathematical terms thrown at you regarding circles, but there are still a few more.

A line fitting case (2) is a **tangent line**, and a line fitting case (3) is a **secant line**. In the figure showing circle Q, lines *m* and $\overleftrightarrow{TZ}$ are secant lines and $\overleftrightarrow{NI}$ is said to be tangent to the circle. Point N is the **point of tangency** for $\overleftrightarrow{NI}$.

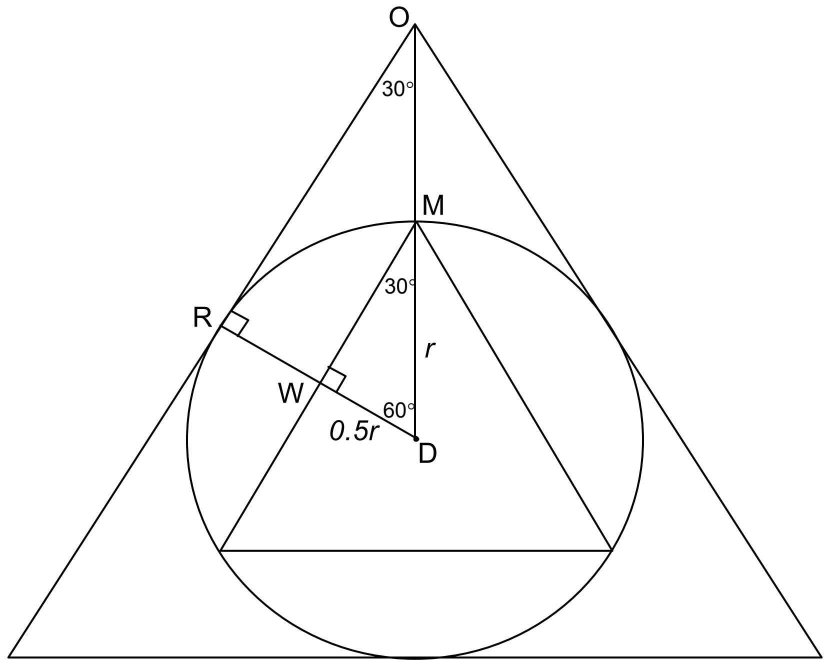


Example:
Given circle O and the congruence $\overline{HU} \cong \overline{HB}$, prove $\angle U \cong \angle B$ without citing the Angle-Side Theorem.

Solution:
Remember that in a circle, the radius has constant length, and thus all radii are congruent. With that reasoning, $\overline{OB} \cong \overline{OU}$ because all radii are congruent. Also, $\overline{OH} \cong \overline{OH}$ because of the reflexive property. We can combine these two statements with the given ($\overline{HU} \cong \overline{HB}$) to assert $\Delta HOB \cong \Delta HOU$ by the SSS theorem. Then $\angle U \cong \angle B$ because CPCTC.

Example:
Any polygon whose vertices all touch a given circle is said to be **inscribed** in the circle; any polygon whose sides are all tangent to a circle is said to **circumscribe** the circle. If equilateral triangles are both inscribed in and circumscribed about a circle, then what is the ratio of the sides between the two triangles?

Solution:
This problem is extremely difficult and will combine many of the concepts thus far. Our diagram will contain two equilateral triangles with a circle between them. One important fact that will help us is that if all of the altitudes are drawn in an equilateral triangle, then six 30-60-90 triangles are formed. In the inscribed triangle, the hypotenuse of these smaller triangles will be equal to the radius. In the circumscribed triangle, the shorter leg of these triangles is equal to the radius.



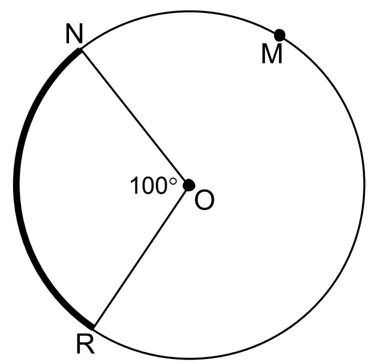
Before we continue, be sure you can prove on your own that these 30-60-90 triangles do indeed form when the appropriate lines are drawn. We can see that an angle similarity forms, giving us ∆ODR ~ ∆MDW. We also know that all radii in a circle are congruent so MD = DR. We can assign an arbitrary length to this radius so I'll call it *r*. Because of the 30-60-90 triangles, DW = 0.5*r*. The proportion between corresponding sides in the large triangles will be the same as the proportion between any corresponding parts, so the ratio we are looking for is the same as the ratio of DW to DR. The sides are in ratio 1 to 2.

THE ANGLE-SECANT INVASION OF NORMANDY

Arc **Central Angle** **Inscribed Angle** **Secant-Secant Angle**
Secant-Tangent Angle **Tangent-Tangent Angle** **Chord-Chord Angle** **Arc Measure**
Chord-Tangent Angle

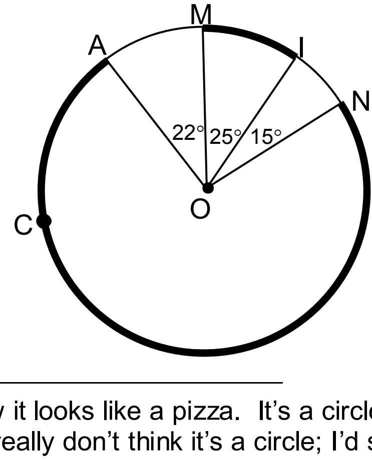
The chords, tangents, and secants that have been covered thus far as they pertain to circles are unfortunately not an end to a means. While these line segments exist and sit comfortably through or next to their circles, they are blissfully unaware that they too (much like triangles and quadrilaterals) are subject to geometry's frustratingly extensive description by properties. Our ultimate goal, at least with regard to circles, is a thorough understanding of all the properties between the lengths of tangents, the lengths of secants, the lengths of chords, and the angles formed by all of them.

We'll begin our investigation with a look at the angles associated with circles, but before we even get to any specific angles, there is one key concept that must be stated: One full rotation comprises 360°. Look at the circle to the right and imagine that N is not fixed but instead moves freely around the edge of the circle. Can you see that if N traveled all the way around the circle, the radius $\overline{ON}$ would pass through or "sweep out" a full rotation? It's that circular rotation that is described in the geo-numeric statement "A circle contains 360°."



Now look again at the diagram, and notice the darkened section. That particular shape is called an **arc**. An arc is mathematically defined as two points on a circle along with all the points connecting them along the circle; arc NR is written similarly to line segment $\overline{NR}$, except that the line above the letters is curved to make a small arc. An arc is a fraction of the circle's circumference; its measure is linear. More important than the arc's length, though, is the **central angle** that intercepts the arc. A central angle in a circle is an angle whose vertex is the center of the circle, and the measure of the central angle is equal to the measure of the intercepted arc. In the circle here, the measure of arc NR is clearly 100°. More accurately, though, I should say that the measure of *minor* arc NR is 100°. Notice that the points N and R actually create two distinct arcs: the smaller, darkened piece that we've been calling arc NR and the longer, undarkened, as-yet-unnamed piece. The official designation of that large arc is *major arc NMR*—when naming a major arc (any arc covering more than half a rotation is "major"), we include a third point to indicate the endpoints of the arc, *along with a point* through which the arc passes. In this particular case, it should be reasonably clear that major arc NMR has measure 260°, because minor arc NR and major arc NMR must together form a complete circle of 360°.

Example:
Find the measure of the shaded arcs in the given circle.[31]



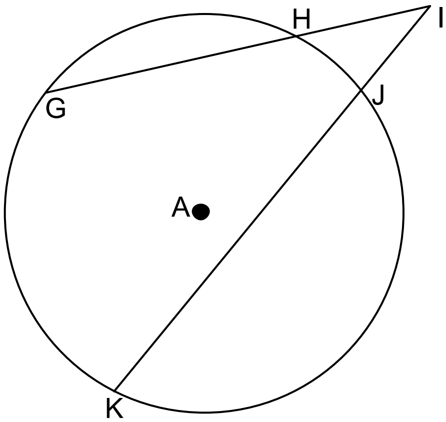
Solution:[32]
We know that the measure of arc MI is the same as m∠MOI, which is given to be 25°. To find the highlighted major arc, we have to find the measure of ∠AON and subtract from 360.
$m\angle AON = 22° + 25° + 15° = 62°$. Thus the measure of arc ACN is 360 – 62, or 298°.

$m\overset{\frown}{MI} = 25°$

$m\overset{\frown}{ACN} = 298°$

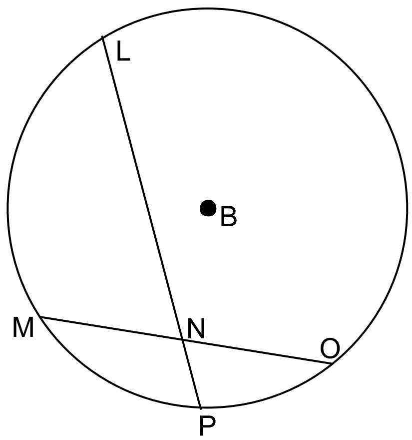
[31] I know it looks like a pizza. It's a circle…….. no, stop that, it really IS a circle - seriously. – Craig
[32] No, I really don't think it's a circle; I'd say it looks more like a pizza. – Daniel

In addition to the central angle, there are several other angles related to circles. If perceptive, you saw the list of vocabulary terms at the beginning of this section. On the next page are six pictures of circle-related angles. See if you can label them on your own before turning the page again to find out what they are. Use the list of vocabulary terms from above: inscribed angle, chord-tangent angle, secant-secant angle, secant-tangent angle, tangent-tangent angle, and chord-chord angle.



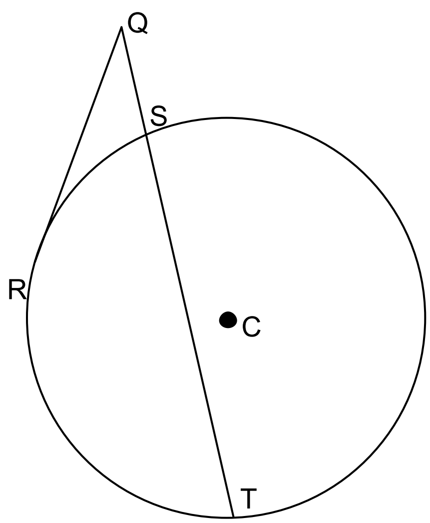
∠I is a ____________________ angle.

m∠I =



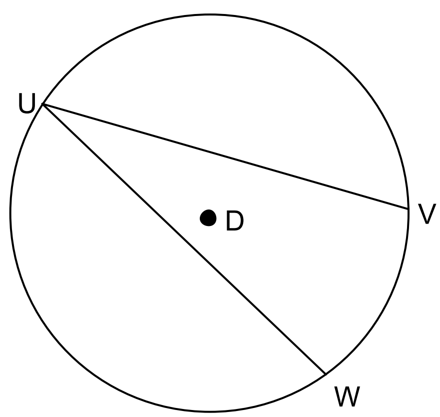
∠LNM is a ____________________ angle.

m∠LNM = m∠PNO =



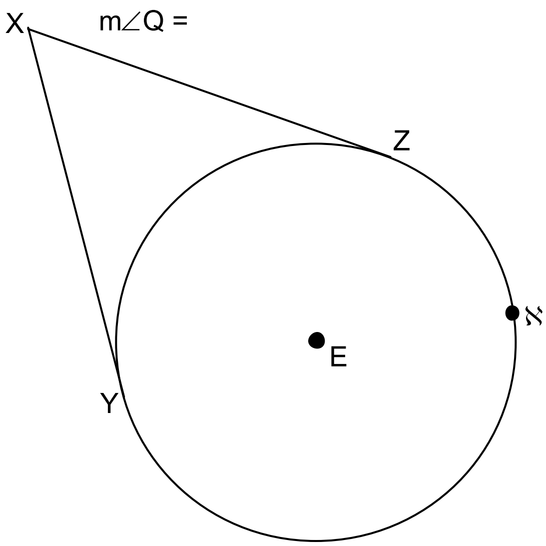
∠Q is a ____________________ angle.

m∠Q =



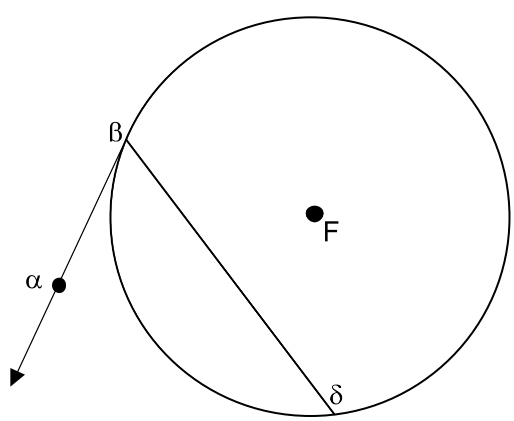
∠U is an ____________________ angle.

m∠U =



∠X is a ____________________ angle.

m∠X = =



∠αβδ is a ____________________ angle.

m∠αβδ =

How many of the angles were you able to identify? The second line of each description was probably a bit hard to complete without the needed information first. The first lines, though, should have all been fairly intuitive based on the definitions of secants, tangents, and chords. At any rate, you should have managed to write down most of the angle classifications.

$\angle$I is a **secant-secant angle**.
$\angle$Q is a **secant-tangent angle**.
$\angle$X is a **tangent-tangent angle**.
$\angle$LNM is a **chord-chord angle**.
$\angle$U is an **inscribed angle**.[33]
$\angle\alpha\beta\delta$ is a **chord-tangent angle**.

If those are not the labels that you selected, then go back and write them in correctly. Now, for our grand anti-climax, the mathematical definition of each type of angle follows.

- A secant-secant angle is an angle with vertex exterior to a circle and sides determined by secants to the circle.
- A secant-tangent angle is an angle with a vertex exterior to a circle and sides determined by a secant and a tangent to the circle.
- A tangent-tangent angle is an angle with a vertex exterior to a circle and sides determined by tangents to the circle.
- A chord-chord angle is an angle formed by two intersecting chords.
- An inscribed angle is an angle with vertex on the circle and sides determined by two chords.
- A chord-tangent angle is an angle with vertex on the circle and sides determined by a tangent and a chord.

Now, we must move on to the measurement of these types of angles. There are three general theorems that deal with the measurements of circle-related angles. The proof supporting the theorems is complicated, but the theorems themselves are not.

- If the vertex of a special circle angle lies in the *exterior* of the circle, the measure of the angle is *half the difference* of the intercepted arcs. [This applies to the secant-secant angle, the secant-tangent angle, and the tangent-tangent angle.]
- If the vertex of a special circle angle lies in the *interior* of the circle, the measure of the angle is *half the sum* of the intercepted arcs. [Note that this only applies to the chord-chord angle.]
- If the vertex of a special circle angle lies *on* the circle, the measure of the angle is *half the intercepted arc*. [This applies to the inscribed angle and the chord-tangent angle.] This is like a special case of the first two properties; the “second arc” now has measure of 0°.

Now you can complete the previous page. Be extra careful to note the alternate equation to find a tangent-tangent angle.

$$m\angle I = \tfrac{1}{2}(m\overset{\frown}{GK} - m\overset{\frown}{HJ})$$

$$m\angle Q = \tfrac{1}{2}(m\overset{\frown}{RT} - m\overset{\frown}{RS})$$

$$m\angle X = \tfrac{1}{2}(m\overset{\frown}{Y\aleph Z} - m\overset{\frown}{YZ}) = 180° - m\overset{\frown}{YZ}$$

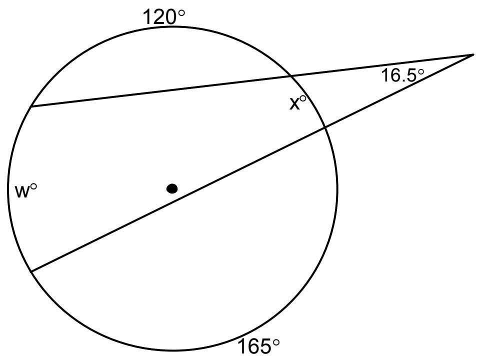
$$m\angle LNM = m\angle PNO = \tfrac{1}{2}(m\overset{\frown}{LM} + m\overset{\frown}{OP})$$

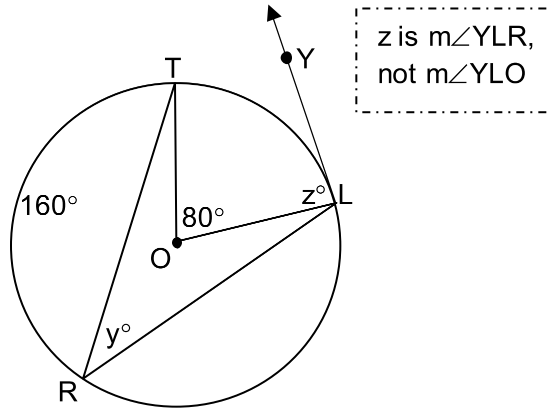
$$m\angle U = \tfrac{1}{2} m\overset{\frown}{VW}$$

$$m\angle\alpha\beta\delta = \tfrac{1}{2} m\overset{\frown}{\beta\delta}$$

[33] I hope you managed to label the inscribed angle on the table through process of elimination. We haven’t defined inscription yet in the resource. I also gave a large hint by writing “an” as the article before the blank. Wasn’t that nice of me? - Craig

Example:
Find w, x, y, and z in the two diagrams given.





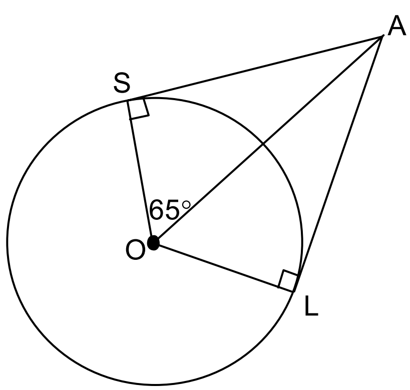
Solution:
In the first circle, we see that w and x are two of the arc measures in a circle. They are also the intercepted arcs of a 16.5° secant-secant angle. The remaining arcs are given as 120° and 165°. This gives us the two-variable system below.[34]

$$\begin{matrix} \frac{1}{2}(w-x)=16.5 \\ w+x+120+165=360 \end{matrix} \Rightarrow \begin{matrix} w-x=33 \\ w+x=45 \end{matrix} \Rightarrow \begin{matrix} 2w=78 \\ w=39 \end{matrix} \Rightarrow x=6$$

In the first problem, w = 39° and x = 6°.

In the second problem, we observe that m∠TOL = 80°. This tells us that the measure of arc TL is also 80°. We hardly have to do any calculations to find y. y is the inscribed angle that intercepts arc TL, and thus y = 40° because an inscribed angle has half the degree measure of the intercepted arc. Now to find z, we note that *z* is the angle measure of a chord-tangent angle, intercepting arc LTR. The measure of arc LTR is 240°, and the value of z must then be 120.
In the second problem, y = 40° and z = 120°.

Example:
Given the drawing of circle O, find m∠SAL.



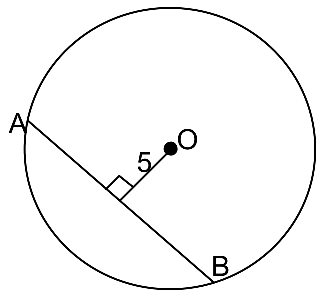
Solution:
We see that m∠SOA = 65°; that means that m∠LOA = 65° also because ΔSOA ≅ ΔLOA.[35] ∠SAL is a tangent-tangent angle; if we remember the alternate formula for a tangent-tangent angle, we know that it is supplementary to minor arc SL. We now know that the minor arc SL has measure of 130°, so m∠SAL = 50°.

[34] I hope you finished studying your algebra. It's this sort of concept overlap that test-writers absolutely adore and test-takers absolutely abhor. This one is not too bad, but problems like this have the potential to become very difficult if you don't completely understand all of the concepts involved.
[35] Can you prove this triangle congruence on your own? Here it will involve the SSS theorem.

THE STORY OF PRINCESS CIRQUE AND MAIDENS CHORDELIA, SECANTIA, AND TANGENTIA[36]

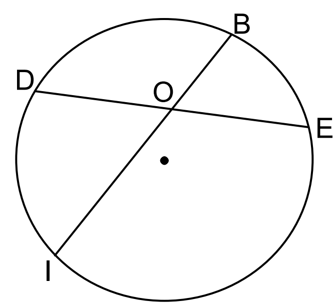
Distance to a Chord from the Center **Chord-Chord Power Theorem** **Secant-Tangent Power Theorem** **Secant-Secant Power Theorem**

When the special angle properties of circles are entirely said and done, there is surprisingly little rote memorization; there are just three possibilities for the placement of the angle's vertex. When we concern ourselves with the lengths of chords, secants, and tangents, however, simplicity is a bit more elusive.



We have already introduced and defined the chord at the beginning of the section concerning circles. Unfortunately, we discussed little about it other than its definition: a chord is a line segment connecting two points on a circle. The chord itself has yet another catalogue of properties that pertain to it. In any circle, the **distance to a chord from the center** is defined as the length of the perpendicular line segment from the center to the chord. For example, in the diagram here, the distance from the center to chord AB is 5. The following is a list of theorems involving chords and their distances from the center.

- If two chords in a circle are the same distance from the center, then they are congruent.
- If two chords in a circle are congruent, then they are the same distance from the center.
- If two chords in a circle are congruent, then their subtended arcs are congruent.
- If two chords are *not* congruent, then the longer chord will be closer to the center.
- If two chords are different distances from the center, the one closer to the center will be longer.
- A circle's longest chord is the diameter (it is a distance of 0 from the center).
- **Chord-Chord Power Theorem:** Two intersecting chords form four line segments such that the product of one chord's line segments equals the product of the other chord's line segments



The last of these theorems may seem a bit wordy and hard to understand, but once you get the words matched with a visual it's usually pretty clear. For the diagram to the right, to say that "the product of one chord's line segments equals the product of the other chord's line segments" means only that $\mathrm{DO} \times \mathrm{OE} = \mathrm{IO} \times \mathrm{OB}$ in this diagram.

The Chord-Chord Power Theorem is used quite frequently. Problems in which chords intersect with specific ratios are fairly common and are not hard to solve. Just remember that the product of one chord's segments equals the product of the other chord's segments. Because you know that secants and tangents also occur with circles, you probably won't be surprised now to learn that there are theorems about those segments, too, all of which deserve memorization. Luckily, the lists of theorems for secants and tangents are shorter and more concise than the list of theorems for chords. There are only four theorems in total for secants and tangents of circles.

Suppose point D is exterior to circle O. The two tangent lines are drawn that pass through D; the points of tangency are L and Y.

- $\overline{DL} \cong \overline{DY}$ - two tangents to a circle from a common point are congruent
- $\overline{OL} \perp \overline{LD}$ and $\overline{OY} \perp \overline{YD}$ - the radius drawn from the circle's center to a point of tangency is perpendicular to that tangent

Now suppose that a circle has a secant and a tangent from a common exterior point.

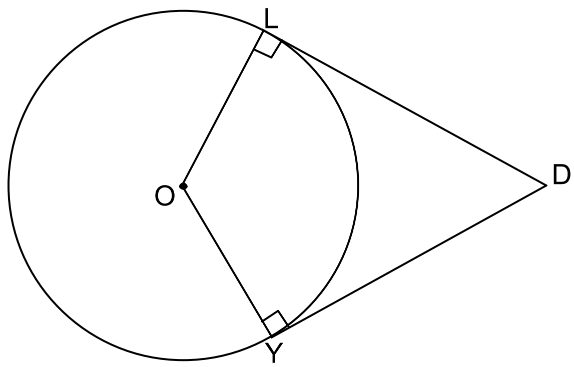
[36] I suppose this title is only creative in one of those sad, twisted ways. Luckily for me, I'm sad and twisted, so it was a perfect match. - Craig

- **Secant-Tangent Power Theorem:** The product of the lengths of the secant and its external part is equal to the square of the length of the tangent.

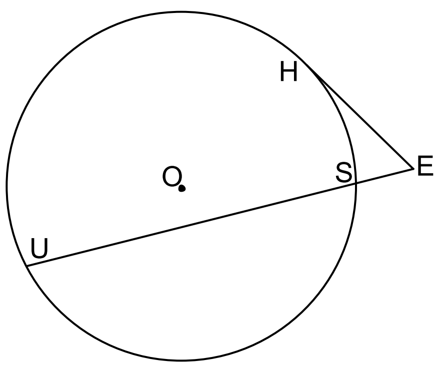
Lastly, suppose that a circle has two secants from a common exterior point.

- **Secant-Secant Power Theorem:** The product of the lengths of one secant and its external part is equal to the product of the lengths of the other secant and its external part.

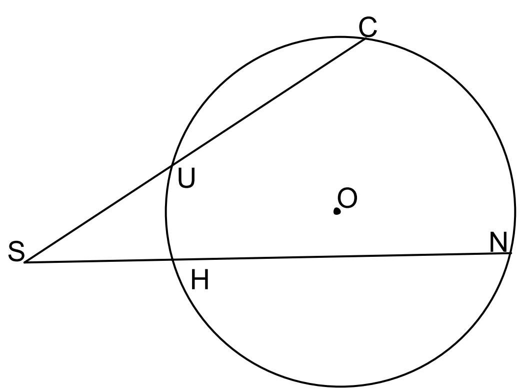
These four theorems concerning secants and tangents are reiterated graphically below.



- Two tangents from a common exterior point are congruent – in this case, $\overline{DL} \cong \overline{DY}$
- Any radius drawn to a point of tangency is perpendicular to the tangent – in this case, $\overline{OL} \perp \overline{DL}$ and $\overline{OY} \perp \overline{DY}$



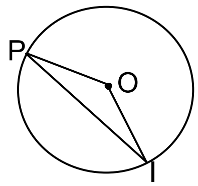
- If a tangent and a secant are drawn from a common exterior point, the product of the secant's length and the length of its external part equals the square of the length of the tangent: in this case, $UE \times SE = (HE)^2$
  [**<u>Secant-Tangent Power Theorem</u>**]



- If two secants are drawn from a common point, the product of the first secant's length and the length of its external part equals the product of the second secant's length and the length of its external part: in this case, $SC \times SU = SN \times SH$
  [**<u>Secant-Secant Power Theorem</u>**]

Example:

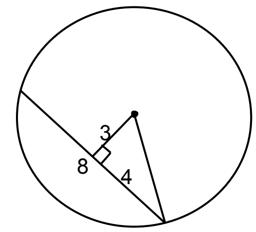
Find the measure of arc PI given that m∠P = 20°.



Solution:

All radii in a circle are congruent, and thus $\overline{OP} \cong \overline{OI}$. By citing the Angle-Side Theorem, we know that ∠P ≅ ∠I so m∠I = 20°. The interior angles of any triangle must add to 180° so m∠POI = 140°. Thus, the measure of arc PI is 140°.

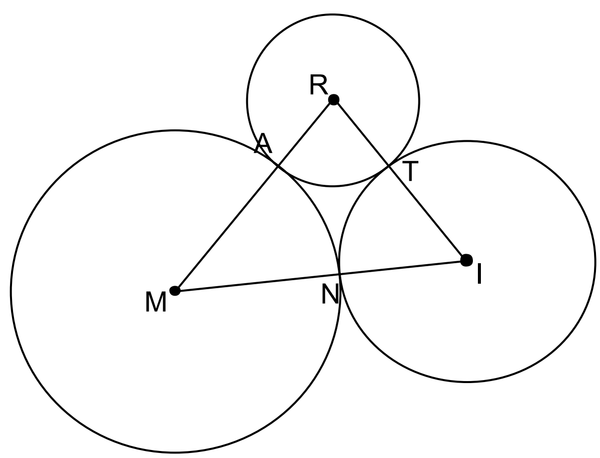
Example:
Find the radius of a circle in which a chord of length 8 is a distance 3 from the center.



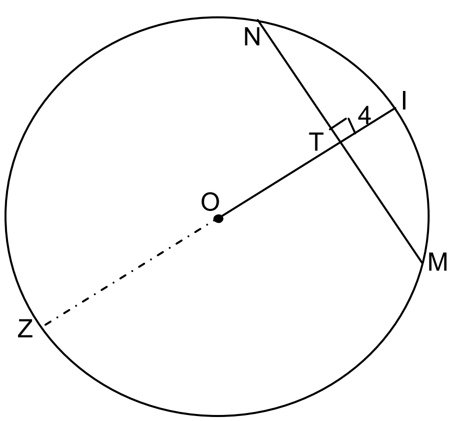
Solution:
The distance to a chord from the center is given to be the perpendicular distance. We then have the drawing here; the radius of the circle is the hypotenuse of a right triangle. It is a recognizable 3, 4, 5 right triangle so the radius of the circle is 5.

Example:
Three circles are externally tangent to each other. A triangle joining the centers of the three circles has sides 9, 11, and 15. What are the radii of the three circles?

Solution:
Without a picture given, we must draw one. We stop to draw three circles that are externally tangent to each other. The two largest circles contain the triangle side of length 15. In the drawing we have, RI = 9, MR = 11, and MI = 15. Let's give RA the value of x and see what we can do. If RA is x, then MA = 11–x since MR = 11. All radii in a circle are congruent so MN = 11–x also. Since MI = 15, we know that NI = 15 – (11–x) = 4 + x. Again, all radii in a circle are congruent, so TI = 4 + x also. Knowing that RI = 9 lastly gives us that RT = 9 – (x + 4) = 5–x. RA = x and RT = 5–x, but they are radii of the same circle so we solve the equation x = 5–x to get the solution x = 2.5. We travel around the circles doing our algebra substitution to get that the radii are 2.5, 8.5, and 6.5 (traveling counter-clockwise).

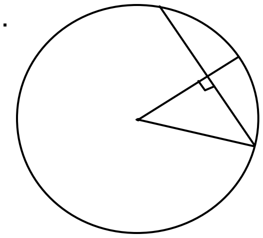


Example:
In this diagram, find the radius of the circle given that the chord has length 16. Solve the problem (a) by using right triangles and (b) by using the Chord-Chord Power Theorem.

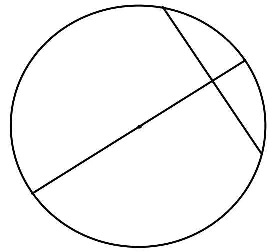


Solution:
We are given that the chord has length 16, so we can begin labeling line segment lengths. We know that NT = MT = 8. We are also given that TI = 4. $\overline{OI}$ is a radius of the circle so OI = r. We then know that OT = r – 4.

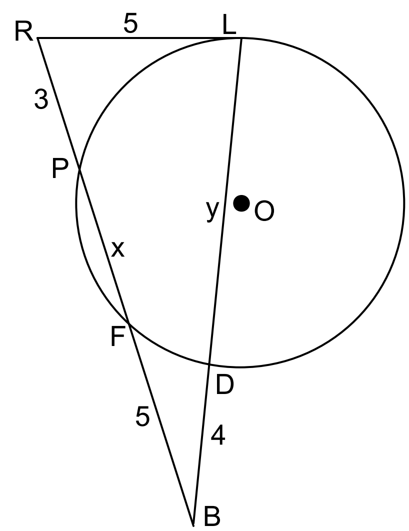
a) To solve for the radius using right triangles, we draw $\overline{OM}$ and label it with length r.
ΔOTM then is a right triangle with hypotenuse $\overline{OM}$.
$OT^2 + TM^2 = OM^2$
$(r - 4)^2 + 82 = r^2$
$r^2 - 8r + 16 + 64 = r^2$
$-8r = -80$
$r = 10$



b) To use the Chord-Chord Power Theorem, we must draw radius $\overline{OZ}$ to create diameter $\overline{ZI}$. ZO = r and OT = r – 4 so we know that ZT = 2r – 4. By the Chord-Chord Power Theorem, we can calculate the radius.
$ZT \times TI = NT \times TM$
$(2r - 4) \times 4 = 8 \times 8$
$8r - 16 = 64$
$8r = 80$
$r = 10$



Example:
Find x and y in the given diagram.



Solution:
The problem requires us to employ the Secant-Secant Power Theorem and the Secant-Tangent Power Theorem in the same problem. First, we should use exterior point R because we will only have to solve for one missing variable instead of two.

We use the Secant-Tangent Power Theorem.

$RF \times RP = RL^2$

$(x + 3) \times 3 = 5^2$

$3x + 9 = 25$

$3x = 16$

$x = \frac{16}{3}$

After we have x, we can find y using the Secant-Secant Power Theorem and exterior point B. We using the Secant-Secant Power Theorem.

$BP \times BF = BL \times BD$

$(5 + \frac{16}{3}) \times 5 = (4 + y) \times 4$

$(\frac{15}{3} + \frac{16}{3}) \times 5 = 16 + 4y$

$\frac{31}{3} \times 5 = 16 + 4y$

$\frac{155}{3} = \frac{48}{3} + 4y$

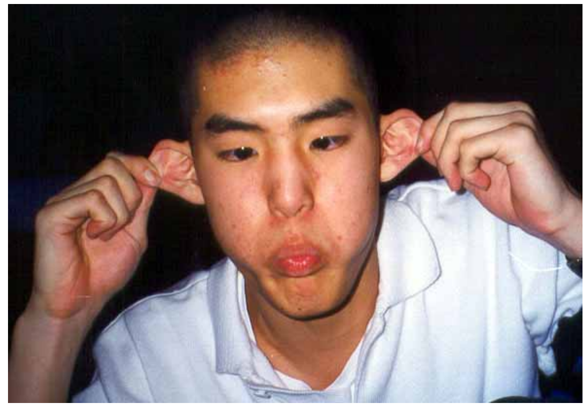
$\frac{107}{3} = 4y$

$\frac{107}{12} = y$

ABOUT THE AUTHOR, PART II

…ged in a broom closet at Nimitz High School. His most noticeable accomplishments include primarily the face found in the picture here, along with passing freshman physics at Caltech.

He is DemiDec's most frequent flyer and may be cont…



Craig in his finest hour.