PLANE AND SPHERICAL TRIGONOMETRY

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PLANE AND SPHERICAL TRIGONOMETRY

BY

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PREFACE TO THE FOURTH EDITION

This edition presents a new set of problems in Plane Trigonometry. The type of problem has been preserved, but the details have been changed. The undersigned acknowledges indebtedness to the members of the Department of Mathematics at the Armour Institute of Technology for valuable suggestions and criticisms. He is especially indebted to Profs. S. F. Bibb and W. A. Spencer for their contribution of many new identities and equations and also expresses thanks to Mr. Clark Palmer, son of the late Dean Palmer, for assisting in checking answers to problems and in proofreading and for offering many constructive criticisms.

CHARLES WILBER LEIGH.

CHICAGO,
June, 1934.

PREFACE TO THE FIRST EDITION

This text has been written because the authors felt the need of a treatment of trigonometry that duly emphasized those parts necessary to a proper understanding of the courses taken in schools of technology. Yet it is hoped that teachers of mathematics in classical colleges and universities as well will find it suited to their needs. It is useless to claim any great originality in treatment or in the selection of subject matter. No attempt has been made to be novel only; but the best ideas and treatment have been used, no matter how often they have appeared in other works on trigonometry.

The following points are to be especially noted:

(1) The measurement of angles is considered at the beginning.

(2) The trigonometric functions are defined at once for any angle, then specialized for the acute angle; not first defined for acute angles, then for obtuse angles, and then for general angles. To do this, use is made of Cartesian coordinates, which are now almost universally taught in elementary algebra.

(3) The treatment of triangles comes in its natural and logical order and is not *forced* to the first pages of the book.

(4) Considerable use is made of the line representation of the trigonometric functions. This makes the proof of certain theorems easier of comprehension and lends itself to many useful applications.

(5) Trigonometric equations are introduced early and used often.

(6) Anti-trigonometric functions are used throughout the work, not placed in a short chapter at the close. They are used in the solutions of equations and triangles. Much stress is laid upon the principal values of anti-trigonometric functions as used later in the more advanced subjects of mathematics.

(7) A limited use is made of the so-called "laboratory method" to impress upon the student certain fundamental ideas.

(8) Numerous carefully graded practical problems are given and an abundance of drill exercises.

(9) There is a chapter on complex numbers, series, and hyperbolic functions.

(10) A very complete treatment is given on the use of logarithmic and trigonometric tables. This is printed in connection with the tables, and so does not break up the continuity of the trigonometry proper.

(11) The tables are carefully compiled and are based upon those of Gauss. Particular attention has been given to the determination of angles near 0 and 90°, and to the functions of such angles. The tables are printed in an unshaded type, and the arrangement on the pages has received careful study.

The authors take this opportunity to express their indebtedness to Prof. D. F. Campbell of the Armour Institute of Technology, Prof. N. C. Riggs of the Carnegie Institute of Technology, and Prof. W. B. Carver of Cornell University, who have read the work in manuscript and proof and have made many valuable suggestions and criticisms.

THE AUTHORS.

CHICAGO,
September, 1914.

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GREEK ALPHABET

A, α	*Alpha*	N, ν	*Nu*
B, β	*Beta*	Ξ, ξ	*Xi*
Γ, γ	*Gamma*	O, o	*Omicron*
Δ, δ	*Delta*	Π, π	*Pi*
E, ε	*Epsilon*	P, ρ	*Rho*
Z, ζ	*Zeta*	Σ, σ	*Sigma*
H, η	*Eta*	T, τ	*Tau*
Θ, θ	*Theta*	Υ, υ	*Upsilon*
I, ι	*Iota*	Φ, φ	*Phi*
K, κ	*Kappa*	X, χ	*Chi*
Λ, λ	*Lambda*	Ψ, ψ	*Psi*
M, μ	*Mu*	Ω, ω	*Omega*

PLANE AND SPHERICAL TRIGONOMETRY

CHAPTER I

INTRODUCTION

GEOMETRY

1. Introductory remarks.—The word trigonometry is derived from two Greek words, *τριγωνον* (trigonon), meaning triangle, and *μετρια* (metria), meaning measurement. While the derivation of the word would seem to confine the subject to triangles, the measurement of triangles is merely a part of the general subject which includes many other investigations involving angles.

Trigonometry is both geometric and algebraic in nature. Historically, trigonometry developed in connection with astronomy, where distances that could not be measured directly were computed by means of angles and lines that could be measured. The beginning of these methods may be traced to Babylon and Ancient Egypt.

The noted Greek astronomer Hipparchus is often called the founder of trigonometry. He did his chief work between 146 and 126 B. C. and developed trigonometry as an aid in measuring angles and lines in connection with astronomy. The subject of trigonometry was separated from astronomy and established as a distinct branch of mathematics by the great mathematician Leonhard Euler, who lived from 1707 to 1783.

To pursue the subject of trigonometry successfully, the student should know the subjects usually treated in algebra up to and including quadratic equations, and be familiar with plane geometry, especially the theorems on triangles and circles.

Frequent use is made of the protractor, compasses, and the straightedge in constructing figures.

While parts of trigonometry can be applied at once to the solution of various interesting and practical problems, much of it is studied because it is very frequently used in more advanced subjects in mathematics.

ANGLES

2. Definitions.—The definition of an angle as given in geometry admits of a clear conception of small angles only. In trigonometry, we wish to consider *positive* and *negative* angles and these of any size whatever; hence we need a more comprehensive definition of an angle.

If a line, starting from the position OX (Fig. 1), is revolved about the point O and always kept in the same plane, we say the line **generates** an angle. If it revolves from the position OX to the position OA, in the direction indicated by the arrow, the angle XOA is generated.

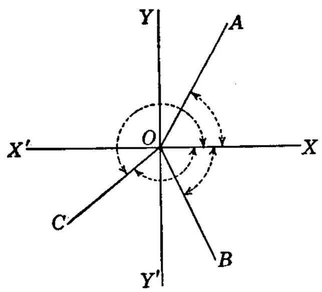

FIG. 1.

The original position OX of the generating line is called the **initial side,** and the final position OA, the **terminal side** of the angle.

If the rotation of the generating line is *counterclockwise*, as already taken, the angle is said to be **positive.** If OX revolves in a *clockwise* direction to a position, as OB, the angle generated is said to be **negative.**

In reading an angle, the letter on the initial side is read first to give the proper sense of direction. If the angle is read in the opposite sense, the negative of the angle is meant.

Thus, $$\angle AOX = -\angle XOA.$$

It is easily seen that this conception of an angle makes it possible to think of an angle as being of any size whatever. Thus, the generating line, when it has reached the position OY, having made a quarter of a revolution in a counterclockwise direction, has generated a right angle; when it has reached the position OX' it has generated two right angles. A complete revolution generates an angle containing four right angles; two revolutions, eight right angles; and so on for any amount of turning.

The right angle is divided into 90 equal parts called degrees (°), each degree is divided into 60 equal parts called minutes (′), and each minute into 60 equal parts called seconds (″).

Starting from any position as initial side, it is evident that for each position of the terminal side, there are two angles less

than 360°, one positive and one negative. Thus, in Fig. 1, OC is the terminal side for the positive angle XOC or for the negative angle XOC.

3. Quadrants.—It is convenient to divide the plane formed by a complete revolution of the generating line into four parts by the two perpendicular lines $X'X$ and $Y'Y$. These parts are called **first, second, third,** and **fourth quadrants,** respectively. They are placed as shown by the Roman numerals in Fig. 2.

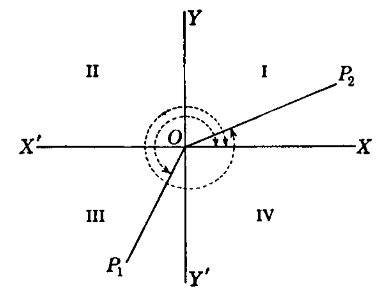

FIG. 2.

If OX is taken as the initial side of an angle, the angle is said to lie in the quadrant in which its terminal side lies. Thus, XOP_1 (Fig. 2) lies in the third quadrant, and XOP_2, formed by more than one revolution, lies in the first quadrant.

An angle lies between two quadrants if its terminal side lies on the line between two quadrants.

4. Graphical addition and subtraction of angles.—Two angles are added by placing them in the same plane with their vertices together and the initial side of the second on the terminal side of the first. The sum is the angle from the initial side of the first to the terminal side of the second.

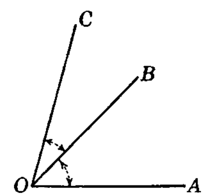

FIG. 3.

Subtraction is performed by adding the negative of the subtrahend to the minuend.

Thus, in Fig. 3,

$$\angle AOB + \angle BOC = \angle AOC.$$
$$\angle AOC - \angle BOC = \angle AOC + \angle COB = \angle AOB.$$
$$\angle BOC - \angle AOC = \angle BOC + \angle COA = \angle BOA.$$

EXERCISES

Use the protractor in laying off the angles in the following exercises:

1. Choose an initial side and lay off the following angles. Indicate each angle by a circular arrow. 75°; 145°; 243°; 729°; 456°; 976°. State the quadrant in which each angle lies.

2. Lay off the following angles and state the quadrant that each is in: −40°; −147°; −295°; −456°; −1048°.

3. Lay off the following pairs of angles, using the same initial side for each pair: 170° and −190°; −40° and 320°; 150° and −210°.

4. Give a positive angle that has the same terminal side as each of the following: 30°; 165°; −90°; −210°; −45°; 395°; −390°.

5. Show by a figure the position of the revolving line when it has generated each of the following: 3 right angles; $2\frac{1}{2}$ right angles; $1\frac{1}{3}$ right angles; $4\frac{2}{3}$ right angles.

Unite graphically, using the protractor:

6. 40° + 70°; 25° + 36°; 95° + 125°; 243° + 725°.

7. 75° − 43°; 125° − 59°; 23° − 49°; 743° − 542°; 90° − 270°.

8. 45° + 30° + 25°; 125° + 46° + 95°; 327° + 25° + 400°.

9. 45° − 56° + 85°; 325° − 256° + 400°.

10. Draw two angles lying in the first quadrant but differing by 360°. Two negative angles in the fourth quadrant and differing by 360°.

11. Draw the following angles and their complements: 30°; 210°; 345°; −45°; −300°; −150°.

5. Angle measurement.—Several systems for measuring angles are in use. The system is chosen that is best adapted to the purpose for which it is used.

(1) *The right angle.*—The most familiar unit of measure of an angle is the right angle. It is easy to construct, enters frequently into the practical uses of life, and is almost always used in geometry. It has no subdivisions and does not lend itself readily to computations.

(2) *The sexagesimal system.*—The **sexagesimal system** has for its fundamental unit the degree, which is defined to be the angle formed by $\frac{1}{360}$ part of a revolution of the generating line. This is the system used by engineers and others in making practical numerical computations. The subdivisions of the degree are the minute and the second, as stated in **Art. 2.** The word "sexagesimal" is derived from the Latin word *sexagesimus*, meaning one-sixtieth.

(3) *The centesimal system.*—Another system for measuring angles was proposed in France somewhat over a century ago. This is the **centesimal system.** In it the right angle is divided into 100 equal parts called **grades,** the grade into 100 equal parts called minutes, and the minute into 100 equal parts called seconds. While this system has many admirable features, its use could not become general without recomputing with a great expenditure of labor many of the existing tables.

(4) *The circular or natural system.*—In the **circular** or **natural system** for measuring angles, sometimes called **radian measure** or **π-measure,** the fundamental unit is the radian.

The radian is defined to be the *angle which, when placed with its vertex at the center of a circle, intercepts an arc equal in length to the radius of the circle.* Or it is defined as *the positive angle generated when a point on the generating line has passed through an arc equal in length to the radius of the circle being formed by that point.*

In Fig. 4, the angles AOB, BOC, · · · FOG are each 1 radian, since the sides of each angle intercept an arc equal in length to the radius of the circle.

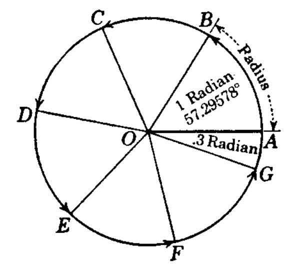

Fig. 4.

The circular system lends itself naturally to the measurement of angles in many theoretical considerations. It is used almost exclusively in the calculus and its applications.

(5) *Other systems.*—Instead of dividing the degree into minutes and seconds, it is sometimes divided into tenths, hundredths, and thousandths. This **decimal scale** has been used more or less ever since decimal fractions were invented in the sixteenth century.

The **mil** is a unit of angle used in artillery practice. The mil is $\frac{1}{6400}$ revolution, or very nearly $\frac{1}{1000}$ radian; hence its name. The scales by means of which the guns in the United States Field Artillery are aimed are graduated in this unit.

6. The radian.—That the circular measure is the natural system to use in measuring an angle is apparent from a consideration of the geometrical basis for the definition of the radian.

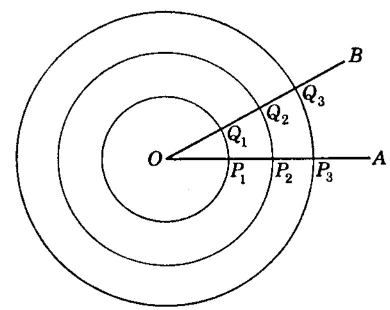

Fig. 5.

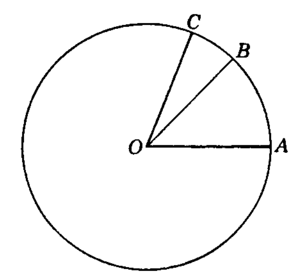

Fig. 6.

(1) Given several concentric circles and an angle AOB at the center as in Fig. 5, then

$$\frac{\text{arc } P_1Q_1}{OP_1} = \frac{\text{arc } P_2Q_2}{OP_2} = \frac{\text{arc } P_3Q_3}{OP_3}, \text{ etc.}$$

That is, the ratio of the intercepted arc to the radius of that arc is a constant for all circles when the angle is the same. *The angle at the center which makes this ratio unity is then a convenient unit for measuring angles. This is* 1 *radian.*

(2) In the same or equal circles, two angles at the center are in the same ratio as their intercepted arcs. That is, in Fig. 6,

$$\frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } AC}.$$

Here, if $\angle AOC$ is unity when arc $AC = r$, $\angle AOB = \dfrac{\text{arc } AB}{r}$, or, in general, $\theta = \dfrac{s}{r}$, where θ is the angle at the center measured in radians, s the arc length, and r the radius of the circle.

7. Relations between radian and degree.—The relations between a degree and a radian can be readily determined from their definitions. Since the circumference of a circle is 2π times the radius,

$$2\pi \text{ radians} = 1 \text{ revolution.}$$

Also $$360° = 1 \text{ revolution.}$$

Then $$2\pi \text{ radians} = 360°.$$

$$\therefore 1 \text{ radian} = \frac{360°}{2\pi} = \frac{180°}{\pi} = 57.29578° -$$
$$= 206264.8'' + = 57°\ 17'\ 44.8'' +.$$

For less accurate work 1 radian is taken as 57.3°.

Conversely, **180° = π radians.**

$$\therefore 1° = \frac{\pi}{180} = 0.0174533 - \text{ radian.}$$

To convert radians to degrees, multiply the number of radians by $\dfrac{180}{\pi}$, *or* 57.29578 −.

To convert degrees to radians, multiply the number of degrees by $\dfrac{\pi}{180}$, *or* 0.017453 +.

In writing an angle in degrees, minutes, and seconds, the signs °, ′, ″ are always expressed. In writing an angle in circular measure, usually no abbreviation is used. Thus, the angle 2 means an angle of 2 radians, the angle $\frac{1}{2}\pi$ means an angle of $\frac{1}{2}\pi$ radians. One should be careful to note that $\frac{1}{2}\pi$ does not denote

an angle, it simply tells how many radians the angle contains. Sometimes radian is abbreviated as follows: 3^r, $3^{(r)}$, 3ρ, or 3 rad. When the word "radians" is omitted, the student should be careful to supply it mentally.

Many of the most frequently used angles are conveniently expressed in radian measure by using π. In this manner the values are expressed accurately and long decimals are avoided. Thus, 180° = π radians, 90° = $\frac{1}{2}\pi$ radians, 60° = $\frac{1}{3}\pi$ radians, 135° = $\frac{3}{4}\pi$ radians, 30° = $\frac{1}{6}\pi$ radians. These forms are more convenient than the decimal form. For instance, $\frac{1}{3}\pi$ radians = 1.0472 radians.

Example 1.—Reduce 2.5 radians to degrees, minutes, and seconds.

Solution.—1 radian = 57.29578°.

Then 2.5 radians = 2.5 × 57.29578° = 143.2394°.

To find the number of minutes, multiply the decimal part of the number of degrees by 60.

$$0.2394° = 60 \times 0.2394 = 14.364'.$$

Likewise, 0.364′ = 60 × 0.364 = 21.8″.

$$\therefore 2.5 \text{ radians} = 143°\ 14'\ 22''.$$

Example 2.—Reduce 22° 36′ 30″ to radians.

Solution.—First, change to degrees and decimal of degree.

This gives $$22°\ 36'\ 30'' = 22.6083°.$$
$$1° = 0.017453 \text{ radian.}$$
$$22.6083° = 22.6083 \times 0.017453 = 0.3946 \text{ radian.}$$
$$\therefore 22°\ 36'\ 30'' = 0.3946 \text{ radian.}$$

EXERCISES

The first eight exercises are to be done orally.

1. Express the angles of the following numbers of radians in degrees: $\frac{1}{2}\pi$; $\frac{2}{3}\pi$; $\frac{3}{2}\pi$; $\frac{5}{6}\pi$; $\frac{4}{3}\pi$; $\frac{3}{4}\pi$; $\frac{11}{6}\pi$; $\frac{7}{6}\pi$.

2. Express the following angles as some number of π radians: 30°; 90°; 180°; 135°; 120°; 240°; 270°; 330°; 225°; 315°; 81°; 360°; 720°.

3. Express the angles of the following numbers of right angles in radians, using π; 2; $\frac{1}{2}$; $\frac{1}{3}$; $\frac{4}{3}$; $3\frac{1}{2}$; $2\frac{1}{3}$; $1\frac{2}{3}$; $3\frac{1}{3}$.

4. Express in radians each angle of an equilateral triangle. Of a regular hexagon. Of an isosceles triangle if the vertex angle is a right angle.

5. How many degrees does the minute hand of a watch turn through in 15 min.? In 20 min.? How many radians in each of these angles?

6. What is the measure of 90° when the right angle is taken as the unit of measure? Of 135°? Of 60°? Of 240°? Of 540°? Of −270°? Of −360°? Of −630°?

7. What is the measure of each of the angles of the previous exercise when the radian is taken as the unit of measure?

8. What is the angular velocity of the second hand of a watch in radians per minute? What is the angular velocity of the minute hand?

Reduce the following angles to degrees, minutes and integral seconds:

9. 2.3 radians. *Ans.* 131° 46′ 49″.
10. 1.42 radians. *Ans.* 81° 21′ 36″.
11. 3.75 radians. *Ans.* 214° 51′ 33″.
12. 0.25 radian. *Ans.* 14° 19′ 26″.
13. $\frac{3}{16}\pi$ radian. *Ans.* 33° 45′.
14. $\frac{11}{4}\pi$ radians. *Ans.* 495°.
15. 0.0074 radian. *Ans.* 25′ 16″.
16. 6.28 radians. *Ans.* 359° 49′ 3″.

Reduce the following angles to radians correct to four decimals, using **Art. 7**:

17. 55°. **18.** 103°. **19.** 265°. **20.** 17°.
21. 24° 37′ 27″. *Ans.* 0.4298.
22. 285° 28′ 56″. *Ans.* 4.9825.
23. 416° 48′ 45″. *Ans.* 7.2746.

Reduce the following angles to radians, using **Table V**, of **Tables.**

24. 25° 14′ 23″. *Ans.* 0.4405162.
25. 175° 42′ 15″. *Ans.* 3.0666162.
26. 78° 15′ 30″. *Ans.* 1.3658655.
27. 243° 35′ 42″. *Ans.* 4.2515348.
28. 69° 25′ 8″. *Ans.* 1.2115882.
29. 9° 9′ 9″. *Ans.* 0.1597412.

30. Compute the equivalents given in **Art. 7.**

31. Show that 1 mil is very nearly 0.001 radian, and find the per cent of error in using 1 mil = 0.001 radian. *Ans.* 1.86 per cent.

32. What is the measure of each of the following angles when the right angle is taken as the unit of measure: 1 radian, 2π radians, 650°, 2.157 radians? *Ans.* 0.6366; 4; 7.222; 1.373.

33. An angular velocity of 10 revolutions per second is how many radians per minute? *Ans.* 3769.91.

34. An angular velocity of 30 revolutions per minute is how many π radians per second? *Ans.* One-π radians.

35. An angular velocity of 80 radians per minute is how many degrees per second? *Ans.* 76.394°.

36. Show that nine-tenths the number of grades in an angle is the number of degrees in that angle.

37. The angles of a triangle are in the ratio of 2:3:7. Express the angles in radians. *Ans.* $\frac{1}{6}\pi$; $\frac{1}{4}\pi$; $\frac{7}{12}\pi$.

38. Express an interior angle of each of the following regular polygons in radians: octagon, pentagon, 16-gon, 59-gon.

39. Express 48° 22′ 25″ in the centesimal system in grades, minutes, and seconds. *Ans.* 53 grades 74 min. 84 sec.

ANGLE AT CENTER OF CIRCLE

8. Relations between angle, arc, and radius.—In **Art. 6,** it is shown that, if the central angle is measured in radians and the arc

length and the radius are measured in the same linear unit, then

$$\textbf{angle} = \frac{\textbf{arc}}{\textbf{radius}}.$$

That is, if θ, s, and r are the measures, respectively, of the angle, arc, and radius (Fig. 7),

$$\boldsymbol{\theta = s \div r,}$$

Solving this for s and then for r,

$$\boldsymbol{s = r\theta,}$$

and

$$\boldsymbol{r = s \div \theta.}$$

These are the simplest geometrical relations between the angle at the center of a circle, the intercepted arc, and the radius. They are of frequent use in mathematics and its applications, and should be remembered.

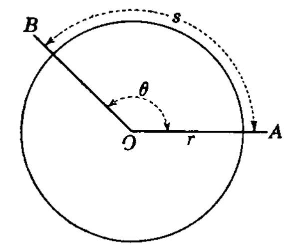

Fig. 7.

Example 1.—The diameter of a graduated circle is 10 ft., and the graduations are 5′ of arc apart; find the length of arc between the graduations in fractions of an inch to three decimal places.

Solution.—By formula, $s = r\theta$.

From the example, $r = 12 \times 5 = 60$ in.,

and $\theta = 0.01745 \times \frac{5}{60} = 0.00145$ radian.

Substituting in the formula, $s = 60 \times 0.00145 = 0.087$.

$\therefore$ length of 5′ arc is 0.087 in.

Example 2.—A train is traveling on a circular curve of $\frac{1}{2}$-mile radius at the rate of 30 miles per hour. Through what angle would the train turn in 45 sec.?

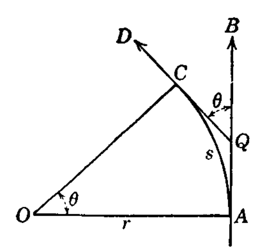

Fig. 8.

Solution.—When at the position A (Fig. 8), the train is moving in the direction AB. After 45 sec. it has reached C, and is then moving in the direction CD. It has then turned through the angle BQC.

But $\angle BQC = \angle AOC = \theta$. Why?

The train travels the arc $s = \frac{3}{8}$ mile in 45 sec.

To find value of θ, use formula

$$\theta = s \div r.$$

$$\therefore\ \theta = \tfrac{3}{8} \div \tfrac{1}{2} = 0.75 \text{ radian} = 42° 58′ 19″.$$

9. Area of circular sector.—In Fig. 9, the area *BOC*, bounded by two radii and an arc of a circle, is a sector. In geometry it is shown that *the area of a sector of a circle equals one-half the arc length times the radius.*

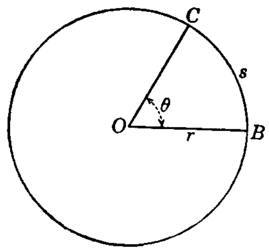

FIG. 9.

That is, $A = \frac{1}{2}rs.$

But $s = r\theta.$

Hence, $A = \frac{1}{2}r^2\theta.$

Example.—Find the area of the sector of a circle having a radius 8 ft. if the central angle is 40°.

Solution.—

$40° = 40 \times 0.01745 = 0.698$ radian.

Using the formula $A = \frac{1}{2}r^2\theta,$

$A = \frac{1}{2} \times 8^2 \times 0.698 = 22.34.$

$\therefore$ area of sector = 22.34 sq. ft.

ORAL EXERCISES

1. How many radians are there in the central angle intercepting an arc of 20 in. on a circle of 5-in. radius?

2. The minute hand of a clock is 4 in. long. Find the distance moved by the outer end when the hand has turned through 3 radians. When it has moved 20 min.

3. A wheel revolves with an angular velocity of 8 radians per second. Find the linear velocity of a point on the circumference if the radius is 6 ft.

4. The velocity of the rim of a flywheel is 75 ft. per second. Find the angular velocity in radians per second if the wheel is 8 ft. in diameter.

5. A pulley carrying a belt is revolving with an angular velocity of 10 radians per second. Find the velocity of the belt if the pulley is 5 ft. in diameter.

6. An angle of 3 mils will intercept what length of arc at 1000 yd.?

7. A freight car 30 ft. in length at right angles to the line of sight intercepts an angle of 2 mils. What is its distance from the observer?

8. A train is traveling on a circular curve of $\frac{1}{2}$-mile radius at the rate of 30 miles an hour. Through what angle does it turn in 15 sec.?

9. A belt traveling 60 ft. per second runs on a pulley 3 ft. in diameter. What is the angular velocity of the pulley in radians per second?

10. A circular target at 3000 yd. subtends an angle of 1 mil at the eye. How large is the target?

WRITTEN EXERCISES

1. The diameter of the drive wheels of a locomotive is 72 in. Find the number of revolutions per minute they make when the engine is going 45 miles per hour. *Ans.* 210.08 r.p.m.

2. A flywheel is revolving at the rate of 456 r.p.m. What angle does a radius of the wheel generate in 1 sec.? Express in degrees and radians. How many π radians are generated in 2.5 sec.? *Ans.* 2736°; 47.752 radians; 38.

3. A flywheel 6 ft. in diameter is revolving at an angular velocity of 30 radians per second. Find the rim velocity in miles per hour. *Ans.* 61.36 miles per hour.

4. The angular velocity of a flywheel is 10π radians per second. Find the circumferential velocity in feet per second if the radius of the wheel is 6 ft. *Ans.* 188.5 ft. per second.

5. A wheel is revolving at an angular velocity of $\frac{5\pi}{3}$ radians per second. Find the number of revolutions per minute. Per hour. *Ans.* 50 r.p.m.; 3000 r.p.h.

6. In a circle of 9-in. radius, how long an arc will have an angle at the center of 2.5 radians? An angle of 155° 36′? *Ans.* 22.5 in.; 24.44 in.

7. An automobile wheel 2.5 ft. in outside diameter rolls along a road, the axle moving at the rate of 45 miles per hour; find the angular velocity in π radians per second. *Ans.* 16.81 π radians.

8. Chicago is at north latitude 41° 59′. Use 3960 miles as the radius of the earth and find the distance from Chicago to the equator. *Ans.* 2901.7 miles.

9. Use 3960 miles as the radius of the earth and find the length in feet of 1″ of arc of the equator. *Ans.* 101.37 ft.

10. A train of cars is running at the rate of 35 miles per hour on a curve of 1000 ft. radius. Find its angular velocity in radians per minute. *Ans.* 3.08 radians per minute.

11. Find the length of arc which at 1 mile will subtend an angle of 1′. An angle of 1″. *Ans.* 1.536 ft.; 0.0253 ft.

12. The radius of the earth's orbit around the sun, which is about 92,700,000 miles, subtends at the star Sirius an angle of about 0.4″. Find the approximate distance of Sirius from the earth. *Ans.* 48 (10^{12}) miles.

13. Assume that the earth moves around the sun in a circle of 93,000,000-mile radius. Find its rate per second, using $365\frac{1}{4}$ days for a revolution. *Ans.* 18.5 miles per second.

14. The earth revolves on its axis once in 24 hours. Use 3960 miles for the radius and find the velocity of a point on the equator in feet per second. Find the angular velocity in radians per hour. In seconds of angle per second of time. *Ans.* 1520.6 ft. per second; 0.262 radian per hour.

15. The circumferential speed generally advised by makers of emery wheels is 5500 ft. per minute. Find the angular velocity in radians per second for a wheel 16 in. in diameter. *Ans.* 137.5 radian per second.

16. Find the area of a circular sector in a circle of 12 in. radius, if the angle is π radians. If 135°. If 5 radians. *Ans.* 226.2 sq. in.; 169.7 sq. in.

17. The perimeter of a sector of a circle is equal to two-thirds the circumference of the circle. Find the angle of the sector in circular measure and in sexagesimal measure. *Ans.* 2.1888 radians; 125° 24.5′.

10. General angles.—In Fig. 10, the angle XOP_1 is 30°; or if the angle is thought of as formed by one complete revolution and 30°, it is 390°; if by two complete revolutions and 30°, it is 750°. So an angle having OX for initial side and OP_1 for terminal side is 30°, $360° + 30°$, $2 \times 360° + 30°$, or, in general, $n \times 360° + 30°$, where n takes the values 0, 1, 2, 3, $\cdots$, that is, n is any integer, zero included.

In radian measure this is $2n\pi + \frac{1}{6}\pi$.

The expression $n \times 360° + 30°$, or $2n\pi + \frac{1}{6}\pi$, is called the **general measure** of all the angles having OX as initial side and OP_1 as terminal side.

If the angle XOP_2 is 30° less than 180°, then the general measure of the angles having OX as initial side and OP_2 as terminal side is an odd number times 180° less 30°; and may be written

$$(2n + 1)180° - 30°,$$

or $$(2n + 1)\pi - \tfrac{1}{6}\pi.$$

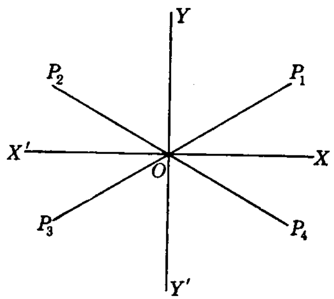

FIG. 10.

Similarly, $n\pi \pm \frac{1}{6}\pi$ means an integral number of times π is taken and then $\frac{1}{6}\pi$ is added or subtracted. This gives the terminal side in one of the four positions shown in Fig. 10 by OP_1, OP_2, OP_3, and OP_4.

It is evident that throughout this article n may have negative as well as positive values, and that any angle θ might be used instead of 30°, or $\frac{1}{6}\pi$.

EXERCISES

1. Use the same initial side for each and draw angles of 50°; $360° + 50°$; $n \cdot 360° + 50°$.
2. Use the same initial side for each and draw angles of 40°; $180° + 40°$; $2 \cdot 180° + 40°$; $3 \cdot 180° + 40°$; $n \cdot 180° + 40°$.
3. Use the same initial side for each and draw angles of 30°; $90° + 30°$; $2 \cdot 90° + 30°$; $3 \cdot 90° + 30°$; $n \cdot 90° + 30°$.
4. Draw the terminal sides for all the angles whose general measure is $2n \cdot 90°$. For all the angles whose general measure is $(2n + 1)90°$.
5. Draw the following angles: $2n\pi$; $(2n + 1)\pi$; $(2n + 1)\frac{1}{2}\pi$; $(4n + 1)\frac{1}{2}\pi$ $(4n + 3)\frac{1}{2}\pi$.
6. Draw the following angles: $2n \times 180° \pm 60°$; $(2n + 1)180° \pm 60°$; $(2n + 1)\pi \pm \frac{1}{3}\pi$; $2n\pi + \frac{1}{3}\pi$; $(2n + 1)\frac{\pi}{2} \pm \frac{\pi}{3}$; $n\pi \pm \frac{1}{4}\pi$; $(4n + 1)\frac{\pi}{2} \pm \frac{\pi}{6}$; $(4n - 1)\frac{\pi}{2} \pm \frac{\pi}{6}$.

7. Give the general measure of all the angles having the lines that bisect the four quadrants as terminal sides. Those that have the lines that trisect the four quadrants as terminal sides.

COORDINATES

11. Directed lines and segments.—For certain purposes in trigonometry it is convenient to give a line a property not often used in plane geometry. This is the property of having *direction.*

In Fig. 11, RQ is a directed straight line if it is thought of as traced by a point moving without change of direction from R toward Q or from Q toward R. The direction is often shown by an arrow.

Let a fixed point O on RQ be taken as a point from which to measure distances. Choose a fixed length as a unit and lay it off on the line RQ beginning at O. The successive points located in this manner will be 1, 2, 3, 4, $\cdots$ times the unit distance

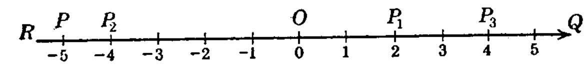

FIG. 11.

from O. These points may be thought of as representing the numbers, or the numbers may be thought of as representing the points.

Since there are two directions from O in which the measurements may be made, it is evident that there are two points equally distant from O. Since there are both positive and negative numbers, we shall *agree* to represent the points to the *right* of O by positive numbers and those to the *left* by negative numbers.

Thus, a point 2 units to the right of O represents the number 2; and, conversely, the number 2 represents a point 2 units to the right of O. A point 4 units to the left of O represents the number -4; and, conversely, the number -4 represents a point 4 units to the left of O.

The point O from which the measurements are made is called the **origin.** It represents the number zero.

A **segment** of a line is a definite part of a directed line.

The segment of a line is read by giving its initial point and its terminal point. Thus, in Fig. 11, OP_1, OP_2, and P_1P_3 are segments. In the last, P_1 is the **initial point** and P_3 the **terminal point.**

The **value** of a segment is determined by its length and direction, and it is defined to be *the number which would represent the terminal point of the segment if the initial point were taken as origin.*

It follows from this definition that the value of a segment read in one direction is the negative of the value if read in the opposite direction.

In Fig. 12, taking O as origin, the values of the segments are as follows:

$$OP_1 = 3,\ OP_3 = 8,\ OP_5 = -5,\ P_2P_3 = 3,\ P_3P_1 = -5.$$
$$P_4P_6 = -6,\ P_6P_5 = 3,\ P_1P_2 = -P_2P_1 = 2.$$

Two segments are equal if they have the same direction and the same length, that is, the same value.

If two segments are so placed that the initial point of the second is on the terminal point of the first, the **sum of the two segments**

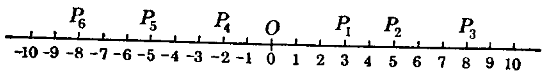

FIG. 12.

is the segment having as initial point the initial point of the first, and as terminal point the terminal point of the second.

The segments are subtracted by reversing the direction of the subtrahend and adding.

Thus, in Fig. 12,

$$P_5P_4 + P_4P_1 = P_5P_1 = 8.$$
$$P_2P_4 + P_4P_6 = P_2P_6 = -13.$$
$$P_1P_3 - P_2P_3 = P_1P_3 + P_3P_2 = P_1P_2 = 2.$$
$$P_2P_3 - P_1P_3 = P_2P_3 + P_3P_1 = P_2P_1 = -2.$$

12. Rectangular coordinates.—Let $X'X$ and $Y'Y$ (Fig. 13) be two fixed directed straight lines, perpendicular to each other and intersecting at the point O. Choose the positive direction towards the right, when parallel to $X'X$; and upwards, when parallel to $Y'Y$. Hence the negative directions are towards the left, and downwards.

The two lines $X'X$ and $Y'Y$ divide the plane into four quadrants, numbered as in **Art. 3.**

Any point P_1 in the plane is located by the segments NP_1 and MP_1 drawn parallel to $X'X$ and $Y'Y$ respectively, for the values of these segments tell how far and in what direction P_1 is from the two lines $X'X$ and $Y'Y$.

It is evident that for any point in the plane there is *one pair of values and only one;* and, conversely, for every pair of values there is *one point and only one.*

The value of the segment NP_1 or OM is called the **abscissa** of the point P_1, and is usually represented by x. The value of the segment MP_1 or ON is called the **ordinate** of the point P_1, and is usually represented by y. Taken together the abscissa x and the ordinate y are called the **coordinates** of the point P_1. They are written, for brevity, within parentheses and separated by a comma, the abscissa always being first, as (x, y).

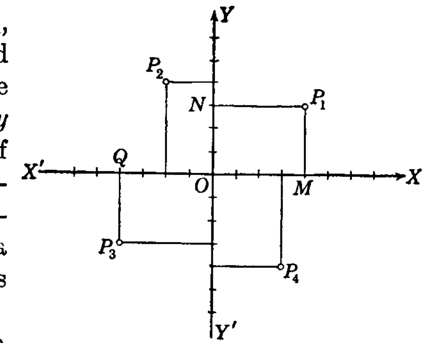

FIG. 13.

The line $X'X$ is called the **axis of abscissas** or the **x-axis.** The line $Y'Y$ is called the **axis of ordinates** or the **y-axis.** Together, these lines are called the **coordinate axes.**

It is evident that, in the first quadrant, both coordinates are positive; in the second quadrant, the abscissa is negative and the ordinate is positive; in the third quadrant, both coordinates are negative; and, in the fourth quadrant, the abscissa is positive and the ordinate is negative. This is shown in the following table:

Quadrant	I	II	III	IV
Abscissa.	+	−	−	+
Ordinate.	+	+	−	−

Thus, in Fig. 13, P_1, P_2, P_3, and P_4 are, respectively, the points $(4, 3)$, $(-2, 4)$, $(-4, -3)$, and $(3, -4)$. The points M, O, N, and Q are, respectively, $(4, 0)$, $(0, 0)$, $(0, 3)$, and $(-4, 0)$.

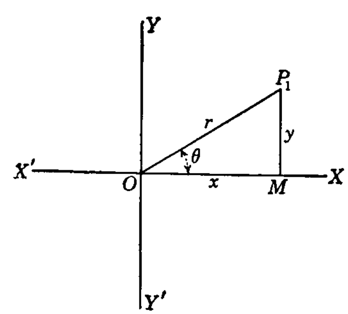

FIG. 14.

13. Polar coordinates.—The point P_1 (Fig. 14) can also be located if the angle θ and the length of the line OP_1 are known. The line OP_1 is called the **radius vector** and is usually represented by r. Since r denotes the distance of the point P_1 from O, it is always considered positive.

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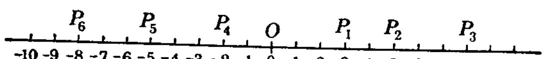

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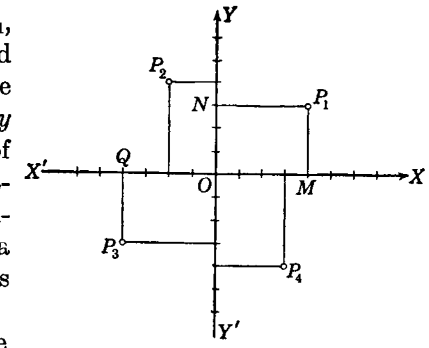

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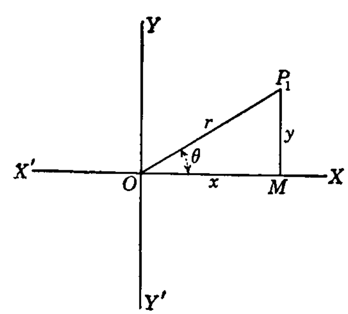

FIG. 14.

13. Polar coordinates.—The point P_1 (Fig. 14) can also be located if the angle θ and the length of the line OP_1 are known. The line OP_1 is called the **radius vector** and is usually represented by r. Since r denotes the distance of the point P_1 from O, it is always considered positive.

Point O is called the pole. The corresponding values of r and θ taken together are called the *polar coordinates* of the point P.

It is seen that r is the hypotenuse of a right triangle of which x and y are the legs; hence $r^2 = x^2 + y^2$, no matter in what quadrant the point is located.

EXERCISES

1. Plot the points (4, 5), (2, 7), (0, 4), (5, 5), (7, 0), (−2, 4), (−4, 5), (−6, −2), (0, −7), (−6, 0), (3, −4), (7, −6).

2. Find the radius vector for each of the points in Exercise 1. Plot in each case. *Ans.* 6.40; 7.28; 4; 7.07.

3. Where are all the points whose abscissas are 5? Whose ordinates are 0? Whose abscissas are −2? Whose *radius vectors* are 3?

4. The positive direction of the x-axis is taken as the initial side of an angle of 60°. A point is taken on the terminal side with a radius vector equal to 12. Find the ordinate and the abscissa of the point.

5. In Exercise 4, what is the ratio of the ordinate to the abscissa? The ratio of the radius vector to the ordinate? Show that you get the same ratios if any other point on the terminal side is taken.

6. With the positive x-axis as initial side, construct angles of 30°, 135°, 240°, 300°. Take a point on the terminal side so that the radius vector is $2a$ in each case, and find the length of the ordinate and the abscissa of the point.

7. The hour hand of a clock is 2 ft. long. Find the coordinates of its outer end when it is twelve o'clock; when three; nine; half-past ten. Use perpendicular and horizontal axes intersecting where the hands are fastened *Ans.* (0, 2); (2, 0); (−2, 0); (−1.414, 1.414).

CHAPTER II

TRIGONOMETRIC FUNCTIONS OF ONE ANGLE

14. Functions of an angle.—Connected with any angle there are six ratios that are of fundamental importance, as upon them is founded the whole subject of trigonometry. They are called **trigonometric ratios** or **trigonometric functions of the angle.**

One of the first things to be done in trigonometry is to investigate the properties of these ratios, and to establish relations

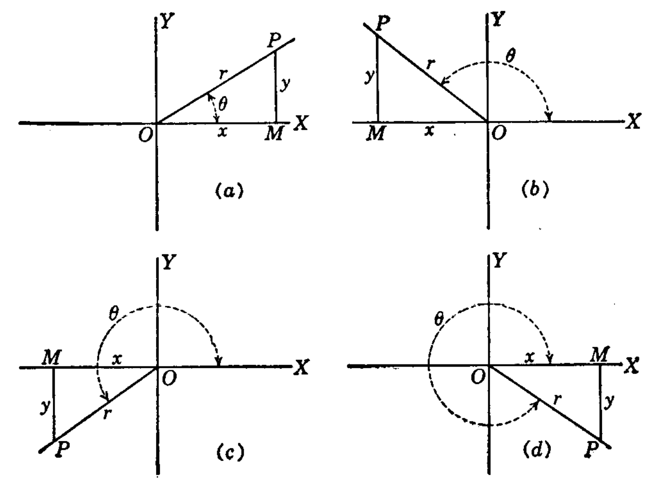

Fig. 15.

between them, as they are the tools by which we work all sorts of problems in trigonometry.

15. Trigonometric ratios.—Draw an angle θ in each of the four quadrants as shown in Fig. 15, each angle having its vertex at the origin and its initial side coinciding with the positive part of the x-axis. Choose any point P (x, y) in the terminal side of such angle at the distance r from the origin. Draw $MP \perp OX$, forming the coordinates $OM = x$ and $MP = y$, and the radius vector, or distance, $OP = r$. Then in whatever quadrant θ is found, the functions are defined as follows:

$$\textbf{sine } \theta \text{ (written } \textbf{sin } \theta) = \frac{\text{ordinate}}{\text{distance}} = \frac{MP}{OP} = \frac{y}{r}.$$
$$\textbf{cosine } \theta \text{ (written } \textbf{cos } \theta) = \frac{\text{abscissa}}{\text{distance}} = \frac{OM}{OP} = \frac{x}{r}.$$
$$\textbf{tangent } \theta \text{ (written } \textbf{tan } \theta) = \frac{\text{ordinate}}{\text{abscissa}} = \frac{MP}{OM} = \frac{y}{x}.$$
$$\textbf{cotangent } \theta \text{ (written } \textbf{cot } \theta) = \frac{\text{abscissa}}{\text{ordinate}} = \frac{OM}{MP} = \frac{x}{y}.$$
$$\textbf{secant } \theta \text{ (written } \textbf{sec } \theta) = \frac{\text{distance}}{\text{abscissa}} = \frac{OP}{OM} = \frac{r}{x}.$$
$$\textbf{cosecant } \theta \text{ (written } \textbf{csc } \theta) = \frac{\text{distance}}{\text{ordinate}} = \frac{OP}{MP} = \frac{r}{y}.$$

Two other functions frequently used are:

$$\textbf{versed sine } \theta \text{ (written } \textbf{vers } \theta) = 1 - \cos \theta.$$
$$\textbf{coversed sine } \theta \text{ (written } \textbf{covers } \theta) = 1 - \sin \theta.$$

The trigonometric functions are pure numbers, that is, abstract numbers, and are subject to the ordinary rules of algebra, such as addition, subtraction, multiplication, and division.

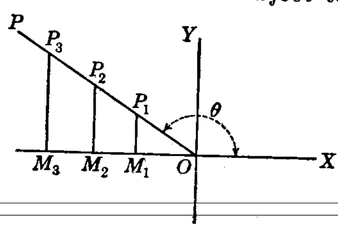

Fig. 16.

16. Correspondence between angles and trigonometric ratios. *To each and every angle there corresponds but one value of each trigonometric ratio.* Draw any angle θ as in Fig. 16. Choose points P_1, P_2, P_3, etc. on the terminal side OP. Draw M_1P_1, M_2P_2, M_3P_3, etc. perpendicular to OX. From the geometry of the figure,

$$\frac{M_1P_1}{OP_1} = \frac{M_2P_2}{OP_2} = \frac{M_3P_3}{OP_3} = \text{etc.} = \sin \theta,$$
$$\frac{OM_1}{OP_1} = \frac{OM_2}{OP_2} = \frac{OM_3}{OP_3} = \text{etc.} = \cos \theta,$$
$$\frac{M_1P_1}{OM_1} = \frac{M_2P_2}{OM_2} = \frac{M_3P_3}{OM_3} = \text{etc.} = \tan \theta,$$

and similarly for the other trigonometric ratios. Hence, the six ratios remain unchanged as long as the value of the angle is unchanged.

It is this exactness of relations between angles and certain lines connected with them that makes it possible to consider a great variety of questions by means of trigonometry which cannot

be handled by methods of geometry. Geometry gives but few relations between angles and lines that can be used in computations, as most of these relations are stated in a comparative manner—for instance, in a triangle, the greater side is opposite the greater angle.

Definition.—When one quantity so depends on another that for every value of the first there are one or more values of the second, the second is said to be a **function** of the first.

Since to every value of the angle there corresponds a value for each of the trigonometric ratios, the ratios are called **trigonometric functions.**

They are also called **natural trigonometric functions** in order to distinguish them from logarithmic trigonometric functions.

A table of natural trigonometric functions for angles 0 to 90° for each minute is given on pages 112 to 134 of **Tables.*** An explanation of the table is given on page 29 of **Tables.**

17. Signs of the trigonometric functions.—The sine of an angle θ has been defined as the ratio of the ordinate to the distance of any point in the terminal side of the angle. Since the distance r is always positive (**Art. 13**), sin θ will have the same algebraic sign as the ordinate of the point. Therefore, sin θ is positive when the angle is in the first or second quadrant, and negative when the angle is in the third or fourth quadrant.

In a similar manner the algebraic signs of the remaining functions of θ are determined. The student should verify the following table:

Quadrant	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

It is very important that one should be able to tell immediately the sign of any trigonometric function in any quadrant. The signs may be remembered by memorizing the table given; but, for most students, they may be more readily remembered by discerning relations between the signs of the functions. One

* The reference is to "Logarithmic and Trigonometric Tables" by the authors.

good scheme is to fix in mind the signs of the sine and cosine. Then if the sine and cosine have like signs, the tangent is plus; and if they have unlike signs, the tangent is negative. The signs of the cosecant, secant, and cotangent always agree respectively with the sine, cosine, and tangent. The scheme shown in Fig. 17 may help in remembering the signs.

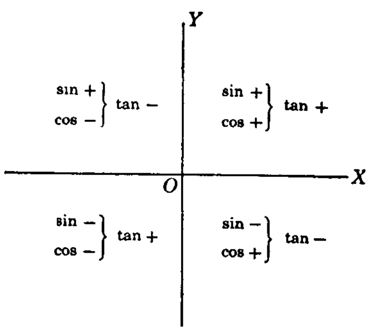

FIG. 17.

EXERCISES

Answer Exercises 1 to 27 orally.

In what quadrant does the terminal side of the angle lie in each of the following cases:

1. When all the functions are positive?

2. When $\sin\theta$ is positive and $\cos\theta$ negative?

3. When $\sin\theta$ is positive and $\tan\theta$ negative?

4. When $\cos\theta$ is positive and $\tan\theta$ negative?

5. When $\sin\theta$ is negative and $\tan\theta$ positive?

6. When $\sin\theta$ is negative and $\cos\theta$ negative?

7. When $\sec\theta$ is negative and $\csc\theta$ negative?

Give the sign of each of the trigonometric functions of the following angles:

8. $120°$.	**12.** $\frac{1}{3}\pi$.	**16.** $\frac{27}{4}\pi$.	**20.** $2n\pi + \frac{1}{3}\pi$.
9. $230°$.	**13.** $\frac{3}{4}\pi$.	**17.** $-27°$.	**21.** $2n\pi - \frac{1}{4}\pi$.
10. $340°$.	**14.** $\frac{11}{6}\pi$.	**18.** $-213°$.	**22.** $(2n + 1)\pi - \frac{1}{6}\pi$.
11. $520°$.	**15.** $\frac{5}{4}\pi$.	**19.** $-700°$.	**23.** $(2n + 1)\pi + \frac{1}{3}\pi$.

24. Show that neither the sine nor the cosine of an angle can be greater than $+1$ or less than -1.

25. Show that neither the secant nor the cosecant of an angle can have a value between -1 and $+1$.

26. Show that the tangent and the cotangent of an angle may have any real value whatever.

27. Is there an angle whose tangent is positive and whose cotangent is negative? Whose secant is positive and whose cosine is negative? Whose secant is positive and whose cosecant is negative?

Construct and measure the following acute angles:

28. Whose sine is $\frac{2}{5}$.

29. Whose tangent is $\frac{2}{5}$.

30. Whose cosine is $\frac{2}{3}$.

31. Whose cotangent is 3.

32. Whose secant is $\frac{4}{3}$.

33. Whose cosecant is $\frac{5}{3}$.

COMPUTATIONS OF TRIGONOMETRIC FUNCTIONS

18. Calculation from measurements. *Example.*—Determine the approximate values of the functions of 25°. By means of the protractor draw angle $XOP = 25°$ (Fig. 18). Choose P in the

terminal side, say, $2\frac{3}{16}$ in. distant from the origin. Draw $MP \perp OX$. By measurement, $OM = 2$ in. and $MP = \frac{15}{16}$ in. From the definitions we have:

$$\sin 25° = \frac{MP}{OP} = \frac{\frac{15}{16}}{2\frac{3}{16}} = 0.43. \quad \cos 25° = \frac{OM}{OP} = \frac{2}{2\frac{3}{16}} = 0.91.$$

$$\tan 25° = \frac{MP}{OM} = \frac{\frac{15}{16}}{2} = 0.47. \quad \cot 25° = \frac{OM}{MP} = \frac{2}{\frac{15}{16}} = 2.13.$$

$$\sec 25° = \frac{OP}{OM} = \frac{2\frac{3}{16}}{2} = 1.09. \quad \csc 25° = \frac{OP}{MP} = \frac{2\frac{3}{16}}{\frac{15}{16}} = 2.33.$$

$$\text{vers } 25° = 1 - \cos 25° = 1 - 0.91 = 0.09.$$

$$\text{covers } 25° = 1 - \sin 25° = 1 - 0.43 = 0.57.$$

In a similar manner any angle can be constructed, measurements taken, and the functions computed; but the results will be only approximate because of the inaccuracy of measurement.

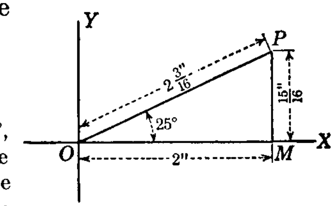

FIG. 18.

EXERCISE

In the same figure construct angles of 10°, 20°, 30°, · · · 80°, with their vertices at the origin and their initial sides on the positive part of the x-axis. Choose the same distance on the terminal side of each angle, draw and measure the coordinates, and calculate the trigonometric functions of each angle to two decimal places. Tabulate the results and compare with Table IV.

19. Calculations from geometric relations.—There are two right triangles for which geometry gives definite relations between sides and angles. These are the right isosceles triangle whose acute angles are each 45°, and the right triangles whose acute angles are 30 and 60°. The functions of any angle for which the abscissa, ordinate, and distance form one of these triangles can readily be computed to any desired degree of accuracy. All such angles, together with 0, 90, 180, 270, and 360°, with their functions are tabulated on page 24. These are very important for future use.

20. Trigonometric functions of 30°.—Draw angle $XOP = 30°$ as in Fig. 19. Choose P in the terminal side and draw $MP \perp OX$. By geometry, MP, the side opposite the 30°-angle, is one-half the hypotenuse OP. Take $y = MP = 1$ unit. Then $r = OP = 2$ units, and $x = OM = \sqrt{3}$. By definition, then, we have:

$$\sin 30° = \frac{y}{r} = \frac{1}{2}. \qquad \cot 30° = \frac{x}{y} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

$$\cos 30° = \frac{x}{r} = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}. \qquad \sec 30° = \frac{r}{x} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}.$$

$$\tan 30° = \frac{y}{x} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1}{3}\sqrt{3}. \qquad \csc 30° = \frac{r}{y} = \frac{2}{1} = 2.$$

21. Trigonometric functions of 45°.—Draw angle $XOP = 45°$ as in Fig. 20. Choose the point P in the terminal side and draw its coordinates OM and MP, which are necessarily equal. Then

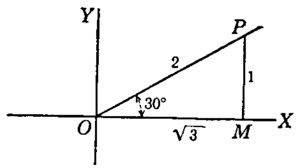

FIG. 19.

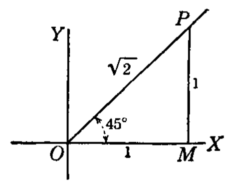

FIG. 20.

the coordinates of P may be taken as (1, 1), and $r = \sqrt{2}$. By definition, then, we have:

$$\sin 45° = \frac{y}{r} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}. \qquad \cot 45° = \frac{x}{y} = \frac{1}{1} = 1.$$

$$\cos 45° = \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}. \qquad \sec 45° = \frac{r}{x} = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

$$\tan 45° = \frac{y}{x} = \frac{1}{1} = 1. \qquad \csc 45° = \frac{r}{y} = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

22. Trigonometric functions of 120°.—Draw angle $XOP = 120°$ as in Fig. 21. Choose any point P in the terminal side and draw its coordinates OM and MP. Triangle MOP is a right triangle with $\angle MOP = 60°$. Then, as in computing the functions of 30°, we may take $OP = 2$, $MO = 1$, and $MP = \sqrt{3}$. But the abscissa of P is $OM = -1$. Then the coordinates of P are $(-1, \sqrt{3})$, and $r = 2$. By definition, then, we have:

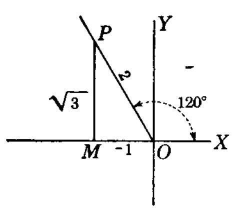

FIG. 21.

$$\sin 120° = \frac{y}{r} = \frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{3}.$$

$$\cos 120° = \frac{x}{r} = \frac{-1}{2} = -\frac{1}{2}.$$

$$\tan 120° = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}.$$

$$\cot 120° = \frac{x}{y} = \frac{-1}{\sqrt{3}} = -\frac{1}{3}\sqrt{3}.$$

$$\sec 120° = \frac{r}{x} = \frac{2}{-1} = -2.$$

$$\csc 120° = \frac{r}{y} = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}.$$

In forming the ratios for the angles whose terminal sides lie on the lines between the quadrants, such as 0, 90, 180, 270, and 360°, the denominator is frequently zero. Strictly speaking, this gives rise to an impossibility for division by zero is meaningless. In all such cases we say that the function has become infinite.

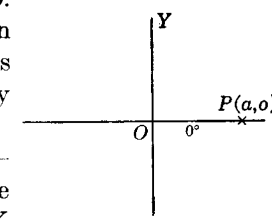

FIG. 22.

23. Trigonometric functions of 0°.—The initial and terminal sides of 0° are both on OX. Choose the point P on OX as in Fig. 22, at the distance of a from O. Then the coordinates of P are $(a, 0)$, and $r = a$. By definition, then, we have:

$$\sin 0° = \frac{y}{r} = \frac{0}{a} = 0. \qquad \tan 0° = \frac{y}{x} = \frac{0}{a} = 0.$$

$$\cos 0° = \frac{x}{r} = \frac{a}{a} = 1. \qquad \sec 0° = \frac{r}{x} = \frac{a}{a} = 1.$$

cot 0° and csc 0° have no meaning.*

24. Trigonometric functions of 90°.—Draw angle $XOY = 90°$ as in Fig. 23. Choose any point P in the terminal side at

* By the expression $\frac{a}{0} = \infty$ is understood the value of $\frac{a}{x}$ as x approaches zero as a limit. For example, $\frac{a}{1} = a$; $\frac{a}{0.1} = 10a$; $\frac{a}{0.01} = 100a$; $\frac{a}{0.001} = 1000a$; $\frac{a}{0.0000001} = 10{,}000{,}000a$; etc. That is, as x gets nearer and nearer to zero $\frac{a}{x}$ gets larger and larger, and can be made to become larger than any number N. The value of $\frac{a}{x}$ is then said to become infinite as x approaches zero. The symbol is ∞ usually read infinity. It should be carefully noted that a is not divided by 0, for division by 0 is meaningless.

Whenever the symbol "∞" is used it should be read "has no meaning."

FREQUENTLY USED ANGLES AND THEIR FUNCTIONS

$\theta°$	θ in radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0°	0	0	1	0	∞	1	∞
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	∞	0	∞	1
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2
180°	π	0	-1	0	∞	-1	∞
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	-2
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	-2	$-\frac{2\sqrt{3}}{3}$
270°	$\frac{3\pi}{2}$	-1	0	∞	0	∞	-1
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2
360°	2π	0	1	0	∞	1	∞

the distance a from the origin. Then the coordinates of P are $(0, a)$, and $r = a$. By definition, then, we have:

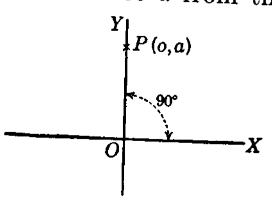

FIG. 23.

$$\sin 90° = \frac{y}{r} = \frac{a}{a} = 1.$$

$$\cot 90° = \frac{x}{y} = \frac{0}{a} = 0.$$

$$\cos 90° = \frac{x}{r} = \frac{0}{a} = 0. \qquad \csc 90° = \frac{r}{y} = \frac{a}{a} = 1.$$

$\tan 90°$ and $\sec 90°$ have no meaning.

EXERCISES

Construct the figure and compute the functions for each of the following angles.

1. 60°
2. 135°.
3. 150°.
4. 180°.
5. 240°.
6. 330°.
7. 270°.
8. 315°.

25. Exponents of trigonometric functions.—When the trigonometric functions are to be raised to powers, they are written $\sin^2\theta$, $\cos^3\theta$, $\tan^4\theta$, etc., instead of $(\sin\theta)^2$, $(\cos\theta)^3$, $(\tan\theta)^4$, etc., except when the exponent is -1. Then the function is enclosed in parentheses. Thus, $(\sin\theta)^{-1} = \frac{1}{\sin\theta}$ (see **Art. 35**).

EXERCISES

Find that the numerical values of each of the Exercises 1 to 10 is unity.

1. $\sin^2 30° + \cos^2 30°$.
2. $\sin^2 60° + \cos^2 60°$.
3. $\sin^2 120° + \cos^2 120°$.
4. $\sin^2 135° + \cos^2 135°$.
5. $\sin^2 300° + \cos^2 300°$.
6. $\sec^2 30° - \tan^2 30°$.
7. $\sec^2 150° - \tan^2 150°$.
8. $\sec^2 330° - \tan^2 330°$.
9. $\csc^2 45° - \cot^2 45°$.
10. $\csc^2 240° - \cot^2 240°$.

Find the numerical values of the following expressions correct to three decimal places:

11. $\sin 45° + 3\cos 60°$. *Ans.* 2.207.
12. $\cos^2 60° + \sin^3 90°$. *Ans.* 1.250.
13. $10\cos^4 30° + \sec 45°$. *Ans.* 7.039.
14. $\sec 0° \cdot \cos 60° + \csc 90° \sec^2 45°$. *Ans.* 2.500.
15. $\cos 120° \cos 270° - \sin 90° \tan^3 135°$. *Ans.* 1.000.

In the following expressions, show that the left-hand member is equal to the right, by using the table on page 24:

16. $\sin 60° \cos 30° + \cos 60° \sin 30° = \sin 90°$.
17. $\cos 45° \cos 135° - \sin 45° \sin 135° = \cos 180°$.
18. $\sin 60° \cos 30° - \cos 60° \sin 30° = \sin 30°$.
19. $\cos 210° \cos 30° - \sin 210° \sin 30° = \cos 240°$.
20. $\sin 300° \cos 30° - \cos 300° \sin 30° = \sin 270°$.
21. $\frac{\tan 240° + \tan 60°}{1 - \tan 240° \tan 60°} = \tan 300°$.
22. $\frac{\tan 120° - \tan 60°}{1 + \tan 120° \tan 60°} = \tan 60°$.

26. Given the function of an acute angle, to construct the angle. *Example* 1.—Given $\sin\theta = \frac{4}{5}$. Construct angle θ and find the other functions.

Solution.—By definition, $\sin\theta = \frac{y}{r} = \frac{4}{5}$. Since we are concerned only with the ratios of the lines, we may take $y = 4$, and $r = 5$ units of any size. Draw AB parallel to OX and 4 units above (Fig. 24), intersecting OY at N. With the origin as a center and

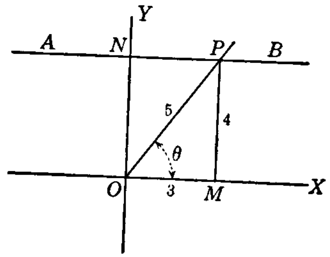

FIG. 24.

a radius of 5 units, draw an arc intersecting AB in the point P. Draw OP forming $\angle XOP$, and draw $MP \perp OX$. Then $OP = 5$, $MP = 4$, and

$$OM = \sqrt{\overline{OP}^2 - \overline{MP}^2} = \sqrt{25 - 16} = 3.$$

$\therefore \angle XOP = \theta$ is the required angle since $\sin\theta = \frac{MP}{OP} = \frac{4}{5}$.

The remaining functions may be written as follows:

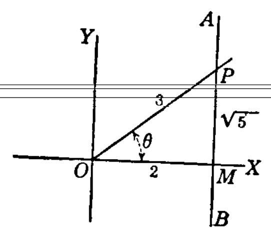

FIG. 25.

$$\cos\theta = \frac{OM}{OP} = \frac{3}{5},\ \tan\theta = \frac{MP}{OM} = \frac{4}{3},\ \cot\theta = \frac{OM}{MP} = \frac{3}{4},$$
$$\sec\theta = \frac{OP}{OM} = \frac{5}{3},\ \csc\theta = \frac{OP}{MP} = \frac{5}{4}.$$

Example 2.—Given $\cos\theta = \frac{2}{3}$. Construct angle θ and find the other functions.

Solution.—By definition, $\cos\theta = \frac{x}{r} = \frac{2}{3}$. Choose $x = 2$ and $r = 3$. Draw $AB \parallel OY$ and 2 units to the right (Fig. 25), intersecting OX at M. With the origin as a center and a radius of 3 units, draw an arc cutting AB at P. Join O and P, forming $\angle XOP$. Then $OP = 3$, $OM = 2$, and

$$MP = \sqrt{\overline{OP}^2 - \overline{OM}^2} = \sqrt{5}.$$

$\therefore \angle XOP = \theta$ is the required angle since $\cos\theta = \frac{OM}{OP} = \frac{2}{3}$.

The remaining functions are as follows:

$$\sin\theta = \frac{MP}{OP} = \frac{\sqrt{5}}{3},\ \tan\theta = \frac{MP}{OM} = \frac{\sqrt{5}}{2},\ \cot\theta = \frac{OM}{MP} = \frac{2}{\sqrt{5}},$$
$$\sec\theta = \frac{OP}{OM} = \frac{3}{2},\ \csc\theta = \frac{OP}{MP} = \frac{3}{\sqrt{5}}.$$

Example 3.—Given $\tan\theta = \frac{2}{5}$. Construct angle θ and find the other functions.

Solution.—By definition, $\tan\theta = \frac{y}{x} = \frac{2}{5}$. Choose $y = 2$ and $x = 5$. Draw $AB \parallel OY$ and 5 units to the right (Fig. 26), intersecting OX at M; also draw $CD \parallel OX$ and 2 units above intersecting AB at P. Then $OM = 5$, $MP = 2$, and $OP = \sqrt{29}$.

$\therefore \angle XOP = \theta$ is the required angle since $\tan\theta = \frac{MP}{OM} = \frac{2}{5}$.

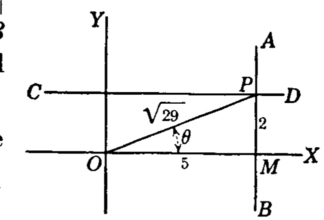

FIG. 26

The other functions are as follows:

$$\sin\theta = \frac{2}{\sqrt{29}},\quad \cos\theta = \frac{5}{\sqrt{29}},\quad \cot\theta = \frac{5}{2},\quad \sec\theta = \frac{\sqrt{29}}{5},$$
$$\csc\theta = \frac{\sqrt{29}}{2}.$$

EXERCISES

In Exercises 1 to 12, construct θ from the given function and find the other functions of θ when in the first quadrant.

1. $\sin\theta = \frac{3}{5}$.
2. $\cos\theta = \frac{3}{4}$.
3. $\tan\theta = 3$.
4. $\cot\theta = 2.5$.
5. $\sin\theta = \frac{1}{2}$.
6. $\cos\theta = \frac{1}{2}\sqrt{3}$.
7. $\tan\theta = \frac{a}{b}$.
8. $\cos\theta = \frac{a}{b}$.
9. $\sec\theta = \sqrt{2}$.
10. $\csc\theta = \frac{4}{3}$.
11. $\tan\theta = 4$.
12. $\cot\theta = \frac{3}{2}$.

13. Find the value of $\sqrt{\frac{\sin\theta\cos\theta}{\sec\theta\csc\theta}}$, when $\tan\theta = \frac{1}{5}$, and θ is an acute angle. *Ans.* $\frac{5}{26}$.

14. Find the value of $\frac{\sec\theta + \tan\theta}{\cos\theta + \text{vers}\,\theta}$, when $\cos\theta = \frac{3}{5}$, and θ is an acute angle. *Ans.* 3.

15. Find the value of $\dfrac{\csc\theta + \sec\theta}{\sin\theta + \cos\theta}$, when $\cos\theta = \dfrac{\sqrt{10}}{10}$, and θ is an acute angle.

Ans. $\frac{10}{3}$.

16. Find the value of $\dfrac{\sin\theta\cot\theta + \cos\theta}{\sec\theta\cot\theta}$, when $\cot\theta = \sqrt{5}$, and θ is an acute angle.

Ans. 0.745.

17. Find the value of $\dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin^2\theta\sec\theta}{\cos^2\theta\tan^2\theta}$, when $\csc\theta = 3$ and θ is an acute angle.

Ans. 1.414.

27. Trigonometric functions applied to right triangles.—When the angle θ is acute, the abscissa, ordinate, and distance for any point in the terminal side form a right triangle, in which the given angle θ is one of the acute angles. On account of the many applications of the right triangle in trigonometry, the definitions of the trigonometric functions will be stated with special reference to the right triangle. These definitions are very important and are frequently the first ones taught, but it should be carefully noted that they are not general because they apply only to *acute* angles.

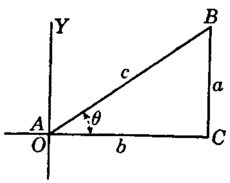

FIG. 27.

Draw the right triangle ABC (Fig. 27), with the vertex A at the origin, and AC on the initial line. Then AC and CB are the coordinates of B in the terminal side AB. Let $AC = b$, $CB = a$, and $AB = c$.

By definition:

$$\sin A = \frac{\text{ordinate}}{\text{distance}} = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}.$$
$$\cos A = \frac{\text{abscissa}}{\text{distance}} = \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}.$$
$$\tan A = \frac{\text{ordinate}}{\text{abscissa}} = \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}}.$$
$$\cot A = \frac{\text{abscissa}}{\text{ordinate}} = \frac{b}{a} = \frac{\text{side adjacent}}{\text{side opposite}}.$$
$$\sec A = \frac{\text{distance}}{\text{abscissa}} = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{side adjacent}}.$$
$$\csc A = \frac{\text{distance}}{\text{ordinate}} = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{side opposite}}.$$

Again, suppose the triangle ABC placed so that $\angle B$ has its vertex at the origin, BC for the initial side, and BA for the

terminal side, as in Fig. 28. The coordinates of A are $BC = a$ and $CA = b$.

By definition:

$$\sin B = \frac{b}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}. \qquad \cot B = \frac{a}{b} = \frac{\text{side adjacent}}{\text{side opposite}}.$$
$$\cos B = \frac{a}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}. \qquad \sec B = \frac{c}{a} = \frac{\text{hypotenuse}}{\text{side adjacent}}.$$
$$\tan B = \frac{b}{a} = \frac{\text{side opposite}}{\text{side adjacent}}. \qquad \csc B = \frac{c}{b} = \frac{\text{hypotenuse}}{\text{side opposite}}.$$

Then, no matter where the right triangle is found, the functions of the acute angles may be written in terms of the legs and the hypotenuse of the right triangle.

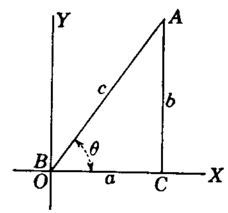

FIG. 28.

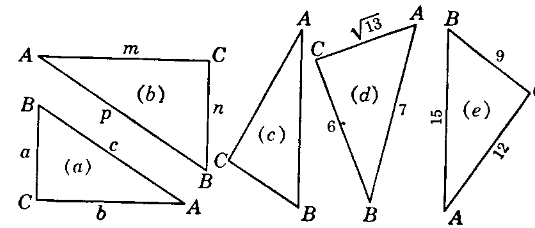

FIG. 29.

EXERCISES

1. Give orally the six trigonometric ratios of each of the acute angles of the right triangles in Fig. 29.

In the right triangle ABC, find the six trigonometric ratios from the following data:

2. $a = \frac{1}{4}c$. **3.** $b = \frac{1}{3}c$. **4.** $a = 4b$.

5. In a right triangle find a if $\sin A = \frac{3}{5}$, and $c = 4.28$.

Ans. $a = 2.568$.

6. In a right triangle find b if $\cos A = \frac{1}{3}$, and $c = 53.16$. *Ans.* $b = 17.72$.

7. In a right triangle find a if $\cot A = \frac{5}{3}$, and $b = 18.7$. *Ans.* $a = 11.22$.

8. In a right triangle find c if $\sin A = \frac{5}{16}$, and $a = 12.65$. *Ans.* $c = 40.48$.

9. In a right triangle find a if $\tan B = 7.5$, and $b = 8.32$. *Ans.* $a = 1.109$.

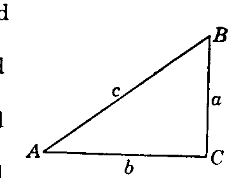

FIG. 30.

10. In a right triangle find b if $\cot B = 4.56$, and $a = 42$.

Ans. $b = 9.21$.

11. In a right triangle find a and c if $\sin B = \frac{2}{3}$, and $b = 22.45$.

Ans. $c = 33.675$; $a = 25.099$.

12. In a right triangle find a and b if $\sin A = 0.236$, and $c = 45$.

Ans. $a = 10.62$; $b = 43.73$.

The following refer to a right triangle:

13. $c = r + s$, $a = \sqrt{2rs}$; find $\tan A$. *Ans.* $\tan A = \sqrt{\dfrac{2rs}{r^2 + s^2}}$.

14. $b = \sqrt{r^2 + s^2}$, $c = r + s$; find $\sin A$. *Ans.* $\sin A = \dfrac{\sqrt{2rs}}{r + s}$.

15. $a = 2rs$, $b = r^2 - s^2$; find $\cos B$. *Ans.* $\cos B = \dfrac{2rs}{r^2 + s^2}$.

16. Construct a right triangle in which $\sin A = 2 \sin B$. In which $\sin A = 3 \cos A$. In which $\tan A = 3 \tan B$.

Construct the angle θ from each of the following data:

17. $\tan \theta = 2 \cot \theta$. **19.** $\sin \theta = 3 \cos \theta$. **21.** $\cot \theta = 3 \tan \theta$.
18. $\cos \theta = 2 \sin \theta$. **20.** $\sec \theta = 2 \csc \theta$. **22.** $\sin \theta = \cos \theta$.

28. Relations between the functions of complementary angles.—From the formulas of **Art. 27,** the following relations are evident:

$$\sin A = \cos B = \frac{a}{c}. \qquad \cot A = \tan B = \frac{b}{a}.$$
$$\cos A = \sin B = \frac{b}{c}. \qquad \sec A = \csc B = \frac{c}{b}.$$
$$\tan A = \cot B = \frac{a}{b}. \qquad \csc A = \sec B = \frac{c}{a}.$$

But angles A and B are complementary; therefore, *the sine, cosine, tangent, cotangent, secant, and cosecant of an angle are, respectively, the cosine, sine, cotangent, tangent, cosecant, and secant of the complement of the angle. They are also called cofunctions.*

For example, $\cos 75° = \sin (90° - 75°) = \sin 15°$;
$\tan 80° = \cot (90° - 80°) = \cot 10°$.

Note.—The term cosine was not used until the beginning of the seventeenth century. Before that time the expression, sine of the complement (Latin, *complementi sinus*) was used instead. Cosine is a contraction of the Latin expression. Similarly, cotangent and cosecant are contractions of *complementi tangens* and *complementi secans* respectively.

The abbreviations, sin, cos, tan, cot, sec, and csc did not come into general use until the middle of the eighteenth century.

EXERCISES

1. Express the following functions as functions of the complements of these angles: sin 60°; cos 25°; tan 15°; cot 65°; sec 10°; csc 42°; $\sin \theta$; $\sin 3\theta$; $\cos (\theta - 90°)$.

2. If $\sin 40° = \cos \theta$, find θ.
3. If $\tan 50° = \cot 2\theta$, find θ.
4. If $\csc 20° = \sec 2\theta$, find θ.
5. If $\cos \theta = \sin 2\theta$, find θ.
6. If $\sin 2\theta = \cos 4\theta$, find θ.
7. If $\tan \theta = \cot 5\theta$, find θ.
8. If $\csc 6\theta = \sec 4\theta$, find θ.
9. If $\cos \frac{1}{3}\theta = \sin \theta$, find θ.
10. If $\cot \frac{1}{4}\theta = \tan \theta$, find θ. *Ans.* $67\frac{1}{2}°$.

11. If $\cos \theta = \sin (45° - \frac{1}{2}\theta)$, find θ. *Ans.* 90°.
12. If $\cot \alpha = \tan (45° + \alpha)$, find α. *Ans.* 22° 30′.
13. If $\csc (60° - \alpha) = \sec (15° + 3\alpha)$, find α. *Ans.* 7° 30′.
14. If $\sin (35° + \beta) = \cos (\beta - 15°)$, find β. *Ans.* 35°.
15. Express each of the following functions as functions of angles less than 45°: sin 68°; cot 88°; sec 75°; csc 47° 58′ 12″; cos 71° 12′ 56″.

29. Given the function of an angle in any quadrant, to construct the angle. *Example* 1.—Given $\sin \theta = \frac{3}{5}$. Construct angle θ and find all the other functions.

Solution.—By definition, $\sin \theta = \dfrac{y}{r}$. Take $y = 3$ units and $r = 5$ units. Draw $AB \parallel OX$ and 3 units above it as in Fig. 31. Construct the arc of a circle with center at O and radius 5 units, intersecting AB at P_1 and P_2. Then for P_1, $x = 4$, $y = 3$, and $r = 5$; for P_2, $x = -4$, $y = 3$, and $r = 5$. Now OP_1 and OP_2 are terminal sides, respectively of $\angle XOP_1 = \theta_1$ and $\angle XOP_2 = \theta_2$, each of which has its sine equal to $\frac{3}{5}$. Then from the definitions of the trigonometric functions we have the following:

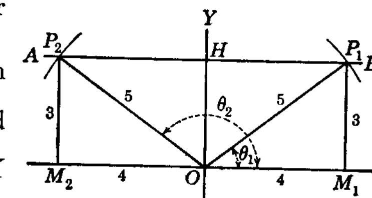

FIG. 31.

Quadrant	Angle	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\cot \theta$	$\sec \theta$	$\csc \theta$
I.	θ_1	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{5}{3}$
II.	θ_2	$\frac{3}{5}$	$-\frac{4}{5}$	$-\frac{3}{4}$	$-\frac{4}{3}$	$-\frac{5}{4}$	$\frac{5}{3}$

Example 2.—Given $\cos \theta = -\frac{2}{3}$. Construct θ and find all the other functions.

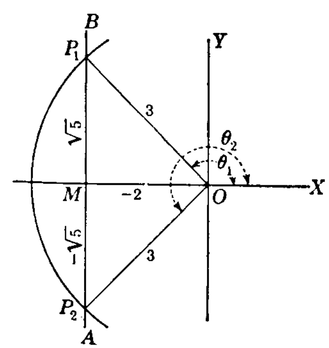

FIG. 32.

Solution.—By definition, $\cos \theta = \dfrac{x}{r} = -\dfrac{2}{3}$. Since r is always positive, we take $x = -2$ units and $r = 3$ units. Draw $AB \parallel OY$ and 2 units to the left as in Fig. 32. Construct a circle of radius 3, with its center at O, and intersecting AB at P_1 and P_2. Draw OP_1 and OP_2. As in Example 1, it may be shown that $\angle XOP_1 = \theta_1$ and $\angle XOP_2 = \theta_2$ are the required angles. The functions are as follows:

Quadrant	Angle	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
II............	θ_1	$\frac{\sqrt{5}}{3}$	$-\frac{2}{3}$	$-\frac{\sqrt{5}}{2}$	$-\frac{2}{\sqrt{5}}$	$-\frac{3}{2}$	$\frac{3}{\sqrt{5}}$
III...........	θ_2	$-\frac{\sqrt{5}}{3}$	$-\frac{2}{3}$	$\frac{\sqrt{5}}{2}$	$\frac{2}{\sqrt{5}}$	$-\frac{3}{2}$	$-\frac{3}{\sqrt{5}}$

Example 3.—Given $\tan\theta = \frac{3}{4}$. Construct angle θ and find all the other functions.

Solution.—By definition, $\tan\theta = \frac{y}{x}$. Hence $\frac{y}{x} = \frac{3}{4} = \frac{-3}{-4}$, and we may take $y = \pm 3$ and $x = \pm 4$. Then

$$r = \sqrt{(\pm 4)^2 + (\pm 3)^2} = 5.$$

With O as a center and 5 as a radius, construct a circle as in Fig. 33. Draw AB and $CD \parallel OY$ and 4 units to the right and left

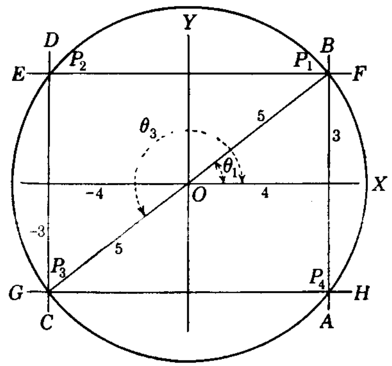

FIG. 33.

respectively of OY. Also draw EF and $GH \parallel OX$ and 3 units above and below OX respectively. These lines and the circle intersect at the points P_1, P_2, P_3, and P_4. Since x and y must both be positive or both negative, the required points must be P_1 and P_3 located in the first and third quadrants. Draw OP_1 and OP_3 forming the angles $XOP_1 = \theta_1$ and $XOP_3 = \theta_3$. The functions are as follows:

Quadrant	Angle	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
I.............	θ_1	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{5}{3}$
III...........	θ_3	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{4}$	$\frac{4}{3}$	$-\frac{5}{4}$	$-\frac{5}{3}$

EXERCISES

Draw the angles less than 360° and tabulate the six trigonometric ratios determined by each of the following:

1. $\cos\theta = -\frac{4}{5}$. **5.** $\cos\theta = 0.6$. **9.** $\csc\theta = -\frac{2}{\sqrt{3}}$.

2. $\sin\theta = -\frac{5}{13}$. **6.** $\cot\theta = 3$. **10.** $\sin\theta = -\frac{4}{7}$.

3. $\tan\theta = -\frac{1}{4}$. **7.** $\tan\theta = -\sqrt{3}$. **11.** $\tan\theta = \frac{5}{2}$.

4. $\sin\theta = \frac{7}{25}$. **8.** $\sec\theta = -4$. **12.** $\csc\theta = 2.4$.

13. What is the greatest value that the sine of an angle may have? The least value? How does the value of the sine change as the angle changes from 0° to 90°? From 90° to 180°? From 180° to 270°? From 270° to 360°?

14. Answer the questions of Exercise 13 for the cosine. For the tangent.

In Exercises 15, 18, and 21 show by substitution that the right-hand member is equal to the left.

15. $(1 + \tan^2\theta)(1 - \cot^2\theta) = \sec^2\theta - \csc^2\theta$, when $\sin\theta = \frac{4}{5}$ and θ is in the second quadrant.

16. Find the value of $\frac{\sin\theta\tan\theta}{\sec\theta}$, when $\cot\theta = -\frac{2}{3}$ and θ is in the fourth quadrant. *Ans.* $\frac{9}{13}$.

17. Find the value of $\frac{\sin\theta + \tan\theta}{\cos\theta + \operatorname{vers}\theta}$, when $\csc\theta = -\frac{5}{4}$ and θ is in the fourth quadrant. *Ans.* -2.133.

18. Cos $\theta\tan\theta + \sin\theta\cot\theta = \sin\theta + \cos\theta$, when $\sec\theta = 2$ and θ is in the fourth quadrant.

19. Find the value of $\frac{\sec\theta - \csc\theta}{\sec\theta + \csc\theta}$, when $\tan\theta = -2$ and θ is in the second quadrant. *Ans.* 3.

20. Find the value of $\frac{\sin\theta + \cot\theta}{\cos\theta + \csc\theta}$, when $\cot\theta = 2\sqrt{2}$ and $\sin\theta = -\frac{1}{3}$. *Ans.* -0.6328.

21. $\cot\theta + \frac{\sin\theta}{1 + \cos\theta} = \csc\theta$, when $\sin\theta = -\frac{\sqrt{3}}{2}$ and θ is in the third quadrant.

CHAPTER III

RELATIONS BETWEEN TRIGONOMETRIC FUNCTIONS

30. Fundamental relations between the functions of an angle.—In handling questions that occur in mathematics a great deal of use is made of relations that exist between trigonometric functions of angles. These relations are numerous, but it is necessary to memorize only a few of them. In this chapter are considered only those relations that exist between functions of one angle. In a later chapter will be found relations where different angles are involved.

From the figures of **Art. 15,** it is evident that for an angle in any quadrant

(1) $$x^2 + y^2 = r^2.$$

Dividing (1) by r^2, $\dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} = \dfrac{r^2}{r^2} = \mathbf{1}.$

But $\dfrac{x}{r} = \cos\theta$ and $\dfrac{y}{r} = \sin\theta$.

[1] $$\therefore \boldsymbol{\sin^2\theta + \cos^2\theta = 1}.$$

Dividing (1) by x^2, $1 + \dfrac{y^2}{x^2} = \dfrac{r^2}{x^2}$.

But $\tan\theta = \dfrac{y}{x}$ and $\sec\theta = \dfrac{r}{x}$.

[2] $$\therefore \boldsymbol{1 + \tan^2\theta = \sec^2\theta}.$$

Dividing (1) by y^2, $\dfrac{x^2}{y^2} + 1 = \dfrac{r^2}{y^2}$.

But $\cot\theta = \dfrac{x}{y}$ and $\csc\theta = \dfrac{r}{y}$.

[3] $$\therefore \boldsymbol{1 + \cot^2\theta = \csc^2\theta}.$$

Also, from the definitions of the trigonometric functions, the following reciprocal relations are evident:

[4] $$\boldsymbol{\csc\theta = \frac{1}{\sin\theta}} \quad \text{and} \quad \boldsymbol{\sin\theta = \frac{1}{\csc\theta}}.$$

[5] $$\boldsymbol{\sec\theta = \frac{1}{\cos\theta}} \quad \text{and} \quad \boldsymbol{\cos\theta = \frac{1}{\sec\theta}}.$$

[6] $$\boldsymbol{\cot\theta = \frac{1}{\tan\theta}} \quad \text{and} \quad \boldsymbol{\tan\theta = \frac{1}{\cot\theta}}.$$

The following formulas are easily derived:

[7] $$\boldsymbol{\tan\theta = \frac{\sin\theta}{\cos\theta}}.$$

[8] $$\boldsymbol{\cot\theta = \frac{\cos\theta}{\sin\theta}}.$$

The eight formulas of this article are **identities,** for they are true for any angle whatever. They are often spoken of as **fundamental identities,** or **formulas.** They should be carefully memorized as they are frequently used.

It will be noted that throughout the book the important formulas are printed in bold-faced type and numbered in square brackets for ready reference.

The following examples make use of the fundamental formulas in computing the other trigonometric functions when one function is given. Compare the work with that of the previous articles where the angles were first constructed.

Example 1.—Given $\tan\theta = \frac{4}{3}$, and θ in the first quadrant, determine the other functions by means of the fundamental formulas.

Solution.—By [2], $\sec\theta = \sqrt{1 + \tan^2\theta} = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$.

By [6], $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{1}{\frac{4}{3}} = \dfrac{3}{4}$.

By [3], $\csc\theta = \sqrt{1 + \cot^2\theta} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$.

By [4], $\sin\theta = \dfrac{1}{\csc\theta} = \dfrac{1}{\frac{5}{4}} = \dfrac{4}{5}$.

By [5], $\cos\theta = \dfrac{1}{\sec\theta} = \dfrac{1}{\frac{5}{3}} = \dfrac{3}{5}$.

Example 2.—Given $\sin\theta = \frac{1}{2}$, and θ in the second quadrant, determine the other functions by means of the fundamental formulas.

Solution.—By [1], $\cos\theta = -\sqrt{1 - \sin^2\theta} = -\sqrt{1 - \frac{1}{4}} = -\frac{1}{2}\sqrt{3}$.

By [7], $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\frac{1}{2}}{-\frac{1}{2}\sqrt{3}} = -\dfrac{1}{3}\sqrt{3}$.

By [6], $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{1}{-\frac{1}{3}\sqrt{3}} = -\sqrt{3}$.

By [5], $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{-\frac{1}{2}\sqrt{3}} = -\dfrac{2}{3}\sqrt{3}.$

By [4], $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{1}{\frac{1}{2}} = 2.$

Note.—The proper algebraic sign is determined by **Art. 17.**

EXERCISES

In Exercises 1 to 10 determine the remaining functions from the given functions by means of the fundamental identities and check by constructing the angle and computing the functions.

1. Given $\sin\theta = -\frac{4}{5}$, and θ in the third quadrant.
2. Given $\tan\theta = -\frac{5}{12}$, and θ in the fourth quadrant.
3. Given $\sec\theta = \sqrt{2}$, and θ in the first quadrant.
4. Given $\cos\theta = \dfrac{\sqrt{3}}{2}$, and θ in the fourth quadrant.
5. Given $\tan\theta = \frac{3}{2}$, and θ in the third quadrant.
6. Given $\cot\theta = 5$, and θ in the first quadrant.
7. Given $\sec\alpha = -\frac{5}{4}$, and α in the third quadrant.
8. Given $\tan\beta = \frac{24}{7}$, and β in the third quadrant.
9. Given $\csc\theta = -\frac{17}{8}$, and θ in the fourth quadrant.
10. Given $\sin\alpha = -\frac{9}{41}$, and α in the third quadrant.
11. If $\cos\frac{1}{2}\alpha = \sqrt{\dfrac{s(s-a)}{bc}}$, where $s = \dfrac{a+b+c}{2}$, show that $\sin\frac{1}{2}\alpha = \sqrt{\dfrac{(s-b)(s-c)}{bc}}.$
12. If $\cos\frac{1}{2}\gamma = \sqrt{\dfrac{s(s-c)}{ab}}$, where $s = \dfrac{a+b+c}{2}$, show that $\tan\frac{1}{2}\gamma = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}.$
13. If $\tan\beta = c$, show that $\csc\beta$ is real for all values of c.
14. Given $\sin\gamma = \dfrac{2mn}{m^2+n^2}$; show that $\tan\gamma = \pm\dfrac{2mn}{m^2-n^2}.$

31. To express one function in terms of each of the other functions.—Any trigonometric function can readily be expressed in terms of any other function by means of the fundamental formulas. While the work cannot be carried out so rapidly as by the method of the following article, it gives needed drill in the use of the formulas.

Example.—Express $\sin\theta$ in terms of each of the other functions.

By [1], $\sin\theta = \sqrt{1-\cos^2\theta}.$

By [5], $\cos\theta = \dfrac{1}{\sec\theta}.$

$$\therefore \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{\sec^2\theta}} = \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta}.$$

By [2], $\sec^2\theta = 1 + \tan^2\theta.$

$$\therefore \sin\theta = \frac{\sqrt{\sec^2\theta-1}}{\sec\theta} = \frac{\sqrt{\tan^2\theta}}{\sqrt{1+\tan^2\theta}} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}.$$

Also $\sin\theta = \dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \dfrac{\dfrac{1}{\cot\theta}}{\sqrt{1+\dfrac{1}{\cot^2\theta}}} = \dfrac{1}{\sqrt{1+\cot^2\theta}}.$

By [4], $\sin\theta = \dfrac{1}{\csc\theta}.$

The algebraic sign of $\sin\theta$ is determined from the quadrant in which θ is found.

32. To express all the functions of an angle in terms of one function of the angle, by means of a triangle.—The scheme outlined in this article can be carried out rapidly and will be found of very great use in future work.

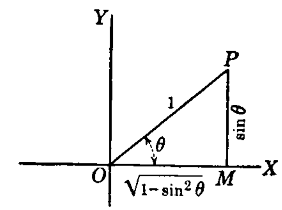

Fig. 34.

Example 1.—Express all the functions of θ in terms of $\sin\theta$.

Solution.—Construct angle θ in the first quadrant (Fig. 34) and choose the point P in the terminal side with coordinates OM and MP. Then, by definition, $\sin\theta = \dfrac{MP}{OP}$, and, if OP is taken equal to 1, $MP = \sin\theta$, and $OM = \sqrt{\overline{OP}^2 - \overline{MP}^2} = \sqrt{1-\sin^2\theta}$.

The remaining functions may then be written as follows:

$$\cos\theta = \frac{OM}{OP} = \sqrt{1-\sin^2\theta}. \qquad \sec\theta = \frac{OP}{OM} = \frac{1}{\sqrt{1-\sin^2\theta}}.$$

$$\tan\theta = \frac{MP}{OM} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}. \qquad \csc\theta = \frac{OP}{MP} = \frac{1}{\sin\theta}.$$

$$\cot\theta = \frac{OM}{MP} = \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}.$$

Example 2.—Express all the functions in terms of $\cos\theta$.

Solution.—Construct angle θ in the first quadrant (Fig. 35) and choose the point P in the terminal side with coordinates

OM and MP. Then, by definition, $\cos\theta = \dfrac{OM}{OP}$, and, if OP is taken equal to 1, $OM = \cos\theta$, and $MP = \sqrt{\overline{OP}^2 - \overline{OM}^2} = \sqrt{1 - \cos^2\theta}$.

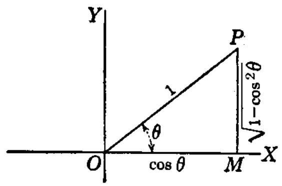

FIG. 35.

The remaining functions may then be written as follows:

$$\sin\theta = \frac{MP}{OP} = \sqrt{1 - \cos^2\theta}.$$

$$\tan\theta = \frac{MP}{OM} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta}. \qquad \sec\theta = \frac{OP}{OM} = \frac{1}{\cos\theta}.$$

$$\cot\theta = \frac{OM}{MP} = \frac{\cos\theta}{\sqrt{1-\cos^2\theta}}. \qquad \csc\theta = \frac{OP}{MP} = \frac{1}{\sqrt{1-\cos^2\theta}}.$$

In the following table, the student is asked to show that each function in the first column is equal to every expression found in the same row with the function:

$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\sqrt{1+\cot^2\theta}}$	$\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\frac{1}{\csc\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\frac{1}{\sqrt{1+\tan^2\theta}}$	$\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{\csc^2\theta-1}}{\csc\theta}$
$\tan\theta$	$\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\frac{1}{\sqrt{\csc^2\theta-1}}$
$\cot\theta$	$\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\csc^2\theta-1}$
$\sec\theta$	$\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\frac{\csc\theta}{\sqrt{\csc^2\theta-1}}$
$\csc\theta$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{1-\cos^2\theta}}$	$\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\csc\theta$

The table has been prepared under the assumption that θ is an acute angle. Should θ be in any other quadrant, the proper sign for each function may then be determined.

33. Transformation of trigonometric expressions.—In all transformations, avoid radicals if possible. Usually, this can best be done by changing to sines and cosines and then simplifying. It will be noticed that, if there are no radicals in an expression, it can be changed to sines and cosines without using radicals. If the expression is in a factored form, it is often desirable to reduce each factor separately and multiply the results.

Example 1.—Express $\dfrac{\cos\theta}{\sin\theta\cot^2\theta}$ in terms of $\tan\theta$.

Solution.—$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$, then $\dfrac{\cos\theta}{\sin\theta\cdot\cot^2\theta}$

$$= \frac{\cos\theta}{\sin\theta\cdot\dfrac{\cos^2\theta}{\sin^2\theta}} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

Example 2.—Express $1 - 2(1 - \text{covers}\,\theta)^2 + \dfrac{\tan^4\theta}{(1+\tan^2\theta)^2}$ in terms of $\cos\theta$.

Solution.—By definition and formulas, $\text{covers}\,\theta = 1 - \sin\theta$, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $1 + \tan^2\theta = \sec^2\theta$, and $\cos\theta = \dfrac{1}{\sec\theta}$.

Substituting these values, we have

$$1 - 2[1 - (1 - \sin\theta)]^2 + \frac{\dfrac{\sin^4\theta}{\cos^4\theta}}{\sec^4\theta} = 1 - 2\sin^2\theta + \frac{\dfrac{\sin^4\theta}{\cos^4\theta}}{\dfrac{1}{\cos^4\theta}}$$

$$= 1 - 2\sin^2\theta + \sin^4\theta = (1 - \sin^2\theta)^2 = (\cos^2\theta)^2 = \cos^4\theta.$$

EXERCISES

Transform the following expressions as indicated:

1. $\sin\theta\cot\theta\sec\theta$ to 1.
2. $\cos\theta\tan\theta\csc\theta$ to 1.
3. $(1 - \sin^2\phi)(1 + \tan^2\phi)$ to 1.
4. $(1 + \cos\theta)\,\text{vers}\,\theta + \text{covers}\,\theta\,(1 + \sin\theta)$ to 1.
5. $\dfrac{1 - \sec^2\phi}{1 - \csc^2\phi}$ to $\tan^4\phi$.
6. $\sec\theta - \tan\theta\sin\theta$ to $\cos\theta$.
7. $\sin^2\phi(1 + \sec^2\phi)$ to $\sec^2\phi - \cos^2\phi$.
8. $\dfrac{\sin\theta}{1 - \cos\theta}$ to $\cot\theta + \csc\theta$.
9. $\sin^4\phi - \cos^4\phi$ to $1 - 2\cos^2\phi$.
10. $\sin^4\theta + \cos^4\theta$ to $1 - 2\sin^2\theta\cos^2\theta$.
11. $\dfrac{\sin^2\phi\sec\phi}{1 + \sec\phi} + \cos\phi$ to 1.
12. $\dfrac{\cos\theta}{2}\sqrt{\dfrac{1+\sin^2\theta}{\cos\theta}}\left[\sqrt{\dfrac{\cos^3\theta}{1+\sin^2\theta}} + \sqrt{\dfrac{1+\sin^2\theta}{\cos\theta}}\right]$ to 1.
13. $\sqrt{\dfrac{\sec^2\phi - 1}{\sec^2\phi(1 + \cot^2\phi)}} + \dfrac{\cot^2\phi}{\csc\phi}\sqrt{\dfrac{\csc^2\phi - 1}{\csc^2\phi}}$ to $(1 + \cot\phi)(1 - \sin\phi\cos\phi)$.
14. $7\sec^2\phi - 6\tan^2\phi + 9\cos^2\phi$ to $\dfrac{(1 + 3\cos^2\phi)^2}{\cos^2\phi}$.
15. $\dfrac{\sin^2\phi\cos^2\phi + \cos^4\phi + 2\cos^2\phi + \sin^2\phi}{1 - \tan^2\phi}$ to $\dfrac{3 + \tan^2\phi}{1 - \tan^4\phi}$.
16. $\dfrac{(1 - \text{vers}^2\,\theta)^2 - (1 - \text{covers}^2\,\theta)^2}{\cos\theta - \sin\theta}$ to $5(\cos\theta + \sin\theta) - 4(1 + \sin\theta\cos\theta)$.

34. Identities.—When two expressions in some letter x are equal for all values of that letter they are said to be **identically equal.**

The equation formed by equating the two expressions is called an **identity.**

The symbol denoting identity is $\equiv$. When there can be no misunderstanding as to the meaning, the sign of equality is often used to denote identity. The symbol $\equiv$ is read "identically equals," or "is identically equal to."

Thus, $x^2 - 1 \equiv (x - 1)(x + 1)$ because the equation is true for all values of x.

Since the fundamental formulas are true for all values of θ, they are identities.

In showing that one trigonometric expression is identically equal to another, we either transform both expressions to the same form, or transform one expression into the other, by means of the fundamental formulas. That is, if A is to be proved identically equal to B, it can be done by

(1) *Changing A to B,*

(2) *Changing B to A, or*

(3) *Changing both A and B to a third form C.*

In the applications of this part of trigonometry, however, one usually knows exactly into what form a certain expression must be transformed. For this reason it is usual to require the student to change the first member of an identity into the second.

It is usually best, especially for the beginner, to express all the functions of the expression which is to be transformed in terms of sine and cosine before attempting to simplify.

Avoid radicals whenever possible.

When the expression that is to be transformed is given in a factored form, it is usually best to simplify each factor separately before multiplying them together.

Example 1.—By transforming the first member into the second prove the identity $\tan\theta\sin\theta + \cos\theta = \sec\theta$.

Proof.—Substituting $\dfrac{\sin\theta}{\cos\theta}$ for $\tan\theta$, we have

$$\frac{\sin\theta}{\cos\theta}\cdot\sin\theta + \cos\theta = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta.$$

Example 2.—By transforming the first member into the second prove the identity $\dfrac{\cot\alpha\cos\alpha}{\cot\alpha + \cos\alpha} = \dfrac{\cot\alpha - \cos\alpha}{\cot\alpha\cos\alpha}$.

Proof.—Substituting $\dfrac{\cos\alpha}{\sin\alpha}$ for $\cot\alpha$, we have

$$\frac{\cot\alpha\cos\alpha}{\cot\alpha + \cos\alpha} = \frac{\dfrac{\cos\alpha}{\sin\alpha}\cdot\cos\alpha}{\dfrac{\cos\alpha}{\sin\alpha} + \cos\alpha} = \frac{\dfrac{\cos^2\alpha}{\sin\alpha}}{\dfrac{\cos\alpha(1 + \sin\alpha)}{\sin\alpha}}$$

$$= \frac{\cos^2\alpha}{\cos\alpha(1 + \sin\alpha)} = \frac{1 - \sin^2\alpha}{\cos\alpha(1 + \sin\alpha)} = \frac{1 - \sin\alpha}{\cos\alpha}.$$

Now multiply the numerator and denominator by $\cot\alpha$, and we have

$$\frac{1 - \sin\alpha}{\cos\alpha}\cdot\frac{\cot\alpha}{\cot\alpha} = \frac{\cot\alpha - \sin\alpha\cot\alpha}{\cos\alpha\cot\alpha} = \frac{\cot\alpha - \cos\alpha}{\cos\alpha\cot\alpha}.$$

EXERCISES

Prove the following identities by transforming the first member of the identity into the second:

1. $\dfrac{\cos\theta\csc\theta}{\cot\theta} = 1.$
2. $\tan\theta\cos\theta = \sin\theta.$
3. $\sec\theta\cot\theta = \csc\theta.$
4. $\dfrac{\sin\theta\sec\theta}{\tan\theta} = 1.$
5. $(1 - \cos^2\phi)\sec^2\phi = \tan^2\phi.$
6. $\sec^2\phi + \csc^2\phi = \sec^2\phi\csc^2\phi.$
7. $\dfrac{1}{\cot^2\phi} - \sin^2\phi = \left(\dfrac{1}{\cot^2\phi}\right)\sin^2\phi.$
8. $\cot^2\phi - \cos^2\phi = \cos^2\phi\cot^2\phi.$
9. $(\sec^2\theta - 1)\csc^2\theta = \sec^2\theta.$
10. $\cot\theta + \tan\theta = \cot\theta\sec^2\theta.$
11. $(\tan\phi + \cot\phi)^2 = \sec^2\phi\csc^2\phi.$
12. $(\cos\theta - \sin\theta)^2 + 2\sin\theta\cos\theta = 1.$
13. $\sec^4\phi - \tan^4\phi = (\sec^2\phi)(2\sin^2\phi + \cos^2\phi).$
14. $\tan\theta(\sin\theta + \cos\theta)^2\cot\theta - 2\sin\theta\cos\theta = 1.$
15. $\dfrac{1 + \csc\theta}{\csc\theta - 1} = \dfrac{1 + \sin\theta}{1 - \sin\theta}.$
16. $\dfrac{\sin\beta}{\sec\phi}\dfrac{\sqrt{\sec^2\phi - 1}}{\sqrt{1 - \sin^2\beta}}\cdot(1 - \sin^2\phi)^{-\frac{1}{2}} = \tan\phi\tan\beta.$
17. $\dfrac{\cos\theta}{1 - \sin\theta} - \dfrac{1 - \sin\theta}{\cos\theta} = 2\tan\theta.$
18. $\dfrac{(1 - \tan\phi)^2}{\sec^2\phi} + 2\sin\phi\cos\phi = 1.$
19. $\dfrac{\sin\theta + \sin\phi}{\sin\theta - \sin\phi} = \dfrac{\csc\phi + \csc\theta}{\csc\phi - \csc\theta}.$
20. $\dfrac{(1 + \sin\phi)}{2\cos\phi}\left[\sqrt{\dfrac{1 - \sin\phi}{1 + \sin\phi}} + \sqrt{\dfrac{1 + \sin\phi}{1 - \sin\phi}}\right]\sqrt{\dfrac{1 - \sin\phi}{1 + \sin\phi}} = \sec\phi.$

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Avoid radicals whenever possible.

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Example 1.—By transforming the first member into the second prove the identity $\tan\theta \sin\theta + \cos\theta = \sec\theta$.

Proof.—Substituting $\dfrac{\sin\theta}{\cos\theta}$ for $\tan\theta$, we have

$$\frac{\sin\theta}{\cos\theta}\cdot\sin\theta + \cos\theta = \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} = \frac{1}{\cos\theta} = \sec\theta.$$

Example 2.—By transforming the first member into the second prove the identity $\dfrac{\cot\alpha\cos\alpha}{\cot\alpha + \cos\alpha} = \dfrac{\cot\alpha - \cos\alpha}{\cot\alpha\cos\alpha}$.

Proof.—Substituting $\dfrac{\cos\alpha}{\sin\alpha}$ for $\cot\alpha$, we have

$$\frac{\cot\alpha\cos\alpha}{\cot\alpha + \cos\alpha} = \frac{\dfrac{\cos\alpha}{\sin\alpha}\cdot\cos\alpha}{\dfrac{\cos\alpha}{\sin\alpha} + \cos\alpha} = \frac{\dfrac{\cos^2\alpha}{\sin\alpha}}{\dfrac{\cos\alpha(1 + \sin\alpha)}{\sin\alpha}}$$

$$= \frac{\cos^2\alpha}{\cos\alpha(1 + \sin\alpha)} = \frac{1 - \sin^2\alpha}{\cos\alpha(1 + \sin\alpha)} = \frac{1 - \sin\alpha}{\cos\alpha}.$$

Now multiply the numerator and denominator by $\cot\alpha$, and we have

$$\frac{1 - \sin\alpha}{\cos\alpha}\cdot\frac{\cot\alpha}{\cot\alpha} = \frac{\cot\alpha - \sin\alpha\cot\alpha}{\cos\alpha\cot\alpha} = \frac{\cot\alpha - \cos\alpha}{\cos\alpha\cot\alpha}.$$

EXERCISES

Prove the following identities by transforming the first member of the identity into the second:

1. $\dfrac{\cos\theta\csc\theta}{\cot\theta} = 1.$
2. $\tan\theta\cos\theta = \sin\theta.$
3. $\sec\theta\cot\theta = \csc\theta.$
4. $\dfrac{\sin\theta\sec\theta}{\tan\theta} = 1.$
5. $(1 - \cos^2\phi)\sec^2\phi = \tan^2\phi.$
6. $\sec^2\phi + \csc^2\phi = \sec^2\phi\csc^2\phi.$
7. $\dfrac{1}{\cot^2\phi} - \sin^2\phi = \left(\dfrac{1}{\cot^2\phi}\right)\sin^2\phi.$
8. $\cot^2\phi - \cos^2\phi = \cos^2\phi\cot^2\phi.$
9. $(\sec^2\theta - 1)\csc^2\theta = \sec^2\theta.$
10. $\cot\theta + \tan\theta = \cot\theta\sec^2\theta.$
11. $(\tan\phi + \cot\phi)^2 = \sec^2\phi\csc^2\phi.$
12. $(\cos\theta - \sin\theta)^2 + 2\sin\theta\cos\theta = 1.$
13. $\sec^4\phi - \tan^4\phi = (\sec^2\phi)(2\sin^2\phi + \cos^2\phi).$
14. $\tan\theta(\sin\theta + \cos\theta)^2\cot\theta - 2\sin\theta\cos\theta = 1.$
15. $\dfrac{1 + \csc\theta}{\csc\theta - 1} = \dfrac{1 + \sin\theta}{1 - \sin\theta}.$
16. $\dfrac{\sin\beta}{\sec\phi}\dfrac{\sqrt{\sec^2\phi - 1}}{\sqrt{1 - \sin^2\beta}}\cdot(1 - \sin^2\phi)^{-\frac{1}{2}} = \tan\phi\tan\beta.$
17. $\dfrac{\cos\theta}{1 - \sin\theta} - \dfrac{1 - \sin\theta}{\cos\theta} = 2\tan\theta.$
18. $\dfrac{(1 - \tan\phi)^2}{\sec^2\phi} + 2\sin\phi\cos\phi = 1.$
19. $\dfrac{\sin\theta + \sin\phi}{\sin\theta - \sin\phi} = \dfrac{\csc\phi + \csc\theta}{\csc\phi - \csc\theta}.$
20. $\dfrac{(1 + \sin\phi)}{2\cos\phi}\left[\sqrt{\dfrac{1 - \sin\phi}{1 + \sin\phi}} + \sqrt{\dfrac{1 + \sin\phi}{1 - \sin\phi}}\right]\sqrt{\dfrac{1 - \sin\phi}{1 + \sin\phi}} = \sec\phi.$

21. $(\tan^2 \theta + 1)\cot^2 \theta = \csc^2 \theta.$

22. $\sin^2 \phi \sec \phi\ (\sin^2 \theta \sec \theta + \cos \theta) + \cos \phi\ (\sin^2 \theta \sec \theta + \cos \theta) = \sec \phi \sec \theta.$

23. $2 \sin \phi \cos \phi + \sin^2 \phi \tan \phi + \cos^2 \phi \cot \phi = \sec \phi \csc \phi.$

24. $\sin \phi \cos \phi\ [2 + (\sec^2 \phi - 1) + (\csc^2 \phi - 1)] = \sec \phi \csc \phi.$

25. $\cos \theta\ (\sec \theta + \csc \theta) + \sin \theta\ (\sec \theta - \csc \theta) = \sec \theta \csc \theta.$

26. $\sqrt{\dfrac{1 + \cos \phi}{1 - \cos \phi}} = \csc \phi + \cot \phi.$

27. $\dfrac{\sec^2 \phi(1 + \cos \phi \tan \phi)}{(\tan \phi + \sec \phi)^2 + 1} = \dfrac{1}{2}.$

28. $2 \tan^2 \theta + 2 \tan \theta \sec \theta + 1 = \sec^2 \theta(1 + \sin \theta)^2.$

29. $1 - 3 \cos^2 \phi \sin^2 \phi + 2 \sin^3 \phi \cos^3 \phi = (\sin^3 \phi + \cos^3 \phi)^2.$

30. $\text{covers}\, B(1 - \cos^3 B) - \text{vers}\, B(1 - \sin^3 B) = \text{vers}\, B\ \text{covers}\, B(\cos B - \sin B)(1 + \sin B + \cos B).$

31. $(2 \sin^2 \theta - \cos^2 \theta)^2 - 9(2 \sin^2 \theta - 1)^2 = (2 - 3 \sin^2 \theta)(2 + 3 \sin \theta)(3 \sin \theta - 2).$

32. $\dfrac{\tan^2 \phi\ (\sec \phi - 1)}{\sec \phi + 1} - \sec^2 \phi = 1 - 2 \sec \phi.$

33. $\dfrac{\sqrt{1 - \sin \phi \cos \phi}}{\sin \phi \cos \phi}\left[\sqrt{\dfrac{1 + \cot \phi}{\sin^2 \phi + \dfrac{\cos^2 \phi}{\tan \phi}}} - \sqrt{\dfrac{1 + \cot^3 \phi}{\csc^2 \phi(1 + \cot \phi)}}\right] = 1.$

34. $\dfrac{\sin^2 \theta[\cos^4 \theta - \sin^2 \theta] + \cos^6 \theta}{\cos^2 \theta[2 \cos^2 \theta - 1]} = \sec^2 \theta.$

35. $\dfrac{\tan \alpha + \tan \beta}{\sec \alpha - \sec \beta} = \dfrac{\sec \alpha + \sec \beta}{\tan \alpha - \tan \beta}.$

35. Inverse trigonometric functions.—The equation

$$\sin \theta = a$$

means that θ is an angle whose sine is a. The expression $\sin^{-1} a$ is an abbreviation for the expression "*an angle whose sine is a.*" Then we may write

$$\theta = \sin^{-1} a.$$

The form $\sin^{-1} a$ is also read "*anti-sine a*," "*inverse-sine a*," "*arc sine a*." It is also written invsin a and arc sin a.

Analogous forms with analogous meanings are given for the other functions.

Illustrations.—$\sin^{-1} \frac{1}{2} = 30$ or $150°$. $\cos^{-1} 1 = 0°$. $\tan^{-1} 1 = 45$ or $225°$.

The notations $\sin^{-1} a$, $\cos^{-1} a$, etc. have the advantage that they are the forms most frequently used in other branches of mathematics and its applications; but they have the disadvantage of conflicting with the customary notation for exponents, and so tend to cause confusion. Thus, $\sin^2\theta$ is usually written for

$(\sin \theta)^2$ and x^{-1} for $\dfrac{1}{x}$, and so the symbol $\sin^{-1} a$ might consistently be taken to mean $\dfrac{1}{\sin a} = \csc a$, which is something entirely different from our meaning of $\sin^{-1} a$ as explained at the beginning of this article.

Example.—Show that $\sin \cos^{-1} \frac{15}{17} = \frac{8}{17}$,

Solution.—Let $\theta = \cos^{-1} \frac{15}{17}$.

Then from the definitions of the inverse functions,

$$\cos \theta = \tfrac{15}{17},$$

By [1], $\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - (\frac{15}{17})^2} = \frac{8}{17}.$

$$\therefore \sin \cos^{-1} \tfrac{15}{17} = \tfrac{8}{17}.$$

This could also be solved by constructing the angle.

EXERCISES

Answer Exercises 1 to 12 orally, considering only angles that are less than 90°.

1. $\sin \cos^{-1} \dfrac{\sqrt{2}}{2}.$ **5.** $\cos \sec^{-1} 5.$ **9.** $\sin \sec^{-1} \frac{13}{12}.$

2. $\sin \sin^{-1} \frac{7}{25}.$ **6.** $\tan \sin^{-1} \frac{24}{25}.$ **10.** $\csc \cot^{-1} \frac{1}{3}.$

3. $\tan \sec^{-1} 2.$ **7.** $\sin \cos^{-1} 0.$ **11.** $\sin \cos^{-1} \frac{3}{5}.$

4. $\cos \csc^{-1} 3.$ **8.** $\sin \tan^{-1} \sqrt{3}.$ **12.** $\cos \sec^{-1} 5.$

Prove the relations in Exercises 13 to 22.

13. $\sin \cos^{-1} a = \pm\sqrt{1 - a^2}.$

14. $\sin \tan^{-1} a = \pm\dfrac{a}{\sqrt{1 + a^2}}.$

15. $\sin \cot^{-1} a = \pm\dfrac{1}{\sqrt{1 + a^2}}.$

16. $\sin \sec^{-1} a = \pm\dfrac{\sqrt{a^2 - 1}}{a}.$

17. $\sin \csc^{-1} a = \dfrac{1}{a}.$

18. $\cos \sin^{-1} a = \pm\sqrt{1 - a^2}.$

19. $\cos \tan^{-1} a = \pm\dfrac{1}{\sqrt{1 + a^2}}.$

20. $\cos \cot^{-1} a = \pm\dfrac{a}{\sqrt{1 + a^2}}.$

21. $\cos \sec^{-1} a = \dfrac{1}{a}.$

22. $\cos \csc^{-1} a = \pm\dfrac{\sqrt{a^2 - 1}}{a}.$

For angles not greater than 90°, show that the following are true:

23. $\sin^{-1} \frac{24}{25} = \cos^{-1} \frac{7}{25}.$ **24.** $\tan^{-1} \frac{16}{63} = \sin^{-1} \frac{16}{65}.$

36. Trigonometric equations.—A trigonometric equation is an equation in which the unknown is involved in a trigonometric function.

The solution of a trigonometric equation is a value of the angle which satisfies the equation.

In general, both algebra and trigonometry are involved in solving a trigonometric equation. Algebra must be used when the trigonometric functions are involved algebraically in a trigonometric equation, for then the equation must first be solved for some trigonometric function.

Thus, $\sin^2\theta - \frac{3}{2}\sin\theta + \frac{1}{2} = 0$ is a quadratic equation in $\sin\theta$; and, algebraically, is solved for $\sin\theta$ exactly as $x^2 - \frac{3}{2}x + \frac{1}{2} = 0$ is solved for x, either by the formula for solving a quadratic equation or by factoring. The solutions for $\sin\theta$ are

$$\sin\theta = \tfrac{1}{2}, \text{ and } \sin\theta = 1.$$

The trigonometry part of the solution is to find θ from these equations. They are solved by knowing the values of θ when $\sin\theta = \frac{1}{2}$ and $\sin\theta = 1$. They give

$$\theta = \sin^{-1}\tfrac{1}{2} = 30°, \text{ and } \theta = \sin^{-1} 1 = 90°.$$

Example 1.—Solve $\sin\theta = \frac{1}{2}\sqrt{2}$ for $\theta < 90°$.

Here all that is necessary is to know the angle less than 90° whose sine is $\frac{1}{2}\sqrt{2}$. From the table on page 24 this is found to be 45°,

$$\therefore \text{ if } \sin\theta = \tfrac{1}{2}\sqrt{2},\ \theta = 45°.$$

Example 2.—Solve $\tan\theta = 0.43654$ for $\theta < 90°$.

This value of θ cannot be found by referring to page 24, as it requires a more extensive table of natural functions. By referring to **Table IV,** θ is found to be 23° 35′.

$$\therefore \text{ if } \tan\theta = 0.43654,\ \theta = 23°\ 35'.$$

Example 3.—Solve $\cos\theta = 0.77467$ for $\theta < 90°$.

From **Table IV,** θ is found to be 39° 13′ 30″.

$$\therefore \text{ if } \cos\theta = 0.77467,\ \theta = 39°\ 13'\ 30''.$$

In using **Table IV** for finding this value of θ, interpolation is required. If the method is not familiar, the explanation will be found on page 30 of the **Tables.**

Example 4.—Solve the equation $\cos^2\alpha + 2\cos\alpha - 3 = 0$ for values of α not greater than 90°.

Solution.—Factoring the equation,

$$(\cos\alpha + 3)(\cos\alpha - 1) = 0.$$

Equating each factor to 0 and solving for $\cos\alpha$,

$$\cos\alpha = 1 \text{ and } -3.$$

$$\therefore \alpha = \cos^{-1} 1 \text{ and } \alpha = \cos^{-1}(-3).$$

Since there is no angle with a cosine equal to -3, the only solution admissible is $\alpha = \cos^{-1} 1 = 0°$.

This can be checked by substituting 0° for α in the original equation.

Example 5.—Solve $7\tan^2\theta - 4\sec^2\theta + 3 = 0$ for values of θ not greater than 90°.

Solution.—First transform so that but a single function is involved. This can be done in many ways, but very readily by changing $\sec^2\theta$ to $1 + \tan^2\theta$, which gives

$$7\tan^2\theta - 4(1 + \tan^2\theta) + 3 = 0.$$

Simplifying, $$3\tan^2\theta - 1 = 0.$$

Solving for $\tan\theta$, $$\tan\theta = \pm\tfrac{1}{3}\sqrt{3}.$$

Or $$\theta = \tan^{-1}\tfrac{1}{3}\sqrt{3}, \text{ and } \theta = \tan^{-1}(-\tfrac{1}{3}\sqrt{3}).$$

The first of these gives $\theta = 30°$, which is the only value of θ less than 90°.

EXERCISES

Solve orally the following trigonometric equations for values of the angles not greater than 90°:

1. $\sin\theta = 1$.
2. $\sin\theta = \frac{1}{2}\sqrt{2}$.
3. $\sin\theta = \frac{1}{2}$.
4. $\cos\theta = 1$.
5. $\cos\theta = \frac{\sqrt{3}}{2}$.
6. $\tan\theta = 1$.
7. $\csc\theta = 2$.
8. $\tan\theta = \sqrt{3}$.
9. $\cot\theta = \frac{1}{\sqrt{3}}$.
10. $\sec\theta = \sqrt{2}$.
11. $\sec\theta = 1$.
12. $\sin\theta = 0$.
13. $\csc\theta = \sqrt{2}$.
14. $\csc\theta = 1$.
15. $\csc\theta = \frac{2}{\sqrt{3}}$.

Solve orally the following anti-trigonometric equations for values of the angles not greater than 90°:

16. $\theta = \cos^{-1}\frac{1}{2}\sqrt{2}$.
17. $\theta = \sin^{-1} 0$.
18. $\theta = \tan^{-1}\sqrt{3}$.
19. $\theta = \sec^{-1}\sqrt{2}$.
20. $\alpha = \tan^{-1}\frac{1}{\sqrt{3}}$.
21. $\alpha = \tan^{-1} 0$.
22. $\alpha = \csc^{-1} 2$.
23. $\alpha = \sec^{-1} 2$.
24. $\beta = \csc^{-1} 1$.
25. $\beta = \cot^{-1} 0$.
26. $\gamma = \cot^{-1}\sqrt{3}$.
27. $\gamma = \csc^{-1}\sqrt{2}$.

Use **Table IV** in solving the following trigonometric equations for values of the angles not greater than 90°:

28. $\sin\theta = 0.50628$.
29. $\cos\theta = 0.85249$.
30. $\tan\theta = 0.58124$.
31. $\cot\theta = 1.6372$.
32. $\sin\theta = 0.27148$.
33. $\cot\theta = 3.6245$.
34. $\sin\theta = 0.74896$.
35. $\cos\theta = 0.61520$.
36. $\cot\theta = 3.2790$.
37. $\cos\theta = 0.57200$.
38. $\cos\theta = \frac{2}{3}$.
39. $\theta = \cos^{-1}\frac{4}{5}$.
40. $\theta = \tan^{-1}\sqrt{2}$.
41. $\theta = \cot^{-1}\frac{1}{4}$.
42. $\theta = \cos^{-1}\frac{5}{13}$.

Solve the following trigonometric equations for values of the angles not greater than 90°:

43. $\sin^2\theta - \sin\theta = 0$. *Ans.* 0°, 90°.
44. $(\cos\theta - 1)(2\cos\theta - 1) = 0$. *Ans.* 0°, 60°.
45. $\tan^4\theta - 9 = 0$. *Ans.* 60°.

46. $\sec^2\theta = 4\tan^2\theta$. *Ans.* 30°.

47. $\sqrt{3}(\tan\theta + \cot\theta) = 4$. *Ans.* 30°, 60°.

48. $3\tan\theta = 2\cos\theta$. *Ans.* 30°.

49. $3\tan^2\theta - 2\sqrt{3}\tan\theta + 1 = 0$. *Ans.* 30°.

50. $4\sin^2\theta - 2(\sqrt{2}+1)\sin\theta + \sqrt{2} = 0$. *Ans.* 30°, 45°.

51. $2\cos^2\theta - (2+\sqrt{2})\cos\theta + \sqrt{2} = 0$. *Ans.* 0°, 45°.

52. $4\cos^2\theta - 2(1+\sqrt{3})\cos\theta + \sqrt{3} = 0$. *Ans.* 30°, 60°.

53. $3\tan^2\theta - 4\sqrt{3}\tan\theta + 4 = 0$. *Ans.* 49° 6.4′.

54. $2\cos\theta - \cot\theta = 0$. *Ans.* 30°, 90°.

55. $4\sin^2\theta - 5\sin\theta + 1 = 0$. *Ans.* 14° 28′ 39″, 90°.

56. $\tan\theta(\sec\theta - \sqrt{2}) = \sqrt{3}(\sec\theta - \sqrt{2})$. *Ans.* 45°, 60°.

57. $4\sin^2\theta - 3\sqrt{6}\sin\theta + 3 = 0$. *Ans.* 37° 45.7′.

58. $7\cos^2\theta - 29\cos\theta + 4 = 0$. *Ans.* 81° 47.2′.

59. $2\cos^2\theta - \sin^2\theta = 0$. *Ans.* 54° 44′ 8″.

60. $\tan\theta + 4 = 2(\sin\theta + \sec\theta)$. *Ans.* 60°.

61. $\sin^3\theta - \cos^3\theta = 0$. *Ans.* 45°.

62. $4\tan^2\theta = 3\sec^2\theta$. *Ans.* 60°.

63. $\tan^2\theta - 4\tan\theta + 1 = 0$. *Ans.* 15°, 75°.

64. $\sec\theta - 1 = (\sqrt{2} - 1)\tan\theta$. *Ans.* 0°, 45°.

CHAPTER IV

RIGHT TRIANGLES

37. General statement.—One of the direct applications of trigonometry is the solution of triangles both right and oblique. It is in this way that the surveyor determines heights and distances that cannot be measured directly; for instance, the height of a mountain or the distance from one point to another where a lake or a mountain prevents direct measurement. It is well to note, however, that the solution of triangles is not the phase of trigonometry that is of most importance to the student who is to pursue more advanced subjects in mathematics. He will more often find use for the relations existing between the different functions, and in transforming one form of an expression involving trigonometric functions into an equivalent one.

It is a recognized fact in all walks of life, and it is certainly ingrained in mathematical science, that every real advance goes hand in hand with the invention of sharper tools and simpler methods. Practical geometry was developed in Egypt to help redetermine boundaries of the land after an overflow of the Nile. At an early date astronomy gave the main incentive for the development of trigonometry.

In attacking the triangle, trigonometry, in many ways, is a more powerful tool than geometry, which makes little use of the angles, while trigonometry makes use of the angles, as well as of the sides, of a triangle.

38. Solution of a triangle.—Every triangle, whether right or oblique, has six parts, *viz.*, three sides and three angles. When certain ones of these are given, the others can be found.

The process of finding the parts not given is called the **solution** of the triangle. By means of trigonometry a triangle can be solved when the parts given are sufficient to make a definite geometrical construction of the triangle. By geometry, a triangle can be constructed when three parts are given, at least one of which is a side. The remaining parts can then be measured and so a solution of the triangle obtained.

There are two ways of solving a triangle:

(1) *The graphical solution.*

(2) *The solution by computation.*

39. The graphical solution.—This consists in drawing a triangle such that its angles are equal to the given angles, and its sides equal to or proportional to the given sides. Of course, it is necessary that the given parts be consistent and sufficient to determine a definite triangle. For instance, two angles must not be given such that their sum is greater than 180°; nor can a construction be made if three sides are given such that one of them is as great as or greater than the sum of the other two.

EXERCISES

1. Construct triangles by means of the straightedge and compasses, having given:

(*a*) Two sides and the included angle.

(*b*) Two angles and the included side.

(*c*) Three sides.

(*d*) Two sides and an angle opposite one of them. Discuss and give drawings for all the possibilities.

(*e*) Three angles. Is the construction definite?

2. Construct right triangles by means of the straightedge and compasses, having given:

(*a*) Two legs.

(*b*) An acute angle and the hypotenuse.

(*c*) An acute angle and one leg.

(*d*) The hypotenuse and a leg.

Use the protractor in measuring the angles and construct the following:

(*a*) A right triangle with an acute angle equal to 42° and adjacent side 3.75 in.

(*b*) An oblique triangle with an angle equal to 35° 16′ and the including sides 9 and 18 in., respectively.

(*c*) A triangle with two angles 41° and 63°, respectively, and the side opposite the first angle 7.5 in.

(*d*) A triangle with sides 7.3, 4.5, and 3.8 in., respectively.

(*e*) A triangle with sides 11.5 and 4.7 ft. and the angle opposite the second side 120°.

40. The solution of right triangles by computation.—In the two previous articles, what was said referred to the oblique triangle as well as to the right triangle; here reference is to the right triangle only.

Since in a right triangle the right angle is always a given part, it is necessary to have given only two other parts, at least one of which is a side.

In what follows a, b, and c represent the altitude, base, and hypotenuse respectively, and A, B, and C, the angles opposite the respective sides.

The solutions depend upon the following relations, the first two of which are from geometry and the last eight from the definitions of trigonometric functions:

(1) $c^2 = a^2 + b^2.$ (6) $\sin B = \frac{b}{c}.$

(2) $A + B = 90°.$ (7) $\tan A = \frac{a}{b}.$

(3) $\sin A = \frac{a}{c}.$ (8) $\cot B = \frac{a}{b}.$

(4) $\cos B = \frac{a}{c}.$ (9) $\cot A = \frac{b}{a}.$

(5) $\cos A = \frac{b}{c}.$ (10) $\tan B = \frac{b}{a}.$

Number (2) shows that no other part can be derived from the two acute angles alone. In each of the other formulas, three parts are involved. If any two of these parts are given, the third can be found. Thus, in (3) if a and A are given, $c = \frac{a}{\sin A}$; if c and A are given, $a = c \sin A$; and if a and c are given, $A = \sin^{-1} \frac{a}{c}.$

Exercise.—Solve each of the above formulas for each letter in terms of the others.

SOLUTION OF RIGHT TRIANGLE BY NATURAL FUNCTIONS

41. Steps in the solution.—In solving a triangle, it is of the greatest importance to follow some regular order. The following is suggested:

(1) *Construct the triangle carefully to scale, using compasses, protractor, and ruler.* The required parts can then be measured and a check obtained on the computed values.

(2) *State the given and the required parts, and write down the formulas which are needed in the solution, solving each for the part required.* In choosing these formulas, select for each part required a formula that shall contain two known parts and one required part. Thus, if A and a are the given parts and c

the required part, then $\sin A = \frac{a}{c}$ contains the given parts and the required part c. This solved for c gives $c = \frac{a}{\sin A}$. In general, avoid the use of $c^2 = a^2 + b^2$ unless a table of squares and square roots is at hand.

(3) *Compute by substituting the given values in the formulas and evaluating.*

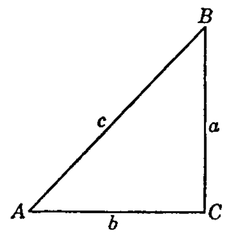
(4) *Arrange the work neatly and systematically, as this conduces to accuracy and therefore speed.*

(5) *Always check.* This can be done by making a careful construction, and also by using other formulas than those used in the solution.

Example 1.—Given $a = 3.25$ and $A = 47° 25.6'$; find b, c, and B.

Solution.

Construction



Given $\begin{cases} a = 3.25. \\ A = 47° 25.6'. \end{cases}$

To find* $\begin{cases} b = 2.986. \\ c = 4.413. \\ B = 42° 34.4'. \end{cases}$

Formulas

$$(1) \qquad \tan A = \frac{a}{b} \qquad \therefore b = \frac{a}{\tan A}.$$

$$(2) \qquad \sin A = \frac{a}{c} \qquad \therefore c = \frac{a}{\sin A}.$$

$$(3) \qquad A + B = 90° \qquad \therefore B = 90° - A.$$

Computation

$$b = \frac{3.25}{1.0885} = 2.986.$$

$$c = \frac{3.25}{0.7364} = 4.413.$$

$$B = 90° - 47° 25.6' = 42° 34.4'.$$

Check

$$a^2 = c^2 - b^2 = (c + b)(c - b).$$

$$3.25^2 = (4.413 + 2.986)(4.413 - 2.986).$$

$$10.5625 = 10.5584.$$

* Results to be inserted when work is completed.

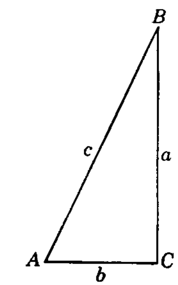
These agree to four significant figures.

The formula $\sin B = \frac{b}{c}$ could also be used in checking.

Example 2.—Given $a = 6.72$ and $b = 3.27$; find c, A, and B.

Solution.

Construction



Given $\begin{cases} a = 6.72. \\ b = 3.27. \end{cases}$

To find $\begin{cases} c = 7.473. \\ A = 64° 3.1'. \\ B = 25° 56.9'. \end{cases}$

Formulas

$$(1) \qquad \tan A = \frac{a}{b} \qquad \therefore A = \tan^{-1}\frac{a}{b}.$$

$$(2) \qquad \cot B = \frac{a}{b} \qquad \therefore B = \cot^{-1}\frac{a}{b}.$$

$$(3) \qquad \sin A = \frac{a}{c} \qquad \therefore c = \frac{a}{\sin A}.$$

Computation

$$A = \tan^{-1}\frac{6.72}{3.27} = \tan^{-1} 2.0550 = 64° 3.1'-.$$

$$B = \cot^{-1}\frac{6.72}{3.27} = \cot^{-1} 2.0550 = 25° 56.9'+.$$

$$c = \frac{6.72}{0.89919} = 7.473+.$$

Check

$$a^2 = c^2 - b^2 = (c + b)(c - b).$$

$$6.72^2 = (7.473 + 3.27)(7.473 - 3.27).$$

$$45.158 = 10.743 \times 4.203 = 45.153.$$

It is to be noted that, in computing c, the angle A was used. Though A was not a given part, it was used to avoid the formula $c = \sqrt{a^2 + b^2}$.

EXERCISES

Solve the following right triangles using natural functions. Use the formulas $(c + a)(c - a) = b^2$ and $(c + b)(c - b) = a^2$ as a check.

	Given	*Find*
1.	$b = 32, B = 35°$;	a, c, A. Check.
2.	$a = 11, A = 43°$;	b, c, B. Check.
3.	$a = 77, A = 72° 30'$;	b, c, B. Check.
4.	$b = 130, B = 67° 15'$;	a, c, A. Check.
5.	$a = 27, c = 45$;	b, A, B. Check.
6.	$b = 100, A = 70°$;	a, c, B. Check.
7.	$c = 30, A = 51°$;	a, b, B. Check.
8.	$c = 130, B = 22° 28'$;	a, b, A. Check.
9.	$a = 40, B = 29°$;	b, c, A. Check.
10.	$a = 48, b = 26$;	c, A, B. Check.
11.	$b = 150, c = 200$;	a, A, B. Check.
12.	$b = 7.636, B = 73° 45.7'$;	$A = 16° 14.3', a = 2.224, c = 7.9534$.
13.	$c = 0.532, B = 50° 21.9'$;	$A = 39° 38.1', a = 0.3394, b = 0.4097$.
14.	$a = 192.56, b = 437.98$;	$A = 23° 44', B = 66° 16', c = 478.44$.
15.	$c = 65.8, A = 47° 59.8'$;	$B = 42° 0.2', a = 48.897, b = 44.032$.
16.	$b = 1.30, B = 79° 27'$;	$A = 10° 33', a = 0.242, c = 1.322$.
17.	$b = 5.21, c = 8.42$;	$A = 51° 46.4', B = 38° 13.6', a = 6.615$.
18.	$b = 52.02, c = 769.96$;	$A = 86° 7.6', B = 3° 52.4', a = 768.22$.
19.	$b = 89.49, A = 3° 47.6'$;	$B = 86° 12.4', a = 5.934, c = 89.685$.
20.	$a = 0.1515, A = 40° 46.9'$;	$B = 49° 13.1', b = 0.1757, c = 0.232$.

21. The shadow of a flagpole 75 ft. high is 98 ft. What is the angle of elevation of the sun at that instant? *Ans.* 37° 25.6'.

22. If side a is three times side b in a right triangle, find angle A. *Ans.* $A = 71° 33.9'$.

23. What angle does a mountain slope make with the horizontal plane if it rises 450 ft. in 60 rods on the horizontal? *Note:* 1 rod = $16\frac{1}{2}$ ft. *Ans.* 24° 26.6'.

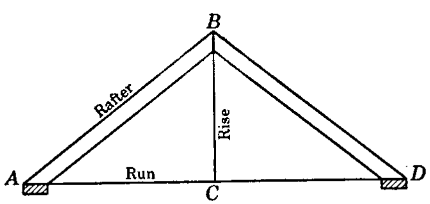


FIG. 36.

24. What is the angle of inclination of a stairway with the floor if the steps have a tread of 10 in. and a rise of 8 in.? *Ans.* 38° 39.6'.

25. What angle does a rafter make with the horizontal if it has a rise of 6 ft. in a run of 15 ft.? *Ans.* 21° 48.1'.

26. Certain lots in a city are laid out by lines perpendicular to B street, and running through to A street as shown in Fig. 37. Required the width of the lots on A street if the angle between the streets is 35° 50'. *Ans.* 123.35 ft.

27. Find the angle between the rafters and the horizontal in roofs of the following pitches: two-thirds, half, third, fourth. *Ans.* 53° 7.75'; 45°; 33° 41.4'; 26° 33.9'.

Note.—By the pitch of a roof is meant the ratio of the rise of the rafters to twice the run, or, in a V-shaped roof, it is the ratio of the distance from the plate to the ridge, to the width of the building.

28. One of the equal sides of an isosceles triangle is 5.74 in. and one of the base angles is 23° 35'; find the altitude and the base. *Ans.* 2.296 in.; 10.521 in.

29. The base of an isosceles triangle is 40 ft. and the vertex angle is 48° 30'; find the equal sides and the base angles. *Ans.* 48.7 ft.; 65° 45'.

30. One side of a regular pentagon inscribed in a circle is 8 in.; find the radius of the circle. *Ans.* 6.8 in.

31. One side of a regular octagon inscribed in a circle is 15 in.; find the radius of the circle. *Ans.* 19.6 in.

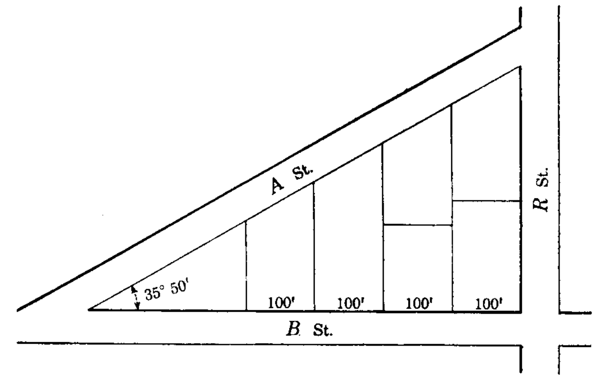


FIG. 37.

32. One side of a regular decagon inscribed in a circle is 8.56 in.; find the radius of the circle. *Ans.* 13.85 in.

33. One side of a regular octagon circumscribed about a circle is 12.8 in.; find the radius of the circle. *Ans.* 15.45 in.

34. The radius of a circle is 24 in.; find the side of a regular inscribed pentagon. Of a regular circumscribed pentagon. *Ans.* 28.2 in.; 34.9 in.

Find the areas of the following isosceles triangles:

35. Altitude 28 ft. and base angles each 55° 27'. *Ans.* 539.85 sq. ft.

36. Base 35.6 ft. and base angles each 64° 51'. *Ans.* 674.85 sq. ft.

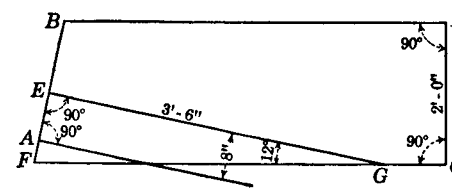


FIG. 38.

37. Equal sides each 10.8 in. and vertex angle 48° 17'. *Ans.* 43.53 sq. in.

38. The radius of a circle is 11 ft. What angle will a chord 14 ft. long subtend at the center? *Ans.* 79° 2.5'.

39. The chord of a circle is 12 ft. long and the angle which it subtends at the center is 41.6°. Find the radius of the circle. *Ans.* 16.9 ft.

40. Five holes are drawn on a piece of steel with their centers equally spaced on the circumference of a circle 10 in. in diameter. Find the distance in a straight line between the centers of two consecutive holes. *Ans.* 5.9 in.

41. Thirty holes are drawn with their centers equally spaced on the circumference of a circle 22 in. in diameter. Find the distance between the centers of two consecutive holes. *Ans.* 2.3 in.

42. Using Fig. 38, with the dimensions as given, find AB. *Ans.* 23.61 in.

SOLUTION OF RIGHT TRIANGLE BY LOGARITHMS

42. Remark on logarithms.—By the use of logarithms, the processes of multiplication, division, raising to powers, and extracting roots may be shortened. In the solution of triangles, logarithms are very advantageous in saving time and labor, and thus conduce to accuracy. The student should bear in mind, however, that logarithms are not necessary for this work. The computer must decide for himself whether or not it will be of advantage to use logarithms in any given problem.

Formulas which have been so arranged that they involve only operations of multiplication, division, raising to powers, and extracting roots are said to be adapted to computation by logarithms.

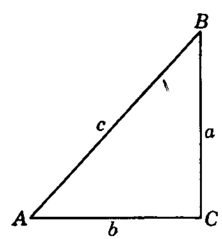
43. Solution of right triangles by logarithmic functions.—The solution of a right triangle is the same by logarithms as by natural functions, except that logarithms are used to avoid the long multiplications and divisions. The tables of logarithmic functions are used instead of the tables of natural functions.

Example 1.—Given $a = 33.75$ and $c = 45.72$; find A, B, and b.

Solution. *Construction*

$$\text{Given}\begin{cases} a = 33.75 \\ c = 45.72. \end{cases}$$

$$\text{To find}^*\begin{cases} A = 47° \ 34.6'. \\ B = 42° \ 25.4'. \\ b = 30.843. \end{cases}$$



Formulas

(1) $\sin A = \dfrac{a}{c}.$

(2) $\cos B = \dfrac{a}{c}.$

(3) $\cos A = \dfrac{b}{c}$, or $b = c \cos A.$

* Results to be inserted when work is completed.

Logarithmic formulas

(1) $\log \sin A = \log a - \log c.$

(2) $\log \cos B = \log a - \log c.$

(3) $\log b = \log c + \log \cos A.$

Computation

$$\begin{aligned} \log a &= 1.52827 \\ \log c &= 1.66011 \\ \log \sin A &= 9.86816 - 10 \\ A &= 47° \ 34.6' \\ \log \cos B &= 9.86816 - 10 \\ B &= 42° \ 25.4' \\ \log c &= 1.66011 \\ \log \cos A &= 9.82905 - 10 \\ \log b &= 1.48916 \\ b &= 30.843 \end{aligned}$$

Check

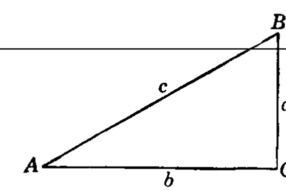
$$\begin{aligned} a^2 &= c^2 - b^2 = (c + b)(c - b) \\ &= 76.563 \times 14.877 \\ \log (c + b) &= 1.88402 \\ \log (c - b) &= 1.17251 \\ \log a^2 &= 3.05653 \\ \log a &= 1.52827 \\ a &= 33.75 \end{aligned}$$

Example 2.—Given $b = 8.724$ and $A = 29° \ 52.3'$; find B, c and a.

Solution. *Construction*

$$\text{Given}\begin{cases} b = 8.724. \\ A = 29° \ 52.3'. \end{cases}$$

$$\text{To find}\begin{cases} B = 60° \ 7.7'. \\ c = 10.061. \\ a = 5.011. \end{cases}$$



Formulas

(1) $A + B = 90°$, or $B = 90° - A.$

(2) $\tan A = \dfrac{a}{b}$, or $a = b \tan A.$

(3) $\cos A = \dfrac{b}{c}$, or $c = \dfrac{b}{\cos A}.$

Logarithmic formulas

(1) $\log a = \log b + \log \tan A.$

(2) $\log c = \log b - \log \cos A.$

Computation	*Check*
$B = 90° - 29° 52.3' = 60° 7.7'$	$b^2 = c^2 - a^2 = (c + a)(c - a)$
$\log b = 0.94072$	$= 15.072 \times 5.050$
$\log \tan A = 9.75919 - 10$	$\log (c + a) = 1.17817$
$\log a = 0.69991$	$\log (c - a) = 0.70329$
$a = 5.0109$	$\log b^2 = 1.88146$
$\log b = 0.94072$	$\log b = 0.94073$
$\log \cos A = 9.93809 - 10$	$b = 8.7242$
$\log c = 1.00263$	
$c = 10.061$	

Note.—It is best to make a full ***skeleton solution*** before proceeding to the use of the **Tables**. The skeleton solution can be seen in this example by erasing the numerical quantities.

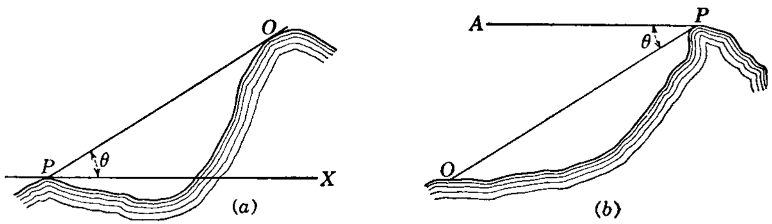


FIG. 39.

In using the **Tables,** plan so as to save time as much as possible. For instance, if both log sine and log cosine of some angle are required, look up both of them while the tables are open at that page.

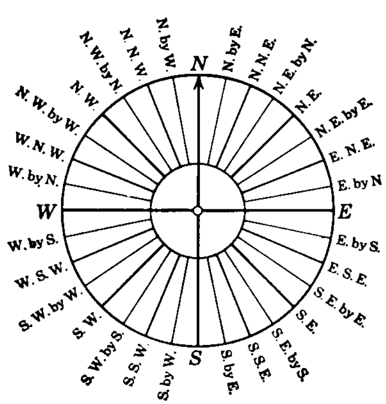


FIG. 40.

44. Definitions.—The **angle of elevation** is the angle between the line of sight and the horizontal plane through the eye when

the object observed is above that horizontal plane. When the object observed is below this horizontal plane, the angle is called the **angle of depression.**

Thus, in Fig. 39*a*, an object O is seen from the point P. The angle θ between the line PO and the horizontal PX is the angle of elevation. In Fig. 39*b* an object O is seen from the point of observation P. The angle θ between the line PO and the horizontal AP is called the angle of depression.

Directions on the surface of the earth are often given by directions as located on the **mariner's compass.** As seen from Fig. 40, these directions are located with reference to the four cardinal points, north, south, east, and west. The directions are often spoken of as **bearings.** Present practice, however, gives the bearing of a line in degrees. The bearing of a line is defined to be the acute angle the line makes with the north-and-south line.

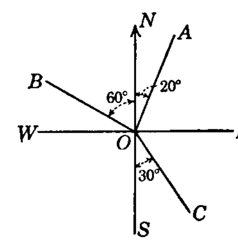


FIG. 41.

Thus, in Fig. 41, if O is the point of observation, the bearing of OA is north, 20° east, written N. 20° E. The bearing of OB is N. 60° W., and that of OC is S. 30° E.

EXERCISES

Solve the following right triangles for the parts not given. The first two parts printed are the given parts. Use logarithms.

1. $a = 31.756$, $A = 54° 43.5'$. Check results.
2. $b = 13.98$, $B = 21° 54'$. Check results.
3. $b = 1676.34$, $c = 5432.8$. Check results.
4. $a = 4.5612$, $B = 43° 3.7'$. Check results.
5. $b = 54.78$, $A = 35° 43.2'$. Check results.
6. $a = 25.13$, $c = 43.412$. Check results.
7. $c = 23.746$, $A = 32° 54.21'$. Check results.
8. $a = 134.90$, $b = 101.43$. Check results.
9. $a = 14.23$, $b = 9.499$. Check results.
10. $c = 143.89$, $B = 39° 54.8'$. Check results.
11. $a = 18.091$, $b = 1378.2$. Check results.
12. $a = 896$, $B = 2° 6' 10''$. Check results.
13. $a = 653$, $c = 680$, $b = 189.7$, $A = 73° 48'$, $B = 16° 12'$.
14. $b = 675.31$, $B = 78° 34.6'$, $a = 136.46$, $c = 688.97$, $A = 11° 25.4'$.
15. $b = 1100$, $c = 1650$, $a = 1229.9$, $A = 48° 11.4'$, $B = 41° 48.6'$.
16. $c = 11.003$, $A = 45° 32' 19''$, $a = 7.8530$, $b = 7.7067$, $B = 44° 27.7'$.
17. $a = 0.001348$, $b = 0.0009896$, $c = 0.0016722$, $A = 53° 43'$, $B = 36° 17'$.

18. A ladder 30 ft. long rests against a building standing on level ground, and makes an angle of 65° 35′ with the ground. Find the distance it reaches up the building. *Ans.* 27.3 ft.

19. A tower stands on level ground. At a point 161.7 ft. distant and 5.5 ft. above the ground the angle of elevation of the top of the tower is 62° 48′. Find the height of the tower to the nearest foot. *Ans.* 320 ft.

20. From the top of a tower 375 ft. high, the angle of depression of a man on the horizontal plane through the foot of the tower is 37° 24.6′. Find the distance the man is from the foot of the tower. *Ans.* 490.3 ft.

21. What is the angle of inclination of a roadbed having a grade of 14 per cent? One with a grade of 26 per cent? (A road with a rise of 14 ft. in 100 ft. on the horizontal has a grade of 14 per cent).
Ans. 7° 58.2′; 14° 34.4′.

22. Locate the centers of the holes B and C (Fig. 42) by finding the distance each is to the right and above the center O. The radius of the circle is 4.5 in. Compute correct to four decimals.
Ans. (3.6406, 2.6450); (1.3906, 4.2798).

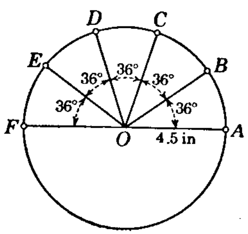


FIG. 42.

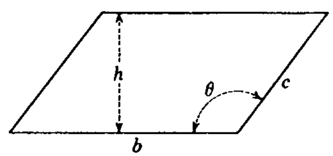


FIG. 43.

23. In the parallelogram of Fig. 43, $b = 33.7$ in., $c = 14.8$ in., and $\theta =$ 126° 15′. Find the altitude h of the parallelogram. *Ans.* 11.94 in.

24. A ladder 32 ft. long is resting against a wall at an angle of 21.7°. If the foot of the ladder is drawn away 4 ft., how far down the wall will the top of the ladder fall? *Ans.* 1.9 ft.

25. A man surveying a mine, measures a length $AB = 1240$ ft. due east with a dip of 6° 15′; then a length $BC = 3425$ ft. due south with a dip of 10° 45′. How much deeper is C than A? *Ans.* 773.84 ft.

26. Find the number of square yards of cloth in a conical tent with a circular base, and vertical angle 78°, the center pole being 12 ft. high.
Ans. 52.4 sq. yd.

Find the areas of the following isosceles triangles:

27. Altitude is 27 ft. and base angles each 55.6°. *Ans.* 499.2 sq. ft.

28. Base is 3 ft. and vertical angle 38° 24′. *Ans.* 6.46 sq. ft.

29. Each leg is 15 ft. and base angles each 63° 18.6′. *Ans.* 90.29 sq. ft.

30. Find the area of a regular pentagon one of whose sides is 10 in.
Ans. 172.05 sq. in.

31. Find the area of a regular octagon one of whose sides is 15 in.
Ans. 1086.4 sq. in.

32. Find the difference in the areas of a regular hexagon and a regular octagon, each of perimeter 80 ft. *Ans.* 20.96. sq. ft.

33. Prove that the area of a right triangle is given by each of the following, where S is the area:

$$S = \tfrac{1}{2}bc \sin A.$$
$$S = \tfrac{1}{4}ac \cos A.$$
$$S = \tfrac{1}{2}c^2 \sin A \cos A.$$

34. If R is the radius of a circle, show that the area of a regular circumscribed polygon of n sides is given by the formula:

$$A = nR^2 \tan \frac{180°}{n}.$$

35. Show that the area of a regular inscribed polygon of n sides is given by the formula:

$$A = nR^2 \sin \frac{180°}{n} \cos \frac{180°}{n} = \frac{1}{2}nR^2 \sin \frac{360°}{n}.$$

36. The radius of a circle is 30 in. Find the perimeter of a regular inscribed pentagon. *Ans.* 176.34 in.

37. Find the area of a regular octagon inscribed in a circle whose radius is 8 in. *Ans.* 181.02 sq. in.

38. What diameter of stock must be chosen so that a hexagonal end 3 in. across the flats may be milled upon it?
Ans. 3.46 in.

Answer the question for an octagon. The meaning of "across the flats" is shown in Fig. 44.
Ans. 3.25 in.

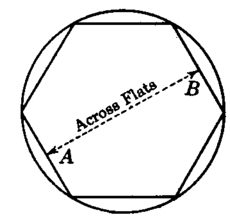


FIG. 44.

39. From a point 460 ft. above the level of a lake the angle of depression of a point on the near shore is 21° 56′, and of a point directly beyond on the opposite shore is 4° 31′. Find the width of the lake.
Ans. 4680.7 ft.

40. Find the angles at the base made by the sides of a tower with the horizontal, if the tower is 47 ft. 6 in. high, has a square base 6 ft. on a side, and a top 8 in. square. *Ans.* 86° 47.2′.

41. Suppose the earth a sphere with a radius of 3960 miles; find the length of the arctic circle which is at latitude 66° 32′. *Ans.* 9908.3 miles.

42. Find the length of 1° of longitude in the latitude of Chicago, 41° 50′, if the earth is a sphere with a radius of 3960 miles. *Ans.* 51.497 miles.

43. A circle 12 in. in diameter is suspended from a point and held in a horizontal position by 12 strings each 8 in. long and equally spaced around the circumference. Find the angle between two consecutive strings.
Ans. 22° 23.2′.

44. A girder to carry a bridge is in the form of a circular arc. The length of the span is 120 ft. and the height of the arch is 30 ft. Find the angle at the center of the circle such that its sides intercept the arc of the girder; and find the radius of the circle. *Ans.* 106° 15.7′; 75 ft.

45. A tree stands upon the same plane as a house whose height is 65 ft. The angle of elevation and depression of the top and base of the tree from the top of the house are 45° and 62°, respectively. Find the height of the tree. *Ans.* 99.6 ft.

46. From a point 20 ft. above the surface of the water, the angle of elevation of a tree standing at the edge of the water is 41° 15′, while the angle of

depression of its image in the water is 58° 45′. Find the height of the tree, and its horizontal distance from the point of observation.
Ans. 51.88 ft.; 65.50 ft.

47. The legs supporting a tank tower are 50 ft. long and 18 ft. apart at the base, forming a square. The angle which the legs make with the horizontal line between the feet diagonally opposite is 83° 30′. How far apart are the tops of the legs? *Ans.* 10 ft.

48. The angle of elevation of a balloon from a point due south of it is 50°, and from another point 1 mile due west of the former the angle of elevation is 40°. Find the height of the balloon. *Ans.* 1.18 miles.

49. At a point P on a level plain the angle of elevation of an airplane that is southwest of P is 38° 35′. At a point Q, 2 miles due south of P, the airplane appears in the northwest. What is the height of the airplane?
Ans. 1.13 miles.

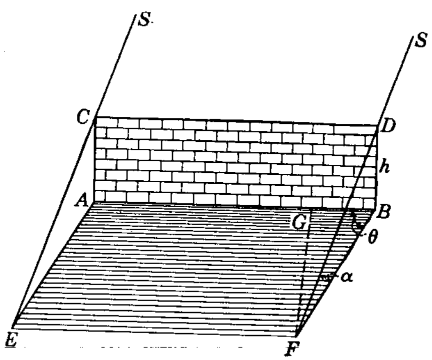


FIG. 45.

50. From a point on a level plain the angle of elevation of the top of a hill is 23° 46′; and a tower 45 ft. high standing on the top of the hill subtends an angle of 5° 16′. Find the height of the hill above the plain. *Ans.* 172.7 ft.

51. A flagstaff stands upon the top of a building 150 ft. high. At a horizontal distance of 225 ft. from the base of the building the flagstaff subtends an angle of 6° 30′. Find the height of the flagstaff.
Ans. 40.07 ft.

52. Two observers are stationed 1 mile apart on a straight east-and-west level road. An airplane flying north passes between them, and, as it is over the road, the angles of elevation are observed to be 72° 30′ and 65° 15′. Find the height of the airplane. *Ans.* 1.29 miles.

53. A ship is sailing due east at 16 miles per hour. A lighthouse is observed due south at 8:30 A.M. At 9:45 A.M. the bearing of the same lighthouse is S. 38° 30′ W. Find the distance the ship is from the lighthouse at the time of the first observation. *Ans.* 25.14 miles.

54. Find the width of the shadow of the wall shown in Fig. 45. If the height of the wall is h ft., the angle of elevation of the sun α, and the angle between the vertical plane through the sun and the plane of the wall θ, show that width of shadow $= h \cot \alpha \sin \theta$.

55. A wall extending east and west is 8 ft. high. The sun has an inclination of 49° 30′ and is 47° 15′ 30″ west of south. Find the width of the shadow of the wall. *Ans.* 4.637 ft.

56. A tripod is made of three sticks, each 5 ft. long, by tying together the ends of the sticks, the other ends resting on the ground 3 ft. apart. Find the height of the tripod. *Ans.* 4.690 ft.

57. At a certain point the angle of elevation of a mountain peak is 40° 30′. At a distance of 3 miles farther away in the same horizontal plane, its angle of elevation is 27° 40′. Find the distance of the top of the mountain above the horizontal plane, and the horizontal distance from the first point of observation to the point directly below the peak. *Ans.* 4.77 miles.

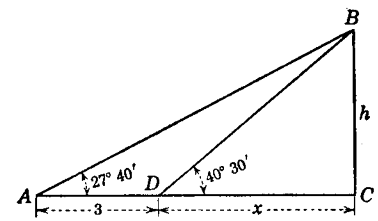


FIG. 46.

Suggestion.—Find two simultaneous equations involving the unknowns h and x representing the distances as shown in Fig. 46. These are $\tan 40° 30' = \frac{h}{x}$ and $\tan 27° 40' = \frac{h}{3 + x}$. Solve these algebraically for h. and x.

58. At a certain point A the angle of elevation of a mountain peak is α; at a point B that is a miles farther away in the same horizontal plane its angle of elevation is β. If h represents the distance the peak is above the plane and x the horizontal distance the peak is from A, derive the formulas:

$$h = \frac{a \tan \alpha \tan \beta}{\tan \alpha - \tan \beta}; \; x = \frac{a \tan \beta}{\tan \alpha - \tan \beta}$$

Note.—In using these formulas, it is convenient to use natural functions. In Exercise 5, page 150, is given a solution of the same problem, obtaining formulas adapted to logarithms.

59. Find the height of a tree if the angle of elevation of its top changes from 35° to 61° 30′ on walking toward it 200 ft. in a horizontal line through its base. *Ans.* 225.93 ft.

60. A man walking on a level plain toward a tower observes that at a certain point the angle of elevation of the top of the tower is 30°, and, on walking 305 ft. directly toward the tower, the angle of elevation of the top is 52°. Find the height of the tower if the point of observation each time is 5 ft. above the ground. *Ans.* 325.8 ft.

61. At a certain point the angle of elevation of the top of a mountain is 36° 15′. At a second point 700 ft. farther away in the same horizontal plane the angle of elevation is 28° 30′. Find the height of the mountain above the horizontal plane. *Ans.* 1464.6 ft.

CHAPTER V

FUNCTIONS OF LARGE ANGLES

45. It is proved in **Art. 16** that, for any angle, each of the trigonometric functions has just one value. On the other hand, it was shown later that a particular value of a function may go with more than one angle. For instance, $\sin^{-1} \frac{1}{2}$ is 30° and 150° and, in fact, may be any one of the other angles whose terminal sides lie in the same positions as the terminal side of 30° or 150°. This would suggest that possibly any function of a large angle may be equal to a function of an angle that is not greater than 90°. Further, it would seem that some such relation must exist for the tables have only the functions of angles of 90° or less tabulated. We shall now proceed to show that a function of a large angle can be expressed as a function of an angle less than 90°.

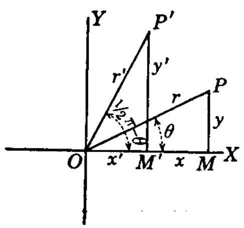


Fig. 47.

46. Functions of $\frac{1}{2}\pi - \theta$ in terms of functions of θ.—It has been shown in previous articles that a function of an acute angle is equal to the *cofunction* of the *complement* of that angle. This will now be proved in a different manner.

Let θ be any acute angle drawn as in Fig. 47. Construct $\frac{1}{2}\pi - \theta$, take $OP' = OP$, and let x, y, and r be the abscissa, ordinate, and distance, respectively, of P; and x', y', and r' those for P'. It is evident that right triangles OMP and $O'M'P'$ are equal.

Then, since $y' = x$, $x' = y$, and $r' = r$,

$$\sin\left(\tfrac{1}{2}\pi - \theta\right) = \frac{y'}{r'} = \frac{x}{r} = \cos\theta.$$

$$\cos\left(\tfrac{1}{2}\pi - \theta\right) = \frac{x'}{r'} = \frac{y}{r} = \sin\theta.$$

$$\tan\left(\tfrac{1}{2}\pi - \theta\right) = \frac{y'}{x'} = \frac{x}{y} = \cot\theta.$$

$$\cot\left(\tfrac{1}{2}\pi - \theta\right) = \frac{x'}{y'} = \frac{y}{x} = \tan\theta.$$

$$\sec\left(\tfrac{1}{2}\pi - \theta\right) = \frac{r'}{x'} = \frac{r}{y} = \csc\theta.$$

$$\csc\left(\tfrac{1}{2}\pi - \theta\right) = \frac{r'}{y'} = \frac{r}{x} = \sec\theta.$$

Notice that in each line the function at the end is the cofunction of the one at the beginning.

47. Functions of $\frac{1}{2}\pi + \theta$ in terms of functions of θ.—In Fig. 48, let θ be any acute angle. Construct $\frac{1}{2}\pi + \theta$, take $OP' = OP$, and represent the other parts as shown.

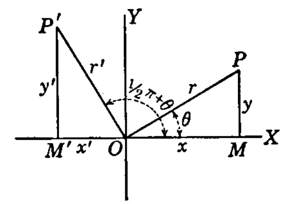


Fig. 48.

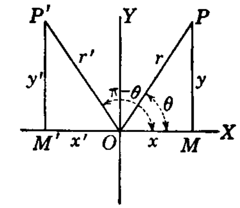


Fig. 49.

Then, since $y' = x$, $x' = -y$, and $r' = r$,

$$\sin\left(\tfrac{1}{2}\pi + \theta\right) = \frac{y'}{r'} = \frac{x}{r} = \cos\theta.$$

$$\cos\left(\tfrac{1}{2}\pi + \theta\right) = \frac{x'}{r'} = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta.$$

$$\tan\left(\tfrac{1}{2}\pi + \theta\right) = \frac{y'}{x'} = \frac{x}{-y} = -\frac{x}{y} = -\cot\theta.$$

$$\cot\left(\tfrac{1}{2}\pi + \theta\right) = \frac{x'}{y'} = \frac{-y}{x} = -\frac{y}{x} = -\tan\theta.$$

$$\sec\left(\tfrac{1}{2}\pi + \theta\right) = \frac{r'}{x'} = \frac{r}{-y} = -\frac{r}{y} = -\csc\theta.$$

$$\csc\left(\tfrac{1}{2}\pi + \theta\right) = \frac{r'}{y'} = \frac{r}{x} = \sec\theta.$$

Notice that here, also, in each line the function at the end is the cofunction of the one at the beginning.

Examples.—$\sin 130° = \sin (90° + 40°) = \cos 40°$.
$\cot 110° = \cot (90° + 20°) = -\tan 20°$.

48. Functions of $\pi - \theta$ in terms of functions of θ.—In Fig. 49, let θ be an acute angle. Construct $\pi - \theta$, take $OP' = OP$, and represent the other parts as shown.

Then, since $x' = -x$, $y' = y$, and $r' = r$,

$$\sin(\pi - \theta) = \frac{y'}{r'} = \frac{y}{r} = \sin\theta.$$

$$\cos(\pi - \theta) = \frac{x'}{r'} = \frac{-x}{r} = -\frac{x}{r} = -\cos\theta.$$

$$\tan(\pi - \theta) = \frac{y'}{x'} = \frac{y}{-x} = -\frac{y}{x} = -\tan\theta.$$

$$\cot(\pi - \theta) = \frac{x'}{y'} = \frac{-x}{y} = -\frac{x}{y} = -\cot\theta.$$

$$\sec(\pi - \theta) = \frac{r'}{x'} = \frac{r}{-x} = -\frac{r}{x} = -\sec\theta.$$

$$\csc(\pi - \theta) = \frac{r'}{y'} = \frac{r}{y} = \csc\theta.$$

Notice that in each line the function at the end is the same function as the one at the beginning.

Examples.—$\cos 160° = \cos(180° - 20°) = -\cos 20°.$
$\csc 140° = \csc(180° - 40°) = \csc 40°.$

49. Functions of $\pi + \theta$ in terms of functions of θ.—In Fig. 50, let θ be an acute angle. Construct $\pi + \theta$, take $OP' = OP$, and represent the other parts as shown.

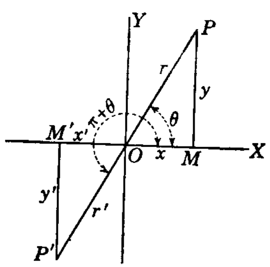


FIG. 50.

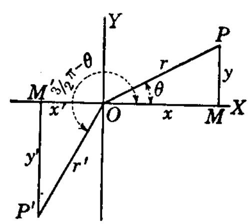


FIG. 51.

Then, since $x' = -x$, $y' = -y$, and $r' = r$,

$$\sin(\pi + \theta) = \frac{y'}{r'} = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta.$$

$$\cos(\pi + \theta) = \frac{x'}{r'} = \frac{-x}{r} = -\frac{x}{r} = -\cos\theta.$$

$$\tan(\pi + \theta) = \frac{y'}{x'} = \frac{-y}{-x} = \frac{y}{x} = \tan\theta.$$

$$\cot(\pi + \theta) = \frac{x'}{y'} = \frac{-x}{-y} = \frac{x}{y} = \cot\theta.$$

$$\sec(\pi + \theta) = \frac{r'}{x'} = \frac{r}{-x} = -\frac{r}{x} = -\sec\theta.$$

$$\csc(\pi + \theta) = \frac{r'}{y'} = \frac{r}{-y} = -\frac{r}{y} = -\csc\theta.$$

Notice that here, also, in each line the function at the end is the same function as the one at the beginning.

Examples.—$\tan 230° = \tan(180° + 50°) = \tan 50°.$
$\cos 205° = \cos(180° + 25°) = -\cos 25°.$

50. Functions of $\frac{3}{2}\pi - \theta$ in terms of functions of θ.—In Fig. 51, let θ be an acute angle. Construct $\frac{3}{2}\pi - \theta$, take $OP' = OP$, and represent the other parts as shown.

Then, since $y' = -x$, $x' = -y$, and $r' = r$,

$$\sin(\tfrac{3}{2}\pi - \theta) = \frac{y'}{r'} = \frac{-x}{r} = -\frac{x}{r} = -\cos\theta.$$

$$\cos(\tfrac{3}{2}\pi - \theta) = \frac{x'}{r'} = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta.$$

$$\tan(\tfrac{3}{2}\pi - \theta) = \frac{y'}{x'} = \frac{x}{y} = \cot\theta.$$

$$\cot(\tfrac{3}{2}\pi - \theta) = \frac{x'}{y'} = \frac{y}{x} = \tan\theta.$$

$$\sec(\tfrac{3}{2}\pi - \theta) = \frac{r'}{x'} = \frac{r}{-y} = -\frac{r}{y} = -\csc\theta.$$

$$\csc(\tfrac{3}{2}\pi - \theta) = \frac{r'}{y'} = \frac{r}{-x} = -\frac{r}{x} = -\sec\theta.$$

Notice that here again in each line the function at the end is the cofunction of the one at the beginning.

Examples.—$\sin 250° = \sin(270° - 20°) = -\cos 20°.$
$\tan 210° = \tan(270° - 60°) = \cot 60°.$

51. Functions of $\frac{3}{2}\pi + \theta$ in terms of functions of θ.—In Fig. 52, let θ be an acute angle. Construct $\frac{3}{2}\pi + \theta$, take $OP' = OP$, and represent the other parts as shown.

Then, since $y' = -x$, $x' = y$, and $r' = r$,

$$\sin(\tfrac{3}{2}\pi + \theta) = \frac{y'}{r'} = \frac{-x}{r} = -\frac{x}{r} = -\cos\theta.$$

$$\cos(\tfrac{3}{2}\pi + \theta) = \frac{x'}{r'} = \frac{y}{r} = \sin\theta.$$

$$\tan(\tfrac{3}{2}\pi + \theta) = \frac{y'}{x'} = \frac{-x}{y} = -\frac{x}{y} = -\cot\theta.$$

$$\cot\left(\tfrac{3}{2}\pi + \theta\right) = \frac{x'}{y'} = \frac{y}{-x} = -\frac{y}{x} = -\tan\theta.$$

$$\sec\left(\tfrac{3}{2}\pi + \theta\right) = \frac{r'}{x'} = \frac{r}{y} = \csc\theta.$$

$$\csc\left(\tfrac{3}{2}\pi + \theta\right) = \frac{r'}{y'} = \frac{r}{-x} = -\frac{r}{x} = -\sec\theta.$$

Notice that here, also, in each line the function at the end is the cofunction of the one at the beginning.

Examples.—$\cot 310° = \cot(270° + 40°) = -\tan 40°.$
$\sec 340° = \sec(270° + 70°) = \csc 70°.$

52. Functions of $-\theta$ or $2\pi - \theta$ in terms of functions of θ.—In Fig. 53, let θ be an acute angle. Construct $2\pi - \theta$, $OP' = OP$, and represent the other parts as shown.

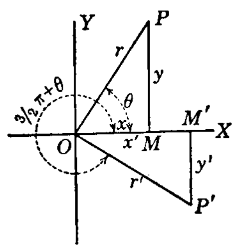


FIG. 52.

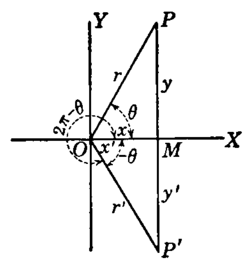


FIG. 53.

Then, since $x' = x$, $y' = -y$, and $r' = r$,

$$\sin(-\theta) = \frac{y'}{r'} = \frac{-y}{r} = -\frac{y}{r} = -\sin\theta.$$

$$\cos(-\theta) = \frac{x'}{r'} = \frac{x}{r} = \cos\theta.$$

$$\tan(-\theta) = \frac{y'}{x'} = \frac{-y}{x} = -\frac{y}{x} = -\tan\theta.$$

$$\cot(-\theta) = \frac{x'}{y'} = \frac{x}{-y} = -\frac{x}{y} = -\cot\theta.$$

$$\sec(-\theta) = \frac{r'}{x'} = \frac{r}{x} = \sec\theta.$$

$$\csc(-\theta) = \frac{r'}{y'} = \frac{r}{-y} = -\frac{r}{y} = -\csc\theta.$$

These formulas can readily be remembered by noting that the functions of the negative angle are the same as those of the positive angle, but of opposite sign, except the cosine and the secant, which are of the same sign.

53. Functions of an angle greater than 2π.—Any angle α greater than 2π has the same trigonometric functions as α minus an integral multiple of 2π, because α and $\alpha - 2n\pi$ have the same initial and terminal sides. That is, the functions of α equal the same functions of $\alpha - 2n\pi$, where n is an integer.

That is, a function of an angle that is larger than 360° can be found by dividing the angle by 360° and finding the required function of the remainder.

54. Summary of the reduction formulas.—The formulas of the previous articles are here collected so as to be convenient for reference. It will be well to memorize the last group, the one expressing the functions of negative angles as functions of positive angles.

$\sin(\frac{1}{2}\pi - \theta) = \cos\theta.$	$\sin(\frac{1}{2}\pi + \theta) = \cos\theta.$
$\cos(\frac{1}{2}\pi - \theta) = \sin\theta.$	$\cos(\frac{1}{2}\pi + \theta) = -\sin\theta.$
$\tan(\frac{1}{2}\pi - \theta) = \cot\theta.$	$\tan(\frac{1}{2}\pi + \theta) = -\cot\theta.$
$\cot(\frac{1}{2}\pi - \theta) = \tan\theta.$	$\cot(\frac{1}{2}\pi + \theta) = -\tan\theta.$
$\sec(\frac{1}{2}\pi - \theta) = \csc\theta.$	$\sec(\frac{1}{2}\pi + \theta) = -\csc\theta.$
$\csc(\frac{1}{2}\pi - \theta) = \sec\theta.$	$\csc(\frac{1}{2}\pi + \theta) = \sec\theta.$
$\sin(\pi - \theta) = \sin\theta.$	$\sin(\pi + \theta) = -\sin\theta.$
$\cos(\pi - \theta) = -\cos\theta.$	$\cos(\pi + \theta) = -\cos\theta.$
$\tan(\pi - \theta) = -\tan\theta.$	$\tan(\pi + \theta) = \tan\theta.$
$\cot(\pi - \theta) = -\cot\theta.$	$\cot(\pi + \theta) = \cot\theta.$
$\sec(\pi - \theta) = -\sec\theta.$	$\sec(\pi + \theta) = -\sec\theta.$
$\csc(\pi - \theta) = \csc\theta.$	$\csc(\pi + \theta) = -\csc\theta.$
$\sin(\frac{3}{2}\pi - \theta) = -\cos\theta.$	$\sin(\frac{3}{2}\pi + \theta) = -\cos\theta.$
$\cos(\frac{3}{2}\pi - \theta) = -\sin\theta.$	$\cos(\frac{3}{2}\pi + \theta) = \sin\theta.$
$\tan(\frac{3}{2}\pi - \theta) = \cot\theta.$	$\tan(\frac{3}{2}\pi + \theta) = -\cot\theta.$
$\cot(\frac{3}{2}\pi - \theta) = \tan\theta.$	$\cot(\frac{3}{2}\pi + \theta) = -\tan\theta.$
$\sec(\frac{3}{2}\pi - \theta) = -\csc\theta.$	$\sec(\frac{3}{2}\pi + \theta) = \csc\theta.$
$\csc(\frac{3}{2}\pi - \theta) = -\sec\theta.$	$\csc(\frac{3}{2}\pi + \theta) = -\sec\theta.$
$\sin(2\pi - \theta) = -\sin\theta.$	$\sin(-\theta) = -\sin\theta.$
$\cos(2\pi - \theta) = \cos\theta.$	$\cos(-\theta) = \cos\theta.$
$\tan(2\pi - \theta) = -\tan\theta.$	$\tan(-\theta) = -\tan\theta.$
$\cot(2\pi - \theta) = -\cot\theta.$	$\cot(-\theta) = -\cot\theta.$
$\sec(2\pi - \theta) = \sec\theta.$	$\sec(-\theta) = \sec\theta.$
$\csc(2\pi - \theta) = -\csc\theta.$	$\csc(-\theta) = -\csc\theta.$

While the proofs of these formulas have all been based upon the assumption that θ is an acute angle, they are true for all values of θ, and can be carried through for any value of θ in exactly the same manner as for θ an acute angle.

Tables of trigonometric functions, in general, do not contain angles greater than 90°. Since the principal application of the reduction formulas is made in determining the numerical values of functions of angles greater than 90°, it will be found convenient to have a rule for the application of the formulas for θ an acute angle. The rule gives a final summary of the preceding articles.

RULE.—*Express the given angle in the form* $n \cdot 90° \pm \theta$, *where* θ *is acute. If n is even, take the same function of* θ *as of the given angle; if n is odd, take the cofunction of* θ. *In either case the final sign is determined by the function of the given angle and the quadrant in which that angle lies.*

If the given angle is negative, first express its function as the function of the given angle with its sign changed, and then proceed as before.

Example 1.—Find cos 825°.

Solution.—By the rule and **Table V,**

$$\cos 825° = \cos (9 \times 90° + 15°) = -\sin 15° = -0.25882.$$

Since $9 \times 90°$ is an odd number times 90°, we take the sine of 15°. It is negative because 825° lies in the second quadrant in which cosine is negative.

Another solution of this is as follows:

$$\begin{aligned}\cos 825° &= \cos (2 \times 360° + 105°) = \cos 105° \\ &= \cos (90° + 15°) = -\sin 15°.\end{aligned}$$

Example 2.—Find cot (−1115°).

Solution.—First express as a positive angle and then apply the rule.

$$\begin{aligned}\cot (-1115°) &= -\cot 1115° = -\cot (12 \times 90° + 35°) \\ &= -\cot 35° = -1.4281.\end{aligned}$$

Example 3.—Find the value of $\dfrac{2 \sec 3\pi - 3 \sin \frac{9}{2}\pi + 2 \cos \frac{7}{2}\pi}{3 \csc \frac{15}{2}\pi + 7 \cos \frac{5}{2}\pi - \sec 7\pi}$.

Solution.—First evaluate each of the functions.

$$\begin{aligned}\sec 3\pi &= \sec (6 \times \tfrac{1}{2}\pi + 0) = -\sec 0 = -1. \\ \sin \tfrac{9}{2}\pi &= \sin (9 \times \tfrac{1}{2}\pi + 0) = \cos 0 = 1. \\ \cos \tfrac{7}{2}\pi &= \cos (7 \times \tfrac{1}{2}\pi + 0) = \sin 0 = 0. \\ \csc \tfrac{15}{2}\pi &= \csc (15 \times \tfrac{1}{2}\pi + 0) = -\sec 0 = -1. \\ \cos \tfrac{5}{2}\pi &= \cos (5 \times \tfrac{1}{2}\pi + 0) = \sin 0 = 0. \\ \sec 7\pi &= \sec (14 \times \tfrac{1}{2}\pi + 0) = -\sec 0 = -1.\end{aligned}$$

Substituting these values,

$$\begin{aligned}\frac{2 \sec 3\pi - 3 \sin \frac{9}{2}\pi + 2 \cos \frac{7}{2}\pi}{3 \csc \frac{15}{2}\pi + 7 \cos \frac{5}{2}\pi - \sec 7\pi} &= \frac{2(-1) - 3 \cdot 1 + 2 \cdot 0}{3(-1) + 7 \cdot 0 - (-1)} \\ &= \frac{-5}{-2} = \frac{5}{2}.\end{aligned}$$

Example 4.—Evaluate $\sin \theta \Big]_{\frac{3}{2}\pi}^{6}$.

Solution.—The notation given is a form frequently used, and means that: (1) the upper number 6 is to be substituted for θ; (2) the lower number $\frac{3}{2}\pi$ is to be substituted for θ; and (3) the result of (2) is to be subtracted from that of (1).

Then $$\sin \theta \Big]_{\frac{3}{2}\pi}^{6} = \sin 6 - \sin \tfrac{3}{2}\pi.$$

Since 6 is a number of radians and

$$\begin{aligned}6 \text{ radians} &= 6(57° \, 17' \, 44.8'') = 343° \, 46' \, 29'', \\ \sin 6 &= \sin 343° \, 46' \, 29'' = -\cos 73° \, 46' \, 29'' = -0.27941. \\ \therefore \sin \theta \Big]_{\frac{3}{2}\pi}^{6} &= -0.27941 - (-1) = 0.72059.\end{aligned}$$

EXERCISES

In Exercises 1 to 47 do the work orally. Express each of the following as a function of θ:

1. sin (720° + θ).
2. sin (720° − θ).
3. sin (630° + θ).
4. cot (630° − θ).
5. sec (1080° + θ).
6. cos (990° + θ).
7. tan (720° − θ).
8. sin (θ − 540°).
9. cos (θ − 1080°).
10. tan (θ − 810°).
11. csc (1890° + θ).
12. sec (2880° − θ).

Express the following functions as functions of acute angles. Give two answers, one where the angle is less than 45° and one where it is greater.

13. sin 150°.
14. cos 100°.
15. tan 210°.
16. cot 265°.
17. cos 320°.
18. cos (−45°).
19. tan 290°.
20. cot 185°.
21. sec 275°.
22. sin (−85°).
23. sin 127° 30′.
24. cos 281° 30′.
25. cot 235° 15′.
26. tan 347° 20′.
27. tan (−68° 30′).

Express the following functions as functions of angles less than 45°.

28. sec 165°.
29. cot 430°.
30. tan 305°.
31. cos 195°.
32. sin 145°.
33. cos 2000°.
34. sec 600°.
35. cot 1050°.
36. csc 840°.
37. sin 700°.
38. sec (−300).
39. cot (−425).
40. tan (−600).
41. sin (−450).
42. cos (−325).

What is the value of each of the following:

43. 3 sin (90° + θ) + 4 cos (180° − θ).
44. 3 sin (360° − θ) − 3 cos (270° + θ).

45. $2 \tan (180° - \theta) - 2 \cot (90° + \theta)$.
46. $5 \sin (270° + \theta) - 3 \sin (270° - \theta)$.
47. $4 \cos (180° - \theta) + 5 \sin (270° + \theta)$.

Show that the following are true equalities:

48. $\tan (225° - \theta) = \tan (45° - \theta)$.
49. $\sin (135° + \theta) = \cos (45° + \theta)$.
50. $\cot (135° + \theta) = - \cot (45° - \theta)$.
51. $\tan (45 \pm \theta) = \cot (45 \mp \theta)$.

By the use of the table of natural functions, find the sine, cosine, tangent, and cotangent of the following angles:

52. 156°.	**56.** 835° 40′.	**60.** −481°.
53. 215°.	**57.** 460° 18′.	**61.** −1301°.
54. 268°.	**58.** 934° 52′.	**62.** −152° 13′.
55. 297°.	**59.** 1045° 25′.	**63.** −209° 24′.

64. Find the sine, cosine, tangent, and cotangent of 135°, 150°, 240°, 330°, 315°, 120°, 210° by expressing them in terms of functions of 30°, 45°, or 60°. Compare the results with the table of values given on page 24.

65. Simplify $\dfrac{\sin (\frac{3}{2}\pi - \theta) \cos (\frac{1}{2}\pi + \theta)}{\tan (\frac{1}{2}\pi + \theta)} - \dfrac{\sin (\frac{3}{2}\pi - \theta)}{\sec (\pi + \theta)}$. *Ans.* -1.

Verify Exercises 66 to 71.

66. $\dfrac{\tan \pi - \tan \theta}{1 + \tan \pi \tan \theta} = \tan (\pi - \theta)$.
67. $\cos \frac{3}{2}\pi \cos \theta - \sin \frac{3}{2}\pi \sin \theta = \cos (\frac{3}{2}\pi + \theta)$.
68. $\sin \frac{3}{2}\pi \cos \theta - \cos \frac{3}{2}\pi \sin \theta = \sin (\frac{3}{2}\pi - \theta)$.
69. $\sin (\frac{1}{2}\pi + \alpha) \cos (\pi - \alpha) + \cos (\frac{1}{2}\pi + \alpha) \sin (\pi - \alpha) = -1$.
70. $\dfrac{\sin (-\theta) + \cos (-\theta)}{\tan (-\theta) - \cot (-\theta)} = \dfrac{\sin (90° + \theta) + \cos (270° - \theta)}{\cot (180° + \theta) + \tan (360° - \theta)}$.
71. $\dfrac{3(\sin \frac{3}{2}\pi - \tan 2\pi + \cos 3\pi)}{4 \csc \frac{3}{2}\pi \cdot \sec 5\pi} = -\dfrac{3}{2}$.

72. Evaluate $12(\frac{1}{2}\theta - \frac{1}{4} \sin 2\theta) \Big]_0^{\pi}$. *Ans.* 6π.

73. Evaluate $(\tan x + \cos x) \Big]_0^{\pi}$. *Ans.* -2.

74. Evaluate $(\frac{1}{3}x - \frac{2}{3} \sin 2x) \Big]_{\frac{1}{3}\pi}^{2\pi}$. *Ans.* 2.323.

75. Evaluate $\frac{1}{2}a^2(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta) \Big]_0^{\frac{3}{2}\pi}$. *Ans.* $2.534a^2$.

76. Evaluate $\pi a^3(\theta - 4\sin\theta + \frac{3}{4}\sin 2\theta + \frac{1}{3}\sin^3\theta) \Big]_0^{\pi}$. *Ans.* $a^3\pi^2$.

77. Evaluate $- \cos x \Big]_2^{\pi} + \cos x \Big]_{\pi}^{4}$. *Ans.* 0.9302.

78. Evaluate $- \sin x \Big]_1^{\frac{3}{4}\pi} + \sin x \Big]_{\frac{3}{4}\pi}^{3}$. *Ans.* -0.4313.

79. If $\sin \theta = -\frac{16}{65}$, with θ in the fourth quadrant, show that $\text{vers} (\theta - \pi) = \frac{128}{65}$.

80. If $\cot 250° = \dfrac{1}{b}$, show that $\tan 160° = -\dfrac{1}{b}$, and $\sec 430° = \sqrt{1 + b^2}$.

81. If $\text{covers } 115° = 1 - \dfrac{1}{c}$ find $\dfrac{\text{vers } 205° \cos 335°}{\cot 245°}$. *Ans.* $\dfrac{\sqrt{c^2 - 1}}{c^3 - c^2}$.

82. If $\tan 200° = c$, find $\dfrac{\sin 110° - \cos 250°}{\csc 160° + \sec 340°}$. *Ans.* $\dfrac{c}{c^2 + 1}$.

83. If $\csc 160° = c$, find $\dfrac{\sin 250° + \tan 290°}{\cot 200° + \cos 340°}$. *Ans.* -1.

Draw the figures and derive the formulas in each of the following:

84. Functions of $90° + \theta$ in terms of functions of θ when θ is in the third quadrant.

85. Functions of $270° - \theta$ in terms of functions of θ when θ is in the second quadrant.

86. Functions of $180° + \theta$ in terms of functions of θ when θ is in the fourth quadrant.

55. Solution of trigonometric equations.—All the angles less than 360° that have the same absolute value for each of the trigonometric functions are called **corresponding angles.** In general, there are four such angles for each trigonometric function. For instance, if $\sin \theta$ is $\frac{1}{2}$ in absolute value, that is, if $\sin \theta = \pm\frac{1}{2}$, then $\theta = 30, 150, 210$, and 330°. These four angles are called the corresponding angles when the absolute value of $\sin \theta$ is $\frac{1}{2}$.

In general, the corresponding angles lie one in each quadrant, and have their terminal sides placed equally above and below the x-axis. The exception is when the angles lie between the quadrants, and then there are but two corresponding angles. Thus, if $\sin \theta = \pm 1$, the corresponding angles are 90 and 270°.

It follows that, if ϕ is the angle lying in the first quadrant, then the other corresponding angles are $180° \pm \phi$ and $360° - \phi$.

If the value of a trigonometric function is given, the angle can be found by the following:

Rule.—*First find the acute angle ϕ by the table of natural functions, using the absolute value of the given function. The remaining, or corresponding, angles which have the same trigonometric function in absolute value are $180° \pm \phi$ and $360° - \phi$. From these four angles the angles can be chosen in the proper quadrants to satisfy the given function.*

That is, if the function is positive, the angle is taken in those quadrants in which that function is positive.

Example 1.—Given $\sin \theta = -\frac{1}{2}$; find $\theta < 360°$.

Solution.—First find $\phi = \sin^{-1} \frac{1}{2} = 30°$.

By the rule, the remaining angles which have their sine equal to $\frac{1}{2}$ in absolute value are $180° - 30° = 150°$,

$$180° + 30° = 210°,$$

and

$$360° - 30° = 330°.$$

Since the sine is negative, θ must be in the third and fourth quadrants.

$$\therefore \theta = 210 \text{ and } 330°.$$

Example 2.—Given $\cos\theta = -\frac{1}{2}\sqrt{2}$; find $\theta < 360°$.

Solution.—Find $\phi = \cos^{-1}\frac{1}{2}\sqrt{2} = 45°$.

The corresponding angles are 135, 225, and 315°.

But the cosine is negative in the second and third quadrants,

$$\therefore \theta = 135 \text{ and } 225°.$$

Example 3.—Given $2\sin\theta + \cos\theta = 2$; solve for $\theta < 360°$.

Solution.—First express all the functions in terms of one function as in **Art. 33.** Then, since $\cos\theta = \sqrt{1 - \sin^2\theta}$, we have

$$2\sin\theta + \sqrt{1 - \sin^2\theta} = 2.$$

Transposing and squaring, $4\sin^2\theta - 8\sin\theta + 4 = 1 - \sin^2\theta$.

Transposing, $5\sin^2\theta - 8\sin\theta + 3 = 0$, which is a quadratic equation in $\sin\theta$.

Solving for $\sin\theta$ by the formula,

$$\sin\theta = \frac{8 \pm \sqrt{64 - 60}}{10} = 1 \text{ or } \frac{3}{5}.$$

Then $\theta = \sin^{-1} 1 = 90°$, and $\theta = \sin^{-1}\frac{3}{5} = 36°\ 52.2'$ or $143°\ 7.8'$.

By substituting these values in the original equation, it will be found that only 90° and 36° 52.2′ satisfy that equation.

$$\therefore \theta = 90° \text{ and } 36°\ 52.2'.$$

Example 4.—Given $\tan\theta\sec\theta = -\sqrt{2}$; solve for $\theta < 2\pi$

Solution.—Substituting $\sec\theta = \sqrt{1 + \tan^2\theta}$,

$$\tan\theta\sqrt{1 + \tan^2\theta} = -\sqrt{2}.$$

Squaring, $\tan^2\theta\,(1 + \tan^2\theta) = 2$.

$$\tan^4\theta + \tan^2\theta - 2 = 0, \text{ a quadratic equation in } \tan^2\theta.$$

Solving,
$$\tan^2\theta = 1 \text{ or } -2$$
$$\tan\theta = \pm 1 \text{ or } \pm\sqrt{-2}.$$
$$\therefore \theta = \tan^{-1}(\pm 1) = \tfrac{1}{4}\pi,\ \tfrac{3}{4}\pi,\ \tfrac{5}{4}\pi,\ \tfrac{7}{4}\pi.$$

Since $\sqrt{-2}$ is imaginary, no such angle as $\tan^{-1}(\pm\sqrt{-2})$ exists.

From the original equation the product of $\tan\theta$ and $\sec\theta$ is negative. Therefore, these functions must be opposite in sign, and the angle θ must be in the third or fourth quadrant. It is necessary, then, to reject $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$.

$$\therefore \theta = \tfrac{5}{4}\pi \text{ and } \tfrac{7}{4}\pi.$$

Example 5.—Given $\tan\theta + \cot\theta = 2$; solve for $\theta < 2\pi$.

Solution.—Expressing in terms of $\cot\theta$,

$$\frac{1}{\cot\theta} + \cot\theta = 2.$$

Solving for $\cot\theta$, $\cot\theta = 1$.

$$\therefore \theta = \cot^{-1} 1 = \tfrac{1}{4}\pi \text{ or } \tfrac{5}{4}\pi.$$

Example 6.—Given $\tan 2\theta = \sqrt{3}$, solve for $\theta < 360°$.

Solution.—$\tan 2\theta = \sqrt{3}$.

Then
$$2\theta = 60°,\ 240°,\ 420°,\ 600°.$$
$$\therefore \theta = 30°,\ 120°,\ 210°,\ 300°.$$

Notice that, in order to find all values of $\theta < 360°$, we take all values of $2\theta < 720°$.

EXERCISES

Give orally the values of the following angles less than 360°:

1. $\sin\theta = \frac{1}{2}\sqrt{2}$.
2. $\cos\theta = \frac{1}{2}\sqrt{2}$.
3. $\cos\theta = -\frac{1}{2}$.
4. $\sin\theta = -1$.
5. $\sin\theta = -\frac{1}{2}$.
6. $\cos\theta = -\frac{1}{\sqrt{2}}$.
7. $\cos\theta = 0$.
8. $\tan\theta = -1$.
9. $\sin\theta = \frac{1}{2}\sqrt{3}$.
10. $\sin\theta = -\frac{1}{2}\sqrt{3}$.
11. $\theta = \cos^{-1}(-\frac{1}{2}\sqrt{3})$.
12. $\theta = \cot^{-1}(\frac{1}{3}\sqrt{3})$.
13. $\theta = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.
14. $\theta = \sin^{-1}\left(-\frac{3}{2}\frac{1}{\sqrt{3}}\right)$.
15. $\theta = \cos^{-1}(-\frac{1}{2}\sqrt{2})$.

Give orally the general measures of the following angles:

16. $\sin\theta = -1$.
17. $\cos\theta = 1$.
18. $\cos\theta = -\frac{1}{\sqrt{2}}$.
19. $\theta = \tan^{-1}\sqrt{3}$.
20. $\theta = \tan^{-1}\frac{\sqrt{3}}{3}$.
21. $\theta = \sin^{-1}(-\frac{1}{2}\sqrt{3})$.

Solve the following for values of $\theta < 360°$:

22. $\tan\theta = -0.69321$. *Ans.* 145° 16′ 11″, 325° 16′ 11″.
23. $\cos\theta = -0.27689$. *Ans.* 106° 4′ 28″, 253° 55′ 32″.
24. $\cos\theta = \pm 0.89613$. *Ans.* 26° 20′ 46″, 153° 39′ 14″, 206° 20′ 46″, 333° 39′ 14″.
25. $\sin\theta = \pm 0.80001$. *Ans.* 53° 7′ 53″, 126° 52′ 7″, 233° 7′ 53″, 306° 52′ 7″.
26. $\cot\theta = 2.1801$. *Ans.* 24° 38′ 26″, 204° 38′ 26″.
27. $\tan\theta = 1.2345$. *Ans.* 50° 59′ 26″, 230° 59′ 26″.
28. $\cos\theta = \pm 0.73218$. *Ans.* 42° 55′ 51″, 137° 4′ 9″, 222° 55′ 51″, 317° 4′ 9″.

29. $\sin\theta = \pm 0.29868.$ *Ans.* $17°\,22'\,42''$, $162°\,37'\,18''$, $197°\,22'\,42''$, $342°\,37'\,18''$.

30. $\cot\theta = 0.81638.$ *Ans.* $50°\,46'\,21''$, $230°\,46'\,21''$.

31. $\sin\frac{1}{2}\theta = \frac{1}{2}.$ *Ans.* $60°$, $300°$.

32. $\cos 2\theta = \frac{1}{2}\sqrt{2}.$ *Ans.* $22°\,30'$, $157°\,30'$, $202°\,30'$, $337°\,30'$.

33. $\tan 3\theta = 1.$ *Ans.* $15°$, $75°$, $135°$, $195°$, $255°$, $315°$.

34. $\sec 2\theta = \pm 2.$ *Ans.* $30°$, $60°$, $120°$, $150°$, $210°$, $240°$, $300°$, $330°$.

35. $\sin 2\theta = 0.65923.$ *Ans.* $20°\,37'\,14''$, $69°\,22'\,46''$. $200°\,37'\,14''$, $249°\,22'\,46''$.

36. $\cos\frac{1}{2}\theta = \pm 0.57916.$ *Ans.* $109°\,13'$, $250°\,47'$.

37. $\tan\frac{1}{2}\theta = 0.51804.$ *Ans.* $54°\,46'\,20''$.

38. $\sin\theta = -\cos\theta.$ *Ans.* $135°$, $315°$.

39. $\tan\theta = \cot\theta.$ *Ans.* $45°$, $135°$, $225°$, $315°$.

40. $2\sin^2\theta - 3\sin\theta = 2.$ *Ans.* $210°$, $330°$.

41. $4\cos^2\theta + 2\sqrt{2}\cos\theta = 2\cos\theta + \sqrt{2}.$ *Ans.* $60°$, $135°$, $225°$, $300°$.

42. $2\sin^2\theta + 3\sin\theta + 1 = 0.$ *Ans.* $210°$, $270°$, $330°$.

43. $2\cos^2\theta + \sqrt{3}\cos\theta = 3(\sqrt{3} + 2\cos\theta).$ *Ans.* $150°$, $210°$.

44. $\csc^2\theta = 1 + \tan^2\theta.$ *Ans.* $45°$, $135°$, $225°$, $315°$.

45. $\sqrt{3}\cos\theta + \sin\theta = 2.$ *Ans.* $30°$.

46. $2\cos^2\theta + 11\cos\theta = 6.$ *Ans.* $60°$, $300°$.

47. $2\cos^2\theta + \sin\theta = 1.$ *Ans.* $90°$, $210°$, $330°$.

48. $\cos 2\theta(1 - 2\sin\theta) = 0.$ *Ans.* $30°$, $45°$, $135°$, $150°$, $225°$, $315°$.

49. $\cos\theta(3 - 4\sin^2 2\theta) = 0.$ *Ans.* $30°$, $60°$, $90°$, $120°$, $150°$, $210°$, $240°$, $270°$, $300°$, $330°$.

50. $\sqrt{3}\tan\theta + 1 = \sqrt{3} + \cot\theta.$ *Ans.* $45°$, $150°$, $225°$, $330°$.

51. Eliminate θ from the equations $\sin^3\theta = x$, and $\cos^3\theta = y$. *Ans.* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1.$

Suggestion.—Find $\sin^2\theta$ and $\cos^2\theta$ and add.

52. Eliminate θ from the equations

$$a\cos\theta + b\sin\theta = c$$
$$d\cos\theta + e\sin\theta = f.$$

Ans. $(bf - ce)^2 + (cd - af)^2 = (bd - ae)^2.$

Suggestion.—Solve for $\sin\theta$ and $\cos\theta$.

$\sin\theta = \dfrac{cd - af}{bd - ae}$, $\cos\theta = \dfrac{bf - ce}{bd - ae}$; substitute these values in $\sin^2\theta + \cos^2\theta = 1$.

53. Given $r\cos\theta = x$, and $r\sin\theta = y$; solve for r and θ.

Ans. $r = \sqrt{x^2 + y^2}$; $\theta = \tan^{-1}\dfrac{y}{x}.$

54. Given $r\sin\theta\cos\varphi = x$, $r\cos\theta\cos\varphi = y$, $r\sin\varphi = z$; solve for r, θ, and φ.

Ans. $r = \sqrt{x^2 + y^2 + z^2}$; $\varphi = \sin^{-1}\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$; $\theta = \tan^{-1}\dfrac{x}{y}.$

55. Find the value of: $\dfrac{\sin(-60°)}{\cos 150°} + \dfrac{\cos(-60°)}{\sin 150°} + \dfrac{\cot(-60°)}{\tan 150°}.$

Ans. 3.

56. Prove that $\dfrac{\tan 230°}{\tan 130°} \cdot \dfrac{\cot 130°}{\cot 230°} \cdot \dfrac{\sec 310°}{\csc 410°} = \tan 50°.$

57. If $\sin 130° = a$, show that $\dfrac{\sec 230° \sin 320°}{\cot 220°} = \dfrac{\sqrt{1 - a^2}}{a}.$

58. If $\cos(\frac{3}{2}\pi + \theta) = \frac{12}{13}$, show that $\cot(\pi - \theta) = -\frac{5}{12}$ and $\csc(\pi + \theta) = -\frac{13}{12}$.

In the following problems, find all values of θ less than $360°$. Check each angle.

59. $\sin 3\theta = -\dfrac{\sqrt{3}}{2}.$

60. $\cos 3\theta = -\frac{1}{2}.$

61. $\cot 3\theta = -1.$

62. $\sec 4\theta = 2.$

CHAPTER VI

GRAPHICAL REPRESENTATION OF TRIGONOMETRIC FUNCTIONS

56. Line representation of the trigonometric functions.—Construct a circle of radius OH, with its center at the origin of coordinates (Fig. 54). Since, in finding the trigonometric functions of an angle with its vertex at the origin of coordinates and its initial side on the positive part of the axis of abscissas, any

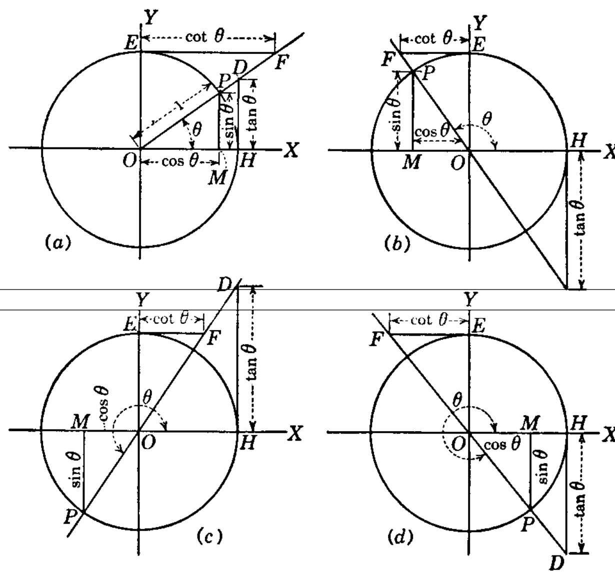


Fig. 54.

point may be chosen in the terminal side of the angle, we may take the point where the terminal side cuts the circumference of the circle. Draw angle θ = angle XOP in each of the four quadrants, and draw $MP \perp OX$ in each case. Now choose OH as the unit of measure, that is, $OH = 1$. Then in each of the four quadrants,

$$\sin\theta = \frac{MP}{OP} = \frac{MP}{OH} = \frac{MP}{1} = MP.$$

$$\cos\theta = \frac{OM}{OP} = \frac{OM}{OH} = \frac{OM}{1} = OM.$$

Stated in words these are as follows:

The sine of an angle θ is represented by the ordinate of the point where the terminal side cuts the circumference of the unit circle.

The cosine of an angle θ is represented by the abscissa of the point where the terminal side cuts the unit circle.

It should be noted that the ordinate gives the value of the sine both in magnitude and in sign. That is, when the point is above the x-axis, the sine is positive, and when below it is negative; likewise, for the cosine with reference to the y-axis. In this way one can visualize the sine and the cosine.

Draw tangents to the circle at H and E (Fig. 54), to meet the terminal side OP extended or produced back through the origin as the position of the angle requires. In each of the four figures, triangles OMP, OHD, and OEF are similar. Assume that HD is positive when measured upward, and negative when measured downward; also that EF is positive when measured to the right, and negative when measured to the left.

From the similar triangles, $\dfrac{MP}{OM} = \dfrac{HD}{OH}$ and $\dfrac{OM}{MP} = \dfrac{EF}{OE}$.

Then in each of the four quadrants,

$$\tan\theta = \frac{MP}{OM} = \frac{HD}{OH} = \frac{HD}{1} = HD.$$
$$\cot\theta = \frac{OM}{MP} = \frac{EF}{OE} = \frac{EF}{1} = EF.$$

Or, in words, these are:

The tangent of an angle θ is represented by the ordinate of the point where the terminal side of θ is cut by a tangent line drawn to the unit circle where the circle cuts the positive part of the axis of abscissas.

The cotangent of an angle θ is represented by the abscissa of the point where the terminal side of θ is cut by a tangent line drawn to the unit circle where the circle cuts the positive part of the axis of ordinates.

Let it be assumed that OD and OF are positive when measured on the terminal side OP of the angle, and that they are negative when measured on OP produced back through the origin. Then in each of the four quadrants,

$$\sec\theta = \frac{OP}{OM} = \frac{OD}{OH} = \frac{OD}{1} = OD.$$

$$\csc\theta = \frac{OP}{MP} = \frac{OF}{OE} = \frac{OF}{1} = OF.$$

Or, in words, these are:

The secant of an angle θ is represented by the segment of the terminal side of θ from the origin to the point where the line representing the tangent of θ cuts the terminal side.

The cosecant of an angle θ is represented by the segment of the terminal side of θ from the origin to the point where the line representing the cotangent of θ cuts the terminal side.

It is not to be understood that the functions are lines; but that, where the radius is taken as the unit of measure, and the lines are expressed in terms of this unit, the numbers which then represent the lines are the functions. Thus, if MP (Fig. 54) is 4 in. and the radius is 7 in., MP in terms of OH is $\frac{4}{7}$, which is then the sine of θ.

Historically, the line definitions of the trigonometric functions were given before the ratio definitions. This graphical way of representing the functions assists in clarifying many questions arising in connection with the functions. For instance, it makes apparent the origin of the terms tangent and secant of an angle. This manner of defining the functions gave rise to the term **circular functions** by which they are often called.

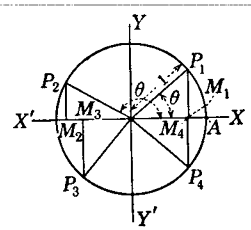


Fig. 55.

EXERCISES

Draw the following angles and represent their trigonometric functions as lines:

1. 30°.	**3.** 245°.	**5.** 90°.
2. 160°.	**4.** 330°.	**6.** 180°.

57. Changes in the value of the sine and cosine as the angle increases from 0 to 360°.—Draw a circle with unit radius (Fig. 55) and construct an angle θ in each of the four quadrants. Since in a unit circle the sine of an angle θ is represented by the ordinate of the point where the terminal side of the angle intersects the circle, the variation in the ordinate will represent the variation in the $\sin\theta$. At 0° the ordinate is 0. As the angle increases from 0 to 90°, the ordinate increases from 0 to 1. As θ increases from 90 to 180°, the ordinate decreases from 1 to 0. From 180 to 270°, the ordinate becomes negative and decreases from

0 to −1. From 270 to 360°, the ordinate increases from −1 to 0. Therefore as the angle varies from 0 to 360°, the sine varies from 0 at 0° to 1 at 90°, to 0 at 180°, to −1 at 270°, and back to 0 at 360°.

The cosine, being represented by the abscissa of the point where the terminal side of the angle intersects the unit circle, will then decrease from 1 to 0 as the angle increases from 0 to 90°. From 90 to 180°, the cosine is negative and decreases from 0 to −1. From 180 to 360°, the cosine increases from −1 through 0 at 270° to 1 at 360°.

EXERCISES

Discuss orally the changes in the following functions as θ varies from 0 to 360°:

1. $\sin 2\theta$.	**7.** $\cos 3\theta$.	**13.** $\cot 2\theta$.
2. $\sin 3\theta$.	**8.** $\cos \frac{1}{2}\theta$.	**14.** $\sec\theta$.
3. $\sin 4\theta$.	**9.** $\cos(-\theta)$.	**15.** $\sin(\theta + 30°)$.
4. $\sin \frac{1}{2}\theta$.	**10.** $\tan\theta$.	**16.** $\cos(\theta + 45°)$.
5. $2\sin\theta$.	**11.** $\tan 2\theta$.	**17.** $\sin(\theta - 45°)$.
6. $\cos\theta$.	**12.** $2\tan\theta$.	**18.** $\sin(\theta + \alpha)$.

19. Trace the changes in $\sin^2\alpha$ as α varies from 0 to 2π.

20. Trace the changes in $\sin\alpha + \cos\alpha$. What is the maximum value? The minimum value? Find the values of α for these values of $\sin\alpha + \cos\alpha$. For what values of α is $\sin\alpha + \cos\alpha = 0$?

TRIGONOMETRIC CURVES

58. Graph of $y = \sin\theta$.—The changes which take place in $\sin\theta$, as indicated in the preceding article, are best shown by a

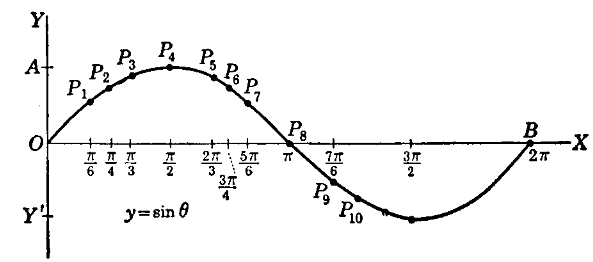


Fig. 56.

graph. Referring again to Fig. 55 (**Art. 57**), let OA be the unit of measure. Then the complete circumference is the measure of 360°, that is, 360° may be represented by a line 2π units in length. Lay off $OB = 6.2832$ on OX (Fig. 56). OB is then the radian measure of 2π, or $OB = 2\pi$. Then lay off the proportional parts as indicated in the figure, using multiples of $\frac{1}{6}\pi$ and $\frac{1}{4}\pi$ only. (Other angles could be used as well as, or in addition to these, making the curve more nearly accurate; but for our purpose the easy proportional parts of 2π are used.) Lay off OA on the y-axis. This will represent the unit for plotting the sines of the angles.

Select various values of θ from 0 to 2π, determine the corresponding values of y, and plot the points of which these values are the coordinates.

Values of θ:	0	$\frac{1}{6}\pi$	$\frac{1}{4}\pi$	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	$\frac{3}{4}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{5}{4}\pi$	etc.
Values of y:	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$	etc.
Points:	O	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	etc.

Draw a curve through these points. *The curve is the graph of* $y = \sin\theta$. It shows the change in $\sin\theta$ as the angle changes from 0 to 2π.

It is evident that the curve will repeat its form if θ were given values from 2π to 4π, from 4π to 6π, etc., or from 0 to -2π, etc. The curve is then periodic.

Here the angle and the function are both plotted to the same unit or scale, that is, the unit on the y-axis is the same length as that to represent 1 radian on the x-axis. The curve so plotted may be called the **proper sine curve.** Often, however, for convenience when plotting on coordinate paper, the angles are plotted according to the divisions on the paper. For example, 1 space = 6° or 10°, or some other convenient angle, depending on the size of the plot.

59. Periodic functions and periodic curves.—In nature, there are many motions that are recurrent. Sound waves, light waves, and water waves are familiar examples. Motions in machines are repeated in a periodic manner. The vibration of a pendulum is a simple case, as is also the piston-rod motion in an engine. Other familiar illustrations are the vibration of a piano string, breathing movements, heart beats, and the motion of tides. An alternating electric current has periodic changes. It increases to a maximum value in one direction, decreases to zero, and on down to a minimum, that is, to a maximum value in the opposite direction, rises again, and repeats these changes. It is thus an *alternating* current passing from a maximum in one direction to a maximum in the other direction, say, 60 times a second.

Before physical quantities that change in a periodic fashion can be dealt with mathematically, it is necessary to find a mathematical statement for such a periodic change.

Definitions.—A curve that repeats in form as illustrated by the sine curve is called a **periodic curve.** The function that gives rise to a periodic curve is called a **periodic function.** The least repeating part of a periodic curve is called a **cycle** of the curve. The change in the value of the variable necessary for a cycle is called the **period** of the function. The greatest absolute value of the ordinates of a periodic function is called the **amplitude** of the function.

Example 1.—Find the period of $\sin n\theta$, and plot $y = \sin 2\theta$.

Since, in finding the value of $\sin n\theta$, the angle θ is multiplied by n before finding the sine, the period is $\frac{2\pi}{n}$.

The curve for $y = \sin 2\theta$ is shown in Fig. 57. The period of the function is π radians, and there are two cycles of the curve in 2π radians.

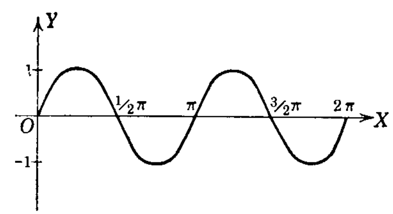


FIG. 57.

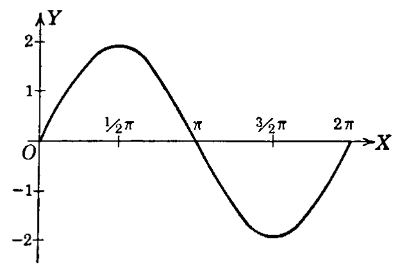


FIG. 58.

The number n in $\sin n\theta$ is called the **periodicity factor.**

Example 2.—Find the amplitude of $b \sin\theta$, and plot $y = 2\sin\theta$.

Since, in finding the value of $b\sin\theta$, $\sin\theta$ is found and then multiplied by b, the amplitude of the function is b, for the greatest value of $\sin\theta$ is 1.

The curve for $y = 2\sin\theta$ is shown in Fig. 58. The amplitude is 2.

The number b in $b\sin\theta$ is sometimes called the **amplitude factor.**

By a proper choice of a periodicity factor and an amplitude factor, a function of any amplitude and any period desired can be found.

While the sine function is perhaps the most frequently used of the periodic functions, the cosine function can be used quite as readily. By a proper choice and combination of sines and cosines a function can be built up that will represent exactly or approximately any periodic phenomenon. Just how this may be done can hardly be explained here.

60. Mechanical construction of graph of sin θ.—On one of the heavy horizontal lines of a sheet of coordinate paper, choose an origin and lay off angles every 15° from 0 to 360°, using each small space to represent 15°, as in Fig. 59. With any convenient point on this horizontal axis as a center, describe a circle with a radius equal to 30 spaces. Choose the initial side of all the angles on the axis of the angles.

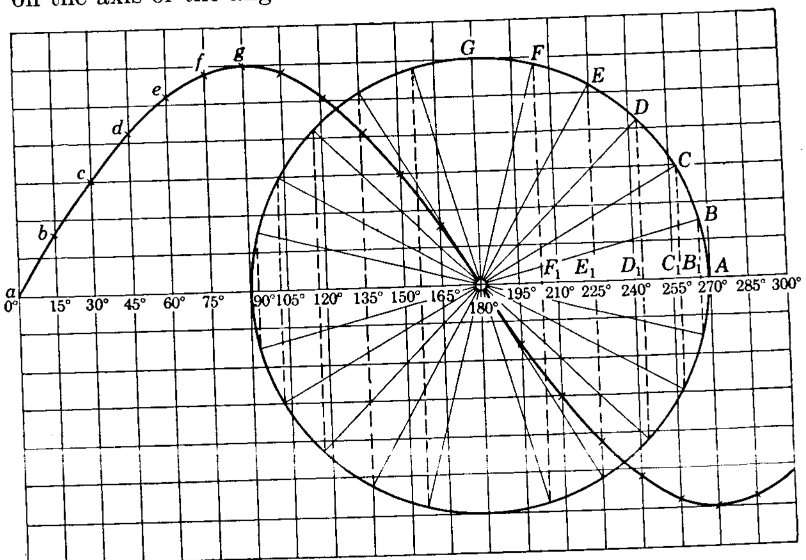


FIG. 59.

By means of the protractor lay off the central angles every 15° from 0 to 360°, such as $\angle AOB$, $\angle AOC$, etc. Let the radius of the circle be the unit of measure. Then the sines of the angles are the ordinates of the points A, B, C, etc. Through B draw a horizontal line to intersect the vertical line through 15° as plotted on the horizontal axis. The point b, thus determined, has as coordinates (15°, sin 15°). In the same way locate c (30°, sin 30°); d (45°, sin 45°); e (60°, sin 60°); etc. Through these points draw a curve.

With the sine curve thus constructed, one can determine the value of the sine of an angle approximately by measurement. For example, find the sin 51°. By measurement, the ordinate for sin 51° is 23.3 spaces. Since the unit is 30 spaces,

$$\sin 51° = \frac{23.3}{30} = 0.7766.$$

From the table of natural functions sin 51° = 0.77715. A comparison of the results with **Table IV** for a number of angles will give an idea of the accuracy of the graph.

Exercise.—Measure the ordinates for the angles given in the following table, compute the sines, and tabulate the results. Find the sines of the same angles from the **Tables** and tabulate. Compare the results.

θ	Sine from curve	Sine from table	Difference
18°			
57°			
78°			
99°			
123°			
138°			
171°			

EXERCISES

1. Plot $y = \sin \theta$, first, using as a unit on the x-axis a length twice as great as that on the y-axis; second, using as a unit on the x-axis a length one-half as great as that on the y-axis. Plot both curves on the same set of axes.

2. Plot $y = \cos \theta$. Give its period and amplitude.

3. Plot $y = \tan \theta$ and $y = \cot \theta$ on the same set of axes.

4. Plot $y = \sec \theta$ and $y = \csc \theta$ on the same set of axes.

5. Plot $y = \sin x + \cos x$.

Suggestion.—Plot $y_1 = \sin x$ and $y_2 = \cos x$ on the same set of axes. Then find the points on the curve $y = \sin x + \cos x$ from the relation $y = y_1 + y_2$, by adding the ordinates for various values of x.

6. Plot $y = \sin^2 x$ and $y = \cos^2 x$ on the same set of axes. Note that the curves never extend below the x-axis.

7. Plot $y = \frac{1}{2} \sin x$, $y = \sin x$, $y = 2 \sin x$, and $y = \frac{3}{2} \sin x$ on the same set of axes. Give the period and the amplitude of each.

8. Plot $y = \sin \frac{1}{2}x$, $y = \sin x$, $y = \sin 2x$, and $y = \sin \frac{3}{2}x$ on the same set of axes. Give the period and the amplitude of each.

61. Projection of a point having uniform circular motion. *Example* 1.—A point P (Fig. 60) moves around a vertical circle of radius 3 in. in a counterclockwise direction. It starts with the point at A and moves with an angular velocity of 1 revolution in 10 sec. Plot a curve showing the distance the projection of P on the vertical diameter is from O at any time t, and find its equation.

Plotting.—Let OP be any position of the radius drawn to the moving point. OP starts from the position OA and at the end of 1 sec. is in position OP_1, having turned through an angle of $36° = 0.6283$ radian. At the end of 2 sec. it has turned to OP_2, through an angle of $72° = 1.2566$ radians, and so on to positions $OP_3, OP_4, \cdots, OP_{10}$.

The points $N_1, N_2, \cdots$ are the projections of $P_1, P_2, \cdots$, respectively, on the vertical diameter.

Produce the horizontal diameter OA through A, and lay off the seconds on this to some scale, taking the origin at A.

For each second plot a point whose ordinate is the corresponding distance of N from O. These points determine a curve of

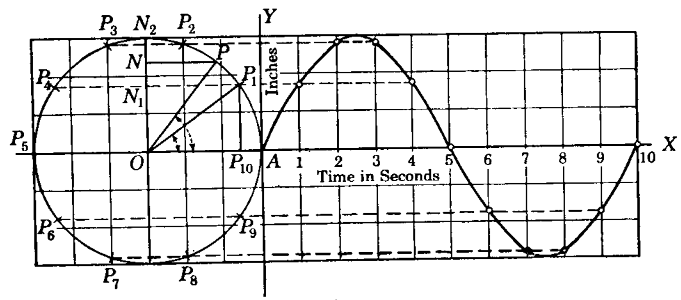


FIG. 60.

which any ordinate y is the distance from the center O of the projection of P upon the vertical diameter at the time t represented by the abscissa of the point.

It is evident that for the second and each successive revolution, the curve repeats, that is, it is a periodic curve.

Since the radius OP turns through 0.6283 radian per second,

$$\text{angle } AOP = 0.6283t \text{ radian,}$$

and

$$ON = OP \cdot \sin 0.6283t,$$

or

$$y = 3 \sin 0.6283t \text{ is the equation of the curve.}$$

In general, then, it is readily seen that, if a straight line of length r starts in a horizontal position when time $t = 0$, and revolves in a vertical plane around one end at a uniform angular velocity ω per unit of time, the projection y of the moving end upon a vertical straight line has a motion represented by the equation

$$y = r \sin \omega t.$$

Similarly, the projection of the moving point upon the horizontal is given by the ordinates of the curve whose equation is

$$y = r \cos \omega t.$$

If the time is counted from some other instant than that from which the above is counted, then the motion is represented by

$$y = r \sin (\omega t + \alpha),$$

where α is the angle that OP makes with the line OA at the instant from which t is counted. As an illustration of this consider the following:

Example 2.—A crank OP (Fig. 61) of length 2 ft. starts from a position making an angle $\alpha = 40° = \frac{2}{9}\pi$ radians with the horizontal line OA when $t = 0$. It rotates in the positive direction at the rate of 2 revolutions per second. Plot the curve

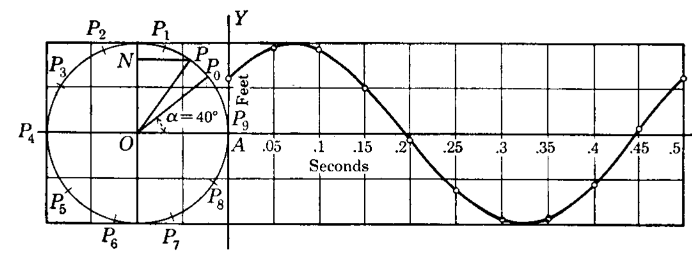


FIG. 61.

showing the projection of P upon a vertical diameter, and write the equation.

Plotting.—The axes are chosen as before, and points are found for each 0.05 sec. The curve is as shown in Fig. 61.

The equation is $y = 2 \sin (4\pi t + \frac{2}{9}\pi)$.

Definitions.—The number of cycles of a periodic curve in a unit of time is called the **frequency.**

It is evident that

$$f = \frac{1}{T},$$

where f is the frequency and T is the period.

In $y = r \sin (\omega t + \alpha)$, $f = \dfrac{\omega}{2\pi}$ and $T = \dfrac{2\pi}{\omega}$.

The angle α is called the **angle of lag.**

62. Summary.—In summary it may be noted again that the equation

$$y = a \sin (nx + \alpha)$$

gives a periodic curve. In this equation there are three arbitrary constants, a, n, and α. A change in any one of these constants will change the curve.

(1) If a is changed, the *amplitude* of the curve is changed.

(2) If n is changed, the *period* of the curve is changed.

(3) If α is changed, the curve is moved without change in shape from left to right or *vice versa*.

63. Simple harmonic motion.—If a point moves at a uniform rate around a circle and the point be projected on a straight line in the plane of the circle, the oscillating motion, that is, the back-and-forth motion, of the projected point is called **simple harmonic motion.** The name is abbreviated s.h.m. In **Art. 61,** the point N of Fig. 61 is the projection of the point P. As P moves around the circle the point N moves back-and-forth along the vertical diameter and performs a simple harmonic motion. It is readily seen that the point N moves more slowly near the ends of the diameter and more rapidly near the center. It thus changes its velocity or is accelerated.

It can be shown that many motions that one wishes to deal with are simple harmonic. Such is the motion of a swinging weight suspended by a string, a pendulum, a vibrating tuning fork, the particles of water in a wave, a coiled wire spring supporting a weight when the weight is pulled downward and released. Also many motions which are not simple harmonic may be treated as resulting from several such motions. Such motions occur in alternating electric currents, in sound waves, and in light waves.

EXERCISES

1. A crank 12 in. long starts from a horizontal position and rotates in the positive direction in a vertical plane at the rate of 2π radians per second. The projection of the moving end of the crank upon a vertical line oscillates with a simple harmonic motion. Construct a curve that represents this motion, and write its equation. *Ans.* $y = 12 \sin 2\pi t$.

2. A crank 6 in. long starts from a position making an angle of 55° with the horizontal, and rotates in a vertical plane in the positive direction at the rate of 1 revolution in 5 sec. Construct a curve showing the projection of the moving end of the crank on a vertical line. Write the equation of the curve and give the period and the frequency.

Ans. $y = 6 \sin (t \cdot 72° + 55)$; 5; $\frac{1}{5}$.

3. Plot the curves that represent the following motions:

$$(a)\ y = 10 \sin (4t + 0.6);$$
$$(b)\ y = 4 \sin (\tfrac{1}{8}\pi t + \tfrac{1}{12}\pi).$$

Give the period and frequency of each. *Ans.* (*a*) 1.571, 0.637; (*b*) 16, $\frac{1}{16}$.

4. Plot $y = r \sin \pi t$ and $y = r \sin (\pi t + \frac{1}{4}\pi)$ on the same set of axes. Notice that the highest points on each are separated by the constant angle $\frac{1}{4}\pi$. Such curves are said to be out of phase. The difference in phase is stated in time or as an angle. In the latter case it is called the phase angle.

5. Plot $y = r \sin \frac{1}{4}\pi t$, $y = r \sin (\frac{1}{4}\pi t - \frac{1}{4}\pi)$, and $y = r \cos \frac{1}{4}\pi t$ all on the same set of axes. What is the difference in phase between these?

6. What is the difference in phase between the curves of $y = \sin x$ and $y = \cos x$? Between $y = \cos x$ and $y = \sin (x + \frac{1}{2}\pi)$.

PRINCIPAL VALUES OF INVERSE FUNCTIONS

64. Inverse functions.—We have seen that $\sin^{-1} t$ means the angle whose sine is t. In **Art. 57,** it was shown that the sine function varied from -1 to $+1$. Then the equation $\theta = \sin^{-1} t$ has real solutions when and only when t is not less than -1 or greater than $+1$. In the same way it can be shown that $\theta = \cos^{-1} t$ has a solution when and only when t is not less than -1, or greater than $+1$.

Since $\tan \theta$ and $\cot \theta$ can have any value from $-\infty$ to $+\infty$, the equations $\theta = \tan^{-1} t$ and $\theta = \cot^{-1} t$ have solutions for all values of t.

The two expressions $\sin \theta = t$ and $\theta = \sin^{-1} t$ both express the same thing, namely, that θ is an angle whose sine is equal to t. In the first expression t is a function of θ and in the second θ is a function of t.

In $\sin \theta = t$, there is but one value of t for every value of θ. $\sin \theta$ is then said to be a **single valued function** of θ.

In $\theta = \sin^{-1} t$ for every value of t, there are an indefinite number of values of θ, as was seen in **Art. 53.** $\sin^{-1} t$ is then said to be a **multiple valued function** of t.

65. Graph of y = sin⁻¹ x, or y = arc sin x.—Stating $y = \sin^{-1} x$ in the form $\sin y = x$, it is readily seen by comparison with $y = \sin x$, that $\sin y = x$ is obtained from $y = \sin x$ by interchanging x and y. Then the graph of $y = \sin^{-1} x$ is obtained by plotting the sine curve on the y-axis instead of the x-axis as in **Art. 58.** The curve is shown in Fig. 62.

In many mathematical operations where $\sin^{-1} x$ enters, it is often desirable and, indeed, necessary to consider a portion of the curve (Fig. 62) for which there will be but one value of y for every value of x.

A glance at the figure will show that for the portion AOC of the curve the function is single-valued. That is, for every value of x between and including -1 and $+1$, y takes values between and including $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

Definition.—The values of $\sin^{-1} x$ between and including $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ for each value of x are called the **principal values** of $\sin^{-1} x$.

To represent the principal value of the function, the s is often written a capital, thus, $\text{Sin}^{-1} x$. The other functions are denoted in a similar manner.

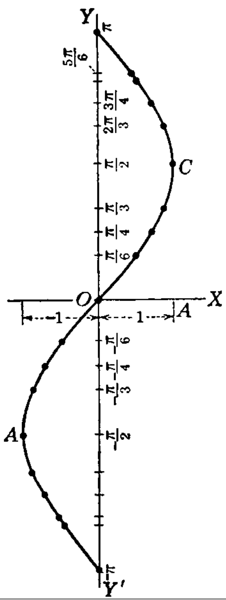


FIG. 62

The notation $\text{Sin}^{-1} x$ denotes the principal values of $\sin^{-1} x$. The values are from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$.

The notation $\text{Cos}^{-1} x$ denotes the principal values of $\cos^{-1} x$. The values are from 0 to π inclusive.

The notation $\text{Tan}^{-1} x$ denotes the principal values of $\tan^{-1} x$. The values are from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$.

The notation Cot^{-1} denotes the principal values of $\cot^{-1} x$. The values are from 0 to π.

Example.—Evaluate $\left(\frac{1}{5}x\sqrt{15^2 - 9x^2} + 15 \sin^{-1} \frac{1}{5}x\right)\Big]_2^5$.

Solution.—The notation is explained in the solution to Example 4, page 69.

Substituting $x = 5$,

$$\tfrac{1}{5} \cdot 5\sqrt{225 - 9 \cdot 5^2} + 15 \sin^{-1} 1 = \tfrac{15}{2}\pi = 23.5620.$$

Substituting $x = 2$,

$$\tfrac{1}{5} \cdot 2\sqrt{225 - 9 \cdot 2^2} + 15 \sin^{-1} \tfrac{2}{5} = 5.4991 + 6.1728 = 11.6719.$$

$$23.5620 - 11.6719 = 11.890. \qquad \textit{Ans.}$$

EXERCISES

Give the results of the following orally:

1. $\text{Cos}^{-1} (-1)$.
2. $\text{Sin}^{-1} (-1)$.
3. $\text{Sin}^{-1} \frac{1}{2}$.
4. $\text{Cos}^{-1} \frac{1}{2}\sqrt{3}$.
5. $\text{Cos}^{-1} (-\frac{1}{2}\sqrt{3})$.
6. $\text{Sin}^{-1} (-\frac{1}{2})$.
7. Arc cot $\frac{1}{3}\sqrt{3}$.
8. Arc tan $\sqrt{3}$.
9. Arc cos $(-\frac{1}{2}\sqrt{2})$.
10. Arc sin $\frac{1}{2}\sqrt{3}$.
11. tan $\text{Cot}^{-1} \sqrt{3}$.
12. cos $\text{Sin}^{-1} (-\frac{1}{2})$.
13. sin $\text{Cos}^{-1} \frac{1}{2}$.
14. sin $\text{Cos}^{-1} (-\frac{1}{2})$.
15. cos $\text{Sin}^{-1} \frac{1}{2}$.

In the following find the numerical values of the given expressions, using the principal values of the angles. In many applications of anti-functions, as in the calculus, they enter into the expressions for areas, volumes, etc., and the angles must be expressed in radians.

16. $\frac{3}{5}[\sin^{-1} 1 - \tan^{-1} (-1)]$. *Ans.* $\frac{9\pi}{20}$.
17. $16[\sin^{-1} (-0.2) - \sin^{-1} (-0.4)]$. *Ans.* 3.363.
18. $\tan^{-1} \frac{1}{2} + \tan^{-1} (-1)$. *Ans.* -0.3217.
19. $\sin^{-1} \left(-\frac{1}{2}\right) + \cos^{-1} \frac{\sqrt{3}}{2}$. *Ans.* 0.
20. $\cos^{-1} 0 - \tan^{-1} (-\sqrt{3})$. *Ans.* $\frac{5\pi}{6}$.
21. $\cos^{-1} \left(-\frac{\sqrt{3}}{2}\right) + \sin^{-1} (-1)$. *Ans.* $\frac{\pi}{3}$.
22. $\cot^{-1} (-\sqrt{3}) + \sin^{-1} \left(-\frac{\sqrt{2}}{2}\right)$. *Ans.* 1.833.
23. $\sin^{-1} \left(-\frac{\sqrt{3}}{2}\right) - \cos^{-1} (-1.)$ *Ans.* $-\frac{4\pi}{3}$.
24. $8[\cos^{-1} (0.2) - \cos^{-1} (0.4)]$. *Ans.* 1.681.
25. $\tan^{-1} \left(-\frac{1}{\sqrt{3}}\right) - \tan^{-1} (-\sqrt{3})$ *Ans.* $\frac{\pi}{6}$.
26. Plot $y = \cos^{-1} x$ and show that for values of y from 0 to π inclusive the values of x range from $+1$ to -1, inclusive.
27. Evaluate $\sin^{-1} \frac{x}{2}\Big]_{-2}^{2}$ *Ans.* π.
28. Evaluate $\frac{1}{2}\left(x\sqrt{9 - x^2} + 9 \sin^{-1} \frac{x}{3}\right)\Big]_0^3$ *Ans.* $\frac{9\pi}{4}$.
29. Evaluate $4\pi b\left[\frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a}\right)\right]_{-a}^{a}$ *Ans.* $2\pi^2a^2b$.
30. Evaluate $10\left(x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a}\right)\Big]_0^a$. *Ans.* $5a^2\pi$.
31. $\frac{1}{2}\left[(x^2 + 1) \tan^{-1} x - x\right]_{-1}^{1}$. *Ans.* 0.5707.
32. $(x - 1) \text{vers}^{-1} x + \sqrt{2x - x^2}\Big]_0^1$. *Ans.* 1.

CHAPTER VII

PRACTICAL APPLICATIONS AND RELATED PROBLEMS

66. Accuracy.—It is of very great importance that one should bear in mind as far as possible the limitations as regards accuracy. The degree of accuracy that can be depended upon in a computation is limited by the accuracy of the tables of trigonometric functions and logarithms used, and by the data involved in the computation.

The greater the number of decimal places in the table, the more accurately, in general, can the angles be determined from the natural or logarithmic functions; but, in a given table, the accuracy is greater the more rapidly the function is changing. Since the cosine of the angle changes slowly when the angle is near 0°, small angles should not be determined from the cosine. For a like reason, the sines should not be used when the angle is near 90°. The tangent and the cotangent change more rapidly throughout the quadrant and so can be used for any angle.

Most of the data used in problems are obtained from measurements made with instruments devised to determine those data more or less accurately. The inability to be precise in the data depends not only upon the instruments used, but upon the person making the measurements and upon the thing measured.

A man in practical work uses instruments which are of such accuracy as to secure results suitable for his purpose. The data given in problems for practice are supposed to be of such accuracy as the instruments that are used in such measurements would warrant.

In the solution of a problem it is useless to carry out the computations with a greater degree of accuracy than that of the data. That is, if the data are accurate only to, say, four significant figures, there is no necessity to compute accurately to more figures than this. If the measuring instrument can be read only to minutes of angle, in the computation there is no object in carrying the work to seconds of angle.

In general, the following is the agreement between the measurement of distances and the related angles:

(1) Distances to two significant figures, angles to the nearest 0.5°.

(2) Distances to three significant figures, angles to the nearest 5′.

(3) Distances to four significant figures, angles to the nearest 1′.

(4) Distances to five significant figures, angles to the nearest 0.1′.

In drill problems, the angles are often expressed as if accurate to seconds when the distances are expressed in five figures. This gives variety in interpolating, but one should not be misled by the implied accuracy. The United States Coast and Geodetic Survey sets the following standards for its finest surveys: A line 1 mile long may turn to the one side or the other not more than $\frac{1}{8}$ in. The average closing error in leveling work must be less than 1 in. in 100 miles. The first gives a variation in the angle of 0.4″ to each side, or a total of 0.8″. In making such accurate computations, a 10-place table is used.

67. Tests of accuracy.—The practical man endeavors in one way or another to check both his measurements and his computations. In our work here we are interested in checks on the computation.

(1) Often a graphical construction to scale will give results that will check the numerical work. If the construction is made free-hand, only the gross mistakes in computation will be discovered; but if the construction is made carefully with accurate instruments, results may be obtained as accurate as the data will warrant. This, then, may be considered a graphical solution of the problem.

(2) Mistakes in the computations may be found by making another computation using a different set of data; or by recomputing, using the same data but using a different set of formulas. Many ways will present themselves to the thoughtful student.

EXERCISES

1. In determining an angle by means of a table of natural functions that is correct to five places, if the angle is near 1° can seconds be determined from the cosine of the angle? Can tenths of minutes? Can minutes?

2. Answer the same questions as in Exercise 1 if the sine is used instead of the cosine. If the tangent is used. If the cotangent.

3. Answer similar questions if the angle is near 89°, 80°, 10°, 20°, 70°, 45°.

4. From the results obtained in the first three exercises, state conclusions as to what sized angles can be determined most accurately from sine, cosine, tangent, and cotangent of the angle.

5. Compare the logarithms of 92.8766 and 92.876; 99.8375, 99.837, and 99.838; 121.575, 121.57, and 121.6.

6. Can a number be determined correct to six figures by using a five-place logarithm table? When? When is it not possible to determine five figures of a number by means of a five-place table of logarithms?

APPLICATION OF RIGHT TRIANGLES TO VECTORS

68. Orthogonal projection.—If from a point P (Fig. 63a), a perpendicular PQ be drawn to any straight line RS, then the

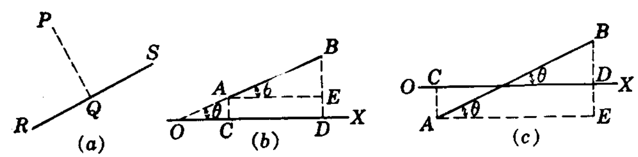


FIG. 63.

foot of the perpendicular Q is said to be the **orthogonal projection,** or simply the **projection,** of P upon RS.

The **projection of a line segment** upon a given straight line is the portion of the given line lying between the projections of the ends of the segment.

In Fig. 63b and c, CD is the projection of AB on OX. In each case $AE = CD$ and $AE = AB \cos \theta$.

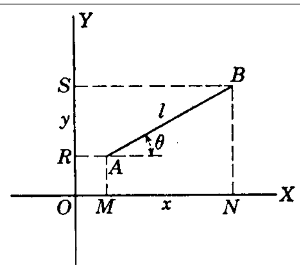


FIG. 64.

The projections usually made are upon a horizontal line OX and a vertical line OY, as in Fig. 64. Hence, if l is the length of the segment of line projected, x the projection on OX, y the projection on OY, and θ the angle of inclination, that is, the angle that the line segment makes with the x-axis, then

[9] $$x = l \cos \theta,$$

and

[10] $$y = l \sin \theta.$$

This may be stated in the following:

THEOREM.—*The projection of any line segment upon a horizontal line equals the length of the segment multiplied by the cosine of the angle of inclination; the projection upon a vertical line*

equals the length of the segment multiplied by the sine of the angle of inclination.

69. Vectors.—In physics and engineering, line segments are often used to represent quantities that have direction as well as magnitude. Velocities, accelerations, and forces are such quantities.

For instance, a force of 100 lb. acting in a northeasterly direction may be represented by a line, say, 10 in. long, drawn in a northeasterly direction. The line is drawn so as to represent the force to some scale; here it is 10 lb. to the inch. An arrow head is put on one end of the line to show its direction.

In Fig. 65, $OP = v$ is a line representing a directed quantity. Such a line is called a **vector.** O is the **beginning** of the vector and P is the **terminal.** $OQ = x$ is the **projection of the vector**

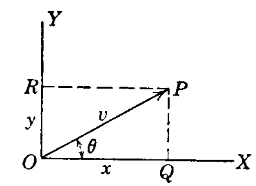


FIG. 65.

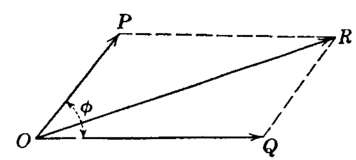


FIG. 66.

on the horizontal OX, $OR = y$ is the **projection on the vertical** OY, and θ is the **inclination** of the vector. The vectors x and y are called **components** of the vector v. As before,

$$x = v \cos \theta,$$

and

$$y = v \sin \theta.$$

Suppose the vectors OQ and OP (Fig. 66), represent the magnitude and direction of two forces acting at the point O, and having any angle ϕ between their lines of action. If the parallelogram $OQRP$ is completed, then the diagonal OR represents in magnitude and direction a force that will produce the same effect as the two given forces.

The vector OR is called the **resultant** of the vectors OQ and OP. The process of finding the resultant of two or more given forces is called **composition of forces.**

Conversely, the vectors OQ and OP are **components** of OR.

Since QR is equal and parallel to OP, it follows that the two components and their resultant form a closed triangle OQR. The relations between forces and their resultant may then be found by solving a triangle which is, in general, an oblique triangle.

Example 1.—Suppose that a weight W is resting on a rough horizontal table as shown in Fig. 67. Suppose that a force of 40 lb. is acting on the weight in the direction OP, making an angle of 20° with the horizontal; then the horizontal pull on the weight is $OQ = 40 \cos 20° = 37.588$ lb., and the vertical lift on the weight is $OR = 40 \sin 20° = 13.68$ lb.

Example 2.—A car is moving up an incline, making an angle of 35° with the horizontal, at the rate of 26 ft. per second. What is its horizontal velocity?

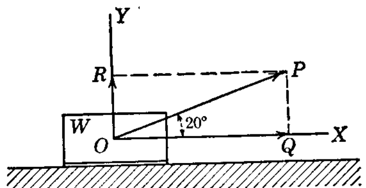


FIG. 67.

Horizontal velocity $= 26 \cos 35° = 21.3$ ft. per second.

Vertical velocity $= 26 \sin 35° = 14.9$ ft. per second.

EXERCISES

1. Find the projection of a line segment 31.2 ft. long upon a straight line making an angle of 34° 16.4′ with the segment. *Ans.* 25.78.

2. The line segment AB, 32.67 in. long makes an angle of 45° 23′ with the line OX. Find the projection on OX. Find the projection on OY perpendicular to OX and in the same plane as OX and AB. *Ans.* 22.95; 23.26.

3. A steamer is moving S. 21° W. at the rate of 28 miles per hour. How fast is it moving in a westerly direction? In a southerly direction? *Ans.* 10.03 miles per hour; 26.14 miles per hour.

4. The direction a force of 1800 lb. is acting, makes an angle of 26° 35′ with the horizontal. Find the horizontal and vertical components of the force. *Ans.* 1610 lb.; 805.5 lb.

5. A ship is sailing at 20.5 miles per hour in a direction N. 24° 35′ E. Find the northerly and easterly components of its speed. *Ans.* 18.64 miles per hour; 8.528 miles per hour.

6. A force of 300 lb. is acting on a body lying on a horizontal plane, in a direction which makes an angle of 20° with the horizontal. What is the force tending to lift the body from the plane? *Ans.* 102.6 lb.

7. A body weighing 58 lb. rests on a horizontal table and is acted upon by a force of 55 lb., acting at an angle of 27° 45′ with the surface of the table. What is the pressure on the table? *Ans.* 32.39 lb.

8. A body weighing 71 lb. rests on a horizontal table and is acted upon by a force of 125 lb. acting at an angle of (−31° 30′) with the surface of the table. What is the pressure on the table? *Ans.* 136.3 lb.

9. The horizontal and vertical components of a force are respectively 234.5 and 654.3 lb. What is the magnitude of the force, and what angle does its line of action make with the horizontal? *Ans.* 695.1 lb.; 70° 16.95′.

10. The horizontal and vertical components of a force are respectively 145.7 and −175.3 lb. What is the magnitude of the force, and what angle does its line of action make with the horizontal? *Ans.* 227.95 lb.; −50° 16.1′.

11. A river runs directly south at 5 miles per hour. A man starts at the west bank and rows directly across at the rate of 4.5 miles per hour. In what direction does his boat move? *Ans.* 41° 59.2′ with bank.

12. A ferryboat at a point on one bank of a river $\frac{3}{4}$ mile wide wishes to reach a point directly across the river. If the river flows 3.75 miles per hour and the ferryboat can steam 8.1 miles per hour, in what direction should the boat be pointed? *Ans.* 27° 34.7′ upstream.

13. Two men are lifting a stone by means of ropes in the same vertical plane. One man pulls 143 lb. in a direction 40° from the vertical and the other 130 lb. in a direction 45° from the other side of the vertical. Determine the weight of the stone. *Ans.* 201.47 lb.

14. Two forces of 245 and 195 lb. act in the same vertical plane upon a heavy body, the first at an angle of 42 with° the horizontal and the second at an angle of 60°. Find the total force tending to move the body horizontally; to lift it vertically. *Ans.* 279.6 or 84.77 lb.; 332.8 lb.

USEFUL AND MORE DIFFICULT PROBLEMS

70. Distance and dip of the horizon. In Fig. 68, let O be the center of the earth, r the radius of the earth, and h the height of a point P above its surface; to find the distance from the point P to the horizon at A.

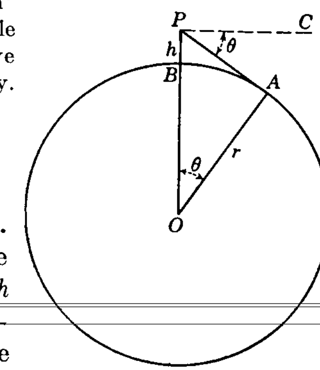


FIG. 68.

By geometry, $\overline{PA}^2 = \overline{PO}^2 - \overline{OA}^2 = (r + h)^2 - r^2 = 2rh + h^2$.

$$\therefore PA = \sqrt{2rh + h^2}.$$

For points above the surface that are reached by man, h^2 is very small compared with $2rh$,

$$\therefore PA = \sqrt{2rh}, \text{ approximately.}$$

In the above, PA, r, and h are in the same units. A very simple formula can be derived, however, if h be taken in feet, r and PA in miles, and $r = 3960$ miles. Then

$$PA = \sqrt{2 \times 3960 \times \frac{h}{5280}} = \sqrt{\frac{3}{2}h} \text{ miles.}$$

The following approximate rules may then be stated:

The distance of the horizon in miles is approximately equal to the square root of $\frac{3}{2}$ times the height of the point of observation in feet.

The height of the point of observation in feet is $\frac{2}{3}$ times the square of the distance of the horizon in miles.

Definition.—The angle $APC = \theta$ in Fig. 68 is called the **dip of the horizon.**

Evidently, $\tan \theta = \dfrac{PA}{r}$.

71. Areas of sector and segment.—Formulas for solving for the areas of the sector and segment of a circle are derived here so that they may be used for reference.

From geometry, the area of the sector of a circle as XOA (Fig. 69) equals the arc XnA times one-half the radius OX.

By **Art. 8,** arc $XnA = OX \times \theta$, where θ is expressed in radians. Hence, using r for the radius and S for the area of sector,

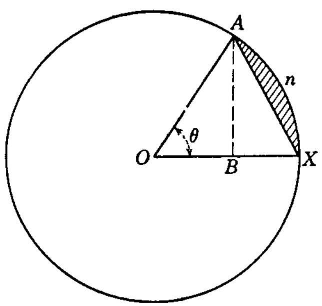


FIG. 69.

$$S = \tfrac{1}{2}r^2\theta.$$

Evidently, the area of the segment $XAn = S -$ area of triangle XOA. But area of triangle $XOA = \frac{1}{2}OX \cdot BA = \frac{1}{2}OX \cdot OA \sin \theta = \frac{1}{2}r^2 \sin \theta$. Hence, using G for area of segment,

$$G = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2 \sin \theta.$$

$$\textbf{[11]} \quad \therefore G = \tfrac{1}{2}r^2(\theta - \sin \theta).$$

As an exercise, the student may later show that this formula holds when θ is an obtuse angle. Also when $\pi < \theta < 2\pi$.

This is the simplest accurate formula for finding the area of a segment of a circle. It is of frequent use in many practical problems. Various approximate formulas for finding the area of a segment are given for the use of practical men not having a knowledge of trigonometry. Two of the best known of these are the following:

$$(1) \qquad A = \tfrac{2}{3}hw + \frac{h^3}{2w},$$

$$(2) \qquad A = \tfrac{4}{3}h^2\sqrt{\frac{2r}{h} - 0.608},$$

where r is the radius, h the height of the segment, and w the length of the chord.

Example 1.—Fine the area of the segment of a circle of radius 16 in., and having a central angle of 78° 30′.

Solution.—By **Table V,** 78° 30′ = 1.3701 radians.

$$\sin 78° 30' = 0.9799.$$

Substituting these values in **[11],**

$$G = \tfrac{1}{2} \times 16^2(1.3701 - 0.9799) = 49.94$$
$$\therefore \text{area of segment} = 49.94 \text{ sq. in.}$$

72. Widening of pavements on curves.—The tendency of a motorist to "cut the corners" is due to his unconscious desire to give the path of his car around a turn the longest possible radius. Many highway engineers recognize this tendency by widening the pavement on the inside of the curve, as shown in Fig. 70. The practice adds much to the attractive appearance of the highway. If the pavement is the same width around the curve as on the tangents, the curved section appears narrower than the normal width; whereas, if the curved section is widened gradually to the midpoint G of the turn, the pavement appears to have a uniform width all the way around.

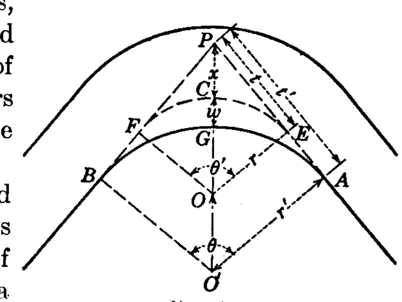


FIG. 70.

In order that the part added may fit the curve properly, it is necessary to have the curve of the inner edge a true arc of a circle, tangent to the edge of the straightaway sections, and therefore it must start before the point E of the curve is reached. The part added may be easily staked out on the ground with transit and tape, by means of data derived from the radius r, the central angle θ of the curve, and the width w. In practice, the width w is taken from 2 to 8 ft. according to the value of r. The area added can be readily computed when values for r, w, and θ are given.

Referring to the figure, derive the following formulas:

$$x = r \sec \tfrac{1}{2}\theta - r = \frac{r}{\cos \frac{1}{2}\theta} - r.$$

$$x + w = r' \sec \tfrac{1}{2}\theta - r' = \frac{r'}{\cos \frac{1}{2}\theta} - r'.$$

$$\therefore r' = \frac{x + w}{\sec \frac{1}{2}\theta - 1} = \frac{(x + w) \cos \frac{1}{2}\theta}{1 - \cos \frac{1}{2}\theta}$$

$$t = r \tan \tfrac{1}{2}\theta.$$

$$t' = r' \tan \tfrac{1}{2}\theta.$$

Area added $= BFCEAG = BPAO' - FPEC - BGAO'$.

$$BPAO' = r't'.$$

$$FPEC = FPEO - FCEO = rt - \frac{\theta}{360}\pi r^2.$$

$$BGAO' = \frac{\theta}{360}\pi r'^2.$$

$$\therefore \text{area added} = r't' - \left(rt - \frac{\theta}{360}\pi r^2\right) - \frac{\theta}{360}\pi r'^2$$

$$= r't' - rt - \frac{\theta}{360}\pi(r' + r)(r' - r).$$

EXERCISES

1. A cliff 2500 ft. high is on the seashore. How far away is the horizon? *Ans.* 61.24 miles.

2. Find the greatest distance at which the lamp of a lighthouse can be seen from the deck of a ship. The lamp is 75 ft. above the surface of the water and the deck of the ship 40 ft. *Ans.* 18.4 miles.

3. Find the radius of one's horizon if located 1000 ft. above the earth. How large when located 2.5 miles above the earth? *Ans.* 38.73 miles; 140.7 miles.

4. How high above the earth must one be to see a point on the surface 35 miles away? *Ans.* 816.7 ft.

5. Two lighthouses, one 100 ft. high and the other 75 ft. are just barely visible from each other over the water. Find how far apart they are. *Ans.* 22.86 miles.

6. In Art. 72, find the area if $r = 300$ ft., $w = 4$ ft., and $\theta = 100°$. *Ans.* 1395 sq. ft.

7. A thin rope is fastened by its ends to two points 25 ft. apart and in a horizontal plane. It has a heavy weight hanging at its midpoint causing it to sag 5 ft., and making the rope from center to ends extend in practically straight lines. Find the angle between one-half of the rope and a horizontal, and find the total length of the rope between the points of support. *Ans.* 21° 48.1′; 26.93 ft.

8. The radius of a circle is 72.52 ft. In this circle a chord subtends an angle of 40° 32.4′ at the center. Find the difference between the length of the chord and the length of its arc. *Ans.* 1.066 ft.

9. Compute the volume for each foot in the depth of a horizontal cylindrical oil tank of length 40 ft. and diameter 4 ft.
Ans. 98.27 cu. ft.; 251.33 cu. ft.; 404.39 cu. ft.; 502.66 cu. ft.

10. A cylindrical tank in a horizontal position is filled with water to within 10 in. of the top. Find the volume of the water if the tank is 10 ft. long and 5 ft. in diameter. *Ans.* 174.84 cu. ft.

11. Find the angle between the diagonal of a cube and one of the diagonals of a face of the cube. *Ans.* 35° 15.8′.

12. If R and r are the radii of two pulleys, D the distance between the centers, and L the length of the belt, show that, when the belt is not crossed

(Fig. 71), the length is given by the following formula where the angle is taken in radians:

$$L = 2\sqrt{D^2 - (R - r)^2} + \pi(R + r) + 2(R - r)\sin^{-1}\frac{R - r}{D}.$$

13. Using the same notation as in Exercise 12, show that, when the belt is crossed (Fig. 72), the length is given by the following formula:

$$L = 2\sqrt{D^2 - (R + r)^2} + (R + r)\left(\pi + 2\sin^{-1}\frac{R + r}{D}\right).$$

Note.—These formulas would seldom be used in practice. An approximate formula would be more convenient, or the length would be measured with a tape line.

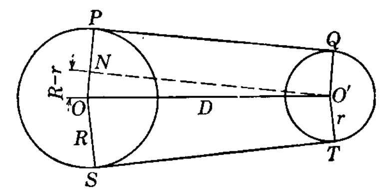


FIG. 71.

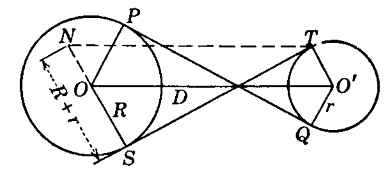


FIG. 72.

A rule often given for finding the length of an uncrossed belt is: Add twice the distance between the centers of the shafts to half the sum of the circumferences of the two pulleys.

14. Using the formula of Exercise 12, and given $R = 16$ in., $r = 8$ in., and $D = 12$ ft., find the length of the belt. Find the length by the approximate rule. *Ans.* 30.32 ft.; 30.28 ft.

15. Use the same values as in Exercise 14, and find by the formula of Exercise 13 the length of the belt when crossed. *Ans.* 30.62 ft.

16. An open belt connects two pulleys of diameters 6 and 2 ft., respectively. If the distance between their centers is 12 ft., find the length of the belt. *ns.* 36.9 ft.

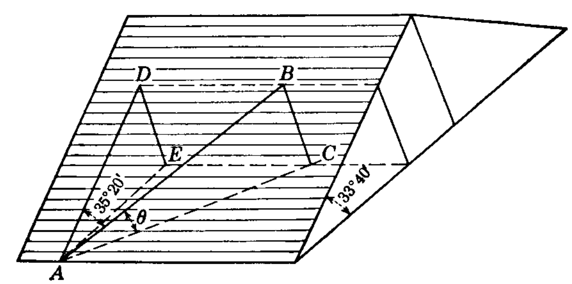


FIG. 73.

17. Two pulleys of diameters 8 and 3 ft., respectively, are connected by a crossed belt. If the centers of pulleys are 14 ft. apart, find the length of the belt. *Ans.* 47.47 ft.

18. The slope of the roof in Fig. 73 is 33° 40′. Find the angle θ which is the inclination to the horizontal of the line AB, drawn in the roof and making an angle of 35° 20′ with the line of greatest slope.

Solution.—$\sin\theta = \dfrac{CB}{AB}$.

$$\cos 35°\,20' = \frac{AD}{AB},\ \text{or}\ AB = \frac{AD}{\cos 35°\,20'}.$$

$$\sin 33°\,40' = \frac{ED}{AD},\ \text{or}\ ED = CB = AD \sin 33°\,40'.$$

Then
$$\sin\theta = AD \sin 33°\,40' \div \frac{AD}{\cos 35°\,20'}$$
$$= \sin 33°\,40' \times \cos 35°\,20'$$
$$= 0.55436 \times 0.81580 = 0.45225.$$
$$\therefore \theta = \sin^{-1} 0.45225 = 26°\,53.3'.$$

19. A hill slopes at an angle of 32° with the horizontal. A path leads up it, making an angle of 47° 30′ with the line of steepest slope; find the inclination of the path with the horizontal. *Ans.* 20° 58′ 40″.

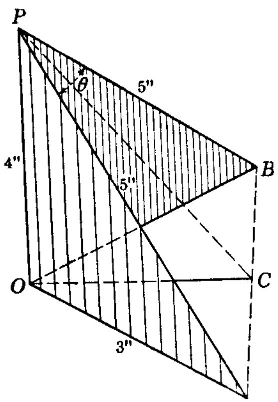


FIG. 74.

20. Two roofs have their ridges at right angles, and each is inclined to the horizontal at an angle of 30°. Find the inclination of their line of intersection to the horizontal. *Ans.* 22° 12′ 28″.

21. A mountain side has a slope of 30°. A road ascending the mountain is to be built and is to have a grade of 7 per cent. Find the angle it will make with the line of greatest slope. *Ans.* 81° 57.2′.

22. Two set squares whose sides are 3, 4, and 5 in. are placed as in Fig. 74, so that their 4-in. sides and right angles coincide, and the angle between the 3-in. sides is 46° 35′. Find the angle θ between the longest sides. *Ans.* 27° 26.9′.

23. Show that placing the carpenter's square as shown in Fig. 75*b* will determine the miter for making a regular pentagonal frame as shown in *a*. What is the angle θ of the miter? *Ans.* θ = 54°.

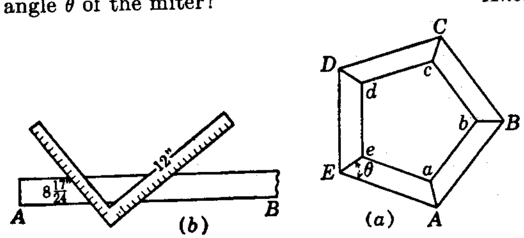


FIG. 75.

24. If 12 in. is taken on one leg of a carpenter's square, how many inches must be taken on the other leg to cut miters for making regular polygons of the following numbers of sides: 3, 4, 6, 8, and 10? Express results to the sixteenth of an inch. *Ans.* $20\frac{13}{16}$; 12; $6\frac{15}{16}$; 5; $3\frac{7}{8}$.

25. In the frame of a tower shown in Fig. 76, determine the distances from *A* and *B*, *C* and *D*, etc., to make the holes in the braces so that they may be bolted at points *a*, *b*, *c*, etc. These distances should be accurate to tenths of an inch. Can these distances be determined by means of geometry?

Ans. *Aa* = 10 ft. 5.3 in., etc.; yes.

26. A street-railway track is to turn a corner on the arc of a circle. If the track is at a distance *a* from the curbstone and the turn is through an angle θ, show that the radius $r = OR = ON$ (Fig. 77) of the curve to pass at a distance *b* from the corner is given by the formula

$$r = \frac{a - b\cos\frac{1}{2}\theta}{1 - \cos\frac{1}{2}\theta}.$$

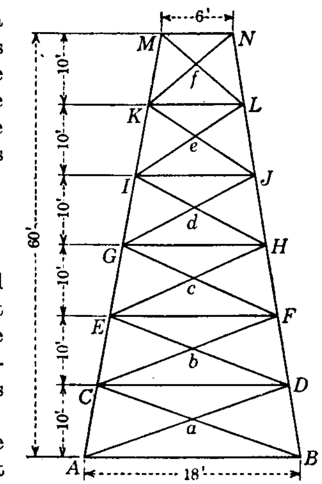


FIG. 76.

27. When the 8-in. crank of a horizontal engine is vertical, the piston is 1.5 in. past the midstroke. What is the length of the connecting rod and what angle does the connecting rod make with the guides at this instant? *Ans.* 22.08 in.; 21° 14.6′.

28. In Fig. 78, *LGA* is an arc of a circle with center at *O*, *LV* and *AV* are tangents at the extremities of the arc, *GF* is tangent to the arc at its center point *G*, and θ is the angle at the center of the circle and intercepting the arc. Derive the following formulas useful in railway surveying:

$t = r\tan\frac{1}{2}\theta.$	$c = 2r\sin\frac{1}{2}\theta.$	$m = r \operatorname{vers}\frac{1}{2}\theta.$
$e = r(\sec\frac{1}{2}\theta - 1).$	$e = t\tan\frac{1}{4}\theta.$	$c = 2m\cot\frac{1}{4}\theta.$
$c = 2t\cos\frac{1}{2}\theta.$	$GA = \frac{1}{2}c\sec\frac{1}{4}\theta.$	

29. A salesman for a wire-screen company wishes formulas for laying out a screen in the form of the frustum of a right circular cone of large diameter

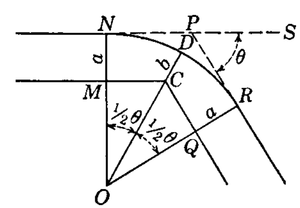


FIG. 77.

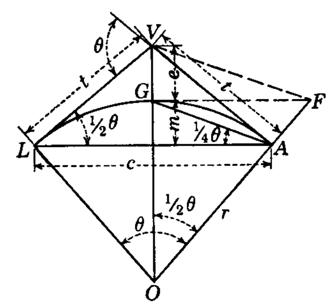


FIG. 78.

D, small diameter *d*, and slant height *s*. He also wishes the dimensions *l* and *w* of the rectangular piece from which the screen is to be cut. The layout is in the form of a section of a ring bounded by two concentric circles of radii *R* and *r*, and having a central angle θ. Determine formulas for *R*,

r, and θ in terms of D, d, and s; and formulas for l and w in terms of R, r, and θ.

Ans. $R = \dfrac{sD}{D - d}$; $r = \dfrac{sd}{D - d}$; $\theta = \dfrac{D - d}{s}180°$;
$l = 2R \sin \frac{1}{2}\theta$; $w = R - r \cos \frac{1}{2}\theta$.

REFLECTION AND REFRACTION OF LIGHT

73. Reflection of a ray of light.—The path of a ray of light in a homogeneous medium as air is a straight line. But when a ray of light strikes a polished surface it is **reflected** according to the well-known law which states that *the angle of incidence is equal to the angle of reflection.*

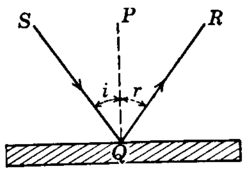


FIG. 79.

Thus, in Fig. 79, the incident ray SQ strikes the polished surface at Q and is reflected in the direction QR. The line QP is perpendicular to the surface at Q. The angle $SQP = i$ is the **angle of incidence,** and the angle $PQR = r$ is the **angle of reflection.** The law states that these two angles are equal.

74. Refraction of a ray of light.—When a ray of light passes from one transparent medium to another which is more or less dense, its direction is changed, that is, the ray of light is **refracted.**

Thus, in Fig. 80, a ray of light SQ, passing through air, meets the surface of a piece of glass at Q and is refracted toward the normal, or perpendicular, QP'. It continues in the direction QT until it meets the other surface of the glass at T, where it is again refracted, but this time away from the normal; and passes into the air in the direction TR. If the two surfaces of the glass are parallel, it has been found by experiment that the direction of TR is the same as that of SQ.

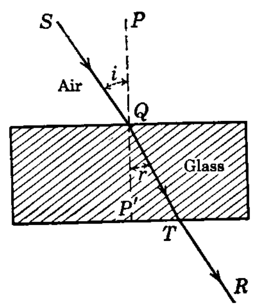


FIG. 80.

The lines QP and QP' are perpendicular to the surface at Q. The angle $SQP = i$ is the **angle of incidence,** and the angle $P'QT = r$ is the **angle of refraction.**

It has been found by experiment that for a given kind of glass the ratio

$$\frac{\sin i}{\sin r} = \mu$$

is constant whatever the angle of incidence may be. This means that, for a certain kind of glass, if the angle of incidence is changed, then the angle of refraction also changes in such a

manner that the ratio of the sines is constant. This ratio for **a ray of light** passing from air to crown glass is very nearly $\frac{3}{2}$, and for water it is $\frac{4}{3}$.

The value of the ratio $\dfrac{\sin i}{\sin r} = \mu$ is called the **index of refraction** of the glass with respect to air.

It follows that the index of refraction of air with respect to glass is the reciprocal of that of glass with respect to air. That is, if the index of refraction of glass with respect to air is μ, then the index of refraction of air with respect to glass is $\dfrac{1}{\mu}$. The same may be stated for any other two transparent substances.

EXERCISES

1. Prove that if a mirror that is reflecting a ray of light is turned through an angle α, the reflected ray is turned through an angle 2α.

2. The eye is 20 in. in front of a mirror and an object appears to be 25 in. back of the mirror, while the line of sight makes an angle of 32° 30′ with the mirror. Find the distance and direction of the object from the eye.

Ans. 70.8 in. in a direction making an angle of 4° 3′ with plane of mirror.

3. A ray of light passes from air into carbon disulphide. Find the angle of refraction if the angle of incidence is 33° 10′ and the index of refraction is 1.758. *Ans.* 18° 7.9′.

4. When $\mu = 1.167$ and the angle of incidence is 19° 30′, find the angle of refraction.

Ans. 16° 37.3′.

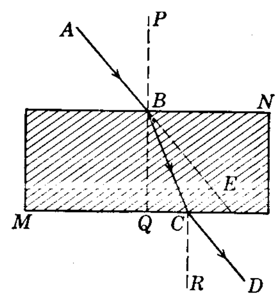


FIG. 81.

5. A ray of light travels the path $ABCD$ (Fig. 81) in passing through the plate glass MN 0.625 in. thick. What is the displacement CE if the ray strikes the glass at an angle $ABP = 42° 10'$ the index of refraction being $\frac{3}{2}$? *Ans.* 0.1887 in.

6. If the eye is at a point under water, what is the greatest angle from the zenith that a star can appear to be? *Ans.* 48° 35.4′.

7. A source of light is under water. What is the greatest angle a ray can make with the normal and pass into the air? For any greater angle the ray is totally reflected. *Ans.* 48° 35.4′.

8. A straight rod is partially immersed in water. The image in the water appears inclined at an angle of 45° with the surface. Find the inclination of the rod to the surface of the water if the index of refraction is $\frac{4}{3}$.

Ans. 70° 31.6′.

SIDES OPPOSITE VERY SMALL ANGLES

75. Relation between sin θ, θ, and tan θ, for small angles.—Draw angle $BOE = \theta$ (Fig. 82). With O as a center and $OB = 1$ as radius, describe the arc BD. Draw $DA \perp$ to OB and BE

tangent to the arc at B. Then $\sin\theta = AD$, $\theta = \text{arc } DB$, and $\tan\theta = BE$. Comparing areas of triangles and sector;

$$\triangle OBD < \text{sector } OBD < \triangle OBE.$$

But $\triangle OBD = \frac{1}{2}OB \times AD$, sector $OBD = \frac{1}{2}\overline{OB}^2 \cdot \theta$, where θ is in radians (see **Art. 8**) and $\triangle OBE = \frac{1}{2}OB \cdot BE$.

Then $$\tfrac{1}{2}OB \cdot AD < \tfrac{1}{2}\overline{OB}^2 \cdot \theta < \tfrac{1}{2}OB \cdot BE.$$

Dividing by $\frac{1}{2}$ and substituting $OB = 1$, $AD = \sin\theta$, and $BE = \tan\theta$,

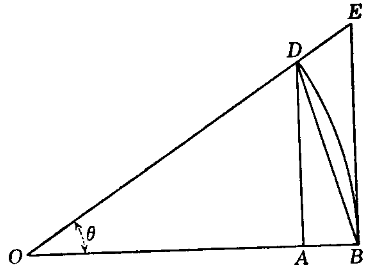


FIG. 82.

$$\sin\theta < \theta < \tan\theta.$$

Dividing by $\sin\theta$ and remembering that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$,

$$1 < \frac{\theta}{\sin\theta} < \sec\theta$$

Now as θ approaches 0 as a limit $\sec\theta$ approaches 1 as a limit, written $\lim_{\theta\to 0} \sec\theta = 1$.

Then, since $\dfrac{\theta}{\sin\theta}$ is always less than a quantity which approaches 1 as a limit, and at the same time is greater than 1, we have

$$\textbf{[12]} \qquad \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = 1.$$

Again, dividing $\sin\theta < \theta < \tan\theta$ by $\tan\theta$ and simplifying, $\cos\theta < \dfrac{\theta}{\tan\theta} < 1$. But $\lim_{\theta\to 0} \cos\theta = 1$; therefore, $\lim_{\theta\to 0} \dfrac{\theta}{\tan\theta} = 1$.

By computing the following table, the student will find **[12]** verified.

Angle in degrees	$\sin\theta$	θ in radians	$\tan\theta$	$\dfrac{\theta}{\sin\theta}$
20°				
10°				
5°				
4°				
3°				
2°				
1°				
½°				

These results show that, for small angles, $\sin\theta$ and $\tan\theta$ may be replaced by θ in radians and the results will be approximately correct.

For example, $\sin 5° 9.4' = 0.0899$,
$5° 9.4' = 0.0900$ radian,
and $\tan 5° 9.4' = 0.0902$.

For a smaller angle the agreement will be still closer.

76. Side opposite small angle given.—When a very small angle and the side opposite are given in a right triangle, another side can be found by means of the sine or tangent of the small angle considered as a number of radians. The short side, considered as an arc, divided by the number of radians in the small angle will give a long side.

Example 1.—A tower is 125 ft. high. The angle of elevation of the top of the tower, from a point in the same horizontal plane as the base, is 1°. Find the distance from the point of observation to the tower.

Solution.—Let x = distance to the tower in feet.

Then $$\tan 1° = \frac{125}{x}, \text{ or } x = \frac{125}{\tan 1°}.$$

But $1° = 0.01745$ radian $= \tan 1°$, approximately.

$$\therefore x = \frac{125}{0.01745} = 7163 \text{ ft.}$$

Example 2.—A railway track has a 2 per cent grade for a certain distance. Find the inclination of the track to the horizontal.

Solution.—The per cent of a grade is the ratio of the number of feet rise to the number of feet on the horizontal. Then, for a 2 per cent grade the tangent of the angle of inclination is 0.02, which is approximately the angle in radians.

By **Table V**, 0.02 radian = 1° 8.7′.

77. Lengths of long sides given.—In a right triangle having two sides, including an acute angle, given, the angle can be found by the formula

$$\tan \tfrac{1}{2}A = \sqrt{\frac{c-b}{c+b}},$$

where A is the angle included by the hypotenuse c and the side b.

This formula is derived as follows:

In Fig. 83, ABC is a right triangle.

Draw AE bisecting angle A, and draw DB perpendicular to AE.

Then $\frac{1}{2}A = \angle DAE = \angle CBD$.

Also $AD = AB = c$, and $CD = c - b$.

In triangle CBD, $\tan CBD = \tan \frac{1}{2}A = \frac{CD}{CB} = \frac{c-b}{a}$.

But $$a = \sqrt{c^2 - b^2} = \sqrt{(c+b)(c-b)}.$$

$$\therefore \tan \tfrac{1}{2}A = \sqrt{\frac{c-b}{c+b}}.$$

When angle A is small, $2 \tan \frac{1}{2}A$ gives approximately the value of A in radians, which can be used as already explained.

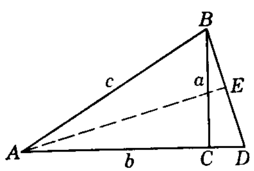


FIG. 83.

Example.—At what distance may a mountain 1 mile high be seen at sea, if the earth's radius is 3960 miles?

Solution.—Let s = distance in miles, and let the angle at the center of the earth between the radius to the mountain and the radius to the point at sea be θ.

By the formula, $\tan \frac{1}{2}\theta = \sqrt{\frac{3961 - 3960}{3961 + 3960}} = \sqrt{\frac{1}{7921}}$.

Then θ in radians $= 2\sqrt{\frac{1}{7921}}$.

By the formula $s = r\theta$ of **Art. 8,**

$$s = 2\sqrt{\frac{1}{7921}} \times 3960 = 89 \text{ miles}.$$

This example can also be computed by the rule given in **Art. 70,** which gives $s = \sqrt{\frac{3}{2}h}$, where h is the height of the mountain in feet.

$$\therefore \sqrt{\tfrac{3}{2} \times 5280} = 89 \text{ miles}.$$

EXERCISES

1. A certain plane is inclined to the horizontal at an angle of 1° 10′. Find the per cent of grade of a railway track constructed on this plane. *Ans.* 2.04 per cent.

2. A railway track rises 100 ft. to the mile. Find the angle of inclination of the track. *Ans.* 1° 5.1′.

3. If the diameter of the earth as seen from the moon makes an angle of 1° 54′, find the distance from the moon to the earth, taking the earth's radius as 3960 miles. *Ans.* 239,000 miles.

4. If the distance from the earth to the sun is 93,000,000 miles and the diameter of the sun makes an angle of 32′ at the earth, find the diameter of the sun in miles. *Ans.* 870,000 miles.

5. Telescopes at the ends of a base line 350 ft. long, on the deck of a ship, are turned upon a distant fort. The lines of sight of the telescopes are found to make angles of 89° 10′ and 89° 40′ with the base line. Find the distance from the ship to the fort. *Ans.* 3.26 miles.

6. The diameter of the moon subtends an angle of 31′ 5″ at the earth. The moon is approximately 240,000 miles from the earth. Find the diameter of the moon in miles. *Ans.* 2170 miles.

7. At what distance may a mountain 2.5 miles high be seen at sea, taking the earth's radius at 3960 miles. *Ans.* 140.7 miles.

CHAPTER VIII

FUNCTIONS INVOLVING MORE THAN ONE ANGLE

78. In the previous chapters, we have worked with, and established the relations between, the functions of a single angle. But, in solving oblique triangles and in many of the applications of trigonometry to other subjects, formulas are used which are derived from the functions of the sums or differences of angles. These functions are expressed in terms of the functions of the individual angles and are as follows for the sine and cosine:

[13] $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$

[14] $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

[15] $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

[16] $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

Formulas [13] and [14] are often called **addition formulas,** and [15] and [16] **subtraction formulas.**

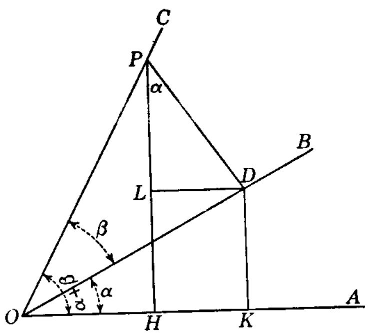


FIG. 84.

79. Derivation of the formulas for the sine and cosine of the sum of two angles.—Let $\angle AOB = \alpha$ and $\angle BOC = \beta$ (Fig. 84), each of which is acute and so chosen that $\alpha + \beta = \angle AOC$ is less than 90°. In order that the functions of α, β, and $\alpha + \beta$ may be involved in the same formula, we may form right triangles which have α, β, and $\alpha + \beta$ as acute angles.

Choose any point P in the terminal side OC. Draw $PH \perp OA$ $PD \perp OB$, $DK \perp OA$, and $DL \perp PH$. $\triangle KOD$ is similar to $\triangle LPD$, since their sides are perpendicular each to each. Then $\angle LPD = \alpha$.

By definition, $\sin(\alpha + \beta) = \dfrac{HP}{OP} = \dfrac{KD + LP}{OP} = \dfrac{KD}{OP} + \dfrac{LP}{OP}.$

Now multiply numerator and denominator of $\dfrac{KD}{OP}$ by OD, the common side of the two triangles of which KD and OP are sides

respectively. Also, multiply $\dfrac{LP}{OP}$ in the same way by PD, the common side of triangles DOP and LPD. Then

$$\sin(\alpha + \beta) = \frac{KD}{OD}\cdot\frac{OD}{OP} + \frac{LP}{PD}\cdot\frac{PD}{OP}.$$

But $\dfrac{KD}{OD} = \sin\alpha$, $\dfrac{OD}{OP} = \cos\beta$, $\dfrac{LP}{PD} = \cos\alpha$, and $\dfrac{PD}{OP} = \sin\beta$.

[13] $\therefore \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$

By definition,

$$\cos(\alpha + \beta) = \frac{OH}{OP} = \frac{OK - HK}{OP} = \frac{OK}{OP} - \frac{LD}{OP} = \frac{OK}{OD}\cdot\frac{OD}{OP} - \frac{LD}{PD}\cdot\frac{PD}{OP}.$$

But $\dfrac{OK}{OD} = \cos\alpha$, $\dfrac{OD}{OP} = \cos\beta$, $\dfrac{LD}{PD} = \sin\alpha$, and $\dfrac{PD}{OP} = \sin\beta$.

[14] $\therefore \cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

80. Derivation of the formulas for the sine and cosine of the difference of two angles.—Let $\angle AOB = \alpha$ and $\angle COB = \beta$ be the two acute angles (Fig. 85). Then angle $AOC = \alpha - \beta$. For reasons similar to those given in the preceding article, choose any point P in the terminal side OC of $(\alpha - \beta)$. Draw $PH \perp OA$, $PR \perp OB$, $RD \perp OA$, and $PE \perp DR$. $\triangle DOR$ is similar to $\triangle PER$ and $\angle ERP = \alpha$.

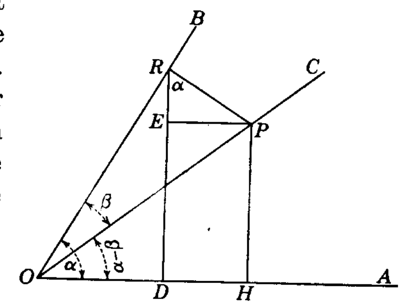


FIG. 85.

By definition,

$$\sin(\alpha - \beta) = \frac{HP}{OP} = \frac{DR - ER}{OP} = \frac{DR}{OP} - \frac{ER}{OP} = \frac{DR}{OR}\cdot\frac{OR}{OP} - \frac{ER}{RP}\cdot\frac{RP}{OP}.$$

But $\dfrac{DR}{OR} = \sin\alpha$, $\dfrac{OR}{OP} = \cos\beta$, $\dfrac{ER}{RP} = \cos\alpha$, and $\dfrac{RP}{OP} = \sin\beta$.

[15] $\therefore \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

By definition,

$$\cos(\alpha - \beta) = \frac{OH}{OP} = \frac{OD + EP}{OP} = \frac{OD}{OP} + \frac{EP}{OP} = \frac{OD}{OR}\cdot\frac{OR}{OP} + \frac{EP}{RP}\cdot\frac{RP}{OP}.$$

[16] $\therefore \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

In the proof of [15] and [16], it was assumed that $\alpha > \beta$. Now suppose $\beta > \alpha$. Then $\alpha - \beta = -(\beta - \alpha)$.

By **Art. 52** $\sin(\alpha - \beta) = \sin[-(\beta - \alpha)] = -\sin(\beta - \alpha)$.
By **[15]** $-\sin(\beta - \alpha) = -(\sin\beta\cos\alpha - \cos\beta\sin\alpha)$.
$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$,

which is the same result as was obtained before.

81. Proof of the addition formulas for other values of the angles.—In **Art. 79** formulas **[13]** and **[14]** were proved when α, β, and $\alpha + \beta$ are each less than 90°. They are, however, true for all values of the angles.

(1) *Suppose that α and β are acute and such that $\alpha = 90° - \phi$ and $\beta = 90° - \gamma$, where ϕ and γ are each less than 45°.* On this assumption, $(\alpha + \beta) > 90°$, $(\phi + \gamma) < 90°$, $\sin\alpha = \cos\phi$, $\cos\alpha = \sin\phi$, $\sin\beta = \cos\gamma$, and $\cos\beta = \sin\gamma$.

$$\begin{aligned}\therefore \sin(\alpha + \beta) &= \sin[(90° - \phi) + (90° - \gamma)]\\ &= \sin[180° - (\phi + \gamma)]\\ &= \sin(\phi + \gamma) = \sin\phi\cos\gamma + \cos\phi\sin\gamma.\end{aligned}$$

Substituting for the functions of ϕ and γ their values in terms of the functions of α and β,

$$\begin{aligned}\sin(\alpha + \beta) &= \cos\alpha\sin\beta + \sin\alpha\cos\beta\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta.\end{aligned}$$

That is, the formula for $\sin(\alpha + \beta)$ is true when $(\alpha + \beta)$ is an angle in the second quadrant and α and β as stated.

In the same way we may show that the formula for $\cos(\alpha + \beta)$ is true for values of the angles as given above.

(2) *Suppose that α is in the second quadrant and β in the third, such that $\alpha = 90° + \phi$ and $\beta = 180° + \gamma$.* On this assumption, $\sin\alpha = \sin(90° + \phi) = \cos\phi$; $\cos\alpha = \cos(90° + \phi) = -\sin\phi$; $\sin\beta = \sin(180° + \gamma) = -\sin\gamma$; $\cos\beta = \cos(180° + \gamma) = -\cos\gamma$;
$\sin(\alpha + \beta) = \sin[(90° + \phi) + (180° + \gamma)] = \sin[270° + (\phi + \gamma)]$
$= -\cos(\phi + \gamma) = -\cos\phi\cos\gamma + \sin\phi\sin\gamma$.

Substituting for the functions of ϕ and γ their values in terms of the functions of α and β,

$$\begin{aligned}\sin(\alpha + \beta) &= -(\sin\alpha)(-\cos\beta) + (-\cos\alpha)(-\sin\beta)\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta.\end{aligned}$$

In the same manner it may be shown that the addition formulas are true for any angles. It will now be assumed that the addition formulas for sine and cosine are true for all values of the angles.

82. Proof of the subtraction formulas for all values of the angles.—Since the addition formulas are true for all values of α and β, they are true when $-\beta$ is put for β. Then

$$\begin{aligned}\sin(\alpha - \beta) &= \sin[\alpha + (-\beta)]\\ &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta),\end{aligned}$$

and

$$\begin{aligned}\cos(\alpha - \beta) &= \cos[\alpha + (-\beta)]\\ &= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta).\end{aligned}$$

But $\sin(-\beta) = -\sin\beta$, and $\cos(-\beta) = \cos\beta$.

$$\therefore \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$

and

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

That is, the subtraction formulas are true in general.

Example 1.—Given $\sin\alpha = \frac{3}{5}$ and $\cos\beta = \frac{5}{13}$; find $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ if α and β are acute.

Solution.—The formulas to be used are

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

and

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

To substitute in these formulas it is necessary first to find $\cos\alpha$ and $\sin\beta$.

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = \sqrt{1 - (\tfrac{3}{5})^2} = \tfrac{4}{5}.$$
$$\sin\beta = \sqrt{1 - \cos^2\beta} = \sqrt{1 - (\tfrac{5}{13})^2} = \tfrac{12}{13}.$$

Substituting in the formulas,

$$\sin(\alpha + \beta) = \tfrac{3}{5}\cdot\tfrac{5}{13} + \tfrac{4}{5}\cdot\tfrac{12}{13} = \tfrac{3}{13} + \tfrac{48}{65} = \tfrac{63}{65}.$$
$$\cos(\alpha + \beta) = \tfrac{4}{5}\cdot\tfrac{5}{13} - \tfrac{3}{5}\cdot\tfrac{12}{13} = \tfrac{4}{13} - \tfrac{36}{65} = -\tfrac{16}{65}.$$

Example 2.—Prove that $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \frac{1}{2}\pi$, using only the principal values of the anti-functions.

Proof.— Let $\alpha = \sin^{-1}\frac{3}{5}$ and $\beta = \sin^{-1}\frac{4}{5}$.
$\therefore \sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{4}{5}$.

Then $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$= \frac{3}{5}\cdot\frac{3}{5} + \frac{4}{5}\cdot\frac{4}{5} = 1.$

But $\sin^{-1} 1 = \frac{1}{2}\pi$.
$\therefore \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \frac{1}{2}\pi$.

EXERCISES

Answer Exercises 1 to 10 orally. Apply the addition and subtraction formulas in the following expansions:

1. Expand sin (15° + 30°).
2. Expand cos (45° + 15°).
3. Expand sin (75° − 60°).
4. Expand cos (23° − 10°).
5. Expand sin (240° − 30°).
6. Expand cos (30° − 120°).
7. Does sin 60° = sin (40° + 20°)?

8. Does $2 \sin 25° = \sin (25° + 25°)$?

9. Does $\sin (40° + 30°) = \sin 40° + \sin 30°$?

10. Does $\cos (360° + 120°) = \cos 120°$?

11. Given $\sin \alpha = \frac{12}{13}$ and $\sin \beta = \frac{7}{25}$, α and β acute; find $\sin (\alpha + \beta)$, $\cos (\alpha + \beta)$, $\sin (\alpha - \beta)$, and $\cos (\alpha - \beta)$. *Ans.* $\frac{323}{325}$; $\frac{36}{325}$; $\frac{253}{325}$; $\frac{204}{325}$.

12. Given $\sin \alpha = \frac{\sqrt{2}}{2}$, and $\tan \beta = \sqrt{3}$, α and β acute; find $\sin (\alpha + \beta)$ and $\cos (\alpha + \beta)$. *Ans.* 0.9659; −0.2588.

13. Given $\sin \alpha = \frac{1}{2}$, and $\tan \beta = \frac{1}{3}$, α and β acute; find $\sin (\alpha + \beta)$, $\cos (\alpha + \beta)$, $\sin (\alpha - \beta)$, and $\cos (\alpha - \beta)$.
Ans. 0.74820; 0.66347; 0.20048; 0.97970.

14. Given $\sin \alpha = -\frac{3}{5}$, $\cos \beta = -\frac{12}{13}$, α in the fourth quadrant and β in the third. Find $\sin (\alpha + \beta)$, $\cos (\alpha + \beta)$, $\cos (\alpha - \beta)$, and $\sin (\alpha - \beta)$.
Ans. $\sin (\alpha + \beta) = \frac{16}{65}$; $\cos (\alpha + \beta) = -\frac{63}{65}$.

15. Given $\cos \alpha = -\frac{4}{5}$, α in the second quadrant, and $\cot \beta = \frac{12}{5}$, β in the third quadrant. Find $\sin (\alpha + \beta)$, $\cos (\alpha + \beta)$, $\sin (\alpha - \beta)$, and $\cos (\alpha - \beta)$. *Ans.* $\sin (\alpha + \beta) = -\frac{16}{65}$; $\cos (\alpha + \beta) = \frac{63}{65}$.

16. Find $\sin 90°$ by using $90° = 60° + 30°$.

17. Find $\sin 90°$ by using $90° = 150° - 60°$.

18. Find $\cos 180°$ by using $180° = 135° + 45°$.

19. Find $\sin 75°$ by using $75° = 120° - 45°$.
Ans. $\frac{1}{4}(\sqrt{2} + \sqrt{6}) = 0.9659$.

20. Find $\sin 150°$ by using (*a*) $150° = 120° + 30°$, (*b*) $150° = 210° - 60°$, (*c*) $150° = 75° + 75°$.

21. Find sin 120 by using (*a*) $120° = 60° + 60°$, (*b*) $120° = 90° + 30°$, (*c*) $120° = 210° - 90°$.

Find the values of the following expressions, using only the principal values of the angles:

22. $\sin (\sin^{-1} \frac{16}{65} + \cos^{-1} \frac{12}{13})$. *Ans.* $\frac{3}{5}$.

23. $\cos \left(\sin^{-1} \frac{1}{\sqrt{5}} + \tan^{-1} \frac{1}{3}\right)$. *Ans.* $\frac{1}{\sqrt{2}}$.

24. $\sin \left(\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}\right)$. *Ans.* $\frac{1}{\sqrt{2}}$.

25. $\cos (\tan^{-1} \frac{1}{2} - \cos^{-1} \frac{1}{5})$. *Ans.* 0.617.

26. $\sin [\tan^{-1} (-\frac{1}{2}) + \sin^{-1} \frac{1}{2}]$. *Ans.* 0.05993.

27. $\sin (\sin^{-1} a + \sin^{-1} b)$. *Ans.* $a\sqrt{1 - b^2} + b\sqrt{1 - a^2}$.

28. $\cos (\sin^{-1} a + \sin^{-1} b)$. *Ans.* $\sqrt{1 - a^2}\sqrt{1 - b^2} - ab$.

29. $\sin (\cos^{-1} a - \sin^{-1} b)$. *Ans.* $\sqrt{1 - a^2}\sqrt{1 - b^2} - ab$.

30. $\cos (\sin^{-1} a - \cos^{-1} b)$. *Ans.* $b\sqrt{1 - a^2} + a\sqrt{1 - b^2}$.

Prove the following by expanding by the addition and subtraction formulas:

31. $\sin (90° + \theta) = \cos \theta$.

32. $\sin (180° - \theta) = \sin \theta$.

33. $\sin (270° - \theta) = -\cos \theta$.

34. $\cos (180° + \theta) = -\cos \theta$.

35. $\cos (270° - \theta) = -\sin \theta$.

36. $\cos (360° - \theta) = \cos \theta$.

37. If $\cos \alpha = \frac{2}{\sqrt{5}}$, and $\tan \beta = \frac{1}{3}$, α and β are acute angles, prove that $\alpha + \beta = 45°$.

In the following use only principal values of the angles:

38. Prove $\sin^{-1} \frac{5}{13} + \sin^{-1} \frac{12}{13} = \frac{\pi}{2}$.

39. Prove $\sin^{-1} \frac{1}{4} + \cos^{-1} \frac{1}{4} = \frac{\pi}{2}$.

40. Prove $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

41. Prove $\sin^{-1} a \pm \sin^{-1} b = \sin^{-1} (a\sqrt{1 - b^2} \pm b\sqrt{1 - a^2})$.

42. Prove $\cos^{-1} a \pm \cos^{-1} b = \cos^{-1} [ab \mp \sqrt{(1 - a^2)(1 - b^2)}]$.

43. Prove that in any right triangle $\cos (A - B) = \frac{2ab}{c^2}$.

Find a value of θ in the following exercises:

44. $\cos (50° + \alpha) \cos (50° - \alpha) + \sin (50° + \alpha) \sin (50° - \alpha) = \cos \theta$.
Ans. $\theta = 2\alpha$.

45. $\cos 30° \cos (105° - \alpha) - \sin 30° \sin (105° - \alpha) = \cos \theta$.
Ans. $\theta = 135° - \alpha$.

46. $\sin (60° + \frac{1}{2}\beta) \cos (60° - \frac{1}{2}\beta) + \cos (60° + \frac{1}{2}\beta) \sin (60° - \frac{1}{2}\beta) = \sin \theta$.
Ans. $\theta = 120°$.

47. $\cos (45° - x) \cos (45° + x) + \sin (45° - x) \sin (45° + x) = \cos \theta$.
Ans. $\theta = -2x$.

48. Prove that $\cos (\alpha - \beta)$ gives the same result whether $\alpha > \beta$ or $\alpha < \beta$

Prove that formulas [13] and [14] are true in the following cases:

49. α in the fourth quadrant and β in the first.

50. α in the third quadrant and β in the third.

51. α in the first quadrant and β in the third.

Expand and derive the formulas expressed in the following:

52. $\sin (\alpha + \beta + \gamma) = \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \beta \cos \gamma + \cos \alpha \cos \beta \sin \gamma - \sin \alpha \sin \beta \sin \gamma$.

Suggestion.—$\sin (\alpha + \beta + \gamma) = \sin [(\alpha + \beta) + \gamma]$.
$= \sin (\alpha + \beta) \cos \gamma + \cos (\alpha + \beta) \sin \gamma$.

53. $\cos (\alpha + \beta + \gamma) = \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \beta \cos \gamma - \sin \alpha \cos \beta \sin \gamma - \cos \alpha \sin \beta \sin \gamma$.

83. Formulas for the tangents of the sum and the difference of two angles.—By [7], [13], and [14],

$$\tan (\alpha + \beta) = \frac{\sin (\alpha + \beta)}{\cos (\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta}.$$

Dividing both numerator and denominator by $\cos \alpha \cos \beta$, and applying [7],

$$\tan (\alpha + \beta) = \frac{\frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

[17] $$\therefore \tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}.$$

[18] Similarly, $\mathbf{tan}\ (\boldsymbol{\alpha} - \boldsymbol{\beta}) = \dfrac{\mathbf{tan}\ \boldsymbol{\alpha} - \mathbf{tan}\ \boldsymbol{\beta}}{\mathbf{1} + \mathbf{tan}\ \boldsymbol{\alpha}\ \mathbf{tan}\ \boldsymbol{\beta}}.$

Since formulas [**13**], [**14**], [**15**], and [**16**] are true for all values of α and β, the formulas [**17**] and [**18**] are true in general. These formulas express the tangent of the sum or of the difference of two angles in terms of the tangents of the individual angles.

EXERCISES

1. Find $\tan 75°$ by using $75° = 45° + 30°$. *Ans.* 3.732.

2. Find $\tan 15°$ by using $15° = 45° - 30°$. *Ans.* 0.268.

3. Given $\tan \alpha = \frac{7}{24}$ and $\tan \beta = \frac{5}{12}$, α and β acute; find $\tan (\alpha + \beta)$ and $\tan (\alpha - \beta)$. *Ans.* 0.8063; −0.1115.

4. Given $\sin \alpha = \frac{3}{4}$ and $\cos \beta = \frac{1}{3}$, α in the second quadrant and β in the first quadrant; find $\tan (\alpha + \beta)$ and $\tan (\alpha - \beta)$. *Ans.* 0.4028; 1.7953.

5. Find $\tan (\tan^{-1} \frac{4}{3} - \tan^{-1} \frac{1}{7})$, using the principal values of the angles. *Ans.* 1.

6. Find $\tan (\tan^{-1} \frac{1}{3} - \cos^{-1} \frac{2}{3})$, using the principal values of the angles. *Ans.* −0.57166.

7. Find $\tan (\tan^{-1} \frac{2}{11} + \cot^{-1} \frac{24}{7})$. *Ans.* $\frac{1}{2}$.

8. Find $\cot (\cot^{-1} \frac{84}{13} + \cot^{-1} \frac{15}{8})$. *Ans.* $\frac{4}{3}$.

9. Prove that $\sec^{-1} \dfrac{5}{3} - \cot^{-1} 7 = \dfrac{\pi}{4}$.

10. Prove that $\tan^{-1} \frac{5}{7} + \tan^{-1} \frac{1}{6} = \frac{1}{4}\pi$.

11. Prove that $\tan^{-1} a \pm \tan^{-1} b = \tan^{-1} \dfrac{a \pm b}{1 \mp ab}$.

12. Prove that $\tan^{-1} \frac{3}{5} + \tan^{-1} \frac{3}{4} - \tan^{-1} \frac{8}{19} = \frac{1}{4}\pi$.

13. Derive formula [**18**].

14. Derive the formula $\cot (\alpha + \beta) = \dfrac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta}$.

15. Derive the formula $\cot (\alpha - \beta) = \dfrac{\cot \alpha \cot \beta + 1}{\cot \beta - \cot \alpha}$.

16. Prove that in any right triangle $\tan (B - A) = \dfrac{(b + a)(b - a)}{2ab}$.

17. Derive the formula

$$\tan (\alpha + \beta + \gamma) = \frac{\tan \alpha + \tan \beta + \tan \gamma - \tan \alpha \tan \beta \tan \gamma}{1 - \tan \alpha \tan \beta - \tan \beta \tan \gamma - \tan \gamma \tan \alpha}.$$

18. Prove that $\tan \alpha + \dfrac{\tan \varphi \sec \alpha}{\cos \alpha - \tan \varphi \sin \alpha} = \tan (\alpha + \varphi)$.

84. Functions of an angle in terms of functions of half the angle.—Since the formulas for the sum of two angles are true for all values of α and β, they are true when $\beta = \alpha$.

Then, $\sin (\alpha + \beta) = \sin (\alpha + \alpha) = \sin \alpha \cos \alpha + \cos \alpha \sin \alpha$. That is,

[19] $$\mathbf{sin}\ \mathbf{2}\boldsymbol{\alpha} = \mathbf{2}\ \mathbf{sin}\ \boldsymbol{\alpha}\ \mathbf{cos}\ \boldsymbol{\alpha}.$$

This formula may be stated as follows:

The sine of any angle is equal to twice the product of the sine and cosine of the half angle.

Thus, $$\sin 40° = 2 \sin 20° \cos 20°.$$
$$\sin \alpha = 2 \sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\alpha.$$

And, conversely, $2 \sin 3\alpha \cos 3\alpha = \sin 2(3\alpha) = \sin 6\alpha$.

$$2 \sin 25° \cos 25° = \sin 2(25°) = \sin 50°.$$

Also

$$\cos (\alpha + \beta) = \cos (\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha$$
$$= \cos^2 \alpha - \sin^2 \alpha = 1 - \sin^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha,$$
or $$= \cos^2 \alpha - (1 - \cos^2 \alpha) = 2 \cos^2 \alpha - 1.$$

That is,

[20] $$\mathbf{cos}\ \mathbf{2}\boldsymbol{\alpha} = \mathbf{cos}^2\ \boldsymbol{\alpha} - \mathbf{sin}^2\ \boldsymbol{\alpha} = \mathbf{1} - \mathbf{2}\ \mathbf{sin}^2\ \boldsymbol{\alpha} = \mathbf{2}\ \mathbf{cos}^2\ \boldsymbol{\alpha} - \mathbf{1}.$$

Thus, $\cos 30° = \cos^2 15° - \sin^2 15°$, or $1 - 2 \sin^2 15°$, etc.

$$\cos 3\theta = \cos^2 \frac{3\theta}{2} - \sin^2 \frac{3\theta}{2}.$$

Also, $\tan (\alpha + \beta) = \tan (\alpha + \alpha) = \dfrac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha}$.

That is,

[21] $$\mathbf{tan}\ \mathbf{2}\boldsymbol{\alpha} = \frac{\mathbf{2}\ \mathbf{tan}\ \boldsymbol{\alpha}}{\mathbf{1} - \mathbf{tan}^2\ \boldsymbol{\alpha}}.$$

Thus, $\tan 60° = \dfrac{2 \tan 30°}{1 - \tan^2 30°}$, $\tan 100° = \dfrac{2 \tan 50°}{1 - \tan^2 50°}$,

$$\tan \theta = \frac{2 \tan \frac{1}{2}\theta}{1 - \tan^2 \frac{1}{2}\theta}.$$

Example 1.—Given the functions of 30°, to find the functions of 60°.

Solution.—$\sin 60° = 2 \sin 30° \cos 30° = 2(\frac{1}{2})(\frac{1}{2}\sqrt{3}) = \frac{1}{2}\sqrt{3}$.

$$\cos 60° = 2 \cos^2 30° - 1 = 2(\tfrac{1}{2}\sqrt{3})^2 - 1 = \tfrac{3}{2} - 1 = \tfrac{1}{2},$$
or $$= 1 - 2 \sin^2 30° = 1 - 2(\tfrac{1}{2})^2 = \tfrac{1}{2}.$$

$$\tan 60° = \frac{2 \tan 30°}{1 - \tan^2 30°} = \frac{\frac{2}{3}\sqrt{3}}{1 - (\frac{1}{3}\sqrt{3})^2} = \sqrt{3}.$$

Example 2.—Prove that $\dfrac{2 \tan \alpha}{1 + \tan^2 \alpha} = \sin 2\alpha$.

Proof.—$\tan \alpha = \dfrac{\sin \alpha}{\cos \alpha}$ and $1 + \tan^2 \alpha = \sec^2 \alpha$.

Then $\dfrac{2\tan\alpha}{1+\tan^2\alpha} = \dfrac{2\dfrac{\sin\alpha}{\cos\alpha}}{\sec^2\alpha} = \dfrac{2\dfrac{\sin\alpha}{\cos\alpha}}{\dfrac{1}{\cos^2\alpha}} = 2\sin\alpha\cos\alpha = \sin 2\alpha.$

EXERCISES

Express the following in terms of functions of half the angle. Answer orally.

1. $\sin 4\alpha$.
2. $\sin \frac{1}{2}\alpha$.
3. $\sin 8\alpha$.
4. $\cos 4\alpha$.
5. $\cos \frac{1}{2}\alpha$.
6. $\sin 90°$.
7. $\sin 20°$.
8. $\cos 90°$.
9. $\tan \frac{1}{2}\theta$.
10. $\tan 4\theta$.
11. $\tan 80°$.
12. $\sin 3\theta$.
13. $\sin (90° + 2\alpha)$.
14. $\sin (2 \sin^{-1} a)$.
15. $\cos (2 \cos^{-1} a)$.
16. $\cos (2 \sin^{-1} a)$.
17. $\sin (2 \cos^{-1} a)$.
18. $\tan (2 \tan^{-1} a)$.
19. Given the functions of 75°; find sine, cosine, and tangent of 150°.

Note.—See Exercise 19, page 112.

20. Given the functions of 150°; find sine, cosine, and tangent of 300°.
21. Given $\sin\theta = \frac{7}{25}$ and θ in the first quadrant; find sine, cosine and tangent of 2θ. *Ans.* 0.5376; 0.8432; 0.6376.
22. Given $\cos\theta = \frac{1}{5}$; find $\cos 2\theta$. *Ans.* −0.92.
23. Given $\sin\frac{1}{2}\theta = \frac{5}{13}$ and $\cos\frac{1}{2}\theta = -\frac{12}{13}$; find $\sin\theta$ and $\cos\theta$. *Ans.* −0.7101; 0.7041.
24. Given $\tan 4\theta = \frac{5}{12}$; find $\tan 8\theta$. *Ans.* 1.0084.

Find the value of the following, using the principal values of the angles:

25. $\sin (2\cos^{-1}\frac{2}{5})$. *Ans.* 0.7332.
26. $\cos (2\sin^{-1}\frac{2}{5})$. *Ans.* 0.68.
27. $\sin\left(2\sin^{-1}\dfrac{1}{\sqrt{1+x^2}}\right)$. *Ans.* $\dfrac{2x}{1+x^2}$.
28. $\cos (2 \text{ arc } \tan \frac{1}{3})$. *Ans.* 0.8.
29. $\tan (2 \text{ invsin } \frac{1}{3})$. *Ans.* 0.8081.
30. If $\tan\dfrac{\theta}{2} = \dfrac{y}{x}$ show that $\sin\theta = \dfrac{2xy}{x^2+y^2}$ and $\sin 2\theta = \dfrac{4xy(x^2-y^2)}{(x^2+y^2)^2}$.
31. If $\tan\theta = \dfrac{y}{x}$, show that $\sqrt{\dfrac{x+y}{x-y}} + \sqrt{\dfrac{x-y}{x+y}} = \dfrac{2\cos\theta}{\sqrt{\cos 2\theta}}$.
32. Prove that $\tan\beta = \dfrac{\sin 2\beta}{1+\cos 2\beta}$.
33. Prove that $\tan\theta - \cot\theta = -2\cot 2\theta$.
34. Prove that $\dfrac{1-\cos\alpha+\cos\beta-\cos(\alpha+\beta)}{1+\cos\alpha-\cos\beta-\cos(\alpha+\beta)} = \left(\tan\dfrac{\alpha}{2}\right)\left(\cot\dfrac{\beta}{2}\right)$.
35. Prove that $\dfrac{1+\tan^2(45°-\theta)}{1-\tan^2(45°-\theta)} = \dfrac{1}{\sin 2\theta}$.
36. Prove that $\dfrac{\tan^2 2\alpha}{\sec^2\alpha - 1} = (1+\sec 2\alpha)^2$.
37. Prove that in any right triangle $\sin 2A = \sin 2B$.
38. Prove that in any right triangle $\cos 2A = \sin (B - A)$.

Derive the formulas given in Exercises 39 to 44.

39. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Suggestion.—In formula [13] let $\alpha = 2\theta$ and $\beta = \theta$.

40. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.
41. $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.
42. $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.
43. $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.
44. $\tan 4\theta = \dfrac{4\tan\theta(1-\tan^2\theta)}{1 - 6\tan^2\theta + \tan^4\theta}$.

85. Functions of an angle in terms of functions of twice the angle.—By [20], $\cos 2\alpha = 1 - 2\sin^2\alpha$. Solving this for $\sin\alpha$, we have $\sin\alpha = \pm\sqrt{\dfrac{1-\cos 2\alpha}{2}}$

Let $\alpha = \frac{1}{2}\theta$ and we have

$$[22] \qquad \sin\frac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{2}}.$$

That is, *the sine of an angle is equal to the square root of one-half of the quantity, one minus the cosine of twice the angle.*

Thus, $\sin 50° = \sqrt{\dfrac{1-\cos 100°}{2}}$, $\sin 10° = \sqrt{\dfrac{1-\cos 20°}{2}}$.

Also by [20], $\cos 2\alpha = 2\cos^2\alpha - 1$. Solving this for $\cos\alpha$ we have $\cos\alpha = \pm\sqrt{\dfrac{1+\cos 2\alpha}{2}}$.

Let $\alpha = \frac{1}{2}\theta$ and we have

$$[23] \qquad \cos\frac{1}{2}\theta = \pm\sqrt{\frac{1+\cos\theta}{2}}.$$

That is, *the cosine of an angle is equal to the square root of one-half of the quantity, one plus the cosine of twice the angle.*

Thus, $\cos 30° = \sqrt{\dfrac{1+\cos 60°}{2}}$, $\cos 50° = \sqrt{\dfrac{1+\cos 100°}{2}}$.

By dividing [22] by [23], we can derive

$$[24] \qquad \tan\frac{1}{2}\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}.$$

The last two forms given in [24] may be obtained as follows: Multiplying numerator and denominator of $\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$ by $\sqrt{1-\cos\theta}$,

$$\tan\frac{1}{2}\theta = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta}} = \frac{\sqrt{(1-\cos\theta)^2}}{\sqrt{1-\cos^2\theta}} = \frac{1-\cos\theta}{\sin\theta}.$$

Again, multiplying numerator and denominator by $\sqrt{1 + \cos\theta}$,

$$\tan\frac{1}{2}\theta = \sqrt{\frac{1 - \cos\theta}{1 + \cos\theta}\cdot\frac{1 + \cos\theta}{1 + \cos\theta}} = \frac{\sqrt{1 - \cos^2\theta}}{\sqrt{(1 + \cos\theta)^2}} = \frac{\sin\theta}{1 + \cos\theta}.$$

$$\text{Thus, } \tan 40^\circ = \sqrt{\frac{1 - \cos 80^\circ}{1 + \cos 80^\circ}} = \frac{1 - \cos 80^\circ}{\sin 80^\circ} = \frac{\sin 80^\circ}{1 + \cos 80^\circ}.$$

Example.—Find the value of $\sin(\frac{1}{2}\cos^{-1}\frac{3}{4})$.

Solution.—Let $\theta = \cos^{-1}\frac{3}{4}$. Then $\cos\theta = \frac{3}{4}$.

$$\sin\left(\frac{1}{2}\cos^{-1}\frac{3}{4}\right) = \sin\frac{1}{2}\theta = \sqrt{\frac{1 - \cos\theta}{2}} = \sqrt{\frac{1 - \frac{3}{4}}{2}} = \sqrt{\frac{1}{8}} = \frac{\sqrt{2}}{4}.$$

EXERCISES

Express the following in terms of functions of twice the angle. Answer orally.

1. $\sin 2\alpha$.
2. $\sin \frac{1}{4}\alpha$.
3. $\cos 4\alpha$.
4. $\cos 50^\circ$.
5. $\sin 70^\circ$.
6. $\tan 90^\circ$.
7. $\tan 2\theta$.
8. $\tan \frac{1}{4}\theta$.
9. $\sin 3\theta$.
10. $\cos 80^\circ$.
11. $\sin(45^\circ + \frac{1}{2}\alpha)$.
12. $\cos(45^\circ - \frac{1}{2}\alpha)$.
13. $\sin(135^\circ + \frac{1}{2}\alpha)$.
14. $\cos(135^\circ - \frac{1}{2}\alpha)$.
15. $\tan(135^\circ - \frac{1}{2}\alpha)$.
16. Given $\cos 60^\circ = \frac{1}{2}$; find $\sin 30^\circ$ and $\cos 30^\circ$.
17. Given $\cos 135^\circ = -\frac{1}{2}\sqrt{2}$; find $\tan 67\frac{1}{2}^\circ$.
18. Given $\cos 270^\circ = 0$; find $\sin 135^\circ$ and $\cos 135^\circ$.
19. Given $\cos 30^\circ = \frac{1}{2}\sqrt{3}$; find $\sin 15^\circ$ and $\cos 15^\circ$.
20. Given $\cos 2\theta = \frac{1}{5}$; find $\sin\theta$ and $\cos\theta$. *Ans.* ± 0.6325; ± 0.7746.
21. Given $\cos\theta = 0.6$; find $\sin\frac{1}{2}\theta$ and $\cos\frac{1}{2}\theta$. *Ans.* ± 0.4462; ± 0.8946.
22. Given $\cos 8\theta = -\frac{1}{2}\sqrt{2}$; find $\sin 4\theta$ and $\cos 4\theta$. *Ans.* ± 0.9239; ± 0.3827.
23. Given $\sin\theta = -\frac{1}{4}$, θ in the third quadrant; find $\sin\frac{1}{2}\theta$, $\cos\frac{1}{2}\theta$, and $\tan\frac{1}{2}\theta$. *Ans.* 0.9920; -0.1260.
24. Given $\cos\theta = \dfrac{b^2 + c^2 - a^2}{2bc}$, and $2s = a + b + c$; prove that

    (*a*) $\sin^2\frac{1}{2}\theta = \dfrac{(s - b)(s - c)}{bc}$;

    (*b*) $\cos^2\frac{1}{2}\theta = \dfrac{s(s - a)}{bc}$;

    (*c*) $\tan^2\frac{1}{2}\theta = \dfrac{(s - b)(s - c)}{s(s - a)}$.

Find the value of the following, using the principal values of the angles:

25. $\cos(\frac{1}{2}\tan^{-1}\frac{16}{63})$. *Ans.* 0.9923.
26. $\sin(\frac{1}{2}\cot^{-1}\frac{3}{4})$. *Ans.* 0.4472.
27. $\tan\left[\frac{1}{2}\sin^{-1}\left(\frac{2\sqrt{x}}{1 + x}\right)\right]$. *Ans.* $\sqrt{x}$, $0 < x < 1$; $\frac{1}{\sqrt{x}}$, $x > 1$.
28. $\sin(\pi + \frac{1}{2}\sin^{-1}\frac{7}{25})$. *Ans.* -0.14142.
29. $\tan(60^\circ + \frac{1}{2}\sec^{-1}\frac{5}{4})$. *Ans.* 4.8867.
30. $\sin(120^\circ + \frac{1}{2}\sin^{-1}\frac{1}{3})$. *Ans.* 0.76901.
31. $\cos(90^\circ - \frac{1}{2}\sin^{-1}\frac{1}{4})$. *Ans.* 0.12597.

32. In any right triangle prove $\tan\frac{1}{2}A = \frac{c - b}{a}$, and $\sin\frac{1}{2}A = \sqrt{\frac{c - b}{2c}}$.
33. Prove that $\tan\frac{1}{2}\theta$ and $\cot\frac{1}{2}\theta$ are the roots of $x^2 - 2x\csc\theta + 1 = 0$.

Prove the following identities:

34. $1 + \tan\beta\tan\dfrac{\beta}{2} = \dfrac{1}{\cos\beta}$.
35. $\tan\left(\dfrac{\alpha + \beta}{2}\right)\cot\left(\dfrac{\alpha - \beta}{2}\right) = \dfrac{(\sin\alpha + \sin\beta)^2}{\sin^2\alpha - \sin^2\beta}$.
36. $\cot\beta = \dfrac{1}{2}\left(\cot\dfrac{\beta}{2} - \tan\dfrac{\beta}{2}\right)$.
37. In [24], show why the sign $\pm$ is not necessary before $\dfrac{1 - \cos\theta}{\sin\theta}$ and $\dfrac{\sin\theta}{1 + \cos\theta}$.

86. To express the sum and difference of two like trigonometric functions as a product.—In this article the following formulas are proved:

[25] $\quad \sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta)$.

[26] $\quad \sin\alpha - \sin\beta = 2\cos\frac{1}{2}(\alpha + \beta)\sin\frac{1}{2}(\alpha - \beta)$.

[27] $\quad \cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta)$.

[28] $\quad \cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha + \beta)\sin\frac{1}{2}(\alpha - \beta)$.

The object of these four relations is to express sums and differences of functions as products. In this manner formulas can be made suitable for logarithmic computations.

Proof of [25] *and* [26].—Let $\alpha = x + y$ and $\beta = x - y$.

Solving simultaneously for x and y,

$$x = \tfrac{1}{2}(\alpha + \beta) \text{ and } y = \tfrac{1}{2}(\alpha - \beta).$$

By [13], $\sin\alpha = \sin(x + y) = \sin x\cos y + \cos x\sin y$. $\quad$ (*a*)

By [15], $\sin\beta = \sin(x - y) = \sin x\cos y - \cos x\sin y$. $\quad$ (*b*)

By adding (*a*) and (*b*), $\sin\alpha + \sin\beta = 2\sin x\cos y$.

Substituting the values of x and y, we have

$$\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\alpha - \beta).$$

Subtracting (*b*) from (*a*) and substituting for x and y,

$$\sin\alpha - \sin\beta = 2\cos x\sin y = 2\cos\tfrac{1}{2}(\alpha + \beta)\sin\tfrac{1}{2}(\alpha - \beta).$$

Proof of [27] *and* [28].—

By [14], $\cos\alpha = \cos(x + y) = \cos x\cos y - \sin x\sin y$. $\quad$ (*c*)

By [16], $\cos\beta = \cos(x - y) = \cos x\cos y + \sin x\sin y$. $\quad$ (*d*)

Adding (*c*) and (*d*) and substituting for x and y,

$$\cos\alpha + \cos\beta = 2\cos x\cos y = 2\cos\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\alpha - \beta).$$

Subtracting (d) from (c) and substituting for x and y,

$$\cos\alpha - \cos\beta = -2\sin x \sin y = -2\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\alpha-\beta).$$

Example 1.—Express $\dfrac{\cos 70° - \cos 30°}{\cos 70° + \cos 30°}$ as a product.

Solution.—

By [**28**], $\cos 70° - \cos 30° = -2\sin\tfrac{1}{2}(70° + 30°)\sin\tfrac{1}{2}(70 - 30°)$
$= -2\sin 50°\sin 20°.$

By [**27**], $\cos 70° + \cos 30° = 2\cos\tfrac{1}{2}(70° + 30°)\cos\tfrac{1}{2}(70° - 30°)$
$= 2\cos 50°\cos 20°.$

$$\text{Then}\quad \frac{\cos 70° - \cos 30°}{\cos 70° + \cos 30°} = \frac{-2\sin 50°\sin 20°}{2\cos 50°\cos 20°} = -\tan 50°\tan 20°.$$

Example 2.—Show that the following equality is true by using the tables to compute each side of the equality:

$$\sin 60° + \sin 40° = 2\sin 50°\cos 10°.$$

Solution.—The right-hand member is best computed by logarithms.

$\sin 60° = 0.8660$	Let	$x = 2\sin 50°\cos 10°$
$\sin 40° = 0.6428$		$\log 2 = 0.30103$
$\sin 60° + \sin 40° = 1.5088$		$\log\sin 50° = 9.88425$
		$\log\cos 10° = 9.99335$
		$\log x = 0.17863$
		$x = 1.5088$

The two results are found to agree.

Example 3.—If $\alpha + \beta + \gamma = 180°$, prove the identity:

$$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma.$$

Proof.—

By [**25**], $\sin\alpha + \sin\beta = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta).$

Now $\gamma = 180° - (\alpha + \beta).$

$\therefore \sin\gamma = \sin[180° - (\alpha+\beta)] = \sin(\alpha+\beta)$ (**Art. 48**).

By [**19**], $\sin(\alpha+\beta) = 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha+\beta).$

$\therefore \sin\alpha + \sin\beta + \sin\gamma$
$= 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta) + 2\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha+\beta)$
$= 2\sin\tfrac{1}{2}(\alpha+\beta)[\cos\tfrac{1}{2}(\alpha-\beta) + \cos\tfrac{1}{2}(\alpha+\beta)].$

But $\cos\tfrac{1}{2}(\alpha-\beta) + \cos\tfrac{1}{2}(\alpha+\beta)$
$= 2\cos\tfrac{1}{2}[\tfrac{1}{2}(\alpha-\beta) + \tfrac{1}{2}(\alpha+\beta)]\cos\tfrac{1}{2}[\tfrac{1}{2}(\alpha-\beta) - \tfrac{1}{2}(\alpha+\beta)]$
$= 2\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta.$

$\therefore \sin\alpha + \sin\beta + \sin\gamma = 2\sin\tfrac{1}{2}(\alpha+\beta)2\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta$
$= 4\sin\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta.$

But $\tfrac{1}{2}(\alpha+\beta) = 90° - \tfrac{1}{2}\gamma$, and $\sin\tfrac{1}{2}(\alpha+\beta) = \cos\tfrac{1}{2}\gamma.$

$$\therefore \sin\alpha + \sin\beta + \sin\gamma = 4\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma.$$

EXERCISES

Express the following sums and differences of functions as products. Answer orally.

1. $\sin 70° + \sin 50°$.
2. $\sin 70° - \sin 50°$.
3. $\cos 70° + \cos 50°$.
4. $\cos 70° - \cos 50°$.
5. $\sin 80° + \sin 140°$.
6. $\cos 140° - \cos 70°$.
7. $\sin 3\theta + \sin\theta$.
8. $\cos 5\theta - \cos 7\theta$.
9. $\cos 7\theta - \sin 3\theta$.
10. $\cos 2\alpha - \cos 2\beta$.
11. $\sin(\alpha+\beta) + \sin(\alpha-\beta)$.
12. $\cos(\alpha+\beta) - \cos(\alpha-\beta)$.

Express the following as products and simplify:

13. $\sin 80° - \sin 40°$. *Ans.* $\sin 20°$.
14. $\cos 80° - \cos 40°$. *Ans.* $-\sqrt{3}\sin 20°$.
15. $\cos 40° + \cos 20°$. *Ans.* $\sqrt{3}\cos 10°$.
16. $\sin 40° + \sin 20°$. *Ans.* $\cos 10°$.
17. $\sin 50° + \sin 70°$. *Ans.* $\sqrt{3}\cos 10°$.
18. $\dfrac{\sin 50° - \sin 30°}{\cos 50° + \cos 30°}$. *Ans.* $\tan 10°$.
19. $\dfrac{\sin 70° + \sin 50°}{\cos 70° - \cos 50°}$. *Ans.* $-\cot 10°$.
20. $\dfrac{\cos 50° + \cos 10°}{\sin 50° + \sin 10°}$. *Ans.* $\cot 30°$.
21. $\dfrac{\cos\alpha - \cos\beta}{\sin\alpha + \sin\beta}$. *Ans.* $-\tan\tfrac{1}{2}(\alpha-\beta)$.
22. $\dfrac{\cos 2\theta + \cos\theta}{\sin 2\theta + \sin\theta}$. *Ans.* $\cot\tfrac{3}{2}\theta$.
23. $\cos(60° + \alpha) + \cos(60° - \alpha)$. *Ans.* $\cos\alpha$.
24. $\cos(\alpha + 30°) + \cos(\alpha - 30°)$. *Ans.* $\sqrt{3}\cos\alpha$.
25. $\sin(\alpha + 60°) + \sin(\alpha - 60°)$. *Ans.* $\sin\alpha$.
26. $\tan(\beta - \tfrac{1}{4}\pi) + \cot(\beta + \tfrac{1}{4}\pi)$. *Ans.* 0.
27. Solve $\cos 3\theta + \sin 2\theta - \cos\theta = 0$ for values of $\theta < 360°$.
    Ans. 0°, 30°, 90°, 150°, 180°, 270°.
28. Solve $\sin 3\theta + \sin 2\theta + \sin\theta = 0$ for values of $\theta < 360°$.
    Ans. 0°, 90°, 120°, 180°, 240°, 270°.
29. Solve $\cos 3\theta - \sin 2\theta + \cos\theta = 0$ for values of $\theta < 360°$.
    Ans. 30°, 90°, 150°, 270°.
30. Solve $\sin 5\theta - \sin 3\theta + \sin\theta = 0$ for values of $\theta < 360°$.
    Ans. 0°, 30°, 60°, 120°, 150°, 180°, 210°, 240°, 300° 330°.

If $\alpha + \beta + \gamma = 180°$, prove the identities in the following exercises:

31. $\sin\alpha + \sin\beta - \sin\gamma = 4\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma$.
32. $\cos\alpha + \cos\beta + \cos\gamma = 4\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma + 1$.
33. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -4\cos\alpha\cos\beta\cos\gamma - 1$.
34. $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$.

87. To change the product of functions of angles into the sum of functions.—From **Art. 78,**

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \qquad (a)$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta. \qquad (b)$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \qquad (c)$$
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \qquad (d)$$

Adding (a) and (b), $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha\cos\beta$.

[29] $\therefore \sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha + \beta) + \frac{1}{2}\sin(\alpha - \beta)$.

Subtracting (b) from (a),

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta.$$

[30] $\therefore \cos\alpha\sin\beta = \frac{1}{2}\sin(\alpha + \beta) - \frac{1}{2}\sin(\alpha - \beta)$.

Adding (c) and (d), $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$.

[31] $\therefore \cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha + \beta) + \frac{1}{2}\cos(\alpha - \beta)$.

Subtracting (d) from (c),

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin\alpha\sin\beta.$$

[32] $\therefore \sin\alpha\sin\beta = -\frac{1}{2}\cos(\alpha + \beta) + \frac{1}{2}\cos(\alpha - \beta)$.

Example 1.—Prove that $\sin 4\theta\cos 2\theta = \frac{1}{2}\sin 6\theta + \frac{1}{2}\sin 2\theta$.

Proof.—Applying **[29]**, where $\alpha = 4\theta$ and $\beta = 2\theta$,

$$\sin 4\theta\cos 2\theta = \tfrac{1}{2}\sin(4\theta + 2\theta) + \tfrac{1}{2}\sin(4\theta - 2\theta)$$
$$= \tfrac{1}{2}\sin 6\theta + \tfrac{1}{2}\sin 2\theta.$$

It is often desirable to express the products and powers of sines and cosines as sums of functions that involve multiples of the angle. The formulas of this article and of **Art. 84** can be used for this purpose (see also **Art. 127**).

Example 2.—Prove that $\sin^2\theta\cos\theta = -\frac{1}{4}\cos 3\theta + \frac{1}{4}\cos\theta$.

Proof.—By **[19]**, $\sin 2\theta = 2\sin\theta\cos\theta$.

Then $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$.

Therefore, $\sin^2\theta\cos\theta = \sin\theta(\sin\theta\cos\theta)$

$$= \tfrac{1}{2}\sin 2\theta\sin\theta$$

By **[32]**, $= \frac{1}{2}[-\frac{1}{2}\cos(2\theta + \theta) + \frac{1}{2}\cos(2\theta - \theta)]$

$$= -\tfrac{1}{4}\cos 3\theta + \tfrac{1}{4}\cos\theta.$$

Example 3.—Prove that $\cos^5\theta = \frac{1}{16}(10\cos\theta + 5\cos 3\theta + \cos 5\theta)$.

Proof.—$\cos^5\theta = (\cos^2\theta)^2\cos\theta = \left(\dfrac{1 + \cos 2\theta}{2}\right)^2\cos\theta$ By **[23]**.

$$= \tfrac{1}{4}(1 + 2\cos 2\theta + \cos^2 2\theta)\cos\theta$$

$$= \tfrac{1}{4}\left(1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\right)\cos\theta \quad \text{By [23].}$$
$$= \tfrac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)\cos\theta$$
$$= \tfrac{1}{8}(3\cos\theta + 4\cos 2\theta\cos\theta + \cos 4\theta\cos\theta)$$
$$= \tfrac{1}{8}(3\cos\theta + 2\cos 3\theta + 2\cos\theta + \tfrac{1}{2}\cos 5\theta + \tfrac{1}{2}\cos 3\theta)$$
$$= \tfrac{1}{16}(10\cos\theta + 5\cos 3\theta + \cos 5\theta).$$

EXERCISES

Apply the formulas of this article to the following. Answer orally.

1. $\sin 50^\circ\sin 40^\circ$.
2. $\sin 40^\circ\cos 20^\circ$.
3. $\cos 20^\circ\sin 50^\circ$.
4. $\sin 70^\circ\cos 10^\circ$.
5. $\sin 6\phi\cos 4\phi$.
6. $\cos 4\phi\sin 8\phi$.
7. $\sin 20\phi\sin 12\phi$.
8. $\cos 10\phi\cos 16\phi$.

Prove the following identities:

9. $4\cos 2\phi\cos 4\phi\cos 6\phi = 1 + \cos 4\phi + \cos 8\phi + \cos 12\phi$.
10. $4\sin 2\phi\sin 4\phi\sin 6\phi = \sin 4\phi + \sin 8\phi - \sin 12\phi$.
11. $\dfrac{\sin 6\phi\cos 3\phi - \sin 8\phi\cos\phi}{\sin 4\phi\sin 3\phi - \cos 2\phi\cos\phi} = \tan 2\phi$.
12. $\sin 3\phi\sin\phi + \sin^2\phi + \cos 3\phi\cos\phi - \cos^2\phi = 0$.
13. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + \cos\frac{3\pi}{13} + \cos\frac{5\pi}{13} = 0$.
14. $\sin 160^\circ\sin 120^\circ\sin 80^\circ\sin 40^\circ = \frac{3}{16}$.
15. $\cos 160^\circ\cos 120^\circ\cos 80^\circ\cos 40^\circ = \frac{1}{16}$.
16. $\cos^2\phi\sin^2\phi = \frac{1}{8}(1 - \cos 4\phi)$.
17. $\sin^3\phi\cos^3\phi = \frac{1}{32}(3\sin 2\phi - \sin 6\phi)$.
18. $\sin^2\phi\cos^3\phi = \frac{1}{8}(\cos\phi - \frac{1}{2}\cos 3\phi - \frac{1}{2}\cos 5\phi)$.
19. $\sin^3\phi = \frac{1}{4}(3\sin\phi - \sin 3\phi)$.
20. $\cos^3\phi = \frac{1}{4}(3\cos\phi + \cos 3\phi)$.
21. $\sin^m\phi\cos^n\phi = \frac{1}{32}(2 - \cos 2\phi - 2\cos 4\phi + \cos 6\phi)$, if $m = 4$, $n = 2$.
22. Work out other combinations of m and n as illustrated in **[21]**.

88. Important trigonometric series.—The trigonometric series given in this article and the following exercises are important, especially in certain problems in electricity. In the series, α and β are angles and n is an integer equal to the number of terms in a series. The two fundamental series with their sums are:

(1) $$\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \cdots \sin[\alpha + (n - 1)\beta] = \frac{\sin[\alpha + \frac{1}{2}(n - 1)\beta]\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}.$$

(2) $$\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \cdots \cos[\alpha + (n - 1)\beta] = \frac{\cos[\alpha + \frac{1}{2}(n - 1)\beta]\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}.$$

In both (1) and (2) $\sin\frac{1}{2}\beta \neq 0$.

Proof of (1).—Let S_n = the sum of n terms of the series.

Multiplying each term by $2 \sin \frac{1}{2}\beta$ and applying **[32]** to each product, we have the following equations, one equation resulting from each term:

$$2 \sin \alpha \sin \tfrac{1}{2}\beta = \cos (\alpha - \tfrac{1}{2}\beta) - \cos (\alpha + \tfrac{1}{2}\beta).$$
$$2 \sin (\alpha + \beta) \sin \tfrac{1}{2}\beta = \cos (\alpha + \tfrac{1}{2}\beta) - \cos (\alpha + \tfrac{3}{2}\beta).$$
$$2 \sin (\alpha + \beta) \sin \tfrac{1}{2}\beta = \cos (\alpha + \tfrac{3}{2}\beta) - \cos (\alpha + \tfrac{5}{2}\beta).$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$2 \sin [\alpha + (n-1)\beta] \sin \tfrac{1}{2}\beta = \cos \left(\alpha + \frac{2n-3}{2}\beta\right) - \cos \left(\alpha + \frac{2n-1}{2}\beta\right).$$

Adding these and noting that the sum of the first members is $S_n \cdot 2 \sin \frac{1}{2}\beta$,

$$S_n \cdot 2 \sin \tfrac{1}{2}\beta = \cos (\alpha - \tfrac{1}{2}\beta) - \cos \left(\alpha + \frac{2n-1}{2}\beta\right).$$

Applying **[28]** to the second member of this,

$$S_n \cdot 2 \sin \tfrac{1}{2}\beta = 2 \sin [\alpha + \tfrac{1}{2}(n-1)\beta \sin \tfrac{1}{2}n\beta.$$
$$\therefore S_n = \frac{\sin [\alpha + \frac{1}{2}(n-1)\beta] \sin \frac{1}{2}n\beta}{\sin \frac{1}{2}\beta},$$

which is true if $\sin \frac{1}{2}\beta \neq 0$.

The proof of (2) can be carried out in an exactly similar manner multiplying by $2 \sin \frac{1}{2}\beta$ and applying **[30]**.

EXERCISES

Prove the following, where, in each, the denominator of the sum must be different from zero:

1. $\sin \alpha + \sin 2\alpha + \sin 3\alpha + \cdots \sin n\alpha = \dfrac{\sin \frac{1}{2}(n+1)\alpha \sin \frac{1}{2}n\alpha}{\sin \frac{1}{2}\alpha}.$

Suggestion.—In (1) put $\beta = \alpha$.

2. $\cos \alpha + \cos 2\alpha + \cos 3\alpha + \cdots \cos n\alpha = \dfrac{\cos \frac{1}{2}(n+1)\alpha \sin \frac{1}{2}n\alpha}{\sin \frac{1}{2}\alpha}.$

3. $\sin \alpha + \sin 3\alpha + \sin 5\alpha + \cdots \sin (2n-1)\alpha = \dfrac{\sin^2 n\alpha}{\sin \alpha}.$

4. $\cos \alpha + \cos 3\alpha + \cos 5\alpha + \cdots \cos (2n-1)\alpha = \dfrac{\sin 2n\alpha}{2 \sin \alpha}.$

5. $\sin \alpha + \sin\left(\alpha + \dfrac{2\pi}{n}\right) + \sin\left(\alpha + \dfrac{4\pi}{n}\right) + \cdots \sin\left[\alpha + \dfrac{2(n-1)\pi}{n}\right] = 0.$

6. $\cos \alpha + \cos\left(\alpha + \dfrac{2\pi}{n}\right) + \cos\left(\alpha + \dfrac{4\pi}{n}\right) + \cdots \cos\left[\alpha + \dfrac{2(n-1)\pi}{n}\right] = 0.$

GENERAL EXERCISES

Prove the following identities by transforming the first member into the second:

1. $\cos 306° + \cos 234° + \cos 162° + \cos 18° = 0.$

2. $\dfrac{\tan x + \tan y}{\tan x - \tan y} = \dfrac{\sin (x+y)}{\sin (x-y)}.$

3. $\dfrac{1 - \tan x \tan y}{1 + \tan x \tan y} = \dfrac{\cos (x+y)}{\cos (x-y)}.$

4. $\tan (45° \pm y) = \dfrac{1 \pm \tan y}{1 \mp \tan y}.$

5. $\cot (45° \pm y) = \dfrac{\cot y \mp 1}{\cot y \pm 1}.$

6. $\cot x \pm \tan y = \dfrac{\cos (x \mp y)}{\sin x \cos y}.$

7. $\dfrac{1}{\sin 10°} - \dfrac{\sqrt{3}}{\cos 10°} = 4.$

8. $\frac{1}{2}[\cot \theta - \tan \theta] = \cot 2\theta.$

9. $4 \cot^2 2\alpha + 2 = \tan^2 \alpha + \cot^2 \alpha.$

10. $\dfrac{(2 - \sec^2 \alpha)(2 - \csc^2 \alpha)}{\sec^2 \alpha \csc^2 \alpha} = -\cos^2 2\alpha.$

11. $2\alpha + 4 \tan^{-1} (\sec \alpha - \tan \alpha) = \pi.$

12. $2\alpha + 4 \cot^{-1} (\sec \alpha + \tan \alpha) = \pi.$

13. $\dfrac{\tan \alpha + \tan \beta}{\cot \alpha + \cot \beta} = \tan \alpha \tan \beta.$

14. $\dfrac{\tan \alpha - \tan \beta}{\cot \alpha - \cot \beta} = -\tan \alpha \tan \beta.$

15. $\tan (45° + \theta) - \tan (45° - \theta) = 2 \tan 2\theta.$

16. $\dfrac{\sec^2 (\frac{1}{4}\pi + \frac{1}{2}\alpha)}{2 \tan (\frac{1}{4}\pi + \frac{1}{2}\alpha)} = \sec \alpha.$

17. $\dfrac{\sec^2 \frac{1}{2}\left(\frac{\pi}{2} + \alpha\right)}{\tan \left(\frac{\pi}{4} + \frac{1}{2}\alpha\right)} = \dfrac{2}{\cos \alpha}.$

18. $\dfrac{\tan \theta}{2 \cos \theta} + \dfrac{1}{4}\left(1 + \tan^2 \dfrac{1}{2}\theta\right)\cot \dfrac{1}{2}\theta = \dfrac{\sec^2 \theta}{2 \sin \theta}.$

19. $\dfrac{1 + \sin \theta - \cos \theta}{1 + \sin \theta + \cos \theta} = \tan \dfrac{1}{2}\theta.$

20. $\dfrac{2 \sin \theta - \sin 2\theta}{2 \sin \theta + \sin 2\theta} = \tan^2 \dfrac{1}{2}\theta.$

21. $8 \sin^4 \frac{1}{2}\theta - 8 \sin^2 \frac{1}{2}\theta + 1 = \cos 2\theta.$

22. $\sin (\alpha + \beta) \sin (\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta = \cos^2 \beta - \cos^2 \alpha.$

23. $\cos (\alpha + \beta) \cos (\alpha - \beta) = \cos^2 \alpha - \sin^2 \beta = \cos^2 \beta - \sin^2 \alpha.$

24. $\dfrac{\cos (\alpha - \beta)}{\cos (\alpha + \beta)} = \dfrac{1 + \tan \alpha \tan \beta}{1 - \tan \alpha \tan \beta}.$

25. $\dfrac{4 \sin \alpha \sin (60° + \alpha)}{\csc (60° - \alpha)} = \sin 3\alpha.$

26. $\dfrac{\sin \alpha - \sqrt{1 + \sin 2\alpha}}{\cos \alpha - \sqrt{1 + \sin 2\alpha}} = \cot \alpha.$

27. $\dfrac{3\sec^2\theta}{3\tan\theta + 1} - \dfrac{\sec^2\theta}{\tan\theta + 3} = \dfrac{8}{3 + 5\sin 2\theta}.$

In the two following exercises α, β, and γ are the angles of any triangle:

28. $\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin\gamma.$

29. $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma.$

30. $\dfrac{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10}{\cos 5\theta + 5\cos 3\theta + 10\cos\theta} = 2\cos\theta.$

Suggestion.—Write the numerator in form $\cos 6\theta + \cos 4\theta + 5\cos 4\theta + 5\cos 2\theta + 10\cos 2\theta + 10$.

Apply [27] and [23], and this becomes $2\cos 5\theta\cos\theta + 10\cos 3\theta\cos\theta + 20\cos^2\theta = 2\cos\theta(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$.

31. $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$

32. $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$

33. $\tan\{\sin^{-1}[\cos(\tan^{-1}x)]\} = \dfrac{1}{x}.$

34. $3\tan^{-1}a = \tan^{-1}\dfrac{3a - a^3}{1 - 3a^2}.$

35. $\tan^{-1}\dfrac{1}{1 - 2a + 4a^2} + \tan^{-1}\dfrac{1}{1 + 2a + 4a^2} = \tan^{-1}\dfrac{1}{2a^2}.$

36. $\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{8} = \dfrac{\pi}{4}.$

37. $\tan^{-1}\dfrac{bx}{a\sqrt{a^2 - b^2 - x^2}} = \sin^{-1}\dfrac{bx}{\sqrt{a^2 - x^2}\sqrt{a^2 - b^2}}.$

38. $2\tan^{-1}(0.2) + \tan^{-1}\left(\dfrac{1}{7}\right) + 2\tan^{-1}\left(\dfrac{1}{8}\right) = \dfrac{\pi}{4}.$

39. $\tan^{-1}\dfrac{1}{1 + a} + \tan^{-1}\dfrac{1}{1 - a} + \tan^{-1}\dfrac{2}{a^2} = n\pi.$

40. $\tan^{-1}\dfrac{2x - y}{y\sqrt{3}} + \tan^{-1}\dfrac{2y - x}{x\sqrt{3}} = \tan^{-1}\sqrt{3}.$

Solve the following equations for values of the angle less than 360°:

41. $\sin 2\theta + 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta = 0.$ *Ans.* 0°, 120°, 180°, 240°.

42. $\sin\theta\cos\theta - \sin\frac{1}{2}\theta\cos\frac{1}{2}\theta = 0.$ *Ans.* 0°, 60°, 180°, 300°.

43. $\sin 2\theta + \cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta = 0.$ *Ans.* 90°, 210°, 270°, 330°.

44. $\sin^2 2\theta - \cos^2 2\theta = 0.$ *Ans.* $22\frac{1}{2}°$, $67\frac{1}{2}°$, $112\frac{1}{2}°$, etc.

45. $\cos 2\theta - \sin^2\theta = 0.$
Ans. 35° 15.9′; 144° 44.1′; 215° 15.9′; 324° 44.1′.

46. $\sin 2\theta + \cos 2\theta + \sin\theta = 1.$ *Ans.* 0°, 65° 42.3′, 180°, 204° 17.7′.

47. $\sin 4\theta + \sin 2\theta + \cos\theta = 0.$
Ans. 70°, 90°, 110°, 190°, 230°, 270°, 310°, 350°.

48. $\theta = \sin^{-1}(\cos 2\theta) - 60°.$ *Ans.* 10°, 130°, 250°, 330°.

49. $\tan(80° - \frac{1}{2}\theta) = \cot\frac{2}{3}\theta.$ *Ans.* 60°.

50. $\cot(40° + \theta) = \tan\frac{1}{2}\theta.$ *Ans.* $33\frac{1}{3}°$, $153\frac{1}{3}°$, $273\frac{1}{3}°$.

51. $(\tan x + \sin x)(\tan x - \sin x) - \cos^2 x = 2(\sec^2 x - 3)\cot 2x.$
Ans. 116° 33.9′, 296° 33.9′.

52. $1 - \sin^2 x + \cos x = \sin 2x.$ *Ans.* 53° 7.8′, 90°, 180°, 270°.

53. $(4\cos^2\theta + 1)\tan^2\theta = 6.$ *Ans.* 60°, 120°, 240°, 300°.

54. $\tan\theta - 1 + \tan(\theta - 45°) - \left(\dfrac{1 + \cos 2\theta}{2}\right) = \sin^2\theta.$
Ans. 60°, 120°, 240°, 300°.

Solve the following equations, giving the values in general measure:

55. $\sin 2\theta + 1 = \tan(\theta + 45°).$ *Ans.* $n\pi + \frac{3}{4}\pi$; $n\pi$.

56. $\cos 5\theta + \cos 3\theta + \cos\theta = 0.$ *Ans.* $\frac{1}{6}(2n + 1)\pi$; $\frac{1}{3}(3n \pm 1)\pi$.

57. $\cos 7\theta - \cos\theta = 0.$ *Ans.* $\frac{1}{4}n\pi$; $\frac{1}{3}n\pi$.

58. $\sin 5\theta + \sin 3\theta = 0.$ *Ans.* $\frac{1}{4}n\pi$; $(2n + 1)\frac{1}{2}\pi$.

59. $\cos 7\theta + \cos 5\theta + \cos 3\theta = 0.$
Ans. $(2n + 1)\frac{1}{10}\pi$; $(2n + 1)\frac{1}{2}\pi \pm \frac{1}{6}\pi$.

60. $\tan(\frac{1}{4}\pi + \theta) + \tan(\frac{1}{4}\pi - \theta) = 4.$ *Ans.* $n\pi \pm \frac{1}{6}\pi$.

Solve the following equations for x:

61. $\sin^{-1}2x - \sin^{-1}\sqrt{3}x = \sin^{-1}x.$

Solution.—Taking the sine of both members of the equation,

$$2x\sqrt{1 - 3x^2} - \sqrt{1 - 4x^2}\cdot\sqrt{3}x = x.$$

Transposing and factoring, $x(2\sqrt{1 - 3x^2} - \sqrt{3}\sqrt{1 - 4x^2} - 1) = 0.$

Equating each factor to 0, $x = 0$, $2\sqrt{1 - 3x^2} - \sqrt{3}\sqrt{1 - 4x^2} - 1 = 0.$

Solving these equations, $x = 0$, and $x = \pm\frac{1}{2}$.

All of these values satisfy the equation when principal values of the angles are used.

62. $\tan(\cos^{-1}x) = \sin\left(\cot^{-1}\dfrac{1}{2}\right).$ *Ans.* $\dfrac{\sqrt{5}}{3}$.

63. $\sin^{-1}\left(\dfrac{5}{x}\right) + \sin^{-1}\left(\dfrac{12}{x}\right) = \dfrac{\pi}{2}.$ *Ans.* 0; 13.

64. $\tan^{-1}2x + \tan^{-1}3x = \frac{1}{4}\pi.$ *Ans.* $\frac{1}{6}$.

65. $\tan^{-1}x + 2\cot^{-1}x = 135°.$ *Ans.* 1.

66. $\sin\left(\dfrac{1}{2}\pi - 2\tan^{-1}\sqrt{\dfrac{1 - x}{1 + x}}\right) = a.$ *Ans.* a.

67. $\cos^{-1}\left(\dfrac{x^2 - 1}{x^2 + 1}\right) + \tan^{-1}\left(\dfrac{2x}{x^2 - 1}\right) = \dfrac{2}{3}\pi.$ *Ans.* $\sqrt{3}$.

68. $\sin^{-1}(\frac{1}{4}x) + \sin^{-1}(\frac{1}{2}x) = \frac{1}{4}\pi.$ *Ans.* $\sqrt{1.022}$.

69. $\cot^{-1}(x - 1) - \cot^{-1}(x + 1) = \frac{1}{12}\pi.$ *Ans.* $\pm(1 + \sqrt{3})$.

70. $\cot^{-1}(x) - \cot^{-1}(x + 2) = 15°.$ *Ans.* $\sqrt{3}$, $-\sqrt{3} - 2$.

71. Given $a\sin\theta + b\cos\theta = c$, and $a\cos\theta - b\sin\theta = d$; eliminate θ.
Ans. $a^2 + b^2 = c^2 + d^2$.

72. Eliminate φ from the following equations:

$x = a\cos\varphi$, $y = b\sin\varphi$. *Ans.* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

73. Eliminate φ from the following equations:

$a\cos\varphi + b\sin\varphi = c$, $b\cos\varphi + c\sin\varphi = a$.
Ans. $(bc - a^2)^2 + (c^2 - ab)^2 = (ac - b^2)^2$.

Suggestion.—Solve for $\sin\varphi$ and $\cos\varphi$, then square and add.

74. Given $P\cos\theta - W\sin\alpha = 0$ and $R + P\sin\theta - W\cos\alpha = 0$; solve for R. *Ans.* $R = \dfrac{W\cos(\alpha + \theta)}{\cos\theta}$.

75. Eliminate θ from the following equations:

$\csc\theta - \sin\theta = a$, $\sec\theta - \cos\theta = b$. *Ans.* $a^{\frac{2}{3}}b^{\frac{2}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}}) = 1$.

Suggestion.—From the first $a = \dfrac{\cos^2\theta}{\sin\theta}$. From the second $b = \dfrac{\sin^2\theta}{\cos\theta}$.
Find $a^{\frac{2}{3}}b^{\frac{2}{3}}(a^{\frac{2}{3}} + b^{\frac{2}{3}})$.

76. Eliminate θ and φ from the following equations:

$$\sin\theta + \sin\varphi = a.$$
$$\cos\theta + \cos\varphi = b.$$
$$\cos(\theta - \varphi) = c. \qquad \textit{Ans. } a^2 + b^2 - 2c = 2.$$

77. Eliminate θ from the following equations:

$$x\sin\theta - y\cos\theta = \sqrt{x^2 + y^2}.$$
$$\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2} = \frac{1}{x^2 + y^2}. \qquad \textit{Ans. } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Suggestion.—Square the first equation and collect the terms in x^2 and y^2. This gives the square of $x\cos\theta + y\sin\theta = 0$. Then $\tan\theta = -\frac{x}{y}$. From this find $\sin\theta$ and $\cos\theta$ and substitute in the second equation.

78. Show that if $\tan(\theta - \alpha)\tan(\theta - \beta) = \tan^2\theta$, $\theta = \frac{1}{2}\tan^{-1}\frac{2\sin\alpha\sin\beta}{\sin(\alpha + \beta)}$

Suggestion.—Write the equation in the form

$$\frac{\sin(\theta - \alpha)\sin(\theta - \beta)}{\cos(\theta - \alpha)\cos(\theta - \beta)} = \frac{\sin^2\theta}{\cos^2\theta}.$$

Applying [**32**] and [**31**],

$$\frac{-\frac{1}{2}\cos(2\theta - \alpha - \beta) + \frac{1}{2}\cos(\alpha - \beta)}{\frac{1}{2}\cos(2\theta - \alpha - \beta) + \frac{1}{2}\cos(\alpha - \beta)} = \frac{\sin^2\theta}{\cos^2\theta}.$$

Clearing of fractions and uniting, or by composition and division,

$$\cos[2\theta - (\alpha + \beta)] = \cos 2\theta\cos(\alpha - \beta).$$

Applying [**16**],

$$\cos 2\theta\cos(\alpha + \beta) + \sin 2\theta\sin(\alpha + \beta) = \cos 2\theta\cos(\alpha - \beta).$$

Then $$\tan 2\theta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{\sin(\alpha + \beta)}.$$

Applying [**28**], $\tan 2\theta = \frac{2\sin\alpha\sin\beta}{\sin(\alpha + \beta)}.$

79. Given $aI^2\cos\theta + bI\cos\theta = c\theta$, and $an^2I^2 - bnI = c\frac{\theta}{\cos\theta}$; prove that $anI^2 = c\frac{\theta}{\cos\theta}$.

Suggestion.—Multiply the first by $\frac{n}{\cos\theta}$ and add the second.

80. Show that if $\tan\varphi = \frac{B}{A}$,

$$A\sin\theta + B\cos\theta = \sqrt{A^2 + B^2}\sin\left(\theta + \tan^{-1}\frac{B}{A}\right).$$

81. Given $I = W\sin\theta$, and $P\cos\theta = W\sin\theta$; show that

$$\frac{1}{I^2} = \frac{1}{P^2} + \frac{1}{W^2}.$$

82. Solve the following equations for x and y:

$$x\cos\theta + y\sin\theta = a. \qquad \textit{Ans. } x = a\cos\theta + b\sin\theta.$$
$$x\sin\theta - y\cos\theta = b. \qquad y = a\sin\theta - b\cos\theta.$$

83. Show that
$$\frac{k\sin\theta}{\sin\cot^{-1}\left(-\frac{2\cot\theta + RC\omega}{\omega\sqrt{4LC - R^2C^2}}\right)} = \frac{2k}{\omega\sqrt{4LC - R^2C^2}}\sqrt{(LC\omega^2 - 1)\sin^2\theta + \frac{1}{2}RC\omega\sin 2\theta + 1}.$$

CHAPTER IX

OBLIQUE TRIANGLES

89. General statement.—In the present chapter methods for solving any triangle will be developed. As pointed out in **Art. 38,** it is possible to solve a triangle whenever there are enough parts given so that the triangle can be constructed. The constructions and, likewise, the solutions fall under four cases, depending upon the parts given and required:

CASE I. *Given one side and two angles.*

CASE II. *Given two sides and an angle opposite one of them.*

CASE III. *Given two sides and the included angle.*

CASE IV. *Given the three sides.*

Since there are *six* parts to a triangle, and, in each of the four cases, *three* parts are given, then, in general, there are *three* unknown parts to be found in solving a triangle. Also, since

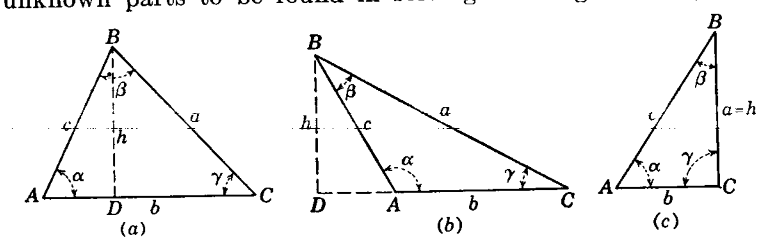


FIG. 86.

three independent equations are necessary and sufficient to determine three unknowns, it is necessary to have three independent formulas or relations connecting the parts of a triangle.

These three relations are:

(1) *The sum of the angles of a triangle is equal to* 180°.

(2) *The sine theorem, or the law of sines.*

(3) *The cosine theorem, or the law of cosines.*

For greater convenience in carrying out the numerical work of the solutions, various other relations are derived from the formulas growing out of the sine theorem and cosine theorem.

90. Law of sines.—*In any triangle the sides are proportional to the sines of the opposite angles.*

First Proof.—In Fig. 86, let ABC be any triangle, and let h be the perpendicular from B to AC. The following applies to each of the triangles (*a*), (*b*), and (*c*); but note that in triangle (*c*) $h = a$.

(1) $$\sin\alpha = \frac{h}{c}.$$

(2) $$\sin\gamma = \frac{h}{a}.$$

Dividing (1) by (2), there results

(3) $$\frac{\sin\alpha}{\sin\gamma} = \frac{a}{c}, \text{ or } \frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}.$$

Similarly, drawing perpendiculars from A to CB,

(4) $$\frac{\sin\beta}{\sin\gamma} = \frac{b}{c}, \text{ or } \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Hence, uniting (3) and (4), there results

[33] $$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

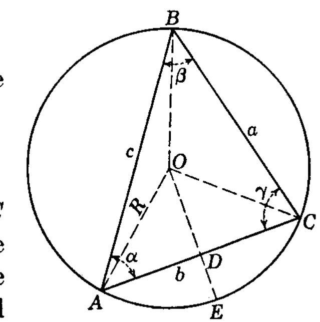


FIG. 87.

Second Proof.—In Fig. 87, let ABC be any triangle. About the triangle circumscribe a circle. Let O be the center. Draw the radii OA, OB, and OC. Draw OD perpendicular to AC.

Then $\angle AOD = \beta$ or is the supplement of β.

In triangle AOD,

$$AD = AO \sin \angle AOD.$$

$$\therefore \tfrac{1}{2}b = R \sin\beta.$$

In a similar manner,

$$\tfrac{1}{2}c = R\sin\gamma,$$

and $$\tfrac{1}{2}a = R\sin\alpha.$$

These give $$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

COROLLARY.—*The constant ratio of a side of the triangle to the sine of the opposite angle is equal to the diameter of the circumscribed circle.*

EXERCISES

1. Derive the proportion $\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$.

2. Derive $2R = \frac{a}{\sin\alpha}$, also $2R = \frac{c}{\sin\gamma}$.

3. Solve $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$ for each part involved.

4. What does the law of sines become when one of the angles, say γ, is a right angle?

91. Law of cosines.—*In any triangle the square of a side equals the sum of the squares of the other sides minus twice the product of these sides by the cosine of their included angle.*

Proof.—In each triangle of Fig. 86,

$$a^2 = h^2 + \overline{DC}^2.$$

But $$h^2 = c^2 - \overline{AD}^2 \text{ and } \overline{DC}^2 = (b - AD)^2.$$

(Notice that in (a) AD is positive, in (b) negative, and in (c) DC is zero because D falls on C.)

$$\begin{aligned}\therefore a^2 &= c^2 - \overline{AD}^2 + (b - AD)^2\\ &= c^2 - \overline{AD}^2 + b^2 - 2b \cdot AD + \overline{AD}^2\\ &= c^2 + b^2 - 2b \cdot AD.\end{aligned}$$

But $AD = c\cos\alpha$,

[34_1] $$\therefore \mathbf{a^2 = b^2 + c^2 - 2bc\cos\alpha}.$$

By similar proofs or by cyclic changes we have,

[34_2] $$\mathbf{b^2 = a^2 + c^2 - 2ac\cos\beta}.$$

[34_3] $$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

The cyclic changes of letters is carried out as follows:

a changes to b.	α changes to β.
b changes to c.	β changes to γ.
c changes to a.	γ changes to α.

EXERCISES

1. Are the formulas [34_1], [34_2], and [34_3] adapted to solving by logarithms?
2. Derive [34_2] and [34_3] independently.
3. Solve each of the three formulas for the angles in terms of the sides.
4. Solve $a^2 = b^2 + c^2 - 2bc\cos\alpha$ for b.

Ans. $b = c\cos\alpha \pm \sqrt{a^2 - c^2\sin^2\alpha}$.

5. What does the law of cosines become when one of the angles, say γ, is a right angle?

92. Case I. The solution of a triangle when one side and two angles are given.—In this case, it is evident that the third angle can always be found from the equation

$$\alpha + \beta + \gamma = 180°.$$

The sides can then be found by using the relations stated in the law of sines, namely,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta},\ \frac{a}{\sin\alpha} = \frac{c}{\sin\gamma},\ \text{and}\ \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

In each of these there are four parts of the triangle involved; therefore, if any three of these parts are known, the fourth can be found. That is, any one of these equations can be solved for any one of the four parts.

Any formula not used in the solution of a triangle may be used in checking the work. One should be certain, however, that the check formula was not involved in the formulas used in solving. For instance, when two equations from the law of sines have been used to find the parts of a triangle, the third equation from the law of sines cannot be used as a check, since the first two equations involve the third.

Two particularly convenient equations for checking the accuracy of the numerical solutions of triangles are the following, known as **Mollweide's equations,** from the German astronomer Karl Brandon Mollweide (1774–1825), though why they should bear his name is not clear, since they were known long before his time, and were used by Newton and others.

(1) $$\frac{a-b}{c} = \frac{\sin\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}\gamma}.$$

(2) $$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}\gamma}.$$

The certainty of these equations as a check lies in the fact that each contains all six parts of a triangle.

Mollweide's equations are readily derived from the law of sines.

Derivation of (1).—From the law of sines,

$$a = \frac{c\sin\alpha}{\sin\gamma}, \qquad b = \frac{c\sin\beta}{\sin\gamma}.$$

Then $$\frac{a-b}{c} = \frac{\frac{c\sin\alpha}{\sin\gamma} - \frac{c\sin\beta}{\sin\gamma}}{c} = \frac{\sin\alpha - \sin\beta}{\sin\gamma}$$

By **[19]**, **[26]**, $$= \frac{2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{2\sin\frac{1}{2}\gamma\cos\frac{1}{2}\gamma}.$$

Now $$\begin{aligned}\sin\tfrac{1}{2}\gamma &= \sin\tfrac{1}{2}[180° - (\alpha+\beta)] = \sin[90° - \tfrac{1}{2}(\alpha+\beta)]\\ &= \cos\tfrac{1}{2}(\alpha+\beta).\end{aligned}$$

$$\therefore \frac{a-b}{c} = \frac{\sin \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}\gamma}.$$

Equation (2) can be derived in a very similar manner.

The same suggestions as were given in **Art. 41** for the solution of right triangles should be carried out here. Draw the triangle, state the formulas, make out a careful scheme for all the work, and, *lastly,* fill in the numerical part by the use of the **Tables.** Remember that in computations *time* and *accuracy* are of very great importance. Time will be saved by carefully planning the arrangement of the work. Accuracy can be secured by checking the work at every step. Verify at every step the additions, subtractions, multiplications, and divisions. Check interpolations when using **Tables,** by repeating the work at each step.

From geometry, the area of a triangle equals one-half the product of the base and altitude. Using b for base, h for altitude, and K for area, $K = \frac{1}{2}bh$. But $h = c \sin \alpha$, and $c = \dfrac{b \sin \gamma}{\sin \beta}$.

[35] $$\therefore K = \frac{b^2 \sin \alpha \sin \gamma}{2 \sin \beta}.$$

Since any side of the triangle can be used as base, or the given side, two other forms for **[35]** can be found. These may be written from the formula given by making the cyclic changes in the parts of the triangle.

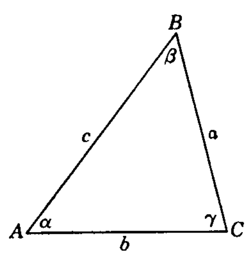
Example.—Given $\alpha = 53° 23.7'$, $\gamma = 75° 46.3'$, and $a = 27.64$; find β, b, and c.

Solution.

Given $\begin{cases} \alpha = 53° 23.7'. \\ \gamma = 75° 46.3'. \\ a = 27.64. \end{cases}$

To find* $\begin{cases} \beta = 50° 50'. \\ b = 26.695. \\ c = 33.375. \end{cases}$

Construction



Formulas

$$\alpha + \beta + \gamma = 180° \qquad \therefore \beta = 180° - (\alpha + \gamma).$$

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} \qquad \therefore b = \frac{a \sin \beta}{\sin \alpha}.$$

* Values to be put in after solving.

$$\frac{a}{\sin \alpha} = \frac{c}{\sin \gamma} \qquad \therefore c = \frac{a \sin \gamma}{\sin \alpha}.$$

Logarithmic formulas

$$\log b = \log a + \log \sin \beta + \text{colog} \sin \alpha.$$
$$\log c = \log a + \log \sin \gamma + \text{colog} \sin \alpha.$$

Computation

$$\beta = 180° - (53° 23.7' + 75° 46.3') = 50° 50'.$$

$\log a = 1.44154$	$\log a = 1.44154$
$\log \sin \beta = 9.88948$	$\log \sin \gamma = 9.98647$
$\text{colog} \sin \alpha = 0.09541$	$\text{colog} \sin \alpha = 0.09541$
$\log b = 1.42643$	$\log c = 1.52342$
$b = 26.695$	$c = 33.375$

Check by Mollweide's equation:

$$\frac{c-b}{a} = \frac{\sin \frac{1}{2}(\gamma - \beta)}{\cos \frac{1}{2}\alpha}, \text{ or } c - b = \frac{a \sin \frac{1}{2}(\gamma - \beta)}{\cos \frac{1}{2}\alpha},$$

$c - b = 6.680$ $\qquad \log a = 1.44154$

Note.—Use Mollweide's equation with the middle-sized side in the denominator.

$a = 27.64$	$\log \sin \frac{1}{2}(\gamma - \beta) = 9.33426$
$\frac{1}{2}(\gamma - \beta) = 12° 28.1'$	$\text{colog} \cos \frac{1}{2}\alpha = 0.04896$
$\frac{1}{2}\alpha = 26° 41.8'$	$\log (c - b) = 0.82476$
	$c - b = 6.680$

EXERCISES

1. Given β, γ, and a; to find α, b, and c. Give formulas and scheme for solution.

2. Give the formula for area when b is the given side. When c is the given side.

3. Given $\alpha = 40° 5.5'$, $\beta = 28° 34.4'$, $c = 267.95$;
find $a = 185.26$, $b = 137.58$, $\gamma = 111° 20.1'$.

4. Given $\alpha = 58° 9'$, $\beta = 41° 41.2'$, $c = 108.85$;
find $a = 93.84$, $b = 73.472$, $\gamma = 80° 9.8'$.

5. Given $\alpha = 23° 4' 8''$, $\gamma = 33° 9' 22''$, $c = 5.94$;
find $a = 4.256$, $b = 9.028$, $\beta = 123° 46' 30''$, $K = 5.265$.

6. Given $\beta = 34° 47.3'$, $\gamma = 109° 26.3'$, $a = 322.4$;
find $b = 314.66$, $c = 520.09$, $\alpha = 35° 46.4'$, $K = 47{,}833$.

7. Given $\beta = 56° 21.3'$, $\gamma = 55° 17' 37''$, $b = 89.042$;
find $a = 99.42$, $c = 87.93$, $\alpha = 68° 21.1'$, $K = 3{,}638.7$.

8. Given $\alpha = 144° 8.4'$, $\beta = 25° 19.2'$, $b = 430.10$;
find $a = 589.14$, $c = 183.96$, $\gamma = 10° 32.4'$, $K = 23{,}174$.

Solve the following and check by Mollweide's equations:

9. Given $\alpha = 47° 16.2'$, $\beta = 75° 41.4'$, $c = 23.53$; find a, b, γ, and K.

10. Given $\alpha = 96° 41.4'$, $\gamma = 23° 13.3'$, $a = 2.458$; find b, c, β and K.

11. Given $\beta = 40° 13' 20''$, $\gamma = 60° 12' 13''$, $b = 22.659$; find a, c, and α.
12. Given $\beta = 18° 22' 26''$, $\gamma = 99° 15' 27''$, $a = 35.863$; find b, c, and α.
13. Given $\alpha = 68° 42' 28''$, $\beta = 35° 42' 18''$, $a = 27.423$; find b, c, and γ,
14. The distance between two points P and Q in a horizontal plane cannot be measured directly. In order to find the distance, a line $PA = 238$ ft. is measured in the same plane, and the angles $APQ = 128° 38'$ and $PAQ = 35° 58'$ are measured. Find PQ. *Ans.* 526.37 ft.
15. To find the width of a river, a line $AB = 600$ ft. is measured on one side parallel to the bank of the stream. A tree C stands on the opposite bank. The angles $ABC = 65° 30'$, and $BAC = 81° 10'$ are measured. Find the width of the stream if line AB is 30 ft. from the bank of the stream. *Ans.* 951.8 ft.
16. Find the area of a triangular plot of ground one side of which is 130 rd., and the angles adjacent to this side are 47° 15′ and 55° 45′. *Ans.* 5264 sq. rods.
17. In a triangle, given c, α, and β; prove that

$$a = \frac{c \sin \alpha}{\sin (\alpha + \beta)}, \text{ and } b = \frac{c \sin \beta}{\sin (\alpha + \beta)}.$$

18. The points A and B are on opposite sides of a river, and the distance AB cannot be measured directly. A point C is chosen on the same side of the river as A and the following measurements made: $AC = 600$ ft., $\angle CAB = 80° 45'$, and $\angle ACB = 60° 10'$. Compute the distance AB. *Ans.* 825.6 ft.

93. Case II. The solution of a triangle when two sides and an angle opposite one of them are given.—It is known from geometry that when two sides and an angle opposite one of them are given the triangle *may not* be uniquely determined.

With these parts given: (1) It may not be possible to construct any triangles; (2) it may be possible to construct just one triangle; (3) it may be possible to construct *two* triangles—the *ambiguous case.*

EXERCISES

Construct carefully the following triangles:
1. (a) $a = 1$ in., $c = 3$ in., and $\alpha = 40°$.
(b) $a = 2$ in., $c = 3$ in., and $\alpha = 140°$.
2. (a) $a = 1$ in., $c = 2$ in., and $\alpha = 30°$.
(b) $a = 3$ in., $c = 2$ in., and $\alpha = 35°$.
(c) $a = 3$ in., $c = 2$ in., and $\alpha = 120°$.
3. $a = 2$ in., $c = 3$ in., and $\alpha = 30°$.

Corresponding to Exercises 1, 2 and 3 above, we have the following, which should be compared with the corresponding constructions in Fig. 88

(1) No solution when:
(a) Angle is acute and opposite side less than adjacent side times the sine of the angle.

(b) Angle is obtuse and opposite side not greater than adjacent side.
(2) One solution when:
(a) Angle is acute and opposite side is equal to adjacent side times the sine of the angle. This gives a right triangle.
(b) Angle of any size and opposite side greater than adjacent side.

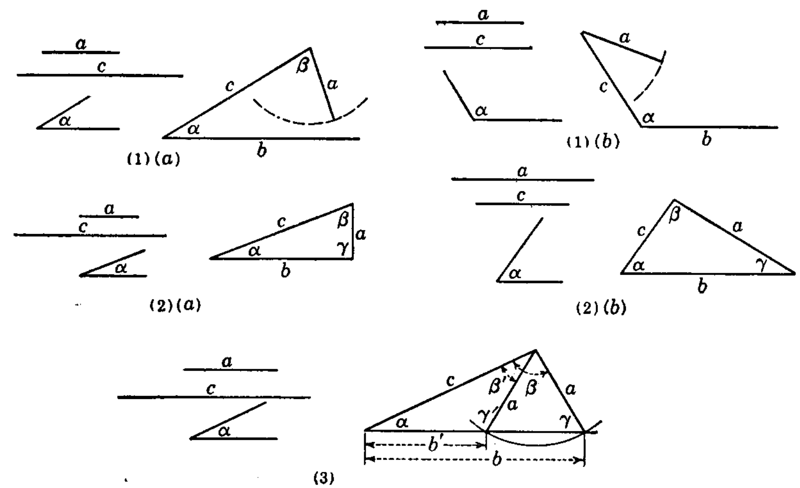


FIG. 88.

(3) Two solutions when angle is acute and the opposite side greater than the adjacent side times the sine of the angle, and less than the adjacent side.

The ambiguity of (3) is also apparent from the solution of γ found from the relation $\sin \gamma = \frac{c \sin \alpha}{a}$. This equation has two values of γ less than 180° each of which may enter into the triangle when α is acute. With each of these values of γ there may be found values of β and b, thus making two triangles.

When logarithms are used, proper conclusions can be drawn from the following, where a, b, and α are given. For other given parts, the proper change can easily be made.

If $\log \sin \beta = 0$, $\sin \beta = 1$, $\beta = 90°$; hence a right triangle.
If $\log \sin \beta > 0$, $\sin \beta > 1$, which is impossible; hence no solution.
If $\log \sin \beta < 0$ and $b < a$, and therefore $\beta < \alpha$, only the acute value of β can be used; hence there is one solution.
If $\log \sin \beta < 0$ and $b > a$, both acute value of β and its supplement may be used; hence there are two solutions.

If the given parts are a, c, and α, with α acute and $a < c$, the formulas for the solution are:

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}, \text{ which gives two values for } \gamma, \text{ say, } \gamma \text{ and } \gamma';$$

$$\beta = 180° - (\alpha + \gamma);\ \beta' = 180° - (\alpha + \gamma');$$

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \text{ gives } b;\ \frac{b'}{\sin\beta'} = \frac{a}{\sin\alpha} \text{ gives } b';$$

$$\text{or } \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \text{ gives } b;\ \frac{b'}{\sin\beta'} = \frac{c}{\sin\gamma'} \text{ gives } b'.$$

The area K can be determined as follows: suppose b, c, and γ are given. Then $K = \frac{1}{2}bh = \frac{1}{2}bc\sin\alpha$, and $\alpha = 180° - (\beta + \gamma)$, where β can be determined from $\sin\beta = \dfrac{b\sin\gamma}{c}$.

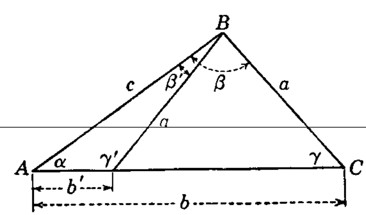
Example.—Solve the triangle when $a = 11.75$, $c = 15.61$, and $\alpha = 34° 15.3'$.

Solution.—Here α is acute, $a < c$, and $a > c\sin\alpha$; hence there are two solutions.

Construction

$$\text{Given}\begin{cases} a = 11.75. \\ c = 15.61. \\ \alpha = 34° 15.3'. \end{cases}$$

$$\text{To find}\begin{cases} \gamma = 48° 23.9'. \\ \beta = 97° 20.8'. \\ b = 20.704. \\ \gamma' = 131° 36.1'. \\ \beta' = 14° 8.6'. \\ b' = 5.1008. \end{cases}$$



Formulas

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma} \qquad \therefore \sin\gamma = \frac{c\sin\alpha}{a} = \sin\gamma'.$$

$$\beta = 180° - (\alpha + \gamma);\ \beta' = 180° - (\alpha + \gamma').$$

$$\frac{b}{\sin\beta} = \frac{a}{\sin\alpha} \qquad \therefore b = \frac{a\sin\beta}{\sin\alpha}.$$

$$\frac{b'}{\sin\beta'} = \frac{a}{\sin\alpha} \qquad \therefore b' = \frac{a\sin\beta'}{\sin\alpha}.$$

Logarithmic formulas

$$\log\sin\gamma = \log c + \log\sin\alpha + \operatorname{colog} a = \log\sin\gamma'.$$

$$\log b = \log a + \log\sin\beta + \operatorname{colog}\sin\alpha.$$

$$\log b' = \log a + \log\sin\beta' + \operatorname{colog}\sin\alpha.$$

Computation

$\log c = 1.19340$	$\log a = 1.07004$
$\log\sin\alpha = 9.75041$	$\log\sin\beta = 9.99642$
$\operatorname{colog} a = 8.92996$	$\operatorname{colog}\sin\alpha = 0.24959$
$\log\sin\gamma = 9.87377$	$\log b = 1.31605$
$\gamma = 48° 23.9'$	$b = 20.704$
$\gamma' = 131° 36.1'$	$\log a = 1.07004$
$\therefore \beta = 97° 20.8'$	$\log\sin\beta' = 9.38801$
$\beta' = 14° 8.6'$	$\operatorname{colog}\sin\alpha = 0.24959$
	$\log b' = 0.70764$
	$b' = 5.1008$

EXERCISES

Apply the tests and determine the number of solutions in Exercises 1 to 5.

1. Given $a = 4$, $c = 5$ and $\alpha = 55°$.

2. Given $a = 25$, $b = 45$ and $\beta = 117°$.

3. Given $b = 72$, $c = 28$ and $\gamma = 21°$.

4. Given $b = 22.5$, $c = 55.3$, and $\beta = 24° 0.5'$.

5. Given $a = 49.7$, $b = 55.3$, and $\alpha = 132°, 20.5'$.

6. Given $a = 78.291$, $c = 111.98$, $\alpha = 38° 21.3'$;
find $\gamma = 62° 34.1'$, $\beta = 79° 4.6'$, $b = 123.88$.
$\gamma' = 117° 25.9'$, $\beta' = 24° 12.8'$, $b' = 51.74$.

7. Given $a = 84.675$, $b = 94.423$, $\beta = 69° 11' 28''$;
find $\gamma = 53° 51' 2''$, $\alpha = 56° 57' 30''$, $c = 81.564$.

8. Given $a = 16.1$, $b = 18.7$, and $\alpha = 22° 18' 23''$;
find $\beta = 26° 9' 29''$, $\gamma = 131° 32' 8''$, $c = 31.752$.
$\beta' = 153° 50' 31''$, $\gamma' = 3° 51' 6''$, $c' = 2.849$.

9. Given $a = 58.345$, $b = 47.654$, and $\beta = 18° 15' 46''$;
find $\alpha = 22° 33.7'$, $\gamma = 139° 10.5'$, $c = 99.415$.
$\alpha' = 157° 26.3'$, $\gamma' = 4° 17.9'$, $c' = 11.4$.

10. Given $a = 248.4$, $b = 96.1$, $\alpha = 66° 31'$;
find $c = 270.5$, $\beta = 20° 47'$, $\gamma = 92° 42'$.

11. Given $a = 462.3$, $b = 535.9$, $\alpha = 42° 32'$;
find $c = 682.07$, $\beta = 51° 35.7'$, $\gamma = 85° 52.3'$.
$c' = 107.7$, $\beta' = 128° 24.3'$, $\gamma' = 9° 3.7'$.

12. Given $b = 160$, $c = 180$, $\beta = 20° 18' 23''$;
find $a = 316.1$, $\alpha = 136° 42' 47''$, $\gamma = 22° 58' 50''$.
$a' = 21.51$, $\alpha' = 2° 40' 27''$, $\gamma' = 157° 1' 10''$.

13. Given $b = 32.597$, $c = 43.465$, and $\beta = 28° 43.6'$;
find $a = 63.14$, $\alpha = 111° 25'$, $\gamma = 39° 51.4'$.
$a' = 13.092$, $\alpha' = 11° 7.8'$, $\gamma' = 140° 8.6'$.

14. Given $b = 46.342$, $c = 65.899$, and $\beta = 21° 15' 18''$
find $a = 101.13$, $\alpha = 127° 42' 50''$, $\gamma = 31° 1' 52''$.
$a' = 21.706$, $\alpha' = 9° 46' 34''$, $\gamma' = 148° 58' 8''$.

15. Given $a = 24.897$, $b = 33.543$, $\alpha = 26° 44.9'$; find c, β, and γ. Check by Mollweide's equations

16. Given $a = 25.34$, $c = 45.76$, $\alpha = 35° 43.8'$; find b, β, and γ. Check.
17. Given $b = 366.62$, $c = 621.35$, $\beta = 154° 38'$; find a, α, and γ. Check.
18. Given $a = 322.22$, $c = 847.36$, $\alpha = 17° 34' 48''$; find b, β, and γ. Check.

94. Case III. The solution of a triangle when two sides and the included angle are given. First method.—Let the given parts be a, b, and γ. Then, from the law of cosines,

$$c = \sqrt{a^2 + b^2 - 2ab \cos \gamma},$$

and α and β may be found from

$$\sin \alpha = \frac{a \sin \gamma}{c} \text{ and } \sin \beta = \frac{b \sin \gamma}{c}, \text{ respectively.}$$

As a check, $\alpha + \beta + \gamma = 180°$ may be used, or use Mollweide's equations.

The area $K = \frac{1}{2}hb = \frac{1}{2}ab \sin \gamma$; or, in words, *the area equals one-half the product of the two sides and the sine of the included angle.*

[36] $$K = \tfrac{1}{2}ab \sin \gamma.$$

It is evident that the formula for finding c is not adapted to the use of logarithms. This method is often convenient, however, when the numbers expressing the sides contain few figures or when only the third side is to be found.

EXERCISES

Solve the following by the first method and check:

1. Given $a = 2$, $b = 3$, $\gamma = 41° 39.8'$; find c, α, β, and K.
2. Given $a = 4$, $c = 8$, $\beta = 105° 32.3'$; find b, α, γ, and K.
3. Given $b = 27$, $c = 80$, $\alpha = 64° 45' 34''$; find a, β, γ, and K.
4. Given $a = 19$, $b = 29$, $\gamma = 76° 24'$; find c, α, β, and K.
5. Given $b = 14$, $c = 16$, $\alpha = 125° 18.9'$; find a, β, γ, and K.

95. Case III. Second method.—For a solution by logarithms when two sides and the included angle are given, the following theorem, known as the law of tangents, is needed.

Law of Tangents.—*In any triangle the difference of any two sides is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.*

Proof.— $\dfrac{a}{b} = \dfrac{\sin \alpha}{\sin \beta}$, from the law of sines.

Then $\dfrac{a - b}{a + b} = \dfrac{\sin \alpha - \sin \beta}{\sin \alpha + \sin \beta}$, by a theorem of proportion.

By [25] and [26], $$= \frac{2 \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}(\alpha - \beta)}{2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta)}$$

$$= \tan \tfrac{1}{2}(\alpha - \beta) \cot \tfrac{1}{2}(\alpha + \beta) = \frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)}.$$

[37] $$\therefore \frac{a - b}{a + b} = \frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)}.$$

This can be put in another form for

$$\alpha + \beta = 180° - \gamma \text{ and } \tfrac{1}{2}(\alpha + \beta) = 90° - \tfrac{1}{2}\gamma,$$
$$\therefore \tan \tfrac{1}{2}(\alpha + \beta) = \tan (90° - \tfrac{1}{2}\gamma) = \cot \tfrac{1}{2}\gamma.$$

Substituting this in [37] gives

[38] $$\tan \frac{1}{2}(\alpha - \beta) = \frac{a - b}{a + b} \cot \frac{1}{2}\gamma.$$

Formula [37] or [38] makes it possible to find $\frac{1}{2}(\alpha - \beta)$ when a, b, and γ are given, while $\frac{1}{2}(\alpha + \beta)$ can readily be found because $\frac{1}{2}(\alpha + \beta) = 90° - \frac{1}{2}\gamma$, therefore α and β can be found from the relations:

$$\alpha = \tfrac{1}{2}(\alpha + \beta) + \tfrac{1}{2}(\alpha - \beta),$$
and
$$\beta = \tfrac{1}{2}(\alpha + \beta) - \tfrac{1}{2}(\alpha - \beta).$$

It is evident that the other side can be found by the law of sines, which may also be used as a check, together with $\alpha + \beta + \gamma = 180°$ after finding $\frac{1}{2}\gamma$. A more certain check formula is one of Mollweide's equations.

A discussion similar to the above can be given when any two sides and the included angle are given. The other formulas can also readily be written by a cyclic change in the letters.

A convenient set of formulas for solving the triangle when a, b, and γ are given is

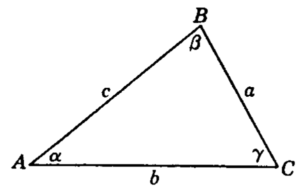
$$\tfrac{1}{2}(\alpha + \beta) = 90° - \tfrac{1}{2}\gamma.$$
$$\tan \frac{1}{2}(\alpha - \beta) = \frac{a - b}{a + b} \cot \frac{1}{2}\gamma.$$
$$\alpha = \tfrac{1}{2}(\alpha + \beta) + \tfrac{1}{2}(\alpha - \beta).$$
$$\beta = \tfrac{1}{2}(\alpha + \beta) - \tfrac{1}{2}(\alpha - \beta).$$
$$c = \frac{a \sin \gamma}{\sin \alpha} = \frac{b \sin \gamma}{\sin \beta}.$$

It should be noted that negatives are avoided if the larger angle and side come first in [38]. Thus, if $\beta > \alpha$ and hence $b > a$, write [38] in form $\tan \dfrac{1}{2}(\beta - \alpha) = \dfrac{b - a}{b + a} \cot \dfrac{1}{2}\gamma$.

Example.—Solve the triangle when $a = 42.367$, $c = 58.964$, and $\beta = 79° 31' 44''$.

Solution.

Construction



Given $\begin{cases} a = 42.367. \\ c = 58.964. \\ \beta = 79° 31' 44''. \end{cases}$

To find $\begin{cases} b = 66.057. \\ \alpha = 39° 6' 1''. \\ \gamma = 61° 22' 15''. \end{cases}$

Formulas

$$\frac{1}{2}(\gamma + \alpha) = 90° - \frac{1}{2}\beta, \ \tan \frac{1}{2}(\gamma - \alpha) = \frac{c - a}{c + a} \cot \frac{1}{2}\beta,$$

$$b = \frac{a \sin \beta}{\sin \alpha}, \text{ and, for a check, } \frac{c - a}{b} = \frac{\sin \frac{1}{2}(\gamma - \alpha)}{\cos \frac{1}{2}\beta}.$$

Computation

$c = 58.964$	$\log a = 1.62703$
$a = 42.367$	$\log \sin \beta = 9.99270$
$c - a = 16.597$	$\text{colog} \sin \alpha = 0.20019$
$c + a = 101.331$	$\log b = 1.81992$
$\frac{1}{2}\beta = 39°45'52''$	$b = 66.057$
	Check
$\log (c - a) = 1.22003$	$\log b = 1.81992$
$\text{colog} (c + a) = 7.99426$	$\log \sin \frac{1}{2}(\gamma - \alpha) = 9.28585$
$\log \cot \frac{1}{2}\beta = 0.07981$	$\text{colog} \cos \frac{1}{2}\beta = 0.11426$
$\log \tan \frac{1}{2}(\gamma - \alpha) = 9.29410$	$\log (c - a) = 1.22003$
$\frac{1}{2}(\gamma - \alpha) = 11° 8' 7''$	$c - a = 16.597$
$\frac{1}{2}(\gamma + \alpha) = 50° 14' 8''$	
$\gamma = 61° 22' 15''$	
$\alpha = 39° 6' 1''$	

EXERCISES

1. Derive formulas like **[38]** for finding $\tan \frac{1}{2}(\alpha - \gamma)$; for $\tan \frac{1}{2}(\gamma - \beta)$.

2. Given $a = 50.35$, $b = 36.54$, $\gamma = 125° 12.3'$;
find $c = 77.405$, $\alpha = 32° 6.4'$, $\beta = 22° 41.3'$.

3. Given $a = 26.548$, $c = 41.654$, $\beta = 61° 0' 33''$;
find $b = 36.986$, $\alpha = 38° 53' 29''$, $\gamma = 80° 5' 57''$.

4. Given $a = 51.455$, $b = 27.345$, and $\gamma = 51° 19.8'$;
find $c = 40.461$, $\alpha = 96° 49.3'$, $\beta = 31° 50.9'$, $K = 549.29$.

5. Given $a = 285.6$; $b = 171.4$, and $\gamma = 65° 41' 10''$;
find $c = 265.78$, $\alpha = 78° 19' 9''$, $\beta = 35° 59' 41''$, $K = 22{,}305$.

6. Given $b = 248.65$, $c = 471.69$, and $\alpha = 139° 8' 46''$;
find $a = 679.52$, $\beta = 13° 50' 55''$, $\gamma = 27° 0' 19''$, $K = 38{,}361$.

7. Given $a = 43.5$, $b = 38.1$, $\gamma = 57° 14.9'$; find c, α, and β. Check by Mollweide's equations.

8. Given $a = 26$, $c = 25$, $\beta = 42° 56.8'$; find b, α, γ, and K. Check.

9. Given $b = 569.59$, $c = 543.76$, $\alpha = 71° 56'$;
find $a = 654.21$, $\beta = 55° 51.9'$, $\gamma = 52° 12.1'$.

10. In an isosceles triangle each of the two equal sides is 23 in. and the included angle is 58° 40′. Find the third side. *Ans.* 22.5 in.

11. The two diagonals of a parallelogram are, respectively, 30 and 25 in., and one of the angles formed by them is 71° 25′. Find the sides of the parallelogram. *Ans.* 16.18 in.; 22.38 in.

12. To find the distance AB through a swamp, a point C was chosen and the following measurements made: $CA = 163$ rd., $CB = 145$ rd., and angle $ACB = 36° 37'$. Compute the distance AB. *Ans.* 98.25 rd.

13. At a certain point the length of a lake subtends an angle of 53° 44.5′, and the distances from this point to the extremities of the lake are 144 and 86.3 rd., respectively. Find the length of the lake. *Ans.* 116.1 rd.

14. Two railroad tracks intersect at an angle of 85° 30′. At a certain time a train going 32 miles an hour passes the point of intersection; 2 min. later a train going 55 miles an hour on the other track passes this point. Write a formula showing their distance apart t min. after the first train passes the intersecting point. How far will they be apart in 25 min.? *Ans.* 24.045 miles, or 25.815 miles.

15. Two headlands P and Q are separated by water. In order to find the distance between them a third point A is chosen from which both P and Q are visible, and the following measurements are made: $AP = 1160$ ft., $AQ = 1945$ ft., and angle $PAQ = 60° 30'$. Find the distance PQ. *Ans.* 1705 ft.

96. Case IV. The solution of a triangle when the three sides are given.—In this case the angles can be found by means of the law of cosines, from which the following formulas are derived:

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}.$$

$$\cos \beta = \frac{a^2 + c^2 - b^2}{2ac}.$$

$$\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab}.$$

These formulas give the cosines of the angles and, therefore, the angles; but they are not adapted to logarithms. They are convenient when the sides are expressed in numbers of few figures, or when tables of squares and products are at hand

A very good check formula is $\alpha + \beta + \gamma = 180°$.

EXERCISES

Find the angles when the sides are given as follows:

1. $a = 3$, $b = 4$, and $c = 5$.
2. $a = 4$, $b = 3$, and $c = 6$.
3. $a = 15$, $b = 19$, and $c = 21$.
4. $a = 12$, $b = 13$, and $c = 16$.
5. $a = 10$, $b = 8$, and $c = 7$.
6. $a = 200$, $b = 300$, and $c = 400$.
7. $a = 12$, $b = 17$, and $c = 14$.
8. $a = 12$, $b = 5$, and $c = 13$.

97. Case IV. Formulas adapted to the use of logarithms.—(1) Start with the equation $\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$ and subtract each member of it from 1. This gives

$$1 - \cos\alpha = 1 - \frac{b^2 + c^2 - a^2}{2bc}.$$

$$\therefore 2\sin^2\frac{1}{2}\alpha = \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} = \frac{(a - b + c)(a + b - c)}{2bc}$$

Let $a + b + c = 2s$. Then $a - b + c = 2(s - b)$, and $a + b - c = 2(s - c)$.

Substituting these values in the above,

$$2\sin^2\frac{1}{2}\alpha = \frac{2(s - b)2(s - c)}{2bc}.$$

[39₁] $$\therefore \sin\frac{1}{2}\alpha = \sqrt{\frac{(s - b)(s - c)}{bc}}.$$

In like manner are obtained the following:

[39₂] $$\sin\frac{1}{2}\beta = \sqrt{\frac{(s - a)(s - c)}{ac}}.$$

[39₃] $$\sin\frac{1}{2}\gamma = \sqrt{\frac{(s - a)(s - b)}{ab}}.$$

(2) By adding each member of the equation $\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$ to 1, and carrying out the work in a manner similar to the above, there are obtained the following:

[40₁] $$\cos\frac{1}{2}\alpha = \sqrt{\frac{s(s - a)}{bc}}.$$

[40₂] $$\cos\frac{1}{2}\beta = \sqrt{\frac{s(s - b)}{ac}}.$$

[40₃] $$\cos\frac{1}{2}\gamma = \sqrt{\frac{s(s - c)}{ab}}.$$

(3) By dividing each formula of the set under (1) by the corresponding formula of the set under (2), there results:

[41₁] $$\tan\frac{1}{2}\alpha = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}}.$$

[41₂] $$\tan\frac{1}{2}\beta = \sqrt{\frac{(s - a)(s - c)}{s(s - b)}}.$$

[41₃] $$\tan\frac{1}{2}\gamma = \sqrt{\frac{(s - a)(s - b)}{s(s - c)}}.$$

These last three can be put in a form slightly more convenient

Since $$\sqrt{\frac{(s - b)(s - c)}{s(s - a)}} = \sqrt{\frac{(s - a)(s - b)(s - c)}{s(s - a)^2}} = \frac{1}{s - a}\sqrt{\frac{(s - a)(s - b)(s - c)}{s}},$$

by writing $$r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}},$$

[42₁] $$\tan\frac{1}{2}\alpha = \frac{r}{s - a}.$$

Similarly, the following are obtained:

[42₂] $$\tan\frac{1}{2}\beta = \frac{r}{s - b}.$$

[42₃] $$\tan\frac{1}{2}\gamma = \frac{r}{s - c}.$$

In using any of these sets of formulas, the work may be checked by

$$\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma = 90°.$$

The area can be found from

$$K = \tfrac{1}{2}bh = \tfrac{1}{2}bc\sin\alpha = bc\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\alpha$$

$$= bc\sqrt{\frac{(s - b)(s - c)}{bc}}\sqrt{\frac{s(s - a)}{bc}} = \sqrt{s(s - a)(s - b)(s - c)}.$$

[43] $$\therefore K = \sqrt{s(s - a)(s - b)(s - c)}.^*$$

Since the sine varies most rapidly for small angles, and the cosine most rapidly for angles near 90°, formulas **[39]** should be used when the angles are small, and **[40]** when the angles are near 90°. In all cases the tangent varies more rapidly than either sine or cosine. Hence, formulas **[41]** or **[42]** are always more nearly accurate than **[39]** or **[40]**.

* Formula **[43]** was discovered by Hero (or Heron) of Alexandria about the beginning of the Christian era.

Again, formulas [**41**] or [**42**] are more convenient, since, for a complete solution of the triangle, they require only *four* logarithms to be taken from the table; while [**39**] and [**40**] require, respectively, six and seven.

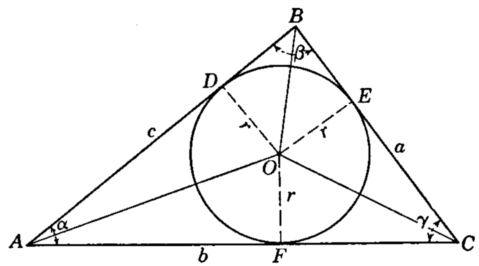


FIG. 89.

Formulas [**42**] may be derived by taking from geometry the fact that the area of a triangle, when the three sides are given, is

$$K = \sqrt{s(s-a)(s-b)(s-c)};$$

and, from Fig. 89, $K = sr$, where r is the radius of the inscribed circle.

$$\therefore sr = \sqrt{s(s-a)(s-b)(s-c)},$$

and
$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Also
$$AF + EC + EB = s.$$
$$\therefore AF = s - (EC + EB) = s - a.$$

But
$$\tan\frac{1}{2}\alpha = \frac{r}{AF}.$$
$$\therefore \tan\frac{1}{2}\alpha = \frac{r}{s-a}.$$

It should be noted that r in the formulas of this article is the radius of the inscribed circle, and the formula given for r is a simple formula for finding the radius of the inscribed circle.

Example.—Solve for the angles when $a = 23.764$, $b = 42.376$, and $c = 31.166$.

Solution.—Use formulas [**42**] with that for r.

$a = 23.764$	$\log(s-a) = 1.39600$
$b = 42.376$	$\log(s-b) = 0.79775$
$c = \underline{31.166}$	$\log(s-c) = 1.24272$
$2s = 97.306$	$\text{colog } s = \underline{8.31289}$

$s = 48.653$	$\log r^2 = 1.74936$
$s - a = 24.889$	$\log r = 0.87468$
$s - b = 6.277$	$\log\tan\frac{1}{2}\alpha = 9.47868$
$s - c = \underline{17.487}$	$\therefore \frac{1}{2}\alpha = 16° 45' 21''$
$2s = 97.306$	$\log\tan\frac{1}{2}\beta = 0.07693$
A check.	$\therefore \frac{1}{2}\beta = 50° 2' 53''$
	$\log\tan\frac{1}{2}\gamma = 9.63196$
	$\therefore \frac{1}{2}\gamma = 23° 11' 45''$

Check.—$\frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2}\gamma = 89° 59' 59''$.

Remark.—The sum of s, $(s-a)$, $(s-b)$, and $(s-c)$ is $2s$, and hence is a check on the additions and subtractions.

To facilitate the subtractions, write the values of s on the margin of a slip of paper, when it can be placed above the values a, b, and c, successively. In like manner $\log r$ can be written on a margin and placed above logs of $(s-a)$, $(s-b)$, and $(s-c)$.

EXERCISES

1. Derive $\sin\frac{1}{2}\gamma = \sqrt{\frac{(s-a)(s-b)}{ab}}$ from the law of cosines.

2. Derive $\cos\frac{1}{2}\beta = \sqrt{\frac{s(s-b)}{ac}}$ from the law of cosines.

3. Derive $K = \sqrt{s(s-a)(s-b)(s-c)}$ by geometry.

4. What is the tabular difference for each of log sine, log cosine, and log tangent when the angle is near 11°? How accurately can the angle be found from each?

5. Answer the same questions for 82° and 46°.

6. In Fig. 89, show that $BE = s - b$.

7. Can $s - a$ be less than **0**. Show why.

8. How many values of α will satisfy $\sin\frac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{bc}}$?

9. In solving the triangle, when two sides and an angle opposite one of them are given, an ambiguity was introduced because from the sine of the angle two values of the angle were found. Why is there not an ambiguity when formulas [**39**] are used?

10. Given $a = 72.392$, $b = 55.678$, $c = 42.364$;
find $\frac{1}{2}\alpha = 47° 6' 10''$, $\frac{1}{2}\beta = 25° 2' 42''$, $\frac{1}{2}\gamma = 17° 51' 11''$.

11. Given $a = 43.294$, $b = 40.526$, $c = 39.945$;
find $\frac{1}{2}\alpha = 32° 32' 45''$, $\frac{1}{2}\beta = 29° 3' 2''$, $\frac{1}{2}\gamma = 28° 24' 11''$.

12. Given $a = 610$, $b = 363$, $c = 493$;
find $\alpha = 89° 33' 50''$, $\beta = 36° 31' 2''$, $\gamma = 53° 55' 6''$.

13. Given $a = 16.47$, $b = 25.49$, $c = 33.77$;
find $\alpha = 28° 5' 2''$, $\beta = 46° 46' 4''$, $\gamma = 105° 8' 51''$.

14. Solve the example in **Art. 97** by using formulas [**39**]. By using formulas [**40**]. Compare the work with that in the solution of the example.

15. Given $a = 98.34$, $b = 353.26$, $c = 276.49$;
find $\alpha = 11° 16' 58''$, $\beta = 135° 20' 27''$, $\gamma = 33° 22' 32''$, $K = 9{,}554.5$.

16. Given $a = 8.363$, $b = 5.473$, $c = 10.373$;
find $\alpha = 53° 27' 12''$, $\beta = 31° 43' 8''$, $\gamma = 94° 49' 40''$, $K = 22.804$.

17. Given $a = 49.63$, $b = 39.65$, $c = 67.54$; find α, β, γ, and K. Check.

18. Given $a = 2.374$, $b = 4.375$, $c = 5.73$; find α, β, and γ. Check.

19. Given $a = 70$, $b = 40$, $c = 35$; find the area of the triangle and the radius of the inscribed circle. *Ans.* $K = 470$, $r = 6.483$.

20. The sides of a triangle are, respectively 28, 16, and 25 ft. Find the area of the triangle and the area of the inscribed circle.
Ans. 198.52 sq. ft., 104.02 sq. ft.

21. Find the radius of the largest circular gas tank that can be constructed on a triangular lot whose sides are 75, 85, and 95 ft., respectively, and locate the center by giving the distance from the ends of the 85-ft. side to the point of tangency on the other sides. *Ans.* $r = 23.85$ ft., 52.5 ft., 32.5 ft.

GENERAL EXERCISES

1. Find the area of a triangle with sides 13.6 and 16.39 ft. and included angle 163° 36′ 16″. *Ans.* 31.459 sq. ft.

2. Find the area of a triangle with the three sides, respectively, 47.45, 36.4, and 36.65 ft. *Ans.* 658.85 sq. ft.

3. Two sides of a parallelogram are 46.3 and 46.36 rd., respectively, and the included angle is 56° 35′. Find the area. *Ans.* 1791.6 sq. rd.

4. The base of a triangle is 62.53 ft. and the two angles at the base are, respectively, 109° 53′, and 36° 16′; find the other two sides and the area of the triangle. *Ans.* 66.407 ft., 105.57 ft., 1952.5 sq. ft.

5. Two angles of a triangle are, respectively, 57° 47′ 14″ and 59° 47′ 43″. If the included side is 14.63 in., find the area. *Ans.* 88.286 sq. in.

6. In a triangle an angle is 52° 16′ and the opposite side is 36 in.; find the diameter of the circumscribed circle. *Ans.* 45.52 in.

7. If the sides of triangle are 4, 6, 7, find the radius of the inscribed circle. *Ans.* 1.41.

8. If the sides of a triangle are 4, 6, 5, find the radius of the circumscribed circle. *Ans.* 3.024.

9. The three sides of a triangle are 8, 12, 15; find the length of median drawn to the side 12. *Ans.* 10.42.

10. In a triangle ABC, angle A is 126° 47′, and AD is the bisector of angle A with D on the side BC. If $b = 24$, and $c = 15$, find AD, BD, and DC.
Ans. $AD = 8.27$, $BD = 13.5$, $DC = 21.6$.

11. The angles of a triangle are in the ratio of 3:5:7; and the longest side is 154 ft. Solve the triangle.
Ans. Angles, 36°, 60°, 84°; sides 91.02, 134.1.

12. The sides of a triangle are in the ratio of 7:4:8; find the sine of the smallest angle. The cosine of the largest angle.
Ans. 0.49992, 0.01786.

13. Solve the following triangle for the parts not given: $K = 7934.2$, $\alpha = 36°$, and $\beta - \gamma = 16°$.
Ans. $a = 102.65$, $b = 171.99$, $c = 156.97$, $\beta = 80°$, $\gamma = 64°$.

14. The sides of a triangular field of which the area is 13 acres are in the ratio of 3:4:6. Find the length of the shortest side. *Ans.* 59.248 rds.

15. Prove that in any triangle $K = \dfrac{b^2 \sin \alpha \sin \gamma}{2 \sin (\alpha + \gamma)}$.

16. Use the corollary of Art. 90, and the formula $K = \frac{1}{2}ab \sin \gamma$, and show that the radius of the circumscribed circle is given by $R = \dfrac{abc}{4K}$. Also show that $K = \dfrac{abc}{4R}$.

17. In a parallelogram given a diagonal $d = 15.36$, and the angles $\alpha =$ 26° 36.4′, and $\beta =$ 36° 32.4′ which this diagonal makes with the sides; find the sides. *Ans.* 10.25, 7.711.

18. In a parallelogram are given a side a, a diagonal d, and the angle θ between the diagonals; find the other diagonal and side.

19. If one side of a parallelogram is 13.52 in., one diagonal is 19.23 in., and one angle between the diagonals is 35° 32′ 35″, find the other diagonal.
Ans. 40.27 or 8.974 in.

20. The two parallel sides of a trapezoid are a and b, and the angles formed by the nonparallel sides at the two ends of one of the parallel sides are, respectively, α and β. Find the lengths of the nonparallel sides.
Ans. $\dfrac{(a - b) \sin \alpha}{\sin (\alpha + \beta)}$ and $\dfrac{(a - b) \sin \beta}{\sin (\alpha + \beta)}$.

21. The two parallel sides of a trapezoid are, respectively, 17.5 and 9.3 ft., and the angles formed by the nonparallel sides at the ends of the first side are respectively 31° 25′, and 52° 36′. Find the lengths of the nonparallel sides. *Ans.* 4.298 ft., 6.55 ft.

22. Show that the area of any quadrilateral is equal to one-half the product of its diagonals and the sine of the included angle.

23. One side of a parallelogram is 46.4 rd., and the angles which the diagonals make with that side are 57° 34′ and 36° 34′. Find the length of the other side. *Ans.* 49.67 rd.

24. Two circles whose radii are 28 and 36 in. intersect. The angle between the tangents at a point of intersection is 36° 35′. Find the distance between their centers. *Ans.* 60.82 in., 21.47 in.

25. B is 48 miles from A in the direction N 71° W, and C is 75 miles from A in the direction N 15° E. What is the position of C relative to B?
Ans. 86.18 miles, N 48° 45′ 16″ E.

26. Given a parallelogram $ABCD$ with $AD = m$, $AC = d$, $AB = n$, $\angle BAD = \phi$, and $\angle DAC = \alpha$; prove that $\dfrac{\sin (\phi - \alpha)}{\sin \phi} = \dfrac{m}{d}$ and that

$$\cot \phi = \cot \alpha - \frac{m}{d \sin \alpha}.$$ If $AD = AB = m$,

prove that $d = 2m \cos \frac{1}{2}\phi$.

27. Given two triangles with data shown in Fig. 90; prove that $p = w \tan 50°$. If $w = 200$, find the values of p, r_1, r_2, and r_3.
Ans. $p = 238.36$, $r_1 = 178.46$, $r_2 = 300.56$, $r_3 = 254.87$.

EXERCISES, APPLICATIONS

1. Two streets intersect at an angle of 86° 36′. The corner lot fronts 100 ft. on one street and 146 ft. on the other, and the other two sides are perpendicular to the streets. Find the area of the lot. *Ans.* 13,696 sq. ft.

2. Along a bank of a river, a line 500 ft. in length is measured. The angles between this line and the lines drawn from its extremities to a point P on the opposite bank of the river are, respectively, $62° 35'$ and $55° 44'$. Find the width of the river. *Ans.* 416.7 ft.

3. A bridge is to be constructed over a valley. If the length of the bridge is l and the inclinations of the two sides of the valley are respectively α and β, find the height of a pier erected at the lowest point of the valley to support the bridge. *Ans.* $\dfrac{l \sin \alpha \sin \beta}{\sin (\alpha + \beta)}$.

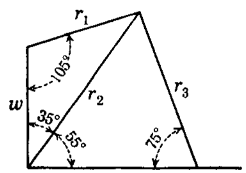


FIG. 90.

4. A ship at a point Q observes two capes A and B; the bearing of A is N $36° 35'$ E, and the bearing of B is N $16° 36'$ W. Find the distance the ship is from each cape if it is known that the distance between the capes is 23.8 miles, and the bearing of B from A is N $58° 40'$ W. *Ans.* 19.92 miles from A; 29.61 miles from B.

5. In Fig. 91, find the height $DC = x$, and the distance $AC = y$ of an inaccessible object, having measured on a horizontal plane the distance a in the line CAB, and the angles α and β.

Suggestion.–

$$AD = \frac{a \sin \beta}{\sin (\alpha - \beta)}.$$

$$CD = AD \sin \alpha = \frac{a \sin \alpha \sin \beta}{\sin (\alpha - \beta)}.$$

$$AC = AD \cos \alpha = \frac{a \cos \alpha \sin \beta}{\sin (\alpha - \beta)}.$$

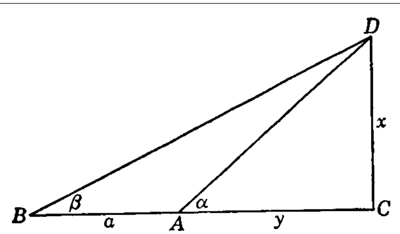


FIG. 91.

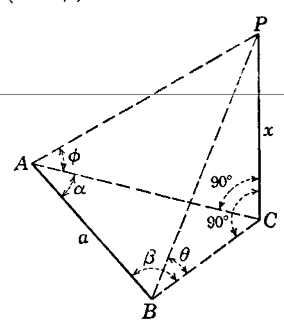


FIG. 92.

6. In Fig. 92, the point P is an inaccessible object above the horizontal plane ABC. The straight line $AB = a$ is measured, also the angles α, β, θ, and ϕ. Find the height x of the point P above the plane, giving two solutions which will check each other. State the result in the form

$$x = \frac{a \sin \beta \tan \phi}{\sin (\alpha + \beta)} = \frac{a \sin \alpha \tan \theta}{\sin (\alpha + \beta)}.$$

7. In Exercise 6, given $a = 465$ ft., $\alpha = 49° 51' 47''$, $\beta = 52° 46' 30''$, $\phi = 39° 16' 14''$, and $\theta = 40° 25' 5''$; find x. Could this exercise be solved if θ were not given? *Ans.* 310.26 ft.

8. Two observers at A and B, 125 rd. apart on a horizontal plane, observe at the same instant an aviator. His angle of elevation at A is $72° 25'$, and at B $64° 34.8'$. The angles made by the projections of the lines of sight on that horizontal plane with the line AB are $40° 27'$ at A and $25° 38'$ at B. Find the height of the aviator. *Ans.* 3080 ft.

9. Compute the inaccessible distance PQ (Fig. 93) when given the line $AB = a$ and the angles α, β, γ, and δ. Are the data sufficient for a check?

10. In Exercise 9, given $a = 330$ ft., $\alpha = 41° 36.5'$, $\beta = 64° 47.5'$, $\gamma = 30° 46.5'$, and $\delta = 35° 53.5'$; find PQ. *Ans.* 271.8 ft.

11. To find the distance between two inaccessible points A and B, a base line $CD = 800$ ft. is measured in the same plane as A and B, and the angles $DCA = 106°$, $DCB = 39°$, $CDB = 122°$ and $CDA = 41°$ are measured. Compute the distance AB. *Ans.* 1924 ft.

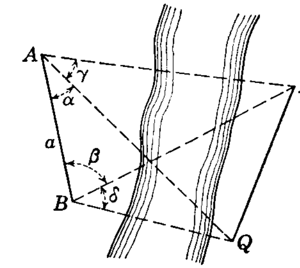


FIG. 93.

12. Two points P and Q are on opposite sides of a stream and invisible from each other on account of an island in the stream. A straight line AB is run through Q and the following measurements taken: $AQ = 824$ ft., $QB = 662$ ft., and $QAP = 42° 34.4'$, and angle $QBP = 57° 45'$. Compute QP. *Ans.* 872.1 ft.

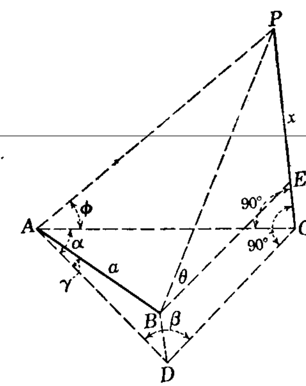


FIG. 94.

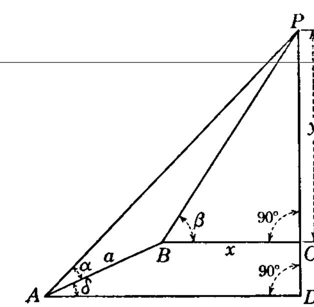


FIG. 95.

13. Two points P and Q on the same side of a river are inaccessible. They are both visible from a single point A only, on the opposite side of the river. From other points on this side of the river only P or only Q can be seen. Show what measurements can be made to compute PQ, and outline the solution.

14. A statue, of height $2h$, standing on the top of a pillar, subtends an angle α at a point on the ground distant d from the foot of the pillar. Prove that the height of the pillar is $\sqrt{h^2 + 2hd \cot \alpha - d^2} - h$.

15. A flagstaff 50 ft. tall stands on the top of the end of a building 105 ft. high. At what distance from the base of the building will the flagstaff subtend an angle of 9°? *Ans.* 251 ft. or 64.9 ft.

16. In taking measurements for finding the height of P (Fig. 94) above the horizontal line AC, a line $AB = a$ was measured in a plane making an angle $DAB = \gamma$ with the horizontal. Other angles measured were $\angle DAC = \alpha$, $\angle ADC = \beta$, $\angle CAP = \phi$, and $\angle EBP = \theta$. Find the height x that P is above C, and put in the form

$$x = \frac{a \cos \gamma \sin \beta \tan \phi}{\sin (\alpha + \beta)} = \frac{a \cos \gamma \sin \alpha \tan \theta}{\sin (\alpha + \beta)} + a \sin \gamma.$$

17. In Exercise 16, given $a = 145$ ft., $\alpha = 47° 60' 33''$, $\beta = 60° 44' 20''$, $\phi = 59° 35' 12''$, $\gamma = 4° 15' 31''$, and $\theta = 63° 45' 43''$; find x. *Ans.* 226.94 ft.

18. From the data given in Fig. 95; find x and y in the forms:

$$x = \frac{a \sin \alpha \cos \beta}{\sin (\beta - \alpha - \delta)}, \quad y = \frac{a \sin \alpha \sin \beta}{\sin (\beta - \alpha - \delta)}.$$

19. From the top of a hill 720 ft. high, the angles of depression of the top and the base of a tower are, respectively, 38° 30′ and 51° 25′. Find the height of the tower. *Ans.* 263.1 ft.

20. A tower 120 ft. high casts a shadow 148 ft. long upon a plane which slopes downward from the base of the tower at the rate of 1 ft. in 12 ft. What is the angle of elevation of the sun? *Ans.* 41° 53.5′.

21. A flagstaff 40 ft. high stands on the top of a wall 29 ft. high. At a point P on the level with the base of the wall and on a line perpendicular to the wall below the flagstaff, the height of the wall and the flagstaff subtend equal angles. Find the distance of P from the wall. *Ans.* 72.63 ft.

22. A tower stands on the top of a hill whose side has a uniform inclination of θ with the horizontal. At a distance of d from the foot of the tower measured down the hill the tower subtends an angle ϕ. Find the height h of the tower. *Ans.* $h = \dfrac{d \sin \phi}{\cos (\phi + \theta)}$.

23. In the preceding exercise, find the height of the tower if $\theta = 18° 45'$, $\phi = 23° 45'$, and $d = 410$ ft. *Ans.* 224 ft.

24. From a point 250 ft. above the level of a lake and to one side, an observer finds the angles of depression of the two ends of the lake to be 4° 15′ and 3° 30′, respectively. The angle between the two lines of sight is 48° 20′. Find the length of the lake. *Ans.* 3128 ft.

25. A man is on a bluff 300 ft. above the surface of a lake. From his position the angles of depression of the two ends of the lake are 10° 30′ and 6° 45′, respectively. The angle between the two lines of sight is 98° 40′. Find the length of the lake. *Ans.* 3239 ft.

26. From a point h ft. above the surface of a lake the angle of elevation of a cloud is observed to be α, and the angle of depression of its reflection in the lake is β. Find that the height of the cloud above the surface of the lake is $h \dfrac{\sin (\beta + \alpha)}{\sin (\beta - \alpha)}$ ft.

27. A kite K, sent up and fastened to the ground at a point A, drifted so that it stands directly over the point B in the same horizontal plane as A

and separated from it by water so that AB cannot be measured directly. To find the height of the kite, a line AC 1000 ft. long is laid off on the horizontal, and the angles $BAK = 46° 35' 52''$, $KAC = 67° 54' 39''$, and $ACK = 65°$ are measured. Compute the vertical height of the kite. *Ans.* 899 ft.

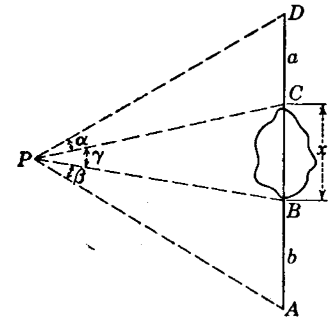


Fig. 96.

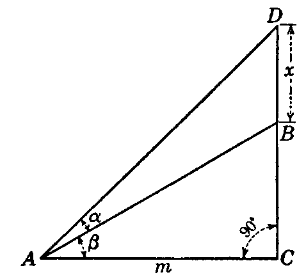


Fig. 97.

28. Given the data as shown in Fig. 96; find the distance x in form:

$$(a + x)(b + x) \sin \alpha \sin \beta = ab \sin (\alpha + \gamma) \sin (\beta + \gamma).$$

After numerical values are substituted, this can be solved as a quadratic equation in x.

29. Given data as shown in Fig. 97; solve for x and state the result in a formula. *Ans.* $x = m[\tan (\alpha + \beta) - \tan \beta]$.

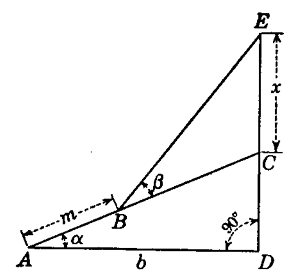


Fig. 98.

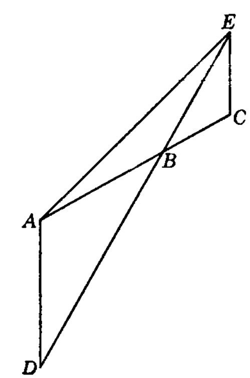


Fig. 99.

30. Given data as shown in Fig. 98; solve for x and state the result in a formula. *Ans.* $x = \dfrac{(b \sec \alpha - m) \sin \beta}{\cos (\alpha + \beta)}$.

31. A flagstaff of known height c stands on the top at the end of a building. At a point P on the level with the base of the building, the building and the flagstaff each subtend an angle α. Find the distance of the point P from the base of the building. *Ans.* $\dfrac{c}{\tan 2\alpha - \tan \alpha}$.

32. At each end of a horizontal base line of length $2a$, the angle of elevation of a mountain peak is β, and at the middle of the base line it is α. Show that the height of the peak above the plane of the base line is

$$\frac{a \sin \alpha \sin \beta}{\sqrt{\sin (\alpha + \beta) \sin (\alpha - \beta)}}.$$

33. Two railway tracks intersect making an angle of 70°. The tracks are connected by a circular Y that is tangent to each of the tracks at points 700 ft. from the intersection. Find the radius of the Y and its length. Neglect the width of the tracks. *Ans.* 490.16 ft., 941.02 ft.

34. In Fig. 99, CE is parallel to DA, $DA = 10$ ft. $AB = 10$ ft., $BC = 5$ ft. and angle $DAB = 120°$. Find AE and angle AEC.
Ans. 18.03 ft., 46° 6.2′.

35. Two forces of 75 and 92 lb., respectively, are acting on a body. What is the resultant force if the angle between the forces is 54° 36′?
Ans. 148.6 lb.

36. Resolve a force of 250 lb. acting along the positive x-axis into two components of 170 and 180 lb., and find the directions of the components with respect to the x-axis. *Ans.* 46° 2′ 23″; −42° 50′.

37. Two forces of 35 lb. each are acting on a body. One is directed downward and the other at a positive angle of 47° with the horizontal. Find the magnitude of the resultant and its direction with reference to the horizontal.
Ans. 25.655 lb.; −21° 30′.

38. Three forces of 18, 22, and 27 lb. respectively, and in the same plane are in equilibrium. Find the angles they make with each other. Check by noting the sum of the angles is 360°.

39. Four forces are acting on the origin of a system of rectangular axes. One of 300 lb. acts along the negative x-axis, one of 175 lb. acts along the positive x-axis, one of 60 lb. acts at an angle of 50° with the x-axis, and one of r lb. acts at an angle θ with the x-axis. If the forces are in equilibrium, find r and θ. *Ans.* −28° 0′ 10″; 97.9 lb.

40. Five forces in equilibrium are acting at the origin of a system of rectangular axes. One of 4000 lb. acts along the negative y-axis, one of 1700 lb. acts along the negative x-axis, one of 1400 lb. acts at an angle of 135° with the x-axis, one of r_1 lb. acts at an angle of 60° with the x-axis, and one of r_2 lb. acts along the positive x-axis. Find r_1 and r_2.
Ans. 3475.8 lb.; 952.1 lb.

41. An automobile is traveling N. 45° W. at 40 miles per hour, and the wind is blowing from the northeast at 30 miles per hour. What velocity and direction does the wind appear to have to the chauffeur?
Ans. 50 miles per hour N 8° 7.8′ W.

42. A train is running at the rate of 40 miles per hour in the direction S. 55° W., and the engine leaves a steam track in the direction N. 80° E. The wind is known to be blowing from the northeast; find its velocity.
Ans. 29.47 miles per hour.

43. In a river flowing due south at 3 miles per hour a boat is drifted by a wind blowing from the southwest at the rate of 15 miles per hour. Determine the position of the boat after 60 minutes if resistance reduces the effect of the wind 60 per cent. *Ans.* 4.42 miles N, 73° 40.8′ E.

44. A ship S is 12 miles to the north of a ship Q. S sails 10 miles per hour and Q 15 miles per hour. Find the distance and direction Q should sail in order to intercept S which is sailing in a northeasterly direction.
Ans. 29.23 miles, N 28° 7.5′ E.

45. A tug that can steam 13 miles per hour is at a point P. It wishes to intercept a steamer as soon as possible that is due east at a point Q and making 21 miles per hour in a direction S. 58° W. Find the direction the tug must steam and the time it will take if Q is 3 miles from P.
Ans. S 31° 7′ 44″ E; 7′ 20.3″.

46. Two poles are 42 ft. apart and one is 6 ft. taller than the other. A cable 48 ft. long is fastened to the tops of the poles and supports a weight of 400 lb. hanging from it by a trolley. When the trolley is at rest find the two segments of the cable and the angle each makes with the horizontal. Suppose the trolley has no friction and that the two segments of the cable are straight lines. *Ans.* 30.2 ft., 17.8 ft.; each angle = 28° 57.3′.

Suggestion.—Tension in cable is same throughout, and horizontal components are equal.

47. An airplane, which is at an altitude of 1800 ft. and moving at the rate of 100 miles per hour in a direction due east, drops a bomb. Disregarding the resistance of the air, where will the bomb strike the ground?
Ans. 1551.7 ft. east of point where bomb was dropped.

Suggestion.—To find the number of seconds it is falling, use the equation

$$\tfrac{1}{2}gt^2 = 1600.$$

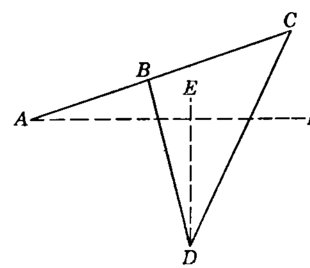


FIG. 100.

48. In a dredge derrick (Fig. 100) the following measurements are made: AF is perpendicular to DE, $AB = 20$ ft., $BC = 25$ ft., $DB = 30$ ft., $\angle FAC = 20°$, $\angle BDE = 15°$. Find $\angle DBC$ and DC.
Ans. $\angle DBC = 95°$; $DC = 40.69$ ft.

49. $ABCD$ is the ground plan of a barn of known dimensions $AB = a$ and $AD = b$. A surveying party, wishing to locate a point P in the same horizontal plane with the barn, measure the angles $DPC = \alpha$ and $BPC = \beta$. Determine the lengths of the lines $PB = x$, $PC = y$, and $PD = z$.

Ans. $x = -\dfrac{b \cos \phi}{\sin \beta}$; $y = \dfrac{a \sin (\phi + \alpha)}{\sin \alpha} = -\dfrac{b \cos (\phi - \beta)}{\sin \beta}$; $z = \dfrac{a \sin \phi}{\sin \alpha}$;

and $\tan \phi = -\dfrac{a + b \cot \beta}{b + a \cot \alpha}$, where ϕ = angle DCP.

50. The jib of a crane makes an angle of 35° with the vertical. If the crane swings through a right angle about its vertical axis, find the angle between the first and the last positions of the jib. *Ans.* 47° 51′ 18″.

51. If the jib of a crane makes an angle ϕ with the vertical and swings about the vertical axis through an angle θ, show that the angle α between the first and last positions of the jib is given by the equation

$$\sin \tfrac{1}{2}\alpha = \sin \phi \sin \tfrac{1}{2}\theta.$$

52. An umbrella is partly open and has n straight ribs each inclined at an angle ϕ with the center stick of the umbrella. Show that the angle θ between consecutive ribs is given by the equation $\sin \frac{1}{2}\theta = \sin \frac{\pi}{n} \sin \phi$.

53. To lay out a pentagon in a circle, draw two perpendicular diameters AB and CD (Fig. 101) and bisect AO at E. With E as a center and ED

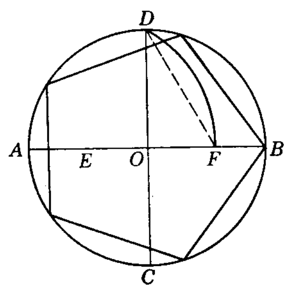


Fig. 101.

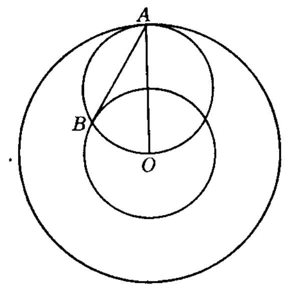


Fig. 102.

as a radius, draw the arc DF. The length of the chord DF is the side of the inscribed pentagon. Prove this.

54. To lay out a regular heptagon in a circle, make a construction as shown in Fig. 102. AB is very nearly the side of the inscribed regular heptagon. Determine the error in one side for a circle with a radius of 10 in. and determine the per cent of error. Determine the angle at the center intercepting the chord found by this process.

Ans. 0.2 per cent too small.

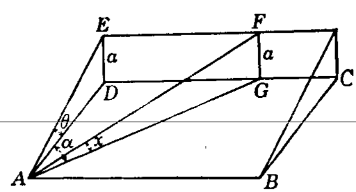


Fig. 103.

55. If the angle of slope of a plane is θ, find the angle of slope x of the line of intersection of this plane with a vertical plane making an angle α with the vertical plane containing the line of greatest slope. (Note the difference between this exercise and Exercise 18, page 99.)

Suggestion.—In Fig. 103, $AD = a \cot \theta$, $AG = a \cot x$.

$$\therefore \cos \alpha = \frac{a \cot \theta}{a \cot x}, \text{ and } \tan x = \tan \theta \cos \alpha.$$

56. Two vertical faces of rock at right angles to each other show sections of a geological stratum which have dips (angles with the horizontal) of α and β respectively. If δ is the true dip (angle between the stratum and a horizontal plane), show that

$$\tan^2 \delta = \tan^2 \alpha + \tan^2 \beta.$$

57. Two vertical planes at right angles to each other intersect a third plane that is inclined at an unknown angle θ to a horizontal plane. If the intersections of the vertical planes with the third plane make angles of α and β, respectively, with the horizontal plane, find the secant of θ.

Ans. $\sec \theta = \sqrt{1 + \tan^2 \alpha + \tan^2 \beta}$.

Note.—The answer to the above is an important formula used in calculus.

58. To determine the dip of a stratum that is under ground, three holes are bored at three angular points of a horizontal square of side a. The depths at which the stratum is struck are, respectively, p, q, and r ft. Show that the dip δ of the stratum is given by the equation

$$\tan \delta = \frac{\sqrt{(p - q)^2 + (q - r)^2}}{a}.$$

CHAPTER X

MISCELLANEOUS TRIGONOMETRIC EQUATIONS

98. Types of equations.—In this chapter equations of the following types will be considered:

(1) Where there is one unknown angle involved in trigonometric functions.

(2) Where the unknown is not an angle but is involved in inverse trigonometric functions.

(3) Where there are other unknowns, as well as unknown angles, involved in simultaneous equations; but only the angle involved trigonometrically.

(4) Where the unknown angle is involved both algebraically and trigonometrically.

It is not possible to give general solutions of equations of all these types. They offer algebraic as well as trigonometric difficulties. Methods of solution are best shown by examples.

EXERCISES

1. Given $\tan 2\theta = \frac{24}{7}$; find $\sin\theta$ and $\cos\theta$ without finding θ, for values in the first and second quadrants.

Solution.—By [**21**], $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

$$\therefore \frac{2\tan\theta}{1-\tan^2\theta} = \frac{24}{7}, \text{ or } 12\tan^2\theta + 7\tan\theta - 12 = 0.$$

Solving for $\tan\theta$, $\tan\theta = \dfrac{-7 \pm \sqrt{49+576}}{24} = -\dfrac{4}{3}$ or $\dfrac{3}{4}$.

When $\tan\theta = \frac{3}{4}$, $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$.

When $\tan\theta = -\frac{4}{3}$, $\sin\theta = \frac{4}{5}$ and $\cos\theta = -\frac{3}{5}$.

The student can easily verify these by triangles or formulas.

2. Given $\tan^{-1}(a+1) + \tan^{-1}(a-1) = \tan^{-1} 2$; find a.

Solution.—Let $\theta = \tan^{-1}(a+1)$; then $\tan\theta = a+1$.
Let $\beta = \tan^{-1}(a-1)$; then $\tan\beta = a-1$.

$$\tan(\theta+\beta) = \frac{\tan\theta+\tan\beta}{1-\tan\theta\tan\beta} = \frac{a+1+a-1}{1-(a+1)(a-1)} = \frac{2a}{2-a^2}.$$

$$\therefore \theta+\beta = \tan^{-1}\frac{2a}{2-a^2} = \tan^{-1} 2.$$

That is, $\dfrac{2a}{2-a^2} = 2$, whence $a^2 + a - 2 = 0$,

or $(a+2)(a-1) = 0$, whence $a = -2$ or 1.

Check.—When $a = -2$, $\tan^{-1}(-1) + \tan^{-1}(-3) = \tan^{-1}\dfrac{-1-3}{1-(-1)(-3)}$

$$= \tan^{-1}\frac{-4}{-2} = \tan^{-1} 2.$$

3. Given $\sin 2\theta = 2\sin\theta$; find $\theta < 360°$. *Ans.* $0, \pi$.

Suggestion.—Use $\sin 2\theta = 2\sin\theta\cos\theta$ and factor.

4. Given $\tan 2\theta = \frac{12}{5}$; find $\sin\theta$ and $\cos\theta$ for θ in quadrants I and II.
Ans. $\dfrac{2}{\sqrt{13}}, \pm\dfrac{3}{\sqrt{13}}$.

5. $2\cos^2 2\theta + \cos 2\theta - 1 = 0$; find θ. *Ans.* $(n \pm \frac{1}{6})\pi$, $(2n+1)\frac{\pi}{2}$.

6. $\sin 2\theta + \sin 4\theta + \sin 6\theta = 0$; find θ. *Ans.* $\dfrac{n\pi}{4}$, $(2n+1\pm\frac{1}{3})\pi$.

Suggestion.—Apply [**25**] to $\sin 2\theta + \sin 6\theta$. Factor the resulting equation and equate each factor to zero.

7. $\cos 2\theta = \sin\theta$; find θ. *Ans.* $(2n+\frac{1}{2}\pm\frac{1}{3})\pi$, $(2n+\frac{3}{2})\pi$.

8. Given $\tan^{-1}(a+1) + \tan^{-1}(a-1) = \tan^{-1}(-\frac{7}{24})$; solve for a.
Ans. 7.137 or −0.280.

9. Given $r\sin\theta = 2$ and $r\cos\theta = 4$; solve for r and θ.
Ans. $r = 2\sqrt{5}$; $\theta = 26° 33' 53''$, $206° 33' 53''$.

Suggestion.—Square both equations and add, to obtain r. Divide the first by the second to obtain θ.

10. Given $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{3}{4}$; find x.
Ans. 0.610 or −3.277.

11. Given $\cos^{-1}(1-a) + \cos^{-1} a = \cos^{-1}(-a)$; solve for a.
Ans. 0 or $\frac{1}{2}$.

12. Given $r\sin(\theta - \tan^{-1}\frac{1}{4}) = 3$, and $r\cos(\theta - \tan^{-1}\frac{1}{4}) = 6$; find r and θ.
Ans. $r = 3\sqrt{5}$; $\theta = 40° 36' 5''$, $220° 36' 5''$.

Suggestion.—Obtain r as in Exercise 9. To obtain θ, divide one equation by the other; expand the functions and solve for $\tan\theta$.

13. $\sin 4\theta = 2\cos 2\theta$; find θ. *Ans.* $(n+\frac{1}{2})\frac{\pi}{2}$.

14. $\tan^{-1}\dfrac{a+1}{a-1} + \tan^{-1}\dfrac{a-1}{a} = \tan^{-1} x$. Find a when (*a*) $x = 1$, (*b*) $x = 2$, (*c*) $x = -7$. *Ans.* (*a*) $a = 0$, (*b*) $a = \frac{1}{2}$ or -1, (*c*) $a = 2$.

15. $\tan 2\alpha \tan\alpha = 1$; find α. *Ans.* $(n \pm \frac{1}{6})\pi$.

16. $\sin(120° - x) - \sin(120° + x) = \frac{1}{2}\sqrt{3}$; find x. *Ans.* 60°, 120°.

17. $\cos(30° + \theta) - \sin(60° + \theta) = -\frac{1}{2}\sqrt{3}$; find θ. *Ans.* 60°, 120°.

18. $\sqrt{1 - \sqrt{\sin^4 x + \sin^2 x}} = \sin x - 1$; solve for x.

19. $\sqrt{7\sin\alpha - 5} + \sqrt{4\sin\alpha - 1} = \sqrt{7\sin\alpha - 4} + \sqrt{4\sin\alpha - 2}$; solve for α. *Ans.* $(2n+\frac{1}{2})\pi$.

20. Given $\tan(80° - \frac{1}{2}\theta) = \cot\frac{2}{3}\theta$; find θ. *Ans.* 60°.

21. Given $3\sin^{-1} x + 2\cos^{-1} x = 240°$; solve for x. *Ans.* $\frac{1}{2}\sqrt{3}$.

22. Given $\tan^{-1} x + 2\cot^{-1} x = 135°$; solve for x. *Ans.* 1.

23. Given $\tan 2x (\tan^2 x - 1) = 2 \sec^2 x - 6$; solve for x.
Ans. 45°, 225°, 116° 33′ 56″, 296° 33′ 56″.

24. Given $10 \cos \theta - 5 \sin \theta = 2$; show that $\theta = 2 \tan^{-1} \frac{1}{3}$.

99. To solve $r \sin \theta + s \cos \theta = t$, for θ, when r, s, and t are known. *Solution.*—Either $\sin \theta$ or $\cos \theta$ can be eliminated by means of the relation $\sin^2 \theta + \cos^2 \theta = 1$, but logarithms are not applicable to this solution. A solution will now be given in which the computations may be done by logarithms.

(1) Let $\qquad m \sin \gamma = r$, and $m \cos \gamma = s$,

where m is a positive constant, and γ an auxiliary angle.

Such an assumption is always permissible, for, squaring both equations of (1) and adding,

$$m^2 \sin^2 \gamma + m^2 \cos^2 \gamma = r^2 + s^2, \text{ or } m^2 = r^2 + s^2, \text{ or } m = \sqrt{r^2 + s^2}.$$

Then m is real if r and s are real quantities.

Dividing the first equation of (1) by the second,

(2) $$\tan \gamma = \frac{r}{s}.$$

Since the tangent may have any real value from $-\infty$ to $+\infty$, when r and s are real, the angle γ will always exist.

Substituting (1) in the original equation,

$m \sin \gamma \, \theta + m \cos \gamma \cos \theta = t$, which, by [16], gives

(3) $$m \cos (\theta - \gamma) = t.$$

Now m and γ can be determined from (1) and (2), and then $\theta - \gamma$ from (3). From this θ is determined.

Example.—Given $3 \sin \theta + 4 \cos \theta = 2$; find θ.

Solution.—This is of the form given in this article, and $r = 3$, $s = 4$, and $t = 2$. $\therefore m = \sqrt{r^2 + s^2} = \sqrt{9 + 16} = 5$.

$$\tan \gamma = \frac{r}{s} = \frac{3}{4} = 0.75, \text{ and } \gamma = 36° \, 52' \, 12'' \text{ or } 216° \, 52' \, 12''.$$

Since r and s are both positive, $\sin \gamma$ and $\cos \gamma$ are positive. Therefore, γ is in the first quadrant, and so must be 36° 52′ 12″ only.

$$\cos (\theta - \gamma) = \frac{t}{m} = \frac{2}{5} = 0.4, \text{ by equation (3).}$$

$$\theta - \gamma = 66° \, 25' \, 18'' \text{ or } 293° \, 34' \, 42''.$$

$$\therefore \theta = 66° \, 25' \, 18'' + 36° \, 52' \, 12'' = 103° \, 17' \, 30'',$$

and $$\theta = 293° \, 34' \, 42'' + 36° \, 52' \, 12'' = 330° \, 26' \, 54''.$$

The method given in this article enables one to combine two simple harmonic motions of the same period into a single simple harmonic motion of the same period.

Thus, $r \sin \theta \pm s \cos \theta$ becomes $m \cos (\theta \mp \gamma)$.

100. Equations in the form $\begin{cases} \rho \sin \alpha \cos \beta = a \\ \rho \sin \alpha \sin \beta = b \\ \rho \cos \alpha = c \end{cases}$

where ρ, α, and β are variables. *Solution.*—Squaring all three equations and adding,

$$\rho^2 \sin^2 \alpha \cos^2 \beta + \rho^2 \sin^2 \alpha \sin^2 \beta + \rho^2 \cos^2 \alpha = a^2 + b^2 + c^2.$$
$$\rho^2 \sin^2 \alpha (\cos^2 \beta + \sin^2 \beta) + \rho^2 \cos^2 \alpha = a^2 + b^2 + c^2.$$
$$\rho^2(\sin^2 \alpha + \cos^2 \alpha) = \rho^2 = a^2 + b^2 + c^2.$$

From the third equation,

$$\cos \alpha = \frac{c}{\rho} = \frac{c}{\pm\sqrt{a^2 + b^2 + c^2}}, \text{ or } \alpha = \cos^{-1} \frac{c}{\pm\sqrt{a^2 + b^2 + c^2}}.$$

Dividing the second equation by the first, $\dfrac{\rho \sin \alpha \sin \beta}{\rho \sin \alpha \cos \beta} = \dfrac{b}{a}.$

Whence, $\tan \beta = \dfrac{b}{a}$; $\beta = \tan^{-1} \dfrac{b}{a}.$

101. Equations in the form $\sin (\alpha + \beta) = c \sin \alpha$, where β and c are known. *Solution.*—Dividing by $\sin \alpha$,

$$\frac{\sin (\alpha + \beta)}{\sin \alpha} = \frac{c}{1}.$$

Taking the proportion by composition and division,

$$\frac{\sin (\alpha + \beta) + \sin \alpha}{\sin (\alpha + \beta) - \sin \alpha} = \frac{c + 1}{c - 1}.$$

By [25] and [26], $$\frac{2 \sin (\alpha + \frac{1}{2}\beta) \cos \frac{1}{2}\beta}{2 \cos (\alpha + \frac{1}{2}\beta) \sin \frac{1}{2}\beta} = \frac{c + 1}{c - 1}.$$

Applying [7], $$\frac{\tan (\alpha + \frac{1}{2}\beta)}{\tan \frac{1}{2}\beta} = \frac{c + 1}{c - 1},$$

or $$\tan \left(\alpha + \frac{1}{2}\beta\right) = \frac{c + 1}{c - 1} \tan \frac{1}{2}\beta.$$

From which, since β and c are known, α may be found.

Example.—Solve $\sin (\alpha + 50°) = 2 \sin \alpha$.

Solution.—Substituting 50° for β and 2 for c in the above formula, we have

$$\tan(\alpha + 25°) = \frac{2+1}{2-1}\tan 25° = 3 \tan 25°.$$

$$\begin{aligned} \log 3 &= 0.47712 \\ \log \tan 25° &= 9.66867 \\ \log \tan(\alpha + 25°) &= 0.14579 \\ \alpha + 25° &= 54° 26' 29'' \text{ or } 234° 26' 29''. \\ \alpha &= 29° 26' 29'' \text{ or } 209° 26' 29''. \end{aligned}$$

102. Equations in the form $\tan(\alpha + \beta) = c \tan \alpha$, where β and c are known. *Solution.*—Dividing by $\tan \alpha$ and taking the resulting proportion by composition and division,

$$\frac{\tan(\alpha+\beta)}{\tan \alpha} = \frac{c}{1}.$$

$$\frac{\tan(\alpha+\beta) + \tan \alpha}{\tan(\alpha+\beta) - \tan\alpha} = \frac{c+1}{c-1}.$$

$$\frac{\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} + \dfrac{\sin\alpha}{\cos\alpha}}{\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} - \dfrac{\sin\alpha}{\cos\alpha}} = \frac{\sin(2\alpha+\beta)}{\sin[(\alpha+\beta)-\alpha]} = \frac{c-1}{c+1}.$$

$$\therefore \sin(2\alpha+\beta) = \frac{c+1}{c-1}\sin\beta.$$

Since c and β are known, α may be found.

Example.—Given $\tan(\alpha + 24°) = 4 \tan \alpha$; find α.

Solution.—Substituting 24° for β and 4 for c in the above formula, we have

$$\begin{aligned} \sin(2\alpha + 24°) &= \tfrac{5}{3}\sin 24°. \\ \log 5 &= 0.69897 \\ \log \sin 24° &= 9.60931 \\ \text{colog } 3 &= 9.52288 \\ \log \sin(2\alpha + 24°) &= 9.83116 \\ 2\alpha + 24° &= 42° 40.7', 137° 19.3', 402° 40.7', 497° 19.3'. \\ 2\alpha &= 18° 40.7', 113° 19.3', 378° 40.7', 473° 19.3'. \\ \therefore \alpha &= 9° 20.4', 56° 39.6', 189° 20.4', 236° 39.6'. \end{aligned}$$

103. Equations of the form $t = \theta + \phi \sin t$, where θ and ϕ are given angles.—First express θ and ϕ in radians if not already so given. Then t must satisfy the relation $t - \theta = \phi \sin t$.

Let $$y_1 = t - \theta \text{ and } y_2 = \phi \sin t.$$

Plot the straight line with equation $y_1 = t - \theta$, and the sine curve $y_2 = \phi \sin t$. An approximate value of t can be determined from the value of t where the line and the curve intersect. The more nearly accurate the sine curve is plotted the more nearly will the value of t come to the solution of the equation.

Example.—Given $t = 2 + \pi \sin t$.

Let $y_1 = t - 2$ and $y_2 = \pi \sin t$.

Now plot $y_1 = t - 2$, giving the line AB, as in Fig. 104. Also plot the modified sine curve with equation $y_2 = \pi \sin t$.

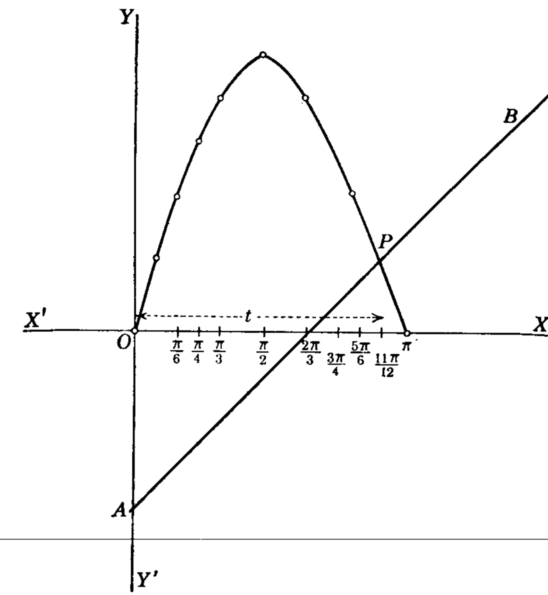


FIG. 104.

By measurement, the abscissa t for the point P of intersection is found to be 2.86 radians, or 164°. This is, therefore, an approximate solution for the equation.

Substituting for t in the original equation,

$$2.86 = 2 + \pi \sin 164° = 2 + 0.8659 = 2.8659.$$

This result shows the value of t to be too small.

Substituting $t = 165°$, $2.88 = 2 + \pi \sin 165° = 2.813$.

This result shows that 165° is too large, which the intersection of the curves also verifies. The correct value may now be approximated by assuming values of t between 164 and 165°, say 164° 10', etc.

EXERCISES

1. Change $3 \cos\theta + 4 \sin\theta$ to the form $m \cos(\theta - \gamma)$.
Ans. $5 \cos(\theta - 53° 7' 45'')$.

2. Change $a \cos \omega t + b \sin \omega t$ to the form $m \cos (\omega t - \gamma)$.

Ans. $\sqrt{a^2 + b^2} \cos \left(\omega t - \tan^{-1} \frac{b}{a}\right)$.

3. Given $5 \sin \theta - 2 \cos \theta = 3$; find θ. *Ans.* 55° 39′ 20″, 167° 56′ 50″.

Suggestion.—$\sin \gamma$ is + and $\cos \gamma$ is −; therefore, γ will be in second quadrant.

4. Given $2 \sin \theta + 5 \cos \theta = -3$; find θ.

Ans. 145° 39′ 20″, 257° 56′ 50″.

5. Given $1.31 \sin \theta - 3.58 \cos \theta = 1.885$; find θ.

Ans. 99° 32′ 10″, 220° 16′.

Suggestion.—Use logarithms in the solution.

6. Given
$$\rho \sin \alpha \cos \beta = 3,$$
$$\rho \sin \alpha \sin \beta = 2,$$
$$\rho \cos \alpha = 1; \text{ find } \alpha, \beta, \text{ and } \rho.$$

Ans. $\rho = \sqrt{14}$; $\alpha = 74° 29' 56''$; $\beta = 33° 41' 24''$.

7. Given
$$\rho \sin \theta \cos \phi = 6,$$
$$\rho \sin \theta \sin \phi = 2,$$
$$\rho \cos \theta = 0; \text{ find } \theta, \phi, \text{ and } \rho.$$

Ans. $\rho = 2\sqrt{10}$; $\theta = 90°$; $\phi = 18° 26' 5''$.

8. Given $\sin (x + 32° 16') = 4 \sin x$; find x.

Ans. 9° 36′ 23″, 189° 36′ 23″.

9. Given $\sin (y - 75°) = 3 \sin y$; find y. *Ans.* 160° 35.3′, 340° 35.3′.

10. Given $\tan (r + 40°) = 5 \tan r$; find r.

Ans. 17° 18.6′, 32° 41.4′, 197° 18.6′, 212° 41.4.

11. Given $\tan (s - 60° 20') = 2 \tan s$; find s.

12. Given $x = 1 + 30° \sin x$; find x approximately.

13. Given $S = 60° + \frac{\pi}{3} \sin S$; find S approximately.

CHAPTER XI

COMPLEX NUMBERS, DEMOIVRE'S THEOREM, SERIES

104. Imaginary numbers.—If the equation $x^2 + 1 = 0$ is solved, we obtain the symbol $\sqrt{-1}$, and from its derivation the square of this symbol must be -1. The symbol $\sqrt{-1}$ is commonly represented by i, and is called the **unit of imaginaries.** It follows that, if i is a number, it is such a number that $i^2 = -1$. This is taken as the definition of i.

As the only property attached to i by its definition is that its square is -1, it may be multiplied by any real number a. The product ai is called an **imaginary number.**

In contradistinction to imaginary numbers, the rational and irrational numbers, including positive and negative integers and fractions, are called **real numbers.**

The name "imaginary number" suggests an unreality that does not exist, for in the present state of the development of mathematics the imaginary number in comparison with whole numbers is no more unreal in the ordinary sense than is the fraction or the irrational number.

The system of numbers created by accepting the imaginary unit is an entirely new system of numbers distinct from the system of real numbers. The operations upon these numbers and their combinations with real numbers present various applications of trigonometry; and, further, the method developed in this field can be used to advantage in deriving various formulas in trigonometry.

105. Square root of a negative number.—By definition, the square root of a number is such a number that, when multiplied by itself, will give the original number. If c is a positive real number, no real number can equal $\sqrt{-c}$, for, by the definition of square root,

$$\sqrt{-c} \cdot \sqrt{-c} = -c.$$

But the square of every real number is positive. Hence, the square root of a negative real number is not a real number. That it is an imaginary number can be shown as follows:

$i^2 = -1$, by definition of unit of imaginaries.

$\sqrt{c} \cdot \sqrt{c} = c$, by definition of square root.

Then $\sqrt{ci} \cdot \sqrt{ci} = c(-1) = -c$, by multiplying.

$\therefore \sqrt{ci} = \sqrt{-c}$, taking the square root of each member.

Therefore, $\sqrt{-c}$ is an imaginary number, being of the form ai of **Art. 104.**

It follows that any number of the form $\sqrt{-c}$, where c is a positive real number can be put in the form $\sqrt{c}i$, which will be called the **proper imaginary form.** It is always best to write an imaginary number in the proper form before performing an operation.

Thus,
$$\sqrt{-4} = \sqrt{4}i = 2i.$$
$$\sqrt{-6} = \sqrt{6}i.$$

106. Operations with imaginary numbers.—It can be shown that, with proper definitions for combining imaginary numbers, they act like real numbers and obey all the laws of algebra, with the exception of the law

$$\sqrt{a}\sqrt{b} = \sqrt{ab}.$$

This law is excepted because it conflicts with the definition of the unit of imaginaries, and a definition is always fundamental.

Thus, if this law did apply, we should have

$$\sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1,$$

whereas, by definition, $\sqrt{-1}\sqrt{-1} = i^2 = -1$.

It, therefore, contradicts the definition of the unit of imaginaries to say

$$\sqrt{-2}\sqrt{-3} = \sqrt{(-2)(-3)} = \sqrt{6}.$$

If the imaginary numbers are first put in the proper form, no trouble will occur.

Thus, $\sqrt{-2}\sqrt{-3} = \sqrt{2}i \cdot \sqrt{3}i = \sqrt{6}i^2 = -\sqrt{6}$.

107. Complex numbers.—In order to solve the general quadratic equation $mx^2 + nx + l = 0$, it is necessary to have numbers of the form $a + bi$. These numbers are formed by adding a real number to an imaginary number.

Thus, the solution of $x^2 - 4x + 13 = 0$ gives the two values for x, $2 + 3i$ and $2 - 3i$.

Numbers of the form $a + bi$, where a and b are real numbers and $i^2 = -1$, are called **complex numbers.**

It should be noted that a real number is a special case of the complex number $a + bi$ where $b = 0$, and an imaginary number is a special case where $a = 0$.

108. Conjugate complex numbers are complex numbers which differ only in the signs of their imaginary parts. Thus, $3 - 4i$ and $3 + 4i$ are conjugate complex numbers. Also $+2i$ and $-2i$ are conjugate.

EXERCISES

Write the complex numbers conjugate to the following:

1. $2 - 3i$.	**3.** $1 + \sqrt{3}i$.	**5.** $-7 - \sqrt{-1}$.	**7.** $\sqrt{6}i$.
2. $3 + i$.	**4.** $\sqrt{2} - \sqrt{-3}$.	**6.** $3\sqrt{-1}$.	**8.** $\sqrt{-4}$.

109. Graphical representation of complex numbers.—Long after imaginary numbers presented themselves in algebraic work, they were rejected by mathematicians as impossible. Indeed, this was the case with any new kind of number. The negative number was disregarded for centuries after it appeared, and was generally accepted only after its graphical representation was introduced by Descartes (1596–1650). The system of representing complex numbers graphically was discovered independently by Wessel, a Norwegian, in 1797; by Argand, a Frenchman, in 1806; and by Gauss, a German, in 1831. More recently the practical importance of complex numbers as graphically represented has been recognized by physicists and engineers. In the field of electricity they have been put to important uses by Steinmetz and others. The system for representing these numbers is as follows:

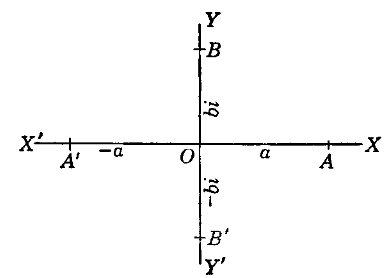


Fig. 105.

Draw the rectangular coordinate axes $X'OX$ and $Y'OY$ (Fig. 105). The real numbers can be represented by points on the line $X'OX$ as follows: The point A, corresponding to the positive number a, is taken a units to the right of O. The point A', corresponding to the negative number $-a$, is taken a units to the left of O. The number a can also be considered as represented by the line segment OA, and the number $-a$ by the line segment OA'.

If the point A represents the number 5, the point A', representing the number -5, can be obtained by rotating OA through 180°. It seems, then, that multiplying a number by -1 acts as if the multiplication rotated the line segment representing the number through 180°.

Multiplying a number twice by i, as $5ii$, is the same as multiplying it by -1. In other words, it rotates the line segment, representing the number 5, through 180°, but does not change the length of the line segment. This suggests that multiplying a number by i should rotate the line segment, representing the number, through 90°. Then an imaginary number like bi would be represented by a point B on the positive y-axis and b units above the x-axis. Likewise, the imaginary number $-bi$ would be represented by a point B' on the negative y-axis and b units below the x-axis. The imaginary number bi can also be considered as represented by the line segment OB, and the imaginary number $-bi$, by the line segment OB'.

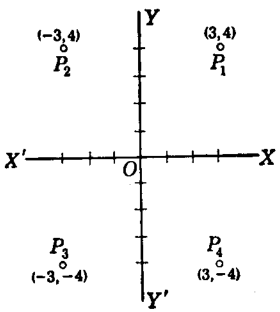


FIG. 106

The line on which the real numbers are represented is called the **axis of reals.** The line on which the imaginary numbers are represented is called the **axis of imaginaries.**

By this system it is easy to represent complex numbers graphically. In Fig. 106, the complex number $3 + 4i$ is represented by the point (3, 4), 3 units to the right of the y-axis and 4 units above the x-axis. Likewise, the complex number $3 - 4i$ is represented by the point $(3, -4)$; the complex number $-3 + 4i$ by the point $(-3, 4)$; and the complex number $-3 - 4i$ by the point $(-3, -4)$.

The figure on which the complex numbers are plotted is called the **Argand diagram.**

EXERCISES

Plot the following complex numbers:

1. $2 + 3i$; $-2 + 3i$; $2 - 3i$; $-2 - 3i$.
2. $4 - i$; $4 + i$; $6i$; $-5i$; $-6 + 3i$; $-3 - 5i$; 4.
3. $\pi + i$; $\pi + \pi i$; $-1 - \pi i$; $e + \pi i$; $\pi - ei$; $e - i$; $-1 - ei$.
4. i; i^2; i^3; i^4; i^5; i^6; i^7; i^8.

Plot the following complex numbers and their conjugates:

5. $1 - i$, $2 + i$; $2i$; $3 - 2i$; $2i - 4$.

110. Powers of i.—It is a very easy matter to compute any power of i.

$i^1 = i$, by the definition of an exponent.
$i^2 = -1$, by the definition of the unit of imaginaries.
$i^3 = i^2 i = (-1)i = -i$.
$i^4 = (i^2)^2 = (-1)^2 = 1$.
$i^5 = (i^2)^2 i = (-1)^2 i = i$.
$i^6 = (i^2)^3 = (-1)^3 = -1$.
. . . .

By continuing this process, it is found that the integral powers of i recur in a cycle of the four different values i, -1, $-i$, and 1.

The powers of i can, perhaps, be made clearer by referring to Fig. 107, where each multiplication by i rotates the line segment of unit length through 90°.

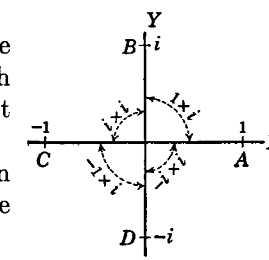


FIG. 107.

The value of any integral power of i can be readily found as illustrated in the following examples:

$$i^{17} = (i^2)^8 i = (-1)^8 i = i.$$
$$i^{18} = (i^2)^9 = (-1)^9 = -1.$$

EXERCISES

Compute the following powers of i:

1. i^7.	6. i^{20}.	11. $(-i)^{10}$.	16. i^{120}.
2. i^9.	7. i^{25}.	12. $(-i)^{11}$.	17. i^{202}.
3. i^{13}.	8. i^{30}.	13. $-i^6$.	18. $-i^{300}$.
4. i^{17}.	9. i^{35}.	14. $-i^8$.	19. $-i^{1000}$.
5. i^{18}.	10. $(-i)^8$.	15. i^{101}.	20. $(-i)^{567}$.

111. Operations on complex numbers.—Complex numbers, under proper definitions for the four fundamental operations, obey all the laws of algebra, with the exception of the law mentioned in **Art. 106.** In fact, complex numbers act the same as real numbers.

The four fundamental operations are defined as follows:

Addition. $(a + bi) + (c + di) = (a + c) + (b + d)i$.

Subtraction. $(a + bi) - (c + di) = (a - c) + (b - d)i$.

Multiplication. $(a + bi)(c + di) = ac + adi + bci + bdi^2$
$= ac - bd + (ad + bc)i$.

Division. $\dfrac{a+bi}{c+di} = \dfrac{(a+bi)(c-di)}{(c+di)(c-di)} = \dfrac{ac+bd+(bc-ad)i}{c^2+d^2}$

$= \dfrac{ac+bd}{c^2+d^2} + \dfrac{(bc-ad)i}{c^2+d^2}.$

Note that division cannot be defined if $c^2 + d^2 = 0$, that is, $c = 0$ and $d = 0$. In this case it will be seen in the next article that the complex number $c + di = 0$. Hence, in the field of complex numbers, division by zero is impossible.

Note also that the four fundamental operations on complex numbers always yield complex numbers.

EXERCISES*

Simplify the following by performing the operations indicated:

1. $(3 + 2i) + (6 - 7i)$.
2. $(1 + i) - (3 + 4i)$.
3. $(7 - i) + (3 + 4i)$.
4. $(1 - 6i) - (-7 + 3i)$.
5. $(3 + 2i)(1 + 5i)$.
6. $(-3 + 2i)(-3 - i)$.
7. $\dfrac{4-i}{2+i}$.
8. $\dfrac{5+2i}{3+4i}$.
9. $\dfrac{6-3i}{7-i}$.
10. $\dfrac{8+2i}{1-2i}$.
11. $\dfrac{\sqrt{2}+3i}{1-\sqrt{2}i}$.
12. $\dfrac{3-\sqrt{6}i}{6+\sqrt{3}i}$.
13. $\dfrac{2+3i}{\sqrt{2}-\sqrt{3}i}$.
14. $\dfrac{5}{\sqrt{2}+\sqrt{3}i}$.
15. $\dfrac{6-7i}{i}$.
16. $\dfrac{1+3i}{5i}$.
17. $(1 + \sqrt{5}i) \div \sqrt{5}i$.
18. $1 \div (1 - i)$.
19. $(1 - i)^2$.
20. $(1 + i)^3$.
21. $(3 - 4i)^2$.
22. $(\sqrt{3} - \sqrt{-2})^3$.
23. $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^3$.
24. $\left(\dfrac{1+\sqrt{-3}}{2}\right)^3$.
25. $\dfrac{\sqrt{-a}+\sqrt{-b}}{\sqrt{-a}-\sqrt{-b}}$.
26. $\dfrac{a+i\sqrt{1-a^2}}{a-i\sqrt{1-a^2}}$.
27. $\dfrac{a+i\sqrt{1+a^2}}{a-i\sqrt{1-a^2}}$.
28. $(i^9 + i^{10} + i^{11} + i^{12})^7$.
29. $(i^7 + i^8 + i^9)^{12}$.
30. $(i^6 + i^9 + i^{12})^5$.
31. $\left(2a^{-2} - \dfrac{a}{\sqrt{-2}}\right)^3$.
32. $\left(\dfrac{a}{\sqrt{3}} + \dfrac{\sqrt{-3}}{a}\right)^3$.
33. Prove that $1 + i$ is a root of the equation

$$2x^3 - x^2 - 2x + 6 = 0.$$

34. Find the value of $x^3 - 2x^2 + 9x + 13$ when $x = 2 + 3i$.

* Answers to above problems will be found at the end of this chapter.

35. Find the value of $(x^2 + 5x)^2 + x(x + 5)$ when $x = \dfrac{-5 + i\sqrt{3}}{2}$.
36. Find the value of $\dfrac{3x^2 - 4x + 12}{x^2 + x + 1}$ when $x = 3 + i$. *Ans.* 2.
37. Find the value of $\dfrac{5x^2 - 6x + 9}{2x^2 - x + 2}$ when $x = 2 + i$. *Ans.* 2.
38. Prove that the sum and the product of two conjugate complex numbers are both real.
39. Prove that, if the sum and the product of two complex numbers are real, the numbers are conjugate complex numbers.

112. Properties of complex numbers. THEOREM I.—*If the complex number* $a + bi = 0$, *then* $a = 0$ *and* $b = 0$.

Proof.—Since the laws of algebra for real numbers hold with one exception for complex numbers, if bi is transposed to the right-hand side, then $a = -bi$. Squaring both sides of this equation gives $a^2 = -b^2$. But a positive number cannot equal a negative number unless both are zero. Hence $a = 0$ and $b = 0$.

THEOREM II.—*If* $a + bi$ *and* $c + di$ *are two complex numbers such that* $a + bi = c + di$, *then* $a = c$ *and* $b = d$.

Proof.—If $c + di$ is transposed to the left-hand side of the equation, $a - c + (b - d)i = 0$. Then, by Theorem I, $a - c = 0$ and $b - d = 0$. Hence, $a = c$ and $b = d$.

THEOREM III.—*If the product of two complex numbers vanishes, at least one of the factors must vanish, and conversely.*

The proof of this theorem is to be given as an exercise.

EXERCISES

Find the real values of x and y for which the following equations are true.

1. $x + y + (2x + 3y)i = 3 + i$. *Ans.* $x = 8$, $y = -5$.

Suggestion.—Apply Theorem II, which gives

$$x + y = 3, \text{ and } 2x + 3y = 1.$$

2. $3x - 2 + (-2)i = y(1 - i)$. *Ans.* $x = 1\frac{1}{3}$; $y = 2$.
3. $y + 16 + 2(y + 1)i = 2x(2 - i)$. *Ans.* $x = 3$; $y = -4$.
4. $x(1 + i) + y(1 - i) = 2$. *Ans.* $x = 1$; $y = 1$.
5. $x^2 + 2xyi + y^2 = 25 + 24i$. *Ans.* $x = \pm 4$ and ± 3; $y = \pm 3$ and ± 4.
6. $x(x + i) + y(y + i) = 5 + 3i$. *Ans.* $x = 1$ and 2; $y = 2$ and 1.
7. The product of two complex numbers is $5 - i$, the sum of their real parts is 3, and the product of their real parts is 2. Find the numbers. *Ans.* $1 + i$ and $2 - 3i$, or $1 - \frac{3}{2}i$ and $2 + 2i$.
8. Find two conjugate complex numbers whose product is 13, and the product of whose imaginary parts is 9. *Ans.* $2 + 3i$ and $2 - 3i$.

113. Complex numbers and vectors.—By means of the Argand diagram the general complex number $a + bi$ is represented by the point P (Fig. 108), with coordinates (a, b). In **Art. 109,**

it was seen that a real number or a pure imaginary number can be represented by either a point or a line segment. This notion can be extended to complex numbers by representing the complex number $a + bi$ by the line segment OP. For, if the segment OP is given, a, the real part of the complex number, equals the projection of OP on the x-axis, and b, the coefficient of i, equals the projection of OP on the y-axis.

Definition.—A quantity that has magnitude as well as direction is called a **vector.**

The line segment OP begins at the origin and ends at P. It, therefore, has a magnitude and a direction. Hence the complex number P can be represented by the vector OP. Hereafter, the word "vector" will be used in place of "line segment."

114. Polar form of complex numbers.—The vectorial representation of a complex number enables it to be written in another form, called the "polar form" of the complex number. Let θ be the angle through which the positive portion of the x-axis would have to be revolved in order to coincide with the vector OP (Fig. 108). The angle θ is called the **amplitude,** or **argument,** of the complex number $a + bi$. Let r be the length, or magnitude, of the vector OP. The number r is called the **modulus** of the complex number $a + bi$, and is always taken *positive.* From the right triangle whose sides are a, b, and r,

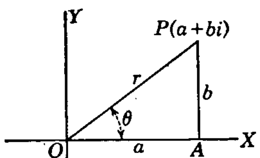


FIG. 108.

$$a = r\cos\theta, \text{ and } b = r\sin\theta.$$

Note that these equations hold no matter in what quadrant θ lies. The complex number $a + bi$ can now be written

$$a + bi = r(\cos\theta + i\sin\theta).$$

The expression $r(\cos\theta + i\sin\theta)$ is called the **polar form** of a complex number.

The expression $a + bi$ is called the **rectangular form** of a complex number.

If θ is increased, or decreased, by multiples of 360°, the sine and the cosine are not changed, then the polar form of a complex number can be written

$$r[\cos(\theta + k \cdot 360°) + i\sin(\theta + k \cdot 360°)],$$

where k is any positive or negative integer. This is called the **complete polar form** of a complex number.

The values of r and θ in terms of a and b can be obtained immediately from Fig. 108. They are

$$r = \sqrt{a^2 + b^2}, \text{ and } \theta = \sin^{-1}\frac{b}{r} = \cos^{-1}\frac{a}{r} = \tan^{-1}\frac{b}{a}.$$

Example 1.—Write $2 + 2\sqrt{3}i$ in the polar form. Plot.

$$r = \sqrt{2^2 + (2\sqrt{3})^2} = 4.$$

$$\theta = \tan^{-1}\frac{2\sqrt{3}}{2} = \tan^{-1}\sqrt{3} = 60°.$$

$$\therefore 2 + 2\sqrt{3}i = 4(\cos 60° + i\sin 60°).$$

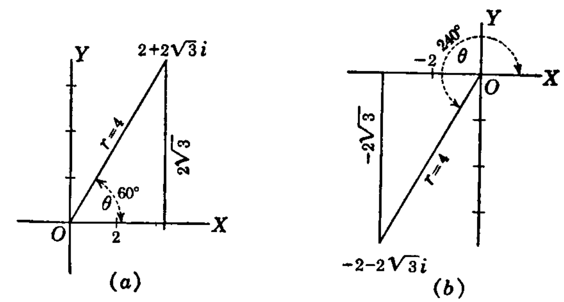


FIG. 109.

The plotting is shown in Fig. 109*a*.

Example 2.—Write $-2 - 2\sqrt{3}i$ in the polar form. Plot.

$$r = \sqrt{(-2)^2 + (-2\sqrt{3})^2} = 4.$$

$$\theta = \tan^{-1}\frac{-2\sqrt{3}}{-2} = \tan^{-1}\sqrt{3} = 240°.$$

Here 240° is taken because both a and b are negative and θ is, therefore, in the third quadrant.

$$\therefore -2 - 2\sqrt{3}i = 4(\cos 240° + i\sin 240°).$$

The plotting is shown in Fig. 109*b*.

Note that, while a and b may be negative numbers, r is always *positive,* and that the signs in front of $\cos\theta$ and $\sin\theta$ are always *plus.*

The complete polar forms of the complex numbers in the preceding examples are

$$2 + 2\sqrt{3}i = 4[\cos(60° + k \cdot 360°) + i\sin(60° + k \cdot 360°)],$$

and

$$-2 - 2\sqrt{3}i = 4[\cos(240° + k \cdot 360°) + i \sin(240° + k \cdot 360°)].$$

Suggestion.—In changing from rectangular form to polar form, it is better first to plot the complex number, and then check the results obtained by computation with those indicated on the graph. Thus, the common error of writing for θ a first-quadrant angle when θ is an angle in some other quadrant may be avoided. Often the values of r and θ can be obtained directly from the figure.

EXERCISES*

Write the following complex numbers in the polar form:

1. $1 + i$. *Ans.* $\sqrt{2}(\cos 45° + i \sin 45°)$.
2. $-1 + i$. *Ans.* $\sqrt{2}(\cos 135° + i \sin 135°)$.
3. $-1 - i$. *Ans.* $\sqrt{2}(\cos 225° + i \sin 225°)$.
4. $1 - i$. *Ans.* $\sqrt{2}(\cos 315° + i \sin 315°)$.
5. $-\sqrt{3} + 3i$.
6. $\sqrt{3} - 3i$.
7. $\sqrt{6} + 3\sqrt{2}i$.
8. $-\sqrt[3]{2} + \sqrt[3]{2}i$.
9. $-3\sqrt{2} - 3\sqrt{6}i$.
10. $\sqrt{2} + \sqrt{6}i$.
11. $\sqrt{5} + \sqrt{15}i$.
12. $-\sqrt{21} + \sqrt{7}i$.
13. 4.
14. $4i$.
15. -6.
16. $-6i$.
17. i.
18. $-i$.
19. 1.
20. -1.
21. $\sqrt{2}$.
22. $\sqrt{-3}$.
23. $2i^2$.
24. $3i^3$.
25. $1 + 3i$.
26. $4 - 2i$.
27. $-4 + 3i$.
28. $\cos 30° - i \sin 30°$.
29. $-\cos 75° + i \sin 75°$.
30. $-3(\cos 10° + i \sin 10°)$.
31. $\cos 30° + i \sin 60°$.
32. $\sin 30° + i \sin 240°$.
33. $-\sin 210° - i \sin 120°$.

Write the following complex numbers in the complete polar form:

34. $3 + 3i$. **35.** $-\sqrt{3} - 3i$. **36.** $-1 + \sqrt{3}i$. **37.** $3i$.

Write the following complex numbers in the rectangular form:

38. $3(\cos 30° + i \sin 30°)$.
39. $\sqrt{2}(\cos 45° + i \sin 45°)$.
40. $4(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)$.
41. $2(\cos \frac{1}{2}\pi + i \sin \frac{1}{2}\pi)$.
42. $4(\cos \frac{7}{4}\pi + i \sin \frac{7}{4}\pi)$.
43. $4(\cos \frac{7}{6}\pi + i \sin \frac{7}{6}\pi)$.
44. $10(\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi)$.
45. $6(\cos 720° + i \sin 720°)$.
46. $2(\cos 150° + i \sin 150°)$.
47. $2(\cos 510° + i \sin 510°)$.
48. $\cos(-210°) + i \sin(-210°)$.
49. $\cos(-570°) + i \sin(-570°)$.
50. $\cos 100° + i \sin 100°$.
51. $2(\cos 200° + i \sin 200°)$.
52. $3(\cos 300° + i \sin 300°)$.
53. $4(\cos 1000° + i \sin 1000°)$.

115. Graphical representation of addition.—Let the vector OP (Fig. 110), represent the complex number $a + bi$, and let the vector OS represent the complex number $c + di$.

* Answers to some of the above problems will be found at the end of this chapter.

In order to represent graphically the sum of $a + bi$ and $c + di$, complete the parallelogram $OPTS$ by drawing PT parallel to OS and ST parallel to OP. Then the vector OT represents the complex number $(a + bi) + (c + di)$.

Proof.—Drop perpendiculars from P and T to the x-axis, and call the feet of these perpendiculars P_1 and T_1, respectively. Also drop perpendiculars from S and T to the y-axis, and call the feet of these perpendiculars S_2 and T_2, respectively.

Then the real part of the complex number represented by the vector OT is $OT_1 = OP_1 + P_1T_1 = a + c$.

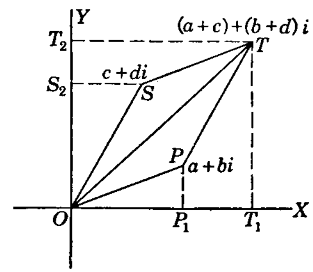


FIG. 110.

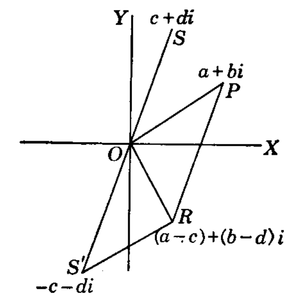


FIG. 111.

The coefficient of i for the complex number represented by the vector OT is $OT_2 = OS_2 + S_2T_2 = b + d$.

Therefore, OT represents the complex number $a + c + (b + d)i$.

116. Graphical representation of subtraction.—In order to represent graphically $(a + bi) - (c + di)$, write the expression in the form $(a + bi) + (-c - di)$. In Fig. 111, produce the line OS through the origin to a point S', so that $S'O = OS$. The vector OS' represents the complex number $-c - di$. Then add the vector OS' to OP precisely as was done in the case of addition.

EXERCISES

Perform the following operations graphically, and check the results algebraically:

1. $(3 + 4i) + (5 + 2i)$.
2. $(-3 + 2i) + (6 - 3i)$.
3. $(2 + i) + i$.
4. $2 + (3 - i)$.
5. $(3 + 4i) - (5 + 2i)$.
6. $(-3 + 2i) - (8 - 3i)$.
7. $(2 + i) - i$.
8. $(1 - i) + (-2 + 3i) + (4 + i)$.
9. $2(\cos 45° + i \sin 45°) + (\cos 135° + i \sin 135°)$.
10. $(\cos 30° + i \sin 30°) + (\cos 60° + i \sin 60°)$.
11. $(\cos 0° + i \sin 0°) + 4(\cos 90° + i \sin 90°)$.
12. $(\cos 60° + i \sin 60°) + (\cos 180° + i \sin 180°) + (\cos 300° + i \sin 300°)$.
13. $2(\cos 120° + i \sin 120°) - 3(\cos 135° + i \sin 135°)$.

117. Multiplication of complex numbers in polar form. THEOREM.—*The modulus of the product of two complex numbers equals the product of their moduli, and the amplitude of their product equals the sum of their amplitudes.*

Proof.—Let $r_1(\cos\theta_1 + i\sin\theta_1)$, and $r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers. Then their product is

$$[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)]$$
$$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$
$$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \qquad \text{By } [\mathbf{13}] \text{ and } [\mathbf{14}].$$

It is evident that this theorem can be generalized to include the product of any number of complex numbers. Thus,

$$[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)]\cdots[r_n(\cos\theta_n + i\sin\theta_n)]$$
$$= r_1r_2\cdots r_n[\cos(\theta_1+\theta_2+\cdots\theta_n) + i\sin(\theta_1+\theta_2+\cdots\theta_n)].$$

Example.—$[5(\cos 15° + i\sin 15°)][6(\cos 20° + i\sin 20°)] = 30(\cos 35° + i\sin 35°)$.

If the result is required in the rectangular form, find the value $\cos 35°$ and $\sin 35°$ in trigonometric tables.

Thus, $30(\cos 35° + i\sin 35°) = 24.575 + 17.207i$.

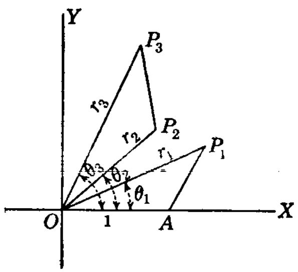


FIG. 112.

118. Graphical representation of multiplication.—Let the vectors OP_1 and OP_2 (Fig. 112) represent the complex numbers $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$, respectively. Let the point A on the positive axis of reals be 1 unit distant from the origin. Join P_1 to A Construct a triangle OP_2P_3 similar to the triangle OAP_1 and similarly situated. Then the vector OP_3 represents $r_3(\cos\theta_3 + i\sin\theta_3)$, the required product.

Proof.—$\theta_3 = \angle AOP_2 + \angle P_2OP_3 = \angle AOP_2 + \angle AOP_1 = \theta_2 + \theta_1$.

Since corresponding sides of similar triangles are in proportion,

$$\frac{OP_3}{OP_2} = \frac{OP_1}{OA}. \quad \text{Whence } \frac{r_3}{r_2} = \frac{r_1}{1}, \text{ or } r_3 = r_2r_1.$$

Therefore,

$$r_3(\cos\theta_3 + i\sin\theta_3) = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$

119. Division of complex numbers in polar form. THEOREM. *The modulus of the quotient of two complex numbers equals the modulus of the dividend divided by the modulus of the divisor,*

and the amplitude of the quotient equals the amplitude of the dividend minus the amplitude of the divisor.

Proof.—

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$
$$= \frac{r_1}{r_2}\,\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$
$$= \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)].$$

Example.—$16(\cos 157° + i\sin 157°) \div 8(\cos 22° + i\sin 22°)$
$= 2(\cos 135° + i\sin 135°) = -\sqrt{2} + \sqrt{2}i$.

120. Graphical representation of division.—To divide graphically the complex number represented by the vector OP_3 by the complex number represented by the vector OP_1 (Fig. 112), construct a triangle OP_3P_2 similar to the triangle OP_1A and similarly situated. Then the complex number represented by the vector OP_2 is the required quotient. The proof is left as an exercise.

EXERCISES

Perform the following operations and express the results in rectangular form:

1. $[3(\cos 15° + i\sin 15°)][6(\cos 75° + i\sin 75°)]$. *Ans.* $18i$.
2. $[4(\cos 127° + i\sin 127°)][3(\cos 203° + i\sin 203°)]$. *Ans.* $6\sqrt{3} - 6i$.
3. $[2(\cos\frac{1}{5}\pi + i\sin\frac{1}{5}\pi)][5(\cos\frac{2}{15}\pi + i\sin\frac{2}{15}\pi)]$. *Ans.* $5 + 5\sqrt{3}i$.
4. $[3(\cos\frac{1}{12}\pi + i\sin\frac{1}{12}\pi)][5(\cos\frac{5}{12}\pi + i\sin\frac{5}{12}\pi)]$. *Ans.* $15i$.
5. $\dfrac{4(\cos 47° + i\sin 47°)}{2(\cos 17° + i\sin 17°)}$. *Ans.* $\sqrt{3} + i$.
6. $\dfrac{12(\cos 26° + i\sin 26°)}{3(\cos 206° + i\sin 206°)}$. *Ans.* -4.

Perform the following multiplications and divisions graphically and check the results algebraically:

7. $(1 + i)(2 + 3i)$.
8. $(10 + 11i) \div (4 + i)$.
9. $(2 + i)(1 - 2i)$.
10. $(3i - 1) \div (1 + i)$.
11. Plot $a + bi$ and $i(a + bi)$. Does multiplying a complex number by i rotate the vector representing the complex number through 90°? Prove.

121. Involution of complex numbers.—If all the factors of the generalized theorem for multiplication of complex numbers (**Art. 117**) are equal, the result is the nth power of $r(\cos\theta + i\sin\theta)$. Hence $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$.

This result is known as DeMoivre's Theorem, discovered by Abraham DeMoivre (1667–1754), who was French by birth but lived in England after the age of seventeen.

Note that the theorem has been established when n is a positive integer only.

Example 1.—$[r(\cos\theta + i\sin\theta)]^3 = r^3(\cos 3\theta + i\sin 3\theta)$.

Example 2.—$[2(\cos 40° + i\sin 40°)]^6 = 2^6(\cos 240° + i\sin 240°)$

$$= 64(-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}i) = -32 - 32\sqrt{3}i.$$

EXERCISES

Simplify the following and express the results in the rectangular form Plot in Exercises 1 to 6.

1. $[3(\cos 15° + i\sin 15°)]^6$. *Ans.* $729i$.
2. $[2(\cos 50° + i\sin 50)]^6$. *Ans.* $32 - 32\sqrt{3}i$.
3. $[2(\cos 120° + i\sin 120°)]^4$. *Ans.* $-8 + 8\sqrt{3}i$.
4. $[2(\cos 315° + i\sin 315°)]^4$. *Ans.* -16.
5. $(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i)^4$. *Ans.* -1.
6. $(\frac{1}{2}\sqrt{3} - \frac{1}{2}i)^5$. *Ans.* $-\frac{1}{2}\sqrt{3} - \frac{1}{2}i$.
7. $(\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i)^{100}$. *Ans.* -1.
8. $(\frac{1}{2}\sqrt{3} + \frac{1}{2}i)^{500}$. *Ans.* $-\frac{1}{2} - \frac{1}{2}\sqrt{3}i$.
9. $(-\frac{1}{2} + \frac{1}{2}\sqrt{3}i)^{1000}$. *Ans.* $-\frac{1}{2} + \frac{1}{2}\sqrt{3}i$.
10. $[\frac{1}{2}\sqrt{2}(-1 - i)]^{200}$. *Ans.* 1.

In Exercises 11 to 15, raise the right-hand side to the indicated power by the binomial theorem, simplify, and then apply Theorem II of **Art. 112**.

11. $\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2$.
12. $\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$.
13. $\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$.
14. $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$.
15. $\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$.
16. Using the results of Exercise 12, express $\tan 3\theta$ in terms of $\tan\theta$.
17. Using the results of Exercise 13, express $\tan 4\theta$ in terms of $\tan\theta$.

122. DeMoivre's theorem for negative and fractional exponents.—DeMoivre's theorem is also true when n is not a positive integer.

CASE I.—*When n is a positive rational number.*

Let $n = \frac{p}{q}$ and let $\theta = q\varphi$.

Then $[r(\cos\theta + i\sin\theta)]^n = [r(\cos\theta + i\sin\theta)]^{\frac{p}{q}}$

$$= [r(\cos q\varphi + i\sin q\varphi)]^{\frac{p}{q}} = [r(\cos\varphi + i\sin\varphi)^q]^{\frac{p}{q}}$$

$$= r^{\frac{p}{q}}(\cos\varphi + i\sin\varphi)^p = r^{\frac{p}{q}}(\cos p\varphi + i\sin p\varphi)$$

$$= r^n(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta) = r^n(\cos n\theta + i\sin n\theta).$$

CASE II.—*When n is a negative rational number.*

Let $n = -m$, where m is a positive integer or fraction.

Then $[r(\cos\theta + i\sin\theta)]^n = [r(\cos\theta + i\sin\theta)]^{-m}$

$$= \frac{1}{[r(\cos\theta + i\sin\theta)]^m} = \frac{1}{r^m(\cos m\theta + i\sin m\theta)}$$

$$= \frac{1(\cos 0° + i\sin 0°)}{r^m(\cos m\theta + i\sin m\theta)} = r^{-m}[\cos(-m\theta) + i\sin(-m\theta)]$$

$$= r^n(\cos n\theta + i\sin n\theta).$$

Hence DeMoivre's theorem is true when n is any rational number. The theorem can also be proved for irrational values of n.

123. Evolution of complex numbers.—By means of DeMoivre's theorem, all the nth roots of a complex number can be found. It will be seen that, in order to get all the roots, it will be necessary to use the complete polar form of the complex number. Thus,

$$\sqrt[n]{r(\cos\theta + i\sin\theta)} = [r(\cos\theta + i\sin\theta)]^{\frac{1}{n}}$$

$$= \{r[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]\}^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right).$$

By $r^{\frac{1}{n}}$ is meant the arithmetical value of $\sqrt[n]{r}$. Giving k the values $0, 1, 2, \cdots n - 1$, in succession, all the nth roots of the complex number can be found.

Example 1.—Find all the cube roots of $1 - i$ in polar form. Plot.

Solution.—Plot $1 - i$ and find r and θ by inspection or computation.

$$1 - i = \sqrt{2}[\cos(315° + k\cdot 360°) + i\sin(315° + k\cdot 360°)].$$

$$\sqrt[3]{1 - i} = \{\sqrt{2}[\cos(315° + k\cdot 360°) + i\sin(315° + k\cdot 360°)]\}^{\frac{1}{3}}$$

$$= \sqrt[6]{2}\left(\cos\frac{315° + k\cdot 360°}{3} + i\sin\frac{315° + k\cdot 360°}{3}\right)$$

$$= \sqrt[6]{2}[\cos(105° + k\cdot 120°) + i\sin(105° + k\cdot 120°)].$$

Giving k in succession the values 0, 1, 2, the required cube roots are found. Representing them by z_1, z_2, and z_3, they are

$$z_1 = \sqrt[6]{2}(\cos 105° + i\sin 105°),$$

$$z_2 = \sqrt[6]{2}(\cos 225° + i\sin 225°),$$

$$z_3 = \sqrt[6]{2}(\cos 345° + i\sin 345°).$$

These are all the cube roots, for, if k should be given values greater than 2, no new cube roots would be found, as every root so found would be either z_1, z_2, or z_3. The plotting is shown in Fig. 113.

The three cube roots can be changed to the rectangular form by using logarithms to express approximately in decimals the products indicated. This gives the following:

$$z_1 = -0.2905 + 1.084i.$$
$$z_2 = -0.7937 - 0.7937i.$$
$$z_3 = 1.084 - 0.2905i.$$

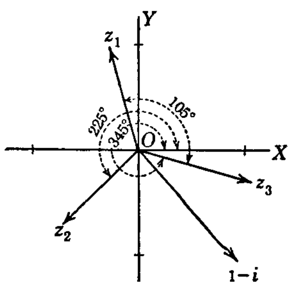


FIG. 113.

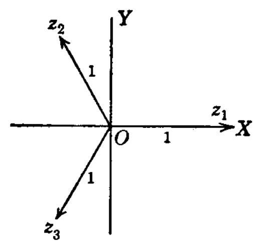


FIG. 114.

Example 2.—Find all the cube roots of 1 in rectangular form. Plot.

Solution.—The modulus of 1 is 1 and its amplitude is 0°.

Then $1 = 1[\cos(0° + k \cdot 360°) + i \sin(0° + k \cdot 360°)]$
$= \cos(k \cdot 360°) + i \sin(k \cdot 360°).$

Hence $\sqrt[3]{1} = [\cos(k \cdot 360°) + i \sin(k \cdot 360°)]^{\frac{1}{3}}$
$= \cos(k \cdot 120°) + i \sin(k \cdot 120°).$

Giving k in succession the values 0, 1, 2, the three cube roots of 1 are as follows:

$$z_1 = \cos 0° + i \sin 0°.$$
$$z_2 = \cos 120° + i \sin 120°.$$
$$z_3 = \cos 240° + i \sin 240°.$$

Changing these to the rectangular form,

$$z_1 = 1,\ z_2 = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}i,\ z_3 = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3}i.$$

The plotting is shown in Fig. 114.

Note that in Examples 1 and 2 each of the cube roots lies at a vertex of an equilateral triangle whose center is at the origin.

The triangle in each case is inscribed in a circle of radius equal to the common modulus of the roots.

Example 3.—Find all the fourth roots of -1 in rectangular form.

Solution.—The modulus of -1 is 1 and its amplitude is 180°.

Then $-1 = 1[\cos(180° + k \cdot 360°) + i \sin(180° + k \cdot 360°)].$

$$\sqrt[4]{-1} = \cos(45° + k \cdot 90°) + i \sin(45° + k \cdot 90°).$$

Giving k in succession the values 0, 1, 2, 3, the four fourth roots of -1 are as follows:

$$z_1 = \cos 45° + i \sin 45° = \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}i.$$
$$z_2 = \cos 135° + i \sin 135° = -\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{2}i.$$
$$z_3 = \cos 225° + i \sin 225° = -\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\sqrt{2}i.$$
$$z_4 = \cos 315° + i \sin 315° = \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\sqrt{2}i.$$

These results, if plotted, would lie at the vertices of a square.

Finding the nth roots of a complex number, $a + bi$, is equivalent to solving the equation $x^n - (a + bi) = 0$. Therefore, DeMoivre's theorem gives a means of solving the general binomial equation. For example, the three roots of Example 1 are the three roots of the cubic equation $x^3 - 1 + i = 0$. The three roots of Example 2 are the solutions of the equation $x^3 - 1 = 0$.

As already pointed out, the n distinct nth roots of a complex number lie at the vertices of a regular n-gon whose center is at the origin, and whose vertices lie on a circle whose radius is r, the common modulus of the roots. This is immediately apparent from the general form of the nth root. Whence it is seen that all the nth roots have the same modulus and hence all lie at the same distance from the origin. Also their amplitudes differ by the constant angle $\frac{2\pi}{n}$, as k is given in succession the values $0, 1, 2, \cdots n - 1$. Therefore, the points representing the roots are equally spaced around a circle.

EXERCISES

Find all the roots in Exercises 1 to 14 and express in polar form.

1. $x^4 = 1 - i.$
2. $x^3 = 1 - i.$
3. $x^3 = -1 - i.$
4. $x^3 = \frac{1}{2}\sqrt{3} + \frac{1}{2}i.$
5. $x^3 = \frac{1}{2}\sqrt{3} - \frac{1}{2}i.$
6. $x^5 = i.$

7. $x^5 = -1.$
8. $x^2 = \cos 20° + i \sin 20°.$
9. $x^2 = \cos 140° + i \sin 140°.$
10. $x^3 = \cos 105° + i \sin 105°.$
11. $x^3 = \cos 300° + i \sin 300°.$
12. $x^5 = -32.$
13. $x^4 = 8 + 8\sqrt{3}i.$
14. $x^4 = -8 - 8\sqrt{3}i.$

Find all the roots in Exercises 15 to 22 in rectangular form and represent them graphically.

15. $x^3 = -1.$
16. $x^3 = -8.$
17. $x^3 = i.$
18. $x^3 = -i.$
19. $x^4 = -1.$
20. $x^6 = -i.$
21. $x^5 = 32\ (\cos 150° + i \sin 150°).$
22. $x^4 = \cos 120 + i \sin 120.$

Solve the following equations, express the roots in rectangular form, and represent them graphically:

23. $x^3 - 1 = 0.$
24. $x^4 - 1 = 0.$
25. $x^5 - 1 = 0.$
26. $x^6 - 64 = 0.$
27. $x^2 + 1 = 0.$
28. $x^5 + 32 = 0.$
29. $x^4 - i = 0.$
30. $x^4 + i = 0.$
31. $x^5 - \sqrt{-32} = 0.$
32. $x^5 - \sqrt{-243} = 0.$

TRIGONOMETRIC SERIES

124. Expansion of sin $n\theta$ and cos $n\theta$.—By DeMoivre's theorem and the binomial theorem,

(1) $$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n = \cos^n\theta + ni\cos^{n-1}\theta\cdot\sin\theta$$
$$- \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta - \frac{in(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta$$
$$+ \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta$$
$$+ \frac{in(n-1)(n-2)(n-3)(n-4)}{5!}\cos^{n-5}\theta\sin^5\theta - \cdots.*$$

Equating the real parts,

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta + \cdots.$$

Let $\alpha = n\theta$, then $\theta = \frac{\alpha}{n}$ and $n = \frac{\alpha}{\theta}$, where α is to be held constant while n and θ are to vary. Substituting these values,

* The symbol $n!$, or $\underline{|n}$, is used to denote the product $1 \cdot 2 \cdot 3 \cdots n$, and is read "factorial n."

$$\cos\alpha = \cos^n\theta - \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)}{2!}\cos^{n-2}\theta\sin^2\theta$$
$$+ \frac{\frac{\alpha}{\theta}\left(\frac{\alpha}{\theta}-1\right)\left(\frac{\alpha}{\theta}-2\right)\left(\frac{\alpha}{\theta}-3\right)}{4!}\cos^{n-4}\theta\sin^4\theta + \cdots.$$
$$= \cos^n\theta - \frac{\alpha(\alpha-\theta)}{2!}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2$$
$$+ \frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)}{4!}\cos^{n-4}\theta\left(\frac{\sin\theta}{\theta}\right)^4 \cdots.$$

Now, as n becomes infinite $\frac{\alpha}{n} = \theta$ approaches zero, $\cos\theta \to 1$, $\frac{\sin\theta}{\theta} \to 1$, and $\alpha - \theta \to \alpha$. Therefore,

(2) $$\boldsymbol{\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \cdots.}$$

Equating the coefficients of the imaginary parts of (1),

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos^{n-5}\theta\sin^5\theta - \cdots.$$

Making the substitutions for θ and n,

$$\sin\alpha = \alpha\cos^{n-1}\theta\left(\frac{\sin\theta}{\theta}\right) - \frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{3!}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3$$
$$+ \frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)(\alpha-4\theta)}{5!}\cos^{n-5}\theta\left(\frac{\sin\theta}{\theta}\right)^5 - \cdots.$$

Then when n becomes infinite

(3) $$\boldsymbol{\sin\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \cdots.}$$

In (2) and (3), α is in radians.

If we divide (3) by (2), we get

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!}\cdots}{1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!}\cdots} = \alpha + \frac{\alpha^3}{3} + \frac{2\alpha^5}{15} + \cdots.$$

125. Computation of trigonometric functions.—Formulas (2) and (3) may be used to compute the functions of angles. Thus, let

$$\alpha = 10° = \tfrac{1}{18}\pi.$$

Then

$$\sin\frac{1}{18}\pi = \frac{1}{18}\pi - \frac{(\frac{1}{18}\pi)^3}{3!} + \frac{(\frac{1}{18}\pi)^5}{5!} - \cdots = a - b + c - \cdots,$$

where $a, b, c, \ldots$ may be computed as follows:

$$\begin{aligned} \log\pi &= 0.49715 \\ \log 18 &= 1.25527 \\ \hline \log a &= 9.24188 - 10 \\ a &= 0.17453 \end{aligned} \qquad \begin{aligned} 3\log\pi &= 1.49145 \\ \text{colog } 18^3 &= 6.23419 - 10 \\ \text{colog } 3! &= 9.22185 - 10 \\ \hline \log b &= 6.94749 - 10 \\ b &= 0.000886 \end{aligned}$$

$$\begin{aligned} 5\log\pi &= 2.48575 \\ \text{colog } 18^5 &= 3.72365 - 10 \\ \text{colog } 5! &= 7.92082 - 10 \\ \hline \log c &= 4.13022 - 10 \\ c &= 0.000001349 \end{aligned}$$

$$\sin 10° = a - b = 0.17453 - 0.000886 = 0.17364.$$

From the table of natural functions, $\sin 10° = 0.17365$. By means of (2), $\cos 10°$ may be computed.

EXERCISES

Compute the following functions correct to the fourth decimal place and compare with the tables:

1. $\sin 20°$.
2. $\cos 25°$.
3. $\tan 30°$.
4. $\sin 45°$.

126. Exponential values of sin θ, cos θ, and tan θ.—In algebra it is proved that if e is the base of the natural system of logarithms, then

$$(1) \qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$

Now if $i\theta$ is substituted for x, where $i = \sqrt{-1}$,

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right).$$

But, by **Art. 124,** the expressions in the first and second parentheses are equal to $\cos\theta$ and $\sin\theta$, respectively.

$$(2) \qquad \therefore e^{i\theta} = \cos\theta + i\sin\theta.$$

Substituting $x = -i\theta$ in (1) and reducing as before, we have

$$(3) \qquad e^{-i\theta} = \cos\theta - i\sin\theta.$$

Subtracting (3) from (2),

$$(4) \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Adding (2) and (3),

$$(5) \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Dividing (4) by (5),

$$(6) \qquad \tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}.$$

Note.—The expressions for $\sin\theta$, $\cos\theta$, and $\tan\theta$, given in (4), (5), and (6), are called *exponential values* of these functions. They are also called *Euler's Equations* after Euler their discoverer. Euler (1707–1783) was one of the greatest of the physicists, astronomers, and mathematicians of the eighteenth century.

EXERCISES

By means of the exponential values prove the following identities:

1. $\sin^2\theta + \cos^2\theta = 1$.
2. $1 + \tan^2\theta = \sec^2\theta$.
3. $1 + \cot^2\theta = \csc^2\theta$.
4. $\sin 2\theta = 2\sin\theta\cos\theta$.
5. $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.
6. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

127. Series for $\sin^n\theta$ and $\cos^n\theta$ in terms of sines or cosines of multiples of θ.—From (4) of **Art. 126,**

$$2i\sin\theta = e^{i\theta} - e^{-i\theta}.$$

Expanding by the binomial theorem,

$$(2i\sin\theta)^n = (e^{i\theta} - e^{-i\theta})^n$$

$$= (e^{i\theta})^n + n(e^{i\theta})^{n-1}(-e^{-i\theta}) + \frac{n(n-1)}{2!}(e^{i\theta})^{n-2}(-e^{-i\theta})^2$$

$$+ \cdots + \frac{n(n-1)}{2!}(e^{i\theta})^2(-e^{-i\theta})^{n-2} + ne^{i\theta}(-e^{-i\theta})^{n-1} + (-e^{-i\theta})^n.$$

When n is *odd,* the number of terms in the series is *even,* and when n is *even,* the number of terms is *odd.* Therefore, when n is odd, the terms can be grouped in pairs, the first with the last, the second with the last but one, etc. But, when n is even, there will be a certain number of pairs and one extra term, which is the middle term of the series.

From this series, general formulas can be derived for expressing $\sin^n\theta$ as a series of sines or cosines of multiples of θ.

By using (5) of **Art. 126,** $\cos^n\theta$ can be dealt with in a similar manner.

Here special cases only will be given. From these and other special cases, however, laws can easily be discovered that will determine the coefficients, and multiples of the angles.

Example 1.—Express $\sin^5\theta$ in sines of multiples of θ.

Since $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$,

$$\sin^5\theta = \frac{1}{2^4}\left[\frac{e^{i5\theta} - 5e^{i3\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-i3\theta} - e^{-i5\theta}}{2i}\right].$$

Grouping in pairs, the first with the last, the second with the last but one, etc.,

$$\sin^5\theta = \frac{1}{2^4}\left[\frac{e^{i5\theta} - e^{-i5\theta}}{2i} - 5\frac{e^{i3\theta} - e^{-i3\theta}}{2i} + 10\frac{e^{i\theta} - e^{-i\theta}}{2i}\right].$$

$$\therefore \sin^5\theta = \tfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$

Example 2.—Express $\sin^6\theta$ in cosines of multiples of θ.

Since $$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

$$\sin^6\theta = -\frac{1}{2^5}\left[\frac{e^{i6\theta} - 6e^{i4\theta} + 15e^{i2\theta} - 20 + 15e^{-i2\theta} - 6e^{-i4\theta} + e^{-i6\theta}}{2}\right].$$

Grouping in pairs,

$$\sin^6\theta = -\frac{1}{2^5}\left[\frac{e^{i6\theta} + e^{-i6\theta}}{2} - 6\frac{e^{i4\theta} + e^{-i4\theta}}{2} + 15\frac{e^{i2\theta} + e^{-i2\theta}}{2} - 10\right].$$

$$\therefore \sin^6\theta = -\tfrac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10).$$

Example 3.—Express $\cos^3\theta$ in cosines of multiples of θ.

Since $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

$$\cos^3\theta = \frac{1}{4}\left[\frac{e^{i3\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-i3\theta}}{2}\right]$$

$$= \frac{1}{4}\left[\frac{e^{i3\theta} + e^{-i3\theta}}{2} + 3\frac{e^{i\theta} + e^{-i\theta}}{2}\right].$$

$$\therefore \cos^3\theta = \tfrac{1}{4}[\cos 3\theta + 3\cos\theta].$$

Example 4.—Express $\cos^4\theta$ in cosines of multiples of θ

Since $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

$$\cos^4\theta = \frac{1}{8}\left[\frac{e^{i4\theta} + 4e^{i2\theta} + 6 + 4e^{-i2\theta} + e^{-i4\theta}}{2}\right]$$

$$= \frac{1}{8}\left[\frac{e^{i4\theta} + e^{-i4\theta}}{2} + 4\frac{e^{i2\theta} + e^{-i2\theta}}{2} + 3\right].$$

$$\therefore \cos^4\theta = \tfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3).$$

EXERCISES

Prove the following identities:

1. $\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$.
2. $\cos^7\theta = \frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$.
3. $128\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35$.
4. $64\sin^7\theta = 35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta$.
5. $\sin^6\theta + \cos^6\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$.

128. Hyperbolic functions.—In **Art. 56,** the trigonometric functions were called circular functions because of their relation to the arc of a circle. There is another set of functions whose properties are very similar to the properties of the trigonometric functions. Because of their relation to the hyperbola, they are called **hyperbolic functions.** They are defined as follows:

(1) Hyperbolic sine x (written $\sinh x$) $= \dfrac{e^x - e^{-x}}{2}$.

(2) Hyperbolic cosine x (written $\cosh x$) $= \dfrac{e^x + e^{-x}}{2}$.

(3) Hyperbolic tangent x (written $\tanh x$) $= \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

(4) Hyperbolic cotangent x (written $\coth x$) $= \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$.

(5) Hyperbolic secant x (written $\operatorname{sech} x$) $= \dfrac{2}{e^x + e^{-x}}$.

(6) Hyperbolic cosecant x (written $\operatorname{csch} x$) $= \dfrac{2}{e^x - e^{-x}}$.

In these formulas e is the base of the Napierian system of logarithms, and so stands for the number $2.7182818 \cdots$.

From the definitions, the following relations are evident:

$$\tanh x = \frac{\sinh x}{\cosh x},\quad \tanh x = \frac{1}{\coth x},\quad \operatorname{sech} x = \frac{1}{\cosh x}.$$

129. Relations between the hyperbolic functions.—Squaring (1) and (2) and subtracting the second from the first,

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1.$$

$$\therefore \cosh^2 x - \sinh^2 x = 1.$$

By analogy, from (1) we may write

$$\sinh (x + y) = \frac{e^{x+y} - e^{-(x+y)}}{2}.$$

Also, $\sinh x \cosh y = \dfrac{e^x - e^{-x}}{2} \cdot \dfrac{e^y + e^{-y}}{2}$

$$= \tfrac{1}{4}[e^{x+y} - e^{-x}e^y + e^x e^{-y} - e^{-(x+y)}].$$

And $\cosh x \sinh y = \dfrac{e^x + e^{-x}}{2} \cdot \dfrac{e^y - e^{-y}}{2}$

$$= \tfrac{1}{4}[e^{x+y} + e^{-x}e^y - e^x e^{-y} - e^{-(x+y)}].$$

Adding the last two,

$$\sinh x \cosh y + \cosh x \sinh y = \tfrac{1}{2}[e^{x+y} - e^{-(x+y)}].$$

Comparing this with the first,

$$\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y.$$

EXERCISES

Prove the following identities:

1. $\operatorname{sech}^2 x + \tanh^2 x = 1.$
2. $\coth^2 x - \operatorname{csch}^2 x = 1.$
3. $\sinh (-x) = -\sinh x.$
4. $\cosh (-x) = \cosh x.$
5. $\sinh (x - y) = \sinh x \cosh y - \cosh x \sinh y.$
6. $\cosh (x + y) = \cosh x \cosh y - \sinh x \sinh y.$
7. $\cosh (x - y) = \cosh x \cosh y + \sinh x \sinh y.$
8. $\tanh (x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$
9. $\sinh 2x = 2 \sinh x \cosh x.$
10. $\cosh 2x = \cosh^2 x + \sinh^2 x.$

130. Relations between the trigonometric and hyperbolic functions.—If in (4) of **Art. 126** we substitute $i\theta$ for θ,

$$i \sin i\theta = \tfrac{1}{2}[e^{i(i\theta)} - e^{-i(i\theta)}] = -\tfrac{1}{2}[e^{\theta} - e^{-\theta}] = -\sinh \theta.$$

(1) $$\therefore \sin i\theta = i \sinh \theta.$$

Substituting $i\theta$ for θ in (5) of **Art. 126,**

$$\cos i\theta = \tfrac{1}{2}[e^{i(i\theta)} + e^{-i(i\theta)}] = \tfrac{1}{2}[e^{\theta} + e^{-\theta}] = \cosh \theta.$$

(2) $$\therefore \cos i\theta = \cosh \theta.$$

Dividing (1) by (2),

(3) $$\tan i\theta = i \tanh \theta.$$

131. Expression of sinh x and cosh x in a series. Computation.—By definition and by (1) of **Art. 126,**

$$\sinh x = \tfrac{1}{2}[e^x - e^{-x}]$$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - \left(1 - x + \frac{x}{2!} - \frac{x^3}{3!} + \cdots\right)\right]$$

$$= \frac{1}{2}\left[2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + \cdots\right] = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

(1) $$\therefore \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

Also, $$\cosh x = \tfrac{1}{2}[e^x + e^{-x}]$$

$$= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\right]$$

$$= \frac{1}{2}\left[2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots\right] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

(2) $$\therefore \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots.$$

Series (1) and (2) for sinh x and cosh x are convergent for all real values of x. Therefore, for any real value of x the hyperbolic functions of x can be computed.

131′. Forces and velocities represented as complex numbers.—Since forces and velocities, to be completely defined, must be known in magnitude and direction, they are vectors and may be expressed in the complex number notation. In Fig. *A*, *OP* represents to scale a force of F lb., making an angle θ with the x-axis. By **Art. 113**

$$OP = F(\cos \theta + i \sin \theta). \qquad [1]$$

Since *OP* locates the point *P* with polar coordinates (F, θ), it suggests the following notation: Force (F, θ) represents a force of magnitude or modulus F lb., with direction or amplitude θ. Force (5, 0°) defines point *A* in Fig. *B*, and is *OA*. Force (5, 135°) locates point *P* and is *OB*.

Example 1.—Locate the following:

Force (10, 90°); Force (8, 180°);
Force (10, 240°); Force (15, 300°).

Similarly, velocity (20 miles, 45°) means that a body is moving 20 miles per hour in a northeasterly direction, as the x-axis will be taken as the East and West line.

Equation (1) may be written

$$OP = F \cos \theta + iF \sin \theta = x + yi$$

where x and y are the rectangular components of OP.

If the problem is to find the resultant or sum of several concurrent forces, first express each of the forces in the rectangular form. Then by **Art. 115,** these complex numbers may be added

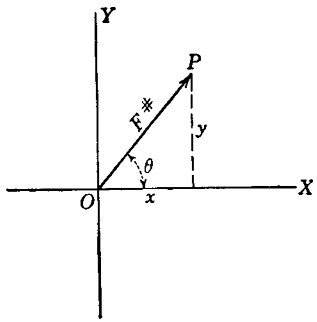


FIG. A.

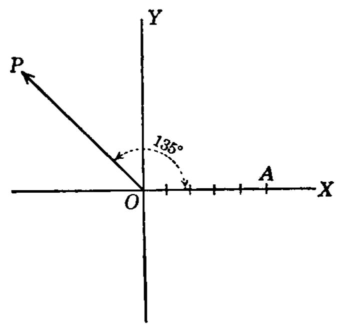


FIG. B.

algebraically. Their sum is a complex number with the x component equal to the sum of the x's of the forces, and the y component equal to the sum of the y's.

Example 2.—Find the resultant of the forces in Example 1.

$$\begin{aligned}
\text{Force } (10,\ 90°) &= 0 + 10i.\\
\text{Force } (\ 8,\ 180°) &= -8 + 0i.\\
\text{Force } (10,\ 240°) &= -5 - 8.66i.\\
\text{Force } (15,\ 300°) &= 7.5 - 13i.\\
\text{Force } (F,\ \theta) &= (0 - 8 - 5 + 7.5) + (10 + 0 - 8.66 - 13)i.\\
&= -5.5 - 11.66i
\end{aligned}$$

Then
$$F = \sqrt{(-5.5)^2 + (-11.66)^2}, \quad = 12.8$$

and
$$\theta = \tan^{-1}\left(\frac{-11.66}{-5.5}\right)$$
$$= 180° + 64°\ 44.8' = 244°\ 44.8'$$

Force (12.8, 244° 44.8′) is the resultant. Notice that the angle θ is in the third quadrant since both components are negative.

Example 3.—The current in a river flowing due south causes a boat to drift three miles per hour. At the same time a wind from

the southwest causes the boat to drift to the northeast at the rate of six miles per hour. Find the resultant velocity of the boat both in magnitude and direction.

Solution.—

$$\begin{aligned}
\text{Velocity } (3 \text{ miles}, 270°) &= 0 - 3i.\\
\text{Velocity } (6 \text{ miles},\ \ 45°) &= 4.24 + 4.24i.\\
\text{Velocity } (V,\ \theta) &= 4.24 + 1.24i.\\
V &= \sqrt{(4.24)^2 + (1.24)^2}\\
&= 4.32 \text{ miles per hour}\\
\theta &= \tan^{-1}\frac{1.24}{4.24}\\
&= 16°\ 18.1'.
\end{aligned}$$

Therefore, the boat moves with a velocity of 4.32 miles per hour in a direction E 16° 18.1′ N.

If a set of concurrent forces are in equilibrium, their resultant must equal zero; that is, $x + yi = 0$. But this can be true when, and only when, $x = 0$ and $y = 0$. The sum of all the x-components of the forces will equal zero. Similarly, the summation of y-components equals zero. This leads to two equations which can be solved simultaneously.

For other problems the student is asked to solve Nos. 1, 2, 3, 4, 5, 6, 9, 11, 12 of **Art. 69,** by the theory of complex numbers. Also 35, 36, 37, 38, 39, 40 of General Exercises at the end of Chapter IX.

ANSWERS TO EXERCISES

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1. $9 - 5i$. **4.** $8 - 9i$. **7.** $\frac{1}{5}(7 - 6i)$.
2. $-2 - 3i$. **5.** $-7 + 17i$. **8.** $\frac{1}{25}(23 - 14i)$.
3. $10 + 3i$. **6.** $11 - 3i$. **9.** $\frac{3}{10}(3 - i)$.
10. $\frac{2}{5}(2 + 9i)$.
11. $\frac{1}{3}(-2\sqrt{2} + 5i)$.
12. $\frac{1}{13}(6 - \sqrt{2}) - \frac{1}{13}(\sqrt{3} + 2\sqrt{6})i$.
13. $\frac{1}{5}(2\sqrt{2} - 3\sqrt{3}) + \frac{1}{5}(2\sqrt{3} + 3\sqrt{2})i$.
14. $\sqrt{2} - \sqrt{3}i$ **17.** $1 - \frac{1}{5}\sqrt{5}i$. **20.** $-2 + 2i$.
15. $-7 - 6i$. **18.** $\frac{1}{2} + \frac{1}{2}i$. **21.** $-7 - 24i$.
16. $\frac{1}{5}(3 - i)$. **19.** $-2i$. **22.** $-3\sqrt{3} - 7\sqrt{2}i$.
23. 1. **24.** -1. **25.** $\dfrac{a + b + 2\sqrt{ab}}{a - b}$,
26. $2a^2 - 1 + 2ai\sqrt{1 - a^2}$.
27. $a^2 - \sqrt{1 - a^4} + ai(\sqrt{1 - a^2} + \sqrt{1 + a^2})$. **28.** 0. **29.** 1.

30. i. **31.** $\frac{8}{a^6} - 3 + \left(\frac{6\sqrt{2}}{a^3} - \frac{\sqrt{2}a^3}{4}\right)i$.

32. $\frac{\sqrt{3}a^3}{9} - \frac{3\sqrt{3}}{a} + \left(\sqrt{3}a - \frac{3\sqrt{3}}{a^3}\right)i$. **34.** $-17 + 12i$.

35. 42.

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6. $2\sqrt{3}(\cos 300° + i \sin 300°)$.
8. $2^{\frac{5}{6}}(\cos 135° + i \sin 135°)$.
10. $2\sqrt{2}(\cos 60° + i \sin 60°)$.
12. $2\sqrt{7}(\cos 150° + i \sin 150°)$.
14. $4(\cos 90° + i \sin 90°)$.
16. $6(\cos 270° + i \sin 270°)$.
18. $\cos 270° + i \sin 270°$.
20. $\cos 180° + i \sin 180°$.
22. $\sqrt{3}(\cos 90° + i \sin 90°)$.
24. $3(\cos 270° + i \sin 270°)$.
26. $2\sqrt{5}(\cos 333° 26' + i \sin 333° 26')$.
34. $3\sqrt{2}[\cos (45° + k \cdot 360°) + i \sin (45° + k \cdot 360°)]$.
37. $3[\cos (90° + k \cdot 360°) + i \sin (90° + k \cdot 360°)]$.
38. $\frac{3}{2}\sqrt{3} + \frac{3}{2}i$.
43. $-2\sqrt{3} - 2i$.
47. $-\sqrt{3} + i$.

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1. $\sqrt[8]{2}(\cos 78° 45' + i \sin 78° 45')$.
$\sqrt[8]{2}(\cos 168° 45' + i \sin 168° 45')$.
$\sqrt[8]{2}(\cos 258° 45' + i \sin 258° 45')$.
$\sqrt[8]{2}(\cos 348° 45' + i \sin 348° 45')$.

4. $\cos 10° + i \sin 10°$;
$\cos 130° + i \sin 130°$;
$\cos 250° + i \sin 250°$.

11. $\cos 100° + i \sin 100°$;
$\cos 220° + i \sin 220°$;
$\cos 340° + i \sin 340°$.

16. $1 + \sqrt{3}i$;
$1 - \sqrt{3}i$;
-2.

20. $0.7071 + 0.7071i$.
$-0.2588 + 0.9659i$.
$-0.9659 + 0.2588i$.
$-0.7071 - 0.7071i$.
$0.2588 - 0.9659i$.
$0.9659 - 0.2588i$.

30. $0.3827 + 0.9239i$.
$-0.9239 + 0.3827i$.
$-0.3827 - 0.9239i$.
$0.9239 - 0.3827i$.

CHAPTER XII

SPHERICAL TRIGONOMETRY

132. Spherical trigonometry investigates the relations that exist between the parts of a spherical triangle.

For convenience, a few of the definitions and theorems of spherical geometry are stated here.

The section of the surface of a sphere made by a plane is a **great circle** if the plane passes through the center of the sphere, and a **small circle** if the plane does not pass through the center of the sphere.

The diameter of a sphere perpendicular to the plane of a circle of the sphere is called the **axis** of that circle. The points where the axis of a circle of a sphere intersects the surface of the sphere are called the **poles** of the circle.

133. Spherical triangle.—A **spherical triangle** is the figure on the surface of a sphere bounded by three arcs of great circles. The three arcs are the sides of the triangle, and the angles formed by the arcs at the points where they meet are the angles of the triangle.

The angle between two intersecting arcs is measured by the angle between the tangents drawn to the arcs at the point of intersection.

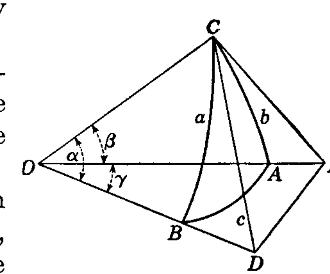


Fig. 115.

If a trihedral angle is placed with its vertex at the center of a sphere, the face planes intersect the surface of the sphere in arcs of great circles which form a spherical triangle. The sides of the spherical triangle measure the face angles of the trihedral angle, and the angles of the triangle are equal to the dihedral angles of the trihedral angle.

In Fig. 115, O is the center of a sphere. O-ABC is a trihedral angle. AB, BC, and CA are arcs of great circles. ABC is a spherical triangle. Arcs a, b, and c are the measures of α, β, and γ

respectively. $\angle BCA$ and $D\text{-}OC\text{-}E$ are measured by the same plane angle, as also are $\angle ABC$ and $E\text{-}OD\text{-}C$, and $\angle CAB$ and $C\text{-}OE\text{-}D$.

The sum of the sides of a spherical triangle is less than 360°.

The sum of the angles of a spherical triangle is greater than 180° *and less than* 540°.

It is evident that the sides and angles of a spherical triangle can be greater than 180°; however, to simplify the subject, it is agreed to consider only those spherical triangles in which the sides and angles are each less than 180°.

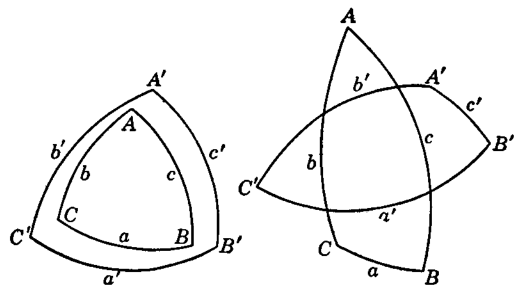


FIG. 116.

134. Polar triangles.—If the vertices of a spherical triangle are used as poles and great circles drawn, another triangle is formed called the **polar triangle** of the first.

Thus in Fig. 116, A is the pole of a', B the pole of b', C the pole of c', and $A'B'C'$ is the polar triangle of ABC.

It is evident that, in general, the great circles drawn as stated will intersect so as to form eight triangles. The one of these is the polar triangle in which A and A', B and B', C and C' lie on the same side of a', b', and c' respectively.

If one triangle is the polar of another, then the latter is the polar triangle of the former.

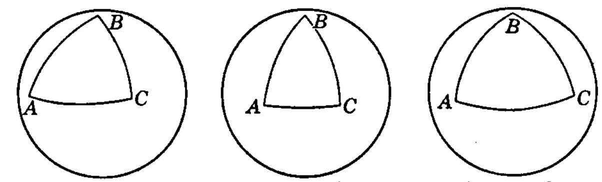
The sides and the angles of a spherical triangle are the supplements, respectively, of the angles and the sides opposite in the polar triangle, and, conversely.

Thus in Fig. 116, $A' = \pi - a$, $B' = \pi - b$, $C' = \pi - c$,
$a' = \pi - A$, $b' = \pi - B$, $c' = \pi - C$.

These relations are of great importance, for, if any general theorem be proved with respect to the sides and angles of any spherical triangle, it can at once be applied to the polar triangle. Thus, any theorem of a spherical triangle may be at once transformed into another by replacing each side, or angle, by the supplement of its opposite angle, or side, in the polar triangle.

Since the side of a spherical triangle and the corresponding face angle of the trihedral angle have the same numerical measure, the plane trigonometric functions may be taken of the arcs as well as of the plane angles. Hence the identities of plane trigonometry are true for the sides of a spherical triangle.

A **right spherical** triangle is one which has an angle equal to 90°. A **birectangular** triangle is one which has two right



Right Birectangular Trirectangular

FIG. 117.

angles. A **trirectangular** triangle is one which has three right angles.

RIGHT SPHERICAL TRIANGLES

135. In a spherical triangle there are six parts, three sides and three angles, besides the radius of the sphere which is supposed known. In general, *if three of these parts are given the other parts can be found.* If the triangle is a right spherical triangle, two given parts in addition to the right angle are sufficient to solve the triangle.

Since there are three unknowns to be found in solving a right triangle, it is necessary to have any two given parts combined with the remaining three in three independent relations or formulas. Now, since the five parts taken three at a time form *ten* combinations, *ten* formulas are necessary and sufficient to solve all right spherical triangles.

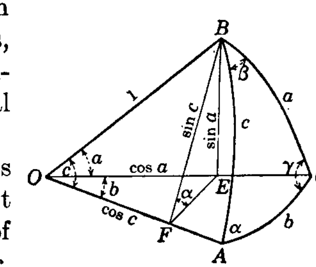


FIG. 118.

If a, b, c, α, and β are the five parts of the triangle, omitting the right angle, then the ten combinations of these taken three at a time are abc, $ab\alpha$, $ab\beta$, $ac\alpha$, $ac\beta$, $a\alpha\beta$, $bc\alpha$, $bc\beta$, $b\alpha\beta$, and $c\alpha\beta$. It is necessary to derive a formula connecting the parts in each of these combinations.

136. Derivation of formulas for the solution of right spherical triangles.—Let O be the center of a sphere of unit radius, and ABC a right spherical triangle, with γ the right angle, formed by the intersection of the three planes AOB, AOC, and COB with the surface of the sphere. Pass the plane BFE through B perpendicular to OA. Then the plane angle BFE measures α, and a, b, and c have, respectively, the same measures as angles COB, AOC, and AOB. Further, $EB = \sin\ a$, $FB = \sin\ c$, $OF = \cos c$, and $OE = \cos a$.

Then
$$FE = EB \cot \alpha = \sin a \cot \alpha. \qquad (a)$$
$$FE = FB \cos \alpha = \sin c \cos \alpha. \qquad (b)$$
$$FE = OE \sin b = \cos a \sin b. \qquad (c)$$
$$FE = OF \tan b = \cos c \tan b. \qquad (d)$$

From (a) and (b), $\sin a \cot \alpha = \sin c \cos \alpha$.

$$\sin a = \sin c \frac{\cos \alpha}{\cot \alpha}, \text{ or}$$

(1) $$\sin a = \sin \alpha \sin c. \qquad (ac\alpha)$$

By analogy, interchanging a and b, α and β,

(2) $$\sin b = \sin \beta \sin c. \qquad (bc\beta)$$

From (a) and (c), $\sin a \cot \alpha = \cos a \sin b$.

$$\therefore \sin b = \frac{\sin a}{\cos a} \cot \alpha, \text{ or}$$

(3) $$\sin b = \tan a \cot \alpha. \qquad (ab\alpha)$$

(4) By analogy, $$\sin a = \tan b \cot \beta. \qquad (ab\beta)$$

From (a) and (d), $\sin a \cot \alpha = \cos c \tan b$.

$$\therefore \cos c = \frac{\sin a \cot \alpha}{\tan b} = \frac{\tan b \cot \beta \cot \alpha}{\tan b}, \text{ or}$$

(5) $$\cos c = \cot \alpha \cot \beta. \qquad (c\alpha\beta)$$

From (b) and (c), $\sin c \cos \alpha = \cos a \sin b$.

$$\therefore \cos \alpha = \frac{\cos a \sin b}{\sin c} = \frac{\cos a \sin c \sin \beta}{\sin c}, \text{ or}$$

(6) $$\cos \alpha = \sin \beta \cos a. \qquad (a\alpha\beta)$$

(7) By analogy, $$\cos \beta = \sin \alpha \cos b. \qquad (b\alpha\beta)$$

From (b) and (d), $\sin c \cos \alpha = \cos c \tan b$.

$$\therefore \cos \alpha = \frac{\cos c}{\sin c} \tan b, \text{ or}$$

(8) $$\cos \alpha = \tan b \cot c. \qquad (bc\alpha)$$

(9) By analogy, $$\cos \beta = \tan a \cot c. \qquad (ac\beta)$$

From (c) and (d), $\cos a \sin b = \cos c \tan b$.

$$\therefore \cos c = \frac{\cos a \sin b}{\tan b}, \text{ or}$$

(10) $$\cos c = \cos a \cos b. \qquad (abc)$$

137. Napier's rules of circular parts.—The preceding ten formulas for the solution of right spherical triangles are included in a theorem first stated and proved by Napier. The theorem is usually stated as two rules known as "Napier's rules of circular parts."

In the right spherical triangle ABC, omit the right angle at C and consider the sides a and b, and the complements of α, β, and c. Call these the circular parts of the triangle and designate them as a, b, co-α, co-β, and co-c.

In the triangle or the circular scheme shown in the figure, any one of these five parts may be selected and called the **middle part**; then the two parts next to it are called **adjacent parts**, and the other two parts the **opposite parts.** For example, if b is chosen as the middle part, then co-α and a are the adjacent parts and co-c and co-β are the opposite parts. Napier's rules are then stated as follows:

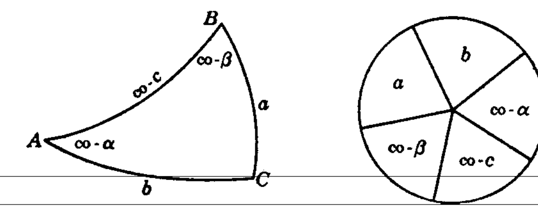


FIG. 119.

(1) *The sine of a middle part equals the product of the* **tangents** *of the* **adjacent** *parts.*

(2) *The sine of a middle part equals the product of the* **cosines** *of the* **opposite** *parts.*

It may assist in remembering the rules to notice the repetition of a in (1) and of o in (2).

Napier's rules may be verified by showing that they give the ten formulas of **Art. 136.** A demonstration of the theorem as given by Napier may be found in Todhunter and Leathem's "Spherical Trigonometry."

Example.—Use co-α as the middle part and apply rule (1).

$$\sin(\text{co-}\alpha) = \tan b \tan(\text{co-}c).$$
$$\therefore \cos\alpha = \tan b \cot c, \text{ which is formula (8).}$$

Exercise.—Verify all the ten formulas by Napier's rules.

Napier's rules thus furnish a very convenient way for the determination of the formulas for the solution of right spherical triangles.

138. Species.—Two angular quantities are said to be of the **same species** when they are both in the same quadrant, and of **different species** when they are in different quadrants.

Since any or all the parts of a right spherical triangle may be less than or greater than 90°, it is necessary to have a method for the determination of the species of the parts. The following rules will be found to cover all cases:

(1) *An oblique angle and its opposite side are always of the same species.*

(2) *If the hypotenuse is less than* 90°, *the two oblique angles and therefore the two sides of the triangle are of the same species; if the hypotenuse is greater than* 90°, *the two oblique angles, and therefore the two sides, are of opposite species.*

These rules are here verified in two cases. As an exercise, the student is asked to verify them in other cases.

Example 1.—By formula (5) (**Art. 136**), $\cos c = \cot\alpha \cot\beta$. If $c < 90°$, $\cos c$ is +. Then the product $\cot\alpha \cot\beta$ must be +, and this will be true if $\cot\alpha$ and $\cot\beta$ are both + or both −; that is, if α and β are both in the same quadrant.

If $c > 90°$, $\cos c$ is −. Then the product $\cot\alpha \cot\beta$ must be −; that is, $\cot\alpha$ and $\cot\beta$ must be opposite in sign, and therefore α and β must be in different quadrants. This verifies rule (2).

Example 2.—From formula (7) (**Art. 136**), $\sin\alpha = \dfrac{\cos\beta}{\cos b}$.

Since $\sin\alpha$ is always +, $\cos\beta$ and $\cos b$ must both be + or both −. Therefore, both β and b are in the same quadrant. This verifies rule (1).

139. Solution of right spherical triangles.—As stated before, when any two parts other than the right angle are given, the remaining parts of the right spherical triangle can be found. The necessary formulas can be obtained by taking each of the unknown parts in turn with the two known parts, and then applying Napier's rules. Or the formulas can be chosen from **Art. 136.**

The quadrant in which the unknown part belongs is determined by the rules of species.

The work may be checked by applying Napier's rules to the three parts obtained by the solution of the triangle.

Example 1.—Given $\alpha = 30° 51.2'$, $\beta = 71° 36'$; find a, b, and c, using Napier's rules.

Formulas / *Construction*

To find a, co-α is the middle part, a and co-β the opposite parts. Then $\sin(\text{co-}\alpha) = \cos a \cos(\text{co-}\beta)$, or $\cos\alpha = \cos a \sin\beta$.

$$\therefore \cos a = \frac{\cos\alpha}{\sin\beta}.$$

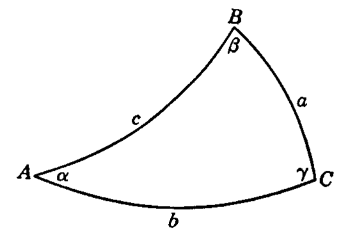


Fig. 120.

To find b, co-β is the middle part, co-α and b the opposite parts. Then $\sin(\text{co-}\beta) = \cos(\text{co-}\alpha) \cos b$, or $\cos\beta = \sin\alpha \cos b$.

$$\therefore \cos b = \frac{\cos\beta}{\sin\alpha}.$$

To find c, co-c is the middle part and co-α and co-β are the adjacent parts. Then $\sin(\text{co-}c) = \tan(\text{co-}\alpha)\tan(\text{co-}\beta)$.

$$\therefore \cos c = \cot\alpha \cot\beta.$$

To check, use co-c as the middle part with a and b as the opposite parts. Then $\sin(\text{co-}c) = \cos a \cos b$, or $\cos c = \cos a \cos b$.

Computation

$\log\cos\alpha = 9.93373$	$\log\cos\beta = 9.49920$
$\log\sin\beta = 9.97721$	$\log\sin\alpha = 9.70998$
$\log\cos a = 9.95652$	$\log\cos b = 9.78922$
$a = 25° 12.8'$	$b = 52° 0.8'$
$\log\cot\alpha = 0.22375$	*Check*
$\log\cot\beta = 9.52200$	$\log\cos a = 9.95652$
$\log\cos c = 9.74575$	$\log\cos b = 9.78922$
$c = 56° 9.6'$	$\log\cos c = 9.74574$

Note.—The formulas used in this solution could have been taken from **Art. 136**, by selecting the formulas for the combinations $(a\alpha\beta)$, $(b\alpha\beta)$, and $(c\alpha\beta)$.

Example 2 (ambiguous case).—Given $a = 24° 8'$, $\alpha = 32° 10'$; find β, b, and c, using Napier's rules.

Formulas

To find β. $\sin(\text{co-}\alpha) = \cos a \cos(\text{co-}\beta)$.

$$\therefore \sin\beta = \frac{\cos\alpha}{\cos a}.$$

To find b. $\sin b = \tan a \tan(\text{co-}\alpha)$.

$$\therefore \sin b = \tan a \cot\alpha.$$

To find c. $\sin a = \cos(\text{co-}\alpha)\cos(\text{co-}c)$.

$$\therefore \sin c = \frac{\sin a}{\sin\alpha}.$$

Construction

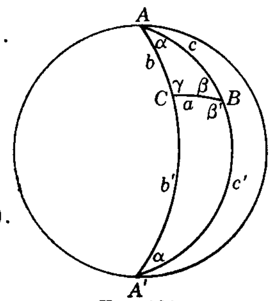


FIG. 121.

Check.—$\sin b = \cos(\text{co-}c)\cos(\text{co-}\beta)$, or $\sin b = \sin c \sin\beta$.

Computation

$\log\cos\alpha = 9.92763$
$\log\cos a = \underline{9.96028}$
$\log\sin\beta = 9.96735$
$\beta = 68° 3' 36''$
$\beta' = \underline{111° 56' 24''}$
$\log\sin a = 9.61158$
$\log\sin\alpha = \underline{9.72622}$
$\log\sin c = 9.88536$
$c = 50° 10' 27''$
$c' = \underline{129° 49' 33''}$

$\log\tan a = 9.65130$
$\log\cot\alpha = \underline{0.20140}$
$\log\sin b = 9.85270$
$b = 45° 25' 40''$
$b' = \underline{134° 34' 20''}$

Check

$\log\sin c = 9.88536$
$\log\sin\beta = \underline{9.96735}$
$\log\sin b = \underline{9.85271}$

Since each of the unknown parts is determined from the sine, there are two values of each unknown part. For this reason it is called the ambiguous case. The proper grouping of the parts may be determined from the rules for species.

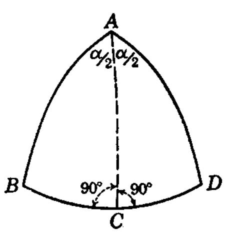


FIG. 122.

By rule (2), when $c < 90°$, α and β must be in the same quadrant. By rule (1), b and β must be in the same quadrant.

$\therefore a, b, c, \alpha$, and β are the parts of one right spherical triangle.

Again by rule (2), when $c > 90°$, α and β are of opposite species.

$\therefore a, b', c', \alpha$ and β' are the parts of a right spherical triangle.

140. Isosceles spherical triangles.—When two sides of a spherical triangle are equal, it is said to be **isosceles.**

By dropping a perpendicular from the vertex of the triangle to the base, the triangle is divided into two symmetrical right triangles, as ACD and ACB in the figure.

The solution of the isosceles spherical triangle is, therefore, made to depend upon the solution of two right spherical triangles.

141. Quadrantal triangles.—When one side of a spherical triangle is equal to 90°, the triangle is called a **quadrantal triangle.**

By taking the polar triangle of a quadrantal triangle a right triangle is obtained, and this can be solved. The supplements of the parts of the right triangle will give the corresponding parts of the quadrantal triangle.

EXERCISES

1. Given $c = 69° 25' 11''$, $\beta = 63° 25' 3''$;
find $a = 50° 0' 0''$, $b = 56° 50' 52''$, $\alpha = 54° 54' 42''$.

2. Given $c = 78° 53' 20''$, $\alpha = 83° 56' 40''$;
find $a = 77° 21' 40''$, $b = 28° 14' 34''$, $\beta = 28° 49' 54''$.

3. Given $c = 61° 4' 56''$, $a = 40° 31' 20''$;
find $b = 50° 29' 48''$, $\beta = 61° 49' 23''$, $\alpha = 47° 55' 35''$.

4. Given $c = 70° 23' 42''$, $b = 48° 39' 16''$;
find $a = 59° 28' 30''$, $\alpha = 66° 7' 22''$, $\beta = 52° 50' 18''$.

5. Given $\alpha = 27° 28' 38''$, $\beta = 73° 27' 11''$;
find $c = 55° 9' 40''$, $a = 22° 15' 10''$, $b = 51° 53' 0''$.

6. Given $\alpha = 83° 56' 40''$, $\beta = 151° 10' 3''$;
find $a = 77° 21' 50''$, $b = 151° 45' 29''$, $c = 101° 6' 40''$.

7. Given $a = 25° 12' 48''$, $b = 52° 0' 45''$;
find $c = 56° 9' 38''$, $\alpha = 30° 51' 16''$, $\beta = 71° 36' 0''$.

8. Given $a = 100°$, $b = 98° 20'$;
find $c = 88° 33.5'$, $\alpha = 99° 53.8'$, $\beta = 98° 12.5'$.

9. Given $\alpha = 92° 8' 23''$, $b = 49° 59' 58''$;
find $a = 92° 47' 34''$, $c = 91° 47' 55''$, $\beta = 50° 2' 0''$.

10. Given $\beta = 54° 35' 17''$, $a = 15° 16' 50''$;
find $b = 20° 20' 20''$, $c = 25° 14' 38''$, $\alpha = 38° 10' 0''$.

11. Given $\beta = 83° 56' 40''$, $b = 77° 21' 40''$;
find $a = 28° 14' 34''$, $c = 78° 53' 20''$, $\alpha = 28° 49' 54''$;
$a' = 151° 45' 29''$, $c' = 101° 6' 40''$, $\alpha' = 151° 10' 3''$.

12. Given $\alpha = 66° 7' 20''$, $a = 59° 28' 27''$;
find $b = 48° 39' 16''$, $c = 70° 23' 42''$, $\beta = 52° 50' 20''$;
$b' = 131° 20' 44''$, $c' = 109° 36' 18''$, $\beta' = 127° 9' 40''$.

13. Solve the isosceles spherical triangle in which the equal sides are each 34° 45.6′, and their included angle 112° 44.6′.

Ans. Equal angles = 38° 59.6′; side = 56° 41′.

14. Solve the isosceles triangle in which the equal angles are each 102° 6.4′, and the base 115° 18′.

Ans. Equal sides = 97° 34′; included angle = 116° 54.5′.

15. In a quadrantal triangle, $c = 90°$, $a = 116° 44' 48''$, $b = 44° 26' 21''$; find $\alpha = 130° 0' 4''$, $\beta = 36° 54' 48''$, $\gamma = 59° 4' 26''$.

16. In a quadrantal triangle, $c = 90°$, $\alpha = 121° 20'$, $\beta = 42° 1'$; find $a = 112° 10' 20''$, $b = 46° 31' 36''$, $\gamma = 67° 16' 22''$.

OBLIQUE SPHERICAL TRIANGLES

142. Sine theorem (law of sines).—*In any spherical triangle, the sines of the angles are proportional to the sines of the opposite sides.*

Proof.—Let ABC (Fig. 123) be a spherical triangle. Construct the great circle arc CD, forming the two right spherical triangles CBD and CAD. Represent the arc CD by h.

By (1) of **Art. 136.**

$$\sin \alpha = \frac{\sin h}{\sin b}, \text{ and } \sin \beta = \frac{\sin h}{\sin a}.$$

By division, $$\frac{\sin \alpha}{\sin \beta} = \frac{\sin a}{\sin b}, \text{ or } \frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b}.$$

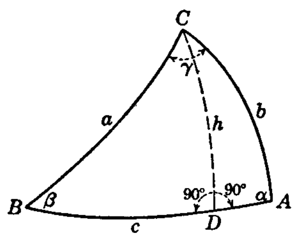


FIG. 123.

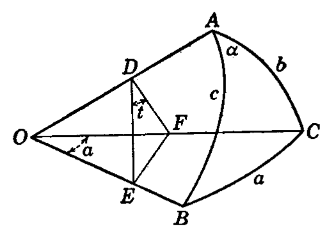


FIG. 124.

In a similar manner it may be proved that

$$\frac{\sin \alpha}{\sin \gamma} = \frac{\sin a}{\sin c}, \text{ or } \frac{\sin \alpha}{\sin a} = \frac{\sin \gamma}{\sin c}.$$

[44] $$\therefore \frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}.$$

Formula [**44**] is useful in solving a spherical triangle when two angles and a side opposite one of them are given, or when two sides and an angle opposite one are given.

Note.—While, in the figure, D falls between A and B, the theorem can be as readily proved if D does not fall between A and B.

143. Cosine theorem (law of cosines).—*In any spherical triangle, the cosine of any side is equal to the product of the cosines of the two other sides, increased by the product of the sines of these sides times the cosine of their included angle.*

Proof.—Let ABC (Fig. 124) be a spherical triangle cut from the surface of a sphere, with center O, and radius OA chosen as unity. At any point D in OA, draw a plane EDF perpendicular to the edge OA and meeting the faces of the trihedral angle in DE, DF, and EF. Then $\angle EDF = t$ is the measure of α.

In the triangle DEF, $\overline{EF}^2 = \overline{ED}^2 + \overline{FD}^2 - 2ED \cdot FD \cos t$.

Also, in triangle EOF, $\overline{EF}^2 = \overline{OE}^2 + \overline{OF}^2 - 2OE \cdot OF \cos a$.

Equating these values of $\overline{EF}^2$,

$$\overline{OE}^2 + OF - 2OE \cdot OF \cos a = \overline{ED}^2 + \overline{FD}^2 - 2ED \cdot FD \cos t.$$

$$\text{Or } 2OE \cdot OF \cos a = \overline{OE}^2 - \overline{ED}^2 + \overline{OF}^2 - \overline{FD}^2 + 2ED \cdot FD \cos t.$$

But OED and OFD are right triangles, then

$$\overline{OE}^2 - \overline{ED}^2 = \overline{OD}^2 \text{ and } \overline{OF}^2 - \overline{FD}^2 = \overline{OD}^2.$$

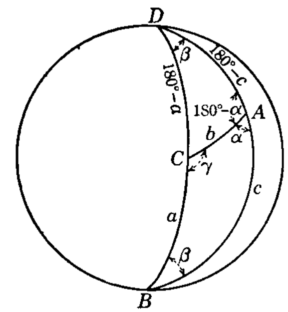


FIG. 125.

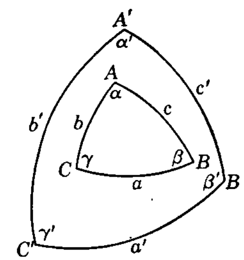


FIG. 126.

Making these substitutions, dividing by the coefficient of $\cos a$, and arranging the factors, there results

$$\cos a = \frac{OD}{ED} \cdot \frac{OD}{OF} + \frac{ED}{OE} \cdot \frac{FD}{OF} \cos t.$$

[45_1] $$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

Similarly,

[45_2] $$\cos b = \cos a \cos c + \sin a \sin c \cos \beta.$$

[45_3] $$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma.$$

In Fig. 124, both b and c are less than 90°, while no restriction is placed upon α or a. The resulting formulas are true, however, in general, as may easily be shown.

In Fig. 125, let ABC be a spherical triangle with $c > 90°$ and $b < 90°$. Complete the great circle arcs to form the triangle DCA, in which $AD = (180° - c) < 90°$. The parts of DCA are $180° - c$, $180° - \alpha$, $180° - a$, β, and b. Then by [45_1],

$$\cos(180° - a) = \cos b \cos(180° - c) + \sin b \sin(180° - c) \cos(180° - \alpha).$$

$$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

Exercise.—Draw a spherical triangle in which the two sides b and c are each greater than 90°, and verify formula [**45_1**].

144. Theorem.—*The cosine of any angle of a spherical triangle is equal to the product of the sines of the two other angles multiplied by the cosine of their included side, diminished by the product of the cosines of the two other angles.*

Let ABC be the spherical triangle of which $A'B'C'$ is the polar triangle. Then $a = 180° - \alpha'$, $b = 180° - \beta'$, $c = 180° - \gamma'$, and $\alpha = 180° - a'$.

Substituting these values in [**45_1**] and simplifying,

$$\cos \alpha' = -\cos \beta' \cos \gamma' + \sin \beta' \sin \gamma' \cos a'.$$

This formula expresses a relation between the parts of a polar triangle. But the relation is true for any triangle, since for every spherical triangle there is a polar triangle and conversely.

[**46_1**] $$\therefore \cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a.$$

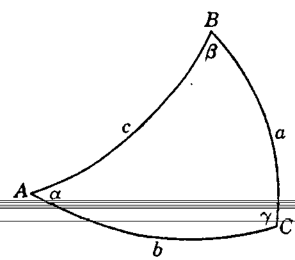


FIG. 127.

Similarly

[**46_2**] $$\cos \beta = -\cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos b.$$

[**46_3**] $$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c.$$

145. Given the three sides to find the angles.—Let ABC be a spherical triangle with given sides a, b, and c.

From [**45_1**],

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \qquad (a)$$

In order to adapt this formula to logarithmic computation, proceed as follows:

(1) Subtracting each member of formula (a) from unity,

$$1 - \cos \alpha = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Then $$2 \sin^2 \frac{1}{2}\alpha = \frac{\cos (b - c) - \cos a}{\sin b \sin c}.$$

But $\cos (b - c) - \cos a = -2 \sin \frac{1}{2}(b - c + a) \sin \frac{1}{2}(b - c - a)$ by [**28**]. Also $\sin \frac{1}{2}(b - c - a) = -\sin \frac{1}{2}(a - b + c)$.

$$\therefore \sin^2 \frac{1}{2}\alpha = \frac{\sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a - b + c)}{\sin b \sin c}.$$

Now let $a + b + c = 2s$.
Then $a + b - c = 2(s - c)$,
and $a - b + c = 2(s - b)$.

[**47_1**] $$\therefore \sin \frac{1}{2}\alpha = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}.$$

In like manner the following are obtained:

[**47_2**] $$\sin \frac{1}{2}\beta = \sqrt{\frac{\sin (s - a) \sin (s - c)}{\sin a \sin c}}.$$

[**47_3**] $$\sin \frac{1}{2}\gamma = \sqrt{\frac{\sin (s - a) \sin (s - b)}{\sin a \sin b}}.$$

(2) By adding each member of formula (a) to unity, and carrying out the work in a similar manner to that in (1), the following are obtained:

[**48_1**] $$\cos \frac{1}{2}\alpha = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}.$$

[**48_2**] $$\cos \frac{1}{2}\beta = \sqrt{\frac{\sin s \sin (s - b)}{\sin a \sin c}}.$$

[**48_3**] $$\cos \frac{1}{2}\gamma = \sqrt{\frac{\sin s \sin (s - c)}{\sin a \sin b}}.$$

(3) By dividing [**47_1**] by [**48_1**],

$$\tan \frac{1}{2}\alpha = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}} = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s \sin^2 (s-a)}}.$$

By writing $r = \sqrt{\dfrac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}$,

[**49_1**] $$\tan \frac{1}{2}\alpha = \frac{r}{\sin (s - a)}.$$

In a like manner the following are obtained:

[**49_2**] $$\tan \frac{1}{2}\beta = \frac{r}{\sin (s - b)}.$$

[**49_3**] $$\tan \frac{1}{2}\gamma = \frac{r}{\sin (s - c)}.$$

146. Given the three angles to find the sides.—If in the formulas [**47**] and [**48**] the parts of the spherical triangle be replaced by their values in terms of the parts of the polar triangle, the following formulas are obtained, where $S = \frac{1}{2}(\alpha + \beta + \gamma)$:

[50_1] $$\cos \frac{1}{2}a = \sqrt{\frac{\cos (S-\beta)\cos (S-\gamma)}{\sin \beta \sin \gamma}}.$$

[50_2] $$\cos \frac{1}{2}b = \sqrt{\frac{\cos (S-\alpha)\cos (S-\gamma)}{\sin \alpha \sin \gamma}}.$$

[50_3] $$\cos \frac{1}{2}c = \sqrt{\frac{\cos (S-\alpha)\cos (S-\beta)}{\sin \alpha \sin \beta}}.$$

[51_1] $$\sin \frac{1}{2}a = \sqrt{-\frac{\cos S \cos (S-\alpha)}{\sin \beta \sin \gamma}}.$$

[51_2] $$\sin \frac{1}{2}b = \sqrt{-\frac{\cos S \cos (S-\beta)}{\sin \alpha \sin \gamma}}.$$

[51_3] $$\sin \frac{1}{2}c = \sqrt{-\frac{\cos S \cos (S-\gamma)}{\sin \alpha \sin \beta}}.$$

Dividing [51_1] by [50_1],

$$\tan \frac{1}{2}a = \sqrt{\frac{-\cos S \cos (S-\alpha)}{\cos (S-\beta)\cos (S-\gamma)}}$$
$$= \cos (S-\alpha)\sqrt{\frac{-\cos S}{\cos (S-\alpha)\cos (S-\beta)\cos (S-\gamma)}}.$$

By writing $R = \sqrt{\dfrac{-\cos S}{\cos (S-\alpha)\cos (S-\beta)\cos (S-\gamma)}}$,

[52_1] $$\tan \tfrac{1}{2}a = R \cos (S-\alpha).$$

In like manner the following are obtained:

[52_2] $$\tan \tfrac{1}{2}b = R \cos (S-\beta).$$
[52_3] $$\tan \tfrac{1}{2}c = R \cos (S-\gamma).$$

Note.—Since $90° < S < 270°$, $\cos S$ is negative. Also, since, in the polar triangle, any side is less than the sum of the two others,

$$\pi - \alpha < (\pi - \beta) + (\pi - \gamma), \text{ or } \beta + \gamma - \alpha < \pi.$$

Further, $$\beta + \gamma - \alpha > -\pi.$$

$$\therefore -\tfrac{1}{2}\pi < S - \alpha < \tfrac{1}{2}\pi, \text{ and } \cos (S-\alpha) \text{ is positive.}$$

Similarly, it can be shown that $\cos (S-\beta)$ and $\cos (S-\gamma)$ are each positive.

This makes the radical expressions of this article real.

Further, the positive sign must be given to the radicals in each case, for $\frac{1}{2}a$, $\frac{1}{2}b$, and $\frac{1}{2}c$ are each less than 90°.

147. Napier's analogies.—Dividing [49_1] by [49_2],

$$\frac{\tan \frac{1}{2}\alpha}{\tan \frac{1}{2}\beta} = \frac{\sin (s-b)}{\sin (s-a)}$$

Taking this proportion by composition and division,

$$\frac{\tan \frac{1}{2}\alpha + \tan \frac{1}{2}\beta}{\tan \frac{1}{2}\alpha - \tan \frac{1}{2}\beta} = \frac{\sin (s-b) + \sin (s-a)}{\sin (s-b) - \sin (s-a)}. \quad (1)$$

But $$\frac{\tan \frac{1}{2}\alpha + \tan \frac{1}{2}\beta}{\tan \frac{1}{2}\alpha - \tan \frac{1}{2}\beta} = \frac{\sin \frac{1}{2}(\alpha+\beta)}{\sin \frac{1}{2}(\alpha-\beta)}.$$

Also $$\frac{\sin (s-b) + \sin (s-a)}{\sin (s-b) - \sin (s-a)} = \frac{2 \sin \frac{1}{2}(2s-a-b)\cos \frac{1}{2}(a-b)}{2 \cos \frac{1}{2}(2s-a-b)\sin \frac{1}{2}(a-b)}$$
$$= \frac{\tan \frac{1}{2}c}{\tan \frac{1}{2}(a-b)}.$$

Substituting these values in formula (1), there results

[53] $$\frac{\tan \frac{1}{2}(a-b)}{\tan \frac{1}{2}c} = \frac{\sin \frac{1}{2}(\alpha-\beta)}{\sin \frac{1}{2}(\alpha+\beta)}.$$

Replacing a, b, c, α, and β by their values in terms of the parts of the polar triangle, [53] becomes

[54] $$\frac{\tan \frac{1}{2}(\alpha-\beta)}{\cot \frac{1}{2}\gamma} = \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)}.$$

Again, multiplying [49_1] by [49_2],

$$\frac{\tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta}{1} = \frac{\sin (s-c)}{\sin s}.$$

Taking this proportion by composition and division,

$$\frac{1 + \tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta}{1 - \tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta} = \frac{\sin s + \sin (s-c)}{\sin s - \sin (s-c)}. \quad (2)$$

But
$$\frac{1 + \tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta}{1 - \tan \frac{1}{2}\alpha \tan \frac{1}{2}\beta} = \frac{\cos \frac{1}{2}\alpha \cos \frac{1}{2}\beta + \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta}{\cos \frac{1}{2}\alpha \cos \frac{1}{2}\beta - \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta} = \frac{\cos \frac{1}{2}(\alpha-\beta)}{\cos \frac{1}{2}(\alpha+\beta)}.$$

Also $$\frac{\sin s + \sin (s-c)}{\sin s - \sin (s-c)} = \frac{2 \sin \frac{1}{2}(2s-c)\cos \frac{1}{2}c}{2 \cos \frac{1}{2}(2s-c)\sin \frac{1}{2}c} = \frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}c}.$$

Substituting these values in formula (2), there results

[55] $$\frac{\tan \frac{1}{2}(a+b)}{\tan \frac{1}{2}c} = \frac{\cos \frac{1}{2}(\alpha-\beta)}{\cos \frac{1}{2}(\alpha+\beta)}.$$

Replacing a, b, c, α, and β by their values in terms of the parts of the polar triangle, [55] becomes

[56] $$\frac{\tan \frac{1}{2}(\alpha+\beta)}{\cot \frac{1}{2}\gamma} = \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)}.$$

Equations [53], [54], [55], and [56] are known as **Napier's analogies.**

By making the proper changes in a, b, c, α, β, and γ, the corresponding formulas may be written for the other parts of the triangle.

148. Gauss's equations.—Taking the values of $\sin \frac{1}{2}\alpha$, $\cos \frac{1}{2}\alpha$, $\sin \frac{1}{2}\beta$, and $\cos \frac{1}{2}\beta$, and combining the functions of $\frac{1}{2}\alpha$ with those of $\frac{1}{2}\beta$, there result the four following forms:

$$\sin \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}} \sqrt{\frac{\sin s \sin (s-b)}{\sin a \sin c}}$$
$$= \frac{\sin (s-b)}{\sin c} \sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}} = \frac{\sin (s-b)}{\sin c} \cos \tfrac{1}{2}\gamma. \quad (1)$$

$$\cos \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}} \sqrt{\frac{\sin (s-a) \sin (s-c)}{\sin a \sin c}}$$
$$= \frac{\sin (s-a)}{\sin c} \sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}} = \frac{\sin (s-a)}{\sin c} \cos \tfrac{1}{2}\gamma. (2)$$

$$\cos \tfrac{1}{2}\alpha \cos \tfrac{1}{2}\beta = \sqrt{\frac{\sin s \sin (s-a)}{\sin b \sin c}} \sqrt{\frac{\sin s \sin (s-b)}{\sin a \sin c}}$$
$$= \frac{\sin s}{\sin c} \sqrt{\frac{\sin (s-a) \sin (s-b)}{\sin a \sin b}} = \frac{\sin s}{\sin c} \sin \tfrac{1}{2}\gamma. \quad (3)$$

$$\sin \tfrac{1}{2}\alpha \sin \tfrac{1}{2}\beta = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}} \sqrt{\frac{\sin (s-a) \sin (s-c)}{\sin a \sin c}}$$
$$= \frac{\sin (s-c)}{\sin c} \sqrt{\frac{\sin (s-a) \sin (s-b)}{\sin a \sin b}} = \frac{\sin (s-c)}{\sin c} \sin \tfrac{1}{2}\gamma. \quad (4)$$

Adding (1) and (2), there results

$$\sin \tfrac{1}{2}(\alpha + \beta) = \frac{\cos \frac{1}{2}\gamma}{\sin c}[\sin (s-b) + \sin (s-a)]$$
$$= \frac{\cos \frac{1}{2}\gamma}{2 \sin \frac{1}{2}c \cos \frac{1}{2}c} 2 \sin \tfrac{1}{2}[2s - (a+b)] \cos \tfrac{1}{2}(a-b).$$

But $\sin \frac{1}{2}[2s - (a+b)] = \sin \frac{1}{2}c$.

[57] $\quad \therefore \cos \frac{1}{2}c \sin \frac{1}{2}(\alpha + \beta) = \cos \frac{1}{2}\gamma \cos \frac{1}{2}(a - b).$

Subtracting (4) from (3), there results

$$\cos \tfrac{1}{2}(\alpha + \beta) = \frac{\sin \frac{1}{2}\gamma}{\sin c}[\sin s - \sin (s-c)]$$
$$= \frac{\sin \frac{1}{2}\gamma}{2 \sin \frac{1}{2}c \cos \frac{1}{2}c} 2 \cos \tfrac{1}{2}(2s - c) \sin \tfrac{1}{2}c.$$

But $\cos \frac{1}{2}(2s - c) = \cos \frac{1}{2}(a + b)$.

[58] $\quad \therefore \cos \frac{1}{2}c \cos \frac{1}{2}(\alpha + \beta) = \sin \frac{1}{2}\gamma \cos \frac{1}{2}(a + b).$

Subtracting (2) from (1), there results

$$\sin \tfrac{1}{2}(\alpha - \beta) = \frac{\cos \frac{1}{2}\gamma}{\sin c}[\sin (s-b) - \sin (s-a)]$$
$$= \frac{\cos \frac{1}{2}\gamma}{2 \sin \frac{1}{2}c \cos \frac{1}{2}c} 2 \cos \tfrac{1}{2}[2s - (a+b)] \sin \tfrac{1}{2}(a-b).$$

[59] $\quad \therefore \sin \frac{1}{2}c \sin \frac{1}{2}(\alpha - \beta) = \cos \frac{1}{2}\gamma \sin \frac{1}{2}(a - b).$

Adding (3) to (4), there results

$$\cos \tfrac{1}{2}(\alpha - \beta) = \frac{\sin \frac{1}{2}\gamma}{\sin c}[\sin s + \sin (s-c)]$$
$$= \frac{\sin \frac{1}{2}\gamma}{2 \sin \frac{1}{2}c \cos \frac{1}{2}c} 2 \sin \tfrac{1}{2}(2s - c) \cos \tfrac{1}{2}c.$$

[60] $\quad \therefore \sin \frac{1}{2}c \cos \frac{1}{2}(\alpha - \beta) = \sin \frac{1}{2}\gamma \sin \frac{1}{2}(a + b).$

Equations [57], [58], [59], and [60] are known as **Gauss's equations,** or **Delambre's analogies.** Geometric proofs of Gauss's equations and Napier's analogies can be found in Todhunter and Leathem's "Spherical Trigonometry."

Exercise.—Derive [60] from [57] by using the parts of the polar triangle.

149. Rules for species in oblique spherical triangles.—(1) *If a side (or an angle) differs from 90° by a larger number of degrees than another side (or angle) in the triangle, it is of the same species as its opposite angle (or side).*

Since all angles and sides of a spherical triangle are each less than 180°, in order to verify this rule, it is necessary to show that $\cos a$ and $\cos \alpha$, for example, have the same sign when

$$|a - 90°| > |b - 90°|.$$

From [45_1], $\quad \cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$

$$\therefore \cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Since $|a - 90°| > |b - 90°|$, a is nearer 0 or 180° than b, and therefore $|\cos a| > |\cos b|$, and, since $\cos c$ cannot exceed unity,

$$\therefore |\cos a| > |\cos b \cos c|.$$

Further, the denominator will always be positive.

Then the sign of $\cos \alpha$ is the sign of the numerator of the fraction. That is, $\cos \alpha$ has the same sign as $\cos a$; therefore a and α are in the same quadrant.

For example, suppose $a = 120°$, $b = 70°$, $c = 130°$. Since $|120° - 90°| > |90° - 70°|$, α is in the second quadrant with a. Also, since $|130° - 90°| > |90° - 70°|$, γ is in the second quadrant with c. This leaves β undetermined in quadrant. It is determined by the second rule, which follows.

(2) *Half the sum of two sides of a spherical triangle must be of the same species as half the sum of the two opposite angles.*

From]**55**], $\tan \frac{1}{2}(a + b) = \tan \frac{1}{2}c \frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)}$.

Since $\frac{1}{2}c < 90°$, $\tan \frac{1}{2}c > 0$. Also, since $(\alpha - \beta) < 180°$, $\cos \frac{1}{2}(\alpha - \beta) > 0$. Therefore, $\tan \frac{1}{2}(a + b)$ and $\cos \frac{1}{2}(\alpha + \beta)$ are of the same sign. But $a + b$ and $\alpha + \beta$ must each be less than 360°, and, therefore, $\frac{1}{2}(a + b)$ and $\frac{1}{2}(\alpha + \beta)$ must each be less than 180°. Then $\frac{1}{2}(a + b)$ and $\frac{1}{2}(\alpha + \beta)$ must both be in the first quadrant or both be in the second quadrant, since $\tan \frac{1}{2}(a + b)$ and $\cos \frac{1}{2}(\alpha + \beta)$ are of the same sign.

150. Cases.—In the solution of oblique spherical triangles, the six following cases arise:

CASE I. *Given the three sides.*
CASE II. *Given the three angles.*
CASE III. *Given two sides and the included angle.*
CASE IV. *Given two angles and the included side.*
CASE V. *Given two sides and an angle opposite one of them.*
CASE VI. *Given two angles and a side opposite one of them.*

Any oblique spherical triangle can be solved by the formulas derived in the previous articles. In selecting a formula choose one which includes the parts given and the one to be found. The following list of formulas, together with the corresponding formulas for other parts, is sufficient for solving any spherical triangle:

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}. \tag{1}$$

$$\tan \frac{1}{2}\alpha = \frac{r}{\sin (s - a)},$$

where
$$r = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}. \tag{2}$$

$$\tan \tfrac{1}{2}a = R \cos (S - \alpha),$$

where
$$R = \sqrt{\frac{-\cos S}{\cos (S - \alpha) \cos (S - \beta) \cos (S - \gamma)}} \tag{3}$$

$$\frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2}c} = \frac{\sin \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha + \beta)}. \tag{4}$$

$$\frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}c} = \frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)}. \tag{5}$$

$$\frac{\tan \frac{1}{2}(\alpha - \beta)}{\cot \frac{1}{2}\gamma} = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)}. \tag{6}$$

$$\frac{\tan \frac{1}{2}(\alpha + \beta)}{\cot \frac{1}{2}\gamma} = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}. \tag{7}$$

151. Case I. Given the three sides to find the three angles. *Example.*—Given $a = 46° 20' 45''$, $b = 65° 18' 15''$, $c = 90° 31' 46''$; to find α, β, and γ.

Formulas

$$\tan \frac{1}{2}\alpha = \frac{r}{\sin (s - a)}.$$

$$\tan \frac{1}{2}\beta = \frac{r}{\sin (s - b)}.$$

$$\tan \frac{1}{2}\gamma = \frac{r}{\sin (s - c)}.$$

Construction

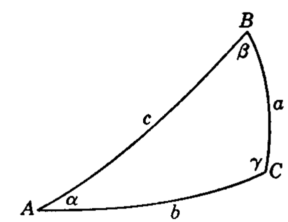


FIG. 128.

$$r = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}.$$

Computation

$a = 46° 20' 45''$	$\log \sin (s - a) = 9.91200$
$b = 65° 18' 15''$	$\log \sin (s - b) = 9.76697$
$c = 90° 31' 46''$	$\log \sin (s - c) = 9.26309$
$2s = 202° 10' 46''$	$\text{colog} \sin s = 0.00819$
$s = 101° \ 5' 23''$	$\log r^2 = 8.95025$
$s - a = 54° 44' 38''$	$\log r = 9.47513$
$s - b = 35° 47' \ 8''$	$\log \tan \frac{1}{2}\alpha = 9.56313$
$s - c = 10° 33' 37''$	$\therefore \frac{1}{2}\alpha = 20° \ 5' 15''$
$2s = 202° 10' 46''$	$\log \tan \frac{1}{2}\beta = 9.70816$
A check	$\therefore \frac{1}{2}\beta = 27° \ 3' 12''$
	$\log \tan \frac{1}{2}\gamma = 0.21204$
	$\therefore \frac{1}{2}\gamma = 58° 27' 45''$

Check by the sine law.

EXERCISES

1. Given $a = 68° 45'$, $b = 53° 15'$, $c = 46° 30'$;
find $\alpha = 94° 52' 40''$, $\beta = 58° 56' 10''$, $\gamma = 50° 50' 52''$.

2. Given $a = 70° 14' 20''$, $b = 49° 24' 10''$, $c = 38° 46' 10''$;
find $\alpha = 110° 51' 16''$, $\beta = 48° 56' 4''$, $\gamma = 38° 26' 48''$.
3. Given $a = 50° 12.1'$, $b = 116° 44.8'$, $c = 129° 11.7'$;
find $\alpha = 59° 4.4'$, $\beta = 94° 23.2'$, $\gamma = 120° 4.8'$.
4. Given $a = 68° 20.4'$, $b = 52° 18.3'$, $c = 96° 20.7'$;
find $\alpha = 56° 16.3'$, $\beta = 45° 4.7'$, $\gamma = 117° 12.3'$.
5. Given $a = 96° 24' 30''$, $b = 68° 27' 26''$, $c = 87° 31' 37''$;
find $\alpha = 97° 53' 0''$, $\beta = 67° 59' 39''$, $\gamma = 84° 46' 40''$.
6. Given $a = 31° 9' 13''$, $b = 84° 18' 28''$, $c = 115° 10' 0''$;
find $\alpha = 4° 23' 35''$, $\beta = 8° 28' 20''$, $\gamma = 172° 17' 56''$.

152. Case II. Given the three angles to find the three sides.—For the solution, use $\tan \frac{1}{2}a = R \cos (S - \alpha)$ and the corresponding forms, and proceed as in Case I.

EXERCISES

1. Given $\alpha = 129° 5' 28''$, $\beta = 142° 12' 42''$, $\gamma = 105° 8' 10''$;
find $a = 135° 49' 20''$, $b = 146° 37' 15''$, $c = 60° 4' 54''$.
2. Given $\alpha = 59° 4' 28''$, $\beta = 94° 23' 12''$, $\gamma = 120° 4' 52''$;
find $a = 50° 12' 4''$, $b = 116° 44' 48''$, $c = 129° 11' 42''$.
3. Given $\alpha = 107° 33' 20''$, $\beta = 127° 22' 0''$, $\gamma = 128° 41' 49''$;
find $a = 82° 47' 34''$, $b = 124° 12' 31''$, $c = 125° 41' 43''$.
4. Given $\alpha = 102° 14' 12''$, $\beta = 54° 32' 24''$, $\gamma = 89° 5' 46''$;
find $a = 104° 25' 8''$, $b = 53° 49' 25''$, $c = 97° 44', 18''$.

153. Case III. Given two sides and the included angle.—The sum and the difference of the two unknown angles can be found by [**54**] and [**56**]. The unknown side can be found by either [**53**] or [**55**]; together, they furnish a check on the work.

Example.—Given $a = 103° 44.7'$, $b = 64° 12.3'$, $\gamma = 98° 33.8'$; find α, β, and c.

Formulas

From [**54**],
$$\tan \frac{1}{2}(\alpha - \beta) = \cot \frac{1}{2}\gamma \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)}.$$
From [**56**],
$$\tan \frac{1}{2}(\alpha + \beta) = \cot \frac{1}{2}\gamma \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}.$$
From [**53**],
$$\tan \frac{1}{2}c = \tan \frac{1}{2}(a - b) \frac{\sin \frac{1}{2}(\alpha + \beta)}{\sin \frac{1}{2}(\alpha - \beta)}.$$
From [**55**],
$$\tan \frac{1}{2}c = \tan \frac{1}{2}(a + b) \frac{\cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)}.$$

Construction

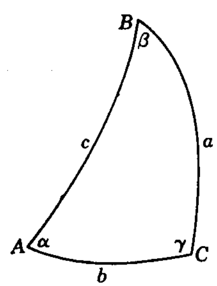


Fig. 129.

Computation

$a = 103° 44.7'$
$b = 64° 12.3'$
$\frac{1}{2}(a + b) = 83° 58.5'$
$\frac{1}{2}(a - b) = 19° 46.2'$
$\frac{1}{2}\gamma = 49° 16.9'$
$\alpha = 98° 55.9'$
$\beta = 66° 18'$

$\log \cot \frac{1}{2}\gamma = 9.93485$
$\log \sin \frac{1}{2}(a - b) = 9.52923$
$\text{colog} \sin \frac{1}{2}(a + b) = 0.00241$
$\log \tan \frac{1}{2}(\alpha - \beta) = 9.46649$
$\therefore \frac{1}{2}(\alpha - \beta) = 16° 19'$

$\log \cot \frac{1}{2}\gamma = 9.93485$
$\log \cos \frac{1}{2}(a - b) = 9.97360$
$\text{colog} \cos \frac{1}{2}(a + b) = 0.97897$
$\log \tan \frac{1}{2}(\alpha + \beta) = 0.88742$
$\therefore \frac{1}{2}(\alpha + \beta) = 82° 37'$

$\log \tan \frac{1}{2}(a - b) = 9.55562$
$\log \sin \frac{1}{2}(\alpha + \beta) = 9.99638$
$\text{colog} \sin \frac{1}{2}(\alpha - \beta) = 0.55138$
$\log \tan \frac{1}{2}c = 0.10338$
$\therefore \frac{1}{2}c = 51° 45.3'$

$\log \tan \frac{1}{2}(a + b) = 0.97657$
$\log \cos \frac{1}{2}(\alpha + \beta) = 9.10893$
$\text{colog} \cos \frac{1}{2}(\alpha - \beta) = 0.01785$
$\log \tan \frac{1}{2}c = 0.10335$
$\therefore \frac{1}{2}c = 51° 45.3'$

EXERCISES

1. Given $b = 99° 40' 48''$, $c = 64° 23' 15''$, $\alpha = 97° 26' 29''$;
find $a = 100° 49' 30''$, $\beta = 95° 38' 4''$, $\gamma = 65° 33' 10''$.
2. Given $a = 88° 21' 20''$, $b = 124° 7' 17''$, $\gamma = 50° 2' 1''$;
find $\alpha = 63° 22' 56''$, $\beta = 132° 13' 58''$, $c = 58° 58' 24''$.
3. Given $b = 156° 12.2'$, $c = 112° 48.6'$, $\alpha = 76° 32.4'$;
find $a = 63° 48.7'$, $\beta = 154° 4.1'$, $\gamma = 87° 27.1'$.
4. Given $a = 70° 20' 50''$, $b = 38° 28'$, $\gamma = 52° 29' 45''$;
find $\alpha = 107° 47' 7''$, $\beta = 38° 58' 27''$, $c = 51° 40' 54''$.
5. Given $a = 135° 49' 20''$, $c = 60° 4' 54''$, $\beta = 142° 12' 42''$;
find $\alpha = 129° 5' 28''$, $\gamma = 105° 8' 10''$, $b = 146° 37' 15''$.

154. Case IV. Given two angles and the included side.—This case, like the preceding, is to be solved by Napier's analogies, using the four forms in a similar manner.

EXERCISES

1. Given $\alpha = 59° 4' 25''$, $\beta = 88° 12' 24''$, $c = 47° 42' 1''$;
find $a = 50° 2' 1''$, $b = 63° 15' 15''$, $\gamma = 55° 52' 42''$.
2. Given $\alpha = 63° 45.6'$, $\beta = 95° 56.7'$, $c = 52° 27.8'$;
find $a = 61° 41.3'$, $b = 77° 29.4'$, $\gamma = 53° 53.5'$.
3. Given $\alpha = 125° 41' 44''$, $\gamma = 82° 47' 35''$, $b = 52° 37' 57''$;
find $a = 128° 41' 46''$, $c = 107° 33' 20''$, $\beta = 55° 47' 40''$.
4. Given $\beta = 34° 29' 30''$, $\gamma = 36° 6' 50''$, $a = 85° 59' 0''$;
find $b = 47° 29' 20''$, $c = 50° 6' 20''$, $\alpha = 129° 58' 30''$.

155. Case V. Given two sides and the angle opposite one of them.—In this case the angle opposite the other side can be found by the sine law, when the other side and angle can be found by Napier's analogies. For example, given a, b, and α, to find c, β, and γ, use the following formulas:

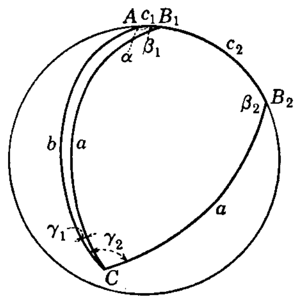


Fig. 130.

$$\sin\beta = \frac{\sin b \sin\alpha}{\sin a}.$$

$$\tan\frac{1}{2}c = \tan\frac{1}{2}(a-b)\,\frac{\sin\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha-\beta)}.$$

$$\tan\frac{1}{2}c = \tan\frac{1}{2}(a+b)\,\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}.$$

$$\cot\frac{1}{2}\gamma = \tan\frac{1}{2}(\alpha-\beta)\,\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}.$$

$$\cot\frac{1}{2}\gamma = \tan\frac{1}{2}(\alpha+\beta)\,\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)}.$$

A check is obtained by the agreement in the values of $\frac{1}{2}c$ and $\frac{1}{2}\gamma$ from the different formulas.

Since β is determined from $\sin\beta$, there will be two values of β less than 180°, both of which may enter into a triangle. By the first rule for species (**Art. 149**), if $|90° - b| > |90° - a|$, b and β must be the same species. This definitely determines β. Otherwise, both values of β may be admissible. The application of the second rule for species will show whether or not two triangles are possible.

Example.—Given $a = 148°\ 34'\ 24''$, $b = 142°\ 11'\ 36''$, and $\alpha = 153°\ 17'\ 36''$; find c, β, and γ.

$$\sin\beta = \frac{\sin b \sin\alpha}{\sin a}.$$

$$\begin{aligned}
\log\sin 142°\ 11'\ 36'' &= 9.78746\\
\log\sin 153°\ 17'\ 36'' &= 9.65265\\
\text{colog}\sin 148°\ 34'\ 24'' &= 0.28282\\
\log\sin\beta &= 9.72293\\
\beta_1 &= 31°\ 53'\ 42''\\
\beta_2 &= 148°\ 6'\ 18''
\end{aligned}$$

Here, since $|90° - 142°\ 11'\ 36''| < |90° - 148°\ 34'\ 24''|$, both values of β may be admissible. Since $\frac{1}{2}(a+b) = 145°\ 23'$ is in the second quadrant, as also are $\frac{1}{2}(\alpha+\beta_1) = 90°\ 35'\ 36''$ and $\frac{1}{2}(\alpha+\beta_2) = 150°\ 42'$, they are of the same species by the second rule for species. Hence, both β_1 and β_2 are admissible values to use.

The student can complete the solution and find the following values:

$$c_1 = 7°\ 18'\ 24''. \qquad \gamma_1 = 6°\ 17'\ 36''.$$
$$c_2 = 62°\ 8'\ 36''. \qquad \gamma_2 = 130°\ 21'\ 12''.$$

This case is the *ambiguous case* in oblique spherical triangles, and is analogous to the ambiguous case in plane trigonometry. In practical applications, some facts about the general shape of the triangle may be known which will determine the values to be chosen without having recourse to the rules for species. A complete discussion of the ambiguous case may be found in Todhunter and Leathem's "Spherical Trigonometry," pages 80 to 85.

EXERCISES

1. Given $a = 46°\ 20'\ 45''$, $b = 65°\ 18'\ 15''$, $\alpha = 40°\ 10'\ 30''$;
find $c_1 = 90°\ 31'\ 46''$, $\beta_1 = 54°\ 6'\ 19''$, $\gamma_1 = 116°\ 55'\ 26''$;
$c_2 = 27°\ 23'\ 14''$, $\beta_2 = 125°\ 53'\ 41''$, $\gamma_2 = 24°\ 12'\ 53''$.

2. Given $a = 99°\ 40'\ 48''$, $b = 64°\ 23'\ 15''$, $\alpha = 95°\ 38'\ 4''$;
find $c = 100°\ 49'\ 30''$, $\beta = 65°\ 33'\ 10''$, $\gamma = 97°\ 26'\ 29''$.

3. Solve and check, $a = 31°\ 40'\ 25''$, $b = 32°\ 30'$, $\alpha = 88°\ 20'$.

4. Solve and check, $a = 149°$, $b = 133°$, $\alpha = 146°$.

5. Given $a = 62°\ 15'\ 24''$, $b = 103°\ 18'\ 47''$, $\alpha = 53°\ 42'\ 38''$;
find $c_1 = 153°\ 9'\ 36''$, $\beta_1 = 62°\ 24'\ 25''$, $\gamma_1 = 155°\ 43'\ 11''$;
$c_2 = 70°\ 25'\ 26''$, $\beta_2 = 117°\ 35'\ 35''$, $\gamma_2 = 59°\ 6'\ 50''$.

156. Case VI. Given two angles and the side opposite one of them.—In this case use the same formulas as in Case V, and apply the rules for species when there is any question as to the number of solutions.

EXERCISES

1. Given $\alpha = 29°\ 2'\ 55''$, $\beta = 45°\ 44'\ 6''$, $a = 35°\ 37'\ 18''$
find $b_1 = 59°\ 12'\ 16''$, $c_1 = 82°\ 17'\ 5''$, $\gamma_1 = 124°\ 17'\ 52''$;
$b_2 = 120°\ 47'\ 44''$, $c_2 = 150°\ 50'\ 51''$, $\gamma_2 = 156°\ 2'\ 24''$.

2. Given $\alpha = 73°\ 11'\ 18''$, $\beta = 61°\ 18'\ 12''$, $a = 46°\ 45'\ 30''$;
find $b = 41°\ 52'\ 35''$, $c = 41°\ 35'\ 4''$, $\gamma = 60°\ 42'\ 47''$.

3. Solve and check, $\alpha = 122°$, $\beta = 71°$, $a = 81°$.

4. Solve and check, $\alpha = 37°\ 42'$, $\beta = 47°\ 20'$, $b = 41°\ 50'$.

5. Given $\alpha = 36°\ 20'\ 20''$, $\beta = 46°\ 30'\ 40''$, $a = 42°\ 15'\ 20''$;
find $b_1 = 55°\ 25'\ 2''$, $c_1 = 81°\ 27'\ 26''$, $\gamma_1 = 119°\ 22'\ 28''$;
$b_2 = 124°\ 34'\ 58''$, $c_2 = 162°\ 34'\ 27''$, $\gamma_2 = 164°\ 41'\ 55''$.

157. Area of a spherical triangle.—Let r be the radius of the sphere on which the triangle is situated, Δ the area of the triangle, and E the spherical excess.

By spherical geometry, the areas of any two spherical triangles are to each other as their spherical excesses. Now the area of a trirectangular triangle is $\frac{1}{2}\pi r^2$, and its spherical excess is 90°.

Then $$\Delta : \tfrac{1}{2}\pi r^2 = E : 90°.$$

$$[61] \qquad \therefore \Delta = \frac{\pi r^2 E}{180°}.$$

When all the angles of the triangle are known, the spherical excess and, therefore, the area are easily computed. If the angles are not all given, but enough data are known for the solution of the triangle, the angles may be found by Napier's analogies, and then the area may be computed by the above formula.

158. L'Huilier's formula.—This is a formula for determining the spherical excess directly in terms of the sides. It may be derived as follows: Since $E = \alpha + \beta + \gamma - 180°$,

$$\tan\frac{1}{4}E = \frac{\sin\frac{1}{4}(\alpha+\beta+\gamma-180°)}{\cos\frac{1}{4}(\alpha+\beta+\gamma-180°)}$$

$$= \frac{2\sin\frac{1}{4}(\alpha+\beta+\gamma-180°)\cos\frac{1}{4}(\alpha+\beta-\gamma+180°)}{2\cos\frac{1}{4}(\alpha+\beta+\gamma-180°)\cos\frac{1}{4}(\alpha+\beta-\gamma+180°)}$$

$$= \frac{\sin\frac{1}{2}(\alpha+\beta) - \cos\frac{1}{2}\gamma}{\cos\frac{1}{2}(\alpha+\beta) + \sin\frac{1}{2}\gamma}, \text{ by [29] and [31].}$$

By [57], $\sin\frac{1}{2}(\alpha+\beta) = \dfrac{\cos\frac{1}{2}\gamma\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}.$

By [58], $\cos\frac{1}{2}(\alpha+\beta) = \dfrac{\sin\frac{1}{2}\gamma\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}.$

By making these substitutions,

$$\tan\frac{1}{4}E = \left(\frac{\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}c}{\cos\frac{1}{2}(a+b) + \cos\frac{1}{2}c}\right)\cot\frac{1}{2}\gamma$$

$$= \left(\frac{-2\sin\frac{1}{4}(a-b+c)\sin\frac{1}{4}(a-b-c)}{2\cos\frac{1}{4}(a+b+c)\cos\frac{1}{4}(a+b-c)}\right)\cot\frac{1}{2}\gamma$$

$$= \frac{\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-a)}{\cos\frac{1}{2}s\cos\frac{1}{2}(s-c)}\sqrt{\frac{\sin s\sin(s-c)}{\sin(s-a)\sin(s-b)}}.$$

$$[62] \; \therefore \tan\tfrac{1}{4}E = \sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}.$$

EXERCISES

1. On a sphere of radius 6 in., $\alpha = 87° 20' 45''$, $\beta = 32° 40' 56''$, and $\gamma = 77° 45' 32''$. Find the area of the triangle. *Ans.* 11.176 sq. in.
2. Given $a = 56° 37'$, $b = 108° 14'$, $c = 75° 29'$; find E. *Ans.* $E = 48° 32' 35''$.
3. Given $a = 47° 18'$, $b = 53° 26'$, $c = 63° 54'$; find E. *Ans.* $E = 24° 29' 50''$.
4. Given $\alpha = 110° 10'$, $b = 33° 1' 45''$, $c = 155° 5' 18''$; find E. *Ans.* $E = 133° 48' 53''$.
5. Given $a = b = c = 60°$, on a sphere of 12-in. radius; find the area of the triangle. *Ans.* 79.38 sq. in.

6. Given $\alpha = 49° 50'$, $\beta = 67° 30'$, $\gamma = 74° 40'$, on a sphere of 10-in. radius; find the area of the triangle. *Ans.* 20.94 sq. in.
7. Given $\alpha = 110°$, $\beta = 94°$, $c = 44°$, on a sphere of 10-ft. radius; find the area of the triangle. *Ans.* 128.15 sq. ft.
8. Given $a = 15° 22' 44''$, $c = 44° 27' 40''$, $\beta = 167° 42' 27''$, on a sphere of radius 100 ft.; find the area of the triangle. *Ans.* 248.32 sq. ft.
9. Find the area of a triangle having sides of 1° each on the surface of the earth. *Ans.* 2070 sq. miles.

APPLICATIONS OF SPHERICAL TRIGONOMETRY

159. Definitions and notations.—In all the applications of spherical trigonometry to the measurements of arcs of great circles on the surface of the earth, and to problems of astronomy, the earth will be treated as a sphere of radius 3956 miles.

A **meridian** is a great circle of the earth drawn through the poles N and S. The meridian NGS passing through Greenwich, England, is called the **principal meridian.**

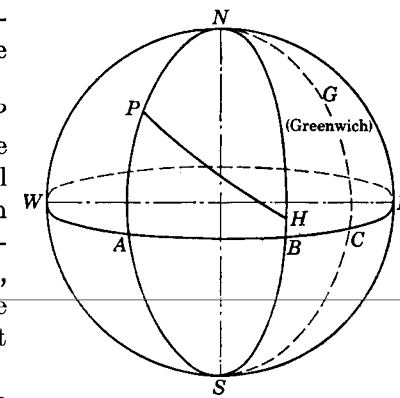


Fig. 131.

The **longitude** of any point P on the earth's surface is the angle between the principal meridian NGS and the meridian NPS through P. It is measured by the great circle arc, CA, of the equator between the points where the meridians cut the equator.

If a point on the surface of the earth is west of the principal meridian, its longitude is *positive.* If east, it is *negative.* A point 70° west of the principal meridian is usually designated as in "longitude 70° W." "Longitude 70° E." means in 70° east longitude. The letter θ is used to designate longitude.

The **latitude** of a point on the surface of the earth is the number of degrees it is north or south of the equator, measured along a meridian. Latitude is *positive* when measured north of the equator, and *negative* when south. The letter ϕ is used to designate latitude.

160. The terrestrial triangle.—Given two points on the earth's surface, H with latitude ϕ_1 and longitude θ_1, and P with latitude ϕ_2 and longitude θ_2; then the arcs $HN = 90° - \phi_1$, $PN = 90° - \phi_2$, and HP, which represents the distance between the points, form a spherical triangle called the **terrestrial triangle.**

The angle $HNP = \theta_2 - \theta_1$, the difference of the longitudes, is known, as are also the arcs HN and PN. Two sides and the included angle of the triangle HPN are then known and the triangle can be solved for the distance HP and for the *bearing* of one point from the other.

The angle NHP is the **bearing** of P from H, and angle NPH the bearing of H from P. The bearing will be represented by γ. The bearing of a line is usually the smallest angle which the line or path makes with the meridian through the point.

A complete solution of the terrestrial triangle by Napier's analogies and the sine law will furnish the bearings of any two points and their distance apart.

EXERCISES

1. Find the shortest distance, in statute miles, between New York, 40° 45′ N., 73° 58′ W., and Chicago, 41° 50′ N., 87° 35′ W. *Ans.* 710 miles.

2. Find the shortest distance, in statute miles, between Chicago, 41° 50′ N., 87° 35′ W., and San Francisco, 37° 40′ N., 122° 28′ W.

Ans. 1859 miles.

Find the shortest distance between the following places, and find the bearing of each from the other:

3. New York, 40° 45′ N., 73° 58′ W., and Rio Janeiro, 22° 54′ S., 43° 10′ W.

Ans. Distance = 69° 47.8′ = 4187.8 geographic miles.
Bearing of New York from Rio Janeiro, N. 24° 24.9′ W.
Bearing of Rio Janeiro from New York, S. 30° 10.5′ E.

4. San Francisco, 37° 48′ N., 122° 28′ W., and Manila, 14° 36′ N., 121° 5′ E.

Ans. Distance = 100° 43.5′ = 6043 geographic miles.
Bearing of San Francisco from Manila, N. 46° 3.4′ E.
Bearing of Manila from San Francisco, N. 61° 51.7′ W.

Find the shortest distance in statute miles between the following places and check the work:

5. Chicago, 41° 50′ N., 87° 35′ W., and Manila, 14° 36′ N., 121° 5′ E.

6. Greenwich, 51° 29′ N., and Valparaiso, 33° 2′ S., 71° 42′ W.

7. Paris, 48° 50′ N., 2° 20′ E., and Calcutta, 22° 35′ N., 88° 27′ E.

8. From a point at 40° N., 8° 15.6′ W. a ship sails on the arc of a great circle a distance of 3000 statute miles, starting in the direction S. 61° 15′ W. Find its latitude and longitude. *Ans.* 12° 18.5′ N., 46° 22.0′ W.

9. What is the shortest distance on the surface of the earth from a point A, 45° N., 74° W., and a point B which is 2000 miles directly west from A?

Ans. 1978 miles.

161. Applications to astronomy.—The daily rotation of the earth about its axis, from west to east, causes the stars to appear to rotate daily from east to west. They move as if attached to the surface of an immense sphere rotating about an axis through its center. This sphere is called the **celestial sphere,** and the center of the earth is taken as its center.

The location of a star, or any heavenly body, is the point where a line drawn from the observer through the star pierces the celestial sphere. The location of one heavenly body with reference to another is thus seen to depend upon the arcs of great circles and spherical angles.

162. Fundamental points, circles of reference, and systems of coordinates.—The **north pole** P and the **south pole** Q of the celestial sphere (Fig. 132) are the points where the earth's axis produced pierces the surface of the sphere.

The **horizon** of any point on the earth is the great circle cut from the celestial sphere by a horizontal plane through the point. Thus HAH' is the horizon.

If at any point on the earth a perpendicular is erected to the horizon at that point, the point where it pierces the celestial sphere above the plane is called the **zenith** of the point, while the piercing point below the plane is called the **nadir** of the point. Thus, Z is the zenith and N the nadir of the point O.

The intersection of the plane of the earth's equator with the celestial sphere is called the **celestial equator.** Thus, EAW is the celestial equator.

The great circle through the north pole and a star is called the **hour circle** of the star. Thus, PSI is the hour circle of the star at S.

The hour circle through the zenith is called the **celestial meridian,** or simply the meridian. Thus, $PZEH$ is the celestial meridian.

The **hour angle** of a star is the angle at the pole between the meridian and the hour circle of the star. Thus, angle SPZ is the hour angle of S. The hour angle is usually expressed as so many hours, minutes, and seconds before or after noon. An hour angle of 1 hr. is $\frac{1}{24}$ of 360°, or 15°.

The **declination** of a star is its angular distance north or south of the equator. The declination of the star S is IS.

The **altitude** of a star is its angular distance above the horizon measured on the great circle through the zenith and the star. The altitude of the star S is MS.

From the definition of latitude in **Art. 159,** it is easily seen that the meridian arc EZ from the equator toward the zenith is the latitude of the point on the earth's surface.

The triangle ZPS is called the **astronomical triangle.** ZP is the colatitude of the observer, SZ is the coaltitude of the star, SP is the codeclination of the star, and the angle SPZ is the hour angle of the star. The answers to many practical problems of astronomy are obtained by solving the astronomical triangle. The determination of correct time, when the declination and altitude of the sun and the latitude of the observer are

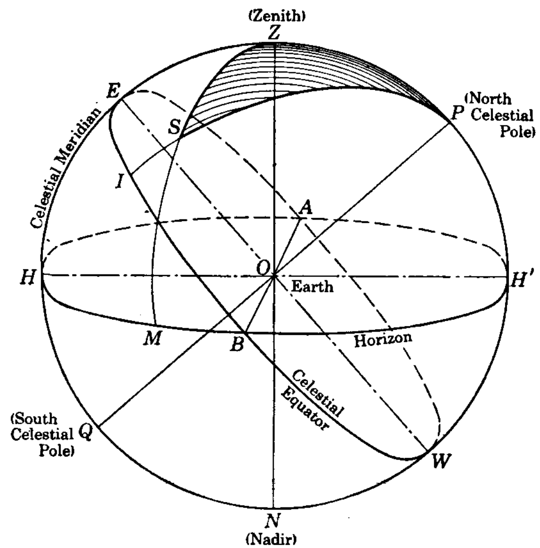


FIG. 132.

known, is obtained by solving for angle SPZ. The time of sunrise, neglecting refraction, may be found from the determination of angle SPZ when the altitude is zero. We then have a quadrantal triangle, and its polar triangle is a right spherical triangle.

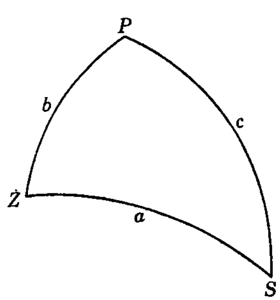


FIG. 133.

Example 1.—The latitude of Boston is 42° 20′ N. A forenoon observation showed the altitude of the sun to be 25° 30′. If the declination of the sun is 15° 50′ N., find the time of observation.

Solution.—In the triangle PZS,

$$PS = 90° - \text{Decl.} = 74° 10'.$$
$$PZ = 90° - \text{Lat.} = 47° 40'.$$
$$ZS = 90° - \text{Alt.} = 64° 30'.$$

To determine $\angle P$, use **[49_1]**,

$$\tan \frac{1}{2}P = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}.$$

$a = 64° 30'$	$\log \sin 45° 30' = 9.85324$
$b = 47° 40'$	$\log \sin 19° = 9.51264$
$c = 74° 10'$	$\text{colog} \sin 93° 10' = 0.00066$
$2s = 186° 20'$	$\text{colog} \sin 28° 40' = 0.31902$
$s = 93° 10'$	$\log \tan^2 \frac{1}{2}P = 9.68556$
$s - a = 28° 40'$	$\log \tan \frac{1}{2}P = 9.84278$
$s - b = 45° 30'$	$\frac{1}{2}P = 34° 50' 55''$
$s - c = 19°$	$P = 69° 41' 50''$

$\therefore \angle P$ in hours = 4 hr. 38 min. 47 sec.

Therefore, the time is 7 hr. 21 min. 13 sec. A.M.

Example 2.—Find the time of sunset for a place, latitude 32° 15′ N., when the declination is 17° 38′ N.

Solution.—In the triangle ZPS,

$$PS = 90° - 17° 38' = 72° 22'.$$
$$PZ = 90° - 32° 15' = 57° 45'.$$
$$ZS = 90°.$$

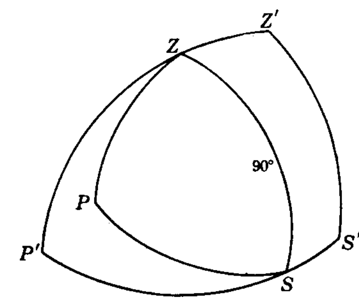


FIG. 134.

Triangle ZPS is then a quadrantal triangle which may be solved by Napier's analogies. A better method, however, is to solve the polar triangle and from that obtain the required parts of the given triangle.

In the polar triangle $P'Z'S'$, $\angle P' = 90°$. $\angle S' = 180° - PZ$, and $\angle Z' = 180° - PS$.

By Napier's rules or by (5) (**Art. 136**),

$$\cos S'Z' = \cot S' \cot Z'.$$
$$\cos S'Z' = \cot(180° - 57° 45')\cot(180° - 72° 22')$$
$$= \tan 32° 15' \tan 17° 38'.$$
$$\log \tan 32° 15' = 9.80000$$
$$\log \tan 17° 38' = 9.50223$$
$$\log \cos S'Z' = 9.30223$$
$$\therefore S'Z' = 78° 25' 50''$$
$$\therefore \angle P = 180° - 78° 25' 50'' = 101° 34' 10''.$$

And $\angle P$ in hours = 6 hr. 46 min. 17 sec.

Therefore the sun sets at 6 hr. 46 min. 17 sec. P.M.

EXERCISES

1. A forenoon observation on the sun showed the altitude to be 41°. If the latitude of the observer is 40° N. and the sun's declination is 20° N., find the time of observation. *Ans.* 8 hr. 29 min. 20 sec.

2. Find the time of sunset at a place, latitude 45° 30′ N., if the declination of the sun is 18° N. *Ans.* 7 hr. 17 min. 14 sec.

3. Find the altitude of the sun at 3 P.M. for a place, latitude 19° 25′ N. when the declination of the sun is 8° 23′ N. *Ans.* 45° 5′.

4. If the latitude of an observer is 8° 57′ N., and the sun's declination is 23° 2′ S., find the time of sunrise. *Ans.* 6 hr. 15 min. 21.6 sec. A.M.

5. Find the time of sunrise at Chicago, latitude 41° 50′, on June 21, when the sun's declination is 23° 30′. *Ans.* 4 hr. 28 min. 23 sec. A.M.

FORMULAS

[1] $\sin^2\theta + \cos^2\theta = 1.$

[2] $1 + \tan^2\theta = \sec^2\theta.$

[3] $1 + \cot^2\theta = \csc^2\theta.$

[4] $\sin\theta = \dfrac{1}{\csc\theta}$ and $\csc\theta = \dfrac{1}{\sin\theta}.$

[5] $\cos\theta = \dfrac{1}{\sec\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}.$

[6] $\tan\theta = \dfrac{1}{\cot\theta}$ and $\cot\theta = \dfrac{1}{\tan\theta}.$

[7] $\tan\theta = \dfrac{\sin\theta}{\cos\theta}.$

[8] $\cot\theta = \dfrac{\cos\theta}{\sin\theta}.$

[9] $x = l\cos\theta.$ (Projection on x-axis.)

[10] $y = l\sin\theta.$ (Projection on y-axis.)

[11] $G = \frac{1}{2}r^2(\theta - \sin\theta).$ (Area of segment.)

[12] $\sin\theta < \theta < \tan\theta.$ $\lim\limits_{\theta\to 0}\left[\dfrac{\theta}{\sin\theta}\right] = 1.$

[13] $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$

[14] $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

[15] $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

[16] $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

[17] $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$

[18] $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$

[19] $\sin 2\theta = 2\sin\theta\cos\theta.$

[20] $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1.$

[21] $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}.$

[22] $\sin\dfrac{1}{2}\theta = \pm\sqrt{\dfrac{1 - \cos\theta}{2}}.$

[23] $\cos\dfrac{1}{2}\theta = \pm\sqrt{\dfrac{1 + \cos\theta}{2}}.$

[24] $\tan\dfrac{1}{2}\theta = \pm\sqrt{\dfrac{1 - \cos\theta}{1 + \cos\theta}} = \dfrac{1 - \cos\theta}{\sin\theta} = \dfrac{\sin\theta}{1 + \cos\theta}.$

[25] $\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta).$

[26] $\sin\alpha - \sin\beta = 2\cos\frac{1}{2}(\alpha + \beta)\sin\frac{1}{2}(\alpha - \beta).$

[27] $\cos\alpha + \cos\beta = 2\cos\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta).$

[28] $\cos\alpha - \cos\beta = -2\sin\frac{1}{2}(\alpha + \beta)\sin\frac{1}{2}(\alpha - \beta).$

[29] $\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha + \beta) + \frac{1}{2}\sin(\alpha - \beta).$

[30] $\cos\alpha\sin\beta = \frac{1}{2}\sin(\alpha + \beta) - \frac{1}{2}\sin(\alpha - \beta).$

[31] $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha + \beta) + \frac{1}{2}\cos(\alpha - \beta).$

[32] $\sin\alpha\sin\beta = -\frac{1}{2}\cos(\alpha + \beta) + \frac{1}{2}\cos(\alpha - \beta).$

[33] $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}.$ (Law of sines.)

[34] $a^2 = b^2 + c^2 - 2bc\cos\alpha.$ (Law of cosines.)

[35] $K = \dfrac{b^2\sin\alpha\sin\gamma}{2\sin\beta}.$

[36] $K = \frac{1}{2}ab\sin\gamma.$

[37] $\dfrac{a + b}{a - b} = \dfrac{\tan\frac{1}{2}(\alpha + \beta)}{\tan\frac{1}{2}(\alpha - \beta)}.$

[38] $\tan\dfrac{1}{2}(\alpha - \beta) = \dfrac{a - b}{a + b}\cot\dfrac{1}{2}\gamma.$

[39] $\sin\dfrac{1}{2}\alpha = \sqrt{\dfrac{(s - b)(s - c)}{bc}}.$

[40] $\cos\dfrac{1}{2}\alpha = \sqrt{\dfrac{s(s - a)}{bc}}.$

[41] $\tan\dfrac{1}{2}\alpha = \sqrt{\dfrac{(s - b)(s - c)}{s(s - a)}}.$

[42] $\tan\dfrac{1}{2}\alpha = \dfrac{r}{s - a}$, where $r = \sqrt{\dfrac{(s - a)(s - b)(s - c)}{s}}.$

[43] $K = \sqrt{s(s - a)(s - b)(s - c)}.$

[44] $\dfrac{\sin\alpha}{\sin a} = \dfrac{\sin\beta}{\sin b} = \dfrac{\sin\gamma}{\sin c}.$

[45] $\cos a = \cos b\cos c + \sin b\sin c\cos\alpha.$

[46] $\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a.$

[47] $\sin \frac{1}{2}\alpha = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}$, where $s = \frac{1}{2}(a + b + c)$.

[48] $\cos \frac{1}{2}\alpha = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}}$.

[49] $\tan \frac{1}{2}\alpha = \frac{r}{\sin (s - a)}$, where

$$r = \sqrt{\frac{\sin (s - a) \sin (s - b) \sin (s - c)}{\sin s}}.$$

[50] $\cos \frac{1}{2}a = \sqrt{\frac{\cos (S - \beta) \cos (S - \gamma)}{\sin \beta \sin \gamma}}$,

where $S = \frac{1}{2}(\alpha + \beta + \gamma)$.

[51] $\sin \frac{1}{2}a = \sqrt{-\frac{\cos S \cos (S - \alpha)}{\sin \beta \sin \gamma}}$.

[52] $\tan \frac{1}{2}a = R \cos (S - \alpha)$, where

$$R = \sqrt{\frac{-\cos S}{\cos (S - \alpha) \cos (S - \beta) \cos (S - \gamma)}}.$$

[53] $\frac{\tan \frac{1}{2}(a - b)}{\tan \frac{1}{2}c} = \frac{\sin \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(\alpha + \beta)}$.

[54] $\frac{\tan \frac{1}{2}(\alpha - \beta)}{\cot \frac{1}{2}\gamma} = \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)}$.

[55] $\frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}c} = \frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)}$.

[56] $\frac{\tan \frac{1}{2}(\alpha + \beta)}{\cot \frac{1}{2}\gamma} = \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}$.

[57] $\cos \frac{1}{2}c \sin \frac{1}{2}(\alpha + \beta) = \cos \frac{1}{2}\gamma \cos \frac{1}{2}(a - b)$.

[58] $\cos \frac{1}{2}c \cos \frac{1}{2}(\alpha + \beta) = \sin \frac{1}{2}\gamma \cos \frac{1}{2}(a + b)$.

[59] $\sin \frac{1}{2}c \sin \frac{1}{2}(\alpha - \beta) = \cos \frac{1}{2}\gamma \sin \frac{1}{2}(a - b)$.

[60] $\sin \frac{1}{2}c \cos \frac{1}{2}(\alpha - \beta) = \sin \frac{1}{2}\gamma \sin \frac{1}{2}(a + b)$.

[61] $\Delta = \frac{\pi r^2 E}{180°}$.

[62] $\tan \frac{1}{4}E = \sqrt{\tan \frac{1}{2}s \tan \frac{1}{2}(s - a) \tan \frac{1}{2}(s - b) \tan \frac{1}{2}(s - c)}$

[63] $[r(\cos \theta + i \sin \theta)]^n = r^n[\cos n\theta + i \sin n\theta]$.

(DeMoivre's theorem, **Art. 121.**)

[64] $\sin \alpha = \alpha - \frac{\alpha^3}{\underline{|3}} + \frac{\alpha^5}{\underline{|5}} - \frac{\alpha^7}{\underline{|7}} + \cdots.$

[65] $\cos \alpha = 1 - \frac{\alpha^2}{\underline{|2}} + \frac{\alpha^4}{\underline{|4}} - \frac{\alpha^6}{\underline{|6}} + \cdots.$

[66] $\tan \alpha = \alpha + \frac{\alpha^3}{3} + \frac{2\alpha^5}{15} + \cdots.$

USEFUL CONSTANTS

1 cu. ft. of water weighs 62.5 lb. = 1000 oz. (Approx.)
1 gal. of water weighs $8\frac{1}{3}$ lb. (Approx.)
1 gal. = 231 cu. in. (by law of Congress).
1 bu. = 2150.42 cu. in. (by law of Congress).
1 bu. = 1.2446− cu. ft. = $\frac{5}{4}$ cu. ft. (Approx.)
1 cu. ft. = $7\frac{1}{2}$ gal. (Approx.)
1 bbl. = 4.211− cu. ft.
1 m. = 39.37 in. (by law of Congress).
1 in. = 25.4 mm.
1 ft. = 30.4801 cm.
1 kg. = 2.20462 lb.
1 g. = 15.432 gr.
1 lb. (avoirdupois) = 453.5924277 g. = 0.45359+ kg.
1 lb. (avoirdupois) = 7000 gr. (by law of Congress).
1 lb. (apothecaries) = 5760 gr. (by law of Congress).
1 l. = 1.05668 qt. (liquid) = 0.90808 qt. (dry).
1 qt. (liquid) = 946.358 cc. = 0.946358 l., or cu. dm.
1 qt. (dry) = 1101.228 cc. = 1.101228 l., or cu. dm.
$\pi = 3.14159265358979 = 3.1416 = \frac{355}{113} = 3\frac{1}{7}$. (All approx.)
1 radian = 57° 17′ 44.8″ = 57.2957795°+.
1° = 0.01745329+ radian.
e = 2.718281828+, the base of the Napierian logarithms.

INDEX

FIVE-PLACE
LOGARITHMIC
AND
TRIGONOMETRIC TABLES

WITH EXPLANATORY CHAPTER

ARRANGED BY

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PREFACE TO THE THIRD EDITION

This new edition has given the author the opportunity to substitute a number of new exercises which, it is hoped, will prove helpful in introducing the student to the use of the tables.

Charles Wilber Leigh.

Chicago, Ill.,
June, 1934.

CONTENTS

LOGARITHMS AND EXPLANATIONS OF TABLES

LOGARITHMS AND EXPLANATION OF TABLES

1. Use of logarithms.—By the use of logarithms, the processes of multiplication, division, raising to a power, and extracting a root of arithmetical numbers are usually much simplified. The process of multiplication is replaced by one of addition, that of division by one of subtraction, that of raising to a power by a simple multiplication, and that of extracting a root by a division.

Many calculations that are difficult or impossible by other mathematical methods are readily carried out by means of logarithms. It was said by the great French astronomer, Laplace, that the method of logarithms, by reducing to a few days the labors of many months, doubled, as it were, the life of an astronomer, besides freeing him from the errors and disgust inseparable from long calculations. Of course, these same advantages are shared by others who find it necessary to perform numerical calculations.*

2. Exponents.—The student is already familiar with the following definitions and theorems from algebra, concerning exponents. For convenience they are restated here.

Definitions.

(1) $a^n = a \cdot a \cdot a \cdot \cdot \cdot$ to n factors. n an integer.

(2) $a^{-n} = \frac{1}{a^n}$.

(3) $a^0 = 1$.

(4) $a^{\frac{n}{m}} = \sqrt[m]{a^n}$.

Theorems.

(1) $a^n \cdot a^m = a^{n+m}$.

(2) $a^n \div a^m = a^{n-m}$.

(3) $(a \cdot b \cdot c \cdot \cdot \cdot r)^n = a^n \cdot b^n \cdot c^n \cdot \cdot \cdot r^n$.

(4) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

(5) $(a^n)^m = a^{nm}$.

* "The miraculous powers of modern calculations are due to three inventions: the Hindu Notation, Decimal Fractions, and Logarithms." (CAJORI, "A History of Elementary Mathematics.")

3. Definitions.—If three numbers N, b, and x have such values that

$$N = b^x,$$

then x is called the **logarithm*** of N to the base b. In words, this gives the following.

DEFINITION.—*The logarithm of a number to a given base is the* **exponent** *by which the base must be affected to produce that number.*

If, in the equation $N = b^x$, all possible positive values are given to N, while b is some positive number other than 1, the corresponding values of x form a **system of logarithms.**

4. Notation.—If 4 is taken as a base, then, in the language, or notation, of exponents,

$$4^3 = 64.$$

In the language, or notation, of logarithms, the same idea is expressed by saying the logarithm of 64 to the base 4 is 3. This is abbreviated and written

$$\log_4 64 = 3.$$

EXPONENT NOTATION	LOGARITHMIC NOTATION
$4^5 = 1024.$	$\log_4 1024 = 5.$
$3^4 = 81.$	$\log_3 81 = 4.$
$5^3 = 125.$	$\log_5 125 = 3.$
$4^{0.5} = 2.$	$\log_4 2 = 0.5.$
$16^{\frac{3}{2}} = 64.$	$\log_{16} 64 = \frac{3}{2}.$
$10^3 = 1000.$	$\log_{10} 1000 = 3.$

EXERCISES

Answer as many as possible orally.

Express in logarithmic notation:

1. $3^3 = 27.$
2. $5^3 = 125.$
3. $7^3 = 343.$
4. $64^{0.5} = 8.$
5. $10^4 = 10{,}000.$
6. $125^{\frac{1}{3}} = 5.$
7. $32^{0.4} = 4.$
8. $10^{2.3979} = 250.$
9. $10^{1.5465} = 35.2.$

Express in exponent notation:

10. $\log_2 256 = 8.$
11. $\log_6 216 = 3.$
12. $\log_{16} 2 = 0.25.$
13. $\log_{10} 643 = 2.8082.$
14. $\log_{10} 429 = 2.6325.$
15. $\log_{10} 99.9 = 1.9996.$

Find the logarithms of the following:

16. $\log_6 36.$
17. $\log_3 243.$
18. $\log_5 3125.$
19. $\log_9 729.$
20. $\log_8 512.$
21. $\log_{10} 100{,}000.$
22. $\log_8 2.$
23. $\log_{16} 128.$
24. $\log_{10} 0.001.$

* The word "logarithm" is derived from *logos* meaning ratio and *arithmos* meaning number.

Find the value of x in the following:

25. $\log_3 x = 4.$
26. $\log_{10} x = 4.$
27. $\log_{16} x = \frac{3}{4}.$
28. $\log_{10} x = 0.$
29. $\log_{10} x = -3.$
30. $\log_3 x = -2.$
31. $\log_{16} x = 0.75.$
32. $\log_8 x = \frac{4}{3}.$
33. $\log_{25} x = \frac{3}{2}.$
34. $\log_{16} x = \frac{3}{2}.$
35. $\log_{125} x = \frac{2}{3}.$
36. $\log_{49} x = \frac{3}{2}.$

Find the value of x in the following:

37. $\log_x 1000 = 3.$
38. $\log_x 81 = 4.$
39. $\log_x 256 = 4.$
40. $\log_x 1024 = 10.$
41. $\log_x 8 = 0.5.$
42. $\log_x 4 = 0.25.$
43. $\log_x 16 = \frac{4}{5}.$
44. $\log_x 18 = 0.5.$
45. $\log_x 27 = 0.75.$
46. $\log_x 2 = 0.125$
47. $\log_x 36 = \frac{2}{3}.$
48. $\log_x 100 = \frac{2}{3}.$

49. What are the logarithms of 2, 4, 8, 16, 32, 64, 128, 256, to the base 2?

50. What are the logarithms of 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$, to the base 2?

51. What are the logarithms of 3, 9, 27, 81, 243, 729, to the base 3? To the base $\frac{1}{3}$?

52. What are the logarithms of 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, $\frac{1}{243}$, to the base 3? To the base $\frac{1}{3}$?

5. Systems of logarithms.*—While, theoretically, any positive number other than 1 may be used as a base for a system of logarithms, in practice only two bases are used.

(1) The **common system,** or Briggs' system, of which the base is 10.

(2) The **natural system,** also called the hyperbolic, or Napierian system, of which the base is a number that to seven decimal places is 2.7182818. This base is usually represented by the letter e.

The common system is the one commonly used in computing, and the natural system in more advanced and theoretical work.

6. Properties of logarithms.—The use of logarithms depends upon the following properties, which are true for any base greater than unity:

(1) *The logarithm of* 1 *is zero.*

Since $b^0 = 1$, for any base, $\log_b 1 = 0$.

(2) *The logarithm of the base of any system is unity.*

Since $b^1 = b$, for any base, $\log_b b = 1$.

* Logarithms were invented by John Napier, Baron of Merchiston, of Scotland, who lived from 1550 to 1617. They were described by him in 1614. A contemporary of Napier, Henry Briggs (1556–1631), professor of Gresham College, London, modified the new invention by using the base 10, and so made it more convenient for practical purposes. (See CAJORI, "A History of Elementary Mathematics," p. 160 *et seq.* For a very complete account of logarithms, see CAJORI, FLORIAN, "History of the Logarithmic and Exponential Concepts.")

If $N = b^x$ and $M = b^y$, then, by the definition of a logarithm, $\log_b N = x$ and $\log_b M = y$. We have also by the definitions and theorems of exponents and logarithms:

(a) $N \times M = b^x \times b^y = b^{x+y}$.

$\therefore \log_b (N \times M) = x + y = \log_b N + \log_b M$.

(b) $N \div M = b^x \div b^y = b^{x-y}$.

$\therefore \log_b (N \div M) = x - y = \log_b N - \log_b M$.

(c) $N^n = (b^x)^n = b^{nx}$. $\therefore \log_b (N^n) = nx = n \log_b N$.

(d) $\sqrt[n]{N} = (b^x)^{\frac{1}{n}} = b^{\frac{1}{n}x}$. $\therefore \log_b \sqrt[n]{N} = \frac{1}{n}x = \frac{1}{n} \log_b N$.

The following theorems are, therefore, established:

(3) *The logarithm of a product equals the sum of the logarithms of the factors.* By (a).

(4) *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.* By (b).

(5) *The logarithm of a power of a number is equal to the exponent of the power times the logarithm of the number.* By (c).

(6) *The logarithm of a root of a number equals the logarithm of the number divided by the index of the root.* By (d).

The truth of the statements in **Art. 1** follows from these theorems. That is, the process of multiplication is replaced by an addition; division by a subtraction; raising to a power, by a multiplication; and extracting a root, by a division.

7. Logarithms to the base 10.—In what follows, if no base is stated, it is understood that the base 10 is used.

When the base is 10 we evidently have the following:

$10^5 = 100{,}000$	$\therefore \log 100{,}000 = 5.$
$10^4 = 10{,}000$	$\therefore 10{,}000 = 4.$
$10^3 = 1000$	$\therefore \log 1000 = 3.$
$10^2 = 100$	$\therefore \log 100 = 2.$
$10^1 = 10$	$\therefore \log 10 = 1.$
$10^0 = 1$	$\therefore \log 1 = 0.$
$10^{-1} = 0.1$	$\therefore \log 0.1 = -1 = 9 - 10.$

$10^{-2} = 0.01$	$\therefore \log 0.01 = -2 = 8 - 10.$
$10^{-3} = 0.001$	$\therefore \log 0.001 = -3 = 7 - 10.$
$10^{-4} = 0.0001$	$\therefore \log 0.0001 = -4 = 6 - 10.$
$10^{-5} = 0.00001$	$\therefore \log 0.00001 = -5 = 5 - 10.$

It is evident that these are the only numbers between 0.00001 and 100,000 which have integers for logarithms. Every other number in this range has, then, for a logarithm an integer plus or minus a fraction. This fraction is put in the form of a decimal.

For instance, the logarithm of any number between 1000 and 10,000 is between 3 and 4, or it is 3 + a decimal. For a number between 100 and 1000 the logarithm is 2 + a decimal. Between 0.01 and 0.1, the logarithm may be −2 + a decimal or −1 − a decimal; but, in order that the fractional part of the logarithm may always be positive, it is agreed to take the logarithm so that the integral part only is negative.

Usually, then, the logarithm of a number consists of two parts, an *integer* and a *fraction*, the fraction being the approximate value of an irrational number.

The integral part is called the **characteristic.**

The fractional part is called the **mantissa.**

The logarithm is the characteristic plus the mantissa.

The mantissas of the positive numbers arranged in order are called a **table of logarithms.**

The logarithm of 3467 consists of the characteristic 3 plus some mantissa, because 3467 lies between 1000 and 10,000. The logarithm of 59,436 is 4 + a decimal, because 59,436 lies between 10,000 and 100,000. The log 0.0236 = −2 + a decimal, because 0.0236 lies between 0.01 and 0.1.

It is readily seen that multiplying a number by 10^n increases the characteristic by n, where n is an integer; and dividing a number by 10^n decreases the characteristic by n, for

$$\log (N \times 10^n) = \log N + \log 10^n = \log N + n \log 10$$
$$= \log N + n, \text{ and}$$
$$\log (N \div 10^n) = \log N - \log 10^n = \log N - n \log 10$$
$$= \log N - n.$$

This establishes the following:

THEOREM.—*The position of the decimal point in the number affects the characteristic of the logarithm only, the mantissa remaining unchanged for the same sequence of figures.*

The advantages in using the base 10 are that the characteristic can be determined by inspection, and that the mantissa remains unchanged for the same sequence of figures.

Thus,

$$\log 934{,}700 = 5.97067.$$
$$\log 9347 = 3.97067.$$
$$\log 9.347 = 0.97067.$$
$$\log 0.009347 = \bar{3}.97067 = 7.97067 - 10.$$

8. Rules for determining the characteristic.—From the foregoing considerations the following rules for determining the characteristic are evident:

(1) *When the number is greater than* 1, *the characteristic is positive, and is one less than the number of digits to the left of the decimal point.*

(2) *When the number is less than* 1 *and expressed decimally, the characteristic is negative, and is one more than the number of zeros immediately at the right of the decimal point.*

When the characteristic is negative the minus sign is placed above the characteristic to show that it alone is negative. Thus, in $\log 0.009347 = \bar{3}.97067$, the $\bar{3}.97067$ means $-3 + 0.97067$. It should not be written -3.97067, for then the minus sign would indicate that both characteristic and mantissa were negative, while we have agreed that the mantissa shall always be considered positive.

In computations involving negative characteristics, to avoid the use of the negative, 10 is usually added to the characteristic and subtracted at the right of the mantissa. In writing logarithms in this form, *the characteristic, when* 10 *is added, is* 9 *minus the number of zeros immediately at the right of the decimal point.*

Thus, in the above, $\log 0.009347 = 7.97067 - 10$.

The characteristics of the following are as given: of 3426 is 3 by rule (1); of 3.2364 is 0 by rule (1); of 0.00639 is -3, or $7 - 10$, by rule (2); of 2.04 is 0 by rule (1); of 0.000067 is -5, or $5 - 10$, by rule (2).

EXERCISES

In each of the following state the characteristic to the base 10:

1. 923.	**6.** 42,376.	**11.** 32.54.	**16.** 89,236.
2. 425.03.	**7.** 3.2067.	**12.** 0.0123.	**17.** 0.2146.
3. 1.111.	**8.** 0.00046.	**13.** 3.004.	**18.** 333.33.
4. 8.004.	**9.** 0.000395.	**14.** 2525.1.	**19.** $10^5 \times 2$.
5. 800.4.	**10.** 0.04762.	**15.** 1000.25.	**20.** $10^{-3} \times 6$.

If $\log 31 = 1.49136$, give:

21. log 310.	**24.** log 0.031.	**27.** log 0.3100.
22. log 31,000.	**25.** log 0.0031.	**28.** $\log (31 \times 10^7)$.
23. log 3.1.	**26.** log 0.000031.	**29.** $\log (31 \times 10^{-5})$.

9. The mantissa.—The determination of the decimal part of a logarithm, the mantissa, is more difficult than the determination of the characteristic. Because of this difficulty the mantissas have been carefully determined and arranged in tables of logarithms.* They are given to three, four, five, or more places of decimals.

The degree of accuracy in computations made by logarithms depends upon the number of places in the table used; the more places in the table the greater the degree of accuracy. The tables generally used are those having from four to seven places.

10. Tables.—Upon examining a five-place table of logarithms (see **Table I**), it is noticed that the first column has the letter N at the top and the bottom. This is an abbreviation for number. The other columns have at top and bottom the numbers 0, 1, 2, 3, · · · 9. **Table I** contains the integers from 1000 to 11,009. Pages 36 to 53 have the numbers from 1000 to 10,009. Here the first three figures are printed in the column marked N and the fourth figure at the top and bottom of another column. Thus, to locate 4756, 475 is found in the N-column on page 43 and 6 in the column headed 6. Pages 54 and 55 contain the numbers from 10,000 to 11,009, where the first four figures are printed in the N-column.

The columns of numbers after the first consist of the mantissas of the numbers located in the N-column and at the top, or bottom, of another column. These mantissas are printed correct to five decimals, except on pages 54 and 55, where they are given to seven places. To save space, the first two figures of the mantissas

* Professor Briggs' tables were computed to 14 places, but were not finished by him. They were completed by Adrian Vlacq (1628), who shortened them to 10 places, and finished a table including the numbers from 1 to 100,000.

Briggs' and Vlacq's tables are essentially the same as those now in use. They have been checked and recomputed in part many times. At present, errors found in tables are typographical. The most complete check was undertaken by the French authorities in 1784. It required the labors of nearly a hundred mathematicians and computers for over two years. They computed to 14 places the logarithms of all integers from 1 to 200,000, besides natural and logarithmic trigonometric functions. These tables were never printed. Two manuscript copies are preserved.

are printed in the 0-column only. Any such two figures go with the other figures to the right and below until another two figures is found in the 0-column. Except that when an asterisk (*) is found before the three figures given in the other columns, the first two figures of the mantissa are taken from the next line below.

When a mantissa ends in a figure 5 it is printed $\bar{5}$ when it is really less than printed; otherwise, a mantissa when ending in 5 is larger than printed.

Thus, if the mantissa is 0.0273496, in contracting it to five places, it is printed $0.0273\bar{5}$. This is to guide one wishing to write the mantissas correct to four places.

For the meaning of the **Prop. Parts,** see **Art. 21.** For the meaning of the numbers at the foot of the pages and connected with **S** and **T,** see **Art. 29.**

Notice that, when advancing in the table, the mantissas increase.

The difference between two consecutive mantissas is called the **tabular difference.**

11. To find the mantissa of the logarithm of a number.—Use **Table I,** pages 36 to 53.

(1) *When the number consists of four significant figures.*

Example.—Find the mantissa of log 4673.

Find the first three figures, 467, of the number in the N-column and the 3 at the top of the page. The mantissa of log 4673 is found to the right of 467 and in the column headed 3.

$$\therefore \text{mantissa of log } 4673 = 0.66960.$$

In like manner find the following:

$$\begin{aligned}
\text{Mant. of log } 4799 &= 0.68115.\\
\text{Mant. of log } 23.78 &= 0.37621.\\
\text{Mant. of log } 2.955 &= 0.47056.\\
\text{Mant. of log } 0.0003964 &= 0.59813.\\
\text{Mant. of log } 3560 &= 0.55145.\\
\text{Mant. of log } 4930 &= 0.69285.\\
\text{Mant. of log } 556{,}700 &= 0.74562.\\
\text{Mant. of log } 0.001001 &= 0.00043.
\end{aligned}$$

(2) *When the number consists of one, two, or three significant figures.* The number is found in the N-column and the mantissa to the right in the 0-column.

$$\begin{aligned}
\text{Thus, Mant. of log } 4.78 &= \text{Mant. of log } 4780 = 0.67943.\\
\text{Mant. of log } 39 &= \text{Mant. of log } 3900 = 0.59106.\\
\text{Mant. of log } 4 &= \text{Mant. of log } 4000 = 0.60206.
\end{aligned}$$

(3) *When the number consists of five or more significant figures.*

Example 1.—Find the mantissa of log 39,467.

Since 39,467 lies between 39,460 and 39,470, its mantissa must lie between the mantissas of these numbers.

$$\begin{aligned}
\text{Mant. of log } 39{,}460 &= 0.59616.\\
\text{Mant. of log } 39{,}470 &= 0.59627.
\end{aligned}$$

The difference between these mantissas is 0.00011, which is the tabular difference. Since an increase of 10 in the number increases the mantissa 0.00011, an increase of 7 in the number will increase the mantissa 0.7 as much, or the increase is

$$0.00011 \times 0.7 = 0.000077 \text{ or } 0.00008.$$
$$\therefore \text{Mant. of log } 39{,}467 = 0.59616 + 0.00008 = 0.59624.$$

The process of finding the mantissa as above is called **interpolation.** As carried out, it is assumed that *the increase of the logarithm is proportional to the increase of the number.* This assumption is not strictly true as will be seen in **Art. 35.**

Example 2.—Find the mantissa of log 792,836.

$$\begin{aligned}
\text{Mant. of log } 792{,}900 &= 0.89922\\
\text{Mant. of log } 792{,}800 &= 0.89916\\
\text{Tabular difference} &= 0.00006
\end{aligned}$$

Since an increase of 100 in the number increases the mantissa 0.00006, an increase of 36 in the number increases the mantissa

$$0.00006 \times 0.36 = 0.00002,$$

correct to the nearest fifth decimal place.

$\therefore$ Mant. of log 792,836 = 0.89916 + 0.00002 = 0.89918.

These processes should seem reasonable; but, since they are to be performed so frequently, it is best to work by rule.

12. Rules for finding the mantissa.—(1) *For a number consisting of four figures, find the first three figures of the number in the N-column and the fourth figure at the head of a column; then read the mantissa in the column under the last figure and at the right of the first three figures.*

(2) *For a number consisting of one, two, or three figures, find the number in the N-column and the mantissa to the right in the column headed* 0.

(3) *For a number consisting of more than four figures, find the mantissa for the first four figures by rule* (1) *and add to this the product of the tabular difference by the remaining figures of the number considered as a decimal number.*

13. Finding the logarithm of a number.—In finding the logarithm of a number, it is best to determine the characteristic first and then look up the mantissa. Perform all the interpolations without the aid of a pencil if possible. The use of the proportional parts is explained in **Art. 21**; but the student is advised to become familiar with interpolating without their help.

Example 1.—Find the logarithm of 92.36.

The characteristic is 1, by rule (1) for characteristics. The mantissa is 0.96548, by rule (1) for mantissas.

$$\therefore \log 92.36 = 1.96548.$$

Example 2.—Find the logarithm of 3.4676.

Rule (1) for characteristic gives 0.

Rule (3) for mantissa gives 0.54003.

$$\therefore \log 3.4676 = 0.54003.$$

Example 3.—Find the logarithm of 0.00039724.

Rule (2) for characteristic gives $\bar{4}$.

Rule (3) for mantissa gives 0.59905.

$$\therefore \log 0.00039724 = \bar{4}.59905 = 6.59905 - 10.$$

EXERCISES

Verify the following by the tables:

1. log 9376 = 3.97202.
2. log 4.236 = 0.62696.
3. log 220 = 2.34242.
4. log 1.11 = 0.04532.
5. log 20 = 1.30103.
6. log 0.02 = $\bar{2}$.30103.
7. log 0.00263 = 7.41996 − 10.
8. log 26.436 = 1.42220.
9. log 3.1416 = 0.49715.
10. log 0.492357 = $\bar{1}$.69228.
11. log 276.392 = 2.44153.
12. log 0.027646 = 8.44164 − 10.
13. log 0.0049643 = $\bar{3}$.69586.
14. log 0.029896 = $\bar{2}$.47561.
15. log 2.71828 = 0.43429.
16. log 99.999 = 2.00000.
17. log 1111.11 = 3.04575.
18. log 100.03 = 2.00013.

14. To find the number corresponding to a logarithm.—If log 31.416 = 1.49715, then 31.416 is the **number corresponding to the logarithm** 1.49715. It is sometimes called the **antilogarithm** and is written 31.416 = $\log^{-1}$ 1.49715.

In nearly every problem involving logarithms, it is not only necessary to find the logarithms of numbers; but the inverse process, that of finding the number corresponding to a logarithm, has to be performed.

Since the position of the decimal point in no way affects the mantissa, we should expect to determine the sequence of figures in the number from the mantissa. And since a change in the position of the decimal point increases or decreases the characteristic, the decimal point may be located in the number when the characteristic of the logarithm of the number is known.

(1) *When the mantissa of the given logarithm is given exactly in the table.*

Example.—Find the number having 2.58939 for its logarithm. Find in the table the mantissa 0.58939. To the left of this mantissa, in the *N*-column, find the first three figures, 388, of the number, and at the head of the page find the fourth figure, 5, of the number. The number then consists of the sequence of figures 3885, but we do not know where the decimal point is until we consider the characteristic, which is 2. Hence there must be three figures at the left of the decimal point.

$$\therefore 2.58939 = \log 388.5.$$

A change in the characteristic changes the location of the decimal point.

Thus,
$$4.58939 = \log 38{,}850,$$
$$\bar{2}.58939 = \log 0.03885,$$
$$7.58939 - 10 = \log 0.003885.$$

(2) *When the mantissa of the given logarithm is not given exactly in the table.* In this case two other consecutive mantissas can always be found between which the mantissa of the given logarithm lies. The number of four figures corresponding to the smaller of these mantissas gives the first four figures of the number sought. The fifth, and often the sixth, figure can then be found by interpolating, assuming that, for comparatively small differences in the numbers, the differences in the numbers are proportional to the differences in the logarithms of the numbers.

For using the proportional parts in interpolating, see **Art. 21.**

Example.—Find the number whose logarithm is 1.49863.

In the table find the mantissas 0.49859 and 0.49872, between which the given mantissa lies. Thinking only of the sequence of figures in the numbers,

$$\begin{array}{rl} 0.49872 = \text{Mant. of log } 3153 \\ 0.49859 = \text{Mant. of log } \underline{3152} \\ \hline 0.00013 \qquad\qquad\qquad 1 \end{array}$$

Hence a difference of 0.00013 in the logarithm makes a difference of 1 in the number. Now the given mantissa is 0.00004 larger than the smaller one. Then the number having 0.49863 as the mantissa of its logarithm is

$$\frac{0.00004}{0.00013} \times 1 = \frac{4}{13} = 0.3$$

larger than 3152. Hence, the sequence of figures for the number having 1.49863 as a logarithm is 31,523. Since the characteristic is 1,

$$1.49863 = \log 31.523.$$

The interpolation should be carried out mentally, leaving out zeros and taking $\frac{4}{13} \times 1 = 0.3$.

This could also be stated as a proportion,

$$13:4 = 1:x, \therefore x = 0.3.$$

15. Rules for finding the number corresponding to a given logarithm.—(1) *When the mantissa of the given logarithm is found exactly in the table, the first three figures of the number are found to the left of the mantissa in the N-column, and the fourth figure is at the head of the column in which the mantissa is found.*

(2) *When the mantissa of the given logarithm is not found exactly in the table, find the mantissa nearest the given mantissa but smaller. The first four figures of the number are those corresponding to this mantissa, and are found by rule* (1). *For another figure, divide the difference between the mantissa found and the given mantissa by the tabular difference.*

In both (1) *and* (2), *place the decimal point so that the rules for determining the characteristic may be applied and give the characteristic of the logarithm.*

Example.—Find the number corresponding to 3.87626.

Mantissa nearest 0.87626 is 0.87622 = Mant. of log 7520.

Tabular difference = 6.

Difference between mantissas = 4.

$$4 \div 6 = 0.7 \text{ to nearest tenth.}$$

$$\therefore 3.87626 = \log 7520.7.$$

EXERCISES

Find the values of x or verify the following:

1. $3.70944 = \log 5122.$

2. $\bar{2}.58377 = \log 0.03835.$

3. $1.74819 = \log x.$

4. $7.94236 - 10 = \log 0.008757.$

5. $0.47712 = \log x.$

6. $3.47954 = \log 3016.7.$

7. $2.57351 = \log 374.55.$

8. $0.92876 = \log x.$

9. $9.23465 - 10 = \log x.$

10. $\bar{4}.92317 = \log 0.00083786.$

11. $8.12112 - 10 = \log 0.013217.$

12. $6.28697 = \log x.$

13. $\bar{6}.89909 = \log x.$

14. $11.46729 = \log x.$

15. $9.92867 - 20 = \log x.$

16. $\bar{3}.88888 = \log x.$

17. $3.33333 = \log x.$

18. $4.0002565 = \log 10005.9.$

19. $2.0331894 = \log 107.9417.$

20. $3.0275278 = \log 1065.437.$

For exercises 18 to 20 use pages 54 and 55 of **Table I.**

16. To multiply by means of logarithms.—Property (4) of **Art. 6** gives the following:

RULE.—*To find the product of two or more factors, find the sum of the logarithms of the factors; the product is the number corresponding to this sum.*

Example 1.—Find the product of 34.796 × 0.0294 × 3.1416.

Process.

$$\begin{aligned} \log 34.796 &= 1.54153 \\ \log 0.0294 &= 8.46835 - 10 \\ \log 3.1416 &= \underline{0.49715} \\ \log \text{of product} &= 0.50703 \\ \therefore \text{product} &= \underline{3.2139} \end{aligned}$$

Example 2.—Find the product of 3.276 × (−4.6243) × (−0.004682).

Process.

$$\begin{aligned} \log 3.276 &= 0.51534 \\ \log 4.6243 &= 0.66505n \\ \log 0.004682 &= \underline{\bar{3}.67043n} \\ \log \text{of product} &= \bar{2}.85082 \\ \therefore \text{product} &= \underline{0.070928} \end{aligned}$$

Since the logarithms cannot take into account the negative numbers, the easiest way to keep count of the negative factors is to place a letter n after their logarithms. In finding a sum or difference of logarithms, write an n after the result only if an odd number of the separate logarithms are affected by an n. This method was introduced by the great mathematician Gauss (1777–1855).

17. To divide by means of logarithms.—Property (5) of **Art. 6** gives the following:

RULE.—*To find the quotient of two numbers, subtract the logarithm of the divisor from the logarithm of the dividend; the quotient is the number corresponding to this difference.*

Example 1.—Find the quotient of 27.634 ÷ 5.427.

Process.

$$\begin{aligned}\log 27.634 &= 1.44144\\ \log 5.427 &= 0.73456\\ \log \text{of quotient} &= 0.70688\\ \therefore \text{quotient} &= 5.0919\end{aligned}$$

Example 2.—Evaluate $\dfrac{7.246 \times 0.8964 \times 5.463}{4.27 \times 0.3987 \times 27.89}$.

Here find logarithm of the numerator then that of the denominator.

Process.

$$\begin{aligned}\log 7.246 &= 0.86010\\ \log 0.8964 &= 9.95250 - 10\\ \log 5.463 &= 0.73743\\ \log \text{of Num.} &= 1.55003\\ \log \text{of Den.} &= 1.67653\\ \log \text{of quotient} &= 9.87350 - 10\\ \therefore \text{quotient} &= 0.74732\end{aligned}$$

$$\begin{aligned}\log 4.27 &= 0.63043\\ \log 0.3987 &= 9.60065 - 10\\ \log 27.89 &= 1.44545\\ \log \text{of Den.} &= 1.67653\end{aligned}$$

18. Cologarithms.—The logarithm of the reciprocal of a number is called the **cologarithm** of the number. It is also called the **arithmetical complement.**

Since the reciprocal of N is $\frac{1}{N}$, $\text{colog}\, N = \log \frac{1}{N} = \log 1 - \log N$.

Also $\log \frac{M}{N} = \log M + \log \frac{1}{N} = \log M + \text{colog}\, N$, that is:

The logarithm of the quotient of two numbers is equal to the logarithm of the dividend plus the cologarithm of the divisor.

That is, subtracting the logarithm of a number is the same as adding the cologarithm of the number. It is evident that, by using cologarithms, the work can often be made more compact than otherwise. It should be noted that it is never necessary to use cologarithms.

To find the cologarithm of a number, subtract the logarithm of the number from 10 − 10. Do the work mentally, beginning at the left and subtracting each figure from 9, except the last significant figure at the right, which is to be taken from 10.

Thus, colog 9.423 = log 1 − log 9.423.

$$\begin{aligned}\log 1 &= 10.00000 - 10\\ \log 9.423 &= 0.97419\\ \therefore \text{colog}\ 9.423 &= 9.02581 - 10\end{aligned}$$

The solution of Example 2 (**Art. 17**) takes the following form:

$$\begin{aligned}\log 7.246 &= 0.86010\\ \log 0.8964 &= 9.95250 - 10\\ \log 5.463 &= 0.73743\\ \text{colog}\ 4.27 &= 9.36957 - 10\\ \text{colog}\ 0.3987 &= 0.39935\\ \text{colog}\ 27.89 &= 8.55455 - 10\\ \log \text{of quotient} &= 9.87350 - 10\\ \therefore \text{quotient} &= 0.74732\end{aligned}$$

19. To find the power of a number by means of logarithms.—Property (6) of **Art. 6** gives the following:

RULE.—*To find the power of a number, multiply the logarithm of the number by the exponent of the power; the number corresponding to this logarithm is the required power.*

Example 1.—Find the value of $(2.378)^6$.

Process.

$$\begin{aligned}\log 2.378 &= 0.37621\\ 6 \times \log 2.378 &= 2.25726 = \log (2.378)^6\\ \therefore (2.378)^6 &= 180.83\end{aligned}$$

Example 2.—Find the value of $(237.45)^{\frac{5}{7}}$

Process.

$$\begin{aligned}\log 237.45 &= 2.37557\\ \tfrac{5}{7} \times \log 237.45 &= 1.69684 = \log (237.45)^{\frac{5}{7}}\\ \therefore (237.45)^{\frac{5}{7}} &= 49.756\end{aligned}$$

20. To find the root of a number by means of logarithms.—Property (7) of **Art. 6** gives the following:

RULE.—*To find the root of a number, divide the logarithm of the number by the index of the root; the number corresponding to this logarithm is the root required.*

Example 1.—Find $\sqrt[5]{27.658}$.

Process.

$$\begin{aligned}\log 27.658 &= 1.44182\\ \tfrac{1}{5} \times \log 27.658 &= 0.28836 = \log \sqrt[5]{27.658}\\ \therefore \sqrt[5]{27.658} &= 1.9425\end{aligned}$$

Example 2.—Find $\sqrt[6]{0.008673}$.

Process.

$$\log 0.008673 = 7.93817 - 10$$

$$\begin{aligned} \log \sqrt[6]{0.008673} &= \tfrac{1}{6}(7.93817 - 10) \\ &= \tfrac{1}{6}(57.93817 - 60) \\ &= 9.65636 - 10 \end{aligned}$$

$$\therefore \sqrt[6]{0.008673} = 0.45327$$

Remark.—When a logarithm with a negative characteristic is to be divided by a number not exactly contained in the characteristic, it is best first to add and subtract such a number of times 10 that, after dividing, there will be a minus 10 at the right. In the above, before dividing (7.93817 − 10) by 6, 50 was added and subtracted. If the divisor had been 3, however, the division could have been performed by writing the logarithm in the form $\bar{3}.93817$ and dividing at once by 3.

21. Proportional parts.—In **Table I,** the marginal tables, marked **Prop. Parts,** contain the products of the tabular differences by 1, 2, 3, · · · 9 tenths. These products are arranged for convenience in interpolating. The work should be done mentally.

Thus, in interpolating, if the tabular difference is 35, then the marginal table is as given. In finding the logarithm, it is required to multiply, say, 35, by some number; and, in finding a number corresponding to a logarithm, it is required to divide some number by 35.

	35
1	3.5
2	7.0
3	10.5
4	14.0
5	17.5
6	21.0
7	24.5
8	28.0
9	31.5

(1) Multiply 35 by 0.68.

Process.

$$\begin{aligned} 35 \times 0.6 &= 21.0 \\ 35 \times 0.08 &= 2.8 \\ \therefore 35 \times 0.68 &= 23.8 \end{aligned}$$

(2) Divide 29 by 35.

Process. Dividend. . 29
Next less. . . . 28.0 giving 0.8
Remainder. . . 10
Next less. . . . 7.0 giving 0.02
Remainder. . . 30
Next less. . . . 28 giving 0.008
Etc. 0.828 · · · = quotient.

In interpolating, the division is usually only to determine the nearest first figure, and therefore can easily be done mentally.

22. Suggestions.—In interpolating, do not carry logarithms beyond the number of decimal places given in the table.

In writing numbers correct to a certain number of figures, take in the last place the figure that is nearest the true result when this is possible. If the next figure after the last one to be taken is 5 followed only by zeros, most computers take the nearest even figure for the last one. Thus, if the number is 0.02467500 it would be taken 0.02468 to five places; and if 0.02468500 it would also be taken 0.02468.

In working with tables, use the pencil as little as possible. Work for accuracy first and then for speed.

Write out a scheme for all logarithmic work before referring to the table. Be sure that your work is arranged so that it could be followed at any time by yourself or another person.

Example.—Write out a scheme for finding the value of

$$x = \frac{9.46 \times (41.6)^2 \times \sqrt{9.462}}{276.2 \times 3.4675}.$$

Scheme.

$$\begin{aligned} \log 9.46 &= \\ 2 \log 41.6 &= \\ \tfrac{1}{2} \log 9.462 &= \\ \text{colog } 276.2 &= \\ \text{colog } 3.4675 &= \\ \log x &= \\ x &= \end{aligned}$$

EXERCISES

Solve by logarithms:

1. 32.758 × 8.3759. *Ans.* 274.37.
2. 9.0083 × 0.072893. *Ans.* 0.65664.
3. (−0.001009) × 52456.7. *Ans.* −52.929.
4. 64.785 × 5.6346 × 0.01025. *Ans.* 3.7416.
5. 2.71828 × 1000 × 0.31461. *Ans.* 855.2.
6. 59.7642 ÷ 5.73894. *Ans.* 10.414.
7. 0.083467 ÷ 0.0046834. *Ans.* 17.822.
8. 11.01101 ÷ 96.15. *Ans.* 0.11452.
9. 579.996 ÷ (−37.16). *Ans.* −15.608.
10. 9.94923 ÷ 429.693. *Ans.* 0.023154.
11. $(1.74)^{17}$. *Ans.* 12284.
12. $(4.43769)^3 \times 0.9746$. *Ans.* 85.174.
13. $(1.5651)^{\frac{1}{4}}$. *Ans.* 1.1185.
14. $\sqrt[6]{17}$. *Ans.* 1.6035.
15. $\sqrt{77} \div \sqrt[3]{15}$. *Ans.* 3.558.
16. $\sqrt[9]{0.0000067}$. *Ans.* 0.26615.
17. $(1.42)^{12}$. *Ans.* 67.217.
18. $\sqrt[3]{\frac{0.002396}{8.926}}$. *Ans.* 0.064507.

19. $\left(\frac{41.73}{249}\right)^{\frac{8}{17}}$. *Ans.* 0.43146.

20. $\sqrt[5]{647.647}$. *Ans.* 3.6498.

21. $(0.1234)^{0.216}$. *Ans.* 0.6364.

22. $(0.1234)^{2.16}$. *Ans.* 0.010896.

23. $(0.1234)^{-0.216}$. *Ans.* 1.5713.

24. $(0.1234)^{-2.16}$. *Ans.* 91.78.

25. $(1.234)^{-0.216}$. *Ans.* 0.95558.

26. $(1.234)^{-2.16}$. *Ans.* 0.63497.

27. $(1.234)^{21.6}$. *Ans.* 93.866.

28. $(0.004)^{0.564}$. *Ans.* 0.04442.

29. $(0.004)^{-0.00564}$. *Ans.* 1.0316.

30. $(5.67)^{5.67}$. *Ans.* 18.741.

31. $(6.0606)^{6.06}$). *Ans.* 55.209.

32. $57.692 \times \sqrt[7]{93.2764}$. *Ans.* 110.29.

33. $\sqrt[8]{61.0061 \times 0.0077079}$. *Ans.* 0.91.

34. $\sqrt[5]{0.00006568} \div \sqrt[4]{0.000888444}$. *Ans.* 0.84398.

35. $\sqrt[7]{0.0064392} \div \sqrt[3]{-0.00965432}$. *Ans.* −2.2842.

36. $\frac{6607 \times 8 \times 91}{0.01002 \times 303.033 \times 8.71}$. *Ans.* 1,818,700.

37. $\frac{576.9 \times 0.98764 \times 57}{98.439 \times 39.846}$. *Ans.* 8.2796.

38. $\frac{8.72 \times \sqrt{72.56} \times (3.2654)^2}{(2.3849)^3 \sqrt[3]{974.681}}$. *Ans.* 5.8891.

39. $\frac{7^{0.63}(10^{0.34} - 0.2^{0.34})}{0.34}$. *Ans.* 16.127.

40. $\frac{10^{10} \times 0.0000000000000006209 \times \sqrt[4]{33.75}}{0.0000067896}$. *Ans.* 2204.2.

41. $-4500 \times 5^{\frac{4}{3}}v^{-\frac{1}{3}}]_4^8$. *Ans.* 5000.

Suggestion.—First substitute 8 for v and then substitute 4; finally, subtract the second result from the first.

42. $\frac{11{,}520 \times 4^{1.41}}{0.41}(4^{-0.41} - 8^{-0.41})$. *Ans.* 27,806.

43. Evaluate $50(e^{\frac{1}{4}} + e^{-\frac{1}{4}})$, where $e = 2.71828$. *Ans.* 103.14.

44. $2088^{\frac{5}{7}} \times 25(14{,}400^{\frac{2}{7}} - 2088^{\frac{2}{7}})$. *Ans.* 38,435.

Solve the following equations:

45. $(0.8)^x = 3$. *Ans.* −4.9233.

46. $(0.036)^x = 36$. *Ans.* −1.078.

47. $7^x - 3(7)^{\frac{1}{2}x} - 18 = 0$. *Ans.* 1.8415.

48. $(0.9)^{2x} + 3(0.9)^x - 10 = 0$. *Ans.* −6.5784.

49. $\log x + \log(x + 9) = 1$. *Ans.* 1.

50. $2 \log x = 2 + \log\left(\frac{110 - x}{100}\right)$. *Ans.* 10.

51. $3^{x+y} = 6^y$, and $2^x = 2(3)^{y+1}$. *Ans.* $x = -1.691$, $y = -2.691$.

52. $(10)^x(0.01)^y = 1$, and $(0.003)^{\frac{3}{4}y} = \frac{3}{4}$. *Ans.* $x = 0.13312$, $y = 0.06656$.

53. $\log x + \log(x + 3) = 1$. *Ans.* 2.

54. $2^x + 5^y = 1.64$, and $4^{2x} = 3.9$. *Ans.* $x = 0.49087$, $y = -0.90060$.

55. $(2.16)^x(1.06)^y = 0.12$, and $(3.8)^x(4.9)^y = 2.7$. *Ans.* $x = -2.9905$, $y = 3.1372$.

56. Evaluate $\sqrt{\frac{s(s - b)(s - c)}{s - a}}$, where $2s = a + b + c$ and $a = 47.236$, $b = 82.798$, and $c = 75.643$. *Ans.* 31.750.

57. Evaluate $\sqrt{s(s - a)(s - b)(s - c)}$, where $2s = a + b + c$ and $a = 4.2763$, $b = 9.9264$, and $c = 8.4399$. *Ans.* 17.904.

58. Using the formula for horsepower, $H = \frac{PLAN}{33{,}000}$; find H when $P = 76.5$, $L = 2.25$, $A = 231.8$, and $N = 116$. *Ans.* 140.25.

59. Given $W = 0.0033 \times 10^{-7}n$, find W when $n = 75{,}000$. *Ans.* 0.000024749.

60. In finding the diameter of a wrought-iron shaft that will transmit 90 hp. when the number of revolutions is 100 per minute, using a factor of safety of 8, it is required to find the diameter d from the formula:

$$d = 68.5\sqrt[3]{\frac{90}{100 \times \frac{50{,}000}{8}}}.$$

Ans. 3.5904.

61. Find the value of M from the formula $M = \frac{Wgl^3}{4bd^3B}$, when $g = 980$, $W = 75$, $l = 50$, $b = 0.98178$, $d = 0.5680$, and $B = 0.01093$. *Ans.* 11.681×10^{11}.

62. Find the value of n from the formula $n = \frac{360\,Lmgl}{\pi^2\theta r^4}$, when $L = 69.6$, $m = 10$, $g = 980$, $l = 28$, $\theta = 1.1955$, and $r = 0.317$. *Ans.* 0.57704×10^{11}.

63. If $m = ar^{-1.16}$, find r when $m = 2.263$ and $a = 0.4086$. *Ans.* 0.22864.

64. Given $p = p_0\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}}$, find the value of p in terms of p_0 if $\gamma = 1.41$. *Ans.* $0.5266p_0$.

65. If an indebtedness is paid in instalments, the payments being equal and each including the interest to the date of the instalment, then the number of instalments necessary to pay the debt is given by the formula:

$$n = \frac{\log p - \log(p - Pr)}{\log(1 + r)},$$

where P equals the total indebtedness, p = the amount of one instalment, and r = the rate per cent for the period between instalments. Find the number of instalments necessary to pay an indebtedness of \$1500 if the interest is 8 per cent per annum and the instalments are \$15 a month. *Ans.* 164.5 nearly.

66. Using the formula of Exercise 65, find the number of instalments if the indebtedness is \$1800, the interest 5 per cent per annum, and the instalments \$5 per week.

TABLE II

23. Changing systems of logarithms.—In what precedes, the computations have been made with logarithms to the base 10. It is often necessary to make computations when the logarithms used are the natural, or Napierian, logarithms, in which the base is $e = 2.71828 \cdots$. It will now be shown how to find the logarithm of a number to the base e from the table of logarithms to the base 10, and *vice versa*, by the help of **Table II,** page 56. For the sake of generality, the relation between the logarithms will be shown for any two bases.

THEOREM.—*Given the logarithm of a number N to the base a; then the logarithm of N to the base b is given by the relation:*

$$\log_b N = \frac{1}{\log_a b} \log_a N.$$

Proof.—Let $x = \log_b N$; then $b^x = N$.

$$\therefore \log_a b^x = \log_a N; \text{ or } x \log_a b = \log_a N.$$

$$\therefore x = \frac{1}{\log_a b} \log_a N.$$

But $x = \log_b N$.

(1) $$\therefore \log_b N = \frac{1}{\log_a b} \log_a N.$$

The constant multiplier $\frac{1}{\log_a b}$ is called the **modulus** of the system of which the base is b with reference to the system of which the base is a.

If a is put for N in formula (1),

$$\log_b a = \frac{1}{\log_a b} \log_a a.$$

(2) $$\therefore \log_b a = \frac{1}{\log_a b}, \text{ or } \log_b a \log_a b = 1.$$

It follows from (1) that the modulus of the *natural system* with reference to the *common system* is $\frac{1}{\log_{10} e}$, and the modulus of the *common system* with reference to the *natural system* is $\frac{1}{\log_e 10}$. That is,

(3) $$\log_e N = \frac{1}{\log_{10} e} \log_{10} N,$$

(4) and $$\log_{10} N = \frac{1}{\log_e 10} \log_e N.$$

The modulus $\frac{1}{\log_e 10} = \log_{10} e$ is usually represented by M.

Hence, by (2), $\frac{1}{\log_{10} e} = \frac{1}{M}$.

But $\log_{10} e = \log_{10} 2.71828 \cdots = 0.43429448.$

$$\therefore M = 0.43429448 \text{ and } \frac{1}{M} = 2.30258509.$$

Using these values for $\frac{1}{M}$ and M, (3) and (4) become the following:

(5) $$\mathbf{\log_e N = 2.3026 \log_{10} N.}$$

(6) $$\mathbf{\log_{10} N = 0.43429 \log_e N.}$$

24. Use of Table II.—In **Table II** are arranged multiples of M and $\frac{1}{M}$ to facilitate changing natural logarithms to common logarithms and *vice versa.*

Example 1.—Find the Napierian logarithm of 225, its common logarithm being 2.35218.

By (3) (**Art. 23**) $\log_e 225 = \frac{1}{M} \log_{10} 225$

$$= \frac{1}{M} \times 2.35218.$$

Table II gives the products:

$$\frac{1}{M} \times 2.3 = 5.295945714$$
$$\frac{1}{M} \times 0.052 = 0.1197344248$$
$$\frac{1}{M} \times 0.00018 = 0.0004144653$$
$$\therefore \frac{1}{M} \times 2.35218 = 5.4160946041$$
$$\therefore \log_e 225 = 5.41609.$$

Example 2.—Find the common logarithm of 762, its natural logarithm being 6.63595.

By (4) (**Art. 23**) $\log_{10} 762 = M \log_e 762 = M \times 6.63594$

$$M \times 6.6 = 2.866343581$$
$$M \times 0.035 = 0.01520030687$$
$$M \times 0.00094 = 0.0004082368$$
$$\therefore M \times 6.63594 = 2.88195212467$$
$$\therefore \log_{10} 762 = 2.88195$$

Example 3.—Find $\log_{0.3} 0.00107$.

By **Art. 23,**
$$\log_{0.3} 0.00107 = \frac{1}{\log_{10} 0.3} \times \log_{10} 0.00107$$
$$= \frac{1}{\bar{1}.47712} \times \bar{3}.02938$$
$$= \frac{-2.97062}{-0.52288} = 5.6814.$$

EXERCISES

Find the following logarithms:

1. $\log_e 426$. *Ans.* 6.05444.
2. $\log_e 1076$. *Ans.* 6.98101.
3. $\log_e 0.0763$. *Ans.* -2.57309.
4. $\log_e 1.467$. *Ans.* 0.38322.
5. $\log_e 0.01352$. *Ans.* -4.3036.
6. $\log_e 0.002457$. *Ans.* -6.0088.
7. $\log_e 5.128$. *Ans.* 1.6347.
8. $\log_e \pi$. *Ans.* 1.1447.
9. $\log_{0.3} 3.16$. *Ans.* $\bar{1}.0444$.
10. $\log_2 9.23$. *Ans.* 3.2064.
11. $\log_{0.3} 1.007$. *Ans.* -0.0057949.
12. $\log_{100} 22{,}843$. *Ans.* 2.17938.
13. Find x if $\log_e x = 6.96319$. *Ans.* 1057.
14. Find x if $\log_e x = -3.46954$. *Ans.* 0.031131.
15. Find x if $\log_e x = -2.45673$. *Ans.* 0.0857.
16. Evaluate $\frac{1}{2}(\log_e 5 - \log_e 4)$. *Ans.* 0.11157.
17. Evaluate $\log_e 12 - \log_e 2$. *Ans.* 1.792.
18. Evaluate $6 \log_e x\Big]_{\frac{1}{2}}^{8}$. *Ans.* 16.636.
19. Evaluate $-6 \log_e(-x)\Big]_{-4}^{-\frac{1}{2}}$. *Ans.* 12.477.
20. Evaluate $6 + \log_e 4 - \log_e 2$. *Ans.* 6.6931.
21. Evaluate $\frac{1}{40}(\log_e 10 - \log_e 4)$. *Ans.* 0.02291.
22. Evaluate $\dfrac{\log_e 10.1 - \log_e 10}{0.0001}$. *Ans.* 99.472.
23. Given $R = 10^6 \cdot \dfrac{t}{C} \cdot \dfrac{1}{\log_e\left(\dfrac{V_0}{V}\right)}$, where $t = 120$, $V_0 = 123$, $V = 115.8$, and $C = 0.082$; find R. *Ans.* 2.426×10^{10}.

24. The work W done by a volume of gas, expanding at a constant temperature from volume V_0 to volume V_1, is given by the formula:
$$W = p_0 V_0 \log_e\left(\frac{V_1}{V_0}\right).$$
Find the value of W if $p_0 = 87.5$, $V_0 = 246$, and $V_1 = 472$. *Ans.* 14,026.

25. Given $q = q_0 e^{kt}$; find k if $q = \frac{1}{2}q_0$ when $t = 1800$; then find q in terms of q_0 when $t = 500$. *Ans.* $k = -\dfrac{\log_e 2}{1800}$, $q = 0.8248q_0$.

26. Evaluate $\frac{1}{2}\left[\frac{1}{2}x\sqrt{4 + x^2} + 2\log_e(x + \sqrt{4 + x^2})\right]_0^4$. *Ans.* 5.916.
27. Evaluate $\frac{3}{8}\left[x\sqrt{x^2 - 16} - 16\log_e(x + \sqrt{x^2 - 16})\right]_4^8$. *Ans.* 12.88.

TABLE III

25. On pages 58 to 65 are arranged the logarithms of sines and tangents of angles from 3 to 7° for every 10″; and the logarithms of cosines and cotangents of angles from 83 to 87° for every 10″. The method of using these pages will follow from the explanation for the remaining pages of the table (see **Art. 28**).

On pages 66 to 110 are arranged the logarithms, to five decimal places, of the trigonometric sines, cosines, tangents, and cotangents, of angles from 0 to 90°, for each minute.

The logarithms in the columns headed **log sin, log cos,** or **log tan** are increased by 10 so as to avoid writing negative characteristics. Those in the column headed **log cot** are printed without this increase. The minus sign is printed over the final 5 in the logarithms, as explained in **Art. 10.**

The columns marked **d** give the tabular differences for the **log sin** and **log cos** columns. The column marked **c.d.** (common difference) gives the tabular differences for both **log tan** and **log cot** columns.

The marginal tables, marked **Prop. Parts,** give $\frac{1}{60}, \frac{2}{60}, \ldots \frac{9}{60}, \frac{10}{60}, \frac{20}{60}, \ldots \frac{50}{60}$ of the tabular differences, and are arranged for convenience in interpolating for seconds. The use of them is similar to that explained in **Art. 21.**

Since $\sec\theta = \dfrac{1}{\cos\theta}$ and $\csc\theta = \dfrac{1}{\sin\theta}$, the logarithms of the secant and cosecant of an angle are the *cologarithms* (arithmetical complements) of those of the cosine and sine respectively.

26. To find the logarithmic function of an acute angle.—

(1) *When the angle is given in degrees and minutes.* If the angle is less than 45°, the degrees are found at the head of the page, the minutes at the left, and the functions are taken as named at the tops of the columns. If the angle is between 45 and 90°, the degrees are found at the foot of the page, the minutes at the right, and the functions are taken as named at the bottoms of the columns. The functions are found in the same line with the minutes. The following should be located in the table.

$\log\sin 17°\,27' = 9.47694 - 10.$ $\log\sin 68°\,23' = 9.96833 - 10.$
$\log\cos 29°\,36' = 9.93927 - 10.$ $\log\cos 76°\,14' = 9.37652 - 10.$
$\log\tan 10°\,16' = 9.25799 - 10.$ $\log\tan 86°\,14' = 1.18154.$
$\log\cot 9°\,46' = 0.76414.$ $\log\cot 56°\,43' = 9.81721 - 10.$

(2) *When the angle contains seconds.* Here the function is found for the degrees and minutes and an interpolation made for the seconds similar to the interpolations in **Table I.** The tabular difference is multiplied by the number of seconds and divided by 60. This product may be taken from the **Prop. Parts** tables.

Since the sine and tangent *increase* as the angle *increases* from 0 to 90°, the correction for the seconds is *added;* but, since the cosine and cotangent *decrease* as the angle *increases* from 0 to 90°, the correction for the seconds is *subtracted.*

Example 1.—Find log sin 51° 26′ 23″.

$$\log \sin 51° 26' = 9.89314 - 10.$$
$$\text{Correction for } 23'' = 10 \times \tfrac{23}{60} = 4$$
$$\therefore \log \sin 51° 26' 23'' = 9.89318 - 10.$$

Example 2.—Find log cos 27° 49′ 37″.

$$\log \cos 27° 49' = 9.94667 - 10.$$
$$\text{Correction for } 37'' = 7 \times \tfrac{37}{60} = 4$$
$$\therefore \log \cos 27° 49' 37'' = 9.94663 - 10.$$

RULE.—*Find the function corresponding to the given degrees and minutes. Multiply the tabular difference by the number of seconds considered as sixtieths. When finding the sine, or tangent, add this product to the function corresponding to the degrees and minutes; but when finding the cosine, or cotangent, subtract this product.*

(3) *When the angle has decimal of minute.* Here the only difference in procedure from that given in (2) is that, in interpolating, the tabular difference is multiplied by the decimal of a minute given. It is evident that the **Prop. Parts** tables cannot be used for this.

27. To find the angle corresponding to a given logarithmic function.—(1) *When the function can be found in the table,* locate the function and read the angle in degrees and minutes at the head and left, or at the foot and right, of the page, as the case may be.

(2) *When the function cannot be found in the table,* the method of procedure can best be shown by examples.

Example 1.—Find the angle θ if $\log \sin \theta = 9.81659 - 10.$ Nearest log sin but less from table, $9.81651 - 10 = \log \sin 40° 57'$.

Tabular difference for a difference of 1′ in angle is 14.

Difference between given function and function found is 8.

Hence, if the increase in the angle necessary to increase the function by 8 is x, then

$$14:8 = 60'' : x''. \quad \therefore x'' = 34''.$$
$$\therefore 9.81659 - 10 = \log \sin 40° 57' 34''.$$
$$\therefore \theta = 40° 57' 34''.$$

Another angle having the same function is in the second quadrant and is $180° - 40° 57' 34'' = 139° 2' 26''$.

Example 2.—Find θ if $\log \cos \theta = 9.23764$.

Since the cosine decreases as the angle increases, locate in the table the nearest log cos but larger than the one given.

$$9.23823 - 10 = \log \cos 80° 2'.$$

Tabular difference is 71.

Difference between the function given and the one found is 59.

$$\tfrac{59}{71} \times 60'' = 50''.$$
$$\therefore \theta = 80° 2' 50''.$$

Another angle having the same function is in the fourth quadrant and is

$$360° - 80° 2' 50'' = 279° 57' 10''.$$

Example 3.—Find θ if $\log \tan \theta = 9.98773$.

From table, $9.98762 = \log \tan 44° 11'$.

Tabular difference, 25.

Difference between the function given and the one found, 11.

Using the **Prop. Parts** table headed 25, find the nearest number to 11 which is 8.3, the difference for 20″. Then subtract 8.3 from 11 leaving 2.7, the difference for 6″.

$$\therefore 9.98773 = \log \tan 44° 11' 26''.$$
$$\therefore \theta = 44° 11' 26'' \text{ and } 224° 11' 26''.$$

Of course, the interpolation should be done mentally when possible.

The method of procedure may be stated in the following rules:

RULE I.—*For a logarithmic sine or tangent:* (1) *find the degrees and minutes corresponding to the function next less than the given function;* (2) *find the difference between the given function and the one next less;* (3) *find the fractional part of 60″ that this difference is of the tabular difference. The required angle is the degrees and minutes corresponding to the function found in the table together with the seconds found.*

RULE II.—*For a logarithmic cosine or cotangent:* (1) *find the degrees and minutes corresponding to the function next greater than* *the given function;* (2) *find the difference between the given function and the one next greater;* (3) *find the fractional part of* 60″ *that this difference is of the tabular difference. The required angle is the degrees and minutes corresponding to the function found in the table together with the seconds found.*

28. Angles near 0 and 90°.—In what precedes, it has been assumed that the variation in the angle is proportional to the variation in the function. In angles near 0°, this is not very accurate with the sine and tangent; and near 90°, it is not very accurate with the cosine and cotangent.

Table III (pages 58 to 65) gives the functions for every 10″ between 3 and 7° and 83 and 87°. This makes the interpolations more nearly accurate for these angles. For the angles less than 3° and greater than 87°, the S and T scheme is convenient.

29. Functions by means of S and T.—The quantities S and T which are used are defined by the equations:

$$S = \log \frac{\sin \alpha}{\alpha}, \text{ or } S = \log \sin \alpha - \log \alpha,$$

and

$$T = \log \frac{\tan \alpha}{\alpha}, \text{ or } T = \log \tan \alpha - \log \alpha,$$

where α is the number of seconds in the angle.

For convenience, the values of S, T, and α for angles from 0° to 3° 4′ are arranged at the bottom of pages 36 to 55.

On pages 66 to 68 are columns headed **cpl S** and **cpl T**. These give the arithmetical complements of the values of S and T.

From the above are derived the following:

FORMULAS FOR THE USE OF S AND T

(1) *For angles near* 0°.

$$\begin{aligned} \log \sin \alpha &= \log \alpha'' + S. \\ \log \tan \alpha &= \log \alpha'' + T. \\ \log \cot \alpha &= \text{cpl} \log \alpha'' + \text{cpl } T \\ &= \text{cpl} \log \tan \alpha. \end{aligned}$$

$$\begin{aligned} \log \alpha'' &= \log \sin \alpha + \text{cpl } S \\ &= \log \tan \alpha + \text{cpl } T \\ &= \text{cpl} \log \cot \alpha + \text{cpl } T. \end{aligned}$$

(2) *For angles near* 90°.

$$\begin{aligned} \log \cos \alpha &= \log (90° - \alpha)'' + S. \\ \log \cot \alpha &= \log (90° - \alpha)'' + T. \\ \log \tan \alpha &= \text{cpl} \log (90° - \alpha)'' + \text{cpl } T \\ &= \text{cpl} \log \cot \alpha. \end{aligned}$$

$$\begin{aligned} \log (90° - \alpha)'' &= \log \cos \alpha + \text{cpl } S \\ &= \log \cot \alpha + \text{cpl } T \\ &= \text{cpl} \log \tan \alpha + \text{cpl } T. \end{aligned}$$

EXAMPLES

1. Find log sin 0° 47′ 19″.

$$\begin{aligned} 47' \, 19'' &= 2839'' \\ \log 2839 &= 3.45317 \\ S &= 4.68556 - 10 \\ \therefore \log \sin 0° \, 47' \, 19'' &= 8.13873 - 10 \end{aligned}$$

2. Find log tan 1° 27′ 14″.

$$\begin{aligned} 1° \, 27' \, 14'' &= 5234'' \\ \log 5234 &= 3.71883 \\ T &= 4.68567 - 10 \\ \therefore \log \tan 1° \, 27' \, 14'' &= 8.40450 - 10 \end{aligned}$$

3. Find log cot 0° 57′ 49″.

$$\begin{aligned} 57' \, 49'' &= 3469'' \\ \text{cpl} \log 3469 &= 6.45980 - 10 \\ \text{cpl } T &= 5.31438 \\ \therefore \log \cot 0° \, 57' \, 49'' &= 1.77418 \end{aligned}$$

4. Find α if log sin α = 7.85387 − 10

$$\begin{aligned} \log \sin \alpha &= 7.85387 - 10 \\ \text{cpl } S &= 5.31443 \\ \therefore \log \alpha'' &= 3.16830 \\ \alpha'' &= 1473.3'' \\ \therefore \alpha &= 0° \, 24' \, 33.3'' \end{aligned}$$

5. Find log cos 89° 27′ 32″.

$$\begin{aligned} 90° - 89° \, 27' \, 32'' &= 0° \, 32' \, 28'' \\ &= 1948'' \\ \log 1948 &= 3.28959 \\ S &= 4.68557 - 10 \\ \therefore \log \cos 89° \, 27' \, 32'' &= 7.97516 - 10 \end{aligned}$$

6. Find log cot 88° 49′ 51″.

$$\begin{aligned} 90° - 88° \, 49' \, 51'' &= 1° \, 10' \, 9'' \\ &= 4209'' \\ \log 4209 &= 3.62418 \\ T &= 4.68563 - 10 \\ \therefore \log \cot 88° \, 49' \, 51'' &= 8.30981 - 10 \end{aligned}$$

7. Find log tan 89° 47′ 33.82″.

$$\begin{aligned} 90° - 89° \, 47' \, 33.82'' &= 12' \, 26.18''. \\ &= 746.18'' \\ \text{colog } 746.18 &= 7.12715 - 10 \\ \text{cpl } T &= 5.31442 \\ \therefore \log \tan 89° \, 47' \, 33.82'' &= 2.44157. \end{aligned}$$

8. Find α if log cot α = 7.86432 − 10

$$\begin{aligned} \log \cot \alpha &= 7.86432 - 10 \\ \text{cpl } T &= 5.31442 \\ \therefore \log (90° - \alpha)'' &= 3.17874 \\ (90° - \alpha)'' &= 1509.2'' \\ 90° - \alpha &= 0° \, 25' \, 9.2'' \\ \therefore \alpha &= 89° \, 34' \, 50.8'' \end{aligned}$$

In Example 4, to find cpl S, locate log sin α on page 66, and read cpl S in the adjoining column. In Example 8, to find cpl T, locate log cot α on page 66.

30. Functions of angles greater than 90°.—In trigonometry there is a rule which says: To find the function of an angle greater than 90°, express the angle as a multiple of 90° plus an acute angle. If this multiple is even, take the same function of the acute angle as the one required; and, if the multiple is odd, take the cofunction of the acute angle. In either case prefix the sign determined by the quadrant the original angle is in.

Thus,

$$\sin 562° = \sin (6 \times 90° + 22°) = -\sin 22°$$
$$\tan 1042° = \tan (11 \times 90° + 52°) = -\cot 52°.$$

As a further convenience in finding the logarithmic functions of angles greater than 90°, there are arranged at the top and bottom of each page of **Table III** other angles. If α is the acute angle of the page, then $180° + \alpha$ is printed in light type and has the same functions as α; while $90° + \alpha$ and $270° + \alpha$ are printed in black type, and for the functions of these angles one must take the co-function of α. In either case proper regard must be paid to the algebraic sign.

Thus,

$$\log \cos 128° = \log \sin 38°n.$$
$$\log \tan 218° = \log \tan 38°.$$
$$\log \sin 308° = \log \cos 38°n.$$

The small letter n is placed after the function to indicate that the natural function is negative. Of course, the logarithm cannot take account of this.

EXERCISES

Verify the following.

1. $\log \sin 61° 41' 31'' = 9.94469.$
2. $\log \cos 31° 47' 27'' = 9.92940.$
3. $\log \tan 15° 14' 36'' = 9.43538.$
4. $\log \sin 45° 43' 28'' = 9.85491.$
5. $\log \cot 5° 50' 47'' = 0.98972.$
6. $\log \tan 80° 58' 17'' = 0.79889.$
7. $\log \sin 0° 29' 47'' = 7.93765.$
8. $\log \tan 1° 14' 27'' = 8.33566.$
9. $\log \cos 88° 47' 13'' = 8.32572.$
10. $\log \cot 89° 12' 18'' = 8.14227.$
11. $\log \cos 216° 14' 33'' = 9.90662n.$
12. $\log \sin 138° 48' 6'' = 9.81867.$
13. $\log \tan 325° 17' 29'' = 9.84052n$
14. $\log \cot 227° 28' 3'' = 9.96253.$

Find the values of θ less than 360° in the following:

15. $\log \sin \theta = 9.28762.$ *Ans.* 11° 10′ 53″ and 168° 49′ 7″.
16. $\log \cos \theta = 9.87642.$ *Ans.* 41° 12′ 22″ and 318° 47′ 38″.
17. $\log \tan \theta = 9.47632.$ *Ans.* 16° 40′ 13″ and 196° 40′ 13″.
18. $\log \cot \theta = 0.49632.$ *Ans.* 17° 41′ 18″ and 197° 41′ 18″.
19. $\log \tan \theta = 0.49936.$ *Ans.* 72° 25′ 38″ and 252° 25′ 38″.
20. $\log \cos \theta = 8.32967.$ *Ans.* 88° 46′ 33″ and 271° 13′ 27″.
21. $\log \sin \theta = 7.99892.$ *Ans.* 0° 34′ 17.5″ and 179° 25′ 42.5″.
22. $\log \sin \theta = 9.98762n.$ *Ans.* 256° 23′ and 283° 37′.
23. $\log \cos \theta = 9.89263n.$ *Ans.* 141° 20′ 54″ and 218° 39′ 6″.
24. $\log \tan \theta = 0.96236n.$ *Ans.* 96° 13′ 25″ and 276° 13′ 25″.

25. $\tan \theta = \dfrac{3.26 \tan 198° 13' \cos 13° 17'}{4.76 \sin 28° 16'}.$
Ans. 24° 51′ 15″ and 204° 51′ 15″.

26. $\sin \theta = \dfrac{17 \sin 283° 19' \tan 47° 16'}{39.2 \cos 183° 6'}.$
Ans. 27° 13′ 26″ and 152° 46′ 34″.

27. $\cos 2\theta = \dfrac{\tan (-2712° 15' 40'') \sec 3050° 40'}{\tan 1522° 46' 30'' \csc 1898° 17'}.$
Ans. 45° 47′ 30″, 134° 12′ 30″, 225° 47′ 30″, and 314° 12′ 30″

28. Find x if $x = \dfrac{1906}{\cot 24° 16' 19''}.$ *Ans.* 859.44.

29. Find x if $x = \dfrac{(122.87)^{\frac{1}{3}} \times 0.00008721}{\tan 18° 19' 20''}.$ *Ans.* 0.0013093.

TABLE IV

31. In this table (pages 112 to 134) are arranged the natural trigonometric sine, cosine, tangent, and cotangent of angles from 0 to 90° for each minute. The values are given correct to five figures.

The arrangement of and the method of using the table are practically the same as for **Table III,** except that there are no differences given and no table of proportional parts.

32. To find the natural function of an angle.—(1) *When the angle is given in degrees and minutes:* If the angle is less than 45°, the degrees are found at the head of the page, the minutes at the left, and the functions are taken as named at the tops of the columns. If the angle is between 45 and 90°, the degrees are found at the foot of the page, the minutes at the right, and the functions are taken as named at the bottoms of the columns. The functions are found in the same line with the minutes. When the angle is greater than 90°, first express as a function of an acute angle, as in **Art. 30.**

Thus,

$$\sin 27° 13' = 0.45736. \qquad \sin 62° 18' = 0.88539.$$
$$\cos 36° 42' = 0.80178. \qquad \cos 83° 47' = 0.10829.$$
$$\tan 11° 17' = 0.19952. \qquad \tan 75° 14' = 3.7938.$$
$$\cot 21° 43' = 2.5108. \qquad \cot 56° 28' = 0.66272.$$
$$\sin 228° 13' = -\sin 48° 13' = -0.74567.$$

(2) *When the angle contains seconds:* Here the function is found for the degrees and minutes as in (1) and an interpolation is made for the seconds similar to the interpolation of **Table I.** The tabular difference is multiplied by the number of seconds and divided by 60. Since the sine and the tangent *increase* as the angle *increases* from 0 to 90°, the correction for the seconds is added; but, since the cosine and the cotangent *decrease* as the angle *increases* from 0 to 90°, the correction for the seconds is subtracted.

Example 1.—Find sin 27° 41′ 16″.

$$
\begin{aligned}
\sin 27° 41' &= 0.46458\\
\text{Correction for } 16'' = 26 \times \tfrac{16}{60} &= 7\\
\therefore \sin 27° 41' 16'' &= 0.46465
\end{aligned}
$$

Example 2.—Find cot 65° 22′ 36″.

$$
\begin{aligned}
\cot 65° 22' &= 0.45854\\
\text{Correction for } 36'' = 35 \times \tfrac{36}{60} &= 21\\
\therefore \cot 65° 22' 36'' &= 0.45833
\end{aligned}
$$

RULE.—*Find the function corresponding to the given degrees and minutes. Multiply the tabular difference by the number of seconds considered as sixtieths. When finding the sine, or tangent, add this product to the function corresponding to the degrees and minutes; but, when finding the cosine, or cotangent, subtract this product.*

(3) *When the angle has decimal of minute:* Here the only difference in procedure from that given in (2) is that, in interpolating, the tabular difference is multiplied by the decimal of a minute given.

33. To find the angle corresponding to a given natural function. (1) *When the function can be found in the table,* locate the function and read the angle in degrees and minutes at the head and left, or at the foot and right, of the page, as the case may be.

(2) *When the function cannot be found in the table,* the method involves interpolation and can best be shown by examples.

Example 1.—Find the values of angle θ if $\sin \theta = 0.53862$. Nearest sine but less from table, 0.53853 = sin 32° 35′. Tabular difference for difference of 1′ in angle is 24. Difference between given function and function found is 9.

The increase in the angle necessary to increase the function by 9 is $\frac{9}{24} \times 60'' = 22\frac{1}{2}''$ or 23″.

$$\therefore 0.53862 = \sin 32° 35' 23''.$$
$$\therefore \theta = 32° 35' 23''.$$

Another angle having the same sine is in the second quadrant and is 180° − 32° 35′ 23″ = 147° 24′ 37″.

Example 2.—Find the value of angle θ if $\cos \theta = 0.32346$.

Since the cosine decreases as the angle increases, locate in the table the nearest cosine but greater than the one given. This is

$$0.32364 = \cos 71° 7'.$$

Tabular difference for difference of 1′ in angle is 27.

Difference between given function and function found is 18.

The increase in the angle necessary to decrease the function by 18 is $\frac{18}{27} \times 60'' = 40''$.

$$\therefore 0.32346 = \cos 71° 7' 40''.$$
$$\therefore \theta = 71° 7' 40''.$$

Another angle having the same cosine is in the fourth quadrant and is 360° − 71° 7′ 40″ = 288° 52′ 20″.

Example 3.—Find the value of angle θ if $\tan \theta = -1.2783$.

Since $\tan \theta$ is negative, θ must lie in the second and the fourth quadrants.

First find the angle θ' in the first quadrant that has its tangent equal numerically to $\tan \theta$. That is, find θ' if $\tan \theta' = 1.2783$.

Then $\theta' = 51° 57' 53''$.

$\theta = 180° - \theta' = 128° 2' 7''$,

or $\theta = 360° - \theta' = 308° 2' 7''$.

EXERCISES

Verify the following:

1. sin 27° 22′ 41″ = 0.45986.
2. cos 36° 14′ 16″ = 0.80657.
3. tan 41° 19′ 26″ = 0.87926.
4. cot 13° 14′ 52″ = 4.2475.
5. cos 72° 28′ 14″ = 0.30119.
6. tan 83° 40′ 30″ = 9.0218.
7. sin 2° 19′ 40″ = 0.04061.
8. tan 156° 42′ 13″ = −0.43059.
9. sin 220° 35′ 30″ = −0.65066.
10. cot 295° 17′ 14″ = −0.47242.
11. cos 314° 14.6′ = 0.69771.
12. sin 126° 23.7′ = 0.80494.
13. sin 342° 43.2′ = −0.29704.
14. cos 142° 19.8′ = −0.79154.

Find the values of θ less than 360° in the following:

15. $\sin \theta = 0.49367$. *Ans.* 29° 34′ 55″ and 150° 25′ 5″.
16. $\sin \theta = 0.82764$. *Ans.* 55° 51′ 26″ and 124° 8′ 34″.
17. $\cos \theta = 0.89672$. *Ans.* 26° 16′ 10″ and 333° 43′ 50″.
18. $\cos \theta = 0.22724$. *Ans.* 76° 51′ 56″ and 283° 8′ 4″.
19. $\tan \theta = 2.4379$. *Ans.* 67° 41′ 49″ and 247° 41′ 49″.
20. $\tan \theta = 0.87623$. *Ans.* 41° 13′ 33″ and 221° 13′ 33″.
21. $\cot \theta = 1.8923$. *Ans.* 27° 51′ 17″ and 207° 51′ 17″.
22. $\cot \theta = 0.43729$. *Ans.* 66° 22′ 51″ and 246° 22′ 51″.
23. $\sin \theta = -0.89723$. *Ans.* 243° 47′ 46″ and 296° 12′ 14″.
24. $\cos \theta = -0.42936$. *Ans.* 115° 25′ 37″ and 244° 34′ 23″.
25. $\tan \theta = -0.92834$. *Ans.* 137° 7′ 41″ and 317° 7′ 41″.
26. $\cot \theta = -2.4376$. *Ans.* 157° 41′ 40″ and 337° 41′ 40″.

TABLE V

34. This table (page 135) can be used to change an angle expressed in degrees to radians, or *vice versa.* It may also be used for finding the arc length in a circle when the angle at the center is given, or *vice versa.*

Example 1.—Express 143° 27′ 36″ in radians.

$$\begin{aligned} 143^\circ &= 2.4958208 \\ 27' &= 0.0078540 \\ 36'' &= 0.0001745 \\ \therefore\ 143^\circ\ 27'\ 36'' &= 2.5038493 \text{ radians.} \end{aligned}$$

The accuracy of the last figure in the sum cannot be relied upon.

Example 2.—Express 3.6678437 radians in degrees, minutes, and seconds.

$$\begin{aligned} \text{Given, } & 3.6678437 \\ \text{Next less in table, } & 3.1415927 = 180^\circ \\ \text{Difference, } & 0.5262510 \\ \text{Next less, } & 0.5235988 = 30^\circ \\ \text{Difference, } & 0.0026522 \\ \text{Next less, } & 0.0026180 = 0^\circ\ 9' \\ \text{Difference, } & 0.0000342 \\ \text{Next less, } & 0.0000339 = 0^\circ\ 0'\ 7'' \\ \text{Difference, } & 0.0000003 \\ \therefore\ 3.6678437 \text{ radians} & = 210^\circ\ 9'\ 7''. \end{aligned}$$

EXERCISES

Verify the following:

1. 216° 44′ 44″ = 3.78292 radians.
2. 47° 23′ 58″ = 0.82728 radian.
3. 725° 19′ 33″ = 12.65932 radians.
4. 3.96423 radians = 227° 8′ 22″.
5. 1.49367 radians = 85° 34′ 52″.
6. 0.0236784 radian = 1° 21′ 24″.

35. Errors of interpolation.—In the process of interpolation in logarithms, values are inserted as if the change in the logarithm between the two nearest tabular values was directly proportional to the change in the number. This would mean that the graph of the equation $y = \log x$ for this interval is a straight line.

If values of x and $y = \log x$ are plotted in the usual manner in rectangular coordinates, the graph of $y = \log x$ is as shown in Fig. 1, where the unit on the y-axis is 10 times as large as the unit on the x-axis.

The values of the logarithms of numbers can be read from this curve, but not to a very high degree of accuracy. The values of x and y given in the table fall so close together on this curve that the interpolating cannot be shown. Suppose, for example, that log 1.7854 is required. Take the portion of the curve near $x = 1.7854$ and magnify it in the ratio of 1 to 20,000 on the x-axis

and 1 to 1000 on the y-axis; the resulting curve is shown in Fig. 2. Referring to this figure, when $x = 1.785$, $y = 0.25164$; and when $x = 1.786$, $y = 0.25188$. These give, respectively, the two points S and T on the curve. When $x = 1.7854$, $y = \log 1.7854$ has the value shown by the point P on the curve; but, by interpolation,

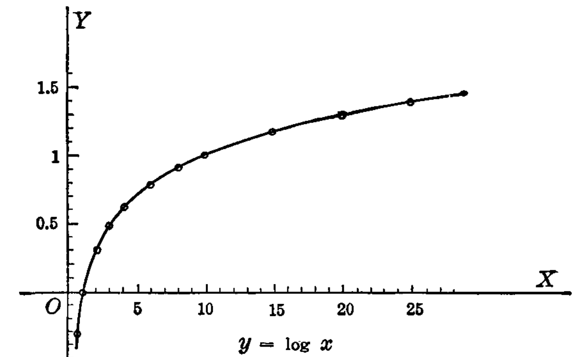

FIG. 1.

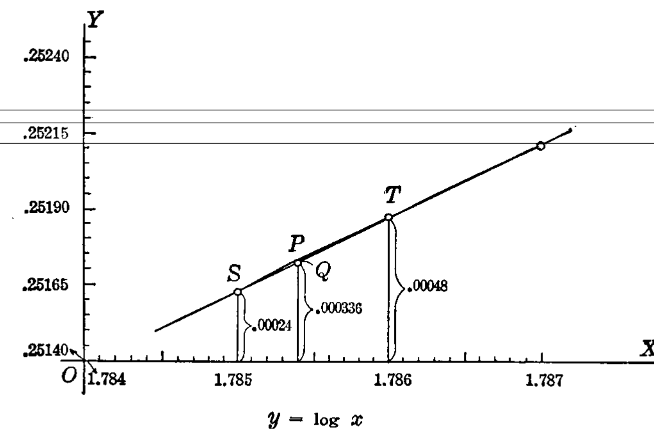

FIG. 2.

the value of log 1.7854 = 0.251736 and is shown by the point Q. Therefore, the interpolation gives an error equal to QP.

By using a higher-place table of logarithms, the value of log 1.7854 = 0.2517355. This shows that the error is such that the logarithm is not affected in the fifth decimal place.

A similar discussion could be given for interpolating in trigonometric functions.

TABLE I

COMMON LOGARITHMS OF NUMBERS

From 1 to 10,000 to five places.
From 10,000 to 11,000 to seven places.
(For explanations, see pages 7 to 12.)

Also values of S. and T. from 0° to 3°4′.
(For explanations, see page 26.)

N.	L. 0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
101	432	475	518	561	604	647	689	732	775	817
102	860	903	945	988	*030	*072	*115	*157	*199	*242
103	01 284	326	368	410	452	494	536	578	620	662
104	703	745	787	828	870	912	953	995	*036	*078
105	02 119	160	202	243	284	325	366	407	449	490
106	531	572	612	653	694	735	776	816	857	898
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03 342	383	423	463	503	543	583	623	663	703
109	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
111	532	571	610	650	689	727	766	805	844	883
112	922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05 308	346	385	423	461	500	538	576	614	652
114	690	729	767	805	843	881	918	956	994	*032
115	06 070	108	145	183	221	258	296	333	371	408
116	446	483	521	558	595	633	670	707	744	781
117	819	856	893	930	967	*004	*041	*078	*115	*151
118	07 188	225	262	298	335	372	408	445	482	518
119	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08 279	314	350	386	422	458	493	529	565	600
122	636	672	707	743	778	814	849	884	920	955
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09 342	377	412	447	482	517	552	587	621	656
125	691	726	760	795	830	864	899	934	968	*003
126	10 037	072	106	140	175	209	243	278	312	346
127	380	415	449	483	517	551	585	619	653	687
128	721	755	789	823	857	890	924	958	992	*025
129	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
131	727	760	793	826	860	893	926	959	992	*024
132	12 057	090	123	156	189	222	254	287	320	352
133	385	418	450	483	516	548	581	613	646	678
134	710	743	775	808	840	872	905	937	969	*001
135	13 033	066	098	130	162	194	226	258	290	322
136	354	386	418	450	481	513	545	577	609	640
137	672	704	735	767	799	830	862	893	925	956
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
141	922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15 229	259	290	320	351	381	412	442	473	503
143	534	564	594	625	655	685	715	746	776	806
144	836	866	897	927	957	987	*017	*047	*077	*107
145	16 137	167	197	227	256	286	316	346	376	406
146	435	465	495	524	554	584	613	643	673	702
147	732	761	791	820	850	879	909	938	967	997
148	17 026	056	085	114	143	173	202	231	260	289
149	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

0 1′ = 60″	S. 4.68 557	T. 4.68 557
0 2 = 120	557	557
0 3 = 180	557	557
0 16 = 960	557	558
0 17 = 1020	557	558
0 18 = 1080	557	558
0° 19′ = 1140″	S. 4.68 557	T. 4.68 558
0 20 = 1200	557	558
0 21 = 1260	557	558
0 22 = 1320	557	558
0 23 = 1380	557	558
0 24 = 1440	557	558
0 25 = 1500	557	558

N.	L. 0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
151	898	926	955	984	*013	*041	*070	*099	*127	*156
152	18 184	213	241	270	298	327	355	384	412	441
153	469	498	526	554	583	611	639	667	696	724
154	752	780	808	837	865	893	921	949	977	*005
155	19 033	061	089	117	145	173	201	229	257	285
156	312	340	368	396	424	451	479	507	535	562
157	590	618	645	673	700	728	756	783	811	838
158	866	893	921	948	976	*003	*030	*058	*085	*112
159	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
161	683	710	737	763	790	817	844	871	898	925
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21 219	245	272	299	325	352	378	405	431	458
164	484	511	537	564	590	617	643	669	696	722
165	748	775	801	827	854	880	906	932	958	985
166	22 011	037	063	089	115	141	167	194	220	246
167	272	298	324	350	376	401	427	453	479	505
168	531	557	583	608	634	660	686	712	737	763
169	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
171	300	325	350	376	401	426	452	477	502	528
172	553	578	603	629	654	679	704	729	754	779
173	805	830	855	880	905	930	955	980	*005	*030
174	24 055	080	105	130	155	180	204	229	254	279
175	304	329	353	378	403	428	452	477	502	527
176	551	576	601	625	650	674	699	724	748	773
177	797	822	846	871	895	920	944	969	993	*018
178	25 042	066	091	115	139	164	188	212	237	261
179	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
181	768	792	816	840	864	888	912	935	959	983
182	26 007	031	055	079	102	126	150	174	198	221
183	245	269	293	316	340	364	387	411	435	458
184	482	505	529	553	576	600	623	647	670	694
185	717	741	764	788	811	834	858	881	905	928
186	951	975	998	*021	*045	*068	*091	*114	*138	*161
187	27 184	207	231	254	277	300	323	346	370	393
188	416	439	462	485	508	531	554	577	600	623
189	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
191	28 103	126	149	171	194	217	240	262	285	307
192	330	353	375	398	421	443	466	488	511	533
193	556	578	601	623	646	668	691	713	735	758
194	780	803	825	847	870	892	914	937	959	981
195	29 003	026	048	070	092	115	137	159	181	203
196	226	248	270	292	314	336	358	380	403	425
197	447	469	491	513	535	557	579	601	623	645
198	667	688	710	732	754	776	798	820	842	863
199	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

0° 2′ = 120″	S. 4.68 557	T. 4.68 557
0 3 = 180	557	557
0 4 = 240	557	558
0 25 = 1500	557	558
0 26 = 1560	557	558
0 27 = 1620	557	558
0° 28′ = 1680″	S. 4.68 557	T. 4.68 558
0 29 = 1740	557	559
0 30 = 1800	557	559
0 31 = 1860	557	559
0 32 = 1920	557	559
0 33 = 1980	557	559
0 34 = 2040	557	559

N.	L. 0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20	19	18	17
1	2.0	1.9	1.8	1.7
2	4.0	3.8	3.6	3.4
3	6.0	5.7	5.4	5.1
4	8.0	7.6	7.2	6.8
5	10.0	9.5	9.0	8.5
6	12.0	11.4	10.8	10.2
7	14.0	13.3	12.6	11.9
8	16.0	15.2	14.4	13.6
9	18.0	17.1	16.2	15.3

0° 3′= 180″	S. 4.68 557	T. 4.68 557
0 4 = 240	557	558
0 5 = 300	557	558
0 33 =1980	557	559
0 34 =2040	557	559
0 35 =2100	557	559

0° 36′=2160″	S. 4.68 557	T. 4.68 559
0 37 =2220	557	559
0 38 =2280	557	559
0 39 =2340	557	559
0 40 =2400	557	559
0 41 =2460	556	560
0 42 =2520	556	560

N.	L. 0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123
252	40 140	157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466
254	483	500	518	535	552	569	586	603	620	637
255	654	671	688	705	722	739	756	773	790	807
256	824	841	858	875	892	909	926	943	960	976
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
258	41 162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
261	664	681	697	714	731	747	764	780	797	814
262	830	847	863	880	896	913	929	946	963	979
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
264	42 160	177	193	210	226	243	259	275	292	308
265	325	341	357	374	390	406	423	439	455	472
266	488	504	521	537	553	570	586	602	619	635
267	651	667	684	700	716	732	749	765	781	797
268	813	830	846	862	878	894	911	927	943	959
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441
272	457	473	489	505	521	537	553	569	584	600
273	616	632	648	664	680	696	712	727	743	759
274	775	791	807	823	838	854	870	886	902	917
275	933	949	965	981	996	*012	*028	*044	*059	*075
276	44 091	107	122	138	154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389
278	404	420	436	451	467	483	498	514	529	545
279	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
281	871	886	902	917	932	948	963	979	994	*010
282	45 025	040	056	071	086	102	117	133	148	163
283	179	194	209	225	240	255	271	286	301	317
284	332	347	362	378	393	408	423	439	454	469
285	484	500	515	530	545	561	576	591	606	621
286	637	652	667	682	697	712	728	743	758	773
287	788	803	818	834	849	864	879	894	909	924
288	939	954	969	984	*000	*015	*030	*045	*060	*075
289	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
291	389	404	419	434	449	464	479	494	509	523
292	538	553	568	583	598	613	627	642	657	672
293	687	702	716	731	746	761	776	790	805	820
294	835	850	864	879	894	909	923	938	953	967
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114
296	47 129	144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407
298	422	436	451	465	480	494	509	524	538	553
299	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	18	17
1	1.8	1.7
2	3.6	3.4
3	5.4	5.1
4	7.2	6.8
5	9.0	8.5
6	10.8	10.2
7	12.6	11.9
8	14.4	13.6
9	16.2	15.3

log e = 0.43429

	16	15	14
1	1.6	1.5	1.4
2	3.2	3.0	2.8
3	4.8	4.5	4.2
4	6.4	6.0	5.6
5	8.0	7.5	7.0
6	9.6	9.0	8.4
7	11.2	10.5	9.8
8	12.8	12.0	11.2
9	14.4	13.5	12.6

0° 4′= 240″	S. 4.68 557	T. 4.68 558
0 5 = 300	557	558
0 41 =2460	556	560
0 42 =2520	556	560
0 43 =2580	556	560
0 44 =2640	556	560

0° 45′=2700″	S. 4.68 556	T 4.68 560
0 46 =2760	556	560
0 47 =2820	556	560
0 48 =2880	556	560
0 49 =2940	556	560
0 50 =3000	556	560

N.	L. 0	1	2	3	4	5	6	7	8	9
300	47 712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	15	14	13	12
1	1.5	1.4	1.3	1.2
2	3.0	2.8	2.6	2.4
3	4.5	4.2	3.9	3.6
4	6.0	5.6	5.2	4.8
5	7.5	7.0	6.5	6.0
6	9.0	8.4	7.8	7.2
7	10.5	9.8	9.1	8.4
8	12.0	11.2	10.4	9.6
9	13.5	12.6	11.7	10.8

$\log \pi = 0.49715$

0° 5′ = 300″	S. 4.68 557	T. 4.68 558	
0 6 = 360	557	558	
0 50 = 3000	556	561	
0 51 = 3060	556	561	
0 52 = 3120	556	561	
0 53 = 3180	556	561	
0° 54′ = 3240″	S. 4.68 556	T. 4.68 561	
0 55 = 3300	556	561	
0 56 = 3360	556	561	
0 57 = 3420	555	561	
0 58 = 3480	555	562	
0 59 = 3540	555	562	

N.	L. 0	1	2	3	4	5	6	7	8	9
350	54 407	419	432	444	456	469	481	494	506	518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	*011
355	55 023	035	047	060	072	084	096	108	121	133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
364	56 110	122	134	146	158	170	182	194	205	217
365	229	241	253	265	277	289	301	312	324	336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	*008	*019	*031	*043
372	57 054	066	078	089	101	113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	403	415	426	438	449	461	473	484	496	507
376	519	530	542	553	565	576	588	600	611	623
377	634	646	657	669	680	692	703	715	726	738
378	749	761	772	784	795	807	818	830	841	852
379	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
381	58 092	104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	546	557	569	580	591	602	614	625	636	647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	660	671	682	693	704	715	726	737	748	759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086
399	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	13	12	11	10
1	1.3	1.2	1.1	1.0
2	2.6	2.4	2.2	2.0
3	3.9	3.6	3.3	3.0
4	5.2	4.8	4.4	4.0
5	6.5	6.0	5.5	5.0
6	7.8	7.2	6.6	6.0
7	9.1	8.4	7.7	7.0
8	10.4	9.6	8.8	8.0
9	11.7	10.8	9.9	9.0

0° 5′ = 300″	S. 4.68 557	T. 4.68 558
0 6 = 360	557	558
0 7 = 420	557	558
0 58 = 3480	555	562
0 59 = 3540	555	562
1 0 = 3600	555	562
1° 1′ = 3660″	S. 4.68 555	T. 4.68 562
1 2 = 3720	555	562
1 3 = 3780	555	562
1 4 = 3840	555	563
1 5 = 3900	555	563
1 6 = 3960	555	563
1 7 = 4020	555	563

N.	L. 0	1	2	3	4	5	6	7	8	9
400	60 206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	*002	*013	*023	*034	*045	*055
408	61 066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
414	700	711	721	731	742	752	763	773	784	794
415	805	815	826	836	847	857	868	878	888	899
416	909	920	930	941	951	962	972	982	993	*003
417	62 014	024	034	045	055	066	076	086	097	107
418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
421	428	439	449	459	469	480	490	500	511	521
422	531	542	552	562	572	583	593	603	613	624
423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
426	941	951	961	972	982	992	*002	*012	*022	*033
427	63 043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
429	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
431	448	458	468	478	488	498	508	518	528	538
432	548	558	568	579	589	599	609	619	629	639
433	649	659	669	679	689	699	709	719	729	739
434	749	759	769	779	789	799	809	819	829	839
435	849	859	869	879	889	899	909	919	929	939
436	949	959	969	979	988	998	*008	*018	*028	*038
437	64 048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
439	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
441	444	454	464	473	483	493	503	513	523	532
442	542	552	562	572	582	591	601	611	621	631
443	640	650	660	670	680	689	699	709	719	729
444	738	748	758	768	777	787	797	807	816	826
445	836	846	856	865	875	885	895	904	914	924
446	933	943	953	963	972	982	992	*002	*011	*021
447	65 031	040	050	060	070	079	089	099	108	118
448	128	137	147	157	167	176	186	196	205	215
449	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	11	10	9
1	1.1	1.0	0.9
2	2.2	2.0	1.8
3	3.3	3.0	2.7
4	4.4	4.0	3.6
5	5.5	5.0	4.5
6	6.6	6.0	5.4
7	7.7	7.0	6.3
8	8.8	8.0	7.2
9	9.9	9.0	8.1

		S.	T.
0° 6′	= 360″	4.68 557	4.68 558
0 7	= 420	557	558
0 8	= 480	557	558
1 6	= 3960	555	563
1 7	= 4020	555	563
1 8	= 4080	555	563
1° 9′	= 4140″	4.68 555	4.68 563
1 10	= 4200	554	563
1 11	= 4260	554	564
1 12	= 4320	554	564
1 13	= 4380	554	564
1 14	= 4440	554	564
1 15	= 4500	554	564

N.	L. 0	1	2	3	4	5	6	7	8	9
450	65 321	331	341	350	360	369	379	389	398	408
451	418	427	437	447	456	466	475	485	495	504
452	514	523	533	543	552	562	571	581	591	600
453	610	619	629	639	648	658	667	677	686	696
454	706	715	725	734	744	753	763	772	782	792
455	801	811	820	830	839	849	858	868	877	887
456	896	906	916	925	935	944	954	963	973	982
457	992	*001	*011	*020	*030	*039	*049	*058	*068	*077
458	66 087	096	106	115	124	134	143	153	162	172
459	181	191	200	210	219	229	238	247	257	266
460	276	285	295	304	314	323	332	342	351	361
461	370	380	389	398	408	417	427	436	445	455
462	464	474	483	492	502	511	521	530	539	549
463	558	567	577	586	596	605	614	624	633	642
464	652	661	671	680	689	699	708	717	727	736
465	745	755	764	773	783	792	801	811	820	829
466	839	848	857	867	876	885	894	904	913	922
467	932	941	950	960	969	978	987	997	*006	*015
468	67 025	034	043	052	062	071	080	089	099	108
469	117	127	136	145	154	164	173	182	191	201
470	210	219	228	237	247	256	265	274	284	293
471	302	311	321	330	339	348	357	367	376	385
472	394	403	413	422	431	440	449	459	468	477
473	486	495	504	514	523	532	541	550	560	569
474	578	587	596	605	614	624	633	642	651	660
475	669	679	688	697	706	715	724	733	742	752
476	761	770	779	788	797	806	815	825	834	843
477	852	861	870	879	888	897	906	916	925	934
478	943	952	961	970	979	988	997	*006	*015	*024
479	68 034	043	052	061	070	079	088	097	106	115
480	124	133	142	151	160	169	178	187	196	205
481	215	224	233	242	251	260	269	278	287	296
482	305	314	323	332	341	350	359	368	377	386
483	395	404	413	422	431	440	449	458	467	476
484	485	494	502	511	520	529	538	547	556	565
485	574	583	592	601	610	619	628	637	646	655
486	664	673	681	690	699	708	717	726	735	744
487	753	762	771	780	789	797	806	815	824	833
488	842	851	860	869	878	886	895	904	913	922
489	931	940	949	958	966	975	984	993	*002	*011
490	69 020	028	037	046	055	064	073	082	090	099
491	108	117	126	135	144	152	161	170	179	188
492	197	205	214	223	232	241	249	258	267	276
493	285	294	302	311	320	329	338	346	355	364
494	373	381	390	399	408	417	425	434	443	452
495	461	469	478	487	496	504	513	522	531	539
496	548	557	566	574	583	592	601	609	618	627
497	636	644	653	662	671	679	688	697	705	714
498	723	732	740	749	758	767	775	784	793	801
499	810	819	827	836	845	854	862	871	880	888
500	897	906	914	923	932	940	949	958	966	975
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	10	9	8
1	1.0	0.9	0.8
2	2.0	1.8	1.6
3	3.0	2.7	2.4
4	4.0	3.6	3.2
5	5.0	4.5	4.0
6	6.0	5.4	4.8
7	7.0	6.3	5.6
8	8.0	7.2	6.4
9	9.0	8.1	7.2

		S.	T.
0° 7′	= 420″	4.68 557	4.68 558
0 8	= 480	557	558
0 9	= 540	557	558
1 15	= 4500	554	564
1 16	= 4560	554	565
1 17	= 4620	554	565
1° 18′	= 4680″	4.68 554	4.68 565
1 19	= 4740	554	565
1 20	= 4800	554	565
1 21	= 4860	553	566
1 22	= 4920	553	566
1 23	= 4980	553	566
1 24	= 5040	553	566

N.	L. 0	1	2	3	4	5	6	7	8	9
500	69 897	906	914	923	932	940	949	958	966	975
501	984	992	*001	*010	*018	*027	*036	*044	*053	*062
502	70 070	079	088	096	105	114	122	131	140	148
503	157	165	174	183	191	200	209	217	226	234
504	243	252	260	269	278	286	295	303	312	321
505	329	338	346	355	364	372	381	389	398	406
506	415	424	432	441	449	458	467	475	484	492
507	501	509	518	526	535	544	552	561	569	578
508	586	595	603	612	621	629	638	646	655	663
509	672	680	689	697	706	714	723	731	740	749
510	757	766	774	783	791	800	808	817	825	834
511	842	851	859	868	876	885	893	902	910	919
512	927	935	944	952	961	969	978	986	995	*003
513	71 012	020	029	037	046	054	063	071	079	088
514	096	105	113	122	130	139	147	155	164	172
515	181	189	198	206	214	223	231	240	248	257
516	265	273	282	290	299	307	315	324	332	341
517	349	357	366	374	383	391	399	408	416	425
518	433	441	450	458	466	475	483	492	500	508
519	517	525	533	542	550	559	567	575	584	592
520	600	609	617	625	634	642	650	659	667	675
521	684	692	700	709	717	725	734	742	750	759
522	767	775	784	792	800	809	817	825	834	842
523	850	858	867	875	883	892	900	908	917	925
524	933	941	950	958	966	975	983	991	999	*008
525	72 016	024	032	041	049	057	066	074	082	090
526	099	107	115	123	132	140	148	156	165	173
527	181	189	198	206	214	222	230	239	247	255
528	263	272	280	288	296	304	313	321	329	337
529	346	354	362	370	378	387	395	403	411	419
530	428	436	444	452	460	469	477	485	493	501
531	509	518	526	534	542	550	558	567	575	583
532	591	599	607	616	624	632	640	648	656	665
533	673	681	689	697	705	713	722	730	738	746
534	754	762	770	779	787	795	803	811	819	827
535	835	843	852	860	868	876	884	892	900	908
536	916	925	933	941	949	957	965	973	981	989
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070
538	73 078	086	094	102	111	119	127	135	143	151
539	159	167	175	183	191	199	207	215	223	231
540	239	247	255	263	272	280	288	296	304	312
541	320	328	336	344	352	360	368	376	384	392
542	400	408	416	424	432	440	448	456	464	472
543	480	488	496	504	512	520	528	536	544	552
544	560	568	576	584	592	600	608	616	624	632
545	640	648	656	664	672	679	687	695	703	711
546	719	727	735	743	751	759	767	775	783	791
547	799	807	815	823	830	838	846	854	862	870
548	878	886	894	902	910	918	926	933	941	949
549	957	965	973	981	989	997	*005	*013	*020	*028
550	74 036	044	052	060	068	076	084	092	099	107

Prop. Parts

	9	8	7
1	0.9	0.8	0.7
2	1.8	1.6	1.4
3	2.7	2.4	2.1
4	3.6	3.2	2.8
5	4.5	4.0	3.5
6	5.4	4.8	4.2
7	6.3	5.6	4.9
8	7.2	6.4	5.6
9	8.1	7.2	6.3

		S.	T.
0° 8′	= 480″	4.68 557	4.68 558
0 9	= 540	557	558
0 10	= 600	557	558
1 23	= 4980	553	566
1 24	= 5040	553	566
1 25	= 5100	553	566
1° 26′	= 5160″	4.68 553	4.68 567
1 27	= 5220	553	567
1 28	= 5280	553	567
1 29	= 5340	553	567
1 30	= 5400	553	567
1 31	= 5460	552	568
1 32	= 5520	552	568

N.	L. 0	1	2	3	4	5	6	7	8	9
550	74 036	044	052	060	068	076	084	092	099	107
551	115	123	131	139	147	155	162	170	178	186
552	194	202	210	218	225	233	241	249	257	265
553	273	280	288	296	304	312	320	327	335	343
554	351	359	367	374	382	390	398	406	414	421
555	429	437	445	453	461	468	476	484	492	500
556	507	515	523	531	539	547	554	562	570	578
557	586	593	601	609	617	624	632	640	648	656
558	663	671	679	687	695	702	710	718	726	733
559	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	850	858	865	873	881	889
561	896	904	912	920	927	935	943	950	958	966
562	974	981	989	997	*005	*012	*020	*028	*035	*043
563	75 051	059	066	074	082	089	097	105	113	120
564	128	136	143	151	159	166	174	182	189	197
565	205	213	220	228	236	243	251	259	266	274
566	282	289	297	305	312	320	328	335	343	351
567	358	366	374	381	389	397	404	412	420	427
568	435	442	450	458	465	473	481	488	496	504
569	511	519	526	534	542	549	557	565	572	580
570	587	595	603	610	618	626	633	641	648	656
571	664	671	679	686	694	702	709	717	724	732
572	740	747	755	762	770	778	785	793	800	808
573	815	823	831	838	846	853	861	868	876	884
574	891	899	906	914	921	929	937	944	952	959
575	967	974	982	989	997	*005	*012	*020	*027	*035
576	76 042	050	057	065	072	080	087	095	103	110
577	118	125	133	140	148	155	163	170	178	185
578	193	200	208	215	223	230	238	245	253	260
579	268	275	283	290	298	305	313	320	328	335
580	343	350	358	365	373	380	388	395	403	410
581	418	425	433	440	448	455	462	470	477	485
582	492	500	507	515	522	530	537	545	552	559
583	567	574	582	589	597	604	612	619	626	634
584	641	649	656	664	671	678	686	693	701	708
585	716	723	730	738	745	753	760	768	775	782
586	790	797	805	812	819	827	834	842	849	856
587	864	871	879	886	893	901	908	916	923	930
588	938	945	953	960	967	975	982	989	997	*004
589	77 012	019	026	034	041	048	056	063	070	078
590	085	093	100	107	115	122	129	137	144	151
591	159	166	173	181	188	195	203	210	217	225
592	232	240	247	254	262	269	276	283	291	298
593	305	313	320	327	335	342	349	357	364	371
594	379	386	393	401	408	415	422	430	437	444
595	452	459	466	474	481	488	495	503	510	517
596	525	532	539	546	554	561	568	576	583	590
597	597	605	612	619	627	634	641	648	656	663
598	670	677	685	692	699	706	714	721	728	735
599	743	750	757	764	772	779	786	793	801	808
600	815	822	830	837	844	851	859	866	873	880

Prop. Parts

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

		S.	T.
0° 9′	= 540″	4.68 557	4.68 558
0 10	= 600	557	558
1 31	= 5460	552	568
1 32	= 5520	552	568
1 33	= 5580	552	568
1 34	= 5640	552	568
1° 35′	= 5700″	4.68 552	4.68 569
1 36	= 5760	552	569
1 37	= 5820	552	569
1 38	= 5880	552	569
1 39	= 5940	551	569
1 40	= 6000	551	570

N.	L. 0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
601	887	895	902	909	916	924	931	938	945	952
602	960	967	974	981	988	996	*003	*010	*017	*025
603	78 032	039	046	053	061	068	075	082	089	097
604	104	111	118	125	132	140	147	154	161	168
605	176	183	190	197	204	211	219	226	233	240
606	247	254	262	269	276	283	290	297	305	312
607	319	326	333	340	347	355	362	369	376	383
608	390	398	405	412	419	426	433	440	447	455
609	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
611	604	611	618	625	633	640	647	654	661	668
612	675	682	689	696	704	711	718	725	732	739
613	746	753	760	767	774	781	789	796	803	810
614	817	824	831	838	845	852	859	866	873	880
615	888	895	902	909	916	923	930	937	944	951
616	958	965	972	979	986	993	*000	*007	*014	*021
617	79 029	036	043	050	057	064	071	078	085	092
618	099	106	113	120	127	134	141	148	155	162
619	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
621	309	316	323	330	337	344	351	358	365	372
622	379	386	393	400	407	414	421	428	435	442
623	449	456	463	470	477	484	491	498	505	511
624	518	525	532	539	546	553	560	567	574	581
625	588	595	602	609	616	623	630	637	644	650
626	657	664	671	678	685	692	699	706	713	720
627	727	734	741	748	754	761	768	775	782	789
628	796	803	810	817	824	831	837	844	851	858
629	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955	962	969	975	982	989	996
631	80 003	010	017	024	030	037	044	051	058	065
632	072	079	085	092	099	106	113	120	127	134
633	140	147	154	161	168	175	182	188	195	202
634	209	216	223	229	236	243	250	257	264	271
635	277	284	291	298	305	312	318	325	332	339
636	346	353	359	366	373	380	387	393	400	407
637	414	421	428	434	441	448	455	462	468	475
638	482	489	496	502	509	516	523	530	536	543
639	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
641	686	693	699	706	713	720	726	733	740	747
642	754	760	767	774	781	787	794	801	808	814
643	821	828	835	841	848	855	862	868	875	882
644	889	895	902	909	916	922	929	936	943	949
645	956	963	969	976	983	990	996	*003	*010	*017
646	81 023	030	037	043	050	057	064	070	077	084
647	090	097	104	111	117	124	131	137	144	151
648	158	164	171	178	184	191	198	204	211	218
649	224	231	238	245	251	258	265	271	278	285
650	291	298	305	311	318	325	331	338	345	351

Prop. Parts

	8	7	6
1	0.8	0.7	0.6
2	1.6	1.4	1.2
3	2.4	2.1	1.8
4	3.2	2.8	2.4
5	4.0	3.5	3.0
6	4.8	4.2	3.6
7	5.6	4.9	4.2
8	6.4	5.6	4.8
9	7.2	6.3	5.4

Arc	S.	T.
0° 10′ = 600″	4.68 557	4.68 558
0 11 = 660	557	558
1 40 = 6000	551	570
1 41 = 6060	551	570
1 42 = 6120	551	570
1 43 = 6180	551	570
1° 44′ = 6240″	4.68 551	4.68 571
1 45 = 6300	551	571
1 46 = 6360	551	571
1 47 = 6420	550	572
1 48 = 6480	550	572
1 49 = 6540	550	572

N.	L. 0	1	2	3	4	5	6	7	8	9
650	81 291	298	305	311	318	325	331	338	345	351
651	358	365	371	378	385	391	398	405	411	418
652	425	431	438	445	451	458	465	471	478	485
653	491	498	505	511	518	525	531	538	544	551
654	558	564	571	578	584	591	598	604	611	617
655	624	631	637	644	651	657	664	671	677	684
656	690	697	704	710	717	723	730	737	743	750
657	757	763	770	776	783	790	796	803	809	816
658	823	829	836	842	849	856	862	869	875	882
659	889	895	902	908	915	921	928	935	941	948
660	954	961	968	974	981	987	994	*000	*007	*014
661	82 020	027	033	040	046	053	060	066	073	079
662	086	092	099	105	112	119	125	132	138	145
663	151	158	164	171	178	184	191	197	204	210
664	217	223	230	236	243	249	256	263	269	276
665	282	289	295	302	308	315	321	328	334	341
666	347	354	360	367	373	380	387	393	400	406
667	413	419	426	432	439	445	452	458	465	471
668	478	484	491	497	504	510	517	523	530	536
669	543	549	556	562	569	575	582	588	595	601
670	607	614	620	627	633	640	646	653	659	666
671	672	679	685	692	698	705	711	718	724	730
672	737	743	750	756	763	769	776	782	789	795
673	802	808	814	821	827	834	840	847	853	860
674	866	872	879	885	892	898	905	911	918	924
675	930	937	943	950	956	963	969	975	982	988
676	995	*001	*008	*014	*020	*027	*033	*040	*046	*052
677	83 059	065	072	078	085	091	097	104	110	117
678	123	129	136	142	149	155	161	168	174	181
679	187	193	200	206	213	219	225	232	238	245
680	251	257	264	270	276	283	289	296	302	308
681	315	321	327	334	340	347	353	359	366	372
682	378	385	391	398	404	410	417	423	429	436
683	442	448	455	461	467	474	480	487	493	499
684	506	512	518	525	531	537	544	550	556	563
685	569	575	582	588	594	601	607	613	620	626
686	632	639	645	651	658	664	670	677	683	689
687	696	702	708	715	721	727	734	740	746	753
688	759	765	771	778	784	790	797	803	809	816
689	822	828	835	841	847	853	860	866	872	879
690	885	891	897	904	910	916	923	929	935	942
691	948	954	960	967	973	979	985	992	998	*004
692	84 011	017	023	029	036	042	048	055	061	067
693	073	080	086	092	098	105	111	117	123	130
694	136	142	148	155	161	167	173	180	186	192
695	198	205	211	217	223	230	236	242	248	255
696	261	267	273	280	286	292	298	305	311	317
697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
699	448	454	460	466	473	479	485	491	497	504
700	510	516	522	528	535	541	547	553	559	566

Prop. Parts

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

Arc	S.	T.
0° 10′ = 600″	4.68 557	4.68 558
0 11 = 660	557	558
0 12 = 720	557	558
1 48 = 6480	550	572
1 49 = 6540	550	572
1 50 = 6600	550	572
1° 51′ = 6660″	4.68 550	4.68 573
1 52 = 6720	550	573
1 53 = 6780	550	573
1 54 = 6840	550	573
1 55 = 6900	549	574
1 56 = 6960	549	574
1 57 = 7020	549	574

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
700	84 510	516	522	528	535	541	547	553	559	566	
701	572	578	584	590	597	603	609	615	621	628	
702	634	640	646	652	658	665	671	677	683	689	
703	696	702	708	714	720	726	733	739	745	751	
704	757	763	770	776	782	788	794	800	807	813	
705	819	825	831	837	844	850	856	862	868	874	**7**
706	880	887	893	899	905	911	917	924	930	936	1 | 0.7
707	942	948	954	960	967	973	979	985	991	997	2 | 1.4
708	85 003	009	016	022	028	034	040	046	052	058	3 | 2.1
709	065	071	077	083	089	095	101	107	114	120	4 | 2.8
710	126	132	138	144	150	156	163	169	175	181	5 | 3.5
711	187	193	199	205	211	217	224	230	236	242	6 | 4.2
712	248	254	260	266	272	278	285	291	297	303	7 | 4.9
713	309	315	321	327	333	339	345	352	358	364	8 | 5.6
714	370	376	382	388	394	400	406	412	418	425	9 | 6.3
715	431	437	443	449	455	461	467	473	479	485	
716	491	497	503	509	516	522	528	534	540	546	
717	552	558	564	570	576	582	588	594	600	606	
718	612	618	625	631	637	643	649	655	661	667	
719	673	679	685	691	697	703	709	715	721	727	
720	733	739	745	751	757	763	769	775	781	788	
721	794	800	806	812	818	824	830	836	842	848	**6**
722	854	860	866	872	878	884	890	896	902	908	1 | 0.6
723	914	920	926	932	938	944	950	956	962	968	2 | 1.2
724	974	980	986	992	998	*004	*010	*016	*022	*028	3 | 1.8
725	86 034	040	046	052	058	064	070	076	082	088	4 | 2.4
726	094	100	106	112	118	124	130	136	141	147	5 | 3.0
727	153	159	165	171	177	183	189	195	201	207	6 | 3.6
728	213	219	225	231	237	243	249	255	261	267	7 | 4.2
729	273	279	285	291	297	303	308	314	320	326	8 | 4.8
730	332	338	344	350	356	362	368	374	380	386	9 | 5.4
731	392	398	404	410	415	421	427	433	439	445	
732	451	457	463	469	475	481	487	493	499	504	
733	510	516	522	528	534	540	546	552	558	564	
734	570	576	581	587	593	599	605	611	617	623	
735	629	635	641	646	652	658	664	670	676	682	
736	688	694	700	705	711	717	723	729	735	741	**5**
737	747	753	759	764	770	776	782	788	794	800	1 | 0.5
738	806	812	817	823	829	835	841	847	853	859	2 | 1.0
739	864	870	876	882	888	894	900	906	911	917	3 | 1.5
740	923	929	935	941	947	953	958	964	970	976	4 | 2.0
741	982	988	994	999	*005	*011	*017	*023	*029	*035	5 | 2.5
742	87 040	046	052	058	064	070	075	081	087	093	6 | 3.0
743	099	105	111	116	122	128	134	140	146	151	7 | 3.5
744	157	163	169	175	181	186	192	198	204	210	8 | 4.0
745	216	221	227	233	239	245	251	256	262	268	9 | 4.5
746	274	280	286	291	297	303	309	315	320	326	
747	332	338	344	349	355	361	367	373	379	384	
748	390	396	402	408	413	419	425	431	437	442	
749	448	454	460	466	471	477	483	489	495	500	
750	506	512	518	523	529	535	541	547	552	558	

N. L. 0 1 2 3 4 5 6 7 8 9 Prop. Parts

0° 11′ = 660″	S. 4.68 557	T. 4.68 558	
0 12 = 720	557	558	
0 13 = 780	557	558	
1 56 = 6960	549	574	
1 57 = 7020	549	574	
1 58 = 7080	549	575	

1° 59′ = 7140″	S. 4.68 549	T. 4.68 575
2 0 = 7200	549	575
2 1 = 7260	549	575
2 2 = 7320	548	576
2 3 = 7380	548	576
2 4 = 7440	548	576
2 5 = 7500	548	577

N.	L. 0	1	2	3	4	5	6	7	8	9	Prop. Parts
750	87 506	512	518	523	529	535	541	547	552	558	
751	564	570	576	581	587	593	599	604	610	616	
752	622	628	633	639	645	651	656	662	668	674	
753	679	685	691	697	703	708	714	720	726	731	
754	737	743	749	754	760	766	772	777	783	789	
755	795	800	806	812	818	823	829	835	841	846	
756	852	858	864	869	875	881	887	892	898	904	
757	910	915	921	927	933	938	944	950	955	961	
758	967	973	978	984	990	996	*001	*007	*013	*018	
759	88 024	030	036	041	047	053	058	064	070	076	
760	081	087	093	098	104	110	116	121	127	133	
761	138	144	150	156	161	167	173	178	184	190	**6**
762	195	201	207	213	218	224	230	235	241	247	1 | 0.6
763	252	258	264	270	275	281	287	292	298	304	2 | 1.2
764	309	315	321	326	332	338	343	349	355	360	3 | 1.8
765	366	372	377	383	389	395	400	406	412	417	4 | 2.4
766	423	429	434	440	446	451	457	463	468	474	5 | 3.0
767	480	485	491	497	502	508	513	519	525	530	6 | 3.6
768	536	542	547	553	559	564	570	576	581	587	7 | 4.2
769	593	598	604	610	615	621	627	632	638	643	8 | 4.8
770	649	655	660	666	672	677	683	689	694	700	9 | 5.4
771	705	711	717	722	728	734	739	745	750	756	
772	762	767	773	779	784	790	795	801	807	812	
773	818	824	829	835	840	846	852	857	863	868	
774	874	880	885	891	897	902	908	913	919	925	
775	930	936	941	947	953	958	964	969	975	981	
776	986	992	997	*003	*009	*014	*020	*025	*031	*037	
777	89 042	048	053	059	064	070	076	081	087	092	
778	098	104	109	115	120	126	131	137	143	148	
779	154	159	165	170	176	182	187	193	198	204	
780	209	215	221	226	232	237	243	248	254	260	
781	265	271	276	282	287	293	298	304	310	315	**5**
782	321	326	332	337	343	348	354	360	365	371	1 | 0.5
783	376	382	387	393	398	404	409	415	421	426	2 | 1.0
784	432	437	443	448	454	459	465	470	476	481	3 | 1.5
785	487	492	498	504	509	515	520	526	531	537	4 | 2.0
786	542	548	553	559	564	570	575	581	586	592	5 | 2.5
787	597	603	609	614	620	625	631	636	642	647	6 | 3.0
788	653	658	664	669	675	680	686	691	697	702	7 | 3.5
789	708	713	719	724	730	735	741	746	752	757	8 | 4.0
790	763	768	774	779	785	790	796	801	807	812	9 | 4.5
791	818	823	829	834	840	845	851	856	862	867	
792	873	878	883	889	894	900	905	911	916	922	
793	927	933	938	944	949	955	960	966	971	977	
794	982	988	993	998	*004	*009	*015	*020	*026	*031	
795	90 037	042	048	053	059	064	069	075	080	086	
796	091	097	102	108	113	119	124	129	135	140	
797	146	151	157	162	168	173	179	184	189	195	
798	200	206	211	217	222	227	233	238	244	249	
799	255	260	266	271	276	282	287	293	298	304	
800	309	314	320	325	331	336	342	347	352	358	

N. L. 0 1 2 3 4 5 6 7 8 9 Prop. Parts

0° 12′ = 720″	S. 4.68 557	T. 4.68 558
0 13 = 780	557	558
0 14 = 840	557	558
2 5 = 7500	548	577
2 6 = 7560	548	577
2 7 = 7620	548	577

2° 8′ = 7680″	S. 4.68 547	T. 4.68 578
2 9 = 7740	547	578
2 10 = 7800	547	578
2 11 = 7860	547	579
2 12 = 7920	547	579
2 13 = 7980	547	579
2 14 = 8040	546	579

N.	L. 0	1	2	3	4	5	6	7	8	9
800	90 309	314	320	325	331	336	342	347	352	358
801	363	369	374	380	385	390	396	401	407	412
802	417	423	428	434	439	445	450	455	461	466
803	472	477	482	488	493	499	504	509	515	520
804	526	531	536	542	547	553	558	563	569	574
805	580	585	590	596	601	607	612	617	623	628
806	634	639	644	650	655	660	666	671	677	682
807	687	693	698	703	709	714	720	725	730	736
808	741	747	752	757	763	768	773	779	784	789
809	795	800	806	811	816	822	827	832	838	843
810	849	854	859	865	870	875	881	886	891	897
811	902	907	913	918	924	929	934	940	945	950
812	956	961	966	972	977	982	988	993	998	*004
813	91 009	014	020	025	030	036	041	046	052	057
814	062	068	073	078	084	089	094	100	105	110
815	116	121	126	132	137	142	148	153	158	164
816	169	174	180	185	190	196	201	206	212	217
817	222	228	233	238	243	249	254	259	265	270
818	275	281	286	291	297	302	307	312	318	323
819	328	334	339	344	350	355	360	365	371	376
820	381	387	392	397	403	408	413	418	424	429
821	434	440	445	450	455	461	466	471	477	482
822	487	492	498	503	508	514	519	524	529	535
823	540	545	551	556	561	566	572	577	582	587
824	593	598	603	609	614	619	624	630	635	640
825	645	651	656	661	666	672	677	682	687	693
826	698	703	709	714	719	724	730	735	740	745
827	751	756	761	766	772	777	782	787	793	798
828	803	808	814	819	824	829	834	840	845	850
829	855	861	866	871	876	882	887	892	897	903
830	908	913	918	924	929	934	939	944	950	955
831	960	965	971	976	981	986	991	997	*002	*007
832	92 012	018	023	028	033	038	044	049	054	059
833	065	070	075	080	085	091	096	101	106	111
834	117	122	127	132	137	143	148	153	158	163
835	169	174	179	184	189	195	200	205	210	215
836	221	226	231	236	241	247	252	257	262	267
837	273	278	283	288	293	298	304	309	314	319
838	324	330	335	340	345	350	355	361	366	371
839	376	381	387	392	397	402	407	412	418	423
840	428	433	438	443	449	454	459	464	469	474
841	480	485	490	495	500	505	511	516	521	526
842	531	536	542	547	552	557	562	567	572	578
843	583	588	593	598	603	609	614	619	624	629
844	634	639	645	650	655	660	665	670	675	681
845	686	691	696	701	706	711	716	722	727	732
846	737	742	747	752	758	763	768	773	778	783
847	788	793	799	804	809	814	819	824	829	834
848	840	845	850	855	860	865	870	875	881	886
849	891	896	901	906	911	916	921	927	932	937
850	942	947	952	957	962	967	973	978	983	988
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

		S. 4.68	T. 4.68
0° 13′	= 780″	557	558
0 14	= 840	557	558
0 15	= 900	557	558
2 13	= 7980	547	579
2 14	= 8040	546	579
2 15	= 8100	546	580
2° 16′	= 8160″	546	580
2 17	= 8220	546	580
2 18	= 8280	546	581
2 19	= 8340	546	581
2 20	= 8400	545	582
2 21	= 8460	545	582
2 22	= 8520	545	582

N.	L. 0	1	2	3	4	5	6	7	8	9
850	92 942	947	952	957	962	967	973	978	983	988
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039
852	93 044	049	054	059	064	069	075	080	085	090
853	095	100	105	110	115	120	125	131	136	141
854	146	151	156	161	166	171	176	181	186	192
855	197	202	207	212	217	222	227	232	237	242
856	247	252	258	263	268	273	278	283	288	293
857	298	303	308	313	318	323	328	334	339	344
858	349	354	359	364	369	374	379	384	389	394
859	399	404	409	414	420	425	430	435	440	445
860	450	455	460	465	470	475	480	485	490	495
861	500	505	510	515	520	526	531	536	541	546
862	551	556	561	566	571	576	581	586	591	596
863	601	606	611	616	621	626	631	636	641	646
864	651	656	661	666	671	676	682	687	692	697
865	702	707	712	717	722	727	732	737	742	747
866	752	757	762	767	772	777	782	787	792	797
867	802	807	812	817	822	827	832	837	842	847
868	852	857	862	867	872	877	882	887	892	897
869	902	907	912	917	922	927	932	937	942	947
870	952	957	962	967	972	977	982	987	992	997
871	94 002	007	012	017	022	027	032	037	042	047
872	052	057	062	067	072	077	082	086	091	096
873	101	106	111	116	121	126	131	136	141	146
874	151	156	161	166	171	176	181	186	191	196
875	201	206	211	216	221	226	231	236	240	245
876	250	255	260	265	270	275	280	285	290	295
877	300	305	310	315	320	325	330	335	340	345
878	349	354	359	364	369	374	379	384	389	394
879	399	404	409	414	419	424	429	433	438	443
880	448	453	458	463	468	473	478	483	488	493
881	498	503	507	512	517	522	527	532	537	542
882	547	552	557	562	567	571	576	581	586	591
883	596	601	606	611	616	621	626	630	635	640
884	645	650	655	660	665	670	675	680	685	689
885	694	699	704	709	714	719	724	729	734	738
886	743	748	753	758	763	768	773	778	783	787
887	792	797	802	807	812	817	822	827	832	836
888	841	846	851	856	861	866	871	876	880	885
889	890	895	900	905	910	915	919	924	929	934
890	939	944	949	954	959	963	968	973	978	983
891	988	993	998	*002	*007	*012	*017	*022	*027	*032
892	95 036	041	046	051	056	061	066	071	075	080
893	085	090	095	100	105	109	114	119	124	129
894	134	139	143	148	153	158	163	168	173	177
895	182	187	192	197	202	207	211	216	221	226
896	231	236	240	245	250	255	260	265	270	274
897	279	284	289	294	299	303	308	313	318	323
898	328	332	337	342	347	352	357	361	366	371
899	376	381	386	390	395	400	405	410	415	419
900	424	429	434	439	444	448	453	458	463	468
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

		S. 4.68	T. 4.68
0° 14′	= 840″	557	558
0 15	= 900	557	558
2 21	= 8460	545	582
2 22	= 8520	545	582
2 23	= 8580	545	583
2 24	= 8640	545	583
2° 25′	= 8700″	545	583
2 26	= 8760	544	584
2 27	= 8820	544	584
2 28	= 8880	544	584
2 29	= 8940	544	585
2 30	= 9000	544	585

N.	L. 0	1	2	3	4	5	6	7	8	9
900	95 424	429	434	439	444	448	453	458	463	468
901	472	477	482	487	492	497	501	506	511	516
902	521	525	530	535	540	545̄	550̄	554	559	564
903	569	574	578	583	588	593	598	602	607	612
904	617	622	626	631	636	641	646	650	655	660
905	665̄	670	674	679	684	689	694	698	703	708
906	713	718	722	727	732	7[illegible]	742	746	751	756
907	761	766	770	775	780	785̄	789	794	799	804
908	809	813	818	823	828	832	837	842	847	852
909	856	861	866	871	875	880	885	890	895̄	899
910	904	909	914	918	923	928	933	938	942	947
911	952	957	961	966	971	976	980	985	990	995̄
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042
913	96 047	052	057	061	066	071	076	080	085	090
914	095̄	099	104	109	114	118	123	128	133	137
915	142	147	152	156	161	166	171	175	180	185̄
916	190	194	199	204	209	213	218	223	227	232
917	237	242	246	251	256	261	265	270	275̄	280
918	284	289	294	298	303	308	313	317	322	327
919	332	336	341	346	350	355	360	365̄	369	374
920	379	384	388	393	398	402	407	412	417	421
921	426	431	435	440	445̄	450̄	454	459	464	468
922	473	478	483	487	492	497	501	506	511	515
923	520	525̄	530	534	539	544	548	553	558	562
924	567	572	577	581	586	591	595	600	605̄	609
925	614	619	624	628	633	638	642	647	652	656
926	661	666	670	675	680	685̄	689	694	699	703
927	708	713	717	722	727	731	736	741	745	750
928	755̄	759	764	769	774	778	783	788	792	797
929	802	806	811	816	820	825̄	830	834	839	844
930	848	853	858	862	867	872	876	881	886	890
931	895̄	900	904	909	914	918	923	928	932	937
932	942	946	951	956	960	965̄	970	974	979	984
933	988	993	997	*002	*007	*011	*016	*021	*025	*030
934	97 035̄	039	044	049	053	058	063	067	072	077
935	081	086	090	095	100	104	109	114	118	123
936	128	132	137	142	146	151	155	160	165̄	169
937	174	179	183	188	192	197	202	206	211	216
938	220	225̄	230	234	239	243	248	253	257	262
939	267	271	276	280	285	290	294	299	304	308
940	313	317	322	327	331	336	340	345	350̄	354
941	359	364	368	373	377	382	387	391	396	400
942	405	410	414	419	424	428	433	437	442	447
943	451	456	460	465̄	470	474	479	483	488	493
944	497	502	506	511	516	520	525̄	529	534	539
945	543	548	552	557	562	566	571	575	580	585̄
946	589	594	598	603	607	612	617	621	626	630
947	635̄	640	644	649	653	658	663	667	672	676
948	681	685	690	695̄	699	704	708	713	717	722
949	727	731	736	740	745̄	749	754	759	763	768
950	772	777	782	786	791	795	800	804	809	813
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

		S.	T.
0° 15′	= 900″	4.68 557	4.68 558
0 16	= 960	557	558
2 30	= 9000	544	585
2 31	= 9060	544	585
2 32	= 9120	543	586
2 33	= 9180	543	586
2° 34′	= 9240″	4.68 543	4.68 587
2 35	= 9300	543	587
2 36	= 9360	543	587
2 37	= 9420	542	588
2 38	= 9480	542	588
2 39	= 9540	542	588

N.	L. 0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
951	818	823	827	832	836	841	845	850	855̄	859
952	864	868	873	877	882	886	891	896	900	905̄
953	909	914	918	923	928	932	937	941	946	950
954	955̄	959	964	968	973	978	982	987	991	996
955	98 000	005̄	009	014	019	023	028	032	037	041
956	046	050	055̄	059	064	068	073	078	082	087
957	091	096	100	105̄	109	114	118	123	127	132
958	137	141	146	150	155̄	159	164	168	173	177
959	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	250̄	254	259	263	268
961	272	277	281	286	290	295̄	299	304	308	313
962	318	322	327	331	336	340	345̄	349	354	358
963	363	367	372	376	381	385	390	394	399	403
964	408	412	417	421	426	430	435̄	439	444	448
965	453	457	462	466	471	475	480	484	489	493
966	498	502	507	511	516	520	525̄	529	534	538
967	543	547	552	556	561	565	570	574	579	583
968	588	592	597	601	605	610	614	619	623	628
969	632	637	641	646	650	655̄	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
971	722	726	731	735	740	744	749	753	758	762
972	767	771	776	780	784	789	793	798	802	807
973	811	816	820	825̄	829	834	838	843	847	851
974	856	860	865̄	869	874	878	883	887	892	896
975	900	905̄	909	914	918	923	927	932	936	941
976	945̄	949	954	958	963	967	972	976	981	985
977	989	994	998	*003	*007	*012	*016	*021	*025	*029
978	99 034	038	043	047	052	056	061	065̄	069	074
979	078	083	087	092	096	100	105̄	109	114	118
980	123	127	131	136	140	145̄	149	154	158	162
981	167	171	176	180	185̄	189	193	198	202	207
982	211	216	220	224	229	233	238	242	247	251
983	255	260	264	269	273	277	282	286	291	295
984	300	304	308	313	317	322	326	330	335̄	339
985	344	348	352	357	361	366	370	374	379	383
986	388	392	396	401	405	410	414	419	423	427
987	432	436	441	445̄	449	454	458	463	467	471
988	476	480	484	489	493	498	502	506	511	515
989	520	524	528	533	537	542	546	550	555̄	559
990	564	568	572	577	581	585	590	594	599	603
991	607	612	616	621	625̄	629	634	638	642	647
992	651	656	660	664	669	673	677	682	686	691
993	695̄	699	704	708	712	717	721	726	730	734
994	739	743	747	752	756	760	765̄	769	774	778
995	782	787	791	795	800	804	808	813	817	822
996	826	830	835̄	839	843	848	852	856	861	865
997	870	874	878	883	887	891	896	900	904	909
998	913	917	922	926	930	935̄	939	944	948	952
999	957	961	965	970	974	978	983	987	991	996
1000	00 000	004	009	013	017	022	026	030	035̄	039
N.	L. 0	1	2	3	4	5	6	7	8	9

Prop. Parts

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

		S.	T.
0° 15′	= 900″	4.68 557	4.68 558
0 16	= 960	557	558
0 17	= 1020	557	558
2 38	= 9480	542	588
2 39	= 9540	542	588
2 40	= 9600	542	589
2° 41′	= 9660″	4.68 542	4.68 589
2 42	= 9720	541	590
2 43	= 9780	541	590
2 44	= 9840	541	590
2 45	= 9900	541	591
2 46	= 9960	541	591
2 47	= 10020	540	592

N.	L.	0	1	2	3	4	5	6	7	8	9
1000	000	0000	0434	0869	1303	1737	2171	2605	3039	3473	3907
1001		4341	4775	5208	5642	6076	6510	6943	7377	7810	8244
1002		8677	9111	9544	9977	*0411	*0844	*1277	*1710	*2143	*2576
1003	001	3009	3442	3875	4308	4741	5174	5607	6039	6472	6905
1004		7337	7770	8202	8635	9067	9499	9932	*0364	*0796	*1228
1005	002	1661	2093	2525	2957	3389	3821	4253	4685	5116	5548
1006		5980	6411	6843	7275	7706	8138	8569	9001	9432	9863
1007	003	0295	0726	1157	1588	2019	2451	2882	3313	3744	4174
1008		4605	5036	5467	5898	6328	6759	7190	7620	8051	8481
1009		8912	9342	9772	*0202	*0633	*1063	*1493	*1924	*2354	*2784
1010	004	3214	3644	4074	4504	4933	5363	5793	6223	6652	7082
1011		7512	7941	8371	8800	9229	9659	*0088	*0517	*0947	*1376
1012	005	1805	2234	2663	3092	3521	3950	4379	4808	5237	5666
1013		6094	6523	6952	7380	7809	8238	8666	9094	9523	9951
1014	006	0380	0808	1236	1664	2092	2521	2949	3377	3805	4233
1015		4660	5088	5516	5944	6372	6799	7227	7655	8082	8510
1016		8937	9365	9792	*0219	*0647	*1074	*1501	*1928	*2355	*2782
1017	007	3210	3637	4064	4490	4917	5344	5771	6198	6624	7051
1018		7478	7904	8331	8757	9184	9610	*0037	*0463	*0889	*1316
1019	008	1742	2168	2594	3020	3446	3872	4298	4724	5150	5576
1020		6002	6427	6853	7279	7704	8130	8556	8981	9407	9832
1021	009	0257	0683	1108	1533	1959	2384	2809	3234	3659	4084
1022		4509	4934	5359	5784	6208	6633	7058	7483	7907	8332
1023		8756	9181	9605	*0030	*0454	*0878	*1303	*1727	*2151	*2575
1024	010	3000	3424	3848	4272	4696	5120	5544	5967	6391	6815
1025		7239	7662	8086	8510	8933	9357	9780	*0204	*0627	*1050
1026	011	1474	1897	2320	2743	3166	3590	4013	4436	4859	5282
1027		5704	6127	6550	6973	7396	7818	8241	8664	9086	9509
1028		9931	*0354	*0776	*1198	*1621	*2043	*2465	*2887	*3310	*3732
1029	012	4154	4576	4998	5420	5842	6264	6685	7107	7529	7951
1030		8372	8794	9215	9637	*0059	*0480	*0901	*1323	*1744	*2165
1031	013	2587	3008	3429	3850	4271	4692	5113	5534	5955	6376
1032		6797	7218	7639	8059	8480	8901	9321	9742	*0162	*0583
1033	014	1003	1424	1844	2264	2685	3105	3525	3945	4365	4785
1034		5205	5625	6045	6465	6885	7305	7725	8144	8564	8984
1035		9403	9823	*0243	*0662	*1082	*1501	*1920	*2340	*2759	*3178
1036	015	3598	4017	4436	4855	5274	5693	6112	6531	6950	7369
1037		7788	8206	8625	9044	9462	9881	*0300	*0718	*1137	*1555
1038	016	1974	2392	2810	3229	3647	4065	4483	4901	5319	5737
1039		6155	6573	6991	7409	7827	8245	8663	9080	9498	9916
1040	017	0333	0751	1168	1586	2003	2421	2838	3256	3673	4090
1041		4507	4924	5342	5759	6176	6593	7010	7427	7844	8260
1042		8677	9094	9511	9927	*0344	*0761	*1177	*1594	*2010	*2427
1043	018	2843	3259	3676	4092	4508	4925	5341	5757	6173	6589
1044		7005	7421	7837	8253	8669	9084	9500	9916	*0332	*0747
1045	019	1163	1578	1994	2410	2825	3240	3656	4071	4486	4902
1046		5317	5732	6147	6562	6977	7392	7807	8222	8637	9052
1047		9467	9882	*0296	*0711	*1126	*1540	*1955	*2369	*2784	*3198
1048	020	3613	4027	4442	4856	5270	5684	6099	6513	6927	7341
1049		7755	8169	8583	8997	9411	9824	*0238	*0652	*1066	*1479
1050	021	1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
N.	L.	0	1	2	3	4	5	6	7	8	9

2° 46′ = 9 960″	S. 4.68 541	T. 4.68 591		2° 51′ = 10 260″	S. 4.68 540	T. 4.68 593
2 47 = 10 020	540	592		2 52 = 10 320	539	594
2 48 = 10 080	540	592		2 53 = 10 380	539	594
2 49 = 10 140	540	592		2 54 = 10 440	539	595
2 50 = 10 200	540	593		2 55 = 10 500	539	595

N.	L.	0	1	2	3	4	5	6	7	8	9
1050	021	1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
1051		6027	6440	6854	7267	7680	8093	8506	8919	9332	9745
1052	022	0157	0570	0983	1396	1808	2221	2634	3046	3459	3871
1053		4284	4696	5109	5521	5933	6345	6758	7170	7582	7994
1054		8406	8818	9230	9642	*0054	*0466	*0878	*1289	*1701	*2113
1055	023	2525	2936	3348	3759	4171	4582	4994	5405	5817	6228
1056		6639	7050	7462	7873	8284	8695	9106	9517	9928	*0339
1057	024	0750	1161	1572	1982	2393	2804	3214	3625	4036	4446
1058		4857	5267	5678	6088	6498	6909	7319	7729	8139	8549
1059		8960	9370	9780	*0190	*0600	*1010	*1419	*1829	*2239	*2649
1060	025	3059	3468	3878	4288	4697	5107	5516	5926	6335	6744
1061		7154	7563	7972	8382	8791	9200	9609	*0018	*0427	*0836
1062	026	1245	1654	2063	2472	2881	3289	3698	4107	4515	4924
1063		5333	5741	6150	6558	6967	7375	7783	8192	8600	9008
1064		9416	9824	*0233	*0641	*1049	*1457	*1865	*2273	*2680	*3088
1065	027	3496	3904	4312	4719	5127	5535	5942	6350	6757	7165
1066		7572	7979	8387	8794	9201	9609	*0016	*0423	*0830	*1237
1067	028	1644	2051	2458	2865	3272	3679	4086	4492	4899	5306
1068		5713	6119	6526	6932	7339	7745	8152	8558	8964	9371
1069		9777	*0183	*0590	*0996	*1402	*1808	*2214	*2620	*3026	*3432
1070	029	3838	4244	4649	5055	5461	5867	6272	6678	7084	7489
1071		7895	8300	8706	9111	9516	9922	*0327	*0732	*1138	*1543
1072	030	1948	2353	2758	3163	3568	3973	4378	4783	5188	5592
1073		5997	6402	6807	7211	7616	8020	8425	8830	9234	9638
1074	031	0043	0447	0851	1256	1660	2064	2468	2872	3277	3681
1075		4085	4489	4893	5296	5700	6104	6508	6912	7315	7719
1076		8123	8526	8930	9333	9737	*0140	*0544	*0947	*1350	*1754
1077	032	2157	2560	2963	3367	3770	4173	4576	4979	5382	5785
1078		6188	6590	6993	7396	7799	8201	8604	9007	9409	9812
1079	033	0214	0617	1019	1422	1824	2226	2629	3031	3433	3835
1080		4238	4640	5042	5444	5846	6248	6650	7052	7453	7855
1081		8257	8659	9060	9462	9864	*0265	*0667	*1068	*1470	*1871
1082	034	2273	2674	3075	3477	3878	4279	4680	5081	5482	5884
1083		6285	6686	7087	7487	7888	8289	8690	9091	9491	9892
1084	035	0293	0693	1094	1495	1895	2296	2696	3096	3497	3897
1085		4297	4698	5098	5498	5898	6298	6698	7098	7498	7898
1086		8298	8698	9098	9498	9898	*0297	*0697	*1097	*1496	*1896
1087	036	2295	2695	3094	3494	3893	4293	4692	5091	5491	5890
1088		6289	6688	7087	7486	7885	8284	8683	9082	9481	9880
1089	037	0279	0678	1076	1475	1874	2272	2671	3070	3468	3867
1090		4265	4663	5062	5460	5858	6257	6655	7053	7451	7849
1091		8248	8646	9044	9442	9839	*0237	*0635	*1033	*1431	*1829
1092	038	2226	2624	3022	3419	3817	4214	4612	5009	5407	5804
1093		6202	6599	6996	7393	7791	8188	8585	8982	9379	9776
1094	039	0173	0570	0967	1364	1761	2158	2554	2951	3348	3745
1095		4141	4538	4934	5331	5727	6124	6520	6917	7313	7709
1096		8106	8502	8898	9294	9690	*0086	*0482	*0878	*1274	*1670
1097	040	2066	2462	2858	3254	3650	4045	4441	4837	5232	5628
1098		6023	6419	6814	7210	7605	8001	8396	8791	9187	9582
1099		9977	*0372	*0767	*1162	*1557	*1952	*2347	*2742	*3137	*3532
1100	041	3927	4322	4716	5111	5506	5900	6295	6690	7084	7479
N.	L.	0	1	2	3	4	5	6	7	8	9

2° 55′ = 10 500″	S. 4.68 539	T. 4.68 595		3° 0′ = 10 800″	S. 4.68 538	T. 4.68 597
2 56 = 10 560	539	595		3 1 = 10 860	537	598
2 57 = 10 620	538	596		3 2 = 10 920	537	598
2 58 = 10 680	538	596		3 3 = 10 980	537	599
2 59 = 10 740	538	597		3 4 = 11 040	537	599

TABLE II.

Base of common (Briggs) logarithms = 10.
Base of natural (Napierian) logarithms (e) = 2.718281828459 · · · ·
Modulus of Com. Logs. = $\log_{10}e = M$ = 0.4342944819 · · · .

$$\frac{1}{M} = \log_e 10 = 2.30258509299 \cdots$$

$$\log_{10}N = M \times \log_e N. \qquad \log_e N = \frac{1}{M} \times \log_{10}N.$$

Multiples of M To Convert from Nat. to Com. Logs				Multiples of $\frac{1}{M}$ To Convert from Com. to Nat. Logs			
0	0.00 000 000	**50**	21.71 472 410	**0**	0.00 000 000	**50**	115.12 925 46$\bar{5}$
1	0.43 429 448	51	22.14 901 858	1	2.30 258 509	51	117.43 183 974
2	0.86 858 896	52	22.58 331 306	2	4.60 517 019	52	119.73 442 484
3	1.30 288 34$\bar{5}$	53	23.01 760 754	3	6.90 775 528	53	122.03 700 993
4	1.73 717 793	54	23.45 190 202	4	9.21 034 037	54	124.33 959 502
5	2.17 147 241	**55**	23.88 619 650	**5**	11.51 292 546	**55**	126.64 218 011
6	2.60 576 689	56	24.32 049 099	6	13.81 551 056	56	128.94 476 521
7	3.04 006 137	57	24.75 478 547	7	16.11 809 565	57	131.24 735 030
8	3.47 435 586	58	25.18 907 995	8	18.42 068 074	58	133.54 993 539
9	3.90 865 034	59	25.62 337 443	9	20.72 326 584	59	135.85 252 049
10	4.34 294 482	**60**	26.05 766 891	**10**	23.02 585 093	**60**	138.15 510 558
11	4.77 723 930	61	26.49 196 340	11	25.32 843 602	61	140.45 769 067
12	5.21 153 378	62	26.92 625 788	12	27.63 102 112	62	142.76 027 577
13	5.64 582 826	63	27.36 055 236	13	29.93 360 621	63	145.06 286 086
14	6.08 012 27$\bar{5}$	64	27.79 484 684	14	32.23 619 130	64	147.36 544 595
15	6.51 441 723	**65**	28.22 914 132	**15**	34.53 877 639	**65**	149.66 803 104
16	6.94 871 171	66	28.66 343 581	16	36.84 136 149	66	151.97 061 614
17	7.38 300 619	67	29.09 773 029	17	39.14 394 658	67	154.27 320 123
18	7.81 730 067	68	29.53 202 477	18	41.44 653 167	68	156.57 578 632
19	8.25 159 516	69	29.96 631 925	19	43.74 911 677	69	158.87 837 142
20	8.68 588 964	**70**	30.40 061 373	**20**	46.05 170 186	**70**	161.18 095 651
21	9.12 018 412	71	30.83 490 822	21	48.35 428 695	71	163.48 354 160
22	9.55 447 860	72	31.26 920 270	22	50.65 687 20$\bar{5}$	72	165.78 612 670
23	9.98 877 308	73	31.70 349 718	23	52.95 945 714	73	168.08 871 179
24	10.42 306 757	74	32.13 779 166	24	55.26 204 223	74	170.39 129 688
25	10.85 736 20$\bar{5}$	**75**	32.57 208 614	**25**	57.56 462 732	**75**	172.69 388 197
26	11.29 165 653	76	33.00 638 062	26	59.86 721 242	76	174.99 646 707
27	11.72 595 101	77	33.44 067 511	27	62.16 979 751	77	177.29 905 216
28	12.16 024 549	78	33.87 496 959	28	64.47 238 260	78	179.60 163 725
29	12.59 453 998	79	34.30 926 407	29	66.77 496 770	79	181.90 422 23$\bar{5}$
30	13.02 883 446	**80**	34.74 355 855	**30**	69.07 755 279	**80**	184.20 680 744
31	13.46 312 894	81	35.17 785 303	31	71.38 013 788	81	186.50 939 253
32	13.89 742 342	82	35.61 214 752	32	73.68 272 298	82	188.81 197 763
33	14.33 171 790	83	36.04 644 200	33	75.98 530 807	83	191.11 456 272
34	14.76 601 238	84	36.48 073 648	34	78.28 789 316	84	193.41 714 781
35	15.20 030 687	**85**	36.91 503 096	**35**	80.59 047 825	**85**	195.71 973 290
36	15.63 460 13$\bar{5}$	86	37.34 932 544	36	82.89 306 33$\bar{5}$	86	198.02 231 800
37	16.06 889 583	87	37.78 361 993	37	85.19 564 844	87	200.32 490 309
38	16.50 319 031	88	38.21 791 441	38	87.49 823 353	88	202.62 748 818
39	16.93 748 479	89	38.65 220 889	39	89.80 081 863	89	204.93 007 328
40	17.37 177 928	**90**	39.08 650 337	**40**	92.10 340 372	**90**	207.23 265 837
41	17.80 607 376	91	39.52 079 785	41	94.40 598 881	91	209.53 524 346
42	18.24 036 824	92	39.95 509 234	42	96.70 857 391	92	211.83 782 856
43	18.67 466 272	93	40.38 938 682	43	99.01 115 900	93	214.14 041 36$\bar{5}$
44	19.10 895 720	94	40.82 368 130	44	101.31 374 409	94	216.44 299 874
45	19.54 325 169	**95**	41.25 797 578	**45**	103.61 632 918	**95**	218.74 558 383
46	19.97 754 617	96	41.69 227 026	46	105.91 891 428	96	221.04 816 893
47	20.41 184 06$\bar{5}$	97	42.12 656 474	47	108.22 149 937	97	223.35 075 402
48	20.84 613 513	98	42.56 085 923	48	110.52 408 446	98	225.65 333 911
49	21.28 042 961	99	42.99 515 371	49	112.82 666 956	99	227.95 592 421
50	21.71 472 410	**100**	43.42 944 819	**50**	115.12 925 46$\bar{5}$	**100**	230.25 850 930

TABLE III

FIVE PLACE LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

(For explanations, see pages 23 to 26.)

Pages 58 to 65 give logarithms of sines and tangents for each 10 seconds from 3° to 7° and of cosines and cotangents for each 10 seconds from 83° to 87°. For values of functions for angles less than 3° and greater than 87° the S. and T. method may be used.

Pages 66 to 110 give logarithms of sines, tangents, cosines, and cotangents for each minute from 0° to 90°.

FORMULAS FOR THE USE OF S. AND T.

(For explanations see page 26.)

(1) For α near zero degrees.

$\log \sin \alpha = \log \alpha'' + \text{S}.$
$\log \tan \alpha = \log \alpha'' + \text{T}.$
$\log \cot \alpha = \text{cpl} \log \alpha'' + \text{cpl T}.$
$\qquad = \text{cpl} \log \tan \alpha.$

$\log \alpha'' = \log \sin \alpha + \text{cpl S}.$
$\qquad = \log \tan \alpha + \text{cpl T}.$
$\qquad = \text{cpl} \log \cot \alpha + \text{cpl T}.$

(2) For α near ninety degrees.

$\log \cos \alpha = \log (90° - \alpha)'' + \text{S}.$
$\log \cot \alpha = \log (90° - \alpha)'' + \text{T}.$
$\log \tan \alpha = \text{cpl} \log (90° - \alpha)'' + \text{cpl T}.$
$\qquad = \text{cpl} \log \cot \alpha.$

$\log (90° - \alpha)'' = \log \cos \alpha + \text{cpl S}.$
$\qquad = \log \cot \alpha + \text{cpl T}.$
$\qquad = \text{cpl} \log \tan \alpha + \text{cpl T}.$

9.99	′	0″	10″	20″	30″	40″	50″	60″			d
940	**0**	8.71 880	920	960	*000	*040	*080	*120	59	940	40
940	1	8.72 120	160	200	240	280	320	359	58	939	40
939	2	359	399	439	478	518	558	597	57	938	40
938	3	597	637	676	716	75$\bar{5}$	794	834	56	938	40
938	4	834	873	912	951	991	*030	*069	**55**	937	39
937	**5**	8.73 069	108	147	186	22$\bar{5}$	264	303	54	936	39
936	6	303	342	380	419	458	497	535	53	936	39
936	7	535	574	613	651	690	728	767	52	935	39
935	8	767	805	844	882	920	959	997	51	934	38
934	9	997	*035	*073	*112	*150	*188	*226	**50**	934	38
934	**10**	8.74 226	264	302	340	378	416	454	49	933	38
933	11	454	491	529	567	60$\bar{5}$	642	680	48	932	38
932	12	680	718	755	793	831	868	906	47	932	38
932	13	906	943	980	*018	*055	*092	*130	46	931	37
931	14	8.75 130	167	204	241	279	316	353	**45**	930	37
930	**15**	353	390	427	464	501	538	47$\bar{5}$	44	929	37
929	16	57$\bar{5}$	612	648	685	722	759	795	43	929	37
929	17	795	832	869	905	942	979	*015	42	928	37
928	18	8.76 015	052	088	12$\bar{5}$	161	197	234	41	927	36
927	19	234	270	306	343	379	41$\bar{5}$	451	**40**	926	36
926	**20**	451	487	523	559	595	631	667	39	926	36
926	21	667	703	739	775	811	847	883	38	92$\bar{5}$	36
92$\bar{5}$	22	883	919	954	990	*026	*061	*097	37	924	36
924	23	8.77 097	133	168	204	239	27$\bar{5}$	310	36	923	36
923	24	310	346	381	416	452	487	522	**35**	923	35
923	**25**	522	558	593	628	663	698	733	34	922	35
922	26	733	768	803	838	873	908	943	33	921	35
921	27	943	978	*013	*048	*083	*118	*152	32	920	35
920	28	8.78 152	187	222	257	291	326	360	31	920	35
920	29	360	395	430	464	499	533	568	**30**	919	35
919	**30**	568	602	636	671	705	739	774	29	918	34
918	31	774	808	842	876	910	94$\bar{5}$	979	28	917	34
917	32	979	*013	*047	*081	*11$\bar{5}$	*149	*183	27	917	34
917	33	8.79 183	217	251	284	318	352	386	26	916	34
916	34	386	420	453	487	521	55$\bar{5}$	588	**25**	915	34
915	**35**	588	622	655	689	722	756	789	24	914	34
914	36	789	823	856	890	923	956	990	23	913	34
913	37	990	*023	*056	*090	*123	*156	*189	22	913	33
913	38	8.80 189	222	255	289	322	35$\bar{5}$	388	21	912	33
912	39	388	421	454	487	519	552	585	**20**	911	33
911	**40**	585	618	651	684	716	749	782	19	910	33
910	41	782	81$\bar{5}$	847	880	913	945	978	18	909	33
909	42	978	*010	*043	*075	*108	*140	*173	17	909	32
909	43	8.81 173	205	237	270	302	334	367	16	908	32
908	44	367	399	431	463	496	528	560	**15**	907	32
907	**45**	560	592	624	656	688	720	752	14	906	32
906	46	752	784	816	848	880	912	944	13	905	32
905	47	944	975	*007	*039	*071	*103	*134	12	904	32
904	48	8.82 134	166	198	229	261	292	324	11	904	32
904	49	324	356	387	419	450	482	513	**10**	903	32
903	**50**	513	544	576	607	639	670	701	9	902	31
902	51	701	732	764	79$\bar{5}$	826	857	888	8	901	31
901	52	888	920	951	982	*013	*044	*075	7	900	31
900	53	8.83 07$\bar{5}$	106	137	168	199	230	261	6	899	31
899	54	261	292	322	353	384	41$\bar{5}$	446	**5**	898	31
898	**55**	446	476	507	538	568	599	630	4	898	31
898	56	630	660	691	721	752	783	813	3	897	30
897	57	813	844	874	904	93$\bar{5}$	965	996	2	896	30
896	58	996	*026	*056	*087	*117	*147	*177	1	89$\bar{5}$	30
89$\bar{5}$	59	8.84 177	208	238	268	298	328	358	**0**	894	30
		60″	50″	40″	30″	20″	10″	0″	′	9.99	d

Prop. Parts

	40	39
1	4.0	3.9
2	8.0	7.8
3	12.0	11.7
4	16.0	15.6
5	20.0	19.5
6	24.0	23.4
7	28.0	27.3
8	32.0	31.2
9	36.0	35.1

	38	37
1	3.8	3.7
2	7.6	7.4
3	11.4	11.1
4	15.2	14.8
5	19.0	18.5
6	22.8	22.2
7	26.6	25.9
8	30.4	29.6
9	34.2	33.3

	36
1	3.6
2	7.2
3	10.8
4	14.4
5	18.0
6	21.6
7	25.2
8	28.8
9	32.4

	35	34
1	3.5	3.4
2	7.0	6.8
3	10.5	10.2
4	14.0	13.6
5	17.5	17.0
6	21.0	20.4
7	24.5	23.8
8	28.0	27.2
9	31.5	30.6

	33	32
1	3.3	3.2
2	6.6	6.4
3	9.9	9.6
4	13.2	12.8
5	16.5	16.0
6	19.8	19.2
7	23.1	22.4
8	26.4	25.6
9	29.7	28.8

	31	30
1	3.1	3.0
2	6.2	6.0
3	9.3	9.0
4	12.4	12.0
5	15.5	15.0
6	18.6	18.0
7	21.7	21.0
8	24.8	24.0
9	27.9	27.0

log cos **86°** L. sin

′	0″	10″	20″	30″	40″	50″	60″		d
0	8.71 940	980	*020	*060	*100	*141	*181	59	40
1	8.72 181	221	261	301	341	380	420	58	40
2	420	460	500	540	579	619	659	57	40
3	659	698	738	777	817	856	896	56	40
4	896	935	97$\bar{5}$	*014	*053	*093	*132	**55**	39
5	8.73 132	171	210	249	288	327	366	54	39
6	366	405	444	483	522	561	600	53	39
7	600	638	677	716	754	793	832	52	39
8	832	870	909	947	986	*024	*063	51	38
9	8.74 063	101	139	178	216	254	292	**50**	38
10	292	330	369	407	44$\bar{5}$	483	521	49	38
11	521	559	597	634	672	710	748	48	38
12	748	786	823	861	899	936	974	47	38
13	974	*012	*049	*087	*124	*162	*199	46	38
14	8.75 199	236	274	311	348	385	423	**45**	37
15	423	460	497	534	571	608	645	44	37
16	645	682	719	756	793	830	867	43	37
17	867	904	940	977	*014	*051	*087	42	37
18	8.76 087	124	160	197	233	270	306	41	36
19	306	343	379	416	452	488	52$\bar{5}$	**40**	36
20	52$\bar{5}$	561	597	633	669	706	742	39	36
21	742	778	814	850	886	922	958	38	36
22	958	994	*030	*065	*101	*137	*173	37	36
23	8.77 173	208	244	280	315	351	387	36	36
24	387	422	458	493	529	564	600	**35**	36
25	600	63$\bar{5}$	670	706	741	776	811	34	35
26	811	847	882	917	952	987	*022	33	35
27	8.78 022	057	092	127	162	197	232	32	35
28	232	267	302	337	371	406	441	31	35
29	441	475	510	54$\bar{5}$	579	614	649	**30**	35
30	649	683	718	752	787	821	855	29	34
31	855	890	924	958	993	*027	*061	28	34
32	8.79 061	096	130	164	198	232	266	27	34
33	266	300	334	368	402	436	470	26	34
34	470	504	538	572	606	639	673	**25**	34
35	673	707	741	774	808	842	875	24	34
36	875	909	942	976	*009	*043	*076	23	34
37	8.80 076	110	143	177	210	243	277	22	34
38	277	310	343	376	409	443	476	21	33
39	476	509	542	575	608	641	674	**20**	33
40	674	707	740	773	806	839	872	19	33
41	872	90$\bar{5}$	937	970	*003	*036	*068	18	33
42	8.81 068	101	134	166	199	232	264	17	33
43	264	297	329	362	394	427	459	16	32
44	459	491	524	556	588	621	653	**15**	32
45	653	685	717	750	782	814	846	14	32
46	846	878	910	942	974	*006	*038	13	32
47	8.82 038	070	102	134	166	198	230	12	32
48	230	262	293	325	357	389	420	11	32
49	420	452	484	515	547	579	610	**10**	32
50	610	642	673	70$\bar{5}$	736	768	799	9	32
51	799	831	862	893	92$\bar{5}$	956	987	8	31
52	987	*019	*050	*081	*112	*144	*17$\bar{5}$	7	31
53	8.83 17$\bar{5}$	206	237	268	299	330	361	6	31
54	361	392	423	454	485	516	547	**5**	31
55	547	578	609	640	671	701	732	4	31
56	732	763	794	824	855	886	916	3	31
57	916	947	978	*008	*039	*069	*100	2	31
58	8.84 100	130	161	191	222	252	282	1	30
59	282	313	343	374	404	434	464	**0**	30
	60″	50″	40″	30″	20″	10″	0″	′	d

Prop. Parts

	41	40
1	4.1	4.0
2	8.2	8.0
3	12.3	12.0
4	16.4	16.0
5	20.5	20.0
6	24.6	24.0
7	28.7	28.0
8	32.8	32.0
9	36.9	36.0

	39	38
1	3.9	3.8
2	7.8	7.6
3	11.7	11.4
4	15.6	15.2
5	19.5	19.0
6	23.4	22.8
7	27.3	26.6
8	31.2	30.4
9	35.1	34.2

	37	36
1	3.7	3.6
2	7.4	7.2
3	11.1	10.8
4	14.8	14.4
5	18.5	18.0
6	22.2	21.6
7	25.9	25.2
8	29.6	28.8
9	33.3	32.4

	35	34
1	3.5	3.4
2	7.0	6.8
3	10.5	10.2
4	14.0	13.6
5	17.5	17.0
6	21.0	20.4
7	24.5	23.8
8	28.0	27.2
9	31.5	30.6

	33	32
1	3.3	3.2
2	6.6	6.4
3	9.9	9.6
4	13.2	12.8
5	16.5	16.0
6	19.8	19.2
7	23.1	22.4
8	26.4	25.6
9	29.7	28.8

	31	30
1	3.1	3.0
2	6.2	6.0
3	9.3	9.0
4	12.4	12.0
5	15.5	15.0
6	18.6	18.0
7	21.7	21.0
8	24.8	24.0
9	27.9	27.0

log cot **86°**

9.99	′	0″	10″	20″	30″	40″	50″	60″			d
894	0	8.84 358	389	419	449	479	509	539	59	893	30
893	1	539	569	599	629	659	688	718	58	892	30
892	2	718	748	778	808	838	867	897	57	891	30
891	3	897	927	957	986	*016	*045	*075	56	891	30
891	4	8.85 075	105	134	164	193	223	252	55	890	30
890	5	252	282	311	341	370	400	429	54	889	30
889	6	429	458	488	517	546	576	605	53	888	29
888	7	605	634	663	693	722	751	780	52	887	29
887	8	780	809	838	867	896	926	955	51	886	29
886	9	955	984	*013	*042	*070	*099	*128	50	885	29
885	10	8.86 128	157	186	215	244	273	301	49	884	29
884	11	301	330	359	388	416	445	474	48	883	29
883	12	474	502	531	560	588	617	645	47	882	29
882	13	645	674	703	731	760	788	816	46	881	29
881	14	816	845	873	902	930	958	987	45	880	28
880	15	987	*015	*043	*072	*100	*128	*156	44	879	28
879	16	8.87 156	185	213	241	269	297	325	43	879	28
879	17	325	354	382	410	438	466	494	42	878	28
878	18	494	522	550	578	606	634	661	41	877	28
877	19	661	689	717	745	773	801	829	40	876	28
876	20	829	856	884	912	940	967	995	39	875	28
875	21	995	*023	*050	*078	*106	*133	*161	38	874	28
874	22	8.88 161	188	216	243	271	298	326	37	873	28
873	23	326	353	381	408	436	463	490	36	872	27
872	24	490	518	545	572	600	627	654	35	871	27
871	25	654	681	709	736	763	790	817	34	870	27
870	26	817	845	872	899	926	953	980	33	869	27
869	27	980	*007	*034	*061	*088	*115	*142	32	868	27
868	28	8.89 142	169	196	223	250	277	304	31	867	27
867	29	304	330	357	384	411	438	464	30	866	27
866	30	464	491	518	545	571	598	625	29	865	27
865	31	625	651	678	704	731	758	784	28	864	26
864	32	784	811	837	864	890	917	943	27	863	26
863	33	943	970	996	*023	*049	*075	*102	26	862	26
862	34	8.90 102	128	154	181	207	233	260	25	861	26
861	35	260	286	312	338	364	391	417	24	860	26
860	36	417	443	469	495	521	548	574	23	859	26
859	37	574	600	626	652	678	704	730	22	558	26
858	38	730	756	782	808	834	859	885	21	857	26
857	39	885	911	937	963	989	*015	*040	20	856	26
856	40	8.91 040	066	092	118	143	169	195	19	855	26
855	41	195	221	246	272	298	323	349	18	854	26
854	42	349	374	400	426	451	477	502	17	853	26
853	43	502	528	553	579	604	630	655	16	852	26
852	44	655	680	706	731	757	782	807	15	851	25
851	45	807	833	858	883	909	934	959	14	850	25
850	46	959	984	*010	*035	*060	*085	*110	13	848	25
848	47	8.92 110	135	161	186	211	236	261	12	847	25
847	48	261	286	311	336	361	386	411	11	846	25
846	49	411	436	461	486	511	536	561	10	845	25
845	50	561	586	611	636	660	685	710	9	844	25
844	51	710	735	760	784	809	834	859	8	843	25
843	52	859	883	908	933	957	982	*007	7	842	25
842	53	8.93 007	031	056	081	105	130	154	6	841	24
841	54	154	179	203	228	253	277	301	5	840	24
840	55	301	326	350	375	399	424	448	4	839	24
839	56	448	472	497	521	546	570	594	3	838	24
838	57	594	619	643	667	691	716	740	2	837	24
837	58	740	764	788	812	837	861	885	1	836	24
836	59	885	909	933	957	981	*006	*030	0	834	24
		60″	50″	40″	30″	20″	10″	0″	′	9.99	d

Prop. Parts

	31	30
1	3.1	3.0
2	6.2	6.0
3	9.3	9.0
4	12.4	12.0
5	15.5	15.0
6	18.6	18.0
7	21.7	21.0
8	24.8	24.0
9	27.9	27.0

	29
1	2.9
2	5.8
3	8.7
4	11.6
5	14.5
6	17.4
7	20.3
8	23.2
9	26.1

	28	27
1	2.8	2.7
2	5.6	5.4
3	8.4	8.1
4	11.2	10.8
5	14.0	13.5
6	16.8	16.2
7	19.6	18.9
8	22.4	21.6
9	25.2	24.3

	26
1	2.6
2	5.2
3	7.8
4	10.4
5	13.0
6	15.6
7	18.2
8	20.8
9	23.4

	25	24
1	2.5	2.4
2	5.0	4.8
3	7.5	7.2
4	10.0	9.6
5	12.5	12.0
6	15.0	14.4
7	17.5	16.8
8	20.0	19.2
9	22.5	21.6

′	0″	10″	20″	30″	40″	50″	60″		d
0	8.84 464	495	525	555	585	615	646	59	30
1	646	676	706	736	766	796	826	58	30
2	826	856	886	916	946	976	*006	57	30
3	8.85 006	036	065	095	125	155	185	56	30
4	185	214	244	274	304	333	363	55	30
5	363	392	422	452	481	511	540	54	30
6	540	570	599	629	658	688	717	53	30
7	717	747	776	805	835	864	893	52	29
8	893	922	952	981	*010	*039	*069	51	29
9	8.86 069	098	127	156	185	214	243	50	29
10	243	272	301	330	359	388	417	49	29
11	417	447	475	504	533	562	591	48	29
12	591	619	648	677	706	734	763	47	29
13	763	792	821	849	878	907	935	46	29
14	935	964	992	*021	*049	*078	*106	45	28
15	8.87 106	135	163	192	220	249	277	44	28
16	277	305	334	362	390	419	447	43	28
17	447	475	503	532	560	588	616	42	28
18	616	644	673	701	729	757	785	41	28
19	785	813	841	869	897	925	953	40	28
20	953	981	*009	*037	*065	*092	*120	39	28
21	8.88 120	148	176	204	231	259	287	38	28
22	287	315	342	370	398	425	453	37	28
23	453	481	508	536	563	591	618	36	28
24	618	646	674	701	728	756	783	35	28
25	783	811	838	866	893	920	948	34	28
26	948	975	*002	*029	*057	*084	*111	33	27
27	8.89 111	138	166	193	220	247	274	32	27
28	274	301	328	355	383	410	437	31	27
29	437	464	491	518	545	571	598	30	27
30	598	625	652	679	706	733	760	29	27
31	760	786	813	840	867	894	920	28	27
32	920	947	974	*000	*027	*054	*080	27	27
33	8.90 080	107	134	160	187	213	240	26	27
34	240	266	293	319	346	372	399	25	26
35	399	425	451	478	504	531	557	24	26
36	557	583	610	636	662	688	715	23	26
37	715	741	767	793	820	846	872	22	26
38	872	898	924	950	976	*002	*029	21	26
39	8.91 029	055	081	107	133	159	185	20	26
40	185	211	236	262	288	314	340	19	26
41	340	366	392	418	443	469	495	18	26
42	495	521	547	572	598	624	650	17	26
43	650	675	701	727	752	778	803	16	26
44	803	829	855	880	906	931	957	15	26
45	957	982	*008	*033	*059	*084	*110	14	26
46	8.92 110	135	160	186	211	237	262	13	25
47	262	287	313	338	363	388	414	12	25
48	414	439	464	489	515	540	565	11	25
49	565	590	615	640	665	691	716	10	25
50	716	741	766	791	816	841	866	9	25
51	866	891	916	941	966	991	*016	8	25
52	8.93 016	040	065	090	115	140	165	7	25
53	165	190	214	239	264	289	313	6	25
54	313	338	363	388	412	437	462	5	25
55	462	486	511	536	560	585	609	4	24
56	609	634	658	683	707	732	756	3	24
57	756	781	805	830	854	879	903	2	24
58	903	928	952	976	*001	*025	*049	1	24
59	8.94 049	074	098	122	147	171	195	0	24
	60″	50″	40″	30″	20″	10″	0″	′	d

Prop. Parts

	31	30
1	3.1	3.0
2	6.2	6.0
3	9.3	9.0
4	12.4	12.0
5	15.5	15.0
6	18.6	18.0
7	21.7	21.0
8	24.8	24.0
9	27.9	27.0

	29
1	2.9
2	5.8
3	8.7
4	11.6
5	14.5
6	17.4
7	20.3
8	23.2
9	26.1

	28	27
1	2.8	2.7
2	5.6	5.4
3	8.4	8.1
4	11.2	10.8
5	14.0	13.5
6	16.8	16.2
7	19.6	18.9
8	22.4	21.6
9	25.2	24.3

	26
1	2.6
2	5.2
3	7.8
4	10.4
5	13.0
6	15.6
7	18.2
8	20.8
9	23.4

	25	24
1	2.5	2.4
2	5.0	4.8
3	7.5	7.2
4	10.0	9.6
5	12.5	12.0
6	15.0	14.4
7	17.5	16.8
8	20.0	19.2
9	22.5	21.6

L. cos Table III **5°** log sin

9.99	′	0″	10″	20″	30″	40″	50″	60″			d
834	**0**	8.94 030	054	078	102	126	150	174	59	833	24
833	1	174	198	222	246	270	294	317	58	832	24
832	2	317	341	365	389	413	437	461	57	831	24
831	3	461	484	508	532	556	580	603	56	830	24
830	4	603	627	651	675	698	722	746	**55**	829	24
829	**5**	746	769	793	817	840	864	887	54	828	24
828	6	887	911	935	958	982	*005	*029	53	827	24
827	7	8.95 029	052	076	099	123	146	170	52	825	24
825	8	170	193	216	240	263	287	310	51	824	23
824	9	310	333	357	380	403	427	450	**50**	823	23
823	**10**	450	473	496	520	543	566	589	49	822	23
822	11	589	613	636	659	682	705	728	48	821	23
821	12	728	752	775	798	821	844	867	47	820	23
820	13	867	890	913	936	959	982	*005	46	819	23
819	14	8.96 005	028	051	074	097	120	143	**45**	817	23
817	**15**	143	166	189	212	234	257	280	44	816	23
816	16	280	303	326	349	371	394	417	43	815	23
815	17	417	440	462	485	508	531	553	42	814	23
814	18	553	576	599	621	644	667	689	41	813	23
813	19	689	712	735	757	780	802	825	**40**	812	23
812	**20**	825	847	870	892	915	937	960	39	810	22
810	21	960	982	*005	*027	*050	*072	*095	38	809	22
809	22	8.97 095	117	139	162	184	207	229	37	808	22
808	23	229	251	274	296	318	341	363	36	807	22
807	24	363	385	407	430	452	474	496	**35**	806	22
806	**25**	496	518	541	563	585	07	629	34	804	22
804	26	629	651	674	696	718	740	762	33	803	22
803	27	762	784	806	828	850	872	894	32	802	22
802	28	894	916	938	960	982	*004	*026	31	801	22
801	29	8.98 026	048	070	092	114	135	157	**30**	800	22
800	**30**	157	179	201	223	245	266	288	29	798	22
798	31	288	310	332	354	375	397	419	28	797	22
797	32	419	441	462	484	506	527	549	27	796	22
796	33	549	571	592	614	636	657	679	26	795	22
795	34	679	701	722	744	765	787	808	**25**	793	22
793	**35**	808	830	851	873	894	916	937	24	792	22
792	36	937	959	980	002	*023	*045	*066	23	791	22
791	37	8.99 066	087	109	130	152	173	194	22	790	21
790	38	194	216	237	258	280	301	322	21	788	21
788	39	322	343	365	386	407	428	450	**20**	787	21
787	**40**	450	471	492	513	534	556	577	19	786	21
786	41	577	598	619	640	661	682	704	18	785	21
785	42	704	725	746	767	788	809	830	17	783	21
783	43	830	851	872	893	914	935	956	16	782	21
782	44	956	977	998	*019	*040	*061	*082	**15**	781	21
781	**45**	9.00 082	103	123	144	165	186	207	14	780	21
780	46	207	228	249	269	290	311	332	13	778	21
778	47	332	353	373	394	415	436	456	12	777	21
777	48	456	477	498	518	539	560	581	11	776	21
776	49	581	601	622	642	663	684	704	**10**	775	21
775	**50**	704	725	746	766	787	807	828	9	773	21
773	51	828	848	869	889	910	930	951	8	772	20
772	52	951	971	992	*012	*033	*053	*074	7	771	20
771	53	9.01 074	094	115	135	155	176	196	6	769	20
769	54	196	217	237	257	278	298	318	**5**	768	20
768	**55**	318	339	359	379	399	420	440	4	767	20
767	56	440	460	480	501	521	541	561	3	765	20
765	57	561	582	602	622	642	662	682	2	764	20
764	58	682	703	723	743	763	783	803	1	763	20
763	59	803	823	843	863	883	903	923	**0**	761	20
		60″	50″	40″	30″	20″	10″	0″	′	9.99	d

Prop. Parts

	24	**23**	**22**	**21**	**20**
1	2.4	2.3	2.2	2.1	2.0
2	4.8	4.6	4.4	4.2	4.0
3	7.2	6.9	6.6	6.3	6.0
4	9.6	9.2	8.8	8.4	8.0
5	12.0	11.5	11.0	10.5	10.0
6	14.4	13.8	13.2	12.6	12.0
7	16.8	16.1	15.4	14.7	14.0
8	19.2	18.4	17.6	16.8	16.0
9	21.6	20.7	19.8	18.9	18.0

log cos **84°** L. sin

′	0″	10″	20″	30″	40″	50″	60″		d
0	8.94 195	219	244	268	292	316	340	59	24
1	340	365	389	413	437	461	485	58	24
2	485	509	533	557	581	606	630	57	24
3	630	654	678	702	725	749	773	56	24
4	773	797	821	845	869	893	917	**55**	24
5	917	941	964	988	*012	*036	*060	54	24
6	8.95 060	083	107	131	155	178	202	53	24
7	202	226	249	273	297	320	344	52	24
8	344	368	391	415	439	462	486	51	24
9	486	509	533	556	580	603	627	**50**	24
10	627	650	674	697	721	744	767	49	23
11	767	791	814	838	861	884	908	48	23
12	908	931	954	977	*001	*024	*047	47	23
13	8.96 047	071	094	117	140	163	187	46	23
14	187	210	233	256	279	302	325	**45**	23
15	325	349	372	395	418	441	464	44	23
16	464	487	510	533	556	579	602	43	23
17	602	625	648	671	694	717	739	42	23
18	739	762	785	808	831	854	877	41	23
19	877	899	922	945	968	991	*013	**40**	23
20	8.97 013	036	059	081	104	127	150	39	23
21	150	172	195	218	240	263	285	38	23
22	285	308	331	353	376	398	421	37	23
23	421	443	466	488	511	533	556	36	22
24	556	578	601	623	646	668	691	**35**	22
25	691	713	735	758	780	802	825	34	22
26	825	847	869	892	914	936	959	33	22
27	959	981	*003	*025	*048	*070	*092	32	22
28	8.98 092	114	136	159	181	203	225	31	22
29	225	247	269	291	314	336	358	**30**	22
30	358	380	402	424	446	468	490	29	22
31	490	512	534	556	578	600	622	28	22
32	622	644	666	687	709	731	753	27	22
33	753	775	797	819	841	862	884	26	22
34	884	906	928	950	971	993	*015	**25**	22
35	8.99 015	037	058	080	102	123	145	24	22
36	145	167	188	210	232	253	275	23	22
37	275	297	318	340	361	383	405	22	22
38	405	426	448	469	491	512	534	21	22
39	534	555	577	598	620	641	662	**20**	21
40	662	684	705	727	748	769	791	19	21
41	791	812	834	855	876	898	919	18	21
42	919	940	961	983	*004	*025	*046	17	21
43	9.00 046	068	089	110	131	153	174	16	21
44	174	195	216	237	258	280	301	**15**	21
45	301	322	343	364	385	406	427	14	21
46	427	448	469	490	511	532	553	13	21
47	553	574	595	616	637	658	679	12	21
48	679	700	721	742	763	784	805	11	21
49	805	826	846	867	888	909	930	**10**	21
50	930	951	971	992	*013	*034	*055	9	21
51	9.01 055	075	096	117	138	158	179	8	21
52	179	200	220	241	262	282	303	7	21
53	303	324	344	365	386	406	427	6	21
54	427	447	468	489	509	530	550	**5**	20
55	550	571	591	612	632	653	673	4	20
56	673	694	714	735	755	776	796	3	20
57	796	816	837	857	878	898	918	2	20
58	918	939	959	979	*000	*020	*040	1	20
59	9 02 040	061	081	101	121	142	162	**0**	20
	60″	50″	40″	30″	20″	10″	0″	′	d

Prop. Parts

	25	**24**	**23**	**22**	**21**	**20**
1	2.5	2.4	2.3	2.2	2.1	2.0
2	5.0	4.8	4.6	4.4	4.2	4.0
3	7.5	7.2	6.9	6.6	6.3	6.0
4	10.0	9.6	9.2	8.8	8.4	8.0
5	12.5	12.0	11.5	11.0	10.5	10.0
6	15.0	14.4	13.8	13.2	12.6	12.0
7	17.5	16.8	16.1	15.4	14.7	14.0
8	20.0	19.2	18.4	17.6	16.8	16.0
9	22.5	21.6	20.7	19.8	18.9	18.0

log cot **84°**

L. cos TABLE III 6° log sin

9.99	′	0″	10″	20″	30″	40″	50″	60″			d
761	0	9.01 923	943	964	984	*004	*024	*043	59	760	20
760	1	9.02 043	063	083	103	123	143	163	58	759	20
759	2	163	183	203	223	243	263	283	57	757	20
757	3	283	302	322	342	362	382	402	56	756	20
756	4	402	421	441	461	481	501	520	55	755̄	20
755̄	5	520	540	560	579	599	619	639	54	753	20
753	6	639	658	678	698	717	737	757	53	752	20
752	7	757	776	796	816	835	855̄	874	52	751	20
751	8	874	894	914	933	953	972	992	51	749	20
749	9	992	*011	*031	*050	*070	*089	*109	50	748	20
748	10	9.03 109	128	148	167	187	206	226	49	747	20
747	11	226	245	265̄	284	303	323	342	48	745	19
745	12	342	361	381	400	420	439	458	47	744	19
744	13	458	478	497	516	535	555̄	574	46	742	19
742	14	574	593	613	632	651	670	690	45	741	19
741	15	690	709	728	747	766	786	805̄	44	740	19
740	16	805̄	824	843	862	881	901	920	43	738	19
738	17	920	939	958	977	996	*015	*034	42	737	19
737	18	9.04 034	053	072	091	110	129	149	41	736	19
736	19	149	168	187	206	225̄	244	262	40	734	19
734	20	262	281	300	319	338	357	376	39	733	19
733	21	376	395	414	433	452	471	490	38	731	19
731	22	490	508	527	546	565̄	584	603	37	730	19
730	23	603	621	640	659	678	697	715	36	728	19
728	24	715	734	753	772	790	809	828	35	727	19
727	25	828	847	865	884	903	921	940	34	726	19
726	26	940	959	977	996	*015̄	*033	*052	33	724	19
724	27	9.05 052	071	089	108	126	145̄	164	32	723	19
723	28	164	182	201	219	238	256	275̄	31	721	18
721	29	275̄	293	312	330	349	367	386	30	720	18
720	30	386	404	423	441	460	478	497	29	718	18
718	31	497	515	533	552	570	589	607	28	717	18
717	32	607	625	644	662	681	699	717	27	716	18
716	33	717	736	754	772	791	809	827	26	714	18
714	34	827	845	864	882	900	918	937	25	713	18
713	35	937	955̄	973	991	*010	*028	*046	24	711	18
711	36	9.06 046	064	082	101	119	137	155	23	710	18
710	37	155	173	191	210	228	246	264	22	708	18
708	38	264	282	300	318	336	354	372	21	707	18
707	39	372	390	408	426	445̄	463	481	20	705	18
705	40	481	499	517	535̄	553	571	589	19	704	18
704	41	589	606	624	642	660	678	696	18	702	18
702	42	696	714	732	750	768	786	804	17	701	18
701	43	804	821	839	857	875	893	911	16	699	18
699	44	911	929	946	964	982	*000	*018	15	698	18
698	45	9.07 018	035	053	071	089	106	124	14	696	18
696	46	124	142	160	177	195	213	231	13	695̄	18
695̄	47	231	248	266	284	301	319	337	12	693	18
693	48	337	354	372	390	407	425̄	442	11	692	18
692	49	442	460	478	495	513	530	548	10	690	18
690	50	548	566	583	601	618	636	653	9	689	18
689	51	653	671	688	706	723	741	758	8	687	18
687	52	758	776	793	811	828	846	863	7	686	18
686	53	863	881	898	915	933	950	968	6	684	18
684	54	968	985	*002	*020	*037	*055̄	*072	5	683	17
683	55	9.08 072	089	107	124	141	159	176	4	681	17
681	56	176	193	211	228	245	262	280	3	680	17
680	57	280	297	314	331	349	366	383	2	678	17
678	58	383	400	418	435̄	452	469	486	1	677	17
677	59	486	504	521	538	555	572	589	0	675	17
		60″	50″	40″	30″	20″	10″	0″	′	9.99	d

Prop. Parts

	21	20	19	18	17
1	2.1	2.0	1.9	1.8	1.7
2	4.2	4.0	3.8	3.6	3.4
3	6.3	6.0	5.7	5.4	5.1
4	8.4	8.0	7.6	7.2	6.8
5	10.5	10.0	9.5	9.0	8.5
6	12.6	12.0	11.4	10.8	10.2
7	14.7	14.0	13.3	12.6	11.9
8	16.8	16.0	15.2	14.4	13.6
9	18.9	18.0	17.1	16.2	15.3

log cos 83° L. sin

TABLE III 6° log tan

′	0″	10″	20″	30″	40″	50″	60″		d
0	9.02 162	182	203	223	243	263	283	59	20
1	283	304	324	344	364	384	404	58	20
2	404	425̄	445̄	465̄	485̄	505	525	57	20
3	525	545	565	585	605	625	645	56	20
4	645	666	686	706	726	746	766	55	20
5	766	785	805	825	845	865	885	54	20
6	885	905	925	945̄	965̄	985̄	*005̄	53	20
7	9.03 005̄	025̄	044	064	084	104	124	52	20
8	124	144	163	183	203	223	242	51	20
9	242	262	282	302	321	341	361	50	20
10	361	381	400	420	440	459	479	49	20
11	479	499	518	538	558	577	597	48	20
12	597	616	636	656	675	695̄	714	47	20
13	714	734	753	773	793	812	832	46	20
14	832	851	871	890	910	929	948	45	19
15	948	968	987	*007	*026	*046	*065	44	19
16	9.04 065	084	104	123	143	162	181	43	19
17	181	201	220	239	259	278	297	42	19
18	297	317	336	355	374	394	413	41	19
19	413	432	451	471	490	509	528	40	19
20	528	548	567	586	605	624	643	39	19
21	643	663	682	701	720	739	758	38	19
22	758	777	796	815	835̄	854	873	37	19
23	873	892	911	930	949	968	987	36	19
24	987	*006	*025̄	*044	*063	*082	*101	35	19
25	9.05 101	120	139	158	177	195	214	34	19
26	214	233	252	271	290	309	328	33	19
27	328	347	365	384	403	422	441	32	19
28	441	460	478	497	516	535̄	553	31	19
29	553	572	591	610	628	647	666	30	19
30	666	685̄	703	722	741	759	778	29	19
31	778	797	815	834	853	871	890	28	19
32	890	909	927	946	964	983	*002	27	19
33	9.06 002	020	039	057	076	094	113	26	18
34	113	132	150	169	187	206	224	25	18
35	224	243	261	279	298	316	335̄	24	18
36	335̄	353	372	390	409	427	445	23	18
37	445	464	482	500	519	537	556	22	18
38	556	574	592	611	629	647	666	21	18
39	666	684	702	720	739	757	775	20	18
40	775	793	812	830	848	866	885̄	19	18
41	885̄	903	921	939	957	976	994	18	18
42	994	*012	*030	*048	*066	*085̄	*103	17	18
43	9.07 103	121	139	157	175	193	211	16	18
44	211	229	247	266	284	302	320	15	18
45	320	338	356	374	392	410	428	14	18
46	428	446	464	482	500	518	536	13	18
47	536	554	572	589	607	625	643	12	18
48	643	661	679	697	715̄	733	751	11	18
49	751	768	786	804	822	840	858	10	18
50	858	875	893	911	929	947	964	9	18
51	964	982	*000	*018	*035	*053	*071	8	18
52	9.08 071	089	106	124	142	160	177	7	18
53	177	195̄	213	230	248	266	283	6	18
54	283	301	319	336	354	371	389	5	18
55	389	407	424	442	460	477	495̄	4	18
56	495̄	512	530	547	565̄	582	600	3	18
57	600	617	635	653	670	688	705	2	18
58	705	722	740	757	775̄	792	810	1	18
59	810	827	845̄	862	880	897	914	0	17
	60″	50″	40″	30″	20″	10″	0′	′	d

Prop. Parts

	21	20	19	18	17
1	2.1	2.0	1.9	1.8	1.7
2	4.2	4.0	3.8	3.6	3.4
3	6.3	6.0	5.7	5.4	5.1
4	8.4	8.0	7.6	7.2	6.8
5	10.5	10.0	9.5	9.0	8.5
6	12.6	12.0	11.4	10.8	10.2
7	14.7	14.0	13.3	12.6	11.9
8	16.8	16.0	15.2	14.4	13.6
9	18.9	18.0	17.1	16.2	15.3

log cot 83°

TABLE III **0°** 90° 180° 270°

″	′	log sin	d	cpl S	cpl T	log tan	c d	log cot	log cos	
0	0	....		....	....	....		....	0.00 000	60
60	1	6.46 373	30103	5.31 443	5.31 443	6.46 373	30103	3.53 627	0.00 000	59
120	2	6.76 476	17609	5.31 443	5.31 443	6.76 476	17609	3.23 524	0.00 000	58
180	3	6.94 08$\bar{5}$	12494	5.31 443	5.31 443	6.94 08$\bar{5}$	12494	3.05 915	0.00 000	57
240	4	7.06 579	9691	5.31 443	5.31 442	7.06 579	9691	2.93 421	0.00 000	56
300	5	7.16 270	7918	5.31 443	5.31 442	7.16 270	7918	2.83 730	0.00 000	55
360	6	7.24 188	6694	5.31 443	5.31 442	7.24 188	6694	2.75 812	0.00 000	54
420	7	7.30 882	5800	5.31 443	5.31 442	7.30 882	5800	2.69 118	0.00 000	53
480	8	7.36 682	5115	5.31 443	5.31 442	7.36 682	5115	2.63 318	0.00 000	52
540	9	7.41 797	4576	5.31 443	5.31 442	7.41 797	4576	2.58 203	0.00 000	51
600	10	7.46 373	4139	5.31 443	5.31 442	7.46 373	4139	2.53 627	0.00 000	50
660	11	7.50 512	3779	5.31 443	5.31 442	7.50 512	3779	2.49 488	0.00 000	49
720	12	7.54 291	3476	5.31 443	5.31 442	7.54 291	3476	2.45 709	0.00 000	48
780	13	7.57 767	3218	5.31 443	5.31 442	7.57 767	3219	2.42 233	0.00 000	47
840	14	7.60 985	2997	5.31 443	5.31 442	7.60 986	2996	2.39 014	0.00 000	46
900	15	7.63 982	2802	5.31 443	5.31 442	7.63 982	2803	2.36 018	0.00 000	45
960	16	7.66 784	2633	5.31 443	5.31 442	7.66 78$\bar{5}$	2633	2.33 215	0.00 000	44
1020	17	7.69 417	2483	5.31 443	5.31 442	7.69 418	2482	2.30 582	9.99 999	43
1080	18	7.71 900	2348	5.31 443	5.31 442	7.71 900	2348	2.28 100	9.99 999	42
1140	19	7.74 248	2227	5.31 443	5.31 442	7.74 248	2228	2.25 752	9.99 999	41
1200	20	7.76 475	2119	5.31 443	5.31 442	7.76 476	2119	2.23 524	9.99 999	40
1260	21	7.78 594	2021	5.31 443	5.31 442	7.78 595	2020	2.21 40$\bar{5}$	9.99 999	39
1320	22	7.80 61$\bar{5}$	1930	5.31 443	5.31 442	7.80 615	1931	2.19 38$\bar{5}$	9.99 999	38
1380	23	7.82 545	1848	5.31 443	5.31 442	7.82 546	1848	2.17 454	9.99 999	37
1440	24	7.84 393	1773	5.31 443	5.31 442	7.84 394	1773	2.15 606	9.99 999	36
1500	25	7.86 166	1704	5.31 443	5.31 442	7.86 167	1704	2.13 833	9.99 999	35
1560	26	7.87 870	1639	5.31 443	5.31 442	7.87 871	1639	2.12 129	9.99 999	34
1620	27	7.89 509	1579	5.31 443	5.31 442	7.89 510	1579	2.10 490	9.99 999	33
1680	28	7.91 088	1524	5.31 443	5.31 442	7.91 089	1524	2.08 911	9.99 999	32
1740	29	7.92 612	1472	5.31 443	5.31 441	7.92 613	1473	2.07 387	9.99 998	31
1800	30	7.94 084	1424	5.31 443	5.31 441	7.94 086	1424	2.05 914	9.99 998	30
1860	31	7.95 508	1379	5.31 443	5.31 441	7.95 510	1379	2.04 490	9.99 998	29
1920	32	7.96 887	1336	5.31 443	5.31 441	7.96 889	1336	2.03 111	9.99 998	28
1980	33	7.98 223	1297	5.31 443	5.31 441	7.98 225	1297	2.01 77$\bar{5}$	9.99 998	27
2040	34	7.99 520	1259	5.31 443	5.31 441	7.99 522	1259	2.00 478	9.99 998	26
2100	35	8.00 779	1223	5.31 443	5.31 441	8.00 781	1223	1.99 219	9.99 998	25
2160	36	8.02 002	1190	5.31 443	5.31 441	8.02 004	1190	1.97 996	9.99 998	24
2220	37	8.03 192	1158	5.31 443	5.31 441	8.03 194	1159	1.96 806	9.99 997	23
2280	38	8.04 350	1128	5.31 443	5.31 441	8.04 353	1128	1.95 647	9.99 997	22
2340	39	8.05 478	1100	5.31 443	5.31 441	8.05 481	1100	1.94 519	9.99 997	21
2400	40	8.06 578	1072	5.31 443	5.31 441	8.06 581	1072	1.93 419	9.99 997	20
2460	41	8.07 6$\bar{5}$0	1046	5.31 444	5.31 440	8.07 653	1047	1.92 347	9.99 997	19
2520	42	8.08 696	1022	5.31 444	5.31 440	8.08 700	1022	1.91 300	9.99 997	18
2580	43	8.09 718	999	5.31 444	5.31 440	8.09 722	998	1.90 278	9.99 997	17
2640	44	8.10 717	976	5.31 444	5.31 440	8.10 720	976	1.89 280	9.99 996	16
2700	45	8.11 693	954	5.31 444	5.31 440	8.11 696	955	1.88 304	9.99 996	15
2760	46	8.12 647	934	5.31 444	5.31 440	8.12 651	934	1.87 349	9.99 996	14
2820	47	8.13 581	914	5.31 444	5.31 440	8.13 585	915	1.86 41$\bar{5}$	9.99 996	13
2880	48	8.14 495	896	5.31 444	5.31 440	8.14 $\bar{5}$00	895	1.85 500	9.99 996	12
2940	49	8.15 391	877	5.31 444	5.31 440	8.15 395	878	1.84 60$\bar{5}$	9.99 996	11
3000	50	8.16 268	860	5.31 444	5.31 439	8.16 273	860	1.83 727	9.99 995	10
3060	51	8.17 128	843	5.31 444	5.31 439	8.17 133	843	1.82 867	9.99 995	9
3120	52	8.17 971	827	5.31 444	5.31 439	8.17 976	828	1.82 024	9.99 995	8
3180	53	8.18 798	812	5.31 444	5.31 439	8.18 804	812	1.81 196	9.99 99$\bar{5}$	7
3240	54	8.19 610	797	5.31 444	5.31 439	8.19 616	797	1.80 384	9.99 99$\bar{5}$	6
3300	55	8.20 407	782	5.31 444	5.31 439	8.20 413	782	1.79 587	9.99 994	5
3360	56	8.21 189	769	5.31 444	5.31 439	8.21 195	769	1.78 80$\bar{5}$	9.99 994	4
3420	57	8.21 958	755	5.31 44$\bar{5}$	5.31 439	8.21 964	756	1.78 036	9.99 994	3
3480	58	8.22 713	743	5.31 44$\bar{5}$	5.31 438	8.22 720	742	1.77 280	9.99 994	2
3540	59	8.23 456	730	5.31 44$\bar{5}$	5.31 438	8.23 462	730	1.76 538	9.99 994	1
3600	60	8.24 186		5.31 44$\bar{5}$	5.31 438	8.24 192		1.75 808	9.99 993	0
		log cos	d			log cot	c d	log tan	log sin	′

179° 269° 359° **89°**

TABLE III **1°** 91° 181° 271°

″	′	log sin	d	cpl S	cpl T	log tan	c d	log cot	log cos	
3600	0	8.24 186	717	5.31 445	5.31 438	8.24 192	718	1.75 808	9.99 993	60
3660	1	8.24 903	706	5.31 44$\bar{5}$	5.31 438	8.24 910	706	1.75 090	9.99 993	59
3720	2	8.25 609	695	5.31 44$\bar{5}$	5.31 438	8.25 616	696	1.74 384	9.99 993	58
3780	3	8.26 304	684	5.31 44$\bar{5}$	5.31 438	8.26 312	684	1.73 688	9.99 993	57
3840	4	8.26 988	673	5.31 445	5.31 437	8.26 996	673	1.73 004	9.99 992	56
3900	5	8.27 661	663	5.31 445	5.31 437	8.27 669	663	1.72 331	9.99 992	55
3960	6	8.28 324	653	5.31 445	5.31 437	8.28 332	654	1.71 668	9.99 992	54
4020	7	8.28 977	644	5.31 445	5.31 437	8.28 986	643	1.71 014	9.99 992	53
4080	8	8.29 621	634	5.31 445	5.31 437	8.29 629	634	1.70 371	9.99 992	52
4140	9	8.30 25$\bar{5}$	624	5.31 445	5.31 437	8.30 263	625	1.69 737	9.99 991	51
4200	10	8.30 879	616	5.31 446	5.31 437	8.30 888	617	1.69 112	9.99 991	50
4260	11	8.31 495	608	5.31 446	5.31 436	8.31 50$\bar{5}$	607	1.68 495	9.99 991	49
4320	12	8.32 103	599	5.31 446	5.31 436	8.32 112	599	1.67 888	9.99 990	48
4380	13	8.32 702	590	5.31 446	5.31 436	8.32 711	591	1.67 289	9.99 990	47
4440	14	8.33 292	583	5.31 446	5.31 436	8.33 302	584	1.66 698	9.99 990	46
4500	15	8.33 875	575	5.31 446	5.31 436	8.33 886	575	1.66 114	9.99 990	45
4560	16	8.34 450	568	5.31 446	5.31 435	8.34 461	568	1.65 539	9.99 989	44
4620	17	8.35 018	560	5.31 446	5.31 435	8.35 029	561	1.64 971	9.99 989	43
4680	18	8.35 578	553	5.31 446	5.31 435	8.35 590	553	1.64 410	9.99 989	42
4740	19	8.36 131	547	5.31 446	5.31 43$\bar{5}$	8.36 143	546	1.63 857	9.99 989	41
4800	20	8.36 678	539	5.31 446	5.31 43$\bar{5}$	8.36 689	540	1.63 311	9.99 988	40
4860	21	8.37 217	533	5.31 447	5.31 434	8.37 229	533	1.62 771	9.99 988	39
4920	22	8.37 7$\bar{5}$0	526	5.31 447	5.31 434	8.37 762	527	1.62 238	9.99 988	38
4980	23	8.38 276	520	5.31 447	5.31 434	8.38 289	520	1.61 711	9.99 987	37
5040	24	8.38 796	514	5.31 447	5.31 434	8.38 809	514	1.61 191	9.99 987	36
5100	25	8.39 310	508	5.31 447	5.31 434	8.39 323	509	1.60 677	9.99 987	35
5160	26	8.39 818	502	5.31 447	5.31 433	8.39 832	502	1.60 168	9.99 986	34
5220	27	8.40 320	496	5.31 447	5.31 433	8.40 334	496	1.59 666	9.99 986	33
5280	28	8.40 816	491	5.31 447	5.31 433	8.40 830	491	1.59 170	9.99 986	32
5340	29	8.41 307	485	5.31 447	5.31 433	8.41 321	486	1.58 679	9.99 985	31
5400	30	8.41 792	480	5.31 447	5.31 433	8.41 807	480	1.58 193	9.99 985	30
5460	31	8.42 272	474	5.31 448	5.31 432	8.42 287	475	1.57 713	9.99 98$\bar{5}$	29
5520	32	8.42 746	470	5.31 448	5.31 432	8.42 762	470	1.57 238	9.99 984	28
5580	33	8.43 216	464	5.31 448	5.31 432	8.43 232	464	1.56 768	9.99 984	27
5640	34	8.43 680	459	5.31 448	5.31 432	8.43 696	460	1.56 304	9.99 984	26
5700	35	8.44 139	455	5.31 448	5.31 431	8.44 156	455	1.55 844	9.99 983	25
5760	36	8.44 594	450	5.31 448	5.31 431	8.44 611	450	1.55 389	9.99 983	24
5820	37	8.45 044	445	5.31 448	5.31 431	8.45 061	446	1.54 939	9.99 983	23
5880	38	8.45 489	441	5.31 448	5.31 431	8.45 507	441	1.54 493	9.99 982	22
5940	39	8.45 930	436	5.31 449	5.31 431	8.45 948	437	1.54 052	9.99 982	21
6000	40	8.46 366	433	5.31 449	5.31 430	8.46 38$\bar{5}$	432	1.53 615	9.99 982	20
6060	41	8.46 799	427	5.31 449	5.31 430	8.46 817	428	1.53 183	9.99 981	19
6120	42	8.47 226	424	5.31 449	5.31 430	8.47 245	424	1.52 75$\bar{5}$	9.99 981	18
6180	43	8.47 6$\bar{5}$0	419	5.31 449	5.31 430	8.47 669	420	1.52 331	9.99 981	17
6240	44	8.48 069	416	5.31 449	5.31 429	8.48 089	416	1.51 911	9.99 980	16
6300	45	8.48 48$\bar{5}$	411	5.31 449	5.31 429	8.48 505	412	1.51 49$\bar{5}$	9.99 980	15
6360	46	8.48 896	408	5.31 449	5.31 429	8.48 917	408	1.51 083	9.99 979	14
6420	47	8.49 304	404	5.31 4$\bar{5}$0	5.31 428	8.49 325	404	1.50 67$\bar{5}$	9.99 979	13
6480	48	8.49 708	400	5.31 4$\bar{5}$0	5.31 428	8.49 729	401	1.50 271	9.99 979	12
6540	49	8.50 108	396	5.31 4$\bar{5}$0	5.31 428	8.50 130	397	1.49 870	9.99 978	11
6600	50	8.50 504	393	5.31 4$\bar{5}$0	5.31 428	8.50 527	393	1.49 473	9.99 978	10
6660	51	8.50 897	390	5.31 450	5.31 427	8.50 920	390	1.49 080	9.99 977	9
6720	52	8.51 287	386	5.31 450	5.31 427	8.51 310	386	1.48 690	9.99 977	8
6780	53	8.51 673	382	5.31 450	5.31 427	8.51 696	383	1.48 304	9.99 977	7
6840	54	8.52 055	379	5.31 450	5.31 427	8.52 079	380	1.47 921	9.99 976	6
6900	55	8.52 434	376	5.31 451	5.31 426	8.52 459	376	1.47 541	9.99 976	5
6960	56	8.52 810	373	5.31 451	5.31 426	8.52 83$\bar{5}$	373	1.47 165	9.99 975	4
7020	57	8.53 183	369	5.31 451	5.31 426	8.53 208	370	1.46 792	9.99 97$\bar{5}$	3
7080	58	8.53 552	367	5.31 451	5.31 425	8.53 578	367	1.46 422	9.99 974	2
7140	59	8.53 919	363	5.31 451	5.31 425	8.53 94$\bar{5}$	363	1.46 055	9.99 974	1
7200	60	8.54 282		5.31 451	5.31 42$\bar{5}$	8.54 308		1.45 692	9.99 974	0
		log cos	d			log cot	c d	log tan	log sin	′

178° 268° 358° **88°**

″	′	log sin	d	cpl S	cpl T	log tan	c d	log cot	log cos	
7200	**0**	8.54 282	360	5.31 451	5.31 425̄	8.54 308	361	1.45 692	9.99 974	**60**
7260	1	8.54 642	357	5.31 451	5.31 425̄	8.54 669	358	1.45 331	9.99 973	59
7320	2	8.54 999	355	5.31 452	5.31 424	8.55 027	355	1.44 973	9.99 973	58
7380	3	8.55 354	351	5.31 452	5.31 424	8.55 382	352	1.44 618	9.99 972	57
7440	4	8.55 705	349	5.31 452	5.31 424	8.55 734	349	1.44 266	9.99 972	56
7500	**5**	8.56 054	346	5.31 452	5.31 423	8.56 083	346	1.43 917	9.99 971	**55**
7560	6	8.56 400	343	5.31 452	5.31 423	8.56 429	344	1.43 571	9.99 971	54
7620	7	8.56 743	341	5.31 452	5.31 423	8.56 773	341	1.43 227	9.99 970	53
7680	8	8.57 084	337	5.31 453	5.31 422	8.57 114	338	1.42 886	9.99 970	52
7740	9	8.57 421	336	5.31 453	5.31 422	8.57 452	336	1.42 548	9.99 969	51
7800	**10**	8.57 757	332	5.31 453	5.31 422	8.57 788	333	1.42 212	9.99 969	**50**
7860	11	8.58 089	330	5.31 453	5.31 421	8.58 121	330	1.41 879	9.99 968	49
7920	12	8.58 419	328	5.31 453	5.31 421	8.58 451	328	1.41 549	9.99 968	48
7980	13	8.58 747	325	5.31 453	5.31 421	8.58 779	326	1.41 221	9.99 967	47
8040	14	8.59 072	323	5.31 454	5.31 421	8.59 105	323	1.40 895̄	9.99 967	46
8100	**15**	8.59 395̄	320	5.31 454	5.31 420	8.59 428	321	1.40 572	9.99 967	**45**
8160	16	8.59 715	318	5.31 454	5.31 420	8.59 749	319	1.40 251	9.99 966	44
8220	17	8.60 033	316	5.31 454	5.31 420	8.60 068	316	1.39 932	9.99 966	43
8280	18	8.60 349	313	5.31 454	5.31 419	8.60 384	314	1.39 616	9.99 965̄	42
8340	19	8.60 662	311	5.31 454	5.31 419	8.60 698	311	1.39 302	9.99 964	41
8400	**20**	8.60 973	309	5.31 455̄	5.31 418	8.61 009	310	1.38 991	9.99 964	**40**
8460	21	8.61 282	307	5.31 455̄	5.31 418	8.61 319	307	1.38 681	9.99 963	39
8520	22	8.61 589	305	5.31 455̄	5.31 418	8.61 626	305	1.38 374	9.99 963	38
8580	23	8.61 894	302	5.31 455	5.31 417	8.61 931	303	1.38 069	9.99 962	37
8640	24	8.62 196	301	5.31 455	5.31 417	8.62 234	301	1.37 766	9.99 962	36
8700	**25**	8.62 497	298	5.31 455	5.31 417	8.62 535	299	1.37 465̄	9.99 961	**35**
8760	26	8.62 795̄	296	5.31 456	5.31 416	8.62 834	297	1.37 166	9.99 961	34
8820	27	8.63 091	294	5.31 456	5.31 416	8.63 131	295	1.36 869	9.99 960	33
8880	28	8.63 385	293	5.31 456	5.31 416	8.63 426	292	1.36 574	9.99 960	32
8940	29	8.63 678	290	5.31 456	5.31 415	8.63 718	291	1.36 282	9.99 959	31
9000	**30**	8.63 968	288	5.31 456	5.31 415̄	8.64 009	289	1.35 991	9.99 959	**30**
9060	31	8.64 256	287	5.31 456	5.31 415̄	8.64 298	287	1.35 702	9.99 958	29
9120	32	8.64 543	284	5.31 457	5.31 414	8.64 585	285	1.35 415̄	9.99 958	28
9180	33	8.64 827	283	5.31 457	5.31 414	8 [illegible]4 870	284	1.35 130	9.99 957	27
9240	34	8.65 110	281	5.31 457	5.31 413	8.65 154	281	1.34 846	9.99 956	26
9300	**35**	8.65 391	279	5.31 457	5.31 413	8.65 435̄	280	1.34 565	9.99 956	**25**
9360	36	8.65 670	277	5.31 457	5.31 413	8.65 715̄	278	1.34 285	9.99 955	24
9420	37	8.65 947	276	5.31 458	5.31 412	8.65 993	276	1.34 007	9.99 955̄	23
9480	38	8.66 223	274	5.31 458	5.31 412	8.66 269	274	1.33 731	9.99 954	22
9540	39	8.66 497	272	5.31 458	5.31 412	8.66 543	273	1.33 457	9.99 954	21
9600	**40**	8.66 769	270	5.31 458	5.31 411	8.66 816	271	1.33 184	9.99 953	**20**
9660	41	8.67 039	269	5.31 458	5.31 411	8.67 087	269	1.32 913	9.99 952	19
9720	42	8.67 308	267	5.31 459	5.31 410	8.67 356	268	1.32 644	9.99 952	18
9780	43	8.67 575	266	5.31 459	5.31 410	8.67 624	266	1.32 376	9.99 951	17
9840	44	8.67 841	263	5.31 459	5.31 410	8.67 890	264	1.32 110	9.99 951	16
9900	**45**	8.68 104	263	5.31 459	5.31 409	8.68 154	263	1.31 846	9.99 950	**15**
9960	46	8.68 367	260	5.31 459	5.31 409	8.68 417	261	1.31 583	9.99 949	14
10020	47	8.68 627	259	5.31 460	5.31 408	8.68 678	260	1.31 322	9.99 949	13
10080	48	8.68 886	258	5.31 460	5.31 408	8.68 938	258	1.31 062	9.99 948	12
10140	49	8.69 144	256	5.31 460	5.31 408	8.69 196	257	1.30 804	9.99 948	11
10200	**50**	8.69 400	254	5.31 460	5.31 407	8.69 453	255	1.30 547	9.99 947	**10**
10260	51	8.69 654	253	5.31 460	5.31 407	8.69 708	254	1.30 292	9.99 946	9
10320	52	8.69 907	252	5.31 461	5.31 406	8.69 962	252	1.30 038	9.99 946	8
10380	53	8.70 159	250	5.31 461	5.31 406	8.70 214	251	1.29 786	9.99 945̄	7
10440	54	8.70 409	249	5.31 461	5.31 405	8.70 465̄	249	1.29 535	9.99 944	6
10500	**55**	8.70 658	247	5.31 461	5.31 405̄	8.70 714	248	1.29 286	9.99 944	**5**
10560	56	8.70 905̄	246	5.31 461	5.31 405̄	8.70 962	246	1.29 038	9.99 943	4
10620	57	8.71 151	244	5.31 462	5.31 404	8.71 208	245	1.28 792	9.99 942	3
10680	58	8.71 395	243	5.31 462	5.31 404	8.71 453	244	1.28 547	9.99 942	2
10740	59	8.71 638	242	5.31 462	5.31 403	8.71 697	243	1.28 303	9.99 941	1
10800	**60**	8.71 880		5.31 462	5.31 403	8.71 940		1.28 060	9.99 940	**0**
″		log cos	d			log cot	c d	log tan	log sin	′

′	log sin	d	log tan	c d	log cot	log cos	
0	8.71 880	240	8.71 940	241	1.28 060	9.99 940	**60**
1	8.72 120	239	8.72 181	239	1.27 819	9.99 940	59
2	8.72 359	238	8.72 420	239	1.27 580	9.99 939	58
3	8.72 597	237	8.72 659	237	1.27 341	9.99 938	57
4	8.72 834	235	8.72 896	236	1.27 104	9.99 938	56
5	8.73 069	234	8.73 132	234	1.26 868	9.99 937	**55**
6	8.73 303	232	8.73 366	234	1.26 634	9.99 936	54
7	8.73 535	232	8.73 600	232	1.26 400	9.99 936	53
8	8.73 767	230	8.73 832	231	1.26 168	9.99 935	52
9	8.73 997	229	8.74 063	229	1.25 937	9.99 934	51
10	8.74 226	228	8.74 292	229	1.25 708	9.99 934	**50**
11	8.74 454	226	8.74 521	227	1.25 479	9.99 933	49
12	8.74 680	226	8.74 748	226	1.25 252	9.99 932	48
13	8.74 906	224	8.74 974	225	1.25 026	9.99 932	47
14	8.75 130	223	8.75 199	224	1.24 801	9.99 931	46
15	8.75 353	222	8.75 423	222	1.24 577	9.99 930	**45**
16	8.75 575̄	220	8.75 645	222	1.24 355̄	9.99 929	44
17	8.75 795	220	8.75 867	220	1.24 133	9.99 929	43
18	8.76 015	219	8.76 087	219	1.23 913	9.99 928	42
19	8.76 234	217	8.76 306	219	1.23 694	9.99 927	41
20	8.76 451	216	8.76 525̄	217	1.23 475	9.99 926	**40**
21	8.76 667	216	8.76 742	216	1.23 258	9.99 926	39
22	8.76 883	214	8.76 958	215	1.23 042	9.99 925̄	38
23	8.77 097	213	8.77 173	214	1.22 827	9.99 924	37
24	8.77 310	212	8.77 387	213	1.22 613	9.99 923	36
25	8.77 522	211	8.77 600	211	1.22 400	9.99 923	**35**
26	8.77 733	210	8.77 811	211	1.22 189	9.99 922	34
27	8.77 943	209	8.78 022	210	1.21 978	9.99 921	33
28	8.78 152	208	8.78 232	209	1.21 768	9.99 920	32
29	8.78 360	208	8.78 441	208	1.21 559	9.99 920	31
30	8.78 568	206	8.78 649	206	1.21 351	9.99 919	**30**
31	8.78 774	205	8.78 855	206	1.21 145̄	9.99 918	29
32	8.78 979	204	8.79 061	205	1.20 939	9.99 917	28
33	8.79 183	203	8.79 266	204	1.20 734	9.99 917	27
34	8.79 386	202	8.79 470	203	1.20 530	9.99 916	26
35	8.79 588	201	8.79 673	202	1.20 327	9.99 915	**25**
36	8.79 789	201	8.79 875	201	1.20 125̄	9.99 914	24
37	8.79 990	199	8.80 076	201	1.19 924	9.99 913	23
38	8.80 189	199	8.80 277	199	1.19 723	9.99 913	22
39	8.80 388	197	8.80 476	198	1.19 524	9.99 912	21
40	8.80 585	197	8.80 674	198	1.19 326	9.99 911	**20**
41	8.80 782	196	8.80 872	196	1.19 128	9.99 910	19
42	8.80 978	195	8.81 068	196	1.18 932	9.99 909	18
43	8.81 173	194	8.81 264	195	1.18 736	9.99 909	17
44	8.81 367	193	8.81 459	194	1.18 541	9.99 908	16
45	8.81 560	192	8.81 653	193	1.18 347	9.99 907	**15**
46	8.81 752	192	8.81 846	192	1.18 154	9.99 906	14
47	8.81 944	190	8.82 038	192	1.17 962	9.99 905	13
48	8.82 134	190	8.82 230	190	1.17 770	9.99 904	12
49	8.82 324	189	8.82 420	190	1.17 580	9.99 904	11
50	8.82 513	188	8.82 610	189	1.17 390	9.99 903	**10**
51	8.82 701	187	8.82 799	188	1.17 201	9.99 902	9
52	8.82 888	187	8.82 987	188	1.17 013	9.99 901	8
53	8.83 075̄	186	8.83 175̄	186	1.16 825	9.99 900	7
54	8.83 261	185	8.83 361	186	1.16 639	9.99 899	6
55	8.83 446	184	8.83 547	185	1.16 453	9.99 898	**5**
56	8.83 630	183	8.83 732	184	1.16 268	9.99 898	4
57	8.83 813	183	8.83 916	184	1.16 084	9.99 897	3
58	8.83 996	181	8.84 100	182	1.15 900	9.99 896	2
59	8.84 177	181	8.84 282	182	1.15 718	9.99 895̄	1
60	8.84 358		8.84 464		1.15 536	9.99 894	**0**
	log cos	d	log cot	c d	log tan	log sin	′

Prop. Parts

	241	**239**	**237**	**235**	**234**
1	4.0	4.0	4.0	3.9	3.9
2	8.0	8.0	7.9	7.8	7.8
3	12.0	12.0	11.8	11.8	11.7
4	16.1	15.9	15.8	15.7	15.6
5	20.1	19.9	19.8	19.6	19.5
6	24.1	23.9	23.7	23.5	23.4
7	28.1	27.9	27.6	27.4	27.3
8	32.1	31.9	31.6	31.3	31.2
9	36.2	35.8	35.6	35.2	35.1
10	40.2	39.8	39.5	39.2	39.0
20	80.3	79.7	79.0	78.3	78.0
30	120.5	119.5	118.5	117.5	117.0
40	160.7	159.3	158.0	156.7	156.0
50	200.8	199.2	197.5	195.8	195.0

	232	**229**	**227**	**225**	**223**
1	3.9	3.8	3.8	3.8	3.7
2	7.7	7.6	7.6	7.5	7.4
3	11.6	11.4	11.4	11.2	11.2
4	15.5	15.3	15.1	15.0	14.9
5	19.3	19.1	18.9	18.8	18.6
6	23.2	22.9	22.7	22.5	22.3
7	27.1	26.7	26.5	26.2	26.0
8	30.9	30.5	30.3	30.0	29.7
9	34.8	34.4	34.0	33.8	33.4
10	38.7	38.2	37.8	37.5	37.2
20	77.3	76.3	75.7	75.0	74.3
30	116.0	114.5	113.5	112.5	111.5
40	154.7	152.7	151.3	150.0	148.7
50	193.3	190.8	189.2	187.5	185.8

	222	**220**	**217**	**215**	**213**
1	3.7	3.7	3.6	3.6	3.6
2	7.4	7.3	7.2	7.2	7.1
3	11.1	11.0	10.8	10.8	10.6
4	14.8	14.7	14.5	14.3	14.2
5	18.5	18.3	18.1	17.9	17.8
6	22.2	22.0	21.7	21.5	21.3
7	25.9	25.7	25.3	25.1	24.8
8	29.6	29.3	28.9	28.7	28.4
9	33.3	33.0	32.6	32.2	32.0
10	37.0	36.7	36.2	35.8	35.5
20	74.0	73.3	72.3	71.7	71.0
30	111.0	110.0	108.5	107.5	106.5
40	148.0	146.7	144.7	143.3	142.0
50	185.0	183.3	180.8	179.2	177.5

	211	**208**	**206**	**203**	**201**
1	3.5	3.5̄	3.4	3.4	3.4
2	7.0	6.9	6.9	6.8	6.7
3	10.6	10.4	10.3	10.2	10.0
4	14.1	13.9	13.7	13.5	13.4
5	17.6	17.3	17.2	16.9	16.8
6	21.1	20.8	20.6	20.3	20.1
7	24.6	24.3	24.0	23.7	23.4
8	28.1	27.7	27.5̄	27.1	26.8
9	31.6	31.2	30.9	30.4	30.2
10	35.2	34.7	34.3	33.8	33.5
20	70.3	69.3	68.7	67.7	67.0
30	105.5	104.0	103.0	101.5	100.5
40	140.7	138.7	137.3	135.3	134.0
50	175.8	173.3	171.7	169.2	167.5

	199	**197**	**195**	**193**	**192**
1	3.3	3.3	3.2	3.2	3.2
2	6.6	6.6	6.5	6.4	6.4
3	10.0	9.8	9.8	9.6	9.6
4	13.3	13.1	13.0	12.9	12.8
5	16.6	16.4	16.2	16.1	16 0
6	19 9	19.7	19.5	19.3	19.2
7	23.2	23.0	22.8	22.5	22.4
8	26.5	26.3	26.0	25.7	25.6
9	29.8	29.6	29.2	29.0	28.8
10	33.2	32.8	32 5	32.2	32.0
20	66.3	65.7	65.0	64.3	64.0
30	99.5	98.5	97.5	96.5	96.0
40	132.7	131.3	130.0	128.7	128.0
50	165.8	164.2	162.5	160.8	160.0

	189	**187**	**185**	**183**	**181**
1	3.2	3.1	3.1	3.0	3.0
2	6.3	6.2	6.2	6.1	6.0
3	9.4	9.4	9.2	9.2	9.0
4	12.6	12.5	12.3	12.2	12.1
5	15.8	15.6	15.4	15.2	15.1
6	18.9	18.7	18.5	18.3	18.1
7	22.0	21.8	21.6	21.4	21.1
8	25.2	24.9	24.7	24.4	24.1
9	28.4	28.0	27.8	27.4	27.2
10	31.5	31.2	30.8	30.5	30.2
20	63.0	62.3	61.7	61.0	60.3
30	94.5	93.5	92.5	91.5	90.5
40	126.0	124.7	123.3	122.0	120.7
50	157.5	155.8	154.2	152.5	150.8

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	
0	8.84 358	181	8.84 464	182	1.15 536	9.99 894	60
1	8.84 539	179	8.84 646	180	1.15 354	9.99 893	59
2	8.84 718	179	8.84 826	180	1.15 174	9.99 892	58
3	8.84 897	178	8.85 006	179	1.14 994	9.99 891	57
4	8.85 075	177	8.85 185	178	1.14 815	9.99 891	56
5	8.85 252	177	8.85 363	177	1.14 637	9.99 890	55
6	8.85 429	176	8.85 540	177	1.14 460	9.99 889	54
7	8.85 605	175	8.85 717	176	1.14 283	9.99 888	53
8	8.85 780	175	8.85 893	176	1.14 107	9.99 887	52
9	8.85 955	173	8.86 069	174	1.13 931	9.99 886	51
10	8.86 128	173	8.86 243	174	1.13 757	9.99 885	50
11	8.86 301	173	8.86 417	174	1.13 583	9.99 884	49
12	8.86 474	171	8.86 591	172	1.13 409	9.99 883	48
13	8.86 645	171	8.86 763	172	1.13 237	9.99 882	47
14	8.86 816	171	8.86 935	171	1.13 065	9.99 881	46
15	8.86 987	169	8.87 106	171	1.12 894	9.99 880	45
16	8.87 156	169	8.87 277	170	1.12 723	9.99 879	44
17	8.87 325	169	8.87 447	169	1.12 553	9.99 879	43
18	8.87 494	167	8.87 616	169	1.12 384	9.99 878	42
19	8.87 661	168	8.87 785	168	1.12 215	9.99 877	41
20	8.87 829	166	8.87 953	167	1.12 047	9.99 876	40
21	8.87 995	166	8.88 120	167	1.11 880	9.99 875	39
22	8.88 161	165	8.88 287	166	1.11 713	9.99 874	38
23	8.88 326	164	8.88 453	165	1.11 547	9.99 873	37
24	8.88 490	164	8.88 618	165	1.11 382	9.99 872	36
25	8.88 654	163	8.88 783	165	1.11 217	9.99 871	35
26	8.88 817	163	8.88 948	163	1.11 052	9.99 870	34
27	8.88 980	162	8.89 111	163	1.10 889	9.99 869	33
28	8.89 142	162	8.89 274	163	1.10 726	9.99 868	32
29	8.89 304	160	8.89 437	161	1.10 563	9.99 867	31
30	8.89 464	161	8.89 598	162	1.10 402	9.99 866	30
31	8.89 625	159	8.89 760	160	1.10 240	9.99 865	29
32	8.89 784	159	8.89 920	160	1.10 080	9.99 864	28
33	8.89 943	159	8.90 080	160	1.09 920	9.99 863	27
34	8.90 102	158	8.90 240	159	1.09 760	9.99 862	26
35	8.90 260	157	8.90 399	158	1.09 601	9.99 861	25
36	8.90 417	157	8.90 557	158	1.09 443	9.99 860	24
37	8.90 574	156	8.90 715	157	1.09 285	9.99 859	23
38	8.90 730	155	8.90 872	157	1.09 128	9.99 858	22
39	8.90 885	155	8.91 029	156	1.08 971	9.99 857	21
40	8.91 040	155	8.91 185	155	1.08 815	9.99 856	20
41	8.91 195	154	8.91 340	155	1.08 660	9.99 855	19
42	8.91 349	153	8.91 495	155	1.08 505	9.99 854	18
43	8.91 502	153	8.91 650	153	1.08 350	9.99 853	17
44	8.91 655	152	8.91 803	154	1.08 197	9.99 852	16
45	8.91 807	152	8.91 957	153	1.08 043	9.99 851	15
46	8.91 959	151	8.92 110	152	1.07 890	9.99 850	14
47	8.92 110	151	8.92 262	152	1.07 738	9.99 848	13
48	8.92 261	150	8.92 414	151	1.07 586	9.99 847	12
49	8.92 411	150	8.92 565	151	1.07 435	9.99 846	11
50	8.92 561	149	8.92 716	150	1.07 284	9.99 845	10
51	8.92 710	149	8.92 866	150	1.07 134	9.99 844	9
52	8.92 859	148	8.93 016	149	1.06 984	9.99 843	8
53	8.93 007	147	8.93 165	148	1.06 835	9.99 842	7
54	8.93 154	147	8.93 313	149	1.06 687	9.99 841	6
55	8.93 301	147	8.93 462	147	1.06 538	9.99 840	5
56	8.93 448	146	8.93 609	147	1.06 391	9.99 839	4
57	8.93 594	146	8.93 756	147	1.06 244	9.99 838	3
58	8.93 740	145	8.93 903	146	1.06 097	9.99 837	2
59	8.93 885	145	8.94 049	146	1.05 951	9.99 836	1
60	8.94 030		8.94 195		1.05 805	9.99 834	0
	log cos	d	log cot	c d	log tan	log sin	′

Prop. Parts

	182	181	179	178	177
1	3.0	3.0	3.0	3.0	3.0
2	6.1	6.0	6.0	5.9	5.9
3	9.1	9.0	9.0	8.9	8.8
4	12.1	12.1	11.9	11.9	11.8
5	15.2	15.1	14.9	14.8	14.8
6	18.2	18.1	17.9	17.8	17.7
7	21.2	21.1	20.9	20.8	20.6
8	24.3	24.1	23.9	23.7	23.6
9	27.3	27.2	26.8	26.7	26.6
10	30.3	30.2	29.8	29.7	29.5
20	60.7	60.3	59.7	59.3	59.0
30	91.0	90.5	89.5	89.0	88.5
40	121.3	120.7	119.3	118.7	118.0
50	151.7	150.8	149.2	148.3	147.5

	176	175	174	173	172
1	2.9	2.9	2.9	2.9	2.9
2	5.9	5.8	5.8	5.8	5.7
3	8.8	8.8	8.7	8.6	8.6
4	11.7	11.7	11.6	11.5	11.5
5	14.7	14.6	14.5	14.4	14.3
6	17.6	17.5	17.4	17.3	17.2
7	20.5	20.4	20.3	20.2	20.1
8	23.5	23.3	23.2	23.1	22.9
9	26.4	26.2	26.1	26.0	25.8
10	29.3	29.2	29.0	28.8	28.7
20	58.7	58.3	58.0	57.7	57.3
30	88.0	87.5	87.0	86.5	86.0
40	117.3	116.7	116.0	115.3	114.7
50	146.7	145.8	145.0	144.2	143.3

	171	170	169	168	167
1	2.8	2.8	2.8	2.8	2.8
2	5.7	5.7	5.6	5.6	5.6
3	8.6	8.5	8.4	8.4	8.4
4	11.4	11.3	11.3	11.2	11.1
5	14.2	14.2	14.1	14.0	13.9
6	17.1	17.0	16.9	16.8	16.7
7	20.0	19.8	19.7	19.6	19.5
8	22.8	22.7	22.5	22.4	22.3
9	25.6	25.5	25.4	25.2	25.0
10	28.5	28.3	28.2	28.0	27.8
20	57.0	56.7	56.3	56.0	55.7
30	85.5	85.0	84.5	84.0	83.5
40	114.0	113.3	112.7	112.0	111.3
50	142.5	141.7	140.8	140.0	139.2

	166	165	164	163	162
1	2.8	2.8	2.7	2.7	2.7
2	5.5	5.5	5.5	5.4	5.4
3	8.3	8.2	8.2	8.2	8.1
4	11.1	11.0	10.9	10.9	10.8
5	13.8	13.8	13.7	13.6	13.5
6	16.6	16.5	16.4	16.3	16.2
7	19.4	19.2	19.1	19.0	18.9
8	22.1	22.0	21.9	21.7	21.6
9	24.9	24.8	24.6	24.4	24.3
10	27.7	27.5	27.3	27.2	27.0
20	55.3	55.0	54.7	54.3	54.0
30	83.0	82.5	82.0	81.5	81.0
40	110.7	110.0	109.3	108.7	108.0
50	138.3	137.5	136.7	135.8	135.0

	161	160	159	158	157
1	2.7	2.7	2.6	2.6	2.6
2	5.4	5.3	5.3	5.3	5.2
3	8.0	8.0	8.0	7.9	7.8
4	10.7	10.7	10.6	10.5	10.5
5	13.4	13.3	13.2	13.2	13.1
6	16.1	16.0	15.9	15.8	15.7
7	18.8	18.7	18.6	18.4	18.3
8	21.5	21.3	21.2	21.1	20.9
9	24.2	24.0	23.8	23.7	23.6
10	26.8	26.7	26.5	26.3	26.2
20	53.7	53.3	53.0	52.7	52.3
30	80.5	80.0	79.5	79.0	78.5
40	107.3	106.7	106.0	105.3	104.7
50	134.2	133.3	132.5	131.7	130.8

	156	155	154	153	152
1	2.6	2.6	2.6	2.6	2.5
2	5.2	5.2	5.1	5.1	5.1
3	7.8	7.8	7.7	7.6	7.6
4	10.4	10.3	10.3	10.2	10.1
5	13.0	12.9	12.8	12.8	12.7
6	15.6	15.5	15.4	15.3	15.2
7	18.2	18.1	18.0	17.8	17.7
8	20.8	20.7	20.5	20.4	20.3
9	23.4	23.2	23.1	23.0	22.8
10	26.0	25.8	25.7	25.5	25.3
20	52.0	51.7	51.3	51.0	50.7
30	78.0	77.5	77.0	76.5	76.0
40	104.0	103.3	102.7	102.0	101.3
50	130.0	129.2	128.3	127.5	126.7

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	
0	8.94 030	144	8.94 195	145	1.05 805	9.99 834	60
1	8.94 174	143	8.94 340	145	1.05 660	9.99 833	59
2	8.94 317	144	8.94 485	145	1.05 515	9.99 832	58
3	8.94 461	142	8.94 630	143	1.05 370	9.99 831	57
4	8.94 603	143	8.94 773	144	1.05 227	9.99 830	56
5	8.94 746	141	8.94 917	143	1.05 083	9.99 829	55
6	8.94 887	142	8.95 060	142	1.04 940	9.99 828	54
7	8.95 029	141	8.95 202	142	1.04 798	9.99 827	53
8	8.95 170	140	8.95 344	142	1.04 656	9.99 825	52
9	8.95 310	140	8.95 486	141	1.04 514	9.99 824	51
10	8.95 450	139	8.95 627	140	1.04 373	9.99 823	50
11	8.95 589	139	8.95 767	141	1.04 233	9.99 822	49
12	8.95 728	139	8.95 908	139	1.04 092	9.99 821	48
13	8.95 867	138	8.96 047	140	1.03 953	9.99 820	47
14	8.96 005	138	8.96 187	138	1.03 813	9.99 819	46
15	8.96 143	137	8.96 325	139	1.03 675	9.99 817	45
16	8.96 280	137	8.96 464	138	1.03 536	9.99 816	44
17	8.96 417	136	8.96 602	137	1.03 398	9.99 815	43
18	8.96 553	136	8.96 739	138	1.03 261	9.99 814	42
19	8.96 689	136	8.96 877	136	1.03 123	9.99 813	41
20	8.96 825	135	8.97 013	137	1.02 987	9.99 812	40
21	8.96 960	135	8.97 150	135	1.02 850	9.99 810	39
22	8.97 095	134	8.97 285	136	1.02 715	9.99 809	38
23	8.97 229	134	8.97 421	135	1.02 579	9.99 808	37
24	8.97 363	133	8.97 556	135	1.02 444	9.99 807	36
25	8.97 496	133	8.97 691	134	1.02 309	9.99 806	35
26	8.97 629	133	8.97 825	134	1.02 175	9.99 804	34
27	8.97 762	132	8.97 959	133	1.02 041	9.99 803	33
28	8.97 894	132	8.98 092	133	1.01 908	9.99 802	32
29	8.98 026	131	8.98 225	133	1.01 775	9.99 801	31
30	8.98 157	131	8.98 358	132	1.01 642	9.99 800	30
31	8.98 288	131	8.98 490	132	1.01 510	9.99 798	29
32	8.98 419	130	8.98 622	131	1.01 378	9.99 797	28
33	8.98 549	130	8.98 753	131	1.01 247	9.99 796	27
34	8.98 679	129	8.98 884	131	1.01 116	9.99 795	26
35	8.98 808	129	8.99 015	130	1.00 985	9.99 793	25
36	8.98 937	129	8.99 145	130	1.00 855	9.99 792	24
37	8.99 066	128	8.99 275	130	1.00 725	9.99 791	23
38	8.99 194	128	8.99 405	129	1.00 595	9.99 790	22
39	8.99 322	128	8.99 534	128	1.00 466	9.99 788	21
40	8.99 450	127	8.99 662	129	1.00 338	9.99 787	20
41	8.99 577	127	8.99 791	128	1.00 209	9.99 786	19
42	8.99 704	126	8.99 919	127	1.00 081	9.99 785	18
43	8.99 830	126	9.00 046	128	0.99 954	9.99 783	17
44	8.99 956	126	9.00 174	127	0.99 826	9.99 782	16
45	9.00 082	125	9.00 301	126	0.99 699	9.99 781	15
46	9.00 207	125	9.00 427	126	0.99 573	9.99 780	14
47	9.00 332	124	9.00 553	126	0.99 447	9.99 778	13
48	9.00 456	125	9.00 679	126	0.99 321	9.99 777	12
49	9.00 581	123	9.00 805	125	0.99 195	9.99 776	11
50	9.00 704	124	9.00 930	125	0.99 070	9.99 775	10
51	9.00 828	123	9.01 055	124	0.98 945	9.99 773	9
52	9.00 951	123	9.01 179	124	0.98 821	9.99 772	8
53	9.01 074	122	9.01 303	124	0.98 697	9.99 771	7
54	9.01 196	122	9.01 427	123	0.98 573	9.99 769	6
55	9.01 318	122	9.01 550	123	0.98 450	9.99 768	5
56	9.01 440	121	9.01 673	123	0.98 327	9.99 767	4
57	9.01 561	121	9.01 796	122	0.98 204	9.99 765	3
58	9.01 682	121	9.01 918	122	0.98 082	9.99 764	2
59	9.01 803	120	9.02 040	122	0.97 960	9.99 763	1
60	9.01 923		9.02 162		0.97 838	9.99 761	0
	log cos	d	log cot	c d	log tan	log sin	′

Prop. Parts

	151	149	148	147	146
1	2.5	2.5	2.5	2.4	2.4
2	5.0	5.0	4.9	4.9	4.9
3	7.6	7.4	7.4	7.4	7.3
4	10.1	9.9	9.9	9.8	9.7
5	12.6	12.4	12.3	12.2	12.2
6	15.1	14.9	14.8	14.7	14.6
7	17.6	17.4	17.3	17.2	17.0
8	20.1	19.9	19.7	19.6	19.5
9	22.6	22.4	22.2	22.0	21.9
10	25.2	24.8	24.7	24.5	24.3
20	50.3	49.7	49.3	49.0	48.7
30	75.5	74.5	74.0	73.5	73.0
40	100.7	99.3	98.7	98.0	97.3
50	125.8	124.2	123.3	122.5	121.7

	145	144	143	142	141
1	2.4	2.4	2.4	2.4	2.4
2	4.8	4.8	4.8	4.7	4.7
3	7.2	7.2	7.2	7.1	7.0
4	9.7	9.6	9.5	9.5	9.4
5	12.1	12.0	11.9	11.8	11.8
6	14.5	14.4	14.3	14.2	14.1
7	16.9	16.8	16.7	16.6	16.4
8	19.3	19.2	19.1	18.9	18.8
9	21.8	21.6	21.4	21.3	21.2
10	24.2	24.0	23.8	23.7	23.5
20	48.3	48.0	47.7	47.3	47.0
30	72.5	72.0	71.5	71.0	70.5
40	96.7	96.0	95.3	94.7	94.0
50	120.8	120.0	119.2	118.3	117.5

	140	139	138	137	136
1	2.3	2.3	2.3	2.3	2.3
2	4.7	4.6	4.6	4.6	4.5
3	7.0	7.0	6.9	6.8	6.8
4	9.3	9.3	9.2	9.1	9.1
5	11.7	11.6	11.5	11.4	11.3
6	14.0	13.9	13.8	13.7	13.6
7	16.3	16.2	16.1	16.0	15.9
8	18.7	18.5	18.4	18.3	18.1
9	21.0	20.8	20.7	20.6	20.4
10	23.3	23.2	23.0	22.8	22.7
20	46.7	46.3	46.0	45.7	45.3
30	70.0	69.5	69.0	68.5	68.0
40	93.3	92.7	92.0	91.3	90.7
50	116.7	115.8	115.0	114.2	113.3

	135	134	133	132	131
1	2.2	2.2	2.2	2.2	2.2
2	4.5	4.5	4.4	4.4	4.4
3	6.8	6.7	6.6	6.6	6.6
4	9.0	8.9	8.9	8.8	8.7
5	11.2	11.2	11.1	11.0	10.9
6	13.5	13.4	13.3	13.2	13.1
7	15.8	15.6	15.5	15.4	15.3
8	18.0	17.9	17.7	17.6	17.5
9	20.2	20.1	20.0	19.8	19.6
10	22.5	22.3	22.2	22.0	21.8
20	45.0	44.7	44.3	44.0	43.7
30	67.5	67.0	66.5	66.0	65.5
40	90.0	89.3	88.7	88.0	87.3
50	112.5	111.7	110.8	110.0	109.2

	130	129	128	127	126
1	2.2	2.2	2.1	2.1	2.1
2	4.3	4.3	4.3	4.2	4.2
3	6.5	6.4	6.4	6.4	6.3
4	8.7	8.6	8.5	8.5	8.4
5	10.8	10.8	10.7	10.6	10.5
6	13.0	12.9	12.8	12.7	12.6
7	15.2	15.0	14.9	14.8	14.7
8	17.3	17.2	17.1	16.9	16.8
9	19.5	19.4	19.2	19.0	18.9
10	21.7	21.5	21.3	21.2	21.0
20	43.3	43.0	42.7	42.3	42.0
30	65.0	64.5	64.0	63.5	63.0
40	86.7	86.0	85.3	84.7	84.0
50	108.3	107.5	106.7	105.8	105.0

	125	124	123	122	121
1	2.1	2.1	2.0	2.0	2.0
2	4.2	4.1	4.1	4.1	4.0
3	6.2	6.2	6.2	6.1	6.0
4	8.3	8.3	8.2	8.1	8.1
5	10.4	10.3	10.2	10.2	10.1
6	12.5	12.4	12.3	12.2	12.1
7	14.6	14.5	14.4	14.2	14.1
8	16.7	16.5	16.4	16.3	16.1
9	18.8	18.6	18.4	18.3	18.2
10	20.8	20.7	20.5	20.3	20.2
20	41.7	41.3	41.0	40.7	40.3
30	62.5	62.0	61.5	61.0	60.5
40	83.3	82.7	82.0	81.3	80.7
50	104.2	103.3	102.5	101.7	100.8

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	
0	9.01 923	120	9.02 162	121	0.97 838	9.99 761	**60**
1	9.02 043	120	9.02 283	121	0.97 717	9.99 760	59
2	9.02 163	120	9.02 404	121	0.97 596	9.99 759	58
3	9.02 283	119	9.02 525	120	0.97 475̄	9.99 757	57
4	9.02 402	118	9.02 645	121	0.97 355̄	9.99 756	56
5	9.02 520	119	9.02 766	119	0.97 234	9.99 755̄	**55**
6	9.02 639	118	9.02 885	120	0.97 115̄	9.99 753	54
7	9.02 757	117	9.03 005̄	119	0.96 995	9.99 752	53
8	9.02 874	118	9.03 124	118	0.96 876	9.99 751	52
9	9.02 992	117	9.03 242	119	0.96 758	9.99 749	51
10	9.03 109	117	9.03 361	118	0.96 639	9.99 748	**50**
11	9.03 226	116	9.03 479	118	0.96 521	9.99 747	49
12	9.03 342	116	9.03 597	117	0.96 403	9.99 745	48
13	9.03 458	116	9.03 714	118	0.96 286	9.99 744	47
14	9.03 574	116	9.03 832	116	0.96 168	9.99 742	46
15	9.03 690	115	9.03 948	117	0.96 052	9.99 741	**45**
16	9.03 805̄	115	9.04 065	116	0.95 935̄	9.99 740	44
17	9.03 920	114	9.04 181	116	0.95 819	9.99 738	43
18	9.04 034	115	9.04 297	116	0.95 703	9.99 737	42
19	9.04 149	113	9.04 413	115	0.95 587	9.99 736	41
20	9.04 262	114	9.04 528	115	0.95 472	9.99 734	**40**
21	9.04 376	114	9.04 643	115	0.95 357	9.99 733	39
22	9.04 490	113	9.04 758	115	0.95 242	9.99 731	38
23	9.04 603	112	9.04 873	114	0.95 127	9.99 730	37
24	9.04 715	113	9.04 987	114	0.95 013	9.99 728	36
25	9.04 828	112	9.05 101	113	0.94 899	9.99 727	**35**
26	9.04 940	112	9.05 214	114	0.94 786	9.99 726	34
27	9.05 052	112	9.05 328	113	0.94 672	9.99 724	33
28	9.05 164	111	9.05 441	112	0.94 559	9.99 723	32
29	9.05 275̄	111	9.05 553	113	0.94 447	9.99 721	31
30	9.05 386	111	9.05 666	112	0.94 334	9.99 720	**30**
31	9.05 497	110	9.05 778	112	0.94 222	9.99 718	29
32	9.05 607	110	9.05 890	112	0.94 110	9.99 717	28
33	9.05 717	110	9.06 002	111	0.93 998	9.99 716	27
34	9.05 827	110	9.06 113	111	0.93 887	9.99 714	26
35	9.05 937	109	9.06 224	111	0.93 776	9.99 713	**25**
36	9.06 046	109	9.06 335̄	110	0.93 665	9.99 711	24
37	9.06 155	109	9.06 445	111	0.93 555̄	9.99 710	23
38	9.06 264	108	9.06 556	110	0.93 444	9.99 708	22
39	9.06 372	109	9.06 666	109	0.93 334	9.99 707	21
40	9.06 481	108	9.06 775̄	110	0.93 225̄	9.99 705	**20**
41	9.06 589	107	9.06 885̄	109	0.93 115	9.99 704	19
42	9.06 696	108	9.06 994	109	0.93 006	9.99 702	18
43	9.06 804	107	9.07 103	108	0.92 897	9.99 701	17
44	9.06 911	107	9.07 211	109	0.92 789	9.99 699	16
45	9.07 018	106	9.07 320	108	0.92 680	9.99 698	**15**
46	9.07 124	107	9.07 428	108	0.92 572	9.99 696	14
47	9.07 231	106	9.07 536	107	0.92 464	9.99 695̄	13
48	9.07 337	105	9.07 643	108	0.92 357	9.99 693	12
49	9.07 442	106	9.07 751	107	0.92 249	9.99 692	11
50	9.07 548	105	9.07 858	106	0.92 142	9.99 690	**10**
51	9.07 653	105	9.07 964	107	0.92 036	9.99 689	9
52	9.07 758	105	9.08 071	106	0.91 929	9.99 687	8
53	9.07 863	105	9.08 177	106	0.91 823	9.99 686	7
54	9.07 968	104	9.08 283	106	0.91 717	9.99 684	6
55	9.08 072	104	9.08 389	106	0.91 611	9.99 683	**5**
56	9.08 176	104	9.08 495̄	105	0.91 505	9.99 681	4
57	9.08 280	103	9.08 600	105	0.91 400	9.99 680	3
58	9.08 383	103	9.08 705	105	0.91 295̄	9.99 678	2
59	9.08 486	103	9.08 810	104	0.91 190	9.99 677	1
60	9.08 589		9.08 914		0.91 086	9.99 675	0
	log cos	d	log cot	c d	log tan	log sin	′

Prop. Parts

	121	**120**	**119**	**118**
1	2.0	2.0	2.0	2.0
2	4.0	4.0	4.0	3.9
3	6.0	6.0	6.0	5.9
4	8.1	8.0	7.9	7.9
5	10.1	10.0	9.9	9.8
6	12.1	12.0	11.9	11.8
7	14.1	14.0	13.9	13.8
8	16.1	16.0	15.9	15.7
9	18.2	18.0	17.8	17.7
10	20.2	20.0	19.8	19.7
20	40.3	40.0	39.7	39.3
30	60.5	60.0	59.5	59.0
40	80.7	80.0	79.3	78.7
50	100.8	100.0	99.2	98.3

	117	**116**	**115**	**114**
1	2.0	1.9	1.9	1.9
2	3.9	3.9	3.8	3.8
3	5.8	5.8	5.8	5.7
4	7.8	7.7	7.7	7.6
5	9.8	9.7	9.6	9.5
6	11.7	11.6	11.5	11.4
7	13.6	13.5	13.4	13.3
8	15.6	15.5̄	15.3	15.2
9	17.6	17.4	17.2	17.1
10	19.5	19.3	19.2	19.0
20	39.0	38.7	38.3	38.0
30	58.5	58.0	57.5	57.0
40	78.0	77.3	76.7	76.0
50	97.5	96.7	95.8	95.0

	113	**112**	**111**	**110**
1	1.9	1.9	1.8	1.8
2	3.8	3.7	3.7	3.7
3	5.6	5.6	5.6	5.5
4	7.5	7.5	7.4	7.3
5	9.4	9.3	9.2	9.2
6	11.3	11.2	11.1	11.0
7	13.2	13.1	13.0	12.8
8	15.1	14.9	14.8	14.7
9	17.0	16.8	16.6	16.5
10	18.8	18.7	18.5	18.3
20	37.7	37.3	37.0	36.7
30	56.5	56.0	55.5	55.0
40	75.3	74.7	74.0	73.3
50	94.2	93.3	92.5	91.7

	109	**108**	**107**	**106**
1	1.8	1.8	1.8	1.8
2	3.6	3.6	3.6	3.5
3	5.4	5.4	5.4	5.3
4	7.3	7.2	7.1	7.1
5	9.1	9.0	8.9	8.8
6	10.9	10.8	10.7	10.6
7	12.7	12.6	12.5̄	12.4
8	14.5	14.4	14.3	14.1
9	16.4	16.2	16.0	15.9
10	18.2	18.0	17.8	17.7
20	36.3	36.0	35.7	35.3
30	54.5	54.0	53.5	53.0
40	72.7	72.0	71.3	70.7
50	90.8	90.0	89.2	88.3

Prop. Parts

173° 263° **353°** **83°**

′	log sin	d	log tan	c d	log cot	log cos	
0	9.08 589	103	9.08 914	105	0.91 086	9.99 675	**60**
1	9.08 692	103	9.09 019	104	0.90 981	9.99 674	59
2	9.08 795̄	102	9.09 123	104	0.90 877	9.99 672	58
3	9.08 897	102	9.09 227	103	0.90 773	9.99 670	57
4	9.08 999	102	9.09 330	104	0.90 670	9.99 669	56
5	9.09 101	101	9.09 434	103	0.90 566	9.99 667	**55**
6	9.09 202	102	9.09 537	103	0.90 463	9.99 666	54
7	9.09 304	101	9.09 640	102	0.90 360	9.99 664	53
8	9.09 405̄	101	9.09 742	103	0.90 258	9.99 663	52
9	9.09 506	100	9.09 845̄	102	0.90 155	9.99 661	51
10	9.09 606	101	9.09 947	102	0.90 053	9.99 659	**50**
11	9.09 707	100	9.10 049	101	0.89 951	9.99 658	49
12	9.09 807	100	9.10 150	102	0.89 850	9.99 656	48
13	9.09 907	99	9.10 252	101	0.89 748	9.99 655̄	47
14	9.10 006	100	9.10 353	101	0.89 647	9.99 653	46
15	9.10 106	99	9.10 454	101	0.89 546	9.99 651	**45**
16	9.10 205̄	99	9.10 555̄	101	0.89 445	9.99 650	44
17	9.10 304	98	9.10 656	100	0.89 344	9.99 648	43
18	9.10 402	99	9.10 756	100	0.89 244	9.99 647	42
19	9.10 501	98	9.10 856	100	0.89 144	9.99 645̄	41
20	9.10 599	98	9.10 956	100	0.89 044	9.99 643	**40**
21	9.10 697	98	9.11 056	99	0.88 944	9.99 642	39
22	9.10 795	98	9.11 155	99	0.88 845̄	9.99 640	38
23	9.10 893	97	9.11 254	99	0.88 746	9.99 638	37
24	9.10 990	97	9.11 353	99	0.88 647	9.99 637	36
25	9.11 087	97	9.11 452	99	0.88 548	9.99 635	**35**
26	9.11 184	97	9.11 551	98	0.88 449	9.99 633	34
27	9.11 281	96	9.11 649	98	0.88 351	9.99 632	33
28	9.11 377	97	9.11 747	98	0.88 253	9.99 630	32
29	9.11 474	96	9.11 845	98	0.88 155̄	9.99 629	31
30	9.11 570	96	9.11 943	97	0.88 057	9.99 627	**30**
31	9.11 666	95	9.12 040	98	0.87 960	9.99 625	29
32	9.11 761	96	9.12 138	97	0.87 862	9.99 624	28
33	9.11 857	95	9.12 235̄	97	0.87 765	9.99 622	27
34	9.11 952	95	9.12 332	96	0.87 668	9.99 620	26
35	9.12 047	95	9.12 428	97	0.87 572	9.99 618	**25**
36	9.12 142	94	9.12 525̄	96	0.87 475	9.99 617	24
37	9.12 236	95	9.12 621	96	0.87 379	9.99 615	23
38	9.12 331	94	9.12 717	96	0.87 283	9.99 613	22
39	9.12 425̄	94	9.12 813	96	0.87 187	9.99 612	21
40	9.12 519	93	9.12 909	95	0.87 091	9.99 610	**20**
41	9.12 612	94	9.13 004	95	0.86 996	9.99 608	19
42	9.12 706	93	9.13 099	95	0.86 901	9.99 607	18
43	9.12 799	93	9.13 194	95	0.86 806	9.99 605̄	17
44	9.12 892	93	9.13 289	95	0.86 711	9.99 603	16
45	9.12 985	93	9.13 384	94	0.86 616	9.99 601	**15**
46	9.13 078	93	9.13 478	95	0.86 522	9.99 600	14
47	9.13 171	92	9.13 573	94	0.86 427	9.99 598	13
48	9.13 263	92	9.13 667	94	0.86 333	9.99 596	12
49	9.13 355	92	9.13 761	93	0.86 239	9.99 595̄	11
50	9.13 447	92	9.13 854	94	0.86 146	9.99 593	**10**
51	9.13 539	91	9.13 948	93	0.86 052	9.99 591	9
52	9.13 630	92	9.14 041	93	0.85 959	9.99 589	8
53	9.13 722	91	9.14 134	93	0.85 866	9.99 588	7
54	9.13 813	91	9.14 227	93	0.85 773	9.99 586	6
55	9.13 904	90	9.14 320	92	0.85 680	9.99 584	**5**
56	9.13 994	91	9.14 412	92	0.85 588	9.99 582	4
57	9.14 085	90	9.14 504	93	0.85 496	9.99 581	3
58	9.14 175	91	9.14 597	91	0.85 403	9.99 579	2
59	9.14 266	90	9.14 688	92	0.85 312	9.99 577	1
60	9.14 356		9.14 780		0.85 220	9.99 575	**0**
	log cos	d	log cot	c d	log tan	log sin	′

Prop. Parts

	105	**104**	**103**	**102**
1	1.8	1.7	1.7	1.7
2	3.5	3.5̄	3.4	3.4
3	5.2	5.2	5.2	5.1
4	7.0	6.9	6.9	6.8
5	8.8	8.7	8.6	8.5
6	10.5	10.4	10.3	10.2
7	12.2	12.1	12.0	11.9
8	14.0	13.9	13.7	13.6
9	15.8	15.6	15.4	15.3
10	17.5	17.3	17.2	17.0
20	35.0	34.7	34.3	34.0
30	52.5	52.0	51.5	51.0
40	70.0	69.3	68.7	68.0
50	87.5	86.7	85.8	85.0

	101	**100**	**99**	**98**
1	1.7	1.7	1.6	1.6
2	3.4	3.3	3.3	3.3
3	5.0	5.0	5.0̄	4.9
4	6.7	6.7	6.6	6.5
5	8.4	8.3	8.2	8.2
6	10.1	10.0	9.9	9.8
7	11.8	11.7	11.6	11.4
8	13.5̄	13.3	13.2	13.1
9	15.2	15.0	14.8	14.7
10	16.8	16.7	16.5	16.3
20	33.7	33.3	33.0	32.7
30	50.5	50.0	49.5	49.0
40	67.3	66.7	66.0	65.3
50	84.2	83.3	82.5	81.7

	97	**96**	**95**	**94**
1	1.6	1.6	1.6	1.6
2	3.2	3.2	3.2	3.1
3	4.8	4.8	4.8	4.7
4	6.5̄	6.4	6.3	6.3
5	8.1	8.0	7.9	7.8
6	9.7	9.6	9.5	9.4
7	11.3	11.2	11.1	11.0
8	12.9	12.8	12.7	12.5
9	14.6	14.4	14.2	14.1
10	16.2	16.0	15.8	15.7
20	32.3	32.0	31.7	31.3
30	48.5	48.0	47.5	47.0
40	64.7	64.0	63.3	62.7
50	80.8	80.0	79.2	78.3

	93	**92**	**91**	**90**
1	1.6	1.5	1.5	1.5
2	3.1	3.1	3.0	3.0
3	4.6	4.6	4.6	4.5
4	6.2	6.1	6.1	6.0
5	7.8	7.7	7.6	7.5
6	9.3	9.2	9.1	9.0
7	10.8	10.7	10.6	10.5
8	12.4	12.3	12.1	12.0
9	14.0	13.8	13.6	13.5
10	15.5	15.3	15.2	15.0
20	31.0	30.7	30.3	30.0
30	46.5	46.0	45.5	45.0
40	62.0	61.3	60.7	60.0
50	77.5	76.7	75.8	75.0

Prop. Parts

172° 262° **352°** **82°**

′	log sin	d	log tan	c d	log cot	log cos	′
0	9.14 356	89	9.14 780	92	0.85 220	9.99 575	**60**
1	9.14 445	90	9.14 872	91	0.85 128	9.99 574	59
2	9.14 535̄	89	9.14 963	91	0.85 037	9.99 572	58
3	9.14 624	90	9.15 054	91	0.84 946	9.99 570	57
4	9.14 714	89	9.15 145	91	0.84 855̄	9.99 568	56
5	9.14 803	88	9.15 236	91	0.84 764	9.99 566	**55**
6	9.14 891	89	9.15 327	90	0.84 673	9.99 565̄	54
7	9.14 980	89	9.15 417	91	0.84 583	9.99 563	53
8	9.15 069	88	9.15 508	90	0.84 492	9.99 561	52
9	9.15 157	88	9.15 598	90	0.84 402	9.99 559	51
10	9.15 245	88	9.15 688	89	0.84 312	9.99 557	**50**
11	9.15 333	88	9.15 777	90	0 84 223	9.99 556	49
12	9.15 421	87	9.15 867	89	0.84 133	9.99 554	48
13	9.15 508	88	9.15 956	90	0.84 044	9.99 552	47
14	9.15 596	87	9.16 046	89	0.83 954	9.99 550	46
15	9.15 683	87	9.16 135̄	89	0.83 865	9.99 548	**45**
16	9.15 770	87	9.16 224	88	0.83 776	9.99 546	44
17	9.15 857	87	9.16 312	89	0.83 688	9.99 545̄	43
18	9.15 944	86	9.16 401	88	0.83 599	9.99 543	42
19	9.16 030	86	9.16 489	88	0.83 511	9.99 541	41
20	9.16 116	87	9.16 577	88	0.83 423	9.99 539	**40**
21	9.16 203	86	9.16 665	88	0.83 335̄	9.99 537	39
22	9.16 289	85	9.16 753	88	0.83 247	9.99 535	38
23	9.16 374	86	9.16 841	87	0.83 159	9.99 533	37
24	9.16 460	85	9.16 928	88	0.83 072	9.99 532	36
25	9.16 545	86	9.17 016	87	0.82 984	9.99 530	**35**
26	9 16 631	85	9.17 103	87	0.82 897	9.99 528	34
27	9.16 716	85	9.17 190	87	0.82 810	9.99 526	33
28	9.16 801	85	9.17 277	86	0.82 723	9.99 524	32
29	9.16 886	84	9.17 363	87	0.82 637	9.99 522	31
30	9.16 970	85	9.17 45̄0	86	0.82 550	9.99 520	**30**
31	9.17 055̄	84	9.17 536	86	0.82 464	9.99 518	29
32	9.17 139	84	9.17 622	86	0.82 378	9.99 517	28
33	9.17 223	84	9.17 708	86	0.82 292	9.99 515̄	27
34	9.17 307	84	9.17 794	86	0.82 206	9.99 513	26
35	9.17 391	83	9.17 880	85	0.82 120	9.99 511	**25**
36	9.17 474	84	9.17 965	86	0.82 035̄	9.99 509	24
37	9.17 558	83	9.18 051	85	0.81 949	9.99 507	23
38	9.17 641	83	9.18 136	85	0.81 864	9.99 505	22
39	9.17 724	83	9.18 221	85	0.81 779	9.99 503	21
40	9.17 807	83	9.18 306	85	0.81 694	9.99 501	**20**
41	9.17 890	83	9.18 391	84	0.81 609	9.99 499	19
42	9.17 973	82	9.18 475	85	0.81 525̄	9.99 497	18
43	9.18 055	82	9.18 560	84	0.81 440	9.99 495	17
44	9.18 137	83	9.18 644	84	0.81 356	9.99 494	16
45	9.18 220	82	9.18 728	84	0.81 272	9.99 492	**15**
46	9.18 302	81	9.18 812	84	0.81 188	9.99 490	14
47	9.18 383	82	9.18 896	83	0.81 104	9.99 488	13
48	9.18 465	82	9.18 979	84	0.81 021	9.99 486	12
49	9.18 547	81	9.19 063	83	0.80 937	9.99 484	11
50	9.18 628	81	9.19 146	83	0.80 854	9.99 482	**10**
51	9.18 709	81	9.19 229	83	0.80 771	9.99 480	9
52	9.18 790	81	9.19 312	83	0.80 688	9.99 478	8
53	9.18 871	81	9.19 395	83	0.80 605̄	9.99 476	7
54	9.18 952	81	9.19 478	83	0.80 522	9.99 474	6
55	9 19 033	80	9.19 561	82	0.80 439	9.99 472	**5**
56	9.19 113	80	9.19 643	82	0.80 357	9.99 470	4
57	9.19 193	80	9.19 725	82	0.80 275̄	9.99 468	3
58	9.19 273	80	9.19 807	82	0.80 193	9.99 466	2
59	9.19 353	80	9.19 889	82	0.80 111	9.99 464	1
60	9.19 433		9.19 971		0.80 029	9.99 462	**0**
	log cos	d	log cot	c d	log tan	log sin	′

Prop. Parts

	92	**91**	**90**
1	1.5	1.5	1.5
2	3.1	3.0	3.0
3	4.6	4.6	4.5
4	6.1	6.1	6.0
5	7.7	7.6	7.5
6	9.2	9.1	9.0
7	10.7	10.6	10.5
8	12.3	12.1	12.0
9	13.8	13.6	13.5
10	15.3	15.2	15.0
20	30.7	30.3	30.0
30	46.0	45.5	45.0
40	61.3	60.7	60.0
50	76.7	75.8	75.0

	89	**88**	**87**
1	1.5̄	1.5̄	1.4
2	3.0	2.9	2.9
3	4.4	4.4	4.4
4	5.9	5.9	5.8
5	7.4	7.3	7.2
6	8.9	8.8	8.7
7	10.4	10.3	10.2
8	11.9	11.7	11.6
9	13.4	13.2	13.0
10	14.8	14.7	14.5
20	29.7	29.3	29.0
30	44.5	44.0	43.5
40	59.3	58.7	58.0
50	74.2	73.3	72.5

	86	**85**	**84**
1	1.4	1.4	1.4
2	2.9	2.8	2.8
3	4.3	4.2	4.2
4	5.7	5.7	5.6
5	7.2	7.1	7.0
6	8.6	8.5	8.4
7	10.0	9.9	9.8
8	11.5̄	11.3	11.2
9	12.9	12.8	12.6
10	14.3	14.2	14.0
20	28.7	28.3	28.0
30	43.0	42.5	42.0
40	57.3	56.7	56.0
50	71.7	70.8	70.0

	83	**82**	**81**
1	1.4	1.4	1.4
2	2.8	2.7	2.7
3	4.2	4.1	4.0
4	5.5	5.5̄	5.4
5	6.9	6.8	6.8
6	8.3	8.2	8.1
7	9.7	9.6	9.4
8	11.1	10.9	10.8
9	12.4	12.3	12.2
10	13.8	13.7	13.5
20	27.7	27.3	27.0
30	41.5	41.0	40.5
40	55.3	54.7	54.0
50	69.2	68.3	67.5

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	′
0	9.19 433	80	9.19 971	82	0.80 029	9.99 462	**60**
1	9.19 513	79	9.20 053	81	0.79 947	9.99 460	59
2	9.19 592	80	9.20 134	82	0.79 866	9.99 458	58
3	9.19 672	79	9.20 216	81	0.79 784	9.99 456	57
4	9.19 751	79	9.20 297	81	0.79 703	9.99 454	56
5	9.19 830	79	9.20 378	81	0.79 622	9.99 452	**55**
6	9.19 909	79	9.20 459	81	0.79 541	9.99 45̄0	54
7	9.19 988	79	9.20 540	81	0.79 460	9.99 448	53
8	9.20 067	78	9.20 621	80	0.79 379	9.99 446	52
9	9.20 145	78	9.20 701	81	0.79 299	9.99 444	51
10	9.20 223	79	9.20 782	80	0.79 218	9.99 442	**50**
11	9.20 302	78	9.20 862	80	0.79 138	9.99 440	49
12	9.20 380	78	°.20 942	80	0.79 058	9.99 438	48
13	9.20 458	77	9.21 022	80	0.78 978	9.99 436	47
14	9.20 535	78	9.21 102	80	0.78 898	9.99 434	46
15	9.20 613	78	9.21 182	79	0.78 818	9.99 432	**45**
16	9.20 691	77	9.21 261	80	0.78 739	9.99 429	44
17	9.20 768	77	9.21 341	79	0.78 659	9.99 427	43
18	9.20 845	77	9.21 420	79	0.78 580	9.99 425	42
19	9.20 922	77	9.21 499	79	0.78 501	9.99 423	41
20	9.20 999	77	9.21 578	79	0.78 422	9.99 421	**40**
21	9.21 076	77	9.21 657	79	0.78 343	9.99 419	39
22	9.21 153	76	9.21 736	78	0.78 264	9.99 417	38
23	9.21 229	77	9.21 814	79	0.78 186	9.99 415̄	37
24	9.21 306	76	9.21 893	78	0.78 107	9.99 413	36
25	9.21 382	76	9.21 971	78	0.78 029	9.99 411	**35**
26	9.21 458	76	9.22 049	78	0.77 951	9.99 409	34
27	9.21 534	76	9.22 127	78	0.77 873	9.99 407	33
28	9.21 610	75	9.22 205	78	0.77 795̄	9.99 404	32
29	9.21 685	76	9.22 283	78	0.77 717	9.99 402	31
30	9.21 761	75	9.22 361	77	0.77 639	9.99 400	**30**
31	9.21 836	76	9.22 438	78	0.77 562	9.99 398	29
32	9.21 912	75	9.22 516	77	0.77 484	9.99 396	28
33	9.21 987	75	9.22 593	77	0.77 407	9.99 394	27
34	9.22 062	75	9.22 670	77	0.77 330	9.99 392	26
35	9.22 137	74	9.22 747	77	0.77 253	9.99 390	**25**
36	9.22 211	75	9.22 824	77	0.77 176	9.99 388	24
37	9.22 286	75	9.22 901	76	0.77 099	9.99 385	23
38	9.22 361	74	9.22 977	77	0.77 023	9.99 383	22
39	9.22 435̄	74	9.23 054	76	0.76 946	9.99 381	21
40	9.22 509	74	9.23 130	76	0.76 870	9.99 379	**20**
41	9.22 583	74	9.23 206	77	0.76 794	9.99 377	19
42	9.22 657	74	9.23 283	76	0.76 717	9.99 375̄	18
43	9.22 731	74	9.23 359	76	0.76 641	9.99 372	17
44	9.22 805̄	73	9.23 435̄	75	0.76 565	9.99 370	16
45	9.22 878	74	9.23 510	76	0.76 490	9.99 368	**15**
46	9.22 952	73	9.23 586	75	0.76 414	9.99 366	14
47	9.23 025	73	9.23 661	76	0.76 339	9.99 364	13
48	9.23 098	73	9.23 737	75	0.76 263	9.99 362	12
49	9.23 171	73	9.23 812	75	0.76 188	9.99 359	11
50	9.23 244	73	9.23 887	75	0.76 113	9.99 357	**10**
51	9.23 317	73	9.23 962	75	0.76 038	9.99 355	9
52	9.23 390	72	9.24 037	75	0.75 963	9.99 353	8
53	9.23 462	73	9.24 112	74	0.75 888	9.99 351	7
54	9.23 535̄	72	9.24 186	75	0.75 814	9.99 348	6
55	9.23 607	72	9.24 261	74	0.75 739	9.99 346	**5**
56	9.23 679	73	9.24 335	75	0.75 665̄	9.99 344	4
57	9.23 752	71	9.24 410	74	0.75 590	9.99 342	3
58	9.23 823	72	9.24 484	74	0.75 516	9.99 340	2
59	9.23 895	72	9.24 558	74	0.75 442	9.99 337	1
60	9.23 967		9.24 632		0.75 368	9.99 335	**0**
	log cos	d	log cot	c d	log tan	log sin	′

Prop. Parts

	80	**79**	**78**	**77**
1	1.3	1.3	1.3	1.3
2	2.7	2.6	2.6	2.6
3	4.0	4.0	3.9	3.8
4	5.3	5.3	5.2	5.1
5	6.7	6.6	6.5	6.4
6	8.0	7.9	7.8	7.7
7	9.3	9.2	9.1	9.0
8	10.7	10.5	10.4	10.3
9	12.0	11.8	11.7	11.6
10	13.3	13.2	13.0	12.8
20	26.7	26.3	26.0	25.7
30	40.0	39.5	39.0	38.5
40	53.3	52.7	52.0	51.3
50	66.7	65.8	65.0	64.2

	76	**75**	**74**	**73**
1	1.3	1.2	1.2	1.2
2	2.5	2.5	2.5̄	2.4
3	3.8	3.8	3.7	3.6
4	5.1	5.0	4.9	4.9
5	6.3	6.2	6.2	6.1
6	7.6	7.5	7.4	7.3
7	8.9	8.8	8.6	8.5
8	10.1	10.0	9.9	9.7
9	11.4	11.2	11.1	11.0
10	12.7	12.5	12.3	12.2
20	25.3	25.0	24.7	24.3
30	38.0	37.5	37.0	36.5
40	50.7	50.0	49.3	48.7
50	63.3	62.5	61.7	60.8

	72	**71**	**3**	**2**
1	1.2	1.2	0.0	0.0
2	2.4	2.4	0.1	0.1
3	3.6	3.6	0.2	0.1
4	4.8	4.7	0.2	0.1
5	6.0	5.9	0.2	0.2
6	7.2	7.1	0.3	0.2
7	8.4	8.3	0.4	0.2
8	9.6	9.5̄	0.4	0.3
9	10.8	10.6	0.4	0.3
10	12.0	11.8	0.5	0.3
20	24.0	23.7	1.0	0.7
30	36.0	35.5	1.5	1.0
40	48.0	47.3	2.0	1.3
50	60.0	59.2	2.5	1.7

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.23 967	72	9.24 632	74	0.75 368	9.99 335	2	60
1	9.24 039	71	9.24 706	73	0.75 294	9.99 333	2	59
2	9.24 110	71	9.24 779	74	0.75 221	9.99 331	3	58
3	9.24 181	72	9.24 853	73	0.75 147	9.99 328	2	57
4	9.24 253	71	9.24 926	74	0.75 074	9.99 326	2	56
5	9.24 324	71	9.25 000	73	0.75 000	9.99 324	2	55
6	9.24 395	71	9.25 073	73	0.74 927	9.99 322	3	54
7	9.24 466	70	9.25 146	73	0.74 854	9.99 319	2	53
8	9.24 536	71	9.25 219	73	0.74 781	9.99 317	2	52
9	9.24 607	70	9.25 292	73	0.74 708	9.99 315	2	51
10	9.24 677	71	9.25 365	72	0.74 635	9.99 313	3	50
11	9.24 748	70	9.25 437	73	0.74 563	9.99 310	2	49
12	9.24 818	70	9.25 510	72	0.74 490	9.99 308	2	48
13	9.24 888	70	9.25 582	73	0.74 418	9.99 306	2	47
14	9.24 958	70	9.25 655	72	0.74 345	9.99 304	3	46
15	9.25 028	70	9.25 727	72	0.74 273	9.99 301	2	45
16	9.25 098	70	9.25 799	72	0.74 201	9.99 299	2	44
17	9.25 168	69	9.25 871	72	0.74 129	9.99 297	3	43
18	9.25 237	70	9.25 943	72	0.74 057	9.99 294	2	42
19	9.25 307	69	9.26 015	71	0.73 985	9.99 292	2	41
20	9.25 376	69	9.26 086	72	0.73 914	9.99 290	2	40
21	9.25 445	69	9.26 158	71	0.73 842	9.99 288	3	39
22	9.25 514	69	9.26 229	72	0.73 771	9.99 285	2	38
23	9.25 583	69	9.26 301	71	0.73 699	9.99 283	2	37
24	9.25 652	69	9.26 372	71	0.73 628	9.99 281	3	36
25	9.25 721	69	9.26 443	71	0.73 557	9.99 278	2	35
26	9.25 790	68	9.26 514	71	0.73 486	9.99 276	2	34
27	9.25 858	69	9.26 585	70	0.73 415	9.99 274	3	33
28	9.25 927	68	9.26 655	71	0.73 345	9.99 271	2	32
29	9.25 995	68	9.26 726	71	0.73 274	9.99 269	2	31
30	9.26 063	68	9.26 797	70	0.73 203	9.99 267	3	30
31	9.26 131	68	9.26 867	70	0.73 133	9.99 264	2	29
32	9.26 199	68	9.26 937	71	0.73 063	9.99 262	2	28
33	9.26 267	68	9.27 008	70	0.72 992	9.99 260	3	27
34	9.26 335	68	9.27 078	70	0.72 922	9.99 257	2	26
35	9.26 403	67	9.27 148	70	0.72 852	9.99 255	3	25
36	9.26 470	68	9.27 218	70	0.72 782	9.99 252	2	24
37	9.26 538	67	9.27 288	69	0.72 712	9.99 250	2	23
38	9.26 605	67	9.27 357	70	0.72 643	9.99 248	3	22
39	9.26 672	67	9.27 427	69	0.72 573	9.99 245	2	21
40	9.26 739	67	9.27 496	70	0.72 504	9.99 243	2	20
41	9.26 806	67	9.27 566	69	0.72 434	9.99 241	3	19
42	9.26 873	67	9.27 635	69	0.72 365	9.99 238	2	18
43	9.26 940	67	9.27 704	69	0.72 296	9.99 236	3	17
44	9.27 007	66	9.27 773	69	0.72 227	9.99 233	2	16
45	9.27 073	67	9.27 842	69	0.72 158	9.99 231	2	15
46	9.27 140	66	9.27 911	69	0.72 089	9.99 229	3	14
47	9.27 206	67	9.27 980	69	0.72 020	9.99 226	2	13
48	9.27 273	66	9.28 049	68	0.71 951	9.99 224	3	12
49	9.27 339	66	9.28 117	69	0.71 883	9.99 221	2	11
50	9.27 405	66	9.28 186	68	0.71 814	9.99 219	2	10
51	9.27 471	66	9.28 254	69	0.71 746	9.99 217	3	9
52	9.27 537	65	9.28 323	68	0.71 677	9.99 214	2	8
53	9.27 602	66	9.28 391	68	0.71 609	9.99 212	3	7
54	9.27 668	66	9.28 459	68	0.71 541	9.99 209	2	6
55	9.27 734	65	9.28 527	68	0.71 473	9.99 207	3	5
56	9.27 799	65	9.28 595	67	0.71 405	9.99 204	2	4
57	9.27 864	66	9.28 662	68	0.71 338	9.99 202	2	3
58	9.27 930	65	9.28 730	68	0.71 270	9.99 200	3	2
59	9.27 995	65	9.28 798	67	0.71 202	9.99 197	2	1
60	9.28 060		9.28 865		0.71 135	9.99 195		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	74	73	72
1	1.2	1.2	1.2
2	2.5	2.4	2.4
3	3.7	3.6	3.6
4	4.9	4.9	4.8
5	6.2	6.1	6.0
6	7.4	7.3	7.2
7	8.6	8.5	8.4
8	9.9	9.7	9.6
9	11.1	11.0	10.8
10	12.3	12.2	12.0
20	24.7	24.3	24.0
30	37.0	36.5	36.0
40	49.3	48.7	48.0
50	61.7	60.8	60.0

	71	70	69
1	1.2	1.2	1.2
2	2.4	2.3	2.3
3	3.6	3.5	3.4
4	4.7	4.7	4.6
5	5.9	5.8	5.8
6	7.1	7.0	6.9
7	8.3	8.2	8.0
8	9.5	9.3	9.2
9	10.6	10.5	10.4
10	11.8	11.7	11.5
20	23.7	23.3	23.0
30	35.5	35.0	34.5
40	47.3	46.7	46.0
50	59.2	58.3	57.5

	68	67	66
1	1.1	1.1	1.1
2	2.3	2.2	2.2
3	3.4	3.4	3.3
4	4.5	4.5	4.4
5	5.7	5.6	5.5
6	6.8	6.7	6.6
7	7.9	7.8	7.7
8	9.1	8.9	8.8
9	10.2	10.0	9.9
10	11.3	11.2	11.0
20	22.7	22.3	22.0
30	34.0	33.5	33.0
40	45.3	44.7	44.0
50	56.7	55.8	55.0

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.28 060	65	9.28 865	68	0.71 135	9.99 195	3	60
1	9.28 125	65	9.28 933	67	0.71 067	9.99 192	2	59
2	9.28 190	64	9.29 000	67	0.71 000	9.99 190	3	58
3	9.28 254	65	9.29 067	67	0.70 933	9.99 187	2	57
4	9.28 319	65	9.29 134	67	0.70 866	9.99 185	3	56
5	9.28 384	64	9.29 201	67	0.70 799	9.99 182	2	55
6	9.28 448	64	9.29 268	67	0.70 732	9.99 180	3	54
7	9.28 512	65	9.29 335	67	0.70 665	9.99 177	2	53
8	9.28 577	64	9.29 402	66	0.70 598	9.99 175	3	52
9	9.28 641	64	9.29 468	67	0.70 532	9.99 172	2	51
10	9.28 705	64	9.29 535	66	0.70 465	9.99 170	3	50
11	9.28 769	64	9.29 601	67	0.70 399	9.99 167	2	49
12	9.28 833	63	9.29 668	66	0.70 332	9.99 165	3	48
13	9.28 896	64	9.29 734	66	0.70 266	9.99 162	2	47
14	9.28 960	64	9.29 800	66	0.70 200	9.99 160	3	46
15	9.29 024	63	9.29 866	66	0.70 134	9.99 157	2	45
16	9.29 087	63	9.29 932	66	0.70 068	9.99 155	3	44
17	9.29 150	64	9.29 998	66	0.70 002	9.99 152	2	43
18	9.29 214	63	9.30 064	66	0.69 936	9.99 150	3	42
19	9.29 277	63	9.30 130	65	0.69 870	9.99 147	2	41
20	9.29 340	63	9.30 195	66	0.69 805	9.99 145	3	40
21	9.29 403	63	9.30 261	65	0.69 739	9.99 142	2	39
22	9.29 466	63	9.30 326	65	0.69 674	9.99 140	3	38
23	9.29 529	62	9.30 391	66	0.69 609	9.99 137	2	37
24	9.29 591	63	9.30 457	65	0.69 543	9.99 135	3	36
25	9.29 654	62	9.30 522	65	0.69 478	9.99 132	2	35
26	9.29 716	63	9.30 587	65	0.69 413	9.99 130	3	34
27	9.29 779	62	9.30 652	65	0.69 348	9.99 127	3	33
28	9.29 841	62	9.30 717	65	0.69 283	9.99 124	2	32
29	9.29 903	63	9.30 782	64	0.69 218	9.99 122	3	31
30	9.29 966	62	9.30 846	65	0.69 154	9.99 119	2	30
31	9.30 028	62	9.30 911	64	0.69 089	9.99 117	3	29
32	9.30 090	61	9.30 975	65	0.69 025	9.99 114	2	28
33	9.30 151	62	9.31 040	64	0.68 960	9.99 112	3	27
34	9.30 213	62	9.31 104	64	0.68 896	9.99 109	3	26
35	9.30 275	61	9.31 168	65	0.68 832	9.99 106	2	25
36	9.30 336	62	9.31 233	64	0.68 767	9.99 104	3	24
37	9.30 398	61	9.31 297	64	0.68 703	9.99 101	2	23
38	9.30 459	62	9.31 361	64	0.68 639	9.99 099	3	22
39	9.30 521	61	9.31 425	64	0.68 575	9.99 096	3	21
40	9.30 582	61	9.31 489	63	0.68 511	9.99 093	2	20
41	9.30 643	61	9.31 552	64	0.68 448	9.99 091	3	19
42	9.30 704	61	9.31 616	63	0.68 384	9.99 088	2	18
43	9.30 765	61	9.31 679	64	0.68 321	9.99 086	3	17
44	9.30 826	61	9.31 743	63	0.68 257	9.99 083	3	16
45	9.30 887	60	9.31 806	64	0.68 194	9.99 080	2	15
46	9.30 947	61	9.31 870	63	0.68 130	9.99 078	3	14
47	9.31 008	60	9.31 933	63	0.68 067	9.99 075	3	13
48	9.31 068	61	9.31 996	63	0.68 004	9.99 072	2	12
49	9.31 129	60	9.32 059	63	0.67 941	9.99 070	3	11
50	9.31 189	61	9.32 122	63	0.67 878	9.99 067	3	10
51	9.31 250	60	9.32 185	63	0.67 815	9.99 064	2	9
52	9.31 310	60	9.32 248	63	0.67 752	9.99 062	3	8
53	9.31 370	60	9.32 311	62	0.67 689	9.99 059	3	7
54	9.31 430	60	9.32 373	63	0.67 627	9.99 056	2	6
55	9.31 490	59	9.32 436	62	0.67 564	9.99 054	3	5
56	9.31 549	60	9.32 498	63	0.67 502	9.99 051	3	4
57	9.31 609	60	9.32 561	62	0.67 439	9.99 048	2	3
58	9.31 669	59	9.32 623	62	0.67 377	9.99 046	3	2
59	9.31 728	60	9.32 685	62	0.67 315	9.99 043	3	1
60	9.31 788		9.32 747		0.67 253	9.99 040		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	65	64	63
1	1.1	1.1	1.0
2	2.2	2.1	2.1
3	3.2	3.2	3.2
4	4.3	4.3	4.2
5	5.4	5.3	5.2
6	6.5	6.4	6.3
7	7.6	7.5	7.4
8	8.7	8.5	8.4
9	9.8	9.6	9.4
10	10.8	10.7	10.5
20	21.7	21.3	21.0
30	32.5	32.0	31.5
40	43.3	42.7	42.0
50	54.2	53.3	52.5

	62	61	60
1	1.0	1.0	1.0
2	2.1	2.0	2.0
3	3.1	3.0	3.0
4	4.1	4.1	4.0
5	5.2	5.1	5.0
6	6.2	6.1	6.0
7	7.2	7.1	7.0
8	8.3	8.1	8.0
9	9.3	9.2	9.0
10	10.3	10.2	10.0
20	20.7	20.3	20.0
30	31.0	30.5	30.0
40	41.3	40.7	40.0
50	51.7	50.8	50.0

	59	3	2
1	1.0	0.0	0.0
2	2.0	0.1	0.1
3	3.0	0.2	0.1
4	3.9	0.2	0.1
5	4.9	0.2	0.2
6	5.9	0.3	0.2
7	6.9	0.4	0.2
8	7.9	0.4	0.3
9	8.8	0.4	0.3
10	9.8	0.5	0.3
20	19.7	1.0	0.7
30	29.5	1.5	1.0
40	39.3	2.0	1.3
50	49.2	2.5	1.7

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.31 788	59	9.32 747	63	0.67 253	9.99 040	2	60
1	9.31 847	60	9.32 810	62	0.67 190	9.99 038	3	59
2	9.31 907	59	9.32 872	61	0.67 128	9.99 035	3	58
3	9.31 966	59	9.32 933	62	0.67 067	9.99 032	2	57
4	9.32 025̄	59	9.32 995	62	0.67 005̄	9.99 030	3	56
5	9.32 084	59	9.33 057	62	0.66 943	9.99 027	3	55
6	9.32 143	59	9.33 119	61	0.66 881	9.99 024	2	54
7	9.32 202	59	9.33 180	62	0.66 820	9.99 022	3	53
8	9.32 261	58	9.33 242	61	0.66 758	9.99 019	3	52
9	9.32 319	59	9.33 303	62	0.66 697	9.99 016	3	51
10	9.32 378	59	9.33 365̄	61	0.66 635	9.99 013	2	50
11	9.32 437	58	9.33 426	61	0.66 574	9.99 011	3	49
12	9.32 495	58	9.33 487	61	0.66 513	9.99 008	3	48
13	9.32 553	59	9.33 548	61	0.66 452	9.99 005	3	47
14	9.32 612	58	9.33 609	61	0.66 391	9.99 002	2	46
15	9.32 670	58	9.33 670	61	0.66 330	9.99 000	3	45
16	9.32 728	58	9.33 731	61	0.66 269	9.98 997	3	44
17	9.32 786	58	9.33 792	61	0.66 208	9.98 994	3	43
18	9.32 844	58	9.33 853	60	0.66 147	9.98 991	2	42
19	9.32 902	58	9.33 913	61	0.66 087	9.98 989	3	41
20	9.32 960	58	9.33 974	60	0.66 026	9.98 986	3	40
21	9.33 018	57	9.34 034	61	0.65 966	9.98 983	3	39
22	9.33 075	58	9.34 095̄	60	0.65 905	9.98 980	2	38
23	9.33 133	57	9.34 155	60	0.65 845̄	9.98 978	3	37
24	9.33 190	58	9.34 215	61	0.65 785̄	9.98 975̄	3	36
25	9.33 248	57	9.34 276	60	0.65 724	9.98 972	3	35
26	9.33 305	57	9.34 336	60	0.65 664	9.98 969	2	34
27	9.33 362	58	9.34 396	60	0.65 604	9.98 967	3	33
28	9.33 420	57	9.34 456	60	0.65 544	9.98 964	3	32
29	9.33 477	57	9.34 516	60	0.65 484	9.98 961	3	31
30	9.33 534	57	9.34 576	59	0.65 424	9.98 958	3	30
31	9.33 591	56	9.34 635	60	0.65 365̄	9.98 955	2	29
32	9.33 647	57	9.34 695̄	60	0.65 305	9.98 953	3	28
33	9.33 704	57	9.34 755̄	59	0.65 245	9.98 950̄	3	27
34	9.33 761	57	9.34 814	60	0.65 186	9.98 947	3	26
35	9.33 818	56	9.34 874	59	0.65 126	9.98 944	3	25
36	9.33 874	57	9.34 933	59	0.65 067	9.98 941	3	24
37	9.33 931	56	9.34 992	59	0.65 008	9.98 938	2	23
38	9.33 987	56	9.35 051	60	0.64 949	9.98 936	3	22
39	9.34 043	57	9.35 111	59	0.64 889	9.98 933	3	21
40	9.34 100	56	9.35 170	59	0.64 830	9.98 930	3	20
41	9.34 156	56	9.35 229	59	0.64 771	9.98 927	3	19
42	9.34 212	56	9.35 288	59	0.64 712	9.98 924	3	18
43	9.34 268	56	9.35 347	58	0.64 653	9.98 921	2	17
44	9.34 324	56	9.35 405	59	0.64 595̄	9.98 919	3	16
45	9.34 380	56	9.35 464	59	0.64 536	9.98 916	3	15
46	9.34 436	55	9.35 523	58	0.64 477	9.98 913	3	14
47	9.34 491	56	9.35 581	59	0.64 419	9.98 910	3	13
48	9.34 547	55	9.35 640	58	0.64 360	9.98 907	3	12
49	9.34 602	56	9.35 698	59	0.64 302	9.98 904	3	11
50	9.34 658	55	9.35 757	58	0.64 243	9.98 901	3	10
51	9.34 713	56	9.35 815̄	58	0.64 185	9.98 898	2	9
52	9.34 769	55	9.35 873	58	0.64 127	9.98 896	3	8
53	9.34 824	55	9.35 931	58	0.64 069	9.98 893	3	7
54	9.34 879	55	9.35 989	58	0.64 011	9.98 890	3	6
55	9.34 934	55	9.36 047	58	0.63 953	9.98 887	3	5
56	9.34 989	55	9.36 105	58	0.63 895̄	9.98 884	3	4
57	9.35 044	55	9.36 163	58	0.63 837	9.98 881	3	3
58	9.35 099	55	9.36 221	58	0.63 779	9.98 878	3	2
59	9.35 154	55	9.36 279	57	0.63 721	9.98 875	3	1
60	9.35 209		9.36 336		0.63 664	9.98 872		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	63	62	61
1	1.0	1.0	1.0
2	2.1	2.1	2.0
3	3.2	3.1	3.0
4	4.2	4.1	4.1
5	5.2	5.2	5.1
6	6.3	6.2	6.1
7	7.4	7.2	7.1
8	8.4	8.3	8.1
9	9.4	9.3	9.2
10	10.5	10.3	10.2
20	21.0	20.7	20.3
30	31.5	31.0	30.5
40	42.0	41.3	40.7
50	52.5	51.7	50.8

	60	59	58
1	1.0	1.0	1.0
2	2.0	2.0	1.9
3	3.0	3.0	2.9
4	4.0	3.9	3.9
5	5.0	4.9	4.8
6	6.0	5.9	5.8
7	7.0	6.9	6.8
8	8.0	7.9	7.7
9	9.0	8.8	8.7
10	10.0	9.8	9.7
20	20.0	19.7	19.3
30	30.0	29.5	29.0
40	40.0	39.3	38.7
50	50.0	49.2	48.3

	57	56	55
1	1.0	0.9	0.9
2	1.9	1.9	1.8
3	2.8	2.8	2.8
4	3.8	3.7	3.7
5	4.8	4.7	4.6
6	5.7	5.6	5.5
7	6.6	6.5	6.4
8	7.6	7.5	7.3
9	8.6	8.4	8.2
10	9.5	9.3	9.2
20	19.0	18.7	18.3
30	28.5	28.0	27.5
40	38.0	37.3	36.7
50	47.5	46.7	45.8

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.35 209	54	9.36 336	58	0.63 664	9.98 872	3	60
1	9.35 263	55	9.36 394	58	0.63 606	9.98 869	2	59
2	9.35 318	55	9.36 452	57	0.63 548	9.98 867	3	58
3	9.35 373	54	9.36 509	57	0.63 491	9.98 864	3	57
4	9.35 427	54	9.36 566	58	0.63 434	9.98 861	3	56
5	9.35 481	55	9.36 624	57	0.63 376	9.98 858	3	55
6	9.35 536	54	9.36 681	57	0.63 319	9.98 855̄	3	54
7	9.35 590	54	9.36 738	57	0.63 262	9.98 852	3	53
8	9.35 644	54	9.36 795	57	0.63 205̄	9.98 849	3	52
9	9.35 698	54	9.36 852	57	0.63 148	9.98 846	3	51
10	9.35 752	54	9.36 909	57	0.63 091	9.98 843	3	50
11	9.35 806	54	9.36 966	57	0.63 034	9.98 840	3	49
12	9.35 860	54	9.37 023	57	0.62 977	9.98 837	3	48
13	9.35 914	54	9.37 080	57	0.62 920	9.98 834	3	47
14	9.35 968	54	9.37 137	56	0.62 863	9.98 831	3	46
15	9.36 022	53	9.37 193	57	0.62 807	9.98 828	3	45
16	9.36 075	54	9.37 250̄	56	0.62 750	9.98 825	3	44
17	9.36 129	53	9.37 306	57	0.62 694	9.98 822	3	43
18	9.36 182	54	9.37 363	56	0.62 637	9.98 819	3	42
19	9.36 236	53	9.37 419	57	0.62 581	9.98 816	3	41
20	9.36 289	53	9.37 476	56	0.62 524	9.98 813	3	40
21	9.36 342	53	9.37 532	56	0.62 468	9.98 810	3	39
22	9.36 395	54	9.37 588	56	0.62 412	9.98 807	3	38
23	9.36 449	53	9.37 644	56	0.62 356	9.98 804	3	37
24	9.36 502	53	9.37 700	56	0.62 300	9.98 801	3	36
25	9.36 555̄	53	9.37 756	56	0.62 244	9.98 798	3	35
26	9.36 608	52	9.37 812	56	0.62 188	9.98 795	3	34
27	9.36 660	53	9.37 868	56	0.62 132	9.98 792	3	33
28	9.36 713	53	9.37 924	56	0.62 076	9.98 789	3	32
29	9.36 766	53	9.37 980	55	0.62 020	9.98 786	3	31
30	9.36 819	52	9.38 035	56	0.61 965̄	9.98 783	3	30
31	9.36 871	53	9.38 091	56	0.61 909	9.98 780	3	29
32	9.36 924	52	9.38 147	55	0.61 853	9.98 777	3	28
33	9.36 976	52	9.38 202	55	0.61 798	9.98 774	3	27
34	9.37 028	53	9.38 257	56	0.61 743	9.98 771	3	26
35	9.37 081	52	9.38 313	55	0.61 687	9.98 768	3	25
36	9.37 133	52	9.38 368	55	0.61 632	9.98 765̄	3	24
37	9.37 185	52	9.38 423	56	0.61 577	9.98 762	3	23
38	9.37 237	52	9.38 479	55	0.61 521	9.98 759	3	22
39	9.37 289	52	9.38 534	55	0.61 466	9.98 756	3	21
40	9.37 341	52	9.38 589	55	0.61 411	9.98 753	3	20
41	9.37 393	52	9.38 644	55	0.61 356	9.98 750̄	4	19
42	9.37 445	52	9.38 699	55	0.61 301	9.98 746	3	18
43	9.37 497	52	9.38 754	54	0.61 246	9.98 743	3	17
44	9.37 549	51	9.38 808	55	0.61 192	9.98 740	3	16
45	9.37 600	52	9.38 863	55	0.61 137	9.98 737	3	15
46	9.37 652	51	9.38 918	54	0.61 082	9.98 734	3	14
47	9.37 703	52	9.38 972	55	0.61 028	9.98 731	3	13
48	9.37 755̄	51	9.39 027	55	0.60 973	9.98 728	3	12
49	9.37 806	52	9.39 082	54	0.60 918	9.98 725̄	3	11
50	9.37 858	51	9.39 136	54	0.60 864	9.98 722	3	10
51	9.37 909	51	9.39 190	55	0.60 810	9.98 719	4	9
52	9.37 960	51	9.39 245	54	0.60 755	9.98 715	3	8
53	9.38 011	51	9.39 299	54	0.60 701	9.98 712	3	7
54	9.38 062	51	9.39 353	54	0.60 647	9.98 709	3	6
55	9.38 113	51	9.39 407	54	0.60 593	9.98 706	3	5
56	9.38 164	51	9.39 461	54	0.60 539	9.98 703	3	4
57	9.38 215	51	9.39 515	54	0.60 485̄	9.98 700	3	3
58	9.38 266	51	9.39 569	54	0.60 431	9.98 697	3	2
59	9.38 317	51	9.39 623	54	0.60 377	9.98 694	4	1
60	9.38 368		9.39 677		0.60 323	9.98 690		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	57	56	55
1	1.0	0.9	0.9
2	1.9	1.9	1.8
3	2.8	2.8	2.8
4	3.8	3.7	3.7
5	4.8	4.7	4.6
6	5.7	5.6	5.5
7	6.6	6.5	6.4
8	7.6	7.5	7.3
9	8.6	8.4	8.2
10	9.5	9.3	9.2
20	19.0	18.7	18.3
30	28.5	28.0	27.5
40	38.0	37.3	36.7
50	47.5	46.7	45.8

	54	53	52
1	0.9	0.9	0.9
2	1.8	1.8	1.7
3	2.7	2.6	2.6
4	3.6	3.5	3.5̄
5	4.5	4.4	4.3
6	5.4	5.3	5.2
7	6.3	6.2	6.1
8	7.2	7.1	6.9
9	8.1	8.0	7.8
10	9.0	8.8	8.7
20	18.0	17.7	17.3
30	27.0	26.5	26.0
40	36.0	35.3	34.7
50	45.0	44.2	43.3

	51	4	3	2
1	0.8	0.1	0.0	0.0
2	1.7	0.1	0.1	0.1
3	2.6	0.2	0.2	0.1
4	3.4	0.3	0.2	0.1
5	4.2	0.3	0.2	0.2
6	5.1	0.4	0.3	0.2
7	6.0	0.5̄	0.4	0.2
8	6.8	0.5	0.4	0.3
9	7.6	0.6	0.4	0.3
10	8.5	0.7	0.5	0.3
20	17.0	1.3	1.0	0.7
30	25.5	2.0	1.5	1.0
40	34.0	2.7	2.0	1.3
50	42.5	3.3	2.5	1.7

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.38 368	50	9.39 677	54	0.60 323	9.98 690	3	60
1	9.38 418	51	9.39 731	54	0.60 269	9.98 687	3	59
2	9.38 469	50	9.39 78$\bar{5}$	53	0.60 215	9.98 684	3	58
3	9.38 519	51	9.39 838	54	0.60 162	9.98 681	3	57
4	9.38 570	50	9.39 892	53	0.60 108	9.98 678	3	56
5	9.38 620	50	9.39 945	54	0.60 05$\bar{5}$	9.98 67$\bar{5}$	4	55
6	9.38 670	51	9.39 999	53	0.60 001	9.98 671	3	54
7	9.38 721	50	9.40 052	54	0.59 948	9.98 668	3	53
8	9.38 771	50	9.40 106	53	0.59 894	9.98 665	3	52
9	9.38 821	50	9.40 159	53	0.59 841	9.98 662	3	51
10	9.38 871	50	9.40 212	54	0.59 788	9.98 659	3	50
11	9.38 921	50	9.40 266	53	0.59 734	9.98 656	4	49
12	9.38 971	50	9.40 319	53	0.59 681	9.98 652	3	48
13	9.39 021	50	9.40 372	53	0.59 628	9.98 649	3	47
14	9.39 071	50	9.40 42$\bar{5}$	53	0.59 575	9.98 646	3	46
15	9.39 121	49	9.40 478	53	0.59 522	9.98 643	3	45
16	9.39 170	50	9.40 531	53	0.59 469	9.98 640	4	44
17	9.39 220	50	9.40 584	52	0.59 416	9.98 636	3	43
18	9.39 270	49	9.40 636	53	0.59 364	9.98 633	3	42
19	9.39 319	50	9.40 689	53	0.59 311	9.98 630	3	41
20	9.39 369	49	9.40 742	53	0.59 258	9.98 627	4	40
21	9.39 418	49	9.40 79$\bar{5}$	52	0.59 205	9.98 623	3	39
22	9.39 467	50	9.40 847	53	0.59 153	9.98 620	3	38
23	9.39 517	49	9.40 900	52	0.59 100	9.98 617	3	37
24	9.39 566	49	9.40 952	53	0.59 048	9.98 614	4	36
25	9.39 61$\bar{5}$	49	9.41 00$\bar{5}$	52	0.58 995	9.98 610	3	35
26	9.39 664	49	9.41 057	52	0.58 943	9.98 607	3	34
27	9.39 713	49	9.41 109	52	0.58 891	9.98 604	3	33
28	9.39 762	49	9.41 161	53	0.58 839	9.98 601	4	32
29	9.39 811	49	9.41 214	52	0.58 786	9.98 597	3	31
30	9.39 860	49	9.41 266	52	0.58 734	9.98 594	3	30
31	9.39 909	49	9.41 318	52	0.58 682	9.98 591	3	29
32	9.39 958	48	9.41 370	52	0.58 630	9.98 588	4	28
33	9.40 006	49	9.41 422	52	0.58 578	9.98 584	3	27
34	9.40 05$\bar{5}$	48	9.41 474	52	0.58 526	9.98 581	3	26
35	9.40 103	49	9.41 526	52	0.58 474	9.98 578	4	25
36	9.40 152	48	9.41 578	51	0.58 422	9.98 574	3	24
37	9.40 200	49	9.41 629	52	0.58 371	9.98 571	3	23
38	9.40 249	48	9.41 681	52	0.58 319	9.98 568	3	22
39	9.40 297	49	9.41 733	51	0.58 267	9.98 56$\bar{5}$	4	21
40	9.40 346	48	9.41 784	52	0.58 216	9.98 561	3	20
41	9.40 394	48	9.41 836	51	0.58 164	9.98 558	3	19
42	9.40 442	48	9.41 887	52	0.58 113	9.98 55$\bar{5}$	4	18
43	9.40 490	48	9.41 939	51	0.58 061	9.98 551	3	17
44	9.40 538	48	9.41 990	51	0.58 010	9.98 548	3	16
45	9.40 586	48	9.42 041	52	0.57 959	9.98 54$\bar{5}$	4	15
46	9.40 634	48	9.42 093	51	0.57 907	9.98 541	3	14
47	9.40 682	48	9.42 144	51	0.57 856	9.98 538	3	13
48	9.40 730	48	9.42 195	51	0.57 805	9.98 53$\bar{5}$	4	12
49	9.40 778	47	9.42 246	51	0.57 754	9.98 531	3	11
50	9.40 825	48	9.42 297	51	0.57 703	9.98 528	3	10
51	9.40 873	48	9.42 348	51	0.57 652	9.98 52$\bar{5}$	4	9
52	9.40 921	47	9.42 399	51	0.57 601	9.98 521	3	8
53	9.40 968	48	9.42 450	51	0.57 550	9.98 518	3	7
54	9.41 016	47	9.42 501	51	0.57 499	9.98 51$\bar{5}$	4	6
55	9.41 063	48	9.42 552	51	0.57 448	9.98 511	3	5
56	9.41 111	47	9.42 603	50	0.57 397	9.98 508	3	4
57	9.41 158	47	9.42 653	51	0.57 347	9.98 50$\bar{5}$	4	3
58	9.41 205	47	9.42 704	51	0.57 296	9.98 501	3	2
59	9.41 252	48	9.42 75$\bar{5}$	50	0.57 245	9.98 498	4	1
60	9.41 300		9.42 805		0.57 19$\bar{5}$	9.98 494		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	54	53	52
1	0.9	0.9	0.9
2	1.8	1.8	1.7
3	2.7	2.6	2.6
4	3.6	3.5	3.$\bar{5}$
5	4.5	4.4	4.3
6	5.4	5.3	5.2
7	6.3	6.2	6.1
8	7.2	7.1	6.9
9	8.1	8.0	7.8
10	9.0	8.8	8.7
20	18.0	17.7	17.3
30	27.0	26.5	26.0
40	36.0	35.3	34.7
50	45.0	44.2	43.3

	51	50	49
1	0.8	0.8	0.8
2	1.7	1.7	1.6
3	2.6	2.5	2.4
4	3.4	3.3	3.3
5	4.2	4.2	4.1
6	5.1	5.0	4.9
7	6.0	5.8	5.7
8	6.8	6.7	6.5
9	7.6	7.5	7.4
10	8.5	8.3	8.2
20	17.0	16.7	16.3
30	25.5	25.0	24.5
40	34.0	33.3	32.7
50	42.5	41.7	40.8

	48	47	4	3
1	0.8	0.8	0.1	0.0
2	1.6	1.6	0.1	0.1
3	2.4	2.4	0.2	0.2
4	3.2	3.1	0.3	0.2
5	4.0	3.9	0.3	0.2
6	4.8	4.7	0.4	0.3
7	5.6	5.$\bar{5}$	0.$\bar{5}$	0.4
8	6.4	6.3	0.5	0.4
9	7.2	7.0	0.6	0.4
10	8.0	7.8	0.7	0.5
20	16.0	15.7	1.3	1.0
30	24.0	23.5	2.0	1.5
40	32.0	31.3	2.7	2.0
50	40.0	39.2	3.3	2.5

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.41 300	47	9.42 805	51	0.57 19$\bar{5}$	9.98 494	3	60
1	9.41 347	47	9.42 856	50	0.57 144	9.98 491	3	59
2	9.41 394	47	9.42 906	51	0.57 094	9.98 488	4	58
3	9.41 441	47	9.42 957	50	0.57 043	9.98 484	3	57
4	9.41 488	47	9.43 007	50	0.56 993	9.98 481	4	56
5	9.41 53$\bar{5}$	47	9.43 057	51	0.56 943	9.98 477	3	55
6	9.41 582	46	9.43 108	50	0.56 892	9.98 474	3	54
7	9.41 628	47	9.43 158	50	0.56 842	9.98 471	4	53
8	9.41 675	47	9.43 208	50	0.56 792	9.98 467	3	52
9	9.41 722	46	9.43 258	50	0.56 742	9.98 464	4	51
10	9.41 768	47	9.43 308	50	0.56 692	9.98 460	3	50
11	9.41 81$\bar{5}$	46	9.43 358	50	0.56 642	9.98 457	4	49
12	9.41 861	47	9.43 408	50	0.56 592	9.98 453	3	48
13	9.41 908	46	9.43 458	50	0.56 542	9.98 450	3	47
14	9.41 954	47	9.43 508	50	0.56 492	9.98 447	4	46
15	9.42 001	46	9.43 558	49	0.56 442	9.98 443	3	45
16	9.42 047	46	9.43 607	50	0.56 393	9.98 440	4	44
17	9.42 093	47	9.43 657	50	0.56 343	9.98 436	3	43
18	9.42 140	46	9.43 707	49	0.56 293	9.98 433	4	42
19	9.42 186	46	9.43 756	50	0.56 244	9.98 429	3	41
20	9.42 232	46	9.43 806	49	0.56 194	9.98 426	4	40
21	9.42 278	46	9.43 855	50	0.56 14$\bar{5}$	9.98 422	3	39
22	9.42 324	46	9.43 90$\bar{5}$	49	0.56 095	9.98 419	4	38
23	9.42 370	46	9.43 954	50	0.56 046	9.98 415	3	37
24	9.42 416	45	9.44 004	49	0.55 996	9.98 412	3	36
25	9.42 461	46	9.44 053	49	0.55 947	9.98 409	4	35
26	9.42 507	46	9.44 102	49	0.55 898	9.98 405	3	34
27	9.42 553	46	9.44 151	50	0.55 849	9.98 402	4	33
28	9.42 599	45	9.44 201	49	0.55 799	9.98 398	3	32
29	9.42 644	46	9.44 2$\bar{5}$0	49	0.55 750	9.98 39$\bar{5}$	4	31
30	9.42 690	45	9.44 299	49	0.55 701	9.98 391	3	30
31	9.42 735	46	9.44 348	49	0.55 652	9.98 388	4	29
32	9.42 781	45	9.44 397	49	0.55 603	9.98 384	3	28
33	9.42 826	46	9.44 446	49	0.55 554	9.98 381	4	27
34	9.42 872	45	9.44 49$\bar{5}$	49	0.55 505	9.98 377	4	26
35	9.42 917	45	9.44 544	48	0.55 456	9.98 373	3	25
36	9.42 962	46	9.44 592	49	0.55 408	9.98 370	4	24
37	9.43 008	45	9.44 641	49	0.55 359	9.98 366	3	23
38	9.43 053	45	9.44 690	48	0.55 310	9.98 363	4	22
39	9.43 098	45	9.44 738	49	0.55 262	9.98 359	3	21
40	9.43 143	45	9.44 787	49	0.55 213	9.98 356	4	20
41	9.43 188	45	9.44 836	48	0.55 164	9.98 352	3	19
42	9.43 233	45	9.44 884	49	0.55 116	9.98 349	4	18
43	9.43 278	45	9.44 933	48	0.55 067	9.98 345	3	17
44	9.43 323	44	9.44 981	48	0.55 019	9.98 342	4	16
45	9.43 367	45	9.45 029	49	0.54 971	9.98 338	4	15
46	9.43 412	45	9.45 078	48	0.54 922	9.98 334	3	14
47	9.43 457	45	9.45 126	48	0.54 874	9.98 331	4	13
48	9.43 502	44	9.45 174	48	0.54 826	9.98 327	3	12
49	9.43 546	45	9.45 222	49	0.54 778	9.98 324	4	11
50	9.43 591	44	9.45 271	48	0.54 729	9.98 320	3	10
51	9.43 635	45	9.45 319	48	0.54 681	9.98 317	4	9
52	9.43 680	44	9.45 367	48	0.54 633	9.98 313	4	8
53	9.43 724	45	9.45 41$\bar{5}$	48	0.54 585	9.98 309	3	7
54	9.43 769	44	9.45 463	48	0.54 537	9.98 306	4	6
55	9.43 813	44	9.45 511	48	0.54 489	9.98 302	3	5
56	9.43 857	44	9.45 559	47	0.54 441	9.98 299	4	4
57	9.43 901	45	9.45 606	48	0.54 394	9.98 295	4	3
58	9.43 946	44	9.45 654	48	0.54 346	9.98 291	3	2
59	9.43 990	44	9.45 702	48	0.54 298	9.98 288	4	1
60	9.44 034		9.45 7$\bar{5}$0		0.54 250	9.98 284		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	51	50	49
1	0.8	0.8	0.8
2	1.7	1.7	1.6
3	2.6	2.5	2.4
4	3.4	3.3	3.3
5	4.2	4.2	4.1
6	5.1	5.0	4.9
7	6.0	5.8	5.7
8	6.8	6.7	6.5
9	7.6	7.5	7.4
10	8.5	8.3	8.2
20	17.0	16.7	16.3
30	25.5	25.0	24.5
40	34.0	33.3	32.7
50	42.5	41.7	40.8

	48	47	46
1	0.8	0.8	0.8
2	1.6	1.6	1.5
3	2.4	2.4	2.3
4	3.2	3.1	3.1
5	4.0	3.9	3.8
6	4.8	4.7	4.6
7	5.6	5.$\bar{5}$	5.4
8	6.4	6.3	6.1
9	7.2	7.0	6.9
10	8.0	7.8	7.7
20	16.0	15.7	15.3
30	24.0	23.5	23.0
40	32.0	31.3	30.7
50	40.0	39.2	38.3

	45	44	4	3
1	0.8	0.7	0.1	0.0
2	1.5	1.$\bar{5}$	0.1	0.1
3	2.2	2.2	0.2	0.2
4	3.0	2.9	0.3	0.2
5	3.8	3.7	0.3	0.2
6	4.5	4.4	0.4	0.3
7	5.2	5.1	0.$\bar{5}$	0.4
8	6.0	5.9	0.5	0.4
9	6.8	6.6	0.6	0.4
10	7.5	7.3	0.7	0.5
20	15.0	14.7	1.3	1.0
30	22.5	22.0	2.0	1.5
40	30.0	29.3	2.7	2.0
50	37.5	36.7	3.3	2.5

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.44 034	44	9.45 75̄0	47	0.54 250	9.98 284	3	**60**
1	9.44 078	44	9.45 797	48	0.54 203	9.98 281	4	59
2	9.44 122	44	9.45 845̄	47	0.54 155	9.98 277	4	58
3	9.44 166	44	9.45 892	48	0.54 108	9.98 273	3	57
4	9.44 210	43	9.45 940	47	0.54 060	9.98 270	4	56
5	9.44 253	44	9.45 987	48	0.54 013	9.98 266	4	**55**
6	9.44 297	44	9.46 035̄	47	0.53 965	9.98 262	3	54
7	9.44 341	44	9.46 082	48	0.53 918	9.98 259	4	53
8	9.44 385̄	43	9.46 130	47	0.53 870	9.98 255	4	52
9	9.44 428	44	9.46 177	47	0.53 823	9.98 251	3	51
10	9.44 472	44	9.46 224	47	0.53 776	9.98 248	4	**50**
11	9.44 516	43	9.46 271	48	0.53 729	9.98 244	4	49
12	9.44 559	43	9.46 319	47	0.53 681	9.98 240	3	48
13	9.44 602	44	9.46 366	47	0.53 634	9.98 237	4	47
14	9.44 646	43	9.46 413	47	0.53 587	9.98 233	4	46
15	9.44 689	44	9.46 460	47	0.53 540	9.98 229	3	**45**
16	9.44 733	43	9.46 507	47	0.53 493	9.98 226	4	44
17	9.44 776	43	9.46 554	47	0.53 446	9.98 222	4	43
18	9.44 819	43	9.46 601	47	0.53 399	9.98 218	3	42
19	9.44 862	43	9.46 648	46	0.53 352	9.98 215̄	4	41
20	9.44 905	43	9.46 694	47	0.53 306	9.98 211	4	**40**
21	9.44 948	44	9.46 741	47	0.53 259	9.98 207	3	39
22	9.44 992	43	9.46 788	47	0.53 212	9.98 204	4	38
23	9.45 035̄	42	9.46 835̄	46	0.53 165	9.98 200	4	37
24	9.45 077	43	9.46 881	47	0.53 119	9.98 196	4	36
25	9.45 120	43	9.46 928	47	0.53 072	9.98 192	3	**35**
26	9.45 163	43	9.46 975̄	46	0.53 025	9.98 189	4	34
27	9.45 206	43	9.47 021	47	0.52 979	9.98 185̄	4	33
28	9.45 249	43	9.47 068	46	0.52 932	9.98 181	4	32
29	9.45 292	42	9.47 114	46	0.52 886	9.98 177	3	31
30	9.45 334	43	9.47 160	47	0.52 840	9.98 174	4	**30**
31	9.45 377	42	9.47 207	46	0.52 793	9.98 170	4	29
32	9.45 419	43	9.47 253	46	0.52 747	9.98 166	4	28
33	9.45 462	42	9.47 299	47	0.52 701	9.98 162	3	27
34	9.45 504	43	9.47 346	46	0.52 654	9.98 159	4	26
35	9.45 547	42	9.47 392	46	0.52 608	9.98 155̄	4	**25**
36	9.45 589	43	9.47 438	46	0.52 562	9.98 151	4	24
37	9.45 632	42	9.47 484	46	0.52 516	9.98 147	3	23
38	9.45 674	42	9.47 530	46	0.52 470	9.98 144	4	22
39	9.45 716	42	9.47 576	46	0.52 424	9.98 140	4	21
40	9.45 758	43	9.47 622	46	0.52 378	9.98 136	4	**20**
41	9.45 801	42	9.47 668	46	0.52 332	9.98 132	3	19
42	9.45 843	42	9.47 714	46	0.52 286	9.98 129	4	18
43	9.45 885̄	42	9.47 760	46	0.52 240	9.98 125	4	17
44	9.45 927	42	9.47 806	46	0.52 194	9.98 121	4	16
45	9.45 969	42	9.47 852	45	0.52 148	9.98 117	4	**15**
46	9.46 011	42	9.47 897	46	0.52 103	9.98 113	3	14
47	9.46 053	42	9.47 943	46	0.52 057	9.98 110	4	13
48	9.46 095̄	41	9.47 989	46	0.52 011	9.98 106	4	12
49	9.46 136	42	9.48 035̄	45	0.51 965	9.98 102	4	11
50	9.46 178	42	9.48 080	46	0.51 920	9.98 098	4	**10**
51	9.46 220	42	9.48 126	45	0.51 874	9.98 094	4	9
52	9.46 262	41	9.48 171	46	0.51 829	9.98 090	3	8
53	9.46 303	42	9.48 217	45	0.51 783	9.98 087	4	7
54	9.46 345̄	41	9.48 262	45	0.51 738	9.98 083	4	6
55	9.46 386	42	9.48 307	46	0.51 693	9.98 079	4	**5**
56	9.46 428	41	9.48 353	45	0.51 647	9.98 075	4	4
57	9.46 469	42	9.48 398	45	0.51 602	9.98 071	4	3
58	9.46 511	41	9.48 443	46	0.51 557	9.98 067	4	2
59	9.46 552	42	9.48 489	45	0.51 511	9.98 063	3	1
60	9.46 594		9.48 534		0.51 466	9.98 060		**0**
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	48	47	46
1	0.8	0.8	0.8
2	1.6	1.6	1.5
3	2.4	2.4	2.3
4	3.2	3.1	3.1
5	4.0	3.9	3.8
6	4.8	4.7	4.6
7	5.6	5.5̄	5.4
8	6.4	6.3	6.1
9	7.2	7.0	6.9
10	8.0	7.8	7.7
20	16.0	15.7	15.3
30	24.0	23.5	23.0
40	32.0	31.3	30.7
50	40.0	39.2	38.3

	45	44	43
1	0.8	0.7	0.7
2	1.5	1.5̄	1.4
3	2.2	2.2	2.2
4	3.0	2.9	2.9
5	3.8	3.7	3.6
6	4.5	4.4	4.3
7	5.2	5.	5.0
8	6.0	5.9	5.7
9	6.8	6.6	6.4
10	7.5	7.3	7.2
20	15.0	14.7	14.3
30	22.5	22.0	21.5
40	30.0	29.3	28.7
50	37.5	36.7	35.8

	42	41	4	3
1	0.7	0.7	0.1	0.0
2	1.4	1.4	1.1	0.1
3	2.1	2.0	0.2	0.2
4	2.8	2.7	0.3	0.2
5	3.5	3.4	0.3	0.2
6	4.2	4.1	0.4	0.3
7	4.9	4.8	0.5̄	0.4
8	5.6	5.5̄	0.5	0.4
9	6.3	6.2	0.6	0.4
10	7.0	6.8	0.7	0.5
20	14.0	13.7	1.3	1.0
30	21.0	20.5	2.0	1.5
40	28.0	27.3	2.7	2.0
50	35.0	34.2	3.3	2.5

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.46 594	41	9.48 534	45	0.51 466	9.98 060	4	**60**
1	9.46 635̄	41	9.48 579	45	0.51 421	9.98 056	4	59
2	9.46 676	41	9.48 624	45	0.51 376	9.98 052	4	58
3	9.46 717	41	9.48 669	45	0.51 331	9.98 048	4	57
4	9.46 758	42	9.48 714	45	0.51 286	9.98 044	4	56
5	9.46 800	41	9.48 759	45	0.51 241	9.98 040	4	**55**
6	9.46 841	41	9.48 804	45	0.51 196	9.98 036	4	54
7	9.46 882	41	9.48 849	45	0.51 151	9.98 032	3	53
8	9.46 923	41	9.48 894	45	0.51 106	9.98 029	4	52
9	9.46 964	41	9.48 939	45	0.51 061	9.98 025̄	4	51
10	9.47 005̄	40	9.48 984	45	0.51 016	9.98 021	4	**50**
11	9.47 045	41	9.49 029	44	0.50 971	9.98 017	4	49
12	9.47 086	41	9.49 073	45	0.50 927	9.98 013	4	48
13	9.47 127	41	9.49 118	45	0.50 882	9.98 009	4	47
14	9.47 168	41	9.49 163	44	0.50 837	9.98 005	4	46
15	9.47 209	40	9.49 207	45	0.50 793	9.98 001	4	**45**
16	9.47 249	41	9.49 252	44	0.50 748	9.97 997	4	44
17	9.47 290	40	9.49 296	45	0.50 704	9.97 993	4	43
18	9.47 330	41	9.49 341	44	0.50 659	9.97 989	3	42
19	9.47 371	40	9.49 385	45	0.50 615̄	9.97 986	4	41
20	9.47 411	41	9.49 430	44	0.50 570	9.97 982	4	**40**
21	9.47 452	40	9.49 474	45	0.50 526	9.97 978	4	39
22	9.47 492	41	9.49 519	44	0.50 481	9.97 974	4	38
23	9.47 533	40	9.49 563	44	0.50 437	9.97 970	4	37
24	9.47 573	40	9.49 607	45	0.50 393	9.97 966	4	36
25	9.47 613	41	9.49 652	44	0.50 348	9.97 962	4	**35**
26	9.47 654	40	9.49 696	44	0.50 304	9.97 958	4	34
27	9.47 694	40	9.49 740	44	0.50 260	9.97 954	4	33
28	9.47 734	40	9.49 784	44	0.50 216	9.97 95̄0	4	32
29	9.47 774	40	9.49 828	44	0.50 172	9.97 946	4	31
30	9.47 814	40	9.49 872	44	0.50 128	9.97 942	4	**30**
31	9.47 854	40	9.49 916	44	0.50 084	9.97 938	4	29
32	9.47 894	40	9.49 960	44	0.50 040	9.97 934	4	28
33	9.47 934	40	9.50 004	44	0.49 996	9.97 930	4	27
34	9.47 974	40	9.50 048	44	0.49 952	9.97 926	4	26
35	9.48 014	40	9.50 092	44	0.49 908	9.97 922	4	**25**
36	9.48 054	40	9.50 136	44	0.49 864	9.97 918	4	24
37	9.48 094	39	9.50 180	43	0.49 820	9.97 914	4	23
38	9.48 133	40	9.50 223	44	0.49 777	9.97 910	4	22
39	9.48 173	40	9.50 267	44	0.49 733	9.97 906	4	21
40	9.48 213	39	9.50 311	44	0.49 689	9.97 902	4	**20**
41	9.48 252	40	9.50 355̄	43	0.49 645	9.97 898	4	19
42	9.48 292	40	9.50 398	44	0.49 602	9.97 894	4	18
43	9.48 332	39	9.50 442	43	0.49 558	9.97 890	4	17
44	9.48 371	40	9.50 485	44	0.49 515̄	9.97 886	4	16
45	9.48 411	39	9.50 529	43	0.49 471	9.97 882	4	**15**
46	9.48 450	40	9.50 572	44	0.49 428	9.97 878	4	14
47	9.48 490	39	9.50 616	43	0.49 384	9.97 874	4	13
48	9.48 529	39	9.50 659	44	0.49 341	9.97 870	4	12
49	9.48 568	39	9.50 703	43	0.49 297	9.97 866	5	11
50	9.48 607	40	9.50 746	43	0.49 254	9.97 861	4	**10**
51	9.48 647	39	9.50 789	44	0.49 211	9.97 857	4	9
52	9.48 686	39	9.50 833	43	0.49 167	9.97 853	4	8
53	9.48 725	39	9.50 876	43	0.49 124	9.97 849	4	7
54	9.48 764	39	9.50 919	43	0.49 081	9.97 845	4	6
55	9.48 803	39	9.50 962	43	0.49 038	9.97 841	4	**5**
56	9.48 842	39	9.51 005	43	0.48 995̄	9.97 837	4	4
57	9.48 881	39	9.51 048	44	0.48 952	9.97 833	4	3
58	9.48 920	39	9.51 092	43	0.48 908	9.97 829	4	2
59	9.48 959	39	9.51 135̄	43	0.48 865	9.97 825̄	4	1
60	9.48 998		9.51 178		0.48 822	9.97 821		**0**
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	45	44	43
1	0.8	0.7	0.7
2	1.5	1.5̄	1.4
3	2.2	2.2	2.2
4	3.0	2.9	2.9
5	3.8	3.7	3.6
6	4.5	4.4	4.3
7	5.2	5.1	5.0
8	6.0	5.9	5.7
9	6.8	6.6	6.4
10	7.5	7.3	7.2
20	15.0	14.7	14.3
30	22.5	22.0	21.5
40	30.0	29.3	28.7
50	37.5	36.7	35.8

	42	41	40
1	0.7	0.7	0.7
2	1.4	1.4	1.3
3	2.1	2.0	2.0
4	2.8	2.7	2.7
5	3.5	3.4	3.3
6	4.2	4.1	4.0
7	4.9	4.8	4.7
8	5.6	5.5̄	5.3
9	6.3	6.2	6.0
10	7.0	6.8	6.7
20	14.0	13.7	13.3
30	21.0	20.5	20.0
40	28.0	27.3	26.7
50	35.0	34.2	33.3

	39	5	4	3
1	0.6	0.1	0.1	0.0
2	1.3	0.2	0.1	0.1
3	2.0	0.2	0.2	0.2
4	2.6	0.3	0.3	0.2
5	3.2	0.4	0.3	0.2
6	3.9	0.5	0.4	0.3
7	4.6	0.6	0.5̄	0.4
8	5.2	0.7	0.5	0.4
9	5.8	0.8	0.6	0.4
10	6.5	0.8	0.7	0.5
20	13.0	1.7	1.3	1.0
30	19.5	2.5	2.0	1.5
40	26.0	3.3	2.7	2.0
50	32.5	4.2	3.3	2.5

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.48 998	39	9.51 178	43	0.48 822	9.97 821	4	60
1	9.49 037	39	9.51 221	43	0.48 779	9.97 817	5	59
2	9.49 076	39	9.51 264	42	0.48 736	9.97 812	4	58
3	9.49 115	38	9.51 306	43	0.48 694	9.97 808	4	57
4	9.49 153	39	9.51 349	43	0.48 651	9.97 804	4	56
5	9.49 192	39	9.51 392	43	0.48 608	9.97 800	4	55
6	9.49 231	38	9.51 435	43	0.48 565	9.97 796	4	54
7	9.49 269	39	9.51 478	42	0.48 522	9.97 792	4	53
8	9.49 308	39	9.51 520	43	0.48 480	9.97 788	4	52
9	9.49 347	38	9.51 563	43	0.48 437	9.97 784	5	51
10	9.49 385	39	9.51 606	42	0.48 394	9.97 779	4	50
11	9.49 424	38	9.51 648	43	0.48 352	9.97 775	4	49
12	9.49 462	38	9.51 691	43	0.48 309	9.97 771	4	48
13	9.49 500	39	9.51 734	42	0.48 266	9.97 767	4	47
14	9.49 539	38	9.51 776	43	0.48 224	9.97 763	4	46
15	9.49 577	38	9.51 819	42	0.48 181	9.97 759	5	45
16	9.49 615	39	9.51 861	42	0.48 139	9.97 754	4	44
17	9.49 654	38	9.51 903	43	0.48 097	9.97 750	4	43
18	9.49 692	38	9.51 946	42	0.48 054	9.97 746	4	42
19	9.49 730	38	9.51 988	43	0.48 012	9.97 742	4	41
20	9.49 768	38	9.52 031	42	0.47 969	9.97 738	4	40
21	9.49 806	38	9.52 073	42	0.47 927	9.97 734	5	39
22	9.49 844	38	9.52 115	42	0.47 885	9.97 729	4	38
23	9.49 882	38	9.52 157	43	0.47 843	9.97 725	4	37
24	9.49 920	38	9.52 200	42	0.47 800	9.97 721	4	36
25	9.49 958	38	9.52 242	42	0.47 758	9.97 717	4	35
26	9.49 996	38	9.52 284	42	0.47 716	9.97 713	5	34
27	9.50 034	38	9.52 326	42	0.47 674	9.97 708	4	33
28	9.50 072	38	9.52 368	42	0.47 632	9.97 704	4	32
29	9.50 110	38	9.52 410	42	0.47 590	9.97 700	4	31
30	9.50 148	37	9.52 452	42	0.47 548	9.97 696	5	30
31	9.50 185	38	9.52 494	42	0.47 506	9.97 691	4	29
32	9.50 223	38	9.52 536	42	0.47 464	9.97 687	4	28
33	9.50 261	37	9.52 578	42	0.47 422	9.97 683	4	27
34	9.50 298	38	9.52 620	41	0.47 380	9.97 679	5	26
35	9.50 336	38	9.52 661	42	0.47 339	9.97 674	4	25
36	9.50 374	37	9.52 703	42	0.47 297	9.97 670	4	24
37	9.50 411	38	9.52 745	42	0.47 255	9.97 666	4	23
38	9.50 449	37	9.52 787	42	0.47 213	9.97 662	5	22
39	9.50 486	37	9.52 829	41	0.47 171	9.97 657	4	21
40	9.50 523	38	9.52 870	42	0.47 130	9.97 653	4	20
41	9.50 561	37	9.52 912	41	0.47 088	9.97 649	4	19
42	9.50 598	37	9.52 953	42	0.47 047	9.97 645	5	18
43	9.50 635	38	9.52 995	42	0.47 005	9.97 640	4	17
44	9.50 673	37	9.53 037	41	0.46 963	9.97 636	4	16
45	9.50 710	37	9.53 078	42	0.46 922	9.97 632	4	15
46	9.50 747	37	9.53 120	41	0.46 880	9.97 628	5	14
47	9.50 784	37	9.53 161	41	0.46 839	9.97 623	4	13
48	9.50 821	37	9.53 202	42	0.46 798	9.97 619	4	12
49	9.50 858	38	9.53 244	41	0.46 756	9.97 615	5	11
50	9.50 896	37	9.53 285	42	0.46 715	9.97 610	4	10
51	9.50 933	37	9.53 327	41	0.46 673	9.97 606	4	9
52	9.50 970	37	9.53 368	41	0.46 632	9.97 602	5	8
53	9.51 007	36	9.53 409	41	0.46 591	9.97 597	4	7
54	9.51 043	37	9.53 450	42	0.46 550	9.97 593	4	6
55	9.51 080	37	9.53 492	41	0.46 508	9.97 589	5	5
56	9.51 117	37	9.53 533	41	0.46 467	9.97 584	4	4
57	9.51 154	37	9.53 574	41	0.46 426	9.97 580	4	3
58	9.51 191	36	9.53 615	41	0.46 385	9.97 576	5	2
59	9.51 227	37	9.53 656	41	0.46 344	9.97 571	4	1
60	9.51 264		9.53 697		0.46 303	9.97 567		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	43	42	41
1	0.7	0.7	0.7
2	1.4	1.4	1.4
3	2.2	2.1	2.0
4	2.9	2.8	2.7
5	3.6	3.5	3.4
6	4.3	4.2	4.1
7	5.0	4.9	4.8
8	5.7	5.6	5.5
9	6.4	6.3	6.2
10	7.2	7.0	6.8
20	14.3	14.0	13.7
30	21.5	21.0	20.5
40	28.7	28.0	27.3
50	35.8	35.0	34.2

	39	38	37
1	0.6	0.6	0.6
2	1.3	1.3	1.2
3	2.0	1.9	1.8
4	2.6	2.5	2.5
5	3.2	3.2	3.1
6	3.9	3.8	3.7
7	4.6	4.4	4.3
8	5.2	5.1	4.9
9	5.8	5.7	5.6
10	6.5	6.3	6.2
20	13.0	12.7	12.3
30	19.5	19.0	18.5
40	26.0	25.3	24.7
50	32.5	31.7	30.8

	36	5	4
1	0.6	0.1	0.1
2	1.2	0.2	0.1
3	1.8	0.2	0.2
4	2.4	0.3	0.3
5	3.0	0.4	0.3
6	3.6	0.5	0.4
7	4.2	0.6	0.5
8	4.8	0.7	0.5
9	5.4	0.8	0.6
10	6.0	0.8	0.7
20	12.0	1.7	1.3
30	18.0	2.5	2.0
40	24.0	3.3	2.7
50	30.0	4.2	3.3

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.51 264	37	9.53 697	41	0.46 303	9.97 567	4	60
1	9.51 301	37	9.53 738	41	0.46 262	9.97 563	5	59
2	9.51 338	36	9.53 779	41	0.46 221	9.97 558	4	58
3	9.51 374	37	9.53 820	41	0.46 180	9.97 554	4	57
4	9.51 411	36	9.53 861	41	0.46 139	9.97 550	5	56
5	9.51 447	37	9.53 902	41	0.46 098	9.97 545	4	55
6	9.51 484	36	9.53 943	41	0.46 057	9.97 541	5	54
7	9.51 520	37	9.53 984	41	0.46 016	9.97 536	4	53
8	9.51 557	36	9.54 025	40	0.45 975	9.97 532	4	52
9	9.51 593	36	9.54 065	41	0.45 935	9.97 528	5	51
10	9.51 629	37	9.54 106	41	0.45 894	9.97 523	4	50
11	9.51 666	36	9.54 147	40	0.45 853	9.97 519	4	49
12	9.51 702	36	9.54 187	41	0.45 813	9.97 515	5	48
13	9.51 738	36	9.54 228	41	0.45 772	9.97 510	4	47
14	9.51 774	37	9.54 269	40	0.45 731	9.97 506	5	46
15	9.51 811	36	9.54 309	41	0.45 691	9.97 501	4	45
16	9.51 847	36	9.54 350	40	0.45 650	9.97 497	5	44
17	9.51 883	36	9.54 390	41	0.45 610	9.97 492	4	43
18	9.51 919	36	9.54 431	40	0.45 569	9.97 488	4	42
19	9.51 955	36	9.54 471	41	0.45 529	9.97 484	5	41
20	9.51 991	36	9.54 512	40	0.45 488	9.97 479	4	40
21	9.52 027	36	9.54 552	41	0.45 448	9.97 475	5	39
22	9.52 063	36	9.54 593	40	0.45 407	9.97 470	4	38
23	9.52 099	36	9.54 633	40	0.45 367	9.97 466	5	37
24	9.52 135	36	9.54 673	41	0.45 327	9.97 461	4	36
25	9.52 171	36	9.54 714	40	0.45 286	9.97 457	4	35
26	9.52 207	35	9.54 754	40	0.45 246	9.97 453	5	34
27	9.52 242	36	9.54 794	41	0.45 206	9.97 448	4	33
28	9.52 278	36	9.54 835	40	0.45 165	9.97 444	5	32
29	9.52 314	36	9.54 875	40	0.45 125	9.97 439	4	31
30	9.52 350	35	9.54 915	40	0.45 085	9.97 435	5	30
31	9.52 385	36	9.54 955	40	0.45 045	9.97 430	4	29
32	9.52 421	35	9.54 995	40	0.45 005	9.97 426	5	28
33	9.52 456	36	9.55 035	40	0.44 965	9.97 421	4	27
34	9.52 492	35	9.55 075	40	0.44 925	9.97 417	5	26
35	9.52 527	36	9.55 115	40	0.44 885	9.97 412	4	25
36	9.52 563	35	9.55 155	40	0.44 845	9.97 408	5	24
37	9.52 598	36	9.55 195	40	0.44 805	9.97 403	4	23
38	9.52 634	35	9.55 235	40	0.44 765	9.97 399	5	22
39	9.52 669	36	9.55 275	40	0.44 725	9.97 394	4	21
40	9.52 705	35	9.55 315	40	0.44 685	9.97 390	5	20
41	9.52 740	35	9.55 355	40	0.44 645	9.97 385	4	19
42	9.52 775	36	9.55 395	39	0.44 605	9.97 381	5	18
43	9.52 811	35	9.55 434	40	0.44 566	9.97 376	4	17
44	9.52 846	35	9.55 474	40	0.44 526	9.97 372	5	16
45	9.52 881	35	9.55 514	40	0.44 486	9.97 367	4	15
46	9.52 916	35	9.55 554	39	0.44 446	9.97 363	5	14
47	9.52 951	35	9.55 593	40	0.44 407	9.97 358	5	13
48	9.52 986	35	9.55 633	40	0.44 367	9.97 353	4	12
49	9.53 021	35	9.55 673	39	0.44 327	9.97 349	5	11
50	9.53 056	36	9.55 712	40	0.44 288	9.97 344	4	10
51	9.53 092	34	9.55 752	39	0.44 248	9.97 340	5	9
52	9.53 126	35	9.55 791	40	0.44 209	9.97 335	4	8
53	9.53 161	35	9.55 831	39	0.44 169	9.97 331	5	7
54	9.53 196	35	9.55 870	40	0.44 130	9.97 326	4	6
55	9.53 231	35	9.55 910	39	0.44 090	9.97 322	5	5
56	9.53 266	35	9.55 949	40	0.44 051	9.97 317	5	4
57	9.53 301	35	9.55 989	39	0.44 011	9.97 312	4	3
58	9.53 336	34	9.56 028	39	0.43 972	9.97 308	5	2
59	9.53 370	35	9.56 067	40	0.43 933	9.97 303	4	1
60	9.53 405		9.56 107		0.43 893	9.97 299		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	41	40	39
1	0.7	0.7	0.6
2	1.4	1.3	1.3
3	2.0	2.0	2.0
4	2.7	2.7	2.6
5	3.4	3.3	3.2
6	4.1	4.0	3.9
7	4.8	4.7	4.6
8	5.5	5.3	5.2
9	6.2	6.0	5.8
10	6.8	6.7	6.5
20	13.7	13.3	13.0
30	20.5	20.0	19.5
40	27.3	26.7	26.0
50	34.2	33.3	32.5

	37	36	35
1	0.6	0.6	0.6
2	1.2	1.2	1.2
3	1.8	1.8	1.8
4	2.5	2.4	2.3
5	3.1	3.0	2.9
6	3.7	3.6	3.5
7	4.3	4.2	4.1
8	4.9	4.8	4.7
9	5.6	5.4	5.2
10	6.2	6.0	5.8
20	12.3	12.0	11.7
30	18.5	18.0	17.5
40	24.7	24.0	23.3
50	30.8	30.0	29.2

	34	5	4
1	0.6	0.1	0.1
2	1.1	0.2	0.1
3	1.7	0.2	0.2
4	2.3	0.3	0.2
5	2.8	0.4	0.3
6	3.4	0.5	0.4
7	4.0	0.6	0.5
8	4.5	0.7	0.5
9	5.1	0.8	0.6
10	5.7	0.8	0.7
20	11.3	1.7	1.3
30	17.0	2.5	2.0
40	22.7	3.3	2.7
50	28.3	4.2	3.3

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.53 405	35	9.56 107	39	0.43 893	9.97 299	5	60
1	9.53 440	35	9.56 146	39	0.43 854	9.97 294	5	59
2	9.53 475	34	9.56 185	39	0.43 815̄	9.97 289	4	58
3	9.53 509	35	9.56 224	40	0.43 776	9.97 285̄	5	57
4	9.53 544	34	9.56 264	39	0.43 736	9.97 280	4	56
5	9.53 578	35	9.56 303	39	0.43 697	9.97 276	5	55
6	9.53 613	34	9.56 342	39	0.43 658	9.97 271	5	54
7	9.53 647	35	9.56 381	39	0.43 619	9.97 266	4	53
8	9.53 682	34	9.56 420	39	0.43 580	9.97 262	5	52
9	9.53 716	35	9.56 459	39	0.43 541	9.97 257	5	51
10	9.53 751	34	9.56 498	39	0.43 502	9.97 252	4	50
11	9.53 785	34	9.56 537	39	0.43 463	9.97 248	5	49
12	9.53 819	35	9.56 576	39	0.43 424	9.97 243	5	48
13	9.53 854	34	9.56 615	39	0.43 385̄	9.97 238	4	47
14	9.53 888	34	9.56 654	39	0.43 346	9.97 234	5	46
15	9.53 922	35	9.56 693	39	0.43 307	9.97 229	5	45
16	9.53 957	34	9.56 732	39	0.43 268	9.97 224	4	44
17	9.53 991	34	9.56 771	39	0.43 229	9.97 220	5	43
18	9.54 025̄	34	9.56 810	39	0.43 190	9.97 215	5	42
19	9.54 059	34	9.56 849	38	0.43 151	9.97 210	4	41
20	9.54 093	34	9.56 887	39	0.43 113	9.97 206	5	40
21	9.54 127	34	9.56 926	39	0.43 074	9.97 201	5	39
22	9.54 161	34	9.56 965̄	39	0.43 035	9.97 196	4	38
23	9.54 195	34	9.57 004	38	0.42 996	9.97 192	5	37
24	9.54 229	34	9.57 042	39	0.42 958	9.97 187	5	36
25	9.54 263	34	9.57 081	39	0.42 919	9.97 182	4	35
26	9.54 297	34	9.57 120	38	0.42 880	9.97 178	5	34
27	9.54 331	34	9.57 158	39	0.42 842	9.97 173	5	33
28	9.54 365̄	34	9.57 197	38	0.42 803	9.97 168	5	32
29	9.54 399	34	9.57 235	39	0.42 765̄	9.97 163	4	31
30	9.54 433	33	9.57 274	38	0.42 726	9.97 159	5	30
31	9.54 466	34	9.57 312	39	0.42 688	9.97 154	5	29
32	9.54 500	34	9.57 351	38	0.42 649	9.97 149	4	28
33	9.54 534	33	9.57 389	39	0.42 611	9.97 145̄	5	27
34	9.54 567	34	9.57 428	38	0.42 572	9.97 140	5	26
35	9.54 601	34	9.57 466	38	0.42 534	9.97 135	5	25
36	9.54 635̄	33	9.57 504	39	0.42 496	9.97 130	4	24
37	9.54 668	34	9.57 543	38	0.42 457	9.97 126	5	23
38	9.54 702	33	9.57 581	38	0.42 419	9.97 121	5	22
39	9.54 735	34	9.57 619	39	0.42 381	9.97 116	5	21
40	9.54 769	33	9.57 658	38	0.42 342	9.97 111	4	20
41	9.54 802	34	9.57 696	38	0.42 304	9.97 107	5	19
42	9.54 836	33	9.57 734	38	0.42 266	9.97 102	5	18
43	9.54 869	34	9.57 772	38	0.42 228	9.97 097	5	17
44	9.54 903	33	9.57 810	39	0.42 190	9.97 092	5	16
45	9.54 936	33	9.57 849	38	0.42 151	9.97 087	4	15
46	9.54 969	34	9.57 887	38	0.42 113	9.97 083	5	14
47	9.55 003	33	9.57 925̄	38	0.42 075	9.97 078	5	13
48	9.55 036	33	9.57 963	38	0.42 037	9.97 073	5	12
49	9.55 069	33	9.58 001	38	0.41 999	9.97 068	5	11
50	9.55 102	34	9.58 039	38	0.41 961	9.97 063	4	10
51	9.55 136	33	9.58 077	38	0.41 923	9.97 059	5	9
52	9.55 169	33	9.58 115̄	38	0.41 885	9.97 054	5	8
53	9.55 202	33	9.58 153	38	0.41 847	9.97 049	5	7
54	9.55 235̄	33	9.58 191	38	0.41 809	9.97 044	5	6
55	9.55 268	33	9.58 229	38	0.41 771	9.97 039	4	5
56	9.55 301	33	9.58 267	37	0.41 733	9.97 035̄	5	4
57	9.55 334	33	9.58 304	38	0.41 696	9.97 030	5	3
58	9.55 367	33	9.58 342	38	0.41 658	9.97 025̄	5	2
59	9.55 400	33	9.58 380	38	0.41 620	9.97 020	5	1
60	9.55 433		9.58 418		0.41 582	9.97 015		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	40	39	38
1	0.7	0.6	0.6
2	1.3	1.3	1.3
3	2.0	2.0	1.9
4	2.7	2.6	2.5
5	3.3	3.2	3.2
6	4.0	3.9	3.8
7	4.7	4.6	4.4
8	5.3	5.2	5.1
9	6.0	5.8	5.7
10	6.7	6.5	6.3
20	13.3	13.0	12.7
30	20.0	19.5	19.0
40	26.7	26.0	25.3
50	33.3	32.5	31.7

	37	35	34
1	0.6	0.6	0.6
2	1.2	1.2	1.1
3	1.8	1.8	1.7
4	2.5̄	2.3	2.3
5	3.1	2.9	2.8
6	3.7	3.5	3.4
7	4.3	4.1	4.0
8	4.9	4.7	4.5
9	5.6	5.2	5.1
10	6.2	5.8	5.7
20	12.3	11.7	11.3
30	18.5	17.5	17.0
40	24.7	23.3	22.7
50	30.8	29.2	28.3

	33	5	4
1	0.6	0.1	0.1
2	1.1	0.2	0.1
3	1.6	0.2	0.2
4	2.2	0.3	0.3
5	2.8	0.4	0.3
6	3.3	0.5	0.4
7	3.8	0.6	0.5̄
8	4.4	0.7	0.5
9	5.0	0.8	0.6
10	5.5	0.8	0.7
20	11.0	1.7	1.3
30	16.5	2.5	2.0
40	22.0	3.3	2.7
50	27.5	4.2	3.3

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.55 433	33	9.58 418	37	0.41 582	9.97 015	5	60
1	9.55 466	33	9.58 455	38	0.41 545̄	9.97 010	5	59
2	9.55 499	33	9.58 493	38	0.41 507	9.97 005	4	58
3	9.55 532	32	9.58 531	38	0.41 469	9.97 001	5	57
4	9.55 564	33	9.58 569	37	0.41 431	9.96 996	5	56
5	9.55 597	33	9.58 606	38	0.41 394	9.96 991	5	55
6	9.55 630	33	9.58 644	37	0.41 356	9.96 986	5	54
7	9.55 663	32	9.58 681	38	0.41 319	9.96 981	5	53
8	9.55 695	33	9.58 719	38	0.41 281	9.96 976	5	52
9	9.55 728	33	9.58 757	37	0.41 243	9.96 971	5	51
10	9.55 761	32	9.58 794	38	0.41 206	9.96 966	4	50
11	9.55 793	33	9.58 832	37	0.41 168	9.96 962	5	49
12	9.55 826	32	9.58 869	38	0.41 131	9.96 957	5	48
13	9.55 858	33	9.58 907	37	0.41 093	9.96 952	5	47
14	9.55 891	32	9.58 944	37	0.41 056	9.96 947	5	46
15	9.55 923	33	9.58 981	38	0.41 019	9.96 942	5	45
16	9.55 956	32	9.59 019	37	0.40 981	9.96 937	5	44
17	9.55 988	33	9.59 056	38	0.40 944	9.96 932	5	43
18	9.56 021	32	9.59 094	37	0.40 906	9.96 927	5	42
19	9.56 053	32	9.59 131	37	0.40 869	9.96 922	5	41
20	9.56 085	33	9.59 168	37	0.40 832	9.96 917	5	40
21	9.56 118	32	9.59 205	38	0.40 795̄	9.96 912	5	39
22	9.56 150	32	9.59 243	37	0.40 757	9.96 907	4	38
23	9.56 182	33	9.59 280	37	0.40 720	9.96 903	5	37
24	9.56 215̄	32	9.59 317	37	0.40 683	9.96 898	5	36
25	9.56 247	32	9.59 354	37	0.40 646	9.96 893	5	35
26	9.56 279	32	9.59 391	38	0.40 609	9.96 888	5	34
27	9.56 311	32	9.59 429	37	0.40 571	9.96 883	5	33
28	9.56 343	32	9.59 466	37	0.40 534	9.96 878	5	32
29	9.56 375	33	9.59 503	37	0.40 497	9.96 873	5	31
30	9.56 408	32	9.59 540	37	0.40 460	9.96 868	5	30
31	9.56 440	32	9.59 577	37	0.40 423	9.96 863	5	29
32	9.56 472	32	9.59 614	37	0.40 386	9.96 858	5	28
33	9.56 504	32	9.59 651	37	0.40 349	9.96 853	5	27
34	9.56 536	32	9.59 688	37	0.40 312	9.96 848	5	26
35	9.56 568	31	9.59 725̄	37	0.40 275	9.96 843	5	25
36	9.56 599	32	9.59 762	37	0.40 238	9.96 838	5	24
37	9.56 631	32	9.59 799	36	0.40 201	9.96 833	5	23
38	9.56 663	32	9.59 835	37	0.40 165̄	9.96 828	5	22
39	9.56 695	32	9.59 872	37	0.40 128	9.96 823	5	21
40	9.56 727	32	9.59 909	37	0.40 091	9.96 818	5	20
41	9.56 759	31	9.59 946	37	0.40 054	9.96 813	5	19
42	9.56 790	32	9.59 983	36	0.40 017	9.96 808	5	18
43	9.56 822	32	9.60 019	37	0.39 981	9.96 803	5	17
44	9.56 854	32	9.60 056	37	0.39 944	9.96 798	5	16
45	9.56 886	31	9.60 093	37	0.39 907	9.96 793	5	15
46	9.56 917	32	9.60 130	36	0.39 870	9.96 788	5	14
47	9.56 949	31	9.60 166	37	0.39 834	9.96 783	5	13
48	9.56 980	32	9.60 203	37	0.39 797	9.96 778	6	12
49	9.57 012	32	9.60 240	36	0.39 760	9.96 772	5	11
50	9.57 044	31	9.60 276	37	0.39 724	9.96 767	5	10
51	9.57 075	32	9.60 313	36	0.39 687	9.96 762	5	9
52	9.57 107	31	9.60 349	37	0.39 651	9.96 757	5	8
53	9.57 138	31	9.60 386	36	0.39 614	9.96 752	5	7
54	9.57 169	32	9.60 422	37	0.39 578	9.96 747	5	6
55	9.57 201	31	9.60 459	36	0.39 541	9.96 742	5	5
56	9.57 232	32	9.60 495	37	0.39 505̄	9.96 737	5	4
57	9.57 264	31	9.60 532	36	0.39 468	9.96 732	5	3
58	9.57 295̄	31	9.60 568	37	0.39 432	9.96 727	5	2
59	9.57 326	32	9.60 605̄	36	0.39 395	9.96 722	5	1
60	9.57 358		9.60 641		0.39 359	9.96 717		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	38	37	36
1	0.6	0.6	0.6
2	1.3	1.2	1.2
3	1.9	1.8	1.8
4	2.5	2.5̄	2.4
5	3.2	3.1	3.0
6	3.8	3.7	3.6
7	4.4	4.3	4.2
8	5.1	4.9	4.8
9	5.7	5.6	5.4
10	6.3	6.2	6.0
20	12.7	12.3	12.0
30	19.0	18.5	18.0
40	25.3	24.7	24.0
50	31.7	30.8	30.0

	33	32	31
1	0.6	0.5	0.5
2	1.1	1.1	1.0
3	1.6	1.6	1.6
4	2.2	2.1	2.1
5	2.8	2.7	2.6
6	3.3	3.2	3.1
7	3.8	3.7	3.6
8	4.4	4.3	4.1
9	5.0	4.8	4.6
10	5.5	5.3	5.2
20	11.0	10.7	10.3
30	16.5	16.0	15.5
40	22.0	21.3	20.7
50	27.5	26.7	25.8

	6	5	4
1	0.1	0.1	0.1
2	0.2	0.2	0.1
3	0.3	0.2	0.2
4	0.4	0.3	0.3
5	0.5	0.4	0.3
6	0.6	0.5	0.4
7	0.7	0.6	0.5̄
8	0.8	0.7	0.5
9	0.9	0.8	0.6
10	1.0	0.8	0.7
20	2.0	1.7	1.3
30	3.0	2.5	2.0
40	4.0	3.3	2.7
50	5.0	4.2	3.3

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.57 358	31	9.60 641	36	0.39 359	9.96 717	6	60
1	9.57 389	31	9.60 677	37	0.39 323	9.96 711	5	59
2	9.57 420	31	9.60 714	36	0.39 286	9.96 706	5	58
3	9.57 451	31	9.60 750	36	0.39 250	9.96 701	5	57
4	9.57 482	32	9.60 786	37	0.39 214	9.96 696	5	56
5	9.57 514	31	9.60 823	36	0.39 177	9.96 691	5	55
6	9.57 545	31	9.60 859	36	0.39 141	9.96 686	5	54
7	9.57 576	31	9.60 895	36	0.39 105	9.96 681	5	53
8	9.57 607	31	9.60 931	36	0.39 069	9.96 676	6	52
9	9.57 638	31	9.60 967	37	0.39 033	9.96 670	5	51
10	9.57 669	31	9.61 004	36	0.38 996	9.96 665	5	50
11	9.57 700	31	9.61 040	36	0.38 960	9.96 660	5	49
12	9.57 731	31	9.61 076	36	0.38 924	9.96 655	5	48
13	9.57 762	31	9.61 112	36	0.38 888	9.96 650	5	47
14	9.57 793	31	9.61 148	36	0.38 852	9.96 645	5	46
15	9.57 824	31	9.61 184	36	0.38 816	9.96 640	6	45
16	9.57 855	30	9.61 220	36	0.38 780	9.96 634	5	44
17	9.57 885	31	9.61 256	36	0.38 744	9.96 629	5	43
18	9.57 916	31	9.61 292	36	0.38 708	9.96 624	5	42
19	9.57 947	31	9.61 328	36	0.38 672	9.96 619	5	41
20	9.57 978	30	9.61 364	36	0.38 636	9.96 614	6	40
21	9.58 008	31	9.61 400	36	0.38 600	9.96 608	5	39
22	9.58 039	31	9.61 436	36	0.38 564	9.96 603	5	38
23	9.58 070	31	9.61 472	36	0.38 528	9.96 598	5	37
24	9.58 101	30	9.61 508	36	0.38 492	9.96 593	5	36
25	9.58 131	31	9.61 544	35	0.38 456	9.96 588	6	35
26	9.58 162	30	9.61 579	36	0.38 421	9.96 582	5	34
27	9.58 192	31	9.61 615	36	0.38 385	9.96 577	5	33
28	9.58 223	30	9.61 651	36	0.38 349	9.96 572	5	32
29	9.58 253	31	9.61 687	35	0.38 313	9.96 567	5	31
30	9.58 284	30	9.61 722	36	0.38 278	9.96 562	6	30
31	9.58 314	31	9.61 758	36	0.38 242	9.96 556	5	29
32	9.58 345	30	9.61 794	36	0.38 206	9.96 551	5	28
33	9.58 375	31	9.61 830	35	0.38 170	9.96 546	5	27
34	9.58 406	30	9.61 865	36	0.38 135	9.96 541	6	26
35	9.58 436	31	9.61 901	35	0.38 099	9.96 535	5	25
36	9.58 467	30	9.61 936	36	0.38 064	9.96 530	5	24
37	9.58 497	30	9.61 972	36	0.38 028	9.96 525	5	23
38	9.58 527	30	9.62 008	35	0.37 992	9.96 520	6	22
39	9.58 557	31	9.62 043	36	0.37 957	9.96 514	5	21
40	9.58 588	30	9.62 079	35	0.37 921	9.96 509	5	20
41	9.58 618	30	9.62 114	36	0.37 886	9.96 504	6	19
42	9.58 648	30	9.62 150	35	0.37 850	9.96 498	5	18
43	9.58 678	31	9.62 185	36	0.37 815	9.96 493	5	17
44	9.58 709	30	9.62 221	35	0.37 779	9.96 488	5	16
45	9.58 739	30	9.62 256	36	0.37 744	9.96 483	6	15
46	9.58 769	30	9.62 292	35	0.37 708	9.96 477	5	14
47	9.58 799	30	9.62 327	35	0.37 673	9.96 472	5	13
48	9.58 829	30	9.62 362	36	0.37 638	9.96 467	6	12
49	9.58 859	30	9.62 398	35	0.37 602	9.96 461	5	11
50	9.58 889	30	9.62 433	35	0.37 567	9.96 456	5	10
51	9.58 919	30	9.62 468	36	0.37 532	9.96 451	6	9
52	9.58 949	30	9.62 504	35	0.37 496	9.96 445	5	8
53	9.58 979	30	9.62 539	35	0.37 461	9.96 440	5	7
54	9.59 009	30	9.62 574	35	0.37 426	9.96 435	6	6
55	9.59 039	30	9.62 609	36	0.37 391	9.96 429	5	5
56	9.59 069	29	9.62 645	35	0.37 355	9.96 424	5	4
57	9.59 098	30	9.62 680	35	0.37 320	9.96 419	6	3
58	9.59 128	30	9.62 715	35	0.37 285	9.96 413	5	2
59	9.59 158	30	9.62 750	35	0.37 250	9.96 408	5	1
60	9.59 188		9.62 785		0.37 215	9.96 403		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	37	36	35
1	0.6	0.6	0.6
2	1.2	1.2	1.2
3	1.8	1.8	1.8
4	2.5	2.4	2.3
5	3.1	3.0	2.9
6	3.7	3.6	3.5
7	4.3	4.2	4.1
8	4.9	4.8	4.7
9	5.6	5.4	5.2
10	6.2	6.0	5.8
20	12.3	12.0	11.7
30	18.5	18.0	17.5
40	24.7	24.0	23.3
50	30.8	30.0	29.2

	32	31	30
1	0.5	0.5	0.5
2	1.1	1.0	1.0
3	1.6	1.6	1.5
4	2.1	2.1	2.0
5	2.7	2.6	2.5
6	3.2	3.1	3.0
7	3.7	3.6	3.5
8	4.3	4.1	4.0
9	4.8	4.6	4.5
10	5.3	5.2	5.0
20	10.7	10.3	10.0
30	16.0	15.5	15.0
40	21.3	20.7	20.0
50	26.7	25.8	25.0

	29	6	5
1	0.5	0.1	0.1
2	1.0	0.2	0.2
3	1.4	0.3	0.2
4	1.9	0.4	0.3
5	2.4	0.5	0.4
6	2.9	0.6	0.5
7	3.4	0.7	0.6
8	3.9	0.8	0.7
9	4.4	0.9	0.8
10	4.8	1.0	0.8
20	9.7	2.0	1.7
30	14.5	3.0	2.5
40	19.3	4.0	3.3
50	24.2	5.0	4.2

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.59 188	30	9.62 785	35	0.37 215	9.96 403	6	60
1	9.59 218	29	9.62 820	35	0.37 180	9.96 397	5	59
2	9.59 247	30	9.62 855	35	0.37 145	9.96 392	5	58
3	9.59 277	30	9.62 890	36	0.37 110	9.96 387	6	57
4	9.59 307	29	9.62 926	35	0.37 074	9.96 381	5	56
5	9.59 336	30	9.62 961	35	0.37 039	9.96 376	6	55
6	9.59 366	30	9.62 996	35	0.37 004	9.96 370	5	54
7	9.59 396	29	9.63 031	35	0.36 969	9.96 365	5	53
8	9.59 425	30	9.63 066	35	0.36 934	9.96 360	6	52
9	9.59 455	29	9.63 101	34	0.36 899	9.96 354	5	51
10	9.59 484	30	9.63 135	35	0.36 865	9.96 349	6	50
11	9.59 514	29	9.63 170	35	0.36 830	9.96 343	5	49
12	9.59 543	30	9.63 205	35	0.36 795	9.96 338	5	48
13	9.59 573	29	9.63 240	35	0.36 760	9.96 333	6	47
14	9.59 602	30	9.63 275	35	0.36 725	9.96 327	5	46
15	9.59 632	29	9.63 310	35	0.36 690	9.96 322	6	45
16	9.59 661	29	9.63 345	34	0.36 655	9.96 316	5	44
17	9.59 690	30	9.63 379	35	0.36 621	9.96 311	6	43
18	9.59 720	29	9.63 414	35	0.36 586	9.96 305	5	42
19	9.59 749	29	9.63 449	35	0.36 551	9.96 300	6	41
20	9.59 778	30	9.63 484	35	0.36 516	9.96 294	5	40
21	9.59 808	29	9.63 519	34	0.36 481	9.96 289	5	39
22	9.59 837	29	9.63 553	35	0.36 447	9.96 284	6	38
23	9.59 866	29	9.63 588	35	0.36 412	9.96 278	5	37
24	9.59 895	29	9.63 623	34	0.36 377	9.96 273	6	36
25	9.59 924	30	9.63 657	35	0.36 343	9.96 267	5	35
26	9.59 954	29	9.63 692	34	0.36 308	9.96 262	6	34
27	9.59 983	29	9.63 726	35	0.36 274	9.96 256	5	33
28	9.60 012	29	9.63 761	35	0.36 239	9.96 251	6	32
29	9.60 041	29	9.63 796	34	0.36 204	9.96 245	5	31
30	9.60 070	29	9.63 830	35	0.36 170	9.96 240	6	30
31	9.60 099	29	9.63 865	34	0.36 135	9.96 234	5	29
32	9.60 128	29	9.63 899	35	0.36 101	9.96 229	6	28
33	9.60 157	29	9.63 934	34	0.36 066	9.96 223	5	27
34	9.60 186	29	9.63 968	35	0.36 032	9.96 218	6	26
35	9.60 215	29	9.64 003	34	0.35 997	9.96 212	5	25
36	9.60 244	29	9.64 037	35	0.35 963	9.96 207	6	24
37	9.60 273	29	9.64 072	34	0.35 928	9.96 201	5	23
38	9.60 302	29	9.64 106	34	0.35 894	9.96 196	6	22
39	9.60 331	28	9.64 140	35	0.35 860	9.96 190	5	21
40	9.60 359	29	9.64 175	34	0.35 825	9.96 185	6	20
41	9.60 388	29	9.64 209	34	0.35 791	9.96 179	5	19
42	9.60 417	29	9.64 243	35	0.35 757	9.96 174	6	18
43	9.60 446	28	9.64 278	34	0.35 722	9.96 168	6	17
44	9.60 474	29	9.64 312	34	0.35 688	9.96 162	5	16
45	9.60 503	29	9.64 346	35	0.35 654	9.96 157	6	15
46	9.60 532	29	9.64 381	34	0.35 619	9.96 151	5	14
47	9.60 561	28	9.64 415	34	0.35 585	9.96 146	6	13
48	9.60 589	29	9.64 449	34	0.35 551	9.96 140	5	12
49	9.60 618	28	9.64 483	34	0.35 517	9.96 135	6	11
50	9.60 646	29	9.64 517	35	0.35 483	9.96 129	6	10
51	9.60 675	29	9.64 552	34	0.35 448	9.96 123	5	9
52	9.60 704	28	9.64 586	34	0.35 414	9.96 118	6	8
53	9.60 732	29	9.64 620	34	0.35 380	9.96 112	5	7
54	9.60 761	28	9.64 654	34	0.35 346	9.96 107	6	6
55	9.60 789	29	9.64 688	34	0.35 312	9.96 101	6	5
56	9.60 818	28	9.64 722	34	0.35 278	9.96 095	5	4
57	9.60 846	29	9.64 756	34	0.35 244	9.96 090	6	3
58	9.60 875	28	9.64 790	34	0.35 210	9.96 084	5	2
59	9.60 903	28	9.64 824	34	0.35 176	9.96 079	6	1
60	9.60 931		9.64 858		0.35 142	9.96 073		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	36	35	34
1	0.6	0.6	0.6
2	1.2	1.2	1.1
3	1.8	1.8	1.7
4	2.4	2.3	2.3
5	3.0	2.9	2.8
6	3.6	3.5	3.4
7	4.2	4.1	4.0
8	4.8	4.7	4.5
9	5.4	5.2	5.1
10	6.0	5.8	5.7
20	12.0	11.7	11.3
30	18.0	17.5	17.0
40	24.0	23.3	22.7
50	30.0	29.2	28.3

	30	29	28
1	0.5	0.5	0.5
2	1.0	1.0	0.9
3	1.5	1.4	1.4
4	2.0	1.9	1.9
5	2.5	2.4	2.3
6	3.0	2.9	2.8
7	3.5	3.4	3.3
8	4.0	3.9	3.7
9	4.5	4.4	4.2
10	5.0	4.8	4.7
20	10.0	9.7	9.3
30	15.0	14.5	14.0
40	20.0	19.3	18.7
50	25.0	24.2	23.3

	6	5
1	0.1	0.1
2	0.2	0.2
3	0.3	0.2
4	0.4	0.3
5	0.5	0.4
6	0.6	0.5
7	0.7	0.6
8	0.8	0.7
9	0.9	0.8
10	1.0	0.8
20	2.0	1.7
30	3.0	2.5
40	4.0	3.3
50	5.0	4.2

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.60 931	29	9.64 858	34	0.35 142	9.96 073	6	**60**
1	9.60 960	28	9.64 892	34	0.35 108	9.96 067	5	59
2	9.60 988	28	9.64 926	34	0.35 074	9.96 062	6	58
3	9.61 016	29	9.64 960	34	0.35 040	9.96 056	6	57
4	9.61 04$\bar{5}$	28	9.64 994	34	0.35 006	9.96 050	5	56
5	9.61 073	28	9.65 028	34	0.34 972	9.96 04$\bar{5}$	6	**55**
6	9.61 101	28	9.65 062	34	0.34 938	9.96 039	5	54
7	9.61 129	29	9.65 096	34	0.34 904	9.96 034	6	53
8	9.61 158	28	9.65 130	34	0.34 870	9.96 028	6	52
9	9.61 186	28	9.65 164	33	0.34 836	9.96 022	5	51
10	9.61 214	28	9.65 197	34	0.34 803	9.96 017	6	**50**
11	9.61 242	28	9.65 231	34	0.34 769	9.96 011	6	49
12	9.61 270	28	9.65 265	34	0.34 73$\bar{5}$	9.96 005	5	48
13	9.61 298	28	9.65 299	34	0.34 701	9.96 000	6	47
14	9.61 326	28	9.65 333	33	0.34 667	9.95 994	6	46
15	9.61 354	28	9.65 366	34	0.34 634	9.95 988	6	**45**
16	9.61 382	29	9.65 400	34	0.34 600	9.95 982	5	44
17	9.61 411	27	9.65 434	33	0.34 566	9.95 977	6	43
18	9.61 438	28	9.65 467	34	0.34 533	9.95 971	6	42
19	9.61 466	28	9.65 501	34	0.34 499	9.95 965	5	41
20	9.61 494	28	9.65 53$\bar{5}$	33	0.34 465	9.95 960	6	**40**
21	9.61 522	28	9.65 568	34	0.34 432	9.95 954	6	39
22	9.61 550	28	9.65 602	34	0.34 398	9.95 948	6	38
23	9.61 578	28	9.65 636	33	0.34 364	9.95 942	5	37
24	9.61 606	28	9.65 669	34	0.34 331	9.95 937	6	36
25	9.61 634	28	9.65 703	33	0.34 297	9.95 931	6	**35**
26	9.61 662	27	9.65 736	34	0.34 264	9.95 925	5	34
27	9.61 689	28	9.65 770	33	0.34 230	9.95 920	6	33
28	9.61 717	28	9.65 803	34	0.34 197	9.95 914	6	32
29	9.61 74$\bar{5}$	28	9.65 837	33	0.34 163	9.95 908	6	31
30	9.61 773	27	9.65 870	34	0.34 130	9.95 902	5	**30**
31	9.61 800	28	9.65 904	33	0.34 096	9.95 897	6	29
32	9.61 828	28	9.65 937	34	0.34 063	9.95 891	6	28
33	9.61 856	27	9.65 971	33	0.34 029	9.95 88$\bar{5}$	6	27
34	9.61 883	28	9.66 004	34	0.33 996	9.95 879	6	26
35	9.61 911	28	9.66 038	33	0.33 962	9.95 873	5	**25**
36	9.61 939	27	9.66 071	33	0.33 929	9.95 868	6	24
37	9.61 966	28	9.66 104	34	0.33 896	9.95 862	6	23
38	9.61 994	27	9.66 138	33	0.33 862	9.95 856	6	22
39	9.62 021	28	9.66 171	33	0.33 829	9.95 850	6	21
40	9.62 049	27	9.66 204	34	0.33 796	9.95 844	5	**20**
41	9.62 076	28	9.66 238	33	0.33 762	9.95 839	6	19
42	9.62 104	27	9.66 271	33	0.33 729	9.95 833	6	18
43	9.62 131	28	9.66 304	33	0.33 696	9.95 827	6	17
44	9.62 159	27	9.66 337	34	0.33 663	9.95 821	6	16
45	9.62 186	28	9.66 371	33	0.33 629	9.95 815	5	**15**
46	9.62 214	27	9.66 404	33	0.33 596	9.95 810	6	14
47	9.62 241	27	9.66 437	33	0.33 563	9.95 804	6	13
48	9.62 268	28	9.66 470	33	0.33 530	9.95 798	6	12
49	9.62 296	27	9.66 503	34	0.33 497	9.95 792	6	11
50	9.62 323	27	9.66 537	33	0.33 463	9.95 786	6	**10**
51	9.62 350	27	9.66 570	33	0.33 430	9.95 780	5	9
52	9.62 377	28	9.66 603	33	0.33 397	9.95 77$\bar{5}$	6	8
53	9.62 40$\bar{5}$	27	9.66 636	33	0.33 364	9.95 769	6	7
54	9.62 432	27	9.66 669	33	0.33 331	9.95 763	6	6
55	9.62 459	27	9.66 702	33	0.33 298	9.95 757	6	**5**
56	9.62 486	27	9.66 735	33	0.33 26$\bar{5}$	9.95 751	6	4
57	9.62 513	28	9.66 768	33	0.33 232	9.95 745	6	3
58	9.62 541	27	9.66 801	33	0.33 199	9.95 739	6	2
59	9.62 568	27	9.66 834	33	0.33 166	9.95 733	5	1
60	9.62 59$\bar{5}$		9.66 867		0.33 133	9.95 728		**0**
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	34	33
1	0.6	0.6
2	1.1	1.1
3	1.7	1.6
4	2.3	2.2
5	2.8	2.8
6	3.4	3.3
7	4.0	3.8
8	4.5	4.4
9	5.1	$\bar{5}$.0
10	5.7	5.5
20	11.3	11.0
30	17.0	16.5
40	22.7	22.0
50	28.3	27.5

	29	28	27
1	0.$\bar{5}$	0.$\bar{5}$	0.4
2	1.0	0.9	0.9
3	1.4	1.4	1.4
4	1.9	1.9	1.8
5	2.4	2.3	2.2
6	2.9	2.8	2.7
7	3.4	3.3	3.2
8	3.9	3.7	3.6
9	4.4	4.2	4.0
10	4.8	4.7	4.5
20	9.7	9.3	9.0
30	14.5	14.0	13.5
40	19.3	18.7	18.0
50	24.2	23.3	22.5

	6	5
1	0.1	0.1
2	0.2	0.2
3	0.3	0.2
4	0.4	0.3
5	0.5	0.4
6	0.6	0.5
7	0.7	0.6
8	0.8	0.7
9	0.9	0.8
10	1.0	0.8
20	2.0	1.7
30	3.0	2.5
40	4.0	3.3
50	5.0	4.2

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.62 59$\bar{5}$	27	9.66 867	33	0.33 133	9.95 728	6	**60**
1	9.62 622	27	9.66 900	33	0.33 100	9.95 722	6	59
2	9.62 649	27	9.66 933	33	0.33 067	9.95 716	6	58
3	9.62 676	27	9.66 966	33	0.33 034	9.95 710	6	57
4	9.62 703	27	9.66 999	33	0.33 001	9.95 704	6	56
5	9.62 730	27	9.67 032	33	0.32 968	9.95 698	6	**55**
6	9.62 757	27	9.67 06$\bar{5}$	33	0.32 935	9.95 692	6	54
7	9.62 784	27	9.67 098	33	0.32 902	9.95 686	6	53
8	9.62 811	27	9.67 131	32	0.32 869	9.95 680	6	52
9	9.62 838	27	9.67 163	33	0.32 837	9.95 674	6	51
10	9.62 86$\bar{5}$	27	9.67 196	33	0.32 804	9.95 668	5	**50**
11	9.62 892	26	9.67 229	33	0.32 771	9.95 663	6	49
12	9.62 918	27	9.67 262	33	0.32 738	9.95 657	6	48
13	9.62 945	27	9.67 29$\bar{5}$	32	0.32 705	9.95 651	6	47
14	9.62 972	27	9.67 327	33	0.32 673	9.95 64$\bar{5}$	6	46
15	9.62 999	27	9.67 360	33	0.32 640	9.95 639	6	**45**
16	9.63 026	26	9.67 393	33	0.32 607	9.95 633	6	44
17	9.63 052	27	9.67 426	32	0.32 574	9.95 627	6	43
18	9.63 079	27	9.67 458	33	0.32 542	9.95 621	6	42
19	9.63 106	27	9.67 491	33	0.32 509	9.95 61$\bar{5}$	6	41
20	9.63 133	26	9.67 524	32	0.32 476	9.95 609	6	**40**
21	9.63 159	27	9.67 556	33	0.32 444	9.95 603	6	39
22	9.63 186	27	9.67 589	33	0.32 411	9.95 597	6	38
23	9.63 213	26	9.67 622	32	0.32 378	9.95 591	6	37
24	9.63 239	27	9.67 654	33	0.32 346	9.95 58$\bar{5}$	6	36
25	9.63 266	26	9.67 687	32	0.32 313	9.95 579	6	**35**
26	9.63 292	27	9.67 719	33	0.32 281	9.95 573	6	34
27	9.63 319	26	9.67 752	33	0.32 248	9.95 567	6	33
28	9.63 345	27	9.67 78$\bar{5}$	32	0.32 215	9.95 561	6	32
29	9.63 372	26	9.67 817	33	0.32 183	9.95 55$\bar{5}$	6	31
30	9.63 398	27	9.67 8$\bar{5}$0	32	0.32 150	9.95 549	6	**30**
31	9.63 42$\bar{5}$	26	9.67 882	33	0.32 118	9.95 543	6	29
32	9.63 451	27	9.67 91$\bar{5}$	32	0.32 085	9.95 537	6	28
33	9.63 478	26	9.67 947	33	0.32 053	9.95 531	6	27
34	9.63 504	27	9.67 980	32	0.32 020	9.95 52$\bar{5}$	6	26
35	9.63 531	26	9.68 012	32	0.31 988	9.95 519	6	**25**
36	9.63 557	26	9.68 044	33	0.31 956	9.95 513	6	24
37	9.63 583	27	9.68 077	32	0.31 923	9.95 507	7	23
38	9.63 610	26	9.68 109	33	0.31 891	9.95 500	6	22
39	9.63 636	26	9.68 142	32	0.31 858	9.95 494	6	21
40	9.63 662	27	9.68 174	32	0.31 826	9.95 488	6	**20**
41	9.63 689	26	9.68 206	33	0.31 794	9.95 482	6	19
42	9.63 71$\bar{5}$	26	9.68 239	32	0.31 761	9.95 476	6	18
43	9.63 741	26	9.68 271	32	0.31 729	9.95 470	6	17
44	9.63 767	27	9.68 303	33	0.31 697	9.95 464	6	16
45	9.63 794	26	9.68 336	32	0.31 664	9.95 458	6	**15**
46	9.63 820	26	9.68 368	32	0.31 632	9.95 452	6	14
47	9.63 846	26	9.68 400	32	0.31 600	9.95 446	6	13
48	9.63 872	26	9.68 432	33	0.31 568	9.95 440	6	12
49	9.63 898	26	9.68 46$\bar{5}$	32	0.31 535	9.95 434	7	11
50	9.63 924	26	9.68 497	32	0.31 503	9.95 427	6	**10**
51	9.63 950	26	9.68 529	32	0.31 471	9.95 421	6	9
52	9.63 976	26	9.68 561	32	0.31 439	9.95 415	6	8
53	9.64 002	26	9.68 593	33	0.31 407	9.95 409	6	7
54	9.64 028	26	9.68 626	32	0.31 374	9.95 403	6	6
55	9.64 054	26	9.68 658	32	0.31 342	9.95 397	6	**5**
56	9.64 080	26	9.68 690	32	0.31 310	9.95 391	7	4
57	9.64 106	26	9.68 722	32	0.31 278	9.95 384	6	3
58	9.64 132	26	9.68 754	32	0.31 246	9.95 378	6	2
59	9.64 158	26	9.68 786	32	0.31 214	9.95 372	6	1
60	9.64 184		9.68 818		0.31 182	9.95 366		**0**
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	33	32
1	0.6	0.5
2	1.1	1.1
3	1.6	1.6
4	2.2	2.1
5	2.8	2.7
6	3.3	3.2
7	3.8	3.7
8	4.4	4.3
9	$\bar{5}$.0	4.8
10	5.5	5.3
20	11.0	10.7
30	16.5	16.0
40	22.0	21.3
50	27.5	26.7

	27	26
1	0.4	0.4
2	0.9	0.9
3	1.4	1.3
4	1.8	1.7
5	2.2	2.2
6	2.7	2.6
7	3.2	3.0
8	3.6	3.$\bar{5}$
9	4.0	3.9
10	4.5	4.3
20	9.0	8.7
30	13.5	13.0
40	18.0	17.3
50	22.5	21.7

	7	6	5
1	0.1	0.1	0.1
2	0.2	0.2	0.2
3	0.4	0.3	0.2
4	0.$\bar{5}$	0.4	0.3
5	0.6	0.5	0.4
6	0.7	0.6	0.5
7	0.8	0.7	0.6
8	0.9	0.8	0.7
9	1.0	0.9	0.8
10	1.2	1.0	0.8
20	2.3	2.0	1.7
30	3.5	3.0	2.5
40	4.7	4.0	3.3
50	5.8	5.0	4.2

Prop. Parts

′	log sin	d	log tan	cd	log cot	log cos	d	
0	9.64 184	26	9.68 818	32	0.31 182	9.95 366	6	60
1	9.64 210	26	9.68 850	32	0.31 1$\bar{5}$0	9.95 360	6	59
2	9.64 236	26	9.68 882	32	0.31 118	9.95 354	6	58
3	9.64 262	26	9.68 914	32	0.31 086	9.95 348	7	57
4	9.64 288	25	9.68 946	32	0.31 054	9.95 341	6	56
5	9.64 313	26	9.68 978	32	0.31 022	9.95 335	6	55
6	9.64 339	26	9.69 010	32	0.30 990	9.95 329	6	54
7	9.64 365	26	9.69 042	32	0.30 958	9.95 323	6	53
8	9.64 391	26	9.69 074	32	0.30 926	9.95 317	7	52
9	9.64 417	25	9.69 106	32	0.30 894	9.95 310	6	51
10	9.64 442	26	9.69 138	32	0.30 862	9.95 304	6	50
11	9.64 468	26	9.69 170	32	0.30 830	9.95 298	6	49
12	9.64 494	25	9.69 202	32	0.30 798	9.95 292	6	48
13	9.64 519	26	9.69 234	32	0.30 766	9.95 286	7	47
14	9.64 54$\bar{5}$	26	9.69 266	32	0.30 734	9.95 279	6	46
15	9.64 571	25	9.69 298	31	0.30 702	9.95 273	6	45
16	9.64 596	26	9.69 329	32	0.30 671	9.95 267	6	44
17	9.64 622	25	9.69 361	32	0.30 639	9.95 261	7	43
18	9.64 647	26	9.69 393	32	0.30 607	9.95 254	6	42
19	9.64 673	25	9.69 42$\bar{5}$	32	0.30 575	9.95 248	6	41
20	9.64 698	26	9.69 457	31	0.30 543	9.95 242	6	40
21	9.64 724	25	9.69 488	32	0.30 512	9.95 236	7	39
22	9.64 749	26	9.69 520	32	0.30 480	9.95 229	6	38
23	9.64 77$\bar{5}$	25	9.69 552	32	0.30 448	9.95 223	6	37
24	9.64 800	26	9.69 584	31	0.30 416	9.95 217	6	36
25	9.64 826	25	9.69 615	32	0.30 38$\bar{5}$	9.95 211	7	35
26	9.64 851	26	9.69 647	32	0.30 353	9.95 204	6	34
27	9.64 877	25	9.69 679	31	0.30 321	9.95 198	6	33
28	9.64 902	25	9.69 710	32	0.30 290	9.95 192	7	32
29	9.64 927	26	9.69 742	32	0.30 258	9.95 185	6	31
30	9.64 953	25	9.69 774	31	0.30 226	9.95 179	6	30
31	9.64 978	25	9.69 805	32	0.30 19$\bar{5}$	9.95 173	6	29
32	9.65 003	26	9.69 837	31	0.30 163	9.95 167	7	28
33	9.65 029	25	9.69 868	32	0.30 132	9.95 160	6	27
34	9.65 054	25	9.69 900	32	0.30 100	9.95 154	6	26
35	9.65 079	25	9.69 932	31	0.30 068	9.95 148	7	25
36	9.65 104	26	9.69 963	32	0.30 037	9.95 141	6	24
37	9.65 130	25	9.69 99$\bar{5}$	31	0.30 005	9.95 13$\bar{5}$	6	23
38	9.65 15$\bar{5}$	25	9.70 026	32	0.29 974	9.95 129	7	22
39	9.65 180	25	9.70 058	31	0.29 942	9.95 122	6	21
40	9.65 205	25	9.70 089	32	0.29 911	9.95 116	6	20
41	9.65 230	25	9.70 121	31	0.29 879	9.95 110	7	19
42	9.65 255	26	9.70 152	32	0.29 848	9.95 103	6	18
43	9.65 281	25	9.70 184	31	0.29 816	9.95 097	7	17
44	9.65 306	25	9.70 215	32	0.29 78$\bar{5}$	9.95 090	6	16
45	9.65 331	25	9.70 247	31	0.29 753	9.95 084	6	15
46	9.65 356	25	9.70 278	31	0.29 722	9.95 078	7	14
47	9.65 381	25	9.70 309	32	0.29 691	9.95 071	6	13
48	9.65 406	25	9.70 341	31	0.29 659	9.95 06$\bar{5}$	6	12
49	9.65 431	25	9.70 372	32	0.29 628	9.95 059	7	11
50	9.65 456	25	9.70 404	31	0.29 596	9.95 052	6	10
51	9.65 481	25	9.70 43$\bar{5}$	31	0.29 565	9.95 046	7	9
52	9.65 506	25	9.70 466	32	0.29 534	9.95 039	6	8
53	9.65 531	25	9.70 498	31	0.29 502	9.95 033	6	7
54	9.65 556	24	9.70 529	31	0.29 471	9.95 027	7	6
55	9.65 580	25	9.70 560	32	0.29 440	9.95 020	6	5
56	9.65 605	25	9.70 592	31	0.29 408	9.95 014	7	4
57	9.65 630	25	9.70 623	31	0.29 377	9.95 007	6	3
58	9.65 655	25	9.70 654	31	0.29 346	9.95 001	6	2
59	9.65 680	25	9.70 685	32	0.29 31$\bar{5}$	9.94 99$\bar{5}$	7	1
60	9.65 70$\bar{5}$		9.70 717		0.29 283	9.94 988		0
	log cos	d	log cot	cd	log tan	log sin	d	′

Prop. Parts

	32	31
1	0.5	0.5
2	1.1	1.0
3	1.6	1.6
4	2.1	2.1
5	2.7	2.6
6	3.2	3.1
7	3.7	3.6
8	4.3	4.1
9	4.8	4.6
10	5.3	5.2
20	10.7	10.3
30	16.0	15.5
40	21.3	20.7
50	26.7	25.8

	26	25	24
1	0.4	0.4	0.4
2	0.9	0.8	0.8
3	1.3	1.2	1.2
4	1.7	1.7	1.6
5	2.2	2.1	2.0
6	2.6	2.5	2.4
7	3.0	2.9	2.8
8	3.$\bar{5}$	3.3	3.2
9	3.9	3.8	3.6
10	4.3	4.2	4.0
20	8.7	8.3	8.0
30	13.0	12.5	12.0
40	17.3	16.7	16.0
50	21.7	20.8	20.0

	7	6
1	0.1	0.1
2	0.2	0.2
3	0.4	0.3
4	0.$\bar{5}$	0.4
5	0.6	0.5
6	0.7	0.6
7	0.8	0.7
8	0.9	0.8
9	1.0	0.9
10	1.2	1.0
20	2.3	2.0
30	3.5	3.0
40	4.7	4.0
50	5.8	5.0

Prop. Parts

′	log sin	d	log tan	cd	log cot	log cos	d	
0	9.65 70$\bar{5}$	24	9.70 717	31	0.29 283	9.94 988	6	60
1	9.65 729	25	9.70 748	31	0.29 252	9.94 982	7	59
2	9.65 754	25	9.70 779	31	0.29 221	9.94 975	6	58
3	9.65 779	25	9.70 810	31	0.29 190	9.94 969	7	57
4	9.65 804	24	9.70 841	32	0.29 159	9.94 962	6	56
5	9.65 828	25	9.70 873	31	0.29 127	9.94 956	7	55
6	9.65 853	25	9.70 904	31	0.29 096	9.94 949	6	54
7	9.65 878	24	9.70 93$\bar{5}$	31	0.29 065	9.94 943	7	53
8	9.65 902	25	9.70 966	31	0.29 034	9.94 936	6	52
9	9.65 927	25	9.70 997	31	0.29 003	9.94 930	7	51
10	9.65 952	24	9.71 028	31	0.28 972	9.94 923	6	50
11	9.65 976	25	9.71 059	31	0.28 941	9.94 917	6	49
12	9.66 001	24	9.71 090	31	0.28 910	9.94 911	7	48
13	9.66 025	25	9.71 121	32	0.28 879	9.94 904	6	47
14	9.66 050	25	9.71 153	31	0.28 847	9.94 898	7	46
15	9.66 07$\bar{5}$	24	9.71 184	31	0.28 816	9.94 891	6	45
16	9.66 099	25	9.71 21$\bar{5}$	31	0.28 785	9.94 88$\bar{5}$	7	44
17	9.66 124	24	9.71 246	31	0.28 754	9.94 878	7	43
18	9.66 148	25	9.71 277	31	0.28 723	9.94 871	6	42
19	9.66 173	24	9.71 308	31	0.28 692	9.94 86$\bar{5}$	7	41
20	9.66 197	24	9.71 339	31	0.28 661	9.94 858	6	40
21	9.66 221	25	9.71 370	31	0.28 630	9.94 852	7	39
22	9.66 246	24	9.71 401	30	0.28 599	9.94 845	6	38
23	9.66 270	25	9.71 431	31	0.28 569	9.94 839	7	37
24	9.66 29$\bar{5}$	24	9.71 462	31	0.28 538	9.94 832	6	36
25	9.66 319	24	9.71 493	31	0.28 507	9.94 826	7	35
26	9.66 343	25	9.71 524	31	0.28 476	9.94 819	6	34
27	9.66 368	24	9.71 555	31	0.28 44$\bar{5}$	9.94 813	7	33
28	9.66 392	24	9.71 586	31	0.28 414	9.94 806	7	32
29	9.66 416	25	9.71 617	31	0.28 383	9.94 799	6	31
30	9.66 441	24	9.71 648	31	0.28 352	9.94 793	7	30
31	9.66 46$\bar{5}$	24	9.71 679	30	0.28 321	9.94 786	6	29
32	9.66 489	24	9.71 709	31	0.28 291	9.94 780	7	28
33	9.66 513	24	9.71 740	31	0.28 260	9.94 773	6	27
34	9.66 537	25	9.71 771	31	0.28 229	9.94 767	7	26
35	9.66 562	24	9.71 802	31	0.28 198	9.94 760	7	25
36	9.66 586	24	9.71 833	30	0.28 167	9.94 753	6	24
37	9.66 610	24	9.71 863	31	0.28 137	9.94 747	7	23
38	9.66 634	24	9.71 894	31	0.28 106	9.94 740	6	22
39	9.66 658	24	9.71 92$\bar{5}$	30	0.28 075	9.94 734	7	21
40	9.66 682	24	9.71 955	31	0.28 04$\bar{5}$	9.94 727	7	20
41	9.66 706	25	9.71 986	31	0.28 014	9.94 720	6	19
42	9.66 731	24	9.72 017	31	0.27 983	9.94 714	7	18
43	9.66 75$\bar{5}$	24	9.72 048	30	0.27 952	9.94 707	7	17
44	9.66 779	24	9.72 078	31	0.27 922	9.94 700	6	16
45	9.66 803	24	9.72 109	31	0.27 891	9.94 694	7	15
46	9.66 827	24	9.72 140	30	0.27 860	9.94 687	7	14
47	9.66 851	24	9.72 170	31	0.27 830	9.94 680	6	13
48	9.66 87$\bar{5}$	24	9.72 201	30	0.27 799	9.94 674	7	12
49	9.66 899	23	9.72 231	31	0.27 769	9.94 667	7	11
50	9.66 922	24	9.72 262	31	0.27 738	9.94 660	6	10
51	9.66 946	24	9.72 293	30	0.27 707	9.94 654	7	9
52	9.66 970	24	9.72 323	31	0.27 677	9.94 647	7	8
53	9.66 994	24	9.72 354	30	0.27 646	9.94 640	6	7
54	9.67 018	24	9.72 384	31	0.27 616	9.94 634	7	6
55	9.67 042	24	9.72 41$\bar{5}$	30	0.27 585	9.94 627	7	5
56	9.67 066	24	9.72 445	31	0.27 55$\bar{5}$	9.94 620	6	4
57	9.67 090	23	9.72 476	30	0.27 524	9.94 614	7	3
58	9.67 113	24	9.72 506	31	0.27 494	9.94 607	7	2
59	9.67 137	24	9.72 537	30	0.27 463	9.94 600	7	1
60	9.67 161		9.72 567		0.27 433	9.94 593		0
	log cos	d	log cot	cd	log tan	log sin	d	′

Prop. Parts

	32	31	30
1	0.5	0.5	0.5
2	1.1	1.0	1.0
3	1.6	1.6	1.5
4	2.1	2.1	2.0
5	2.7	2.6	2.5
6	3.2	3.1	3.0
7	3.7	3.6	3.5
8	4.3	4.1	4.0
9	4.8	4.6	4.5
10	5.3	5.2	5.0
20	10.7	10.3	10.0
30	16.0	15.5	15.0
40	21.3	20.7	20.0
50	26.7	25.8	25.0

	25	24	23
1	0.4	0.4	0.4
2	0.8	0.8	0.8
3	1.2	1.2	1.2
4	1.7	1.6	1.5
5	2.1	2.0	1.9
6	2.5	2.4	2.3
7	2.9	2.8	2.7
8	3.3	3.2	3.1
9	3.8	3.6	3.4
10	4.2	4.0	3.8
20	8.3	8.0	7.7
30	12.5	12.0	11.5
40	16.7	16.0	15.3
50	20.8	20.0	19.2

	7	6
1	0.1	0.1
2	0.2	0.2
3	0.4	0.3
4	0.$\bar{5}$	0.4
5	0.6	0.5
6	0.7	0.6
7	0.8	0.7
8	0.9	0.8
9	1.0	0.9
10	1.2	1.0
20	2.3	2.0
30	3.5	3.0
40	4.7	4.0
50	5.8	5.0

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.67 161	24	9.72 567	31	0.27 433	9.94 593	6	60
1	9.67 185̄	23	9.72 598	30	0.27 402	9.94 587	7	59
2	9.67 208	24	9.72 628	31	0.27 372	9.94 580	7	58
3	9.67 232	24	9.72 659	30	0.27 341	9.94 573	6	57
4	9.67 256	24	9.72 689	31	0.27 311	9.94 567	7	56
5	9.67 280	23	9.72 720	30	0.27 280	9.94 560	7	55
6	9.67 303	24	9.72 750	30	0.27 250	9.94 553	7	54
7	9.67 327	23	9.72 780	31	0.27 220	9.94 546	6	53
8	9.67 350	24	9.72 811	30	0.27 189	9.94 540	7	52
9	9.67 374	24	9.72 841	31	0.27 159	9.94 533	7	51
10	9.67 398	23	9.72 872	30	0.27 128	9.94 526	7	50
11	9.67 421	24	9.72 902	30	0.27 098	9.94 519	6	49
12	9.67 445̄	23	9.72 932	31	0.27 068	9.94 513	7	48
13	9.67 468	24	9.72 963	30	0.27 037	9.94 506	7	47
14	9.67 492	23	9.72 993	30	0.27 007	9.94 499	7	46
15	9.67 515	24	9.73 023	31	0.26 977	9.94 492	7	45
16	9.67 539	23	9.73 054	30	0.26 946	9.94 485	6	44
17	9.67 562	24	9.73 084	30	0.26 916	9.94 479	7	43
18	9.67 586	23	9.73 114	30	0.26 886	9.94 472	7	42
19	9.67 609	24	9.73 144	31	0.26 856	9.94 465	7	41
20	9.67 633	23	9.73 175̄	30	0.26 825	9.94 458	7	40
21	9.67 656	24	9.73 205̄	30	0.26 795	9.94 451	6	39
22	9.67 680	23	9.73 235	30	0.26 765̄	9.94 445̄	7	38
23	9.67 703	23	9.73 265	30	0.26 735̄	9.94 438	7	37
24	9.67 726	24	9.73 295	31	0.26 705̄	9.94 431	7	36
25	9.67 750̄	23	9.73 326	30	0.26 674	9.94 424	7	35
26	9.67 773	23	9.73 356	30	0.26 644	9.94 417	7	34
27	9.67 796	24	9.73 386	30	0.26 614	9.94 410	6	33
28	9.67 820	23	9.73 416	30	0.26 584	9.94 404	7	32
29	9.67 843	23	9.73 446	30	0.26 554	9.94 397	7	31
30	9.67 866	24	9.73 476	31	0.26 524	9.94 390	7	30
31	9.67 890	23	9.73 507	30	0.26 493	9.94 383	7	29
32	9.67 913	23	9.73 537	30	0.26 463	9.94 376	7	28
33	9.67 936	23	9.73 567	30	0.26 433	9.94 369	7	27
34	9.67 959	23	9.73 597	30	0.26 403	9.94 362	7	26
35	9.67 982	24	9.73 627	30	0.26 373	9.94 355	6	25
36	9.68 006	23	9.73 657	30	0.26 343	9.94 349	7	24
37	9.68 029	23	9.73 687	30	0.26 313	9.94 342	7	23
38	9.68 052	23	9.73 717	30	0.26 283	9.94 335̄	7	22
39	9.68 075	23	9.73 747	30	0.26 253	9.94 328	7	21
40	9.68 098	23	9.73 777	30	0.26 223	9.94 321	7	20
41	9.68 121	23	9.73 807	30	0.26 193	9.94 314	7	19
42	9.68 144	23	9.73 837	30	0.26 163	9.94 307	7	18
43	9.68 167	23	9.73 867	30	0.26 133	9.94 300	7	17
44	9.68 190	23	9.73 897	30	0.26 103	9.94 293	7	16
45	9.68 213	24	9.73 927	30	0.26 073	9.94 286	7	15
46	9.68 237	23	9.73 957	30	0.26 043	9.94 279	6	14
47	9.68 260	23	9.73 987	30	0.26 013	9.94 273	7	13
48	9.68 283	22	9.74 017	30	0.25 983	9.94 266	7	12
49	9.68 305	23	9.74 047	30	0.25 953	9.94 259	7	11
50	9.68 328	23	9.74 077	30	0.25 923	9.94 252	7	10
51	9.68 351	23	9.74 107	30	0.25 893	9.94 245̄	7	9
52	9.68 374	23	9.74 137	29	0.25 863	9.94 238	7	8
53	9.68 397	23	9.74 166	30	0.25 834	9.94 231	7	7
54	9.68 420	23	9.74 196	30	0.25 804	9.94 224	7	6
55	9.68 443	23	9.74 226	30	0.25 774	9.94 217	7	5
56	9.68 466	23	9.74 256	30	0.25 744	9.94 210	7	4
57	9.68 489	23	9.74 286	30	0.25 714	9.94 203	7	3
58	9.68 512	22	9.74 316	29	0.25 684	9.94 196	7	2
59	9.68 534	23	9.74 345	30	0.25 655	9.94 189	7	1
60	9.68 557		9.74 375		0.25 625̄	9.94 182		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	31	30	29
1	0.5	0.5	0.5̄
2	1.0	1.0	1.0
3	1.6	1.5	1.4
4	2.1	2.0	1.9
5	2.6	2.5	2.4
6	3.1	3.0	2.9
7	3.6	3.5	3.4
8	4.1	4.0	3.9
9	4.6	4.5	4.4
10	5.2	5.0	4.8
20	10.3	10.0	9.7
30	15.5	15.0	14.5
40	20.7	20.0	19.3
50	25.8	25.0	24.2

	24	23	22
1	0.4	0.4	0.4
2	0.8	0.8	0.7
3	1.2	1.2	1.1
4	1.6	1.5	1.5̄
5	2.0	1.9	1.8
6	2.4	2.3	2.2
7	2.8	2.7	2.6
8	3.2	3.1	2.9
9	3.6	3.4	3.3
10	4.0	3.8	3.7
20	8.0	7.7	7.3
30	12.0	11.5	11.0
40	16.0	15.3	14.7
50	20.0	19.2	18.3

	7	6
1	0.1	0.1
2	0.2	0.2
3	0.4	0.3
4	0.5̄	0.4
5	0.6	0.5
6	0.7	0.6
7	0.8	0.7
8	0.9	0.8
9	1.0	0.9
10	1.2	1.0
20	2.3	2.0
30	3.5	3.0
40	4.7	4.0
50	5.8	5.0

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.68 557	23	9.74 375	30	0.25 625̄	9.94 182	7	60
1	9.68 580	23	9.74 405̄	30	0.25 595	9.94 175̄	7	59
2	9.68 603	22	9.74 435̄	30	0.25 565	9.94 168	7	58
3	9.68 625	23	9.74 465̄	29	0.25 535	9.94 161	7	57
4	9.68 648	23	9.74 494	30	0.25 506	9.94 154	7	56
5	9.68 671	23	9.74 524	30	0.25 476	9.94 147	7	55
6	9.68 694	22	9.74 554	29	0.25 446	9.94 140	7	54
7	9.68 716	23	9.74 583	30	0.25 417	9.94 133	7	53
8	9.68 739	23	9.74 613	30	0.25 387	9.94 126	7	52
9	9.68 762	22	9.74 643	30	0.25 357	9.94 119	7	51
10	9.68 784	23	9.74 673	29	0.25 327	9.94 112	7	50
11	9.68 807	22	9.74 702	30	0.25 298	9.94 105̄	7	49
12	9.68 829	23	9.74 732	30	0.25 268	9.94 098	8	48
13	9.68 852	23	9.74 762	29	0.25 238	9.94 090	7	47
14	9.68 875̄	22	9.74 791	30	0.25 209	9.94 083	7	46
15	9.68 897	23	9.74 821	30	0.25 179	9.94 076	7	45
16	9.68 920	22	9.74 851	29	0.25 149	9.94 069	7	44
17	9.68 942	23	9.74 880	30	0.25 120	9.94 062	7	43
18	9.68 965̄	22	9.74 910	29	0.25 090	9.94 055	7	42
19	9.68 987	23	9.74 939	30	0.25 061	9.94 048	7	41
20	9.69 010	22	9.74 969	29	0.25 031	9.94 041	7	40
21	9.69 032	23	9.74 998	30	0.25 002	9.94 034	7	39
22	9.69 055̄	22	9.75 028	30	0.24 972	9.94 027	7	38
23	9.69 077	23	9.75 058	29	0.24 942	9.94 020	8	37
24	9.69 100	22	9.75 087	30	0.24 913	9.94 012	7	36
25	9.69 122	22	9.75 117	29	0.24 883	9.94 005	7	35
26	9.69 144	23	9.75 146	30	0.24 854	9.93 998	7	34
27	9.69 167	22	9.75 176	29	0.24 824	9.93 991	7	33
28	9.69 189	23	9.75 205	30	0.24 795̄	9.93 984	7	32
29	9.69 212	22	9.75 235̄	29	0.24 765	9.93 977	7	31
30	9.69 234	22	9.75 264	30	0.24 736	9.93 970	7	30
31	9.69 256	23	9.75 294	29	0.24 706	9.93 963	8	29
32	9.69 279	22	9.75 323	30	0.24 677	9.93 955	7	28
33	9.69 301	22	9.75 353	29	0.24 647	9.93 948	7	27
34	9.69 323	22	9.75 382	29	0.24 618	9.93 941	7	26
35	9.69 345	23	9.75 411	30	0.24 589	9.93 934	7	25
36	9.69 368	22	9.75 441	29	0.24 559	9.93 927	7	24
37	9.69 390	22	9.75 470	30	0.24 530	9.93 920	8	23
38	9.69 412	22	9.75 500̄	29	0.24 500	9.93 912	7	22
39	9.69 434	22	9.75 529	29	0.24 471	9.93 905	7	21
40	9.69 456	23	9.75 558	30	0.24 442	9.93 898	7	20
41	9.69 479	22	9.75 588	29	0.24 412	9.93 891	7	19
42	9.69 501	22	9.75 617	30	0.24 383	9.93 884	8	18
43	9.69 523	22	9.75 647	29	0.24 353	9.93 876	7	17
44	9.69 545	22	9.75 676	29	0.24 324	9.93 869	7	16
45	9.69 567	22	9.75 705	30	0.24 295̄	9.93 862	7	15
46	9.69 589	22	9.75 735̄	29	0.24 265	9.93 855̄	8	14
47	9.69 611	22	9.75 764	29	0.24 236	9.93 847	7	13
48	9.69 633	22	9.75 793	29	0.24 207	9.93 840	7	12
49	9.69 655	22	9.75 822	30	0.24 178	9.93 833	7	11
50	9.69 677	22	9.75 852	29	0.24 148	9.93 826	7	10
51	9.69 699	22	9.75 881	29	0.24 119	9.93 819	8	9
52	9.69 721	22	9.75 910	29	0.24 090	9.93 811	7	8
53	9.69 743	22	9.75 939	30	0.24 061	9.93 804	7	7
54	9.69 765	22	9.75 969	29	0.24 031	9.93 797	8	6
55	9.69 787	22	9.75 998	29	0.24 002	9.93 789	7	5
56	9.69 809	22	9.76 027	29	0.23 973	9.93 782	7	4
57	9.69 831	22	9.76 056	30	0.23 944	9.93 775̄	7	3
58	9.69 853	22	9.76 086	29	0.23 914	9.93 768	8	2
59	9.69 875	22	9.76 115̄	29	0.23 885	9.93 760	7	1
60	9.69 897		9.76 144		0.23 856	9.93 753		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	30	29
1	0.5	0.5̄
2	1.0	1.0
3	1.5	1.4
4	2.0	1.9
5	2.5	2.4
6	3.0	2.9
7	3.5	3.4
8	4.0	3.9
9	4.5	4.4
10	5.0	4.8
20	10.0	9.7
30	15.0	14.5
40	20.0	19.3
50	25.0	24.2

	23	22
1	0.4	0.4
2	0.8	0.7
3	1.2	1.1
4	1.5	1.5̄
5	1.9	1.8
6	2.3	2.2
7	2.7	2.6
8	3.1	2.9
9	3.4	3.3
10	3.8	3.7
20	7.7	7.3
30	11.5	11.0
40	15.3	14.7
50	19.2	18.3

	8	7
1	0.1	0.1
2	0.3	0.2
3	0.4	0.4
4	0.5	0.5̄
5	0.7	0.6
6	0.8	0.7
7	0.9	0.8
8	1.1	0.9
9	1.2	1.0
10	1.3	1.2
20	2.7	2.3
30	4.0	3.5
40	5.3	4.7
50	6.7	5.8

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.69 897	22	9.76 144	29	0.23 856	9.93 753	7	**60**
1	9.69 919	22	9.76 173	29	0.23 827	9.93 746	8	59
2	9.69 941	22	9.76 202	29	0.23 798	9.93 738	7	58
3	9.69 963	21	9.76 231	30	0.23 769	9.93 731	7	57
4	9.69 984	22	9.76 261	29	0.23 739	9.93 724	7	56
5	9.70 006	22	9.76 290	29	0.23 710	9.93 717	8	**55**
6	9.70 028	22	9.76 319	29	0.23 681	9.93 709	7	54
7	9.70 050	22	9.76 348	29	0.23 652	9.93 702	7	53
8	9.70 072	21	9.76 377	29	0.23 623	9.93 695	8	52
9	9.70 093	22	9.76 406	29	0.23 594	9.93 687	7	51
10	9.70 115	22	9.76 435	29	0.23 565̄	9.93 680	7	**50**
11	9.70 137	22	9.76 464	29	0.23 536	9.93 673	8	49
12	9.70 159	21	9.76 493	29	0.23 507	9.93 665	7	48
13	9.70 180	22	9.76 522	29	0.23 478	9.93 658	8	47
14	9.70 202	22	9.76 551	29	0.23 449	9.93 650	7	46
15	9.70 224	21	9.76 580	29	0.23 420	9.93 643	7	**45**
16	9.70 245	22	9.76 609	30	0.23 391	9.93 636	8	44
17	9.70 267	21	9.76 639	29	0.23 361	9.93 628	7	43
18	9.70 288	22	9.76 668	29	0.23 332	9.93 621	7	42
19	9.70 310	22	9.76 697	28	0.23 303	9.93 614	8	41
20	9.70 332	21	9.76 725	29	0.23 275̄	9.93 606	7	**40**
21	9.70 353	22	9.76 754	29	0.23 246	9.93 599	8	39
22	9.70 375̄	21	9.76 783	29	0.23 217	9.93 591	7	38
23	9.70 396	22	9.76 812	29	0.23 188	9.93 584	7	37
24	9.70 418	21	9.76 841	29	0.23 159	9.93 577	8	36
25	9.70 439	22	9.76 870	29	0.23 130	9.93 569	7	**35**
26	9.70 461	21	9.76 899	29	0.23 101	9.93 562	8	34
27	9.70 482	22	9.76 928	29	0.23 072	9.93 554	7	33
28	9.70 504	21	9.76 957	29	0.23 043	9.93 547	8	32
29	9.70 525	22	9.76 986	29	0.23 014	9.93 539	7	31
30	9.70 547	21	9.77 015̄	29	0.22 985	9.93 532	7	**30**
31	9.70 568	22	9.77 044	29	0.22 956	9.93 525̄	8	29
32	9.70 590	21	9.77 073	28	0.22 927	9.93 517	7	28
33	9.70 611	22	9.77 101	29	0.22 899	9.93 510	8	27
34	9.70 633	21	9.77 130	29	0.22 870	9.93 502	7	26
35	9.70 654	21	9.77 159	29	0.22 841	9.93 495̄	8	**25**
36	9.70 675	22	9.77 188	29	0.22 812	9.93 487	7	24
37	9.70 697	21	9.77 217	29	0.22 783	9.93 480	8	23
38	9.70 718	21	9.77 246	28	0.22 754	9.93 472	7	22
39	9.70 739	22	9.77 274	29	0.22 726	9.93 465̄	8	21
40	9.70 761	21	9.77 303	29	0.22 697	9.93 457	7	**20**
41	9.70 782	21	9.77 332	29	0.22 668	9.93 450	8	19
42	9.70 803	21	9.77 361	29	0.22 639	9.93 442	7	18
43	9.70 824	22	9.77 390	28	0.22 610	9.93 435̄	8	17
44	9.70 846	21	9.77 418	29	0.22 582	9.93 427	7	16
45	9.70 867	21	9.77 447	29	0.22 553	9.93 420	8	**15**
46	9.70 888	21	9.77 476	29	0.22 524	9.93 412	7	14
47	9.70 909	22	9.77 505̄	28	0.22 495	9.93 405̄	8	13
48	9.70 931	21	9.77 533	29	0.22 467	9.93 397	7	12
49	9.70 952	21	9.77 562	29	0.22 438	9.93 390	8	11
50	9.70 973	21	9.77 591	28	0.22 409	9.93 382	7	**10**
51	9.70 994	21	9.77 619	29	0.22 381	9.93 375̄	8	9
52	9.71 015	21	9.77 648	29	0.22 352	9.93 367	7	8
53	9.71 036	22	9.77 677	29	0.22 323	9.93 360	8	7
54	9.71 058	21	9.77 706	28	0.22 294	9.93 352	8	6
55	9.71 079	21	9.77 734	29	0.22 266	9.93 344	7	**5**
56	9.71 100	21	9.77 763	28	0.22 237	9.93 337	8	4
57	9.71 121	21	9.77 791	29	0.22 209	9.93 329	7	3
58	9.71 142	21	9.77 820	29	0.22 180	9.93 322	8	2
59	9.71 163	21	9.77 849	28	0.22 151	9.93 314	7	1
60	9.71 184		9.77 877		0.22 123	9.93 307		**0**
	og cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	30	**29**	**28**
1	0.5	0.5̄	0.5̄
2	1.0	1.0	0.9
3	1.5	1.4	1.4
4	2.0	1.9	1.9
5	2.5	2 4	2.3
6	3.0	2.9	2.8
7	3.5	3 4	3.3
8	4.0	3 9	3.7
9	4.5	4 4	4.2
10	5.0	4.8	4.7
20	10.0	9.7	9 3
30	15.0	14.5	14.0
40	20.0	19.3	18.7
50	25.0	24.2	23.3

	22	**21**
1	0.4	0.4
2	0.7	0.7
3	1.1	1.0
4	1.5̄	1.4
5	1.8	1.8
6	2.2	2.1
7	2.6	2.4
8	2.9	2.8
9	3.3	3.2
10	3.7	3.5
20	7.3	7.0
30	11.0	10.5
40	14.7	14.0
50	18.3	17.5

	8	**7**
1	0.1	0.1
2	0.3	0.2
3	0.4	0.4
4	0.5	0.5̄
5	0.7	0.6
6	0.8	0.7
7	0.9	0.8
8	1.1	0.9
9	1.2	1.0
10	1.3	1.2
20	2.7	2.3
30	4.0	3.5
40	5.3	4.7
50	6.7	5.8

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.71 184	21	9.77 877	29	0.22 123	9.93 307	8	**60**
1	9.71 205̄	21	9.77 906	29	0.22 094	9.93 299	8	59
2	9.71 226	21	9.77 935̄	28	0.22 065	9.93 291	7	58
3	9.71 247	21	9.77 963	29	0.22 037	9.93 284	8	57
4	9.71 268	21	9.77 992	28	0.22 008	9.93 276	7	56
5	9.71 289	21	9.78 020	29	0.21 980	9.93 269	8	**55**
6	9.71 310	21	9.78 049	28	0.21 951	9.93 261	8	54
7	9.71 331	21	9.78 077	29	0.21 923	9.93 253	7	53
8	9.71 352	21	9.78 106	29	0.21 894	9.93 246	8	52
9	9.71 373	20	9.78 135̄	28	0.21 865	9.93 238	8	51
10	9.71 393	21	9.78 163	29	0.21 837	9.93 230	7	**50**
11	9.71 414	21	9.78 192	28	0.21 808	9.93 223	8	49
12	9.71 435	21	9.78 220	29	0.21 780	9.93 215	8	48
13	9.71 456	21	9.78 249	28	0.21 751	9.93 207	7	47
14	9.71 477	21	9.78 277	29	0.21 723	9.93 200	8	46
15	9.71 498	21	9.78 306	28	0.21 694	9.93 192	8	**45**
16	9.71 519	20	9.78 334	29	0.21 666	9.93 184	7	44
17	9.71 539	21	9.78 363	28	0.21 637	9.93 177	8	43
18	9.71 560	21	9.78 391	28	0.21 609	9.93 169	8	42
19	9.71 581	21	9.78 419	29	0.21 581	9.93 161	7	41
20	9.71 602	20	9.78 448	28	0.21 552	9.93 154	8	**40**
21	9.71 622	21	9.78 476	29	0.21 524	9.93 146	8	39
22	9.71 643	21	9.78 505̄	28	0.21 495	9.93 138	7	38
23	9.71 664	21	9.78 533	29	0.21 467	9.93 131	8	37
24	9.71 685̄	20	9.78 562	28	0.21 438	9.93 123	8	36
25	9.71 705	21	9.78 590	28	0.21 410	9.93 115	7	**35**
26	9.71 726	21	9.78 618	29	0.21 382	9.93 108	8	34
27	9.71 747	20	9.78 647	28	0.21 353	9.93 100	8	33
28	9.71 767	21	9.78 675	29	0.21 325̄	9.93 092	8	32
29	9.71 788	21	9.78 704	28	0.21 296	9.93 084	7	31
30	9.71 809	20	9.78 732	28	0.21 268	9.93 077	8	**30**
31	9.71 829	21	9.78 760	29	0.21 240	9.93 069	8	29
32	9.71 850̄	20	9.78 789	28	0.21 211	9.93 061	8	28
33	9.71 870	21	9.78 817	28	0.21 183	9.93 053	7	27
34	9.71 891	20	9.78 845	29	0.21 155̄	9.93 046	8	26
35	9.71 911	21	9.78 874	28	0.21 126	9.93 038	8	**25**
36	9.71 932	20	9.78 902	28	0.21 098	9.93 030	8	24
37	9.71 952	21	9.78 930	29	0.21 070	9.93 022	8	23
38	9.71 973	21	9.78 959	28	0.21 041	9.93 014	7	22
39	9.71 994	20	9.78 987	28	0.21 013	9.93 007	8	21
40	9.72 014	20	9.79 015	28	0.20 985̄	9.92 999	8	**20**
41	9.72 034	21	9.79 043	29	0.20 957	9.92 991	8	19
42	9.72 055̄	20	9.79 072	28	0.20 928	9.92 983	7	18
43	9.72 075	21	9.79 100	28	0.20 900	9.92 976	8	17
44	9.72 096	20	9.79 128	28	0.20 872	9.92 968	8	16
45	9.72 116	21	9.79 156	29	0.20 844	9.92 960	8	**15**
46	9.72 137	20	9.79 185̄	28	0.20 815	9.92 952	8	14
47	9.72 157	20	9.79 213	28	0.20 787	9.92 944	8	13
48	9.72 177	21	9.79 241	28	0.20 759	9.92 936	7	12
49	9.72 198	20	9.79 269	28	0.20 731	9.92 929	8	11
50	9.72 218	20	9.79 297	29	0.20 703	9.92 921	8	**10**
51	9.72 238	21	9.79 326	28	0.20 674	9.92 913	8	9
52	9.72 259	20	9.79 354	28	0.20 646	9.92 905	8	8
53	9.72 279	20	9.79 382	28	0.20 618	9.92 897	8	7
54	9.72 299	21	9.79 410	28	0.20 590	9.92 889	8	6
55	9.72 320	20	9.79 438	28	0.20 562	9.92 881	7	**5**
56	9.72 340	20	9.79 466	29	0.20 534	9.92 874	8	4
57	9.72 360	21	9.79 495̄	28	0.20 505	9.92 866	8	3
58	9.72 381	20	9.79 523	28	0.20 477	9.92 858	8	2
59	9.72 401	20	9.79 551	28	0.20 449	9.92 850̄	8	1
60	9.72 421		9.79 579		0.20 421	9.92 842		**0**
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	29	**28**
1	0.5̄	0.5̄
2	1.0	0.9
3	1.4	1.4
4	1.9	1.9
5	2.4	2.3
6	2.9	2.8
7	3.4	3.3
8	3.9	3.7
9	4.4	4.2
10	4.8	4.7
20	9.7	9.3
30	14.5	14.0
40	19.3	18.7
50	24.2	23.3

	21	**20**
1	0.4	0.3
2	0.7	0.7
3	1.0	1.0
4	1.4	1.3
5	1.8	1.7
6	2.1	2.0
7	2.4	2.3
8	2.8	2.7
9	3.2	3.0
10	3.5	3.3
20	7.0	6.7
30	10.5	10.0
40	14.0	13.3
50	17.5	16.7

	8	**7**
1	0.1	0.1
2	0.3	0.2
3	0.4	0.4
4	0.5	0.5̄
5	0.7	0.6
6	0.8	0.7
7	0.9	0.8
8	1.1	0.9
9	1.2	1.0
10	1.3	1.2
20	2.7	2.3
30	4.0	3.5
40	5.3	4.7
50	6.7	5.8

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.72 421	20	9.79 579	28	0.20 421	9.92 842	8	60
1	9.72 441	20	9.79 607	28	0.20 393	9.92 834	8	59
2	9.72 461	21	9.79 635	28	0.20 365	9.92 826	8	58
3	9.72 482	20	9.79 663	28	0.20 337	9.92 818	8	57
4	9.72 502	20	9.79 691	28	0.20 309	9.92 810	7	56
5	9.72 522	20	9.79 719	28	0.20 281	9.92 803	8	55
6	9.72 542	20	9.79 747	29	0.20 253	9.92 795	8	54
7	9.72 562	20	9.79 776	28	0.20 224	9.92 787	8	53
8	9.72 582	20	9.79 804	28	0.20 196	9.92 779	8	52
9	9.72 602	20	9.79 832	28	0.20 168	9.92 771	8	51
10	9.72 622	21	9.79 860	28	0.20 140	9.92 763	8	50
11	9.72 643	20	9.79 888	28	0.20 112	9.92 755	8	49
12	9.72 663	20	9.79 916	28	0.20 084	9.92 747	8	48
13	9.72 683	20	9.79 944	28	0.20 056	9.92 739	8	47
14	9.72 703	20	9.79 972	28	0.20 028	9.92 731	8	46
15	9.72 723	20	9.80 000	28	0.20 000	9.92 723	8	45
16	9.72 743	20	9.80 028	28	0.19 972	9.92 715	8	44
17	9.72 763	20	9.80 056	28	0.19 944	9.92 707	8	43
18	9.72 783	20	9.80 084	28	0.19 916	9.92 699	8	42
19	9.72 803	20	9.80 112	28	0.19 888	9.92 691	8	41
20	9.72 823	20	9.80 140	28	0.19 860	9.92 683	8	40
21	9.72 843	20	9.80 168	27	0.19 832	9.92 675	8	39
22	9.72 863	20	9.80 195	28	0.19 805	9.92 667	8	38
23	9.72 883	19	9.80 223	28	0.19 777	9.92 659	8	37
24	9.72 902	20	9.80 251	28	0.19 749	9.92 651	8	36
25	9.72 922	20	9.80 279	28	0.19 721	9.92 643	8	35
26	9.72 942	20	9.80 307	28	0.19 693	9.92 635	8	34
27	9.72 962	20	9.80 335	28	0.19 665	9.92 627	8	33
28	9.72 982	20	9.80 363	28	0.19 637	9.92 619	8	32
29	9.73 002	20	9.80 391	28	0.19 609	9.92 611	8	31
30	9.73 022	19	9.80 419	28	0.19 581	9.92 603	8	30
31	9.73 041	20	9.80 447	27	0.19 553	9.92 595	8	29
32	9.73 061	20	9.80 474	28	0.19 526	9.92 587	8	28
33	9.73 081	20	9.80 502	28	0.19 498	9.92 579	8	27
34	9.73 101	20	9.80 530	28	0.19 470	9.92 571	8	26
35	9.73 121	19	9.80 558	28	0.19 442	9.92 563	8	25
36	9.73 140	20	9.80 586	28	0.19 414	9.92 555	9	24
37	9.73 160	20	9.80 614	28	0.19 386	9.92 546	8	23
38	9.73 180	20	9.80 642	27	0.19 358	9.92 538	8	22
39	9.73 200	19	9.80 669	28	0.19 331	9.92 530	8	21
40	9.73 219	20	9.80 697	28	0.19 303	9.92 522	8	20
41	9.73 239	20	9.80 725	28	0.19 275	9.92 514	8	19
42	9.73 259	19	9.80 753	28	0.19 247	9.92 506	8	18
43	9.73 278	20	9.80 781	27	0.19 219	9.92 498	8	17
44	9.73 298	20	9.80 808	28	0.19 192	9.92 490	8	16
45	9.73 318	19	9.80 836	28	0.19 164	9.92 482	9	15
46	9.73 337	20	9.80 864	28	0.19 136	9.92 473	8	14
47	9.73 357	20	9.80 892	27	0.19 108	9.92 465	8	13
48	9.73 377	19	9.80 919	28	0.19 081	9.92 457	8	12
49	9.73 396	20	9.80 947	28	0.19 053	9.92 449	8	11
50	9.73 416	19	9.80 975	28	0.19 025	9.92 441	8	10
51	9.73 435	20	9.81 003	27	0.18 997	9.92 433	8	9
52	9.73 455	19	9.81 030	28	0.18 970	9.92 425	9	8
53	9.73 474	20	9.81 058	28	0.18 942	9.92 416	8	7
54	9.73 494	19	9.81 086	27	0.18 914	9.92 408	8	6
55	9.73 513	20	9.81 113	28	0.18 887	9.92 400	8	5
56	9.73 533	19	9.81 141	28	0.18 859	9.92 392	8	4
57	9.73 552	20	9.81 169	27	0.18 831	9.92 384	8	3
58	9.73 572	19	9.81 196	28	0.18 804	9.92 376	9	2
59	9.73 591	20	9.81 224	28	0.18 776	9.92 367	8	1
60	9.73 611		9.81 252		0.18 748	9.92 359		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	29	28	27
1	0.5	0.5	0.4
2	1.0	0.9	0.9
3	1.4	1.4	1.4
4	1.9	1.9	1.8
5	2.4	2.3	2.2
6	2.9	2.8	2.7
7	3.4	3.3	3.2
8	3.9	3.7	3.6
9	4.4	4.2	4.0
10	4.8	4.7	4.5
20	9.7	9.3	9.0
30	14.5	14.0	13.5
40	19.3	18.7	18.0
50	24.2	23.3	22.5

	21	20	19
1	0.4	0.3	0.3
2	0.7	0.7	0.6
3	1.0	1.0	1.0
4	1.4	1.3	1.3
5	1.8	1.7	1.6
6	2.1	2.0	1.9
7	2.4	2.3	2.2
8	2.8	2.7	2.5
9	3.2	3.0	2.8
10	3.5	3.3	3.2
20	7.0	6.7	6.3
30	10.5	10.0	9.5
40	14.0	13.3	12.7
50	17.5	16.7	15.8

	9	8	7
1	0.2	0.1	0.1
2	0.3	0.3	0.2
3	0.4	0.4	0.4
4	0.6	0.5	0.5
5	0.8	0.7	0.6
6	0.9	0.8	0.7
7	1.0	0.9	0.8
8	1.2	1.1	0.9
9	1.4	1.2	1.0
10	1.5	1.3	1.2
20	3.0	2.7	2.3
30	4.5	4.0	3.5
40	6.0	5.3	4.7
50	7.5	6.7	5.8

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.73 611	19	9.81 252	27	0.18 748	9.92 359	8	60
1	9.73 630	20	9.81 279	28	0.18 721	9.92 351	8	59
2	9.73 650	19	9.81 307	28	0.18 693	9.92 343	8	58
3	9.73 669	20	9.81 335	27	0.18 665	9.92 335	9	57
4	9.73 689	19	9.81 362	28	0.18 638	9.92 326	8	56
5	9.73 708	19	9.81 390	28	0.18 610	9.92 318	8	55
6	9.73 727	20	9.81 418	27	0.18 582	9.92 310	8	54
7	9.73 747	19	9.81 445	28	0.18 555	9.92 302	9	53
8	9.73 766	19	9.81 473	27	0.18 527	9.92 293	8	52
9	9.73 785	20	9.81 500	28	0.18 500	9.92 285	8	51
10	9.73 805	19	9.81 528	28	0.18 472	9.92 277	8	50
11	9.73 824	19	9.81 556	27	0.18 444	9.92 269	9	49
12	9.73 843	20	9.81 583	28	0.18 417	9.92 260	8	48
13	9.73 863	19	9.81 611	27	0.18 389	9.92 252	8	47
14	9.73 882	19	9.81 638	28	0.18 362	9.92 244	9	46
15	9.73 901	20	9.81 666	27	0.18 334	9.92 235	8	45
16	9.73 921	19	9.81 693	28	0.18 307	9.92 227	8	44
17	9.73 940	19	9.81 721	27	0.18 279	9.92 219	8	43
18	9.73 959	19	9.81 748	28	0.18 252	9.92 211	9	42
19	9.73 978	19	9.81 776	27	0.18 224	9.92 202	8	41
20	9.73 997	20	9.81 803	28	0.18 197	9.92 194	8	40
21	9.74 017	19	9.81 831	27	0.18 169	9.92 186	9	39
22	9.74 036	19	9.81 858	28	0.18 142	9.92 177	8	38
23	9.74 055	19	9.81 886	27	0.18 114	9.92 169	8	37
24	9.74 074	19	9.81 913	28	0.18 087	9.92 161	9	36
25	9.74 093	20	9.81 941	27	0.18 059	9.92 152	8	35
26	9.74 113	19	9.81 968	28	0.18 032	9.92 144	8	34
27	9.74 132	19	9.81 996	27	0.18 004	9.92 136	9	33
28	9.74 151	19	9.82 023	28	0.17 977	9.92 127	8	32
29	9.74 170	19	9.82 051	27	0.17 949	9.92 119	8	31
30	9.74 189	19	9.82 078	28	0.17 922	9.92 111	9	30
31	9.74 208	19	9.82 106	27	0.17 894	9.92 102	8	29
32	9.74 227	19	9.82 133	28	0.17 867	9.92 094	8	28
33	9.74 246	19	9.82 161	27	0.17 839	9.92 086	9	27
34	9.74 265	19	9.82 188	27	0.17 812	9.92 077	8	26
35	9.74 284	19	9.82 215	28	0.17 785	9.92 069	9	25
36	9.74 303	19	9.82 243	27	0.17 757	9.92 060	8	24
37	9.74 322	19	9.82 270	28	0.17 730	9.92 052	8	23
38	9.74 341	19	9.82 298	27	0.17 702	9.92 044	9	22
39	9.74 360	19	9.82 325	27	0.17 675	9.92 035	8	21
40	9.74 379	19	9.82 352	28	0.17 648	9.92 027	9	20
41	9.74 398	19	9.82 380	27	0.17 620	9.92 018	8	19
42	9.74 417	19	9.82 407	28	0.17 593	9.92 010	8	18
43	9.74 436	19	9.82 435	27	0.17 565	9.92 002	9	17
44	9.74 455	19	9.82 462	27	0.17 538	9.91 993	8	16
45	9.74 474	19	9.82 489	28	0.17 511	9.91 985	9	15
46	9.74 493	19	9.82 517	27	0.17 483	9.91 976	8	14
47	9.74 512	19	9.82 544	27	0.17 456	9.91 968	9	13
48	9.74 531	18	9.82 571	28	0.17 429	9.91 959	8	12
49	9.74 549	19	9.82 599	27	0.17 401	9.91 951	9	11
50	9.74 568	19	9.82 626	27	0.17 374	9.91 942	8	10
51	9.74 587	19	9.82 653	28	0.17 347	9.91 934	9	9
52	9.74 606	19	9.82 681	27	0.17 319	9.91 925	8	8
53	9.74 625	19	9.82 708	27	0.17 292	9.91 917	9	7
54	9.74 644	18	9.82 735	27	0.17 265	9.91 908	8	6
55	9.74 662	19	9.82 762	28	0.17 238	9.91 900	9	5
56	9.74 681	19	9.82 790	27	0.17 210	9.91 891	8	4
57	9.74 700	19	9.82 817	27	0.17 183	9.91 883	9	3
58	9.74 719	18	9.82 844	27	0.17 156	9.91 874	8	2
59	9.74 737	19	9.82 871	28	0.17 129	9.91 866	9	1
60	9.74 756		9.82 899		0.17 101	9.91 857		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	28	27
1	0.5	0.4
2	0.9	0.9
3	1.4	1.4
4	1.9	1.8
5	2.3	2.2
6	2.8	2.7
7	3.3	3.2
8	3.7	3.6
9	4.2	4.0
10	4.7	4.5
20	9.3	9.0
30	14.0	13.5
40	18.7	18.0
50	23.3	22.5

	20	19	18
1	0.3	0.3	0.3
2	0.7	0.6	0.6
3	1.0	1.0	0.9
4	1.3	1.3	1.2
5	1.7	1.6	1.5
6	2.0	1.9	1.8
7	2.3	2.2	2.1
8	2.7	2.5	2.4
9	3.0	2.8	2.7
10	3.3	3.2	3.0
20	6.7	6.3	6.0
30	10.0	9.5	9.0
40	13.3	12.7	12.0
50	16.7	15.8	15.0

	9	8
1	0.2	0.1
2	0.3	0.3
3	0.4	0.4
4	0.6	0.5
5	0.8	0.7
6	0.9	0.8
7	1.0	0.9
8	1.2	1.1
9	1.4	1.2
10	1.5	1.3
20	3.0	2.7
30	4.5	4.0
40	6.0	5.3
50	7.5	6.7

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.74 756	19	9.82 899	27	0.17 101	9.91 857	8	60
1	9.74 775	19	9.82 926	27	0.17 074	9.91 849	9	59
2	9.74 794	18	9.82 953	27	0.17 047	9.91 840	8	58
3	9.74 812	19	9.82 980	28	0.17 020	9.91 832	9	57
4	9.74 831	19	9.83 008	27	0.16 992	9.91 823	8	56
5	9.74 850	18	9.83 035	27	0.16 965	9.91 815	9	55
6	9.74 868	19	9.83 062	27	0.16 938	9.91 806	8	54
7	9.74 887	19	9.83 089	28	0.16 911	9.91 798	9	53
8	9.74 906	18	9.83 117	27	0.16 883	9.91 789	8	52
9	9.74 924	19	9.83 144	27	0.16 856	9.91 781	9	51
10	9.74 943	18	9.83 171	27	0.16 829	9.91 772	9	50
11	9.74 961	19	9.83 198	27	0.16 802	9.91 763	8	49
12	9.74 980	19	9.83 225	27	0.16 775	9.91 755	9	48
13	9.74 999	18	9.83 252	28	0.16 748	9.91 746	8	47
14	9.75 017	19	9.83 280	27	0.16 720	9.91 738	9	46
15	9.75 036	18	9.83 307	27	0.16 693	9.91 729	9	45
16	9.75 054	19	9.83 334	27	0.16 666	9.91 720	8	44
17	9.75 073	18	9.83 361	27	0.16 639	9.91 712	9	43
18	9.75 091	19	9.83 388	27	0.16 612	9.91 703	8	42
19	9.75 110	18	9.83 415	27	0.16 585	9.91 695	9	41
20	9.75 128	19	9.83 442	28	0.16 558	9.91 686	9	40
21	9.75 147	18	9.83 470	27	0.16 530	9.91 677	8	39
22	9.75 165	19	9.83 497	27	0.16 503	9.91 669	9	38
23	9.75 184	18	9.83 524	27	0.16 476	9.91 660	9	37
24	9.75 202	19	9.83 551	27	0.16 449	9.91 651	8	36
25	9.75 221	18	9.83 578	27	0.16 422	9.91 643	9	35
26	9.75 239	19	9.83 605	27	0.16 395	9.91 634	9	34
27	9.75 258	18	9.83 632	27	0.16 368	9.91 625	8	33
28	9.75 276	18	9.83 659	27	0.16 341	9.91 617	9	32
29	9.75 294	19	9.83 686	27	0.16 314	9.91 608	9	31
30	9.75 313	18	9.83 713	27	0.16 287	9.91 599	8	30
31	9.75 331	19	9.83 740	28	0.16 260	9.91 591	9	29
32	9.75 350	18	9.83 768	27	0.16 232	9.91 582	9	28
33	9.75 368	18	9.83 795	27	0.16 205	9.91 573	8	27
34	9.75 386	19	9.83 822	27	0.16 178	9.91 565	9	26
35	9.75 405	18	9.83 849	27	0.16 151	9.91 556	9	25
36	9.75 423	18	9.83 876	27	0.16 124	9.91 547	9	24
37	9.75 441	18	9.83 903	27	0.16 097	9.91 538	8	23
38	9.75 459	19	9.83 930	27	0.16 070	9.91 530	9	22
39	9.75 478	18	9.83 957	27	0.16 043	9.91 521	9	21
40	9.75 496	18	9.83 984	27	0.16 016	9.91 512	8	20
41	9.75 514	19	9.84 011	27	0.15 989	9.91 504	9	19
42	9.75 533	18	9.84 038	27	0.15 962	9.91 495	9	18
43	9.75 551	18	9.84 065	27	0.15 935	9.91 486	9	17
44	9.75 569	18	9.84 092	27	0.15 908	9.91 477	8	16
45	9.75 587	18	9.84 119	27	0.15 881	9.91 469	9	15
46	9.75 605	19	9.84 146	27	0.15 854	9.91 460	9	14
47	9.75 624	18	9.84 173	27	0.15 827	9.91 451	9	13
48	9.75 642	18	9.84 200	27	0.15 800	9.91 442	9	12
49	9.75 660	18	9.84 227	27	0.15 773	9.91 433	8	11
50	9.75 678	18	9.84 254	26	0.15 746	9.91 425	9	10
51	9.75 696	18	9.84 280	27	0.15 720	9.91 416	9	9
52	9.75 714	19	9.84 307	27	0.15 693	9.91 407	9	8
53	9.75 733	18	9.84 334	27	0.15 666	9.91 398	9	7
54	9.75 751	18	9.84 361	27	0.15 639	9.91 389	8	6
55	9.75 769	18	9.84 388	27	0.15 612	9.91 381	9	5
56	9.75 787	18	9.84 415	27	0.15 585	9.91 372	9	4
57	9.75 805	18	9.84 442	27	0.15 558	9.91 363	9	3
58	9.75 823	18	9.84 469	27	0.15 531	9.91 354	9	2
59	9.75 841	18	9.84 496	27	0.15 504	9.91 345	9	1
60	9.75 859		9.84 523		0.15 477	9.91 336		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	28	27	26
1	0.5	0.4	0.4
2	0.9	0.9	0.9
3	1.4	1.4	1.3
4	1.9	1.8	1.7
5	2.3	2.2	2.2
6	2.8	2.7	2.6
7	3.3	3.2	3.0
8	3.7	3.6	3.5
9	4.2	4.0	3.9
10	4.7	4.5	4.3
20	9.3	9.0	8.7
30	14.0	13.5	13.0
40	18.7	18.0	17.3
50	23.3	22.5	21.7

	19	18
1	0.3	0.3
2	0.6	0.6
3	1.0	0.9
4	1.3	1.2
5	1.6	1.5
6	1.9	1.8
7	2.2	2.1
8	2.5	2.4
9	2.8	2.7
10	3.2	3.0
20	6.3	6.0
30	9.5	9.0
40	12.7	12.0
50	15.8	15.0

	9	8
1	0.2	0.1
2	0.3	0.3
3	0.4	0.4
4	0.6	0.5
5	0.8	0.7
6	0.9	0.8
7	1.0	0.9
8	1.2	1.1
9	1.4	1.2
10	1.5	1.3
20	3.0	2.7
30	4.5	4.0
40	6.0	5.3
50	7.5	6.7

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.75 859	18	9.84 523	27	0.15 477	9.91 336	8	60
1	9.75 877	18	9.84 550	26	0.15 450	9.91 328	9	59
2	9.75 895	18	9.84 576	27	0.15 424	9.91 319	9	58
3	9.75 913	18	9.84 603	27	0.15 397	9.91 310	9	57
4	9.75 931	18	9.84 630	27	0.15 370	9.91 301	9	56
5	9.75 949	18	9.84 657	27	0.15 343	9.91 292	9	55
6	9.75 967	18	9.84 684	27	0.15 316	9.91 283	9	54
7	9.75 985	18	9.84 711	27	0.15 289	9.91 274	8	53
8	9.76 003	18	9.84 738	26	0.15 262	9.91 266	9	52
9	9.76 021	18	9.84 764	27	0.15 236	9.91 257	9	51
10	9.76 039	18	9.84 791	27	0.15 209	9.91 248	9	50
11	9.76 057	18	9.84 818	27	0.15 182	9.91 239	9	49
12	9.76 075	18	9.84 845	27	0.15 155	9.91 230	9	48
13	9.76 093	18	9.84 872	27	0.15 128	9.91 221	9	47
14	9.76 111	18	9.84 899	26	0.15 101	9.91 212	9	46
15	9.76 129	17	9.84 925	27	0.15 075	9.91 203	9	45
16	9.76 146	18	9.84 952	27	0.15 048	9.91 194	9	44
17	9.76 164	18	9.84 979	27	0.15 021	9.91 185	9	43
18	9.76 182	18	9.85 006	27	0.14 994	9.91 176	9	42
19	9.76 200	18	9.85 033	26	0.14 967	9.91 167	9	41
20	9.76 218	18	9.85 059	27	0.14 941	9.91 158	9	40
21	9.76 236	17	9.85 086	27	0.14 914	9.91 149	8	39
22	9.76 253	18	9.85 113	27	0.14 887	9.91 141	9	38
23	9.76 271	18	9.85 140	26	0.14 860	9.91 132	9	37
24	9.76 289	18	9.85 166	27	0.14 834	9.91 123	9	36
25	9.76 307	17	9.85 193	27	0.14 807	9.91 114	9	35
26	9.76 324	18	9.85 220	27	0.14 780	9.91 105	9	34
27	9.76 342	18	9.85 247	26	0.14 753	9.91 096	9	33
28	9.76 360	18	9.85 273	27	0.14 727	9.91 087	9	32
29	9.76 378	17	9.85 300	27	0.14 700	9.91 078	9	31
30	9.76 395	18	9.85 327	27	0.14 673	9.91 069	9	30
31	9.76 413	18	9.85 354	26	0.14 646	9.91 060	9	29
32	9.76 431	17	9.85 380	27	0.14 620	9.91 051	9	28
33	9.76 448	18	9.85 407	27	0.14 593	9.91 042	9	27
34	9.76 466	18	9.85 434	26	0.14 566	9.91 033	10	26
35	9.76 484	17	9.85 460	27	0.14 540	9.91 023	9	25
36	9.76 501	18	9.85 487	27	0.14 513	9.91 014	9	24
37	9.76 519	18	9.85 514	26	0.14 486	9.91 005	9	23
38	9.76 537	17	9.85 540	27	0.14 460	9.90 996	9	22
39	9.76 554	18	9.85 567	27	0.14 433	9.90 987	9	21
40	9.76 572	18	9.85 594	26	0.14 406	9.90 978	9	20
41	9.76 590	17	9.85 620	27	0.14 380	9.90 969	9	19
42	9.76 607	18	9.85 647	27	0.14 353	9.90 960	9	18
43	9.76 625	17	9.85 674	26	0.14 326	9.90 951	9	17
44	9.76 642	18	9.85 700	27	0.14 300	9.90 942	9	16
45	9.76 660	17	9.85 727	27	0.14 273	9.90 933	9	15
46	9.76 677	18	9.85 754	26	0.14 246	9.90 924	9	14
47	9.76 695	17	9.85 780	27	0.14 220	9.90 915	9	13
48	9.76 712	18	9.85 807	27	0.14 193	9.90 906	10	12
49	9.76 730	17	9.85 834	26	0.14 166	9.90 896	9	11
50	9.76 747	18	9.85 860	27	0.14 140	9.90 887	9	10
51	9.76 765	17	9.85 887	26	0.14 113	9.90 878	9	9
52	9.76 782	18	9.85 913	27	0.14 087	9.90 869	9	8
53	9.76 800	17	9.85 940	27	0.14 060	9.90 860	9	7
54	9.76 817	18	9.85 967	26	0.14 033	9.90 851	9	6
55	9.76 835	17	9.85 993	27	0.14 007	9.90 842	10	5
56	9.76 852	18	9.86 020	26	0.13 980	9.90 832	9	4
57	9.76 870	17	9.86 046	27	0.13 954	9.90 823	9	3
58	9.76 887	17	9.86 073	27	0.13 927	9.90 814	9	2
59	9.76 904	18	9.86 100	26	0.13 900	9.90 805	9	1
60	9.76 922		9.86 126		0.13 874	9.90 796		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	27	26
1	0.4	0.4
2	0.9	0.9
3	1.4	1.3
4	1.8	1.7
5	2.2	2.2
6	2.7	2.6
7	3.2	3.0
8	3.6	3.5
9	4.0	3.9
10	4.5	4.3
20	9.0	8.7
30	13.5	13.0
40	18.0	17.3
50	22.5	21.7

	18	17	10
1	0.3	0.3	0.2
2	0.6	0.6	0.3
3	0.9	0.8	0.5
4	1.2	1.1	0.7
5	1.5	1.4	0.8
6	1.8	1.7	1.0
7	2.1	2.0	1.2
8	2.4	2.3	1.3
9	2.7	2.6	1.5
10	3.0	2.8	1.7
20	6.0	5.7	3.3
30	9.0	8.5	5.0
40	12.0	11.3	6.7
50	15.0	14.2	8.3

	9	8
1	0.2	0.1
2	0.3	0.3
3	0.4	0.4
4	0.6	0.5
5	0.8	0.7
6	0.9	0.8
7	1.0	0.9
8	1.2	1.1
9	1.4	1.2
10	1.5	1.3
20	3.0	2.7
30	4.5	4.0
40	6.0	5.3
50	7.5	6.7

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.76 922	17	9.86 126	27	0.13 874	9.90 796	9	60
1	9.76 939	18	9.86 153	26	0.13 847	9.90 787	10	59
2	9.76 957	17	9.86 179	27	0.13 821	9.90 777	9	58
3	9.76 974	17	9.86 206	26	0.13 794	9.90 768	9	57
4	9.76 991	18	9.86 232	27	0.13 768	9.90 759	9	56
5	9.77 009	17	9.86 259	26	0.13 741	9.90 750	9	55
6	9.77 026	17	9.86 285	27	0.13 715̄	9.90 741	10	54
7	9.77 043	18	9.86 312	26	0.13 688	9.90 731	9	53
8	9.77 061	17	9.86 338	27	0.13 662	9.90 722	9	52
9	9.77 078	17	9.86 365	27	0.13 635̄	9.90 713	9	51
10	9.77 095	17	9.86 392	26	0.13 608	9.90 704	10	50
11	9.77 112	18	9.86 418	27	0.13 582	9.90 694	9	49
12	9.77 130	17	9.86 445̄	26	0.13 555	9.90 685	9	48
13	9.77 147	17	9.86 471	27	0.13 529	9.90 676	9	47
14	9.77 164	17	9.86 498	26	0.13 502	9.90 667	10	46
15	9.77 181	18	9.86 524	27	0.13 476	9.90 657	9	45
16	9.77 199	17	9.86 551	26	0.13 449	9.90 648	9	44
17	9.77 216	17	9.86 577	26	0.13 423	9.90 639	9	43
18	9.77 233	17	9.86 603	27	0.13 397	9.90 630	10	42
19	9.77 250	18	9.86 630	26	0.13 370	9.90 620	9	41
20	9.77 268	17	9.86 656	27	0.13 344	9.90 611	9	40
21	9.77 285̄	17	9.86 683	26	0.13 317	9.90 602	10	39
22	9.77 302	17	9.86 709	27	0.13 291	9.90 592	9	38
23	9.77 319	17	9.86 736	26	0.13 264	9.90 583	9	37
24	9.77 336	17	9.86 762	27	0.13 238	9.90 574	9	36
25	9.77 353	17	9.86 789	26	0.13 211	9.90 565̄	10	35
26	9.77 370	17	9.86 815	27	0.13 185̄	9.90 555	9	34
27	9.77 387	18	9.86 842	26	0.13 158	9.90 546	9	33
28	9.77 405̄	17	9.86 868	26	0.13 132	9.90 537	10	32
29	9.77 422	17	9.86 894	27	0.13 106	9.90 527	9	31
30	9.77 439	17	9.86 921	26	0.13 079	9.90 518	9	30
31	9.77 456	17	9.86 947	27	0.13 053	9.90 509	10	29
32	9.77 473	17	9.86 974	26	0.13 026	9.90 499	9	28
33	9.77 490	17	9.87 000	27	0.13 000	9.90 490	10	27
34	9.77 507	17	9.87 027	26	0.12 973	9.90 480	9	26
35	9.77 524	17	9.87 053	26	0.12 947	9.90 471	9	25
36	9.77 541	17	9.87 079	27	0.12 921	9.90 462	10	24
37	9.77 558	17	9.87 106	26	0.12 894	9.90 452	9	23
38	9.77 575	17	9.87 132	26	0.12 868	9.90 443	9	22
39	9.77 592	17	9.87 158	27	0.12 842	9.90 434	10	21
40	9.77 609	17	9.87 185̄	26	0.12 815	9.90 424	9	20
41	9.77 626	17	9.87 211	27	0.12 789	9.90 415̄	10	19
42	9.77 643	17	9.87 238	26	0.12 762	9.90 405	9	18
43	9.77 660	17	9.87 264	26	0.12 736	9.90 396	10	17
44	9.77 677	17	9.87 290	27	0.12 710	9.90 386	9	16
45	9.77 694	17	9.87 317	26	0.12 683	9.90 377	9	15
46	9.77 711	17	9.87 343	26	0.12 657	9.90 368	10	14
47	9.77 728	16	9.87 369	27	0.12 631	9.90 358	9	13
48	9.77 744	17	9.87 396	26	0.12 604	9.90 349	10	12
49	9.77 761	17	9.87 422	26	0.12 578	9.90 339	9	11
50	9.77 778	17	9.87 448	27	0.12 552	9.90 330	10	10
51	9.77 795	17	9.87 475̄	26	0.12 525	9.90 320	9	9
52	9.77 812	17	9.87 501	26	0.12 499	9.90 311	10	8
53	9.77 829	17	9.87 527	27	0.12 473	9.90 301	9	7
54	9.77 846	16	9.87 554	26	0.12 446	9.90 292	10	6
55	9.77 862	17	9.87 580	26	0.12 420	9.90 282	9	5
56	9.77 879	17	9.87 606	27	0.12 394	9.90 273	10	4
57	9.77 896	17	9.87 633	26	0.12 367	9.90 263	9	3
58	9.77 913	17	9.87 659	26	0.12 341	9.90 254	10	2
59	9.77 930	16	9.87 685	26	0.12 315̄	9.90 244	9	1
60	9.77 946		9.87 711		0.12 289	9.90 235̄		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	27	26
1	0.4	0.4
2	0.9	0.9
3	1.4	1.3
4	1.8	1.7
5	2.2	2.2
6	2.7	2.6
7	3.2	3.0
8	3.6	3.5̄
9	4.0	3.9
10	4.5	4.3
20	9.0	8.7
30	13.5	13.0
40	18.0	17.3
50	22.5	21.7

	18	17	16
1	0.3	0.3	0.3
2	0.6	0.6	0.5
3	0.9	0.8	0.8
4	1.2	1.1	1.1
5	1.5	1.4	1.3
6	1.8	1.7	1.6
7	2.1	2.0	1.9
8	2.4	2.3	2.1
9	2.7	2.6	2.4
10	3.0	2.8	2.7
20	6.0	5.7	5.3
30	9.0	8.5	8.0
40	12.0	11.3	10.7
50	15.0	14.2	13.3

	10	9
1	0.2	0.2
2	0.3	0.3
3	0.5	0.4
4	0.7	0.6
5	0.8	0.8
6	1.0	0.9
7	1.2	1.0
8	1.3	1.2
9	1.5	1.4
10	1.7	1.5
20	3.3	3.0
30	5.0	4.5
40	6.7	6.0
50	8.3	7.5

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.77 946	17	9.87 711	27	0.12 289	9.90 235̄	10	60
1	9.77 963	17	9.87 738	26	0.12 262	9.90 225	9	59
2	9.77 980	17	9.87 764	26	0.12 236	9.90 216	10	58
3	9.77 997	16	9.87 790	27	0.12 210	9.90 206	9	57
4	9.78 013	17	9.87 817	26	0.12 183	9.90 197	10	56
5	9.78 030	17	9.87 843	26	0.12 157	9.90 187	9	55
6	9.78 047	16	9.87 869	26	0.12 131	9.90 178	10	54
7	9.78 063	17	9.87 895	27	0.12 105̄	9.90 168	9	53
8	9.78 080	17	9.87 922	26	0.12 078	9.90 159	10	52
9	9.78 097	16	9.87 948	26	0.12 052	9.90 149	10	51
10	9.78 113	17	9.87 974	26	0.12 026	9.90 139	9	50
11	9.78 130	17	9.88 000	27	0.12 000	9.90 130	10	49
12	9.78 147	16	9.88 027	26	0.11 973	9.90 120	9	48
13	9.78 163	17	9.88 053	26	0.11 947	9.90 111	10	47
14	9.78 180	17	9.88 079	26	0.11 921	9.90 101	10	46
15	9.78 197	16	9.88 105	26	0.11 895̄	9.90 091	9	45
16	9.78 213	17	9.88 131	27	0.11 869	9.90 082	10	44
17	9.78 230	16	9.88 158	26	0.11 842	9.90 072	9	43
18	9.78 246	17	9.88 184	26	0.11 816	9.90 063	10	42
19	9.78 263	17	9.88 210	26	0.11 790	9.90 053	10	41
20	9.78 280	16	9.88 236	26	0.11 764	9.90 043	9	40
21	9.78 296	17	9.88 262	27	0.11 738	9.90 034	10	39
22	9.78 313	16	9.88 289	26	0.11 711	9.90 024	10	38
23	9.78 329	17	9.88 315̄	26	0.11 685	9.90 014	9	37
24	9.78 346	16	9.88 341	26	0.11 659	9.90 005̄	10	36
25	9.78 362	17	9.88 367	26	0.11 633	9.89 995	10	35
26	9.78 379	16	9.88 393	27	0.11 607	9.89 985	9	34
27	9.78 395	17	9.88 420	26	0.11 580	9.89 976	10	33
28	9.78 412	16	9.88 446	26	0.11 554	9.89 966	10	32
29	9.78 428	17	9.88 472	26	0.11 528	9.89 956	9	31
30	9.78 445̄	16	9.88 498	26	0.11 502	9.89 947	10	30
31	9.78 461	17	9.88 524	26	0.11 476	9.89 937	10	29
32	9.78 478	16	9.88 550	27	0.11 450̄	9.89 927	9	28
33	9.78 494	16	9.88 577	26	0.11 423	9.89 918	10	27
34	9.78 510	17	9.88 603	26	0.11 397	9.89 908	10	26
35	9.78 527	16	9.88 629	26	0.11 371	9.89 898	10	25
36	9.78 543	17	9.88 655̄	26	0.11 345	9.89 888	9	24
37	9.78 560	16	9.88 681	26	0.11 319	9.89 879	10	23
38	9.78 576	16	9.88 707	26	0.11 293	9.89 869	10	22
39	9.78 592	17	9.88 733	26	0.11 267	9.89 859	10	21
40	9.78 609	16	9.88 759	27	0.11 241	9.89 849	9	20
41	9.78 625	17	9.88 786	26	0.11 214	9.89 840	10	19
42	9.78 642	16	9.88 812	26	0.11 188	9.89 830	10	18
43	9.78 658	16	9.88 838	26	0.11 162	9.89 820	10	17
44	9.78 674	17	9.88 864	26	0.11 136	9.89 810	9	16
45	9.78 691	16	9.88 890	26	0.11 110	9.89 801	10	15
46	9.78 707	16	9.88 916	26	0.11 084	9.89 791	10	14
47	9.78 723	16	9.88 942	26	0.11 058	9.89 781	10	13
48	9.78 739	17	9.88 968	26	0.11 032	9.89 771	10	12
49	9.78 756	16	9.88 994	26	0.11 006	9.89 761	9	11
50	9.78 772	16	9.89 020	26	0.10 980	9.89 752	10	10
51	9.78 788	17	9.89 046	27	0.10 954	9.89 742	10	9
52	9.78 805̄	16	9.89 073	26	0.10 927	9.89 732	10	8
53	9.78 821	16	9.89 099	26	0.10 901	9.89 722	10	7
54	9.78 837	16	9.89 125̄	26	0.10 875	9.89 712	10	6
55	9.78 853	16	9.89 151	26	0.10 849	9.89 702	9	5
56	9.78 869	17	9.89 177	26	0.10 823	9.89 693	10	4
57	9.78 886	16	9.89 203	26	0.10 797	9.89 683	10	3
58	9.78 902	16	9.89 229	26	0.10 771	9.89 673	10	2
59	9.78 918	16	9.89 255	26	0.10 745	9.89 663	10	1
60	9.78 934		9.89 281		0.10 719	9.89 653		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	27	26
1	0.4	0.4
2	0.9	0.9
3	1.4	1.3
4	1.8	1.7
5	2.2	2.2
6	2.7	2.6
7	3.2	3.0
8	3.6	3.5
9	4.0	3.9
10	4.5	4.3
20	9.0	8.7
30	13.5	13.0
40	18.0	17.3
50	22.5	21.7

	17	16
1	0.3	0.3
2	0.6	0.5
3	0.8	0.8
4	1.1	1.1
5	1.4	1.3
6	1.7	1.6
7	2.0	1.9
8	2.3	2.1
9	2.6	2.4
10	2.8	2.7
20	5.7	5.3
30	8.5	8.0
40	11.3	10.7
50	14.2	13.3

	10	9
1	0.2	0.2
2	0.3	0.3
3	0.5	0.4
4	0.7	0.6
5	0.8	0.8
6	1.0	0.9
7	1.2	1.0
8	1.3	1.2
9	1.5	1.4
10	1.7	1.5
20	3.3	3.0
30	5.0	4.5
40	6.7	6.0
50	8.3	7.5

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.78 934	16	9.89 281	26	0.10 719	9.89 653	10	60
1	9.78 950	17	9.89 307	26	0.10 693	9.89 643	10	59
2	9.78 967	16	9.89 333	26	0.10 667	9.89 633	9	58
3	9.78 983	16	9.89 359	26	0.10 641	9.89 624	10	57
4	9.78 999	16	9.89 385	26	0.10 615	9.89 614	10	56
5	9.79 015	16	9.89 411	26	0.10 589	9.89 604	10	55
6	9.79 031	16	9.89 437	26	0.10 563	9.89 594	10	54
7	9.79 047	16	9.89 463	26	0.10 537	9.89 584	10	53
8	9.79 063	16	9.89 489	26	0.10 511	9.89 574	10	52
9	9.79 079	16	9.89 515	26	0.10 485	9.89 564	10	51
10	9.79 095	16	9.89 541	26	0.10 459	9.89 554	10	50
11	9.79 111	17	9.89 567	26	0.10 433	9.89 544	10	49
12	9.79 128	16	9.89 593	26	0.10 407	9.89 534	10	48
13	9.79 144	16	9.89 619	26	0.10 381	9.89 524	10	47
14	9.79 160	16	9.89 645	26	0.10 355	9.89 514	10	46
15	9.79 176	16	9.89 671	26	0.10 329	9.89 504	9	45
16	9.79 192	16	9.89 697	26	0.10 303	9.89 495	10	44
17	9.79 208	16	9.89 723	26	0.10 277	9.89 485	10	43
18	9.79 224	16	9.89 749	26	0.10 251	9.89 475	10	42
19	9.79 240	16	9.89 775	26	0.10 225	9.89 465	10	41
20	9.79 256	16	9.89 801	26	0.10 199	9.89 455	10	40
21	9.79 272	16	9.89 827	26	0.10 173	9.89 445	10	39
22	9.79 288	16	9.89 853	26	0.10 147	9.89 435	10	38
23	9.79 304	15	9.89 879	26	0.10 121	9.89 425	10	37
24	9.79 319	16	9.89 905	26	0.10 095	9.89 415	10	36
25	9.79 335	16	9.89 931	26	0.10 069	9.89 405	10	35
26	9.79 351	16	9.89 957	26	0.10 043	9.89 395	10	34
27	9.79 367	16	9.89 983	26	0.10 017	9.89 385	10	33
28	9.79 383	16	9.90 009	26	0.09 991	9.89 375	11	32
29	9.79 399	16	9.90 035	26	0.09 965	9.89 364	10	31
30	9.79 415	16	9.90 061	25	0.09 939	9.89 354	10	30
31	9.79 431	16	9.90 086	26	0.09 914	9.89 344	10	29
32	9.79 447	16	9.90 112	26	0.09 888	9.89 334	10	28
33	9.79 463	15	9.90 138	26	0.09 862	9.89 324	10	27
34	9.79 478	16	9.90 164	26	0.09 836	9.89 314	10	26
35	9.79 494	16	9.90 190	26	0.09 810	9.89 304	10	25
36	9.79 510	16	9.90 216	26	0.09 784	9.89 294	10	24
37	9.79 526	16	9.90 242	26	0.09 758	9.89 284	10	23
38	9.79 542	16	9.90 268	26	0.09 732	9.89 274	10	22
39	9.79 558	15	9.90 294	26	0.09 706	9.89 264	10	21
40	9.79 573	16	9.90 320	26	0.09 680	9.89 254	10	20
41	9.79 589	16	9.90 346	25	0.09 654	9.89 244	11	19
42	9.79 605	16	9.90 371	26	0.09 629	9.89 233	10	18
43	9.79 621	15	9.90 397	26	0.09 603	9.89 223	10	17
44	9.79 636	16	9.90 423	26	0.09 577	9.89 213	10	16
45	9.79 652	16	9.90 449	26	0.09 551	9.89 203	10	15
46	9.79 668	16	9.90 475	26	0.09 525	9.89 193	10	14
47	9.79 684	15	9.90 501	26	0.09 499	9.89 183	10	13
48	9.79 699	16	9.90 527	26	0.09 473	9.89 173	11	12
49	9.79 715	16	9.90 553	25	0.09 447	9.89 162	10	11
50	9.79 731	15	9.90 578	26	0.09 422	9.89 152	10	10
51	9.79 746	16	9.90 604	26	0.09 396	9.89 142	10	9
52	9.79 762	16	9.90 630	26	0.09 370	9.89 132	10	8
53	9.79 778	15	9.90 656	26	0.09 344	9.89 122	10	7
54	9.79 793	16	9.90 682	26	0.09 318	9.89 112	11	6
55	9.79 809	16	9.90 708	26	0.09 292	9.89 101	10	5
56	9.79 825	15	9.90 734	25	0.09 266	9.89 091	10	4
57	9.79 840	16	9.90 759	26	0.09 241	9.89 081	10	3
58	9.79 856	16	9.90 785	26	0.09 215	9.89 071	11	2
59	9.79 872	15	9.90 811	26	0.09 189	9.89 060	10	1
60	9.79 887		9.90 837		0.09 163	9.89 050		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	26	25
1	0.4	0.4
2	0.9	0.8
3	1.3	1.2
4	1.7	1.7
5	2.2	2.1
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

	17	16	15
1	0.3	0.3	0.2
2	0.6	0.5	0.5
3	0.8	0.8	0.8
4	1.1	1.1	1.0
5	1.4	1.3	1.2
6	1.7	1.6	1.5
7	2.0	1.9	1.8
8	2.3	2.1	2.0
9	2.6	2.4	2.2
10	2.8	2.7	2.5
20	5.7	5.3	5.0
30	8.5	8.0	7.5
40	11.3	10.7	10.0
50	14.2	13.3	12.5

	11	10	9
1	0.2	0.2	0.2
2	0.4	0.3	0.3
3	0.6	0.5	0.4
4	0.7	0.7	0.6
5	0.9	0.8	0.8
6	1.1	1.0	0.9
7	1.3	1.2	1.0
8	1.5	1.3	1.2
9	1.6	1.5	1.4
10	1.8	1.7	1.5
20	3.7	3.3	3.0
30	5.5	5.0	4.5
40	7.3	6.7	6.0
50	9.2	8.3	7.5

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.79 887	16	9.90 837	26	0.09 163	9.89 050	10	60
1	9.79 903	15	9.90 863	26	0.09 137	9.89 040	10	59
2	9.79 918	16	9.90 889	25	0.09 111	9.89 030	10	58
3	9.79 934	16	9.90 914	26	0.09 086	9.89 020	11	57
4	9.79 950	15	9.90 940	26	0.09 060	9.89 009	10	56
5	9.79 965	16	9.90 966	26	0.09 034	9.88 999	10	55
6	9.79 981	15	9.90 992	26	0.09 008	9.88 989	11	54
7	9.79 996	16	9.91 018	25	0.08 982	9.88 978	10	53
8	9.80 012	15	9.91 043	26	0.08 957	9.88 968	10	52
9	9.80 027	16	9.91 069	26	0.08 931	9.88 958	10	51
10	9.80 043	15	9.91 095	26	0.08 905	9.88 948	11	50
11	9.80 058	16	9.91 121	26	0.08 879	9.88 937	10	49
12	9.80 074	15	9.91 147	25	0.08 853	9.88 927	10	48
13	9.80 089	16	9.91 172	26	0.08 828	9.88 917	11	47
14	9.80 105	15	9.91 198	26	0.08 802	9.88 906	10	46
15	9.80 120	16	9.91 224	26	0.08 776	9.88 896	10	45
16	9.80 136	15	9.91 250	26	0.08 750	9.88 886	11	44
17	9.80 151	15	9.91 276	25	0.08 724	9.88 875	10	43
18	9.80 166	16	9.91 301	26	0.08 699	9.88 865	10	42
19	9.80 182	15	9.91 327	26	0.08 673	9.88 855	11	41
20	9.80 197	16	9.91 353	26	0.08 647	9.88 844	10	40
21	9.80 213	15	9.91 379	25	0.08 621	9.88 834	10	39
22	9.80 228	16	9.91 404	26	0.08 596	9.88 824	11	38
23	9.80 244	15	9.91 430	26	0.08 570	9.88 813	10	37
24	9.80 259	15	9.91 456	26	0.08 544	9.88 803	10	36
25	9.80 274	16	9.91 482	25	0.08 518	9.88 793	11	35
26	9.80 290	15	9.91 507	26	0.08 493	9.88 782	10	34
27	9.80 305	15	9.91 533	26	0.08 467	9.88 772	11	33
28	9.80 320	16	9.91 559	26	0.08 441	9.88 761	10	32
29	9.80 336	15	9.91 585	25	0.08 415	9.88 751	10	31
30	9.80 351	15	9.91 610	26	0.08 390	9.88 741	11	30
31	9.80 366	16	9.91 636	26	0.08 364	9.88 730	10	29
32	9.80 382	15	9.91 662	26	0.08 338	9.88 720	11	28
33	9.80 397	15	9.91 688	25	0.08 312	9.88 709	10	27
34	9.80 412	16	9.91 713	26	0.08 287	9.88 699	11	26
35	9.80 428	15	9.91 739	26	0.08 261	9.88 688	10	25
36	9.80 443	15	9.91 765	26	0.08 235	9.88 678	10	24
37	9.80 458	15	9.91 791	25	0.08 209	9.88 668	11	23
38	9.80 473	16	9.91 816	26	0.08 184	9.88 657	10	22
39	9.80 489	15	9.91 842	26	0.08 158	9.88 647	11	21
40	9.80 504	15	9.91 868	25	0.08 132	9.88 636	10	20
41	9.80 519	15	9.91 893	26	0.08 107	9.88 626	11	19
42	9.80 534	16	9.91 919	26	0.08 081	9.88 615	10	18
43	9.80 550	15	9.91 945	26	0.08 055	9.88 605	11	17
44	9.80 565	15	9.91 971	25	0.08 029	9.88 594	10	16
45	9.80 580	15	9.91 996	26	0.08 004	9.88 584	11	15
46	9.80 595	15	9.92 022	26	0.07 978	9.88 573	10	14
47	9.80 610	15	9.92 048	25	0.07 952	9.88 563	11	13
48	9.80 625	16	9.92 073	26	0.07 927	9.88 552	10	12
49	9.80 641	15	9.92 099	26	0.07 901	9.88 542	11	11
50	9.80 656	15	9.92 125	25	0.07 875	9.88 531	10	10
51	9.80 671	15	9.92 150	26	0.07 850	9.88 521	11	9
52	9.80 686	15	9.92 176	26	0.07 824	9.88 510	11	8
53	9.80 701	15	9.92 202	25	0.07 798	9.88 499	10	7
54	9.80 716	15	9.92 227	26	0.07 773	9.88 489	11	6
55	9.80 731	15	9.92 253	26	0.07 747	9.88 478	10	5
56	9.80 746	16	9.92 279	25	0.07 721	9.88 468	11	4
57	9.80 762	15	9.92 304	26	0.07 696	9.88 457	10	3
58	9.80 777	15	9.92 330	26	0.07 670	9.88 447	11	2
59	9.80 792	15	9.92 356	25	0.07 644	9.88 436	11	1
60	9.80 807		9.92 381		0.07 619	9.88 425		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	26	25
1	0.4	0.4
2	0.9	0.8
3	1.3	1.2
4	1.7	1.7
5	2.2	2.1
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

	16	15
1	0.3	0.2
2	0.5	0.5
3	0.8	0.8
4	1.1	1.0
5	1.3	1.2
6	1.6	1.5
7	1.9	1.8
8	2.1	2.0
9	2.4	2.2
10	2.7	2.5
20	5.3	5.0
30	8.0	7.5
40	10.7	10.0
50	13.3	12.5

	11	10
1	0.2	0.2
2	0.4	0.3
3	0.6	0.5
4	0.7	0.7
5	0.9	0.8
6	1.1	1.0
7	1.3	1.2
8	1.5	1.3
9	1.6	1.5
10	1.8	1.7
20	3.7	3.3
30	5.5	5.0
40	7.3	6.7
50	9.2	8.3

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.80 807	15	9.92 381	26	0.07 619	9.88 425	10	60
1	9.80 822	15	9.92 407	26	0.07 593	9.88 415̄	11	59
2	9.80 837	15	9.92 433	25	0.07 567	9.88 404	10	58
3	9.80 852	15	9.92 458	26	0.07 542	9.88 394	11	57
4	9.80 867	15	9.92 484	26	0.07 516	9.88 383	11	56
5	9.80 882	15	9.92 510	25	0.07 490	9.88 372	10	55
6	9.80 897	15	9.92 535	26	0.07 465̄	9.88 362	11	54
7	9.80 912	15	9.92 561	26	0.07 439	9.88 351	11	53
8	9.80 927	15	9.92 587	25	0.07 413	9.88 340	10	52
9	9.80 942	15	9.92 612	26	0.07 388	9.88 330	11	51
10	9.80 957	15	9.92 638	25	0.07 362	9.88 319	11	50
11	9.80 972	15	9.92 663	26	0.07 337	9.88 308	10	49
12	9.80 987	15	9.92 689	26	0.07 311	9.88 298	11	48
13	9.81 002	15	9.92 715̄	25	0.07 285	9.88 287	11	47
14	9.81 017	15	9.92 740	26	0.07 260	9.88 276	10	46
15	9.81 032	15	9.92 766	26	0.07 234	9.88 266	11	45
16	9.81 047	14	9.92 792	25	0.07 208	9.88 255̄	11	44
17	9.81 061	15	9.92 817	26	0.07 183	9.88 244	10	43
18	9.81 076	15	9.92 843	25	0.07 157	9.88 234	11	42
19	9.81 091	15	9.92 868	26	0.07 132	9.88 223	11	41
20	9.81 106	15	9.92 894	26	0.07 106	9.88 212	11	40
21	9.81 121	15	9.92 920	25	0.07 080	9.88 201	10	39
22	9.81 136	15	9.92 945	26	0.07 055̄	9.88 191	11	38
23	9.81 151	15	9.92 971	25	0.07 029	9.88 180	11	37
24	9.81 166	14	9.92 996	26	0.07 004	9.88 169	11	36
25	9.81 180	15	9.93 022	26	0.06 978	9.88 158	10	35
26	9.81 195	15	9.93 048	25	0.06 952	9.88 148	11	34
27	9.81 210	15	9.93 073	26	0.06 927	9.88 137	11	33
28	9.81 225̄	15	9.93 099	25	0.06 901	9.88 126	11	32
29	9.81 240	14	9.93 124	26	0.06 876	9.88 115	10	31
30	9.81 254	15	9.93 150̄	25	0.06 850	9.88 105̄	11	30
31	9.81 269	15	9.93 175	26	0.06 825̄	9.88 094	11	29
32	9.81 284	15	9.93 201	26	0.06 799	9.88 083	11	28
33	9.81 299	15	9.93 227	25	0.06 773	9.88 072	11	27
34	9.81 314	14	9.93 252	26	0.06 748	9.88 061	10	26
35	9.81 328	15	9.93 278	25	0.06 722	9.88 051	11	25
36	9.81 343	15	9.93 303	26	0.06 697	9.88 040	11	24
37	9.81 358	14	9.93 329	25	0.06 671	9.88 029	11	23
38	9.81 372	15	9.93 354	26	0.06 646	9.88 018	11	22
39	9.81 387	15	9.93 380	26	0.06 620	9.88 007	11	21
40	9.81 402	15	9.93 406	25	0.06 594	9.87 996	11	20
41	9.81 417	14	9.93 431	26	0.06 569	9.87 985	10	19
42	9.81 431	15	9.93 457	25	0.06 543	9.87 975̄	11	18
43	9.81 446	15	9.93 482	26	0.06 518	9.87 964	11	17
44	9.81 461	14	9.93 508	25	0.06 492	9.87 953	11	16
45	9.81 475	15	9.93 533	26	0.06 467	9.87 942	11	15
46	9.81 490	15	9.93 559	25	0.06 441	9.87 931	11	14
47	9.81 505̄	14	9.93 584	26	0.06 416	9.87 920	11	13
48	9.81 519	15	9.93 610	26	0.06 390	9.87 909	11	12
49	9.81 534	15	9.93 636	25	0.06 364	9.87 898	11	11
50	9.81 549	14	9.93 661	26	0.06 339	9.87 887	10	10
51	9.81 563	15	9.93 687	25	0.06 313	9.87 877	11	9
52	9.81 578	14	9.93 712	26	0.06 288	9.87 866	11	8
53	9.81 592	15	9.93 738	25	0.06 262	9.87 855̄	11	7
54	9.81 607	15	9.93 763	26	0.06 237	9.87 844	11	6
55	9.81 622	14	9.93 789	25	0.06 211	9.87 833	11	5
56	9.81 636	15	9.93 814	26	0.06 186	9.87 822	11	4
57	9.81 651	14	9.93 840	25	0.06 160	9.87 811	11	3
58	9.81 665	15	9.93 865	26	0.06 135̄	9.87 800	11	2
59	9.81 680	14	9.93 891	25	0.06 109	9.87 789	11	1
60	9.81 694		9.93 916		0.06 084	9.87 778		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	26	25
1	0.4	0.4
2	0.9	0.8
3	1.3	1.2
4	1.7	1.7
5	2.2	2.1
6	2.6	2.5
7	3.0	2.9
8	3.5̄	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

	15	14
1	0.2	0.2
2	0.5	0.5
3	0.8	0.7
4	1.0	0.9
5	1.2	1.2
6	1.5	1.4
7	1.8	1.6
8	2.0	1.9
9	2.2	2.1
10	2.5	2.3
20	5.0	4.7
30	7.5	7.0
40	10.0	9.3
50	12.5	11.7

	11	10
1	0.2	0.2
2	0.4	0.3
3	0.6	0.5
4	0.7	0.7
5	0.9	0.8
6	1.1	1.0
7	1.3	1.2
8	1.5̄	1.3
9	1.6	1.5
10	1.8	1.7
20	3.7	3.3
30	5.5	5.0
40	7.3	6.7
50	9.2	8.3

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	
0	9.81 694	15	9.93 916	26	0.06 084	9.87 778	11	60
1	9.81 709	14	9.93 942	25	0.06 058	9.87 767	11	59
2	9.81 723	15	9.93 967	26	0.06 033	9.87 756	11	58
3	9.81 738	14	9.93 993	25	0.06 007	9.87 745	11	57
4	9.81 752	15	9.94 018	26	0.05 982	9.87 734	11	56
5	9.81 767	14	9.94 044	25	0.05 956	9.87 723	11	55
6	9.81 781	15	9.94 069	26	0.05 931	9.87 712	11	54
7	9.81 796	14	9.94 095̄	25	0.05 905	9.87 701	11	53
8	9.81 810	15	9.94 120	26	0.05 880	9.87 690	11	52
9	9.81 825̄	14	9.94 146	25	0.05 854	9.87 679	11	51
10	9.81 839	15	9.94 171	26	0.05 829	9.87 668	11	50
11	9.81 854	14	9.94 197	25	0.05 803	9.87 657	11	49
12	9.81 868	14	9.94 222	26	0.05 778	9.87 646	11	48
13	9.81 882	15	9.94 248	25	0.05 752	9.87 635̄	11	47
14	9.81 897	14	9.94 273	26	0.05 727	9.87 624	11	46
15	9.81 911	15	9.94 299	25	0.05 701	9.87 613	12	45
16	9.81 926	14	9.94 324	26	0.05 676	9.87 601	11	44
17	9.81 940	15	9.94 350̄	25	0.05 650	9.87 590	11	43
18	9.81 955̄	14	9.94 375	26	0.05 625̄	9.87 579	11	42
19	9.81 969	14	9.94 401	25	0.05 599	9.87 568	11	41
20	9.81 983	15	9.94 426	26	0.05 574	9.87 557	11	40
21	9.81 998	14	9.94 452	25	0.05 548	9.87 546	11	39
22	9.82 012	14	9.94 477	26	0.05 523	9.87 535̄	11	38
23	9.82 026	15	9.94 503	25	0.05 497	9.87 524	11	37
24	9.82 041	14	9.94 528	26	0.05 472	9.87 513	12	36
25	9.82 055̄	14	9.94 554	25	0.05 446	9.87 501	11	35
26	9.82 069	15	9.94 579	25	0.05 421	9.87 490	11	34
27	9.82 084	14	9.94 604	26	0.05 396	9.87 479	11	33
28	9.82 098	14	9.94 630	25	0.05 370	9.87 468	11	32
29	9.82 112	14	9.94 655	26	0.05 345̄	9.87 457	11	31
30	9.82 126	15	9.94 681	25	0.05 319	9.87 446	12	30
31	9.82 141	14	9.94 706	26	0.05 294	9.87 434	11	29
32	9.82 155̄	14	9.94 732	25	0.05 268	9.87 423	11	28
33	9.82 169	15	9.94 757	26	0.05 243	9.87 412	11	27
34	9.82 184	14	9.94 783	25	0.05 217	9.87 401	11	26
35	9.82 198	14	9.94 808	26	0.05 192	9.87 390	12	25
36	9.82 212	14	9.94 834	25	0.05 166	9.87 378	11	24
37	9.82 226	14	9.94 859	25	0.05 141	9.87 367	11	23
38	9.82 240	15	9.94 884	26	0.05 116	9.87 356	11	22
39	9.82 255̄	14	9.94 910	25	0.05 090	9.87 345̄	11	21
40	9.82 269	14	9.94 935	26	0.05 065̄	9.87 334	12	20
41	9.82 283	14	9.94 961	25	0.05 039	9.87 322	11	19
42	9.82 297	14	9.94 986	26	0.05 014	9.87 311	11	18
43	9.82 311	15	9.95 012	25	0.04 988	9.87 300	12	17
44	9.82 326	14	9.95 037	25	0.04 963	9.87 288	11	16
45	9.82 340	14	9.95 062	26	0.04 938	9.87 277	11	15
46	9.82 354	14	9.95 088	25	0.04 912	9.87 266	11	14
47	9.82 368	14	9.95 113	26	0.04 887	9.87 255̄	12	13
48	9.82 382	14	9.95 139	25	0.04 861	9.87 243	11	12
49	9.82 396	14	9.95 164	26	0.04 836	9.87 232	11	11
50	9.82 410	14	9.95 190	25	0.04 810	9.87 221	12	10
51	9.82 424	15	9.95 215	25	0.04 785̄	9.87 209	11	9
52	9.82 439	14	9.95 240	26	0.04 760	9.87 198	11	8
53	9.82 453	14	9.95 266	25	0.04 734	9.87 187	12	7
54	9.82 467	14	9.95 291	26	0.04 709	9.87 175	11	6
55	9.82 481	14	9.95 317	25	0.04 683	9.87 164	11	5
56	9.82 495̄	14	9.95 342	26	0.04 658	9.87 153	12	4
57	9.82 509	14	9.95 368	25	0.04 632	9.87 141	11	3
58	9.82 523	14	9.95 393	25	0.04 607	9.87 130	11	2
59	9.82 537	14	9.95 418	26	0.04 582	9.87 119	12	1
60	9.82 551		9.95 444		0.04 556	9.87 107		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	26	25
1	0.4	0.4
2	0.9	0.8
3	1.3	1.2
4	1.7	1.7
5	2.2	2.1
6	2.6	2.5
7	3.0	2.9
8	3.5̄	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

	15	14
1	0.2	0.2
2	0.5	0.5
3	0.8	0.7
4	1.0	0.9
5	1.2	1.2
6	1.5	1.4
7	1.8	1.6
8	2.0	1.9
9	2.2	2.1
10	2.5	2.3
20	5.0	4.7
30	7.5	7.0
40	10.0	9.3
50	12.5	11.7

	12	11
1	0.2	0.2
2	0.4	0.4
3	0.6	0.6
4	0.8	0.7
5	1.0	0.9
6	1.2	1.1
7	1.4	1.3
8	1.6	1.5̄
9	1.8	1.6
10	2.0	1.8
20	4.0	3.7
30	6.0	5.5
40	8.0	7.3
50	10.0	9.2

Prop. Parts

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.82 551	14	9.95 444	25	0.04 556	9.87 107	11	60
1	9.82 565	14	9.95 469	26	0.04 531	9.87 096	11	59
2	9.82 579	14	9.95 495	25	0.04 505	9.87 085	12	58
3	9.82 593	14	9.95 520	25	0.04 480	9.87 073	11	57
4	9.82 607	14	9.95 545	26	0.04 455	9.87 062	12	56
5	9.82 621	14	9.95 571	25	0.04 429	9.87 050	11	55
6	9.82 635	14	9.95 596	26	0.04 404	9.87 039	11	54
7	9.82 649	14	9.95 622	25	0.04 378	9.87 028	12	53
8	9.82 663	14	9.95 647	25	0.04 353	9.87 016	11	52
9	9.82 677	14	9.95 672	26	0.04 328	9.87 005	12	51
10	9.82 691	14	9.95 698	25	0.04 302	9.86 993	11	50
11	9.82 705	14	9.95 723	25	0.04 277	9.86 982	12	49
12	9.82 719	14	9.95 748	26	0.04 252	9.86 970	11	48
13	9.82 733	14	9.95 774	25	0.04 226	9.86 959	12	47
14	9.82 747	14	9.95 799	26	0.04 201	9.86 947	11	46
15	9.82 761	14	9.95 825	25	0.04 175	9.86 936	12	45
16	9.82 775	13	9.95 850	25	0.04 150	9.86 924	11	44
17	9.82 788	14	9.95 875	26	0.04 125	9.86 913	11	43
18	9.82 802	14	9.95 901	25	0.04 099	9.86 902	12	42
19	9.82 816	14	9.95 926	26	0.04 074	9.86 890	11	41
20	9.82 830	14	9.95 952	25	0.04 048	9.86 879	12	40
21	9.82 844	14	9.95 977	25	0.04 023	9.86 867	12	39
22	9.82 858	14	9.96 002	26	0.03 998	9.86 855	11	38
23	9.82 872	13	9.96 028	25	0.03 972	9.86 844	12	37
24	9.82 885	14	9.96 053	25	0.03 947	9.86 832	11	36
25	9.82 899	14	9.96 078	26	0.03 922	9.86 821	12	35
26	9.82 913	14	9.96 104	25	0.03 896	9.86 809	11	34
27	9.82 927	14	9.96 129	26	0.03 871	9.86 798	12	33
28	9.82 941	14	9.96 155	25	0.03 845	9.86 786	11	32
29	9.82 955	13	9.96 180	25	0.03 820	9.86 775	12	31
30	9.82 968	14	9.96 205	26	0.03 795	9.86 763	11	30
31	9.82 982	14	9.96 231	25	0.03 769	9.86 752	12	29
32	9.82 996	14	9.96 256	25	0.03 744	9.86 740	12	28
33	9.83 010	13	9.96 281	26	0.03 719	9.86 728	11	27
34	9.83 023	14	9.96 307	25	0.03 693	9.86 717	12	26
35	9.83 037	14	9.96 332	25	0.03 668	9.86 705	11	25
36	9.83 051	14	9.96 357	26	0.03 643	9.86 694	12	24
37	9.83 065	13	9.96 383	25	0.03 617	9.86 682	12	23
38	9.83 078	14	9.96 408	25	0.03 592	9.86 670	11	22
39	9.83 092	14	9.96 433	26	0.03 567	9.86 659	12	21
40	9.83 106	14	9.96 459	25	0.03 541	9.86 647	12	20
41	9.83 120	13	9.96 484	26	0.03 516	9.86 635	11	19
42	9.83 133	14	9.96 510	25	0.03 490	9.86 624	12	18
43	9.83 147	14	9.96 535	25	0.03 465	9.86 612	12	17
44	9.83 161	13	9.96 560	26	0.03 440	9.86 600	11	16
45	9.83 174	14	9.96 586	25	0.03 414	9.86 589	12	15
46	9.83 188	14	9.96 611	25	0.03 389	9.86 577	12	14
47	9.83 202	13	9.96 636	26	0.03 364	9.86 565	11	13
48	9.83 215	14	9.96 662	25	0.03 338	9.86 554	12	12
49	9.83 229	13	9.96 687	25	0.03 313	9.86 542	12	11
50	9.83 242	14	9.96 712	26	0.03 288	9.86 530	12	10
51	9.83 256	14	9.96 738	25	0.03 262	9.86 518	11	9
52	9.83 270	13	9.96 763	25	0.03 237	9.86 507	12	8
53	9.83 283	14	9.96 788	26	0.03 212	9.86 495	12	7
54	9.83 297	13	9.96 814	25	0.03 186	9.86 483	11	6
55	9.83 310	14	9.96 839	25	0.03 161	9.86 472	12	5
56	9.83 324	14	9.96 864	26	0.03 136	9.86 460	12	4
57	9.83 338	13	9.96 890	25	0.03 110	9.86 448	12	3
58	9.83 351	14	9.96 915	25	0.03 085	9.86 436	11	2
59	9.83 365	13	9.96 940	26	0.03 060	9.86 425	12	1
60	9.83 378		9.96 966		0.03 034	9.86 413		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	26	25
1	0.4	0.4
2	0.9	0.8
3	1.3	1.2
4	1.7	1.7
5	2.2	2.1
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

	14	13
1	0.2	0.2
2	0.5	0.4
3	0.7	0.6
4	0.9	0.9
5	1.2	1.1
6	1.4	1.3
7	1.6	1.5
8	1.9	1.7
9	2.1	2.0
10	2.3	2.2
20	4.7	4.3
30	7.0	6.5
40	9.3	8.7
50	11.7	10.8

	12	11
1	0.2	0.2
2	0.4	0.4
3	0.6	0.6
4	0.8	0.7
5	1.0	0.9
6	1.2	1.1
7	1.4	1.3
8	1.6	1.5
9	1.8	1.6
10	2.0	1.8
20	4.0	3.7
30	6.0	5.5
40	8.0	7.3
50	10.0	9.2

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.83 378	14	9.96 966	25	0.03 034	9.86 413	12	60
1	9.83 392	13	9.96 991	25	0.03 009	9.86 401	12	59
2	9.83 405	14	9.97 016	26	0.02 984	9.86 389	12	58
3	9.83 419	13	9.97 042	25	0.02 958	9.86 377	11	57
4	9.83 432	14	9.97 067	25	0.02 933	9.86 366	12	56
5	9.83 446	13	9.97 092	26	0.02 908	9.86 354	12	55
6	9.83 459	14	9.97 118	25	0.02 882	9.86 342	12	54
7	9.83 473	13	9.97 143	25	0.02 857	9.86 330	12	53
8	9.83 486	14	9.97 168	25	0.02 832	9.86 318	12	52
9	9.83 500	13	9.97 193	26	0.02 807	9.86 306	11	51
10	9.83 513	14	9.97 219	25	0.02 781	9.86 295	12	50
11	9.83 527	13	9.97 244	25	0.02 756	9.86 283	12	49
12	9.83 540	14	9.97 269	26	0.02 731	9.86 271	12	48
13	9.83 554	13	9.97 295	25	0.02 705	9.86 259	12	47
14	9.83 567	14	9.97 320	25	0.02 680	9.86 247	12	46
15	9.83 581	13	9.97 345	26	0.02 655	9.86 235	12	45
16	9.83 594	14	9.97 371	25	0.02 629	9.86 223	12	44
17	9.83 608	13	9.97 396	25	0.02 604	9.86 211	11	43
18	9.83 621	13	9.97 421	26	0.02 579	9.86 200	12	42
19	9.83 634	14	9.97 447	25	0.02 553	9.86 188	12	41
20	9.83 648	13	9.97 472	25	0.02 528	9.86 176	12	40
21	9.83 661	13	9.97 497	26	0.02 503	9.86 164	12	39
22	9.83 674	14	9.97 523	25	0.02 477	9.86 152	12	38
23	9.83 688	13	9.97 548	25	0.02 452	9.86 140	12	37
24	9.83 701	14	9.97 573	25	0.02 427	9.86 128	12	36
25	9.83 715	13	9.97 598	26	0.02 402	9.86 116	12	35
26	9.83 728	13	9.97 624	25	0.02 376	9.86 104	12	34
27	9.83 741	14	9.97 649	25	0.02 351	9.86 092	12	33
28	9.83 755	13	9.97 674	26	0.02 326	9.86 080	12	32
29	9.83 768	13	9.97 700	25	0.02 300	9.86 068	12	31
30	9.83 781	14	9.97 725	25	0.02 275	9.86 056	12	30
31	9.83 795	13	9.97 750	26	0.02 250	9.86 044	12	29
32	9.83 808	13	9.97 776	25	0.02 224	9.86 032	12	28
33	9.83 821	13	9.97 801	25	0.02 199	9.86 020	12	27
34	9.83 834	14	9.97 826	25	0.02 174	9.86 008	12	26
35	9.83 848	13	9.97 851	26	0.02 149	9.85 996	12	25
36	9.83 861	13	9.97 877	25	0.02 123	9.85 984	12	24
37	9.83 874	13	9.97 902	25	0.02 098	9.85 972	12	23
38	9.83 887	14	9.97 927	26	0.02 073	9.85 960	12	22
39	9.83 901	13	9.97 953	25	0.02 047	9.85 948	12	21
40	9.83 914	13	9.97 978	25	0.02 022	9.85 936	12	20
41	9.83 927	13	9.98 003	26	0.01 997	9.85 924	12	19
42	9.83 940	14	9.98 029	25	0.01 971	9.85 912	12	18
43	9.83 954	13	9.98 054	25	0.01 946	9.85 900	12	17
44	9.83 967	13	9.98 079	25	0.01 921	9.85 888	12	16
45	9.83 980	13	9.98 104	26	0.01 896	9.85 876	12	15
46	9.83 993	13	9.98 130	25	0.01 870	9.85 864	13	14
47	9.84 006	14	9.98 155	25	0.01 845	9.85 851	12	13
48	9.84 020	13	9.98 180	26	0.01 820	9.85 839	12	12
49	9.84 033	13	9.98 206	25	0.01 794	9.85 827	12	11
50	9.84 046	13	9.98 231	25	0.01 769	9.85 815	12	10
51	9.84 059	13	9.98 256	25	0.01 744	9.85 803	12	9
52	9.84 072	13	9.98 281	26	0.01 719	9.85 791	12	8
53	9.84 085	13	9.98 307	25	0.01 693	9.85 779	13	7
54	9.84 098	14	9.98 332	25	0.01 668	9.85 766	12	6
55	9.84 112	13	9.98 357	26	0.01 643	9.85 754	12	5
56	9.84 125	13	9.98 383	25	0.01 617	9.85 742	12	4
57	9.84 138	13	9.98 408	25	0.01 592	9.85 730	12	3
58	9.84 151	13	9.98 433	25	0.01 567	9.85 718	12	2
59	9.84 164	13	9.98 458	26	0.01 542	9.85 706	13	1
60	9.84 177		9.98 484		0.01 516	9.85 693		0
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	26	25
1	0.4	0.4
2	0.9	0.8
3	1.3	1.2
4	1.7	1.7
5	2.2	2.1
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

	14	13
1	0.2	0.2
2	0.5	0.4
3	0.7	0.6
4	0.9	0.9
5	1.2	1.1
6	1.4	1.3
7	1.6	1.5
8	1.9	1.7
9	2.1	2.0
10	2.3	2.2
20	4.7	4.3
30	7.0	6.5
40	9.3	8.7
50	11.7	10.8

	12	11
1	0.2	0.2
2	0.4	0.4
3	0.6	0.6
4	0.8	0.7
5	1.0	0.9
6	1.2	1.1
7	1.4	1.3
8	1.6	1.5
9	1.8	1.6
10	2.0	1.8
20	4.0	3.7
30	6.0	5.5
40	8.0	7.3
50	10.0	9.2

′	log sin	d	log tan	c d	log cot	log cos	d	′
0	9.84 177	13	9.98 484	25	0.01 516	9.85 693	12	**60**
1	9.84 190	13	9.98 509	25	0.01 491	9.85 681	12	59
2	9.84 203	13	9.98 534	26	0.01 466	9.85 669	12	58
3	9.84 216	13	9.98 560	25	0.01 440	9.85 657	12	57
4	9.84 229	13	9.98 58$\bar{5}$	25	0.01 415	9.85 64$\bar{5}$	13	56
5	9.84 242	13	9.98 610	25	0.01 390	9.85 632	12	**55**
6	9.84 255	14	9.98 635	26	0.01 36$\bar{5}$	9.85 620	12	54
7	9.84 269	13	9.98 661	25	0.01 339	9.85 608	12	53
8	9.84 282	13	9.98 686	25	0.01 314	9.85 596	13	52
9	9.84 29$\bar{5}$	13	9.98 711	26	0.01 289	9.85 583	12	51
10	9.84 308	13	9.98 737	25	0.01 263	9.85 571	12	**50**
11	9.84 321	13	9.98 762	25	0.01 238	9.85 559	12	49
12	9.84 334	13	9.98 787	25	0.01 213	9.85 547	13	48
13	9.84 347	13	9.98 812	26	0.01 188	9.85 534	12	47
14	9.84 360	13	9.98 838	25	0.01 162	9.85 522	12	46
15	9.84 373	12	9.98 863	25	0.01 137	9.85 510	13	**45**
16	9.84 385	13	9.98 888	25	0.01 112	9.85 497	12	44
17	9.84 398	13	9.98 913	26	0.01 087	9.85 48$\bar{5}$	12	43
18	9.84 411	13	9.98 939	25	0.01 061	9.85 473	13	42
19	9.84 424	13	9.98 964	25	0.01 036	9.85 460	12	41
20	9.84 437	13	9.98 989	26	0.01 011	9.85 448	12	**40**
21	9.84 450	13	9.99 01$\bar{5}$	25	0.00 985	9.85 436	13	39
22	9.84 463	13	9.99 040	25	0.00 960	9.85 423	12	38
23	9.84 476	13	9.99 065	25	0.00 93$\bar{5}$	9.85 411	12	37
24	9.84 489	13	9.99 090	26	0.00 910	9.85 399	13	36
25	9.84 502	13	9.99 116	25	0.00 884	9.85 386	12	**35**
26	9.84 51$\bar{5}$	13	9.99 141	25	0.00 859	9.85 374	13	34
27	9.84 528	12	9.99 166	25	0.00 834	9.85 361	12	33
28	9.84 540	13	9.99 191	26	0.00 809	9.85 349	12	32
29	9.84 553	13	9.99 217	25	0.00 783	9.85 337	13	31
30	9.84 566	13	9.99 242	25	0.00 758	9.85 324	12	**30**
31	9.84 579	13	9.99 267	26	0.00 733	9.85 312	13	29
32	9.84 592	13	9.99 293	25	0.00 707	9.85 299	12	28
33	9.84 60$\bar{5}$	13	9.99 318	25	0.00 682	9.85 287	13	27
34	9.84 618	12	9.99 343	25	0.00 657	9.85 274	12	26
35	9.84 630	13	9.99 368	26	0.00 632	9.85 262	12	**25**
36	9.84 643	13	9.99 394	25	0.00 606	9.85 25$\bar{0}$	13	24
37	9.84 656	13	9.99 419	25	0.00 581	9.85 237	12	23
38	9.84 669	13	9.99 444	25	0.00 556	9.85 22$\bar{5}$	13	22
39	9.84 682	12	9.99 469	26	0.00 531	9.85 212	12	21
40	9.84 694	13	9.99 49$\bar{5}$	25	0.00 505	9.85 200	13	**20**
41	9.84 707	13	9.99 520	25	0.00 480	9.85 187	12	19
42	9.84 720	13	9.99 545	25	0.00 45$\bar{5}$	9.85 17$\bar{5}$	13	18
43	9.84 733	12	9.99 570	26	0.00 430	9.85 162	12	17
44	9.84 745	13	9.99 596	25	0.00 404	9.85 15$\bar{0}$	13	16
45	9.84 758	13	9.99 621	25	0.00 379	9.85 137	12	**15**
46	9.84 771	13	9.99 646	26	0.00 354	9.85 12$\bar{5}$	13	14
47	9.84 784	12	9.99 672	25	0.00 328	9.85 112	12	13
48	9.84 796	13	9.99 697	25	0.00 303	9.85 100	13	12
49	9.84 809	13	9.99 722	25	0.00 278	9.85 087	13	11
50	9.84 822	13	9.99 747	26	0.00 253	9.85 074	12	**10**
51	9.84 83$\bar{5}$	12	9.99 773	25	0.00 227	9.85 062	13	9
52	9.84 847	13	9.99 798	25	0.00 202	9.85 049	12	8
53	9.84 860	13	9.99 823	25	0.00 177	9.85 037	13	7
54	9.84 873	12	9.99 848	26	0.00 152	9.85 024	12	6
55	9.84 885	13	9.99 874	25	0.00 126	9.85 012	13	**5**
56	9.84 898	13	9.99 899	25	0.00 101	9.84 999	13	4
57	9.84 911	12	9.99 924	25	0.00 076	9.84 986	12	3
58	9.84 923	13	9.99 949	26	0.00 051	9.84 974	13	2
59	9.84 936	13	9.99 97$\bar{5}$	25	0.00 025	9.84 961	12	1
60	9.84 949		0.00 000		0.00 000	9.84 949		**0**
	log cos	d	log cot	c d	log tan	log sin	d	′

Prop. Parts

	26	**25**
1	0.4	0.4
2	0.9	0.8
3	1.3	1.2
4	1.7	1.7
5	2.2	2.1
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

	14	**13**	**12**
1	0.2	0.2	0.2
2	0.5	0.4	0.4
3	0.7	0.6	0.6
4	0.9	0.9	0.8
5	1.2	1.1	1.0
6	1.4	1.3	1.2
7	1.6	1.5	1.4
8	1.9	1.7	1.6
9	2.1	2.0	1.8
10	2.3	2.2	2.0
20	4.7	4.3	4.0
30	7.0	6.5	6.0
40	9.3	8.7	8.0
50	11.7	10.8	10.0

Prop. Parts

TABLE IV

NATURAL TRIGONOMETRIC FUNCTIONS

Of angles for each minute from 0° to 90°, correct to five significant figures

(For explanation, see page 29.)

0°

′	sin	tan	cot	cos	
0	.00000	.00000	∞	1.0000	60
1	029	029	3437.7	000	59
2	058	058	1718.9	000	58
3	087	087	1145.9	000	57
4	116	116	859.44	000	56
5	.00145	.00145	687.55	1.0000	55
6	175	175	572.96	000	54
7	204	204	491.11	000	53
8	233	233	429.72	000	52
9	262	262	381.97	000	51
10	.00291	.00291	343.77	1.0000	50
11	320	320	312.52	.99999	49
12	349	349	286.48	999	48
13	378	378	264.44	999	47
14	407	407	245.55	999	46
15	.00436	.00436	229.18	.99999	45
16	465	465	214.86	999	44
17	495	495	202.22	999	43
18	524	524	190.98	999	42
19	553	553	180.93	998	41
20	.00582	.00582	171.89	.99998	40
21	611	611	163.70	998	39
22	640	640	156.26	998	38
23	669	669	149.47	998	37
24	698	698	143.24	998	36
25	.00727	.00727	137.51	.99997	35
26	756	756	132.22	997	34
27	785	785	127.32	997	33
28	814	815	122.77	997	32
29	844	844	118.54	996	31
30	.00873	.00873	114.59	.99996	30
31	902	902	110.89	996	29
32	931	931	107.43	996	28
33	960	960	104.17	995	27
34	.00989	.00989	101.11	995	26
35	.01018	.01018	98.218	.99995	25
36	047	047	95.489	995	24
37	076	076	92.908	994	23
38	105	105	90.463	994	22
39	134	135	88.144	994	21
40	.01164	.01164	85.940	.99993	20
41	193	193	83.844	993	19
42	222	222	81.847	993	18
43	251	251	79.943	992	17
44	280	280	78.126	992	16
45	.01309	.01309	76.390	.99991	15
46	338	338	74.729	991	14
47	367	367	73.139	991	13
48	396	396	71.615	990	12
49	425	425	70.153	990	11
50	.01454	.01455	68.750	.99989	10
51	483	484	67.402	989	9
52	513	513	66.105	989	8
53	542	542	64.858	988	7
54	571	571	63.657	988	6
55	.01600	.01600	62.499	.99987	5
56	629	629	61.383	987	4
57	658	658	60.306	986	3
58	687	687	59.266	986	2
59	716	716	58.261	985	1
60	.01745	.01746	57.290	.99985	0
	cos	cot	tan	sin	′

89°

1°

′	sin	tan	cot	cos	
0	.01745	.01746	57.290	.99985	60
1	774	775	56.351	984	59
2	803	804	55.442	984	58
3	832	833	54.561	983	57
4	862	862	53.709	983	56
5	.01891	.01891	52.882	.99982	55
6	920	920	52.081	982	54
7	949	949	51.303	981	53
8	.01978	.01978	50.549	980	52
9	.02007	.02007	49.816	980	51
10	.02036	.02036	49.104	.99979	50
11	065	066	48.412	979	49
12	094	095	47.740	978	48
13	123	124	47.085	977	47
14	152	153	46.449	977	46
15	.02181	.02182	45.829	.99976	45
16	211	211	45.226	976	44
17	240	240	44.639	975	43
18	269	269	44.066	974	42
19	298	298	43.508	974	41
20	.02327	.02328	42.964	.99973	40
21	356	357	42.433	972	39
22	385	386	41.916	972	38
23	414	415	41.411	971	37
24	443	444	40.917	970	36
25	.02472	.02473	40.436	.99969	35
26	501	502	39.965	969	34
27	530	531	39.506	968	33
28	560	560	39.057	967	32
29	589	589	38.618	966	31
30	.02618	.02619	38.188	.99966	30
31	647	648	37.769	965	29
32	676	677	37.358	964	28
33	705	706	36.956	963	27
34	734	735	36.563	963	26
35	.02763	.02764	36.178	.99962	25
36	792	793	35.801	961	24
37	821	822	35.431	960	23
38	850	851	35.070	959	22
39	879	881	34.715	959	21
40	.02908	.02910	34.368	.99958	20
41	938	939	34.027	957	19
42	967	968	33.694	956	18
43	.02996	.02997	33.366	955	17
44	.03025	.03026	33.045	954	16
45	.03054	.03055	32.730	.99953	15
46	083	084	32.421	952	14
47	112	114	32.118	952	13
48	141	143	31.821	951	12
49	170	172	31.528	950	11
50	.03199	.03201	31.242	.99949	10
51	228	230	30.960	948	9
52	257	259	30.683	947	8
53	286	288	30.412	946	7
54	316	317	30.145	945	6
55	.03345	.03346	29.882	.99944	5
56	374	376	29.624	943	4
57	403	405	29.371	942	3
58	432	434	29.122	941	2
59	461	463	28.877	940	1
60	.03490	.03492	28.636	.99939	0
	cos	cot	tan	sin	′

88°

2°

′	sin	tan	cot	cos	
0	.03490	.03492	28.636	.99939	60
1	519	521	.399	938	59
2	548	550	28.166	937	58
3	577	579	27.937	936	57
4	606	609	.712	935	56
5	.03635	.03638	27.490	.99934	55
6	664	667	.271	933	54
7	693	696	27.057	932	53
8	723	725	26.845	931	52
9	752	754	.637	930	51
10	.03781	.03783	26.432	.99929	50
11	810	812	.230	927	49
12	839	842	26.031	926	48
13	868	871	25.835	925	47
14	897	900	.642	924	46
15	.03926	.03929	25.452	.99923	45
16	955	958	.264	922	44
17	.03984	.03987	25.080	921	43
18	.04013	.04016	24.898	919	42
19	042	046	.719	918	41
20	.04071	.04075	24.542	.99917	40
21	100	104	.368	916	39
22	129	133	.196	915	38
23	159	162	24.026	913	37
24	188	191	23.859	912	36
25	.04217	04220	23.695	.99911	35
26	246	250	.532	910	34
27	275	279	.372	909	33
28	304	308	.214	907	32
29	333	337	23.058	906	31
30	.04362	.04366	22.904	.99905	30
31	391	395	.752	904	29
32	420	424	.602	902	28
33	449	454	.454	901	27
34	478	483	.308	900	26
35	.04507	.04512	22.164	.99898	25
36	536	541	22.022	897	24
37	565	570	21.881	896	23
38	594	599	.743	894	22
39	623	628	.606	893	21
40	.04653	.04658	21.470	.99892	20
41	682	687	.337	890	19
42	711	716	.205	889	18
43	740	745	21.075	888	17
44	769	774	20.946	886	16
45	.04798	.04803	20.819	.99885	15
46	827	833	.693	883	14
47	856	862	.569	882	13
48	885	891	.446	881	12
49	914	920	.325	879	11
50	.04943	.04949	20.206	.99878	10
51	.04972	.04978	20.087	876	9
52	.05001	.05007	19.970	875	8
53	030	037	.855	873	7
54	059	066	.740	872	6
55	.05088	.05095	19.627	.99870	5
56	117	124	.516	869	4
57	146	153	.405	867	3
58	175	182	.296	866	2
59	205	212	.188	864	1
60	.05234	.05241	19.081	.99863	0
	cos	cot	tan	sin	′

87°

3°

′	sin	tan	cot	cos	
0	.05234	.05241	19.081	.99863	60
1	263	270	18.976	861	59
2	292	299	.871	860	58
3	321	328	.768	858	57
4	350	357	.666	857	56
5	.05379	.05387	18.564	.99855	55
6	408	416	.464	854	54
7	437	445	.366	852	53
8	466	474	.268	851	52
9	495	503	.171	849	51
10	.05524	.05533	18.075	.99847	50
11	553	562	17.980	846	49
12	582	591	.886	844	48
13	611	620	.793	842	47
14	640	649	.702	841	46
15	.05669	.05678	17.611	.99839	45
16	698	708	.521	838	44
17	727	737	.431	836	43
18	756	766	.343	834	42
19	785	795	.256	833	41
20	.05814	.05824	17.169	.99831	40
21	844	854	17.084	829	39
22	873	883	16.999	827	38
23	902	912	.915	826	37
24	931	941	.832	824	36
25	.05960	.05970	16.750	.99822	35
26	.05989	.05999	.668	821	34
27	.06018	.06029	.587	819	33
28	047	058	.507	817	32
29	076	087	.428	815	31
30	.06105	.06116	16.350	.99813	30
31	134	145	.272	812	29
32	163	175	.195	810	28
33	192	204	.119	808	27
34	221	233	16.043	806	26
35	.06250	.06262	15.969	.99804	25
36	279	291	.895	803	24
37	308	321	.821	801	23
38	337	350	.748	799	22
39	366	379	.676	797	21
40	.06395	.06408	15.605	.99795	20
41	424	438	.534	793	19
42	453	467	.464	792	18
43	482	496	.394	790	17
44	511	525	.325	788	16
45	.06540	.06554	15.257	.99786	15
46	569	584	.189	784	14
47	598	613	.122	782	13
48	627	642	15.056	780	12
49	656	671	14.990	778	11
50	.06685	.06700	14.924	.99776	10
51	714	730	.860	774	9
52	743	759	.795	772	8
53	773	788	.732	770	7
54	802	817	.669	768	6
55	.06831	.06847	14.606	.99766	5
56	860	876	.544	764	4
57	889	905	.482	762	3
58	918	934	.421	760	2
59	947	963	.361	758	1
60	.06976	.06993	14.301	.99756	0
	cos	cot	tan	sin	′

86°

4°

′	sin	tan	cot	cos	
0	.06976	.06993	14.301	.99756	60
1	.07005	.07022	.241	754	59
2	034	051	.182	752	58
3	063	080	.124	750	57
4	092	110	.065	748	56
5	.07121	.07139	14.008	.99746	55
6	150	168	13.951	744	54
7	179	197	.894	742	53
8	208	227	.838	740	52
9	237	256	.782	738	51
10	.07266	.07285	13.727	.99736	50
11	295	314	.672	734	49
12	324	344	.617	731	48
13	353	373	.563	729	47
14	382	402	.510	727	46
15	.07411	.07431	13.457	.99725	45
16	440	461	.404	723	44
17	469	490	.352	721	43
18	498	519	.300	719	42
19	527	548	.248	716	41
20	.07556	.07578	13.197	.99714	40
21	585	607	.146	712	39
22	614	636	.096	710	38
23	643	665	13.046	708	37
24	672	695	12.996	705	36
25	.07701	.07724	12.947	.99703	35
26	730	753	.898	701	34
27	759	782	.850	699	33
28	788	812	.801	696	32
29	817	841	.754	694	31
30	.07846	.07870	12.706	.99692	30
31	875	899	.659	689	29
32	904	929	.612	687	28
33	933	958	.566	685	27
34	962	.07987	.520	683	26
35	.07991	.08017	12.474	.99680	25
36	.08020	046	.429	678	24
37	049	075	.384	676	23
38	078	104	.339	673	22
39	107	134	.295	671	21
40	.08136	.08163	12.251	.99668	20
41	165	192	.207	666	19
42	194	221	.163	664	18
43	223	251	.120	661	17
44	252	280	.077	659	16
45	.08281	.08309	12.035	.99657	15
46	310	339	11.992	654	14
47	339	368	.950	652	13
48	368	397	.909	649	12
49	397	427	.867	647	11
50	.08426	.08456	11.826	.99644	10
51	455	485	.785	642	9
52	484	514	.745	639	8
53	513	544	.705	637	7
54	542	573	.664	635	6
55	.08571	.08602	11.625	.99632	5
56	600	632	.585	630	4
57	629	661	.546	627	3
58	658	690	.507	625	2
59	687	720	.468	622	1
60	.08716	.08749	11.430	.99619	0
	cos	cot	tan	sin	′

85°

5°

′	sin	tan	cot	cos	
0	.08716	.08749	11.430	.99619	60
1	745	778	.392	617	59
2	774	807	.354	614	58
3	803	837	.316	612	57
4	831	866	.279	609	56
5	.08860	.08895	11.242	.99607	55
6	889	925	.205	604	54
7	918	954	.168	602	53
8	947	.08983	.132	599	52
9	.08976	.09013	.095	596	51
10	.09005	.09042	11.059	.99594	50
11	034	071	11.024	591	49
12	063	101	10.988	588	48
13	092	130	.953	586	47
14	121	159	.918	583	46
15	.09150	.09189	10.883	.99580	45
16	179	218	.848	578	44
17	208	247	.814	575	43
18	237	277	.780	572	42
19	266	306	.746	570	41
20	.09295	.09335	10.712	.99567	40
21	324	365	.678	564	39
22	353	394	.645	562	38
23	382	423	.612	559	37
24	411	453	.579	556	36
25	.09440	.09482	10.546	.99553	35
26	469	511	.514	551	34
27	498	541	.481	548	33
28	527	570	.449	545	32
29	556	600	.417	542	31
30	.09585	.09629	10.385	.99540	30
31	614	658	.354	537	29
32	642	688	.322	534	28
33	671	717	.291	531	27
34	700	746	.260	528	26
35	.09729	.09776	10.229	.99526	25
36	758	805	.199	523	24
37	787	834	.168	520	23
38	816	864	.138	517	22
39	845	893	.108	514	21
40	.09874	.09923	10.078	.99511	20
41	903	952	.048	508	19
42	932	.09981	10.019	506	18
43	961	.10011	9.9893	503	17
44	.09990	040	.9601	500	16
45	.10019	.10069	9.9310	.99497	15
46	048	099	.9021	494	14
47	077	128	.8734	491	13
48	106	158	.8448	488	12
49	135	187	.8164	485	11
50	.10164	.10216	9.7882	.99482	10
51	192	246	.7601	479	9
52	221	275	.7322	476	8
53	250	305	.7044	473	7
54	279	334	.6768	470	6
55	.10308	.10363	9.6493	.99467	5
56	337	393	.6220	464	4
57	366	422	.5949	461	3
58	395	452	.5679	458	2
59	424	481	.5411	455	1
60	.10453	.10510	9.5144	.99452	0
	cos	cot	tan	sin	′

84°

6°

′	sin	tan	cot	cos	
0	.10453	.10510	9.5144	.99452	60
1	482	540	.4878	449	59
2	511	569	.4614	446	58
3	540	599	.4352	443	57
4	569	628	.4090	440	56
5	.10597	.10657	9.3831	.99437	55
6	626	687	.3572	434	54
7	655	716	.3315	431	53
8	684	746	.3060	428	52
9	713	775	.2806	424	51
10	.10742	.10805	9.2553	.99421	50
11	771	834	.2302	418	49
12	800	863	.2052	415	48
13	829	893	.1803	412	47
14	858	922	.1555	409	46
15	.10887	.10952	9.1309	.99406	45
16	916	.10981	.1065	402	44
17	945	.11011	.0821	399	43
18	.10973	040	.0579	396	42
19	.11002	070	.0338	393	41
20	.11031	.11099	9.0098	.99390	40
21	060	128	8.9860	386	39
22	089	158	.9623	383	38
23	118	187	.9387	380	37
24	147	217	.9152	377	36
25	.11176	.11246	8.8919	.99374	35
26	205	276	.8686	370	34
27	234	305	.8455	367	33
28	263	335	.8225	364	32
29	291	364	.7996	360	31
30	.11320	.11394	8.7769	.99357	30
31	349	423	.7542	354	29
32	378	452	.7317	351	28
33	407	482	.7093	347	27
34	436	511	.6870	344	26
35	.11465	.11541	8.6648	.99341	25
36	494	570	.6427	337	24
37	523	600	.6208	334	23
38	552	629	.5989	331	22
39	580	659	.5772	327	21
40	.11609	.11688	8.5555	.99324	20
41	638	718	.5340	320	19
42	667	747	.5126	317	18
43	696	777	.4913	314	17
44	725	806	.4701	310	16
45	.11754	.11836	8.4490	.99307	15
46	783	865	.4280	303	14
47	812	895	.4071	300	13
48	840	924	.3863	297	12
49	869	954	.3656	293	11
50	.11898	.11983	8.3450	.99290	10
51	927	.12013	.3245	286	9
52	956	042	.3041	283	8
53	.11985	072	.2838	279	7
54	.12014	101	.2636	276	6
55	.12043	.12131	8.2434	.99272	5
56	071	160	.2234	269	4
57	100	190	.2035	265	3
58	129	219	.1837	262	2
59	158	249	.1640	258	1
60	.12187	.12278	8.1443	.99255	0
	cos	cot	tan	sin	′

83°

7°

′	sin	tan	cot	cos	
0	.12187	.12278	8.1443	.99255	60
1	216	308	.1248	251	59
2	245	338	.1054	248	58
3	274	367	.0860	244	57
4	302	397	.0667	240	56
5	.12331	.12426	8.0476	.99237	55
6	360	456	.0285	233	54
7	389	485	8.0095	230	53
8	418	515	7.9906	226	52
9	447	544	.9718	222	51
10	.12476	.12574	7.9530	.99219	50
11	504	603	.9344	215	49
12	533	633	.9158	211	48
13	562	662	.8973	208	47
14	591	692	.8789	204	46
15	.12620	.12722	7.8606	.99200	45
16	649	751	.8424	197	44
17	678	781	.8243	193	43
18	706	810	.8062	189	42
19	735	840	.7882	186	41
20	.12764	.12869	7.7704	.99182	40
21	793	899	.7525	178	39
22	822	929	.7348	175	38
23	851	958	.7171	171	37
24	880	.12988	.6996	167	36
25	.12908	.13017	7.6821	.99163	35
26	937	047	.6647	160	34
27	966	076	.6473	156	33
28	.12995	106	.6301	152	32
29	.13024	136	.6129	148	31
30	.13053	.13165	7.5958	.99144	30
31	081	195	.5787	141	29
32	110	224	.5618	137	28
33	139	254	.5449	133	27
34	168	284	.5281	129	26
35	.13197	.13313	7.5113	.99125	25
36	226	343	.4947	122	24
37	254	372	.4781	118	23
38	283	402	.4615	114	22
39	312	432	.4451	110	21
40	.13341	.13461	7.4287	.99106	20
41	370	491	.4124	102	19
42	399	521	.3962	098	18
43	427	550	.3800	094	17
44	456	580	.3639	091	16
45	.13485	.13609	7.3479	.99087	15
46	514	639	.3319	083	14
47	543	669	.3160	079	13
48	572	698	.3002	075	12
49	600	728	.2844	071	11
50	.13629	.13758	7.2687	.99067	10
51	658	787	.2531	063	9
52	687	817	.2375	059	8
53	716	846	.2220	055	7
54	744	876	.2066	051	6
55	.13773	.13906	7.1912	.99047	5
56	802	935	.1759	043	4
57	831	965	.1607	039	3
58	860	.13995	.1455	035	2
59	889	.14024	.1304	031	1
60	.13917	.14054	7.1154	.99027	0
	cos	cot	tan	sin	′

82°

′	sin	tan	cot	cos	
0	.13917	.14054	7.1154	.99027	60
1	946	084	.1004	023	59
2	.13975	113	.0855	019	58
3	.14004	143	.0706	015	57
4	033	173	.0558	011	56
5	.14061	.14202	7.0410	.99006	55
6	090	232	.0264	.99002	54
7	119	262	7.0117	.98998	53
8	148	291	6.9972	994	52
9	177	321	.9827	990	51
10	.14205	.14351	6.9682	.98986	50
11	234	381	.9538	982	49
12	263	410	.9395	978	48
13	292	440	.9252	973	47
14	320	470	.9110	969	46
15	.14349	.14499	6.8969	.98965	45
16	378	529	.8828	961	44
17	407	559	.8687	957	43
18	436	588	.8548	953	42
19	464	618	.8408	948	41
20	.14493	.14648	6.8269	.98944	40
21	522	678	.8131	940	39
22	551	707	.7994	936	38
23	580	737	.7856	931	37
24	608	767	.7720	927	36
25	.14637	.14796	6.7584	.98923	35
26	666	826	.7448	919	34
27	695	856	.7313	914	33
28	723	886	.7179	910	32
29	752	915	.7045	906	31
30	.14781	.14945	6.6912	.98902	30
31	810	.14975	.6779	897	29
32	838	.15005	.6646	893	28
33	867	034	.6514	889	27
34	895	064	.6383	884	26
35	.14925	.15094	6.6252	.98880	25
36	954	124	.6122	876	24
37	.14982	153	.5992	871	23
38	.15011	183	.5863	867	22
39	040	213	.5734	863	21
40	.15069	.15243	6.5606	.98858	20
41	097	272	.5478	854	19
42	126	302	.5350	849	18
43	155	332	.5223	845	17
44	184	362	.5097	841	16
45	.15212	.15391	6.4971	.98836	15
46	241	421	.4846	832	14
47	270	451	.4721	827	13
48	299	481	.4596	823	12
49	327	511	.4472	818	11
50	.15356	.15540	6.4348	.98814	10
51	385	570	.4225	809	9
52	414	600	.4103	805	8
53	442	630	.3980	800	7
54	471	660	.3859	796	6
55	.15500	.15689	6.3737	.98791	5
56	529	719	.3617	787	4
57	557	749	.3496	782	3
58	586	779	.3376	778	2
59	615	809	.3257	773	1
60	.15643	.15838	6.3138	.98769	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.15643	.15838	6.3138	.98769	60
1	672	868	.3019	764	59
2	701	898	.2901	760	58
3	730	928	.2783	755	57
4	758	958	.2666	751	56
5	.15787	.15988	6.2549	.98746	55
6	816	.16017	.2432	741	54
7	845	047	.2316	737	53
8	873	077	.2200	732	52
9	902	107	.2085	728	51
10	.15931	.16137	6.1970	.98723	50
11	959	167	.1856	718	49
12	.15988	196	.1742	714	48
13	.16017	226	.1628	709	47
14	046	256	.1515	704	46
15	.16074	.16286	6.1402	.98700	45
16	103	316	.1290	695	44
17	132	346	.1178	690	43
18	160	376	.1066	686	42
19	189	405	.0955	681	41
20	.16218	.16435	6.0844	.98676	40
21	246	465	.0734	671	39
22	275	495	.0624	667	38
23	304	525	.0514	662	37
24	333	555	.0405	657	36
25	.16361	.16585	6.0296	.98652	35
26	390	615	.0188	648	34
27	419	645	6.0080	643	33
28	447	674	5.9972	638	32
29	476	704	.9865	633	31
30	.16505	.16734	5.9758	.98629	30
31	533	764	.9651	624	29
32	562	794	.9545	619	28
33	591	824	.9439	614	27
34	620	854	.9333	609	26
35	.16648	.16884	5.9228	.98604	25
36	677	914	.9124	600	24
37	706	944	.9019	595	23
38	734	.16974	8915	590	22
39	763	.17004	.8811	585	21
40	.16792	.17033	5.8708	.98580	20
41	820	063	.8605	575	19
42	849	093	.8502	570	18
43	878	123	.8400	565	17
44	906	153	.8298	561	16
45	.16935	.17183	5.8197	.98556	15
46	964	213	.8095	551	14
47	16992	243	.7994	546	13
48	.17021	273	.7894	541	12
49	050	303	.7794	536	11
50	.17078	.17333	5.7694	.98531	10
51	107	363	.7594	526	9
52	136	393	.7495	521	8
53	164	423	.7396	516	7
54	193	453	.7297	511	6
55	.17222	.17483	5.7199	.98506	5
56	250	513	.7101	501	4
57	279	543	.7004	496	3
58	308	573	.6906	491	2
59	336	603	.6809	486	1
60	.17365	.17633	5.6713	.98481	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.17365	.17633	5.6713	.98481	60
1	393	663	.6617	476	59
2	422	693	.6521	471	58
3	451	723	.6425	466	57
4	479	753	.6329	461	56
5	.17508	.17783	5.6234	.98455	55
6	537	813	.6140	450	54
7	565	843	.6045	445	53
8	594	873	.5951	440	52
9	623	903	.5857	435	51
10	.17651	.17933	5.5764	.98430	50
11	680	963	.5671	425	49
12	708	.17993	.5578	420	48
13	737	.18023	.5485	414	47
14	766	053	.5393	409	46
15	.17794	.18083	5.5301	.98404	45
16	823	113	.5209	399	44
17	852	143	.5118	394	43
18	880	173	.5026	389	42
19	909	203	.4936	383	41
20	.17937	.18233	5.4845	.98378	40
21	966	263	.4755	373	39
22	.17995	293	.4665	368	38
23	.18023	323	.4575	362	37
24	052	353	.4486	357	36
25	.18081	.18384	5.4397	.98352	35
26	109	414	.4308	347	34
27	138	444	.4219	341	33
28	166	474	.4131	336	32
29	195	504	.4043	331	31
30	.18224	.18534	5.3955	.98325	30
31	252	564	.3868	320	29
32	281	594	.3781	315	28
33	309	624	.3694	310	27
34	338	654	.3607	304	26
35	.18367	.18684	5.3521	.98299	25
36	395	714	.3435	294	24
37	424	745	.3349	288	23
38	452	775	.3263	283	22
39	481	805	.3178	277	21
40	.18509	.18835	5.3093	.98272	20
41	538	865	.3008	267	19
42	567	895	.2924	261	18
43	595	925	.2839	256	17
44	624	955	.2755	250	16
45	.18652	.18986	5.2672	.98245	15
46	681	.19016	.2588	240	14
47	710	046	.2505	234	13
48	738	076	.2422	229	12
49	767	106	.2339	223	11
50	.18795	.19136	5.2257	.98218	10
51	824	166	.2174	212	9
52	852	197	.2092	207	8
53	881	227	.2011	201	7
54	910	257	.1929	196	6
55	.18938	.19287	5.1848	.98190	5
56	967	317	.1767	185	4
57	.18995	347	.1686	179	3
58	.19024	378	.1606	174	2
59	052	408	.1526	168	1
60	.19081	.19438	5.1446	.98163	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.19081	.19438	5.1446	.98163	60
1	109	468	.1366	157	59
2	138	498	.1286	152	58
3	167	529	.1207	146	57
4	195	559	.1128	140	56
5	.19224	.19589	5.1049	.98135	55
6	252	619	.0970	129	54
7	281	649	.0892	124	53
8	309	680	.0814	118	52
9	338	710	.0736	112	51
10	.19366	.19740	5.0658	.98107	50
11	395	770	.0581	101	49
12	423	801	.0504	096	48
13	452	831	.0427	090	47
14	481	861	.0350	084	46
15	.19509	.19891	5.0273	.98079	45
16	538	921	.0197	073	44
17	566	952	.0121	067	43
18	595	.19982	5.0045	061	42
19	623	.20012	4.9969	056	41
20	.19652	.20042	4.9894	.98050	40
21	680	073	.9819	044	39
22	709	103	.9744	039	38
23	737	133	.9669	033	37
24	766	164	.9594	027	36
25	.19794	.20194	4.9520	.98021	35
26	823	224	.9446	016	34
27	851	254	.9372	010	33
28	880	285	.9298	.98004	32
29	908	315	.9225	.97998	31
30	.19937	.20345	4.9152	.97992	30
31	965	376	.9078	987	29
32	.19994	406	.9006	981	28
33	.20022	436	.8933	975	27
34	051	466	.8860	969	26
35	.20079	.20497	4.8788	.97963	25
36	108	527	.8716	958	24
37	136	557	.8644	952	23
38	165	588	.8573	946	22
39	193	618	.8501	940	21
40	.20222	.20648	4.8430	.97934	20
41	250	679	.8359	928	19
42	279	709	.8288	922	18
43	307	739	.8218	916	17
44	336	770	.8147	910	16
45	.20364	.20800	4.8077	.97905	15
46	393	830	.8007	899	14
47	421	861	.7937	893	13
48	450	891	.7867	887	12
49	478	921	.7798	881	11
50	.20507	.20952	4.7729	.97875	10
51	535	.20982	.7659	869	9
52	563	.21013	.7591	863	8
53	592	043	.7522	857	7
54	620	073	.7453	851	6
55	.20649	.21104	4.7385	.97845	5
56	677	134	.7317	839	4
57	706	164	.7249	833	3
58	734	195	.7181	827	2
59	763	225	.7114	821	1
60	.20791	.21256	4.7046	.97815	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.20791	.21256	4.7046	.97815	60
1	820	286	.6979	809	59
2	848	316	.6912	803	58
3	877	347	.6845	797	57
4	905	377	.6779	791	56
5	.20933	.21408	4.6712	.97784	55
6	962	438	.6646	778	54
7	.20990	469	.6580	772	53
8	.21019	499	.6514	766	52
9	047	529	.6448	760	51
10	.21076	.21560	4.6382	.97754	50
11	104	590	.6317	748	49
12	132	621	.6252	742	48
13	161	651	.6187	735	47
14	189	682	.6122	729	46
15	.21218	.21712	4.6057	.97723	45
16	246	743	.5993	717	44
17	275	773	.5928	711	43
18	303	804	.5864	705	42
19	331	834	.5800	698	41
20	.21360	.21864	4.5736	.97692	40
21	388	895	.5673	686	39
22	417	925	.5609	680	38
23	445	956	.5546	673	37
24	474	.21986	.5483	667	36
25	.21502	.22017	4.5420	.97661	35
26	530	047	.5357	655	34
27	559	078	.5294	648	33
28	587	108	.5232	642	32
29	616	139	.5169	636	31
30	.21644	.22169	4.5107	.97630	30
31	672	200	.5045	623	29
32	701	231	.4983	617	28
33	729	261	.4922	611	27
34	758	292	.4860	604	26
35	.21786	.22322	4.4799	.97598	25
36	814	353	.4737	592	24
37	843	383	.4676	585	23
38	871	414	.4615	579	22
39	899	444	.4555	573	21
40	.21928	.22475	4.4494	.97566	20
41	956	505	.4434	560	19
42	.21985	536	.4373	553	18
43	.22013	567	.4313	547	17
44	041	597	.4253	541	16
45	.22070	.22628	4.4194	.97534	15
46	098	658	.4134	528	14
47	126	689	.4075	521	13
48	155	719	.4015	515	12
49	183	750	.3956	508	11
50	.22212	.22781	4.3897	.97502	10
51	240	811	.3838	496	9
52	268	842	.3779	489	8
53	297	872	.3721	483	7
54	325	903	.3662	476	6
55	.22353	.22934	4.3604	.97470	5
56	382	964	.3546	463	4
57	410	.22995	.3488	457	3
58	438	.23026	.3430	450	2
59	467	056	.3372	444	1
60	.22495	.23087	4.3315	.97437	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.22495	.23087	4.3315	.97437	60
1	523	117	.3257	430	59
2	552	148	.3200	424	58
3	580	179	.3143	417	57
4	608	209	.3086	411	56
5	.22637	.23240	4.3029	.97404	55
6	665	271	.2972	398	54
7	693	301	.2916	391	53
8	722	332	.2859	384	52
9	750	363	.2803	378	51
10	.22778	.23393	4.2747	.97371	50
11	807	424	.2691	365	49
12	835	455	.2635	358	48
13	863	485	.2580	351	47
14	892	516	.2524	345	46
15	.22920	.23547	4.2468	.97338	45
16	948	578	.2413	331	44
17	.22977	608	.2358	325	43
18	.23005	639	.2303	318	42
19	033	670	.2248	311	41
20	.23062	.23700	4.2193	.97304	40
21	090	731	.2139	298	39
22	118	762	.2084	291	38
23	146	793	.2030	284	37
24	175	823	.1976	278	36
25	.23203	.23854	4.1922	.97271	35
26	231	885	.1868	264	34
27	260	916	.1814	257	33
28	288	946	.1760	251	32
29	316	.23977	.1706	244	31
30	.23345	.24008	4.1653	.97237	30
31	373	039	.1600	230	29
32	401	069	.1547	223	28
33	429	100	.1493	217	27
34	458	131	.1441	210	26
35	.23486	.24162	4.1388	.97203	25
36	514	193	.1335	196	24
37	542	223	.1282	189	23
38	571	254	.1230	182	22
39	599	285	.1178	176	21
40	.23627	.24[illegible]6	4.1126	.97169	20
41	656	347	.1074	162	19
42	684	377	.1022	155	18
43	712	408	.0970	148	17
44	740	439	.0918	141	16
45	.23769	.24470	4.0867	.97134	15
46	797	501	.0815	127	14
47	825	532	.0764	120	13
48	853	562	.0713	113	12
49	882	593	.0662	106	11
50	.23910	.24624	4.0611	.97100	10
51	938	655	.0560	093	9
52	966	686	.0509	086	8
53	.23995	717	.0459	079	7
54	.24023	747	.0408	072	6
55	.24051	.24778	4.0358	.97065	5
56	079	809	.0308	058	4
57	108	840	.0257	051	3
58	136	871	.0207	044	2
59	164	902	.0158	037	1
60	.24192	.24933	4.0108	.97030	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.24192	.24933	4.0108	.97030	60
1	220	964	.0058	023	59
2	249	.24995	4.0009	015	58
3	277	.25026	3.9959	008	57
4	305	056	.9910	.97001	56
5	.24333	.25087	3.9861	.96994	55
6	362	118	.9812	987	54
7	390	149	.9763	980	53
8	418	180	.9714	973	52
9	446	211	.9665	966	51
10	.24474	.25242	3.9617	.96959	50
11	503	273	.9568	952	49
12	531	304	.9520	945	48
13	559	335	.9471	937	47
14	587	366	.9423	930	46
15	.24615	.25397	3.9375	.96923	45
16	644	428	.9327	916	44
17	672	459	.9279	909	43
18	700	490	.9232	902	42
19	728	521	.9184	894	41
20	.24756	.25552	3.9136	.96887	40
21	784	583	.9089	880	39
22	813	614	.9042	873	38
23	841	645	.8995	866	37
24	869	676	.8947	858	36
25	.24897	.25707	3.8900	.96851	35
26	925	738	.8854	844	34
27	954	769	.8807	837	33
28	.24982	800	.8760	829	32
29	.25010	831	.8714	822	31
30	.25038	.25862	3.8667	.96815	30
31	066	893	.8621	807	29
32	094	924	.8575	800	28
33	122	955	.8528	793	27
34	151	.25986	.8482	786	26
35	.25179	.26017	3.8436	.96778	25
36	207	048	.8391	771	24
37	235	079	.8345	764	23
38	263	110	.8299	756	22
39	291	141	.8254	749	21
40	.25320	.26172	3.8208	.96742	20
41	348	203	.8163	734	19
42	376	235	.8118	727	18
43	404	266	.8073	719	17
44	432	297	.8028	712	16
45	.25460	.26328	3.7983	.96705	15
46	488	359	.7938	697	14
47	516	390	.7893	690	13
48	545	421	.7848	682	12
49	573	452	.7804	675	11
50	.25601	.26483	3.7760	.96667	10
51	629	515	.7715	660	9
52	657	546	.7671	653	8
53	685	577	.7627	645	7
54	713	608	.7583	638	6
55	.25741	.26639	3.7539	.96630	5
56	769	670	.7495	623	4
57	798	701	.7451	615	3
58	826	733	.7408	608	2
59	854	764	.7364	600	1
60	.25882	.26795	3.7321	.96593	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.25882	.26795	3.7321	.96593	60
1	910	826	.7277	585	59
2	938	857	.7234	578	58
3	966	888	.7191	570	57
4	.25994	920	.7148	562	56
5	.26022	.26951	3.7105	.96555	55
6	050	.26982	.7062	547	54
7	079	.27013	.7019	540	53
8	107	044	.6976	532	52
9	135	076	.6933	524	51
10	.26163	.27107	3.6891	.96517	50
11	191	138	.6848	509	49
12	219	169	.6806	502	48
13	247	201	.6764	494	47
14	275	232	.6722	486	46
15	.26303	.27263	3.6680	.96479	45
16	331	294	.6638	471	44
17	359	326	.6596	463	43
18	387	357	.6554	456	42
19	415	388	.6512	448	41
20	.26443	.27419	3.6470	.96440	40
21	471	451	.6429	433	39
22	500	482	.6387	425	38
23	528	513	.6346	417	37
24	556	545	.6305	410	36
25	.26584	.27576	3.6264	.96402	35
26	612	607	.6222	394	34
27	640	638	.6181	386	33
28	668	670	.6140	379	32
29	696	701	.6100	371	31
30	.26724	.27732	3.6059	.96363	30
31	752	764	.6018	355	29
32	780	795	.5978	347	28
33	808	826	.5937	340	27
34	836	858	.5897	332	26
35	.26864	.27889	3.5856	.96324	25
36	892	921	.5816	316	24
37	920	952	.5776	308	23
38	948	.27983	.5736	301	22
39	.26976	.28015	.5696	293	21
40	.27004	.28046	3.5656	.96285	20
41	032	077	.5616	277	19
42	060	109	.5576	269	18
43	088	140	.5536	261	17
44	116	172	.5497	253	16
45	.27144	.28203	3.5457	.96246	15
46	172	234	.5418	238	14
47	200	266	.5379	230	13
48	228	297	.5339	222	12
49	256	329	.5300	214	11
50	.27284	.28360	3.5261	.96206	10
51	312	391	.5222	198	9
52	340	423	.5183	190	8
53	368	454	.5144	182	7
54	396	486	.5105	174	6
55	.27424	.28517	3.5067	.96166	5
56	452	549	.5028	158	4
57	480	580	.4989	150	3
58	508	612	.4951	142	2
59	536	643	.4912	134	1
60	.27564	.28675	3.4874	.96126	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.27564	.28675	3.4874	.96126	60
1	592	706	.4836	118	59
2	620	738	.4798	110	58
3	648	769	.4760	102	57
4	676	801	.4722	094	56
5	.27704	.28832	3.4684	.96086	55
6	731	864	.4646	078	54
7	759	895	.4608	070	53
8	787	927	.4570	062	52
9	815	958	.4533	054	51
10	.27843	.28990	3.4495	.96046	50
11	871	.29021	.4458	037	49
12	899	053	.4420	029	48
13	927	084	.4383	021	47
14	955	116	.4346	013	46
15	.27983	.29147	3.4308	.96005	45
16	.28011	179	.4271	.95997	44
17	039	210	.4234	989	43
18	067	242	.4197	981	42
19	095	274	.4160	972	41
20	.28123	.29305	3.4124	.95964	40
21	150	337	.4087	956	39
22	178	368	.4050	948	38
23	206	400	.4014	940	37
24	234	432	.3977	931	36
25	.28262	.29463	3.3941	.95923	35
26	290	495	.3904	915	34
27	318	526	.3868	907	33
28	346	558	.3832	898	32
29	374	590	.3796	890	31
30	.28402	.29621	3.3759	.95882	30
31	429	653	.3723	874	29
32	457	685	.3687	865	28
33	485	716	.3652	857	27
34	513	748	.3616	849	26
35	.28541	.29780	3.3580	.95841	25
36	569	811	.3544	832	24
37	597	843	.3509	824	23
38	625	875	.3473	816	22
39	652	906	.3438	807	21
40	.28680	.29938	3.3402	.95799	20
41	708	.29970	.3367	791	19
42	736	.30001	.3332	782	18
43	764	033	.3297	774	17
44	792	065	.3261	766	16
45	.28820	.30097	3.3226	.95757	15
46	847	128	.3191	749	14
47	875	160	.3156	740	13
48	903	192	.3122	732	12
49	931	224	.3087	724	11
50	.28959	.30255	3.3052	.95715	10
51	.28987	287	.3017	707	9
52	.29015	319	.2983	698	8
53	042	351	.2948	690	7
54	070	382	.2914	681	6
55	.29098	.30414	3.2879	.95673	5
56	126	446	.2845	664	4
57	154	478	.2811	656	3
58	182	509	.2777	647	2
59	209	541	.2743	639	1
60	.29237	.30573	3.2709	.95630	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.29237	.30573	3.2709	.95630	60
1	265	605	.2675	622	59
2	293	637	.2641	613	58
3	321	669	.2607	605	57
4	348	700	.2573	596	56
5	.29376	.30732	3.2539	.95588	55
6	404	764	.2506	579	54
7	432	796	.2472	571	53
8	460	828	.2438	562	52
9	487	860	.2405	554	51
10	.29515	.30891	3.2371	.95545	50
11	543	923	.2338	536	49
12	571	955	.2305	528	48
13	599	.30987	.2272	519	47
14	626	.31019	.2238	511	46
15	.29654	.31051	3.2205	.95502	45
16	682	083	.2172	493	44
17	710	115	.2139	485	43
18	737	147	.2106	476	42
19	765	178	.2073	467	41
20	.29793	.31210	3.2041	.95459	40
21	821	242	.2008	450	39
22	849	274	.1975	441	38
23	876	306	.1943	433	37
24	904	338	.1910	424	36
25	.29932	.31370	3.1878	.95415	35
26	960	402	.1845	407	34
27	.29987	434	.1813	398	33
28	.30015	466	.1780	389	32
29	043	498	.1748	380	31
30	.30071	.31530	3.1716	.95372	30
31	098	562	.1684	363	29
32	126	594	.1652	354	28
33	154	626	.1620	345	27
34	182	658	.1588	337	26
35	.30209	.31690	3.1556	.95328	25
36	237	722	.1524	319	24
37	265	754	.1492	310	23
38	292	786	.1460	301	22
39	320	[illegible]	.1429	293	21
40	.30348	.31850	3.1397	.95284	20
41	376	882	.1366	275	19
42	403	914	.1334	260	18
43	431	946	.1303	257	17
44	459	.31978	.1271	248	16
45	.30486	.32010	3.1240	.95240	15
46	514	042	.1209	231	14
47	542	074	.1178	222	13
48	570	106	.1146	213	12
49	597	139	.1115	204	11
50	.30625	.32171	3.1084	.95195	10
51	653	203	.1053	186	9
52	680	235	.1022	177	8
53	708	267	.0991	168	7
54	736	299	.0961	159	6
55	.30763	.32331	3.0930	.95150	5
56	791	363	.0899	142	4
57	819	396	.0868	133	3
58	846	428	.0838	124	2
59	874	460	.0807	115	1
60	.30902	.32492	3.0777	.95106	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.30902	.32492	3.0777	.95106	60
1	929	524	.0746	097	59
2	957	556	.0716	088	58
3	.30985	588	.0686	079	57
4	.31012	621	.0655	070	56
5	.31040	.32653	3.0625	.95061	55
6	068	685	.0595	052	54
7	095	717	.0565	043	53
8	123	749	.0535	033	52
9	151	782	.0505	024	51
10	.31178	.32814	3.0475	.95015	50
11	206	846	.0445	.95006	49
12	233	878	.0415	.94997	48
13	261	911	.0385	988	47
14	289	943	.0356	979	46
15	.31316	.32975	3.0326	.94970	45
16	344	.33007	.0296	961	44
17	372	040	.0267	952	43
18	399	072	.0237	943	42
19	427	104	.0208	933	41
20	.31454	.33136	3.0178	.94924	40
21	482	169	.0149	915	39
22	510	201	.0120	906	38
23	537	233	.0090	897	37
24	565	266	.0061	888	36
25	.31593	.33298	3.0032	.94878	35
26	620	330	3.0003	869	34
27	648	363	2.9974	860	33
28	675	395	.9945	851	32
29	703	427	.9916	842	31
30	.31730	.33460	2.9887	.94832	30
31	758	492	.9858	823	29
32	786	524	.9829	814	28
33	813	557	.9800	805	27
34	841	589	.9772	795	26
35	.31868	.33621	2.9743	.94786	25
36	896	654	.9714	777	24
37	923	686	.9686	768	23
38	951	718	.9657	758	22
39	.31979	751	.9629	749	21
40	.32006	33783	2.9600	.94740	20
41	034	816	.9572	730	19
42	061	848	.9544	721	18
43	089	881	.9515	712	17
44	116	913	.9487	702	16
45	.32144	.33945	2.9459	.94693	15
46	171	.33978	.9431	684	14
47	199	.34010	.9403	674	13
48	227	043	.9375	665	12
49	254	075	.9347	656	11
50	.32282	.34108	2.9319	.94646	10
51	309	140	.9291	637	9
52	337	173	.9263	627	8
53	364	205	.9235	618	7
54	392	238	.9208	609	6
55	.32419	.34270	2.9180	.94599	5
56	447	303	.9152	590	4
57	474	335	.9125	580	3
58	502	368	.9097	571	2
59	529	400	.9070	561	1
60	.32557	.34433	2.9042	.94552	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.32557	.34433	2.9042	.94552	60
1	584	465	.9015	542	59
2	612	498	.8987	533	58
3	639	530	.8960	523	57
4	667	563	.8933	514	56
5	.32694	.34596	2.8905	.94504	55
6	722	628	.8878	495	54
7	749	661	.8851	485	53
8	777	693	.8824	476	52
9	804	726	.8797	466	51
10	.32832	.34758	2.8770	.94457	50
11	859	791	.8743	447	49
12	887	824	.8716	438	48
13	914	856	.8689	428	47
14	942	889	.8662	418	46
15	.32969	.34922	2.8636	.94409	45
16	.32997	954	.8609	399	44
17	.33024	.34987	.8582	390	43
18	051	.35020	.8556	380	42
19	079	052	.8529	370	41
20	.33106	.35085	2.8502	.94361	40
21	134	118	.8476	351	39
22	161	150	.8449	342	38
23	189	183	.8423	332	37
24	216	216	.8397	322	36
25	.33244	.35248	2.8370	.94313	35
26	271	281	.8344	303	34
27	298	314	.8318	293	33
28	326	346	.8291	284	32
29	353	379	.8265	274	31
30	.33381	.35412	2.8239	.94264	30
31	408	445	.8213	254	29
32	436	477	.8187	245	28
33	463	510	.8161	235	27
34	490	543	.8135	225	26
35	.33518	.35576	2.8109	.94215	25
36	545	608	.8083	206	24
37	573	641	.8057	196	23
38	600	674	.8032	186	22
39	627	707	.8006	176	21
40	.33655	.35740	2.7980	.94167	20
41	682	772	.7955	157	19
42	710	805	.7929	147	18
43	737	838	.7903	137	17
44	764	871	.7878	127	16
45	.33792	.35904	2.7852	.94118	15
46	819	937	.7827	108	14
47	846	.35969	.7801	098	13
48	874	.36002	.7776	088	12
49	901	035	.7751	078	11
50	.33929	.36068	2.7725	.94068	10
51	956	101	.7700	058	9
52	.33983	134	.7675	049	8
53	.34011	167	.7650	039	7
54	038	199	.7625	029	6
55	.34065	.36232	2.7600	.94019	5
56	093	265	.7575	.94009	4
57	120	298	.7550	.93999	3
58	147	331	.7525	989	2
59	175	364	.7500	979	1
60	.34202	.36397	2.7475	.93969	0
	cos	cot	tan	sin	′

110° 200° 290° **20°** TABLE IV

′	sin	tan	cot	cos	
0	.34202	.36397	2.7475̄	.93969	**60**
1	229	430	.7450	959	59
2	257	463	.7425	949	58
3	284	496	.7400	939	57
4	311	529	.7376	929	56
5	.34339	.36562	2.7351	.93919	**55**
6	366	595̄	.7326	909	54
7	393	628	.7302	899	53
8	421	661	.7277	889	52
9	448	694	.7253	879	51
10	.34475	.36727	2.7228	.93869	**50**
11	503	760	.7204	859	49
12	530	793	.7179	849	48
13	557	826	.7155̄	839	47
14	584	859	.7130	829	46
15	.34612	.36892	2.7106	.93819	**45**
16	639	925̄	.7082	809	44
17	666	958	.7058	799	43
18	694	.36991	.7034	789	42
19	721	.37024	.7009	779	41
20	.34748	.37057	2.6985	.93769	**40**
21	775	090	.6961	759	39
22	803	123	.6937	748	38
23	830	157	.6913	738	37
24	857	190	.6889	728	36
25	.34884	.37223	2.6865	.93718	**35**
26	912	256	.6841	708	34
27	939	289	.6818	698	33
28	966	322	.6794	688	32
29	.34993	355	.6770	677	31
30	.35021	.37388	2.6746	.93667	**30**
31	048	422	.6723	657	29
32	075	455̄	.6699	647	28
33	102	488	.6675	637	27
34	130	521	.6652	626	26
35	.35157	.37554	2.6628	.93616	**25**
36	184	588	.6605̄	606	24
37	211	621	.6581	596	23
38	239	654	.6558	585	22
39	266	687	.6534	575	21
40	.35293	.37720	2.6511	.93565̄	**20**
41	320	754	.6488	555̄	19
42	347	787	.6464	544	18
43	375̄	820	.6441	534	17
44	402	853	.6418	524	16
45	.35429	.37887	2.6395̄	.93514	**15**
46	456	920	.6371	503	14
47	484	953	.6348	493	13
48	511	.37986	.6325	483	12
49	538	.38020	.6302	472	11
50	.35565	.38053	2.6279	.93462	**10**
51	592	086	.6256	452	9
52	619	120	.6233	441	8
53	647	153	.6210	431	7
54	674	186	.6187	420	6
55	.35701	.38220	2.6165̄	.93410	**5**
56	728	253	.6142	400	4
57	755	286	.6119	389	3
58	782	320	.6096	379	2
59	810	353	.6074	368	1
60	.35837	.38386	2.6051	.93358	**0**
	cos	cot	tan	sin	′

159° 249° 339° **69°**

21° 111° 201° 291°

′	sin	tan	cot	cos	
0	.35837	.38386	2.6051	.93358	**60**
1	864	420	.6028	348	59
2	891	453	.6006	337	58
3	918	487	.5983	327	57
4	945	520	.5961	316	56
5	.35973	.38553	2.5938	.93306	**55**
6	.36000	587	.5916	295	54
7	027	620	.5893	285̄	53
8	054	654	.5871	274	52
9	081	687	.5848	264	51
10	.36108	.38721	2.5826	.93253	**50**
11	135	754	.5804	243	49
12	162	787	.5782	232	48
13	190	821	.5759	222	47
14	217	854	.5737	211	46
15	.36244	.38888	2.5715̄	.93201	**45**
16	271	921	.5693	190	44
17	298	955̄	.5671	180	43
18	325	.38988	.5649	169	42
19	352	.39022	.5627	159	41
20	.36379	.39055	2.5605̄	.93148	**40**
21	406	089	.5583	137	39
22	434	122	.5561	127	38
23	461	156	.5539	116	37
24	488	190	.5517	106	36
25	.36515̄	.39223	2.5495	.93095̄	**35**
26	542	257	.5473	084	34
27	569	290	.5452	074	33
28	596	324	.5430	063	32
29	623	357	.5408	052	31
30	.36650	.39391	2.5386	.93042	**30**
31	677	425̄	.5365̄	031	29
32	704	458	.5343	020	28
33	731	492	.5322	.93010	27
34	758	526	.5300	.92999	26
35	.36785	.39559	2.5279	.92988	**25**
36	812	593	.5257	978	24
37	839	626	.5236	967	23
38	867	660	.5214	956	22
39	894	694	.5193	945	21
40	.36921	.39727	2.5172	.92935̄	**20**
41	948	761	.5150	924	19
42	.36975̄	795̄	.5129	913	18
43	.37002	829	.5108	902	17
44	029	862	.5086	892	16
45	.37056	.39896	2.5065	.92881	**15**
46	083	930	.5044	870	14
47	110	963	.5023	859	13
48	137	.39997	.5002	849	12
49	164	.40031	.4981	838	11
50	.37191	.40065̄	2.4960	.92827	**10**
51	218	098	.4939	816	9
52	245̄	132	.4918	805	8
53	272	166	.4897	794	7
54	299	200	.4876	784	6
55	.37326	.40234	2.4855̄	.92773	**5**
56	353	267	.4834	762	4
57	380	301	.4813	751	3
58	407	335̄	.4792	740	2
59	434	369	.4772	729	1
60	.37461	.40403	2.4751	.92718	**0**
	cos	cot	tan	sin	′

68° 158° 248° 338°

112° 202° 292° **22°** TABLE IV

′	sin	tan	cot	cos	
0	.37461	.40403	2.4751	.92718	**60**
1	488	436	.4730	707	59
2	515̄	470	.4709	697	58
3	542	504	.4689	686	57
4	569	538	.4668	675̄	56
5	.37595	.40572	2.4648	.92664	**55**
6	622	606	.4627	653	54
7	649	640	.4606	642	53
8	676	674	.4586	631	52
9	703	707	.4566	620	51
10	.37730	.40741	2.4545	.92609	**50**
11	757	775	.4525̄	598	49
12	784	809	.4504	587	48
13	811	843	.4484	576	47
14	838	877	.4464	565	46
15	.37865̄	.40911	2.4443	.92554	**45**
16	892	945	.4423	543	44
17	919	.40979	.4403	532	43
18	946	.41013	.4383	521	42
19	973	047	.4362	510	41
20	.37999	.41081	2.4342	.92499	**40**
21	.38026	115̄	.4322	488	39
22	053	149	.4302	477	38
23	080	183	.4282	466	37
24	107	217	.4262	455̄	36
25	.38134	.41251	2.4242	.92444	**35**
26	161	285	.4222	432	34
27	188	319	.4202	421	33
28	215̄	353	.4182	410	32
29	241	387	.4162	399	31
30	.38268	.41421	2.4142	.92388	**30**
31	295	455	.4122	377	29
32	322	490	.4102	366	28
33	349	524	.4083	355̄	27
34	376	558	.4063	343	26
35	.38403	.41592	2.4043	.92332	**25**
36	430	626	.4023	321	24
37	456	660	.4004	310	23
38	483	694	.3984	299	22
39	510	728	.3964	287	21
40	.38537	.41763	2.3945̄	.92276	**20**
41	564	797	.3925	265	19
42	591	831	.3906	254	18
43	617	865	.3886	243	17
44	644	899	.3867	231	16
45	.38671	.41933	2.3847	.92220	**15**
46	698	.41968	.3828	209	14
47	725̄	.42002	.3808	198	13
48	752	036	.3789	186	12
49	778	070	.3770	175	11
50	.38805	.42105̄	2.3750	.92164	**10**
51	832	139	.3731	152	9
52	859	173	.3712	141	8
53	886	207	.3693	130	7
54	912	242	.3673	119	6
55	.38939	.42276	2.3654	.92107	**5**
56	966	310	.3635̄	096	4
57	.38993	345̄	.3616	085̄	3
58	.39020	379	.3597	073	2
59	046	413	.3578	062	1
60	.39073	.42447	2.3559	.92050	**0**
	cos	cot	tan	sin	′

157° 247° 337° **67°**

23° 113° 203° 293°

′	sin	tan	cot	cos	
0	.39073	.42447	2.3559	.92050	**60**
1	100	482	.3539	039	59
2	127	516	.3520	028	58
3	153	551	.3501	016	57
4	180	585̄	.3483	.92005̄	56
5	.39207	.42619	2.3464	.91994	**55**
6	234	654	.3445̄	982	54
7	260	688	.3426	971	53
8	287	722	.3407	959	52
9	314	757	.3388	948	51
10	.39341	.42791	2.3369	.91936	**50**
11	367	826	.3351	925̄	49
12	394	860	.3332	914	48
13	421	894	.3313	902	47
14	448	929	.3294	891	46
15	.39474	.42963	2.3276	.91879	**45**
16	501	.42998	.3257	868	44
17	528	.43032	.3238	856	43
18	555̄	067	.3220	845̄	42
19	581	101	.3201	833	41
20	.39608	.43136	2.3183	.91822	**40**
21	635̄	170	.3164	810	39
22	661	205̄	.3146	799	38
23	688	239	.3127	787	37
24	715̄	274	.3109	775	36
25	.39741	.43308	2.3090	.91764	**35**
26	768	343	.3072	752	34
27	795̄	378	.3053	741	33
28	822	412	.3035	729	32
29	848	447	.3017	718	31
30	.39875̄	.43481	2.2998	.91706	**30**
31	902	516	.2980	694	29
32	928	550	.2962	683	28
33	955̄	585	.2944	671	27
34	.39982	620	.2925	660	26
35	.40008	.43654	2.2907	.91648	**25**
36	035̄	689	.2889	636	24
37	062	724	.2871	625̄	23
38	088	758	.2853	613	22
39	115̄	793	.2835̄	601	21
40	.40141	.43828	2.2817	.91590	**20**
41	168	862	.2799	578	19
42	195̄	897	.2781	566	18
43	221	932	.2763	555̄	17
44	248	.43966	.2745̄	543	16
45	.40275̄	.44001	2.2727	.91531	**15**
46	301	036	.2709	519	14
47	328	071	.2691	508	13
48	355̄	105	.2673	496	12
49	381	140	.2655	484	11
50	.40408	.44175̄	2.2637	.91472	**10**
51	434	210	.2620	461	9
52	461	244	.2602	449	8
53	488	279	.2584	437	7
54	514	314	.2566	425	6
55	.40541	.44349	2.2549	.91414	**5**
56	567	384	.2531	402	4
57	594	418	.2513	390	3
58	621	453	.2496	378	2
59	647	488	.2478	366	1
60	.40674	.44523	2.2460	.91355̄	**0**
	cos	cot	tan	sin	′

66° 156° 246° 336°

′	sin	tan	cot	cos	
0	.40674	.44523	2.2460	.91355	60
1	700	558	.2443	343	59
2	727	593	.2425	331	58
3	753	627	.2408	319	57
4	780	662	.2390	307	56
5	.40806	.44697	2.2373	.91295	55
6	833	732	.2355	283	54
7	860	767	.2338	272	53
8	886	802	.2320	260	52
9	913	837	.2303	248	51
10	.40939	.44872	2.2286	.91236	50
11	966	907	.2268	224	49
12	.40992	942	.2251	212	48
13	.41019	.44977	.2234	200	47
14	045	.45012	.2216	188	46
15	.41072	.45047	2.2199	.91176	45
16	098	082	.2182	164	44
17	125	117	.2165	152	43
18	151	152	.2148	140	42
19	178	187	.2130	128	41
20	.41204	.45222	2.2113	.91116	40
21	231	257	.2096	104	39
22	257	292	.2079	092	38
23	284	327	.2062	080	37
24	310	362	.2045	068	36
25	.41337	.45397	2.2028	.91056	35
26	363	432	.2011	044	34
27	390	467	.1994	032	33
28	416	502	.1977	020	32
29	443	538	.1960	.91008	31
30	.41469	.45573	2.1943	.90996	30
31	496	608	.1926	984	29
32	522	643	.1909	972	28
33	549	678	.1892	960	27
34	575	713	.1876	948	26
35	.41602	.45748	2.1859	.90936	25
36	628	784	.1842	924	24
37	655	819	.1825	911	23
38	681	854	.1808	899	22
39	707	889	.1792	887	21
40	.41734	.45924	2.1775	.90875	20
41	760	960	.1758	863	19
42	787	.45995	.1742	851	18
43	813	.46030	.1725	839	17
44	840	065	.1708	826	16
45	.41866	.46101	2.1692	.90814	15
46	892	136	.1675	802	14
47	919	171	.1659	790	13
48	945	206	.1642	778	12
49	972	242	.1625	766	11
50	.41998	.46277	2.1609	.90753	10
51	.42024	312	.1592	741	9
52	051	348	.1576	729	8
53	077	383	.1560	717	7
54	104	418	.1543	704	6
55	.42130	.46454	2.1527	.90692	5
56	156	489	.1510	680	4
57	183	525	.1494	668	3
58	209	560	.1478	655	2
59	235	595	.1461	643	1
60	.42262	.46631	2.1445	.90631	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.42262	.46631	2.1445	.90631	60
1	288	666	.1429	618	59
2	315	702	.1413	606	58
3	341	737	.1396	594	57
4	367	772	.1380	582	56
5	.42394	.46808	2.1364	.90569	55
6	420	843	.1348	557	54
7	446	879	.1332	545	53
8	473	914	.1315	532	52
9	499	950	.1299	520	51
10	.42525	.46985	2.1283	.90507	50
11	552	.47021	.1267	495	49
12	578	056	.1251	483	48
13	604	092	.1235	470	47
14	631	128	.1219	458	46
15	.42657	.47163	2.1203	.90446	45
16	683	199	.1187	433	44
17	709	234	.1171	421	43
18	736	270	.1155	408	42
19	762	305	.1139	396	41
20	.42788	.47341	2.1123	.90383	40
21	815	377	.1107	371	39
22	841	412	.1092	358	38
23	867	448	.1076	346	37
24	894	483	.1060	334	36
25	.42920	.47519	2.1044	.90321	35
26	946	555	.1028	309	34
27	972	590	.1013	296	33
28	.42999	626	.0997	284	32
29	.43025	662	.0981	271	31
30	.43051	.47698	2.0965	.90259	30
31	077	733	.0950	246	29
32	104	769	.0934	233	28
33	130	805	.0918	221	27
34	156	840	.0903	208	26
35	.43182	.47876	2.0887	.90196	25
36	209	912	.0872	183	24
37	235	948	.0856	171	23
38	261	.47984	.0840	158	22
39	287	.48019	.0825	146	21
40	.43313	.48055	2.0809	.90133	20
41	340	091	.0794	120	19
42	366	127	.0778	108	18
43	392	163	.0763	095	17
44	418	198	.0748	082	16
45	.43445	.48234	2.0732	.90070	15
46	471	270	.0717	057	14
47	497	306	.0701	045	13
48	523	342	.0686	032	12
49	549	378	.0671	019	11
50	.43575	.48414	2.0655	.90007	10
51	602	450	.0640	.89994	9
52	628	486	.0625	981	8
53	654	521	.0609	968	7
54	680	557	.0594	956	6
55	.43706	.48593	2.0579	.89943	5
56	733	629	.0564	930	4
57	759	665	.0549	918	3
58	785	701	.0533	905	2
59	811	737	.0518	892	1
60	.43837	.48773	2.0503	.89879	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.43837	.48773	2.0503	.89879	60
1	863	809	.0488	867	59
2	889	845	.0473	854	58
3	916	881	.0458	841	57
4	942	917	.0443	828	56
5	.43968	.48953	2.0428	.89816	55
6	.43994	.48989	.0413	803	54
7	.44020	.49026	.0398	790	53
8	046	062	.0383	777	52
9	072	098	.0368	764	51
10	.44098	.49134	2.0353	.89752	50
11	124	170	.0338	739	49
12	151	206	.0323	726	48
13	177	242	.0308	713	47
14	203	278	.0293	700	46
15	.44229	.49315	2.0278	.89687	45
16	255	351	.0263	674	44
17	281	387	.0248	662	43
18	307	423	.0233	649	42
19	333	459	.0219	636	41
20	.44359	.49495	2.0204	.89623	40
21	385	532	.0189	610	39
22	411	568	.0174	597	38
23	437	604	.0160	584	37
24	464	640	.0145	571	36
25	.44490	.49677	2.0130	.89558	35
26	516	713	.0115	545	34
27	542	749	.0101	532	33
28	568	786	.0086	519	32
29	594	822	.0072	506	31
30	.44620	.49858	2.0057	.89493	30
31	646	894	.0042	480	29
32	672	931	.0028	467	28
33	698	.49967	2.0013	454	27
34	724	.50004	1.9999	441	26
35	.44750	.50040	1.9984	.89428	25
36	776	076	.9970	415	24
37	802	113	.9955	402	23
38	828	149	.9941	389	22
39	854	185	.9926	376	21
40	.44880	.50222	1.9912	.89363	20
41	906	258	.9897	350	19
42	932	295	.9883	337	18
43	958	331	.9868	324	17
44	.44984	368	.9854	311	16
45	.45010	.50404	1.9840	.89298	15
46	036	441	.9825	285	14
47	062	477	.9811	272	13
48	088	514	.9797	259	12
49	114	550	.9782	245	11
50	.45140	.50587	1.9768	.89232	10
51	166	623	.9754	219	9
52	192	660	.9740	206	8
53	218	696	.9725	193	7
54	243	733	.9711	180	6
55	.45269	.50769	1.9697	.89167	5
56	295	806	.9683	153	4
57	321	843	.9669	140	3
58	347	879	.9654	127	2
59	373	916	.9640	114	1
60	.45399	.50953	1.9626	.89101	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.45399	.50953	1.9626	.89101	60
1	425	.50989	.9612	087	59
2	451	.51026	.9598	074	58
3	477	063	.9584	061	57
4	503	099	.9570	048	56
5	.45529	.51136	1.9556	.89035	55
6	554	173	.9542	021	54
7	580	209	.9528	.89008	53
8	606	246	.9514	.88995	52
9	632	283	.9500	981	51
10	.45658	.51319	1.9486	.88968	50
11	684	356	.9472	955	49
12	710	393	.9458	942	48
13	736	430	.9444	928	47
14	762	467	.9430	915	46
15	.45787	.51503	1.9416	.88902	45
16	813	540	.9402	888	44
17	839	577	.9388	875	43
18	865	614	.9375	862	42
19	891	651	.9361	848	41
20	.45917	.51688	1.9347	.88835	40
21	942	724	.9333	822	39
22	968	761	.9319	808	38
23	.45994	798	.9306	795	37
24	.46020	835	.9292	782	36
25	.46046	.51872	1.9278	.88768	35
26	072	909	.9265	755	34
27	097	946	.9251	741	33
28	123	.51983	.9237	728	32
29	149	.52020	.9223	715	31
30	.46175	.52057	1.9210	.88701	30
31	201	094	.9196	688	29
32	226	131	.9183	674	28
33	252	168	.9169	661	27
34	278	205	.9155	647	26
35	.46304	.52242	1.9142	.88634	25
36	330	279	.9128	620	24
37	355	316	.9115	607	23
38	381	353	.9101	593	22
39	407	390	.9088	580	21
40	.46433	.52427	1.9074	.88566	20
41	458	464	.9061	553	19
42	484	501	9047	539	18
43	510	538	9034	526	17
44	536	575	.9020	512	16
45	.46561	.52613	1.9007	.88499	15
46	587	650	.8993	485	14
47	613	687	.8980	472	13
48	639	724	.8967	458	12
49	664	761	.8953	445	11
50	.46690	.52798	1.8940	.88431	10
51	716	836	.8927	417	9
52	742	873	.8913	404	8
53	767	910	.8900	390	7
54	793	947	.8887	377	6
55	.46819	.52985	1.8873	.88363	5
56	844	.53022	.8860	349	4
57	870	059	.8847	336	3
58	896	096	.8834	322	2
59	921	134	.8820	308	1
60	.46947	.53171	1.8807	.88295	0
	cos	cot	tan	sin	′

118° 208° 298° **28°** TABLE IV

′	sin	tan	cot	cos	
0	.46947	.53171	1.8807	.88295	60
1	973	208	.8794	281	59
2	.46999	246	.8781	267	58
3	.47024	283	.8768	254	57
4	050	320	.8755	240	56
5	.47076	.53358	1.8741	.88226	55
6	101	395	.8728	213	54
7	127	432	.8715	199	53
8	153	470	.8702	185	52
9	178	507	.8689	172	51
10	.47204	.53545	1.8676	.88158	50
11	229	582	.8663	144	49
12	255	620	.8650	130	48
13	281	657	.8637	117	47
14	306	694	.8624	103	46
15	.47332	.53732	1.8611	.88089	45
16	358	769	.8598	075	44
17	383	807	.8585	062	43
18	409	844	.8572	048	42
19	434	882	.8559	034	41
20	.47460	.53920	1.8546	.88020	40
21	486	957	.8533	.88006	39
22	511	.53995	.8520	.87993	38
23	537	.54032	.8507	979	37
24	562	070	.8495	965	36
25	.47588	.54107	1.8482	.87951	35
26	614	145	.8469	937	34
27	639	183	.8456	923	33
28	665	220	.8443	909	32
29	690	258	.8430	896	31
30	.47716	.54296	1.8418	.87882	30
31	741	333	.8405	868	29
32	767	371	.8392	854	28
33	793	409	.8379	840	27
34	818	446	.8367	826	26
35	.47844	.54484	1.8354	.87812	25
36	869	522	.8341	798	24
37	895	560	.8329	784	23
38	920	597	.8316	770	22
39	946	635	.8303	756	21
40	.47971	.54673	1.8291	.87743	20
41	.47997	711	.8278	729	19
42	.48022	748	.8265	715	18
43	048	786	.8253	701	17
44	073	824	.8240	687	16
45	.48099	.54862	1.8228	.87673	15
46	124	900	.8215	659	14
47	150	938	.8202	645	13
48	175	.54975	.8190	631	12
49	201	.55013	.8177	617	11
50	.48226	.55051	1.8165	.87603	10
51	252	089	.8152	589	9
52	277	127	.8140	575	8
53	303	165	.8127	561	7
54	328	203	.8115	546	6
55	.48354	.55241	1.8103	.87532	5
56	379	279	.8090	518	4
57	405	317	.8078	504	3
58	430	355	.8065	490	2
59	456	393	.8053	476	1
60	.48481	.55431	1.8040	.87462	0
	cos	cot	tan	sin	′

151° 241° 331° **61°**

29° 119° 209° 299°

′	sin	tan	cot	cos	′
0	.48481	.55431	1.8040	.87462	60
1	506	469	.8028	448	59
2	532	507	.8016	434	58
3	557	545	.8003	420	57
4	583	583	.7991	406	56
5	.48608	.55621	1.7979	.87391	55
6	634	659	.7966	377	54
7	659	697	.7954	363	53
8	684	736	.7942	349	52
9	710	774	.7930	335	51
10	.48735	.55812	1.7917	.87321	50
11	761	850	.7905	306	49
12	786	888	.7893	292	48
13	811	926	.7881	278	47
14	837	.55964	.7868	264	46
15	.48862	.56003	1.7856	.87250	45
16	888	041	.7844	235	44
17	913	079	.7832	221	43
18	938	117	.7820	207	42
19	964	156	.7808	193	41
20	.48989	.56194	1.7796	.87178	40
21	.49014	232	.7783	164	39
22	040	270	.7771	150	38
23	065	309	.7759	136	37
24	090	347	.7747	121	36
25	.49116	.56385	1.7735	.87107	35
26	141	424	.7723	093	34
27	166	462	.7711	079	33
28	192	501	.7699	064	32
29	217	539	.7687	050	31
30	.49242	.56577	1.7675	.87036	30
31	268	616	.7663	021	29
32	293	654	.7651	.87007	28
33	318	693	.7639	.86993	27
34	344	731	7627	978	26
35	.49369	.56769	1.7615	.86964	25
36	394	808	.7603	949	24
37	419	846	.7591	935	23
38	445	885	.7579	921	22
39	470	[illegible]	.7567	906	21
40	.49495	.56962	1.7556	.86892	20
41	521	.57000	.7544	878	19
42	546	039	.7532	863	18
43	571	078	.7520	849	17
44	596	116	.7508	834	16
45	.49622	.57155	1.7496	.86820	15
46	647	193	.7485	805	14
47	672	232	.7473	791	13
48	697	271	.7461	777	12
49	723	309	.7449	762	11
50	.49748	.57348	1.7437	.86748	10
51	773	386	.7426	733	9
52	798	425	.7414	719	8
53	824	464	.7402	704	7
54	849	503	.7391	690	6
55	.49874	.57541	1.7379	.86675	5
56	899	580	.7367	661	4
57	924	619	.7355	646	3
58	950	657	.7344	632	2
59	.49975	696	.7332	617	1
60	.50000	.57735	1.7321	.86603	0
	cos	cot	tan	sin	′

60° 150° 240° 330°

120° 210° 300° **30°** TABLE IV

	sin	tan	cot	cos	
0	.50000	.57735	1.7321	.86603	60
1	025	774	.7309	588	59
2	050	813	.7297	573	58
3	076	851	.7286	559	57
4	101	890	.7274	544	56
5	.50126	.57929	1.7262	.86530	55
6	151	.57968	.7251	515	54
7	176	.58007	.7239	501	53
8	201	046	.7228	486	52
9	227	085	.7216	471	51
10	.50252	.58124	1.7205	.86457	50
11	277	162	.7193	442	49
12	302	201	.7182	427	48
13	327	240	.7170	413	47
14	352	279	.7159	398	46
15	.50377	.58318	1.7147	.86384	45
16	403	357	.7136	369	44
17	428	396	.7124	354	43
18	453	435	.7113	340	42
19	478	474	.7102	325	41
20	.50503	.58513	1.7090	.86310	40
21	528	552	.7079	295	39
22	553	591	.7067	281	38
23	578	631	.7056	266	37
24	603	670	.7045	251	36
25	.50628	.58709	1.7033	.86237	35
26	654	748	.7022	222	34
27	679	787	.7011	207	33
28	704	826	.6999	192	32
29	729	865	.6988	178	31
30	.50754	.58905	1.6977	.86163	30
31	779	944	.6965	148	29
32	804	.58983	.6954	133	28
33	829	.59022	.6943	119	27
34	854	061	.6932	104	26
35	.50879	.59101	1.6920	.86089	25
36	904	140	.6909	074	24
37	929	179	.6898	059	23
38	954	218	.6887	045	22
39	.50979	258	.6875	030	21
40	.51004	.59297	1.6864	.86015	20
41	029	336	.6853	.86000	19
42	054	376	.6842	.85985	18
43	079	415	.6831	970	17
44	104	454	.6820	956	16
45	.51129	.59494	1.6808	.85941	15
46	154	533	.6797	926	14
47	179	573	.6786	911	13
48	204	612	.6775	896	12
49	229	651	.6764	881	11
50	.51254	.59691	1.6753	.85866	10
51	279	730	.6742	851	9
52	304	770	.6731	836	8
53	329	809	.6720	821	7
54	354	849	.6709	806	6
55	.51379	.59888	1.6698	.85792	5
56	404	928	.6687	777	4
57	429	.59967	.6676	762	3
58	454	.60007	.6665	747	2
59	479	046	.6654	732	1
60	.51504	.60086	1.6643	.85717	0
′	cos	cot	tan	sin	

149° 239° 329° **59°**

31° 121° 211° 301°

′	sin	tan	cot	cos	′
0	.51504	.60086	1.6643	.85717	60
1	529	126	.6632	702	59
2	554	165	.6621	687	58
3	579	205	.6610	672	57
4	604	245	.6599	657	56
5	.51628	.60284	1.6588	.85642	55
6	653	324	.6577	627	54
7	678	364	.6566	612	53
8	703	403	.6555	597	52
9	728	443	.6545	582	51
10	.51753	.60483	1.6534	.85567	50
11	778	522	.6523	551	49
12	803	562	.6512	536	48
13	828	602	.6501	521	47
14	852	642	.6490	506	46
15	.51877	.60681	1.6479	.85491	45
16	902	721	.6469	476	44
17	927	761	.6458	461	43
18	952	801	.6447	446	42
19	.51977	841	.6436	431	41
20	.52002	.60881	1.6426	.85416	40
21	026	921	.6415	401	39
22	051	.60960	.6404	385	38
23	076	.61000	.6393	370	37
24	101	040	.6383	355	36
25	.52126	.61080	1.6372	.85340	35
26	151	120	.6361	325	34
27	175	160	.6351	310	33
28	200	200	.6340	294	32
29	225	240	.6329	279	31
30	.52250	.61280	1.6319	.85264	30
31	275	320	.6308	249	29
32	299	360	.6297	234	28
33	324	400	.6287	218	27
34	349	440	.6276	203	26
35	.52374	.61480	1.6265	.85188	25
36	399	520	.6255	173	24
37	423	561	.6244	157	23
38	448	601	.6234	142	22
39	473	641	.6223	127	21
40	.52498	.61681	1.6212	.85112	20
41	522	721	.6202	096	19
42	547	761	.6191	081	18
43	572	801	.6181	066	17
44	597	842	.6170	051	16
45	.52621	.61882	1.6160	.85035	15
46	646	922	.6149	020	14
47	671	.61962	.6139	.85005	13
48	696	.62003	.6128	.84989	12
49	720	043	.6118	974	11
50	.52745	.62083	1.6107	.84959	10
51	770	124	.6097	943	9
52	794	164	.6087	928	8
53	819	204	.6076	913	7
54	844	245	.6066	897	6
55	.52869	.62285	1.6055	.84882	5
56	893	325	.6045	866	4
57	918	366	.6034	851	3
58	943	406	.6024	836	2
59	967	446	.6014	820	1
60	.52992	.62487	1.6003	.84805	0
′	cos	cot	tan	sin	

58° 148° 238° 328°

′	sin	tan	cot	cos	
0	.52992	.62487	1.6003	.84805	60
1	.53017	527	.5993	789	59
2	041	568	.5983	774	58
3	066	608	.5972	759	57
4	091	649	.5962	743	56
5	.53115	.62689	1.5952	.84728	55
6	140	730	.5941	712	54
7	164	770	.5931	697	53
8	189	811	.5921	681	52
9	214	852	.5911	666	51
10	.53238	.62892	1.5900	.84650	50
11	263	933	.5890	635	49
12	288	.62973	.5880	619	48
13	312	.63014	.5869	604	47
14	337	055	.5859	588	46
15	.53361	.63095	1.5849	.84573	45
16	386	136	.5839	557	44
17	411	177	.5829	542	43
18	435	217	.5818	526	42
19	460	258	.5808	511	41
20	.53484	.63299	1.5798	.84495	40
21	509	340	.5788	480	39
22	534	380	.5778	464	38
23	558	421	.5768	448	37
24	583	462	.5757	433	36
25	.53607	.63503	1.5747	.84417	35
26	632	544	.5737	402	34
27	656	584	.5727	386	33
28	681	625	.5717	370	32
29	705	666	.5707	355	31
30	.53730	.63707	1.5697	.84339	30
31	754	748	.5687	324	29
32	779	789	.5677	308	28
33	804	830	.5667	292	27
34	828	871	.5657	277	26
35	.53853	.63912	1.5647	.84261	25
36	877	953	.5637	245	24
37	902	.63994	.5627	230	23
38	926	.64035	.5617	214	22
39	951	076	.5607	198	21
40	.53975	.64117	1.5597	.84182	20
41	.54000	158	.5587	167	19
42	024	199	.5577	151	18
43	049	240	.5567	135	17
44	073	281	.5557	120	16
45	.54097	.64322	1.5547	.84104	15
46	122	363	.5537	088	14
47	146	404	.5527	072	13
48	171	446	.5517	057	12
49	195	487	.5507	041	11
50	.54220	.64528	1.5497	.84025	10
51	244	569	.5487	.84009	9
52	269	610	.5477	.83994	8
53	293	652	.5468	978	7
54	317	693	.5458	962	6
55	.54342	.64734	1.5448	.83946	5
56	366	775	.5438	930	4
57	391	817	.5428	915	3
58	415	858	.5418	899	2
59	440	899	.5408	883	1
60	.54464	.64941	1.5399	.83867	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.54464	.64941	1.5399	.83867	60
1	488	.64982	.5389	851	59
2	513	.65024	.5379	835	58
3	537	065	.5369	819	57
4	561	106	.5359	804	56
5	.54586	.65148	1.5350	.83788	55
6	610	189	.5340	772	54
7	635	231	.5330	756	53
8	659	272	.5320	740	52
9	683	314	.5311	724	51
10	.54708	.65355	1.5301	.83708	50
11	732	397	.5291	692	49
12	756	438	.5282	676	48
13	781	480	.5272	660	47
14	805	521	.5262	645	46
15	.54829	.65563	1.5253	.83629	45
16	854	604	.5243	613	44
17	878	646	.5233	597	43
18	902	688	.5224	581	42
19	927	729	.5214	565	41
20	.54951	.65771	1.5204	.83549	40
21	975	813	.5195	533	39
22	.54999	854	.5185	517	38
23	.55024	896	.5175	501	37
24	048	938	.5166	485	36
25	.55072	.65980	1.5156	.83469	35
26	097	.66021	.5147	453	34
27	121	063	.5137	437	33
28	145	105	.5127	421	32
29	169	147	.5118	405	31
30	.55194	.66189	1.5108	.83389	30
31	218	230	.5099	373	29
32	242	272	.5089	356	28
33	266	314	.5080	340	27
34	291	356	.5070	324	26
35	.55315	.66398	1.5061	.83308	25
36	339	440	.5051	292	24
37	363	482	.5042	276	23
38	388	524	.5032	260	22
39	412	566	.5023	244	21
40	.55436	.66608	1.5013	.83228	20
41	460	650	.5004	212	19
42	484	692	.4994	195	18
43	509	734	.4985	179	17
44	533	776	.4975	163	16
45	.55557	.66818	1.4966	.83147	15
46	581	860	.4957	131	14
47	605	902	.4947	115	13
48	630	944	.4938	098	12
49	654	.66986	.4928	082	11
50	.55678	.67028	1.4919	.83066	10
51	702	071	.4910	050	9
52	726	113	.4900	034	8
53	750	155	.4891	017	7
54	775	197	.4882	.83001	6
55	.55799	.67239	1.4872	.82985	5
56	823	282	.4863	969	4
57	847	324	.4854	953	3
58	871	366	.4844	936	2
59	895	409	.4835	920	1
60	.55919	.67451	1.4826	.82904	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.55919	.67451	1.4826	.82904	60
1	943	493	.4816	887	59
2	968	536	.4807	871	58
3	.55992	578	.4798	855	57
4	.56016	620	.4788	839	56
5	.56040	.67663	1.4779	.82822	55
6	064	705	.4770	806	54
7	088	748	.4761	790	53
8	112	790	.4751	773	52
9	136	832	.4742	757	51
10	.56160	.67875	1.4733	.82741	50
11	184	917	.4724	724	49
12	208	.67960	.4715	708	48
13	232	.68002	.4705	692	47
14	256	045	.4696	675	46
15	.56280	.68088	1.4687	.82659	45
16	305	130	.4678	643	44
17	329	173	.4669	626	43
18	353	215	.4659	610	42
19	377	258	.4650	593	41
20	.56401	.68301	1.4641	.82577	40
21	425	343	.4632	561	39
22	449	386	.4623	544	38
23	473	429	.4614	528	37
24	497	471	.4605	511	36
25	.56521	.68514	1.4596	.82495	35
26	545	557	.4586	478	34
27	569	600	.4577	462	33
28	593	642	.4568	446	32
29	617	685	.4559	429	31
30	.56641	.68728	1.4550	.82413	30
31	665	771	.4541	396	29
32	689	814	.4532	380	28
33	713	857	.4523	363	27
34	736	900	.4514	347	26
35	.56760	.68942	1.4505	.82330	25
36	784	.68985	.4496	314	24
37	808	.69028	.4487	297	23
38	832	071	.4478	281	22
39	856	114	.4469	264	21
40	.56880	.69157	1.4460	.82248	20
41	904	200	.4451	231	19
42	928	243	.4442	214	18
43	952	286	.4433	198	17
44	.56976	329	.4424	181	16
45	.57000	.69372	1.4415	.82165	15
46	024	416	.4406	148	14
47	047	459	.4397	132	13
48	071	502	.4388	115	12
49	095	545	.4379	098	11
50	.57119	.69588	1.4370	.82082	10
51	143	631	.4361	065	9
52	167	675	.4352	048	8
53	191	718	.4344	032	7
54	215	761	.4335	.82015	6
55	.57238	.69804	1.4326	.81999	5
56	262	847	.4317	982	4
57	286	891	.4308	965	3
58	310	934	.4299	949	2
59	334	.69977	.4290	932	1
60	.57358	.70021	1.4281	.81915	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.57358	.70021	1.4281	.81915	60
1	381	064	.4273	899	59
2	405	107	.4264	882	58
3	429	151	.4255	865	57
4	453	194	.4246	848	56
5	.57477	.70238	1.4237	.81832	55
6	501	281	.4229	815	54
7	524	325	.4220	798	53
8	548	368	.4211	782	52
9	572	412	.4202	765	51
10	.57596	.70455	1.4193	.81748	50
11	619	499	.4185	731	49
12	643	542	.4176	714	48
13	667	586	.4167	698	47
14	691	629	.4158	681	46
15	.57715	.70673	1.4150	.81664	45
16	738	717	.4141	647	44
17	762	760	.4132	631	43
18	786	804	.4124	614	42
19	810	848	.4115	597	41
20	.57833	.70891	1.4106	.81580	40
21	857	935	.4097	563	39
22	881	.70979	.4089	546	38
23	904	.71023	.4080	530	37
24	928	066	.4071	513	36
25	.57952	.71110	1.4063	.81496	35
26	976	154	.4054	479	34
27	.57999	198	.4045	462	33
28	.58023	242	.4037	445	32
29	047	285	.4028	428	31
30	.58070	.71329	1.4019	.81412	30
31	094	373	.4011	395	29
32	118	417	.4002	378	28
33	141	461	.3994	361	27
34	165	505	.3985	344	26
35	.58189	.71549	1.3976	.81327	25
36	212	593	.3968	310	24
37	236	637	.3959	293	23
38	260	681	.3951	276	22
39	283	725	.3942	259	21
40	.58307	.71769	1.3934	.81242	20
41	330	813	.3925	225	19
42	354	857	.3916	208	18
43	378	901	.3908	191	17
44	401	946	.3899	174	16
45	.58425	.71990	1.3891	.81157	15
46	449	.72034	.3882	140	14
47	472	078	.3874	123	13
48	496	122	.3865	106	12
49	519	167	.3857	089	11
50	.58543	.72211	1.3848	.81072	10
51	567	255	.3840	055	9
52	590	299	.3831	038	8
53	614	344	.3823	021	7
54	637	388	.3814	.81004	6
55	.58661	.72432	1.3806	80987	5
56	684	477	.3798	970	4
57	708	521	.3789	953	3
58	731	565	.3781	936	2
59	755	610	.3772	919	1
60	.58779	.72654	1.3764	.80902	0
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.58779	.72654	1.3764	.80902	**60**
1	802	699	.3755	885	59
2	826	743	.3747	867	58
3	849	788	.3739	850	57
4	873	832	.3730	833	56
5	.58896	.72877	1.3722	.80816	**55**
6	920	921	.3713	799	54
7	943	.72966	.3705	782	53
8	967	.73010	.3697	765	52
9	.58990	055	.3688	748	51
10	.59014	.73100	1.3680	.80730	**50**
11	037	144	.3672	713	49
12	061	189	.3663	696	48
13	084	234	.3655	679	47
14	108	278	.3647	662	46
15	.59131	.73323	1.3638	.80644	**45**
16	154	368	.3630	627	44
17	178	413	.3622	610	43
18	201	457	.3613	593	42
19	225	502	.3605	576	41
20	.59248	.73547	1.3597	.80558	**40**
21	272	592	.3588	541	39
22	295	637	.3580	524	38
23	318	681	.3572	507	37
24	342	726	.3564	489	36
25	.59365	.73771	1.3555	.80472	**35**
26	389	816	.3547	455	34
27	412	861	.3539	438	33
28	436	906	.3531	420	32
29	459	951	.3522	403	31
30	.59482	.73996	1.3514	.80386	**30**
31	506	74041	.3506	368	29
32	529	086	.3498	351	28
33	552	131	.3490	334	27
34	576	176	.3481	316	26
35	.59599	.74221	1.3473	.80299	**25**
36	622	267	.3465	282	24
37	646	312	.3457	264	23
38	669	357	.3449	247	22
39	693	402	.3440	230	21
40	.59716	.74447	1.3432	.80212	**20**
41	739	492	.3424	195	19
42	763	538	.3416	178	18
43	786	583	.3408	160	17
44	809	628	.3400	143	16
45	.59832	.74674	1.3392	.80125	**15**
46	856	719	.3384	108	14
47	879	764	.3375	091	13
48	902	810	.3367	073	12
49	926	855	.3359	056	11
50	59949	.74900	1.3351	.80038	**10**
51	972	946	.3343	021	9
52	.59995	.74991	.3335	.80003	8
53	.60019	.75037	.3327	.79986	7
54	042	082	.3319	968	6
55	.60065	.75128	1.3311	.79951	**5**
56	089	173	.3303	934	4
57	112	219	.3295	916	3
58	135	264	.3287	899	2
59	158	310	.3278	881	1
60	.60182	.75355	1.3270	.79864	**0**
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.60182	.75355	1.3270	.79864	**60**
1	205	401	.3262	846	59
2	228	447	.3254	829	58
3	251	492	.3246	811	57
4	274	538	.3238	793	56
5	.60298	.75584	1.3230	.79776	**55**
6	321	629	.3222	758	54
7	344	675	.3214	741	53
8	367	721	.3206	723	52
9	390	767	.3198	706	51
10	.60414	.75812	1.3190	.79688	**50**
11	437	858	.3182	671	49
12	460	904	.3175	653	48
13	483	950	.3167	635	47
14	506	.75996	.3159	618	46
15	.60529	.76042	1.3151	.79600	**45**
16	553	088	.3143	583	44
17	576	134	.3135	565	43
18	599	180	.3127	547	42
19	622	226	.3119	530	41
20	.60645	.76272	1.3111	.79512	**40**
21	668	318	.3103	494	39
22	691	364	.3095	477	38
23	714	410	.3087	459	37
24	738	456	.3079	441	36
25	.60761	.76502	1.3072	.79424	**35**
26	784	548	.3064	406	34
27	807	594	.3056	388	33
28	830	640	.3048	371	32
29	853	686	.3040	353	31
30	.60876	.76733	1.3032	.79335	**30**
31	899	779	.3024	318	29
32	922	825	.3017	300	28
33	945	871	.3009	282	27
34	968	918	.3001	264	26
35	.60991	.76964	1.2993	.79247	**25**
36	.61015	.77010	.2985	229	24
37	038	057	.2977	211	23
38	061	103	.2970	193	22
39	084	149	.2962	176	21
40	.61107	.77196	1.2954	.79158	**20**
41	130	242	.2946	140	19
42	153	289	.2938	122	18
43	176	335	.2931	105	17
44	199	382	.2923	087	16
45	.61222	.77428	1.2915	.79069	**15**
46	245	475	.2907	051	14
47	268	521	.2900	033	13
48	291	568	.2892	.79016	12
49	314	615	.2884	.78998	11
50	.61337	.77661	1.2876	.78980	**10**
51	360	708	.2869	962	9
52	383	754	.2861	944	8
53	406	801	.2853	926	7
54	429	848	.2846	908	6
55	.61451	.77895	1.2838	.78891	**5**
56	474	941	.2830	873	4
57	497	.77988	.2822	855	3
58	520	.78035	.2815	837	2
59	543	082	.2807	819	1
60	.61566	.78129	1.2799	.78801	**0**
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.61566	.78129	1.2799	.78801	**60**
1	589	175	.2792	783	59
2	612	222	.2784	765	58
3	635	269	.2776	747	57
4	658	316	.2769	729	56
5	.61681	.78363	1.2761	.78711	**55**
6	704	410	.2753	694	54
7	726	457	.2746	676	53
8	749	504	.2738	658	52
9	772	551	.2731	640	51
10	.61795	.78598	1.2723	.78622	**50**
11	818	645	.2715	604	49
12	841	692	.2708	586	48
13	864	739	.2700	568	47
14	887	786	.2693	550	46
15	.61909	.78834	1.2685	.78532	**45**
16	932	881	.2677	514	44
17	955	928	.2670	496	43
18	.61978	.78975	.2662	478	42
19	.62001	.79022	.2655	460	41
20	.62024	.79070	1.2647	.78442	**40**
21	046	117	.2640	424	39
22	069	164	.2632	405	38
23	092	212	.2624	387	37
24	115	259	.2617	369	36
25	.62138	.79306	1.2609	.78351	**35**
26	160	354	.2602	333	34
27	183	401	.2594	315	33
28	206	449	.2587	297	32
29	229	496	.2579	279	31
30	.62251	.79544	1.2572	.78261	**30**
31	274	591	.2564	243	29
32	297	639	.2557	225	28
33	320	686	.2549	206	27
34	342	734	.2542	188	26
35	.62365	.79781	1.2534	.78170	**25**
36	388	829	.2527	152	24
37	411	877	.2519	134	23
38	433	924	.2512	116	22
39	456	.79972	.2504	098	21
40	.62479	.80020	1.2497	.78079	**20**
41	502	067	.2489	061	19
42	524	115	.2482	043	18
43	547	163	.2475	025	17
44	570	211	.2467	.78007	16
45	.62592	.80258	1.2460	.77988	**15**
46	615	306	.2452	970	14
47	638	354	.2445	952	13
48	660	402	.2437	934	12
49	683	450	.2430	916	11
50	.62706	.80498	1.2423	.77897	**10**
51	728	546	.2415	879	9
52	751	594	.2408	861	8
53	774	642	.2401	843	7
54	796	690	.2393	824	6
55	.62819	.80738	1.2386	.77806	**5**
56	842	786	.2378	788	4
57	864	834	.2371	769	3
58	887	882	.2364	751	2
59	909	930	.2356	733	1
60	62932	.80978	1.2349	.77715	**0**
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.62932	.80978	1.2349	.77715	**60**
1	955	.81027	.2342	696	59
2	.62977	075	.2334	678	58
3	.63000	123	.2327	660	57
4	022	171	.2320	641	56
5	.63045	.81220	1.2312	.77623	**55**
6	068	268	.2305	605	54
7	090	316	.2298	586	53
8	113	364	.2290	568	52
9	135	413	.2283	550	51
10	.63158	.81461	1.2276	.77531	**50**
11	180	510	.2268	513	49
12	203	558	.2261	494	48
13	225	606	.2254	476	47
14	248	655	.2247	458	46
15	.63271	.81703	1.2239	.77439	**45**
16	293	752	.2232	421	44
17	316	800	.2225	402	43
18	338	849	.2218	384	42
19	361	898	.2210	366	41
20	.63383	.81946	1.2203	.77347	**40**
21	406	.81995	.2196	329	39
22	428	.82044	.2189	310	38
23	451	092	.2181	292	37
24	473	141	.2174	273	36
25	.63496	.82190	1.2167	.77255	**35**
26	518	238	.2160	236	34
27	540	287	.2153	218	33
28	563	336	.2145	199	32
29	585	385	.2138	181	31
30	.63608	.82434	1.2131	.77162	**30**
31	630	483	.2124	144	29
32	653	531	.2117	125	28
33	675	580	.2109	107	27
34	698	629	.2102	088	26
35	.63720	.82678	1.2095	.77070	**25**
36	742	727	.2088	051	24
37	765	776	.2081	033	23
38	787	825	.2074	.77014	22
39	810	874	.2066	.76996	21
40	.63832	.82923	1.2059	.76977	**20**
41	854	.82972	.2052	959	19
42	877	.83022	.2045	940	18
43	899	071	.2038	921	17
44	922	120	.2031	903	16
45	.63944	.83169	1.2024	.76884	**15**
46	966	218	.2017	866	14
47	.63989	268	.2009	847	13
48	.64011	317	.2002	828	12
49	033	366	.1995	810	11
50	.64056	.83415	1.1988	.76791	**10**
51	078	465	.1981	772	9
52	100	514	.1974	754	8
53	123	564	.1967	735	7
54	145	613	.1960	717	6
55	.64167	.83662	1.1953	.76698	**5**
56	190	712	.1946	679	4
57	212	761	.1939	661	3
58	234	811	.1932	642	2
59	256	860	.1925	623	1
60	.64279	.83910	1.1918	.76604	**0**
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.64279	.83910	1.1918	.76604	**60**
1	301	.83960	.1910	586	59
2	323	.84009	.1903	567	58
3	346	059	.1896	548	57
4	368	108	.1889	530	56
5	.64390	.84158	1.1882	.76511	**55**
6	412	208	.1875	492	54
7	435	258	.1868	473	53
8	457	307	.1861	455	52
9	479	357	.1854	436	51
10	.64501	.84407	1.1847	.76417	**50**
11	524	457	.1840	398	49
12	546	507	.1833	380	48
13	568	556	.1826	361	47
14	590	606	.1819	342	46
15	.64612	.84656	1.1812	.76323	**45**
16	635	706	.1806	304	44
17	657	756	.1799	286	43
18	679	806	.1792	267	42
19	701	856	.1785	248	41
20	.64723	.84906	1.1778	.76229	**40**
21	746	.84956	.1771	210	39
22	768	.85006	.1764	192	38
23	790	057	.1757	173	37
24	812	107	.1750	154	36
25	.64834	.85157	1.1743	.76135	**35**
26	856	207	.1736	116	34
27	878	257	.1729	097	33
28	901	308	.1722	078	32
29	923	358	.1715	059	31
30	.64945	.85408	1.1708	.76041	**30**
31	967	458	.1702	022	29
32	.64989	509	.1695	.76003	28
33	.65011	559	.1688	.75984	27
34	033	609	.1681	965	26
35	.65055	.85660	1.1674	.75946	**25**
36	077	710	.1667	927	24
37	100	761	.1660	908	23
38	122	811	.1653	889	22
39	144	862	.1647	870	21
40	.65166	.85912	1.1640	.75851	**20**
41	188	.85963	.1633	832	19
42	210	.86014	.1626	813	18
43	232	064	.1619	794	17
44	254	115	.1612	775	16
45	.65276	.86166	1.1606	.75756	**15**
46	298	216	.1599	738	14
47	320	267	.1592	719	13
48	342	318	.1585	700	12
49	364	368	.1578	680	11
50	.65386	.86419	1.1571	.75661	**10**
51	408	470	.1565	642	9
52	430	521	.1558	623	8
53	452	572	.1551	604	7
54	474	623	.1544	585	6
55	.65496	.86674	1.1538	.75566	**5**
56	518	725	.1531	547	4
57	540	776	.1524	528	3
58	562	827	.1517	509	2
59	584	878	.1510	490	1
60	.65606	.86929	1.1504	.75471	**0**
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.65606	.86929	1.1504	.75471	**60**
1	628	.86980	.1497	452	59
2	650	.87031	.1490	433	58
3	672	082	.1483	414	57
4	694	133	.1477	395	56
5	.65716	.87184	1.1470	.75375	**55**
6	738	236	.1463	356	54
7	759	287	.1456	337	53
8	781	338	.1450	318	52
9	803	389	.1443	299	51
10	.65825	.87441	1.1436	.75280	**50**
11	847	492	.1430	261	49
12	869	543	.1423	241	48
13	891	595	.1416	222	47
14	913	646	.1410	203	46
15	.65935	.87698	1.1403	.75184	**45**
16	956	749	.1396	165	44
17	.65978	801	.1389	146	43
18	.66000	852	.1383	126	42
19	022	904	.1376	107	41
20	.66044	.87955	1.1369	.75088	**40**
21	066	.88007	.1363	069	39
22	088	059	.1356	050	38
23	109	110	.1349	030	37
24	131	162	.1343	.75011	36
25	.66153	.88214	1.1336	.74992	**35**
26	175	265	.1329	973	34
27	197	317	.1323	953	33
28	218	369	.1316	934	32
29	240	421	.1310	915	31
30	.66262	.88473	1.1303	.74896	**30**
31	284	524	.1296	876	29
32	306	576	.1290	857	28
33	327	628	.1283	838	27
34	349	680	.1276	818	26
35	.66371	.88732	1.1270	.74799	**25**
36	393	784	.1263	780	24
37	414	836	.1257	760	23
38	436	888	.1250	741	22
39	458	940	.1243	722	21
40	.66480	.88992	1.1237	.74703	**20**
41	501	.89045	.1230	683	19
42	523	097	.1224	664	18
43	545	149	.1217	644	17
44	566	201	.1211	625	16
45	.66588	.89253	1.1204	.74606	**15**
46	610	306	.1197	586	14
47	632	358	.1191	567	13
48	653	410	.1184	548	12
49	675	463	.1178	528	11
50	.66697	.89515	1.1171	.74509	**10**
51	718	567	.1165	489	9
52	740	620	.1158	470	8
53	762	672	.1152	451	7
54	783	725	.1145	431	6
55	.66805	.89777	1.1139	.74412	**5**
56	827	830	.1132	392	4
57	848	883	.1126	373	3
58	870	935	.1119	353	2
59	891	.89988	.1113	334	1
60	.66913	.90040	1.1106	.74314	**0**
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.66913	.90040	1.1106	.74314	**60**
1	935	093	.1100	295	59
2	956	146	.1093	276	58
3	978	199	.1087	256	57
4	.66999	251	.1080	237	56
5	.67021	.90304	1.1074	.74217	**55**
6	043	357	.1067	198	54
7	064	410	.1061	178	53
8	086	463	.1054	159	52
9	107	516	.1048	139	51
10	.67129	.90569	1.1041	.74120	**50**
11	151	621	.1035	100	49
12	172	674	.1028	080	48
13	194	727	.1022	061	47
14	215	781	.1016	041	46
15	.67237	.90834	1.1009	.74022	**45**
16	258	887	.1003	.74002	44
17	280	940	.0996	.73983	43
18	301	.90993	.0990	963	42
19	323	.91046	.0983	944	41
20	.67344	.91099	1.0977	.73924	**40**
21	366	153	.0971	904	39
22	387	206	.0964	885	38
23	409	259	.0958	865	37
24	430	313	.0951	846	36
25	.67452	.91366	1.0945	.73826	**35**
26	473	419	.0939	806	34
27	495	473	.0932	787	33
28	516	526	.0926	767	32
29	538	580	.0919	747	31
30	.67559	.91633	1.0913	.73728	**30**
31	580	687	.0907	708	29
32	602	740	.0900	688	28
33	623	794	.0894	669	27
34	645	847	.0888	649	26
35	.67666	.91901	1.0881	.73629	**25**
36	688	.91955	.0875	610	24
37	709	.92008	.0869	590	23
38	730	062	.0862	570	22
39	752	116	.0856	551	21
40	.67773	.92170	1.0850	.73531	**20**
41	795	224	.0843	511	19
42	816	277	.0837	491	18
43	837	331	.0831	472	17
44	859	385	.0824	452	16
45	.67880	.92439	1.0818	.73432	**15**
46	901	493	.0812	413	14
47	923	547	.0805	393	13
48	944	601	.0799	373	12
49	965	655	.0793	353	11
50	.67987	.92709	1.0786	.73333	**10**
51	.68008	763	.0780	314	9
52	029	817	.0774	294	8
53	051	872	.0768	274	7
54	072	926	.0761	254	6
55	.68093	.92980	1.0755	.73234	**5**
56	115	.93034	.0749	215	4
57	136	088	.0742	195	3
58	157	143	.0736	175	2
59	179	197	.0730	155	1
60	.68200	.93252	1.0724	.73135	**0**
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.68200	.93252	1.0724	.73135	**60**
1	221	306	.0717	116	59
2	242	360	.0711	096	58
3	264	415	.0705	076	57
4	285	469	.0699	056	56
5	.68306	.93524	1.0692	.73036	**55**
6	327	578	.0686	.73016	54
7	349	633	.0680	.72996	53
8	370	688	.0674	976	52
9	391	742	.0668	957	51
10	.68412	.93797	1.0661	.72937	**50**
11	434	852	.0655	917	49
12	455	906	.0649	897	48
13	476	.93961	.0643	877	47
14	497	.94016	.0637	857	46
15	.68518	.94071	1.0630	.72837	**45**
16	539	125	.0624	817	44
17	561	180	.0618	797	43
18	582	235	.0612	777	42
19	603	290	.0606	757	41
20	.68624	.94345	1.0599	.72737	**40**
21	645	400	.0593	717	39
22	666	455	.0587	697	38
23	688	510	.0581	677	37
24	709	565	.0575	657	36
25	.68730	.94620	1.0569	.72637	**35**
26	751	676	.0562	617	34
27	772	731	.0556	597	33
28	793	786	.0550	577	32
29	814	841	.0544	557	31
30	.68835	.94896	1.0538	.72537	**30**
31	857	.94952	.0532	517	29
32	878	.95007	.0526	497	28
33	899	062	.0519	477	27
34	920	118	.0513	457	26
35	.68941	.95173	1.0507	.72437	**25**
36	962	229	.0501	417	24
37	.68983	284	.0495	397	23
38	.69004	340	.0489	377	22
39	025	395	.0483	357	21
40	.69046	.95451	1.0477	.72337	**20**
41	067	506	.0470	317	19
42	088	562	.0464	297	18
43	109	618	.0458	277	17
44	130	673	.0452	257	16
45	.69151	.95729	1.0446	.72236	**15**
46	172	785	.0440	216	14
47	193	841	.0434	196	13
48	214	897	.0428	176	12
49	235	.95952	.0422	156	11
50	.69256	.96008	1.0416	.72136	**10**
51	277	064	.0410	116	9
52	298	120	.0404	095	8
53	319	176	.0398	075	7
54	340	232	.0392	055	6
55	.69361	.96288	1.0385	.72035	**5**
56	382	344	.0379	.72015	4
57	403	400	.0373	.71995	3
58	424	457	.0367	974	2
59	445	513	.0361	954	1
60	.69466	.96569	1.0355	.71934	**0**
	cos	cot	tan	sin	′

′	sin	tan	cot	cos	
0	.69466	.96569	1.0355	.71934	**60**
1	487	625	.0349	914	59
2	508	681	.0343	894	58
3	529	738	.0337	873	57
4	549	794	.0331	853	56
5	.69570	.96850	1.0325	.71833	**55**
6	591	907	.0319	813	54
7	612	.96963	.0313	792	53
8	633	.97020	.0307	772	52
9	654	076	.0301	752	51
10	.69675	.97133	1.0295	.71732	**50**
11	696	189	.0289	711	49
12	717	246	.0283	691	48
13	737	302	.0277	671	47
14	758	359	.0271	650	46
15	.69779	.97416	1.0265	.71630	**45**
16	800	472	.0259	610	44
17	821	529	.0253	590	43
18	842	586	.0247	569	42
19	862	643	.0241	549	41
20	.69883	.97700	1.0235	.71529	**40**
21	904	756	.0230	508	39
22	925	813	.0224	488	38
23	946	870	.0218	468	37
24	966	927	.0212	447	36
25	.69987	.97984	1.0206	.71427	**35**
26	.70008	.98041	.0200	407	34
27	029	098	.0194	386	33
28	049	155	.0188	366	32
29	070	213	.0182	345	31
30	.70091	.98270	1.0176	.71325	**30**
31	112	327	.0170	305	29
32	132	384	.0164	284	28
33	153	441	.0158	264	27
34	174	499	.0152	243	26
35	.70195	.98556	1.0147	.71223	**25**
36	215	613	.0141	203	24
37	236	671	.0135	182	23
38	257	728	.0129	162	22
39	277	786	.0123	141	21
40	.70298	.98843	1.0117	.71121	**20**
41	319	901	.0111	100	19
42	339	.98958	.0105	080	18
43	360	.99016	.0099	059	17
44	381	073	.0094	039	16
45	.70401	.99131	1.0088	.71019	**15**
46	422	189	.0082	.70998	14
47	443	247	.0076	978	13
48	463	304	.0070	957	12
49	484	362	.0064	937	11
50	.70505	.99420	1.0058	.70916	**10**
51	525	478	.0052	896	9
52	546	536	.0047	875	8
53	567	594	.0041	855	7
54	587	652	.0035	834	6
55	.70608	.99710	1.0029	.70813	**5**
56	628	768	.0023	793	4
57	649	826	.0017	772	3
58	670	884	.0012	752	2
59	690	.99942	.0006	731	1
60	.70711	1.0000	1.0000	.70711	**0**
	cos	cot	tan	sin	′

TABLE V. RADIAN MEASURE, 0° TO 180°, RADIUS = 1.

Degrees						Minutes		Seconds	
0°	0.00000 00	**60°**	1.04719 76	**120°**	2.09439 51	**0′**	0.00000 00	**0″**	0.00000 00
1	0.01745 33	61	1.06465 08	121	2.11184 84	1	0.00029 09	1	0.00000 48
2	0.03490 66	62	1.08210 41	122	2.12930 17	2	0.00058 18	2	0.00000 97
3	0.05235 99	63	1.09955 74	123	2.14675 50	3	0.00087 27	3	0.00001 45
4	0.06981 32	64	1.11701 07	124	2.16420 83	4	0.00116 36	4	0.00001 94
5	0.08726 65	**65**	1.13446 40	**125**	2.18166 16	**5**	0.00145 44	**5**	0.00002 42
6	0.10471 98	66	1.15191 73	126	2.19911 49	6	0.00174 53	6	0.00002 91
7	0.12217 30	67	1.16937 06	127	2.21656 82	7	0.00203 62	7	0.00003 39
8	0.13962 63	68	1.18682 39	128	2.23402 14	8	0.00232 71	8	0.00003 88
9	0.15707 96	69	1.20427 72	129	2.25147 47	9	0.00261 80	9	0.00004 36
10	0.17453 29	**70**	1.22173 05	**130**	2.26892 80	**10**	0.00290 89	**10**	0.00004 85
11	0.19198 62	71	1.23918 38	131	2.28638 13	11	0.00319 98	11	0.00005 33
12	0.20943 95	72	1.25663 71	132	2.30383 46	12	0.00349 07	12	0.00005 82
13	0.22689 28	73	1.27409 04	133	2.32128 79	13	0.00378 15	13	0.00006 30
14	0.24434 61	74	1.29154 36	134	2.33874 12	14	0.00407 24	14	0.00006 79
15	0.26179 94	**75**	1.30899 69	**135**	2.35619 45	**15**	0.00436 33	**15**	0.00007 27
16	0.27925 27	76	1.32645 02	136	2.37364 78	16	0.00465 42	16	0.00007 76
17	0.29670 60	77	1.34390 35	137	2.39110 11	17	0.00494 51	17	0.00008 24
18	0.31415 93	78	1.36135 68	138	2.40855 44	18	0.00523 60	18	0.00008 73
19	0.33161 26	79	1.37881 01	139	2.42600 77	19	0.00552 69	19	0.00009 21
20	0.34906 59	**80**	1.39626 34	**140**	2.44346 10	**20**	0.00581 78	**20**	0.00009 70
21	0.36651 91	81	1.41371 67	141	2.46091 42	21	0.00610 87	21	0.00010 18
22	0.38397 24	82	1.43117 00	142	2.47836 75	22	0.00639 95	22	0.00010 67
23	0.40142 57	83	1.44862 33	143	2.49582 08	23	0.00669 04	23	0.00011 15
24	0.41887 90	84	1.46607 66	144	2.51327 41	24	0.00698 13	24	0.00011 64
25	0.43633 23	85	1.48352 99	**145**	2.53072 74	**25**	0.00727 22	**25**	0.00012 12
26	0.45378 56	86	1.50098 32	146	2.54818 07	26	0.00756 31	26	0.00012 61
27	0.47123 89	87	1.51843 64	147	2.56563 40	27	0.00785 40	27	0.00013 09
28	0.48869 22	88	1.53588 97	148	2.58308 73	28	0.00814 49	28	0.00013 57
29	0.50614 55	89	1.55334 30	149	2.60054 06	29	0.00843 58	29	0.00014 06
30	0.52359 88	**90**	1.57079 63	**150**	2.61799 39	**30**	0.00872 66	**30**	0.00014 54
31	0.54105 21	91	1.58824 96	151	2.63544 72	31	0.00901 75	31	0.00015 03
32	0.55850 54	92	1.60570 29	152	2.65290 05	32	0.00930 84	32	0.00015 51
33	0.57595 87	93	1.62315 62	153	2.67035 38	33	0.00959 93	33	0.00016 00
34	0.59341 19	94	1.64060 95	154	2.68780 70	34	0.00989 02	34	0.00016 48
35	0.61086 52	**95**	1.65806 28	**155**	2.70526 03	**35**	0.01018 11	**35**	0.00016 97
36	0.62831 85	96	1.67551 61	156	2.72271 36	36	0.01047 20	36	0.00017 45
37	0.64577 18	97	1.69296 94	157	2.74016 69	37	0.01076 29	37	0.00017 94
38	0.66322 51	98	1.71042 27	158	2.75762 02	38	0.01105 38	38	0.00018 42
39	0.68067 84	99	1.72787 60	159	2.77507 35	39	0.01134 46	39	0.00018 91
40	0.69813 17	**100**	1.74532 93	**160**	2.79252 68	**40**	0.01163 55	**40**	0.00019 39
41	0.71558 50	101	1.76278 25	161	2.80998 01	41	0.01192 64	41	0.00019 88
42	0.73303 83	102	1.78023 58	162	2.82743 34	42	0.01221 73	42	0.00020 36
43	0.75049 16	103	1.79768 91	163	2.84488 67	43	0.01250 82	43	0.00020 85
44	0.76794 49	104	1.81514 24	164	2.86234 00	44	0.01279 91	44	0.00021 33
45	0.78539 82	**105**	1.83259 57	**165**	2.87979 33	**45**	0.01309 00	**45**	0.00021 82
46	0.80285 15	106	1.85004 90	166	2.89724 66	46	0.01338 09	46	0.00022 30
47	0.82030 47	107	1.86750 23	167	2.91469 99	47	0.01367 17	47	0.00022 79
48	0.83775 80	108	1.88495 56	168	2.93215 31	48	0.01396 26	48	0.00023 27
49	0.85521 13	109	1.90240 89	169	2.94960 64	49	0.01425 35	49	0.00023 76
50	0.87266 46	**110**	1.91986 22	**170**	2.96705 97	**50**	0.01454 44	**50**	0.00024 24
51	0.89011 79	111	1.93731 55	171	2.98451 30	51	0.01483 53	51	0.00024 73
52	0.90757 12	112	1.95476 88	172	3.00196 63	52	0.01512 62	52	0.00025 21
53	0.92502 45	113	1.97222 21	173	3.01941 96	53	0.01541 71	53	0.00025 70
54	0.94247 78	114	1.98967 53	174	3.03687 29	54	0.01570 80	54	0.00026 18
55	0.95993 11	**115**	2.00712 86	**175**	3.05432 62	**55**	0.01599 89	**55**	0.00026 66
56	0.97738 44	116	2.02458 19	176	3.07177 95	56	0.01628 97	56	0.00027 15
57	0.99483 77	117	2.04203 52	177	3.08923 28	57	0.01658 06	57	0.00027 63
58	1.01229 10	118	2.05948 85	178	3.10668 61	58	0.01687 15	58	0.00028 12
59	1.02974 43	119	2.07694 18	179	3.12413 94	59	0.01716 24	59	0.00028 60
60	1.04719 76	**120**	2.09439 51	**180**	3.14159 27	**60**	0.01745 33	**60**	0.00029 09
Degrees						Minutes		Seconds	

TABLE VI. CONSTANTS AND THEIR LOGARITHMS.

		Logarithm
Circumference of a circle in degrees.........	= 360	2.556 3025
Circumference of a circle in minutes.........	= 21,600	4.334 4538
Circumference of a circle in seconds.........	= 1,296,000	6.112 6050
Number of radians in one degree............	= 0.017 4533	8.241 8774 − 10
Number of radians in one minute............	= 0.000 2909	6.463 7261 − 10
Number of radians in one second............	= 0.000 0048	4.685 5749 − 10
Number of degrees in one radian............	= 57.2957795	1.758 1226
Number of minutes in one radian............	= 3437.7468	3.536 2739
Number of seconds in one radian............	= 206,264.806	5.314 4251
$\pi = 3.141\,592653\,589793$....................................		0.497 1499

Also:	Logarithm		Logarithm
		$\sqrt{\pi} = 1.772\,4539$	0.248 5749
$2\pi = 6.283\,1853$	0.798 1799	$\frac{1}{\sqrt{\pi}} = 0.564\,1896$	9.751 4251 − 10
$4\pi = 12.566\,3706$	1.099 2099	$M = 0.434\,2945$	9.637 7843 − 10
$\frac{1}{2}\pi = 1.570\,7963$	0.196 1199	$\frac{1}{M} = 2.302\,5851$	0.362 2157
$\frac{4}{3}\pi = 4.188\,7902$	0.622 0886	$e = 2.718\,2818$	0.434 2945
$\frac{1}{4}\pi = 0.785\,3982$	9.895 0899 − 10	$\frac{1}{e} = 0.367\,8794$	9.565 7055 − 10
$\frac{1}{6}\pi = 0.523\,5988$	9.718 9986 − 10	$\sqrt{2} = 1.414\,2136$	0.150 5150
$\frac{1}{\pi} = 0.318\,3099$	9.502 8501 − 10	$\sqrt{3} = 1.732\,0508$	0.238 5606
$\pi^2 = 9.869\,6044$	0.994 2997	$\sqrt{5} = 2.236\,0680$	0.349 4850

1 mile per hour = 1.466 667 feet per second.
1 foot per second = 0.681 818 miles per hour.
1 cu. ft. of water weighs 62.5 lb. = 1000 oz. (approximate).
1 gal. of water weighs $8\frac{1}{3}$ lb. (approximate).
1 gal. = 231 cu. in. (by law of Congress).
1 bu. = 2150.42 cu.in. (by law of Congress).
1 bu. = 1.2446 cu. ft. = $\frac{5}{4}$ cu. ft. (approximate).
1 cu. ft. = $7\frac{1}{2}$ gal. (approximate).
1 bbl. = 4.211 cu. ft. (approximate).
1 meter = 39.37 inches (by law of Congress).
1 ft. = 30.4801 cm.
1 kg. = 2.20462 lb.
1 gram = 15.432 grains.
1 lb. (avoirdupois) = 453.592 4277 grams = 0.45359 kg.
1 lb. (avoirdupois) = 7000 grains (by law of Congress).
1 lb. (apothecaries) = 5760 grains (by law of Congress).
1 liter = 1.05668 qt. (liquid) = 0.90808 qt. (dry).
1 qt. (liquid) = 946.358 cc. = 0.946 358 liters, or cu. dm.
1 qt. (dry) = 1101.228 cc. = 1.101 228 liters, or cu. dm.