Topics In Analysis

Kuttler

December 18, 2006

Contents

Part I

Review Of Advanced Calculus

Set Theory

1.1 Basic Definitions

A set is a collection of things called elements of the set. For example, the set of integers, the collection of signed whole numbers such as 1,2,-4, etc. This set whose existence will be assumed is denoted by $\mathbb{Z}$. Other sets could be the set of people in a family or the set of donuts in a display case at the store. Sometimes parentheses, $\{\ \}$ specify a set by listing the things which are in the set between the parentheses. For example the set of integers between -1 and 2, including these numbers could be denoted as $\{-1, 0, 1, 2\}$. The notation signifying x is an element of a set S, is written as $x \in S$. Thus, $1 \in \{-1, 0, 1, 2, 3\}$. Here are some axioms about sets. Axioms are statements which are accepted, not proved.

1. Two sets are equal if and only if they have the same elements.
2. To every set, A, and to every condition $S(x)$ there corresponds a set, B, whose elements are exactly those elements x of A for which $S(x)$ holds.
3. For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.
4. The Cartesian product of a nonempty family of nonempty sets is nonempty.
5. If A is a set there exists a set, $\mathcal{P}(A)$ such that $\mathcal{P}(A)$ is the set of all subsets of A. This is called the power set.

These axioms are referred to as the axiom of extension, axiom of specification, axiom of unions, axiom of choice, and axiom of powers respectively.

It seems fairly clear you should want to believe in the axiom of extension. It is merely saying, for example, that $\{1, 2, 3\} = \{2, 3, 1\}$ since these two sets have the same elements in them. Similarly, it would seem you should be able to specify a new set from a given set using some "condition" which can be used as a test to determine whether the element in question is in the set. For example, the set of all integers which are multiples of 2. This set could be specified as follows.

$$\{x \in \mathbb{Z} : x = 2y \text{ for some } y \in \mathbb{Z}\}.$$

In this notation, the colon is read as "such that" and in this case the condition is being a multiple of 2.

Another example of political interest, could be the set of all judges who are not judicial activists. I think you can see this last is not a very precise condition since there is no way to determine to everyone's satisfaction whether a given judge is an activist. Also, just because something is grammatically correct does not mean it makes any sense. For example consider the following nonsense.

$$S = \{x \in \text{set of dogs } : \text{ it is colder in the mountains than in the winter}\}.$$

So what is a condition?

We will leave these sorts of considerations and assume our conditions make sense. The axiom of unions states that for any collection of sets, there is a set consisting of all the elements in each of the sets in the collection. Of course this is also open to further consideration. What is a collection? Maybe it would be better to say "set of sets" or, given a set whose elements are sets there exists a set whose elements consist of exactly those things which are elements of at least one of these sets. If $\mathcal{S}$ is such a set whose elements are sets,

$$\cup\{A : A \in \mathcal{S}\} \text{ or } \cup\mathcal{S}$$

signify this union.

Something is in the Cartesian product of a set or "family" of sets if it consists of a single thing taken from each set in the family. Thus $(1,2,3) \in \{1,4,.2\} \times \{1,2,7\} \times \{4,3,7,9\}$ because it consists of exactly one element from each of the sets which are separated by $\times$. Also, this is the notation for the Cartesian product of finitely many sets. If $\mathcal{S}$ is a set whose elements are sets,

$$\prod_{A \in \mathcal{S}} A$$

signifies the Cartesian product.

The Cartesian product is the set of choice functions, a choice function being a function which selects exactly one element of each set of $\mathcal{S}$. You may think the axiom of choice, stating that the Cartesian product of a nonempty family of nonempty sets is nonempty, is innocuous but there was a time when many mathematicians were ready to throw it out because it implies things which are very hard to believe, things which never happen without the axiom of choice.

A is a subset of B, written $A \subseteq B$, if every element of A is also an element of B. This can also be written as $B \supseteq A$. A is a proper subset of B, written $A \subset B$ or $B \supset A$ if A is a subset of B but A is not equal to $B, A \neq B$. $A \cap B$ denotes the intersection of the two sets, A and B and it means the set of elements of A which are also elements of B. The axiom of specification shows this is a set. The empty set is the set which has no elements in it, denoted as $\emptyset$. $A \cup B$ denotes the union of the two sets, A and B and it means the set of all elements which are in either of the sets. It is a set because of the axiom of unions.

The complement of a set, (the set of things which are not in the given set) must be taken with respect to a given set called the universal set which is a set which contains the one whose complement is being taken. Thus, the complement of A, denoted as A^C (or more precisely as $X \setminus A$) is a set obtained from using the axiom of specification to write

$$A^C \equiv \{x \in X : x \notin A\}$$

The symbol $\notin$ means: "is not an element of". Note the axiom of specification takes place relative to a given set. Without this universal set it makes no sense to use the axiom of specification to obtain the complement.

Words such as "all" or "there exists" are called quantifiers and they must be understood relative to some given set. For example, the set of all integers larger than 3. Or there exists an integer larger than 7. Such statements have to do with a given set, in this case the integers. Failure to have a reference set when quantifiers are used turns out to be illogical even though such usage may be grammatically correct. Quantifiers are used often enough that there are symbols for them. The symbol $\forall$ is read as "for all" or "for every" and the symbol $\exists$ is read as "there exists". Thus $\forall\forall\exists\exists$ could mean for every upside down A there exists a backwards E.

DeMorgan's laws are very useful in mathematics. Let $\mathcal{S}$ be a set of sets each of which is contained in some universal set, U. Then

$$\cup \left\{A^C : A \in \mathcal{S}\right\} = \left(\cap \{A : A \in \mathcal{S}\}\right)^C$$

and

$$\cap \left\{A^C : A \in \mathcal{S}\right\} = \left(\cup \{A : A \in \mathcal{S}\}\right)^C.$$

These laws follow directly from the definitions. Also following directly from the definitions are:

Let $\mathcal{S}$ be a set of sets then

$$B \cup \cup \{A : A \in \mathcal{S}\} = \cup \{B \cup A : A \in \mathcal{S}\}.$$

and: Let $\mathcal{S}$ be a set of sets show

$$B \cap \cup \{A : A \in \mathcal{S}\} = \cup \{B \cap A : A \in \mathcal{S}\}.$$

Unfortunately, there is no single universal set which can be used for all sets. Here is why: Suppose there were. Call it S. Then you could consider A the set of all elements of S which are not elements of themselves, this from the axiom of specification. If A is an element of itself, then it fails to qualify for inclusion in A. Therefore, it must not be an element of itself. However, if this is so, it qualifies for inclusion in A so it is an element of itself and so this can't be true either. Thus the most basic of conditions you could imagine, that of being an element of, is meaningless and so allowing such a set causes the whole theory to be meaningless. The solution is to not allow a universal set. As mentioned by Halmos in Naive set theory, "Nothing contains everything". Always beware of statements involving quantifiers wherever they occur, even this one.

1.2 The Schroder Bernstein Theorem

It is very important to be able to compare the size of sets in a rational way. The most useful theorem in this context is the Schroder Bernstein theorem which is the main result to be presented in this section. The Cartesian product is discussed above. The next definition reviews this and defines the concept of a function.

Definition 1.1 *Let X and Y be sets.*

$$X \times Y \equiv \{(x, y) : x \in X \text{ and } y \in Y\}$$

A relation is defined to be a subset of $X \times Y$. A function, f, also called a mapping, is a relation which has the property that if (x, y) and (x, y_1) are both elements of the f, then $y = y_1$. The domain of f is defined as

$$D(f) \equiv \{x : (x, y) \in f\},$$

written as $f : D(f) \rightarrow Y$.

It is probably safe to say that most people do not think of functions as a type of relation which is a subset of the Cartesian product of two sets. A function is like a machine which takes inputs, x and makes them into a unique output, $f(x)$. Of course, that is what the above definition says with more precision. An ordered pair, (x, y) which is an element of the function or mapping has an input, x and a unique output, y,denoted as $f(x)$ while the name of the function is f. "mapping" is often a noun meaning function. However, it also is a verb as in "f is mapping A to B ". That which a function is thought of as doing is also referred to using the word "maps" as in: f maps X to Y. However, a set of functions may be called a set of maps so this word might also be used as the plural of a noun. There is no help for it. You just have to suffer with this nonsense.

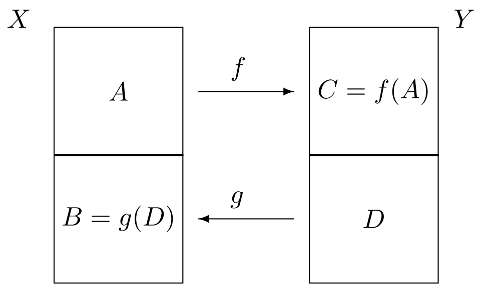
The following theorem which is interesting for its own sake will be used to prove the Schroder Bernstein theorem.

Theorem 1.2 *Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be two functions. Then there exist sets A, B, C, D, such that*

$$A \cup B = X, \ C \cup D = Y, \ A \cap B = \emptyset, \ C \cap D = \emptyset,$$

$$f(A) = C, \ g(D) = B.$$

The following picture illustrates the conclusion of this theorem.



Proof: Consider the empty set, $\emptyset \subseteq X$. If $y \in Y \setminus f(\emptyset)$, then $g(y) \notin \emptyset$ because $\emptyset$ has no elements. Also, if A, B, C, and D are as described above, A also would have this same property that the empty set has. However, A is probably larger. Therefore, say $A_0 \subseteq X$ satisfies $\mathcal{P}$ if whenever $y \in Y \setminus f(A_0)$, $g(y) \notin A_0$.

$$\mathcal{A} \equiv \{A_0 \subseteq X : A_0 \text{ satisfies } \mathcal{P}\}.$$

Let $A = \cup\mathcal{A}$. If $y \in Y \setminus f(A)$, then for each $A_0 \in \mathcal{A}$, $y \in Y \setminus f(A_0)$ and so $g(y) \notin A_0$. Since $g(y) \notin A_0$ for all $A_0 \in \mathcal{A}$, it follows $g(y) \notin A$. Hence A satisfies $\mathcal{P}$ and is the largest subset of X which does so. Now define

$$C \equiv f(A), \ D \equiv Y \setminus C, \ B \equiv X \setminus A.$$

It only remains to verify that $g(D) = B$.

Suppose $x \in B = X \setminus A$. Then $A \cup \{x\}$ does not satisfy $\mathcal{P}$ and so there exists $y \in Y \setminus f(A \cup \{x\}) \subseteq D$ such that $g(y) \in A \cup \{x\}$. But $y \notin f(A)$ and so since A satisfies $\mathcal{P}$, it follows $g(y) \notin A$. Hence $g(y) = x$ and so $x \in g(D)$ and this proves the theorem.

Theorem 1.3 *(Schroder Bernstein) If $f : X \to Y$ and $g : Y \to X$ are one to one, then there exists $h : X \to Y$ which is one to one and onto.*

Proof: Let A, B, C, D be the sets of Theorem1.2 and define

$$h(x) \equiv \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in B \end{cases}$$

Then h is the desired one to one and onto mapping.

Recall that the Cartesian product may be considered as the collection of choice functions.

Definition 1.4 *Let I be a set and let X_i be a set for each $i \in I$. f is a choice function written as*

$$f \in \prod_{i \in I} X_i$$

if $f(i) \in X_i$ for each $i \in I$.

The axiom of choice says that if $X_i \neq \emptyset$ for each $i \in I$, for I a set, then

$$\prod_{i \in I} X_i \neq \emptyset.$$

Sometimes the two functions, f and g are onto but not one to one. It turns out that with the axiom of choice, a similar conclusion to the above may be obtained.

Corollary 1.5 *If $f : X \to Y$ is onto and $g : Y \to X$ is onto, then there exists $h : X \to Y$ which is one to one and onto.*

Proof: For each $y \in Y$, $f^{-1}(y) \equiv \{x \in X : f(x) = y\} \neq \emptyset$. Therefore, by the axiom of choice, there exists $f_0^{-1} \in \prod_{y \in Y} f^{-1}(y)$ which is the same as saying that for each $y \in Y$, $f_0^{-1}(y) \in f^{-1}(y)$. Similarly, there exists $g_0^{-1}(x) \in g^{-1}(x)$ for all $x \in X$. Then f_0^{-1} is one to one because if $f_0^{-1}(y_1) = f_0^{-1}(y_2)$, then

$$y_1 = f\left(f_0^{-1}(y_1)\right) = f\left(f_0^{-1}(y_2)\right) = y_2.$$

Similarly g_0^{-1} is one to one. Therefore, by the Schroder Bernstein theorem, there exists $h : X \to Y$ which is one to one and onto.

Definition 1.6 *A set S, is finite if there exists a natural number n and a map θ which maps $\{1, \cdots, n\}$ one to one and onto S. S is infinite if it is not finite. A set S, is called countable if there exists a map θ mapping $\mathbb{N}$ one to one and onto S.(When θ maps a set A to a set B, this will be written as $\theta : A \to B$ in the future.) Here $\mathbb{N} \equiv \{1, 2, \cdots\}$, the natural numbers. S is at most countable if there exists a map $\theta : \mathbb{N} \to S$ which is onto.*

The property of being at most countable is often referred to as being countable because the question of interest is normally whether one can list all elements of the set, designating a first, second, third etc. in such a way as to give each element of the set a natural number. The possibility that a single element of the set may be counted more than once is often not important.

Theorem 1.7 *If X and Y are both at most countable, then $X \times Y$ is also at most countable. If either X or Y is countable, then $X \times Y$ is also countable.*

Proof: It is given that there exists a mapping $\eta : \mathbb{N} \to X$ which is onto. Define $\eta(i) \equiv x_i$ and consider X as the set $\{x_1, x_2, x_3, \cdots\}$. Similarly, consider Y as the set $\{y_1, y_2, y_3, \cdots\}$. It follows the elements of $X \times Y$ are included in the following rectangular array.

$$\begin{array}{ccccl}
(x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots & \leftarrow \text{Those which have } x_1 \text{ in first slot.} \\
(x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots & \leftarrow \text{Those which have } x_2 \text{ in first slot.} \\
(x_3, y_1) & (x_3, y_2) & (x_3, y_3) & \cdots & \leftarrow \text{Those which have } x_3 \text{ in first slot.} \\
\vdots & \vdots & \vdots & & \qquad\qquad \vdots
\end{array}.$$

Follow a path through this array as follows.

$$\begin{array}{ccccc}
(x_1, y_1) & \rightarrow & (x_1, y_2) & & (x_1, y_3) \quad \rightarrow \\
& \swarrow & & \nearrow & \\
(x_2, y_1) & & (x_2, y_2) & & \\
\downarrow & \nearrow & & & \\
(x_3, y_1) & & & &
\end{array}$$

Thus the first element of $X \times Y$ is (x_1, y_1), the second element of $X \times Y$ is (x_1, y_2), the third element of $X \times Y$ is (x_2, y_1) etc. This assigns a number from $\mathbb{N}$ to each element of $X \times Y$. Thus $X \times Y$ is at most countable.

It remains to show the last claim. Suppose without loss of generality that X is countable. Then there exists $\alpha : \mathbb{N} \to X$ which is one to one and onto. Let $\beta : X \times Y \to \mathbb{N}$ be defined by $\beta((x, y)) \equiv \alpha^{-1}(x)$. Thus β is onto $\mathbb{N}$. By the first part there exists a function from $\mathbb{N}$ onto $X \times Y$. Therefore, by Corollary 1.5, there exists a one to one and onto mapping from $X \times Y$ to $\mathbb{N}$. This proves the theorem.

Theorem 1.8 *If X and Y are at most countable, then $X \cup Y$ is at most countable. If either X or Y are countable, then $X \cup Y$ is countable.*

Proof: As in the preceding theorem,

$$X = \{x_1, x_2, x_3, \cdots\}$$

and

$$Y = \{y_1, y_2, y_3, \cdots\}.$$

Consider the following array consisting of $X \cup Y$ and path through it.

$$\begin{array}{cccccc} x_1 & \rightarrow & x_2 & & x_3 & \rightarrow \\ & \swarrow & & \nearrow & & \\ y_1 & \rightarrow & y_2 & & & \end{array}$$

Thus the first element of $X \cup Y$ is x_1, the second is x_2 the third is y_1 the fourth is y_2 etc.

Consider the second claim. By the first part, there is a map from $\mathbb{N}$ onto $X \times Y$. Suppose without loss of generality that X is countable and $\alpha : \mathbb{N} \to X$ is one to one and onto. Then define $\beta(y) \equiv 1$, for all $y \in Y$,and $\beta(x) \equiv \alpha^{-1}(x)$. Thus, β maps $X \times Y$ onto $\mathbb{N}$ and this shows there exist two onto maps, one mapping $X \cup Y$ onto $\mathbb{N}$ and the other mapping $\mathbb{N}$ onto $X \cup Y$. Then Corollary 1.5 yields the conclusion. This proves the theorem.

1.3 Equivalence Relations

There are many ways to compare elements of a set other than to say two elements are equal or the same. For example, in the set of people let two people be equivalent if they have the same weight. This would not be saying they were the same person, just that they weighed the same. Often such relations involve considering one characteristic of the elements of a set and then saying the two elements are equivalent if they are the same as far as the given characteristic is concerned.

Definition 1.9 *Let S be a set. $\sim$ is an equivalence relation on S if it satisfies the following axioms.*

1. *$x \sim x \quad$ for all $x \in S$. (Reflexive)*
2. *If $x \sim y$ then $y \sim x$. (Symmetric)*
3. *If $x \sim y$ and $y \sim z$, then $x \sim z$. (Transitive)*

Definition 1.10 *$[x]$ denotes the set of all elements of S which are equivalent to x and $[x]$ is called the equivalence class determined by x or just the equivalence class of x.*

With the above definition one can prove the following simple theorem.

Theorem 1.11 *Let $\sim$ be an equivalence class defined on a set, S and let $\mathcal{H}$ denote the set of equivalence classes. Then if $[x]$ and $[y]$ are two of these equivalence classes, either $x \sim y$ and $[x] = [y]$ or it is not true that $x \sim y$ and $[x] \cap [y] = \emptyset$.*

1.4 Partially Ordered Sets

Definition 1.12 *Let $\mathcal{F}$ be a nonempty set. $\mathcal{F}$ is called a partially ordered set if there is a relation, denoted here by $\leq$, such that*

$$x \leq x \text{ for all } x \in \mathcal{F}.$$

$$\text{If } x \leq y \text{ and } y \leq z \text{ then } x \leq z.$$

$\mathcal{C} \subseteq \mathcal{F}$ is said to be a chain if every two elements of $\mathcal{C}$ are related. This means that if $x, y \in \mathcal{C}$, then either $x \leq y$ or $y \leq x$. Sometimes a chain is called a totally ordered set. $\mathcal{C}$ is said to be a maximal chain if whenever $\mathcal{D}$ is a chain containing $\mathcal{C}$, $\mathcal{D} = \mathcal{C}$.

The most common example of a partially ordered set is the power set of a given set with $\subseteq$ being the relation. It is also helpful to visualize partially ordered sets as trees. Two points on the tree are related if they are on the same branch of the tree and one is higher than the other. Thus two points on different branches would not be related although they might both be larger than some point on the trunk. You might think of many other things which are best considered as partially ordered sets. Think of food for example. You might find it difficult to determine which of two favorite pies you like better although you may be able to say very easily that you would prefer either pie to a dish of lard topped with whipped cream and mustard. The following theorem is equivalent to the axiom of choice. For a discussion of this, see the appendix on the subject.

Theorem 1.13 *(Hausdorff Maximal Principle) Let $\mathcal{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.*

Continuous Functions Of One Variable

There is a theorem about the integral of a continuous function which requires the notion of uniform continuity. This is discussed in this section. Consider the function $f(x) = \frac{1}{x}$ for $x \in (0,1)$. This is a continuous function because, it is continuous at every point of $(0,1)$. However, for a given $\varepsilon > 0$, the δ needed in the ε, δ definition of continuity becomes very small as x gets close to 0. The notion of uniform continuity involves being able to choose a single δ which works on the whole domain of f. Here is the definition.

Definition 2.1 *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. Then f is uniformly continuous if for every $\varepsilon > 0$, there exists a δ* ***depending only on*** *ε such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.*

It is an amazing fact that under certain conditions continuity implies uniform continuity.

Definition 2.2 *A set, $K \subseteq \mathbb{R}$ is sequentially compact if whenever $\{a_n\} \subseteq K$ is a sequence, there exists a subsequence, $\{a_{n_k}\}$ such that this subsequence converges to a point of K.*

The following theorem is part of the Heine Borel theorem.

Theorem 2.3 *Every closed interval, $[a, b]$ is sequentially compact.*

Proof: Let $\{x_n\} \subseteq [a, b] \equiv I_0$. Consider the two intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$ each of which has length $(b-a)/2$. At least one of these intervals contains x_n for infinitely many values of n. Call this interval I_1. Now do for I_1 what was done for I_0. Split it in half and let I_2 be the interval which contains x_n for infinitely many values of n. Continue this way obtaining a sequence of nested intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \cdots$ where the length of I_n is $(b-a)/2^n$. Now pick n_1 such that $x_{n_1} \in I_1$, n_2 such that $n_2 > n_1$ and $x_{n_2} \in I_2, n_3$ such that $n_3 > n_2$ and $x_{n_3} \in I_3$, etc. (This can be done because in each case the intervals contained x_n for infinitely many values of n.) By the nested interval lemma there exists a point, c contained in all these intervals. Furthermore,

$$|x_{n_k} - c| < (b-a)\, 2^{-k}$$

and so $\lim_{k\to\infty} x_{n_k} = c \in [a,b]$. This proves the theorem.

Theorem 2.4 *Let $f : K \to \mathbb{R}$ be continuous where K is a sequentially compact set in $\mathbb{R}$. Then f is uniformly continuous on K.*

Proof: If this is not true, there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists a pair of points, x_δ and y_δ such that even though $|x_\delta - y_\delta| < \delta$, $|f(x_\delta) - f(y_\delta)| \geq \varepsilon$. Taking a succession of values for δ equal to $1, 1/2, 1/3, \cdots$, and letting the exceptional pair of points for $\delta = 1/n$ be denoted by x_n and y_n,

$$|x_n - y_n| < \frac{1}{n}, |f(x_n) - f(y_n)| \geq \varepsilon.$$

Now since K is sequentially compact, there exists a subsequence, $\{x_{n_k}\}$ such that $x_{n_k} \to z \in K$. Now $n_k \geq k$ and so

$$|x_{n_k} - y_{n_k}| < \frac{1}{k}.$$

Consequently, $y_{n_k} \to z$ also. (x_{n_k} is like a person walking toward a certain point and y_{n_k} is like a dog on a leash which is constantly getting shorter. Obviously y_{n_k} must also move toward the point also. You should give a precise proof of what is needed here.) By continuity of f

$$0 = |f(z) - f(z)| = \lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon,$$

an obvious contradiction. Therefore, the theorem must be true.

The following corollary follows from this theorem and Theorem 2.3.

Corollary 2.5 *Suppose I is a closed interval, $I = [a,b]$ and $f : I \to \mathbb{R}$ is continuous. Then f is uniformly continuous.*

2.1 Exercises

1. A function, $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous or just Lipschitz for short if there exists a constant, K such that

   $$|f(x) - f(y)| \leq K\,|x-y|$$

   for all $x, y \in D$. Show every Lipschitz function is uniformly continuous.

2. If $|x_n - y_n| \to 0$ and $x_n \to z$, show that $y_n \to z$ also.

3. Consider $f : (1,\infty) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$. Show f is uniformly continuous even though the set on which f is defined is not sequentially compact.

4. If f is uniformly continuous, does it follow that $|f|$ is also uniformly continuous? If $|f|$ is uniformly continuous does it follow that f is uniformly continuous? Answer the same questions with "uniformly continuous" replaced with "continuous". Explain why.

2.2 Theorems About Continuous Functions

In this section, proofs of some theorems which have not been proved yet are given.

Theorem 2.6 *The following assertions are valid*

1. *The function, $af + bg$ is continuous at x when f, g are continuous at $x \in D(f) \cap D(g)$ and $a, b \in \mathbb{R}$.*

2. *If and f and g are each real valued functions continuous at x, then fg is continuous at x. If, in addition to this, $g(x) \neq 0$, then f/g is continuous at x.*

3. *If f is continuous at x, $f(x) \in D(g) \subseteq \mathbb{R}$, and g is continuous at $f(x)$,then $g \circ f$ is continuous at x.*

4. *The function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = |x|$ is continuous.*

Proof: First consider 1.) Let $\varepsilon > 0$ be given. By assumption, there exist $\delta_1 > 0$ such that whenever $|x - y| < \delta_1$, it follows $|f(x) - f(y)| < \frac{\varepsilon}{2(|a|+|b|+1)}$ and there exists $\delta_2 > 0$ such that whenever $|x - y| < \delta_2$, it follows that $|g(x) - g(y)| < \frac{\varepsilon}{2(|a|+|b|+1)}$. Then let $0 < \delta \leq \min(\delta_1, \delta_2)$. If $|x - y| < \delta$, then everything happens at once. Therefore, using the triangle inequality

$$|af(x) + bf(x) - (ag(y) + bg(y))|$$

$$\begin{aligned}
&\leq |a| |f(x) - f(y)| + |b| |g(x) - g(y)| \\
&< |a| \left(\frac{\varepsilon}{2(|a| + |b| + 1)} \right) + |b| \left(\frac{\varepsilon}{2(|a| + |b| + 1)} \right) < \varepsilon.
\end{aligned}$$

Now consider 2.) There exists $\delta_1 > 0$ such that if $|y - x| < \delta_1$, then

$$|f(x) - f(y)| < 1.$$

Therefore, for such y,

$$|f(y)| < 1 + |f(x)|.$$

It follows that for such y,

$$|fg(x) - fg(y)| \leq |f(x) g(x) - g(x) f(y)| + |g(x) f(y) - f(y) g(y)|$$

$$\leq |g(x)| |f(x) - f(y)| + |f(y)| |g(x) - g(y)|$$
$$\leq (1 + |g(x)| + |f(y)|) [|g(x) - g(y)| + |f(x) - f(y)|].$$

Now let $\varepsilon > 0$ be given. There exists δ_2 such that if $|x - y| < \delta_2$, then

$$|g(x) - g(y)| < \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)},$$

and there exists δ_3 such that if $|x-y| < \delta_3$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)}$$

Now let $0 < \delta \leq \min(\delta_1, \delta_2, \delta_3)$. Then if $|x-y| < \delta$, all the above hold at once and so

$$|fg(x) - fg(y)| \leq$$
$$(1 + |g(x)| + |f(y)|) [|g(x) - g(y)| + |f(x) - f(y)|]$$
$$< (1 + |g(x)| + |f(y)|) \left(\frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)} + \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)} \right) = \varepsilon.$$

This proves the first part of 2.) To obtain the second part, let δ_1 be as described above and let $\delta_0 > 0$ be such that for $|x-y| < \delta_0$,

$$|g(x) - g(y)| < |g(x)| / 2$$

and so by the triangle inequality,

$$-|g(x)| / 2 \leq |g(y)| - |g(x)| \leq |g(x)| / 2$$

which implies $|g(y)| \geq |g(x)| / 2$, and $|g(y)| < 3 |g(x)| / 2$.

Then if $|x-y| < \min(\delta_0, \delta_1)$,

$$\left| \frac{f(x)}{g(x)} - \frac{f(y)}{g(y)} \right| = \left| \frac{f(x) g(y) - f(y) g(x)}{g(x) g(y)} \right|$$
$$\leq \frac{|f(x) g(y) - f(y) g(x)|}{\left(\frac{|g(x)|^2}{2} \right)}$$
$$= \frac{2 |f(x) g(y) - f(y) g(x)|}{|g(x)|^2}$$

$$\leq \frac{2}{|g(x)|^2} [|f(x) g(y) - f(y) g(y) + f(y) g(y) - f(y) g(x)|]$$
$$\leq \frac{2}{|g(x)|^2} [|g(y)| |f(x) - f(y)| + |f(y)| |g(y) - g(x)|]$$
$$\leq \frac{2}{|g(x)|^2} \left[\frac{3}{2} |g(x)| |f(x) - f(y)| + (1 + |f(x)|) |g(y) - g(x)| \right]$$
$$\leq \frac{2}{|g(x)|^2} (1 + 2|f(x)| + 2|g(x)|) [|f(x) - f(y)| + |g(y) - g(x)|]$$
$$\equiv M [|f(x) - f(y)| + |g(y) - g(x)|]$$

where M is defined by

$$M \equiv \frac{2}{|g(x)|^2}\left(1+2|f(x)|+2|g(x)|\right)$$

Now let δ_2 be such that if $|x-y|<\delta_2$, then

$$|f(x)-f(y)|<\frac{\varepsilon}{2}M^{-1}$$

and let δ_3 be such that if $|x-y|<\delta_3$, then

$$|g(y)-g(x)|<\frac{\varepsilon}{2}M^{-1}.$$

Then if $0<\delta\leq\min(\delta_0,\delta_1,\delta_2,\delta_3)$, and $|x-y|<\delta$, everything holds and

$$\left|\frac{f(x)}{g(x)}-\frac{f(y)}{g(y)}\right|\leq M\left[|f(x)-f(y)|+|g(y)-g(x)|\right]$$

$$< M\left[\frac{\varepsilon}{2}M^{-1}+\frac{\varepsilon}{2}M^{-1}\right]=\varepsilon.$$

This completes the proof of the second part of 2.)

Note that in these proofs no effort is made to find some sort of "best" δ. The problem is one which has a yes or a no answer. Either is it or it is not continuous.

Now consider 3.). If f is continuous at x, $f(x)\in D(g)\subseteq\mathbb{R}^p$, and g is continuous at $f(x)$,then $g\circ f$ is continuous at x. Let $\varepsilon>0$ be given. Then there exists $\eta>0$ such that if $|y-f(x)|<\eta$ and $y\in D(g)$, it follows that $|g(y)-g(f(x))|<\varepsilon$. From continuity of f at x, there exists $\delta>0$ such that if $|x-z|<\delta$ and $z\in D(f)$, then $|f(z)-f(x)|<\eta$. Then if $|x-z|<\delta$ and $z\in D(g\circ f)\subseteq D(f)$, all the above hold and so

$$|g(f(z))-g(f(x))|<\varepsilon.$$

This proves part 3.)

To verify part 4.), let $\varepsilon>0$ be given and let $\delta=\varepsilon$. Then if $|x-y|<\delta$, the triangle inequality implies

$$|f(x)-f(y)| = ||x|-|y||$$
$$\leq |x-y|<\delta=\varepsilon.$$

This proves part 4.) and completes the proof of the theorem.

Next here is a proof of the intermediate value theorem.

Theorem 2.7 *Suppose $f:[a,b]\to\mathbb{R}$ is continuous and suppose $f(a)<c<f(b)$. Then there exists $x\in(a,b)$ such that $f(x)=c$.*

Proof: Let $d=\frac{a+b}{2}$ and consider the intervals $[a,d]$ and $[d,b]$. If $f(d)\geq c$, then on $[a,d]$, the function is $\leq c$ at one end point and $\geq c$ at the other. On the other hand, if $f(d)\leq c$, then on $[d,b]$ $f\geq 0$ at one end point and ≤ 0 at the other. Pick the interval on which f has values which are at least as large as c and values no larger than c. Now consider that interval, divide it in half as was done for the original interval and argue that on one of these smaller intervals, the function has values at least as large as c and values no larger than c. Continue in this way. Next apply the nested interval lemma to get x in all these intervals. In the n^{th} interval, let x_n, y_n be elements of this interval such that $f(x_n) \leq c, f(y_n) \geq c$. Now $|x_n - x| \leq (b-a) 2^{-n}$ and $|y_n - x| \leq (b-a) 2^{-n}$ and so $x_n \to x$ and $y_n \to x$. Therefore,

$$f(x) - c = \lim_{n\to\infty} (f(x_n) - c) \leq 0$$

while

$$f(x) - c = \lim_{n\to\infty} (f(y_n) - c) \geq 0.$$

Consequently $f(x) = c$ and this proves the theorem.

Lemma 2.8 *Let $\phi : [a,b] \to \mathbb{R}$ be a continuous function and suppose ϕ is $1-1$ on (a,b). Then ϕ is either strictly increasing or strictly decreasing on $[a,b]$.*

Proof: First it is shown that ϕ is either strictly increasing or strictly decreasing on (a,b).

If ϕ is not strictly decreasing on (a,b), then there exists $x_1 < y_1$, $x_1, y_1 \in (a,b)$ such that

$$(\phi(y_1) - \phi(x_1))(y_1 - x_1) > 0.$$

If for some other pair of points, $x_2 < y_2$ with $x_2, y_2 \in (a,b)$, the above inequality does not hold, then since ϕ is $1-1$,

$$(\phi(y_2) - \phi(x_2))(y_2 - x_2) < 0.$$

Let $x_t \equiv tx_1 + (1-t)x_2$ and $y_t \equiv ty_1 + (1-t)y_2$. Then $x_t < y_t$ for all $t \in [0,1]$ because

$$tx_1 \leq ty_1 \text{ and } (1-t)x_2 \leq (1-t)y_2$$

with strict inequality holding for at least one of these inequalities since not both t and $(1-t)$ can equal zero. Now define

$$h(t) \equiv (\phi(y_t) - \phi(x_t))(y_t - x_t).$$

Since h is continuous and $h(0) < 0$, while $h(1) > 0$, there exists $t \in (0,1)$ such that $h(t) = 0$. Therefore, both x_t and y_t are points of (a,b) and $\phi(y_t) - \phi(x_t) = 0$ contradicting the assumption that ϕ is one to one. It follows ϕ is either strictly increasing or strictly decreasing on (a,b).

This property of being either strictly increasing or strictly decreasing on (a,b) carries over to $[a,b]$ by the continuity of ϕ. Suppose ϕ is strictly increasing on (a,b), a similar argument holding for ϕ strictly decreasing on (a,b). If $x > a$, then pick $y \in (a,x)$ and from the above, $\phi(y) < \phi(x)$. Now by continuity of ϕ at a,

$$\phi(a) = \lim_{x\to a+} \phi(z) \leq \phi(y) < \phi(x).$$

Therefore, $\phi(a) < \phi(x)$ whenever $x \in (a,b)$. Similarly $\phi(b) > \phi(x)$ for all $x \in (a,b)$. This proves the lemma.

Corollary 2.9 *Let $f : (a, b) \to \mathbb{R}$ be one to one and continuous. Then $f(a, b)$ is an open interval, (c, d) and $f^{-1} : (c, d) \to (a, b)$ is continuous.*

Proof: Since f is either strictly increasing or strictly decreasing, it follows that $f(a, b)$ is an open interval, (c, d). Assume f is decreasing. Now let $x \in (a, b)$. Why is f^{-1} is continuous at $f(x)$? Since f is decreasing, if $f(x) < f(y)$, then $y \equiv f^{-1}(f(y)) < x \equiv f^{-1}(f(x))$ and so f^{-1} is also decreasing. Let $\varepsilon > 0$ be given. Let $\varepsilon > \eta > 0$ and $(x - \eta, x + \eta) \subseteq (a, b)$. Then $f(x) \in (f(x + \eta), f(x - \eta))$. Let $\delta = \min(f(x) - f(x + \eta), f(x - \eta) - f(x))$. Then if

$$|f(z) - f(x)| < \delta,$$

it follows

$$z \equiv f^{-1}(f(z)) \in (x - \eta, x + \eta) \subseteq (x - \varepsilon, x + \varepsilon)$$

so

$$\left|f^{-1}(f(z)) - x\right| = \left|f^{-1}(f(z)) - f^{-1}(f(x))\right| < \varepsilon.$$

This proves the theorem in the case where f is strictly decreasing. The case where f is increasing is similar.

The Riemann Stieltjes Integral

The integral originated in attempts to find areas of various shapes and the ideas involved in finding integrals are much older than the ideas related to finding derivatives. In fact, Archimedes[1] was finding areas of various curved shapes about 250 B.C. using the main ideas of the integral. What is presented here is a generalization of these ideas. The main interest is in the Riemann integral but if it is easy to generalize to the so called Stieltjes integral in which the length of an interval, $[x,y]$ is replaced with an expression of the form $F(y) - F(x)$ where F is an increasing function, then the generalization is given. However, there is much more that can be written about Stieltjes integrals than what is presented here. A good source for this is the book by Apostol, [3].

3.1 Upper And Lower Riemann Stieltjes Sums

The Riemann integral pertains to bounded functions which are defined on a bounded interval. Let $[a,b]$ be a closed interval. A set of points in $[a,b]$, $\{x_0, \cdot \cdot \cdot, x_n\}$ is a partition if

$$a = x_0 < x_1 < \cdot \cdot \cdot < x_n = b.$$

Such partitions are denoted by P or Q. For f a bounded function defined on $[a,b]$, let

$$M_i(f) \equiv \sup\{f(x) : x \in [x_{i-1}, x_i]\},$$
$$m_i(f) \equiv \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

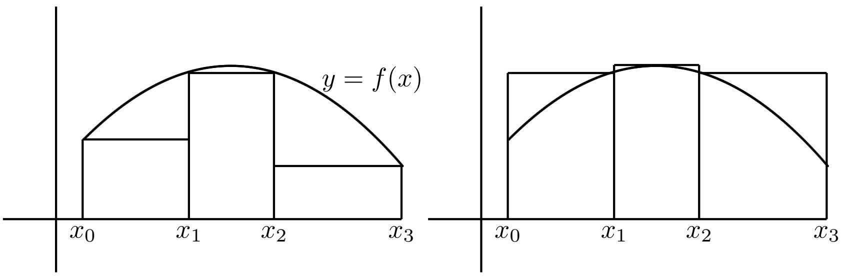
[1] Archimedes 287-212 B.C. found areas of curved regions by stuffing them with simple shapes which he knew the area of and taking a limit. He also made fundamental contributions to physics. The story is told about how he determined that a gold smith had cheated the king by giving him a crown which was not solid gold as had been claimed. He did this by finding the amount of water displaced by the crown and comparing with the amount of water it should have displaced if it had been solid gold.

Definition 3.1 *Let F be an increasing function defined on $[a,b]$ and let $\Delta F_i \equiv F(x_i) - F(x_{i-1})$. Then define upper and lower sums as*

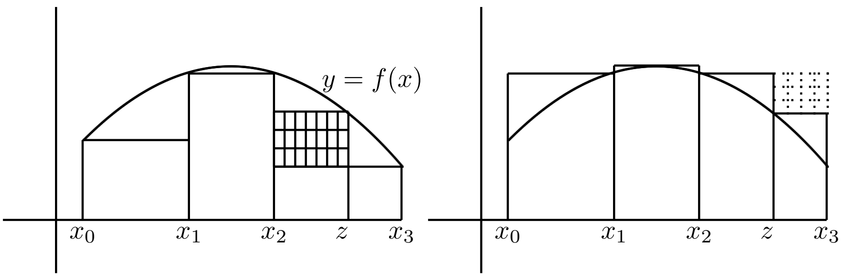
$$U(f,P) \equiv \sum_{i=1}^{n} M_i(f)\,\Delta F_i \text{ and } L(f,P) \equiv \sum_{i=1}^{n} m_i(f)\,\Delta F_i$$

respectively. The numbers, $M_i(f)$ and $m_i(f)$, are well defined real numbers because f is assumed to be bounded and $\mathbb{R}$ is complete. Thus the set $S = \{f(x) : x \in [x_{i-1}, x_i]\}$ is bounded above and below.

In the following picture, the sum of the areas of the rectangles in the picture on the left is a lower sum for the function in the picture and the sum of the areas of the rectangles in the picture on the right is an upper sum for the same function which uses the same partition. In these pictures the function, F is given by $F(x) = x$ and these are the ordinary upper and lower sums from calculus.



What happens when you add in more points in a partition? The following pictures illustrate in the context of the above example. In this example a single additional point, labeled z has been added in.



Note how the lower sum got larger by the amount of the area in the shaded rectangle and the upper sum got smaller by the amount in the rectangle shaded by dots. In general this is the way it works and this is shown in the following lemma.

Lemma 3.2 *If $P \subseteq Q$ then*

$$U(f,Q) \leq U(f,P), \text{ and } L(f,P) \leq L(f,Q).$$

Proof: This is verified by adding in one point at a time. Thus let

$$P = \{x_0, \cdots, x_n\}$$

and let

$$Q = \{x_0, \cdots, x_k, y, x_{k+1}, \cdots, x_n\}.$$

Thus exactly one point, y, is added between x_k and x_{k+1}. Now the term in the upper sum which corresponds to the interval $[x_k, x_{k+1}]$ in $U(f, P)$ is

$$\sup\{f(x) : x \in [x_k, x_{k+1}]\} (F(x_{k+1}) - F(x_k)) \tag{3.1}$$

and the term which corresponds to the interval $[x_k, x_{k+1}]$ in $U(f, Q)$ is

$$\begin{aligned} &\sup\{f(x) : x \in [x_k, y]\} (F(y) - F(x_k)) \\ &+ \sup\{f(x) : x \in [y, x_{k+1}]\} (F(x_{k+1}) - F(y)) \\ &\equiv M_1 (F(y) - F(x_k)) + M_2 (F(x_{k+1}) - F(y)) \end{aligned} \tag{3.2}$$

All the other terms in the two sums coincide. Now $\sup\{f(x) : x \in [x_k, x_{k+1}]\} \geq \max(M_1, M_2)$ and so the expression in 3.2 is no larger than

$$\begin{aligned} &\sup\{f(x) : x \in [x_k, x_{k+1}]\} (F(x_{k+1}) - F(y)) \\ &+ \sup\{f(x) : x \in [x_k, x_{k+1}]\} (F(y) - F(x_k)) \\ &= \sup\{f(x) : x \in [x_k, x_{k+1}]\} (F(x_{k+1}) - F(x_k)), \end{aligned}$$

the term corresponding to the interval, $[x_k, x_{k+1}]$ and $U(f, P)$. This proves the first part of the lemma pertaining to upper sums because if $Q \supseteq P$, one can obtain Q from P by adding in one point at a time and each time a point is added, the corresponding upper sum either gets smaller or stays the same. The second part about lower sums is similar and is left as an exercise.

Lemma 3.3 *If P and Q are two partitions, then*

$$L(f, P) \leq U(f, Q).$$

Proof: By Lemma 3.2,

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q).$$

Definition 3.4

$$\overline{I} \equiv \inf\{U(f, Q) \text{ where } Q \text{ is a partition}\}$$

$$\underline{I} \equiv \sup\{L(f, P) \text{ where } P \text{ is a partition}\}.$$

Note that $\underline{I}$ and $\overline{I}$ are well defined real numbers.

Theorem 3.5 $\underline{I} \leq \overline{I}$.

Proof: From Lemma 3.3,

$$\underline{I} = \sup\{L(f,P) \text{ where } P \text{ is a partition}\} \leq U(f,Q)$$

because $U(f,Q)$ is an upper bound to the set of all lower sums and so it is no smaller than the least upper bound. Therefore, since Q is arbitrary,

$$\begin{aligned}\underline{I} &= \sup\{L(f,P) \text{ where } P \text{ is a partition}\} \\ &\leq \inf\{U(f,Q) \text{ where } Q \text{ is a partition}\} \equiv \overline{I}\end{aligned}$$

where the inequality holds because it was just shown that $\underline{I}$ is a lower bound to the set of all upper sums and so it is no larger than the greatest lower bound of this set. This proves the theorem.

Definition 3.6 *A bounded function f is Riemann Stieltjes integrable, written as*

$$f \in R([a,b])$$

if

$$\underline{I} = \overline{I}$$

and in this case,

$$\int_a^b f(x)\, dF \equiv \underline{I} = \overline{I}.$$

When $F(x) = x$, the integral is called the Riemann integral and is written as

$$\int_a^b f(x)\, dx.$$

Thus, in words, the Riemann integral is the unique number which lies between all upper sums and all lower sums if there is such a unique number.

Recall the following Proposition which comes from the definitions.

Proposition 3.7 *Let S be a nonempty set and suppose* $\sup(S)$ *exists. Then for every $\delta > 0$,*

$$S \cap (\sup(S) - \delta, \sup(S)] \neq \emptyset.$$

If $\inf(S)$ *exists, then for every $\delta > 0$,*

$$S \cap [\inf(S), \inf(S) + \delta) \neq \emptyset.$$

This proposition implies the following theorem which is used to determine the question of Riemann Stieltjes integrability.

Theorem 3.8 *A bounded function f is Riemann integrable if and only if for all $\varepsilon > 0$, there exists a partition P such that*

$$U(f,P) - L(f,P) < \varepsilon. \tag{3.3}$$

Proof: First assume f is Riemann integrable. Then let P and Q be two partitions such that

$$U(f,Q) < \overline{I} + \varepsilon/2,\ L(f,P) > \underline{I} - \varepsilon/2.$$

Then since $\underline{I} = \overline{I}$,

$$U(f, Q\cup P) - L(f, P\cup Q) \le U(f,Q) - L(f,P) < \overline{I} + \varepsilon/2 - (\underline{I} - \varepsilon/2) = \varepsilon.$$

Now suppose that for all $\varepsilon > 0$ there exists a partition such that 3.3 holds. Then for given ε and partition P corresponding to ε

$$\overline{I} - \underline{I} \le U(f,P) - L(f,P) \le \varepsilon.$$

Since ε is arbitrary, this shows $\underline{I} = \overline{I}$ and this proves the theorem.

The condition described in the theorem is called the Riemann criterion .

Not all bounded functions are Riemann integrable. For example, let $F(x) = x$ and

$$f(x) \equiv \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \tag{3.4}$$

Then if $[a,b] = [0,1]$ all upper sums for f equal 1 while all lower sums for f equal 0. Therefore the Riemann criterion is violated for $\varepsilon = 1/2$.

3.2 Exercises

1. Prove the second half of Lemma 3.2 about lower sums.
2. Verify that for f given in 3.4, the lower sums on the interval $[0,1]$ are all equal to zero while the upper sums are all equal to one.
3. Let $f(x) = 1 + x^2$ for $x \in [-1,3]$ and let $P = \left\{-1, -\frac{1}{3}, 0, \frac{1}{2}, 1, 2\right\}$. Find $U(f,P)$ and $L(f,P)$ for $F(x) = x$ and for $F(x) = x^3$.
4. Show that if $f \in R([a,b])$ for $F(x) = x$, there exists a partition, $\{x_0, \cdots, x_n\}$ such that for any $z_k \in [x_k, x_{k+1}]$,

   $$\left| \int_a^b f(x)\ dx - \sum_{k=1}^n f(z_k)(x_k - x_{k-1}) \right| < \varepsilon$$

   This sum, $\sum_{k=1}^n f(z_k)(x_k - x_{k-1})$, is called a Riemann sum and this exercise shows that the Riemann integral can always be approximated by a Riemann sum. For the general Riemann Stieltjes case, does anything change?
5. Let $P = \left\{1, 1\frac{1}{4}, 1\frac{1}{2}, 1\frac{3}{4}, 2\right\}$ and $F(x) = x$. Find upper and lower sums for the function, $f(x) = \frac{1}{x}$ using this partition. What does this tell you about $\ln(2)$?
6. If $f \in R([a,b])$ with $F(x) = x$ and f is changed at finitely many points, show the new function is also in $R([a,b])$. Is this still true for the general case where F is only assumed to be an increasing function? Explain.

7. In the case where $F(x) = x$, define a "left sum" as

$$\sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1})$$

and a "right sum",

$$\sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}).$$

Also suppose that all partitions have the property that $x_k - x_{k-1}$ equals a constant, $(b-a)/n$ so the points in the partition are equally spaced, and define the integral to be the number these right and left sums get close to as n gets larger and larger. Show that for f given in 3.4, $\int_0^x f(t)\,dt = 1$ if x is rational and $\int_0^x f(t)\,dt = 0$ if x is irrational. It turns out that the correct answer should always equal zero for that function, regardless of whether x is rational. This is shown when the Lebesgue integral is studied. This illustrates why this method of defining the integral in terms of left and right sums is total nonsense. Show that even though this is the case, it makes no difference if f is continuous.

3.3 Functions Of Riemann Integrable Functions

It is often necessary to consider functions of Riemann integrable functions and a natural question is whether these are Riemann integrable. The following theorem gives a partial answer to this question. This is not the most general theorem which will relate to this question but it will be enough for the needs of this book.

Theorem 3.9 *Let f, g be bounded functions and let*

$$f([a,b]) \subseteq [c_1, d_1],\ g([a,b]) \subseteq [c_2, d_2].$$

Let $H : [c_1, d_1] \times [c_2, d_2] \to \mathbb{R}$ satisfy,

$$|H(a_1, b_1) - H(a_2, b_2)| \leq K[|a_1 - a_2| + |b_1 - b_2|]$$

for some constant K. Then if $f, g \in R([a,b])$ it follows that $H \circ (f, g) \in R([a,b])$.

Proof: In the following claim, $M_i(h)$ and $m_i(h)$ have the meanings assigned above with respect to some partition of $[a,b]$ for the function, h.

Claim: The following inequality holds.

$$|M_i(H \circ (f,g)) - m_i(H \circ (f,g))| \leq$$

$$K[|M_i(f) - m_i(f)| + |M_i(g) - m_i(g)|].$$

Proof of the claim: By the above proposition, there exist $x_1, x_2 \in [x_{i-1}, x_i]$ be such that

$$H(f(x_1), g(x_1)) + \eta > M_i(H \circ (f,g)),$$

and

$$H\left(f\left(x_2\right), g\left(x_2\right)\right)-\eta<m_i(H \circ(f, g)) .$$

Then

$$\left|M_i(H \circ(f, g))-m_i(H \circ(f, g))\right|$$

$$\begin{aligned}
& <2 \eta+\left|H\left(f\left(x_1\right), g\left(x_1\right)\right)-H\left(f\left(x_2\right), g\left(x_2\right)\right)\right| \\
& <2 \eta+K\left[\left|f\left(x_1\right)-f\left(x_2\right)\right|+\left|g\left(x_1\right)-g\left(x_2\right)\right|\right] \\
& \leq 2 \eta+K\left[\left|M_i(f)-m_i(f)\right|+\left|M_i(g)-m_i(g)\right|\right] .
\end{aligned}$$

Since $\eta>0$ is arbitrary, this proves the claim.

Now continuing with the proof of the theorem, let P be such that

$$\sum_{i=1}^n\left(M_i(f)-m_i(f)\right) \Delta F_i<\frac{\varepsilon}{2 K}, \sum_{i=1}^n\left(M_i(g)-m_i(g)\right) \Delta F_i<\frac{\varepsilon}{2 K} .$$

Then from the claim,

$$\sum_{i=1}^n\left(M_i(H \circ(f, g))-m_i(H \circ(f, g))\right) \Delta F_i$$

$$<\sum_{i=1}^n K\left[\left|M_i(f)-m_i(f)\right|+\left|M_i(g)-m_i(g)\right|\right] \Delta F_i<\varepsilon .$$

Since $\varepsilon>0$ is arbitrary, this shows $H \circ(f, g)$ satisfies the Riemann criterion and hence $H \circ(f, g)$ is Riemann integrable as claimed. This proves the theorem.

This theorem implies that if f, g are Riemann Stieltjes integrable, then so is $a f+b g,|f|, f^2$, along with infinitely many other such continuous combinations of Riemann Stieltjes integrable functions. For example, to see that $|f|$ is Riemann integrable, let $H(a, b)=|a|$. Clearly this function satisfies the conditions of the above theorem and so $|f|=H(f, f) \in R([a, b])$ as claimed. The following theorem gives an example of many functions which are Riemann integrable.

Theorem 3.10 *Let $f:[a, b] \rightarrow \mathbb{R}$ be either increasing or decreasing on $[a, b]$ and suppose F is continuous. Then $f \in R([a, b])$.*

Proof: Let $\varepsilon>0$ be given and let

$$x_i=a+i\left(\frac{b-a}{n}\right), i=0, \cdots, n .$$

Since F is continuous, it follows from Corollary 2.5 on Page 26 that it is uniformly continuous. Therefore, if n is large enough, then for all i,

$$F\left(x_i\right)-F\left(x_{i-1}\right)<\frac{\varepsilon}{f(b)-f(a)+1}$$

Then since f is increasing,

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right)\left(F(x_i) - F(x_{i-1})\right) \\ &\leq \frac{\varepsilon}{f(b) - f(a) + 1} \sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right) \\ &= \frac{\varepsilon}{f(b) - f(a) + 1} \left(f(b) - f(a)\right) < \varepsilon. \end{aligned}$$

Thus the Riemann criterion is satisfied and so the function is Riemann Stieltjes integrable. The proof for decreasing f is similar.

Corollary 3.11 *Let $[a,b]$ be a bounded closed interval and let $\phi : [a,b] \to \mathbb{R}$ be Lipschitz continuous and suppose F is continuous. Then $\phi \in R([a,b])$. Recall that a function, ϕ, is Lipschitz continuous if there is a constant, K, such that for all x, y,*

$$|\phi(x) - \phi(y)| < K|x - y|.$$

Proof: Let $f(x) = x$. Then by Theorem 3.10, f is Riemann Stieltjes integrable. Let $H(a,b) \equiv \phi(a)$. Then by Theorem 3.9 $H \circ (f,f) = \phi \circ f = \phi$ is also Riemann Stieltjes integrable. This proves the corollary.

In fact, it is enough to assume ϕ is continuous, although this is harder. This is the content of the next theorem which is where the difficult theorems about continuity and uniform continuity are used. This is the main result on the existence of the Riemann Stieltjes integral for this book.

Theorem 3.12 *Suppose $f : [a,b] \to \mathbb{R}$ is continuous and F is just an increasing function defined on $[a,b]$. Then $f \in R([a,b])$.*

Proof: By Corollary 2.5 on Page 26, f is uniformly continuous on $[a,b]$. Therefore, if $\varepsilon > 0$ is given, there exists a $\delta > 0$ such that if $|x_i - x_{i-1}| < \delta$, then $M_i - m_i < \frac{\varepsilon}{F(b)-F(a)+1}$. Let

$$P \equiv \{x_0, \cdots, x_n\}$$

be a partition with $|x_i - x_{i-1}| < \delta$. Then

$$\begin{aligned} U(f,P) - L(f,P) &< \sum_{i=1}^{n} (M_i - m_i)\left(F(x_i) - F(x_{i-1})\right) \\ &< \frac{\varepsilon}{F(b) - F(a) + 1}\left(F(b) - F(a)\right) < \varepsilon. \end{aligned}$$

By the Riemann criterion, $f \in R([a,b])$. This proves the theorem.

3.4 Properties Of The Integral

The integral has many important algebraic properties. First here is a simple lemma.

Lemma 3.13 *Let S be a nonempty set which is bounded above and below. Then if $-S \equiv \{-x : x \in S\}$,*

$$\sup(-S) = -\inf(S) \tag{3.5}$$

and

$$\inf(-S) = -\sup(S). \tag{3.6}$$

Proof: Consider 3.5. Let $x \in S$. Then $-x \leq \sup(-S)$ and so $x \geq -\sup(-S)$. If follows that $-\sup(-S)$ is a lower bound for S and therefore, $-\sup(-S) \leq \inf(S)$. This implies $\sup(-S) \geq -\inf(S)$. Now let $-x \in -S$. Then $x \in S$ and so $x \geq \inf(S)$ which implies $-x \leq -\inf(S)$. Therefore, $-\inf(S)$ is an upper bound for $-S$ and so $-\inf(S) \geq \sup(-S)$. This shows 3.5. Formula 3.6 is similar and is left as an exercise.

In particular, the above lemma implies that for $M_i(f)$ and $m_i(f)$ defined above $M_i(-f) = -m_i(f)$, and $m_i(-f) = -M_i(f)$.

Lemma 3.14 *If $f \in R([a,b])$ then $-f \in R([a,b])$ and*

$$-\int_a^b f(x)\, dF = \int_a^b -f(x)\, dF.$$

Proof: The first part of the conclusion of this lemma follows from Theorem 3.10 since the function $\phi(y) \equiv -y$ is Lipschitz continuous. Now choose P such that

$$\int_a^b -f(x)\, dF - L(-f,P) < \varepsilon.$$

Then since $m_i(-f) = -M_i(f)$,

$$\varepsilon > \int_a^b -f(x)\, dF - \sum_{i=1}^n m_i(-f)\,\Delta F_i = \int_a^b -f(x)\, dF + \sum_{i=1}^n M_i(f)\,\Delta F_i$$

which implies

$$\varepsilon > \int_a^b -f(x)\, dF + \sum_{i=1}^n M_i(f)\,\Delta F_i \geq \int_a^b -f(x)\, dF + \int_a^b f(x)\, dF.$$

Thus, since ε is arbitrary,

$$\int_a^b -f(x)\, dF \leq -\int_a^b f(x)\, dF$$

whenever $f \in R([a,b])$. It follows

$$\int_a^b -f(x)\, dF \leq -\int_a^b f(x)\, dF = -\int_a^b -(-f(x))\, dF \leq \int_a^b -f(x)\, dF$$

and this proves the lemma.

Theorem 3.15 *The integral is linear,*

$$\int_a^b (\alpha f + \beta g)(x)\ dF = \alpha \int_a^b f(x)\ dF + \beta \int_a^b g(x)\ dF.$$

whenever $f, g \in R([a,b])$ *and* $\alpha, \beta \in \mathbb{R}$.

Proof: First note that by Theorem 3.9, $\alpha f + \beta g \in R([a,b])$. To begin with, consider the claim that if $f, g \in R([a,b])$ then

$$\int_a^b (f+g)(x)\ dF = \int_a^b f(x)\ dF + \int_a^b g(x)\ dF. \tag{3.7}$$

Let P_1,Q_1 be such that

$$U(f, Q_1) - L(f, Q_1) < \varepsilon/2,\ U(g, P_1) - L(g, P_1) < \varepsilon/2.$$

Then letting $P \equiv P_1 \cup Q_1$, Lemma 3.2 implies

$$U(f, P) - L(f, P) < \varepsilon/2, \text{ and } U(g, P) - U(g, P) < \varepsilon/2.$$

Next note that

$$m_i(f+g) \geq m_i(f) + m_i(g),\ M_i(f+g) \leq M_i(f) + M_i(g).$$

Therefore,

$$L(g+f, P) \geq L(f, P) + L(g, P),\ U(g+f, P) \leq U(f, P) + U(g, P).$$

For this partition,

$$\int_a^b (f+g)(x)\ dF \in [L(f+g, P), U(f+g, P)]$$
$$\subseteq [L(f, P) + L(g, P), U(f, P) + U(g, P)]$$

and

$$\int_a^b f(x)\ dF + \int_a^b g(x)\ dF \in [L(f, P) + L(g, P), U(f, P) + U(g, P)].$$

Therefore,

$$\left| \int_a^b (f+g)(x)\ dF - \left(\int_a^b f(x)\ dF + \int_a^b g(x)\ dF \right) \right| \leq$$

$$U(f, P) + U(g, P) - (L(f, P) + L(g, P)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This proves 3.7 since ε is arbitrary.

It remains to show that

$$\alpha \int_a^b f(x)\ dF = \int_a^b \alpha f(x)\ dF.$$

Suppose first that $\alpha \geq 0$. Then

$$\int_a^b \alpha f(x)\ dF \equiv \sup\{L(\alpha f, P) : P \text{ is a partition}\} =$$

$$\alpha \sup\{L(f, P) : P \text{ is a partition}\} \equiv \alpha \int_a^b f(x)\ dF.$$

If $\alpha < 0$, then this and Lemma 3.14 imply

$$\int_a^b \alpha f(x)\ dF = \int_a^b (-\alpha)(-f(x))\ dF$$

$$= (-\alpha) \int_a^b (-f(x))\ dF = \alpha \int_a^b f(x)\ dF.$$

This proves the theorem.

In the next theorem, suppose F is defined on $[a, b] \cup [b, c]$.

Theorem 3.16 *If $f \in R([a, b])$ and $f \in R([b, c])$, then $f \in R([a, c])$ and*

$$\int_a^c f(x)\ dF = \int_a^b f(x)\ dF + \int_b^c f(x)\ dF. \tag{3.8}$$

Proof: Let P_1 be a partition of $[a, b]$ and P_2 be a partition of $[b, c]$ such that

$$U(f, P_i) - L(f, P_i) < \varepsilon/2,\ i = 1, 2.$$

Let $P \equiv P_1 \cup P_2$. Then P is a partition of $[a, c]$ and

$$U(f, P) - L(f, P)$$
$$= U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{3.9}$$

Thus, $f \in R([a, c])$ by the Riemann criterion and also for this partition,

$$\int_a^b f(x)\ dF + \int_b^c f(x)\ dF \in [L(f, P_1) + L(f, P_2), U(f, P_1) + U(f, P_2)]$$
$$= [L(f, P), U(f, P)]$$

and

$$\int_a^c f(x)\ dF \in [L(f, P), U(f, P)].$$

Hence by 3.9,

$$\left| \int_a^c f(x)\ dF - \left(\int_a^b f(x)\ dF + \int_b^c f(x)\ dF \right) \right| < U(f, P) - L(f, P) < \varepsilon$$

which shows that since ε is arbitrary, 3.8 holds. This proves the theorem.

Corollary 3.17 *Let F be continuous and let $[a,b]$ be a closed and bounded interval and suppose that*

$$a = y_1 < y_2 \cdots < y_l = b$$

and that f is a bounded function defined on $[a,b]$ which has the property that f is either increasing on $[y_j, y_{j+1}]$ or decreasing on $[y_j, y_{j+1}]$ for $j = 1, \cdots, l-1$. Then $f \in R([a,b])$.

Proof: This follows from Theorem 3.16 and Theorem 3.10.

The symbol, $\int_a^b f(x)\, dF$ when $a > b$ has not yet been defined.

Definition 3.18 *Let $[a,b]$ be an interval and let $f \in R([a,b])$. Then*

$$\int_b^a f(x)\, dF \equiv -\int_a^b f(x)\, dF.$$

Note that with this definition,

$$\int_a^a f(x)\, dF = -\int_a^a f(x)\, dF$$

and so

$$\int_a^a f(x)\, dF = 0.$$

Theorem 3.19 *Assuming all the integrals make sense,*

$$\int_a^b f(x)\, dF + \int_b^c f(x)\, dF = \int_a^c f(x)\, dF.$$

Proof: This follows from Theorem 3.16 and Definition 3.18. For example, assume

$$c \in (a,b).$$

Then from Theorem 3.16,

$$\int_a^c f(x)\, dF + \int_c^b f(x)\, dF = \int_a^b f(x)\, dF$$

and so by Definition 3.18,

$$\begin{aligned}\int_a^c f(x)\, dF &= \int_a^b f(x)\, dF - \int_c^b f(x)\, dF \\ &= \int_a^b f(x)\, dF + \int_b^c f(x)\, dF.\end{aligned}$$

The other cases are similar.

The following properties of the integral have either been established or they follow quickly from what has been shown so far.

$$\text{If } f \in R([a,b]) \text{ then if } c \in [a,b]\,,\ f \in R([a,c])\,, \tag{3.10}$$

$$\int_a^b \alpha\, dF = \alpha\left(F(b) - F(a)\right), \tag{3.11}$$

$$\int_a^b (\alpha f + \beta g)(x)\ dF = \alpha \int_a^b f(x)\ dF + \beta \int_a^b g(x)\ dF, \tag{3.12}$$

$$\int_a^b f(x)\ dF + \int_b^c f(x)\ dF = \int_a^c f(x)\ dF, \tag{3.13}$$

$$\int_a^b f(x)\ dF \geq 0 \text{ if } f(x) \geq 0 \text{ and } a < b, \tag{3.14}$$

$$\left|\int_a^b f(x)\ dF\right| \leq \left|\int_a^b |f(x)|\ dF\right|. \tag{3.15}$$

The only one of these claims which may not be completely obvious is the last one. To show this one, note that

$$|f(x)| - f(x) \geq 0,\ |f(x)| + f(x) \geq 0.$$

Therefore, by 3.14 and 3.12, if $a < b$,

$$\int_a^b |f(x)|\ dF \geq \int_a^b f(x)\ dF$$

and

$$\int_a^b |f(x)|\ dF \geq -\int_a^b f(x)\ dF.$$

Therefore,

$$\int_a^b |f(x)|\ dF \geq \left|\int_a^b f(x)\ dF\right|.$$

If $b < a$ then the above inequality holds with a and b switched. This implies 3.15.

3.5 Fundamental Theorem Of Calculus

In this section $F(x) = x$ so things are specialized to the ordinary Riemann integral. With these properties, it is easy to prove the fundamental theorem of calculus[2].

[2]This theorem is why Newton and Liebnitz are credited with inventing calculus. The integral had been around for thousands of years and the derivative was by their time well known. However the connection between these two ideas had not been fully made although Newton's predecessor, Isaac Barrow had made some progress in this direction.

Let $f \in R([a,b])$. Then by 3.10 $f \in R([a,x])$ for each $x \in [a,b]$. The first version of the fundamental theorem of calculus is a statement about the derivative of the function

$$x \to \int_a^x f(t)\, dt.$$

Theorem 3.20 *Let $f \in R([a,b])$ and let*

$$F(x) \equiv \int_a^x f(t)\, dt.$$

Then if f is continuous at $x \in (a,b)$,

$$F'(x) = f(x).$$

Proof: Let $x \in (a,b)$ be a point of continuity of f and let h be small enough that $x+h \in [a,b]$. Then by using 3.13,

$$h^{-1}(F(x+h) - F(x)) = h^{-1} \int_x^{x+h} f(t)\, dt.$$

Also, using 3.11,

$$f(x) = h^{-1} \int_x^{x+h} f(x)\, dt.$$

Therefore, by 3.15,

$$\left|h^{-1}(F(x+h) - F(x)) - f(x)\right| = \left|h^{-1} \int_x^{x+h} (f(t) - f(x))\, dt\right|$$

$$\leq \left|h^{-1} \int_x^{x+h} |f(t) - f(x)|\, dt\right|.$$

Let $\varepsilon > 0$ and let $\delta > 0$ be small enough that if $|t - x| < \delta$, then

$$|f(t) - f(x)| < \varepsilon.$$

Therefore, if $|h| < \delta$, the above inequality and 3.11 shows that

$$\left|h^{-1}(F(x+h) - F(x)) - f(x)\right| \leq |h|^{-1} \varepsilon |h| = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows

$$\lim_{h \to 0} h^{-1}(F(x+h) - F(x)) = f(x)$$

and this proves the theorem.

Note this gives existence for the initial value problem,

$$F'(x) = f(x),\ F(a) = 0$$

whenever f is Riemann integrable and continuous.[3]

The next theorem is also called the fundamental theorem of calculus.

Theorem 3.21 *Let $f \in R([a,b])$ and suppose there exists an antiderivative for f, G, such that*

$$G'(x) = f(x)$$

for every point of (a,b) and G is continuous on $[a,b]$. Then

$$\int_a^b f(x)\, dx = G(b) - G(a). \tag{3.16}$$

Proof: Let $P = \{x_0, \cdots, x_n\}$ be a partition satisfying

$$U(f,P) - L(f,P) < \varepsilon.$$

Then

$$\begin{aligned} G(b) - G(a) &= G(x_n) - G(x_0) \\ &= \sum_{i=1}^{n} G(x_i) - G(x_{i-1}). \end{aligned}$$

By the mean value theorem,

$$\begin{aligned} G(b) - G(a) &= \sum_{i=1}^{n} G'(z_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f(z_i)\, \Delta x_i \end{aligned}$$

where z_i is some point in $[x_{i-1}, x_i]$. It follows, since the above sum lies between the upper and lower sums, that

$$G(b) - G(a) \in [L(f,P), U(f,P)],$$

and also

$$\int_a^b f(x)\, dx \in [L(f,P), U(f,P)].$$

Therefore,

$$\left| G(b) - G(a) - \int_a^b f(x)\, dx \right| < U(f,P) - L(f,P) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, 3.16 holds. This proves the theorem.

[3] Of course it was proved that if f is continuous on a closed interval, $[a,b]$, then $f \in R([a,b])$ but this is a hard theorem using the difficult result about uniform continuity.

The following notation is often used in this context. Suppose F is an antiderivative of f as just described with F continuous on $[a,b]$ and $F' = f$ on (a,b). Then

$$\int_a^b f(x)\ dx = F(b) - F(a) \equiv F(x)|_a^b.$$

Definition 3.22 *Let f be a bounded function defined on a closed interval $[a,b]$ and let $P \equiv \{x_0, \cdots, x_n\}$ be a partition of the interval. Suppose $z_i \in [x_{i-1}, x_i]$ is chosen. Then the sum*

$$\sum_{i=1}^{n} f(z_i)(x_i - x_{i-1})$$

is known as a Riemann sum. Also,

$$||P|| \equiv \max\{|x_i - x_{i-1}| : i = 1, \cdots, n\}.$$

Proposition 3.23 *Suppose $f \in R([a,b])$. Then there exists a partition, $P \equiv \{x_0, \cdots, x_n\}$ with the property that for any choice of $z_k \in [x_{k-1}, x_k]$,*

$$\left|\int_a^b f(x)\ dx - \sum_{k=1}^{n} f(z_k)(x_k - x_{k-1})\right| < \varepsilon.$$

Proof: Choose P such that $U(f,P) - L(f,P) < \varepsilon$ and then both $\int_a^b f(x)\ dx$ and $\sum_{k=1}^n f(z_k)(x_k - x_{k-1})$ are contained in $[L(f,P), U(f,P)]$ and so the claimed inequality must hold. This proves the proposition.

It is significant because it gives a way of approximating the integral.

The definition of Riemann integrability given in this chapter is also called Darboux integrability and the integral defined as the unique number which lies between all upper sums and all lower sums which is given in this chapter is called the Darboux integral . The definition of the Riemann integral in terms of Riemann sums is given next.

Definition 3.24 *A bounded function, f defined on $[a,b]$ is said to be Riemann integrable if there exists a number, I with the property that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if*

$$P \equiv \{x_0, x_1, \cdots, x_n\}$$

is any partition having $||P|| < \delta$, and $z_i \in [x_{i-1}, x_i]$,

$$\left|I - \sum_{i=1}^{n} f(z_i)(x_i - x_{i-1})\right| < \varepsilon.$$

The number $\int_a^b f(x)\ dx$ is defined as I.

Thus, there are two definitions of the Riemann integral. It turns out they are equivalent which is the following theorem of of Darboux.

Theorem 3.25 *A bounded function defined on $[a,b]$ is Riemann integrable in the sense of Definition 3.24 if and only if it is integrable in the sense of Darboux. Furthermore the two integrals coincide.*

The proof of this theorem is left for the exercises in Problems 10 - 12. It isn't essential that you understand this theorem so if it does not interest you, leave it out. Note that it implies that given a Riemann integrable function f in either sense, it can be approximated by Riemann sums whenever $||P||$ is sufficiently small. Both versions of the integral are obsolete but entirely adequate for most applications and as a point of departure for a more up to date and satisfactory integral. The reason for using the Darboux approach to the integral is that all the existence theorems are easier to prove in this context.

3.6 Exercises

1. Let $F(x) = \int_{x^2}^{x^3} \frac{t^5+7}{t^7+87t^6+1}\,dt$. Find $F'(x)$.

2. Let $F(x) = \int_2^x \frac{1}{1+t^4}\,dt$. Sketch a graph of F and explain why it looks the way it does.

3. Let a and b be positive numbers and consider the function,
$$F(x) = \int_0^{ax} \frac{1}{a^2+t^2}\,dt + \int_b^{a/x} \frac{1}{a^2+t^2}\,dt.$$
Show that F is a constant.

4. Solve the following initial value problem from ordinary differential equations which is to find a function y such that
$$y'(x) = \frac{x^7+1}{x^6+97x^5+7},\ y(10) = 5.$$

5. If $F, G \in \int f(x)\,dx$ for all $x \in \mathbb{R}$, show $F(x) = G(x) + C$ for some constant, C. Use this to give a different proof of the fundamental theorem of calculus which has for its conclusion $\int_a^b f(t)\,dt = G(b) - G(a)$ where $G'(x) = f(x)$.

6. Suppose f is Riemann integrable on $[a,b]$ and continuous. (In fact continuous implies Riemann integrable.) Show there exists $c \in (a,b)$ such that
$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$
Hint: You might consider the function $F(x) \equiv \int_a^x f(t)\,dt$ and use the mean value theorem for derivatives and the fundamental theorem of calculus.

7. Suppose f and g are continuous functions on $[a,b]$ and that $g(x) \neq 0$ on (a,b). Show there exists $c \in (a,b)$ such that
$$f(c)\int_a^b g(x)\,dx = \int_a^b f(x)\,g(x)\,dx.$$
Hint: Define $F(x) \equiv \int_a^x f(t)\,g(t)\,dt$ and let $G(x) \equiv \int_a^x g(t)\,dt$. Then use the Cauchy mean value theorem on these two functions.

8. Consider the function
$$f(x) \equiv \begin{cases} \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}.$$
Is f Riemann integrable? Explain why or why not.

9. Prove the second part of Theorem 3.10 about decreasing functions.

10. Suppose f is a bounded function defined on $[a,b]$ and $|f(x)| < M$ for all $x \in [a,b]$. Now let Q be a partition having n points, $\{x_0^*, \cdots, x_n^*\}$ and let P be any other partition. Show that
$$|U(f,P) - L(f,P)| \leq 2Mn\,||P|| + |U(f,Q) - L(f,Q)|.$$
Hint: Write the sum for $U(f,P) - L(f,P)$ and split this sum into two sums, the sum of terms for which $[x_{i-1}, x_i]$ contains at least one point of Q, and terms for which $[x_{i-1}, x_i]$ does not contain any points of Q. In the latter case, $[x_{i-1}, x_i]$ must be contained in some interval, $\left[x_{k-1}^*, x_k^*\right]$. Therefore, the sum of these terms should be no larger than $|U(f,Q) - L(f,Q)|$.

11. ↑ If $\varepsilon > 0$ is given and f is a Darboux integrable function defined on $[a,b]$, show there exists $\delta > 0$ such that whenever $||P|| < \delta$, then
$$|U(f,P) - L(f,P)| < \varepsilon.$$

12. ↑ Prove Theorem 3.25.

Some Important Linear Algebra

This chapter contains some important linear algebra as distinguished from that which is normally presented in undergraduate courses consisting mainly of uninteresting things you can do with row operations.

The notation, $\mathbb{C}^n$ refers to the collection of ordered lists of n complex numbers. Since every real number is also a complex number, this simply generalizes the usual notion of $\mathbb{R}^n$, the collection of all ordered lists of n real numbers. In order to avoid worrying about whether it is real or complex numbers which are being referred to, the symbol $\mathbb{F}$ will be used. If it is not clear, always pick $\mathbb{C}$.

Definition 4.1 *Define*

$$\mathbb{F}^n \equiv \{(x_1, \cdots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \cdots, n\}.$$

$(x_1, \cdots, x_n) = (y_1, \cdots, y_n)$ *if and only if for all* $j = 1, \cdots, n$, $x_j = y_j$. *When*

$$(x_1, \cdots, x_n) \in \mathbb{F}^n,$$

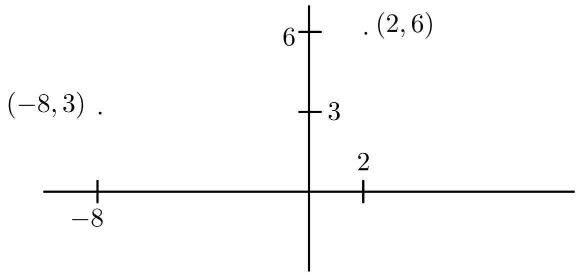
it is conventional to denote $(x_1, \cdots, x_n)$ *by the single bold face letter,* $\mathbf{x}$. *The numbers,* x_j *are called the coordinates. The set*

$$\{(0, \cdots, 0, t, 0, \cdots, 0) : t \in \mathbb{F}\}$$

for t *in the* i^{th} *slot is called the* i^{th} *coordinate axis. The point* $\mathbf{0} \equiv (0, \cdots, 0)$ *is called the origin.*

Thus $(1, 2, 4i) \in \mathbb{F}^3$ and $(2, 1, 4i) \in \mathbb{F}^3$ but $(1, 2, 4i) \neq (2, 1, 4i)$ because, even though the same numbers are involved, they don't match up. In particular, the first entries are not equal.

The geometric significance of $\mathbb{R}^n$ for $n \leq 3$ has been encountered already in calculus or in precalculus. Here is a short review. First consider the case when $n = 1$. Then from the definition, $\mathbb{R}^1 = \mathbb{R}$. Recall that $\mathbb{R}$ is identified with the points of a line. Look at the number line again. Observe that this amounts to identifying a point on this line with a real number. In other words a real number determines where you are on this line. Now suppose $n = 2$ and consider two lines which intersect each other at right angles as shown in the following picture.



Notice how you can identify a point shown in the plane with the ordered pair, $(2, 6)$. You go to the right a distance of 2 and then up a distance of 6. Similarly, you can identify another point in the plane with the ordered pair $(-8, 3)$. Go to the left a distance of 8 and then up a distance of 3. The reason you go to the left is that there is a $-$ sign on the eight. From this reasoning, every ordered pair determines a unique point in the plane. Conversely, taking a point in the plane, you could draw two lines through the point, one vertical and the other horizontal and determine unique points, x_1 on the horizontal line in the above picture and x_2 on the vertical line in the above picture, such that the point of interest is identified with the ordered pair, (x_1, x_2). In short, points in the plane can be identified with ordered pairs similar to the way that points on the real line are identified with real numbers. Now suppose $n = 3$. As just explained, the first two coordinates determine a point in a plane. Letting the third component determine how far up or down you go, depending on whether this number is positive or negative, this determines a point in space. Thus, $(1, 4, -5)$ would mean to determine the point in the plane that goes with $(1, 4)$ and then to go below this plane a distance of 5 to obtain a unique point in space. You see that the ordered triples correspond to points in space just as the ordered pairs correspond to points in a plane and single real numbers correspond to points on a line.

You can't stop here and say that you are only interested in $n \leq 3$. What if you were interested in the motion of two objects? You would need three coordinates to describe where the first object is and you would need another three coordinates to describe where the other object is located. Therefore, you would need to be considering $\mathbb{R}^6$. If the two objects moved around, you would need a time coordinate as well. As another example, consider a hot object which is cooling and suppose you want the temperature of this object. How many coordinates would be needed? You would need one for the temperature, three for the position of the point in the object and one more for the time. Thus you would need to be considering $\mathbb{R}^5$. Many other examples can be given. Sometimes n is very large. This is often the case in applications to business when they are trying to maximize profit subject to constraints. It also occurs in numerical analysis when people try to solve hard problems on a computer.

There are other ways to identify points in space with three numbers but the one presented is the most basic. In this case, the coordinates are known as Cartesian coordinates after Descartes[1] who invented this idea in the first half of the seventeenth century. I will often not bother to draw a distinction between the point in n dimensional space and its Cartesian coordinates.

The geometric significance of $\mathbb{C}^n$ for $n > 1$ is not available because each copy of $\mathbb{C}$ corresponds to the plane or $\mathbb{R}^2$.

4.1 Algebra in $\mathbb{F}^n$

There are two algebraic operations done with elements of $\mathbb{F}^n$. One is addition and the other is multiplication by numbers, called scalars. In the case of $\mathbb{C}^n$ the scalars are complex numbers while in the case of $\mathbb{R}^n$ the only allowed scalars are real numbers. Thus, the scalars always come from $\mathbb{F}$ in either case.

Definition 4.2 *If $\mathbf{x} \in \mathbb{F}^n$ and $a \in \mathbb{F}$, also called a scalar, then $a\mathbf{x} \in \mathbb{F}^n$ is defined by*

$$a\mathbf{x} = a(x_1, \cdots, x_n) \equiv (ax_1, \cdots, ax_n). \tag{4.1}$$

This is known as scalar multiplication. If $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ then $\mathbf{x} + \mathbf{y} \in \mathbb{F}^n$ and is defined by

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1, \cdots, x_n) + (y_1, \cdots, y_n) \\ &\equiv (x_1 + y_1, \cdots, x_n + y_n) \end{aligned} \tag{4.2}$$

With this definition, the algebraic properties satisfy the conclusions of the following theorem.

Theorem 4.3 *For $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$ and α, β scalars, (real numbers), the following hold.*

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}, \tag{4.3}$$

the commutative law of addition,

$$(\mathbf{v} + \mathbf{w}) + \mathbf{z} = \mathbf{v} + (\mathbf{w} + \mathbf{z}), \tag{4.4}$$

the associative law for addition,

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \tag{4.5}$$

the existence of an additive identity,

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}, \tag{4.6}$$

[1] René Descartes 1596-1650 is often credited with inventing analytic geometry although it seems the ideas were actually known much earlier. He was interested in many different subjects, physiology, chemistry, and physics being some of them. He also wrote a large book in which he tried to explain the book of Genesis scientifically. Descartes ended up dying in Sweden.

the existence of an additive inverse, Also

$$\alpha(\mathbf{v}+\mathbf{w}) = \alpha\mathbf{v}+\alpha\mathbf{w}, \tag{4.7}$$

$$(\alpha+\beta)\,\mathbf{v} = \alpha\mathbf{v}+\beta\mathbf{v}, \tag{4.8}$$

$$\alpha(\beta\mathbf{v}) = \alpha\beta(\mathbf{v}), \tag{4.9}$$

$$1\mathbf{v} = \mathbf{v}. \tag{4.10}$$

In the above $\mathbf{0} = (0,\cdots,0)$.

You should verify these properties all hold. For example, consider 4.7

$$\begin{aligned}\alpha(\mathbf{v}+\mathbf{w}) &= \alpha(v_1+w_1,\cdots,v_n+w_n)\\ &= (\alpha(v_1+w_1),\cdots,\alpha(v_n+w_n))\\ &= (\alpha v_1+\alpha w_1,\cdots,\alpha v_n+\alpha w_n)\\ &= (\alpha v_1,\cdots,\alpha v_n)+(\alpha w_1,\cdots,\alpha w_n)\\ &= \alpha\mathbf{v}+\alpha\mathbf{w}.\end{aligned}$$

As usual subtraction is defined as $\mathbf{x}-\mathbf{y}\equiv\mathbf{x}+(-\mathbf{y})$.

4.2 Exercises

1. Verify all the properties 4.3-4.10.
2. Compute $5(1,2+3i,3,-2)+6(2-i,1,-2,7)$.
3. Draw a picture of the points in $\mathbb{R}^2$ which are determined by the following ordered pairs.
   (a) $(1,2)$
   (b) $(-2,-2)$
   (c) $(-2,3)$
   (d) $(2,-5)$
4. Does it make sense to write $(1,2)+(2,3,1)$? Explain.
5. Draw a picture of the points in $\mathbb{R}^3$ which are determined by the following ordered triples.
   (a) $(1,2,0)$
   (b) $(-2,-2,1)$
   (c) $(-2,3,-2)$

4.3 Subspaces Spans And Bases

Definition 4.4 *Let* $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ *be vectors in* $\mathbb{F}^n$. *A linear combination is any expression of the form*

$$\sum_{i=1}^{p} c_i \mathbf{x}_i$$

where the c_i *are scalars. The set of all linear combinations of these vectors is called* $\operatorname{span}(\mathbf{x}_1, \cdots, \mathbf{x}_n)$. *If* $V \subseteq \mathbb{F}^n$, *then* V *is called a subspace if whenever* α, β *are scalars and* $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors of* V, *it follows* $\alpha\mathbf{u} + \beta\mathbf{v} \in V$. *That is, it is "closed under the algebraic operations of vector addition and scalar multiplication". A linear combination of vectors is said to be trivial if all the scalars in the linear combination equal zero. A set of vectors is said to be linearly independent if the only linear combination of these vectors which equals the zero vector is the trivial linear combination. Thus* $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ *is called linearly independent if whenever*

$$\sum_{k=1}^{p} c_k \mathbf{x}_k = \mathbf{0}$$

it follows that all the scalars, c_k *equal zero. A set of vectors,* $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$, *is called linearly dependent if it is not linearly independent. Thus the set of vectors is linearly dependent if there exist scalars,* $c_i, i = 1, \cdots, n$, *not all zero such that* $\sum_{k=1}^{p} c_k \mathbf{x}_k = \mathbf{0}$.

Lemma 4.5 *A set of vectors* $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ *is linearly independent if and only if none of the vectors can be obtained as a linear combination of the others.*

Proof: Suppose first that $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ is linearly independent. If

$$\mathbf{x}_k = \sum_{j \neq k} c_j \mathbf{x}_j,$$

then

$$\mathbf{0} = 1\mathbf{x}_k + \sum_{j \neq k} (-c_j) \mathbf{x}_j,$$

a nontrivial linear combination, contrary to assumption. This shows that if the set is linearly independent, then none of the vectors is a linear combination of the others.

Now suppose no vector is a linear combination of the others. Is $\{\mathbf{x}_1, \cdots, \mathbf{x}_p\}$ linearly independent? If it is not there exist scalars, c_i, not all zero such that

$$\sum_{i=1}^{p} c_i \mathbf{x}_i = \mathbf{0}.$$

Say $c_k \neq 0$. Then you can solve for $\mathbf{x}_k$ as

$$\mathbf{x}_k = \sum_{j \neq k} (-c_j) / c_k \mathbf{x}_j$$

contrary to assumption. This proves the lemma.

The following is called the exchange theorem.

Theorem 4.6 *(Exchange Theorem) Let* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *be a linearly independent set of vectors such that each* $\mathbf{x}_i$ *is in span*$(\mathbf{y}_1, \cdots, \mathbf{y}_s)$. *Then* $r \leq s$.

Proof: Define $\text{span}\{\mathbf{y}_1, \cdots, \mathbf{y}_s\} \equiv V$, it follows there exist scalars, $c_1, \cdots, c_s$ such that

$$\mathbf{x}_1 = \sum_{i=1}^{s} c_i \mathbf{y}_i. \tag{4.11}$$

Not all of these scalars can equal zero because if this were the case, it would follow that $\mathbf{x}_1 = 0$ and so $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ would not be linearly independent. Indeed, if $\mathbf{x}_1 = \mathbf{0}$, $1\mathbf{x}_1 + \sum_{i=2}^{r} 0\mathbf{x}_i = \mathbf{x}_1 = \mathbf{0}$ and so there would exist a nontrivial linear combination of the vectors $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ which equals zero.

Say $c_k \neq 0$. Then solve (4.11) for $\mathbf{y}_k$ and obtain

$$\mathbf{y}_k \in \text{span}\left(\mathbf{x}_1, \overbrace{\mathbf{y}_1, \cdots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \cdots, \mathbf{y}_s}^{\text{s-1 vectors here}}\right).$$

Define $\{\mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$ by

$$\{\mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\} \equiv \{\mathbf{y}_1, \cdots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \cdots, \mathbf{y}_s\}$$

Therefore, $\text{span}\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\} = V$ because if $\mathbf{v} \in V$, there exist constants $c_1, \cdots, c_s$ such that

$$\mathbf{v} = \sum_{i=1}^{s-1} c_i \mathbf{z}_i + c_s \mathbf{y}_k.$$

Now replace the $\mathbf{y}_k$ in the above with a linear combination of the vectors,

$$\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}$$

to obtain

$$\mathbf{v} \in \text{span}\{\mathbf{x}_1, \mathbf{z}_1, \cdots, \mathbf{z}_{s-1}\}.$$

The vector $\mathbf{y}_k$, in the list $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$, has now been replaced with the vector $\mathbf{x}_1$ and the resulting modified list of vectors has the same span as the original list of vectors, $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$.

Now suppose that $r > s$ and that

$$\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_l, \mathbf{z}_1, \cdots, \mathbf{z}_p) = V$$

where the vectors, $\mathbf{z}_1, \cdots, \mathbf{z}_p$ are each taken from the set, $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$ and $l + p = s$. This has now been done for $l = 1$ above. Then since $r > s$, it follows that $l \leq s < r$

and so $l+1 \leq r$. Therefore, $\mathbf{x}_{l+1}$ is a vector not in the list, $\{\mathbf{x}_1, \cdots, \mathbf{x}_l\}$ and since $\text{span}\{\mathbf{x}_1, \cdots, \mathbf{x}_l, \mathbf{z}_1, \cdots, \mathbf{z}_p\} = V$, there exist scalars, c_i and d_j such that

$$\mathbf{x}_{l+1} = \sum_{i=1}^{l} c_i \mathbf{x}_i + \sum_{j=1}^{p} d_j \mathbf{z}_j. \tag{4.12}$$

Now not all the d_j can equal zero because if this were so, it would follow that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ would be a linearly dependent set because one of the vectors would equal a linear combination of the others. Therefore, (4.12) can be solved for one of the $\mathbf{z}_i$, say $\mathbf{z}_k$, in terms of $\mathbf{x}_{l+1}$ and the other $\mathbf{z}_i$ and just as in the above argument, replace that $\mathbf{z}_i$ with $\mathbf{x}_{l+1}$ to obtain

$$\text{span}\left(\mathbf{x}_1, \cdots \mathbf{x}_l, \mathbf{x}_{l+1}, \overbrace{\mathbf{z}_1, \cdots \mathbf{z}_{k-1}, \mathbf{z}_{k+1}, \cdots, \mathbf{z}_p}^{\text{p-1 vectors here}}\right) = V.$$

Continue this way, eventually obtaining

$$\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_s) = V.$$

But then $\mathbf{x}_r \in \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_s)$ contrary to the assumption that $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ is linearly independent. Therefore, $r \leq s$ as claimed.

Definition 4.7 *A finite set of vectors, $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ is a basis for $\mathbb{F}^n$ if*

$$\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_r) = \mathbb{F}^n$$

and $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ is linearly independent.

Corollary 4.8 *Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ and $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$ be two bases[2] of $\mathbb{F}^n$. Then $r = s = n$.*

Proof: From the exchange theorem, $r \leq s$ and $s \leq r$. Now note the vectors,

$$\mathbf{e}_i = \overbrace{(0, \cdots, 0, 1, 0 \cdots, 0)}^{1 \text{ is in the } i^{th} \text{ slot}}$$

for $i = 1, 2, \cdots, n$ are a basis for $\mathbb{F}^n$. This proves the corollary.

Lemma 4.9 *Let $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ be a set of vectors. Then $V \equiv \text{span}(\mathbf{v}_1, \cdots, \mathbf{v}_r)$ is a subspace.*

[2]This is the plural form of basis. We could say basiss but it would involve an inordinate amount of hissing as in "The sixth shiek's sixth sheep is sick". This is the reason that bases is used instead of basiss.

Proof: Suppose α, β are two scalars and let $\sum_{k=1}^{r} c_k \mathbf{v}_k$ and $\sum_{k=1}^{r} d_k \mathbf{v}_k$ are two elements of V. What about

$$\alpha \sum_{k=1}^{r} c_k \mathbf{v}_k + \beta \sum_{k=1}^{r} d_k \mathbf{v}_k?$$

Is it also in V?

$$\alpha \sum_{k=1}^{r} c_k \mathbf{v}_k + \beta \sum_{k=1}^{r} d_k \mathbf{v}_k = \sum_{k=1}^{r} (\alpha c_k + \beta d_k) \mathbf{v}_k \in V$$

so the answer is yes. This proves the lemma.

Definition 4.10 *A finite set of vectors,* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *is a basis for a subspace,* V *of* $\mathbb{F}^n$ *if* $\operatorname{span}(\mathbf{x}_1, \cdots, \mathbf{x}_r) = V$ *and* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *is linearly independent.*

Corollary 4.11 *Let* $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ *and* $\{\mathbf{y}_1, \cdots, \mathbf{y}_s\}$ *be two bases for* V*. Then* $r = s$*.*

Proof: From the exchange theorem, $r \leq s$ and $s \leq r$. Therefore, this proves the corollary.

Definition 4.12 *Let* V *be a subspace of* $\mathbb{F}^n$*. Then* $\dim(V)$ *read as the dimension of* V *is the number of vectors in a basis.*

Of course you should wonder right now whether an arbitrary subspace even has a basis. In fact it does and this is in the next theorem. First, here is an interesting lemma.

Lemma 4.13 *Suppose* $\mathbf{v} \notin \operatorname{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k)$ *and* $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ *is linearly independent. Then* $\{\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{v}\}$ *is also linearly independent.*

Proof: Suppose $\sum_{i=1}^{k} c_i \mathbf{u}_i + d\mathbf{v} = \mathbf{0}$. It is required to verify that each $c_i = 0$ and that $d = 0$. But if $d \neq 0$, then you can solve for $\mathbf{v}$ as a linear combination of the vectors, $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$,

$$\mathbf{v} = -\sum_{i=1}^{k} \left(\frac{c_i}{d}\right) \mathbf{u}_i$$

contrary to assumption. Therefore, $d = 0$. But then $\sum_{i=1}^{k} c_i \mathbf{u}_i = 0$ and the linear independence of $\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$ implies each $c_i = 0$ also. This proves the lemma.

Theorem 4.14 *Let* V *be a nonzero subspace of* $\mathbb{F}^n$*. Then* V *has a basis.*

Proof: Let $\mathbf{v}_1 \in V$ where $\mathbf{v}_1 \neq 0$. If $\operatorname{span}\{\mathbf{v}_1\} = V$, stop. $\{\mathbf{v}_1\}$ is a basis for V. Otherwise, there exists $\mathbf{v}_2 \in V$ which is not in $\operatorname{span}\{\mathbf{v}_1\}$. By Lemma 4.13 $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set of vectors. If $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} = V$ stop, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V. If $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \neq V$, then there exists $\mathbf{v}_3 \notin \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a larger linearly independent set of vectors. Continuing this way, the process must stop before $n+1$ steps because if not, it would be possible to obtain $n+1$ linearly independent vectors contrary to the exchange theorem. This proves the theorem.

In words the following corollary states that any linearly independent set of vectors can be enlarged to form a basis.

Corollary 4.15 *Let V be a subspace of $\mathbb{F}^n$ and let $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ be a linearly independent set of vectors in V. Then either it is a basis for V or there exist vectors, $\mathbf{v}_{r+1}, \cdots, \mathbf{v}_s$ such that $\{\mathbf{v}_1, \cdots, \mathbf{v}_r, \mathbf{v}_{r+1}, \cdots, \mathbf{v}_s\}$ is a basis for V.*

Proof: This follows immediately from the proof of Theorem 31.23. You do exactly the same argument except you start with $\{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ rather than $\{\mathbf{v}_1\}$.

It is also true that any spanning set of vectors can be restricted to obtain a basis.

Theorem 4.16 *Let V be a subspace of $\mathbb{F}^n$ and suppose $\operatorname{span}(\mathbf{u}_1 \cdots, \mathbf{u}_p) = V$ where the $\mathbf{u}_i$ are nonzero vectors. Then there exist vectors, $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ such that $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\} \subseteq \{\mathbf{u}_1 \cdots, \mathbf{u}_p\}$ and $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ is a basis for V.*

Proof: Let r be the smallest positive integer with the property that for some set, $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\} \subseteq \{\mathbf{u}_1 \cdots, \mathbf{u}_p\}$,

$$\operatorname{span}(\mathbf{v}_1 \cdots, \mathbf{v}_r) = V.$$

Then $r \leq p$ and it must be the case that $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ is linearly independent because if it were not so, one of the vectors, say $\mathbf{v}_k$ would be a linear combination of the others. But then you could delete this vector from $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ and the resulting list of $r-1$ vectors would still span V contrary to the definition of r. This proves the theorem.

4.4 An Application To Matrices

The following is a theorem of major significance.

Theorem 4.17 *Suppose A is an $n \times n$ matrix. Then A is one to one if and only if A is onto. Also, if B is an $n \times n$ matrix and $AB = I$, then it follows $BA = I$.*

Proof: First suppose A is one to one. Consider the vectors, $\{A\mathbf{e}_1, \cdots, A\mathbf{e}_n\}$ where $\mathbf{e}_k$ is the column vector which is all zeros except for a 1 in the k^{th} position. This set of vectors is linearly independent because if

$$\sum_{k=1}^{n} c_k A\mathbf{e}_k = \mathbf{0},$$

then since A is linear,

$$A\left(\sum_{k=1}^{n} c_k \mathbf{e}_k\right) = \mathbf{0}$$

and since A is one to one, it follows

$$\sum_{k=1}^{n} c_k \mathbf{e}_k = \mathbf{0}^3$$

which implies each $c_k = 0$. Therefore, $\{A\mathbf{e}_1, \cdots, A\mathbf{e}_n\}$ must be a basis for $\mathbb{F}^n$ because if not there would exist a vector, $\mathbf{y} \notin \text{span}(A\mathbf{e}_1, \cdots, A\mathbf{e}_n)$ and then by Lemma 4.13, $\{A\mathbf{e}_1, \cdots, A\mathbf{e}_n, \mathbf{y}\}$ would be an independent set of vectors having $n+1$ vectors in it, contrary to the exchange theorem. It follows that for $\mathbf{y} \in \mathbb{F}^n$ there exist constants, c_i such that

$$\mathbf{y} = \sum_{k=1}^{n} c_k A\mathbf{e}_k = A\left(\sum_{k=1}^{n} c_k \mathbf{e}_k\right)$$

showing that, since $\mathbf{y}$ was arbitrary, A is onto.

Next suppose A is onto. This means the span of the columns of A equals $\mathbb{F}^n$. If these columns are not linearly independent, then by Lemma 4.5 on Page 55, one of the columns is a linear combination of the others and so the span of the columns of A equals the span of the $n-1$ other columns. This violates the exchange theorem because $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ would be a linearly independent set of vectors contained in the span of only $n-1$ vectors. Therefore, the columns of A must be independent and this equivalent to saying that $A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. This implies A is one to one because if $A\mathbf{x} = A\mathbf{y}$, then $A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$ and so $\mathbf{x} - \mathbf{y} = \mathbf{0}$.

Now suppose $AB = I$. Why is $BA = I$? Since $AB = I$ it follows B is one to one since otherwise, there would exist, $\mathbf{x} \neq \mathbf{0}$ such that $B\mathbf{x} = \mathbf{0}$ and then $AB\mathbf{x} = A\mathbf{0} = \mathbf{0} \neq I\mathbf{x}$. Therefore, from what was just shown, B is also onto. In addition to this, A must be one to one because if $A\mathbf{y} = \mathbf{0}$, then $\mathbf{y} = B\mathbf{x}$ for some $\mathbf{x}$ and then $\mathbf{x} = AB\mathbf{x} = A\mathbf{y} = \mathbf{0}$ showing $\mathbf{y} = \mathbf{0}$. Now from what is given to be so, it follows $(AB)A = A$ and so using the associative law for matrix multiplication,

$$A(BA) - A = A(BA - I) = 0.$$

But this means $(BA - I)\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$ since otherwise, A would not be one to one. Hence $BA = I$ as claimed. This proves the theorem.

This theorem shows that if an $n \times n$ matrix, B acts like an inverse when multiplied on one side of A it follows that $B = A^{-1}$and it will act like an inverse on both sides of A.

The conclusion of this theorem pertains to square matrices only. For example, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \tag{4.13}$$

Then

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

but

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

4.5 The Mathematical Theory Of Determinants

It is assumed the reader is familiar with matrices. However, the topic of determinants is often neglected in linear algebra books these days. Therefore, I will give a fairly quick and grubby treatment of this topic which includes all the main results. Two books which give a good introduction to determinants are Apostol [3] and Rudin [44]. A recent book which also has a good introduction is Baker [7]

Let $(i_1, \cdots, i_n)$ be an ordered list of numbers from $\{1, \cdots, n\}$. This means the order is important so $(1, 2, 3)$ and $(2, 1, 3)$ are different.

The following Lemma will be essential in the definition of the determinant.

Lemma 4.18 *There exists a unique function,* sgn_n *which maps each list of* n *numbers from* $\{1, \cdots, n\}$ *to one of the three numbers,* $0, 1,$ *or* -1 *which also has the following properties.*

$$\operatorname{sgn}_n (1, \cdots, n) = 1 \tag{4.14}$$

$$\operatorname{sgn}_n (i_1, \cdots, p, \cdots, q, \cdots, i_n) = -\operatorname{sgn}_n (i_1, \cdots, q, \cdots, p, \cdots, i_n) \tag{4.15}$$

In words, the second property states that if two of the numbers are switched, the value of the function is multiplied by -1. *Also, in the case where* $n > 1$ *and* $\{i_1, \cdots, i_n\} = \{1, \cdots, n\}$ *so that every number from* $\{1, \cdots, n\}$ *appears in the ordered list,* $(i_1, \cdots, i_n)$,

$$\operatorname{sgn}_n (i_1, \cdots, i_{\theta-1}, n, i_{\theta+1}, \cdots, i_n) \equiv$$

$$(-1)^{n-\theta} \operatorname{sgn}_{n-1} (i_1, \cdots, i_{\theta-1}, i_{\theta+1}, \cdots, i_n) \tag{4.16}$$

where $n = i_\theta$ *in the ordered list,* $(i_1, \cdots, i_n)$.

Proof: To begin with, it is necessary to show the existence of such a function. This is clearly true if $n = 1$. Define $\operatorname{sgn}_1 (1) \equiv 1$ and observe that it works. No switching is possible. In the case where $n = 2$, it is also clearly true. Let $\operatorname{sgn}_2 (1, 2) = 1$ and $\operatorname{sgn}_2 (2, 1) = 0$ while $\operatorname{sgn}_2 (2, 2) = \operatorname{sgn}_2 (1, 1) = 0$ and verify it works. Assuming such a function exists for n, sgn_{n+1} will be defined in terms of sgn_n. If there are any repeated numbers in $(i_1, \cdots, i_{n+1})$, $\operatorname{sgn}_{n+1} (i_1, \cdots, i_{n+1}) \equiv 0$. If there are no repeats, then $n + 1$ appears somewhere in the ordered list. Let θ be the position of the number $n + 1$ in the list. Thus, the list is of the form $(i_1, \cdots, i_{\theta-1}, n + 1, i_{\theta+1}, \cdots, i_{n+1})$. From 4.16 it must be that

$$\operatorname{sgn}_{n+1} (i_1, \cdots, i_{\theta-1}, n + 1, i_{\theta+1}, \cdots, i_{n+1}) \equiv$$

$$(-1)^{n+1-\theta} \operatorname{sgn}_n (i_1, \cdots, i_{\theta-1}, i_{\theta+1}, \cdots, i_{n+1}).$$

It is necessary to verify this satisfies 4.14 and 4.15 with n replaced with $n + 1$. The first of these is obviously true because

$$\operatorname{sgn}_{n+1} (1, \cdots, n, n + 1) \equiv (-1)^{n+1-(n+1)} \operatorname{sgn}_n (1, \cdots, n) = 1.$$

If there are repeated numbers in $(i_1, \cdots, i_{n+1})$, then it is obvious 4.15 holds because both sides would equal zero from the above definition. It remains to verify 4.15 in the case where there are no numbers repeated in $(i_1, \cdots, i_{n+1})$. Consider

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{p}, \cdots, \overset{s}{q}, \cdots, i_{n+1}\right),$$

where the r above the p indicates the number, p is in the r^{th} position and the s above the q indicates that the number, q is in the s^{th} position. Suppose first that $r < \theta < s$. Then

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{p}, \cdots, n \overset{\theta}{+} 1, \cdots, \overset{s}{q}, \cdots, i_{n+1}\right) \equiv$$

$$(-1)^{n+1-\theta} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{p}, \cdots, \overset{s-1}{q}, \cdots, i_{n+1}\right)$$

while

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{q}, \cdots, n \overset{\theta}{+} 1, \cdots, \overset{s}{p}, \cdots, i_{n+1}\right) =$$

$$(-1)^{n+1-\theta} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{q}, \cdots, \overset{s-1}{p}, \cdots, i_{n+1}\right)$$

and so, by induction, a switch of p and q introduces a minus sign in the result. Similarly, if $\theta > s$ or if $\theta < r$ it also follows that 4.15 holds. The interesting case is when $\theta = r$ or $\theta = s$. Consider the case where $\theta = r$ and note the other case is entirely similar.

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, n \overset{r}{+} 1, \cdots, \overset{s}{q}, \cdots, i_{n+1}\right) =$$

$$(-1)^{n+1-r} \operatorname{sgn}_n\left(i_1, \cdots, \overset{s-1}{q}, \cdots, i_{n+1}\right) \tag{4.17}$$

while

$$\operatorname{sgn}_{n+1}\left(i_1, \cdots, \overset{r}{q}, \cdots, n \overset{s}{+} 1, \cdots, i_{n+1}\right) =$$

$$(-1)^{n+1-s} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1}\right). \tag{4.18}$$

By making $s - 1 - r$ switches, move the q which is in the $s - 1^{th}$ position in 4.17 to the r^{th} position in 4.18. By induction, each of these switches introduces a factor of -1 and so

$$\operatorname{sgn}_n\left(i_1, \cdots, \overset{s-1}{q}, \cdots, i_{n+1}\right) = (-1)^{s-1-r} \operatorname{sgn}_n\left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1}\right).$$

Therefore,

$$\begin{aligned} &\operatorname{sgn}_{n+1}\left(i_1, \cdots, n \overset{r}{+} 1, \cdots, \overset{s}{q}, \cdots, i_{n+1}\right) \\ = \;&(-1)^{n+1-r} \operatorname{sgn}_n\left(i_1, \cdots, \overset{s-1}{q}, \cdots, i_{n+1}\right) \end{aligned}$$

$$= (-1)^{n+1-r} (-1)^{s-1-r} \operatorname{sgn}_n \left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1}\right)$$

$$= \quad (-1)^{n+s} \operatorname{sgn}_n \left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1}\right)$$

$$= \quad (-1)^{2s-1} (-1)^{n+1-s} \operatorname{sgn}_n \left(i_1, \cdots, \overset{r}{q}, \cdots, i_{n+1}\right)$$

$$= -\operatorname{sgn}_{n+1} \left(i_1, \cdots, \overset{r}{q}, \cdots, n \overset{s}{+} 1, \cdots, i_{n+1}\right).$$

This proves the existence of the desired function.

To see this function is unique, note that you can obtain any ordered list of distinct numbers from a sequence of switches. If there exist two functions, f and g both satisfying 4.14 and 4.15, you could start with $f(1, \cdots, n) = g(1, \cdots, n)$ and applying the same sequence of switches, eventually arrive at $f(i_1, \cdots, i_n) = g(i_1, \cdots, i_n)$. If any numbers are repeated, then 4.15 gives both functions are equal to zero for that ordered list. This proves the lemma.

In what follows sgn will often be used rather than sgn_n because the context supplies the appropriate n.

Definition 4.19 *Let f be a real valued function which has the set of ordered lists of numbers from $\{1, \cdots, n\}$ as its domain. Define*

$$\sum_{(k_1, \cdots, k_n)} f(k_1 \cdots k_n)$$

to be the sum of all the

$$f(k_1 \cdots k_n)$$

for all possible choices of ordered lists

$$(k_1, \cdots, k_n)$$

of numbers of

$$\{1, \cdots, n\}.$$

For example,

$$\sum_{(k_1, k_2)} f(k_1, k_2) = f(1, 2) + f(2, 1) + f(1, 1) + f(2, 2).$$

Definition 4.20 *Let $(a_{ij}) = A$ denote an $n \times n$ matrix. The determinant of A, denoted by $\det(A)$ is defined by*

$$\det(A) \equiv \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(k_1, \cdots, k_n) a_{1k_1} \cdots a_{nk_n}$$

where the sum is taken over all ordered lists of numbers from $\{1, \cdots, n\}$. Note it suffices to take the sum over only those ordered lists in which there are no repeats because if there are, $\operatorname{sgn}(k_1, \cdots, k_n) = 0$ and so that term contributes 0 to the sum.

Let A be an $n \times n$ matrix, $A = (a_{ij})$ and let $(r_1, \cdots, r_n)$ denote an ordered list of n numbers from $\{1, \cdots, n\}$. Let $A(r_1, \cdots, r_n)$ denote the matrix whose k^{th} row is the r_k row of the matrix, A. Thus

$$\det(A(r_1, \cdots, r_n)) = \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(k_1, \cdots, k_n)\, a_{r_1 k_1} \cdots a_{r_n k_n} \tag{4.19}$$

and

$$A(1, \cdots, n) = A.$$

Proposition 4.21 *Let*

$$(r_1, \cdots, r_n)$$

be an ordered list of numbers from $\{1, \cdots, n\}$. *Then*

$$\operatorname{sgn}(r_1, \cdots, r_n) \det(A)$$

$$= \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(k_1, \cdots, k_n)\, a_{r_1 k_1} \cdots a_{r_n k_n} \tag{4.20}$$

$$= \det(A(r_1, \cdots, r_n)). \tag{4.21}$$

Proof: Let $(1, \cdots, n) = (1, \cdots, r, \cdots s, \cdots, n)$ so $r < s$.

$$\det(A(1, \cdots, r, \cdots, s, \cdots, n)) = \tag{4.22}$$

$$\sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(k_1, \cdots, k_r, \cdots, k_s, \cdots, k_n)\, a_{1k_1} \cdots a_{rk_r} \cdots a_{sk_s} \cdots a_{nk_n},$$

and renaming the variables, calling k_s, k_r and k_r, k_s, this equals

$$= \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(k_1, \cdots, k_s, \cdots, k_r, \cdots, k_n)\, a_{1k_1} \cdots a_{rk_s} \cdots a_{sk_r} \cdots a_{nk_n}$$

$$= \sum_{(k_1, \cdots, k_n)} -\operatorname{sgn}\left(k_1, \cdots, \overbrace{k_r, \cdots, k_s}^{\text{These got switched}}, \cdots, k_n\right) a_{1k_1} \cdots a_{sk_r} \cdots a_{rk_s} \cdots a_{nk_n}$$

$$= -\det(A(1, \cdots, s, \cdots, r, \cdots, n)). \tag{4.23}$$

Consequently,

$$\det(A(1, \cdots, s, \cdots, r, \cdots, n)) =$$

$$-\det(A(1, \cdots, r, \cdots, s, \cdots, n)) = -\det(A)$$

Now letting $A(1, \cdots, s, \cdots, r, \cdots, n)$ play the role of A, and continuing in this way, switching pairs of numbers,

$$\det(A(r_1, \cdots, r_n)) = (-1)^p \det(A)$$

where it took p switches to obtain$(r_1, \cdots, r_n)$ from $(1, \cdots, n)$. By Lemma 4.18, this implies

$$\det(A(r_1, \cdots, r_n)) = (-1)^p \det(A) = \operatorname{sgn}(r_1, \cdots, r_n) \det(A)$$

and proves the proposition in the case when there are no repeated numbers in the ordered list, $(r_1, \cdots, r_n)$. However, if there is a repeat, say the r^{th} row equals the s^{th} row, then the reasoning of 4.22 -4.23 shows that $A(r_1, \cdots, r_n) = 0$ and also $\operatorname{sgn}(r_1, \cdots, r_n) = 0$ so the formula holds in this case also.

Observation 4.22 *There are $n!$ ordered lists of distinct numbers from $\{1, \cdots, n\}$.*

To see this, consider n slots placed in order. There are n choices for the first slot. For each of these choices, there are $n - 1$ choices for the second. Thus there are $n(n-1)$ ways to fill the first two slots. Then for each of these ways there are $n - 2$ choices left for the third slot. Continuing this way, there are $n!$ ordered lists of distinct numbers from $\{1, \cdots, n\}$ as stated in the observation.

With the above, it is possible to give a more symmetric description of the determinant from which it will follow that $\det(A) = \det(A^T)$.

Corollary 4.23 *The following formula for $\det(A)$ is valid.*

$$\det(A) = \frac{1}{n!} \cdot$$

$$\sum_{(r_1, \cdots, r_n)} \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(r_1, \cdots, r_n) \operatorname{sgn}(k_1, \cdots, k_n) a_{r_1 k_1} \cdots a_{r_n k_n}. \tag{4.24}$$

And also $\det(A^T) = \det(A)$ where A^T is the transpose of A. (Recall that for $A^T = (a_{ij}^T)$, $a_{ij}^T = a_{ji}$.)

Proof: From Proposition 4.21, if the r_i are distinct,

$$\det(A) = \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(r_1, \cdots, r_n) \operatorname{sgn}(k_1, \cdots, k_n) a_{r_1 k_1} \cdots a_{r_n k_n}.$$

Summing over all ordered lists, $(r_1, \cdots, r_n)$ where the r_i are distinct, (If the r_i are not distinct, $\operatorname{sgn}(r_1, \cdots, r_n) = 0$ and so there is no contribution to the sum.)

$$n! \det(A) =$$

$$\sum_{(r_1, \cdots, r_n)} \sum_{(k_1, \cdots, k_n)} \operatorname{sgn}(r_1, \cdots, r_n) \operatorname{sgn}(k_1, \cdots, k_n) a_{r_1 k_1} \cdots a_{r_n k_n}.$$

This proves the corollary since the formula gives the same number for A as it does for A^T.

Corollary 4.24 *If two rows or two columns in an $n\times n$ matrix, A, are switched, the determinant of the resulting matrix equals (-1) times the determinant of the original matrix. If A is an $n\times n$ matrix in which two rows are equal or two columns are equal then* $\det(A)=0$. *Suppose the i^{th} row of A equals $(xa_1+yb_1,\cdots,xa_n+yb_n)$. Then*

$$\det(A)=x\det(A_1)+y\det(A_2)$$

where the i^{th} row of A_1 is $(a_1,\cdots,a_n)$ and the i^{th} row of A_2 is $(b_1,\cdots,b_n)$, all other rows of A_1 and A_2 coinciding with those of A. In other words, det *is a linear function of each row A. The same is true with the word "row" replaced with the word "column".*

Proof: By Proposition 4.21 when two rows are switched, the determinant of the resulting matrix is (-1) times the determinant of the original matrix. By Corollary 4.23 the same holds for columns because the columns of the matrix equal the rows of the transposed matrix. Thus if A_1 is the matrix obtained from A by switching two columns,

$$\det(A)=\det\left(A^T\right)=-\det\left(A_1^T\right)=-\det(A_1).$$

If A has two equal columns or two equal rows, then switching them results in the same matrix. Therefore, $\det(A)=-\det(A)$ and so $\det(A)=0$.

It remains to verify the last assertion.

$$\det(A)\equiv\sum_{(k_1,\cdots,k_n)}\operatorname{sgn}(k_1,\cdots,k_n)\,a_{1k_1}\cdots(xa_{k_i}+yb_{k_i})\cdots a_{nk_n}$$

$$=x\sum_{(k_1,\cdots,k_n)}\operatorname{sgn}(k_1,\cdots,k_n)\,a_{1k_1}\cdots a_{k_i}\cdots a_{nk_n}$$

$$+y\sum_{(k_1,\cdots,k_n)}\operatorname{sgn}(k_1,\cdots,k_n)\,a_{1k_1}\cdots b_{k_i}\cdots a_{nk_n}$$

$$\equiv x\det(A_1)+y\det(A_2).$$

The same is true of columns because $\det\left(A^T\right)=\det(A)$ and the rows of A^T are the columns of A.

Definition 4.25 *A vector,* $\mathbf{w}$, *is a linear combination of the vectors $\{\mathbf{v}_1,\cdots,\mathbf{v}_r\}$ if there exists scalars, $c_1,\cdots c_r$ such that $\mathbf{w}=\sum_{k=1}^r c_k\mathbf{v}_k$. This is the same as saying* $\mathbf{w}\in\operatorname{span}\{\mathbf{v}_1,\cdots,\mathbf{v}_r\}$.

The following corollary is also of great use.

Corollary 4.26 *Suppose A is an $n\times n$ matrix and some column (row) is a linear combination of r other columns (rows). Then* $\det(A)=0$.

Proof: Let $A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{pmatrix}$ be the columns of A and suppose the condition that one column is a linear combination of r of the others is satisfied. Then by using Corollary 4.24 you may rearrange the columns to have the n^{th} column a linear combination of the first r columns. Thus $\mathbf{a}_n = \sum_{k=1}^{r} c_k \mathbf{a}_k$ and so

$$\det(A) = \det \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_r & \cdots & \mathbf{a}_{n-1} & \sum_{k=1}^{r} c_k \mathbf{a}_k \end{pmatrix}.$$

By Corollary 4.24

$$\det(A) = \sum_{k=1}^{r} c_k \det \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_r & \cdots & \mathbf{a}_{n-1} & \mathbf{a}_k \end{pmatrix} = 0.$$

The case for rows follows from the fact that $\det(A) = \det\left(A^T\right)$. This proves the corollary.

Recall the following definition of matrix multiplication.

Definition 4.27 *If A and B are $n \times n$ matrices, $A = (a_{ij})$ and $B = (b_{ij})$, $AB = (c_{ij})$ where*

$$c_{ij} \equiv \sum_{k=1}^{n} a_{ik} b_{kj}.$$

One of the most important rules about determinants is that the determinant of a product equals the product of the determinants.

Theorem 4.28 *Let A and B be $n \times n$ matrices. Then*

$$\det(AB) = \det(A)\det(B).$$

Proof: Let c_{ij} be the ij^{th} entry of AB. Then by Proposition 4.21,

$$\det(AB) =$$

$$\begin{aligned}
& \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, c_{1k_1} \cdots c_{nk_n} \\
= & \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n) \left(\sum_{r_1} a_{1r_1} b_{r_1 k_1}\right) \cdots \left(\sum_{r_n} a_{nr_n} b_{r_n k_n}\right) \\
= & \sum_{(r_1\cdots,r_n)} \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}(k_1,\cdots,k_n)\, b_{r_1k_1} \cdots b_{r_nk_n} \left(a_{1r_1} \cdots a_{nr_n}\right) \\
= & \sum_{(r_1\cdots,r_n)} \operatorname{sgn}(r_1 \cdots r_n)\, a_{1r_1} \cdots a_{nr_n} \det(B) = \det(A)\det(B).
\end{aligned}$$

This proves the theorem.

Lemma 4.29 *Suppose a matrix is of the form*

$$M = \begin{pmatrix} A & * \\ \mathbf{0} & a \end{pmatrix} \tag{4.25}$$

or

$$M = \begin{pmatrix} A & \mathbf{0} \\ * & a \end{pmatrix} \tag{4.26}$$

where a is a number and A is an $(n-1) \times (n-1)$ matrix and $*$ denotes either a column or a row having length $n-1$ and the $\mathbf{0}$ denotes either a column or a row of length $n-1$ consisting entirely of zeros. Then

$$\det(M) = a \det(A).$$

Proof: Denote M by (m_{ij}). Thus in the first case, $m_{nn} = a$ and $m_{ni} = 0$ if $i \neq n$ while in the second case, $m_{nn} = a$ and $m_{in} = 0$ if $i \neq n$. From the definition of the determinant,

$$\det(M) \equiv \sum_{(k_1,\cdots,k_n)} \operatorname{sgn}_n(k_1,\cdots,k_n)\, m_{1k_1}\cdots m_{nk_n}$$

Letting θ denote the position of n in the ordered list, $(k_1,\cdots,k_n)$ then using the earlier conventions used to prove Lemma 4.18, $\det(M)$ equals

$$\sum_{(k_1,\cdots,k_n)} (-1)^{n-\theta} \operatorname{sgn}_{n-1}\left(k_1,\cdots,k_{\theta-1},\overset{\theta}{k_{\theta+1}},\cdots,\overset{n-1}{k_n}\right) m_{1k_1}\cdots m_{nk_n}$$

Now suppose 4.26. Then if $k_n \neq n$, the term involving m_{nk_n} in the above expression equals zero. Therefore, the only terms which survive are those for which $\theta = n$ or in other words, those for which $k_n = n$. Therefore, the above expression reduces to

$$a \sum_{(k_1,\cdots,k_{n-1})} \operatorname{sgn}_{n-1}(k_1,\cdots k_{n-1})\, m_{1k_1}\cdots m_{(n-1)k_{n-1}} = a \det(A).$$

To get the assertion in the situation of 4.25 use Corollary 4.23 and 4.26 to write

$$\det(M) = \det\left(M^T\right) = \det\left(\begin{pmatrix} A^T & \mathbf{0} \\ * & a \end{pmatrix}\right) = a \det\left(A^T\right) = a \det(A).$$

This proves the lemma.

In terms of the theory of determinants, arguably the most important idea is that of Laplace expansion along a row or a column. This will follow from the above definition of a determinant.

Definition 4.30 *Let $A = (a_{ij})$ be an $n \times n$ matrix. Then a new matrix called the cofactor matrix, $\operatorname{cof}(A)$ is defined by $\operatorname{cof}(A) = (c_{ij})$ where to obtain c_{ij} delete the i^{th} row and the j^{th} column of A, take the determinant of the $(n-1)\times(n-1)$ matrix which results, (This is called the ij^{th} minor of A.) and then multiply this number by $(-1)^{i+j}$. To make the formulas easier to remember, $\operatorname{cof}(A)_{ij}$ will denote the ij^{th} entry of the cofactor matrix.*

The following is the main result. Earlier this was given as a definition and the outrageous totally unjustified assertion was made that the same number would be obtained by expanding the determinant along any row or column. The following theorem proves this assertion.

Theorem 4.31 *Let A be an $n \times n$ matrix where $n \geq 2$. Then*

$$\det(A) = \sum_{j=1}^{n} a_{ij} \operatorname{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij} \operatorname{cof}(A)_{ij}. \tag{4.27}$$

The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

Proof: Let $(a_{i1}, \cdots, a_{in})$ be the i^{th} row of A. Let B_j be the matrix obtained from A by leaving every row the same except the i^{th} row which in B_j equals $(0, \cdots, 0, a_{ij}, 0, \cdots, 0)$. Then by Corollary 4.24,

$$\det(A) = \sum_{j=1}^{n} \det(B_j)$$

Denote by A^{ij} the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and the j^{th} column of A. Thus $\operatorname{cof}(A)_{ij} \equiv (-1)^{i+j} \det(A^{ij})$. At this point, recall that from Proposition 4.21, when two rows or two columns in a matrix, M, are switched, this results in multiplying the determinant of the old matrix by -1 to get the determinant of the new matrix. Therefore, by Lemma 4.29,

$$\begin{aligned} \det(B_j) &= (-1)^{n-j} (-1)^{n-i} \det\left(\begin{pmatrix} A^{ij} & * \\ \mathbf{0} & a_{ij} \end{pmatrix}\right) \\ &= (-1)^{i+j} \det\left(\begin{pmatrix} A^{ij} & * \\ \mathbf{0} & a_{ij} \end{pmatrix}\right) = a_{ij} \operatorname{cof}(A)_{ij}. \end{aligned}$$

Therefore,

$$\det(A) = \sum_{j=1}^{n} a_{ij} \operatorname{cof}(A)_{ij}$$

which is the formula for expanding $\det(A)$ along the i^{th} row. Also,

$$\begin{aligned} \det(A) &= \det(A^T) = \sum_{j=1}^{n} a_{ij}^T \operatorname{cof}(A^T)_{ij} \\ &= \sum_{j=1}^{n} a_{ji} \operatorname{cof}(A)_{ji} \end{aligned}$$

which is the formula for expanding $\det(A)$ along the i^{th} column. This proves the theorem.

Note that this gives an easy way to write a formula for the inverse of an $n \times n$ matrix.

Theorem 4.32 *A^{-1} exists if and only if* $\det(A) \neq 0$. *If* $\det(A) \neq 0$, *then* $A^{-1} = \left(a_{ij}^{-1}\right)$ *where*

$$a_{ij}^{-1} = \det(A)^{-1} \operatorname{cof}(A)_{ji}$$

for $\operatorname{cof}(A)_{ij}$ *the* ij^{th} *cofactor of* A.

Proof: By Theorem 4.31 and letting $(a_{ir}) = A$, if $\det(A) \neq 0$,

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ir} \det(A)^{-1} = \det(A) \det(A)^{-1} = 1.$$

Now consider

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}$$

when $k \neq r$. Replace the k^{th} column with the r^{th} column to obtain a matrix, B_k whose determinant equals zero by Corollary 4.24. However, expanding this matrix along the k^{th} column yields

$$0 = \det(B_k) \det(A)^{-1} = \sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}$$

Summarizing,

$$\sum_{i=1}^{n} a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1} = \delta_{rk}.$$

Using the other formula in Theorem 4.31, and similar reasoning,

$$\sum_{j=1}^{n} a_{rj} \operatorname{cof}(A)_{kj} \det(A)^{-1} = \delta_{rk}$$

This proves that if $\det(A) \neq 0$, then A^{-1} exists with $A^{-1} = \left(a_{ij}^{-1}\right)$, where

$$a_{ij}^{-1} = \operatorname{cof}(A)_{ji} \det(A)^{-1}.$$

Now suppose A^{-1} exists. Then by Theorem 4.28,

$$1 = \det(I) = \det\left(AA^{-1}\right) = \det(A) \det\left(A^{-1}\right)$$

so $\det(A) \neq 0$. This proves the theorem.

The next corollary points out that if an $n \times n$ matrix, A has a right or a left inverse, then it has an inverse.

Corollary 4.33 *Let A be an $n \times n$ matrix and suppose there exists an $n \times n$ matrix, B such that $BA = I$. Then A^{-1} exists and $A^{-1} = B$. Also, if there exists C an $n \times n$ matrix such that $AC = I$, then A^{-1} exists and $A^{-1} = C$.*

Proof: Since $BA = I$, Theorem 4.28 implies

$$\det B \det A = 1$$

and so $\det A \neq 0$. Therefore from Theorem 4.32, A^{-1} exists. Therefore,

$$A^{-1} = (BA) A^{-1} = B \left(AA^{-1} \right) = BI = B.$$

The case where $CA = I$ is handled similarly.

The conclusion of this corollary is that left inverses, right inverses and inverses are all the same in the context of $n \times n$ matrices.

Theorem 4.32 says that to find the inverse, take the transpose of the cofactor matrix and divide by the determinant. The transpose of the cofactor matrix is called the adjugate or sometimes the classical adjoint of the matrix A. It is an abomination to call it the adjoint although you do sometimes see it referred to in this way. In words, A^{-1} is equal to one over the determinant of A times the adjugate matrix of A.

In case you are solving a system of equations, $A\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, it follows that if A^{-1} exists,

$$\mathbf{x} = \left(A^{-1} A \right) \mathbf{x} = A^{-1} (A\mathbf{x}) = A^{-1}\mathbf{y}$$

thus solving the system. Now in the case that A^{-1} exists, there is a formula for A^{-1} given above. Using this formula,

$$x_i = \sum_{j=1}^{n} a_{ij}^{-1} y_j = \sum_{j=1}^{n} \frac{1}{\det (A)} \operatorname{cof} (A)_{ji} \, y_j .$$

By the formula for the expansion of a determinant along a column,

$$x_i = \frac{1}{\det (A)} \det \begin{pmatrix} * & \cdots & y_1 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & y_n & \cdots & * \end{pmatrix},$$

where here the i^{th} column of A is replaced with the column vector, $(y_1 \cdots \cdot, y_n)^T$, and the determinant of this modified matrix is taken and divided by $\det (A)$. This formula is known as Cramer's rule.

Definition 4.34 *A matrix M, is upper triangular if $M_{ij} = 0$ whenever $i > j$. Thus such a matrix equals zero below the main diagonal, the entries of the form M_{ii} as shown.*

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix}$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

With this definition, here is a simple corollary of Theorem 4.31.

Corollary 4.35 *Let M be an upper (lower) triangular matrix. Then* $\det(M)$ *is obtained by taking the product of the entries on the main diagonal.*

Definition 4.36 *A submatrix of a matrix A is the rectangular array of numbers obtained by deleting some rows and columns of A. Let A be an $m \times n$ matrix. The **determinant rank** of the matrix equals r where r is the largest number such that some $r \times r$ submatrix of A has a non zero determinant. The **row rank** is defined to be the dimension of the span of the rows. The **column rank** is defined to be the dimension of the span of the columns.*

Theorem 4.37 *If A has determinant rank, r, then there exist r rows of the matrix such that every other row is a linear combination of these r rows.*

Proof: Suppose the determinant rank of $A = (a_{ij})$ equals r. If rows and columns are interchanged, the determinant rank of the modified matrix is unchanged. Thus rows and columns can be interchanged to produce an $r \times r$ matrix in the upper left corner of the matrix which has non zero determinant. Now consider the $r+1 \times r+1$ matrix, M,

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} & a_{1p} \\ \vdots & & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{rp} \\ a_{l1} & \cdots & a_{lr} & a_{lp} \end{pmatrix}$$

where C will denote the $r \times r$ matrix in the upper left corner which has non zero determinant. I claim $\det(M) = 0$.

There are two cases to consider in verifying this claim. First, suppose $p > r$. Then the claim follows from the assumption that A has determinant rank r. On the other hand, if $p < r$, then the determinant is zero because there are two identical columns. Expand the determinant along the last column and divide by $\det(C)$ to obtain

$$a_{lp} = -\sum_{i=1}^{r} \frac{\operatorname{cof}(M)_{ip}}{\det(C)} a_{ip}.$$

Now note that $\operatorname{cof}(M)_{ip}$ does not depend on p. Therefore the above sum is of the form

$$a_{lp} = \sum_{i=1}^{r} m_i a_{ip}$$

which shows the l^{th} row is a linear combination of the first r rows of A. Since l is arbitrary, this proves the theorem.

Corollary 4.38 *The determinant rank equals the row rank.*

Proof: From Theorem 4.37, the row rank is no larger than the determinant rank. Could the row rank be smaller than the determinant rank? If so, there exist p rows for $p < r$ such that the span of these p rows equals the row space. But this implies that the $r \times r$ submatrix whose determinant is nonzero also has row rank no larger than p which is impossible if its determinant is to be nonzero because at least one row is a linear combination of the others.

Corollary 4.39 *If A has determinant rank, r, then there exist r columns of the matrix such that every other column is a linear combination of these r columns. Also the column rank equals the determinant rank.*

Proof: This follows from the above by considering A^T. The rows of A^T are the columns of A and the determinant rank of A^T and A are the same. Therefore, from Corollary 4.38, column rank of A = row rank of A^T = determinant rank of A^T = determinant rank of A.

The following theorem is of fundamental importance and ties together many of the ideas presented above.

Theorem 4.40 *Let A be an $n \times n$ matrix. Then the following are equivalent.*

1. $\det(A) = 0$.
2. *A, A^T are not one to one.*
3. *A is not onto.*

Proof: Suppose $\det(A) = 0$. Then the determinant rank of $A = r < n$. Therefore, there exist r columns such that every other column is a linear combination of these columns by Theorem 4.37. In particular, it follows that for some m, the m^{th} column is a linear combination of all the others. Thus letting $A = \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_m & \cdots & \mathbf{a}_n \end{pmatrix}$ where the columns are denoted by $\mathbf{a}_i$, there exists scalars, α_i such that

$$\mathbf{a}_m = \sum_{k \neq m} \alpha_k \mathbf{a}_k.$$

Now consider the column vector, $\mathbf{x} \equiv \begin{pmatrix} \alpha_1 & \cdots & -1 & \cdots & \boldsymbol{\alpha}_n \end{pmatrix}^T$. Then

$$A\mathbf{x} = -\mathbf{a}_m + \sum_{k \neq m} \alpha_k \mathbf{a}_k = \mathbf{0}.$$

Since also $A\mathbf{0} = \mathbf{0}$, it follows A is not one to one. Similarly, A^T is not one to one by the same argument applied to A^T. This verifies that 1.) implies 2.).

Now suppose 2.). Then since A^T is not one to one, it follows there exists $\mathbf{x} \neq \mathbf{0}$ such that

$$A^T\mathbf{x} = \mathbf{0}.$$

Taking the transpose of both sides yields

$$\mathbf{x}^T A = \mathbf{0}$$

where the $\mathbf{0}$ is a $1 \times n$ matrix or row vector. Now if $A\mathbf{y} = \mathbf{x}$, then

$$|\mathbf{x}|^2 = \mathbf{x}^T (A\mathbf{y}) = \left(\mathbf{x}^T A\right) \mathbf{y} = \mathbf{0}\mathbf{y} = 0$$

contrary to $\mathbf{x} \neq \mathbf{0}$. Consequently there can be no $\mathbf{y}$ such that $A\mathbf{y} = \mathbf{x}$ and so A is not onto. This shows that 2.) implies 3.).

Finally, suppose 3.). If 1.) does not hold, then $\det(A) \neq 0$ but then from Theorem 4.32 A^{-1} exists and so for every $\mathbf{y} \in \mathbb{F}^n$ there exists a unique $\mathbf{x} \in \mathbb{F}^n$ such that $A\mathbf{x} = \mathbf{y}$. In fact $\mathbf{x} = A^{-1}\mathbf{y}$. Thus A would be onto contrary to 3.). This shows 3.) implies 1.) and proves the theorem.

Corollary 4.41 *Let A be an $n \times n$ matrix. Then the following are equivalent.*

1. *det$(A) \neq 0$.*
2. *A and A^T are one to one.*
3. *A is onto.*

Proof: This follows immediately from the above theorem.

4.6 Exercises

1. Let $m < n$ and let A be an $m \times n$ matrix. Show that A is **not** one to one. **Hint:** Consider the $n \times n$ matrix, A_1 which is of the form

$$A_1 \equiv \begin{pmatrix} A \\ 0 \end{pmatrix}$$

where the 0 denotes an $(n - m) \times n$ matrix of zeros. Thus $\det A_1 = 0$ and so A_1 is not one to one. Now observe that $A_1\mathbf{x}$ is the vector,

$$A_1\mathbf{x} = \begin{pmatrix} A\mathbf{x} \\ \mathbf{0} \end{pmatrix}$$

which equals zero if and only if $A\mathbf{x} = \mathbf{0}$.

4.7 The Cayley Hamilton Theorem

Definition 4.42 *Let A be an $n \times n$ matrix. The characteristic polynomial is defined as*

$$p_A(t) \equiv \det(tI - A)$$

and the solutions to $p_A(t) = 0$ are called eigenvalues. For A a matrix and $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$, denote by $p(A)$ the matrix defined by

$$p(A) \equiv A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I.$$

The explanation for the last term is that A^0 is interpreted as I, the identity matrix.

The Cayley Hamilton theorem states that every matrix satisfies its characteristic equation, that equation defined by $P_A(t) = 0$. It is one of the most important theorems in linear algebra. The following lemma will help with its proof.

Lemma 4.43 *Suppose for all $|\lambda|$ large enough,*

$$A_0 + A_1\lambda + \cdots + A_m\lambda^m = 0,$$

where the A_i are $n \times n$ matrices. Then each $A_i = 0$.

Proof: Multiply by λ^{-m} to obtain

$$A_0\lambda^{-m} + A_1\lambda^{-m+1} + \cdots + A_{m-1}\lambda^{-1} + A_m = 0.$$

Now let $|\lambda| \to \infty$ to obtain $A_m = 0$. With this, multiply by λ to obtain

$$A_0\lambda^{-m+1} + A_1\lambda^{-m+2} + \cdots + A_{m-1} = 0.$$

Now let $|\lambda| \to \infty$ to obtain $A_{m-1} = 0$. Continue multiplying by λ and letting $\lambda \to \infty$ to obtain that all the $A_i = 0$. This proves the lemma.

With the lemma, here is a simple corollary.

Corollary 4.44 *Let A_i and B_i be $n \times n$ matrices and suppose*

$$A_0 + A_1\lambda + \cdots + A_m\lambda^m = B_0 + B_1\lambda + \cdots + B_m\lambda^m$$

for all $|\lambda|$ large enough. Then $A_i = B_i$ for all i. Consequently if λ is replaced by any $n \times n$ matrix, the two sides will be equal. That is, for C any $n \times n$ matrix,

$$A_0 + A_1C + \cdots + A_mC^m = B_0 + B_1C + \cdots + B_mC^m.$$

Proof: Subtract and use the result of the lemma.

With this preparation, here is a relatively easy proof of the Cayley Hamilton theorem.

Theorem 4.45 *Let A be an $n \times n$ matrix and let $p(\lambda) \equiv \det(\lambda I - A)$ be the characteristic polynomial. Then $p(A) = 0$.*

Proof: Let $C(\lambda)$ equal the transpose of the cofactor matrix of $(\lambda I - A)$ for $|\lambda|$ large. (If $|\lambda|$ is large enough, then λ cannot be in the finite list of eigenvalues of A and so for such λ, $(\lambda I - A)^{-1}$ exists.) Therefore, by Theorem 4.32

$$C(\lambda) = p(\lambda)(\lambda I - A)^{-1}.$$

Note that each entry in $C(\lambda)$ is a polynomial in λ having degree no more than $n-1$. Therefore, collecting the terms,

$$C(\lambda) = C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}$$

for C_j some $n \times n$ matrix. It follows that for all $|\lambda|$ large enough,

$$(A - \lambda I)\left(C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}\right) = p(\lambda) I$$

and so Corollary 4.44 may be used. It follows the matrix coefficients corresponding to equal powers of λ are equal on both sides of this equation. Therefore, if λ is replaced with A, the two sides will be equal. Thus

$$0 = (A - A)\left(C_0 + C_1 A + \cdots + C_{n-1}A^{n-1}\right) = p(A) I = p(A).$$

This proves the Cayley Hamilton theorem.

4.8 An Identity Of Cauchy

There is a very interesting identity for determinants due to Cauchy.

Theorem 4.46 *The following identity holds.*

$$\prod_{i,j}(a_i + b_j)\begin{vmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_n} \\ \vdots & & \vdots \\ \frac{1}{a_n+b_1} & \cdots & \frac{1}{a_n+b_n} \end{vmatrix} = \prod_{j<i}(a_i - a_j)(b_i - b_j). \tag{4.28}$$

Proof: What is the exponent of a_2 on the right? It occurs in $(a_2 - a_1)$ and in $(a_m - a_2)$ for $m > 2$. Therefore, there are exactly $n - 1$ factors which contain a_2. Therefore, a_2 has an exponent of $n - 1$. Similarly, each a_k is raised to the $n - 1$ power and the same holds for the b_k as well. Therefore, the right side of 4.28 is of the form

$$ca_1^{n-1}a_2^{n-1}\cdots a_n^{n-1}b_1^{n-1}\cdots b_n^{n-1}$$

where c is some constant. Now consider the left side of 4.28.

This is of the form

$$\frac{1}{n!}\prod_{i,j}(a_i + b_j)\sum_{i_1\cdots i_n, j_1,\cdots j_n}\operatorname{sgn}(i_1\cdots i_n)\operatorname{sgn}(j_1\cdots j_n)\cdot \frac{1}{a_{i_1}+b_{j_1}}\frac{1}{a_{i_2}+b_{j_2}}\cdots\frac{1}{a_{i_n}+b_{j_n}}.$$

For a given $i_1\cdots i_n, j_1,\cdots j_n$, let

$$S(i_1\cdots i_n, j_1,\cdots j_n) \equiv \{(i_1, j_1), (i_2, j_2)\cdots, (i_n, j_n)\}.$$

This equals

$$\frac{1}{n!}\sum_{i_1\cdots i_n, j_1,\cdots j_n}\operatorname{sgn}(i_1\cdots i_n)\operatorname{sgn}(j_1\cdots j_n)\prod_{(i,j)\notin\{(i_1,j_1),(i_2,j_2)\cdots,(i_n,j_n)\}}(a_i + b_j)$$

where you can assume the i_k are all distinct and the j_k are also all distinct because otherwise sgn will produce a 0. Therefore, in

$$\prod_{(i,j)\notin\{(i_1,j_1),(i_2,j_2)\cdots,(i_n,j_n)\}} (a_i + b_j),$$

there are exactly $n-1$ factors which contain a_k for each k and similarly, there are exactly $n-1$ factors which contain b_k for each k. Therefore, the left side of 4.28 is of the form

$$da_1^{n-1}a_2^{n-1}\cdots a_n^{n-1}b_1^{n-1}\cdots b_n^{n-1}$$

and it remains to verify that $c = d$. Using the properties of determinants, the left side of 4.28 is of the form

$$\prod_{i\neq j}(a_i+b_j)\begin{vmatrix} 1 & \frac{a_1+b_1}{a_1+b_2} & \cdots & \frac{a_1+b_1}{a_1+b_n} \\ \frac{a_2+b_2}{a_2+b_1} & 1 & \cdots & \frac{a_2+b_2}{a_2+b_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_n+b_n}{a_n+b_1} & \frac{a_n+b_n}{a_n+b_2} & \cdots & 1 \end{vmatrix}$$

Let $a_k \to -b_k$. Then this converges to $\prod_{i\neq j}(-b_i+b_j)$. The right side of 4.28 converges to

$$\prod_{j<i}(-b_i+b_j)(b_i-b_j) = \prod_{i\neq j}(-b_i+b_j).$$

Therefore, $d = c$ and this proves the identity.

4.9 Block Multiplication Of Matrices

Suppose A is a matrix of the form

$$\begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{r1} & \cdots & A_{rm} \end{pmatrix} \tag{4.29}$$

where A_{ij} is a $s_i \times p_j$ matrix where s_i does not depend on j and p_j does not depend on i. Such a matrix is called a **block matrix**. Let $n = \sum_j p_j$ and $k = \sum_i s_i$ so A is an $k \times n$ matrix. What is $A\mathbf{x}$ where $\mathbf{x} \in \mathbb{F}^n$? From the process of multiplying a matrix times a vector, the following lemma follows.

Lemma 4.47 *Let A be an $m \times n$ block matrix as in 4.29 and let $\mathbf{x} \in \mathbb{F}^n$. Then $A\mathbf{x}$ is of the form*

$$A\mathbf{x} = \begin{pmatrix} \sum_j A_{1j}\mathbf{x}_j \\ \vdots \\ \sum_j A_{rj}\mathbf{x}_j \end{pmatrix}$$

where $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_m)^T$ and $\mathbf{x}_i \in \mathbb{F}^{p_i}$.

Suppose also that B is a $l \times k$ block matrix of the form

$$\begin{pmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mp} \end{pmatrix} \tag{4.30}$$

and that for all i, j, it makes sense to multiply $B_{is}A_{sj}$ for all $s \in \{1, \cdots, m\}$. (That is the two matrices are conformable.) and that for each $s, B_{is}A_{sj}$ is the same size so that it makes sense to write $\sum_s B_{is}A_{sj}$.

Theorem 4.48 *Let B be an $l \times k$ block matrix as in 4.30 and let A be a $k \times n$ block matrix as in 4.29 such that B_{is} is conformable with A_{sj} and each product, $B_{is}A_{sj}$ is of the same size so they can be added. Then BA is a $l \times n$ block matrix having rp blocks such that the ij^{th} block is of the form*

$$\sum_s B_{is}A_{sj}. \tag{4.31}$$

Proof: Let B_{is} be a $q_i \times p_s$ matrix and A_{sj} be a $p_s \times r_j$ matrix. Also let $\mathbf{x} \in \mathbb{F}^n$ and let $\mathbf{x} = (\mathbf{x}_1, \cdots, \mathbf{x}_m)^T$ and $\mathbf{x}_i \in \mathbb{F}^{r_i}$ so it makes sense to multiply $A_{sj}\mathbf{x}_j$. Then from the associative law of matrix multiplication and Lemma 4.47 applied twice,

$$(BA)\mathbf{x} = B(A\mathbf{x})$$

$$= \begin{pmatrix} B_{11} & \cdots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{m1} & \cdots & B_{mp} \end{pmatrix} \begin{pmatrix} \sum_j A_{1j}\mathbf{x}_j \\ \vdots \\ \sum_j A_{rj}\mathbf{x}_j \end{pmatrix}$$

$$= \begin{pmatrix} \sum_s \sum_j B_{1s}A_{sj}\mathbf{x}_j \\ \vdots \\ \sum_s \sum_j B_{ms}A_{sj}\mathbf{x}_j \end{pmatrix} = \begin{pmatrix} \sum_j \left(\sum_s B_{1s}A_{sj}\right)\mathbf{x}_j \\ \vdots \\ \sum_j \left(\sum_s B_{ms}A_{sj}\right)\mathbf{x}_j \end{pmatrix}.$$

By Lemma 4.47, this shows that $(BA)\mathbf{x}$ equals the block matrix whose ij^{th} entry is given by 4.31 times $\mathbf{x}$. Since $\mathbf{x}$ is an arbitrary vector in $\mathbb{F}^n$, this proves the theorem.

The message of this theorem is that you can formally multiply block matrices as though the blocks were numbers. You just have to pay attention to the preservation of order.

This simple idea of block multiplication turns out to be very useful later. For now here is an interesting and significant application. In this theorem, $p_M(t)$ denotes the polynomial, $\det(tI - M)$. Thus the zeros of this polynomial are the eigenvalues of the matrix, M.

Theorem 4.49 *Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix for $m \leq n$. Then*

$$p_{BA}(t) = t^{n-m}p_{AB}(t),$$

so the eigenvalues of BA and AB are the same including multiplicities except that BA has $n - m$ extra zero eigenvalues.

Proof: Use block multiplication to write

$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}$$

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$$

By Problem 11 of Page 80, it follows that $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$ and $\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}$ have the same characteristic polynomials. Therefore, noting that BA is an $n \times n$ matrix and AB is an $m \times m$ matrix,

$$t^m \det(tI - BA) = t^n \det(tI - AB)$$

and so $\det(tI - BA) = p_{BA}(t) = t^{n-m} \det(tI - AB) = t^{n-m} p_{AB}(t)$. This proves the theorem.

4.10 Exercises

1. Show that matrix multiplication is associative. That is, $(AB)C = A(BC)$.

2. Show the inverse of a matrix, if it exists, is unique. Thus if $AB = BA = I$, then $B = A^{-1}$.

3. In the proof of Theorem 4.32 it was claimed that $\det(I) = 1$. Here $I = (\delta_{ij})$. Prove this assertion. Also prove Corollary 4.35.

4. Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be vectors in $\mathbb{F}^n$ and let $M(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ denote the matrix whose i^{th} column equals $\mathbf{v}_i$. Define

$$d(\mathbf{v}_1, \cdots, \mathbf{v}_n) \equiv \det(M(\mathbf{v}_1, \cdots, \mathbf{v}_n)).$$

Prove that d is linear in each variable, (multilinear), that

$$d(\mathbf{v}_1, \cdots, \mathbf{v}_i, \cdots, \mathbf{v}_j, \cdots, \mathbf{v}_n) = -d(\mathbf{v}_1, \cdots, \mathbf{v}_j, \cdots, \mathbf{v}_i, \cdots, \mathbf{v}_n), \tag{4.32}$$

and

$$d(\mathbf{e}_1, \cdots, \mathbf{e}_n) = 1 \tag{4.33}$$

where here $\mathbf{e}_j$ is the vector in $\mathbb{F}^n$ which has a zero in every position except the j^{th} position in which it has a one.

5. Suppose $f : \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to \mathbb{F}$ satisfies 4.32 and 4.33 and is linear in each variable. Show that $f = d$.

6. Show that if you replace a row (column) of an $n\times n$ matrix A with itself added to some multiple of another row (column) then the new matrix has the same determinant as the original one.

7. If $A=(a_{ij})$, show $\det(A)=\sum_{(k_1,\cdots,k_n)}\operatorname{sgn}(k_1,\cdots,k_n)\,a_{k_1 1}\cdots a_{k_n n}$.

8. Use the result of Problem 6 to evaluate by hand the determinant

$$\det\begin{pmatrix}1 & 2 & 3 & 2\\ -6 & 3 & 2 & 3\\ 5 & 2 & 2 & 3\\ 3 & 4 & 6 & 4\end{pmatrix}.$$

9. Find the inverse if it exists of the matrix,

$$\begin{pmatrix}e^t & \cos t & \sin t\\ e^t & -\sin t & \cos t\\ e^t & -\cos t & -\sin t\end{pmatrix}.$$

10. Let $Ly=y^{(n)}+a_{n-1}(x)\,y^{(n-1)}+\cdots+a_1(x)\,y'+a_0(x)\,y$ where the a_i are given continuous functions defined on a closed interval, (a,b) and y is some function which has n derivatives so it makes sense to write Ly. Suppose $Ly_k=0$ for $k=1,2,\cdots,n$. The Wronskian of these functions, y_i is defined as

$$W(y_1,\cdots,y_n)(x)\equiv\det\begin{pmatrix}y_1(x) & \cdots & y_n(x)\\ y_1'(x) & \cdots & y_n'(x)\\ \vdots & & \vdots\\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x)\end{pmatrix}$$

Show that for $W(x)=W(y_1,\cdots,y_n)(x)$ to save space,

$$W'(x)=\det\begin{pmatrix}y_1(x) & \cdots & y_n(x)\\ y_1'(x) & \cdots & y_n'(x)\\ \vdots & & \vdots\\ y_1^{(n)}(x) & \cdots & y_n^{(n)}(x)\end{pmatrix}.$$

Now use the differential equation, $Ly=0$ which is satisfied by each of these functions, y_i and properties of determinants presented above to verify that $W'+a_{n-1}(x)\,W=0$. Give an explicit solution of this linear differential equation, Abel's formula, and use your answer to verify that the Wronskian of these solutions to the equation, $Ly=0$ either vanishes identically on (a,b) or never.

11. Two $n\times n$ matrices, A and B, are similar if $B=S^{-1}AS$ for some invertible $n\times n$ matrix, S. Show that if two matrices are similar, they have the same characteristic polynomials.

12. Suppose the characteristic polynomial of an $n \times n$ matrix, A is of the form
$$t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$
and that $a_0 \neq 0$. Find a formula A^{-1} in terms of powers of the matrix, A. Show that A^{-1} exists if and only if $a_0 \neq 0$.

13. In constitutive modeling of the stress and strain tensors, one sometimes considers sums of the form $\sum_{k=0}^{\infty} a_k A^k$ where A is a 3×3 matrix. Show using the Cayley Hamilton theorem that if such a thing makes any sense, you can always obtain it as a finite sum having no more than n terms.

4.11 Shur's Theorem

Every matrix is related to an upper triangular matrix in a particularly significant way. This is Shur's theorem and it is the most important theorem in the spectral theory of matrices.

Lemma 4.50 *Let*
$$\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$$
be a basis for $\mathbb{F}^n$*. Then there exists an orthonormal basis for* $\mathbb{F}^n$*,*
$$\{\mathbf{u}_1, \cdots, \mathbf{u}_n\}$$
which has the property that for each $k \leq n$*,*
$$\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k).$$

Proof: Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$ be a basis for $\mathbb{F}^n$. Let $\mathbf{u}_1 \equiv \mathbf{x}_1 / |\mathbf{x}_1|$. Thus for $k = 1$, $\text{span}(\mathbf{u}_1) = \text{span}(\mathbf{x}_1)$ and $\{\mathbf{u}_1\}$ is an orthonormal set. Now suppose for some $k < n$, $\mathbf{u}_1, \cdots, \mathbf{u}_k$ have been chosen such that $(\mathbf{u}_j \cdot \mathbf{u}_l) = \delta_{jl}$ and $\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k)$. Then define
$$\mathbf{u}_{k+1} \equiv \frac{\mathbf{x}_{k+1} - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j) \mathbf{u}_j}{\left| \mathbf{x}_{k+1} - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j) \mathbf{u}_j \right|}, \tag{4.34}$$
where the denominator is not equal to zero because the $\mathbf{x}_j$ form a basis and so
$$\mathbf{x}_{k+1} \notin \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k)$$
Thus by induction,
$$\mathbf{u}_{k+1} \in \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{x}_{k+1}) = \text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{x}_{k+1}).$$
Also, $\mathbf{x}_{k+1} \in \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{u}_{k+1})$ which is seen easily by solving 4.34 for $\mathbf{x}_{k+1}$ and it follows
$$\text{span}(\mathbf{x}_1, \cdots, \mathbf{x}_k, \mathbf{x}_{k+1}) = \text{span}(\mathbf{u}_1, \cdots, \mathbf{u}_k, \mathbf{u}_{k+1}).$$

If $l \leq k$,

$$\begin{aligned}
(\mathbf{u}_{k+1} \cdot \mathbf{u}_l) &= C\left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)(\mathbf{u}_j \cdot \mathbf{u}_l) \right) \\
&= C\left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - \sum_{j=1}^{k} (\mathbf{x}_{k+1} \cdot \mathbf{u}_j)\, \delta_{lj} \right) \\
&= C\left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - (\mathbf{x}_{k+1} \cdot \mathbf{u}_l) \right) = 0.
\end{aligned}$$

The vectors, $\{\mathbf{u}_j\}_{j=1}^n$, generated in this way are therefore an orthonormal basis because each vector has unit length.

The process by which these vectors were generated is called the Gram Schmidt process. Recall the following definition.

Definition 4.51 *An $n \times n$ matrix, U, is unitary if $UU^* = I = U^*U$ where U^* is defined to be the transpose of the conjugate of U.*

Theorem 4.52 *Let A be an $n \times n$ matrix. Then there exists a unitary matrix, U such that*

$$U^*AU = T, \tag{4.35}$$

where T is an upper triangular matrix having the eigenvalues of A on the main diagonal listed according to multiplicity as roots of the characteristic equation.

Proof: Let $\mathbf{v}_1$ be a unit eigenvector for A . Then there exists λ_1 such that

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \ |\mathbf{v}_1| = 1.$$

Extend $\{\mathbf{v}_1\}$ to a basis and then use Lemma 4.50 to obtain $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$, an orthonormal basis in $\mathbb{F}^n$. Let U_0 be a matrix whose i^{th} column is $\mathbf{v}_i$. Then from the above, it follows U_0 is unitary. Then $U_0^* A U_0$ is of the form

$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where A_1 is an $n-1 \times n-1$ matrix. Repeat the process for the matrix, A_1 above. There exists a unitary matrix $\widetilde{U}_1$ such that $\widetilde{U}_1^* A_1 \widetilde{U}_1$ is of the form

$$\begin{pmatrix} \lambda_2 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{pmatrix}.$$

Now let U_1 be the $n \times n$ matrix of the form

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix}.$$

This is also a unitary matrix because by block multiplication,

$$\begin{aligned} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix}^* \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix} &= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1^* \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \widetilde{U}_1^* \widetilde{U}_1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} \end{aligned}$$

Then using block multiplication, $U_1^* U_0^* A U_0 U_1$ is of the form

$$\begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & & & \\ \vdots & \vdots & & A_2 & \\ 0 & 0 & & & \end{pmatrix}$$

where A_2 is an $n-2 \times n-2$ matrix. Continuing in this way, there exists a unitary matrix, U given as the product of the U_i in the above construction such that

$$U^* A U = T$$

where T is some upper triangular matrix. Since the matrix is upper triangular, the characteristic equation is $\prod_{i=1}^{n} (\lambda - \lambda_i)$ where the λ_i are the diagonal entries of T. Therefore, the λ_i are the eigenvalues.

What if A is a real matrix and you only want to consider real unitary matrices?

Theorem 4.53 *Let A be a real $n \times n$ matrix. Then there exists a real unitary matrix, Q and a matrix T of the form*

$$T = \begin{pmatrix} P_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & P_r \end{pmatrix} \tag{4.36}$$

where P_i equals either a real 1×1 matrix or P_i equals a real 2×2 matrix having two complex eigenvalues of A such that $Q^T A Q = T$. The matrix, T is called the real Schur form of the matrix A.

Proof: Suppose

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \ |\mathbf{v}_1| = 1$$

where λ_1 is real. Then let $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ be an orthonormal basis of vectors in $\mathbb{R}^n$. Let Q_0 be a matrix whose i^{th} column is $\mathbf{v}_i$. Then $Q_0^* A Q_0$ is of the form

$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where A_1 is a real $n-1 \times n-1$ matrix. This is just like the proof of Theorem 4.52 up to this point.

Now in case $\lambda_1 = \alpha + i\beta$, it follows since A is real that $\mathbf{v}_1 = \mathbf{z}_1 + i\mathbf{w}_1$ and that $\overline{\mathbf{v}}_1 = \mathbf{z}_1 - i\mathbf{w}_1$ is an eigenvector for the eigenvalue, $\alpha - i\beta$. Here $\mathbf{z}_1$ and $\mathbf{w}_1$ are real vectors. It is clear that $\{\mathbf{z}_1, \mathbf{w}_1\}$ is an independent set of vectors in $\mathbb{R}^n$. Indeed,$\{\mathbf{v}_1, \overline{\mathbf{v}}_1\}$ is an independent set and it follows $\text{span}(\mathbf{v}_1, \overline{\mathbf{v}}_1) = \text{span}(\mathbf{z}_1, \mathbf{w}_1)$. Now using the Gram Schmidt theorem in $\mathbb{R}^n$, there exists $\{\mathbf{u}_1, \mathbf{u}_2\}$, an orthonormal set of real vectors such that $\text{span}(\mathbf{u}_1, \mathbf{u}_2) = \text{span}(\mathbf{v}_1, \overline{\mathbf{v}}_1)$. Now let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ be an orthonormal basis in $\mathbb{R}^n$ and let Q_0 be a unitary matrix whose i^{th} column is $\mathbf{u}_i$. Then $A\mathbf{u}_j$ are both in $\text{span}(\mathbf{u}_1, \mathbf{u}_2)$ for $j = 1, 2$ and so $\mathbf{u}_k^T A\mathbf{u}_j = 0$ whenever $k \geq 3$. It follows that $Q_0^* A Q_0$ is of the form

$$\begin{pmatrix} * & * & \cdots & * \\ * & * & & \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where A_1 is now an $n-2 \times n-2$ matrix. In this case, find $\widetilde{Q}_1$ an $n-2 \times n-2$ matrix to put A_1 in an appropriate form as above and come up with A_2 either an $n-4 \times n-4$ matrix or an $n-3 \times n-3$ matrix. Then the only other difference is to let

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \widetilde{Q}_1 & \\ 0 & 0 & & & \end{pmatrix}$$

thus putting a 2×2 identity matrix in the upper left corner rather than a one. Repeating this process with the above modification for the case of a complex eigenvalue leads eventually to 4.36 where Q is the product of real unitary matrices Q_i above. Finally,

$$\lambda I - T = \begin{pmatrix} \lambda I_1 - P_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & \lambda I_r - P_r \end{pmatrix}$$

where I_k is the 2×2 identity matrix in the case that P_k is 2×2 and is the number 1 in the case where P_k is a 1×1 matrix. Now, it follows that $\det(\lambda I - T) = \prod_{k=1}^{r} \det(\lambda I_k - P_k)$. Therefore, λ is an eigenvalue of T if and only if it is an eigenvalue of some P_k. This proves the theorem since the eigenvalues of T are the same as those of A because they have the same characteristic polynomial due to the similarity of A and T.

Definition 4.54 *When a linear transformation, A, mapping a linear space, V to V has a basis of eigenvectors, the linear transformation is called non defective.*

Otherwise it is called defective. An $n\times n$ matrix, A, is called normal if $AA^* = A^*A$. An important class of normal matrices is that of the Hermitian or self adjoint matrices. An $n \times n$ matrix, A is self adjoint or Hermitian if $A = A^*$.

The next lemma is the basis for concluding that every normal matrix is unitarily similar to a diagonal matrix.

Lemma 4.55 *If T is upper triangular and normal, then T is a diagonal matrix.*

Proof: Since T is normal, $T^*T = TT^*$. Writing this in terms of components and using the description of the adjoint as the transpose of the conjugate, yields the following for the ik^{th} entry of $T^*T = TT^*$.

$$\sum_j t_{ij}t_{jk}^* = \sum_j t_{ij}\overline{t_{kj}} = \sum_j t_{ij}^* t_{jk} = \sum_j \overline{t_{ji}}t_{jk}.$$

Now use the fact that T is upper triangular and let $i = k = 1$ to obtain the following from the above.

$$\sum_j |t_{1j}|^2 = \sum_j |t_{j1}|^2 = |t_{11}|^2$$

You see, $t_{j1} = 0$ unless $j = 1$ due to the assumption that T is upper triangular. This shows T is of the form

$$\begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix}.$$

Now do the same thing only this time take $i = k = 2$ and use the result just established. Thus, from the above,

$$\sum_j |t_{2j}|^2 = \sum_j |t_{j2}|^2 = |t_{22}|^2,$$

showing that $t_{2j} = 0$ if $j > 2$ which means T has the form

$$\begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

Next let $i = k = 3$ and obtain that T looks like a diagonal matrix in so far as the first 3 rows and columns are concerned. Continuing in this way it follows T is a diagonal matrix.

Theorem 4.56 *Let A be a normal matrix. Then there exists a unitary matrix, U such that U^*AU is a diagonal matrix.*

Proof: From Theorem 4.52 there exists a unitary matrix, U such that U^*AU equals an upper triangular matrix. The theorem is now proved if it is shown that the property of being normal is preserved under unitary similarity transformations. That is, verify that if A is normal and if $B = U^*AU$, then B is also normal. But this is easy.

$$\begin{aligned} B^*B &= U^*A^*UU^*AU = U^*A^*AU \\ &= U^*AA^*U = U^*AUU^*A^*U = BB^*. \end{aligned}$$

Therefore, U^*AU is a normal and upper triangular matrix and by Lemma 4.55 it must be a diagonal matrix. This proves the theorem.

Corollary 4.57 *If A is Hermitian, then all the eigenvalues of A are real and there exists an orthonormal basis of eigenvectors.*

Proof: Since A is normal, there exists unitary, U such that $U^*AU = D$, a diagonal matrix whose diagonal entries are the eigenvalues of A. Therefore, $D^* = U^*A^*U = U^*AU = D$ showing D is real.

Finally, let

$$U = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{pmatrix}$$

where the $\mathbf{u}_i$ denote the columns of U and

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

The equation, $U^*AU = D$ implies

$$\begin{aligned} AU &= \begin{pmatrix} A\mathbf{u}_1 & A\mathbf{u}_2 & \cdots & A\mathbf{u}_n \end{pmatrix} \\ &= UD = \begin{pmatrix} \lambda_1\mathbf{u}_1 & \lambda_2\mathbf{u}_2 & \cdots & \lambda_n\mathbf{u}_n \end{pmatrix} \end{aligned}$$

where the entries denote the columns of AU and UD respectively. Therefore, $A\mathbf{u}_i = \lambda_i\mathbf{u}_i$ and since the matrix is unitary, the ij^{th} entry of U^*U equals δ_{ij} and so

$$\delta_{ij} = \overline{\mathbf{u}}_i^T\mathbf{u}_j = \overline{\mathbf{u}_i^T\overline{\mathbf{u}}_j} = \overline{\mathbf{u}_i \cdot \mathbf{u}_j}.$$

This proves the corollary because it shows the vectors $\{\mathbf{u}_i\}$ form an orthonormal basis.

Corollary 4.58 *If A is a real symmetric matrix, then A is Hermitian and there exists a real unitary matrix, U such that $U^TAU = D$ where D is a diagonal matrix.*

Proof: This follows from Theorem 4.53 and Corollary 4.57.

4.12 The Right Polar Decomposition

This is on the right polar decomposition.

Theorem 4.59 *Let F be an $m \times n$ matrix where $m \geq n$. Then there exists an $m \times n$ matrix R and a $n \times n$ matrix U such that*

$$F = RU,\ U = U^*,$$

all eigenvalues of U are non negative,

$$U^2 = F^* F,\ R^* R = I,$$

and $|R\mathbf{x}| = |\mathbf{x}|$.

Proof: $(F^*F)^* = F^*F$ and so F^*F is self adjoint. Also,

$$(F^*Fx, x) = (Fx, Fx) \geq 0.$$

Therefore, all eigenvalues of F^*F must be nonnegative because if $F^*F\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x} \neq \mathbf{0}$,

$$0 \leq (Fx, Fx) = (F^*F\mathbf{x}, \mathbf{x}) = (\lambda\mathbf{x}, \mathbf{x}) = \lambda |\mathbf{x}|^2 .$$

From linear algebra, there exists Q such that $Q^*Q = I$ and $F^*F = Q^*DQ$ where D is a diagonal matrix of the form

$$\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

where each $\lambda_i \geq 0$. Therefore, you can consider

$$D^{1/2} \equiv \begin{pmatrix} \lambda_1^{1/2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^{1/2} \end{pmatrix} \equiv \begin{pmatrix} \mu_1 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & & & v \\ \vdots & & \mu_r & & & \vdots \\ \vdots & & & 0 & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{4.37}$$

where the μ_i are the positive eigenvalues of $D^{1/2}$.

Let $U \equiv Q^* D^{1/2} Q$. This matrix is the square root of F^*F because

$$\left(Q^* D^{1/2} Q\right)\left(Q^* D^{1/2} Q\right) = Q^* D^{1/2} D^{1/2} Q = Q^* DQ = F^* F$$

It is self adjoint because $\left(Q^* D^{1/2} Q\right)^* = Q^* D^{1/2} Q^{**} = Q^* D^{1/2} Q$.

Let $\{\mathbf{x}_1, \cdots, \mathbf{x}_r\}$ be an orthogonal set of eigenvectors such that $U\mathbf{x}_i = \mu_i \mathbf{x}_i$ and normalize so that

$$\{\mu_1 \mathbf{x}_1, \cdots, \mu_r \mathbf{x}_r\} = \{U\mathbf{x}_1, \cdots, U\mathbf{x}_r\}$$

is an orthonormal set of vectors. By 4.37 it follows rank $(U) = r$ and so

$$\{U\mathbf{x}_1, \cdots, U\mathbf{x}_r\}$$

is also an orthonormal basis for $U(\mathbb{F}^n)$.

Then $\{F\mathbf{x}_r, \cdots, F\mathbf{x}_r\}$ is also an orthonormal set of vectors in $\mathbb{F}^m$ because

$$(F\mathbf{x}_i, F\mathbf{x}_j) = (F^* F\mathbf{x}_i, \mathbf{x}_j) = \left(U^2 \mathbf{x}_i, \mathbf{x}_j\right) = (U\mathbf{x}_i, U\mathbf{x}_j) = \delta_{ij}.$$

Let

$$\{U\mathbf{x}_1, \cdots, U\mathbf{x}_r, \mathbf{y}_{r+1}, \cdots, \mathbf{y}_n\}$$

be an orthonormal basis for $\mathbb{F}^n$ and let

$$\{F\mathbf{x}_r, \cdots, F\mathbf{x}_r, \mathbf{z}_{r+1}, \cdots, \mathbf{z}_n, \cdots, \mathbf{z}_m\}$$

be an orthonormal basis for $\mathbb{F}^m$. Then a typical vector of $\mathbb{F}^n$ is of the form

$$\sum_{k=1}^r a_k U\mathbf{x}_k + \sum_{j=r+1}^n b_j \mathbf{y}_j.$$

Define

$$R\left(\sum_{k=1}^r a_k U\mathbf{x}_k + \sum_{j=r+1}^n b_j \mathbf{y}_j\right) \equiv \sum_{k=1}^r a_k F\mathbf{x}_k + \sum_{j=r+1}^n b_j \mathbf{z}_j$$

Then since

$$\{U\mathbf{x}_1, \cdots, U\mathbf{x}_r, \mathbf{y}_{r+1}, \cdots, \mathbf{y}_n\}$$

and

$$\{F\mathbf{x}_r, \cdots, F\mathbf{x}_r, \mathbf{z}_{r+1}, \cdots, \mathbf{z}_n, \cdots, \mathbf{z}_m\}$$

are orthonormal,

$$\begin{aligned}
\left|R\left(\sum_{k=1}^r a_k U\mathbf{x}_k + \sum_{j=r+1}^n b_j \mathbf{y}_j\right)\right|^2 &= \left|\sum_{k=1}^r a_k F\mathbf{x}_k + \sum_{j=r+1}^n b_j \mathbf{z}_j\right|^2 \\
&= \sum_{k=1}^r |a_k|^2 + \sum_{j=r+1}^n |b_j|^2 \\
&= \left|\sum_{k=1}^r a_k U\mathbf{x}_k + \sum_{j=r+1}^n b_j \mathbf{y}_j\right|^2.
\end{aligned}$$

Therefore, R preserves distances.

Letting $\mathbf{x} \in \mathbb{F}^n$,

$$U\mathbf{x} = \sum_{k=1}^{r} a_k U\mathbf{x}_k \tag{4.38}$$

for some unique choice of scalars, a_k because $\{U\mathbf{x}_1, \cdots, U\mathbf{x}_r\}$ is a basis for $U(\mathbb{F}^n)$. Therefore,

$$RU\mathbf{x} = R\left(\sum_{k=1}^{r} a_k U\mathbf{x}_k\right) \equiv \sum_{k=1}^{r} a_k F\mathbf{x}_k = F\left(\sum_{k=1}^{r} a_k \mathbf{x}_k\right).$$

Is $F\left(\sum_{k=1}^{r} a_k \mathbf{x}_k\right) = F(\mathbf{x})$? Using 4.38,

$$\begin{aligned} F^*F\left(\sum_{k=1}^{r} a_k \mathbf{x}_k - \mathbf{x}\right) &= U^2\left(\sum_{k=1}^{r} a_k \mathbf{x}_k - \mathbf{x}\right) \\ &= \sum_{k=1}^{r} a_k \mu_k^2 \mathbf{x}_k - U(U\mathbf{x}) \\ &= \sum_{k=1}^{r} a_k \mu_k^2 \mathbf{x}_k - U\left(\sum_{k=1}^{r} a_k U\mathbf{x}_k\right) \\ &= \sum_{k=1}^{r} a_k \mu_k^2 \mathbf{x}_k - U\left(\sum_{k=1}^{r} a_k \mu_k \mathbf{x}_k\right) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \left|F\left(\sum_{k=1}^{r} a_k \mathbf{x}_k - \mathbf{x}\right)\right|^2 &= \left(F\left(\sum_{k=1}^{r} a_k \mathbf{x}_k - \mathbf{x}\right), F\left(\sum_{k=1}^{r} a_k \mathbf{x}_k - \mathbf{x}\right)\right) \\ &= \left(F^*F\left(\sum_{k=1}^{r} a_k \mathbf{x}_k - \mathbf{x}\right), \left(\sum_{k=1}^{r} a_k \mathbf{x}_k - \mathbf{x}\right)\right) = 0 \end{aligned}$$

and so $F\left(\sum_{k=1}^{r} a_k \mathbf{x}_k\right) = F(\mathbf{x})$ as hoped. Thus $RU = F$ on $\mathbb{F}^n$.

Since R preserves distances,

$$\begin{aligned} |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2(\mathbf{x}, \mathbf{y}) &= |\mathbf{x} + \mathbf{y}|^2 = |R(\mathbf{x} + \mathbf{y})|^2 \\ &= |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2(R\mathbf{x}, R\mathbf{y}). \end{aligned}$$

Therefore,

$$(\mathbf{x}, \mathbf{y}) = (R^*R\mathbf{x}, \mathbf{y})$$

for all $\mathbf{x}, \mathbf{y}$ and so $R^*R = I$ as claimed. This proves the theorem.

Multi-variable Calculus

5.1 Continuous Functions

In what follows, $\mathbb{F}$ will denote either $\mathbb{R}$ or $\mathbb{C}$. It turns out it is more efficient to not make a distinction. However, the main interest is in $\mathbb{R}$ so if you like, you can think $\mathbb{R}$ whenever you see $\mathbb{F}$.

5.1.1 Distance In $\mathbb{F}^n$

It is necessary to give a generalization of the dot product for vectors in $\mathbb{C}^n$. This definition reduces to the usual one in the case the components of the vector are real.

Definition 5.1 *Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Thus $\mathbf{x} = (x_1, \cdots, x_n)$ where each $x_k \in \mathbb{C}$ and a similar formula holding for $\mathbf{y}$. Then the dot product of these two vectors is defined to be*

$$\mathbf{x} \cdot \mathbf{y} \equiv \sum_j x_j \overline{y_j} \equiv x_1 \overline{y_1} + \cdots + x_n \overline{y_n}.$$

Notice how you put the conjugate on the entries of the vector, $\mathbf{y}$. It makes no difference if the vectors happen to be real vectors but with complex vectors you must do it this way. The reason for this is that when you take the dot product of a vector with itself, you want to get the square of the length of the vector, a positive number. Placing the conjugate on the components of $\mathbf{y}$ in the above definition assures this will take place. Thus

$$\mathbf{x} \cdot \mathbf{x} = \sum_j x_j \overline{x_j} = \sum_j |x_j|^2 \geq 0.$$

If you didn't place a conjugate as in the above definition, things wouldn't work out correctly. For example,

$$(1 + i)^2 + 2^2 = 4 + 2i$$

and this is not a positive number.

The following properties of the dot product follow immediately from the definition and you should verify each of them.

Properties of the dot product:

1. $\mathbf{u}\cdot\mathbf{v}=\overline{\mathbf{v}\cdot\mathbf{u}}$.
2. If a,b are numbers and $\mathbf{u},\mathbf{v},\mathbf{z}$ are vectors then $(a\mathbf{u}+b\mathbf{v})\cdot\mathbf{z}=a(\mathbf{u}\cdot\mathbf{z})+b(\mathbf{v}\cdot\mathbf{z})$.
3. $\mathbf{u}\cdot\mathbf{u}\geq 0$ and it equals 0 if and only if $\mathbf{u}=\mathbf{0}$.

The norm is defined in the usual way.

Definition 5.2 *For* $\mathbf{x}\in\mathbb{C}^n$,

$$|\mathbf{x}|\equiv\left(\sum_{k=1}^{n}|x_k|^2\right)^{1/2}=(\mathbf{x}\cdot\mathbf{x})^{1/2}$$

Here is a fundamental inequality called the **Cauchy Schwarz inequality** which is stated here in $\mathbb{C}^n$. First here is a simple lemma.

Lemma 5.3 *If* $z\in\mathbb{C}$ *there exists* $\theta\in\mathbb{C}$ *such that* $\theta z=|z|$ *and* $|\theta|=1$.

Proof: Let $\theta=1$ if $z=0$ and otherwise, let $\theta=\dfrac{\overline{z}}{|z|}$. Recall that for $z=x+iy,\overline{z}=x-iy$ and $\overline{z}z=|z|^2$.

Theorem 5.4 *(Cauchy Schwarz)The following inequality holds for* x_i *and* $y_i\in\mathbb{C}$.

$$|(\mathbf{x}\cdot\mathbf{y})|=\left|\sum_{i=1}^{n}x_i\overline{y}_i\right|\leq\left(\sum_{i=1}^{n}|x_i|^2\right)^{1/2}\left(\sum_{i=1}^{n}|y_i|^2\right)^{1/2}=|\mathbf{x}|\,|\mathbf{y}| \tag{5.1}$$

Proof: Let $\theta\in\mathbb{C}$ such that $|\theta|=1$ and

$$\theta\sum_{i=1}^{n}x_i\overline{y}_i=\left|\sum_{i=1}^{n}x_i\overline{y}_i\right|$$

Thus

$$\theta\sum_{i=1}^{n}x_i\overline{y}_i=\sum_{i=1}^{n}x_i\overline{\left(\overline{\theta}y_i\right)}=\left|\sum_{i=1}^{n}x_i\overline{y}_i\right|.$$

Consider $p(t)\equiv\sum_{i=1}^{n}\left(x_i+t\overline{\theta}y_i\right)\left(\overline{x_i+t\overline{\theta}y_i}\right)$ where $t\in\mathbb{R}$.

$$\begin{aligned}0 &\leq p(t)=\sum_{i=1}^{n}|x_i|^2+2t\operatorname{Re}\left(\theta\sum_{i=1}^{n}x_i\overline{y}_i\right)+t^2\sum_{i=1}^{n}|y_i|^2\\ &= |\mathbf{x}|^2+2t\left|\sum_{i=1}^{n}x_i\overline{y}_i\right|+t^2|\mathbf{y}|^2\end{aligned}$$

If $|\mathbf{y}| = 0$ then 5.1 is obviously true because both sides equal zero. Therefore, assume $|\mathbf{y}| \neq 0$ and then $p(t)$ is a polynomial of degree two whose graph opens up. Therefore, it either has no zeroes, two zeros or one repeated zero. If it has two zeros, the above inequality must be violated because in this case the graph must dip below the x axis. Therefore, it either has no zeros or exactly one. From the quadratic formula this happens exactly when

$$4\left|\sum_{i=1}^{n} x_i \overline{y}_i\right|^2 - 4|\mathbf{x}|^2 |\mathbf{y}|^2 \leq 0$$

and so

$$\left|\sum_{i=1}^{n} x_i \overline{y}_i\right| \leq |\mathbf{x}| |\mathbf{y}|$$

as claimed. This proves the inequality.

By analogy to the case of $\mathbb{R}^n$, length or magnitude of vectors in $\mathbb{C}^n$ can be defined.

Definition 5.5 *Let $\mathbf{z} \in \mathbb{C}^n$. Then $|\mathbf{z}| \equiv (\mathbf{z} \cdot \mathbf{z})^{1/2}$. Also numbers in $\mathbb{F}$ will often be referred to as scalars.*

Theorem 5.6 *For length defined in Definition 5.5, the following hold.*

$$|\mathbf{z}| \geq 0 \text{ and } |\mathbf{z}| = 0 \text{ if and only if } \mathbf{z} = \mathbf{0} \tag{5.2}$$

$$\text{If } \alpha \text{ is a scalar, } |\alpha \mathbf{z}| = |\alpha| |\mathbf{z}| \tag{5.3}$$

$$|\mathbf{z} + \mathbf{w}| \leq |\mathbf{z}| + |\mathbf{w}|. \tag{5.4}$$

Proof: The first two claims are left as exercises. To establish the third,

$$\begin{aligned} |\mathbf{z} + \mathbf{w}|^2 &= (\mathbf{z} + \mathbf{w}, \mathbf{z} + \mathbf{w}) \\ &= \mathbf{z} \cdot \mathbf{z} + \mathbf{w} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{z} + \mathbf{z} \cdot \mathbf{w} \\ &= |\mathbf{z}|^2 + |\mathbf{w}|^2 + 2 \operatorname{Re} \mathbf{w} \cdot \mathbf{z} \\ &\leq |\mathbf{z}|^2 + |\mathbf{w}|^2 + 2 |\mathbf{w} \cdot \mathbf{z}| \\ &\leq |\mathbf{z}|^2 + |\mathbf{w}|^2 + 2 |\mathbf{w}| |\mathbf{z}| = (|\mathbf{z}| + |\mathbf{w}|)^2. \end{aligned}$$

The main difference between $\mathbb{C}^n$ and $\mathbb{R}^n$ is that the scalars are complex numbers.

Definition 5.7 *Suppose you have a vector space, V and for $\mathbf{z}, \mathbf{w} \in V$ and α a scalar a norm is a way of measuring distance or magnitude which satisfies the properties 5.2 - 5.4. Thus a norm is something which does the following.*

$$||\mathbf{z}|| \geq 0 \text{ and } ||\mathbf{z}|| = 0 \text{ if and only if } \mathbf{z} = \mathbf{0} \tag{5.5}$$

$$\text{If } \alpha \text{ is a scalar, } ||\alpha \mathbf{z}|| = |\alpha| \, ||\mathbf{z}|| \tag{5.6}$$

$$||\mathbf{z} + \mathbf{w}|| \leq ||\mathbf{z}|| + ||\mathbf{w}||. \tag{5.7}$$

Here is is understood that for all $\mathbf{z} \in V, ||\mathbf{z}|| \in [0, \infty)$.

Note that $|\cdot|$ provides a norm on $\mathbb{F}^n$ from the above.

5.2 Open And Closed Sets

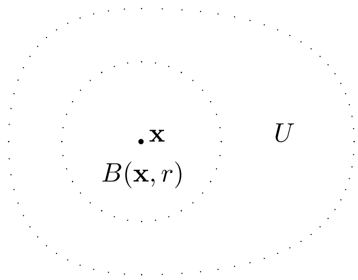
Eventually, one must consider functions which are defined on subsets of $\mathbb{F}^n$ and their properties. The next definition will end up being quite important. It describe a type of subset of $\mathbb{F}^n$ with the property that if $\mathbf{x}$ is in this set, then so is $\mathbf{y}$ whenever $\mathbf{y}$ is close enough to $\mathbf{x}$.

Definition 5.8 *Let $U \subseteq \mathbb{F}^n$. U is an **open set** if whenever $\mathbf{x} \in U$, there exists $r > 0$ such that $B(\mathbf{x}, r) \subseteq U$. More generally, if U is any subset of $\mathbb{F}^n$, $\mathbf{x} \in U$ is an **interior point** of U if there exists $r > 0$ such that $\mathbf{x} \in B(\mathbf{x}, r) \subseteq U$. In other words U is an open set exactly when every point of U is an interior point of U.*

If there is something called an open set, surely there should be something called a closed set and here is the definition of one.

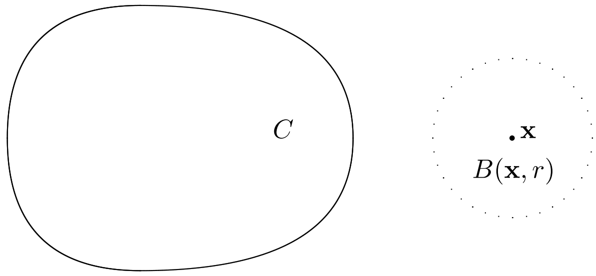
Definition 5.9 *A subset, C, of $\mathbb{F}^n$ is called a **closed set** if $\mathbb{F}^n \setminus C$ is an open set. They symbol, $\mathbb{F}^n \setminus C$ denotes everything in $\mathbb{F}^n$ which is not in C. It is also called the **complement** of C. The symbol, S^C is a short way of writing $\mathbb{F}^n \setminus S$.*

To illustrate this definition, consider the following picture.



You see in this picture how the edges are dotted. This is because an open set, can not include the edges or the set would fail to be open. For example, consider what would happen if you picked a point out on the edge of U in the above picture. Every open ball centered at that point would have in it some points which are outside U. Therefore, such a point would violate the above definition. You also see the edges of $B(\mathbf{x}, r)$ dotted suggesting that $B(\mathbf{x}, r)$ ought to be an open set. This is intuitively clear but does require a proof. This will be done in the next theorem and will give examples of open sets. Also, you can see that if $\mathbf{x}$ is close to the edge of U, you might have to take r to be very small.

It is roughly the case that open sets don't have their skins while closed sets do. Here is a picture of a closed set, C.



Note that $\mathbf{x} \notin C$ and since $\mathbb{F}^n \setminus C$ is open, there exists a ball, $B(\mathbf{x}, r)$ contained entirely in $\mathbb{F}^n \setminus C$. If you look at $\mathbb{F}^n \setminus C$, what would be its skin? It can't be in $\mathbb{F}^n \setminus C$ and so it must be in C. This is a rough heuristic explanation of what is going on with these definitions. Also note that $\mathbb{F}^n$ and $\emptyset$ are both open and closed. Here is why. If $\mathbf{x} \in \emptyset$, then there must be a ball centered at $\mathbf{x}$ which is also contained in $\emptyset$. This must be considered to be true because there is nothing in $\emptyset$ so there can be no example to show it false[1]. Therefore, from the definition, it follows $\emptyset$ is open. It is also closed because if $\mathbf{x} \notin \emptyset$, then $B(\mathbf{x}, 1)$ is also contained in $\mathbb{F}^n \setminus \emptyset = \mathbb{F}^n$. Therefore, $\emptyset$ is both open and closed. From this, it follows $\mathbb{F}^n$ is also both open and closed.

Theorem 5.10 *Let* $\mathbf{x} \in \mathbb{F}^n$ *and let* $r \geq 0$. *Then* $B(\mathbf{x}, r)$ *is an open set. Also,*

$$D(\mathbf{x}, r) \equiv \{\mathbf{y} \in \mathbb{F}^n : |\mathbf{y} - \mathbf{x}| \leq r\}$$

is a closed set.

Proof: Suppose $\mathbf{y} \in B(\mathbf{x}, r)$. It is necessary to show there exists $r_1 > 0$ such that $B(\mathbf{y}, r_1) \subseteq B(\mathbf{x}, r)$. Define $r_1 \equiv r - |\mathbf{x} - \mathbf{y}|$. Then if $|\mathbf{z} - \mathbf{y}| < r_1$, it follows from the above triangle inequality that

$$\begin{aligned} |\mathbf{z} - \mathbf{x}| &= |\mathbf{z} - \mathbf{y} + \mathbf{y} - \mathbf{x}| \\ &\leq |\mathbf{z} - \mathbf{y}| + |\mathbf{y} - \mathbf{x}| \\ &< r_1 + |\mathbf{y} - \mathbf{x}| = r - |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{x}| = r. \end{aligned}$$

Note that if $r = 0$ then $B(\mathbf{x}, r) = \emptyset$, the empty set. This is because if $\mathbf{y} \in \mathbb{F}^n$, $|\mathbf{x} - \mathbf{y}| \geq 0$ and so $\mathbf{y} \notin B(\mathbf{x}, 0)$. Since $\emptyset$ has no points in it, it must be open because

[1] To a mathematician, the statment: Whenever a pig is born with wings it can fly must be taken as true. We do not consider biological or aerodynamic considerations in such statements. There is no such thing as a winged pig and therefore, all winged pigs must be superb flyers since there can be no example of one which is not. On the other hand we would also consider the statement: Whenever a pig is born with wings it can't possibly fly, as equally true. The point is, you can say anything you want about the elements of the empty set and no one can gainsay your statement. Therefore, such statements are considered as true by default. You may say this is a very strange way of thinking about truth and ultimately this is because mathematics is not about truth. It is more about consistency and logic.

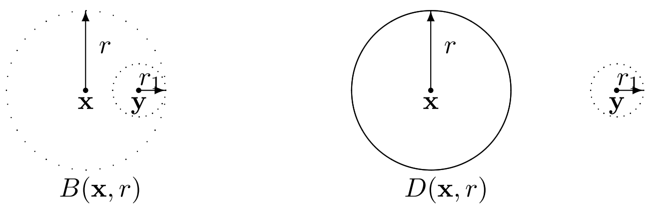
every point in it, (There are none.) satisfies the desired property of being an interior point.

Now suppose $\mathbf{y} \notin D(\mathbf{x}, r)$. Then $|\mathbf{x}-\mathbf{y}| > r$ and defining $\delta \equiv |\mathbf{x}-\mathbf{y}| - r$, it follows that if $\mathbf{z} \in B(\mathbf{y}, \delta)$, then by the triangle inequality,

$$\begin{aligned} |\mathbf{x}-\mathbf{z}| &\geq |\mathbf{x}-\mathbf{y}| - |\mathbf{y}-\mathbf{z}| > |\mathbf{x}-\mathbf{y}| - \delta \\ &= |\mathbf{x}-\mathbf{y}| - (|\mathbf{x}-\mathbf{y}| - r) = r \end{aligned}$$

and this shows that $B(\mathbf{y}, \delta) \subseteq \mathbb{F}^n \setminus D(\mathbf{x}, r)$. Since $\mathbf{y}$ was an arbitrary point in $\mathbb{F}^n \setminus D(\mathbf{x}, r)$, it follows $\mathbb{F}^n \setminus D(\mathbf{x}, r)$ is an open set which shows from the definition that $D(\mathbf{x}, r)$ is a closed set as claimed.

A picture which is descriptive of the conclusion of the above theorem which also implies the manner of proof is the following.



5.3 Continuous Functions

With the above definition of the norm in $\mathbb{F}^p$, it becomes possible to define continuity.

Definition 5.11 *A function* $\mathbf{f} : D(\mathbf{f}) \subseteq \mathbb{F}^p \to \mathbb{F}^q$ *is continuous at* $\mathbf{x} \in D(\mathbf{f})$ *if for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that whenever* $\mathbf{y} \in D(\mathbf{f})$ *and*

$$|\mathbf{y}-\mathbf{x}| < \delta$$

it follows that

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon.$$

$\mathbf{f}$ *is continuous if it is continuous at every point of* $D(\mathbf{f})$.

Note the total similarity to the scalar valued case.

5.3.1 Sufficient Conditions For Continuity

The next theorem is a fundamental result which will allow us to worry less about the ε δ definition of continuity.

Theorem 5.12 *The following assertions are valid.*

1. *The function,* $a\mathbf{f} + b\mathbf{g}$ *is continuous at* $\mathbf{x}$ *whenever* $\mathbf{f}$, $\mathbf{g}$ *are continuous at* $\mathbf{x} \in D(\mathbf{f}) \cap D(\mathbf{g})$ *and* $a, b \in \mathbb{F}$.

2. *If* $\mathbf{f}$ *is continuous at* $\mathbf{x}$, $\mathbf{f}(\mathbf{x}) \in D(\mathbf{g}) \subseteq \mathbb{F}^p$, *and* $\mathbf{g}$ *is continuous at* $\mathbf{f}(\mathbf{x})$ *,then* $\mathbf{g} \circ \mathbf{f}$ *is continuous at* $\mathbf{x}$.

3. *If* $\mathbf{f} = (f_1, \cdots, f_q) : D(\mathbf{f}) \to \mathbb{F}^q$, *then* $\mathbf{f}$ *is continuous if and only if each* f_k *is a continuous* $\mathbb{F}$ *valued function.*

4. *The function* $f : \mathbb{F}^p \to \mathbb{F}$, *given by* $f(\mathbf{x}) = |\mathbf{x}|$ *is continuous.*

The proof of this theorem is in the last section of this chapter. Its conclusions are not surprising. For example the first claim says that $(a\mathbf{f} + b\mathbf{g})(\mathbf{y})$ is close to $(a\mathbf{f} + b\mathbf{g})(\mathbf{x})$ when $\mathbf{y}$ is close to $\mathbf{x}$ provided the same can be said about $\mathbf{f}$ and $\mathbf{g}$. For the second claim, if $\mathbf{y}$ is close to $\mathbf{x}$, $\mathbf{f}(\mathbf{x})$ is close to $\mathbf{f}(\mathbf{y})$ and so by continuity of $\mathbf{g}$ at $\mathbf{f}(\mathbf{x})$, $\mathbf{g}(\mathbf{f}(\mathbf{y}))$ is close to $\mathbf{g}(\mathbf{f}(\mathbf{x}))$. To see the third claim is likely, note that closeness in $\mathbb{F}^p$ is the same as closeness in each coordinate. The fourth claim is immediate from the triangle inequality.

For functions defined on $\mathbb{F}^n$, there is a notion of polynomial just as there is for functions defined on $\mathbb{R}$.

Definition 5.13 *Let* α *be an* n *dimensional multi-index. This means*

$$\alpha = (\alpha_1, \cdots, \alpha_n)$$

where each α_i *is a natural number or zero. Also, let*

$$|\alpha| \equiv \sum_{i=1}^{n} |\alpha_i|$$

The symbol, $\mathbf{x}^\alpha$*,means*

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_3^{\alpha_n}.$$

An n *dimensional polynomial of degree* m *is a function of the form*

$$p(\mathbf{x}) = \sum_{|\alpha| \leq m} d_\alpha \mathbf{x}^\alpha.$$

where the d_α *are complex or real numbers.*

The above theorem implies that polynomials are all continuous.

5.4 Exercises

1. Let $\mathbf{f}(t) = (t, \sin t)$. Show f is continuous at every point t.

2. Suppose $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|$ where K is a constant. Show that $\mathbf{f}$ is everywhere continuous. Functions satisfying such an inequality are called Lipschitz functions.

3. Suppose $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|^\alpha$ where K is a constant and $\alpha \in (0, 1)$. Show that $\mathbf{f}$ is everywhere continuous.

4. Suppose $f : \mathbb{F}^3 \to \mathbb{F}$ is given by $f(\mathbf{x}) = 3x_1x_2 + 2x_3^2$. Use Theorem 5.12 to verify that f is continuous. **Hint:** You should first verify that the function, $\pi_k : \mathbb{F}^3 \to \mathbb{F}$ given by $\pi_k(\mathbf{x}) = x_k$ is a continuous function.

5. Generalize the previous problem to the case where $f : \mathbb{F}^q \to \mathbb{F}$ is a polynomial.

6. State and prove a theorem which involves quotients of functions encountered in the previous problem.

5.5 Limits Of A Function

As in the case of scalar valued functions of one variable, a concept closely related to continuity is that of the limit of a function. The notion of limit of a function makes sense at points, $\mathbf{x}$, which are limit points of $D(\mathbf{f})$ and this concept is defined next.

Definition 5.14 *Let $A \subseteq \mathbb{F}^m$ be a set. A point, $\mathbf{x}$, is a limit point of A if $B(\mathbf{x}, r)$ contains infinitely many points of A for every $r > 0$.*

Definition 5.15 *Let $\mathbf{f} : D(\mathbf{f}) \subseteq \mathbb{F}^p \to \mathbb{F}^q$ be a function and let $\mathbf{x}$ be a limit point of $D(\mathbf{f})$. Then*

$$\lim_{\mathbf{y}\to\mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{L}$$

if and only if the following condition holds. For all $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |\mathbf{y} - \mathbf{x}| < \delta, \text{ and } \mathbf{y} \in D(\mathbf{f})$$

then,

$$|\mathbf{L} - \mathbf{f}(\mathbf{y})| < \varepsilon.$$

Theorem 5.16 *If $\lim_{\mathbf{y}\to\mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{L}$ and $\lim_{y\to x} \mathbf{f}(\mathbf{y}) = \mathbf{L}_1$, then $\mathbf{L} = \mathbf{L}_1$.*

Proof: Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that if $0 < |\mathbf{y} - \mathbf{x}| < \delta$ and $\mathbf{y} \in D(\mathbf{f})$, then

$$|\mathbf{f}(\mathbf{y}) - \mathbf{L}| < \varepsilon, \ |\mathbf{f}(\mathbf{y}) - \mathbf{L}_1| < \varepsilon.$$

Pick such a $\mathbf{y}$. There exists one because $\mathbf{x}$ is a limit point of $D(\mathbf{f})$. Then

$$|\mathbf{L} - \mathbf{L}_1| \leq |\mathbf{L} - \mathbf{f}(\mathbf{y})| + |\mathbf{f}(\mathbf{y}) - \mathbf{L}_1| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows $\mathbf{L} = \mathbf{L}_1$.

As in the case of functions of one variable, one can define what it means for $\lim_{\mathbf{y}\to\mathbf{x}} f(\mathbf{x}) = \pm\infty$.

Definition 5.17 *If $f(\mathbf{x}) \in \mathbb{F}$, $\lim_{\mathbf{y}\to\mathbf{x}} f(\mathbf{x}) = \infty$ if for every number l, there exists $\delta > 0$ such that whenever $|\mathbf{y} - \mathbf{x}| < \delta$ and $\mathbf{y} \in D(\mathbf{f})$, then $f(\mathbf{x}) > l$.*

The following theorem is just like the one variable version presented earlier.

Theorem 5.18 *Suppose* $\lim_{\mathbf{y}\to\mathbf{x}}\mathbf{f}(\mathbf{y})=\mathbf{L}$ *and* $\lim_{\mathbf{y}\to\mathbf{x}}\mathbf{g}(\mathbf{y})=\mathbf{K}$ *where* $\mathbf{K},\mathbf{L}\in\mathbb{F}^q$. *Then if* $a,\ b\in\mathbb{F}$,

$$\lim_{\mathbf{y}\to\mathbf{x}}(a\mathbf{f}(\mathbf{y})+b\mathbf{g}(\mathbf{y}))=a\mathbf{L}+b\mathbf{K}, \tag{5.8}$$

$$\lim_{y\to x}\mathbf{f}\cdot\mathbf{g}(y)=\mathbf{LK} \tag{5.9}$$

and if g *is scalar valued with* $\lim_{\mathbf{y}\to\mathbf{x}}g(\mathbf{y})=K\neq 0$,

$$\lim_{\mathbf{y}\to\mathbf{x}}\mathbf{f}(\mathbf{y})\,g(\mathbf{y})=\mathbf{L}K. \tag{5.10}$$

Also, if $\mathbf{h}$ *is a continuous function defined near* $\mathbf{L}$, *then*

$$\lim_{\mathbf{y}\to\mathbf{x}}\mathbf{h}\circ\mathbf{f}(\mathbf{y})=\mathbf{h}(\mathbf{L}). \tag{5.11}$$

Suppose $\lim_{\mathbf{y}\to\mathbf{x}}\mathbf{f}(\mathbf{y})=\mathbf{L}$. *If* $|\mathbf{f}(\mathbf{y})-\mathbf{b}|\leq r$ *for all* $\mathbf{y}$ *sufficiently close to* $\mathbf{x}$, *then* $|\mathbf{L}-\mathbf{b}|\leq r$ *also.*

Proof: The proof of 5.8 is left for you. It is like a corresponding theorem for continuous functions. Now 5.9is to be verified. Let $\varepsilon>0$ be given. Then by the triangle inequality,

$$\begin{aligned}|\mathbf{f}\cdot\mathbf{g}(\mathbf{y})-\mathbf{L}\cdot\mathbf{K}|&\leq|\mathbf{fg}(\mathbf{y})-\mathbf{f}(\mathbf{y})\cdot\mathbf{K}|+|\mathbf{f}(\mathbf{y})\cdot\mathbf{K}-\mathbf{L}\cdot\mathbf{K}|\\&\leq|\mathbf{f}(\mathbf{y})|\,|\mathbf{g}(\mathbf{y})-\mathbf{K}|+|\mathbf{K}|\,|\mathbf{f}(\mathbf{y})-\mathbf{L}|.\end{aligned}$$

There exists δ_1 such that if $0<|\mathbf{y}-\mathbf{x}|<\delta_1$ and $\mathbf{y}\in D(\mathbf{f})$, then

$$|\mathbf{f}(\mathbf{y})-\mathbf{L}|<1,$$

and so for such $\mathbf{y}$, the triangle inequality implies, $|\mathbf{f}(\mathbf{y})|<1+|\mathbf{L}|$. Therefore, for $0<|\mathbf{y}-\mathbf{x}|<\delta_1$,

$$|\mathbf{f}\cdot\mathbf{g}(\mathbf{y})-\mathbf{L}\cdot\mathbf{K}|\leq(1+|\mathbf{K}|+|\mathbf{L}|)\left[|\mathbf{g}(\mathbf{y})-\mathbf{K}|+|\mathbf{f}(\mathbf{y})-\mathbf{L}|\right]. \tag{5.12}$$

Now let $0<\delta_2$ be such that if $\mathbf{y}\in D(\mathbf{f})$ and $0<|\mathbf{x}-\mathbf{y}|<\delta_2$,

$$|\mathbf{f}(\mathbf{y})-\mathbf{L}|<\frac{\varepsilon}{2(1+|\mathbf{K}|+|\mathbf{L}|)},\ |\mathbf{g}(\mathbf{y})-\mathbf{K}|<\frac{\varepsilon}{2(1+|\mathbf{K}|+|\mathbf{L}|)}.$$

Then letting $0<\delta\leq\min(\delta_1,\delta_2)$, it follows from 5.12 that

$$|\mathbf{f}\cdot\mathbf{g}(\mathbf{y})-\mathbf{L}\cdot\mathbf{K}|<\varepsilon$$

and this proves 5.9.

The proof of 5.10 is left to you.

Consider 5.11. Since $\mathbf{h}$ is continuous near $\mathbf{L}$, it follows that for $\varepsilon>0$ given, there exists $\eta>0$ such that if $|\mathbf{y}-\mathbf{L}|<\eta$, then

$$|\mathbf{h}(\mathbf{y})-\mathbf{h}(\mathbf{L})|<\varepsilon$$

Now since $\lim_{\mathbf{y}\to\mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{L}$, there exists $\delta > 0$ such that if $0 < |\mathbf{y} - \mathbf{x}| < \delta$, then

$$|\mathbf{f}(\mathbf{y}) - \mathbf{L}| < \eta.$$

Therefore, if $0 < |\mathbf{y} - \mathbf{x}| < \delta$,

$$|\mathbf{h}(\mathbf{f}(\mathbf{y})) - \mathbf{h}(\mathbf{L})| < \varepsilon.$$

It only remains to verify the last assertion. Assume $|\mathbf{f}(\mathbf{y}) - \mathbf{b}| \leq r$ for all $\mathbf{y}$ close enough to $\mathbf{x}$. It is required to show that $|\mathbf{L} - \mathbf{b}| \leq r$. If this is not true, then $|\mathbf{L} - \mathbf{b}| > r$. Consider $B(\mathbf{L}, |\mathbf{L} - \mathbf{b}| - r)$. Since $\mathbf{L}$ is the limit of $\mathbf{f}$, it follows $\mathbf{f}(\mathbf{y}) \in B(\mathbf{L}, |\mathbf{L} - \mathbf{b}| - r)$ whenever $\mathbf{y} \in D(\mathbf{f})$ is close enough to $\mathbf{x}$. Thus, by the triangle inequality,

$$|\mathbf{f}(\mathbf{y}) - \mathbf{L}| < |\mathbf{L} - \mathbf{b}| - r$$

and so

$$\begin{aligned} r &< |\mathbf{L} - \mathbf{b}| - |\mathbf{f}(\mathbf{y}) - \mathbf{L}| \leq ||\mathbf{b} - \mathbf{L}| - |\mathbf{f}(\mathbf{y}) - \mathbf{L}|| \\ &\leq |\mathbf{b} - \mathbf{f}(\mathbf{y})|, \end{aligned}$$

a contradiction to the assumption that $|\mathbf{b} - \mathbf{f}(\mathbf{y})| \leq r$.

Theorem 5.19 *For $\mathbf{f} : D(\mathbf{f}) \to \mathbb{F}^q$ and $\mathbf{x} \in D(\mathbf{f})$ a limit point of $D(\mathbf{f})$, $\mathbf{f}$ is continuous at $\mathbf{x}$ if and only if*

$$\lim_{\mathbf{y}\to\mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x}).$$

Proof: First suppose $\mathbf{f}$ is continuous at $\mathbf{x}$ a limit point of $D(\mathbf{f})$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{y} - \mathbf{x}| < \delta$ and $\mathbf{y} \in D(\mathbf{f})$, then $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$. In particular, this holds if $0 < |\mathbf{x} - \mathbf{y}| < \delta$ and this is just the definition of the limit. Hence $\mathbf{f}(\mathbf{x}) = \lim_{\mathbf{y}\to\mathbf{x}} \mathbf{f}(\mathbf{y})$.

Next suppose $\mathbf{x}$ is a limit point of $D(\mathbf{f})$ and $\lim_{\mathbf{y}\to\mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x})$. This means that if $\varepsilon > 0$ there exists $\delta > 0$ such that for $0 < |\mathbf{x} - \mathbf{y}| < \delta$ and $\mathbf{y} \in D(\mathbf{f})$, it follows $|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| < \varepsilon$. However, if $\mathbf{y} = \mathbf{x}$, then $|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| = |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| = 0$ and so whenever $\mathbf{y} \in D(\mathbf{f})$ and $|\mathbf{x} - \mathbf{y}| < \delta$, it follows $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$, showing $\mathbf{f}$ is continuous at $\mathbf{x}$.

The following theorem is important.

Theorem 5.20 *Suppose $\mathbf{f} : D(\mathbf{f}) \to \mathbb{F}^q$. Then for $\mathbf{x}$ a limit point of $D(\mathbf{f})$,*

$$\lim_{\mathbf{y}\to\mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{L} \tag{5.13}$$

if and only if

$$\lim_{\mathbf{y}\to\mathbf{x}} f_k(\mathbf{y}) = L_k \tag{5.14}$$

where $\mathbf{f}(\mathbf{y}) \equiv (f_1(\mathbf{y}), \cdots, f_p(\mathbf{y}))$ and $\mathbf{L} \equiv (L_1, \cdots, L_p)$.

Proof: Suppose 5.13. Then letting $\varepsilon > 0$ be given there exists $\delta > 0$ such that if $0 < |\mathbf{y} - \mathbf{x}| < \delta$, it follows

$$|f_k(\mathbf{y}) - L_k| \leq |\mathbf{f}(\mathbf{y}) - \mathbf{L}| < \varepsilon$$

which verifies 5.14.

Now suppose 5.14 holds. Then letting $\varepsilon > 0$ be given, there exists δ_k such that if $0 < |\mathbf{y} - \mathbf{x}| < \delta_k$, then

$$|f_k(\mathbf{y}) - L_k| < \frac{\varepsilon}{\sqrt{p}}.$$

Let $0 < \delta < \min(\delta_1, \cdots, \delta_p)$. Then if $0 < |\mathbf{y} - \mathbf{x}| < \delta$, it follows

$$\begin{aligned} |\mathbf{f}(\mathbf{y}) - \mathbf{L}| &= \left(\sum_{k=1}^{p} |f_k(\mathbf{y}) - L_k|^2 \right)^{1/2} \\ &< \left(\sum_{k=1}^{p} \frac{\varepsilon^2}{p} \right)^{1/2} = \varepsilon. \end{aligned}$$

This proves the theorem.

This theorem shows it suffices to consider the components of a vector valued function when computing the limit.

Example 5.21 *Find* $\lim_{(x,y)\to(3,1)} \left(\frac{x^2-9}{x-3}, y \right)$.

It is clear that $\lim_{(x,y)\to(3,1)} \frac{x^2-9}{x-3} = 6$ and $\lim_{(x,y)\to(3,1)} y = 1$. Therefore, this limit equals $(6, 1)$.

Example 5.22 *Find* $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$.

First of all observe the domain of the function is $\mathbb{F}^2 \setminus \{(0,0)\}$, every point in $\mathbb{F}^2$ except the origin. Therefore, $(0,0)$ is a limit point of the domain of the function so it might make sense to take a limit. However, just as in the case of a function of one variable, the limit may not exist. In fact, this is the case here. To see this, take points on the line $y = 0$. At these points, the value of the function equals 0. Now consider points on the line $y = x$ where the value of the function equals $1/2$. Since arbitrarily close to $(0,0)$ there are points where the function equals $1/2$ and points where the function has the value 0, it follows there can be no limit. Just take $\varepsilon = 1/10$ for example. You can't be within $1/10$ of $1/2$ and also within $1/10$ of 0 at the same time.

Note it is necessary to rely on the definition of the limit much more than in the case of a function of one variable and it is the case there are no easy ways to do limit problems for functions of more than one variable. It is what it is and you will not deal with these concepts without agony.

5.6 Exercises

1. Find the following limits if possible

   (a) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

   (b) $\lim_{(x,y)\to(0,0)} \frac{x\left(x^2-y^2\right)}{(x^2+y^2)}$

   (c) $\lim_{(x,y)\to(0,0)} \frac{\left(x^2-y^4\right)^2}{(x^2+y^4)^2}$ **Hint:** Consider along $y = 0$ and along $x = y^2$.

   (d) $\lim_{(x,y)\to(0,0)} x \sin\left(\frac{1}{x^2+y^2}\right)$

   (e) The limit as $(x, y) \to (1, 2)$ of the expression

   $$\frac{-2yx^2 + 8yx + 34y + 3y^3 - 18y^2 + 6x^2 - 13x - 20 - xy^2 - x^3}{-y^2 + 4y - 5 - x^2 + 2x}.$$

   Hint: It might help to write this in terms of the variables $(s, t) = (x - 1, y - 2)$.

2. In the definition of limit, why must $\mathbf{x}$ be a limit point of $D(\mathbf{f})$? **Hint:** If $\mathbf{x}$ were not a limit point of $D(\mathbf{f})$, show there exists $\delta > 0$ such that $B(\mathbf{x}, \delta)$ contains no points of $D(\mathbf{f})$ other than possibly $\mathbf{x}$ itself. Argue that 33.3 is a limit and that so is 22 and 7 and 11. In other words the concept is totally worthless.

5.7 The Limit Of A Sequence

As in the case of real numbers, one can consider the limit of a sequence of points in $\mathbb{F}^p$.

Definition 5.23 *A sequence* $\{\mathbf{a}_n\}_{n=1}^{\infty}$ *converges to* $\mathbf{a}$, *and write*

$$\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a} \text{ or } \mathbf{a}_n \to \mathbf{a}$$

if and only if for every $\varepsilon > 0$ *there exists* n_ε *such that whenever* $n \geq n_\varepsilon$,

$$|\mathbf{a}_n - \mathbf{a}| < \varepsilon.$$

In words the definition says that given any measure of closeness, ε, the terms of the sequence are eventually all this close to $\mathbf{a}$. There is absolutely no difference between this and the definition for sequences of numbers other than here bold face is used to indicate $\mathbf{a}_n$ and $\mathbf{a}$ are points in $\mathbb{F}^p$.

Theorem 5.24 *If* $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$ *and* $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}_1$ *then* $\mathbf{a}_1 = \mathbf{a}$.

Proof: Suppose $\mathbf{a}_1 \neq \mathbf{a}$. Then let $0 < \varepsilon < |\mathbf{a}_1 - \mathbf{a}|/2$ in the definition of the limit. It follows there exists n_ε such that if $n \geq n_\varepsilon$, then $|\mathbf{a}_n - \mathbf{a}| < \varepsilon$ and $|\mathbf{a}_n - \mathbf{a}_1| < \varepsilon$. Therefore, for such n,

$$\begin{aligned} |\mathbf{a}_1 - \mathbf{a}| &\leq |\mathbf{a}_1 - \mathbf{a}_n| + |\mathbf{a}_n - \mathbf{a}| \\ &< \varepsilon + \varepsilon < |\mathbf{a}_1 - \mathbf{a}|/2 + |\mathbf{a}_1 - \mathbf{a}|/2 = |\mathbf{a}_1 - \mathbf{a}|, \end{aligned}$$

a contradiction.

As in the case of a vector valued function, it suffices to consider the components. This is the content of the next theorem.

Theorem 5.25 *Let* $\mathbf{a}_n = \left(a_1^n, \cdot \cdot \cdot, a_p^n\right) \in \mathbb{F}^p$. *Then* $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a} \equiv (a_1, \cdot \cdot \cdot, a_p)$ *if and only if for each* $k = 1, \cdot \cdot \cdot, p$,

$$\lim_{n\to\infty} a_k^n = a_k. \tag{5.15}$$

Proof: First suppose $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$. Then given $\varepsilon > 0$ there exists n_ε such that if $n > n_\varepsilon$, then

$$|a_k^n - a_k| \leq |\mathbf{a}_n - \mathbf{a}| < \varepsilon$$

which establishes 5.15.

Now suppose 5.15 holds for each k. Then letting $\varepsilon > 0$ be given there exist n_k such that if $n > n_k$,

$$|a_k^n - a_k| < \varepsilon/\sqrt{p}.$$

Therefore, letting $n_\varepsilon > \max(n_1, \cdot \cdot \cdot, n_p)$, it follows that for $n > n_\varepsilon$,

$$|\mathbf{a}_n - \mathbf{a}| = \left(\sum_{k=1}^n |a_k^n - a_k|^2\right)^{1/2} < \left(\sum_{k=1}^n \frac{\varepsilon^2}{p}\right)^{1/2} = \varepsilon,$$

showing that $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$. This proves the theorem.

Example 5.26 *Let* $\mathbf{a}_n = \left(\frac{1}{n^2+1}, \frac{1}{n}\sin(n), \frac{n^2+3}{3n^2+5n}\right)$.

It suffices to consider the limits of the components according to the following theorem. Thus the limit is $(0, 0, 1/3)$.

Theorem 5.27 *Suppose* $\{\mathbf{a}_n\}$ *and* $\{\mathbf{b}_n\}$ *are sequences and that*

$$\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a} \text{ and } \lim_{n\to\infty} \mathbf{b}_n = \mathbf{b}.$$

Also suppose x *and* y *are numbers in* $\mathbb{F}$. *Then*

$$\lim_{n\to\infty} x\mathbf{a}_n + y\mathbf{b}_n = x\mathbf{a} + y\mathbf{b} \tag{5.16}$$

$$\lim_{n\to\infty} \mathbf{a}_n \cdot \mathbf{b}_n = \mathbf{a} \cdot \mathbf{b} \tag{5.17}$$

If $b_n \in \mathbb{F}$, *then*

$$\mathbf{a}_n b_n \to \mathbf{a} b.$$

Proof: The first of these claims is left for you to do. To do the second, let $\varepsilon > 0$ be given and choose n_1 such that if $n \geq n_1$ then

$$|\mathbf{a}_n - \mathbf{a}| < 1.$$

Then for such n, the triangle inequality and Cauchy Schwarz inequality imply

$$\begin{aligned} |\mathbf{a}_n \cdot \mathbf{b}_n - \mathbf{a} \cdot \mathbf{b}| &\leq |\mathbf{a}_n \cdot \mathbf{b}_n - \mathbf{a}_n \cdot \mathbf{b}| + |\mathbf{a}_n \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b}| \\ &\leq |\mathbf{a}_n| \, |\mathbf{b}_n - \mathbf{b}| + |\mathbf{b}| \, |\mathbf{a}_n - \mathbf{a}| \\ &\leq (|\mathbf{a}| + 1) \, |\mathbf{b}_n - \mathbf{b}| + |\mathbf{b}| \, |\mathbf{a}_n - \mathbf{a}| \, . \end{aligned}$$

Now let n_2 be large enough that for $n \geq n_2$,

$$|\mathbf{b}_n - \mathbf{b}| < \frac{\varepsilon}{2(|\mathbf{a}| + 1)}, \text{ and } |\mathbf{a}_n - \mathbf{a}| < \frac{\varepsilon}{2(|\mathbf{b}| + 1)}.$$

Such a number exists because of the definition of limit. Therefore, let

$$n_\varepsilon > \max(n_1, n_2).$$

For $n \geq n_\varepsilon$,

$$\begin{aligned} |\mathbf{a}_n \cdot \mathbf{b}_n - \mathbf{a} \cdot \mathbf{b}| &\leq (|\mathbf{a}| + 1) \, |\mathbf{b}_n - \mathbf{b}| + |\mathbf{b}| \, |\mathbf{a}_n - \mathbf{a}| \\ &< (|\mathbf{a}| + 1) \frac{\varepsilon}{2(|\mathbf{a}| + 1)} + |\mathbf{b}| \frac{\varepsilon}{2(|\mathbf{b}| + 1)} \leq \varepsilon. \end{aligned}$$

This proves 5.16. The proof of 5.17 is entirely similar and is left for you.

5.7.1 Sequences And Completeness

Recall the definition of a Cauchy sequence.

Definition 5.28 $\{\mathbf{a}_n\}$ *is a Cauchy sequence if for all* $\varepsilon > 0$, *there exists* n_ε *such that whenever* $n, m \geq n_\varepsilon$,

$$|\mathbf{a}_n - \mathbf{a}_m| < \varepsilon.$$

A sequence is Cauchy means the terms are "bunching up to each other" as m, n get large.

Theorem 5.29 *Let* $\{\mathbf{a}_n\}_{n=1}^\infty$ *be a Cauchy sequence in* $\mathbb{F}^p$. *Then there exists a unique* $\mathbf{a} \in \mathbb{F}^p$ *such that* $\mathbf{a}_n \to \mathbf{a}$.

Proof: Let $\mathbf{a}_n = \left(a_1^n, \cdots, a_p^n\right)$. Then

$$|a_k^n - a_k^m| \leq |\mathbf{a}_n - \mathbf{a}_m|$$

which shows for each $k = 1, \cdots, p$, it follows $\{a_k^n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{F}$. This requires that both the real and imaginary parts of a_k^n are Cauchy sequences in $\mathbb{R}$ which means the real and imaginary parts converge in $\mathbb{R}$. This shows $\{a_k^n\}_{n=1}^\infty$

must converge to some a_k. That is $\lim_{n\to\infty} a_k^n = a_k$. Letting $\mathbf{a} = (a_1, \cdots, a_p)$, it follows from Theorem 5.25 that

$$\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}.$$

This proves the theorem.

Theorem 5.30 *The set of terms in a Cauchy sequence in $\mathbb{F}^p$ is bounded in the sense that for all n, $|\mathbf{a}_n| < M$ for some $M < \infty$.*

Proof: Let $\varepsilon = 1$ in the definition of a Cauchy sequence and let $n > n_1$. Then from the definition,

$$|\mathbf{a}_n - \mathbf{a}_{n_1}| < 1.$$

It follows that for all $n > n_1$,

$$|\mathbf{a}_n| < 1 + |\mathbf{a}_{n_1}|.$$

Therefore, for all n,

$$|\mathbf{a}_n| \leq 1 + |\mathbf{a}_{n_1}| + \sum_{k=1}^{n_1} |\mathbf{a}_k|.$$

This proves the theorem.

Theorem 5.31 *If a sequence $\{\mathbf{a}_n\}$ in $\mathbb{F}^p$ converges, then the sequence is a Cauchy sequence.*

Proof: Let $\varepsilon > 0$ be given and suppose $\mathbf{a}_n \to \mathbf{a}$. Then from the definition of convergence, there exists n_ε such that if $n > n_\varepsilon$, it follows that

$$|\mathbf{a}_n - \mathbf{a}| < \frac{\varepsilon}{2}$$

Therefore, if $m, n \geq n_\varepsilon + 1$, it follows that

$$|\mathbf{a}_n - \mathbf{a}_m| \leq |\mathbf{a}_n - \mathbf{a}| + |\mathbf{a} - \mathbf{a}_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

showing that, since $\varepsilon > 0$ is arbitrary, $\{\mathbf{a}_n\}$ is a Cauchy sequence.

5.7.2 Continuity And The Limit Of A Sequence

Just as in the case of a function of one variable, there is a very useful way of thinking of continuity in terms of limits of sequences found in the following theorem. In words, it says a function is continuous if it takes convergent sequences to convergent sequences whenever possible.

Theorem 5.32 *A function $\mathbf{f} : D(\mathbf{f}) \to \mathbb{F}^q$ is continuous at $\mathbf{x} \in D(\mathbf{f})$ if and only if, whenever $\mathbf{x}_n \to \mathbf{x}$ with $\mathbf{x}_n \in D(\mathbf{f})$, it follows $\mathbf{f}(\mathbf{x}_n) \to \mathbf{f}(\mathbf{x})$.*

Proof: Suppose first that $\mathbf{f}$ is continuous at $\mathbf{x}$ and let $\mathbf{x}_n \rightarrow \mathbf{x}$. Let $\varepsilon > 0$ be given. By continuity, there exists $\delta > 0$ such that if $|\mathbf{y} - \mathbf{x}| < \delta$, then $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$. However, there exists n_δ such that if $n \geq n_\delta$, then $|\mathbf{x}_n - \mathbf{x}| < \delta$ and so for all n this large,

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_n)| < \varepsilon$$

which shows $\mathbf{f}(\mathbf{x}_n) \rightarrow \mathbf{f}(\mathbf{x})$.

Now suppose the condition about taking convergent sequences to convergent sequences holds at $\mathbf{x}$. Suppose $\mathbf{f}$ fails to be continuous at $\mathbf{x}$. Then there exists $\varepsilon > 0$ and $\mathbf{x}_n \in D(f)$ such that $|\mathbf{x} - \mathbf{x}_n| < \frac{1}{n}$, yet

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_n)| \geq \varepsilon.$$

But this is clearly a contradiction because, although $\mathbf{x}_n \rightarrow \mathbf{x}$, $\mathbf{f}(\mathbf{x}_n)$ fails to converge to $\mathbf{f}(\mathbf{x})$. It follows $\mathbf{f}$ must be continuous after all. This proves the theorem.

5.8 Properties Of Continuous Functions

Functions of p variables have many of the same properties as functions of one variable. First there is a version of the extreme value theorem generalizing the one dimensional case.

Theorem 5.33 *Let C be closed and bounded and let $f : C \rightarrow \mathbb{R}$ be continuous. Then f achieves its maximum and its minimum on C. This means there exist, $\mathbf{x}_1, \mathbf{x}_2 \in C$ such that for all $\mathbf{x} \in C$,*

$$f(\mathbf{x}_1) \leq f(\mathbf{x}) \leq f(\mathbf{x}_2).$$

There is also the long technical theorem about sums and products of continuous functions. These theorems are proved in the next section.

Theorem 5.34 *The following assertions are valid*

1. *The function, $a\mathbf{f} + b\mathbf{g}$ is continuous at $\mathbf{x}$ when $\mathbf{f}$, $\mathbf{g}$ are continuous at $\mathbf{x} \in D(\mathbf{f}) \cap D(\mathbf{g})$ and $a, b \in \mathbb{F}$.*

2. *If and f and g are each $\mathbb{F}$ valued functions continuous at $\mathbf{x}$, then fg is continuous at $\mathbf{x}$. If, in addition to this, $g(\mathbf{x}) \neq 0$, then f/g is continuous at $\mathbf{x}$.*

3. *If $\mathbf{f}$ is continuous at $\mathbf{x}$, $\mathbf{f}(\mathbf{x}) \in D(\mathbf{g}) \subseteq \mathbb{F}^p$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$.*

4. *If $\mathbf{f} = (f_1, \cdots, f_q) : D(\mathbf{f}) \rightarrow \mathbb{F}^q$, then $\mathbf{f}$ is continuous if and only if each f_k is a continuous $\mathbb{F}$ valued function.*

5. *The function $f : \mathbb{F}^p \rightarrow \mathbb{F}$, given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous.*

5.9 Exercises

1. $\mathbf{f} : D \subseteq \mathbb{F}^p \to \mathbb{F}^q$ is Lipschitz continuous or just Lipschitz for short if there exists a constant, K such that
$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|$$
for all $\mathbf{x}, \mathbf{y} \in D$. Show every Lipschitz function is uniformly continuous which means that given $\varepsilon > 0$ there exists $\delta > 0$ independent of $\mathbf{x}$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, then $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$.

2. If $\mathbf{f}$ is uniformly continuous, does it follow that $|\mathbf{f}|$ is also uniformly continuous? If $|\mathbf{f}|$ is uniformly continuous does it follow that $\mathbf{f}$ is uniformly continuous? Answer the same questions with "uniformly continuous" replaced with "continuous". Explain why.

5.10 Proofs Of Theorems

This section contains the proofs of the theorems which were just stated without proof.

Theorem 5.35 *The following assertions are valid*

1. *The function, $a\mathbf{f} + b\mathbf{g}$ is continuous at $\mathbf{x}$ when $\mathbf{f}$, $\mathbf{g}$ are continuous at $\mathbf{x} \in D(\mathbf{f}) \cap D(\mathbf{g})$ and $a, b \in \mathbb{F}$.*

2. *If and f and g are each $\mathbb{F}$ valued functions continuous at $\mathbf{x}$, then fg is continuous at $\mathbf{x}$. If, in addition to this, $g(\mathbf{x}) \neq 0$, then f/g is continuous at $\mathbf{x}$.*

3. *If $\mathbf{f}$ is continuous at $\mathbf{x}$, $\mathbf{f}(\mathbf{x}) \in D(\mathbf{g}) \subseteq \mathbb{F}^p$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$,then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$.*

4. *If $\mathbf{f} = (f_1, \cdots, f_q) : D(\mathbf{f}) \to \mathbb{F}^q$, then $\mathbf{f}$ is continuous if and only if each f_k is a continuous $\mathbb{F}$ valued function.*

5. *The function $f : \mathbb{F}^p \to \mathbb{F}$, given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous.*

Proof: Begin with 1.) Let $\varepsilon > 0$ be given. By assumption, there exist $\delta_1 > 0$ such that whenever $|\mathbf{x} - \mathbf{y}| < \delta_1$, it follows $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \frac{\varepsilon}{2(|a|+|b|+1)}$ and there exists $\delta_2 > 0$ such that whenever $|\mathbf{x} - \mathbf{y}| < \delta_2$, it follows that $|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})| < \frac{\varepsilon}{2(|a|+|b|+1)}$. Then let $0 < \delta \leq \min(\delta_1, \delta_2)$. If $|\mathbf{x} - \mathbf{y}| < \delta$, then everything happens at once. Therefore, using the triangle inequality
$$|a\mathbf{f}(\mathbf{x}) + b\mathbf{f}(\mathbf{x}) - (a\mathbf{g}(\mathbf{y}) + b\mathbf{g}(\mathbf{y}))|$$

$$\leq |a| |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| + |b| |\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})|$$
$$< |a| \left(\frac{\varepsilon}{2(|a| + |b| + 1)} \right) + |b| \left(\frac{\varepsilon}{2(|a| + |b| + 1)} \right) < \varepsilon.$$

Now begin on 2.) There exists $\delta_1 > 0$ such that if $|\mathbf{y} - \mathbf{x}| < \delta_1$, then

$$|f(\mathbf{x}) - f(\mathbf{y})| < 1.$$

Therefore, for such $\mathbf{y}$,

$$|f(\mathbf{y})| < 1 + |f(\mathbf{x})|.$$

It follows that for such $\mathbf{y}$,

$$|fg(\mathbf{x}) - fg(\mathbf{y})| \leq |f(\mathbf{x}) g(\mathbf{x}) - g(\mathbf{x}) f(\mathbf{y})| + |g(\mathbf{x}) f(\mathbf{y}) - f(\mathbf{y}) g(\mathbf{y})|$$

$$\leq |g(\mathbf{x})| |f(\mathbf{x}) - f(\mathbf{y})| + |f(\mathbf{y})| |g(\mathbf{x}) - g(\mathbf{y})|$$
$$\leq (1 + |g(\mathbf{x})| + |f(\mathbf{y})|) [|g(\mathbf{x}) - g(\mathbf{y})| + |f(\mathbf{x}) - f(\mathbf{y})|].$$

Now let $\varepsilon > 0$ be given. There exists δ_2 such that if $|\mathbf{x} - \mathbf{y}| < \delta_2$, then

$$|g(\mathbf{x}) - g(\mathbf{y})| < \frac{\varepsilon}{2(1 + |g(\mathbf{x})| + |f(\mathbf{y})|)},$$

and there exists δ_3 such that if $|\mathbf{x} - \mathbf{y}| < \delta_3$, then

$$|f(\mathbf{x}) - f(\mathbf{y})| < \frac{\varepsilon}{2(1 + |g(\mathbf{x})| + |f(\mathbf{y})|)}$$

Now let $0 < \delta \leq \min(\delta_1, \delta_2, \delta_3)$. Then if $|\mathbf{x} - \mathbf{y}| < \delta$, all the above hold at once and

$$|fg(\mathbf{x}) - fg(\mathbf{y})| \leq$$

$$(1 + |g(\mathbf{x})| + |f(\mathbf{y})|) [|g(\mathbf{x}) - g(\mathbf{y})| + |f(\mathbf{x}) - f(\mathbf{y})|]$$
$$< (1 + |g(\mathbf{x})| + |f(\mathbf{y})|) \left(\frac{\varepsilon}{2(1 + |g(\mathbf{x})| + |f(\mathbf{y})|)} + \frac{\varepsilon}{2(1 + |g(\mathbf{x})| + |f(\mathbf{y})|)} \right) = \varepsilon.$$

This proves the first part of 2.) To obtain the second part, let δ_1 be as described above and let $\delta_0 > 0$ be such that for $|\mathbf{x} - \mathbf{y}| < \delta_0$,

$$|g(\mathbf{x}) - g(\mathbf{y})| < |g(\mathbf{x})| / 2$$

and so by the triangle inequality,

$$-|g(\mathbf{x})| / 2 \leq |g(\mathbf{y})| - |g(\mathbf{x})| \leq |g(\mathbf{x})| / 2$$

which implies $|g(\mathbf{y})| \geq |g(\mathbf{x})| / 2$, and $|g(\mathbf{y})| < 3 |g(\mathbf{x})| / 2$.

Then if $|\mathbf{x}-\mathbf{y}| < \min(\delta_0, \delta_1)$,

$$\begin{aligned}\left|\frac{f(\mathbf{x})}{g(\mathbf{x})} - \frac{f(\mathbf{y})}{g(\mathbf{y})}\right| &= \left|\frac{f(\mathbf{x})g(\mathbf{y}) - f(\mathbf{y})g(\mathbf{x})}{g(\mathbf{x})g(\mathbf{y})}\right| \\ &\leq \frac{|f(\mathbf{x})g(\mathbf{y}) - f(\mathbf{y})g(\mathbf{x})|}{\left(\frac{|g(\mathbf{x})|^2}{2}\right)} \\ &= \frac{2|f(\mathbf{x})g(\mathbf{y}) - f(\mathbf{y})g(\mathbf{x})|}{|g(\mathbf{x})|^2}\end{aligned}$$

$$\begin{aligned}&\leq \frac{2}{|g(\mathbf{x})|^2}\left[|f(\mathbf{x})g(\mathbf{y}) - f(\mathbf{y})g(\mathbf{y}) + f(\mathbf{y})g(\mathbf{y}) - f(\mathbf{y})g(\mathbf{x})|\right] \\ &\leq \frac{2}{|g(\mathbf{x})|^2}\left[|g(\mathbf{y})||f(\mathbf{x}) - f(\mathbf{y})| + |f(\mathbf{y})||g(\mathbf{y}) - g(\mathbf{x})|\right] \\ &\leq \frac{2}{|g(\mathbf{x})|^2}\left[\frac{3}{2}|\mathbf{g}(\mathbf{x})||f(\mathbf{x}) - f(\mathbf{y})| + (1 + |f(\mathbf{x})|)|g(\mathbf{y}) - g(\mathbf{x})|\right] \\ &\leq \frac{2}{|g(\mathbf{x})|^2}(1 + 2|f(\mathbf{x})| + 2|g(\mathbf{x})|)\left[|f(\mathbf{x}) - f(\mathbf{y})| + |g(\mathbf{y}) - g(\mathbf{x})|\right] \\ &\equiv M\left[|f(\mathbf{x}) - f(\mathbf{y})| + |g(\mathbf{y}) - g(\mathbf{x})|\right]\end{aligned}$$

where

$$M \equiv \frac{2}{|g(\mathbf{x})|^2}(1 + 2|f(\mathbf{x})| + 2|g(\mathbf{x})|)$$

Now let δ_2 be such that if $|\mathbf{x}-\mathbf{y}| < \delta_2$, then

$$|f(\mathbf{x}) - f(\mathbf{y})| < \frac{\varepsilon}{2}M^{-1}$$

and let δ_3 be such that if $|\mathbf{x}-\mathbf{y}| < \delta_3$, then

$$|g(\mathbf{y}) - g(\mathbf{x})| < \frac{\varepsilon}{2}M^{-1}.$$

Then if $0 < \delta \leq \min(\delta_0, \delta_1, \delta_2, \delta_3)$, and $|\mathbf{x}-\mathbf{y}| < \delta$, everything holds and

$$\left|\frac{f(\mathbf{x})}{g(\mathbf{x})} - \frac{f(\mathbf{y})}{g(\mathbf{y})}\right| \leq M\left[|f(\mathbf{x}) - f(\mathbf{y})| + |g(\mathbf{y}) - g(\mathbf{x})|\right]$$

$$< M\left[\frac{\varepsilon}{2}M^{-1} + \frac{\varepsilon}{2}M^{-1}\right] = \varepsilon.$$

This completes the proof of the second part of 2.) Note that in these proofs no effort is made to find some sort of "best" δ. The problem is one which has a yes or a no answer. Either it is or it is not continuous.

Now begin on 3.). If $\mathbf{f}$ is continuous at $\mathbf{x}$, $\mathbf{f}(\mathbf{x}) \in D(\mathbf{g}) \subseteq \mathbb{F}^p$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$. Let $\varepsilon > 0$ be given. Then there exists $\eta > 0$

such that if $|\mathbf{y}-\mathbf{f}(\mathbf{x})|<\eta$ and $\mathbf{y}\in D(\mathbf{g})$, it follows that $|\mathbf{g}(\mathbf{y})-\mathbf{g}(\mathbf{f}(\mathbf{x}))|<\varepsilon$. It follows from continuity of $\mathbf{f}$ at $\mathbf{x}$ that there exists $\delta>0$ such that if $|\mathbf{x}-\mathbf{z}|<\delta$ and $\mathbf{z}\in D(\mathbf{f})$, then $|\mathbf{f}(\mathbf{z})-\mathbf{f}(\mathbf{x})|<\eta$. Then if $|\mathbf{x}-\mathbf{z}|<\delta$ and $\mathbf{z}\in D(\mathbf{g}\circ\mathbf{f})\subseteq D(\mathbf{f})$, all the above hold and so

$$|\mathbf{g}(\mathbf{f}(\mathbf{z}))-\mathbf{g}(\mathbf{f}(\mathbf{x}))|<\varepsilon.$$

This proves part 3.)

Part 4.) says: If $\mathbf{f}=(f_1,\cdot\cdot\cdot,f_q):D(\mathbf{f})\to\mathbb{F}^q$, then $\mathbf{f}$ is continuous if and only if each f_k is a continuous $\mathbb{F}$ valued function. Then

$$\begin{aligned}|f_k(\mathbf{x})-f_k(\mathbf{y})| &\leq |\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})| \\ &\equiv \left(\sum_{i=1}^{q}|f_i(\mathbf{x})-f_i(\mathbf{y})|^2\right)^{1/2} \\ &\leq \sum_{i=1}^{q}|f_i(\mathbf{x})-f_i(\mathbf{y})|. \end{aligned} \tag{5.18}$$

Suppose first that $\mathbf{f}$ is continuous at $\mathbf{x}$. Then there exists $\delta>0$ such that if $|\mathbf{x}-\mathbf{y}|<\delta$, then $|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})|<\varepsilon$. The first part of the above inequality then shows that for each $k=1,\cdot\cdot\cdot,q$, $|f_k(\mathbf{x})-f_k(\mathbf{y})|<\varepsilon$. This shows the only if part. Now suppose each function, f_k is continuous. Then if $\varepsilon>0$ is given, there exists $\delta_k>0$ such that whenever $|\mathbf{x}-\mathbf{y}|<\delta_k$

$$|f_k(\mathbf{x})-f_k(\mathbf{y})|<\varepsilon/q.$$

Now let $0<\delta\leq\min(\delta_1,\cdot\cdot\cdot,\delta_q)$. For $|\mathbf{x}-\mathbf{y}|<\delta$, the above inequality holds for all k and so the last part of 5.18 implies

$$\begin{aligned}|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{y})| &\leq \sum_{i=1}^{q}|f_i(\mathbf{x})-f_i(\mathbf{y})| \\ &< \sum_{i=1}^{q}\frac{\varepsilon}{q}=\varepsilon.\end{aligned}$$

This proves part 4.)

To verify part 5.), let $\varepsilon>0$ be given and let $\delta=\varepsilon$. Then if $|\mathbf{x}-\mathbf{y}|<\delta$, the triangle inequality implies

$$\begin{aligned}|f(\mathbf{x})-f(\mathbf{y})| &= ||\mathbf{x}|-|\mathbf{y}|| \\ &\leq |\mathbf{x}-\mathbf{y}|<\delta=\varepsilon.\end{aligned}$$

This proves part 5.) and completes the proof of the theorem.

Here is a multidimensional version of the nested interval lemma.

The following definition is similar to that given earlier. It defines what is meant by a sequentially compact set in $\mathbb{F}^p$.

Definition 5.36 *A set, $K \subseteq \mathbb{F}^p$ is sequentially compact if and only if whenever $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is a sequence of points in K, there exists a point, $\mathbf{x} \in K$ and a subsequence, $\{\mathbf{x}_{n_k}\}_{k=1}^{\infty}$ such that $\mathbf{x}_{n_k} \to \mathbf{x}$.*

It turns out the sequentially compact sets in $\mathbb{F}^p$are exactly those which are closed and bounded. Only half of this result will be needed in this book and this is proved next. First note that $\mathbb{C}$ can be considered as $\mathbb{R}^2$. Therefore, $\mathbb{C}^p$ may be considered as $\mathbb{R}^{2p}$.

Theorem 5.37 *Let $C \subseteq \mathbb{F}^p$ be closed and bounded. Then C is sequentially compact.*

Proof: Let $\{\mathbf{a}_n\} \subseteq C$. Then let $\mathbf{a}_n = \left(a_1^n, \cdots, a_p^n\right)$. It follows the real and imaginary parts of the terms of the sequence, $\left\{a_j^n\right\}_{n=1}^{\infty}$ are each contained in some sufficiently large closed bounded interval. By Theorem 2.3 on Page 25, there is a subsequence of the sequence of real parts of $\left\{a_j^n\right\}_{n=1}^{\infty}$ which converges. Also there is a further subsequence of the imaginary parts of $\left\{a_j^n\right\}_{n=1}^{\infty}$ which converges. Thus there is a subsequence, n_k with the property that $a_j^{n_k}$ converges to a point, $a_j \in \mathbb{F}$. Taking further subsequences, one obtains the existence of a subsequence, still called n_k such that for each $r = 1, \cdots, p$, $a_r^{n_k}$ converges to a point, $a_r \in \mathbb{F}$ as $k \to \infty$. Therefore, letting $\mathbf{a} \equiv (a_1, \cdots, a_p)$, $\lim_{k\to\infty} \mathbf{a}^{n_k} = \mathbf{a}$. Since C is closed, it follows $\mathbf{a} \in C$. This proves the theorem.

Here is a proof of the extreme value theorem.

Theorem 5.38 *Let C be closed and bounded and let $f : C \to \mathbb{R}$ be continuous. Then f achieves its maximum and its minimum on C. This means there exist, $\mathbf{x}_1, \mathbf{x}_2 \in C$ such that for all $\mathbf{x} \in C$,*

$$f(\mathbf{x}_1) \le f(\mathbf{x}) \le f(\mathbf{x}_2).$$

Proof: Let $M = \sup\{f(\mathbf{x}) : \mathbf{x} \in C\}$. Recall this means $+\infty$ if f is not bounded above and it equals the least upper bound of these values of f if f is bounded above. Then there exists a sequence, $\{\mathbf{x}_n\}$ such that $f(\mathbf{x}_n) \to M$. Since C is sequentially compact, there exists a subsequence, $\mathbf{x}_{n_k}$, and a point, $\mathbf{x} \in C$ such that $\mathbf{x}_{n_k} \to \mathbf{x}$. But then since f is continuous at $\mathbf{x}$, it follows from Theorem 5.32 on Page 105 that $f(\mathbf{x}) = \lim_{k\to\infty} f(\mathbf{x}_{n_k}) = M$. This proves f achieves its maximum and also shows its maximum is less than ∞. Let $\mathbf{x}_2 = \mathbf{x}$. The case of a minimum is handled similarly.

Recall that a function is uniformly continuous if the following definition holds.

Definition 5.39 *Let $\mathbf{f} : D(\mathbf{f}) \to \mathbb{F}^q$. Then $\mathbf{f}$ is uniformly continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $|\mathbf{x} - \mathbf{y}| < \delta$, it follows $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$.*

Theorem 5.40 *Let $\mathbf{f} : C \to \mathbb{F}^q$ be continuous where C is a closed and bounded set in $\mathbb{F}^p$. Then $\mathbf{f}$ is uniformly continuous on C.*

Proof: If this is not so, there exists $\varepsilon > 0$ and pairs of points, $\mathbf{x}_n$ and $\mathbf{y}_n$ satisfying $|\mathbf{x}_n - \mathbf{y}_n| < 1/n$ but $|\mathbf{f}(\mathbf{x}_n) - \mathbf{f}(\mathbf{y}_n)| \ge \varepsilon$. Since C is sequentially compact, there exists $\mathbf{x} \in C$ and a subsequence, $\{\mathbf{x}_{n_k}\}$ satisfying $\mathbf{x}_{n_k} \to \mathbf{x}$. But $|\mathbf{x}_{n_k} - \mathbf{y}_{n_k}| < 1/k$ and so $\mathbf{y}_{n_k} \to \mathbf{x}$ also. Therefore, from Theorem 5.32 on Page 105,

$$\varepsilon \leq \lim_{k\to\infty} |\mathbf{f}(\mathbf{x}_{n_k}) - \mathbf{f}(\mathbf{y}_{n_k})| = |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| = 0,$$

a contradiction. This proves the theorem.

5.11 The Space $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$

Definition 5.41 *The symbol,* $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ *will denote the set of linear transformations mapping* $\mathbb{F}^n$ *to* $\mathbb{F}^m$*. Thus* $L \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ *means that for* α, β *scalars and* $\mathbf{x}, \mathbf{y}$ *vectors in* $\mathbb{F}^n$,

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha L(\mathbf{x}) + \beta L(\mathbf{y}).$$

It is convenient to give a norm for the elements of $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. This will allow the consideration of questions such as whether a function having values in this space of linear transformations is continuous.

5.11.1 The Operator Norm

How do you measure the distance between linear transformations defined on $\mathbb{F}^n$? It turns out there are many ways to do this but I will give the most common one here.

Definition 5.42 $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ *denotes the space of linear transformations mapping* $\mathbb{F}^n$ *to* $\mathbb{F}^m$*. For* $A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$*, the* ***operator norm*** *is defined by*

$$||A|| \equiv \max\{|Ax|_{\mathbb{F}^m} : |x|_{\mathbb{F}^n} \leq 1\} < \infty.$$

Theorem 5.43 *Denote by* $|\cdot|$ *the norm on either* $\mathbb{F}^n$ *or* $\mathbb{F}^m$*. Then* $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ *with this operator norm is a* ***complete normed linear space*** *of dimension* nm *with*

$$||A\mathbf{x}|| \leq ||A|| \, |\mathbf{x}|.$$

Here ***Completeness*** *means that every Cauchy sequence converges.*

Proof: It is necessary to show the norm defined on $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ really is a norm. This means it is necessary to verify

$$||A|| \geq 0 \text{ and equals zero if and only if } A = 0.$$

For α a scalar,

$$||\alpha A|| = |\alpha| \, ||A||,$$

and for $A, B \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$,

$$||A + B|| \leq ||A|| + ||B||$$

The first two properties are obvious but you should verify them. It remains to verify the norm is well defined and also to verify the triangle inequality above. First if $|\mathbf{x}| \leq 1$, and (A_{ij}) is the matrix of the linear transformation with respect to the usual basis vectors, then

$$\begin{aligned} ||A|| &= \max\left\{\left(\sum_i |(A\mathbf{x})_i|^2\right)^{1/2} : |\mathbf{x}| \leq 1\right\} \\ &= \max\left\{\left(\sum_i \left|\sum_j A_{ij}x_j\right|^2\right)^{1/2} : |\mathbf{x}| \leq 1\right\} \end{aligned}$$

which is a finite number by the extreme value theorem.

It is clear that a basis for $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ consists of linear transformations whose matrices are of the form E_{ij} where E_{ij} consists of the $m \times n$ matrix having all zeros except for a 1 in the ij^{th} position. In effect, this considers $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ as $\mathbb{F}^{nm}$. Think of the $m \times n$ matrix as a long vector folded up.

If $\mathbf{x} \neq \mathbf{0}$,

$$|A\mathbf{x}| \frac{1}{|\mathbf{x}|} = \left|A\frac{\mathbf{x}}{|\mathbf{x}|}\right| \leq ||A|| \tag{5.19}$$

It only remains to verify completeness. Suppose then that $\{A_k\}$ is a Cauchy sequence in $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Then from 5.19 $\{A_k\mathbf{x}\}$ is a Cauchy sequence for each $\mathbf{x} \in \mathbb{F}^n$. This follows because

$$|A_k\mathbf{x} - A_l\mathbf{x}| \leq ||A_k - A_l|| \, |\mathbf{x}|$$

which converges to 0 as $k, l \to \infty$. Therefore, by completeness of $\mathbb{F}^m$, there exists $A\mathbf{x}$, the name of the thing to which the sequence, $\{A_k\mathbf{x}\}$ converges such that

$$\lim_{k\to\infty} A_k\mathbf{x} = A\mathbf{x}.$$

Then A is linear because

$$\begin{aligned} A(a\mathbf{x} + b\mathbf{y}) &\equiv \lim_{k\to\infty} A_k(a\mathbf{x} + b\mathbf{y}) \\ &= \lim_{k\to\infty} (aA_k\mathbf{x} + bA_k\mathbf{y}) \\ &= a\lim_{k\to\infty} A_k\mathbf{x} + b\lim_{k\to\infty} A_k\mathbf{y} \\ &= aA\mathbf{x} + bA\mathbf{y}. \end{aligned}$$

By the first part of this argument, $||A|| < \infty$ and so $A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. This proves the theorem.

Proposition 5.44 *Let $A(\mathbf{x}) \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ for each $\mathbf{x} \in U \subseteq \mathbb{F}^p$. Then letting $(A_{ij}(\mathbf{x}))$ denote the matrix of $A(\mathbf{x})$ with respect to the standard basis, it follows A_{ij} is continuous at $\mathbf{x}$ for each i, j if and only if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, then $||A(\mathbf{x}) - A(\mathbf{y})|| < \varepsilon$. That is, A is a continuous function having values in $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ at $\mathbf{x}$.*

Proof: Suppose first the second condition holds. Then from the material on linear transformations,

$$\begin{aligned}|A_{ij}(\mathbf{x})-A_{ij}(\mathbf{y})| &= |\mathbf{e}_i\cdot(A(\mathbf{x})-A(\mathbf{y}))\mathbf{e}_j| \\ &\leq |\mathbf{e}_i|\,|(A(\mathbf{x})-A(\mathbf{y}))\mathbf{e}_j| \\ &\leq ||A(\mathbf{x})-A(\mathbf{y})||.\end{aligned}$$

Therefore, the second condition implies the first.

Now suppose the first condition holds. That is each A_{ij} is continuous at $\mathbf{x}$. Let $|\mathbf{v}|\leq 1$.

$$\begin{aligned}|(A(\mathbf{x})-A(\mathbf{y}))(\mathbf{v})| &= \left(\sum_i\left|\sum_j (A_{ij}(\mathbf{x})-A_{ij}(\mathbf{y}))v_j\right|^2\right)^{1/2} \qquad (5.20)\\ &\leq \left(\sum_i\left(\sum_j |A_{ij}(\mathbf{x})-A_{ij}(\mathbf{y})|\,|v_j|\right)^2\right)^{1/2}.\end{aligned}$$

By continuity of each A_{ij}, there exists a $\delta>0$ such that for each i,j

$$|A_{ij}(\mathbf{x})-A_{ij}(\mathbf{y})|<\frac{\varepsilon}{n\sqrt{m}}$$

whenever $|\mathbf{x}-\mathbf{y}|<\delta$. Then from 5.20, if $|\mathbf{x}-\mathbf{y}|<\delta$,

$$\begin{aligned}|(A(\mathbf{x})-A(\mathbf{y}))(\mathbf{v})| &< \left(\sum_i\left(\sum_j \frac{\varepsilon}{n\sqrt{m}}|\mathbf{v}|\right)^2\right)^{1/2} \\ &\leq \left(\sum_i\left(\sum_j \frac{\varepsilon}{n\sqrt{m}}\right)^2\right)^{1/2}=\varepsilon\end{aligned}$$

This proves the proposition.

5.12 The Frechet Derivative

Let U be an open set in $\mathbb{F}^n$, and let $\mathbf{f}:U\to\mathbb{F}^m$ be a function.

Definition 5.45 *A function* $\mathbf{g}$ *is* $o(\mathbf{v})$ *if*

$$\lim_{|\mathbf{v}|\to 0}\frac{\mathbf{g}(\mathbf{v})}{|\mathbf{v}|}=\mathbf{0} \qquad (5.21)$$

A function $\mathbf{f} : U \to \mathbb{F}^m$ *is differentiable at* $\mathbf{x} \in U$ *if there exists a linear transformation* $L \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ *such that*

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) = \mathbf{f}(\mathbf{x}) + L\mathbf{v} + o(\mathbf{v})$$

This linear transformation L *is the definition of* $D\mathbf{f}(\mathbf{x})$*. This derivative is often called the Frechet derivative. .*

Usually no harm is occasioned by thinking of this linear transformation as its matrix taken with respect to the usual basis vectors.

The definition 5.21 means that the error,

$$\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}) - L\mathbf{v}$$

converges to $\mathbf{0}$ faster than $|\mathbf{v}|$. Thus the above definition is equivalent to saying

$$\lim_{|\mathbf{v}|\to 0} \frac{|\mathbf{f}(\mathbf{x}+\mathbf{v}) - \mathbf{f}(\mathbf{x}) - L\mathbf{v}|}{|\mathbf{v}|} = 0 \tag{5.22}$$

or equivalently,

$$\lim_{\mathbf{y}\to\mathbf{x}} \frac{|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x})(\mathbf{y}-\mathbf{x})|}{|\mathbf{y}-\mathbf{x}|} = 0. \tag{5.23}$$

Now it is clear this is just a generalization of the notion of the derivative of a function of one variable because in this more specialized situation,

$$\lim_{|v|\to 0} \frac{|f(x+v) - f(x) - f'(x)v|}{|v|} = 0,$$

due to the definition which says

$$f'(x) = \lim_{v\to 0} \frac{f(x+v) - f(x)}{v}.$$

For functions of n variables, you can't define the derivative as the limit of a difference quotient like you can for a function of one variable because you can't divide by a vector. That is why there is a need for a more general definition.

The term $o(\mathbf{v})$ is notation that is descriptive of the behavior in 5.21 and it is only this behavior that is of interest. Thus, if t and k are constants,

$$o(\mathbf{v}) = o(\mathbf{v}) + o(\mathbf{v}),\ o(t\mathbf{v}) = o(\mathbf{v}),\ ko(\mathbf{v}) = o(\mathbf{v})$$

and other similar observations hold. The sloppiness built in to this notation is useful because it ignores details which are not important. It may help to think of $o(\mathbf{v})$ as an adjective describing what is left over after approximating $\mathbf{f}(\mathbf{x}+\mathbf{v})$ by $\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})\mathbf{v}$.

Theorem 5.46 *The derivative is well defined.*

Proof: First note that for a fixed vector, $\mathbf{v}$, $o(t\mathbf{v}) = o(t)$. Now suppose both L_1 and L_2 work in the above definition. Then let $\mathbf{v}$ be any vector and let t be a real scalar which is chosen small enough that $t\mathbf{v} + \mathbf{x} \in U$. Then

$$\mathbf{f}(\mathbf{x} + t\mathbf{v}) = \mathbf{f}(\mathbf{x}) + L_1 t\mathbf{v} + o(t\mathbf{v}), \ \mathbf{f}(\mathbf{x} + t\mathbf{v}) = \mathbf{f}(\mathbf{x}) + L_2 t\mathbf{v} + o(t\mathbf{v}).$$

Therefore, subtracting these two yields $(L_2 - L_1)(t\mathbf{v}) = o(t\mathbf{v}) = o(t)$. Therefore, dividing by t yields $(L_2 - L_1)(\mathbf{v}) = \frac{o(t)}{t}$. Now let $t \to 0$ to conclude that $(L_2 - L_1)(\mathbf{v}) = 0$. Since this is true for all $\mathbf{v}$, it follows $L_2 = L_1$. This proves the theorem.

Lemma 5.47 *Let* $\mathbf{f}$ *be differentiable at* $\mathbf{x}$*. Then* $\mathbf{f}$ *is continuous at* $\mathbf{x}$ *and in fact, there exists* $K > 0$ *such that whenever* $|\mathbf{v}|$ *is small enough,*

$$|\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})| \leq K|\mathbf{v}|$$

Proof: From the definition of the derivative, $\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x}) = D\mathbf{f}(\mathbf{x})\mathbf{v} + o(\mathbf{v})$. Let $|\mathbf{v}|$ be small enough that $\frac{o(|\mathbf{v}|)}{|\mathbf{v}|} < 1$ so that $|o(\mathbf{v})| \leq |\mathbf{v}|$. Then for such $\mathbf{v}$,

$$\begin{aligned} |\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})| &\leq |D\mathbf{f}(\mathbf{x})\mathbf{v}| + |\mathbf{v}| \\ &\leq (|D\mathbf{f}(\mathbf{x})| + 1)|\mathbf{v}| \end{aligned}$$

This proves the lemma with $K = |D\mathbf{f}(\mathbf{x})| + 1$.

Theorem 5.48 *(The chain rule) Let* U *and* V *be open sets,* $U \subseteq \mathbb{F}^n$ *and* $V \subseteq \mathbb{F}^m$*. Suppose* $\mathbf{f} : U \to V$ *is differentiable at* $\mathbf{x} \in U$ *and suppose* $\mathbf{g} : V \to \mathbb{F}^q$ *is differentiable at* $\mathbf{f}(\mathbf{x}) \in V$*. Then* $\mathbf{g} \circ \mathbf{f}$ *is differentiable at* $\mathbf{x}$ *and*

$$D(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = D(\mathbf{g}(\mathbf{f}(\mathbf{x})))\, D(\mathbf{f}(\mathbf{x})).$$

Proof: This follows from a computation. Let $B(\mathbf{x},r) \subseteq U$ and let r also be small enough that for $|\mathbf{v}| \leq r$, it follows that $\mathbf{f}(\mathbf{x} + \mathbf{v}) \in V$. Such an r exists because $\mathbf{f}$ is continuous at $\mathbf{x}$. For $|\mathbf{v}| < r$, the definition of differentiability of $\mathbf{g}$ and $\mathbf{f}$ implies

$$\mathbf{g}(\mathbf{f}(\mathbf{x} + \mathbf{v})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) =$$

$$\begin{aligned} & D\mathbf{g}(\mathbf{f}(\mathbf{x}))(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})) + o(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})) \\ = \ & D\mathbf{g}(\mathbf{f}(\mathbf{x}))[D\mathbf{f}(\mathbf{x})\mathbf{v} + o(\mathbf{v})] + o(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})) \\ = \ & D(\mathbf{g}(\mathbf{f}(\mathbf{x})))\, D(\mathbf{f}(\mathbf{x}))\mathbf{v} + o(\mathbf{v}) + o(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})). \end{aligned} \tag{5.24}$$

It remains to show $o(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})) = o(\mathbf{v})$.

By Lemma 5.47, with K given there, letting $\varepsilon > 0$, it follows that for $|\mathbf{v}|$ small enough,

$$|o(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x}))| \leq (\varepsilon/K)|\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})| \leq (\varepsilon/K) K |\mathbf{v}| = \varepsilon|\mathbf{v}|.$$

Since $\varepsilon > 0$ is arbitrary, this shows $o(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x})) = o(\mathbf{v})$ because whenever $|\mathbf{v}|$ is small enough,

$$\frac{|o(\mathbf{f}(\mathbf{x}+\mathbf{v})-\mathbf{f}(\mathbf{x}))|}{|\mathbf{v}|} \leq \varepsilon.$$

By 5.24, this shows

$$\mathbf{g}(\mathbf{f}(\mathbf{x}+\mathbf{v})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) = D(\mathbf{g}(\mathbf{f}(\mathbf{x})))\, D(\mathbf{f}(\mathbf{x}))\,\mathbf{v} + o(\mathbf{v})$$

which proves the theorem.

The derivative is a linear transformation. What is the matrix of this linear transformation taken with respect to the usual basis vectors? Let $\mathbf{e}_i$ denote the vector of $\mathbb{F}^n$ which has a one in the i^{th} entry and zeroes elsewhere. Then the matrix of the linear transformation is the matrix whose i^{th} column is $D\mathbf{f}(\mathbf{x})\,\mathbf{e}_i$. What is this? Let $t \in \mathbb{R}$ such that $|t|$ is sufficiently small.

$$\begin{aligned} \mathbf{f}(\mathbf{x}+t\mathbf{e}_i) - \mathbf{f}(\mathbf{x}) &= D\mathbf{f}(\mathbf{x})\, t\mathbf{e}_i + \mathbf{o}(t\mathbf{e}_i) \\ &= D\mathbf{f}(\mathbf{x})\, t\mathbf{e}_i + \mathbf{o}(t). \end{aligned}$$

Then dividing by t and taking a limit,

$$D\mathbf{f}(\mathbf{x})\,\mathbf{e}_i = \lim_{t\to 0} \frac{\mathbf{f}(\mathbf{x}+t\mathbf{e}_i)-\mathbf{f}(\mathbf{x})}{t} \equiv \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}).$$

Thus the matrix of $D\mathbf{f}(\mathbf{x})$ with respect to the usual basis vectors is the matrix of the form

$$\begin{pmatrix} f_{1,x_1}(\mathbf{x}) & f_{1,x_2}(\mathbf{x}) & \cdots & f_{1,x_n}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ f_{m,x_1}(\mathbf{x}) & f_{m,x_2}(\mathbf{x}) & \cdots & f_{m,x_n}(\mathbf{x}) \end{pmatrix}.$$

As mentioned before, there is no harm in referring to this matrix as $D\mathbf{f}(\mathbf{x})$ but it may also be referred to as $J\mathbf{f}(\mathbf{x})$.

This is summarized in the following theorem.

Theorem 5.49 *Let $\mathbf{f} : \mathbb{F}^n \to \mathbb{F}^m$ and suppose $\mathbf{f}$ is differentiable at $\mathbf{x}$. Then all the partial derivatives $\frac{\partial f_i(\mathbf{x})}{\partial x_j}$ exist and if $J\mathbf{f}(\mathbf{x})$ is the matrix of the linear transformation with respect to the standard basis vectors, then the ij^{th} entry is given by $f_{i,j}$ or $\frac{\partial f_i}{\partial x_j}(\mathbf{x})$.*

What if all the partial derivatives of $\mathbf{f}$ exist? Does it follow that $\mathbf{f}$ is differentiable? Consider the following function.

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}.$$

Then from the definition of partial derivatives,

$$\lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

and

$$\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

However f is not even continuous at $(0,0)$ which may be seen by considering the behavior of the function along the line $y=x$ and along the line $x=0$. By Lemma 5.47 this implies f is not differentiable. Therefore, it is necessary to consider the correct definition of the derivative given above if you want to get a notion which generalizes the concept of the derivative of a function of one variable in such a way as to preserve continuity whenever the function is differentiable.

5.13 C^1 Functions

However, there are theorems which can be used to get differentiability of a function based on existence of the partial derivatives.

Definition 5.50 *When all the partial derivatives exist and are continuous the function is called a C^1 function.*

Because of Proposition 5.44 on Page 113 and Theorem 5.49 which identifies the entries of $J\mathbf{f}$ with the partial derivatives, the following definition is equivalent to the above.

Definition 5.51 *Let $U\subseteq\mathbb{F}^n$ be an open set. Then $\mathbf{f}:U\to\mathbb{F}^m$ is $C^1(U)$ if $\mathbf{f}$ is differentiable and the mapping*

$$\mathbf{x}\to D\mathbf{f}(\mathbf{x}),$$

is continuous as a function from U to $\mathcal{L}(\mathbb{F}^n,\mathbb{F}^m)$.

The following is an important abstract generalization of the familiar concept of partial derivative.

Definition 5.52 *Let $\mathbf{g}:U\subseteq\mathbb{F}^n\times\mathbb{F}^m\to\mathbb{F}^q$, where U is an open set in $\mathbb{F}^n\times\mathbb{F}^m$. Denote an element of $\mathbb{F}^n\times\mathbb{F}^m$ by $(\mathbf{x},\mathbf{y})$ where $\mathbf{x}\in\mathbb{F}^n$ and $\mathbf{y}\in\mathbb{F}^m$. Then the map $\mathbf{x}\to\mathbf{g}(\mathbf{x},\mathbf{y})$ is a function from the open set in X,*

$$\{\mathbf{x}:(\mathbf{x},\mathbf{y})\in U\}$$

to $\mathbb{F}^q$. When this map is differentiable, its derivative is denoted by

$$D_1\mathbf{g}(\mathbf{x},\mathbf{y}),\ \text{or sometimes by } D_{\mathbf{x}}\mathbf{g}(\mathbf{x},\mathbf{y}).$$

Thus,

$$\mathbf{g}(\mathbf{x}+\mathbf{v},\mathbf{y})-\mathbf{g}(\mathbf{x},\mathbf{y})=D_1\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{v}+o(\mathbf{v}).$$

A similar definition holds for the symbol $D_{\mathbf{y}}\mathbf{g}$ or $D_2\mathbf{g}$. The special case seen in beginning calculus courses is where $\mathbf{g}:U\to\mathbb{F}^q$ and

$$\mathbf{g}_{x_i}(\mathbf{x})\equiv\frac{\partial\mathbf{g}(\mathbf{x})}{\partial x_i}\equiv\lim_{h\to 0}\frac{\mathbf{g}(\mathbf{x}+h\mathbf{e}_i)-\mathbf{g}(\mathbf{x})}{h}.$$

The following theorem will be very useful in much of what follows. It is a version of the mean value theorem.

Theorem 5.53 *Suppose* U *is an open subset of* $\mathbb{F}^n$ *and* $\mathbf{f} : U \to \mathbb{F}^m$ *has the property that* $D\mathbf{f}(\mathbf{x})$ *exists for all* $\mathbf{x}$ *in* U *and that,* $\mathbf{x}+t(\mathbf{y}-\mathbf{x}) \in U$ *for all* $t \in [0,1]$*. (The line segment joining the two points lies in* U*.) Suppose also that for all points on this line segment,*

$$||D\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))|| \leq M.$$

Then

$$|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| \leq M|\mathbf{y}-\mathbf{x}|.$$

Proof: Let

$$S \equiv \{t \in [0,1] : \text{ for all } s \in [0,t],$$
$$|\mathbf{f}(\mathbf{x}+s(\mathbf{y}-\mathbf{x})) - \mathbf{f}(\mathbf{x})| \leq (M+\varepsilon) s |\mathbf{y}-\mathbf{x}|\}.$$

Then $0 \in S$ and by continuity of $\mathbf{f}$, it follows that if $t \equiv \sup S$, then $t \in S$ and if $t < 1$,

$$|\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) - \mathbf{f}(\mathbf{x})| = (M+\varepsilon) t |\mathbf{y}-\mathbf{x}|. \tag{5.25}$$

If $t < 1$, then there exists a sequence of positive numbers, $\{h_k\}_{k=1}^{\infty}$ converging to 0 such that

$$|\mathbf{f}(\mathbf{x}+(t+h_k)(\mathbf{y}-\mathbf{x})) - \mathbf{f}(\mathbf{x})| > (M+\varepsilon)(t+h_k)|\mathbf{y}-\mathbf{x}|$$

which implies that

$$|\mathbf{f}(\mathbf{x}+(t+h_k)(\mathbf{y}-\mathbf{x})) - \mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))|$$
$$+|\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) - \mathbf{f}(\mathbf{x})| > (M+\varepsilon)(t+h_k)|\mathbf{y}-\mathbf{x}|.$$

By 5.25, this inequality implies

$$|\mathbf{f}(\mathbf{x}+(t+h_k)(\mathbf{y}-\mathbf{x})) - \mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))| > (M+\varepsilon) h_k |\mathbf{y}-\mathbf{x}|$$

which yields upon dividing by h_k and taking the limit as $h_k \to 0$,

$$|D\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(\mathbf{y}-\mathbf{x})| \geq (M+\varepsilon)|\mathbf{y}-\mathbf{x}|.$$

Now by the definition of the norm of a linear operator,

$$\begin{aligned} M|\mathbf{y}-\mathbf{x}| &\geq ||D\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))|| \, |\mathbf{y}-\mathbf{x}| \\ &\geq |D\mathbf{f}(\mathbf{x}+t(\mathbf{y}-\mathbf{x}))(\mathbf{y}-\mathbf{x})| \geq (M+\varepsilon)|\mathbf{y}-\mathbf{x}|, \end{aligned}$$

a contradiction. Therefore, $t = 1$ and so

$$|\mathbf{f}(\mathbf{x}+(\mathbf{y}-\mathbf{x})) - \mathbf{f}(\mathbf{x})| \leq (M+\varepsilon)|\mathbf{y}-\mathbf{x}|.$$

Since $\varepsilon > 0$ is arbitrary, this proves the theorem.

The next theorem proves that if the partial derivatives exist and are continuous, then the function is differentiable.

Theorem 5.54 *Let* $\mathbf{g} : U \subseteq \mathbb{F}^n \times \mathbb{F}^m \rightarrow \mathbb{F}^q$*. Then* $\mathbf{g}$ *is* $C^1(U)$ *if and only if* $D_1\mathbf{g}$ *and* $D_2\mathbf{g}$ *both exist and are continuous on* U*. In this case,*

$$D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{u},\mathbf{v}) = D_1\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{u}+D_2\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{v}.$$

Proof: Suppose first that $\mathbf{g} \in C^1(U)$. Then if $(\mathbf{x},\mathbf{y}) \in U$,

$$\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}) - \mathbf{g}(\mathbf{x},\mathbf{y}) = D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{u},\mathbf{0}) + o(\mathbf{u}).$$

Therefore, $D_1\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{u} = D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{u},\mathbf{0})$. Then

$$|(D_1\mathbf{g}(\mathbf{x},\mathbf{y}) - D_1\mathbf{g}(\mathbf{x}',\mathbf{y}'))(\mathbf{u})| =$$
$$|(D\mathbf{g}(\mathbf{x},\mathbf{y}) - D\mathbf{g}(\mathbf{x}',\mathbf{y}'))(\mathbf{u},\mathbf{0})| \leq$$
$$||D\mathbf{g}(\mathbf{x},\mathbf{y}) - D\mathbf{g}(\mathbf{x}',\mathbf{y}')||\,|(\mathbf{u},\mathbf{0})|.$$

Therefore,

$$|D_1\mathbf{g}(\mathbf{x},\mathbf{y}) - D_1\mathbf{g}(\mathbf{x}',\mathbf{y}')| \leq ||D\mathbf{g}(\mathbf{x},\mathbf{y}) - D\mathbf{g}(\mathbf{x}',\mathbf{y}')||.$$

A similar argument applies for $D_2\mathbf{g}$ and this proves the continuity of the function, $(\mathbf{x},\mathbf{y}) \rightarrow D_i\mathbf{g}(\mathbf{x},\mathbf{y})$ for $i = 1, 2$. The formula follows from

$$\begin{aligned} D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{u},\mathbf{v}) &= D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{u},\mathbf{0}) + D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{0},\mathbf{v}) \\ &\equiv D_1\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{u}+D_2\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{v}. \end{aligned}$$

Now suppose $D_1\mathbf{g}(\mathbf{x},\mathbf{y})$ and $D_2\mathbf{g}(\mathbf{x},\mathbf{y})$ exist and are continuous.

$$\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v}) - \mathbf{g}(\mathbf{x},\mathbf{y}) = \mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v}) - \mathbf{g}(\mathbf{x},\mathbf{y}+\mathbf{v})$$
$$+\mathbf{g}(\mathbf{x},\mathbf{y}+\mathbf{v}) - \mathbf{g}(\mathbf{x},\mathbf{y})$$
$$= \mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}) - \mathbf{g}(\mathbf{x},\mathbf{y}) + \mathbf{g}(\mathbf{x},\mathbf{y}+\mathbf{v}) - \mathbf{g}(\mathbf{x},\mathbf{y}) +$$
$$[\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v}) - \mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}) - (\mathbf{g}(\mathbf{x},\mathbf{y}+\mathbf{v}) - \mathbf{g}(\mathbf{x},\mathbf{y}))]$$
$$= D_1\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{u} + D_2\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{v} + o(\mathbf{v}) + o(\mathbf{u}) +$$
$$[\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v}) - \mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}) - (\mathbf{g}(\mathbf{x},\mathbf{y}+\mathbf{v}) - \mathbf{g}(\mathbf{x},\mathbf{y}))]. \tag{5.26}$$

Let $\mathbf{h}(\mathbf{x},\mathbf{u}) \equiv \mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v}) - \mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y})$. Then the expression in [ ] is of the form,

$$\mathbf{h}(\mathbf{x},\mathbf{u}) - \mathbf{h}(\mathbf{x},\mathbf{0}).$$

Also

$$D_2\mathbf{h}(\mathbf{x},\mathbf{u}) = D_1\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v}) - D_1\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y})$$

and so, by continuity of $(\mathbf{x},\mathbf{y}) \rightarrow D_1\mathbf{g}(\mathbf{x},\mathbf{y})$,

$$||D_2\mathbf{h}(\mathbf{x},\mathbf{u})|| < \varepsilon$$

whenever $||(\mathbf{u},\mathbf{v})||$ is small enough. By Theorem 5.53 on Page 119, there exists $\delta > 0$ such that if $||(\mathbf{u},\mathbf{v})|| < \delta$, the norm of the last term in 5.26 satisfies the inequality,

$$||\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v}) - \mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}) - (\mathbf{g}(\mathbf{x},\mathbf{y}+\mathbf{v}) - \mathbf{g}(\mathbf{x},\mathbf{y}))|| < \varepsilon\,||\mathbf{u}||\,. \tag{5.27}$$

Therefore, this term is $o((\mathbf{u},\mathbf{v}))$. It follows from 5.27 and 5.26 that

$$\mathbf{g}(\mathbf{x}+\mathbf{u},\mathbf{y}+\mathbf{v}) =$$

$$\mathbf{g}(\mathbf{x},\mathbf{y}) + D_1\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{u} + D_2\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{v} + o(\mathbf{u}) + o(\mathbf{v}) + o((\mathbf{u},\mathbf{v}))$$
$$= \mathbf{g}(\mathbf{x},\mathbf{y}) + D_1\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{u} + D_2\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{v} + o((\mathbf{u},\mathbf{v}))$$

Showing that $D\mathbf{g}(\mathbf{x},\mathbf{y})$ exists and is given by

$$D\mathbf{g}(\mathbf{x},\mathbf{y})(\mathbf{u},\mathbf{v}) = D_1\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{u} + D_2\mathbf{g}(\mathbf{x},\mathbf{y})\,\mathbf{v}.$$

The continuity of $(\mathbf{x},\mathbf{y}) \to D\mathbf{g}(\mathbf{x},\mathbf{y})$ follows from the continuity of $(\mathbf{x},\mathbf{y}) \to D_i\mathbf{g}(\mathbf{x},\mathbf{y})$. This proves the theorem.

Not surprisingly, it can be generalized to many more factors.

Definition 5.55 *Let $\mathbf{g} : U \subseteq \prod_{i=1}^n \mathbb{F}^{r_i} \to \mathbb{F}^q$, where U is an open set. Then the map $\mathbf{x}_i \to \mathbf{g}(\mathbf{x})$ is a function from the open set in $\mathbb{F}^{r_i}$,*

$$\{\mathbf{x}_i : \mathbf{x} \in U\}$$

to $\mathbb{F}^q$. When this map is differentiable, its derivative is denoted by $D_i\mathbf{g}(\mathbf{x})$. To aid in the notation, for $\mathbf{v} \in X_i$, let $\theta_i\mathbf{v} \in \prod_{i=1}^n \mathbb{F}^{r_i}$ be the vector $(\mathbf{0},\cdots,\mathbf{v},\cdots,\mathbf{0})$ where the $\mathbf{v}$ is in the i^{th} slot and for $\mathbf{v} \in \prod_{i=1}^n \mathbb{F}^{r_i}$, let $\mathbf{v}_i$ denote the entry in the i^{th} slot of $\mathbf{v}$. Thus by saying $\mathbf{x}_i \to \mathbf{g}(\mathbf{x})$ is differentiable is meant that for $\mathbf{v} \in X_i$ sufficiently small,

$$\mathbf{g}(\mathbf{x}+\theta_i\mathbf{v}) - \mathbf{g}(\mathbf{x}) = D_i\mathbf{g}(\mathbf{x})\,\mathbf{v} + o(\mathbf{v})\,.$$

Here is a generalization of Theorem 5.54.

Theorem 5.56 *Let $\mathbf{g}, U, \prod_{i=1}^n \mathbb{F}^{r_i}$, be given as in Definition 5.55. Then $\mathbf{g}$ is $C^1(U)$ if and only if $D_i\mathbf{g}$ exists and is continuous on U for each i. In this case,*

$$D\mathbf{g}(\mathbf{x})(\mathbf{v}) = \sum_k D_k\mathbf{g}(\mathbf{x})\,\mathbf{v}_k \tag{5.28}$$

Proof: The only if part of the proof is left for you. Suppose then that $D_i\mathbf{g}$ exists and is continuous for each i. Note that $\sum_{j=1}^k \theta_j\mathbf{v}_j = (\mathbf{v}_1,\cdots,\mathbf{v}_k,\mathbf{0},\cdots,\mathbf{0})$. Thus $\sum_{j=1}^n \theta_j\mathbf{v}_j = \mathbf{v}$ and define $\sum_{j=1}^0 \theta_j\mathbf{v}_j \equiv \mathbf{0}$. Therefore,

$$\mathbf{g}(\mathbf{x}+\mathbf{v}) - \mathbf{g}(\mathbf{x}) = \sum_{k=1}^n \left[\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^k \theta_j\mathbf{v}_j\right) - \mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1}\theta_j\mathbf{v}_j\right)\right] \tag{5.29}$$

Consider the terms in this sum.

$$\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k}\theta_j\mathbf{v}_j\right)-\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1}\theta_j\mathbf{v}_j\right)=\mathbf{g}\left(\mathbf{x}+\theta_k\mathbf{v}_k\right)-\mathbf{g}\left(\mathbf{x}\right)+ \tag{5.30}$$

$$\left(\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k}\theta_j\mathbf{v}_j\right)-\mathbf{g}\left(\mathbf{x}+\theta_k\mathbf{v}_k\right)\right)-\left(\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1}\theta_j\mathbf{v}_j\right)-\mathbf{g}\left(\mathbf{x}\right)\right) \tag{5.31}$$

and the expression in 5.31 is of the form $\mathbf{h}\left(\mathbf{v}_k\right)-\mathbf{h}\left(\mathbf{0}\right)$ where for small $\mathbf{w}\in\mathbb{F}^{r_k}$,

$$\mathbf{h}\left(\mathbf{w}\right)\equiv\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1}\theta_j\mathbf{v}_j+\theta_k\mathbf{w}\right)-\mathbf{g}\left(\mathbf{x}+\theta_k\mathbf{w}\right).$$

Therefore,

$$D\mathbf{h}\left(\mathbf{w}\right)=D_k\mathbf{g}\left(\mathbf{x}+\sum_{j=1}^{k-1}\theta_j\mathbf{v}_j+\theta_k\mathbf{w}\right)-D_k\mathbf{g}\left(\mathbf{x}+\theta_k\mathbf{w}\right)$$

and by continuity, $||D\mathbf{h}\left(\mathbf{w}\right)||<\varepsilon$ provided $|\mathbf{v}|$ is small enough. Therefore, by Theorem 5.53, whenever $|\mathbf{v}|$ is small enough, $|\mathbf{h}\left(\theta_k\mathbf{v}_k\right)-\mathbf{h}\left(\mathbf{0}\right)|\leq\varepsilon\left|\theta_k\mathbf{v}_k\right|\leq\varepsilon\left|\mathbf{v}\right|$ which shows that since ε is arbitrary, the expression in 5.31 is $o\left(\mathbf{v}\right)$. Now in 5.30 $\mathbf{g}\left(\mathbf{x}+\theta_k\mathbf{v}_k\right)-\mathbf{g}\left(\mathbf{x}\right)=D_k\mathbf{g}\left(\mathbf{x}\right)\mathbf{v}_k+o\left(\mathbf{v}_k\right)=D_k\mathbf{g}\left(\mathbf{x}\right)\mathbf{v}_k+o\left(\mathbf{v}\right)$. Therefore, referring to 5.29,

$$\mathbf{g}\left(\mathbf{x}+\mathbf{v}\right)-\mathbf{g}\left(\mathbf{x}\right)=\sum_{k=1}^{n}D_k\mathbf{g}\left(\mathbf{x}\right)\mathbf{v}_k+o\left(\mathbf{v}\right)$$

which shows $D\mathbf{g}$ exists and equals the formula given in 5.28.

The way this is usually used is in the following corollary, a case of Theorem 5.56 obtained by letting $\mathbb{F}^{r_j}=\mathbb{F}$ in the above theorem.

Corollary 5.57 *Let U be an open subset of $\mathbb{F}^n$ and let $\mathbf{f}:U\to\mathbb{F}^m$ be C^1 in the sense that all the partial derivatives of $\mathbf{f}$ exist and are continuous. Then $\mathbf{f}$ is differentiable and*

$$\mathbf{f}\left(\mathbf{x}+\mathbf{v}\right)=\mathbf{f}\left(\mathbf{x}\right)+\sum_{k=1}^{n}\frac{\partial\mathbf{f}}{\partial x_k}\left(\mathbf{x}\right)v_k+\mathbf{o}\left(\mathbf{v}\right).$$

5.14 C^k Functions

Recall the notation for partial derivatives in the following definition.

Definition 5.58 *Let $\mathbf{g}:U\to\mathbb{F}^n$. Then*

$$\mathbf{g}_{x_k}\left(\mathbf{x}\right)\equiv\frac{\partial\mathbf{g}}{\partial x_k}\left(\mathbf{x}\right)\equiv\lim_{h\to0}\frac{\mathbf{g}\left(\mathbf{x}+h\mathbf{e}_k\right)-\mathbf{g}\left(\mathbf{x}\right)}{h}$$

Higher order partial derivatives are defined in the usual way.

$$\mathbf{g}_{x_k x_l}(\mathbf{x}) \equiv \frac{\partial^2 \mathbf{g}}{\partial x_l \partial x_k}(\mathbf{x})$$

and so forth.

To deal with higher order partial derivatives in a systematic way, here is a useful definition.

Definition 5.59 *$\alpha = (\alpha_1, \cdots, \alpha_n)$ for $\alpha_1 \cdots \alpha_n$ positive integers is called a multi-index. For α a multi-index, $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n$ and if $\mathbf{x} \in \mathbb{F}^n$,*

$$\mathbf{x} = (x_1, \cdots, x_n),$$

and $\mathbf{f}$ a function, define

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},\ D^\alpha \mathbf{f}(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} \mathbf{f}(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

The following is the definition of what is meant by a C^k function.

Definition 5.60 *Let U be an open subset of $\mathbb{F}^n$ and let $\mathbf{f} : U \to \mathbb{F}^m$. Then for k a nonnegative integer, $\mathbf{f}$ is C^k if for every $|\alpha| \leq k$, $D^\alpha \mathbf{f}$ exists and is continuous.*

5.15 Mixed Partial Derivatives

Under certain conditions the **mixed partial derivatives** will always be equal. This astonishing fact is due to Euler in 1734.

Theorem 5.61 *Suppose $f : U \subseteq \mathbb{F}^2 \to \mathbb{R}$ where U is an open set on which f_x, f_y, f_{xy} and f_{yx} exist. Then if f_{xy} and f_{yx} are continuous at the point $(x, y) \in U$, it follows*

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Proof: Since U is open, there exists $r > 0$ such that $B((x, y), r) \subseteq U$. Now let $|t|, |s| < r/2, t, s$ real numbers and consider

$$\Delta(s, t) \equiv \frac{1}{st}\{\overbrace{f(x + t, y + s) - f(x + t, y)}^{h(t)} - \overbrace{(f(x, y + s) - f(x, y))}^{h(0)}\}. \tag{5.32}$$

Note that $(x + t, y + s) \in U$ because

$$\begin{aligned} |(x + t, y + s) - (x, y)| &= |(t, s)| = \left(t^2 + s^2\right)^{1/2} \\ &\leq \left(\frac{r^2}{4} + \frac{r^2}{4}\right)^{1/2} = \frac{r}{\sqrt{2}} < r. \end{aligned}$$

As implied above, $h(t) \equiv f(x+t, y+s) - f(x+t, y)$. Therefore, by the mean value theorem from calculus and the (one variable) chain rule,

$$\begin{aligned}\Delta(s,t) &= \frac{1}{st}(h(t) - h(0)) = \frac{1}{st}h'(\alpha t)\,t \\ &= \frac{1}{s}(f_x(x+\alpha t, y+s) - f_x(x+\alpha t, y))\end{aligned}$$

for some $\alpha \in (0,1)$. Applying the mean value theorem again,

$$\Delta(s,t) = f_{xy}(x+\alpha t, y+\beta s)$$

where $\alpha, \beta \in (0,1)$.

If the terms $f(x+t,y)$ and $f(x,y+s)$ are interchanged in 5.32, $\Delta(s,t)$ is unchanged and the above argument shows there exist $\gamma, \delta \in (0,1)$ such that

$$\Delta(s,t) = f_{yx}(x+\gamma t, y+\delta s).$$

Letting $(s,t) \to (0,0)$ and using the continuity of f_{xy} and f_{yx} at (x,y),

$$\lim_{(s,t)\to(0,0)} \Delta(s,t) = f_{xy}(x,y) = f_{yx}(x,y).$$

This proves the theorem.

The following is obtained from the above by simply fixing all the variables except for the two of interest.

Corollary 5.62 *Suppose U is an open subset of $\mathbb{F}^n$ and $f : U \to \mathbb{R}$ has the property that for two indices, k, l, f_{x_k}, $f_{x_l}, f_{x_l x_k}$, and $f_{x_k x_l}$ exist on U and $f_{x_k x_l}$ and $f_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $f_{x_k x_l}(\mathbf{x}) = f_{x_l x_k}(\mathbf{x})$.*

By considering the real and imaginary parts of f in the case where f has values in $\mathbb{F}$ you obtain the following corollary.

Corollary 5.63 *Suppose U is an open subset of $\mathbb{F}^n$ and $f : U \to \mathbb{F}$ has the property that for two indices, k, l, f_{x_k}, $f_{x_l}, f_{x_l x_k}$, and $f_{x_k x_l}$ exist on U and $f_{x_k x_l}$ and $f_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $f_{x_k x_l}(\mathbf{x}) = f_{x_l x_k}(\mathbf{x})$.*

Finally, by considering the components of $\mathbf{f}$ you get the following generalization.

Corollary 5.64 *Suppose U is an open subset of $\mathbb{F}^n$ and $\mathbf{f} : U \to \mathbb{F}^{\,m}$ has the property that for two indices, k, l, $\mathbf{f}_{x_k}$, $\mathbf{f}_{x_l}, \mathbf{f}_{x_l x_k}$, and $\mathbf{f}_{x_k x_l}$ exist on U and $\mathbf{f}_{x_k x_l}$ and $\mathbf{f}_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $\mathbf{f}_{x_k x_l}(\mathbf{x}) = \mathbf{f}_{x_l x_k}(\mathbf{x})$.*

It is necessary to assume the mixed partial derivatives are continuous in order to assert they are equal. The following is a well known example [5].

Example 5.65 *Let*

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} \text{ if } (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}$$

From the definition of partial derivatives it follows immediately that $f_x(0,0) = f_y(0,0) = 0$. Using the standard rules of differentiation, for $(x,y) \neq (0,0)$,

$$f_x = y\frac{x^4 - y^4 + 4x^2y^2}{(x^2+y^2)^2},\ f_y = x\frac{x^4 - y^4 - 4x^2y^2}{(x^2+y^2)^2}$$

Now

$$\begin{aligned} f_{xy}(0,0) &\equiv \lim_{y\to 0}\frac{f_x(0,y) - f_x(0,0)}{y} \\ &= \lim_{y\to 0}\frac{-y^4}{(y^2)^2} = -1 \end{aligned}$$

while

$$\begin{aligned} f_{yx}(0,0) &\equiv \lim_{x\to 0}\frac{f_y(x,0) - f_y(0,0)}{x} \\ &= \lim_{x\to 0}\frac{x^4}{(x^2)^2} = 1 \end{aligned}$$

showing that although the mixed partial derivatives do exist at $(0,0)$, they are not equal there.

5.16 Implicit Function Theorem

The implicit function theorem is one of the greatest theorems in mathematics. There are many versions of this theorem. However, I will give a very simple proof valid in finite dimensional spaces.

Theorem 5.66 *(implicit function theorem) Suppose U is an open set in $\mathbb{R}^n \times \mathbb{R}^m$. Let $\mathbf{f} : U \to \mathbb{R}^n$ be in $C^1(U)$ and suppose*

$$\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0},\ D_1\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n). \tag{5.33}$$

Then there exist positive constants, δ, η, such that for every $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ there exists a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ such that

$$\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}. \tag{5.34}$$

Furthermore, the mapping, $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ is in $C^1(B(\mathbf{y}_0, \eta))$.

Proof: Let

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} f_1(\mathbf{x}, \mathbf{y}) \\ f_2(\mathbf{x}, \mathbf{y}) \\ \vdots \\ f_n(\mathbf{x}, \mathbf{y}) \end{pmatrix}.$$

Define for $\left(\mathbf{x}^1, \cdots, \mathbf{x}^n\right) \in \overline{B(\mathbf{x}_0, \delta)}^n$ and $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ the following matrix.

$$J\left(\mathbf{x}^1, \cdots, \mathbf{x}^n, \mathbf{y}\right) \equiv \begin{pmatrix} f_{1,x_1}\left(\mathbf{x}^1, \mathbf{y}\right) & \cdots & f_{1,x_n}\left(\mathbf{x}^1, \mathbf{y}\right) \\ \vdots & & \vdots \\ f_{n,x_1}\left(\mathbf{x}^n, \mathbf{y}\right) & \cdots & f_{n,x_n}\left(\mathbf{x}^n, \mathbf{y}\right) \end{pmatrix}.$$

Then by the assumption of continuity of all the partial derivatives, there exists $\delta_0 > 0$ and $\eta_0 > 0$ such that if $\delta < \delta_0$ and $\eta < \eta_0$, it follows that for all $\left(\mathbf{x}^1, \cdots, \mathbf{x}^n\right) \in \overline{B(\mathbf{x}_0, \delta)}^n$ and $\mathbf{y} \in B(\mathbf{y}_0, \eta)$,

$$\det\left(J\left(\mathbf{x}^1, \cdots, \mathbf{x}^n, \mathbf{y}\right)\right) > r > 0. \tag{5.35}$$

and $\overline{B(\mathbf{x}_0, \delta_0)} \times \overline{B(\mathbf{y}_0, \eta_0)} \subseteq U$. Pick $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ and suppose there exist $\mathbf{x}, \mathbf{z} \in \overline{B(\mathbf{x}_0, \delta)}$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{z}, \mathbf{y}) = \mathbf{0}$. Consider f_i and let

$$h(t) \equiv f_i(\mathbf{x} + t(\mathbf{z} - \mathbf{x}), \mathbf{y}).$$

Then $h(1) = h(0)$ and so by the mean value theorem, $h'(t_i) = 0$ for some $t_i \in (0, 1)$. Therefore, from the chain rule and for this value of t_i,

$$h'(t_i) = Df_i(\mathbf{x} + t_i(\mathbf{z} - \mathbf{x}), \mathbf{y})(\mathbf{z} - \mathbf{x}) = 0. \tag{5.36}$$

Then denote by $\mathbf{x}^i$ the vector, $\mathbf{x} + t_i(\mathbf{z} - \mathbf{x})$. It follows from 5.36 that

$$J\left(\mathbf{x}^1, \cdots, \mathbf{x}^n, \mathbf{y}\right)(\mathbf{z} - \mathbf{x}) = \mathbf{0}$$

and so from 5.35 $\mathbf{z} - \mathbf{x} = \mathbf{0}$. Now it will be shown that if η is chosen sufficiently small, then for all $\mathbf{y} \in B(\mathbf{y}_0, \eta)$, there exists a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ such that $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$.

Claim: If η is small enough, then the function, $h_{\mathbf{y}}(\mathbf{x}) \equiv |\mathbf{f}(\mathbf{x}, \mathbf{y})|^2$ achieves its minimum value on $\overline{B(\mathbf{x}_0, \delta)}$ at a point of $B(\mathbf{x}_0, \delta)$.

Proof of claim: Suppose this is not the case. Then there exists a sequence $\eta_k \to 0$ and for some $\mathbf{y}_k$ having $|\mathbf{y}_k - \mathbf{y}_0| < \eta_k$, the minimum of $h_{\mathbf{y}_k}$ occurs on a point of the boundary of $\overline{B(\mathbf{x}_0, \delta)}$, $\mathbf{x}_k$ such that $|\mathbf{x}_0 - \mathbf{x}_k| = \delta$. Now taking a subsequence, still denoted by k, it can be assumed that $\mathbf{x}_k \to \mathbf{x}$ with $|\mathbf{x} - \mathbf{x}_0| = \delta$ and $\mathbf{y}_k \to \mathbf{y}_0$. Let $\varepsilon > 0$. Then for k large enough, $h_{\mathbf{y}_k}(\mathbf{x}_0) < \varepsilon$ because $\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}$. Therefore, from the definition of $\mathbf{x}_k$, $h_{\mathbf{y}_k}(\mathbf{x}_k) < \varepsilon$. Passing to the limit yields $h_{\mathbf{y}_0}(\mathbf{x}) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $h_{\mathbf{y}_0}(\mathbf{x}) = 0$ which contradicts the first part of the argument in which it was shown that for $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ there is at most one point, $\mathbf{x}$ of $\overline{B(\mathbf{x}_0, \delta)}$ where $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. Here two have been obtained, $\mathbf{x}_0$ and $\mathbf{x}$. This proves the claim.

Choose $\eta < \eta_0$ and also small enough that the above claim holds and let $\mathbf{x}(\mathbf{y})$ denote a point of $B(\mathbf{x}_0, \delta)$ at which the minimum of $h_{\mathbf{y}}$ on $\overline{B(\mathbf{x}_0, \delta)}$ is achieved. Since $\mathbf{x}(\mathbf{y})$ is an interior point, you can consider $h_{\mathbf{y}}(\mathbf{x}(\mathbf{y}) + t\mathbf{v})$ for $|t|$ small and conclude this function of t has a zero derivative at $t = 0$. Thus

$$Dh_{\mathbf{y}}(\mathbf{x}(\mathbf{y}))\mathbf{v} = 0 = 2\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^T D_1\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})\mathbf{v}$$

for every vector $\mathbf{v}$. But from 5.35 and the fact that $\mathbf{v}$ is arbitrary, it follows $\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y}) = \mathbf{0}$. This proves the existence of the function $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ such that $\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y}) = \mathbf{0}$ for all $\mathbf{y} \in B(\mathbf{y}_0,\eta)$.

It remains to verify this function is a C^1 function. To do this, let $\mathbf{y}_1$ and $\mathbf{y}_2$ be points of $B(\mathbf{y}_0,\eta)$. Then as before, consider the i^{th} component of $\mathbf{f}$ and consider the same argument using the mean value theorem to write

$$\begin{aligned} 0 &= f_i(\mathbf{x}(\mathbf{y}_1),\mathbf{y}_1) - f_i(\mathbf{x}(\mathbf{y}_2),\mathbf{y}_2) \\ &= f_i(\mathbf{x}(\mathbf{y}_1),\mathbf{y}_1) - f_i(\mathbf{x}(\mathbf{y}_2),\mathbf{y}_1) + f_i(\mathbf{x}(\mathbf{y}_2),\mathbf{y}_1) - f_i(\mathbf{x}(\mathbf{y}_2),\mathbf{y}_2) \\ &= D_1 f_i\left(\mathbf{x}^i,\mathbf{y}_1\right)(\mathbf{x}(\mathbf{y}_1) - \mathbf{x}(\mathbf{y}_2)) + D_2 f_i\left(\mathbf{x}(\mathbf{y}_2),\mathbf{y}^i\right)(\mathbf{y}_1 - \mathbf{y}_2). \end{aligned}$$

Therefore,

$$J\left(\mathbf{x}^1,\cdots,\mathbf{x}^n,\mathbf{y}_1\right)(\mathbf{x}(\mathbf{y}_1) - \mathbf{x}(\mathbf{y}_2)) = -M(\mathbf{y}_1 - \mathbf{y}_2) \tag{5.37}$$

where M is the matrix whose i^{th} row is $D_2 f_i\left(\mathbf{x}(\mathbf{y}_2),\mathbf{y}^i\right)$. Then from 5.35 there exists a constant, C independent of the choice of $\mathbf{y} \in B(\mathbf{y}_0,\eta)$ such that

$$\left|\left|J\left(\mathbf{x}^1,\cdots,\mathbf{x}^n,\mathbf{y}\right)^{-1}\right|\right| < C$$

whenever $\left(\mathbf{x}^1,\cdots,\mathbf{x}^n\right) \in \overline{B(\mathbf{x}_0,\delta)}^n$. By continuity of the partial derivatives of $\mathbf{f}$ it also follows there exists a constant, C_1 such that $||D_2 f_i(\mathbf{x},\mathbf{y})|| < C_1$ whenever, $(\mathbf{x},\mathbf{y}) \in \overline{B(\mathbf{x}_0,\delta)} \times B(\mathbf{y}_0,\eta)$. Hence $||M||$ must also be bounded independent of the choice of $\mathbf{y}_1$ and $\mathbf{y}_2$ in $B(\mathbf{y}_0,\eta)$. From 5.37, it follows there exists a constant, C such that for all $\mathbf{y}_1,\mathbf{y}_2$ in $B(\mathbf{y}_0,\eta)$,

$$|\mathbf{x}(\mathbf{y}_1) - \mathbf{x}(\mathbf{y}_2)| \leq C|\mathbf{y}_1 - \mathbf{y}_2|. \tag{5.38}$$

It follows as in the proof of the chain rule that

$$\mathbf{o}(\mathbf{x}(\mathbf{y}+\mathbf{v}) - \mathbf{x}(\mathbf{y})) = \mathbf{o}(\mathbf{v}). \tag{5.39}$$

Now let $\mathbf{y} \in B(\mathbf{y}_0,\eta)$ and let $|\mathbf{v}|$ be sufficiently small that $\mathbf{y}+\mathbf{v} \in B(\mathbf{y}_0,\eta)$. Then

$$\begin{aligned} \mathbf{0} &= \mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}),\mathbf{y}+\mathbf{v}) - \mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y}) \\ &= \mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}),\mathbf{y}+\mathbf{v}) - \mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}),\mathbf{y}) + \mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}),\mathbf{y}) - \mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y}) \end{aligned}$$

$$= D_2\mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}),\mathbf{y})\mathbf{v} + D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})(\mathbf{x}(\mathbf{y}+\mathbf{v}) - \mathbf{x}(\mathbf{y})) + \mathbf{o}(|\mathbf{x}(\mathbf{y}+\mathbf{v}) - \mathbf{x}(\mathbf{y})|)$$

$$\begin{aligned} &= D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})\mathbf{v} + D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})(\mathbf{x}(\mathbf{y}+\mathbf{v}) - \mathbf{x}(\mathbf{y})) + \\ &\quad \mathbf{o}(|\mathbf{x}(\mathbf{y}+\mathbf{v}) - \mathbf{x}(\mathbf{y})|) + (D_2\mathbf{f}(\mathbf{x}(\mathbf{y}+\mathbf{v}),\mathbf{y})\mathbf{v} - D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})\mathbf{v}) \\ &= D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})\mathbf{v} + D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})(\mathbf{x}(\mathbf{y}+\mathbf{v}) - \mathbf{x}(\mathbf{y})) + \mathbf{o}(\mathbf{v}). \end{aligned}$$

Therefore,

$$\mathbf{x}(\mathbf{y}+\mathbf{v}) - \mathbf{x}(\mathbf{y}) = -D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})^{-1} D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})\mathbf{v} + \mathbf{o}(\mathbf{v})$$

which shows that $D\mathbf{x}(\mathbf{y}) = -D_1\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})^{-1} D_2\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y})$ and $\mathbf{y} \to D\mathbf{x}(\mathbf{y})$ is continuous. This proves the theorem.

In practice, how do you verify the condition, $D_1\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n,\mathbb{F}^n)$?

$$\mathbf{f}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} f_1(x_1,\cdots,x_n,y_1,\cdots,y_n) \\ \vdots \\ f_n(x_1,\cdots,x_n,y_1,\cdots,y_n) \end{pmatrix}.$$

The matrix of the linear transformation, $D_1\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)$ is then

$$\begin{pmatrix} \frac{\partial f_1(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_1} & \cdots & \frac{\partial f_1(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_1} & \cdots & \frac{\partial f_n(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_n} \end{pmatrix}$$

and from linear algebra, $D_1\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n,\mathbb{F}^n)$ exactly when the above matrix has an inverse. In other words when

$$\det\begin{pmatrix} \frac{\partial f_1(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_1} & \cdots & \frac{\partial f_1(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_1} & \cdots & \frac{\partial f_n(x_1,\cdots,x_n,y_1,\cdots,y_n)}{\partial x_n} \end{pmatrix} \neq 0$$

at $(\mathbf{x}_0,\mathbf{y}_0)$. The above determinant is important enough that it is given special notation. Letting $\mathbf{z} = \mathbf{f}(\mathbf{x},\mathbf{y})$, the above determinant is often written as

$$\frac{\partial(z_1,\cdots,z_n)}{\partial(x_1,\cdots,x_n)}.$$

Of course you can replace $\mathbb{R}$ with $\mathbb{F}$ in the above by applying the above to the situation in which each $\mathbb{F}$ is replaced with $\mathbb{R}^2$.

Corollary 5.67 *(implicit function theorem) Suppose U is an open set in $\mathbb{F}^n \times \mathbb{F}^m$. Let $\mathbf{f} : U \to \mathbb{F}^n$ be in $C^1(U)$ and suppose*

$$\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0) = \mathbf{0},\ D_1\mathbf{f}(\mathbf{x}_0,\mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n,\mathbb{F}^n). \tag{5.40}$$

Then there exist positive constants, δ,η, such that for every $\mathbf{y} \in B(\mathbf{y}_0,\eta)$ there exists a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0,\delta)$ such that

$$\mathbf{f}(\mathbf{x}(\mathbf{y}),\mathbf{y}) = \mathbf{0}. \tag{5.41}$$

Furthermore, the mapping, $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ is in $C^1(B(\mathbf{y}_0,\eta))$.

The next theorem is a very important special case of the implicit function theorem known as the inverse function theorem. Actually one can also obtain the implicit function theorem from the inverse function theorem. It is done this way in [36] and in [3].

Theorem 5.68 *(inverse function theorem) Let* $\mathbf{x}_0 \in U \subseteq \mathbb{F}^n$ *and let* $\mathbf{f} : U \to \mathbb{F}^n$ *. Suppose*

$$\mathbf{f} \text{ is } C^1(U) \ , \quad \text{and} \quad D\mathbf{f}(\mathbf{x}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n). \tag{5.42}$$

Then there exist open sets, W*, and* V *such that*

$$\mathbf{x}_0 \in W \subseteq U, \tag{5.43}$$

$$\mathbf{f} : W \to V \text{ is one to one and onto}, \tag{5.44}$$

$$\mathbf{f}^{-1} \text{ is } C^1. \tag{5.45}$$

Proof: Apply the implicit function theorem to the function

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) \equiv \mathbf{f}(\mathbf{x}) - \mathbf{y}$$

where $\mathbf{y}_0 \equiv \mathbf{f}(\mathbf{x}_0)$. Thus the function $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ defined in that theorem is $\mathbf{f}^{-1}$. Now let

$$W \equiv B(\mathbf{x}_0, \delta) \cap \mathbf{f}^{-1}(B(\mathbf{y}_0, \eta))$$

and

$$V \equiv B(\mathbf{y}_0, \eta).$$

This proves the theorem.

5.16.1 More Continuous Partial Derivatives

Corollary 5.67 will now be improved slightly. If $\mathbf{f}$ is C^k, it follows that the function which is implicitly defined is also in C^k, not just C^1. Since the inverse function theorem comes as a case of the implicit function theorem, this shows that the inverse function also inherits the property of being C^k.

Theorem 5.69 *(implicit function theorem) Suppose* U *is an open set in* $\mathbb{F}^n \times \mathbb{F}^m$*. Let* $\mathbf{f} : U \to \mathbb{F}^n$ *be in* $C^k(U)$ *and suppose*

$$\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}, \ D_1\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n). \tag{5.46}$$

Then there exist positive constants, δ, η*, such that for every* $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ *there exists a unique* $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ *such that*

$$\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}. \tag{5.47}$$

Furthermore, the mapping, $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ *is in* $C^k(B(\mathbf{y}_0, \eta))$*.*

Proof: From Corollary 5.67 $\mathbf{y} \to \mathbf{x}(\mathbf{y})$ is C^1. It remains to show it is C^k for $k > 1$ assuming that $\mathbf{f}$ is C^k. From 5.47

$$\frac{\partial \mathbf{x}}{\partial y^l} = -D_1(\mathbf{x}, \mathbf{y})^{-1} \frac{\partial \mathbf{f}}{\partial y^l}.$$

Thus the following formula holds for $q = 1$ and $|\alpha| = q$.

$$D^{\alpha}\mathbf{x}(\mathbf{y}) = \sum_{|\beta| \leq q} M_{\beta}(\mathbf{x},\mathbf{y}) D^{\beta}\mathbf{f}(\mathbf{x},\mathbf{y}) \tag{5.48}$$

where M_{β} is a matrix whose entries are differentiable functions of $D^{\gamma}(\mathbf{x})$ for $|\gamma| < q$ and $D^{\tau}\mathbf{f}(\mathbf{x},\mathbf{y})$ for $|\tau| \leq q$. This follows easily from the description of $D_1(\mathbf{x},\mathbf{y})^{-1}$ in terms of the cofactor matrix and the determinant of $D_1(\mathbf{x},\mathbf{y})$. Suppose 5.48 holds for $|\alpha| = q < k$. Then by induction, this yields $\mathbf{x}$ is C^q. Then

$$\frac{\partial D^{\alpha}\mathbf{x}(\mathbf{y})}{\partial y^p} = \sum_{|\beta| \leq |\alpha|} \frac{\partial M_{\beta}(\mathbf{x},\mathbf{y})}{\partial y^p} D^{\beta}\mathbf{f}(\mathbf{x},\mathbf{y}) + M_{\beta}(\mathbf{x},\mathbf{y}) \frac{\partial D^{\beta}\mathbf{f}(\mathbf{x},\mathbf{y})}{\partial y^p}.$$

By the chain rule $\frac{\partial M_{\beta}(\mathbf{x},\mathbf{y})}{\partial y^p}$ is a matrix whose entries are differentiable functions of $D^{\tau}\mathbf{f}(\mathbf{x},\mathbf{y})$ for $|\tau| \leq q+1$ and $D^{\gamma}(\mathbf{x})$ for $|\gamma| < q+1$. It follows since y^p was arbitrary that for any $|\alpha| = q+1$, a formula like 5.48 holds with q being replaced by $q+1$. By induction, $\mathbf{x}$ is C^k. This proves the theorem.

As a simple corollary this yields an improved version of the inverse function theorem.

Theorem 5.70 *(inverse function theorem) Let $\mathbf{x}_0 \in U \subseteq \mathbb{F}^n$ and let $\mathbf{f} : U \to \mathbb{F}^n$. Suppose for k a positive integer,*

$$\mathbf{f} \text{ is } C^k(U), \quad \text{and} \quad D\mathbf{f}(\mathbf{x}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n). \tag{5.49}$$

Then there exist open sets, W, and V such that

$$\mathbf{x}_0 \in W \subseteq U, \tag{5.50}$$

$$\mathbf{f} : W \to V \text{ is one to one and onto}, \tag{5.51}$$

$$\mathbf{f}^{-1} \text{ is } C^k. \tag{5.52}$$

5.17 The Method Of Lagrange Multipliers

As an application of the implicit function theorem, consider the method of Lagrange multipliers from calculus. Recall the problem is to maximize or minimize a function subject to equality constraints. Let $f : U \to \mathbb{R}$ be a C^1 function where $U \subseteq \mathbb{R}^n$ and let

$$g_i(\mathbf{x}) = 0, \ i = 1, \cdots, m \tag{5.53}$$

be a collection of equality constraints with $m < n$. Now consider the system of nonlinear equations

$$\begin{aligned} f(\mathbf{x}) &= a \\ g_i(\mathbf{x}) &= 0, \ i = 1, \cdots, m. \end{aligned}$$

$\mathbf{x}_0$ is a local maximum if $f(\mathbf{x}_0) \geq f(\mathbf{x})$ for all $\mathbf{x}$ near $\mathbf{x}_0$ which also satisfies the constraints 5.53. A local minimum is defined similarly. Let $\mathbf{F} : U \times \mathbb{R} \to \mathbb{R}^{m+1}$ be defined by

$$\mathbf{F}(\mathbf{x},a) \equiv \begin{pmatrix} f(\mathbf{x}) - a \\ g_1(\mathbf{x}) \\ \vdots \\ g_m(\mathbf{x}) \end{pmatrix}. \tag{5.54}$$

Now consider the $m+1 \times n$ Jacobian matrix,

$$\begin{pmatrix} f_{x_1}(\mathbf{x}_0) & \cdots & f_{x_n}(\mathbf{x}_0) \\ g_{1x_1}(\mathbf{x}_0) & \cdots & g_{1x_n}(\mathbf{x}_0) \\ \vdots & & \vdots \\ g_{mx_1}(\mathbf{x}_0) & \cdots & g_{mx_n}(\mathbf{x}_0) \end{pmatrix}.$$

If this matrix has rank $m+1$ then some $m+1 \times m+1$ submatrix has nonzero determinant. It follows from the implicit function theorem that there exist $m+1$ variables, $x_{i_1}, \cdots, x_{i_{m+1}}$ such that the system

$$\mathbf{F}(\mathbf{x},a) = \mathbf{0} \tag{5.55}$$

specifies these $m+1$ variables as a function of the remaining $n-(m+1)$ variables and a in an open set of $\mathbb{R}^{n-m}$. Thus there is a solution $(\mathbf{x},a)$ to 5.55 for some $\mathbf{x}$ close to $\mathbf{x}_0$ whenever a is in some open interval. Therefore, $\mathbf{x}_0$ cannot be either a local minimum or a local maximum. It follows that if $\mathbf{x}_0$ is either a local maximum or a local minimum, then the above matrix must have rank less than $m+1$ which requires the rows to be linearly dependent. Thus, there exist m scalars,

$$\lambda_1, \cdots, \lambda_m,$$

and a scalar μ, not all zero such that

$$\mu \begin{pmatrix} f_{x_1}(\mathbf{x}_0) \\ \vdots \\ f_{x_n}(\mathbf{x}_0) \end{pmatrix} = \lambda_1 \begin{pmatrix} g_{1x_1}(\mathbf{x}_0) \\ \vdots \\ g_{1x_n}(\mathbf{x}_0) \end{pmatrix} + \cdots + \lambda_m \begin{pmatrix} g_{mx_1}(\mathbf{x}_0) \\ \vdots \\ g_{mx_n}(\mathbf{x}_0) \end{pmatrix}. \tag{5.56}$$

If the column vectors

$$\begin{pmatrix} g_{1x_1}(\mathbf{x}_0) \\ \vdots \\ g_{1x_n}(\mathbf{x}_0) \end{pmatrix}, \cdots \begin{pmatrix} g_{mx_1}(\mathbf{x}_0) \\ \vdots \\ g_{mx_n}(\mathbf{x}_0) \end{pmatrix} \tag{5.57}$$

are linearly independent, then, $\mu \neq 0$ and dividing by μ yields an expression of the form

$$\begin{pmatrix} f_{x_1}(\mathbf{x}_0) \\ \vdots \\ f_{x_n}(\mathbf{x}_0) \end{pmatrix} = \lambda_1 \begin{pmatrix} g_{1x_1}(\mathbf{x}_0) \\ \vdots \\ g_{1x_n}(\mathbf{x}_0) \end{pmatrix} + \cdots + \lambda_m \begin{pmatrix} g_{mx_1}(\mathbf{x}_0) \\ \vdots \\ g_{mx_n}(\mathbf{x}_0) \end{pmatrix} \tag{5.58}$$

at every point $\mathbf{x}_0$ which is either a local maximum or a local minimum. This proves the following theorem.

Theorem 5.71 *Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be a C^1 function. Then if $\mathbf{x}_0 \in U$ is either a local maximum or local minimum of f subject to the constraints 5.53, then 5.56 must hold for some scalars $\mu, \lambda_1, \cdots, \lambda_m$ not all equal to zero. If the vectors in 5.57 are linearly independent, it follows that an equation of the form 5.58 holds.*

Metric Spaces And General Topological Spaces

6.1 Metric Space

Definition 6.1 *A metric space is a set, X and a function $d : X \times X \to [0, \infty)$ which satisfies the following properties.*

$$\begin{gathered} d(x, y) = d(y, x) \\ d(x, y) \geq 0 \text{ and } d(x, y) = 0 \text{ if and only if } x = y \\ d(x, y) \leq d(x, z) + d(z, y). \end{gathered}$$

You can check that $\mathbb{R}^n$ and $\mathbb{C}^n$ are metric spaces with $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$. However, there are many others. The definitions of open and closed sets are the same for a metric space as they are for $\mathbb{R}^n$.

Definition 6.2 *A set, U in a metric space is open if whenever $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$. As before, $B(x, r) \equiv \{y : d(x, y) < r\}$. Closed sets are those whose complements are open. A point p is a limit point of a set, S if for every $r > 0, B(p, r)$ contains infinitely many points of S. A sequence, $\{x_n\}$ converges to a point x if for every $\varepsilon > 0$ there exists N such that if $n \geq N$, then $d(x, x_n) < \varepsilon$. $\{x_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists N such that if $m, n \geq N$, then $d(x_n, x_m) < \varepsilon$.*

Lemma 6.3 *In a metric space, X every ball, $B(x, r)$ is open. A set is closed if and only if it contains all its limit points. If p is a limit point of S, then there exists a sequence of distinct points of $S, \{x_n\}$ such that $\lim_{n\to\infty} x_n = p$.*

Proof: Let $z \in B(x, r)$. Let $\delta = r - d(x, z)$. Then if $w \in B(z, \delta)$,

$$d(w, x) \leq d(x, z) + d(z, w) < d(x, z) + r - d(x, z) = r.$$

Therefore, $B(z, \delta) \subseteq B(x, r)$ and this shows $B(x, r)$ is open.

The properties of balls are presented in the following theorem.

Theorem 6.4 *Suppose (X,d) is a metric space. Then the sets $\{B(x,r) : r > 0,\ x \in X\}$ satisfy*

$$\cup \{B(x,r) : r > 0,\ x \in X\} = X \tag{6.1}$$

If $p \in B(x,r_1) \cap B(z,r_2)$, there exists $r > 0$ such that

$$B(p,r) \subseteq B(x,r_1) \cap B(z,r_2). \tag{6.2}$$

Proof: Observe that the union of these balls includes the whole space, X so 6.1 is obvious. Consider 6.2. Let $p \in B(x,r_1) \cap B(z,r_2)$. Consider

$$r \equiv \min(r_1 - d(x,p), r_2 - d(z,p))$$

and suppose $y \in B(p,r)$. Then

$$d(y,x) \leq d(y,p) + d(p,x) < r_1 - d(x,p) + d(x,p) = r_1$$

and so $B(p,r) \subseteq B(x,r_1)$. By similar reasoning, $B(p,r) \subseteq B(z,r_2)$. This proves the theorem.

Let K be a closed set. This means $K^C \equiv X \setminus K$ is an open set. Let p be a limit point of K. If $p \in K^C$, then since K^C is open, there exists $B(p,r) \subseteq K^C$. But this contradicts p being a limit point because there are no points of K in this ball. Hence all limit points of K must be in K.

Suppose next that K contains its limit points. Is K^C open? Let $p \in K^C$. Then p is not a limit point of K. Therefore, there exists $B(p,r)$ which contains at most finitely many points of K. Since $p \notin K$, it follows that by making r smaller if necessary, $B(p,r)$ contains no points of K. That is $B(p,r) \subseteq K^C$ showing K^C is open. Therefore, K is closed.

Suppose now that p is a limit point of S. Let $x_1 \in (S \setminus \{p\}) \cap B(p,1)$. If $x_1, \cdots, x_k$ have been chosen, let

$$r_{k+1} \equiv \min\left\{d(p,x_i), i = 1, \cdot\cdot\cdot, k, \frac{1}{k+1}\right\}.$$

Let $x_{k+1} \in (S \setminus \{p\}) \cap B(p, r_{k+1})$. This proves the lemma.

Lemma 6.5 *If $\{x_n\}$ is a Cauchy sequence in a metric space, X and if some subsequence, $\{x_{n_k}\}$ converges to x, then $\{x_n\}$ converges to x. Also if a sequence converges, then it is a Cauchy sequence.*

Proof: Note first that $n_k \geq k$ because in a subsequence, the indices, $n_1, n_2, \cdots$ are strictly increasing. Let $\varepsilon > 0$ be given and let N be such that for $k > N, d(x, x_{n_k}) < \varepsilon/2$ and for $m, n \geq N, d(x_m, x_n) < \varepsilon/2$. Pick $k > n$. Then if $n > N$,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Finally, suppose $\lim_{n\to\infty} x_n = x$. Then there exists N such that if $n > N$, then $d(x_n, x) < \varepsilon/2$. it follows that for $m, n > N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the lemma.

A useful idea is the idea of distance from a point to a set.

Definition 6.6 *Let (X, d) be a metric space and let S be a nonempty set in X. Then*

$$\text{dist}(x, S) \equiv \inf \{d(x, y) : y \in S\}.$$

The following lemma is the fundamental result.

Lemma 6.7 *The function, $x \to \text{dist}(x, S)$ is continuous and in fact satisfies*

$$|\text{dist}(x, S) - \text{dist}(y, S)| \leq d(x, y).$$

Proof: Suppose dist (x, y) is as least as large as dist (y, S). Then pick $z \in S$ such that $d(y, z) \leq \text{dist}(y, S) + \varepsilon$. Then

$$\begin{aligned} |\text{dist}(x, S) - \text{dist}(y, S)| &= \text{dist}(x, S) - \text{dist}(y, S) \\ &\leq d(x, z) - (d(y, z) - \varepsilon) \\ &= d(x, z) - d(y, z) + \varepsilon \\ &\leq d(x, y) + d(y, z) - d(y, z) + \varepsilon \\ &= d(x, y) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves the lemma.

6.2 Compactness In Metric Space

Many existence theorems in analysis depend on some set being compact. Therefore, it is important to be able to identify compact sets. The purpose of this section is to describe compact sets in a metric space.

Definition 6.8 *Let A be a subset of X. A is compact if whenever A is contained in the union of a set of open sets, there exists finitely many of these open sets whose union contains A. (Every open cover admits a finite subcover.) A is "sequentially compact" means every sequence has a convergent subsequence converging to an element of A.*

In a metric space compact is not the same as closed and bounded!

Example 6.9 *Let X be any infinite set and define $d(x, y) = 1$ if $x \neq y$ while $d(x, y) = 0$ if $x = y$.*

You should verify the details that this is a metric space because it satisfies the axioms of a metric. The set X is closed and bounded because its complement is $\emptyset$ which is clearly open because every point of $\emptyset$ is an interior point. (There are none.) Also X is bounded because $X = B(x, 2)$. However, X is clearly not compact because $\left\{B\left(x, \frac{1}{2}\right) : x \in X\right\}$ is a collection of open sets whose union contains X but since they are all disjoint and nonempty, there is no finite subset of these whose union contains X. In fact $B\left(x, \frac{1}{2}\right) = \{x\}$.

From this example it is clear something more than closed and bounded is needed. If you are not familiar with the issues just discussed, ignore them and continue.

Definition 6.10 *In any metric space, a set E is totally bounded if for every $\varepsilon > 0$ there exists a finite set of points $\{x_1, \cdots, x_n\}$ such that*

$$E \subseteq \cup_{i=1}^{n} B(x_i, \varepsilon).$$

This finite set of points is called an ε net.

The following proposition tells which sets in a metric space are compact. First here is an interesting lemma.

Lemma 6.11 *Let X be a metric space and suppose D is a countable dense subset of X. In other words, it is being assumed X is a separable metric space. Consider the open sets of the form $B(d, r)$ where r is a positive rational number and $d \in D$. Denote this countable collection of open sets by $\mathcal{B}$. Then every open set is the union of sets of $\mathcal{B}$. Furthermore, if $\mathcal{C}$ is any collection of open sets, there exists a countable subset, $\{U_n\} \subseteq \mathcal{C}$ such that $\cup_n U_n = \cup \mathcal{C}$.*

Proof: Let U be an open set and let $x \in U$. Let $B(x, \delta) \subseteq U$. Then by density of D, there exists $d \in D \cap B(x, \delta/4)$. Now pick $r \in \mathbb{Q} \cap (\delta/4, 3\delta/4)$ and consider $B(d, r)$. Clearly, $B(d, r)$ contains the point x because $r > \delta/4$. Is $B(d, r) \subseteq B(x, \delta)$? if so, this proves the lemma because x was an arbitrary point of U. Suppose $z \in B(d, r)$. Then

$$d(z, x) \leq d(z, d) + d(d, x) < r + \frac{\delta}{4} < \frac{3\delta}{4} + \frac{\delta}{4} = \delta$$

Now let $\mathcal{C}$ be any collection of open sets. Each set in this collection is the union of countably many sets of $\mathcal{B}$. Let $\mathcal{B}'$ denote the sets of $\mathcal{B}$ which are contained in some set of $\mathcal{C}$. Thus $\cup \mathcal{B}' = \cup \mathcal{C}$. Then for each $B \in \mathcal{B}'$, pick $U_B \in \mathcal{C}$ such that $B \subseteq U_B$. Then $\{U_B : B \in \mathcal{B}'\}$ is a countable collection of sets of $\mathcal{C}$ whose union equals $\cup \mathcal{C}$. Therefore, this proves the lemma.

Proposition 6.12 *Let (X, d) be a metric space. Then the following are equivalent.*

$$(X, d) \text{ is compact,} \tag{6.3}$$

$$(X, d) \text{ is sequentially compact,} \tag{6.4}$$

$$(X, d) \text{ is complete and totally bounded.} \tag{6.5}$$

Proof: Suppose 6.3 and let $\{x_k\}$ be a sequence. Suppose $\{x_k\}$ has no convergent subsequence. If this is so, then by Lemma 6.3, $\{x_k\}$ has no limit point and no value of the sequence is repeated more than finitely many times. Thus the set

$$C_n = \cup \{x_k : k \geq n\}$$

is a closed set because it has no limit points and if

$$U_n = C_n^C,$$

then

$$X = \cup_{n=1}^{\infty} U_n$$

but there is no finite subcovering, because no value of the sequence is repeated more than finitely many times. This contradicts compactness of (X, d). This shows 6.3 implies 6.4.

Now suppose 6.4 and let $\{x_n\}$ be a Cauchy sequence. Is $\{x_n\}$ convergent? By sequential compactness $x_{n_k} \to x$ for some subsequence. By Lemma 6.5 it follows that $\{x_n\}$ also converges to x showing that (X, d) is complete. If (X, d) is not totally bounded, then there exists $\varepsilon > 0$ for which there is no ε net. Hence there exists a sequence $\{x_k\}$ with $d(x_k, x_l) \geq \varepsilon$ for all $l \neq k$. By Lemma 6.5 again, this contradicts 6.4 because no subsequence can be a Cauchy sequence and so no subsequence can converge. This shows 6.4 implies 6.5.

Now suppose 6.5. What about 6.4? Let $\{p_n\}$ be a sequence and let $\{x_i^n\}_{i=1}^{m_n}$ be a 2^{-n} net for $n = 1, 2, \cdots$. Let

$$B_n \equiv B\left(x_{i_n}^n, 2^{-n}\right)$$

be such that B_n contains p_k for infinitely many values of k and $B_n \cap B_{n+1} \neq \emptyset$. To do this, suppose B_n contains p_k for infinitely many values of k. Then one of the sets which intersect $B_n, B\left(x_i^{n+1}, 2^{-(n+1)}\right)$ must contain p_k for infinitely many values of k because all these indices of points from $\{p_n\}$ contained in B_n must be accounted for in one of finitely many sets, $B\left(x_i^{n+1}, 2^{-(n+1)}\right)$. Thus there exists a strictly increasing sequence of integers, n_k such that

$$p_{n_k} \in B_k.$$

Then if $k \geq l$,

$$\begin{aligned} d\left(p_{n_k}, p_{n_l}\right) &\leq \sum_{i=l}^{k-1} d\left(p_{n_{i+1}}, p_{n_i}\right) \\ &< \sum_{i=l}^{k-1} 2^{-(i-1)} < 2^{-(l-2)}. \end{aligned}$$

Consequently $\{p_{n_k}\}$ is a Cauchy sequence. Hence it converges because the metric space is complete. This proves 6.4.

Now suppose 6.4 and 6.5 which have now been shown to be equivalent. Let D_n be a n^{-1} net for $n = 1, 2, \cdots$ and let

$$D = \cup_{n=1}^{\infty} D_n.$$

Thus D is a countable dense subset of (X, d).

Now let $\mathcal{C}$ be any set of open sets such that $\cup\mathcal{C} \supseteq X$. By Lemma 6.11, there exists a countable subset of $\mathcal{C}$,

$$\widetilde{\mathcal{C}} = \{U_n\}_{n=1}^{\infty}$$

such that $\cup\widetilde{\mathcal{C}} = \cup\mathcal{C}$. If $\mathcal{C}$ admits no finite subcover, then neither does $\widetilde{\mathcal{C}}$ and there exists $p_n \in X \setminus \cup_{k=1}^{n} U_k$. Then since X is sequentially compact, there is a subsequence $\{p_{n_k}\}$ such that $\{p_{n_k}\}$ converges. Say

$$p = \lim_{k\to\infty} p_{n_k}.$$

All but finitely many points of $\{p_{n_k}\}$ are in $X \setminus \cup_{k=1}^{n} U_k$. Therefore $p \in X \setminus \cup_{k=1}^{n} U_k$ for each n. Hence

$$p \notin \cup_{k=1}^{\infty} U_k$$

contradicting the construction of $\{U_n\}_{n=1}^{\infty}$ which required that $\cup_{n=1}^{\infty} U_n \supseteq X$. Hence X is compact. This proves the proposition.

Consider $\mathbb{R}^n$. In this setting totally bounded and bounded are the same. This will yield a proof of the Heine Borel theorem from advanced calculus.

Lemma 6.13 *A subset of* $\mathbb{R}^n$ *is totally bounded if and only if it is bounded.*

Proof: Let A be totally bounded. Is it bounded? Let $\mathbf{x}_1, \cdots, \mathbf{x}_p$ be a 1 net for A. Now consider the ball $B(\mathbf{0}, r+1)$ where $r > \max(|\mathbf{x}_i| : i = 1, \cdots, p)$. If $\mathbf{z} \in A$, then $\mathbf{z} \in B(\mathbf{x}_j, 1)$ for some j and so by the triangle inequality,

$$|\mathbf{z} - \mathbf{0}| \leq |\mathbf{z} - \mathbf{x}_j| + |\mathbf{x}_j| < 1 + r.$$

Thus $A \subseteq B(\mathbf{0}, r+1)$ and so A is bounded.

Now suppose A is bounded and suppose A is not totally bounded. Then there exists $\varepsilon > 0$ such that there is no ε net for A. Therefore, there exists a sequence of points $\{a_i\}$ with $|a_i - a_j| \geq \varepsilon$ if $i \neq j$. Since A is bounded, there exists $r > 0$ such that

$$A \subseteq [-r, r)^n.$$

($\mathbf{x} \in [-r, r)^n$ means $x_i \in [-r, r)$ for each i.) Now define $\mathcal{S}$ to be all cubes of the form

$$\prod_{k=1}^{n} [a_k, b_k)$$

where

$$a_k = -r + i2^{-p}r,\ b_k = -r + (i+1)\, 2^{-p} r,$$

for $i \in \{0, 1, \cdots, 2^{p+1} - 1\}$. Thus $\mathcal{S}$ is a collection of $\left(2^{p+1}\right)^n$ non overlapping cubes whose union equals $[-r, r)^n$ and whose diameters are all equal to $2^{-p} r\sqrt{n}$. Now choose p large enough that the diameter of these cubes is less than ε. This yields a contradiction because one of the cubes must contain infinitely many points of $\{a_i\}$. This proves the lemma.

The next theorem is called the Heine Borel theorem and it characterizes the compact sets in $\mathbb{R}^n$.

Theorem 6.14 *A subset of* $\mathbb{R}^n$ *is compact if and only if it is closed and bounded.*

Proof: Since a set in $\mathbb{R}^n$ is totally bounded if and only if it is bounded, this theorem follows from Proposition 6.12 and the observation that a subset of $\mathbb{R}^n$ is closed if and only if it is complete. This proves the theorem.

6.3 Some Applications Of Compactness

The following corollary is an important existence theorem which depends on compactness.

Corollary 6.15 *Let* X *be a compact metric space and let* $f : X \to \mathbb{R}$ *be continuous. Then* $\max \{f(x) : x \in X\}$ *and* $\min \{f(x) : x \in X\}$ *both exist.*

Proof: First it is shown $f(X)$ is compact. Suppose $\mathcal{C}$ is a set of open sets whose union contains $f(X)$. Then since f is continuous $f^{-1}(U)$ is open for all $U \in \mathcal{C}$. Therefore, $\left\{f^{-1}(U) : U \in \mathcal{C}\right\}$ is a collection of open sets whose union contains X. Since X is compact, it follows finitely many of these, $\left\{f^{-1}(U_1), \cdots, f^{-1}(U_p)\right\}$ contains X in their union. Therefore, $f(X) \subseteq \cup_{k=1}^{p} U_k$ showing $f(X)$ is compact as claimed.

Now since $f(X)$ is compact, Theorem 6.14 implies $f(X)$ is closed and bounded. Therefore, it contains its inf and its sup. Thus f achieves both a maximum and a minimum.

Definition 6.16 *Let* X, Y *be metric spaces and* $f : X \to Y$ *a function.* f *is uniformly continuous if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that whenever* x_1 *and* x_2 *are two points of* X *satisfying* $d(x_1, x_2) < \delta$*, it follows that* $d(f(x_1), f(x_2)) < \varepsilon$*.*

A very important theorem is the following.

Theorem 6.17 *Suppose* $f : X \to Y$ *is continuous and* X *is compact. Then* f *is uniformly continuous.*

Proof: Suppose this is not true and that f is continuous but not uniformly continuous. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exist points, p_δ and q_δ such that $d(p_\delta, q_\delta) < \delta$ and yet $d(f(p_\delta), f(q_\delta)) \geq \varepsilon$. Let p_n and q_n be the points which go with $\delta = 1/n$. By Proposition 6.12 $\{p_n\}$ has a convergent subsequence, $\{p_{n_k}\}$ converging to a point, $x \in X$. Since $d(p_n, q_n) < \frac{1}{n}$, it follows that $q_{n_k} \to x$ also. Therefore,

$$\varepsilon \leq d(f(p_{n_k}), f(q_{n_k})) \leq d(f(p_{n_k}), f(x)) + d(f(x), f(q_{n_k}))$$

but by continuity of f, both $d(f(p_{n_k}), f(x))$ and $d(f(x), f(q_{n_k}))$ converge to 0 as $k \to \infty$ contradicting the above inequality. This proves the theorem.

Another important property of compact sets in a metric space concerns the finite intersection property.

Definition 6.18 *If every finite subset of a collection of sets has nonempty intersection, the collection has the finite intersection property.*

Theorem 6.19 *Suppose $\mathcal{F}$ is a collection of compact sets in a metric space, X which has the finite intersection property. Then there exists a point in their intersection. ($\cap\mathcal{F} \neq \emptyset$).*

Proof: If this were not so, $\cup\left\{F^C : F \in \mathcal{F}\right\} = X$ and so, in particular, picking some $F_0 \in \mathcal{F}$, $\left\{F^C : F \in \mathcal{F}\right\}$ would be an open cover of F_0. Since F_0 is compact, some finite subcover, $F_1^C, \cdot \cdot \cdot, F_m^C$ exists. But then $F_0 \subseteq \cup_{k=1}^m F_k^C$ which means $\cap_{k=0}^\infty F_k = \emptyset$, contrary to the finite intersection property.

Theorem 6.20 *Let X_i be a compact metric space with metric d_i. Then $\prod_{i=1}^m X_i$ is also a compact metric space with respect to the metric, $d(\mathbf{x}, \mathbf{y}) \equiv \max_i (d_i(x_i, y_i))$.*

Proof: This is most easily seen from sequential compactness. Let $\left\{\mathbf{x}^k\right\}_{k=1}^\infty$ be a sequence of points in $\prod_{i=1}^m X_i$. Consider the i^{th} component of $\mathbf{x}^k$, x_i^k. It follows $\left\{x_i^k\right\}$ is a sequence of points in X_i and so it has a convergent subsequence. Compactness of X_1 implies there exists a subsequence of $\mathbf{x}^k$, denoted by $\left\{\mathbf{x}^{k_1}\right\}$ such that

$$\lim_{k_1 \to \infty} x_1^{k_1} \to x_1 \in X_1.$$

Now there exists a further subsequence, denoted by $\left\{\mathbf{x}^{k_2}\right\}$ such that in addition to this, $x_2^{k_2} \to x_2 \in X_2$. After taking m such subsequences, there exists a subsequence, $\left\{\mathbf{x}^l\right\}$ such that $\lim_{l\to\infty} x_i^l = x_i \in X_i$ for each i. Therefore, letting $\mathbf{x} \equiv (x_1, \cdot \cdot \cdot, x_m)$, $\mathbf{x}^l \to \mathbf{x}$ in $\prod_{i=1}^m X_i$. This proves the theorem.

6.4 Ascoli Arzela Theorem

Definition 6.21 *Let (X, d) be a complete metric space. Then it is said to be locally compact if $\overline{B(x, r)}$ is compact for each $r > 0$.*

Thus if you have a locally compact metric space, then if $\{a_n\}$ is a bounded sequence, it must have a convergent subsequence.

Let K be a compact subset of $\mathbb{R}^n$ and consider the continuous functions which have values in a locally compact metric space, (X, d) where d denotes the metric on X. Denote this space as $C(K, X)$.

Definition 6.22 *For $f, g \in C(K, X)$, where K is a compact subset of $\mathbb{R}^n$ and X is a locally compact complete metric space define*

$$\rho_K(f, g) \equiv \sup\{d(f(\mathbf{x}), g(\mathbf{x})) : \mathbf{x} \in K\}.$$

Then ρ_K provides a distance which makes $C(K, X)$ into a metric space.

The Ascoli Arzela theorem is a major result which tells which subsets of $C(K, X)$ are sequentially compact.

Definition 6.23 *Let $A \subseteq C(K, X)$ for K a compact subset of $\mathbb{R}^n$. Then A is said to be uniformly equicontinuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $\mathbf{x}, \mathbf{y} \in K$ with $|\mathbf{x} - \mathbf{y}| < \delta$ and $f \in A$,*

$$d(f(\mathbf{x}), f(\mathbf{y})) < \varepsilon.$$

The set, A is said to be uniformly bounded if for some $M < \infty$, and $a \in X$,

$$f(\mathbf{x}) \in B(a, M)$$

for all $f \in A$ and $\mathbf{x} \in K$.

Uniform equicontinuity is like saying that the whole set of functions, A, is uniformly continuous on K uniformly for $f \in A$. The version of the Ascoli Arzela theorem I will present here is the following.

Theorem 6.24 *Suppose K is a nonempty compact subset of $\mathbb{R}^n$ and $A \subseteq C(K, X)$ is uniformly bounded and uniformly equicontinuous. Then if $\{f_k\} \subseteq A$, there exists a function, $f \in C(K, X)$ and a subsequence, f_{k_l} such that*

$$\lim_{l \to \infty} \rho_K(f_{k_l}, f) = 0.$$

To give a proof of this theorem, I will first prove some lemmas.

Lemma 6.25 *If K is a compact subset of $\mathbb{R}^n$, then there exists $D \equiv \{\mathbf{x}_k\}_{k=1}^{\infty} \subseteq K$ such that D is dense in K. Also, for every $\varepsilon > 0$ there exists a finite set of points, $\{\mathbf{x}_1, \cdots, \mathbf{x}_m\} \subseteq K$, called an ε net such that*

$$\cup_{i=1}^{m} B(\mathbf{x}_i, \varepsilon) \supseteq K.$$

Proof: For $m \in \mathbb{N}$, pick $x_1^m \in K$. If every point of K is within $1/m$ of x_1^m, stop. Otherwise, pick

$$x_2^m \in K \setminus B(x_1^m, 1/m).$$

If every point of K contained in $B(x_1^m, 1/m) \cup B(x_2^m, 1/m)$, stop. Otherwise, pick

$$x_3^m \in K \setminus (B(x_1^m, 1/m) \cup B(x_2^m, 1/m)).$$

If every point of K is contained in $B(x_1^m, 1/m) \cup B(x_2^m, 1/m) \cup B(x_3^m, 1/m)$, stop. Otherwise, pick

$$x_4^m \in K \setminus (B(x_1^m, 1/m) \cup B(x_2^m, 1/m) \cup B(x_3^m, 1/m))$$

Continue this way until the process stops, say at $N(m)$. It must stop because if it didn't, there would be a convergent subsequence due to the compactness of K. Ultimately all terms of this convergent subsequence would be closer than $1/m$, violating the manner in which they are chosen. Then $D = \cup_{m=1}^{\infty} \cup_{k=1}^{N(m)} \{x_k^m\}$. This is countable because it is a countable union of countable sets. If $\mathbf{y} \in K$ and $\varepsilon > 0$, then for some m, $2/m < \varepsilon$ and so $B(\mathbf{y}, \varepsilon)$ must contain some point of $\{x_k^m\}$ since otherwise, the process stopped too soon. You could have picked $\mathbf{y}$. This proves the lemma.

Lemma 6.26 *Suppose D is defined above and $\{g_m\}$ is a sequence of functions of A having the property that for every $\mathbf{x}_k \in D$,*

$$\lim_{m\to\infty} g_m(\mathbf{x}_k) \text{ exists.}$$

Then there exists $g \in C(K, X)$ such that

$$\lim_{m\to\infty} \rho(g_m, g) = 0.$$

Proof: Define g first on D.

$$g(\mathbf{x}_k) \equiv \lim_{m\to\infty} g_m(\mathbf{x}_k).$$

Next I show that $\{g_m\}$ converges at every point of K. Let $\mathbf{x} \in K$ and let $\varepsilon > 0$ be given. Choose $\mathbf{x}_k$ such that for all $f \in A$,

$$d(f(\mathbf{x}_k), f(\mathbf{x})) < \frac{\varepsilon}{3}.$$

I can do this by the equicontinuity. Now if p, q are large enough, say $p, q \geq M$,

$$d(g_p(\mathbf{x}_k), g_q(\mathbf{x}_k)) < \frac{\varepsilon}{3}.$$

Therefore, for $p, q \geq M$,

$$\begin{aligned} d(g_p(\mathbf{x}), g_q(\mathbf{x})) &\leq d(g_p(\mathbf{x}), g_p(\mathbf{x}_k)) + d(g_p(\mathbf{x}_k), g_q(\mathbf{x}_k)) + d(g_q(\mathbf{x}_k), g_q(\mathbf{x})) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

It follows that $\{g_m(\mathbf{x})\}$ is a Cauchy sequence having values X. Therefore, it converges. Let $g(\mathbf{x})$ be the name of the thing it converges to.

Let $\varepsilon > 0$ be given and pick $\delta > 0$ such that whenever $\mathbf{x}, \mathbf{y} \in K$ and $|\mathbf{x} - \mathbf{y}| < \delta$, it follows $d(f(\mathbf{x}), f(\mathbf{y})) < \frac{\varepsilon}{3}$ for all $f \in A$. Now let $\{\mathbf{x}_1, \cdots, \mathbf{x}_m\}$ be a δ net for K as in Lemma 6.25. Since there are only finitely many points in this δ net, it follows that there exists N such that for all $p, q \geq N$,

$$d(g_q(\mathbf{x}_i), g_p(\mathbf{x}_i)) < \frac{\varepsilon}{3}$$

for all $\{\mathbf{x}_1, \cdots, \mathbf{x}_m\}\cdot$ Therefore, for arbitrary $\mathbf{x} \in K$, pick $\mathbf{x}_i \in \{\mathbf{x}_1, \cdots, \mathbf{x}_m\}$ such that $|\mathbf{x}_i - \mathbf{x}| < \delta$. Then

$$\begin{aligned} d(g_q(\mathbf{x}), g_p(\mathbf{x})) &\leq d(g_q(\mathbf{x}), g_q(\mathbf{x}_i)) + d(g_q(\mathbf{x}_i), g_p(\mathbf{x}_i)) + d(g_p(\mathbf{x}_i), g_p(\mathbf{x})) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since N does not depend on the choice of $\mathbf{x}$, it follows this sequence $\{g_m\}$ is uniformly Cauchy. That is, for every $\varepsilon > 0$, there exists N such that if $p, q \geq N$, then

$$\rho(g_p, g_q) < \varepsilon.$$

Next, I need to verify that the function, g is a continuous function. Let N be large enough that whenever $p, q \geq N$, the above holds. Then for all $\mathbf{x} \in K$,

$$d(g(\mathbf{x}), g_p(\mathbf{x})) \leq \frac{\varepsilon}{3} \tag{6.6}$$

whenever $p \geq N$. This follows from observing that for $p, q \geq N$,

$$d(g_q(\mathbf{x}), g_p(\mathbf{x})) < \frac{\varepsilon}{3}$$

and then taking the limit as $q \to \infty$ to obtain 6.6. In passing to the limit, you can use the following simple claim.

Claim: In a metric space, if $a_n \to a$, then $d(a_n, b) \to d(a, b)$.

Proof of the claim: You note that by the triangle inequality, $d(a_n, b) - d(a, b) \leq d(a_n, a)$ and $d(a, b) - d(a_n, b) \leq d(a_n, a)$ and so

$$|d(a_n, b) - d(a, b)| \leq d(a_n, a).$$

Now let p satisfy 6.6 for all $\mathbf{x}$ whenever $p > N$. Also pick $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, then

$$d(g_p(\mathbf{x}), g_p(\mathbf{y})) < \frac{\varepsilon}{3}.$$

Then if $|\mathbf{x} - \mathbf{y}| < \delta$,

$$\begin{aligned} d(g(\mathbf{x}), g(\mathbf{y})) &\leq d(g(\mathbf{x}), g_p(\mathbf{x})) + d(g_p(\mathbf{x}), g_p(\mathbf{y})) + d(g_p(\mathbf{y}), g(\mathbf{y})) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since ε was arbitrary, this shows that g is continuous.

It only remains to verify that $\rho(g, g_k) \to 0$. But this follows from 6.6. This proves the lemma.

With these lemmas, it is time to prove Theorem 6.24.

Proof of Theorem 6.24: Let $D = \{\mathbf{x}_k\}$ be the countable dense set of K gauranteed by Lemma 6.25 and let

$$\{(1,1), (1,2), (1,3), (1,4), (1,5), \cdots\}$$

be a subsequence of $\mathbb{N}$ such that

$$\lim_{k\to\infty} f_{(1,k)}(\mathbf{x}_1) \text{ exists.}$$

This is where the local compactness of X is being used. Now let

$$\{(2,1), (2,2), (2,3), (2,4), (2,5), \cdots\}$$

be a subsequence of

$$\{(1,1), (1,2), (1,3), (1,4), (1,5), \cdots\}$$

which has the property that

$$\lim_{k\to\infty} f_{(2,k)}(\mathbf{x}_2) \text{ exists.}$$

Thus it is also the case that

$$f_{(2,k)}(\mathbf{x}_1) \text{ converges to } \lim_{k\to\infty} f_{(1,k)}(\mathbf{x}_1).$$

because every subsequence of a convergent sequence converges to the same thing as the convergent sequence. Continue this way and consider the array

$$\begin{array}{c} f_{(1,1)}, f_{(1,2)}, f_{(1,3)}, f_{(1,4)}, \cdots \text{ converges at } \mathbf{x}_1 \\ f_{(2,1)}, f_{(2,2)}, f_{(2,3)}, f_{(2,4)} \cdots \text{ converges at } \mathbf{x}_1 \text{ and } \mathbf{x}_2 \\ f_{(3,1)}, f_{(3,2)}, f_{(3,3)}, f_{(3,4)} \cdots \text{ converges at } \mathbf{x}_1,\ \mathbf{x}_2, \text{ and } \mathbf{x}_3 \\ \vdots \end{array}$$

Now let $g_k \equiv f_{(k,k)}$. Thus g_k is ultimately a subsequence of $\{f_{(m,k)}\}$ whenever $k > m$ and therefore, $\{g_k\}$ converges at each point of D. By Lemma 6.26 it follows there exists $g \in C(K)$ such that

$$\lim_{k\to\infty} \rho(g, g_k) = 0.$$

This proves the Ascoli Arzela theorem.

Actually there is an if and only if version of it but the most useful case is what is presented here. The process used to get the subsequence in the proof is called the Cantor diagonalization procedure.

6.5 The Tietze Extension Theorem

It turns out that if H is a closed subset of a metric space, (X, d) and if $f : H \to [a, b]$ is continuous, then there exists g defined on all of X such that $g = f$ on H and g is continuous. This is called the Tietze extension theorem. First it is well to recall continuity in the context of metric space.

Definition 6.27 *Let (X, d) be a metric space and suppose $f : X \to Y$ is a function where (Y, ρ) is also a metric space. For example, $Y = \mathbb{R}$. Then f is continuous at $x \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(f(x), f(z)) < \varepsilon$ whenever $d(x, z) < \delta$. As is usual in such definitions, f is said to be continuous if it is continuous at every point of X.*

The following lemma gives an important example of a continuous real valued function defined on a metric space, (X, d).

Lemma 6.28 *Let (X, d) be a metric space and let $S \subseteq X$ be a nonempty subset. Define*

$$\operatorname{dist}(x, S) \equiv \inf\{d(x, y) : y \in S\}.$$

Then $x \to \text{dist}(x, S)$ is a continuous function satisfying the inequality,

$$|\text{dist}(x, S) - \text{dist}(y, S)| \leq d(x, y). \tag{6.7}$$

Proof: The continuity of $x \to \text{dist}(x, S)$ is obvious if the inequality 6.7 is established. So let $x, y \in X$. Without loss of generality, assume $\text{dist}(x, S) \geq \text{dist}(y, S)$ and pick $z \in S$ such that $d(y, z) - \varepsilon < \text{dist}(y, S)$. Then

$$\begin{aligned} |\text{dist}(x, S) - \text{dist}(y, S)| &= \text{dist}(x, S) - \text{dist}(y, S) \leq d(x, z) - (d(y, z) - \varepsilon) \\ &\leq d(z, y) + d(x, y) - d(y, z) + \varepsilon = d(x, y) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves 6.7.

Lemma 6.29 *Let H, K be two nonempty disjoint closed subsets of a metric space, (X, d). Then there exists a continuous function, $g : X \to [-1, 1]$ such that $g(H) = -1/3$, $g(K) = 1/3, g(X) \subseteq [-1/3, 1/3]$.*

Proof: Let

$$f(x) \equiv \frac{\text{dist}(x, H)}{\text{dist}(x, H) + \text{dist}(x, K)}.$$

The denominator is never equal to zero because if $\text{dist}(x, H) = 0$, then $x \in H$ becasue H is closed. (To see this, pick $h_k \in B(x, 1/k) \cap H$. Then $h_k \to x$ and since H is closed, $x \in H$.) Similarly, if $\text{dist}(x, K) = 0$, then $x \in K$ and so the denominator is never zero as claimed. Hence, by Lemma 6.28, f is continuous and from its definition, $f = 0$ on H and $f = 1$ on K. Now let $g(x) \equiv \frac{2}{3}\left(h(x) - \frac{1}{2}\right)$. Then g has the desired properties.

Definition 6.30 *For f a real or complex valued bounded continuous function defined on a metric space, M*

$$||f||_M \equiv \sup\{|f(x)| : x \in M\}.$$

Lemma 6.31 *Suppose M is a closed set in X where (X, d) is a metric space and suppose $f : M \to [-1, 1]$ is continuous at every point of M. Then there exists a function, g which is defined and continuous on all of X such that $||f - g||_M < \frac{2}{3}$.*

Proof: Let $H = f^{-1}([-1, -1/3]), K = f^{-1}([1/3, 1])$. Thus H and K are disjoint closed subsets of M. Suppose first H, K are both nonempty. Then by Lemma 6.29 there exists g such that g is a continuous function defined on all of X and $g(H) = -1/3$, $g(K) = 1/3$, and $g(X) \subseteq [-1/3, 1/3]$. It follows $||f - g||_M < 2/3$. If $H = \emptyset$, then f has all its values in $[-1/3, 1]$ and so letting $g \equiv 1/3$, the desired condition is obtained. If $K = \emptyset$, let $g \equiv -1/3$. This proves the lemma.

Lemma 6.32 *Suppose M is a closed set in X where (X, d) is a metric space and suppose $f : M \to [-1, 1]$ is continuous at every point of M. Then there exists a function, g which is defined and continuous on all of X such that $g = f$ on M and g has its values in $[-1, 1]$.*

Proof: Let g_1 be such that $g_1(X) \subseteq [-1/3, 1/3]$ and $||f - g_1||_M \leq \frac{2}{3}$. Suppose $g_1, \cdots, g_m$ have been chosen such that $g_j(X) \subseteq [-1/3, 1/3]$ and

$$\left|\left| f - \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} g_i \right|\right|_M < \left(\frac{2}{3}\right)^m . \tag{6.8}$$

Then

$$\left|\left| \left(\frac{3}{2}\right)^m \left(f - \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} g_i \right) \right|\right|_M \leq 1$$

and so $\left(\frac{3}{2}\right)^m \left(f - \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} g_i \right)$ can play the role of f in the first step of the proof. Therefore, there exists g_{m+1} defined and continuous on all of X such that its values are in $[-1/3, 1/3]$ and

$$\left|\left| \left(\frac{3}{2}\right)^m \left(f - \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} g_i \right) - g_{m+1} \right|\right|_M \leq \frac{2}{3}.$$

Hence

$$\left|\left| \left(f - \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} g_i \right) - \left(\frac{2}{3}\right)^m g_{m+1} \right|\right|_M \leq \left(\frac{2}{3}\right)^{m+1} .$$

It follows there exists a sequence, $\{g_i\}$ such that each has its values in $[-1/3, 1/3]$ and for every m 6.8 holds. Then let

$$g(x) \equiv \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} g_i(x) .$$

It follows

$$|g(x)| \leq \left| \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} g_i(x) \right| \leq \sum_{i=1}^{m} \left(\frac{2}{3}\right)^{i-1} \frac{1}{3} \leq 1$$

and since convergence is uniform, g must be continuous. The estimate 6.8 implies $f = g$ on M.

The following is the Tietze extension theorem.

Theorem 6.33 *Let M be a closed nonempty subset of a metric space (X, d) and let $f : M \to [a, b]$ is continuous at every point of M. Then there exists a function, g continuous on all of X which coincides with f on M such that $g(X) \subseteq [a, b]$.*

Proof: Let $f_1(x) = 1 + \frac{2}{b-a}(f(x) - b)$. Then f_1 satisfies the conditions of Lemma 6.32 and so there exists $g_1 : X \to [-1, 1]$ such that g is continuous on X and equals f_1 on M. Let $g(x) = (g_1(x) - 1)\left(\frac{b-a}{2}\right) + b$. This works.

6.6 General Topological Spaces

It turns out that metric spaces are not sufficiently general for some applications. This section is a brief introduction to general topology. In making this generalization, the properties of balls which are the conclusion of Theorem 6.4 on Page 134 are stated as axioms for a subset of the power set of a given set which will be known as a basis for the topology. More can be found in [35] and the references listed there.

Definition 6.34 *Let X be a nonempty set and suppose $\mathcal{B} \subseteq \mathcal{P}(X)$. Then $\mathcal{B}$ is a basis for a topology if it satisfies the following axioms.*

1.) Whenever $p \in A \cap B$ for $A, B \in \mathcal{B}$, it follows there exists $C \in \mathcal{B}$ such that $p \in C \subseteq A \cap B$.

2.) $\cup \mathcal{B} = X$.

Then a subset, U, of X is an open set if for every point, $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Thus the open sets are exactly those which can be obtained as a union of sets of $\mathcal{B}$. Denote these subsets of X by the symbol τ and refer to τ as the topology or the set of open sets.

Note that this is simply the analog of saying a set is open exactly when every point is an interior point.

Proposition 6.35 *Let X be a set and let $\mathcal{B}$ be a basis for a topology as defined above and let τ be the set of open sets determined by $\mathcal{B}$. Then*

$$\emptyset \in \tau, \ X \in \tau, \tag{6.9}$$

$$\text{If}\, \mathcal{C} \subseteq \tau, \text{ then } \cup \mathcal{C} \in \tau \tag{6.10}$$

$$\text{If } A, B \in \tau, \text{ then } A \cap B \in \tau. \tag{6.11}$$

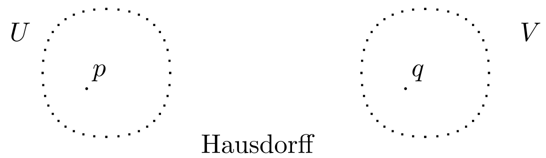
Proof: If $p \in \emptyset$ then there exists $B \in \mathcal{B}$ such that $p \in B \subseteq \emptyset$ because there are no points in $\emptyset$. Therefore, $\emptyset \in \tau$. Now if $p \in X$, then by part 2.) of Definition 6.34 $p \in B \subseteq X$ for some $B \in \mathcal{B}$ and so $X \in \tau$.

If $\mathcal{C} \subseteq \tau$, and if $p \in \cup \mathcal{C}$, then there exists a set, $B \in \mathcal{C}$ such that $p \in B$. However, B is itself a union of sets from $\mathcal{B}$ and so there exists $C \in \mathcal{B}$ such that $p \in C \subseteq B \subseteq \cup \mathcal{C}$. This verifies 6.10.

Finally, if $A, B \in \tau$ and $p \in A \cap B$, then since A and B are themselves unions of sets of $\mathcal{B}$, it follows there exists $A_1, B_1 \in \mathcal{B}$ such that $A_1 \subseteq A, B_1 \subseteq B$, and $p \in A_1 \cap B_1$. Therefore, by 1.) of Definition 6.34 there exists $C \in \mathcal{B}$ such that $p \in C \subseteq A_1 \cap B_1 \subseteq A \cap B$, showing that $A \cap B \in \tau$ as claimed. Of course if $A \cap B = \emptyset$, then $A \cap B \in \tau$. This proves the proposition.

Definition 6.36 *A set X together with such a collection of its subsets satisfying 6.9-6.11 is called a topological space. τ is called the topology or set of open sets of X.*

Definition 6.37 *A topological space is said to be Hausdorff if whenever p and q are distinct points of X, there exist disjoint open sets U, V such that $p \in U$, $q \in V$. In other words points can be separated with open sets.*



Hausdorff

Definition 6.38 *A subset of a topological space is said to be closed if its complement is open. Let p be a point of X and let $E \subseteq X$. Then p is said to be a limit point of E if every open set containing p contains a point of E distinct from p.*

Note that if the topological space is Hausdorff, then this definition is equivalent to requiring that every open set containing p contains infinitely many points from E. Why?

Theorem 6.39 *A subset, E, of X is closed if and only if it contains all its limit points.*

Proof: Suppose first that E is closed and let x be a limit point of E. Is $x \in E$? If $x \notin E$, then E^C is an open set containing x which contains no points of E, a contradiction. Thus $x \in E$.

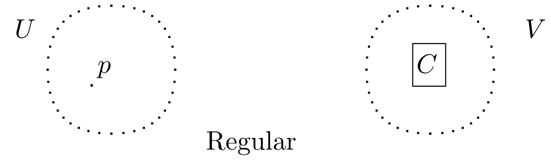
Now suppose E contains all its limit points. Is the complement of E open? If $x \in E^C$, then x is not a limit point of E because E has all its limit points and so there exists an open set, U containing x such that U contains no point of E other than x. Since $x \notin E$, it follows that $x \in U \subseteq E^C$ which implies E^C is an open set because this shows E^C is the union of open sets.

Theorem 6.40 *If (X, τ) is a Hausdorff space and if $p \in X$, then $\{p\}$ is a closed set.*

Proof: If $x \neq p$, there exist open sets U and V such that $x \in U, p \in V$ and $U \cap V = \emptyset$. Therefore, $\{p\}^C$ is an open set so $\{p\}$ is closed.

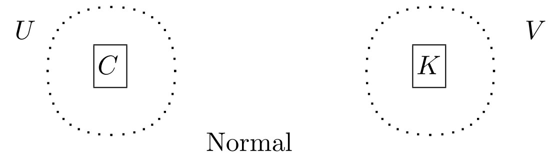
Note that the Hausdorff axiom was stronger than needed in order to draw the conclusion of the last theorem. In fact it would have been enough to assume that if $x \neq y$, then there exists an open set containing x which does not intersect y.

Definition 6.41 *A topological space (X, τ) is said to be regular if whenever C is a closed set and p is a point not in C, there exist disjoint open sets U and V such that $p \in U$, $C \subseteq V$. Thus a closed set can be separated from a point not in the closed set by two disjoint open sets.*



Regular

Definition 6.42 *The topological space, (X, τ) is said to be normal if whenever C and K are disjoint closed sets, there exist disjoint open sets U and V such that $C \subseteq U$, $K \subseteq V$. Thus any two disjoint closed sets can be separated with open sets.*



Normal

Definition 6.43 *Let E be a subset of X. $\overline{E}$ is defined to be the smallest closed set containing E.*

Lemma 6.44 *The above definition is well defined.*

Proof: Let $\mathcal{C}$ denote all the closed sets which contain E. Then $\mathcal{C}$ is nonempty because $X \in \mathcal{C}$.

$$\left(\cap \{A : A \in \mathcal{C}\}\right)^C = \cup \left\{A^C : A \in \mathcal{C}\right\},$$

an open set which shows that $\cap \mathcal{C}$ is a closed set and is the smallest closed set which contains E.

Theorem 6.45 $\overline{E} = E \cup \{$*limit points of* $E\}$.

Proof: Let $x \in \overline{E}$ and suppose that $x \notin E$. If x is not a limit point either, then there exists an open set, U,containing x which does not intersect E. But then U^C is a closed set which contains E which does not contain x, contrary to the definition that $\overline{E}$ is the intersection of all closed sets containing E. Therefore, x must be a limit point of E after all.

Now $E \subseteq \overline{E}$ so suppose x is a limit point of E. Is $x \in \overline{E}$? If H is a closed set containing E, which does not contain x, then H^C is an open set containing x which contains no points of E other than x negating the assumption that x is a limit point of E.

The following is the definition of continuity in terms of general topological spaces. It is really just a generalization of the ε - δ definition of continuity given in calculus.

Definition 6.46 *Let (X, τ) and (Y, η) be two topological spaces and let $f : X \to Y$. f is continuous at $x \in X$ if whenever V is an open set of Y containing $f(x)$, there exists an open set $U \in \tau$ such that $x \in U$ and $f(U) \subseteq V$. f is continuous if $f^{-1}(V) \in \tau$ whenever $V \in \eta$.*

You should prove the following.

Proposition 6.47 *In the situation of Definition 6.46 f is continuous if and only if f is continuous at every point of X.*

Definition 6.48 *Let (X_i, τ_i) be topological spaces. $\prod_{i=1}^n X_i$ is the Cartesian product. Define a product topology as follows. Let $\mathcal{B} = \prod_{i=1}^n A_i$ where $A_i \in \tau_i$. Then $\mathcal{B}$ is a basis for the product topology.*

Theorem 6.49 *The set $\mathcal{B}$ of Definition 6.48 is a basis for a topology.*

Proof: Suppose $\mathbf{x} \in \prod_{i=1}^n A_i \cap \prod_{i=1}^n B_i$ where A_i and B_i are open sets. Say

$$\mathbf{x} = (x_1, \cdots, x_n).$$

Then $x_i \in A_i \cap B_i$ for each i. Therefore, $\mathbf{x} \in \prod_{i=1}^n A_i \cap B_i \in \mathcal{B}$ and $\prod_{i=1}^n A_i \cap B_i \subseteq \prod_{i=1}^n A_i$.

The definition of compactness is also considered for a general topological space. This is given next.

Definition 6.50 *A subset, E, of a topological space (X, τ) is said to be compact if whenever $\mathcal{C} \subseteq \tau$ and $E \subseteq \cup\mathcal{C}$, there exists a finite subset of $\mathcal{C}$, $\{U_1 \cdots U_n\}$, such that $E \subseteq \cup_{i=1}^n U_i$. (Every open covering admits a finite subcovering.) E is precompact if $\overline{E}$ is compact. A topological space is called locally compact if it has a basis $\mathcal{B}$, with the property that $\overline{B}$ is compact for each $B \in \mathcal{B}$.*

A useful construction when dealing with locally compact Hausdorff spaces is the notion of the one point compactification of the space.

Definition 6.51 *Suppose (X, τ) is a locally compact Hausdorff space. Then let $\widetilde{X} \equiv X \cup \{\infty\}$ where ∞ is just the name of some point which is not in X which is called the point at infinity. A basis for the topology $\widetilde{\tau}$ for $\widetilde{X}$ is*

$$\tau \cup \left\{K^C \text{ where } K \text{ is a compact subset of } X\right\}.$$

The complement is taken with respect to $\widetilde{X}$ and so the open sets, K^C are basic open sets which contain ∞.

The reason this is called a compactification is contained in the next lemma.

Lemma 6.52 *If (X, τ) is a locally compact Hausdorff space, then $\left(\widetilde{X}, \widetilde{\tau}\right)$ is a compact Hausdorff space.*

Proof: Since (X, τ) is a locally compact Hausdorff space, it follows $\left(\widetilde{X}, \widetilde{\tau}\right)$ is a Hausdorff topological space. The only case which needs checking is the one of $p \in X$ and ∞. Since (X, τ) is locally compact, there exists an open set of τ, U having compact closure which contains p. Then $p \in U$ and $\infty \in \overline{U}^C$ and these are disjoint open sets containing the points, p and ∞ respectively. Now let $\mathcal{C}$ be an open cover of $\widetilde{X}$ with sets from $\widetilde{\tau}$. Then ∞ must be in some set, U_∞ from $\mathcal{C}$, which must contain a set of the form K^C where K is a compact subset of X. Then there exist sets from $\mathcal{C}$, $U_1, \cdots, U_r$ which cover K. Therefore, a finite subcover of $\widetilde{X}$ is $U_1, \cdots, U_r, U_\infty$.

In general topological spaces there may be no concept of "bounded". Even if there is, closed and bounded is not necessarily the same as compactness. However, in any Hausdorff space every compact set must be a closed set.

Theorem 6.53 *If (X, τ) is a Hausdorff space, then every compact subset must also be a closed set.*

Proof: Suppose $p \notin K$. For each $x \in X$, there exist open sets, U_x and V_x such that

$$x \in U_x,\ p \in V_x,$$

and

$$U_x \cap V_x = \emptyset.$$

If K is assumed to be compact, there are finitely many of these sets, $U_{x_1}, \cdots, U_{x_m}$ which cover K. Then let $V \equiv \cap_{i=1}^m V_{x_i}$. It follows that V is an open set containing p which has empty intersection with each of the U_{x_i}. Consequently, V contains no points of K and is therefore not a limit point of K. This proves the theorem.

Definition 6.54 *If every finite subset of a collection of sets has nonempty intersection, the collection has the finite intersection property.*

Theorem 6.55 *Let $\mathcal{K}$ be a set whose elements are compact subsets of a Hausdorff topological space, (X, τ). Suppose $\mathcal{K}$ has the finite intersection property. Then $\emptyset \neq \cap\mathcal{K}$.*

Proof: Suppose to the contrary that $\emptyset = \cap\mathcal{K}$. Then consider

$$\mathcal{C} \equiv \left\{K^C : K \in \mathcal{K}\right\}.$$

It follows $\mathcal{C}$ is an open cover of K_0 where K_0 is any particular element of $\mathcal{K}$. But then there are finitely many $K \in \mathcal{K}$, $K_1, \cdots, K_r$ such that $K_0 \subseteq \cup_{i=1}^r K_i^C$ implying that $\cap_{i=0}^r K_i = \emptyset$, contradicting the finite intersection property.

Lemma 6.56 *Let (X, τ) be a topological space and let $\mathcal{B}$ be a basis for τ. Then K is compact if and only if every open cover of basic open sets admits a finite subcover.*

Proof: Suppose first that X is compact. Then if $\mathcal{C}$ is an open cover consisting of basic open sets, it follows it admits a finite subcover because these are open sets in $\mathcal{C}$.

Next suppose that every basic open cover admits a finite subcover and let $\mathcal{C}$ be an open cover of X. Then define $\widetilde{\mathcal{C}}$ to be the collection of basic open sets which are contained in some set of $\mathcal{C}$. It follows $\widetilde{\mathcal{C}}$ is a basic open cover of X and so it admits a finite subcover, $\{U_1, \cdots, U_p\}$. Now each U_i is contained in an open set of $\mathcal{C}$. Let O_i be a set of $\mathcal{C}$ which contains U_i. Then $\{O_1, \cdots, O_p\}$ is an open cover of X. This proves the lemma.

In fact, much more can be said than Lemma 6.56. However, this is all which I will present here.

6.7 Connected Sets

Stated informally, connected sets are those which are in one piece. More precisely,

Definition 6.57 *A set, S in a general topological space is separated if there exist sets, A, B such that*

$$S = A \cup B, \ A, B \neq \emptyset, \ \text{and } \overline{A} \cap B = \overline{B} \cap A = \emptyset.$$

In this case, the sets A and B are said to separate S. A set is connected if it is not separated.

One of the most important theorems about connected sets is the following.

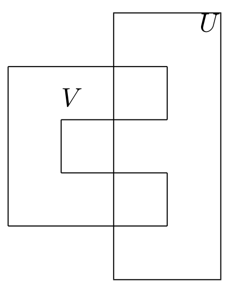
Theorem 6.58 *Suppose U and V are connected sets having nonempty intersection. Then $U \cup V$ is also connected.*

Proof: Suppose $U \cup V = A \cup B$ where $\overline{A} \cap B = \overline{B} \cap A = \emptyset$. Consider the sets, $A \cap U$ and $B \cup U$. Since

$$\overline{(A \cap U)} \cap (B \cap U) = (A \cap U) \cap \left(\overline{B \cap U}\right) = \emptyset,$$

It follows one of these sets must be empty since otherwise, U would be separated. It follows that U is contained in either A or B. Similarly, V must be contained in either A or B. Since U and V have nonempty intersection, it follows that both V and U are contained in one of the sets, A, B. Therefore, the other must be empty and this shows $U \cup V$ cannot be separated and is therefore, connected.

The intersection of connected sets is not necessarily connected as is shown by the following picture.



Theorem 6.59 *Let $f : X \to Y$ be continuous where X and Y are topological spaces and X is connected. Then $f(X)$ is also connected.*

Proof: To do this you show $f(X)$ is not separated. Suppose to the contrary that $f(X) = A \cup B$ where A and B separate $f(X)$. Then consider the sets, $f^{-1}(A)$ and $f^{-1}(B)$. If $z \in f^{-1}(B)$, then $f(z) \in B$ and so $f(z)$ is not a limit point of A. Therefore, there exists an open set, U containing $f(z)$ such that $U \cap A = \emptyset$. But then, the continuity of f implies that $f^{-1}(U)$ is an open set containing z such that $f^{-1}(U) \cap f^{-1}(A) = \emptyset$. Therefore, $f^{-1}(B)$ contains no limit points of $f^{-1}(A)$. Similar reasoning implies $f^{-1}(A)$ contains no limit points of $f^{-1}(B)$. It follows that X is separated by $f^{-1}(A)$ and $f^{-1}(B)$, contradicting the assumption that X was connected.

An arbitrary set can be written as a union of maximal connected sets called connected components. This is the concept of the next definition.

Definition 6.60 *Let S be a set and let $p \in S$. Denote by C_p the union of all connected subsets of S which contain p. This is called the connected component determined by p.*

Theorem 6.61 *Let C_p be a connected component of a set S in a general topological space. Then C_p is a connected set and if $C_p \cap C_q \neq \emptyset$, then $C_p = C_q$.*

Proof: Let $\mathcal{C}$ denote the connected subsets of S which contain p. If $C_p = A \cup B$ where

$$\overline{A} \cap B = \overline{B} \cap A = \emptyset,$$

then p is in one of A or B. Suppose without loss of generality $p \in A$. Then every set of $\mathcal{C}$ must also be contained in A also since otherwise, as in Theorem 6.58, the set would be separated. But this implies B is empty. Therefore, C_p is connected. From this, and Theorem 6.58, the second assertion of the theorem is proved.

This shows the connected components of a set are equivalence classes and partition the set.

A set, I is an interval in $\mathbb{R}$ if and only if whenever $x, y \in I$ then $(x, y) \subseteq I$. The following theorem is about the connected sets in $\mathbb{R}$.

Theorem 6.62 *A set, C in $\mathbb{R}$ is connected if and only if C is an interval.*

Proof: Let C be connected. If C consists of a single point, p, there is nothing to prove. The interval is just $[p, p]$. Suppose $p < q$ and $p, q \in C$. You need to show $(p, q) \subseteq C$. If

$$x \in (p, q) \setminus C$$

let $C \cap (-\infty, x) \equiv A$, and $C \cap (x, \infty) \equiv B$. Then $C = A \cup B$ and the sets, A and B separate C contrary to the assumption that C is connected.

Conversely, let I be an interval. Suppose I is separated by A and B. Pick $x \in A$ and $y \in B$. Suppose without loss of generality that $x < y$. Now define the set,

$$S \equiv \{t \in [x, y] : [x, t] \subseteq A\}$$

and let l be the least upper bound of S. Then $l \in \overline{A}$ so $l \notin B$ which implies $l \in A$. But if $l \notin \overline{B}$, then for some $\delta > 0$,

$$(l, l+\delta) \cap B = \emptyset$$

contradicting the definition of l as an upper bound for S. Therefore, $l \in \overline{B}$ which implies $l \notin A$ after all, a contradiction. It follows I must be connected.

The following theorem is a very useful description of the open sets in $\mathbb{R}$.

Theorem 6.63 *Let U be an open set in $\mathbb{R}$. Then there exist countably many disjoint open sets, $\{(a_i, b_i)\}_{i=1}^{\infty}$ such that $U = \cup_{i=1}^{\infty} (a_i, b_i)$.*

Proof: Let $p \in U$ and let $z \in C_p$, the connected component determined by p. Since U is open, there exists, $\delta > 0$ such that $(z-\delta, z+\delta) \subseteq U$. It follows from Theorem 6.58 that

$$(z-\delta, z+\delta) \subseteq C_p.$$

This shows C_p is open. By Theorem 6.62, this shows C_p is an open interval, (a, b) where $a, b \in [-\infty, \infty]$. There are therefore at most countably many of these connected components because each must contain a rational number and the rational numbers are countable. Denote by $\{(a_i, b_i)\}_{i=1}^{\infty}$ the set of these connected components. This proves the theorem.

Definition 6.64 *A topological space, E is arcwise connected if for any two points, $p, q \in E$, there exists a closed interval, $[a, b]$ and a continuous function, $\gamma : [a, b] \to E$ such that $\gamma(a) = p$ and $\gamma(b) = q$. E is locally connected if it has a basis of connected open sets. E is locally arcwise connected if it has a basis of arcwise connected open sets.*

An example of an arcwise connected topological space would be the any subset of $\mathbb{R}^n$ which is the continuous image of an interval. Locally connected is not the same as connected. A well known example is the following.

$$\left\{ \left(x, \sin \frac{1}{x} \right) : x \in (0, 1] \right\} \cup \{(0, y) : y \in [-1, 1]\} \tag{6.12}$$

You can verify that this set of points considered as a metric space with the metric from $\mathbb{R}^2$ is not locally connected or arcwise connected but is connected.

Proposition 6.65 *If a topological space is arcwise connected, then it is connected.*

Proof: Let X be an arcwise connected space and suppose it is separated. Then $X = A \cup B$ where A, B are two separated sets. Pick $p \in A$ and $q \in B$. Since X is given to be arcwise connected, there must exist a continuous function $\gamma : [a, b] \to X$ such that $\gamma(a) = p$ and $\gamma(b) = q$. But then we would have $\gamma([a, b]) = (\gamma([a, b]) \cap A) \cup (\gamma([a, b]) \cap B)$ and the two sets, $\gamma([a, b]) \cap A$ and $\gamma([a, b]) \cap B$ are separated thus showing that $\gamma([a, b])$ is separated and contradicting Theorem 6.62 and Theorem 6.59. It follows that X must be connected as claimed.

Theorem 6.66 *Let U be an open subset of a locally arcwise connected topological space, X. Then U is arcwise connected if and only if U if connected. Also the connected components of an open set in such a space are open sets, hence arcwise connected.*

Proof: By Proposition 6.65 it is only necessary to verify that if U is connected and open in the context of this theorem, then U is arcwise connected. Pick $p \in U$. Say $x \in U$ satisfies $\mathcal{P}$ if there exists a continuous function, $\gamma : [a, b] \rightarrow U$ such that $\gamma(a) = p$ and $\gamma(b) = x$.

$$A \equiv \{x \in U \text{ such that } x \text{ satisfies } \mathcal{P}.\}$$

If $x \in A$, there exists, according to the assumption that X is locally arcwise connected, an open set, V, containing x and contained in U which is arcwise connected. Thus letting $y \in V$, there exist intervals, $[a, b]$ and $[c, d]$ and continuous functions having values in U, γ, η such that $\gamma(a) = p, \gamma(b) = x, \eta(c) = x$, and $\eta(d) = y$. Then let $\gamma_1 : [a, b + d - c] \rightarrow U$ be defined as

$$\gamma_1(t) \equiv \begin{cases} \gamma(t) & \text{if } t \in [a, b] \\ \eta(t) & \text{if } t \in [b, b + d - c] \end{cases}$$

Then it is clear that γ_1 is a continuous function mapping p to y and showing that $V \subseteq A$. Therefore, A is open. $A \neq \emptyset$ because there is an open set, V containing p which is contained in U and is arcwise connected.

Now consider $B \equiv U \setminus A$. This is also open. If B is not open, there exists a point $z \in B$ such that every open set containing z is not contained in B. Therefore, letting V be one of the basic open sets chosen such that $z \in V \subseteq U$, there exist points of A contained in V. But then, a repeat of the above argument shows $z \in A$ also. Hence B is open and so if $B \neq \emptyset$, then $U = B \cup A$ and so U is separated by the two sets, B and A contradicting the assumption that U is connected.

It remains to verify the connected components are open. Let $z \in C_p$ where C_p is the connected component determined by p. Then picking V an arcwise connected open set which contains z and is contained in U, $C_p \cup V$ is connected and contained in U and so it must also be contained in C_p. This proves the theorem.

As an application, consider the following corollary.

Corollary 6.67 *Let $f : \Omega \rightarrow \mathbb{Z}$ be continuous where Ω is a connected open set. Then f must be a constant.*

Proof: Suppose not. Then it achieves two different values, k and $l \neq k$. Then $\Omega = f^{-1}(l) \cup f^{-1}(\{m \in \mathbb{Z} : m \neq l\})$ and these are disjoint nonempty open sets which separate Ω. To see they are open, note

$$f^{-1}(\{m \in \mathbb{Z} : m \neq l\}) = f^{-1}\left(\cup_{m \neq l}\left(m - \frac{1}{6}, n + \frac{1}{6}\right)\right)$$

which is the inverse image of an open set.

Weierstrass Approximation Theorem

7.1 The Bernstein Polynomials

This short chapter is on the important Weierstrass approximation theorem. It is about approximating an arbitrary continuous function uniformly by a polynomial. It will be assumed only that f has values in $\mathbb{C}$ and that all scalars are in $\mathbb{C}$. First here is some notation.

Definition 7.1 *$\alpha = (\alpha_1, \cdots, \alpha_n)$ for $\alpha_1 \cdots \alpha_n$ positive integers is called a multi-index. For α a multi-index, $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n$ and if $\mathbf{x} \in \mathbb{R}^n$,*

$$\mathbf{x} = (x_1, \cdots, x_n),$$

and f a function, define

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

A polynomial in n variables of degree m is a function of the form

$$p(\mathbf{x}) = \sum_{|\alpha| \leq m} a_\alpha \mathbf{x}^\alpha.$$

Here α is a multi-index as just described.

The following estimate will be the basis for the Weierstrass approximation theorem. It is actually a statement about the variance of a binomial random variable.

Lemma 7.2 *The following estimate holds for $x \in [0,1]$.*

$$\sum_{k=0}^{m} \binom{m}{k} (k - mx)^2 x^k (1-x)^{m-k} \leq \frac{1}{4} m$$

Proof: By the Binomial theorem,

$$\sum_{k=0}^{m}\binom{m}{k}\left(e^{t}x\right)^{k}(1-x)^{m-k}=\left(1-x+e^{t}x\right)^{m}. \tag{7.1}$$

Differentiating both sides with respect to t and then evaluating at $t=0$ yields

$$\sum_{k=0}^{m}\binom{m}{k}kx^{k}(1-x)^{m-k}=mx.$$

Now doing two derivatives of 7.1 with respect to t yields

$$\begin{aligned}\textstyle\sum_{k=0}^{m}\binom{m}{k}k^{2}\left(e^{t}x\right)^{k}(1-x)^{m-k}&=m(m-1)\left(1-x+e^{t}x\right)^{m-2}e^{2t}x^{2}\\&+m\left(1-x+e^{t}x\right)^{m-1}xe^{t}.\end{aligned}$$

Evaluating this at $t=0$,

$$\sum_{k=0}^{m}\binom{m}{k}k^{2}(x)^{k}(1-x)^{m-k}=m(m-1)x^{2}+mx.$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}(k-mx)^{2}x^{k}(1-x)^{m-k}&=m(m-1)x^{2}+mx-2m^{2}x^{2}+m^{2}x^{2}\\&=m\left(x-x^{2}\right)\leq\frac{1}{4}m.\end{aligned}$$

This proves the lemma.

Now for $\mathbf{x}=(x_1,\cdots,x_n)\in[0,1]^n$ consider the polynomial,

$$p_m(\mathbf{x})\equiv\sum_{k_1=1}^{m}\cdots\sum_{k_n=1}^{m}\binom{m}{k_1}\binom{m}{k_2}\cdots\binom{m}{k_n}x_1^{k_1}(1-x_1)^{m-k_1}x_2^{k_2}(1-x_2)^{m-k_2}$$
$$\cdots x_n^{k_n}(1-x_n)^{m-k_n}f\left(\frac{k_1}{m},\cdots,\frac{k_n}{m}\right). \tag{7.2}$$

Also define if I is a set in $\mathbb{R}^n$

$$||h||_I\equiv\sup\{|h(\mathbf{x})|:\mathbf{x}\in I\}.$$

Thus p_m converges uniformly to f on a set, I if

$$\lim_{m\to\infty}||p_m-f||_I=0.$$

Also to simplify the notation, let $\mathbf{k}=(k_1,\cdots,k_n)$ where each $k_i\in[0,m]$, $\frac{\mathbf{k}}{\mathbf{m}}\equiv\left(\frac{k_1}{m},\cdots,\frac{k_n}{m}\right)$, and let

$$\binom{\mathbf{m}}{\mathbf{k}}\equiv\binom{m}{k_1}\binom{m}{k_2}\cdots\binom{m}{k_n}.$$

Also define

$$||\mathbf{k}||_\infty \equiv \max\{k_i, i = 1, 2, \cdot\cdot\cdot, n\}$$

$$\mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \equiv x_1^{k_1}(1-x_1)^{m-k_1} x_2^{k_2}(1-x_2)^{m-k_2}\cdots x_n^{k_n}(1-x_n)^{m-k_n}.$$

Thus in terms of this notation,

$$p_m(\mathbf{x}) = \sum_{||\mathbf{k}||_\infty \le m} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} f\left(\frac{\mathbf{k}}{\mathbf{m}}\right)$$

Lemma 7.3 *For* $\mathbf{x} \in [0,1]^n$, f *a continuous function defined on* $[0,1]^n$, *and* p_m *given in 7.2,* p_m *converges uniformly to* f *on* $[0,1]^n$ *as* $m \to \infty$.

Proof: The function, f is uniformly continuous because it is continuous on a compact set. Therefore, there exists $\delta > 0$ such that if $|\mathbf{x}-\mathbf{y}| < \delta$, then

$$|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon.$$

Denote by G the set of $\mathbf{k}$ such that $(k_i - mx_i)^2 < \eta^2 m^2$ for each i where $\eta = \delta/\sqrt{n}$. Note this condition is equivalent to saying that for each i, $\left|\frac{k_i}{m} - x_i\right| < \eta$. By the binomial theorem,

$$\sum_{||\mathbf{k}||_\infty \le m} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} = 1$$

and so for $\mathbf{x} \in [0,1]^n$,

$$\begin{aligned} |p_m(\mathbf{x}) - f(\mathbf{x})| &\le \sum_{||\mathbf{k}||_\infty \le m} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \left| f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) - f(\mathbf{x})\right| \\ &\le \sum_{\mathbf{k}\in G} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \left| f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) - f(\mathbf{x})\right| \\ &+ \sum_{\mathbf{k}\in G^C} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \left| f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) - f(\mathbf{x})\right| \end{aligned} \tag{7.3}$$

Now for $\mathbf{k} \in G$ it follows that for each i

$$\left|\frac{k_i}{m} - x_i\right| < \frac{\delta}{\sqrt{n}} \tag{7.4}$$

and so $\left|f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) - f(\mathbf{x})\right| < \varepsilon$ because the above implies $\left|\frac{\mathbf{k}}{\mathbf{m}} - \mathbf{x}\right| < \delta$. Therefore, the first sum on the right in 7.3 is no larger than

$$\sum_{\mathbf{k}\in G} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \varepsilon \le \sum_{||\mathbf{k}||_\infty \le m} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}}(\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \varepsilon = \varepsilon.$$

Letting $M \geq \max\{|f(\mathbf{x})| : \mathbf{x} \in [0,1]^n\}$ it follows

$$\begin{aligned}
&|p_m(\mathbf{x}) - f(\mathbf{x})| \\
\leq\ & \varepsilon + 2M \sum_{\mathbf{k}\in G^C} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \\
\leq\ & \varepsilon + 2M \left(\frac{1}{\eta^2 m^2}\right)^n \sum_{\mathbf{k}\in G^C} \binom{\mathbf{m}}{\mathbf{k}} \prod_{j=1}^n (k_j - m x_j)^2 \mathbf{x}^{\mathbf{k}} (\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}} \\
\leq\ & \varepsilon + 2M \left(\frac{1}{\eta^2 m^2}\right)^n \sum_{||\mathbf{k}||_\infty \leq m} \binom{\mathbf{m}}{\mathbf{k}} \prod_{j=1}^n (k_j - m x_j)^2 \mathbf{x}^{\mathbf{k}} (\mathbf{1}-\mathbf{x})^{\mathbf{m}-\mathbf{k}}
\end{aligned}$$

because on G^C,

$$\frac{(k_j - m x_j)^2}{\eta^2 m^2} < 1,\ j = 1, \cdot\cdot\cdot, n.$$

Now by Lemma 7.2,

$$|p_m(\mathbf{x}) - f(\mathbf{x})| \leq \varepsilon + 2M \left(\frac{1}{\eta^2 m^2}\right)^n \left(\frac{m}{4}\right)^n.$$

Therefore, since the right side does not depend on $\mathbf{x}$, it follows

$$\lim\sup_{m\to\infty} ||p_m - f||_{[0,1]^n} \leq \varepsilon$$

and since ε is arbitrary, this shows $\lim_{m\to\infty} ||p_m - f||_{[0,1]^n} = 0$. This proves the lemma.

The following is not surprising.

Lemma 7.4 *Let f be a continuous function defined on $[-M, M]^n$. Then there exists a sequence of polynomials, $\{p_m\}$ converging uniformly to f on $[-M, M]^n$.*

Proof: Let $h(t) = -M + 2Mt$ so $h : [0,1] \to [-M, M]$ and let $\mathbf{h}(\mathbf{t}) \equiv (h(t_1), \cdot\cdot\cdot, h(t_n))$. Therefore, $f \circ \mathbf{h}$ is a continuous function defined on $[0,1]^n$. From Lemma 7.3 there exists a polynomial, $p(\mathbf{t})$ such that $||p_m - f \circ \mathbf{h}||_{[0,1]^n} < \frac{1}{m}$. Now for $\mathbf{x} \in [-M, M]^n$, $\mathbf{h}^{-1}(\mathbf{x}) = (h^{-1}(x_1), \cdot\cdot\cdot, h^{-1}(x_n))$ and so

$$\left|\left|p_m \circ \mathbf{h}^{-1} - f\right|\right|_{[-M,M]^n} = ||p_m - f \circ \mathbf{h}||_{[0,1]^n} < \frac{1}{m}.$$

But $h^{-1}(x) = \frac{x}{2M} + \frac{1}{2}$ and so p_m is still a polynomial. This proves the lemma.

The classical version of the Weierstrass approximation theorem involved showing that a continuous function of one variable defined on a closed and bounded interval is the uniform limit of a sequence of polynomials. This is certainly included as a special case of the above. Now recall the Tietze extension theorem found on Page 146. In the general version about to be presented, the set on which f is defined is just a compact subset of $\mathbb{R}^n$, not the Cartesian product of intervals. For convenience here is the Tietze extension theorem.

Theorem 7.5 *Let M be a closed nonempty subset of a metric space (X, d) and let $f : M \to [a, b]$ is continuous at every point of M. Then there exists a function, g continuous on all of X which coincides with f on M such that $g(X) \subseteq [a, b]$.*

The Weierstrass approximation theorem follows.

Theorem 7.6 *Let K be a compact set in $\mathbb{R}^n$ and let f be a continuous function defined on K. Then there exists a sequence of polynomials $\{p_m\}$ converging uniformly to f on K.*

Proof: Choose M large enough that $K \subseteq [-M, M]^n$ and let $\widetilde{f}$ denote a continuous function defined on all of $[-M, M]^n$ such that $\widetilde{f} = f$ on K. Such an extension exists by the Tietze extension theorem, Theorem 7.5 applied to the real and imaginary parts of f. By Lemma 7.4 there exists a sequence of polynomials, $\{p_m\}$ defined on $[-M, M]^n$ such that $\left|\left|\widetilde{f} - p_m\right|\right|_{[-M,M]^n} \to 0$. Therefore, $\left|\left|\widetilde{f} - p_m\right|\right|_K \to 0$ also. This proves the theorem.

7.2 Stone Weierstrass Theorem

7.2.1 The Case Of Compact Sets

There is a profound generalization of the Weierstrass approximation theorem due to Stone.

Definition 7.7 *$\mathcal{A}$ is an algebra of functions if $\mathcal{A}$ is a vector space and if whenever $f, g \in \mathcal{A}$ then $fg \in \mathcal{A}$.*

To begin with assume that the field of scalars is $\mathbb{R}$. This will be generalized later. Theorem 7.6 implies the following very special case.

Corollary 7.8 *The polynomials are dense in $C([a, b])$.*

The next result is the key to the profound generalization of the Weierstrass theorem due to Stone in which an interval will be replaced by a compact or locally compact set and polynomials will be replaced with elements of an algebra satisfying certain axioms.

Corollary 7.9 *On the interval $[-M, M]$, there exist polynomials p_n such that*

$$p_n(0) = 0$$

and

$$\lim_{n\to\infty} ||p_n - |\cdot|||_\infty = 0.$$

Proof: By Corollary 7.8 there exists a sequence of polynomials, $\{\tilde{p}_n\}$ such that $\tilde{p}_n \to |\cdot|$ uniformly. Then let $p_n(t) \equiv \tilde{p}_n(t) - \tilde{p}_n(0)$. This proves the corollary.

Definition 7.10 *An algebra of functions, $\mathcal{A}$ defined on A, annihilates no point of A if for all $x \in A$, there exists $g \in \mathcal{A}$ such that $g(x) \neq 0$. The algebra separates points if whenever $x_1 \neq x_2$, then there exists $g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$.*

The following generalization is known as the Stone Weierstrass approximation theorem.

Theorem 7.11 *Let A be a compact topological space and let $\mathcal{A} \subseteq C(A;\mathbb{R})$ be an algebra of functions which separates points and annihilates no point. Then $\mathcal{A}$ is dense in $C(A;\mathbb{R})$.*

Proof: First here is a lemma.

Lemma 7.12 *Let c_1 and c_2 be two real numbers and let $x_1 \neq x_2$ be two points of A. Then there exists a function $f_{x_1x_2}$ such that*

$$f_{x_1x_2}(x_1) = c_1,\ f_{x_1x_2}(x_2) = c_2.$$

Proof of the lemma: Let $g \in \mathcal{A}$ satisfy

$$g(x_1) \neq g(x_2).$$

Such a g exists because the algebra separates points. Since the algebra annihilates no point, there exist functions h and k such that

$$h(x_1) \neq 0,\ k(x_2) \neq 0.$$

Then let

$$u \equiv gh - g(x_2)h,\ v \equiv gk - g(x_1)k.$$

It follows that $u(x_1) \neq 0$ and $u(x_2) = 0$ while $v(x_2) \neq 0$ and $v(x_1) = 0$. Let

$$f_{x_1x_2} \equiv \frac{c_1 u}{u(x_1)} + \frac{c_2 v}{v(x_2)}.$$

This proves the lemma. Now continue the proof of Theorem 7.11.

First note that $\overline{\mathcal{A}}$ satisfies the same axioms as $\mathcal{A}$ but in addition to these axioms, $\overline{\mathcal{A}}$ is closed. The closure of $\mathcal{A}$ is taken with respect to the usual norm on $C(A)$,

$$||f||_\infty \equiv \max\{|f(x)| : x \in A\}.$$

Suppose $f \in \overline{\mathcal{A}}$ and suppose M is large enough that

$$||f||_\infty < M.$$

Using Corollary 7.9, let p_n be a sequence of polynomials such that

$$||p_n - |\cdot|||_\infty \to 0,\ p_n(0) = 0.$$

It follows that $p_n \circ f \in \overline{\mathcal{A}}$ and so $|f| \in \overline{\mathcal{A}}$ whenever $f \in \overline{\mathcal{A}}$. Also note that

$$\max(f,g) = \frac{|f-g| + (f+g)}{2}$$

$$\min(f,g) = \frac{(f+g) - |f-g|}{2}.$$

Therefore, this shows that if $f, g \in \overline{\mathcal{A}}$ then

$$\max(f,g),\ \min(f,g) \in \overline{\mathcal{A}}.$$

By induction, if $f_i, i = 1, 2, \cdots, m$ are in $\overline{\mathcal{A}}$ then

$$\max(f_i, i = 1, 2, \cdots, m),\ \min(f_i, i = 1, 2, \cdots, m) \in \overline{\mathcal{A}}.$$

Now let $h \in C(A;\mathbb{R})$ and let $x \in A$. Use Lemma 7.12 to obtain f_{xy}, a function of $\overline{\mathcal{A}}$ which agrees with h at x and y. Letting $\varepsilon > 0$, there exists an open set $U(y)$ containing y such that

$$f_{xy}(z) > h(z) - \varepsilon \text{ if } z \in U(y).$$

Since A is compact, let $U(y_1), \cdots, U(y_l)$ cover A. Let

$$f_x \equiv \max(f_{xy_1}, f_{xy_2}, \cdots, f_{xy_l}).$$

Then $f_x \in \overline{\mathcal{A}}$ and

$$f_x(z) > h(z) - \varepsilon$$

for all $z \in A$ and $f_x(x) = h(x)$. This implies that for each $x \in A$ there exists an open set $V(x)$ containing x such that for $z \in V(x)$,

$$f_x(z) < h(z) + \varepsilon.$$

Let $V(x_1), \cdots, V(x_m)$ cover A and let

$$f \equiv \min(f_{x_1}, \cdots, f_{x_m}).$$

Therefore,

$$f(z) < h(z) + \varepsilon$$

for all $z \in A$ and since $f_x(z) > h(z) - \varepsilon$ for all $z \in A$, it follows

$$f(z) > h(z) - \varepsilon$$

also and so

$$|f(z) - h(z)| < \varepsilon$$

for all z. Since ε is arbitrary, this shows $h \in \overline{\mathcal{A}}$ and proves $\overline{\mathcal{A}} = C(A;\mathbb{R})$. This proves the theorem.

7.2.2 The Case Of Locally Compact Sets

Definition 7.13 *Let (X, τ) be a locally compact Hausdorff space. $C_0(X)$ denotes the space of real or complex valued continuous functions defined on X with the property that if $f \in C_0(X)$, then for each $\varepsilon > 0$ there exists a compact set K such that $|f(x)| < \varepsilon$ for all $x \notin K$. Define*

$$||f||_{\infty} = \sup\{|f(x)| : x \in X\}.$$

Lemma 7.14 *For (X, τ) a locally compact Hausdorff space with the above norm, $C_0(X)$ is a complete space.*

Proof: Let $\left(\widetilde{X}, \widetilde{\tau}\right)$ be the one point compactification described in Lemma 6.52.

$$D \equiv \left\{f \in C\left(\widetilde{X}\right) : f(\infty) = 0\right\}.$$

Then D is a closed subspace of $C\left(\widetilde{X}\right)$. For $f \in C_0(X)$,

$$\widetilde{f}(x) \equiv \begin{cases} f(x) \text{ if } x \in X \\ 0 \text{ if } x = \infty \end{cases}$$

and let $\theta : C_0(X) \to D$ be given by $\theta f = \widetilde{f}$. Then θ is one to one and onto and also satisfies $||f||_{\infty} = ||\theta f||_{\infty}$. Now D is complete because it is a closed subspace of a complete space and so $C_0(X)$ with $||\cdot||_{\infty}$ is also complete. This proves the lemma.

The above refers to functions which have values in $\mathbb{C}$ but the same proof works for functions which have values in any complete normed linear space.

In the case where the functions in $C_0(X)$ all have real values, I will denote the resulting space by $C_0(X; \mathbb{R})$ with similar meanings in other cases.

With this lemma, the generalization of the Stone Weierstrass theorem to locally compact sets is as follows.

Theorem 7.15 *Let $\mathcal{A}$ be an algebra of functions in $C_0(X; \mathbb{R})$ where (X, τ) is a locally compact Hausdorff space which separates the points and annihilates no point. Then $\mathcal{A}$ is dense in $C_0(X; \mathbb{R})$.*

Proof: Let $\left(\widetilde{X}, \widetilde{\tau}\right)$ be the one point compactification as described in Lemma 6.52. Let $\widetilde{\mathcal{A}}$ denote all finite linear combinations of the form

$$\left\{\sum_{i=1}^{n} c_i \widetilde{f}_i + c_0 : f \in \mathcal{A},\ c_i \in \mathbb{R}\right\}$$

where for $f \in C_0(X; \mathbb{R})$,

$$\widetilde{f}(x) \equiv \begin{cases} f(x) \text{ if } x \in X \\ 0 \text{ if } x = \infty \end{cases}.$$

Then $\widetilde{\mathcal{A}}$ is obviously an algebra of functions in $C\left(\widetilde{X};\mathbb{R}\right)$. It separates points because this is true of $\mathcal{A}$. Similarly, it annihilates no point because of the inclusion of c_0 an arbitrary element of $\mathbb{R}$ in the definition above. Therefore from Theorem 7.11, $\widetilde{\mathcal{A}}$ is dense in $C\left(\widetilde{X};\mathbb{R}\right)$. Letting $f \in C_0(X;\mathbb{R})$, it follows $\widetilde{f} \in C\left(\widetilde{X};\mathbb{R}\right)$ and so there exists a sequence $\{h_n\} \subseteq \widetilde{\mathcal{A}}$ such that h_n converges uniformly to $\widetilde{f}$. Now h_n is of the form $\sum_{i=1}^{n} c_i^n \widetilde{f_i^n} + c_0^n$ and since $\widetilde{f}(\infty) = 0$, you can take each $c_0^n = 0$ and so this has shown the existence of a sequence of functions in $\mathcal{A}$ such that it converges uniformly to f. This proves the theorem.

7.2.3 The Case Of Complex Valued Functions

What about the general case where $C_0(X)$ consists of complex valued functions and the field of scalars is $\mathbb{C}$ rather than $\mathbb{R}$? The following is the version of the Stone Weierstrass theorem which applies to this case. You have to assume that for $f \in \mathcal{A}$ it follows $\overline{f} \in \mathcal{A}$. Such an algebra is called self adjoint.

Theorem 7.16 *Suppose $\mathcal{A}$ is an algebra of functions in $C_0(X)$, where X is a locally compact Hausdorff space, which separates the points, annihilates no point, and has the property that if $f \in \mathcal{A}$, then $\overline{f} \in \mathcal{A}$. Then $\mathcal{A}$ is dense in $C_0(X)$.*

Proof: Let $\operatorname{Re}\mathcal{A} \equiv \{\operatorname{Re} f : f \in \mathcal{A}\}$, $\operatorname{Im}\mathcal{A} \equiv \{\operatorname{Im} f : f \in \mathcal{A}\}$. First I will show that $\mathcal{A} = \operatorname{Re}\mathcal{A} + i\operatorname{Im}\mathcal{A} = \operatorname{Im}\mathcal{A} + i\operatorname{Re}\mathcal{A}$. Let $f \in \mathcal{A}$. Then

$$f = \frac{1}{2}\left(f + \overline{f}\right) + \frac{1}{2}\left(f - \overline{f}\right) = \operatorname{Re} f + i\operatorname{Im} f \in \operatorname{Re}\mathcal{A} + i\operatorname{Im}\mathcal{A}$$

and so $\mathcal{A} \subseteq \operatorname{Re}\mathcal{A} + i\operatorname{Im}\mathcal{A}$. Also

$$f = \frac{1}{2i}\left(if + i\overline{f}\right) - \frac{i}{2}\left(if + \overline{(if)}\right) = \operatorname{Im}(if) + i\operatorname{Re}(if) \in \operatorname{Im}\mathcal{A} + i\operatorname{Re}\mathcal{A}$$

This proves one half of the desired equality. Now suppose $h \in \operatorname{Re}\mathcal{A} + i\operatorname{Im}\mathcal{A}$. Then $h = \operatorname{Re} g_1 + i\operatorname{Im} g_2$ where $g_i \in \mathcal{A}$. Then since $\operatorname{Re} g_1 = \frac{1}{2}\left(g_1 + \overline{g_1}\right)$, it follows $\operatorname{Re} g_1 \in \mathcal{A}$. Similarly $\operatorname{Im} g_2 \in \mathcal{A}$. Therefore, $h \in \mathcal{A}$. The case where $h \in \operatorname{Im}\mathcal{A} + i\operatorname{Re}\mathcal{A}$ is similar. This establishes the desired equality.

Now $\operatorname{Re}\mathcal{A}$ and $\operatorname{Im}\mathcal{A}$ are both real algebras. I will show this now. First consider $\operatorname{Im}\mathcal{A}$. It is obvious this is a real vector space. It only remains to verify that the product of two functions in $\operatorname{Im}\mathcal{A}$ is in $\operatorname{Im}\mathcal{A}$. Note that from the first part, $\operatorname{Re}\mathcal{A}, \operatorname{Im}\mathcal{A}$ are both subsets of $\mathcal{A}$ because, for example, if $u \in \operatorname{Im}\mathcal{A}$ then $u + 0 \in \operatorname{Im}\mathcal{A} + i\operatorname{Re}\mathcal{A} = \mathcal{A}$. Therefore, if $v, w \in \operatorname{Im}\mathcal{A}$, both iv and w are in $\mathcal{A}$ and so $\operatorname{Im}(ivw) = vw$ and $ivw \in \mathcal{A}$. Similarly, $\operatorname{Re}\mathcal{A}$ is an algebra.

Both $\operatorname{Re}\mathcal{A}$ and $\operatorname{Im}\mathcal{A}$ must separate the points. Here is why: If $x_1 \neq x_2$, then there exists $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$. If $\operatorname{Im} f(x_1) \neq \operatorname{Im} f(x_2)$, this shows there is a function in $\operatorname{Im}\mathcal{A}$, $\operatorname{Im} f$ which separates these two points. If $\operatorname{Im} f$ fails to separate the two points, then $\operatorname{Re} f$ must separate the points and so you could consider $\operatorname{Im}(if)$ to get a function in $\operatorname{Im}\mathcal{A}$ which separates these points. This shows $\operatorname{Im}\mathcal{A}$ separates the points. Similarly $\operatorname{Re}\mathcal{A}$ separates the points.

Neither $\operatorname{Re}\mathcal{A}$ nor $\operatorname{Im}\mathcal{A}$ annihilate any point. This is easy to see because if x is a point there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$. Thus either $\operatorname{Re} f(x) \neq 0$ or $\operatorname{Im} f(x) \neq 0$. If $\operatorname{Im} f(x) \neq 0$, this shows this point is not annihilated by $\operatorname{Im}\mathcal{A}$. If $\operatorname{Im} f(x) = 0$, consider $\operatorname{Im}(if)(x) = \operatorname{Re} f(x) \neq 0$. Similarly, $\operatorname{Re}\mathcal{A}$ does not annihilate any point.

It follows from Theorem 7.15 that $\operatorname{Re}\mathcal{A}$ and $\operatorname{Im}\mathcal{A}$ are dense in the real valued functions of $C_0(X)$. Let $f \in C_0(X)$. Then there exists $\{h_n\} \subseteq \operatorname{Re}\mathcal{A}$ and $\{g_n\} \subseteq \operatorname{Im}\mathcal{A}$ such that $h_n \to \operatorname{Re} f$ uniformly and $g_n \to \operatorname{Im} f$ uniformly. Therefore, $h_n + ig_n \in \mathcal{A}$ and it converges to f uniformly. This proves the theorem.

7.3 Exercises

1. Let $(X,\tau), (Y,\eta)$ be topological spaces and let $A \subseteq X$ be compact. Then if $f : X \to Y$ is continuous, show that $f(A)$ is also compact.

2. $\uparrow$ In the context of Problem 1, suppose $\mathbb{R} = Y$ where the usual topology is placed on $\mathbb{R}$. Show f achieves its maximum and minimum on A.

3. Let V be an open set in $\mathbb{R}^n$. Show there is an increasing sequence of compact sets, K_m, such that $V = \cup_{m=1}^{\infty} K_m$. **Hint:** Let
$$C_m \equiv \left\{ \mathbf{x} \in \mathbb{R}^n : \operatorname{dist}\left(\mathbf{x}, V^C\right) \geq \frac{1}{m} \right\}$$
where
$$\operatorname{dist}(\mathbf{x}, S) \equiv \inf\{|\mathbf{y} - \mathbf{x}| \text{ such that } \mathbf{y} \in S\}.$$
Consider $K_m \equiv C_m \cap \overline{B(\mathbf{0}, m)}$.

4. Let $B(X;\mathbb{R}^n)$ be the space of functions $\mathbf{f}$, mapping X to $\mathbb{R}^n$ such that
$$\sup\{|\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X\} < \infty.$$
Show $B(X;\mathbb{R}^n)$ is a complete normed linear space if
$$||\mathbf{f}|| \equiv \sup\{|\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X\}.$$

5. Let $\alpha \in [0,1]$. Define, for X a compact subset of $\mathbb{R}^p$,
$$C^{\alpha}(X;\mathbb{R}^n) \equiv \{\mathbf{f} \in C(X;\mathbb{R}^n) : \rho_{\alpha}(\mathbf{f}) + ||\mathbf{f}|| \equiv ||\mathbf{f}||_{\alpha} < \infty\}$$
where
$$||\mathbf{f}|| \equiv \sup\{|\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X\}$$
and
$$\rho_{\alpha}(\mathbf{f}) \equiv \sup\{\frac{|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^{\alpha}} : \mathbf{x}, \mathbf{y} \in X,\ \mathbf{x} \neq \mathbf{y}\}.$$
Show that $\left(C^{\alpha}(X;\mathbb{R}^n), ||\cdot||_{\alpha}\right)$ is a complete normed linear space.

6. Let $\{\mathbf{f}_n\}_{n=1}^{\infty} \subseteq C^{\alpha}(X;\mathbb{R}^n)$ where X is a compact subset of $\mathbb{R}^p$ and suppose
$$||\mathbf{f}_n||_{\alpha} \leq M$$
for all n. Show there exists a subsequence, n_k, such that $\mathbf{f}_{n_k}$ converges in $C(X;\mathbb{R}^n)$. The given sequence is called precompact when this happens. (This also shows the embedding of $C^{\alpha}(X;\mathbb{R}^n)$ into $C(X;\mathbb{R}^n)$ is a compact embedding.)

7. Use the general Stone Weierstrass approximation theorem to prove Theorem 7.6.

8. Let (X, d) be a metric space where d is a bounded metric. Let $\mathcal{C}$ denote the collection of closed subsets of X. For $A, B \in \mathcal{C}$, define
$$\rho(A, B) \equiv \inf\{\delta > 0 : A_{\delta} \supseteq B \text{ and } B_{\delta} \supseteq A\}$$
where for a set S,
$$S_{\delta} \equiv \{x : \text{dist}(x, S) \equiv \inf\{d(x, s) : s \in S\} \leq \delta\}.$$
Show $x \to \text{dist}(x, S)$ is continuous and that therefore, S_{δ} is a closed set containing S. Also show that ρ is a metric on $\mathcal{C}$. This is called the Hausdorff metric.

9. $\uparrow$Suppose (X, d) is a compact metric space. Show $(\mathcal{C}, \rho)$ is a complete metric space. **Hint:** Show first that if $W_n \downarrow W$ where W_n is closed, then $\rho(W_n, W) \to 0$. Now let $\{A_n\}$ be a Cauchy sequence in $\mathcal{C}$. Then if $\varepsilon > 0$ there exists N such that when $m, n \geq N$, then $\rho(A_n, A_m) < \varepsilon$. Therefore, for each $n \geq N$,
$$(A_n)_{\varepsilon} \supseteq \overline{\cup_{k=n}^{\infty} A_k}.$$
Let $A \equiv \cap_{n=1}^{\infty} \overline{\cup_{k=n}^{\infty} A_k}$. By the first part, there exists $N_1 > N$ such that for $n \geq N_1$,
$$\rho\left(\overline{\cup_{k=n}^{\infty} A_k}, A\right) < \varepsilon, \text{ and } (A_n)_{\varepsilon} \supseteq \overline{\cup_{k=n}^{\infty} A_k}.$$
Therefore, for such n, $A_{\varepsilon} \supseteq W_n \supseteq A_n$ and $(W_n)_{\varepsilon} \supseteq (A_n)_{\varepsilon} \supseteq A$ because
$$(A_n)_{\varepsilon} \supseteq \overline{\cup_{k=n}^{\infty} A_k} \supseteq A.$$

10. $\uparrow$ Let X be a compact metric space. Show $(\mathcal{C}, \rho)$ is compact. **Hint:** Let $\mathcal{D}_n$ be a 2^{-n} net for X. Let $\mathcal{K}_n$ denote finite unions of sets of the form $\overline{B(p, 2^{-n})}$ where $p \in \mathcal{D}_n$. Show $\mathcal{K}_n$ is a $2^{-(n-1)}$ net for $(\mathcal{C}, \rho)$.

Part II

Real And Abstract Analysis

Abstract Measure And Integration

8.1 σ Algebras

This chapter is on the basics of measure theory and integration. A measure is a real valued mapping from some subset of the power set of a given set which has values in $[0, \infty]$. Many apparently different things can be considered as measures and also there is an integral defined. By discussing this in terms of axioms and in a very abstract setting, many different topics can be considered in terms of one general theory. For example, it will turn out that sums are included as an integral of this sort. So is the usual integral as well as things which are often thought of as being in between sums and integrals.

Let Ω be a set and let $\mathcal{F}$ be a collection of subsets of Ω satisfying

$$\emptyset \in \mathcal{F},\ \Omega \in \mathcal{F}, \tag{8.1}$$

$$E \in \mathcal{F} \text{ implies } E^C \equiv \Omega \setminus E \in \mathcal{F},$$

$$\text{If } \{E_n\}_{n=1}^{\infty} \subseteq \mathcal{F}, \text{ then } \cup_{n=1}^{\infty} E_n \in \mathcal{F}. \tag{8.2}$$

Definition 8.1 *A collection of subsets of a set, Ω, satisfying Formulas 8.1-8.2 is called a σ algebra.*

As an example, let Ω be any set and let $\mathcal{F} = \mathcal{P}(\Omega)$, the set of all subsets of Ω (power set). This obviously satisfies Formulas 8.1-8.2.

Lemma 8.2 *Let $\mathcal{C}$ be a set whose elements are σ algebras of subsets of Ω. Then $\cap\mathcal{C}$ is a σ algebra also.*

Be sure to verify this lemma. It follows immediately from the above definitions but it is important for you to check the details.

Example 8.3 *Let τ denote the collection of all open sets in $\mathbb{R}^n$ and let $\sigma(\tau) \equiv$ intersection of all σ algebras that contain τ. $\sigma(\tau)$ is called the σ algebra of Borel sets . In general, for a collection of sets, Σ, $\sigma(\Sigma)$ is the smallest σ algebra which contains Σ.*

This is a very important σ algebra and it will be referred to frequently as the Borel sets. Attempts to describe a typical Borel set are more trouble than they are worth and it is not easy to do so. Rather, one uses the definition just given in the example. Note, however, that all countable intersections of open sets and countable unions of closed sets are Borel sets. Such sets are called G_δ and F_σ respectively.

Definition 8.4 *Let $\mathcal{F}$ be a σ algebra of sets of Ω and let $\mu : \mathcal{F} \to [0, \infty]$. μ is called a measure if*

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i) \tag{8.3}$$

whenever the E_i are disjoint sets of $\mathcal{F}$. The triple, $(\Omega, \mathcal{F}, \mu)$ is called a measure space and the elements of $\mathcal{F}$ are called the measurable sets. $(\Omega, \mathcal{F}, \mu)$ is a finite measure space when $\mu(\Omega) < \infty$.

The following theorem is the basis for most of what is done in the theory of measure and integration. It is a very simple result which follows directly from the above definition.

Theorem 8.5 *Let $\{E_m\}_{m=1}^{\infty}$ be a sequence of measurable sets in a measure space $(\Omega, \mathcal{F}, \mu)$. Then if $\cdots E_n \subseteq E_{n+1} \subseteq E_{n+2} \subseteq \cdots$,*

$$\mu(\cup_{i=1}^{\infty} E_i) = \lim_{n \to \infty} \mu(E_n) \tag{8.4}$$

and if $\cdots E_n \supseteq E_{n+1} \supseteq E_{n+2} \supseteq \cdots$ and $\mu(E_1) < \infty$, then

$$\mu(\cap_{i=1}^{\infty} E_i) = \lim_{n \to \infty} \mu(E_n). \tag{8.5}$$

Stated more succinctly, $E_k \uparrow E$ implies $\mu(E_k) \uparrow \mu(E)$ and $E_k \downarrow E$ with $\mu(E_1) < \infty$ implies $\mu(E_k) \downarrow \mu(E)$.

Proof: First note that $\cap_{i=1}^{\infty} E_i = (\cup_{i=1}^{\infty} E_i^C)^C \in \mathcal{F}$ so $\cap_{i=1}^{\infty} E_i$ is measurable. Also note that for A and B sets of $\mathcal{F}$, $A \setminus B \equiv \left(A^C \cup B\right)^C \in \mathcal{F}$. To show 8.4, note that 8.4 is obviously true if $\mu(E_k) = \infty$ for any k. Therefore, assume $\mu(E_k) < \infty$ for all k. Thus

$$\mu(E_{k+1} \setminus E_k) + \mu(E_k) = \mu(E_{k+1})$$

and so

$$\mu(E_{k+1} \setminus E_k) = \mu(E_{k+1}) - \mu(E_k).$$

Also,

$$\bigcup_{k=1}^{\infty} E_k = E_1 \cup \bigcup_{k=1}^{\infty} (E_{k+1} \setminus E_k)$$

and the sets in the above union are disjoint. Hence by 8.3,

$$\mu(\cup_{i=1}^{\infty} E_i) = \mu(E_1) + \sum_{k=1}^{\infty} \mu(E_{k+1} \setminus E_k) = \mu(E_1)$$

$$+\sum_{k=1}^{\infty}\mu(E_{k+1})-\mu(E_k)$$

$$=\mu(E_1)+\lim_{n\to\infty}\sum_{k=1}^{n}\mu(E_{k+1})-\mu(E_k)=\lim_{n\to\infty}\mu(E_{n+1}).$$

This shows part 8.4.

To verify 8.5,

$$\mu(E_1)=\mu(\cap_{i=1}^{\infty}E_i)+\mu(E_1\setminus\cap_{i=1}^{\infty}E_i)$$

since $\mu(E_1)<\infty$, it follows $\mu(\cap_{i=1}^{\infty}E_i)<\infty$. Also, $E_1\setminus\cap_{i=1}^{n}E_i\uparrow E_1\setminus\cap_{i=1}^{\infty}E_i$ and so by 8.4,

$$\mu(E_1)-\mu(\cap_{i=1}^{\infty}E_i)=\mu(E_1\setminus\cap_{i=1}^{\infty}E_i)=\lim_{n\to\infty}\mu(E_1\setminus\cap_{i=1}^{n}E_i)$$

$$=\mu(E_1)-\lim_{n\to\infty}\mu(\cap_{i=1}^{n}E_i)=\mu(E_1)-\lim_{n\to\infty}\mu(E_n),$$

Hence, subtracting $\mu(E_1)$ from both sides,

$$\lim_{n\to\infty}\mu(E_n)=\mu(\cap_{i=1}^{\infty}E_i).$$

This proves the theorem.

It is convenient to allow functions to take the value $+\infty$. You should think of $+\infty$, usually referred to as ∞ as something out at the right end of the real line and its only importance is the notion of sequences converging to it. $x_n\to\infty$ exactly when for all $l\in\mathbb{R}$, there exists N such that if $n\geq N$, then

$$x_n>l.$$

This is what it means for a sequence to converge to ∞. Don't think of ∞ as a number. It is just a convenient symbol which allows the consideration of some limit operations more simply. Similar considerations apply to $-\infty$ but this value is not of very great interest. In fact the set of most interest is the complex numbers or some vector space. Therefore, this topic is not considered.

Lemma 8.6 *Let $f:\Omega\to(-\infty,\infty]$ where $\mathcal{F}$ is a σ algebra of subsets of Ω. Then the following are equivalent.*

$$f^{-1}((d,\infty])\in\mathcal{F}\text{ for all finite }d,$$

$$f^{-1}((-\infty,d))\in\mathcal{F}\text{ for all finite }d,$$

$$f^{-1}([d,\infty])\in\mathcal{F}\text{ for all finite }d,$$

$$f^{-1}((-\infty,d])\in\mathcal{F}\text{ for all finite }d,$$

$$f^{-1}((a,b))\in\mathcal{F}\text{ for all }a<b,-\infty<a<b<\infty.$$

Proof: First note that the first and the third are equivalent. To see this, observe

$$f^{-1}([d,\infty]) = \cap_{n=1}^{\infty} f^{-1}((d-1/n,\infty]),$$

and so if the first condition holds, then so does the third.

$$f^{-1}((d,\infty]) = \cup_{n=1}^{\infty} f^{-1}([d+1/n,\infty]),$$

and so if the third condition holds, so does the first.

Similarly, the second and fourth conditions are equivalent. Now

$$f^{-1}((-\infty,d]) = (f^{-1}((d,\infty]))^C$$

so the first and fourth conditions are equivalent. Thus the first four conditions are equivalent and if any of them hold, then for $-\infty < a < b < \infty$,

$$f^{-1}((a,b)) = f^{-1}((-\infty,b)) \cap f^{-1}((a,\infty]) \in \mathcal{F}.$$

Finally, if the last condition holds,

$$f^{-1}\left([d,\infty]\right) = \left(\cup_{k=1}^{\infty} f^{-1}\left((-k+d,d)\right)\right)^C \in \mathcal{F}$$

and so the third condition holds. Therefore, all five conditions are equivalent. This proves the lemma.

This lemma allows for the following definition of a measurable function having values in $(-\infty,\infty]$.

Definition 8.7 *Let $(\Omega,\mathcal{F},\mu)$ be a measure space and let $f:\Omega\to(-\infty,\infty]$. Then f is said to be measurable if any of the equivalent conditions of Lemma 8.6 hold. When the σ algebra, $\mathcal{F}$ equals the Borel σ algebra, $\mathcal{B}$, the function is called Borel measurable. More generally, if $f:\Omega\to X$ where X is a topological space, f is said to be measurable if $f^{-1}(U)\in\mathcal{F}$ whenever U is open.*

Theorem 8.8 *Let f_n and f be functions mapping Ω to $(-\infty,\infty]$ where $\mathcal{F}$ is a σ algebra of measurable sets of Ω. Then if f_n is measurable, and $f(\omega)=\lim_{n\to\infty} f_n(\omega)$, it follows that f is also measurable. (Pointwise limits of measurable functions are measurable.)*

Proof: First is is shown $f^{-1}((a,b))\in\mathcal{F}$. Let $V_m \equiv \left(a+\frac{1}{m}, b-\frac{1}{m}\right)$ and $\overline{V}_m = \left[a+\frac{1}{m}, b-\frac{1}{m}\right]$. Then for all m, $V_m \subseteq (a,b)$ and

$$(a,b) = \cup_{m=1}^{\infty} V_m = \cup_{m=1}^{\infty} \overline{V}_m.$$

Note that $V_m \neq \emptyset$ for all m large enough. Since f is the pointwise limit of f_n,

$$f^{-1}(V_m) \subseteq \{\omega : f_k(\omega)\in V_m \text{ for all } k \text{ large enough}\} \subseteq f^{-1}(\overline{V}_m).$$

You should note that the expression in the middle is of the form

$$\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m).$$

Therefore,

$$f^{-1}((a,b)) = \cup_{m=1}^{\infty} f^{-1}(V_m) \subseteq \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m)$$
$$\subseteq \cup_{m=1}^{\infty} f^{-1}(\overline{V}_m) = f^{-1}((a,b)).$$

It follows $f^{-1}((a,b)) \in \mathcal{F}$ because it equals the expression in the middle which is measurable. This shows f is measurable.

The following theorem considers the case of functions which have values in a metric space. Its proof is similar to the proof of the above.

Theorem 8.9 *Let $\{f_n\}$ be a sequence of measurable functions mapping Ω to (X,d) where (X,d) is a metric space and $(\Omega, \mathcal{F})$ is a measure space. Suppose also that $f(\omega) = \lim_{n\to\infty} f_n(\omega)$ for all ω. Then f is also a measurable function.*

Proof: It is required to show $f^{-1}(U)$ is measurable for all U open. Let

$$V_m \equiv \left\{ x \in U : \text{dist}\left(x, U^C\right) > \frac{1}{m} \right\}.$$

Thus

$$V_m \subseteq \left\{ x \in U : \text{dist}\left(x, U^C\right) \geq \frac{1}{m} \right\}$$

and $V_m \subseteq \overline{V_m} \subseteq V_{m+1}$ and $\cup_m V_m = U$. Then since V_m is open,

$$f^{-1}(V_m) = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m)$$

and so

$$\begin{aligned} f^{-1}(U) &= \cup_{m=1}^{\infty} f^{-1}(V_m) \\ &= \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m) \\ &\subseteq \cup_{m=1}^{\infty} f^{-1}\left(\overline{V_m}\right) = f^{-1}(U) \end{aligned}$$

which shows $f^{-1}(U)$ is measurable.

Now here is a simple observation.

Observation 8.10 *Let $f : \Omega \to X$ where X is some topological space. Suppose*

$$f(\omega) = \sum_{k=1}^{m} x_k \mathcal{X}_{A_k}(\omega)$$

where each $x_k \in X$ and the A_k are disjoint measurable sets. (Such functions are often referred to as simple functions.) Then f is measurable.

Proof: Letting U be open, $f^{-1}(U) = \cup \{A_k : x_k \in U\}$, a finite union of measurable sets.

There is also a very interesting theorem due to Kuratowski [34] which is presented next.

Theorem 8.11 *Let E be a compact metric space and let $(\Omega, \mathcal{F})$ be a measure space. Suppose $\psi : E \times \Omega \to \mathbb{R}$ has the property that $x \to \psi(x,\omega)$ is continuous and $\omega \to \psi(x,\omega)$ is measurable. Then there exists a measurable function, f having values in E such that*

$$\psi(f(\omega), \omega) = \sup_{x \in E} \psi(x, \omega).$$

Furthermore, $\omega \to \psi(f(\omega), \omega)$ is measurable.

Proof: Let C_1 be a 2^{-1} net of E. Suppose $C_1, \cdots, C_m$ have been chosen such that C_k is a 2^{-k} net and $C_{i+1} \supseteq C_i$ for all i. Then consider $E \setminus \cup \left\{B\left(x, 2^{-(m+1)}\right) : x \in C_m\right\}$. If this set is empty, let $C_{m+1} = C_m$. If it is nonempty, let $\{y_i\}_{i=1}^r$ be a $2^{-(m+1)}$ net for this compact set. Then let $C_{m+1} = C_m \cup \{y_i\}_{i=1}^r$. It follows $\{C_m\}_{m=1}^{\infty}$ satisfies C_m is a 2^{-m} net and $C_m \subseteq C_{m+1}$.

Let $\left\{x_k^1\right\}_{k=1}^{m(1)}$ equal C_1. Let

$$A_1^1 \equiv \left\{\omega : \psi\left(x_1^1, \omega\right) = \max_k \psi\left(x_k^1, \omega\right)\right\}$$

For $\omega \in A_1^1$, define $s_1(\omega) \equiv x_1^1$. Next let

$$A_2^1 \equiv \left\{\omega \notin A_1^1 : \psi\left(x_2^1, \omega\right) = \max_k \psi\left(x_k^1, \omega\right)\right\}$$

and let $s_1(\omega) \equiv x_2^1$ on A_2^1. Continue in this way to obtain a simple function, s_1 such that

$$\psi(s_1(\omega), \omega) = \max\{\psi(x, \omega) : x \in C_1\}$$

and s_1 has values in C_1.

Suppose $s_1(\omega), s_2(\omega), \cdots, s_m(\omega)$ are simple functions with the property that if $m > 1$,

$$\begin{aligned} |s_k(\omega) - s_{k+1}(\omega)| &< 2^{-k}, \\ \psi(s_k(\omega), \omega) &= \max\{\psi(x, \omega) : x \in C_k\} \\ & \quad s_k \text{ has values in } C_k \end{aligned}$$

for each $k + 1 \leq m$, only the second and third assertions holding if $m = 1$. Letting $C_m = \{x_k\}_{k=1}^N$, it follows $s_m(\omega)$ is of the form

$$s_m(\omega) = \sum_{k=1}^{N} x_k \mathcal{X}_{A_k}(\omega), \ A_i \cap A_j = \emptyset. \tag{8.6}$$

Denote by $\{y_{1i}\}_{i=1}^{n_1}$ those points of C_{m+1} which are contained in $B(x_1, 2^{-m})$. Letting A_k play the role of Ω in the first step in which s_1 was constructed, for each $\omega \in A_1$ let $s_{m+1}(\omega)$ be a simple function which has one of the values y_{1i} and satisfies

$$\psi(s_{m+1}(\omega), \omega) = \max_{i \leq n_1} \psi(y_{1i}, \omega)$$

for each $\omega \in A_1$. Next let $\{y_{2i}\}_{i=1}^{n_2}$ be those points of C_{m+1} different than $\{y_{1i}\}_{i=1}^{n_1}$ which are contained in $B(x_2, 2^{-m})$. Then define $s_{m+1}(\omega)$ on A_2 to have values taken from $\{y_{2i}\}_{i=1}^{n_2}$ and

$$\psi(s_{m+1}(\omega), \omega) = \max_{i \leq n_2} \psi(y_{2i}, \omega)$$

for each $\omega \in A_2$. Continuing this way defines s_{m+1} on all of Ω and it satisfies

$$|s_m(\omega) - s_{m+1}(\omega)| < 2^{-m} \text{ for all } \omega \in \Omega \tag{8.7}$$

It remains to verify

$$\psi(s_{m+1}(\omega), \omega) = \max\{\psi(x, \omega) : x \in C_{m+1}\}. \tag{8.8}$$

To see this is so, pick $\omega \in \Omega$. Let

$$\max\{\psi(x, \omega) : x \in C_{m+1}\} = \psi(y_j, \omega) \tag{8.9}$$

where $y_j \in C_{m+1}$ and out of all the balls $B(x_l, 2^{-m})$, the first one which contains y_j is $B(x_k, 2^{-m})$. Then by the construction, $s_{m+1}(\omega) = y_j$. This and 8.9 verifies 8.8.

From 8.7 it follows $s_m(\omega)$ converges uniformly on Ω to a measurable function, $f(\omega)$. Then from the construction, $\psi(f(\omega), \omega) \geq \psi(s_m(\omega), \omega)$ for all m and ω. Now pick $\omega \in \Omega$ and let z be such that $\psi(z, \omega) = \max_{x \in E} \psi(x, \omega)$. Letting $y_k \to z$ where $y_k \in C_k$, it follows from continuity of ψ in the first argument that

$$\begin{aligned} \max_{x \in E} \psi(x, \omega) &= \psi(z, \omega) = \lim_{k \to \infty} \psi(y_k, \omega) \\ &\leq \lim_{m \to \infty} \psi(s_m(\omega), \omega) = \psi(f(\omega), \omega) \leq \max_{x \in E} \psi(x, \omega). \end{aligned}$$

To show $\omega \to \psi(f(\omega), \omega)$ is measurable, note that since E is compact, there exists a countable dense subset, D. Then using continuity of ψ in the first argument,

$$\begin{aligned} \psi(f(\omega), \omega) &= \sup_{x \in E} \psi(x, \omega) \\ &= \sup_{x \in D} \psi(x, \omega) \end{aligned}$$

which equals a measurable function of ω because D is countable. This proves the theorem.

Theorem 8.12 *Let $\mathcal{B}$ consist of open cubes of the form*

$$Q_{\mathbf{x}} \equiv \prod_{i=1}^{n} (x_i - \delta, x_i + \delta)$$

where δ is a positive rational number and $\mathbf{x} \in \mathbb{Q}^n$. Then every open set in $\mathbb{R}^n$ can be written as a countable union of open cubes from $\mathcal{B}$. Furthermore, $\mathcal{B}$ is a countable set.

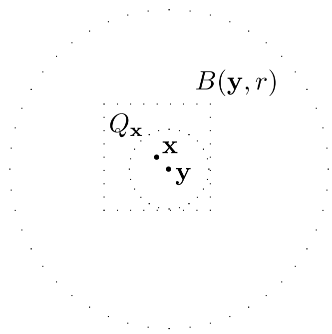
Proof: Let U be an open set and let $\mathbf{y} \in U$. Since U is open, $B(\mathbf{y}, r) \subseteq U$ for some $r > 0$ and it can be assumed $r/\sqrt{n} \in \mathbb{Q}$. Let

$$\mathbf{x} \in B\left(\mathbf{y}, \frac{r}{10\sqrt{n}}\right) \cap \mathbb{Q}^n$$

and consider the cube, $Q_{\mathbf{x}} \in \mathcal{B}$ defined by

$$Q_{\mathbf{x}} \equiv \prod_{i=1}^{n} (x_i - \delta, x_i + \delta)$$

where $\delta = r/4\sqrt{n}$. The following picture is roughly illustrative of what is taking place.



Then the diameter of $Q_{\mathbf{x}}$ equals

$$\left(n\left(\frac{r}{2\sqrt{n}}\right)^2\right)^{1/2} = \frac{r}{2}$$

and so, if $\mathbf{z} \in Q_{\mathbf{x}}$, then

$$\begin{aligned} |\mathbf{z} - \mathbf{y}| &\leq |\mathbf{z} - \mathbf{x}| + |\mathbf{x} - \mathbf{y}| \\ &< \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

Consequently, $Q_{\mathbf{x}} \subseteq U$. Now also,

$$\left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2} < \frac{r}{10\sqrt{n}}$$

and so it follows that for each i,

$$|x_i - y_i| < \frac{r}{4\sqrt{n}}$$

since otherwise the above inequality would not hold. Therefore, $\mathbf{y} \in Q_{\mathbf{x}} \subseteq U$. Now let $\mathcal{B}_U$ denote those sets of $\mathcal{B}$ which are contained in U. Then $\cup \mathcal{B}_U = U$.

To see $\mathcal{B}$ is countable, note there are countably many choices for $\mathbf{x}$ and countably many choices for δ. This proves the theorem.

Recall that $g : \mathbb{R}^n \to \mathbb{R}$ is continuous means g^{-1} (open set) = an open set. In particular $g^{-1}((a,b))$ must be an open set.

Theorem 8.13 *Let $f_i : \Omega \to \mathbb{R}$ for $i = 1, \cdots, n$ be measurable functions and let $g : \mathbb{R}^n \to \mathbb{R}$ be continuous where $\mathbf{f} \equiv (f_1 \cdots f_n)^T$. Then $g \circ \mathbf{f}$ is a measurable function from Ω to $\mathbb{R}$.*

Proof: First it is shown

$$(g \circ \mathbf{f})^{-1}((a,b)) \in \mathcal{F}.$$

Now $(g \circ \mathbf{f})^{-1}((a,b)) = \mathbf{f}^{-1}\left(g^{-1}((a,b))\right)$ and since g is continuous, it follows that $g^{-1}((a,b))$ is an open set which is denoted as U for convenience. Now by Theorem 8.12 above, it follows there are countably many open cubes, $\{Q_k\}$ such that

$$U = \cup_{k=1}^{\infty} Q_k$$

where each Q_k is a cube of the form

$$Q_k = \prod_{i=1}^{n} (x_i - \delta, x_i + \delta).$$

Now

$$\mathbf{f}^{-1}\left(\prod_{i=1}^{n} (x_i - \delta, x_i + \delta)\right) = \cap_{i=1}^{n} f_i^{-1}((x_i - \delta, x_i + \delta)) \in \mathcal{F}$$

and so

$$\begin{aligned} (g \circ \mathbf{f})^{-1}((a,b)) &= \mathbf{f}^{-1}\left(g^{-1}((a,b))\right) = \mathbf{f}^{-1}(U) \\ &= \mathbf{f}^{-1}\left(\cup_{k=1}^{\infty} Q_k\right) = \cup_{k=1}^{\infty} \mathbf{f}^{-1}(Q_k) \in \mathcal{F}. \end{aligned}$$

This proves the theorem.

Corollary 8.14 *Sums, products, and linear combinations of measurable functions are measurable.*

Proof: To see the product of two measurable functions is measurable, let $g(x,y) = xy$, a continuous function defined on $\mathbb{R}^2$. Thus if you have two measurable functions, f_1 and f_2 defined on Ω,

$$g \circ (f_1, f_2)(\omega) = f_1(\omega) f_2(\omega)$$

and so $\omega \to f_1(\omega) f_2(\omega)$ is measurable. Similarly you can show the sum of two measurable functions is measurable by considering $g(x,y) = x + y$ and you can show a linear combination of two measurable functions is measurable by considering $g(x,y) = ax + by$. More than two functions can also be considered as well.

The message of this corollary is that starting with measurable real valued functions you can combine them in pretty much any way you want and you end up with a measurable function.

Here is some notation which will be used whenever convenient.

Definition 8.15 *Let $f : \Omega \to [-\infty, \infty]$. Define*

$$[\alpha < f] \equiv \{\omega \in \Omega : f(\omega) > \alpha\} \equiv f^{-1}((\alpha, \infty])$$

with obvious modifications for the symbols $[\alpha \leq f], [\alpha \geq f], [\alpha \geq f \geq \beta]$, etc.

Definition 8.16 *For a set E,*

$$\mathcal{X}_E(\omega) = \begin{cases} 1 \text{ if } \omega \in E, \\ 0 \text{ if } \omega \notin E. \end{cases}$$

This is called the characteristic function of E. Sometimes this is called the indicator function which I think is better terminology since the term characteristic function has another meaning. Note that this "indicates" whether a point, ω is contained in E. It is exactly when the function has the value 1.

Theorem 8.17 *(Egoroff) Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space,*

$$(\mu(\Omega) < \infty)$$

and let f_n, f be complex valued functions such that $\operatorname{Re} f_n, \operatorname{Im} f_n$ are all measurable and

$$\lim_{n\to\infty} f_n(\omega) = f(\omega)$$

for all $\omega \notin E$ where $\mu(E) = 0$. Then for every $\varepsilon > 0$, there exists a set,

$$F \supseteq E, \ \mu(F) < \varepsilon,$$

such that f_n converges uniformly to f on F^C.

Proof: First suppose $E = \emptyset$ so that convergence is pointwise everywhere. It follows then that $\operatorname{Re} f$ and $\operatorname{Im} f$ are pointwise limits of measurable functions and are therefore measurable. Let $E_{km} = \{\omega \in \Omega : |f_n(\omega) - f(\omega)| \geq 1/m$ for some $n > k\}$. Note that

$$|f_n(\omega) - f(\omega)| = \sqrt{(\operatorname{Re} f_n(\omega) - \operatorname{Re} f(\omega))^2 + (\operatorname{Im} f_n(\omega) - \operatorname{Im} f(\omega))^2}$$

and so, By Theorem 8.13,

$$\left[|f_n - f| \geq \frac{1}{m}\right]$$

is measurable. Hence E_{km} is measurable because

$$E_{km} = \cup_{n=k+1}^{\infty} \left[|f_n - f| \geq \frac{1}{m} \right].$$

For fixed $m, \cap_{k=1}^{\infty} E_{km} = \emptyset$ because f_n converges to f . Therefore, if $\omega \in \Omega$ there exists k such that if $n > k$, $|f_n(\omega) - f(\omega)| < \frac{1}{m}$ which means $\omega \notin E_{km}$. Note also that

$$E_{km} \supseteq E_{(k+1)m}.$$

Since $\mu(E_{1m}) < \infty$, Theorem 8.5 on Page 172 implies

$$0 = \mu(\cap_{k=1}^{\infty} E_{km}) = \lim_{k\to\infty} \mu(E_{km}).$$

Let $k(m)$ be chosen such that $\mu(E_{k(m)m}) < \varepsilon 2^{-m}$ and let

$$F = \bigcup_{m=1}^{\infty} E_{k(m)m}.$$

Then $\mu(F) < \varepsilon$ because

$$\mu(F) \leq \sum_{m=1}^{\infty} \mu\left(E_{k(m)m}\right) < \sum_{m=1}^{\infty} \varepsilon 2^{-m} = \varepsilon$$

Now let $\eta > 0$ be given and pick m_0 such that $m_0^{-1} < \eta$. If $\omega \in F^C$, then

$$\omega \in \bigcap_{m=1}^{\infty} E_{k(m)m}^C.$$

Hence $\omega \in E_{k(m_0)m_0}^C$ so

$$|f_n(\omega) - f(\omega)| < 1/m_0 < \eta$$

for all $n > k(m_0)$. This holds for all $\omega \in F^C$and so f_n converges uniformly to f on F^C.

Now if $E \neq \emptyset$, consider $\{\mathcal{X}_{E^C} f_n\}_{n=1}^{\infty}$. Each $\mathcal{X}_{E^C} f_n$ has real and imaginary parts measurable and the sequence converges pointwise to $\mathcal{X}_E f$ everywhere. Therefore, from the first part, there exists a set of measure less than ε, F such that on $F^C, \{\mathcal{X}_{E^C} f_n\}$ converges uniformly to $\mathcal{X}_{E^C} f$. Therefore, on $(E \cup F)^C$, $\{f_n\}$ converges uniformly to f. This proves the theorem.

Finally here is a comment about notation.

Definition 8.18 *Something happens for μ a.e. ω said as μ almost everywhere, if there exists a set E with $\mu(E) = 0$ and the thing takes place for all $\omega \notin E$. Thus $f(\omega) = g(\omega)$ a.e. if $f(\omega) = g(\omega)$ for all $\omega \notin E$ where $\mu(E) = 0$. A measure space, $(\Omega, \mathcal{F}, \mu)$ is σ finite if there exist measurable sets, Ω_n such that $\mu(\Omega_n) < \infty$ and $\Omega = \cup_{n=1}^{\infty} \Omega_n$.*

8.2 Exercises

1. Let $\Omega = \mathbb{N} = \{1, 2, \cdots\}$. Let $\mathcal{F} = \mathcal{P}(\mathbb{N})$ and let $\mu(S) =$ number of elements in S. Thus $\mu(\{1\}) = 1 = \mu(\{2\}),\ \mu(\{1,2\}) = 2$, etc. Show $(\Omega, \mathcal{F}, \mu)$ is a measure space. It is called counting measure. What functions are measurable in this case?

2. Let Ω be any uncountable set and let $\mathcal{F} = \{A \subseteq \Omega : \text{either } A \text{ or } A^C \text{ is countable}\}$. Let $\mu(A) = 1$ if A is uncountable and $\mu(A) = 0$ if A is countable. Show $(\Omega, \mathcal{F}, \mu)$ is a measure space. This is a well known bad example.

3. Let $\mathcal{F}$ be a σ algebra of subsets of Ω and suppose $\mathcal{F}$ has infinitely many elements. Show that $\mathcal{F}$ is uncountable. **Hint:** You might try to show there exists a countable sequence of disjoint sets of $\mathcal{F}$, $\{A_i\}$. It might be easiest to verify this by contradiction if it doesn't exist rather than a direct construction. Once this has been done, you can define a map, θ, from $\mathcal{P}(\mathbb{N})$ into $\mathcal{F}$ which is one to one by $\theta(S) = \cup_{i \in S} A_i$. Then argue $\mathcal{P}(\mathbb{N})$ is uncountable and so $\mathcal{F}$ is also uncountable.

4. Prove Lemma 8.2.

5. g is Borel measurable if whenever U is open, $g^{-1}(U)$ is Borel. Let $\mathbf{f} : \Omega \to \mathbb{R}^n$ and let $g : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{F}$ is a σ algebra of sets of Ω. Suppose $\mathbf{f}$ is measurable and g is Borel measurable. Show $g \circ \mathbf{f}$ is measurable. To say g is Borel measurable means $g^{-1}(\text{open set}) = (\text{Borel set})$ where a Borel set is one of those sets in the smallest σ algebra containing the open sets of $\mathbb{R}^n$. See Lemma 8.2. **Hint:** You should show, using Theorem 8.12 that $\mathbf{f}^{-1}(\text{open set}) \in \mathcal{F}$. Now let
$$\mathcal{S} \equiv \left\{E \subseteq \mathbb{R}^n : \mathbf{f}^{-1}(E) \in \mathcal{F}\right\}$$
By what you just showed, $\mathcal{S}$ contains the open sets. Now verify $\mathcal{S}$ is a σ algebra. Argue that from the definition of the Borel sets, it follows $\mathcal{S}$ contains the Borel sets.

6. Let $(\Omega, \mathcal{F})$ be a measure space and suppose $f : \Omega \to \mathbb{C}$. Then f is said to be mesurable if
$$f^{-1}(\text{open set}) \in \mathcal{F}.$$
Show f is measurable if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable real-valued functions. Thus it suffices to define a complex valued function to be measurable if the real and imaginary parts are measurable. **Hint:** Argue that $f^{-1}(((a,b) + i(c,d))) = (\operatorname{Re} f)^{-1}((a,b)) \cap (\operatorname{Im} f)^{-1}((c,d))$. Then use Theorem 8.12 to verify that if $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable, it follows f is. Conversely, argue that $(\operatorname{Re} f)^{-1}((a,b)) = f^{-1}((a,b) + i\mathbb{R})$ with a similar formula holding for $\operatorname{Im} f$.

7. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define $\overline{\mu} : \mathcal{P}(\Omega) \to [0, \infty]$ by
$$\overline{\mu}(A) = \inf\{\mu(B) : B \supseteq A,\ B \in \mathcal{F}\}.$$

Show $\overline{\mu}$ satisfies

$$\begin{aligned}\overline{\mu}(\emptyset) &= 0, \text{ if } A \subseteq B,\ \overline{\mu}(A) \leq \overline{\mu}(B), \\ \overline{\mu}(\cup_{i=1}^{\infty} A_i) &\leq \sum_{i=1}^{\infty} \overline{\mu}(A_i),\ \mu(A) = \overline{\mu}(A) \text{ if } A \in \mathcal{F}.\end{aligned}$$

If $\overline{\mu}$ satisfies these conditions, it is called an outer measure. This shows every measure determines an outer measure on the power set.

8. Let $\{E_i\}$ be a sequence of measurable sets with the property that

$$\sum_{i=1}^{\infty} \mu(E_i) < \infty.$$

Let $S = \{\omega \in \Omega$ such that $\omega \in E_i$ for infinitely many values of $i\}$. Show $\mu(S) = 0$ and S is measurable. This is part of the Borel Cantelli lemma. **Hint:** Write S in terms of intersections and unions. Something is in S means that for every n there exists $k > n$ such that it is in E_k. Remember the tail of a convergent series is small.

9. $\uparrow$ Let f_n, f be measurable functions. f_n converges in measure if

$$\lim_{n\to\infty} \mu(x \in \Omega : |f(x) - f_n(x)| \geq \varepsilon) = 0$$

for each fixed $\varepsilon > 0$. Prove the theorem of F. Riesz. If f_n converges to f in measure, then there exists a subsequence $\{f_{n_k}\}$ which converges to f a.e. **Hint:** Choose n_1 such that

$$\mu(x : |f(x) - f_{n_1}(x)| \geq 1) < 1/2.$$

Choose $n_2 > n_1$ such that

$$\mu(x : |f(x) - f_{n_2}(x)| \geq 1/2) < 1/2^2,$$

$n_3 > n_2$ such that

$$\mu(x : |f(x) - f_{n_3}(x)| \geq 1/3) < 1/2^3,$$

etc. Now consider what it means for $f_{n_k}(x)$ to fail to converge to $f(x)$. Then use Problem 8.

8.3 The Abstract Lebesgue Integral

8.3.1 Preliminary Observations

This section is on the Lebesgue integral and the major convergence theorems which are the reason for studying it. In all that follows μ will be a measure defined on a

σ algebra $\mathcal{F}$ of subsets of Ω. $0 \cdot \infty = 0$ is always defined to equal zero. This is a meaningless expression and so it can be defined arbitrarily but a little thought will soon demonstrate that this is the right definition in the context of measure theory. To see this, consider the zero function defined on $\mathbb{R}$. What should the integral of this function equal? Obviously, by an analogy with the Riemann integral, it should equal zero. Formally, it is zero times the length of the set or infinity. This is why this convention will be used.

Lemma 8.19 *Let $f(a,b) \in [-\infty,\infty]$ for $a \in A$ and $b \in B$ where A, B are sets. Then*

$$\sup_{a\in A}\sup_{b\in B} f(a,b) = \sup_{b\in B}\sup_{a\in A} f(a,b).$$

Proof: Note that for all a,b, $f(a,b) \leq \sup_{b\in B}\sup_{a\in A} f(a,b)$ and therefore, for all a,

$$\sup_{b\in B} f(a,b) \leq \sup_{b\in B}\sup_{a\in A} f(a,b).$$

Therefore,

$$\sup_{a\in A}\sup_{b\in B} f(a,b) \leq \sup_{b\in B}\sup_{a\in A} f(a,b).$$

Repeating the same argument interchanging a and b, gives the conclusion of the lemma.

Lemma 8.20 *If $\{A_n\}$ is an increasing sequence in $[-\infty,\infty]$, then* $\sup\{A_n\} = \lim_{n\to\infty} A_n$.

The following lemma is useful also and this is a good place to put it. First $\{b_j\}_{j=1}^{\infty}$ is an enumeration of the a_{ij} if

$$\cup_{j=1}^{\infty}\{b_j\} = \cup_{i,j}\{a_{ij}\}.$$

In other words, the countable set, $\{a_{ij}\}_{i,j=1}^{\infty}$ is listed as $b_1, b_2, \cdots$.

Lemma 8.21 *Let $a_{ij} \geq 0$. Then $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$. Also if $\{b_j\}_{j=1}^{\infty}$ is any enumeration of the a_{ij}, then $\sum_{j=1}^{\infty} b_j = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$.*

Proof: First note there is no trouble in defining these sums because the a_{ij} are all nonnegative. If a sum diverges, it only diverges to ∞ and so ∞ is written as the answer.

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} \geq \sup_n \sum_{j=1}^{\infty}\sum_{i=1}^{n} a_{ij} = \sup_n \lim_{m\to\infty}\sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}$$

$$= \sup_n \lim_{m\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} = \sup_n \sum_{i=1}^{n}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}. \tag{8.10}$$

Interchanging the i and j in the above argument the first part of the lemma is proved.

Finally, note that for all p,

$$\sum_{j=1}^{p} b_j \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$$

and so $\sum_{j=1}^{\infty} b_j \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$. Now let $m, n > 1$ be given. Then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \leq \sum_{j=1}^{p} b_j$$

where p is chosen large enough that $\{b_1, \cdot\cdot\cdot, b_p\} \supseteq \{a_{ij} : i \leq m \text{ and } j \leq n\}$. Therefore, since such a p exists for any choice of m, n,it follows that for any m, n,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \leq \sum_{j=1}^{\infty} b_j.$$

Therefore, taking the limit as $n \to \infty$,

$$\sum_{i=1}^{m} \sum_{j=1}^{\infty} a_{ij} \leq \sum_{j=1}^{\infty} b_j$$

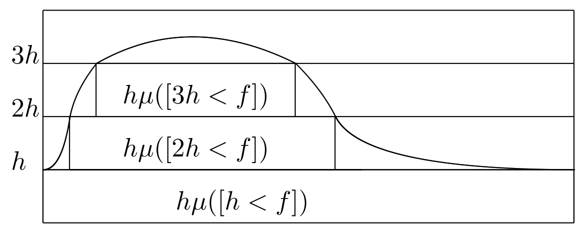
and finally, taking the limit as $m \to \infty$,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \leq \sum_{j=1}^{\infty} b_j$$

proving the lemma.

8.3.2 Definition Of The Lebesgue Integral For Nonnegative Measurable Functions

The following picture illustrates the idea used to define the Lebesgue integral to be like the area under a curve.



You can see that by following the procedure illustrated in the picture and letting h get smaller, you would expect to obtain better approximations to the area under the curve[1] although all these approximations would likely be too small. Therefore, define

$$\int f d\mu \equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu\left([ih < f]\right)$$

Lemma 8.22 *The following inequality holds.*

$$\sum_{i=1}^{\infty} h\mu\left([ih < f]\right) \leq \sum_{i=1}^{\infty} \frac{h}{2}\mu\left(\left[i\frac{h}{2} < f\right]\right).$$

Also, it suffices to consider only h smaller than a given positive number in the above definition of the integral.

Proof:
Let $N \in \mathbb{N}$.

$$\sum_{i=1}^{2N} \frac{h}{2}\mu\left(\left[i\frac{h}{2} < f\right]\right) = \sum_{i=1}^{2N} \frac{h}{2}\mu\left([ih < 2f]\right)$$

$$= \sum_{i=1}^{N} \frac{h}{2}\mu\left([(2i-1)h < 2f]\right) + \sum_{i=1}^{N} \frac{h}{2}\mu\left([(2i)h < 2f]\right)$$

$$= \sum_{i=1}^{N} \frac{h}{2}\mu\left(\left[\frac{(2i-1)}{2}h < f\right]\right) + \sum_{i=1}^{N} \frac{h}{2}\mu\left([ih < f]\right)$$

$$\geq \sum_{i=1}^{N} \frac{h}{2}\mu\left([ih < f]\right) + \sum_{i=1}^{N} \frac{h}{2}\mu\left([ih < f]\right) = \sum_{i=1}^{N} h\mu\left([ih < f]\right).$$

Now letting $N \to \infty$ yields the claim of the lemma.

To verify the last claim, suppose $M < \int f d\mu$ and let $\delta > 0$ be given. Then there exists $h > 0$ such that

$$M < \sum_{i=1}^{\infty} h\mu\left([ih < f]\right) \leq \int f d\mu.$$

By the first part of this lemma,

$$M < \sum_{i=1}^{\infty} \frac{h}{2}\mu\left(\left[i\frac{h}{2} < f\right]\right) \leq \int f d\mu$$

[1] Note the difference between this picture and the one usually drawn in calculus courses where the little rectangles are upright rather than on their sides. This illustrates a fundamental philosophical difference between the Riemann and the Lebesgue integrals. With the Riemann integral intervals are measured. With the Lebesgue integral, it is inverse images of intervals which are measured.

and continuing to apply the first part,

$$M < \sum_{i=1}^{\infty} \frac{h}{2^n} \mu \left(\left[i \frac{h}{2^n} < f \right] \right) \leq \int f d\mu.$$

Choose n large enough that $h/2^n < \delta$. It follows

$$M < \sup_{\delta > h > 0} \sum_{i=1}^{\infty} h\mu \left([ih < f] \right) \leq \int f d\mu.$$

Since M is arbitrary, this proves the last claim.

8.3.3 The Lebesgue Integral For Nonnegative Simple Functions

Definition 8.23 *A function, s, is called simple if it is a measurable real valued function and has only finitely many values. These values will never be $\pm\infty$. Thus a simple function is one which may be written in the form*

$$s(\omega) = \sum_{i=1}^{n} c_i \mathcal{X}_{E_i}(\omega)$$

where the sets, E_i are disjoint and measurable. s takes the value c_i at E_i.

Note that by taking the union of some of the E_i in the above definition, you can assume that the numbers, c_i are the distinct values of s. Simple functions are important because it will turn out to be very easy to take their integrals as shown in the following lemma.

Lemma 8.24 *Let $s(\omega) = \sum_{i=1}^{p} a_i \mathcal{X}_{E_i}(\omega)$ be a nonnegative simple function with the a_i the distinct non zero values of s. Then*

$$\int s d\mu = \sum_{i=1}^{p} a_i \mu(E_i). \tag{8.11}$$

Also, for any nonnegative measurable function, f, if $\lambda \geq 0$, then

$$\int \lambda f d\mu = \lambda \int f d\mu. \tag{8.12}$$

Proof: Consider 8.11 first. Without loss of generality, you can assume $0 < a_1 < a_2 < \cdots < a_p$ and that $\mu(E_i) < \infty$. Let $\varepsilon > 0$ be given and let

$$\delta_1 \sum_{i=1}^{p} \mu(E_i) < \varepsilon.$$

Pick $\delta < \delta_1$ such that for $h < \delta$ it is also true that

$$h < \frac{1}{2} \min (a_1, a_2 - a_1, a_3 - a_2, \cdots, a_n - a_{n-1}).$$

Then for $0 < h < \delta$

$$\begin{aligned}
\sum_{k=1}^{\infty} h\mu([s > kh]) &= \sum_{k=1}^{\infty} h \sum_{i=k}^{\infty} \mu([ih < s \leq (i+1) h]) \\
&= \sum_{i=1}^{\infty} \sum_{k=1}^{i} h\mu([ih < s \leq (i+1) h]) \\
&= \sum_{i=1}^{\infty} ih\mu([ih < s \leq (i+1) h]). \qquad (8.13)
\end{aligned}$$

Because of the choice of h there exist positive integers, i_k such that $i_1 < i_2 < \cdots, < i_p$ and

$$\begin{aligned}
i_1 h &< a_1 \leq (i_1 + 1) h < \cdots < i_2 h < a_2 < \\
&< (i_2 + 1) h < \cdots < i_p h < a_p \leq (i_p + 1) h
\end{aligned}$$

Then in the sum of 8.13 the only terms which are nonzero are those for which $i \in \{i_1, i_2 \cdots, i_p\}$. From the above, you see that

$$\mu([i_k h < s \leq (i_k + 1) h]) = \mu(E_k).$$

Therefore,

$$\sum_{k=1}^{\infty} h\mu([s > kh]) = \sum_{k=1}^{p} i_k h\mu(E_k).$$

It follows that for all h this small,

$$\begin{aligned}
0 &< \sum_{k=1}^{p} a_k \mu(E_k) - \sum_{k=1}^{\infty} h\mu([s > kh]) \\
&= \sum_{k=1}^{p} a_k \mu(E_k) - \sum_{k=1}^{p} i_k h\mu(E_k) \leq h \sum_{k=1}^{p} \mu(E_k) < \varepsilon.
\end{aligned}$$

Taking the inf for h this small and using Lemma 8.22,

$$\begin{aligned}
0 &\leq \sum_{k=1}^{p} a_k \mu(E_k) - \sup_{\delta > h > 0} \sum_{k=1}^{\infty} h\mu([s > kh]) \\
&= \sum_{k=1}^{p} a_k \mu(E_k) - \int s d\mu \leq \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves the first part.

To verify 8.12 Note the formula is obvious if $\lambda = 0$ because then $[ih < \lambda f] = \emptyset$ for all $i > 0$. Assume $\lambda > 0$. Then

$$\begin{aligned} \int \lambda f d\mu &\equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu\left([ih < \lambda f]\right) \\ &= \sup_{h>0} \sum_{i=1}^{\infty} h\mu\left([ih/\lambda < f]\right) \\ &= \sup_{h>0} \lambda \sum_{i=1}^{\infty} (h/\lambda)\, \mu\left([i\,(h/\lambda) < f]\right) \\ &= \lambda \int f d\mu. \end{aligned}$$

This proves the lemma.

Lemma 8.25 *Let the nonnegative simple function, s be defined as*

$$s(\omega) = \sum_{i=1}^{n} c_i \mathcal{X}_{E_i}(\omega)$$

where the c_i are not necessarily distinct but the E_i are disjoint. It follows that

$$\int s = \sum_{i=1}^{n} c_i \mu(E_i).$$

Proof: Let the values of s be $\{a_1, \cdots, a_m\}$. Therefore, since the E_i are disjoint, each a_i equal to one of the c_j. Let $A_i \equiv \cup \{E_j : c_j = a_i\}$. Then from Lemma 8.24 it follows that

$$\begin{aligned} \int s &= \sum_{i=1}^{m} a_i \mu(A_i) = \sum_{i=1}^{m} a_i \sum_{\{j : c_j = a_i\}} \mu(E_j) \\ &= \sum_{i=1}^{m} \sum_{\{j : c_j = a_i\}} c_j \mu(E_j) = \sum_{i=1}^{n} c_i \mu(E_i). \end{aligned}$$

This proves the lemma.

Note that $\int s$ could equal $+\infty$ if $\mu(A_k) = \infty$ and $a_k > 0$ for some k, but $\int s$ is well defined because $s \geq 0$. Recall that $0 \cdot \infty = 0$.

Lemma 8.26 *If $a, b \geq 0$ and if s and t are nonnegative simple functions, then*

$$\int as + bt = a \int s + b \int t.$$

Proof: Let

$$s(\omega)=\sum_{i=1}^{n}\alpha_i\mathcal{X}_{A_i}(\omega),\ t(\omega)=\sum_{i=1}^{m}\beta_j\mathcal{X}_{B_j}(\omega)$$

where α_i are the distinct values of s and the β_j are the distinct values of t. Clearly $as+bt$ is a nonnegative simple function because it is measurable and has finitely many values. Also,

$$(as+bt)(\omega)=\sum_{j=1}^{m}\sum_{i=1}^{n}(a\alpha_i+b\beta_j)\mathcal{X}_{A_i\cap B_j}(\omega)$$

where the sets $A_i\cap B_j$ are disjoint. By Lemma 8.25,

$$\begin{aligned}\int as+bt &= \sum_{j=1}^{m}\sum_{i=1}^{n}(a\alpha_i+b\beta_j)\mu(A_i\cap B_j)\\ &= a\sum_{i=1}^{n}\alpha_i\mu(A_i)+b\sum_{j=1}^{m}\beta_j\mu(B_j)\\ &= a\int s+b\int t.\end{aligned}$$

This proves the lemma.

8.3.4 Simple Functions And Measurable Functions

There is a fundamental theorem about the relationship of simple functions to measurable functions given in the next theorem.

Theorem 8.27 *Let $f\geq 0$ be measurable. Then there exists a sequence of nonnegative simple functions $\{s_n\}$ satisfying*

$$0\leq s_n(\omega) \tag{8.14}$$

$$\cdots s_n(\omega)\leq s_{n+1}(\omega)\cdots$$

$$f(\omega)=\lim_{n\to\infty}s_n(\omega)\ \textit{for all}\ \omega\in\Omega. \tag{8.15}$$

If f is bounded the convergence is actually uniform.

Proof: Letting $I\equiv\{\omega : f(\omega)=\infty\}$, define

$$t_n(\omega)=\sum_{k=0}^{2^n}\frac{k}{n}\mathcal{X}_{[k/n\leq f<(k+1)/n]}(\omega)+n\mathcal{X}_I(\omega).$$

Then $t_n(\omega)\leq f(\omega)$ for all ω and $\lim_{n\to\infty}t_n(\omega)=f(\omega)$ for all ω. This is because $t_n(\omega)=n$ for $\omega\in I$ and if $f(\omega)\in[0,\frac{2^n+1}{n})$, then

$$0\leq f(\omega)-t_n(\omega)\leq\frac{1}{n}. \tag{8.16}$$

Thus whenever $\omega \notin I$, the above inequality will hold for all n large enough. Let

$$s_1 = t_1,\ s_2 = \max(t_1, t_2),\ s_3 = \max(t_1, t_2, t_3), \cdots.$$

Then the sequence $\{s_n\}$ satisfies 8.14-8.15.

To verify the last claim, note that in this case the term $n\mathcal{X}_I(\omega)$ is not present. Therefore, for all n large enough, 8.16 holds for all ω. Thus the convergence is uniform. This proves the theorem.

Although it is not needed here, there is a similar theorem which applies to measurable functions which have values in a separable metric space. In this context, a simple function is one which is of the form

$$\sum_{k=1}^{m} x_k \mathcal{X}_{E_k}(\omega)$$

where the E_k are disjoint measurable sets and the x_k are in X.

Theorem 8.28 *Let $(\Omega, \mathcal{F})$ be a measure space and let $f : \Omega \to X$ where (X, d) is a separable metric space. Then f be a measurable function if and only if there exists a sequence of simple functions,$\{f_n\}$ such that for each $\omega \in \Omega$ and $n \in \mathbb{N}$,*

$$d(f_n(\omega), f(\omega)) \geq d(f_{n+1}(\omega), f(\omega)) \tag{8.17}$$

and

$$\lim_{n\to\infty} d(f_n(\omega), f(\omega)) = 0. \tag{8.18}$$

Proof: Let $D = \{x_k\}_{k=1}^{\infty}$ be a countable dense subset of X. First suppose f is measurable. Then since in a metric space every open set is the countable intersection of closed sets, it follows $f^{-1}(\text{closed set}) \in \mathcal{F}$. Now let $D_n = \{x_k\}_{k=1}^{n}$. Let

$$A_1 \equiv \left\{\omega : d(x_1, f(\omega)) = \min_{k\leq n} d(x_k, f(\omega))\right\}$$

That is, A_1 are those ω such that $f(\omega)$ is approximated best out of D_n by x_1. Why is this a measurable set? It is because $\omega \to d(x, f(\omega))$ is a real valued measurable function, being the composition of a continuous function, $y \to d(x, y)$ and a measurable function, $\omega \to f(\omega)$. Next let

$$A_2 \equiv \left\{\omega \notin A_1 : d(x_2, f(\omega)) = \min_{k\leq n} d(x_k, f(\omega))\right\}$$

and continue in this manner obtaining disjoint measurable sets, $\{A_k\}_{k=1}^{n}$ such that for $\omega \in A_k$ the best approximation to $f(\omega)$ from D_n is x_k. Then

$$f_n(\omega) \equiv \sum_{k=1}^{n} x_k \mathcal{X}_{A_k}(\omega).$$

Note

$$\min_{k\leq n+1} d\left(x_k, f\left(\omega\right)\right) \leq \min_{k\leq n} d\left(x_k, f\left(\omega\right)\right)$$

and so this verifies 8.17. It remains to verify 8.18.

Let $\varepsilon > 0$ be given and pick $\omega \in \Omega$. Then there exists $x_n \in D$ such that $d\left(x_n, f\left(\omega\right)\right) < \varepsilon$. It follows from the construction that $d\left(f_n\left(\omega\right), f\left(\omega\right)\right) \leq d\left(x_n, f\left(\omega\right)\right) < \varepsilon$. This proves the first half.

Now suppose the existence of the sequence of simple functions as described above. Each f_n is a measurable function because $f_n^{-1}\left(U\right) = \cup\left\{A_k : x_k \in U\right\}$. Therefore, the conclusion that f is measurable follows from Theorem 8.9 on Page 175.

8.3.5 The Monotone Convergence Theorem

The following is called the monotone convergence theorem. This theorem and related convergence theorems are the reason for using the Lebesgue integral.

Theorem 8.29 *(Monotone Convergence theorem) Let f have values in $[0,\infty]$ and suppose $\{f_n\}$ is a sequence of nonnegative measurable functions having values in $[0,\infty]$ and satisfying*

$$\lim_{n\to\infty} f_n(\omega) = f(\omega) \text{ for each } \omega.$$

$$\cdots f_n(\omega) \leq f_{n+1}(\omega) \cdots$$

Then f is measurable and

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

Proof: From Lemmas 8.19 and 8.20,

$$\begin{aligned}
\int f d\mu &\equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu\left(\left[ih < f\right]\right) \\
&= \sup_{h>0} \sup_{k} \sum_{i=1}^{k} h\mu\left(\left[ih < f\right]\right) \\
&= \sup_{h>0} \sup_{k} \sup_{m} \sum_{i=1}^{k} h\mu\left(\left[ih < f_m\right]\right) \\
&= \sup_{m} \sup_{h>0} \sum_{i=1}^{\infty} h\mu\left(\left[ih < f_m\right]\right) \\
&\equiv \sup_{m} \int f_m d\mu \\
&= \lim_{m\to\infty} \int f_m d\mu.
\end{aligned}$$

The third equality follows from the observation that

$$\lim_{m\to\infty}\mu\left(\left[ih<f_m\right]\right)=\mu\left(\left[ih<f\right]\right)$$

which follows from Theorem 8.5 since the sets, $[ih<f_m]$ are increasing in m and their union equals $[ih<f]$. This proves the theorem.

To illustrate what goes wrong without the Lebesgue integral, consider the following example.

Example 8.30 *Let $\{r_n\}$ denote the rational numbers in $[0,1]$ and let*

$$f_n(t)\equiv\begin{cases}1 \text{ if } t\notin\{r_1,\cdots,r_n\}\\ 0 \text{ otherwise}\end{cases}$$

Then $f_n(t)\uparrow f(t)$ where f is the function which is one on the rationals and zero on the irrationals. Each f_n is Riemann integrable (why?) but f is not Riemann integrable. Therefore, you can't write $\int f dx=\lim_{n\to\infty}\int f_n dx$.

A meta-mathematical observation related to this type of example is this. If you can choose your functions, you don't need the Lebesgue integral. The Riemann integral is just fine. It is when you can't choose your functions and they come to you as pointwise limits that you really need the superior Lebesgue integral or at least something more general than the Riemann integral. The Riemann integral is entirely adequate for evaluating the seemingly endless lists of boring problems found in calculus books.

8.3.6 Other Definitions

To review and summarize the above, if $f\geq 0$ is measurable,

$$\int f d\mu\equiv\sup_{h>0}\sum_{i=1}^{\infty}h\mu\left(\left[f>ih\right]\right) \tag{8.19}$$

another way to get the same thing for $\int f d\mu$ is to take an increasing sequence of nonnegative simple functions, $\{s_n\}$ with $s_n(\omega)\to f(\omega)$ and then by monotone convergence theorem,

$$\int f d\mu=\lim_{n\to\infty}\int s_n$$

where if $s_n(\omega)=\sum_{j=1}^{m}c_i\mathcal{X}_{E_i}(\omega)$,

$$\int s_n d\mu=\sum_{i=1}^{m}c_i m\left(E_i\right).$$

Similarly this also shows that for such nonnegative measurable function,

$$\int f d\mu=\sup\left\{\int s: 0\leq s\leq f,\ s \text{ simple}\right\}$$

which is the usual way of defining the Lebesgue integral for nonnegative simple functions in most books. I have done it differently because this approach led to an easier proof of the Monotone convergence theorem. Here is an equivalent definition of the integral. The fact it is well defined has been discussed above.

Definition 8.31 *For s a nonnegative simple function,*

$$s(\omega) = \sum_{k=1}^{n} c_k \mathcal{X}_{E_k}(\omega), \int s = \sum_{k=1}^{n} c_k \mu(E_k).$$

For f a nonnegative measurable function,

$$\int f d\mu = \sup \left\{ \int s : 0 \leq s \leq f, \ s \text{ simple} \right\}.$$

8.3.7 Fatou's Lemma

Sometimes the limit of a sequence does not exist. There are two more general notions known as lim sup and lim inf which do always exist in some sense. These notions are dependent on the following lemma.

Lemma 8.32 *Let $\{a_n\}$ be an increasing (decreasing) sequence in $[-\infty, \infty]$. Then $\lim_{n\to\infty} a_n$ exists.*

Proof: Suppose first $\{a_n\}$ is increasing. Recall this means $a_n \leq a_{n+1}$ for all n. If the sequence is bounded above, then it has a least upper bound and so $a_n \to a$ where a is its least upper bound. If the sequence is not bounded above, then for every $l \in \mathbb{R}$, it follows l is not an upper bound and so eventually, $a_n > l$. But this is what is meant by $a_n \to \infty$. The situation for decreasing sequences is completely similar.

Now take any sequence, $\{a_n\} \subseteq [-\infty, \infty]$ and consider the sequence $\{A_n\}$ where

$$A_n \equiv \inf \{a_k : k \geq n\}.$$

Then as n increases, the set of numbers whose inf is being taken is getting smaller. Therefore, A_n is an increasing sequence and so it must converge. Similarly, if $B_n \equiv \sup \{a_k : k \geq n\}$, it follows B_n is decreasing and so $\{B_n\}$ also must converge. With this preparation, the following definition can be given.

Definition 8.33 *Let $\{a_n\}$ be a sequence of points in $[-\infty, \infty]$. Then define*

$$\liminf_{n\to\infty} a_n \equiv \lim_{n\to\infty} \inf \{a_k : k \geq n\}$$

and

$$\limsup_{n\to\infty} a_n \equiv \lim_{n\to\infty} \sup \{a_k : k \geq n\}$$

In the case of functions having values in $[-\infty, \infty]$,

$$\left(\liminf_{n\to\infty} f_n\right)(\omega) \equiv \liminf_{n\to\infty} (f_n(\omega)).$$

A similar definition applies to $\limsup_{n\to\infty} f_n$.

Lemma 8.34 *Let* $\{a_n\}$ *be a sequence in* $[-\infty, \infty]$. *Then* $\lim_{n\to\infty} a_n$ *exists if and only if*

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

and in this case, the limit equals the common value of these two numbers.

Proof: Suppose first $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Then, letting $\varepsilon > 0$ be given, $a_n \in (a-\varepsilon, a+\varepsilon)$ for all n large enough, say $n \geq N$. Therefore, both $\inf\{a_k : k \geq n\}$ and $\sup\{a_k : k \geq n\}$ are contained in $[a-\varepsilon, a+\varepsilon]$ whenever $n \geq N$. It follows $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are both in $[a-\varepsilon, a+\varepsilon]$, showing

$$\left|\liminf_{n\to\infty} a_n - \limsup_{n\to\infty} a_n\right| < 2\varepsilon.$$

Since ε is arbitrary, the two must be equal and they both must equal a. Next suppose $\lim_{n\to\infty} a_n = \infty$. Then if $l \in \mathbb{R}$, there exists N such that for $n \geq N$,

$$l \leq a_n$$

and therefore, for such n,

$$l \leq \inf\{a_k : k \geq n\} \leq \sup\{a_k : k \geq n\}$$

and this shows, since l is arbitrary that

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \infty.$$

The case for $-\infty$ is similar.

Conversely, suppose $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = a$. Suppose first that $a \in \mathbb{R}$. Then, letting $\varepsilon > 0$ be given, there exists N such that if $n \geq N$,

$$\sup\{a_k : k \geq n\} - \inf\{a_k : k \geq n\} < \varepsilon$$

therefore, if $k, m > N$, and $a_k > a_m$,

$$|a_k - a_m| = a_k - a_m \leq \sup\{a_k : k \geq n\} - \inf\{a_k : k \geq n\} < \varepsilon$$

showing that $\{a_n\}$ is a Cauchy sequence. Therefore, it converges to $a \in \mathbb{R}$, and as in the first part, the lim inf and lim sup both equal a. If $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \infty$, then given $l \in \mathbb{R}$, there exists N such that for $n \geq N$,

$$\inf_{n>N} a_n > l.$$

Therefore, $\lim_{n\to\infty} a_n = \infty$. The case for $-\infty$ is similar. This proves the lemma.

The next theorem, known as Fatou's lemma is another important theorem which justifies the use of the Lebesgue integral.

Theorem 8.35 *(Fatou's lemma) Let f_n be a nonnegative measurable function with values in $[0,\infty]$. Let $g(\omega) = \liminf_{n\to\infty} f_n(\omega)$. Then g is measurable and*

$$\int g d\mu \leq \lim\inf_{n\to\infty} \int f_n d\mu.$$

In other words,

$$\int \left(\lim\inf_{n\to\infty} f_n\right) d\mu \leq \lim\inf_{n\to\infty} \int f_n d\mu$$

Proof: Let $g_n(\omega) = \inf\{f_k(\omega) : k \geq n\}$. Then

$$g_n^{-1}([a,\infty]) = \cap_{k=n}^{\infty} f_k^{-1}([a,\infty]) \in \mathcal{F}.$$

Thus g_n is measurable by Lemma 8.6 on Page 173. Also $g(\omega) = \lim_{n\to\infty} g_n(\omega)$ so g is measurable because it is the pointwise limit of measurable functions. Now the functions g_n form an increasing sequence of nonnegative measurable functions so the monotone convergence theorem applies. This yields

$$\int g d\mu = \lim_{n\to\infty} \int g_n d\mu \leq \lim\inf_{n\to\infty} \int f_n d\mu.$$

The last inequality holding because

$$\int g_n d\mu \leq \int f_n d\mu.$$

(Note that it is not known whether $\lim_{n\to\infty} \int f_n d\mu$ exists.) This proves the Theorem.

8.3.8 The Righteous Algebraic Desires Of The Lebesgue Integral

The monotone convergence theorem shows the integral wants to be linear. This is the essential content of the next theorem.

Theorem 8.36 *Let f, g be nonnegative measurable functions and let a, b be nonnegative numbers. Then*

$$\int (af + bg)\, d\mu = a \int f d\mu + b \int g d\mu. \tag{8.20}$$

Proof: By Theorem 8.27 on Page 190 there exist sequences of nonnegative simple functions, $s_n \to f$ and $t_n \to g$. Then by the monotone convergence theorem and Lemma 8.26,

$$\begin{aligned} \int (af + bg)\, d\mu &= \lim_{n\to\infty} \int a s_n + b t_n d\mu \\ &= \lim_{n\to\infty} \left(a \int s_n d\mu + b \int t_n d\mu \right) \\ &= a \int f d\mu + b \int g d\mu. \end{aligned}$$

As long as you are allowing functions to take the value $+\infty$, you cannot consider something like $f+(-g)$ and so you can't very well expect a satisfactory statement about the integral being linear until you restrict yourself to functions which have values in a vector space. This is discussed next.

8.4 The Space L^1

The functions considered here have values in $\mathbb{C}$, a vector space.

Definition 8.37 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and suppose $f : \Omega \to \mathbb{C}$. Then f is said to be measurable if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable real valued functions.*

Definition 8.38 *A complex simple function will be a function which is of the form*

$$s(\omega) = \sum_{k=1}^{n} c_k \mathcal{X}_{E_k}(\omega)$$

where $c_k \in \mathbb{C}$ and $\mu(E_k) < \infty$. For s a complex simple function as above, define

$$I(s) \equiv \sum_{k=1}^{n} c_k \mu(E_k).$$

Lemma 8.39 *The definition, 8.38 is well defined. Furthermore, I is linear on the vector space of complex simple functions. Also the triangle inequality holds,*

$$|I(s)| \leq I(|s|).$$

Proof: Suppose $\sum_{k=1}^{n} c_k \mathcal{X}_{E_k}(\omega) = 0$. Does it follow that $\sum_k c_k \mu(E_k) = 0$? The supposition implies

$$\sum_{k=1}^{n} \operatorname{Re} c_k \mathcal{X}_{E_k}(\omega) = 0, \ \sum_{k=1}^{n} \operatorname{Im} c_k \mathcal{X}_{E_k}(\omega) = 0. \tag{8.21}$$

Choose λ large and positive so that $\lambda + \operatorname{Re} c_k \geq 0$. Then adding $\sum_k \lambda \mathcal{X}_{E_k}$ to both sides of the first equation above,

$$\sum_{k=1}^{n} (\lambda + \operatorname{Re} c_k) \mathcal{X}_{E_k}(\omega) = \sum_{k=1}^{n} \lambda \mathcal{X}_{E_k}$$

and by Lemma 8.26 on Page 189, it follows upon taking $\int$ of both sides that

$$\sum_{k=1}^{n} (\lambda + \operatorname{Re} c_k) \mu(E_k) = \sum_{k=1}^{n} \lambda \mu(E_k)$$

which implies $\sum_{k=1}^{n} \operatorname{Re} c_k \mu(E_k) = 0$. Similarly, $\sum_{k=1}^{n} \operatorname{Im} c_k \mu(E_k) = 0$ and so $\sum_{k=1}^{n} c_k \mu(E_k) = 0$. Thus if

$$\sum_j c_j \mathcal{X}_{E_j} = \sum_k d_k \mathcal{X}_{F_k}$$

then $\sum_j c_j \mathcal{X}_{E_j} + \sum_k (-d_k) \mathcal{X}_{F_k} = 0$ and so the result just established verifies $\sum_j c_j \mu(E_j) - \sum_k d_k \mu(F_k) = 0$ which proves I is well defined.

That I is linear is now obvious. It only remains to verify the triangle inequality.

Let s be a simple function,

$$s = \sum_j c_j \mathcal{X}_{E_j}$$

Then pick $\theta \in \mathbb{C}$ such that $\theta I(s) = |I(s)|$ and $|\theta| = 1$. Then from the triangle inequality for sums of complex numbers,

$$\begin{aligned} |I(s)| &= \theta I(s) = I(\theta s) = \sum_j \theta c_j \mu(E_j) \\ &= \left| \sum_j \theta c_j \mu(E_j) \right| \leq \sum_j |\theta c_j| \, \mu(E_j) = I(|s|). \end{aligned}$$

This proves the lemma.

With this lemma, the following is the definition of $L^1(\Omega)$.

Definition 8.40 $f \in L^1(\Omega)$ *means there exists a sequence of complex simple functions,* $\{s_n\}$ *such that*

$$\begin{gathered} s_n(\omega) \to f(\omega) \text{ for all } \omega \in \Omega \\ \lim_{m,n\to\infty} I(|s_n - s_m|) = \lim_{n,m\to\infty} \int |s_n - s_m| \, d\mu = 0 \end{gathered} \tag{8.22}$$

Then

$$I(f) \equiv \lim_{n\to\infty} I(s_n). \tag{8.23}$$

Lemma 8.41 *Definition 8.40 is well defined.*

Proof: There are several things which need to be verified. First suppose 8.22. Then by Lemma 8.39

$$|I(s_n) - I(s_m)| = |I(s_n - s_m)| \leq I(|s_n - s_m|)$$

and for m, n large enough this last is given to be small so $\{I(s_n)\}$ is a Cauchy sequence in $\mathbb{C}$ and so it converges. This verifies the limit in 8.23 at least exists. It remains to consider another sequence $\{t_n\}$ having the same properties as $\{s_n\}$ and verifying $I(f)$ determined by this other sequence is the same. By Lemma 8.39 and Fatou's lemma, Theorem 8.35 on Page 196,

$$\begin{aligned}|I(s_n) - I(t_n)| &\leq I(|s_n - t_n|) = \int |s_n - t_n|\, d\mu \\ &\leq \int |s_n - f| + |f - t_n|\, d\mu \\ &\leq \lim\inf_{k\to\infty} \int |s_n - s_k|\, d\mu + \lim\inf_{k\to\infty} \int |t_n - t_k|\, d\mu < \varepsilon\end{aligned}$$

whenever n is large enough. Since ε is arbitrary, this shows the limit from using the t_n is the same as the limit from using s_n. This proves the lemma.

What if f has values in $[0,\infty)$? Earlier $\int f d\mu$ was defined for such functions and now $I(f)$ has been defined. Are they the same? If so, I can be regarded as an extension of $\int d\mu$ to a larger class of functions.

Lemma 8.42 *Suppose f has values in $[0,\infty)$ and $f \in L^1(\Omega)$. Then f is measurable and*

$$I(f) = \int f d\mu.$$

Proof: Since f is the pointwise limit of a sequence of complex simple functions, $\{s_n\}$ having the properties described in Definition 8.40, it follows $f(\omega) = \lim_{n\to\infty} \operatorname{Re} s_n(\omega)$ and so f is measurable. Also

$$\int \left|(\operatorname{Re} s_n)^+ - (\operatorname{Re} s_m)^+\right| d\mu \leq \int |\operatorname{Re} s_n - \operatorname{Re} s_m|\, d\mu \leq \int |s_n - s_m|\, d\mu$$

where $x^+ \equiv \frac{1}{2}(|x| + x)$, the positive part of the real number, x. [2]Thus there is no loss of generality in assuming $\{s_n\}$ is a sequence of complex simple functions having values in $[0,\infty)$. Then since for such complex simple functions, $I(s) = \int s d\mu$,

$$\left|I(f) - \int f d\mu\right| \leq |I(f) - I(s_n)| + \left|\int s_n d\mu - \int f d\mu\right| < \varepsilon + \int |s_n - f|\, d\mu$$

whenever n is large enough. But by Fatou's lemma, Theorem 8.35 on Page 196, the last term is no larger than

$$\lim\inf_{k\to\infty} \int |s_n - s_k|\, d\mu < \varepsilon$$

whenever n is large enough. Since ε is arbitrary, this shows $I(f) = \int f d\mu$ as claimed.

As explained above, I can be regarded as an extension of $\int d\mu$ so from now on, the usual symbol, $\int d\mu$ will be used. It is now easy to verify $\int d\mu$ is linear on $L^1(\Omega)$.

[2]The negative part of the real number x is defined to be $x^- \equiv \frac{1}{2}(|x| - x)$. Thus $|x| = x^+ + x^-$ and $x = x^+ - x^-$. .

Theorem 8.43 *$\int d\mu$ is linear on $L^1(\Omega)$ and $L^1(\Omega)$ is a complex vector space. If $f \in L^1(\Omega)$, then $\operatorname{Re} f, \operatorname{Im} f$, and $|f|$ are all in $L^1(\Omega)$. Furthermore, for $f \in L^1(\Omega)$,*

$$\int f d\mu = \int (\operatorname{Re} f)^+ d\mu - \int (\operatorname{Re} f)^- d\mu + i\left(\int (\operatorname{Im} f)^+ d\mu - \int (\operatorname{Im} f)^- d\mu\right)$$

Also the triangle inequality holds,

$$\left|\int f d\mu\right| \leq \int |f| \, d\mu$$

Proof: First it is necessary to verify that $L^1(\Omega)$ is really a vector space because it makes no sense to speak of linear maps without having these maps defined on a vector space. Let f, g be in $L^1(\Omega)$ and let $a, b \in \mathbb{C}$. Then let $\{s_n\}$ and $\{t_n\}$ be sequences of complex simple functions associated with f and g respectively as described in Definition 8.40. Consider $\{as_n + bt_n\}$, another sequence of complex simple functions. Then $as_n(\omega) + bt_n(\omega) \to af(\omega) + bg(\omega)$ for each ω. Also, from Lemma 8.39

$$\int |as_n + bt_n - (as_m + bt_m)| \, d\mu \leq |a| \int |s_n - s_m| \, d\mu + |b| \int |t_n - t_m| \, d\mu$$

and the sum of the two terms on the right converge to zero as $m, n \to \infty$. Thus $af + bg \in L^1(\Omega)$. Also

$$\begin{aligned} \int (af + bg) \, d\mu &= \lim_{n\to\infty} \int (as_n + bt_n) \, d\mu \\ &= \lim_{n\to\infty} \left(a \int s_n d\mu + b \int t_n d\mu\right) \\ &= a \lim_{n\to\infty} \int s_n d\mu + b \lim_{n\to\infty} \int t_n d\mu \\ &= a \int f d\mu + b \int g d\mu. \end{aligned}$$

If $\{s_n\}$ is a sequence of complex simple functions described in Definition 8.40 corresponding to f, then $\{|s_n|\}$ is a sequence of complex simple functions satisfying the conditions of Definition 8.40 corresponding to $|f|$. This is because $|s_n(\omega)| \to |f(\omega)|$ and

$$\int ||s_n| - |s_m|| \, d\mu \leq \int |s_m - s_n| \, d\mu$$

with this last expression converging to 0 as $m, n \to \infty$. Thus $|f| \in L^1(\Omega)$. Also, by similar reasoning, $\{\operatorname{Re} s_n\}$ and $\{\operatorname{Im} s_n\}$ correspond to $\operatorname{Re} f$ and $\operatorname{Im} f$ respectively in the manner described by Definition 8.40 showing that $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $L^1(\Omega)$. Now $(\operatorname{Re} f)^+ = \frac{1}{2}(|\operatorname{Re} f| + \operatorname{Re} f)$ and $(\operatorname{Re} f)^- = \frac{1}{2}(|\operatorname{Re} f| - \operatorname{Re} f)$ so both of these functions are in $L^1(\Omega)$. Similar formulas establish that $(\operatorname{Im} f)^+$ and $(\operatorname{Im} f)^-$ are in $L^1(\Omega)$.

The formula follows from the observation that

$$f = (\operatorname{Re} f)^+ - (\operatorname{Re} f)^- + i\left((\operatorname{Im} f)^+ - (\operatorname{Im} f)^-\right)$$

and the fact shown first that $\int d\mu$ is linear.

To verify the triangle inequality, let $\{s_n\}$ be complex simple functions for f as in Definition 8.40. Then

$$\left|\int f d\mu\right| = \lim_{n\to\infty}\left|\int s_n d\mu\right| \leq \lim_{n\to\infty}\int |s_n|\, d\mu = \int |f|\, d\mu.$$

This proves the theorem.

The following description of $L^1(\Omega)$ is the version most often used because it is easy to verify the conditions for it.

Corollary 8.44 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and let $f : \Omega \to \mathbb{C}$. Then $f \in L^1(\Omega)$ if and only if f is measurable and $\int |f|\, d\mu < \infty$.*

Proof: Suppose $f \in L^1(\Omega)$. Then from Definition 8.40, it follows both real and imaginary parts of f are measurable. Just take real and imaginary parts of s_n and observe the real and imaginary parts of f are limits of the real and imaginary parts of s_n respectively. By Theorem 8.43 this shows the only if part.

The more interesting part is the if part. Suppose then that f is measurable and $\int |f|\, d\mu < \infty$. Suppose first that f has values in $[0, \infty)$. It is necessary to obtain the sequence of complex simple functions. By Theorem 8.27, there exists a sequence of nonnegative simple functions, $\{s_n\}$ such that $s_n(\omega) \uparrow f(\omega)$. Then by the monotone convergence theorem,

$$\lim_{n\to\infty}\int (2f - (f - s_n))\, d\mu = \int 2f d\mu$$

and so

$$\lim_{n\to\infty}\int (f - s_n)\, d\mu = 0.$$

Letting m be large enough, it follows $\int (f - s_m)\, d\mu < \varepsilon$ and so if $n > m$

$$\int |s_m - s_n|\, d\mu \leq \int |f - s_m|\, d\mu < \varepsilon.$$

Therefore, $f \in L^1(\Omega)$ because $\{s_n\}$ is a suitable sequence.

The general case follows from considering positive and negative parts of real and imaginary parts of f. These are each measurable and nonnegative and their integral is finite so each is in $L^1(\Omega)$ by what was just shown. Thus

$$f = \operatorname{Re} f^+ - \operatorname{Re} f^- + i\left(\operatorname{Im} f^+ - \operatorname{Im} f^-\right)$$

and so $f \in L^1(\Omega)$. This proves the corollary.

Theorem 8.45 *(Dominated Convergence theorem) Let* $f_n \in L^1(\Omega)$ *and suppose*

$$f(\omega) = \lim_{n\to\infty} f_n(\omega),$$

and there exists a measurable function g*, with values in* $[0,\infty]$*,*[3] *such that*

$$|f_n(\omega)| \leq g(\omega) \text{ and } \int g(\omega)d\mu < \infty.$$

Then $f \in L^1(\Omega)$ *and*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

Proof: f is measurable by Theorem 8.8. Since $|f| \leq g$, it follows that

$$f \in L^1(\Omega) \text{ and } |f - f_n| \leq 2g.$$

By Fatou's lemma (Theorem 8.35),

$$\begin{aligned} \int 2g d\mu &\leq \lim\inf_{n\to\infty} \int 2g - |f - f_n| d\mu \\ &= \int 2g d\mu - \lim\sup_{n\to\infty} \int |f - f_n| d\mu. \end{aligned}$$

Subtracting $\int 2g d\mu$,

$$0 \leq -\lim\sup_{n\to\infty} \int |f - f_n| d\mu.$$

Hence

$$\begin{aligned} 0 &\geq \lim\sup_{n\to\infty} (\int |f - f_n| d\mu) \geq \lim\sup_{n\to\infty} |\int f d\mu - \int f_n d\mu| \\ &\geq \lim\inf_{n\to\infty} |\int f d\mu - \int f_n d\mu| \geq 0. \end{aligned}$$

This proves the theorem by Lemma 8.34 on Page 195 because the lim sup and lim inf are equal.

Corollary 8.46 *Suppose* $f_n \in L^1(\Omega)$ *and* $f(\omega) = \lim_{n\to\infty} f_n(\omega)$*. Suppose also there exist measurable functions,* g_n*,* g *with values in* $[0,\infty]$ *such that*

$$\lim_{n\to\infty} \int g_n d\mu = \int g d\mu,$$

$g_n(\omega) \to g(\omega)$ μ *a.e. and both* $\int g_n d\mu$ *and* $\int g d\mu$ *are finite. Also suppose* $|f_n(\omega)| \leq g_n(\omega)$*. Then*

$$\lim_{n\to\infty} \int |f - f_n| d\mu = 0.$$

[3] Note that, since g is allowed to have the value ∞, it is not known that $g \in L^1(\Omega)$.

Proof: It is just like the above. This time $g + g_n - |f - f_n| \geq 0$ and so by Fatou's lemma,

$$\int 2g d\mu - \lim\sup_{n\to\infty} \int |f - f_n| \, d\mu =$$

$$\begin{aligned} & \lim\inf_{n\to\infty} \int (g_n + g) - \lim\sup_{n\to\infty} \int |f - f_n| \, d\mu \\ = & \lim\inf_{n\to\infty} \int ((g_n + g) - |f - f_n|) \, d\mu \geq \int 2g d\mu \end{aligned}$$

and so $-\lim\sup_{n\to\infty} \int |f - f_n| \, d\mu \geq 0$.

Definition 8.47 *Let E be a measurable subset of Ω.*

$$\int_E f d\mu \equiv \int f \mathcal{X}_E d\mu.$$

If $L^1(E)$ is written, the σ algebra is defined as

$$\{E \cap A : A \in \mathcal{F}\}$$

and the measure is μ restricted to this smaller σ algebra. Clearly, if $f \in L^1(\Omega)$, then

$$f\mathcal{X}_E \in L^1(E)$$

and if $f \in L^1(E)$, then letting $\tilde{f}$ be the 0 extension of f off of E, it follows $\tilde{f} \in L^1(\Omega)$.

8.5 Vitali Convergence Theorem

The Vitali convergence theorem is a convergence theorem which in the case of a finite measure space is superior to the dominated convergence theorem.

Definition 8.48 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathfrak{S} \subseteq L^1(\Omega)$. $\mathfrak{S}$ is uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathfrak{S}$*

$$|\int_E f d\mu| < \varepsilon \text{ whenever } \mu(E) < \delta.$$

Lemma 8.49 *If $\mathfrak{S}$ is uniformly integrable, then $|\mathfrak{S}| \equiv \{|f| : f \in \mathfrak{S}\}$ is uniformly integrable. Also $\mathfrak{S}$ is uniformly integrable if $\mathfrak{S}$ is finite.*

Proof: Let $\varepsilon > 0$ be given and suppose $\mathfrak{S}$ is uniformly integrable. First suppose the functions are real valued. Let δ be such that if $\mu(E) < \delta$, then

$$\left| \int_E f d\mu \right| < \frac{\varepsilon}{2}$$

for all $f \in \mathfrak{S}$. Let $\mu(E) < \delta$. Then if $f \in \mathfrak{S}$,

$$\begin{aligned}\int_E |f|\, d\mu &\leq \int_{E\cap[f\leq 0]} (-f)\, d\mu + \int_{E\cap[f>0]} f d\mu \\ &= \left|\int_{E\cap[f\leq 0]} f d\mu\right| + \left|\int_{E\cap[f>0]} f d\mu\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

In general, if $\mathfrak{S}$ is a uniformly integrable set of complex valued functions, the inequalities,

$$\left|\int_E \operatorname{Re} f d\mu\right| \leq \left|\int_E f d\mu\right|, \left|\int_E \operatorname{Im} f d\mu\right| \leq \left|\int_E f d\mu\right|,$$

imply $\operatorname{Re}\mathfrak{S} \equiv \{\operatorname{Re} f : f \in \mathfrak{S}\}$ and $\operatorname{Im}\mathfrak{S} \equiv \{\operatorname{Im} f : f \in \mathfrak{S}\}$ are also uniformly integrable. Therefore, applying the above result for real valued functions to these sets of functions, it follows $|\mathfrak{S}|$ is uniformly integrable also.

For the last part, is suffices to verify a single function in $L^1(\Omega)$ is uniformly integrable. To do so, note that from the dominated convergence theorem,

$$\lim_{R\to\infty} \int_{[|f|>R]} |f|\, d\mu = 0.$$

Let $\varepsilon > 0$ be given and choose R large enough that $\int_{[|f|>R]} |f|\, d\mu < \frac{\varepsilon}{2}$. Now let $\mu(E) < \frac{\varepsilon}{2R}$. Then

$$\begin{aligned}\int_E |f|\, d\mu &= \int_{E\cap[|f|\leq R]} |f|\, d\mu + \int_{E\cap[|f|>R]} |f|\, d\mu \\ &< R\mu(E) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This proves the lemma.

The following theorem is Vitali's convergence theorem.

Theorem 8.50 *Let $\{f_n\}$ be a uniformly integrable set of complex valued functions, $\mu(\Omega) < \infty$, and $f_n(x) \to f(x)$ a.e. where f is a measurable complex valued function. Then $f \in L^1(\Omega)$ and*

$$\lim_{n\to\infty} \int_\Omega |f_n - f| d\mu = 0. \tag{8.24}$$

Proof: First it will be shown that $f \in L^1(\Omega)$. By uniform integrability, there exists $\delta > 0$ such that if $\mu(E) < \delta$, then

$$\int_E |f_n|\, d\mu < 1$$

for all n. By Egoroff's theorem, there exists a set, E of measure less than δ such that on E^C, $\{f_n\}$ converges uniformly. Therefore, for p large enough, and $n > p$,

$$\int_{E^C} |f_p - f_n| \, d\mu < 1$$

which implies

$$\int_{E^C} |f_n| \, d\mu < 1 + \int_\Omega |f_p| \, d\mu.$$

Then since there are only finitely many functions, f_n with $n \leq p$, there exists a constant, M_1 such that for all n,

$$\int_{E^C} |f_n| \, d\mu < M_1.$$

But also,

$$\begin{aligned} \int_\Omega |f_m| \, d\mu &= \int_{E^C} |f_m| \, d\mu + \int_E |f_m| \\ &\leq M_1 + 1 \equiv M. \end{aligned}$$

Therefore, by Fatou's lemma,

$$\int_\Omega |f| \, d\mu \leq \lim \inf_{n\to\infty} \int |f_n| \, d\mu \leq M,$$

showing that $f \in L^1$ as hoped.

Now $\mathfrak{S} \cup \{f\}$ is uniformly integrable so there exists $\delta_1 > 0$ such that if $\mu(E) < \delta_1$, then $\int_E |g| \, d\mu < \varepsilon/3$ for all $g \in \mathfrak{S} \cup \{f\}$. By Egoroff's theorem, there exists a set, F with $\mu(F) < \delta_1$ such that f_n converges uniformly to f on F^C. Therefore, there exists N such that if $n > N$, then

$$\int_{F^C} |f - f_n| \, d\mu < \frac{\varepsilon}{3}.$$

It follows that for $n > N$,

$$\begin{aligned} \int_\Omega |f - f_n| \, d\mu &\leq \int_{F^C} |f - f_n| \, d\mu + \int_F |f| \, d\mu + \int_F |f_n| \, d\mu \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which verifies 8.24.

8.6 Exercises

1. Let $\Omega = \mathbb{N} = \{1, 2, \cdots\}$ and $\mu(S) =$ number of elements in S. If

   $$f : \Omega \to \mathbb{C}$$

   what is meant by $\int f d\mu$? Which functions are in $L^1(\Omega)$? Which functions are measurable?

2. Show that for $f \geq 0$ and measurable, $\int f d\mu \equiv \lim_{h\to 0+} \sum_{i=1}^{\infty} h\mu\left([ih < f]\right)$.

3. For the measure space of Problem 1, give an example of a sequence of nonnegative measurable functions $\{f_n\}$ converging pointwise to a function f, such that inequality is obtained in Fatou's lemma.

4. Fill in all the details of the proof of Lemma 8.49.

5. Let $\sum_{i=1}^{n} c_i \mathcal{X}_{E_i}(\omega) = s(\omega)$ be a nonnegative simple function for which the c_i are the distinct nonzero values. Show with the aid of the monotone convergence theorem that the two definitions of the Lebesgue integral given in the chapter are equivalent.

6. Suppose (Ω, μ) is a finite measure space and $\mathfrak{S} \subseteq L^1(\Omega)$. Show $\mathfrak{S}$ is uniformly integrable and bounded in $L^1(\Omega)$ if there exists an increasing function h which satisfies
$$\lim_{t\to\infty} \frac{h(t)}{t} = \infty,\ \sup\left\{\int_\Omega h(|f|)\, d\mu : f \in \mathfrak{S}\right\} < \infty.$$
$\mathfrak{S}$ is bounded if there is some number, M such that
$$\int |f|\, d\mu \leq M$$
for all $f \in \mathfrak{S}$.

7. Let $\{a_n\}, \{b_n\}$ be sequences in $[-\infty, \infty]$ and $a \in \mathbb{R}$. Show
$$\lim\inf_{n\to\infty} (a - a_n) = a - \lim\sup_{n\to\infty} a_n.$$
This was used in the proof of the Dominated convergence theorem. Also show
$$\lim\sup_{n\to\infty} (-a_n) = -\lim\inf_{n\to\infty} (a_n)$$
$$\lim\sup_{n\to\infty} (a_n + b_n) \leq \lim\sup_{n\to\infty} a_n + \lim\sup_{n\to\infty} b_n$$
provided no sum is of the form $\infty - \infty$. Also show strict inequality can hold in the inequality. State and prove corresponding statements for lim inf.

8. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $f, g : \Omega \to (-\infty, \infty]$ are measurable. Prove the sets
$$\{\omega : f(\omega) < g(\omega)\} \text{ and } \{\omega : f(\omega) = g(\omega)\}$$
are measurable. **Hint:** The easy way to do this is to write
$$\{\omega : f(\omega) < g(\omega)\} = \cup_{r\in\mathbb{Q}} [f < r] \cap [g > r].$$
Note that $l(x, y) = x - y$ is not continuous on $(-\infty, \infty]$ so the obvious idea doesn't work.

9. Let $\{f_n\}$ be a sequence of real or complex valued measurable functions. Let

$$S = \{\omega : \{f_n(\omega)\} \text{ converges}\}.$$

Show S is measurable. **Hint:** You might try to exhibit the set where f_n converges in terms of countable unions and intersections using the definition of a Cauchy sequence.

10. Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and let f be a nonnegative measurable function defined on Ω. Also let $\phi : [0, \infty) \to [0, \infty)$ be strictly increasing and have a continuous derivative and $\phi(0) = 0$. Suppose f is bounded and that $0 \leq \phi(f(\omega)) \leq M$ for some number, M. Show that

$$\int_\Omega \phi(f)\, d\mu = \int_0^\infty \phi'(s)\, \mu([s < f])\, ds,$$

where the integral on the right is the ordinary improper Riemann integral. **Hint:** First note that $s \to \phi'(s)\, \mu([s < f])$ is Riemann integrable because ϕ' is continuous and $s \to \mu([s < f])$ is a nonincreasing function, hence Riemann integrable. From the second description of the Lebesgue integral and the assumption that $\phi(f(\omega)) \leq M$, argue that for $[M/h]$ the greatest integer less than M/h,

$$\begin{aligned}
\int_\Omega \phi(f)\, d\mu &= \sup_{h>0} \sum_{i=1}^{[M/h]} h\mu([ih < \phi(f)]) \\
&= \sup_{h>0} \sum_{i=1}^{[M/h]} h\mu\left(\left[\phi^{-1}(ih) < f\right]\right) \\
&= \sup_{h>0} \sum_{i=1}^{[M/h]} \frac{h\Delta_i}{\Delta_i} \mu\left(\left[\phi^{-1}(ih) < f\right]\right)
\end{aligned}$$

where $\Delta_i = \left(\phi^{-1}(ih) - \phi^{-1}((i-1)h)\right)$. Now use the mean value theorem to write

$$\begin{aligned}
\Delta_i &= \left(\phi^{-1}\right)'(t_i)\, h \\
&= \frac{1}{\phi'\left(\phi^{-1}(t_i)\right)} h
\end{aligned}$$

for some t_i between $(i-1)h$ and ih. Therefore, the right side is of the form

$$\sup_h \sum_{i=1}^{[M/h]} \phi'\left(\phi^{-1}(t_i)\right) \Delta_i \mu\left(\left[\phi^{-1}(ih) < f\right]\right)$$

where $\phi^{-1}(t_i) \in \left(\phi^{-1}((i-1)h), \phi^{-1}(ih)\right)$. Argue that if t_i were replaced with ih, this would be a Riemann sum for the Riemann integral

$$\int_0^{\phi^{-1}(M)} \phi'(t)\, \mu([t < f])\, dt = \int_0^\infty \phi'(t)\, \mu([t < f])\, dt.$$

11. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose f_n converges uniformly to f and that f_n is in $L^1(\Omega)$. When is

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu?$$

12. Suppose $u_n(t)$ is a differentiable function for $t \in (a,b)$ and suppose that for $t \in (a,b)$,

$$|u_n(t)|,\ |u_n'(t)| < K_n$$

where $\sum_{n=1}^{\infty} K_n < \infty$. Show

$$(\sum_{n=1}^{\infty} u_n(t))' = \sum_{n=1}^{\infty} u_n'(t).$$

Hint: This is an exercise in the use of the dominated convergence theorem and the mean value theorem.

13. Show that $\{\sum_{i=1}^{\infty} 2^{-n}\mu\left([i2^{-n} < f]\right)\}$ for f a nonnegative measurable function is an increasing sequence. Could you define

$$\int f d\mu \equiv \lim_{n\to\infty} \sum_{i=1}^{\infty} 2^{-n}\mu\left(\left[i2^{-n} < f\right]\right)$$

and would it be equivalent to the above definitions of the Lebesgue integral?

14. Suppose $\{f_n\}$ is a sequence of nonnegative measurable functions defined on a measure space, $(\Omega, \mathcal{S}, \mu)$. Show that

$$\int \sum_{k=1}^{\infty} f_k d\mu = \sum_{k=1}^{\infty} \int f_k d\mu.$$

Hint: Use the monotone convergence theorem along with the fact the integral is linear.

The Construction Of Measures

9.1 Outer Measures

What are some examples of measure spaces? In this chapter, a general procedure is discussed called the method of outer measures. It is due to Caratheodory (1918). This approach shows how to obtain measure spaces starting with an outer measure. This will then be used to construct measures determined by positive linear functionals.

Definition 9.1 *Let Ω be a nonempty set and let $\mu : \mathcal{P}(\Omega) \to [0, \infty]$ satisfy*

$$\mu(\emptyset) = 0,$$

$$\text{If } A \subseteq B, \text{ then } \mu(A) \leq \mu(B),$$

$$\mu(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

Such a function is called an outer measure. For $E \subseteq \Omega$, E is μ measurable if for all $S \subseteq \Omega$,

$$\mu(S) = \mu(S \setminus E) + \mu(S \cap E). \tag{9.1}$$

To help in remembering 9.1, think of a measurable set, E, as a process which divides a given set into two pieces, the part in E and the part not in E as in 9.1. In the Bible, there are four incidents recorded in which a process of division resulted in more stuff than was originally present.[1] Measurable sets are exactly

[1] 1 Kings 17, 2 Kings 4, Mathew 14, and Mathew 15 all contain such descriptions. The stuff involved was either oil, bread, flour or fish. In mathematics such things have also been done with sets. In the book by Bruckner Bruckner and Thompson there is an interesting discussion of the Banach Tarski paradox which says it is possible to divide a ball in $\mathbb{R}^3$ into five disjoint pieces and assemble the pieces to form two disjoint balls of the same size as the first. The details can be found in: The Banach Tarski Paradox by Wagon, Cambridge University press. 1985. It is known that all such examples must involve the axiom of choice.

those for which no such miracle occurs. You might think of the measurable sets as the nonmiraculous sets. The idea is to show that they form a σ algebra on which the outer measure, μ is a measure.

First here is a definition and a lemma.

Definition 9.2 *$(\mu\lfloor S)(A) \equiv \mu(S \cap A)$ for all $A \subseteq \Omega$. Thus $\mu\lfloor S$ is the name of a new outer measure, called μ restricted to S.*

The next lemma indicates that the property of measurability is not lost by considering this restricted measure.

Lemma 9.3 *If A is μ measurable, then A is $\mu\lfloor S$ measurable.*

Proof: Suppose A is μ measurable. It is desired to to show that for all $T \subseteq \Omega$,

$$(\mu\lfloor S)(T) = (\mu\lfloor S)(T \cap A) + (\mu\lfloor S)(T \setminus A).$$

Thus it is desired to show

$$\mu(S \cap T) = \mu(T \cap A \cap S) + \mu(T \cap S \cap A^C). \tag{9.2}$$

But 9.2 holds because A is μ measurable. Apply Definition 9.1 to $S \cap T$ instead of S.

If A is $\mu\lfloor S$ measurable, it does not follow that A is μ measurable. Indeed, if you believe in the existence of non measurable sets, you could let $A = S$ for such a μ non measurable set and verify that S is $\mu\lfloor S$ measurable.

The next theorem is the main result on outer measures. It is a very general result which applies whenever one has an outer measure on the power set of any set. This theorem will be referred to as Caratheodory's procedure in the rest of the book.

Theorem 9.4 *The collection of μ measurable sets, $\mathcal{S}$, forms a σ algebra and*

$$\text{If } F_i \in \mathcal{S},\ F_i \cap F_j = \emptyset,\ \text{then } \mu(\cup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mu(F_i). \tag{9.3}$$

If $\cdots F_n \subseteq F_{n+1} \subseteq \cdots$, then if $F = \cup_{n=1}^{\infty} F_n$ and $F_n \in \mathcal{S}$, it follows that

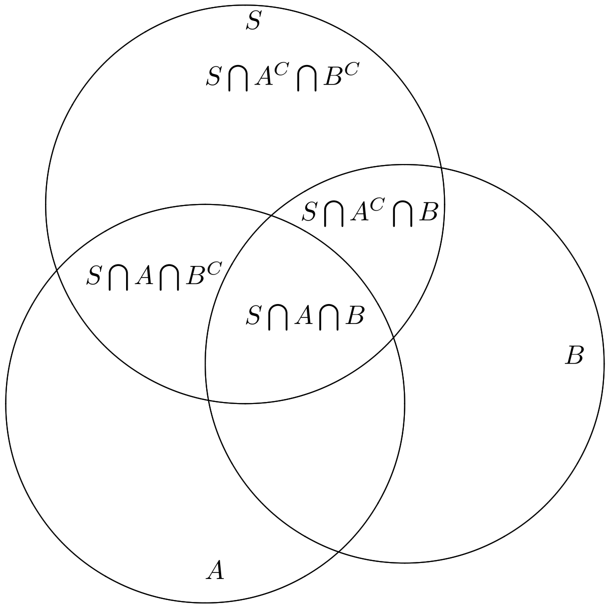
$$\mu(F) = \lim_{n\to\infty} \mu(F_n). \tag{9.4}$$

If $\cdots F_n \supseteq F_{n+1} \supseteq \cdots$, and if $F = \cap_{n=1}^{\infty} F_n$ for $F_n \in \mathcal{S}$ then if $\mu(F_1) < \infty$,

$$\mu(F) = \lim_{n\to\infty} \mu(F_n). \tag{9.5}$$

Also, $(\mathcal{S}, \mu)$ is complete. By this it is meant that if $F \in \mathcal{S}$ and if $E \subseteq \Omega$ with $\mu(E \setminus F) + \mu(F \setminus E) = 0$, then $E \in \mathcal{S}$.

Proof: First note that $\emptyset$ and Ω are obviously in $\mathcal{S}$. Now suppose $A, B \in \mathcal{S}$. I will show $A \setminus B \equiv A \cap B^C$ is in $\mathcal{S}$. To do so, consider the following picture.



Since μ is subadditive,

$$\mu(S) \leq \mu\left(S \cap A \cap B^C\right) + \mu(A \cap B \cap S) + \mu\left(S \cap B \cap A^C\right) + \mu\left(S \cap A^C \cap B^C\right).$$

Now using $A, B \in \mathcal{S}$,

$$\begin{aligned} \mu(S) &\leq \mu\left(S \cap A \cap B^C\right) + \mu(S \cap A \cap B) + \mu\left(S \cap B \cap A^C\right) + \mu\left(S \cap A^C \cap B^C\right) \\ &= \mu(S \cap A) + \mu\left(S \cap A^C\right) = \mu(S) \end{aligned}$$

It follows equality holds in the above. Now observe using the picture if you like that

$$(A \cap B \cap S) \cup \left(S \cap B \cap A^C\right) \cup \left(S \cap A^C \cap B^C\right) = S \setminus (A \setminus B)$$

and therefore,

$$\begin{aligned} \mu(S) &= \mu\left(S \cap A \cap B^C\right) + \mu(A \cap B \cap S) + \mu\left(S \cap B \cap A^C\right) + \mu\left(S \cap A^C \cap B^C\right) \\ &\geq \mu(S \cap (A \setminus B)) + \mu(S \setminus (A \setminus B)). \end{aligned}$$

Therefore, since S is arbitrary, this shows $A \setminus B \in \mathcal{S}$.

Since $\Omega \in \mathcal{S}$, this shows that $A \in \mathcal{S}$ if and only if $A^C \in \mathcal{S}$. Now if $A, B \in \mathcal{S}$, $A \cup B = (A^C \cap B^C)^C = (A^C \setminus B)^C \in \mathcal{S}$. By induction, if $A_1, \cdots, A_n \in \mathcal{S}$, then so is $\cup_{i=1}^n A_i$. If $A, B \in \mathcal{S}$, with $A \cap B = \emptyset$,

$$\mu(A \cup B) = \mu((A \cup B) \cap A) + \mu((A \cup B) \setminus A) = \mu(A) + \mu(B).$$

By induction, if $A_i \cap A_j = \emptyset$ and $A_i \in \mathcal{S}$, $\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$.

Now let $A = \cup_{i=1}^\infty A_i$ where $A_i \cap A_j = \emptyset$ for $i \neq j$.

$$\sum_{i=1}^\infty \mu(A_i) \geq \mu(A) \geq \mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i).$$

Since this holds for all n, you can take the limit as $n \to \infty$ and conclude,

$$\sum_{i=1}^\infty \mu(A_i) = \mu(A)$$

which establishes 9.3. Part 9.4 follows from part 9.3 just as in the proof of Theorem 8.5 on Page 172. That is, letting $F_0 \equiv \emptyset$, use part 9.3 to write

$$\begin{aligned} \mu(F) &= \mu\left(\cup_{k=1}^\infty (F_k \setminus F_{k-1})\right) = \sum_{k=1}^\infty \mu(F_k \setminus F_{k-1}) \\ &= \lim_{n\to\infty} \sum_{k=1}^n \left(\mu(F_k) - \mu(F_{k-1})\right) = \lim_{n\to\infty} \mu(F_n). \end{aligned}$$

In order to establish 9.5, let the F_n be as given there. Then, since $(F_1 \setminus F_n)$ increases to $(F_1 \setminus F)$, 9.4 implies

$$\lim_{n\to\infty} \left(\mu(F_1) - \mu(F_n)\right) = \mu(F_1 \setminus F).$$

Now $\mu(F_1 \setminus F) + \mu(F) \geq \mu(F_1)$ and so $\mu(F_1 \setminus F) \geq \mu(F_1) - \mu(F)$. Hence

$$\lim_{n\to\infty} \left(\mu(F_1) - \mu(F_n)\right) = \mu(F_1 \setminus F) \geq \mu(F_1) - \mu(F)$$

which implies

$$\lim_{n\to\infty} \mu(F_n) \leq \mu(F).$$

But since $F \subseteq F_n$,

$$\mu(F) \leq \lim_{n\to\infty} \mu(F_n)$$

and this establishes 9.5. Note that it was assumed $\mu(F_1) < \infty$ because $\mu(F_1)$ was subtracted from both sides.

It remains to show $\mathcal{S}$ is closed under countable unions. Recall that if $A \in \mathcal{S}$, then $A^C \in \mathcal{S}$ and $\mathcal{S}$ is closed under finite unions. Let $A_i \in \mathcal{S}$, $A = \cup_{i=1}^\infty A_i$, $B_n = \cup_{i=1}^n A_i$. Then

$$\begin{aligned} \mu(S) &= \mu(S \cap B_n) + \mu(S \setminus B_n) \\ &= (\mu\lfloor S)(B_n) + (\mu\lfloor S)(B_n^C). \end{aligned} \tag{9.6}$$

By Lemma 9.3 B_n is $(\mu\lfloor S)$ measurable and so is B_n^C. I want to show $\mu(S) \geq \mu(S \setminus A) + \mu(S \cap A)$. If $\mu(S) = \infty$, there is nothing to prove. Assume $\mu(S) < \infty$.

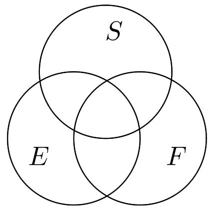
Then apply Parts 9.5 and 9.4 to the outer measure, $\mu\lfloor S$ in 9.6 and let $n \to \infty$. Thus

$$B_n \uparrow A,\ \ B_n^C \downarrow A^C$$

and this yields $\mu(S) = (\mu\lfloor S)(A) + (\mu\lfloor S)(A^C) = \mu(S \cap A) + \mu(S \setminus A)$.

Therefore $A \in \mathcal{S}$ and this proves Parts 9.3, 9.4, and 9.5. It remains to prove the last assertion about the measure being complete.

Let $F \in \mathcal{S}$ and let $\mu(E \setminus F) + \mu(F \setminus E) = 0$. Consider the following picture.



Then referring to this picture and using $F \in \mathcal{S}$,

$$\begin{aligned}
\mu(S) &\leq \mu(S \cap E) + \mu(S \setminus E) \\
&\leq \mu(S \cap E \cap F) + \mu((S \cap E) \setminus F) + \mu(S \setminus F) + \mu(F \setminus E) \\
&\leq \mu(S \cap F) + \mu(E \setminus F) + \mu(S \setminus F) + \mu(F \setminus E) \\
&= \mu(S \cap F) + \mu(S \setminus F) = \mu(S)
\end{aligned}$$

Hence $\mu(S) = \mu(S \cap E) + \mu(S \setminus E)$ and so $E \in \mathcal{S}$. This shows that $(\mathcal{S}, \mu)$ is complete and proves the theorem.

Completeness usually occurs in the following form. $E \subseteq F \in \mathcal{S}$ and $\mu(F) = 0$. Then $E \in \mathcal{S}$.

Where do outer measures come from? One way to obtain an outer measure is to start with a measure μ, defined on a σ algebra of sets, $\mathcal{S}$, and use the following definition of the outer measure induced by the measure.

Definition 9.5 *Let μ be a measure defined on a σ algebra of sets, $\mathcal{S} \subseteq \mathcal{P}(\Omega)$. Then the outer measure induced by μ, denoted by $\overline{\mu}$ is defined on $\mathcal{P}(\Omega)$ as*

$$\overline{\mu}(E) = \inf\{\mu(F)\ : F \in \mathcal{S} \text{ and } F \supseteq E\}.$$

A measure space, $(\mathcal{S}, \Omega, \mu)$ is σ finite if there exist measurable sets, Ω_i with $\mu(\Omega_i) < \infty$ and $\Omega = \cup_{i=1}^{\infty} \Omega_i$.

You should prove the following lemma.

Lemma 9.6 *If $(\mathcal{S}, \Omega, \mu)$ is σ finite then there exist disjoint measurable sets, $\{B_n\}$ such that $\mu(B_n) < \infty$ and $\cup_{n=1}^{\infty} B_n = \Omega$.*

The following lemma deals with the outer measure generated by a measure which is σ finite. It says that if the given measure is σ finite and complete then no new measurable sets are gained by going to the induced outer measure and then considering the measurable sets in the sense of Caratheodory.

Lemma 9.7 *Let* $(\Omega, \mathcal{S}, \mu)$ *be any measure space and let* $\overline{\mu} : \mathcal{P}(\Omega) \to [0, \infty]$ *be the outer measure induced by* μ. *Then* $\overline{\mu}$ *is an outer measure as claimed and if* $\overline{\mathcal{S}}$ *is the set of* $\overline{\mu}$ *measurable sets in the sense of Caratheodory, then* $\overline{\mathcal{S}} \supseteq \mathcal{S}$ *and* $\overline{\mu} = \mu$ *on* $\mathcal{S}$. *Furthermore, if* μ *is* σ *finite and* $(\Omega, \mathcal{S}, \mu)$ *is complete, then* $\overline{\mathcal{S}} = \mathcal{S}$.

Proof: It is easy to see that $\overline{\mu}$ is an outer measure. Let $E \in \mathcal{S}$. The plan is to show $E \in \overline{\mathcal{S}}$ and $\overline{\mu}(E) = \mu(E)$. To show this, let $S \subseteq \Omega$ and then show

$$\overline{\mu}(S) \geq \overline{\mu}(S \cap E) + \overline{\mu}(S \setminus E). \tag{9.7}$$

This will verify that $E \in \overline{\mathcal{S}}$. If $\overline{\mu}(S) = \infty$, there is nothing to prove, so assume $\overline{\mu}(S) < \infty$. Thus there exists $T \in \mathcal{S}$, $T \supseteq S$, and

$$\begin{aligned} \overline{\mu}(S) &> \mu(T) - \varepsilon = \mu(T \cap E) + \mu(T \setminus E) - \varepsilon \\ &\geq \overline{\mu}(T \cap E) + \overline{\mu}(T \setminus E) - \varepsilon \\ &\geq \overline{\mu}(S \cap E) + \overline{\mu}(S \setminus E) - \varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves 9.7 and verifies $\mathcal{S} \subseteq \overline{\mathcal{S}}$. Now if $E \in \mathcal{S}$ and $V \supseteq E$ with $V \in \mathcal{S}$, $\mu(E) \leq \mu(V)$. Hence, taking inf, $\mu(E) \leq \overline{\mu}(E)$. But also $\mu(E) \geq \overline{\mu}(E)$ since $E \in \mathcal{S}$ and $E \supseteq E$. Hence

$$\overline{\mu}(E) \leq \mu(E) \leq \overline{\mu}(E).$$

Next consider the claim about not getting any new sets from the outer measure in the case the measure space is σ finite and complete.

Claim 1: If $E, D \in \mathcal{S}$, and $\mu(E \setminus D) = 0$, then if $D \subseteq F \subseteq E$, it follows $F \in \mathcal{S}$.

Proof of claim 1:

$$F \setminus D \subseteq E \setminus D \in \mathcal{S},$$

and $E \setminus D$ is a set of measure zero. Therefore, since $(\Omega, \mathcal{S}, \mu)$ is complete, $F \setminus D \in \mathcal{S}$ and so

$$F = D \cup (F \setminus D) \in \mathcal{S}.$$

Claim 2: Suppose $F \in \overline{\mathcal{S}}$ and $\overline{\mu}(F) < \infty$. Then $F \in \mathcal{S}$.

Proof of the claim 2: From the definition of $\overline{\mu}$, it follows there exists $E \in \mathcal{S}$ such that $E \supseteq F$ and $\mu(E) = \overline{\mu}(F)$. Therefore,

$$\overline{\mu}(E) = \overline{\mu}(E \setminus F) + \overline{\mu}(F)$$

so

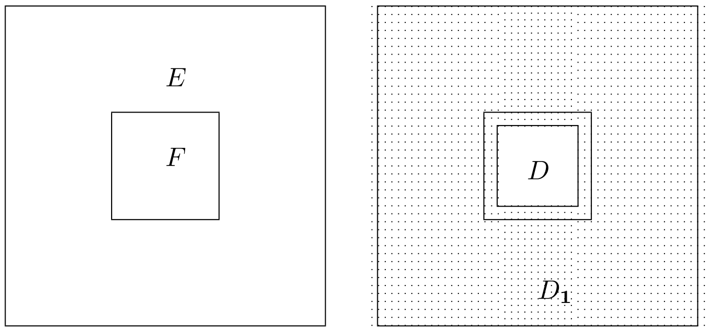
$$\overline{\mu}(E \setminus F) = 0. \tag{9.8}$$

Similarly, there exists $D_1 \in \mathcal{S}$ such that

$$D_1 \subseteq E, \ D_1 \supseteq (E \setminus F), \ \mu(D_1) = \overline{\mu}(E \setminus F).$$

and

$$\overline{\mu}(D_1 \setminus (E \setminus F)) = 0. \tag{9.9}$$



Now let $D = E \setminus D_1$. It follows $D \subseteq F$ because if $x \in D$, then $x \in E$ but $x \notin (E \setminus F)$ and so $x \in F$. Also $F \setminus D = D_1 \setminus (E \setminus F)$ because both sides equal $D_1 \cap F \setminus E$.

Then from 9.8 and 9.9,

$$\begin{aligned} \mu(E \setminus D) &\leq \overline{\mu}(E \setminus F) + \overline{\mu}(F \setminus D) \\ &= \overline{\mu}(E \setminus F) + \overline{\mu}(D_1 \setminus (E \setminus F)) = 0. \end{aligned}$$

By Claim 1, it follows $F \in \mathcal{S}$. This proves Claim 2.

Now since $(\Omega, \mathcal{S}, \mu)$ is σ finite, there are sets of $\mathcal{S}$, $\{B_n\}_{n=1}^{\infty}$ such that $\mu(B_n) < \infty, \cup_n B_n = \Omega$. Then $F \cap B_n \in \mathcal{S}$ by Claim 2. Therefore, $F = \cup_{n=1}^{\infty} F \cap B_n \in \mathcal{S}$ and so $\mathcal{S} = \overline{\mathcal{S}}$. This proves the lemma.

9.2 Regular Measures

Usually Ω is not just a set. It is also a topological space. It is very important to consider how the measure is related to this topology.

Definition 9.8 *Let μ be a measure on a σ algebra $\mathcal{S}$, of subsets of Ω, where (Ω, τ) is a topological space. μ is a Borel measure if $\mathcal{S}$ contains all Borel sets. μ is called outer regular if μ is Borel and for all $E \in \mathcal{S}$,*

$$\mu(E) = \inf\{\mu(V) : V \text{ is open and } V \supseteq E\}.$$

μ is called inner regular if μ is Borel and

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, \text{ and } K \text{ is compact}\}.$$

If the measure is both outer and inner regular, it is called regular.

It will be assumed in what follows that (Ω, τ) is a locally compact Hausdorff space. This means it is Hausdorff: If $p, q \in \Omega$ such that $p \neq q$, there exist open sets, U_p and U_q containing p and q respectively such that $U_p \cap U_q = \emptyset$ and Locally compact: There exists a basis of open sets for the topology, $\mathcal{B}$ such that for each $U \in \mathcal{B}$, $\overline{U}$ is compact. Recall $\mathcal{B}$ is a basis for the topology if $\cup \mathcal{B} = \Omega$ and if every open set in τ is the union of sets of $\mathcal{B}$. Also recall a Hausdorff space is normal if whenever H and C are two closed sets, there exist disjoint open sets, U_H and U_C containing H and C respectively. A regular space is one which has the property that if p is a point not in H, a closed set, then there exist disjoint open sets, U_p and U_H containing p and H respectively.

9.3 Urysohn's lemma

Urysohn's lemma which characterizes normal spaces is a very important result which is useful in general topology and in the construction of measures. Because it is somewhat technical a proof is given for the part which is needed.

Theorem 9.9 *(Urysohn) Let (X, τ) be normal and let $H \subseteq U$ where H is closed and U is open. Then there exists $g : X \to [0, 1]$ such that g is continuous, $g(x) = 1$ on H and $g(x) = 0$ if $x \notin U$.*

Proof: Let $D \equiv \{r_n\}_{n=1}^{\infty}$ be the rational numbers in $(0, 1)$. Choose V_{r_1} an open set such that

$$H \subseteq V_{r_1} \subseteq \overline{V}_{r_1} \subseteq U.$$

This can be done by applying the assumption that X is normal to the disjoint closed sets, H and U^C, to obtain open sets V and W with

$$H \subseteq V, \ U^C \subseteq W, \text{and } V \cap W = \emptyset.$$

Then

$$H \subseteq V \subseteq \overline{V}, \ \overline{V} \cap U^C = \emptyset$$

and so let $V_{r_1} = V$.

Suppose $V_{r_1}, \cdots, V_{r_k}$ have been chosen and list the rational numbers $r_1, \cdots, r_k$ in order,

$$r_{l_1} < r_{l_2} < \cdots < r_{l_k} \text{ for } \{l_1, \cdots, l_k\} = \{1, \cdots, k\}.$$

If $r_{k+1} > r_{l_k}$ then letting $p = r_{l_k}$, let $V_{r_{k+1}}$ satisfy

$$\overline{V}_p \subseteq V_{r_{k+1}} \subseteq \overline{V}_{r_{k+1}} \subseteq U.$$

If $r_{k+1} \in (r_{l_i}, r_{l_{i+1}})$, let $p = r_{l_i}$ and let $q = r_{l_{i+1}}$. Then let $V_{r_{k+1}}$ satisfy

$$\overline{V}_p \subseteq V_{r_{k+1}} \subseteq \overline{V}_{r_{k+1}} \subseteq V_q.$$

If $r_{k+1} < r_{l_1}$, let $p = r_{l_1}$ and let $V_{r_{k+1}}$ satisfy

$$H \subseteq V_{r_{k+1}} \subseteq \overline{V}_{r_{k+1}} \subseteq V_p.$$

Thus there exist open sets V_r for each $r \in \mathbb{Q} \cap (0,1)$ with the property that if $r < s$,

$$H \subseteq V_r \subseteq \overline{V}_r \subseteq V_s \subseteq \overline{V}_s \subseteq U.$$

Now let

$$f(x) = \inf\{t \in D : x \in V_t\},\ f(x) \equiv 1 \text{ if } x \notin \bigcup_{t \in D} V_t.$$

I claim f is continuous.

$$f^{-1}([0,a)) = \cup\{V_t : t < a, t \in D\},$$

an open set.

Next consider $x \in f^{-1}([0,a])$ so $f(x) \leq a$. If $t > a$, then $x \in V_t$ because if not, then

$$\inf\{t \in D : x \in V_t\} > a.$$

Thus

$$f^{-1}([0,a]) = \cap\{V_t : t > a\} = \cap\{\overline{V}_t : t > a\}$$

which is a closed set. If $a = 1$, $f^{-1}([0,1]) = f^{-1}([0,a]) = X$. Therefore,

$$f^{-1}((a,1]) = X \setminus f^{-1}([0,a]) = \text{open set}.$$

It follows f is continuous. Clearly $f(x) = 0$ on H. If $x \in U^C$, then $x \notin V_t$ for any $t \in D$ so $f(x) = 1$ on U^C. Let $g(x) = 1 - f(x)$. This proves the theorem.

In any metric space there is a much easier proof of the conclusion of Urysohn's lemma which applies.

Lemma 9.10 *Let S be a nonempty subset of a metric space, (X,d). Define*

$$f(x) \equiv \operatorname{dist}(x,S) \equiv \inf\{d(x,y) : y \in S\}.$$

Then f is continuous.

Proof: Consider $|f(x) - f(x_1)|$and suppose without loss of generality that $f(x_1) \geq f(x)$. Then choose $y \in S$ such that $f(x) + \varepsilon > d(x,y)$. Then

$$\begin{aligned}
|f(x_1) - f(x)| &= f(x_1) - f(x) \leq f(x_1) - d(x,y) + \varepsilon \\
&\leq d(x_1,y) - d(x,y) + \varepsilon \\
&\leq d(x,x_1) + d(x,y) - d(x,y) + \varepsilon \\
&= d(x_1,x) + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, it follows that $|f(x_1) - f(x)| \leq d(x_1,x)$ and this proves the lemma.

Theorem 9.11 *(Urysohn's lemma for metric space) Let H be a closed subset of an open set, U in a metric space, (X,d). Then there exists a continuous function, $g : X \to [0,1]$ such that $g(x) = 1$ for all $x \in H$ and $g(x) = 0$ for all $x \notin U$.*

Proof: If $x \notin C$, a closed set, then $\text{dist}(x, C) > 0$ because if not, there would exist a sequence of points of C converging to x and it would follow that $x \in C$. Therefore, $\text{dist}(x, H) + \text{dist}\left(x, U^C\right) > 0$ for all $x \in X$. Now define a continuous function, g as

$$g(x) \equiv \frac{\text{dist}\left(x, U^C\right)}{\text{dist}(x, H) + \text{dist}(x, U^C)}.$$

It is easy to see this verifies the conclusions of the theorem and this proves the theorem.

Theorem 9.12 *Every compact Hausdorff space is normal.*

Proof: First it is shown that X, is regular. Let H be a closed set and let $p \notin H$. Then for each $h \in H$, there exists an open set U_h containing p and an open set V_h containing h such that $U_h \cap V_h = \emptyset$. Since H must be compact, it follows there are finitely many of the sets V_h, $V_{h_1} \cdots V_{h_n}$ such that $H \subseteq \cup_{i=1}^n V_{h_i}$. Then letting $U = \cap_{i=1}^n U_{h_i}$ and $V = \cup_{i=1}^n V_{h_i}$, it follows that $p \in U$, $H \in V$ and $U \cap V = \emptyset$. Thus X is regular as claimed.

Next let K and H be disjoint nonempty closed sets.Using regularity of X, for every $k \in K$, there exists an open set U_k containing k and an open set V_k containing H such that these two open sets have empty intersection. Thus $H \cap \overline{U}_k = \emptyset$. Finitely many of the U_k, $U_{k_1}, \cdots, U_{k_p}$ cover K and so $\cup_{i=1}^p \overline{U}_{k_i}$ is a closed set which has empty intersection with H. Therefore, $K \subseteq \cup_{i=1}^p U_{k_i}$ and $H \subseteq \left(\cup_{i=1}^p \overline{U}_{k_i}\right)^C$. This proves the theorem.

A useful construction when dealing with locally compact Hausdorff spaces is the notion of the one point compactification of the space discussed earler. However, it is reviewed here for the sake of convenience or in case you have not read the earlier treatment.

Definition 9.13 *Suppose (X, τ) is a locally compact Hausdorff space. Then let $\widetilde{X} \equiv X \cup \{\infty\}$ where ∞ is just the name of some point which is not in X which is called the point at infinity. A basis for the topology $\widetilde{\tau}$ for $\widetilde{X}$ is*

$$\tau \cup \left\{K^C \text{ where } K \text{ is a compact subset of } X\right\}.$$

The complement is taken with respect to $\widetilde{X}$ and so the open sets, K^C are basic open sets which contain ∞.

The reason this is called a compactification is contained in the next lemma.

Lemma 9.14 *If (X, τ) is a locally compact Hausdorff space, then $\left(\widetilde{X}, \widetilde{\tau}\right)$ is a compact Hausdorff space.*

Proof: Since (X, τ) is a locally compact Hausdorff space, it follows $\left(\widetilde{X}, \widetilde{\tau}\right)$ is a Hausdorff topological space. The only case which needs checking is the one of $p \in X$ and ∞. Since (X, τ) is locally compact, there exists an open set of τ, U

having compact closure which contains p. Then $p \in U$ and $\infty \in \overline{U}^C$ and these are disjoint open sets containing the points, p and ∞ respectively. Now let $\mathcal{C}$ be an open cover of $\widetilde{X}$ with sets from $\widetilde{\tau}$. Then ∞ must be in some set, U_∞ from $\mathcal{C}$, which must contain a set of the form K^C where K is a compact subset of X. Then there exist sets from $\mathcal{C}$, $U_1, \cdots, U_r$ which cover K. Therefore, a finite subcover of $\widetilde{X}$ is $U_1, \cdots, U_r, U_\infty$.

Theorem 9.15 *Let X be a locally compact Hausdorff space, and let K be a compact subset of the open set V. Then there exists a continuous function, $f : X \to [0,1]$, such that f equals 1 on K and $\overline{\{x : f(x) \neq 0\}} \equiv \operatorname{spt}(f)$ is a compact subset of V.*

Proof: Let $\widetilde{X}$ be the space just described. Then K and V are respectively closed and open in $\widetilde{\tau}$. By Theorem 9.12 there exist open sets in $\widetilde{\tau}$, U, and W such that $K \subseteq U, \infty \in V^C \subseteq W$, and $U \cap W = U \cap (W \setminus \{\infty\}) = \emptyset$. Thus $W \setminus \{\infty\}$ is an open set in the original topological space which contains V^C, U is an open set in the original topological space which contains K, and $W \setminus \{\infty\}$ and U are disjoint.

Now for each $x \in K$, let U_x be a basic open set whose closure is compact and such that

$$x \in U_x \subseteq U.$$

Thus $\overline{U_x}$ must have empty intersection with V^C because the open set, $W \setminus \{\infty\}$ contains no points of U_x. Since K is compact, there are finitely many of these sets, $U_{x_1}, U_{x_2}, \cdots, U_{x_n}$ which cover K. Now let $H \equiv \cup_{i=1}^n U_{x_i}$.

Claim: $\overline{H} = \cup_{i=1}^n \overline{U_{x_i}}$

Proof of claim: Suppose $p \in \overline{H}$. If $p \notin \cup_{i=1}^n \overline{U_{x_i}}$ then if follows $p \notin \overline{U_{x_i}}$ for each i. Therefore, there exists an open set, R_i containing p such that R_i contains no other points of U_{x_i}. Therefore, $R \equiv \cap_{i=1}^n R_i$ is an open set containing p which contains no other points of $\cup_{i=1}^n U_{x_i} = W$, a contradiction. Therefore, $\overline{H} \subseteq \cup_{i=1}^n \overline{U_{x_i}}$. On the other hand, if $p \in \overline{U_{x_i}}$ then p is obviously in $\overline{H}$ so this proves the claim.

From the claim, $K \subseteq H \subseteq \overline{H} \subseteq V$ and $\overline{H}$ is compact because it is the finite union of compact sets. Repeating the same argument, there exists an open set, I such that $\overline{H} \subseteq I \subseteq \overline{I} \subseteq V$ with $\overline{I}$ compact. Now $(\overline{I}, \tau_I)$ is a compact topological space where τ_I is the topology which is obtained by taking intersections of open sets in X with $\overline{I}$. Therefore, by Urysohn's lemma, there exists $f : \overline{I} \to [0,1]$ such that f is continuous at every point of $\overline{I}$ and also $f(K) = 1$ while $f(\overline{I} \setminus H) = 0$. Extending f to equal 0 on $\overline{I}^C$, it follows that f is continuous on X, has values in $[0,1]$, and satisfies $f(K) = 1$ and $\operatorname{spt}(f)$ is a compact subset contained in $\overline{I} \subseteq V$. This proves the theorem.

In fact, the conclusion of the above theorem could be used to prove that the topological space is locally compact. However, this is not needed here.

Definition 9.16 *Define* $\operatorname{spt}(f)$ *(support of f) to be the closure of the set $\{x : f(x) \neq 0\}$. If V is an open set, $C_c(V)$ will be the set of continuous functions f, defined on Ω having* $\operatorname{spt}(f) \subseteq V$. *Thus in Theorem 9.15, $f \in C_c(V)$.*

Definition 9.17 *If K is a compact subset of an open set, V, then $K \prec \phi \prec V$ if*

$$\phi \in C_c(V),\ \phi(K) = \{1\},\ \phi(\Omega) \subseteq [0,1],$$

where Ω denotes the whole topological space considered. Also for $\phi \in C_c(\Omega)$, $K \prec \phi$ if

$$\phi(\Omega) \subseteq [0,1] \text{ and } \phi(K) = 1.$$

and $\phi \prec V$ if

$$\phi(\Omega) \subseteq [0,1] \text{ and } \operatorname{spt}(\phi) \subseteq V.$$

Theorem 9.18 *(Partition of unity) Let K be a compact subset of a locally compact Hausdorff topological space satisfying Theorem 9.15 and suppose*

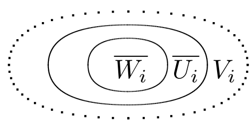
$$K \subseteq V = \cup_{i=1}^{n} V_i,\ V_i \text{ open.}$$

Then there exist $\psi_i \prec V_i$ with

$$\sum_{i=1}^{n} \psi_i(x) = 1$$

for all $x \in K$.

Proof: Let $K_1 = K \setminus \cup_{i=2}^{n} V_i$. Thus K_1 is compact and $K_1 \subseteq V_1$. Let $K_1 \subseteq W_1 \subseteq \overline{W}_1 \subseteq V_1$ with $\overline{W}_1$compact. To obtain W_1, use Theorem 9.15 to get f such that $K_1 \prec f \prec V_1$ and let $W_1 \equiv \{x : f(x) \neq 0\}$. Thus $W_1, V_2, \cdots V_n$ covers K and $\overline{W}_1 \subseteq V_1$. Let $K_2 = K \setminus (\cup_{i=3}^{n} V_i \cup W_1)$. Then K_2 is compact and $K_2 \subseteq V_2$. Let $K_2 \subseteq W_2 \subseteq \overline{W}_2 \subseteq V_2$ $\overline{W}_2$ compact. Continue this way finally obtaining $W_1, \cdots, W_n$, $K \subseteq W_1 \cup \cdots \cup W_n$, and $\overline{W}_i \subseteq V_i$ $\overline{W}_i$ compact. Now let $\overline{W}_i \subseteq U_i \subseteq \overline{U}_i \subseteq V_i$,$\overline{U}_i$ compact.



By Theorem 9.15, let $\overline{U}_i \prec \phi_i \prec V_i$, $\cup_{i=1}^{n} \overline{W}_i \prec \gamma \prec \cup_{i=1}^{n} U_i$. Define

$$\psi_i(x) = \begin{cases} \gamma(x)\phi_i(x) / \sum_{j=1}^{n} \phi_j(x) \text{ if } \sum_{j=1}^{n} \phi_j(x) \neq 0, \\ 0 \text{ if } \sum_{j=1}^{n} \phi_j(x) = 0. \end{cases}$$

If x is such that $\sum_{j=1}^{n} \phi_j(x) = 0$, then $x \notin \cup_{i=1}^{n} \overline{U}_i$. Consequently $\gamma(y) = 0$ for all y near x and so $\psi_i(y) = 0$ for all y near x. Hence ψ_i is continuous at such x. If $\sum_{j=1}^{n} \phi_j(x) \neq 0$, this situation persists near x and so ψ_i is continuous at such points. Therefore ψ_i is continuous. If $x \in K$, then $\gamma(x) = 1$ and so $\sum_{j=1}^{n} \psi_j(x) = 1$. Clearly $0 \leq \psi_i(x) \leq 1$ and $\operatorname{spt}(\psi_j) \subseteq V_j$. This proves the theorem.

The following corollary won't be needed immediately but is of considerable interest later.

Corollary 9.19 *If H is a compact subset of V_i, there exists a partition of unity such that $\psi_i(x) = 1$ for all $x \in H$ in addition to the conclusion of Theorem 9.18.*

Proof: Keep V_i the same but replace V_j with $\widetilde{V_j} \equiv V_j \setminus H$. Now in the proof above, applied to this modified collection of open sets, if $j \neq i, \phi_j(x) = 0$ whenever $x \in H$. Therefore, $\psi_i(x) = 1$ on H.

9.4 Positive Linear Functionals

Definition 9.20 *Let (Ω, τ) be a topological space. $L : C_c(\Omega) \to \mathbb{C}$ is called a positive linear functional if L is linear,*

$$L(af_1 + bf_2) = aLf_1 + bLf_2,$$

and if $Lf \geq 0$ whenever $f \geq 0$.

Theorem 9.21 *(Riesz representation theorem) Let (Ω, τ) be a locally compact Hausdorff space and let L be a positive linear functional on $C_c(\Omega)$. Then there exists a σ algebra $\mathcal{S}$ containing the Borel sets and a unique measure μ, defined on $\mathcal{S}$, such that*

$$\mu \text{ is complete,} \tag{9.10}$$

$$\mu(K) < \infty \text{ for all } K \text{ compact,} \tag{9.11}$$

$$\mu(F) = \sup\{\mu(K) : K \subseteq F,\ K \text{ compact}\},$$

for all F open and for all $F \in \mathcal{S}$ with $\mu(F) < \infty$,

$$\mu(F) = \inf\{\mu(V) : V \supseteq F,\ V \text{ open}\}$$

for all $F \in \mathcal{S}$, and

$$\int f d\mu = Lf \text{ for all } f \in C_c(\Omega). \tag{9.12}$$

The plan is to define an outer measure and then to show that it, together with the σ algebra of sets measurable in the sense of Caratheodory, satisfies the conclusions of the theorem. Always, K will be a compact set and V will be an open set.

Definition 9.22 *$\mu(V) \equiv \sup\{Lf : f \prec V\}$ for V open, $\mu(\emptyset) = 0$. $\mu(E) \equiv \inf\{\mu(V) : V \supseteq E\}$ for arbitrary sets E.*

Lemma 9.23 *μ is a well-defined outer measure.*

Proof: First it is necessary to verify μ is well defined because there are two descriptions of it on open sets. Suppose then that $\mu_1(V) \equiv \inf\{\mu(U) : U \supseteq V$ and U is open$\}$. It is required to verify that $\mu_1(V) = \mu(V)$ where μ is given as $\sup\{Lf : f \prec V\}$. If $U \supseteq V$, then $\mu(U) \geq \mu(V)$ directly from the definition. Hence from the definition of μ_1, it follows $\mu_1(V) \geq \mu(V)$. On the other hand, $V \supseteq V$ and so $\mu_1(V) \leq \mu(V)$. This verifies μ is well defined.

It remains to show that μ is an outer measure. Let $V = \cup_{i=1}^{\infty} V_i$ and let $f \prec V$. Then $\text{spt}(f) \subseteq \cup_{i=1}^{n} V_i$ for some n. Let $\psi_i \prec V_i$, $\sum_{i=1}^{n} \psi_i = 1$ on $\text{spt}(f)$.

$$Lf = \sum_{i=1}^{n} L(f\psi_i) \leq \sum_{i=1}^{n} \mu(V_i) \leq \sum_{i=1}^{\infty} \mu(V_i).$$

Hence

$$\mu(V) \leq \sum_{i=1}^{\infty} \mu(V_i)$$

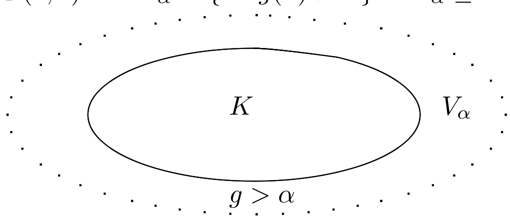
since $f \prec V$ is arbitrary. Now let $E = \cup_{i=1}^{\infty} E_i$. Is $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$? Without loss of generality, it can be assumed $\mu(E_i) < \infty$ for each i since if not so, there is nothing to prove. Let $V_i \supseteq E_i$ with $\mu(E_i) + \varepsilon 2^{-i} > \mu(V_i)$.

$$\mu(E) \leq \mu(\cup_{i=1}^{\infty} V_i) \leq \sum_{i=1}^{\infty} \mu(V_i) \leq \varepsilon + \sum_{i=1}^{\infty} \mu(E_i).$$

Since ε was arbitrary, $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$ which proves the lemma.

Lemma 9.24 *Let K be compact, $g \geq 0$, $g \in C_c(\Omega)$, and $g = 1$ on K. Then $\mu(K) \leq Lg$. Also $\mu(K) < \infty$ whenever K is compact.*

Proof: Let $\alpha \in (0,1)$ and $V_\alpha = \{x : g(x) > \alpha\}$ so $V_\alpha \supseteq K$ and let $h \prec V_\alpha$.



Then $h \leq 1$ on V_α while $g\alpha^{-1} \geq 1$ on V_α and so $g\alpha^{-1} \geq h$ which implies $L(g\alpha^{-1}) \geq Lh$ and that therefore, since L is linear,

$$Lg \geq \alpha Lh.$$

Since $h \prec V_\alpha$ is arbitrary, and $K \subseteq V_\alpha$,

$$Lg \geq \alpha\mu(V_\alpha) \geq \alpha\mu(K).$$

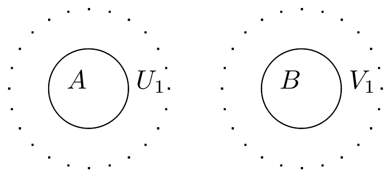
Letting $\alpha \uparrow 1$ yields $Lg \geq \mu(K)$. This proves the first part of the lemma. The second assertion follows from this and Theorem 9.15. If K is given, let

$$K \prec g \prec \Omega$$

and so from what was just shown, $\mu(K) \leq Lg < \infty$. This proves the lemma.

Lemma 9.25 *If A and B are disjoint compact subsets of Ω, then $\mu(A \cup B) = \mu(A) + \mu(B)$.*

Proof: By Theorem 9.15, there exists $h \in C_c(\Omega)$ such that $A \prec h \prec B^C$. Let $U_1 = h^{-1}((\frac{1}{2}, 1])$, $V_1 = h^{-1}([0, \frac{1}{2}))$. Then $A \subseteq U_1, B \subseteq V_1$ and $U_1 \cap V_1 = \emptyset$.



From Lemma 9.24 $\mu(A \cup B) < \infty$ and so there exists an open set, W such that

$$W \supseteq A \cup B, \ \mu(A \cup B) + \varepsilon > \mu(W).$$

Now let $U = U_1 \cap W$ and $V = V_1 \cap W$. Then

$$U \supseteq A, \ V \supseteq B, \ U \cap V = \emptyset, \text{and } \mu(A \cup B) + \varepsilon \geq \mu(W) \geq \mu(U \cup V).$$

Let $A \prec f \prec U, \ B \prec g \prec V$. Then by Lemma 9.24,

$$\mu(A \cup B) + \varepsilon \geq \mu(U \cup V) \geq L(f + g) = Lf + Lg \geq \mu(A) + \mu(B).$$

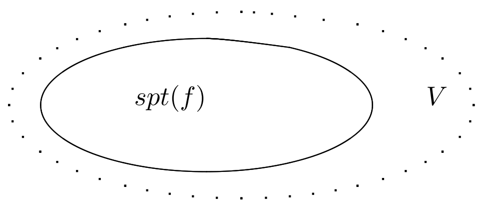
Since $\varepsilon > 0$ is arbitrary, this proves the lemma.

From Lemma 9.24 the following lemma is obtained.

Lemma 9.26 *Let $f \in C_c(\Omega)$, $f(\Omega) \subseteq [0, 1]$. Then $\mu(\text{spt}(f)) \geq Lf$. Also, every open set, V satisfies*

$$\mu(V) = \sup \{\mu(K) : K \subseteq V\}.$$

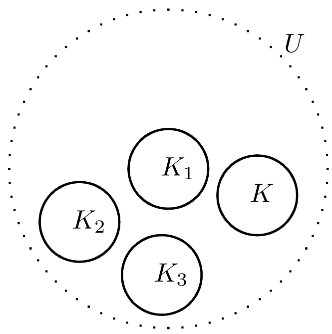
Proof: Let $V \supseteq \text{spt}(f)$ and let $\text{spt}(f) \prec g \prec V$. Then $Lf \leq Lg \leq \mu(V)$ because $f \leq g$. Since this holds for all $V \supseteq \text{spt}(f)$, $Lf \leq \mu(\text{spt}(f))$ by definition of μ.



Finally, let V be open and let $l < \mu(V)$. Then from the definition of μ, there exists $f \prec V$ such that $L(f) > l$. Therefore, $l < \mu(\text{spt}(f)) \leq \mu(V)$ and so this shows the claim about inner regularity of the measure on an open set.

Lemma 9.27 *If K is compact there exists V open, $V \supseteq K$, such that $\mu(V \setminus K) \leq \varepsilon$. If V is open with $\mu(V) < \infty$, then there exists a compact set, $K \subseteq V$ with $\mu(V \setminus K) \leq \varepsilon$.*

Proof: Let K be compact. Then from the definition of μ, there exists an open set U, with $\mu(U) < \infty$ and $U \supseteq K$. Suppose for every open set, V, containing $K, \mu(V \setminus K) > \varepsilon$. Then there exists $f \prec U \setminus K$ with $Lf > \varepsilon$. Consequently, $\mu((f)) > Lf > \varepsilon$. Let $K_1 = \text{spt}(f)$ and repeat the construction with $U \setminus K_1$ in place of U.



Continuing in this way yields a sequence of disjoint compact sets, $K, K_1, \cdots$ contained in U such that $\mu(K_i) > \varepsilon$. By Lemma 9.25

$$\mu(U) \geq \mu(K \cup \cup_{i=1}^{r} K_i) = \mu(K) + \sum_{i=1}^{r} \mu(K_i) \geq r\varepsilon$$

for all r, contradicting $\mu(U) < \infty$. This demonstrates the first part of the lemma.

To show the second part, employ a similar construction. Suppose $\mu(V \setminus K) > \varepsilon$ for all $K \subseteq V$. Then $\mu(V) > \varepsilon$ so there exists $f \prec V$ with $Lf > \varepsilon$. Let $K_1 = \text{spt}(f)$ so $\mu(\text{spt}(f)) > \varepsilon$. If $K_1 \cdots K_n$, disjoint, compact subsets of V have been chosen, there must exist $g \prec (V \setminus \cup_{i=1}^{n} K_i)$ be such that $Lg > \varepsilon$. Hence $\mu(\text{spt}(g)) > \varepsilon$. Let $K_{n+1} = \text{spt}(g)$. In this way there exists a sequence of disjoint compact subsets of V, $\{K_i\}$ with $\mu(K_i) > \varepsilon$. Thus for any m, $K_1 \cdots K_m$ are all contained in V and are disjoint and compact. By Lemma 9.25

$$\mu(V) \geq \mu(\cup_{i=1}^{m} K_i) = \sum_{i=1}^{m} \mu(K_i) > m\varepsilon$$

for all m, a contradiction to $\mu(V) < \infty$. This proves the second part.

Lemma 9.28 *Let $\mathcal{S}$ be the σ algebra of μ measurable sets in the sense of Caratheodory. Then $\mathcal{S} \supseteq$ Borel sets and μ is inner regular on every open set and for every $E \in \mathcal{S}$ with $\mu(E) < \infty$.*

Proof: Define

$$\mathcal{S}_1 = \{E \subseteq \Omega : E \cap K \in \mathcal{S}\}$$

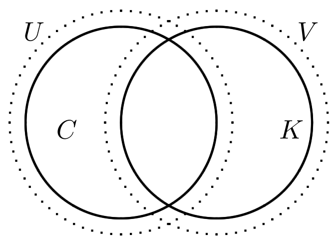
for all compact K.

Let C be a compact set. The idea is to show that $C \in \mathcal{S}$. From this it will follow that the closed sets are in $\mathcal{S}_1$ because if C is only closed, $C \cap K$ is compact. Hence $C \cap K = (C \cap K) \cap K \in \mathcal{S}$. The steps are to first show the compact sets are in $\mathcal{S}$ and this implies the closed sets are in $\mathcal{S}_1$. Then you show $\mathcal{S}_1$ is a σ algebra and so it contains the Borel sets. Finally, it is shown that $\mathcal{S}_1 = \mathcal{S}$ and then the inner regularity conclusion is established.

Let V be an open set with $\mu(V) < \infty$. I will show that

$$\mu(V) \geq \mu(V \setminus C) + \mu(V \cap C).$$

By Lemma 9.27, there exists an open set U containing C and a compact subset of V, K, such that $\mu(V \setminus K) < \varepsilon$ and $\mu(U \setminus C) < \varepsilon$.



Then by Lemma 9.25,

$$\begin{aligned} \mu(V) &\geq \mu(K) \geq \mu((K \setminus U) \cup (K \cap C)) \\ &= \mu(K \setminus U) + \mu(K \cap C) \\ &\geq \mu(V \setminus C) + \mu(V \cap C) - 3\varepsilon \end{aligned}$$

Since ε is arbitrary,

$$\mu(V) = \mu(V \setminus C) + \mu(V \cap C) \tag{9.13}$$

whenever C is compact and V is open. (If $\mu(V) = \infty$, it is obvious that $\mu(V) \geq \mu(V \setminus C) + \mu(V \cap C)$ and it is always the case that $\mu(V) \leq \mu(V \setminus C) + \mu(V \cap C)$.)

Of course 9.13 is exactly what needs to be shown for arbitrary S in place of V. It suffices to consider only S having $\mu(S) < \infty$. If $S \subseteq \Omega$, with $\mu(S) < \infty$, let $V \supseteq S$, $\mu(S) + \varepsilon > \mu(V)$. Then from what was just shown, if C is compact,

$$\begin{aligned} \varepsilon + \mu(S) &> \mu(V) = \mu(V \setminus C) + \mu(V \cap C) \\ &\geq \mu(S \setminus C) + \mu(S \cap C). \end{aligned}$$

Since ε is arbitrary, this shows the compact sets are in $\mathcal{S}$. As discussed above, this verifies the closed sets are in $\mathcal{S}_1$.

Therefore, $\mathcal{S}_1$ contains the closed sets and $\mathcal{S}$ contains the compact sets. Therefore, if $E \in \mathcal{S}$ and K is a compact set, it follows $K \cap E \in \mathcal{S}$ and so $\mathcal{S}_1 \supseteq \mathcal{S}$.

To see that $\mathcal{S}_1$ is closed with respect to taking complements, let $E \in \mathcal{S}_1$.

$$K = (E^C \cap K) \cup (E \cap K).$$

Then from the fact, just established, that the compact sets are in $\mathcal{S}$,

$$E^C \cap K = K \setminus (E \cap K) \in \mathcal{S}.$$

Similarly $\mathcal{S}_1$ is closed under countable unions. Thus $\mathcal{S}_1$ is a σ algebra which contains the Borel sets since it contains the closed sets.

The next task is to show $\mathcal{S}_1 = \mathcal{S}$. Let $E \in \mathcal{S}_1$ and let V be an open set with $\mu(V) < \infty$ and choose $K \subseteq V$ such that $\mu(V \setminus K) < \varepsilon$. Then since $E \in \mathcal{S}_1$, it follows $E \cap K \in \mathcal{S}$ and

$$\begin{aligned} \mu(V) &= \mu(V \setminus (K \cap E)) + \mu(V \cap (K \cap E)) \\ &\geq \mu(V \setminus E) + \mu(V \cap E) - \varepsilon \end{aligned}$$

because

$$\mu(V \cap (K \cap E)) + \overbrace{\mu(V \setminus K)}^{<\varepsilon} \geq \mu(V \cap E)$$

Since ε is arbitrary,

$$\mu(V) = \mu(V \setminus E) + \mu(V \cap E).$$

Now let $S \subseteq \Omega$. If $\mu(S) = \infty$, then $\mu(S) = \mu(S \cap E) + \mu(S \setminus E)$. If $\mu(S) < \infty$, let

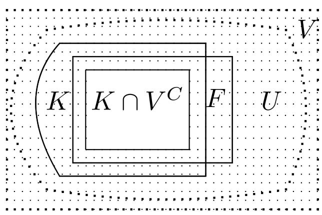
$$V \supseteq S,\ \mu(S) + \varepsilon \geq \mu(V).$$

Then

$$\mu(S) + \varepsilon \geq \mu(V) = \mu(V \setminus E) + \mu(V \cap E) \geq \mu(S \setminus E) + \mu(S \cap E).$$

Since ε is arbitrary, this shows that $E \in \mathcal{S}$ and so $\mathcal{S}_1 = \mathcal{S}$. Thus $\mathcal{S} \supseteq$ Borel sets as claimed.

From Lemma 9.26 and the definition of μ it follows μ is inner regular on all open sets. It remains to show that $\mu(F) = \sup\{\mu(K) : K \subseteq F\}$ for all $F \in \mathcal{S}$ with $\mu(F) < \infty$. It might help to refer to the following crude picture to keep things straight.



In this picture the shaded area is V.

Let U be an open set, $U \supseteq F$, $\mu(U) < \infty$. Let V be open, $V \supseteq U \setminus F$, and $\mu(V \setminus (U \setminus F)) < \varepsilon$. This can be obtained because μ is a measure on $\mathcal{S}$. Thus from outer regularity there exists $V \supseteq U \setminus F$ such that $\mu(U \setminus F) + \varepsilon > \mu(V)$. Then

$$\mu(V \setminus (U \setminus F)) + \mu(U \setminus F) = \mu(V)$$

and so

$$\mu\left(V \setminus (U \setminus F)\right) = \mu\left(V\right) - \mu\left(U \setminus F\right) < \varepsilon.$$

Also,

$$\begin{aligned} V \setminus (U \setminus F) &= V \cap \left(U \cap F^C\right)^C \\ &= V \cap \left[U^C \cup F\right] \\ &= (V \cap F) \cup \left(V \cap U^C\right) \\ &\supseteq V \cap F \end{aligned}$$

and so

$$\mu(V \cap F) \leq \mu\left(V \setminus (U \setminus F)\right) < \varepsilon.$$

Since $V \supseteq U \cap F^C$, $V^C \subseteq U^C \cup F$ so $U \cap V^C \subseteq U \cap F = F$. Hence $U \cap V^C$ is a subset of F. Now let $K \subseteq U$, $\mu(U \setminus K) < \varepsilon$. Thus $K \cap V^C$ is a compact subset of F and

$$\begin{aligned} \mu(F) &= \mu(V \cap F) + \mu(F \setminus V) \\ &< \varepsilon + \mu(F \setminus V) \leq \varepsilon + \mu(U \cap V^C) \leq 2\varepsilon + \mu(K \cap V^C). \end{aligned}$$

Since ε is arbitrary, this proves the second part of the lemma. Formula 9.11 of this theorem was established earlier.

It remains to show μ satisfies 9.12.

Lemma 9.29 $\int f d\mu = Lf$ *for all* $f \in C_c(\Omega)$.

Proof: Let $f \in C_c(\Omega)$, f real-valued, and suppose $f(\Omega) \subseteq [a,b]$. Choose $t_0 < a$ and let $t_0 < t_1 < \cdots < t_n = b$, $t_i - t_{i-1} < \varepsilon$. Let

$$E_i = f^{-1}((t_{i-1}, t_i]) \cap \operatorname{spt}(f). \tag{9.14}$$

Note that $\cup_{i=1}^n E_i$ is a closed set and in fact

$$\cup_{i=1}^n E_i = \operatorname{spt}(f) \tag{9.15}$$

since $\Omega = \cup_{i=1}^n f^{-1}((t_{i-1}, t_i])$. Let $V_i \supseteq E_i, V_i$ is open and let V_i satisfy

$$f(x) < t_i + \varepsilon \text{ for all } x \in V_i, \tag{9.16}$$

$$\mu(V_i \setminus E_i) < \varepsilon/n.$$

By Theorem 9.18 there exists $h_i \in C_c(\Omega)$ such that

$$h_i \prec V_i, \ \sum_{i=1}^n h_i(x) = 1 \text{ on } \operatorname{spt}(f).$$

Now note that for each i,

$$f(x)h_i(x) \leq h_i(x)(t_i + \varepsilon).$$

(If $x \in V_i$, this follows from 9.16. If $x \notin V_i$ both sides equal 0.) Therefore,

$$\begin{aligned} Lf &= L(\sum_{i=1}^{n} fh_i) \leq L(\sum_{i=1}^{n} h_i(t_i+\varepsilon)) \\ &= \sum_{i=1}^{n}(t_i+\varepsilon)L(h_i) \\ &= \sum_{i=1}^{n}(|t_0|+t_i+\varepsilon)L(h_i) - |t_0|L\left(\sum_{i=1}^{n} h_i\right). \end{aligned}$$

Now note that $|t_0|+t_i+\varepsilon \geq 0$ and so from the definition of μ and Lemma 9.24, this is no larger than

$$\sum_{i=1}^{n}(|t_0|+t_i+\varepsilon)\mu(V_i) - |t_0|\mu(\text{spt}(f))$$

$$\leq \sum_{i=1}^{n}(|t_0|+t_i+\varepsilon)(\mu(E_i)+\varepsilon/n) - |t_0|\mu(\text{spt}(f))$$

$$\leq |t_0|\sum_{i=1}^{n}\mu(E_i) + |t_0|\varepsilon + \sum_{i=1}^{n} t_i\mu(E_i) + \varepsilon(|t_0|+|b|)$$

$$+\varepsilon\sum_{i=1}^{n}\mu(E_i) + \varepsilon^2 - |t_0|\mu(\text{spt}(f)).$$

From 9.15 and 9.14, the first and last terms cancel. Therefore this is no larger than

$$(2|t_0|+|b|+\mu(\text{spt}(f))+\varepsilon)\varepsilon + \sum_{i=1}^{n} t_{i-1}\mu(E_i) + \varepsilon\mu(\text{spt}(f))$$

$$\leq \int f d\mu + (2|t_0|+|b|+2\mu(\text{spt}(f))+\varepsilon)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$Lf \leq \int f d\mu \tag{9.17}$$

for all $f \in C_c(\Omega)$, f real. Hence equality holds in 9.17 because $L(-f) \leq -\int f d\mu$ so $L(f) \geq \int f d\mu$. Thus $Lf = \int f d\mu$ for all $f \in C_c(\Omega)$. Just apply the result for real functions to the real and imaginary parts of f. This proves the Lemma.

This gives the existence part of the Riesz representation theorem.

It only remains to prove uniqueness. Suppose both μ_1 and μ_2 are measures on $\mathcal{S}$ satisfying the conclusions of the theorem. Then if K is compact and $V \supseteq K$, let $K \prec f \prec V$. Then

$$\mu_1(K) \leq \int f d\mu_1 = Lf = \int f d\mu_2 \leq \mu_2(V).$$

Thus $\mu_1(K) \leq \mu_2(K)$ for all K. Similarly, the inequality can be reversed and so it follows the two measures are equal on compact sets. By the assumption of inner regularity on open sets, the two measures are also equal on all open sets. By outer regularity, they are equal on all sets of $\mathcal{S}$. This proves the theorem.

An important example of a locally compact Hausdorff space is any metric space in which the closures of balls are compact. For example, $\mathbb{R}^n$ with the usual metric is an example of this. Not surprisingly, more can be said in this important special case.

Theorem 9.30 *Let (Ω, τ) be a metric space in which the closures of the balls are compact and let L be a positive linear functional defined on $C_c(\Omega)$. Then there exists a measure representing the positive linear functional which satisfies all the conclusions of Theorem 9.15 and in addition the property that μ is regular. The same conclusion follows if (Ω, τ) is a compact Hausdorff space.*

Theorem 9.31 *Let (Ω, τ) be a metric space in which the closures of the balls are compact and let L be a positive linear functional defined on $C_c(\Omega)$. Then there exists a measure representing the positive linear functional which satisfies all the conclusions of Theorem 9.15 and in addition the property that μ is regular. The same conclusion follows if (Ω, τ) is a compact Hausdorff space.*

Proof: Let μ and $\mathcal{S}$ be as described in Theorem 9.21. The outer regularity comes automatically as a conclusion of Theorem 9.21. It remains to verify inner regularity. Let $F \in \mathcal{S}$ and let $l < k < \mu(F)$. Now let $z \in \Omega$ and $\Omega_n = \overline{B(z, n)}$ for $n \in \mathbb{N}$. Thus $F \cap \Omega_n \uparrow F$. It follows that for n large enough,

$$k < \mu(F \cap \Omega_n) \leq \mu(F).$$

Since $\mu(F \cap \Omega_n) < \infty$ it follows there exists a compact set, K such that $K \subseteq F \cap \Omega_n \subseteq F$ and

$$l < \mu(K) \leq \mu(F).$$

This proves inner regularity. In case (Ω, τ) is a compact Hausdorff space, the conclusion of inner regularity follows from Theorem 9.21. This proves the theorem.

The proof of the above yields the following corollary.

Corollary 9.32 *Let (Ω, τ) be a locally compact Hausdorff space and suppose μ defined on a σ algebra, $\mathcal{S}$ represents the positive linear functional L where L is defined on $C_c(\Omega)$ in the sense of Theorem 9.15. Suppose also that there exist $\Omega_n \in \mathcal{S}$ such that $\Omega = \cup_{n=1}^{\infty} \Omega_n$ and $\mu(\Omega_n) < \infty$. Then μ is regular.*

The following is on the uniqueness of the σ algebra in some cases.

Definition 9.33 *Let (Ω, τ) be a locally compact Hausdorff space and let L be a positive linear functional defined on $C_c(\Omega)$ such that the complete measure defined by the Riesz representation theorem for positive linear functionals is inner regular. Then this is called a Radon measure. Thus a Radon measure is complete, and regular.*

Corollary 9.34 *Let (Ω, τ) be a locally compact Hausdorff space which is also σ compact meaning*

$$\Omega = \cup_{n=1}^{\infty} \Omega_n, \ \Omega_n \text{ is compact,}$$

and let L be a positive linear functional defined on $C_c(\Omega)$. Then if $(\mu_1, \mathcal{S}_1)$, and $(\mu_2, \mathcal{S}_2)$ are two Radon measures, together with their σ algebras which represent L then the two σ algebras are equal and the two measures are equal.

Proof: Suppose $(\mu_1, \mathcal{S}_1)$ and $(\mu_2, \mathcal{S}_2)$ both work. It will be shown the two measures are equal on every compact set. Let K be compact and let V be an open set containing K. Then let $K \prec f \prec V$. Then

$$\mu_1(K) = \int_K d\mu_1 \leq \int f d\mu_1 = L(f) = \int f d\mu_2 \leq \mu_2(V).$$

Therefore, taking the infimum over all V containing K implies $\mu_1(K) \leq \mu_2(K)$. Reversing the argument shows $\mu_1(K) = \mu_2(K)$. This also implies the two measures are equal on all open sets because they are both inner regular on open sets. It is being assumed the two measures are regular. Now let $F \in \mathcal{S}_1$ with $\mu_1(F) < \infty$. Then there exist sets, H, G such that $H \subseteq F \subseteq G$ such that H is the countable union of compact sets and G is a countable intersection of open sets such that $\mu_1(G) = \mu_1(H)$ which implies $\mu_1(G \setminus H) = 0$. Now $G \setminus H$ can be written as the countable intersection of sets of the form $V_k \setminus K_k$ where V_k is open, $\mu_1(V_k) < \infty$ and K_k is compact. From what was just shown, $\mu_2(V_k \setminus K_k) = \mu_1(V_k \setminus K_k)$ so it follows $\mu_2(G \setminus H) = 0$ also. Since μ_2 is complete, and G and H are in $\mathcal{S}_2$, it follows $F \in \mathcal{S}_2$ and $\mu_2(F) = \mu_1(F)$. Now for arbitrary F possibly having $\mu_1(F) = \infty$, consider $F \cap \Omega_n$. From what was just shown, this set is in $\mathcal{S}_2$ and $\mu_2(F \cap \Omega_n) = \mu_1(F \cap \Omega_n)$. Taking the union of these $F \cap \Omega_n$ gives $F \in \mathcal{S}_2$ and also $\mu_1(F) = \mu_2(F)$. This shows $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Similarly, $\mathcal{S}_2 \subseteq \mathcal{S}_1$.

The following lemma is often useful.

Lemma 9.35 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space where Ω is a metric space having closed balls compact or more generally a topological space. Suppose μ is a Radon measure and f is measurable with respect to $\mathcal{F}$. Then there exists a Borel measurable function, g, such that $g = f$ a.e.*

Proof: Assume without loss of generality that $f \geq 0$. Then let $s_n \uparrow f$ pointwise. Say

$$s_n(\omega) = \sum_{k=1}^{P_n} c_k^n \mathcal{X}_{E_k^n}(\omega)$$

where $E_k^n \in \mathcal{F}$. By the outer regularity of μ, there exists a Borel set, $F_k^n \supseteq E_k^n$ such that $\mu(F_k^n) = \mu(E_k^n)$. In fact F_k^n can be assumed to be a G_δ set. Let

$$t_n(\omega) \equiv \sum_{k=1}^{P_n} c_k^n \mathcal{X}_{F_k^n}(\omega).$$

Then t_n is Borel measurable and $t_n(\omega) = s_n(\omega)$ for all $\omega \notin N_n$ where $N_n \in \mathcal{F}$ is a set of measure zero. Now let $N \equiv \cup_{n=1}^{\infty} N_n$. Then N is a set of measure zero and if $\omega \notin N$, then $t_n(\omega) \to f(\omega)$. Let $N' \supseteq N$ where N' is a Borel set and $\mu(N') = 0$. Then $t_n \mathcal{X}_{(N')^C}$ converges pointwise to a Borel measurable function, g, and $g(\omega) = f(\omega)$ for all $\omega \notin N'$. Therefore, $g = f$ a.e. and this proves the lemma.

9.5 One Dimensional Lebesgue Measure

To obtain one dimensional Lebesgue measure, you use the positive linear functional L given by

$$Lf = \int f(x)\, dx$$

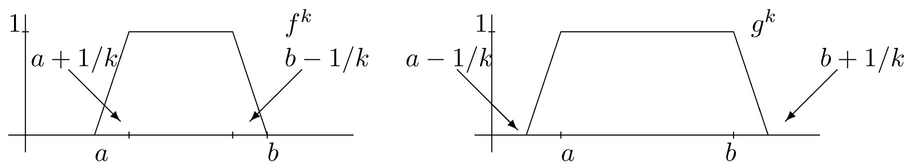
whenever $f \in C_c(\mathbb{R})$. Lebesgue measure, denoted by m is the measure obtained from the Riesz representation theorem such that

$$\int f dm = Lf = \int f(x)\, dx.$$

From this it is easy to verify that

$$m([a,b]) = m((a,b)) = b - a. \tag{9.18}$$

This will be done in general a little later but for now, consider the following picture of functions, f^k and g^k converging pointwise as $k \to \infty$ to $\mathcal{X}_{[a,b]}$.



Then

$$\begin{aligned}\left(b - a - \frac{2}{k}\right) &\leq \int f^k dx = \int f^k dm \leq m((a,b)) \leq m([a,b]) \\ &= \int \mathcal{X}_{[a,b]} dm \leq \int g^k dm = \int g^k dx \leq \left(b - a + \frac{2}{k}\right).\end{aligned}$$

From this the claim in 9.18 follows.

9.6 The Distribution Function

There is an interesting connection between the Lebesgue integral of a nonnegative function with something called the distribution function.

Definition 9.36 *Let $f \geq 0$ and suppose f is measurable. The distribution function is the function defined by*

$$t \to \mu([t < f]) .$$

Lemma 9.37 *If $\{f_n\}$ is an increasing sequence of functions converging pointwise to f then*

$$\mu([f > t]) = \lim_{n\to\infty} \mu([f_n > t])$$

Proof: The sets, $[f_n > t]$ are increasing and their union is $[f > t]$ because if $f(\omega) > t$, then for all n large enough, $f_n(\omega) > t$ also. Therefore, from Theorem 8.5 on Page 172 the desired conclusion follows.

Lemma 9.38 *Suppose $s \geq 0$ is a measurable simple function,*

$$s(\omega) \equiv \sum_{k=1}^{n} a_k \mathcal{X}_{E_k}(\omega)$$

where the a_k are the distinct nonzero values of $s, a_1 < a_2 < \cdots < a_n$. Suppose ϕ is a C^1 function defined on $[0,\infty)$ which has the property that $\phi(0) = 0, \phi'(t) > 0$ for all t. Then

$$\int_0^\infty \phi'(t)\,\mu([s > t])\,dm = \int \phi(s)\,d\mu.$$

Proof: First note that if $\mu(E_k) = \infty$ for any k then both sides equal ∞ and so without loss of generality, assume $\mu(E_k) < \infty$ for all k. Letting $a_0 \equiv 0$, the left side equals

$$\begin{aligned}
\sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} \phi'(t)\,\mu([s>t])\,dm &= \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} \phi'(t) \sum_{i=k}^{n} \mu(E_i)\,dm \\
&= \sum_{k=1}^{n} \sum_{i=k}^{n} \mu(E_i) \int_{a_{k-1}}^{a_k} \phi'(t)\,dm \\
&= \sum_{k=1}^{n} \sum_{i=k}^{n} \mu(E_i)\,(\phi(a_k) - \phi(a_{k-1})) \\
&= \sum_{i=1}^{n} \mu(E_i) \sum_{k=1}^{i} (\phi(a_k) - \phi(a_{k-1})) \\
&= \sum_{i=1}^{n} \mu(E_i)\,\phi(a_i) = \int \phi(s)\,d\mu.
\end{aligned}$$

This proves the lemma.

With this lemma the next theorem which is the main result follows easily.

Theorem 9.39 *Let $f \geq 0$ be measurable and let ϕ be a C^1 function defined on $[0,\infty)$ which satisfies $\phi'(t) > 0$ for all $t > 0$ and $\phi(0) = 0$. Then*

$$\int \phi(f)\,d\mu = \int_0^\infty \phi'(t)\,\mu([f > t])\,dt.$$

Proof: By Theorem 8.27 on Page 190 there exists an increasing sequence of nonnegative simple functions, $\{s_n\}$ which converges pointwise to f. By the monotone convergence theorem and Lemma 9.37,

$$\begin{aligned}\int \phi(f)\,d\mu &= \lim_{n\to\infty}\int \phi(s_n)\,d\mu = \lim_{n\to\infty}\int_0^\infty \phi'(t)\,\mu([s_n>t])\,dm \\ &= \int_0^\infty \phi'(t)\,\mu([f>t])\,dm\end{aligned}$$

This proves the theorem.

9.7 Product Measures

Let $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{T},\nu)$ be two complete measure spaces. In this section consider the problem of defining a product measure, $\overline{\mu\times\nu}$ which is defined on a σ algebra of sets of $X\times Y$ such that $(\overline{\mu\times\nu})(E\times F)=\mu(E)\,\nu(F)$ whenever $E\in\mathcal{S}$ and $F\in\mathcal{T}$. I found the following approach to product measures in [20] and they say they got it from [22].

Definition 9.40 *Let $\mathcal{R}$ denote the set of countable unions of sets of the form $A\times B$, where $A\in\mathcal{S}$ and $B\in\mathcal{T}$ (Sets of the form $A\times B$ are referred to as measurable rectangles) and also let*

$$\rho(A\times B)=\mu(A)\,\nu(B) \tag{9.19}$$

More generally, define

$$\rho(E)\equiv\int\int \mathcal{X}_E(x,y)\,d\mu d\nu \tag{9.20}$$

whenever E is such that

$$x\to\mathcal{X}_E(x,y)\ \text{ is } \mu \text{ measurable for all } y \tag{9.21}$$

and

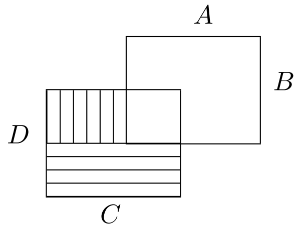
$$y\to\int\mathcal{X}_E(x,y)\,d\mu \text{ is } \nu \text{ measurable.} \tag{9.22}$$

Note that if $E=A\times B$ as above, then

$$\begin{aligned}\int\int \mathcal{X}_E(x,y)\,d\mu d\nu &= \int\int \mathcal{X}_{A\times B}(x,y)\,d\mu d\nu \\ &= \int\int \mathcal{X}_A(x)\,\mathcal{X}_B(y)\,d\mu d\nu = \mu(A)\,\nu(B)=\rho(E)\end{aligned}$$

and so there is no contradiction between 9.20 and 9.19.

The first goal is to show that for $Q\in\mathcal{R}$, 9.21 and 9.22 both hold. That is, $x\to\mathcal{X}_Q(x,y)$ is μ measurable for all y and $y\to\int\mathcal{X}_Q(x,y)\,d\mu$ is ν measurable. This is done so that it is possible to speak of $\rho(Q)$. The following lemma will be the fundamental result which will make this possible. First here is a picture.



Lemma 9.41 *Given $C \times D$ and $\{A_i \times B_i\}_{i=1}^n$, there exist finitely many disjoint rectangles, $\{C_i' \times D_i'\}_{i=1}^p$ such that none of these sets intersect any of the $A_i \times B_i$, each set is contained in $C \times D$ and*

$$\left(\cup_{i=1}^n A_i \times B_i\right) \cup \left(\cup_{k=1}^p C_k' \times D_k'\right) = (C \times D) \cup \left(\cup_{i=1}^n A_i \times B_i\right).$$

Proof: From the above picture, you see that

$$(C \times D) \setminus (A_1 \times B_1) = C \times (D \setminus B_1) \cup (C \setminus A_1) \times (D \cap B_1)$$

and these last two sets are disjoint, have empty intersection with $A_1 \times B_1$, and

$$(C \times (D \setminus B_1) \cup (C \setminus A_1) \times (D \cap B_1)) \cup \left(\cup_{i=1}^n A_i \times B_i\right) = (C \times D) \cup \left(\cup_{i=1}^n A_i \times B_i\right)$$

Now suppose disjoint sets, $\left\{\widetilde{C}_i \times \widetilde{D}_i\right\}_{i=1}^m$ have been obtained, each being a subset of $C \times D$ such that

$$\left(\cup_{i=1}^n A_i \times B_i\right) \cup \left(\cup_{k=1}^m \widetilde{C}_k \times \widetilde{D}_k\right) = \left(\cup_{i=1}^n A_i \times B_i\right) \cup (C \times D)$$

and for all k, $\widetilde{C}_k \times \widetilde{D}_k$ has empty intersection with each set of $\{A_i \times B_i\}_{i=1}^p$. Then using the same procedure, replace each of $\widetilde{C}_k \times \widetilde{D}_k$ with finitely many disjoint rectangles such that none of these intersect $A_{p+1} \times B_{p+1}$ while preserving the union of all the sets involved. The process stops when you have gotten to n. This proves the lemma.

Lemma 9.42 *If $Q = \cup_{i=1}^\infty A_i \times B_i \in \mathcal{R}$, then there exist disjoint sets, of the form $A_i' \times B_i'$ such that $Q = \cup_{i=1}^\infty A_i' \times B_i'$, each $A_i' \times B_i'$ is a subset of some $A_i \times B_i$, and $A_i' \in \mathcal{S}$ while $B_i' \in \mathcal{T}$. Also, the intersection of finitely many sets of $\mathcal{R}$ is a set of $\mathcal{R}$. For ρ defined in 9.20, it follows that 9.21 and 9.22 hold for any element of $\mathcal{R}$. Furthermore,*

$$\rho(Q) = \sum_i \mu(A_i')\,\nu(B_i') = \sum_i \rho(A_i' \times B_i').$$

Proof: Let Q be given as above. Let $A_1' \times B_1' = A_1 \times B_1$. By Lemma 9.41, it is possible to replace $A_2 \times B_2$ with finitely many disjoint rectangles, $\{A_i' \times B_i'\}_{i=2}^{m_2}$ such that none of these rectangles intersect $A_1' \times B_1'$, each is a subset of $A_2 \times B_2$, and

$$\cup_{i=1}^\infty A_i \times B_i = \left(\cup_{i=1}^{m_2} A_i' \times B_i'\right) \cup \left(\cup_{k=3}^\infty A_k \times B_k\right)$$

Now suppose disjoint rectangles, $\{A_i' \times B_i'\}_{i=1}^{m_p}$ have been obtained such that each rectangle is a subset of $A_k \times B_k$ for some $k \leq p$ and

$$\cup_{i=1}^{\infty} A_i \times B_i = \left(\cup_{i=1}^{m_p} A_i' \times B_i'\right) \cup \left(\cup_{k=p+1}^{\infty} A_k \times B_k\right).$$

By Lemma 9.41 again, there exist disjoint rectangles $\{A_i' \times B_i'\}_{i=m_p+1}^{m_{p+1}}$ such that each is contained in $A_{p+1} \times B_{p+1}$, none have intersection with any of $\{A_i' \times B_i'\}_{i=1}^{m_p}$ and

$$\cup_{i=1}^{\infty} A_i \times B_i = \left(\cup_{i=1}^{m_{p+1}} A_i' \times B_i'\right) \cup \left(\cup_{k=p+2}^{\infty} A_k \times B_k\right).$$

Note that no change is made in $\{A_i' \times B_i'\}_{i=1}^{m_p}$. Continuing this way proves the existence of the desired sequence of disjoint rectangles, each of which is a subset of at least one of the original rectangles and such that

$$Q = \cup_{i=1}^{\infty} A_i' \times B_i'.$$

It remains to verify $x \to \mathcal{X}_Q(x, y)$ is μ measurable for all y and

$$y \to \int \mathcal{X}_Q(x, y)\, d\mu$$

is ν measurable whenever $Q \in \mathcal{R}$. Let $Q \equiv \cup_{i=1}^{\infty} A_i \times B_i \in \mathcal{R}$. Then by the first part of this lemma, there exists $\{A_i' \times B_i'\}_{i=1}^{\infty}$ such that the sets are disjoint and $\cup_{i=1}^{\infty} A_i' \times B_i' = Q$. Therefore, since the sets are disjoint,

$$\mathcal{X}_Q(x, y) = \sum_{i=1}^{\infty} \mathcal{X}_{A_i' \times B_i'}(x, y) = \sum_{i=1}^{\infty} \mathcal{X}_{A_i'}(x)\, \mathcal{X}_{B_i'}(y).$$

It follows $x \to \mathcal{X}_Q(x, y)$ is measurable. Now by the monotone convergence theorem,

$$\begin{aligned}
\int \mathcal{X}_Q(x, y)\, d\mu &= \int \sum_{i=1}^{\infty} \mathcal{X}_{A_i'}(x)\, \mathcal{X}_{B_i'}(y)\, d\mu \\
&= \sum_{i=1}^{\infty} \mathcal{X}_{B_i'}(y) \int \mathcal{X}_{A_i'}(x)\, d\mu \\
&= \sum_{i=1}^{\infty} \mathcal{X}_{B_i'}(y)\, \mu(A_i').
\end{aligned}$$

It follows $y \to \int \mathcal{X}_Q(x, y)\, d\mu$ is measurable and so by the monotone convergence theorem again,

$$\begin{aligned}
\int \int \mathcal{X}_Q(x, y)\, d\mu d\nu &= \int \sum_{i=1}^{\infty} \mathcal{X}_{B_i'}(y)\, \mu(A_i')\, d\nu \\
&= \sum_{i=1}^{\infty} \int \mathcal{X}_{B_i'}(y)\, \mu(A_i')\, d\nu \\
&= \sum_{i=1}^{\infty} \nu(B_i')\, \mu(A_i'). \qquad (9.23)
\end{aligned}$$

This shows the measurability conditions, 9.21 and 9.22 hold for $Q \in \mathcal{R}$ and also establishes the formula for $\rho(Q)$, 9.23.

If $\cup_i A_i \times B_i$ and $\cup_j C_j \times D_j$ are two sets of $\mathcal{R}$, then their intersection is

$$\cup_i \cup_j (A_i \cap C_j) \times (B_i \cap D_j)$$

a countable union of measurable rectangles. Thus finite intersections of sets of $\mathcal{R}$ are in $\mathcal{R}$. This proves the lemma.

Now note that from the definition of $\mathcal{R}$ if you have a sequence of elements of $\mathcal{R}$ then their union is also in $\mathcal{R}$. The next lemma will enable the definition of an outer measure.

Lemma 9.43 *Suppose $\{R_i\}_{i=1}^{\infty}$ is a sequence of sets of $\mathcal{R}$ then*

$$\rho(\cup_{i=1}^{\infty} R_i) \leq \sum_{i=1}^{\infty} \rho(R_i).$$

Proof: Let $R_i = \cup_{j=1}^{\infty} A_j^i \times B_j^i$. Using Lemma 9.42, let $\{A'_m \times B'_m\}_{m=1}^{\infty}$ be a sequence of disjoint rectangles each of which is contained in some $A_j^i \times B_j^i$ for some i, j such that

$$\cup_{i=1}^{\infty} R_i = \cup_{m=1}^{\infty} A'_m \times B'_m.$$

Now define

$$S_i \equiv \left\{m : A'_m \times B'_m \subseteq A_j^i \times B_j^i \text{ for some } j\right\}.$$

It is not important to consider whether some m might be in more than one S_i. The important thing to notice is that

$$\cup_{m \in S_i} A'_m \times B'_m \subseteq \cup_{j=1}^{\infty} A_j^i \times B_j^i = R_i.$$

Then by Lemma 9.42,

$$\begin{aligned}
\rho(\cup_{i=1}^{\infty} R_i) &= \sum_m \rho(A'_m \times B'_m) \\
&\leq \sum_{i=1}^{\infty} \sum_{m \in S_i} \rho(A'_m \times B'_m) \\
&\leq \sum_{i=1}^{\infty} \rho(\cup_{m \in S_i} A'_m \times B'_m) \leq \sum_{i=1}^{\infty} \rho(R_i).
\end{aligned}$$

This proves the lemma.

So far, there is no measure and no σ algebra. However, the next step is to define an outer measure which will lead to a measure on a σ algebra of measurable sets from the Caratheodory procedure. When this is done, it will be shown that this measure can be computed using ρ which implies the important Fubini theorem.

Now it is possible to define an outer measure.

Definition 9.44 *For $S \subseteq X \times Y$, define*

$$\left(\overline{\mu \times \nu}\right)(S) \equiv \inf \{\rho(R) : S \subseteq R,\ R \in \mathcal{R}\}. \tag{9.24}$$

The following proposition is interesting but is not needed in the development which follows. It gives a different description of $\left(\overline{\mu \times \nu}\right)$.

Proposition 9.45 $\left(\overline{\mu \times \nu}\right)(S) = \inf \{\sum_{i=1}^{\infty} \mu(A_i)\,\nu(B_i) : S \subseteq \cup_{i=1}^{\infty} A_i \times B_i\}$

Proof: Let $\lambda(S) \equiv \inf \{\sum_{i=1}^{\infty} \mu(A_i)\,\nu(B_i) : S \subseteq \cup_{i=1}^{\infty} A_i \times B_i\}$. Suppose $S \subseteq \cup_{i=1}^{\infty} A_i \times B_i \equiv Q \in \mathcal{R}$. Then by Lemma 9.42, $Q = \cup_i A_i' \times B_i'$ where these rectangles are disjoint. Thus by this lemma, $\rho(Q) = \sum_{i=1}^{\infty} \mu(A_i')\,\nu(B_i') \geq \lambda(S)$ and so $\lambda(S) \leq \left(\overline{\mu \times \nu}\right)(S)$. If $\lambda(S) = \infty$, this shows $\lambda(S) = \left(\overline{\mu \times \nu}\right)(S)$. Suppose then that $\lambda(S) < \infty$ and $\lambda(S) + \varepsilon > \sum_{i=1}^{\infty} \mu(A_i)\,\nu(B_i)$ where $Q = \cup_{i=1}^{\infty} A_i \times B_i \supseteq S$. Then by Lemma 9.42 again, $\cup_{i=1}^{\infty} A_i \times B_i = \cup_{i=1}^{\infty} A_i' \times B_i'$ where the primed rectangles are disjoint, each is a subset of some $A_i \times B_i$ and so

$$\lambda(S) + \varepsilon \geq \sum_{i=1}^{\infty} \mu(A_i)\,\nu(B_i) \geq \sum_{i=1}^{\infty} \mu(A_i')\,\nu(B_i') = \rho(Q) \geq \left(\overline{\mu \times \nu}\right)(S).$$

Since ε is arbitrary, this shows $\lambda(S) \geq \left(\overline{\mu \times \nu}\right)(S)$ and this proves the proposition.

Lemma 9.46 *$\overline{\mu \times \nu}$ is an outer measure on $X \times Y$ and for $R \in \mathcal{R}$*

$$\left(\overline{\mu \times \nu}\right)(R) = \rho(R). \tag{9.25}$$

Proof: First consider 9.25. Since $R \supseteq R$, it follows $\rho(R) \geq \left(\overline{\mu \times \nu}\right)(R)$. On the other hand, if $Q \in \mathcal{R}$ and $Q \supseteq R$, then $\rho(Q) \geq \rho(R)$ and so, taking the infimum on the left yields $\left(\overline{\mu \times \nu}\right)(R) \geq \rho(R)$. This shows 9.25.

It is necessary to show that if $S \subseteq T$, then

$$\left(\overline{\mu \times \nu}\right)(S) \leq \left(\overline{\mu \times \nu}\right)(T), \tag{9.26}$$

$$\left(\overline{\mu \times \nu}\right)(\cup_{i=1}^{\infty} S_i) \leq \sum_{i=1}^{\infty} \left(\overline{\mu \times \nu}\right)(S_i). \tag{9.27}$$

To do this, note that 9.26 is obvious. To verify 9.27, note that it is obvious if $\left(\overline{\mu \times \nu}\right)(S_i) = \infty$ for any i. Therefore, assume $\left(\overline{\mu \times \nu}\right)(S_i) < \infty$. Then letting $\varepsilon > 0$ be given, there exist $R_i \in \mathcal{R}$ such that

$$\left(\overline{\mu \times \nu}\right)(S_i) + \frac{\varepsilon}{2^i} > \rho(R_i),\ R_i \supseteq S_i.$$

Then by Lemma 9.43, 9.25, and the observation that $\cup_{i=1}^{\infty} R_i \in \mathcal{R}$,

$$\begin{aligned}\left(\overline{\mu \times \nu}\right)\left(\cup_{i=1}^{\infty} S_i\right) &\leq \left(\overline{\mu \times \nu}\right)\left(\cup_{i=1}^{\infty} R_i\right) \\ &= \rho\left(\cup_{i=1}^{\infty} R_i\right) \leq \sum_{i=1}^{\infty} \rho\left(R_i\right) \\ &\leq \sum_{i=1}^{\infty}\left(\left(\overline{\mu \times \nu}\right)\left(S_i\right)+\frac{\varepsilon}{2^i}\right) \\ &= \left(\sum_{i=1}^{\infty}\left(\overline{\mu \times \nu}\right)\left(S_i\right)\right)+\varepsilon .\end{aligned}$$

Since ε is arbitrary, this proves the lemma.

By Caratheodory's procedure, it follows there is a σ algebra of subsets of $X \times Y$, denoted here by $\overline{\mathcal{S} \times \mathcal{T}}$ such that $\left(\overline{\mu \times \nu}\right)$ is a complete measure on this σ algebra. The first thing to note is that every rectangle is in this σ algebra.

Lemma 9.47 *Every rectangle is $\left(\overline{\mu \times \nu}\right)$ measurable.*

Proof: Let $S \subseteq X \times Y$. The following inequality must be established.

$$\left(\overline{\mu \times \nu}\right)(S) \geq \left(\overline{\mu \times \nu}\right)(S \cap (A \times B))+\left(\overline{\mu \times \nu}\right)(S \setminus (A \times B)) . \tag{9.28}$$

The following claim will be used to establish this inequality.

Claim: Let $P, A \times B \in \mathcal{R}$. Then

$$\rho(P \cap (A \times B))+\rho(P \setminus (A \times B))=\rho(P) .$$

Proof of the claim: From Lemma 9.42, $P=\cup_{i=1}^{\infty} A_i^{\prime} \times B_i^{\prime}$ where the $A_i^{\prime} \times B_i^{\prime}$ are disjoint. Therefore,

$$P \cap(A \times B)=\bigcup_{i=1}^{\infty}\left(A \cap A_i^{\prime}\right) \times\left(B \cap B_i^{\prime}\right)$$

while

$$P \setminus(A \times B)=\bigcup_{i=1}^{\infty}\left(A_i^{\prime} \setminus A\right) \times B_i^{\prime} \cup \bigcup_{i=1}^{\infty}\left(A \cap A_i^{\prime}\right) \times\left(B_i^{\prime} \setminus B\right) .$$

Since all of the sets in the above unions are disjoint,

$$\rho(P \cap(A \times B))+\rho(P \setminus(A \times B))=$$

$$\int \int \sum_{i=1}^{\infty} \mathcal{X}_{\left(A \cap A_i^{\prime}\right)}(x) \mathcal{X}_{B \cap B_i^{\prime}}(y)\, d\mu d\nu+\int \int \sum_{i=1}^{\infty} \mathcal{X}_{\left(A_i^{\prime} \setminus A\right)}(x) \mathcal{X}_{B_i^{\prime}}(y)\, d\mu d\nu$$

$$+\int \int \sum_{i=1}^{\infty} \mathcal{X}_{A \cap A_i^{\prime}}(x) \mathcal{X}_{B_i^{\prime} \setminus B}(y)\, d\mu d\nu$$

$$= \sum_{i=1}^{\infty} \mu(A \cap A_i') \nu(B \cap B_i') + \mu(A_i' \setminus A) \nu(B_i') + \mu(A \cap A_i') \nu(B_i' \setminus B)$$
$$= \sum_{i=1}^{\infty} \mu(A \cap A_i') \nu(B_i') + \mu(A_i' \setminus A) \nu(B_i') = \sum_{i=1}^{\infty} \mu(A_i') \nu(B_i') = \rho(P).$$

This proves the claim.

Now continuing to verify 9.28, without loss of generality, $(\overline{\mu \times \nu})(S)$ can be assumed finite. Let $P \supseteq S$ for $P \in \mathcal{R}$ and

$$(\overline{\mu \times \nu})(S) + \varepsilon > \rho(P).$$

Then from the claim,

$$\begin{aligned}(\overline{\mu \times \nu})(S) + \varepsilon &> \rho(P) = \rho(P \cap (A \times B)) + \rho(P \setminus (A \times B)) \\ &\geq (\overline{\mu \times \nu})(S \cap (A \times B)) + (\overline{\mu \times \nu})(S \setminus (A \times B)).\end{aligned}$$

Since $\varepsilon > 0$ this shows $A \times B$ is $\overline{\mu \times \nu}$ measurable as claimed.

Lemma 9.48 *Let $\mathcal{R}_1$ be defined as the set of all countable intersections of sets of $\mathcal{R}$. Then if $S \subseteq X \times Y$, there exists $R \in \mathcal{R}_1$ for which it makes sense to write $\rho(R)$ because 9.21 and 9.22 hold such that*

$$(\overline{\mu \times \nu})(S) = \rho(R). \tag{9.29}$$

Also, every element of $\mathcal{R}_1$ is $\overline{\mu \times \nu}$ measurable.

Proof: Consider 9.29. Let $S \subseteq X \times Y$. If $(\overline{\mu \times \nu})(S) = \infty$, let $R = X \times Y$ and it follows $\rho(X \times Y) = \infty = (\overline{\mu \times \nu})(S)$. Assume then that $(\overline{\mu \times \nu})(S) < \infty$.

Therefore, there exists $P_n \in \mathcal{R}$ such that $P_n \supseteq S$ and

$$(\overline{\mu \times \nu})(S) \leq \rho(P_n) < (\overline{\mu \times \nu})(S) + 1/n. \tag{9.30}$$

Let $Q_n = \cap_{i=1}^{n} P_i \in \mathcal{R}$. Define

$$P \equiv \cap_{i=1}^{\infty} Q_i \supseteq S.$$

Then 9.30 holds with Q_n in place of P_n. It is clear that

$$x \to \mathcal{X}_P(x, y) \text{ is } \mu \text{ measurable}$$

because this function is the pointwise limit of functions for which this is so. It remains to consider whether $y \to \int \mathcal{X}_P(x, y) \, d\mu$ is ν measurable. First observe $Q_n \supseteq Q_{n+1}$, $\mathcal{X}_{Q_i} \leq \mathcal{X}_{P_i}$, and

$$\rho(Q_1) = \rho(P_1) = \int \int \mathcal{X}_{P_1}(x, y) \, d\mu d\nu < \infty. \tag{9.31}$$

Therefore, there exists a set of ν measure 0, N, such that if $y \notin N$, then

$$\int \mathcal{X}_{P_1}(x,y)\, d\mu < \infty.$$

It follows from the dominated convergence theorem that

$$\lim_{n\to\infty} \mathcal{X}_{N^C}(y) \int \mathcal{X}_{Q_n}(x,y)\, d\mu = \mathcal{X}_{N^C}(y) \int \mathcal{X}_P(x,y)\, d\mu$$

and so

$$y \to \mathcal{X}_{N^C}(y) \int \mathcal{X}_P(x,y)\, d\mu$$

is also measurable. By completeness of ν,

$$y \to \int \mathcal{X}_P(x,y)\, d\mu$$

must also be ν measurable and so it makes sense to write

$$\int\int \mathcal{X}_P(x,y)\, d\mu d\nu$$

for every $P \in \mathcal{R}_1$. Also, by the dominated convergence theorem,

$$\begin{aligned}
\int\int \mathcal{X}_P(x,y)\, d\mu d\nu &= \int \mathcal{X}_{N^C}(y) \int \mathcal{X}_P(x,y)\, d\mu d\nu \\
&= \lim_{n\to\infty} \int \mathcal{X}_{N^C}(y) \int \mathcal{X}_{Q_n}(x,y)\, d\mu d\nu \\
&= \lim_{n\to\infty} \int\int \mathcal{X}_{Q_n}(x,y)\, d\mu d\nu \\
&= \lim_{n\to\infty} \rho(Q_n) \in \left[\left(\overline{\mu\times\nu}\right)(S), \left(\overline{\mu\times\nu}\right)(S) + 1/n\right]
\end{aligned}$$

for all n. Therefore,

$$\rho(P) \equiv \int\int \mathcal{X}_P(x,y)\, d\mu d\nu = \left(\overline{\mu\times\nu}\right)(S).$$

The sets of $\mathcal{R}_1$ are $\overline{\mu\times\nu}$ measurable because these sets are countable intersections of countable unions of rectangles and Lemma 9.47 verifies the rectangles are $\overline{\mu\times\nu}$ measurable. This proves the Lemma.

The following theorem is the main result.

Theorem 9.49 *Let $E \subseteq X \times Y$ be $\overline{\mu\times\nu}$ measurable and suppose $\left(\overline{\mu\times\nu}\right)(E) < \infty$. Then*

$$x \to \mathcal{X}_E(x,y) \text{ is } \mu \text{ measurable a.e. } y.$$

Modifying $\mathcal{X}_E$ on a set of measure zero, it is possible to write

$$\int \mathcal{X}_E(x,y)\, d\mu.$$

The function,

$$y \to \int \mathcal{X}_E(x,y)\,d\mu$$

is ν measurable and

$$(\overline{\mu \times \nu})(E) = \int \int \mathcal{X}_E(x,y)\,d\mu d\nu.$$

Similarly,

$$(\overline{\mu \times \nu})(E) = \int \int \mathcal{X}_E(x,y)\,d\nu d\mu.$$

Proof: By Lemma 9.48, there exists $R \in \mathcal{R}_1$ such that

$$\rho(R) = (\overline{\mu \times \nu})(E), \ R \supseteq E.$$

Therefore, since R is $\overline{\mu \times \nu}$ measurable and $\rho(R) = (\overline{\mu \times \nu})(R)$, it follows

$$(\overline{\mu \times \nu})(R \setminus E) = 0.$$

By Lemma 9.48 again, there exists $P \supseteq R \setminus E$ with $P \in \mathcal{R}_1$ and

$$\rho(P) = (\overline{\mu \times \nu})(R \setminus E) = 0.$$

Thus

$$\int \int \mathcal{X}_P(x,y)\,d\mu d\nu = 0. \tag{9.32}$$

Since $P \in \mathcal{R}_1$ Lemma 9.48 implies $x \to \mathcal{X}_P(x,y)$ is μ measurable and it follows from the above there exists a set of ν measure zero, N such that if $y \notin N$, then $\int \mathcal{X}_P(x,y)\,d\mu = 0$. Therefore, by completeness of ν,

$$x \to \mathcal{X}_{N^C}(y)\,\mathcal{X}_{R \setminus E}(x,y)$$

is μ measurable and

$$\int \mathcal{X}_{N^C}(y)\,\mathcal{X}_{R \setminus E}(x,y)\,d\mu = 0. \tag{9.33}$$

Now also

$$\mathcal{X}_{N^C}(y)\,\mathcal{X}_R(x,y) = \mathcal{X}_{N^C}(y)\,\mathcal{X}_{R \setminus E}(x,y) + \mathcal{X}_{N^C}(y)\,\mathcal{X}_E(x,y) \tag{9.34}$$

and this shows that

$$x \to \mathcal{X}_{N^C}(y)\,\mathcal{X}_E(x,y)$$

is μ measurable because it is the difference of two functions with this property. Then by 9.33 it follows

$$\int \mathcal{X}_{N^C}(y)\,\mathcal{X}_E(x,y)\,d\mu = \int \mathcal{X}_{N^C}(y)\,\mathcal{X}_R(x,y)\,d\mu.$$

The right side of this equation equals a ν measurable function and so the left side which equals it is also a ν measurable function. It follows from completeness of ν that $y \to \int \mathcal{X}_E(x,y)\,d\mu$ is ν measurable because for y outside of a set of ν measure zero, N it equals $\int \mathcal{X}_R(x,y)\,d\mu$. Therefore,

$$\begin{aligned}\int\int \mathcal{X}_E(x,y)\,d\mu d\nu &= \int\int \mathcal{X}_{N^C}(y)\,\mathcal{X}_E(x,y)\,d\mu d\nu \\ &= \int\int \mathcal{X}_{N^C}(y)\,\mathcal{X}_R(x,y)\,d\mu d\nu \\ &= \int\int \mathcal{X}_R(x,y)\,d\mu d\nu \\ &= \rho(R) = (\overline{\mu\times\nu})(E).\end{aligned}$$

In all the above there would be no change in writing $d\nu d\mu$ instead of $d\mu d\nu$. The same result would be obtained. This proves the theorem.

Now let $f : X \times Y \to [0,\infty]$ be $\overline{\mu\times\nu}$ measurable and

$$\int f d(\overline{\mu\times\nu}) < \infty. \tag{9.35}$$

Let $s(x,y) \equiv \sum_{i=1}^m c_i \mathcal{X}_{E_i}(x,y)$ be a nonnegative simple function with c_i being the nonzero values of s and suppose

$$0 \leq s \leq f.$$

Then from the above theorem,

$$\int s d(\overline{\mu\times\nu}) = \int\int s d\mu d\nu$$

In which

$$\int s d\mu = \int \mathcal{X}_{N^C}(y)\,s d\mu$$

for N a set of ν measure zero such that $y \to \int \mathcal{X}_{N^C}(y)\,s d\mu$ is ν measurable. This follows because 9.35 implies $(\overline{\mu\times\nu})(E_i) < \infty$. Now let $s_n \uparrow f$ where s_n is a nonnegative simple function and

$$\int s_n d(\overline{\mu\times\nu}) = \int\int \mathcal{X}_{N_n^C}(y)\,s_n(x,y)\,d\mu d\nu$$

where

$$y \to \int \mathcal{X}_{N_n^C}(y)\,s_n(x,y)\,d\mu$$

is ν measurable. Then let $N \equiv \cup_{n=1}^\infty N_n$. It follows N is a set of ν measure zero. Thus

$$\int s_n d(\overline{\mu\times\nu}) = \int\int \mathcal{X}_{N^C}(y)\,s_n(x,y)\,d\mu d\nu$$

and letting $n \to \infty$, the monotone convergence theorem implies

$$\begin{aligned}\int f d\left(\overline{\mu \times \nu}\right) &= \int \int \mathcal{X}_{N^C}(y) f(x,y)\, d\mu d\nu \\ &= \int \int f(x,y)\, d\mu d\nu\end{aligned}$$

because of completeness of the measures, μ and ν. This proves Fubini's theorem.

Theorem 9.50 *(Fubini) Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be complete measure spaces and let*

$$\left(\overline{\mu \times \nu}\right)(E) \equiv \inf \left\{ \int \int \mathcal{X}_R(x,y)\, d\mu d\nu : E \subseteq R \in \mathcal{R} \right\}$$ [2]

where $A_i \in \mathcal{S}$ and $B_i \in \mathcal{T}$. Then $\overline{\mu \times \nu}$ is an outer measure on the subsets of $X \times Y$ and the σ algebra of $\overline{\mu \times \nu}$ measurable sets, $\overline{\mathcal{S} \times \mathcal{T}}$, contains all measurable rectangles. If $f \geq 0$ is a $\overline{\mu \times \nu}$ measurable function satisfying

$$\int_{X \times Y} f d\left(\overline{\mu \times \nu}\right) < \infty, \tag{9.36}$$

then

$$\int_{X \times Y} f d\left(\overline{\mu \times \nu}\right) = \int_Y \int_X f d\mu d\nu,$$

where the iterated integral on the right makes sense because for ν a.e. y, $x \to f(x,y)$ is μ measurable and $y \to \int f(x,y)\, d\mu$ is ν measurable. Similarly,

$$\int_{X \times Y} f d\left(\overline{\mu \times \nu}\right) = \int_X \int_Y f d\nu d\mu.$$

In the case where $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are both σ finite, it is not necessary to assume 9.36.

Corollary 9.51 *(Fubini) Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be complete measure spaces such that $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are both σ finite and let*

$$\left(\overline{\mu \times \nu}\right)(E) \equiv \inf \left\{ \int \int \mathcal{X}_R(x,y)\, d\mu d\nu : E \subseteq R \in \mathcal{R} \right\}$$

where $A_i \in \mathcal{S}$ and $B_i \in \mathcal{T}$. Then $\overline{\mu \times \nu}$ is an outer measure. If $f \geq 0$ is a $\overline{\mu \times \nu}$ measurable function then

$$\int_{X \times Y} f d\left(\overline{\mu \times \nu}\right) = \int_Y \int_X f d\mu d\nu,$$

[2] Recall this is the same as

$$\inf \left\{ \sum_{i=1}^{\infty} \mu(A_i)\, \nu(B_i) : E \subseteq \cup_{i=1}^{\infty} A_i \times B_i \right\}$$

in which the A_i and B_i are measurable.

where the iterated integral on the right makes sense because for ν a.e. y, $x \to f(x,y)$ is μ measurable and $y \to \int f(x,y)\,d\mu$ is ν measurable. Similarly,

$$\int_{X\times Y} f d\left(\overline{\mu\times\nu}\right) = \int_X \int_Y f d\nu d\mu.$$

Proof: Let $\cup_{n=1}^{\infty} X_n = X$ and $\cup_{n=1}^{\infty} Y_n = Y$ where $X_n \in \mathcal{S}$, $Y_n \in \mathcal{T}$, $X_n \subseteq X_{n+1}, Y_n \subseteq Y_{n+1}$ for all n and $\mu(X_n) < \infty, \nu(Y_n) < \infty$. From Theorem 9.50 applied to X_n, Y_n and $f_m \equiv \min(f,m)$,

$$\int_{X_n\times Y_n} f_m d\left(\overline{\mu\times\nu}\right) = \int_{Y_n}\int_{X_n} f_m d\mu d\nu$$

Now take $m \to \infty$ and use the monotone convergence theorem to obtain

$$\int_{X_n\times Y_n} f d\left(\overline{\mu\times\nu}\right) = \int_{Y_n}\int_{X_n} f d\mu d\nu.$$

Then use the monotone convergence theorem again letting $n \to \infty$ to obtain the desired conclusion. The argument for the other order of integration is similar.

Corollary 9.52 *If $f \in L^1(X\times Y)$, then*

$$\int f d\left(\overline{\mu\times\nu}\right) = \int\int f(x,y)\,d\mu d\nu = \int\int f(x,y)\,d\nu d\mu.$$

If μ and ν are σ finite, then if f is $\overline{\mu\times\nu}$ measurable having complex values and either $\int\int |f|\,d\mu d\nu < \infty$ or $\int\int |f|\,d\nu d\mu < \infty$, then $\int |f|\,d\left(\overline{\mu\times\nu}\right) < \infty$ so $f \in L^1(X\times Y)$.

Proof: Without loss of generality, it can be assumed that f has real values. Then

$$f = \frac{|f| + f - (|f| - f)}{2}$$

and both $f^+ \equiv \frac{|f|+f}{2}$ and $f^- \equiv \frac{|f|-f}{2}$ are nonnegative and are less than $|f|$. Therefore, $\int g d\left(\overline{\mu\times\nu}\right) < \infty$ for $g = f^+$ and $g = f^-$ so the above theorem applies and

$$\begin{aligned}
\int f d\left(\overline{\mu\times\nu}\right) &\equiv \int f^+ d\left(\overline{\mu\times\nu}\right) - \int f^- d\left(\overline{\mu\times\nu}\right) \\
&= \int\int f^+ d\mu d\nu - \int\int f^- d\mu d\nu \\
&= \int\int f d\mu d\nu.
\end{aligned}$$

It remains to verify the last claim. Suppose s is a simple function,

$$s(x,y) \equiv \sum_{i=1}^{m} c_i \mathcal{X}_{E_i} \leq |f|(x,y)$$

where the c_i are the nonzero values of s. Then

$$s\mathcal{X}_{R_n} \leq |f| \mathcal{X}_{R_n}$$

where $R_n \equiv X_n \times Y_n$ where $X_n \uparrow X$ and $Y_n \uparrow Y$ with $\mu(X_n) < \infty$ and $\nu(Y_n) < \infty$. It follows, since the nonzero values of $s\mathcal{X}_{R_n}$ are achieved on sets of finite measure,

$$\int s\mathcal{X}_{R_n} d\left(\overline{\mu \times \nu}\right) = \int \int s\mathcal{X}_{R_n} d\mu d\nu.$$

Letting $n \to \infty$ and applying the monotone convergence theorem, this yields

$$\int sd\left(\overline{\mu \times \nu}\right) = \int \int sd\mu d\nu. \tag{9.37}$$

Now let $s_n \uparrow |f|$ where s_n is a nonnegative simple function. From 9.37,

$$\int s_n d\left(\overline{\mu \times \nu}\right) = \int \int s_n d\mu d\nu.$$

Letting $n \to \infty$ and using the monotone convergence theorem, yields

$$\int |f| \, d\left(\overline{\mu \times \nu}\right) = \int \int |f| \, d\mu d\nu < \infty$$

9.8 Alternative Treatment Of Product Measure

9.8.1 Monotone Classes And Algebras

Measures are defined on σ algebras which are closed under countable unions. It is for this reason that the theory of measure and integration is so useful in dealing with limits of sequences. However, there is a more basic notion which involves only finite unions and differences.

Definition 9.53 *$\mathcal{A}$ is said to be an algebra of subsets of a set, Z if $Z \in \mathcal{A}$, $\emptyset \in \mathcal{A}$, and when $E, F \in \mathcal{A}$, $E \cup F$ and $E \setminus F$ are both in $\mathcal{A}$.*

It is important to note that if $\mathcal{A}$ is an algebra, then it is also closed under finite intersections. This is because $E \cap F = (E^C \cup F^C)^C \in \mathcal{A}$ since $E^C = Z \setminus E \in \mathcal{A}$ and $F^C = Z \setminus F \in \mathcal{A}$. Note that every σ algebra is an algebra but not the other way around.

Something satisfying the above definition is called an algebra because union is like addition, the set difference is like subtraction and intersection is like multiplication. Furthermore, only finitely many operations are done at a time and so there is nothing like a limit involved.

How can you recognize an algebra when you see one? The answer to this question is the purpose of the following lemma.

Lemma 9.54 *Suppose $\mathcal{R}$ and $\mathcal{E}$ are subsets of $\mathcal{P}(Z)$*[3] *such that $\mathcal{E}$ is defined as the set of all finite disjoint unions of sets of $\mathcal{R}$. Suppose also*

$$\emptyset, Z \in \mathcal{R}$$

$$A \cap B \in \mathcal{R} \quad \text{whenever } A, B \in \mathcal{R},$$

$$A \setminus B \in \mathcal{E} \quad \text{whenever } A, B \in \mathcal{R}.$$

Then $\mathcal{E}$ is an algebra of sets of Z.

Proof: Note first that if $A \in \mathcal{R}$, then $A^C \in \mathcal{E}$ because $A^C = Z \setminus A$.

Now suppose that $E_1$and E_2 are in $\mathcal{E}$,

$$E_1 = \cup_{i=1}^m R_i, \;\; E_2 = \cup_{j=1}^n R_j$$

where the R_i are disjoint sets in $\mathcal{R}$ and the R_j are disjoint sets in $\mathcal{R}$. Then

$$E_1 \cap E_2 = \cup_{i=1}^m \cup_{j=1}^n R_i \cap R_j$$

which is clearly an element of $\mathcal{E}$ because no two of the sets in the union can intersect and by assumption they are all in $\mathcal{R}$. Thus by induction, finite intersections of sets of $\mathcal{E}$ are in $\mathcal{E}$. Consider the difference of two elements of $\mathcal{E}$ next.

If $E = \cup_{i=1}^n R_i \in \mathcal{E}$,

$$E^C = \cap_{i=1}^n R_i^C = \text{ finite intersection of sets of } \mathcal{E}$$

which was just shown to be in $\mathcal{E}$. Now, if $E_1, E_2 \in \mathcal{E}$,

$$E_1 \setminus E_2 = E_1 \cap E_2^C \in \mathcal{E}$$

from what was just shown about finite intersections.

Finally consider finite unions of sets of $\mathcal{E}$. Let E_1 and E_2 be sets of $\mathcal{E}$. Then

$$E_1 \cup E_2 = (E_1 \setminus E_2) \cup E_2 \in \mathcal{E}$$

because $E_1 \setminus E_2$ consists of a finite disjoint union of sets of $\mathcal{R}$ and these sets must be disjoint from the sets of $\mathcal{R}$ whose union yields E_2 because $(E_1 \setminus E_2) \cap E_2 = \emptyset$. This proves the lemma.

The following corollary is particularly helpful in verifying the conditions of the above lemma.

Corollary 9.55 *Let $(Z_1, \mathcal{R}_1, \mathcal{E}_1)$ and $(Z_2, \mathcal{R}_2, \mathcal{E}_2)$ be as described in Lemma 9.54. Then $(Z_1 \times Z_2, \mathcal{R}, \mathcal{E})$ also satisfies the conditions of Lemma 9.54 if $\mathcal{R}$ is defined as*

$$\mathcal{R} \equiv \{R_1 \times R_2 : R_i \in \mathcal{R}_i\}$$

and

$$\mathcal{E} \equiv \{ \text{ finite disjoint unions of sets of } \mathcal{R}\}.$$

Consequently, $\mathcal{E}$ is an algebra of sets.

[3] Set of all subsets of Z

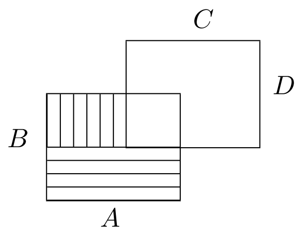
Proof: It is clear $\emptyset, Z_1 \times Z_2 \in \mathcal{R}$. Let $A \times B$ and $C \times D$ be two elements of $\mathcal{R}$.

$$A \times B \cap C \times D = A \cap C \times B \cap D \in \mathcal{R}$$

by assumption.

$$A \times B \setminus (C \times D) =$$
$$A \times \overbrace{(B \setminus D)}^{\in \mathcal{E}_2} \cup \overbrace{(A \setminus C)}^{\in \mathcal{E}_1} \times \overbrace{(D \cap B)}^{\in \mathcal{R}_2}$$
$$= (A \times Q) \cup (P \times R)$$

where $Q \in \mathcal{E}_2$, $P \in \mathcal{E}_1$, and $R \in \mathcal{R}_2$.



Since $A \times Q$ and $P \times R$ do not intersect, it follows the above expression is in $\mathcal{E}$ because each of these terms are. This proves the corollary.

Definition 9.56 *$\mathcal{M} \subseteq \mathcal{P}(Z)$ is called a monotone class if*
a.) $\cdots E_n \supseteq E_{n+1} \cdots$, $E = \cap_{n=1}^{\infty} E_n$, *and* $E_n \in \mathcal{M}$, *then* $E \in \mathcal{M}$.
b.) $\cdots E_n \subseteq E_{n+1} \cdots$, $E = \cup_{n=1}^{\infty} E_n$, *and* $E_n \in \mathcal{M}$, *then* $E \in \mathcal{M}$.
(In simpler notation, $E_n \downarrow E$ *and* $E_n \in \mathcal{M}$ *implies* $E \in \mathcal{M}$. $E_n \uparrow E$ *and* $E_n \in \mathcal{M}$ *implies* $E \in \mathcal{M}$.*)*

Theorem 9.57 *(Monotone Class theorem) Let $\mathcal{A}$ be an algebra of subsets of Z and let $\mathcal{M}$ be a monotone class containing $\mathcal{A}$. Then $\mathcal{M} \supseteq \sigma(\mathcal{A})$, the smallest σ-algebra containing $\mathcal{A}$.*

Proof: Consider all monotone classes which contain $\mathcal{A}$, and take their intersection. The result is still a monotone class which contains $\mathcal{A}$ and is therefore the smallest monotone class containing $\mathcal{A}$. Therefore, assume without loss of generality that $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}$ because if it is shown the smallest monotone class containing $\mathcal{A}$ contains $\sigma(\mathcal{A})$, then the given monotone class does also. To avoid more notation, let $\mathcal{M}$ denote this smallest monotone class.

The plan is to show $\mathcal{M}$ is a σ-algebra. It will then follow $\mathcal{M} \supseteq \sigma(\mathcal{A})$ because $\sigma(\mathcal{A})$ is defined as the intersection of all σ algebras which contain $\mathcal{A}$. For $A \in \mathcal{A}$, define

$$\mathcal{M}_A \equiv \{B \in \mathcal{M} \text{ such that } A \cup B \in \mathcal{M}\}.$$

Clearly $\mathcal{M}_A$ is a monotone class containing $\mathcal{A}$. Hence $\mathcal{M}_A \supseteq \mathcal{M}$ because $\mathcal{M}$ is the smallest such monotone class. But by construction, $\mathcal{M}_A \subseteq \mathcal{M}$. Therefore,

$\mathcal{M} = \mathcal{M}_A$. This shows that $A \cup B \in \mathcal{M}$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{M}$. Now pick $B \in \mathcal{M}$ and define

$$\mathcal{M}_B \equiv \{D \in \mathcal{M} \text{ such that } D \cup B \in \mathcal{M}\}.$$

It was just shown that $\mathcal{A} \subseteq \mathcal{M}_B$. It is clear that $\mathcal{M}_B$ is a monotone class. Thus by a similar argument, $\mathcal{M}_B = \mathcal{M}$ and it follows that $D \cup B \in \mathcal{M}$ whenever $D \in \mathcal{M}$ and $B \in \mathcal{M}$. This shows $\mathcal{M}$ is closed under finite unions.

Next consider the diference of two sets. Let $A \in \mathcal{A}$

$$\mathcal{M}_A \equiv \{B \in \mathcal{M} \text{ such that } B \setminus A \text{ and } A \setminus B \in \mathcal{M}\}.$$

Then $\mathcal{M}_A$, is a monotone class containing $\mathcal{A}$. As before, $\mathcal{M} = \mathcal{M}_A$. Thus $B \setminus A$ and $A \setminus B$ are both in $\mathcal{M}$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{M}$. Now pick $A \in \mathcal{M}$ and consider

$$\mathcal{M}_A \equiv \{B \in \mathcal{M} \text{ such that } B \setminus A \text{ and } A \setminus B \in \mathcal{M}\}.$$

It was just shown $\mathcal{M}_A$ contains $\mathcal{A}$. Now $\mathcal{M}_A$ is a monotone class and so $\mathcal{M}_A = \mathcal{M}$ as before.

Thus $\mathcal{M}$ is both a monotone class and an algebra. Hence, if $E \in \mathcal{M}$ then $Z \setminus E \in \mathcal{M}$. Next consider the question of whether $\mathcal{M}$ is a σ-algebra. If $E_i \in \mathcal{M}$ and $F_n = \cup_{i=1}^n E_i$, then $F_n \in \mathcal{M}$ and $F_n \uparrow \cup_{i=1}^\infty E_i$. Since $\mathcal{M}$ is a monotone class, $\cup_{i=1}^\infty E_i \in \mathcal{M}$ and so $\mathcal{M}$ is a σ-algebra. This proves the theorem.

9.8.2 Product Measure

Definition 9.58 *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ be two measure spaces. A measurable rectangle is a set $A \times B \subseteq X \times Y$ where $A \in \mathcal{S}$ and $B \in \mathcal{F}$. An elementary set will be any subset of $X \times Y$ which is a finite union of disjoint measurable rectangles. $\mathcal{S} \times \mathcal{F}$ will denote the smallest σ algebra of sets in $\mathcal{P}(X \times Y)$ containing all elementary sets.*

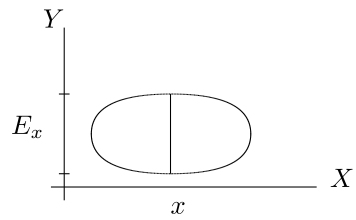
Example 9.59 *It follows from Lemma 9.54 or more easily from Corollary 9.55 that the elementary sets form an algebra.*

Definition 9.60 *Let $E \subseteq X \times Y$,*

$$E_x = \{y \in Y : (x, y) \in E\},$$

$$E^y = \{x \in X : (x, y) \in E\}.$$

These are called the x and y sections.



Theorem 9.61 *If $E \in \mathcal{S} \times \mathcal{F}$, then $E_x \in \mathcal{F}$ and $E^y \in \mathcal{S}$ for all $x \in X$ and $y \in Y$.*

Proof: Let

$$\mathcal{M} = \{E \subseteq \mathcal{S} \times \mathcal{F} \text{ such that for all } x \in X, \quad E_x \in \mathcal{F},$$

$$\text{and for all } y \in Y, \ E^y \in \mathcal{S}.\}$$

Then $\mathcal{M}$ contains all measurable rectangles. If $E_i \in \mathcal{M}$,

$$\left(\cup_{i=1}^{\infty} E_i\right)_x = \cup_{i=1}^{\infty} (E_i)_x \in \mathcal{F}.$$

Similarly,

$$\left(\cup_{i=1}^{\infty} E_i\right)^y = \cup_{i=1}^{\infty} E_i^y \in \mathcal{S}.$$

It follows $\mathcal{M}$ is closed under countable unions.

If $E \in \mathcal{M}$,

$$\left(E^C\right)_x = (E_x)^C \in \mathcal{F}.$$

Similarly, $\left(E^C\right)^y \in \mathcal{S}$. Thus $\mathcal{M}$ is closed under complementation. Therefore $\mathcal{M}$ is a σ-algebra containing the elementary sets. Hence, $\mathcal{M} \supseteq \mathcal{S} \times \mathcal{F}$because $\mathcal{S} \times \mathcal{F}$ is the smallest σ algebra containing these elementary sets. But $\mathcal{M} \subseteq \mathcal{S} \times \mathcal{F}$by definition and so $\mathcal{M} = \mathcal{S} \times \mathcal{F}$. This proves the theorem.

It follows from Lemma 9.54 that the elementary sets form an algebra because clearly the intersection of two measurable rectangles is a measurable rectangle and

$$(A \times B) \setminus (A_0 \times B_0) = (A \setminus A_0) \times B \cup (A \cap A_0) \times (B \setminus B_0),$$

an elementary set.

Theorem 9.62 *If $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ are both finite measure spaces ($\mu(X)$, $\lambda(Y) < \infty$), then for every $E \in \mathcal{S} \times \mathcal{F}$,*

a.) $x \to \lambda(E_x)$ is μ measurable, $y \to \mu(E^y)$ is λ measurable

b.) $\int_X \lambda(E_x) d\mu = \int_Y \mu(E^y) d\lambda$.

Proof: Let

$$\mathcal{M} = \{E \in \mathcal{S} \times \mathcal{F} \text{ such that both } a.) \text{ and } b.) \text{ hold}\}.$$

Since μ and λ are both finite, the monotone convergence and dominated convergence theorems imply $\mathcal{M}$ is a monotone class.

Next I will argue $\mathcal{M}$ contains the elementary sets. Let

$$E = \cup_{i=1}^{n} A_i \times B_i$$

where the measurable rectangles, $A_i \times B_i$ are disjoint. Then

$$\begin{aligned} \lambda(E_x) &= \int_Y \mathcal{X}_E(x,y)\, d\lambda = \int_Y \sum_{i=1}^{n} \mathcal{X}_{A_i \times B_i}(x,y)\, d\lambda \\ &= \sum_{i=1}^{n} \int_Y \mathcal{X}_{A_i \times B_i}(x,y)\, d\lambda = \sum_{i=1}^{n} \mathcal{X}_{A_i}(x)\, \lambda(B_i) \end{aligned}$$

which is clearly μ measurable. Furthermore,

$$\int_X \lambda(E_x)\, d\mu = \int_X \sum_{i=1}^{n} \mathcal{X}_{A_i}(x)\, \lambda(B_i)\, d\mu = \sum_{i=1}^{n} \mu(A_i)\, \lambda(B_i).$$

Similarly,

$$\int_Y \mu(E^y)\, d\lambda = \sum_{i=1}^{n} \mu(A_i)\, \lambda(B_i)$$

and $y \to \mu(E^y)$ is λ measurable and this shows $\mathcal{M}$ contains the algebra of elementary sets. By the monotone class theorem, $\mathcal{M} = \mathcal{S} \times \mathcal{F}$. This proves the theorem.

One can easily extend this theorem to the case where the measure spaces are σ finite.

Theorem 9.63 *If $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ are both σ finite measure spaces, then for every $E \in \mathcal{S} \times \mathcal{F}$,*

a.) $x \to \lambda(E_x)$ is μ measurable, $y \to \mu(E^y)$ is λ measurable.

b.) $\int_X \lambda(E_x) d\mu = \int_Y \mu(E^y) d\lambda$.

Proof: Let $X = \cup_{n=1}^{\infty} X_n, Y = \cup_{n=1}^{\infty} Y_n$ where,

$$X_n \subseteq X_{n+1}, Y_n \subseteq Y_{n+1}, \mu(X_n) < \infty, \lambda(Y_n) < \infty.$$

Let

$$\mathcal{S}_n = \{A \cap X_n : A \in \mathcal{S}\},\ \mathcal{F}_n = \{B \cap Y_n : B \in \mathcal{F}\}.$$

Thus $(X_n, \mathcal{S}_n, \mu)$ and $(Y_n, \mathcal{F}_n, \lambda)$ are both finite measure spaces.

Claim: If $E \in \mathcal{S} \times \mathcal{F}$, then $E \cap (X_n \times Y_n) \in \mathcal{S}_n \times \mathcal{F}_n$.

Proof: Let

$$\mathcal{M}_n = \{E \in \mathcal{S} \times \mathcal{F} : E \cap (X_n \times Y_n) \in \mathcal{S}_n \times \mathcal{F}_n\}.$$

Clearly $\mathcal{M}_n$ contains the algebra of elementary sets. It is also clear that $\mathcal{M}_n$ is a monotone class. Thus $\mathcal{M}_n = \mathcal{S} \times \mathcal{F}$.

Now let $E \in \mathcal{S} \times \mathcal{F}$. By Theorem 9.62,

$$\int_{X_n} \lambda((E \cap (X_n \times Y_n))_x) d\mu = \int_{Y_n} \mu((E \cap (X_n \times Y_n))^y) d\lambda \tag{9.38}$$

where the integrands are measurable. Also

$$(E \cap (X_n \times Y_n))_x = \emptyset$$

if $x \notin X_n$ and a similar observation holds for the second integrand in 9.38 if $y \notin Y_n$. Therefore,

$$\begin{aligned} \int_X \lambda((E \cap (X_n \times Y_n))_x) d\mu &= \int_{X_n} \lambda((E \cap (X_n \times Y_n))_x) d\mu \\ &= \int_{Y_n} \mu((E \cap (X_n \times Y_n))^y) d\lambda \\ &= \int_Y \mu((E \cap (X_n \times Y_n))^y) d\lambda. \end{aligned}$$

Then letting $n \to \infty$, the monotone convergence theorem implies b.) and the measurability assertions of a.) are valid because

$$\begin{aligned} \lambda(E_x) &= \lim_{n\to\infty} \lambda((E \cap (X_n \times Y_n))_x) \\ \mu(E^y) &= \lim_{n\to\infty} \mu((E \cap (X_n \times Y_n))^y). \end{aligned}$$

This proves the theorem.

This theorem makes it possible to define product measure.

Definition 9.64 *For $E \in \mathcal{S} \times \mathcal{F}$ and $(X, \mathcal{S}, \mu), (Y, \mathcal{F}, \lambda)$ σ finite, $(\mu \times \lambda)(E) \equiv \int_X \lambda(E_x) d\mu = \int_Y \mu(E^y) d\lambda$.*

This definition is well defined because of Theorem 9.63.

Theorem 9.65 *If $A \in \mathcal{S}$, $B \in \mathcal{F}$, then $(\mu \times \lambda)(A \times B) = \mu(A)\lambda(B)$, and $\mu \times \lambda$ is a measure on $\mathcal{S} \times \mathcal{F}$ called product measure.*

Proof: The first assertion about the measure of a measurable rectangle was established above. Now suppose $\{E_i\}_{i=1}^{\infty}$ is a disjoint collection of sets of $\mathcal{S} \times \mathcal{F}$. Then using the monotone convergence theorem along with the observation that

$(E_i)_x \cap (E_j)_x = \emptyset$,

$$\begin{aligned}
(\mu \times \lambda)\left(\cup_{i=1}^{\infty} E_i\right) &= \int_X \lambda((\cup_{i=1}^{\infty} E_i)_x) d\mu \\
&= \int_X \lambda\left(\cup_{i=1}^{\infty} (E_i)_x\right) d\mu = \int_X \sum_{i=1}^{\infty} \lambda\left((E_i)_x\right) d\mu \\
&= \sum_{i=1}^{\infty} \int_X \lambda\left((E_i)_x\right) d\mu \\
&= \sum_{i=1}^{\infty} (\mu \times \lambda)(E_i)
\end{aligned}$$

This proves the theorem.

The next theorem is one of several theorems due to Fubini and Tonelli. These theorems all have to do with interchanging the order of integration in a multiple integral.

Theorem 9.66 *Let $f : X \times Y \to [0, \infty]$ be measurable with respect to $\mathcal{S} \times \mathcal{F}$ and suppose μ and λ are σ finite. Then*

$$\int_{X \times Y} f d(\mu \times \lambda) = \int_X \int_Y f(x, y) d\lambda d\mu = \int_Y \int_X f(x, y) d\mu d\lambda \tag{9.39}$$

and all integrals make sense.

Proof: For $E \in \mathcal{S} \times \mathcal{F}$,

$$\int_Y \mathcal{X}_E(x, y) d\lambda = \lambda(E_x), \int_X \mathcal{X}_E(x, y) d\mu = \mu(E^y).$$

Thus from Definition 9.64, 9.39 holds if $f = \mathcal{X}_E$. It follows that 9.39 holds for every nonnegative simple function. By Theorem 8.27 on Page 190, there exists an increasing sequence, $\{f_n\}$, of simple functions converging pointwise to f. Then

$$\int_Y f(x, y) d\lambda = \lim_{n \to \infty} \int_Y f_n(x, y) d\lambda,$$

$$\int_X f(x, y) d\mu = \lim_{n \to \infty} \int_X f_n(x, y) d\mu.$$

This follows from the monotone convergence theorem. Since

$$x \to \int_Y f_n(x, y) d\lambda$$

is measurable with respect to $\mathcal{S}$, it follows that $x \to \int_Y f(x, y) d\lambda$ is also measurable with respect to $\mathcal{S}$. A similar conclusion can be drawn about $y \to \int_X f(x, y) d\mu$. Thus the two iterated integrals make sense. Since 9.39 holds for f_n, another application of the Monotone Convergence theorem shows 9.39 holds for f. This proves the theorem.

Corollary 9.67 *Let $f : X \times Y \to \mathbb{C}$ be $\mathcal{S} \times \mathcal{F}$ measurable. Suppose either $\int_X \int_Y |f|\, d\lambda d\mu$ or $\int_Y \int_X |f|\, d\mu d\lambda < \infty$. Then $f \in L^1(X \times Y, \mu \times \lambda)$ and*

$$\int_{X\times Y} f d(\mu \times \lambda) = \int_X \int_Y f d\lambda d\mu = \int_Y \int_X f d\mu d\lambda \tag{9.40}$$

with all integrals making sense.

Proof: Suppose first that f is real valued. Apply Theorem 9.66 to $f^+$and f^-. 9.40 follows from observing that $f = f^+ - f^-$; and that all integrals are finite. If f is complex valued, consider real and imaginary parts. This proves the corollary.

Suppose f is product measurable. From the above discussion, and breaking f down into a sum of positive and negative parts of real and imaginary parts and then using Theorem 8.27 on Page 190 on approximation by simple functions, it follows that whenever f is $\mathcal{S} \times \mathcal{F}$ measurable, $x \to f(x, y)$ is μ measurable, $y \to f(x, y)$ is λ measurable.

9.9 Completion Of Measures

Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space. Then it is always possible to enlarge the σ algebra and define a new measure $\overline{\mu}$ on this larger σ algebra such that $\left(\Omega, \overline{\mathcal{F}}, \overline{\mu}\right)$ is a complete measure space. Recall this means that if $N \subseteq N' \in \overline{\mathcal{F}}$ and $\overline{\mu}(N') = 0$, then $N \in \overline{\mathcal{F}}$. The following theorem is the main result. The new measure space is called the completion of the measure space.

Theorem 9.68 *Let $(\Omega, \mathcal{F}, \mu)$ be a σ finite measure space. Then there exists a unique measure space, $\left(\Omega, \overline{\mathcal{F}}, \overline{\mu}\right)$ satisfying*

1. *$\left(\Omega, \overline{\mathcal{F}}, \overline{\mu}\right)$ is a complete measure space.*
2. *$\overline{\mu} = \mu$ on $\mathcal{F}$*
3. *$\overline{\mathcal{F}} \supseteq \mathcal{F}$*
4. *For every $E \in \overline{\mathcal{F}}$ there exists $G \in \mathcal{F}$ such that $G \supseteq E$ and $\mu(G) = \overline{\mu}(E)$.*
5. *For every $E \in \overline{\mathcal{F}}$ there exists $F \in \mathcal{F}$ such that $F \subseteq E$ and $\mu(F) = \overline{\mu}(E)$.*

Also for every $E \in \overline{\mathcal{F}}$ there exist sets $G, F \in \mathcal{F}$ such that $G \supseteq E \supseteq F$ and

$$\mu(G \setminus F) = \overline{\mu}(G \setminus F) = 0 \tag{9.41}$$

Proof: First consider the claim about uniqueness. Suppose $(\Omega, \mathcal{F}_1, \nu_1)$ and $(\Omega, \mathcal{F}_2, \nu_2)$ both work and let $E \in \mathcal{F}_1$. Also let $\mu(\Omega_n) < \infty$, $\cdots \Omega_n \subseteq \Omega_{n+1} \cdots$, and $\cup_{n=1}^{\infty} \Omega_n = \Omega$. Define $E_n \equiv E \cap \Omega_n$. Then pick $G_n \supseteq E_n \supseteq F_n$ such that $\mu(G_n) =$

$\mu(F_n) = \nu_1(E_n)$. It follows $\mu(G_n \setminus F_n) = 0$. Then letting $G = \cup_n G_n, F \equiv \cup_n F_n$, it follows $G \supseteq E \supseteq F$ and

$$\begin{aligned} \mu(G \setminus F) &\leq \mu(\cup_n (G_n \setminus F_n)) \\ &\leq \sum_n \mu(G_n \setminus F_n) = 0. \end{aligned}$$

It follows that $\nu_2(G \setminus F) = 0$ also. Now $E \setminus F \subseteq G \setminus F$ and since $(\Omega, \mathcal{F}_2, \nu_2)$ is complete, it follows $E \setminus F \in \mathcal{F}_2$. Since $F \in \mathcal{F}_2$, it follows $E = (E \setminus F) \cup F \in \mathcal{F}_2$. Thus $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Similarly $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Now it only remains to verify $\nu_1 = \nu_2$. Thus let $E \in \mathcal{F}_1 = \mathcal{F}_2$ and let G and F be as just described. Since $\nu_i = \mu$ on $\mathcal{F}$,

$$\begin{aligned} \mu(F) &\leq \nu_1(E) \\ &= \nu_1(E \setminus F) + \nu_1(F) \\ &\leq \nu_1(G \setminus F) + \nu_1(F) \\ &= \nu_1(F) = \mu(F) \end{aligned}$$

Similarly $\nu_2(E) = \mu(F)$. This proves uniqueness. The construction has also verified 9.41.

Next define an outer measure, $\overline{\mu}$ on $\mathcal{P}(\Omega)$ as follows. For $S \subseteq \Omega$,

$$\overline{\mu}(S) \equiv \inf \{\mu(E) : E \in \mathcal{F}\}.$$

Then it is clear $\overline{\mu}$ is increasing. It only remains to verify $\overline{\mu}$ is subadditive. Then let $S = \cup_{i=1}^{\infty} S_i$. If any $\overline{\mu}(S_i) = \infty$, there is nothing to prove so suppose $\overline{\mu}(S_i) < \infty$ for each i. Then there exist $E_i \in \mathcal{F}$ such that $E_i \supseteq S_i$ and

$$\overline{\mu}(S_i) + \varepsilon/2^i > \mu(E_i).$$

Then

$$\begin{aligned} \overline{\mu}(S) &= \overline{\mu}(\cup_i S_i) \\ &\leq \mu(\cup_i E_i) \leq \sum_i \mu(E_i) \\ &\leq \sum_i \left(\overline{\mu}(S_i) + \varepsilon/2^i\right) = \sum_i \overline{\mu}(S_i) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this verifies $\overline{\mu}$ is subadditive and is an outer measure as claimed.

Denote by $\overline{\mathcal{F}}$ the σ algebra of measurable sets in the sense of Caratheodory. Then it follows from the Caratheodory procedure, Theorem 9.4, on Page 210 that $\left(\Omega, \overline{\mathcal{F}}, \overline{\mu}\right)$ is a complete measure space. This verifies 1.

Now let $E \in \mathcal{F}$. Then from the definition of $\overline{\mu}$, it follows

$$\overline{\mu}(E) \equiv \inf \{\mu(F) : F \in \mathcal{F} \text{ and } F \supseteq E\} \leq \mu(E).$$

If $F \supseteq E$ and $F \in \mathcal{F}$, then $\mu(F) \geq \mu(E)$ and so $\mu(E)$ is a lower bound for all such $\mu(F)$ which shows that

$$\overline{\mu}(E) \equiv \inf \{\mu(F) : F \in \mathcal{F} \text{ and } F \supseteq E\} \geq \mu(E).$$

This verifies 2.

Next consider 3. Let $E \in \mathcal{F}$ and let S be a set. I must show

$$\overline{\mu}(S) \geq \overline{\mu}(S \setminus E) + \overline{\mu}(S \cap E).$$

If $\overline{\mu}(S) = \infty$ there is nothing to show. Therefore, suppose $\overline{\mu}(S) < \infty$. Then from the definition of $\overline{\mu}$ there exists $G \supseteq S$ such that $G \in \mathcal{F}$ and $\mu(G) = \overline{\mu}(S)$. Then from the definition of $\overline{\mu}$,

$$\begin{aligned}\overline{\mu}(S) &\leq \overline{\mu}(S \setminus E) + \overline{\mu}(S \cap E) \\ &\leq \mu(G \setminus E) + \mu(G \cap E) \\ &= \mu(G) = \overline{\mu}(S)\end{aligned}$$

This verifies 3.

Claim 4 comes by the definition of $\overline{\mu}$ as used above. The only other case is when $\overline{\mu}(S) = \infty$. However, in this case, you can let $G = \Omega$.

It only remains to verify 5. Let the Ω_n be as described above and let $E \in \overline{\mathcal{F}}$ such that $E \subseteq \Omega_n$. By 4 there exists $H \in \mathcal{F}$ such that $H \subseteq \Omega_n$, $H \supseteq \Omega_n \setminus E$, and

$$\mu(H) = \overline{\mu}(\Omega_n \setminus E). \tag{9.42}$$

Then let $F \equiv \Omega_n \cap H^C$. It follows $F \subseteq E$ and

$$\begin{aligned}E \setminus F &= E \cap F^C = E \cap \left(H \cup \Omega_n^C\right) \\ &= E \cap H = H \setminus (\Omega_n \setminus E)\end{aligned}$$

Hence from 9.42

$$\overline{\mu}(E \setminus F) = \overline{\mu}(H \setminus (\Omega_n \setminus E)) = 0.$$

It follows

$$\overline{\mu}(E) = \overline{\mu}(F) = \mu(F).$$

In the case where $E \in \overline{\mathcal{F}}$ is arbitrary, not necessarily contained in some Ω_n, it follows from what was just shown that there exists $F_n \in \mathcal{F}$ such that $F_n \subseteq E \cap \Omega_n$ and

$$\mu(F_n) = \overline{\mu}(E \cap \Omega_n).$$

Letting $F \equiv \cup_n F_n$

$$\overline{\mu}(E \setminus F) \leq \overline{\mu}(\cup_n (E \cap \Omega_n \setminus F_n)) \leq \sum_n \overline{\mu}(E \cap \Omega_n \setminus F_n) = 0.$$

Therefore, $\overline{\mu}(E) = \mu(F)$ and this proves 5. This proves the theorem.

Now here is an interesting theorem about complete measure spaces.

Theorem 9.69 *Let $(\Omega, \mathcal{F}, \mu)$ be a complete measure space and let $f \leq g \leq h$ be functions having values in $[0, \infty]$. Suppose also that $f(\omega) = h(\omega)$ a.e. ω and that f and h are measurable. Then g is also measurable. If $\left(\Omega, \overline{\mathcal{F}}, \overline{\mu}\right)$ is the completion* *of a σ finite measure space $(\Omega, \mathcal{F}, \mu)$ as described above in Theorem 9.68 then if f is measurable with respect to $\overline{\mathcal{F}}$ having values in $[0, \infty]$, it follows there exists g measurable with respect to $\mathcal{F}$, $g \leq f$, and a set $N \in \mathcal{F}$ with $\mu(N) = 0$ and $g = f$ on N^C. There also exists h measurable with respect to $\mathcal{F}$ such that $h \geq f$, and a set of measure zero, $M \in \mathcal{F}$ such that $f = h$ on M^C.*

Proof: Let $\alpha \in \mathbb{R}$.

$$[f > \alpha] \subseteq [g > \alpha] \subseteq [h > \alpha]$$

Thus

$$[g > \alpha] = [f > \alpha] \cup ([g > \alpha] \setminus [f > \alpha])$$

and $[g > \alpha] \setminus [f > \alpha]$ is a measurable set because it is a subset of the set of measure zero,

$$[h > \alpha] \setminus [f > \alpha].$$

Now consider the last assertion. By Theorem 8.27 on Page 190 there exists an increasing sequence of nonnegative simple functions, $\{s_n\}$ measurable with respect to $\overline{\mathcal{F}}$ which converges pointwise to f. Letting

$$s_n(\omega) = \sum_{k=1}^{m_n} c_k^n \mathcal{X}_{E_k^n}(\omega) \tag{9.43}$$

be one of these simple functions, it follows from Theorem 9.68 there exist sets, $F_k^n \in \mathcal{F}$ such that $F_k^n \subseteq E_k^n$ and $\mu(F_k^n) = \overline{\mu}(E_k^n)$. Then let

$$t_n(\omega) \equiv \sum_{k=1}^{m_n} c_k^n \mathcal{X}_{F_k^n}(\omega).$$

Thus $t_n = s_n$ off a set of measure zero, $N_n \in \overline{\mathcal{F}}, t_n \leq s_n$. Let $N' \equiv \cup_n N_n$. Then by Theorem 9.68 again, there exists $N \in \mathcal{F}$ such that $N \supseteq N'$ and $\mu(N) = 0$. Consider the simple functions,

$$s_n'(\omega) \equiv t_n(\omega) \mathcal{X}_{N^C}(\omega).$$

It is an increasing sequence so let $g(\omega) = \lim_{n\to\infty} s_{n'}(\omega)$. It follows g is mesurable with respect to $\mathcal{F}$ and equals f off N.

Finally, to obtain the function, $h \geq f$, in 9.43 use Theorem 9.68 to obtain the existence of $F_k^n \in \mathcal{F}$ such that $F_k^n \supseteq E_k^n$ and $\mu(F_k^n) = \overline{\mu}(E_k^n)$. Then let

$$t_n(\omega) \equiv \sum_{k=1}^{m_n} c_k^n \mathcal{X}_{F_k^n}(\omega).$$

Thus $t_n = s_n$ off a set of measure zero, $M_n \in \overline{\mathcal{F}}, t_n \geq s_n$, and t_n is measurable with respect to $\mathcal{F}$. Then define

$$s_n' = \max_{k \leq n} t_n.$$

It follows s_n' is an increasing sequence of $\mathcal{F}$ measurable nonnegative simple functions. Since each $s_n' \geq s_n$, it follows that if $h(\omega) = \lim_{n\to\infty} s_n'(\omega)$,then $h(\omega) \geq f(\omega)$. Also if $h(\omega) > f(\omega)$, then $\omega \in \cup_n M_n \equiv M'$, a set of $\overline{\mathcal{F}}$ having measure zero. By Theorem 9.68, there exists $M \supseteq M'$ such that $M \in \mathcal{F}$ and $\mu(M) = 0$. It follows $h = f$ off M. This proves the theorem.

9.10 Another Version Of Product Measures

9.10.1 General Theory

Given two finite measure spaces, $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{S}, \nu)$, there is a way to define a σ algebra of subsets of $X \times Y$, denoted by $\mathcal{F} \times \mathcal{S}$ and a measure, denoted by $\mu \times \nu$ defined on this σ algebra such that

$$\mu \times \nu (A \times B) = \mu (A) \lambda (B)$$

whenever $A \in \mathcal{F}$ and $B \in \mathcal{S}$. This is naturally related to the concept of iterated integrals similar to what is used in calculus to evaluate a multiple integral. The approach is based on something called a π system, [15].

Definition 9.70 *Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{S}, \nu)$ be two measure spaces. A measurable rectangle is a set of the form $A \times B$ where $A \in \mathcal{F}$ and $B \in \mathcal{S}$.*

Definition 9.71 *Let Ω be a set and let $\mathcal{K}$ be a collection of subsets of Ω. Then $\mathcal{K}$ is called a π system if $\emptyset \in \mathcal{K}$ and whenever $A, B \in \mathcal{K}$, it follows $A \cap B \in \mathcal{K}$.*

Obviously an example of a π system is the set of measurable rectangles because

$$A \times B \cap A' \times B' = (A \cap A') \times (B \cap B').$$

The following is the fundamental lemma which shows these π systems are useful.

Lemma 9.72 *Let $\mathcal{K}$ be a π system of subsets of Ω, a set. Also let $\mathcal{G}$ be a collection of subsets of Ω which satisfies the following three properties.*

1. *$\mathcal{K} \subseteq \mathcal{G}$*
2. *If $A \in \mathcal{G}$, then $A^C \in \mathcal{G}$*
3. *If $\{A_i\}_{i=1}^{\infty}$ is a sequence of disjoint sets from $\mathcal{G}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{G}$.*

Then $\mathcal{G} \supseteq \sigma(\mathcal{K})$, where $\sigma(\mathcal{K})$ is the smallest σ algebra which contains $\mathcal{K}$.

Proof: First note that if

$$\mathcal{H} \equiv \{\mathcal{G} : 1 \text{ - } 3 \text{ all hold}\}$$

then $\cap\mathcal{H}$ yields a collection of sets which also satisfies 1 - 3. Therefore, I will assume in the argument that $\mathcal{G}$ is the smallest collection satisfying 1 - 3. Let $A \in \mathcal{K}$ and define

$$\mathcal{G}_A \equiv \{B \in \mathcal{G} : A \cap B \in \mathcal{G}\}.$$

I want to show $\mathcal{G}_A$ satisfies 1 - 3 because then it must equal $\mathcal{G}$ since $\mathcal{G}$ is the smallest collection of subsets of Ω which satisfies 1 - 3. This will give the conclusion that for $A \in \mathcal{K}$ and $B \in \mathcal{G}$, $A \cap B \in \mathcal{G}$. This information will then be used to show that if

$A, B \in \mathcal{G}$ then $A \cap B \in \mathcal{G}$. From this it will follow very easily that $\mathcal{G}$ is a σ algebra which will imply it contains $\sigma(\mathcal{K})$. Now here are the details of the argument.

Since $\mathcal{K}$ is given to be a π system, $\mathcal{K} \subseteq \mathcal{G}_A$. Property 3 is obvious because if $\{B_i\}$ is a sequence of disjoint sets in $\mathcal{G}_A$, then

$$A \cap \cup_{i=1}^{\infty} B_i = \cup_{i=1}^{\infty} A \cap B_i \in \mathcal{G}$$

because $A \cap B_i \in \mathcal{G}$ and the property 3 of $\mathcal{G}$.

It remains to verify Property 2 so let $B \in \mathcal{G}_A$. I need to verify that $B^C \in \mathcal{G}_A$. In other words, I need to show that $A \cap B^C \in \mathcal{G}$. However,

$$A \cap B^C = \left(A^C \cup (A \cap B)\right)^C \in \mathcal{G}$$

Here is why. Since $B \in \mathcal{G}_A$, $A \cap B \in \mathcal{G}$ and since $A \in \mathcal{K} \subseteq \mathcal{G}$ it follows $A^C \in \mathcal{G}$. It follows the union of the disjoint sets, A^C and $(A \cap B)$ is in $\mathcal{G}$ and then from 2 the complement of their union is in $\mathcal{G}$. Thus $\mathcal{G}_A$ satisfies 1 - 3 and this implies since $\mathcal{G}$ is the smallest such, that $\mathcal{G}_A \supseteq \mathcal{G}$. However, $\mathcal{G}_A$ is constructed as a subset of $\mathcal{G}$. This proves that for every $B \in \mathcal{G}$ and $A \in \mathcal{K}$, $A \cap B \in \mathcal{G}$. Now pick $B \in \mathcal{G}$ and consider

$$\mathcal{G}_B \equiv \{A \in \mathcal{G} : A \cap B \in \mathcal{G}\}.$$

I just proved $\mathcal{K} \subseteq \mathcal{G}_B$. The other arguments are identical to show $\mathcal{G}_B$ satisfies 1 - 3 and is therefore equal to $\mathcal{G}$. This shows that whenever $A, B \in \mathcal{G}$ it follows $A \cap B \in \mathcal{G}$.

This implies $\mathcal{G}$ is a σ algebra. To show this, all that is left is to verify $\mathcal{G}$ is closed under countable unions because then it follows $\mathcal{G}$ is a σ algebra. Let $\{A_i\} \subseteq \mathcal{G}$. Then let $A_1' = A_1$ and

$$\begin{aligned} A_{n+1}' &\equiv A_{n+1} \setminus \left(\cup_{i=1}^{n} A_i\right) \\ &= A_{n+1} \cap \left(\cap_{i=1}^{n} A_i^C\right) \\ &= \cap_{i=1}^{n} \left(A_{n+1} \cap A_i^C\right) \in \mathcal{G} \end{aligned}$$

because finite intersections of sets of $\mathcal{G}$ are in $\mathcal{G}$. Since the A_i' are disjoint, it follows

$$\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} A_i' \in \mathcal{G}$$

Therefore, $\mathcal{G} \supseteq \sigma(\mathcal{K})$ and this proves the Lemma.

With this lemma, it is easy to define product measure.

Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{S}, \nu)$ be two finite measure spaces. Define $\mathcal{K}$ to be the set of measurable rectangles, $A \times B$, $A \in \mathcal{F}$ and $B \in \mathcal{S}$. Let

$$\mathcal{G} \equiv \left\{E \subseteq X \times Y : \int_Y \int_X \mathcal{X}_E \, d\mu d\nu = \int_X \int_Y \mathcal{X}_E \, d\nu d\mu \right\} \tag{9.44}$$

where in the above, part of the requirement is for all integrals to make sense.

Then $\mathcal{K} \subseteq \mathcal{G}$. This is obvious.

Next I want to show that if $E \in \mathcal{G}$ then $E^C \in \mathcal{G}$. Observe $\mathcal{X}_{E^C} = 1 - \mathcal{X}_E$ and so

$$\begin{aligned}\int_Y \int_X \mathcal{X}_{E^C} d\mu d\nu &= \int_Y \int_X (1 - \mathcal{X}_E)\, d\mu d\nu \\ &= \int_X \int_Y (1 - \mathcal{X}_E)\, d\nu d\mu \\ &= \int_X \int_Y \mathcal{X}_{E^C} d\nu d\mu\end{aligned}$$

which shows that if $E \in \mathcal{G}$, then $E^C \in \mathcal{G}$.

Next I want to show $\mathcal{G}$ is closed under countable unions of disjoint sets of $\mathcal{G}$. Let $\{A_i\}$ be a sequence of disjoint sets from $\mathcal{G}$. Then

$$\begin{aligned}\int_Y \int_X \mathcal{X}_{\cup_{i=1}^\infty A_i} d\mu d\nu &= \int_Y \int_X \sum_{i=1}^\infty \mathcal{X}_{A_i} d\mu d\nu \\ &= \int_Y \sum_{i=1}^\infty \int_X \mathcal{X}_{A_i} d\mu d\nu \\ &= \sum_{i=1}^\infty \int_Y \int_X \mathcal{X}_{A_i} d\mu d\nu \\ &= \sum_{i=1}^\infty \int_X \int_Y \mathcal{X}_{A_i} d\nu d\mu \\ &= \int_X \sum_{i=1}^\infty \int_Y \mathcal{X}_{A_i} d\nu d\mu \\ &= \int_X \int_Y \sum_{i=1}^\infty \mathcal{X}_{A_i} d\nu d\mu \\ &= \int_X \int_Y \mathcal{X}_{\cup_{i=1}^\infty A_i} d\nu d\mu, \qquad (9.45)\end{aligned}$$

the interchanges between the summation and the integral depending on the monotone convergence theorem. Thus $\mathcal{G}$ is closed with respect to countable disjoint unions.

From Lemma 9.72, $\mathcal{G} \supseteq \sigma(\mathcal{K})$. Also the computation in 9.45 implies that on $\sigma(\mathcal{K})$ one can define a measure, denoted by $\mu \times \nu$ and that for every $E \in \sigma(\mathcal{K})$,

$$(\mu \times \nu)(E) = \int_Y \int_X \mathcal{X}_E d\mu d\nu = \int_X \int_Y \mathcal{X}_E d\nu d\mu. \qquad (9.46)$$

Now here is Fubini's theorem.

Theorem 9.73 *Let $f : X \times Y \to [0, \infty]$ be measurable with respect to the σ algebra, $\sigma(\mathcal{K})$ just defined and let $\mu \times \nu$ be the product measure of 9.46 where μ and ν are finite measures on $(X, \mathcal{F})$ and $(Y, \mathcal{S})$ respectively. Then*

$$\int_{X \times Y} f d(\mu \times \nu) = \int_Y \int_X f d\mu d\nu = \int_X \int_Y f d\nu d\mu.$$

Proof: Let $\{s_n\}$ be an increasing sequence of $\sigma(\mathcal{K})$ measurable simple functions which converges pointwise to f. The above equation holds for s_n in place of f from what was shown above. The final result follows from passing to the limit and using the monotone convergence theorem. This proves the theorem.

The symbol, $\mathcal{F} \times \mathcal{S}$ denotes $\sigma(\mathcal{K})$.

Of course one can generalize right away to measures which are only σ finite.

Theorem 9.74 *Let $f : X \times Y \to [0, \infty]$ be measurable with respect to the σ algebra, $\sigma(\mathcal{K})$ just defined and let $\mu \times \nu$ be the product measure of 9.46 where μ and ν are σ finite measures on $(X, \mathcal{F})$ and $(Y, \mathcal{S})$ respectively. Then*

$$\int_{X \times Y} f d(\mu \times \nu) = \int_Y \int_X f d\mu d\nu = \int_X \int_Y f d\nu d\mu.$$

Proof: Since the measures are σ finite, there exist increasing sequences of sets, $\{X_n\}$ and $\{Y_n\}$ such that $\mu(X_n) < \infty$ and $\mu(Y_n) < \infty$. Then μ and ν restricted to X_n and Y_n respectively are finite. Then from Theorem 9.73,

$$\int_{Y_n} \int_{X_n} f d\mu d\nu = \int_{X_n} \int_{Y_n} f d\nu d\mu$$

Passing to the limit yields

$$\int_Y \int_X f d\mu d\nu = \int_X \int_Y f d\nu d\mu$$

whenever f is as above. In particular, you could take $f = \mathcal{X}_E$ where $E \in \mathcal{F} \times \mathcal{S}$ and define

$$(\mu \times \nu)(E) \equiv \int_Y \int_X \mathcal{X}_E d\mu d\nu = \int_X \int_Y \mathcal{X}_E d\nu d\mu.$$

Then just as in the proof of Theorem 9.73, the conclusion of this theorem is obtained. This proves the theorem.

It is also useful to note that all the above holds for $\prod_{i=1}^n X_i$ in place of $X \times Y$. You would simply modify the definition of $\mathcal{G}$ in 9.44 including all permutations for the iterated integrals and for $\mathcal{K}$ you would use sets of the form $\prod_{i=1}^n A_i$ where A_i is measurable. Everything goes through exactly as above. Thus the following is obtained.

Theorem 9.75 *Let $\{(X_i, \mathcal{F}_i, \mu_i)\}_{i=1}^n$ be σ finite measure spaces and let $\prod_{i=1}^n \mathcal{F}_i$ denote the smallest σ algebra which contains the measurable boxes of the form $\prod_{i=1}^n A_i$ where $A_i \in \mathcal{F}_i$. Then there exists a measure, λ defined on $\prod_{i=1}^n \mathcal{F}_i$ such that if $f : \prod_{i=1}^n X_i \to [0, \infty]$ is $\prod_{i=1}^n \mathcal{F}_i$ measurable, and $(i_1, \cdots, i_n)$ is any permutation of $(1, \cdots, n)$, then*

$$\int f d\lambda = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f d\mu_{i_1} \cdots d\mu_{i_n}$$

9.10.2 Completion Of Product Measure Spaces

Using Theorem 9.69 it is easy to give a generalization to yield a theorem for the completion of product spaces.

Theorem 9.76 *Let $\{(X_i, \mathcal{F}_i, \mu_i)\}_{i=1}^n$ be σ finite measure spaces and let $\prod_{i=1}^n \mathcal{F}_i$ denote the smallest σ algebra which contains the measurable boxes of the form $\prod_{i=1}^n A_i$ where $A_i \in \mathcal{F}_i$. Then there exists a measure, λ defined on $\prod_{i=1}^n \mathcal{F}_i$ such that if $f : \prod_{i=1}^n X_i \to [0, \infty]$ is $\prod_{i=1}^n \mathcal{F}_i$ measurable, and $(i_1, \cdots, i_n)$ is any permutation of $(1, \cdots, n)$, then*

$$\int f d\lambda = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f d\mu_{i_1} \cdots d\mu_{i_n}$$

Let $\left(\prod_{i=1}^n X_i, \overline{\prod_{i=1}^n \mathcal{F}_i}, \overline{\lambda}\right)$ denote the completion of this product measure space and let

$$f : \prod_{i=1}^n X_i \to [0, \infty]$$

be $\overline{\prod_{i=1}^n \mathcal{F}_i}$ measurable. Then there exists $N \in \prod_{i=1}^n \mathcal{F}_i$ such that $\lambda(N) = 0$ and a nonnegative function, f_1 measurable with respect to $\prod_{i=1}^n \mathcal{F}_i$ such that $f_1 = f$ off N and if $(i_1, \cdots, i_n)$ is any permutation of $(1, \cdots, n)$, then

$$\int f d\overline{\lambda} = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f_1 d\mu_{i_1} \cdots d\mu_{i_n}.$$

Furthermore, f_1 may be chosen to satisfy either $f_1 \leq f$ or $f_1 \geq f$.

Proof: This follows immediately from Theorem 9.75 and Theorem 9.69. By the second theorem, there exists a function $f_1 \geq f$ such that $f_1 = f$ for all $(x_1, \cdots, x_n) \notin N$, a set of $\prod_{i=1}^n \mathcal{F}_i$ having measure zero. Then by Theorem 9.68 and Theorem 9.75

$$\int f d\overline{\lambda} = \int f_1 d\lambda = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f_1 d\mu_{i_1} \cdots d\mu_{i_n}.$$

To get $f_1 \leq f$, just use that part of Theorem 9.69.

Since $f_1 = f$ off a set of measure zero, I will dispense with the subscript. Also it is customary to write

$$\lambda = \mu_1 \times \cdots \times \mu_n$$

and

$$\overline{\lambda} = \overline{\mu_1 \times \cdots \times \mu_n}.$$

Thus in more standard notation, one writes

$$\int f d\left(\overline{\mu_1 \times \cdots \times \mu_n}\right) = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f d\mu_{i_1} \cdots d\mu_{i_n}$$

This theorem is often referred to as Fubini's theorem. The next theorem is also called this.

Corollary 9.77 *Suppose* $f \in L^1\left(\prod_{i=1}^n X_i, \overline{\prod_{i=1}^n \mathcal{F}_i}, \overline{\mu_1 \times \cdots \times \mu_n}\right)$ *where each* X_i *is a* σ *finite measure space. Then if* $(i_1, \cdots, i_n)$ *is any permutation of* $(1, \cdots, n)$, *it follows*

$$\int f d\left(\overline{\mu_1 \times \cdots \times \mu_n}\right) = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f d\mu_{i_1} \cdots d\mu_{i_n}.$$

Proof: Just apply Theorem 9.76 to the positive and negative parts of the real and imaginary parts of f. This proves the theorem.

Here is another easy corollary.

Corollary 9.78 *Suppose in the situation of Corollary 9.77,* $f = f_1$ *off* N, *a set of* $\prod_{i=1}^n \mathcal{F}_i$ *having* $\mu_1 \times \cdots \times \mu_n$ *measure zero and that* f_1 *is a complex valued function measurable with respect to* $\prod_{i=1}^n \mathcal{F}_i$. *Suppose also that for some permutation of* $(1, 2, \cdots, n), (j_1, \cdots, j_n)$

$$\int_{X_{j_n}} \cdots \int_{X_{j_1}} |f_1| \, d\mu_{j_1} \cdots d\mu_{j_n} < \infty.$$

Then

$$f \in L^1\left(\prod_{i=1}^n X_i, \overline{\prod_{i=1}^n \mathcal{F}_i}, \overline{\mu_1 \times \cdots \times \mu_n}\right)$$

and the conclusion of Corollary 9.77 holds.

Proof: Since $|f_1|$ is $\prod_{i=1}^n \mathcal{F}_i$ measurable, it follows from Theorem 9.75 that

$$\begin{aligned} \infty &> \int_{X_{j_n}} \cdots \int_{X_{j_1}} |f_1| \, d\mu_{j_1} \cdots d\mu_{j_n} \\ &= \int |f_1| \, d\left(\mu_1 \times \cdots \times \mu_n\right) \\ &= \int |f_1| \, d\left(\overline{\mu_1 \times \cdots \times \mu_n}\right) \\ &= \int |f| \, d\left(\overline{\mu_1 \times \cdots \times \mu_n}\right). \end{aligned}$$

Thus $f \in L^1\left(\prod_{i=1}^n X_i, \overline{\prod_{i=1}^n \mathcal{F}_i}, \overline{\mu_1 \times \cdots \times \mu_n}\right)$ as claimed and the rest follows from Corollary 9.77. This proves the corollary.

The following lemma is also useful.

Lemma 9.79 *Let* $(X, \mathcal{F}, \mu)$ *and* $(Y, \mathcal{S}, \nu)$ *be* σ *finite complete measure spaces and suppose* $f \geq 0$ *is* $\overline{\mathcal{F} \times \mathcal{S}}$ *measurable. Then for a.e.* x,

$$y \to f(x, y)$$

is $\mathcal{S}$ *measurable. Similarly for a.e.* y,

$$x \to f(x, y)$$

is $\mathcal{F}$ *measurable.*

Proof: By Theorem 9.69, there exist $\mathcal{F} \times \mathcal{S}$ measurable functions, g and h and a set, $N \in \mathcal{F} \times \mathcal{S}$ of $\mu \times \lambda$ measure zero such that $g \leq f \leq h$ and for $(x, y) \notin N$, it follows that $g(x, y) = h(x, y)$. Then

$$\int_X \int_Y g d\nu d\mu = \int_X \int_Y h d\nu d\mu$$

and so for a.e. x,

$$\int_Y g d\nu = \int_Y h d\nu.$$

Then it follows that for these values of $x, g(x, y) = h(x, y)$ and so by Theorem 9.69 again and the assumption that $(Y, \mathcal{S}, \nu)$ is complete, $y \to f(x, y)$ is $\mathcal{S}$ measurable. The other claim is similar. This proves the lemma.

9.11 Disturbing Examples

There are examples which help to define what can be expected of product measures and Fubini type theorems. Three such examples are given in Rudin [45]. The theorems given above are more general than those in this reference but the same examples are still useful for showing that the hypotheses of the above theorems are all necessary.

Example 9.80 *Let $\{a_n\}$ be an increasing sequence of numbers in $(0, 1)$ which converges to 1. Let $g_n \in C_c(a_n, a_{n+1})$ such that $\int g_n dx = 1$. Now for $(x, y) \in [0, 1) \times [0, 1)$ define*

$$f(x, y) \equiv \sum_{k=1}^{\infty} g_n(y) (g_n(x) - g_{n+1}(x)).$$

Note this is actually a finite sum for each such (x, y). Therefore, this is a continuous function on $[0, 1) \times [0, 1)$. Now for a fixed y,

$$\int_0^1 f(x, y) dx = \sum_{k=1}^{\infty} g_n(y) \int_0^1 (g_n(x) - g_{n+1}(x)) dx = 0$$

showing that $\int_0^1 \int_0^1 f(x, y) dxdy = \int_0^1 0 dy = 0$. Next fix x.

$$\int_0^1 f(x, y) dy = \sum_{k=1}^{\infty} (g_n(x) - g_{n+1}(x)) \int_0^1 g_n(y) dy = g_1(x).$$

Hence $\int_0^1 \int_0^1 f(x, y) dydx = \int_0^1 g_1(x) dx = 1$. The iterated integrals are not equal. Note the function, g is not nonnegative even though it is measurable. In addition, neither $\int_0^1 \int_0^1 |f(x, y)| dxdy$ nor $\int_0^1 \int_0^1 |f(x, y)| dydx$ is finite and so you can't apply Corollary 9.52. The problem here is the function is not nonnegative and is not absolutely integrable.

Example 9.81 *This time let $\mu = m$, Lebesgue measure on $[0,1]$ and let ν be counting measure on $[0,1]$, in this case, the σ algebra is $\mathcal{P}([0,1])$. Let l denote the line segment in $[0,1] \times [0,1]$ which goes from $(0,0)$ to $(1,1)$. Thus $l = (x,x)$ where $x \in [0,1]$. Consider the outer measure of l in $\overline{m \times \nu}$. Let $l \subseteq \cup_k A_k \times B_k$ where A_k is Lebesgue measurable and B_k is a subset of $[0,1]$. Let $\mathcal{B} \equiv \{k \in \mathbb{N} : \nu(B_k) = \infty\}$. If $m(\cup_{k\in\mathcal{B}} A_k)$ has measure zero, then there are uncountably many points of $[0,1]$ outside of $\cup_{k\in\mathcal{B}} A_k$. For p one of these points, $(p,p) \in A_i \times B_i$ and $i \notin \mathcal{B}$. Thus each of these points is in $\cup_{i\notin\mathcal{B}} B_i$, a countable set because these B_i are each finite. But this is a contradiction because there need to be uncountably many of these points as just indicated. Thus $m(A_k) > 0$ for some $k \in \mathcal{B}$ and so $\overline{m \times \nu}(A_k \times B_k) = \infty$. It follows $\overline{m \times \nu}(l) = \infty$ and so l is $\overline{m \times \nu}$ measurable. Thus $\int \mathcal{X}_l(x,y)\, d\,\overline{m \times \nu} = \infty$ and so you cannot apply Fubini's theorem, Theorem 9.50. Since ν is not σ finite, you cannot apply the corollary to this theorem either. Thus there is no contradiction to the above theorems in the following observation.*

$$\int \int \mathcal{X}_l(x,y)\, d\nu dm = \int 1 dm = 1, \quad \int \int \mathcal{X}_l(x,y)\, dm d\nu = \int 0 d\nu = 0.$$

The problem here is that you have neither $\int f d\,\overline{m \times \nu} < \infty$ not σ finite measure spaces.

The next example is far more exotic. It concerns the case where both iterated integrals make perfect sense but are unequal. In 1877 Cantor conjectured that the cardinality of the real numbers is the next size of infinity after countable infinity. This hypothesis is called the continuum hypothesis and it has never been proved or disproved[4]. Assuming this continuum hypothesis will provide the basis for the following example. It is due to Sierpinski.

Example 9.82 *Let X be an uncountable set. It follows from the well ordering theorem which says every set can be well ordered which is presented in the appendix that X can be well ordered. Let $\omega \in X$ be the first element of X which is preceded by uncountably many points of X. Let Ω denote $\{x \in X : x < \omega\}$. Then Ω is uncountable but there is no smaller uncountable set. Thus by the continuum hypothesis, there exists a one to one and onto mapping, j which maps $[0,1]$ onto Ω. Thus, for $x \in [0,1], j(x)$ is preceeded by countably many points. Let $Q \equiv \left\{(x,y) \in [0,1]^2 : j(x) < j(y)\right\}$ and let $f(x,y) = \mathcal{X}_Q(x,y)$. Then*

$$\int_0^1 f(x,y)\, dy = 1, \quad \int_0^1 f(x,y)\, dx = 0$$

In each case, the integrals make sense. In the first, for fixed x, $f(x,y) = 1$ for all but countably many y so the function of y is Borel measurable. In the second where

[4] In 1940 it was shown by Godel that the continuum hypothesis cannot be disproved. In 1963 it was shown by Cohen that the continuum hypothesis cannot be proved. These assertions are based on the axiom of choice and the Zermelo Frankel axioms of set theory. This topic is far outside the scope of this book and this is only a hopefully interesting historical observation.

y is fixed, $f(x,y)=0$ *for all but countably many* x. *Thus*

$$\int_0^1\int_0^1 f(x,y)\,dydx=1,\quad \int_0^1\int_0^1 f(x,y)\,dxdy=0.$$

The problem here must be that f *is not* $\overline{m\times m}$ *measurable.*

9.12 Exercises

1. Let $\Omega=\mathbb{N}$, the natural numbers and let $d(p,q)=|p-q|$, the usual distance in $\mathbb{R}$. Show that (Ω,d) the closures of the balls are compact. Now let $\Lambda f\equiv\sum_{k=1}^{\infty}f(k)$ whenever $f\in C_c(\Omega)$. Show this is a well defined positive linear functional on the space $C_c(\Omega)$. Describe the measure of the Riesz representation theorem which results from this positive linear functional. What if $\Lambda(f)=f(1)$? What measure would result from this functional? Which functions are measurable?

2. Verify that $\overline{\mu}$ defined in Lemma 9.7 is an outer measure.

3. Let $F:\mathbb{R}\to\mathbb{R}$ be increasing and right continuous. Let $\Lambda f\equiv\int fdF$ where the integral is the Riemann Stieltjes integral of f. Show the measure μ from the Riesz representation theorem satisfies

$$\begin{aligned}\mu([a,b]) &= F(b)-F(a-),\mu((a,b])=F(b)-F(a),\\ \mu([a,a]) &= F(a)-F(a-).\end{aligned}$$

4. Let Ω be a metric space with the closed balls compact and suppose μ is a measure defined on the Borel sets of Ω which is finite on compact sets. Show there exists a unique Radon measure, $\overline{\mu}$ which equals μ on the Borel sets.

5. $\uparrow$ Random vectors are measurable functions, $\mathbf{X}$, mapping a probability space, $(\Omega,P,\mathcal{F})$ to $\mathbb{R}^n$. Thus $\mathbf{X}(\omega)\in\mathbb{R}^n$ for each $\omega\in\Omega$ and P is a probability measure defined on the sets of $\mathcal{F}$, a σ algebra of subsets of Ω. For E a Borel set in $\mathbb{R}^n$, define

$$\mu(E)\equiv P\left(\mathbf{X}^{-1}(E)\right)\equiv\text{ probability that }\mathbf{X}\in E.$$

Show this is a well defined measure on the Borel sets of $\mathbb{R}^n$ and use Problem 4 to obtain a Radon measure, $\lambda_{\mathbf{X}}$ defined on a σ algebra of sets of $\mathbb{R}^n$ including the Borel sets such that for E a Borel set, $\lambda_{\mathbf{X}}(E)=$Probability that $(\mathbf{X}\in E)$.

6. Suppose X and Y are metric spaces having compact closed balls. Show

$$(X\times Y,d_{X\times Y})$$

is also a metric space which has the closures of balls compact. Here

$$d_{X\times Y}((x_1,y_1),(x_2,y_2))\equiv\max(d(x_1,x_2),d(y_1,y_2)).$$

Let
$$\mathcal{A} \equiv \{E \times F : E \text{ is a Borel set in } X, F \text{ is a Borel set in } Y\}.$$

Show $\sigma(\mathcal{A})$, the smallest σ algebra containing $\mathcal{A}$ contains the Borel sets. **Hint:** Show every open set in a metric space which has closed balls compact can be obtained as a countable union of compact sets. Next show this implies every open set can be obtained as a countable union of open sets of the form $U \times V$ where U is open in X and V is open in Y.

7. Suppose $(\Omega, \mathcal{S}, \mu)$ is a measure space which may not be complete. Could you obtain a complete measure space, $\left(\Omega, \overline{\mathcal{S}}, \mu_1\right)$ by simply letting $\overline{\mathcal{S}}$ consist of all sets of the form E where there exists $F \in \mathcal{S}$ such that $(F \setminus E) \cup (E \setminus F) \subseteq N$ for some $N \in \mathcal{S}$ which has measure zero and then let $\mu(E) = \mu_1(F)$?

8. If μ and ν are Radon measures defined on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, show $\overline{\mu \times \nu}$ is also a radon measure on $\mathbb{R}^{n+m}$. **Hint:** Show the $\overline{\mu \times \nu}$ measurable sets include the open sets using the observation that every open set in $\mathbb{R}^{n+m}$ is the countable union of sets of the form $U \times V$ where U and V are open in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Next verify outer regularity by considering $A \times B$ for A, B measurable. Argue sets of $\mathcal{R}$ defined above have the property that they can be approximated in measure from above by open sets. Then verify the same is true of sets of $\mathcal{R}_1$. Finally conclude using an appropriate lemma that $\overline{\mu \times \nu}$ is inner regular as well.

9. Let $(\Omega, \mathcal{S}, \mu)$ be a σ finite measure space and let $f : \Omega \to [0, \infty)$ be measurable. Define
$$A \equiv \{(x, y) : y < f(x)\}$$
Verify that A is $\overline{\mu \times m}$ measurable. Show that
$$\int f d\mu = \int \int \mathcal{X}_A(x, y)\, d\mu dm = \int \mathcal{X}_A d\overline{\mu \times m}.$$

Lebesgue Measure

10.1 Basic Properties

Definition 10.1 *Define the following positive linear functional for* $f \in C_c(\mathbb{R}^n)$.

$$\Lambda f \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\mathbf{x})\, dx_1 \cdots dx_n.$$

Then the measure representing this functional is Lebesgue measure.

The following lemma will help in understanding Lebesgue measure.

Lemma 10.2 *Every open set in* $\mathbb{R}^n$ *is the countable disjoint union of half open boxes of the form*

$$\prod_{i=1}^{n} (a_i, a_i + 2^{-k}]$$

where $a_i = l2^{-k}$ *for some integers,* l, k. *The sides of these boxes are of equal length. One could also have half open boxes of the form*

$$\prod_{i=1}^{n} [a_i, a_i + 2^{-k})$$

and the conclusion would be unchanged.

Proof: Let

$$\mathcal{C}_k = \{\text{All half open boxes } \prod_{i=1}^{n} (a_i, a_i + 2^{-k}] \text{ where}$$

$$a_i = l2^{-k} \text{ for some integer } l.\}$$

Thus $\mathcal{C}_k$ consists of a countable disjoint collection of boxes whose union is $\mathbb{R}^n$. This is sometimes called a tiling of $\mathbb{R}^n$. Think of tiles on the floor of a bathroom and you will get the idea. Note that each box has diameter no larger than $2^{-k}\sqrt{n}$. This is because if

$$\mathbf{x},\mathbf{y}\in\prod_{i=1}^{n}(a_i,a_i+2^{-k}],$$

then $|x_i-y_i|\le 2^{-k}$. Therefore,

$$|\mathbf{x}-\mathbf{y}|\le\left(\sum_{i=1}^{n}\left(2^{-k}\right)^2\right)^{1/2}=2^{-k}\sqrt{n}.$$

Let U be open and let $\mathcal{B}_1\equiv$ all sets of $\mathcal{C}_1$ which are contained in U. If $\mathcal{B}_1,\cdots,\mathcal{B}_k$ have been chosen, $\mathcal{B}_{k+1}\equiv$ all sets of $\mathcal{C}_{k+1}$ contained in

$$U\setminus\cup\left(\cup_{i=1}^{k}\mathcal{B}_i\right).$$

Let $\mathcal{B}_\infty=\cup_{i=1}^{\infty}\mathcal{B}_i$. In fact $\cup\mathcal{B}_\infty=U$. Clearly $\cup\mathcal{B}_\infty\subseteq U$ because every box of every $\mathcal{B}_i$ is contained in U. If $p\in U$, let k be the smallest integer such that p is contained in a box from $\mathcal{C}_k$ which is also a subset of U. Thus

$$p\in\cup\mathcal{B}_k\subseteq\cup\mathcal{B}_\infty.$$

Hence $\mathcal{B}_\infty$ is the desired countable disjoint collection of half open boxes whose union is U. The last assertion about the other type of half open rectangle is obvious. This proves the lemma.

Now what does Lebesgue measure do to a rectangle, $\prod_{i=1}^{n}(a_i,b_i]$?

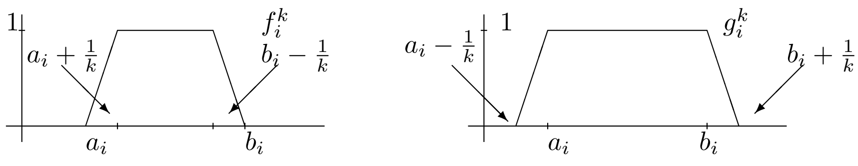
Lemma 10.3 *Let $R=\prod_{i=1}^{n}[a_i,b_i]$, $R_0=\prod_{i=1}^{n}(a_i,b_i)$. Then*

$$m_n(R_0)=m_n(R)=\prod_{i=1}^{n}(b_i-a_i).$$

Proof: Let k be large enough that

$$a_i+1/k<b_i-1/k$$

for $i=1,\cdots,n$ and consider functions g_i^k and f_i^k having the following graphs.



Let

$$g^k(\mathbf{x})=\prod_{i=1}^{n}g_i^k(x_i),\quad f^k(\mathbf{x})=\prod_{i=1}^{n}f_i^k(x_i).$$

Then by elementary calculus along with the definition of Λ,

$$\prod_{i=1}^{n}(b_i - a_i + 2/k) \geq \Lambda g^k = \int g^k dm_n \geq m_n(R) \geq m_n(R_0)$$

$$\geq \int f^k dm_n = \Lambda f^k \geq \prod_{i=1}^{n}(b_i - a_i - 2/k).$$

Letting $k \to \infty$, it follows that

$$m_n(R) = m_n(R_0) = \prod_{i=1}^{n}(b_i - a_i).$$

This proves the lemma.

Lemma 10.4 *Let U be an open or closed set. Then $m_n(U) = m_n(\mathbf{x} + U)$.*

Proof: By Lemma 10.2 there is a sequence of disjoint half open rectangles, $\{R_i\}$ such that $\cup_i R_i = U$. Therefore, $\mathbf{x} + U = \cup_i (\mathbf{x} + R_i)$ and the $\mathbf{x} + R_i$ are also disjoint rectangles which are identical to the R_i but translated. From Lemma 10.3, $m_n(U) = \sum_i m_n(R_i) = \sum_i m_n(\mathbf{x} + R_i) = m_n(\mathbf{x} + U)$.

It remains to verify the lemma for a closed set. Let H be a closed bounded set first. Then $H \subseteq B(\mathbf{0},R)$ for some R large enough. First note that $\mathbf{x}+H$ is a closed set. Thus

$$\begin{aligned} m_n(B(\mathbf{x},R)) &= m_n(\mathbf{x}+H) + m_n((B(\mathbf{0},R)+\mathbf{x}) \setminus (\mathbf{x}+H)) \\ &= m_n(\mathbf{x}+H) + m_n((B(\mathbf{0},R) \setminus H) + \mathbf{x}) \\ &= m_n(\mathbf{x}+H) + m_n((B(\mathbf{0},R) \setminus H)) \\ &= m_n(B(\mathbf{0},R)) - m_n(H) + m_n(\mathbf{x}+H) \\ &= m_n(B(\mathbf{x},R)) - m_n(H) + m_n(\mathbf{x}+H) \end{aligned}$$

the last equality because of the first part of the lemma which implies $m_n(B(\mathbf{x},R)) = m_n(B(\mathbf{0},R))$. Therefore, $m_n(\mathbf{x}+H) = m_n(H)$ as claimed. If H is not bounded, consider $H_m \equiv \overline{B(\mathbf{0},m)} \cap H$. Then $m_n(\mathbf{x}+H_m) = m_n(H_m)$. Passing to the limit as $m \to \infty$ yields the result in general.

Theorem 10.5 *Lebesgue measure is translation invariant. That is*

$$m_n(E) = m_n(\mathbf{x}+E)$$

for all E Lebesgue measurable.

Proof: Suppose $m_n(E) < \infty$. By regularity of the measure, there exist sets G, H such that G is a countable intersection of open sets, H is a countable union of compact sets, $m_n(G \setminus H) = 0$, and $G \supseteq E \supseteq H$. Now $m_n(G) = m_n(G+\mathbf{x})$ and

$m_n(H) = m_n(H + \mathbf{x})$ which follows from Lemma 10.4 applied to the sets which are either intersected to form G or unioned to form H. Now

$$\mathbf{x} + H \subseteq \mathbf{x} + E \subseteq \mathbf{x} + G$$

and both $\mathbf{x}+H$ and $\mathbf{x}+G$ are measurable because they are either countable unions or countable intersections of measurable sets. Furthermore,

$$m_n(\mathbf{x} + G \setminus \mathbf{x} + H) = m_n(\mathbf{x} + G) - m_n(\mathbf{x} + H) = m_n(G) - m_n(H) = 0$$

and so by completeness of the measure, $\mathbf{x} + E$ is measurable. It follows

$$\begin{aligned} m_n(E) &= m_n(H) = m_n(\mathbf{x} + H) \leq m_n(\mathbf{x} + E) \\ &\leq m_n(\mathbf{x} + G) = m_n(G) = m_n(E). \end{aligned}$$

If $m_n(E)$ is not necessarily less than ∞, consider $E_m \equiv B(\mathbf{0}, m) \cap E$. Then $m_n(E_m) = m_n(E_m + \mathbf{x})$ by the above. Letting $m \to \infty$ it follows $m_n(E_m) = m_n(E_m + \mathbf{x})$. This proves the theorem.

Corollary 10.6 *Let D be an $n \times n$ diagonal matrix and let U be an open set. Then*

$$m_n(DU) = |\det(D)| \, m_n(U).$$

Proof: If any of the diagonal entries of D equals 0 there is nothing to prove because then both sides equal zero. Therefore, it can be assumed none are equal to zero. Suppose these diagonal entries are $k_1, \cdots, k_n$. From Lemma 10.2 there exist half open boxes, $\{R_i\}$ having all sides equal such that $U = \cup_i R_i$. Suppose one of these is $R_i = \prod_{j=1}^{n} (a_j, b_j]$, where $b_j - a_j = l_i$. Then $DR_i = \prod_{j=1}^{n} I_j$ where $I_j = (k_j a_j, k_j b_j]$ if $k_j > 0$ and $I_j = [k_j b_j, k_j a_j)$ if $k_j < 0$. Then the rectangles, DR_i are disjoint because D is one to one and their union is DU. Also,

$$m_n(DR_i) = \prod_{j=1}^{n} |k_j| \, l_i = |\det D| \, m_n(R_i).$$

Therefore,

$$m_n(DU) = \sum_{i=1}^{\infty} m_n(DR_i) = |\det(D)| \sum_{i=1}^{\infty} m_n(R_i) = |\det(D)| \, m_n(U).$$

and this proves the corollary.

From this the following corollary is obtained.

Corollary 10.7 *Let $M > 0$. Then $m_n(B(\mathbf{a}, Mr)) = M^n m_n(B(\mathbf{0}, r))$.*

Proof: By Lemma 10.4 there is no loss of generality in taking $\mathbf{a} = \mathbf{0}$. Let D be the diagonal matrix which has M in every entry of the main diagonal so $|\det(D)| = M^n$. Note that $DB(\mathbf{0}, r) = B(\mathbf{0}, Mr)$. By Corollary 10.6 $m_n(B(\mathbf{0}, Mr)) = m_n(DB(\mathbf{0}, r)) = M^n m_n(B(\mathbf{0}, r))$.

There are many norms on $\mathbb{R}^n$. Other common examples are

$$||\mathbf{x}||_\infty \equiv \max\{|x_k| : \mathbf{x} = (x_1, \cdots, x_n)\}$$

or

$$||\mathbf{x}||_p \equiv \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}.$$

With $||\cdot||$ any norm for $\mathbb{R}^n$ you can define a corresponding ball in terms of this norm.

$$B(\mathbf{a}, r) \equiv \{\mathbf{x} \in \mathbb{R}^n \text{ such that } ||\mathbf{x} - \mathbf{a}|| < r\}$$

It follows from general considerations involving metric spaces presented earlier that these balls are open sets. Therefore, Corollary 10.7 has an obvious generalization.

Corollary 10.8 *Let $||\cdot||$ be a norm on $\mathbb{R}^n$. Then for $M > 0$, $m_n(B(\mathbf{a}, Mr)) = M^n m_n(B(\mathbf{0}, r))$ where these balls are defined in terms of the norm $||\cdot||$.*

10.2 The Vitali Covering Theorem

The Vitali covering theorem is concerned with the situation in which a set is contained in the union of balls. You can imagine that it might be very hard to get disjoint balls from this collection of balls which would cover the given set. However, it is possible to get disjoint balls from this collection of balls which have the property that if each ball is enlarged appropriately, the resulting enlarged balls do cover the set. When this result is established, it is used to prove another form of this theorem in which the disjoint balls do not cover the set but they only miss a set of measure zero.

Recall the Hausdorff maximal principle, Theorem 1.13 on Page 24 which is proved to be equivalent to the axiom of choice in the appendix. For convenience, here it is:

Theorem 10.9 *(Hausdorff Maximal Principle) Let $\mathcal{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.*

I will use this Hausdorff maximal principle to give a very short and elegant proof of the Vitali covering theorem. This follows the treatment in Evans and Gariepy [20] which they got from another book. I am not sure who first did it this way but it is very nice because it is so short. In the following lemma and theorem, the balls will be either open or closed and determined by some norm on $\mathbb{R}^n$. When pictures are drawn, I shall draw them as though the norm is the usual norm but the results are unchanged for any norm. Also, I will write (in this section only) $B(\mathbf{a}, r)$ to indicate a set which satisfies

$$\{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{a}|| < r\} \subseteq B(\mathbf{a}, r) \subseteq \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{a}|| \leq r\}$$

and $\widehat{B}(\mathbf{a}, r)$ to indicate the usual ball but with radius 5 times as large,

$$\{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{a}|| < 5r\}.$$

Lemma 10.10 *Let $||\cdot||$ be a norm on $\mathbb{R}^n$ and let $\mathcal{F}$ be a collection of balls determined by this norm. Suppose*

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0$$

and $k \in (0, \infty)$. Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that

$$\text{if } B(\mathbf{p}, r) \in \mathcal{G} \text{ then } r > k, \tag{10.1}$$

$$\text{if } B_1, B_2 \in \mathcal{G} \text{ then } B_1 \cap B_2 = \emptyset, \tag{10.2}$$

$$\mathcal{G} \text{ is maximal with respect to 10.1 and 10.2.}$$

Note that if there is no ball of $\mathcal{F}$ which has radius larger than k then $\mathcal{G} = \emptyset$.

Proof: Let $\mathcal{H} = \{\mathcal{B} \subseteq \mathcal{F}$ such that 10.1 and 10.2 hold$\}$. If there are no balls with radius larger than k then $\mathcal{H} = \emptyset$ and you let $\mathcal{G} = \emptyset$. In the other case, $\mathcal{H} \neq \emptyset$ because there exists $B(\mathbf{p}, r) \in \mathcal{F}$ with $r > k$. In this case, partially order $\mathcal{H}$ by set inclusion and use the Hausdorff maximal principle (see the appendix on set theory) to let $\mathcal{C}$ be a maximal chain in $\mathcal{H}$. Clearly $\cup\mathcal{C}$ satisfies 10.1 and 10.2 because if B_1 and B_2 are two balls from $\cup\mathcal{C}$ then since $\mathcal{C}$ is a chain, it follows there is some element of $\mathcal{C}, \mathcal{B}$ such that both B_1 and B_2 are elements of $\mathcal{B}$ and $\mathcal{B}$ satisfies 10.1 and 10.2. If $\cup\mathcal{C}$ is not maximal with respect to these two properties, then $\mathcal{C}$ was not a maximal chain because then there would exist $\mathcal{B} \supsetneq \cup\mathcal{C}$, that is, $\mathcal{B}$ contains $\mathcal{C}$ as a proper subset and $\{\mathcal{C}, \mathcal{B}\}$ would be a strictly larger chain in $\mathcal{H}$. Let $\mathcal{G} = \cup\mathcal{C}$.

Theorem 10.11 *(Vitali) Let $\mathcal{F}$ be a collection of balls and let*

$$A \equiv \cup\{B : B \in \mathcal{F}\}.$$

Suppose

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0.$$

Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and

$$A \subseteq \cup\{\widehat{B} : B \in \mathcal{G}\}.$$

Proof: Using Lemma 10.10, there exists $\mathcal{G}_1 \subseteq \mathcal{F} \equiv \mathcal{F}_0$ which satisfies

$$B(\mathbf{p}, r) \in \mathcal{G}_1 \text{ implies } r > \frac{M}{2}, \tag{10.3}$$

$$B_1, B_2 \in \mathcal{G}_1 \text{ implies } B_1 \cap B_2 = \emptyset, \tag{10.4}$$

$$\mathcal{G}_1 \text{ is maximal with respect to 10.3, and 10.4.}$$

Suppose $\mathcal{G}_1, \cdots, \mathcal{G}_m$ have been chosen, $m \geq 1$. Let

$$\mathcal{F}_m \equiv \{B \in \mathcal{F} : B \subseteq \mathbb{R}^n \setminus \cup\{\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m\}\}.$$

Using Lemma 10.10, there exists $\mathcal{G}_{m+1} \subseteq \mathcal{F}_m$ such that

$$B(\mathbf{p}, r) \in \mathcal{G}_{m+1} \text{ implies } r > \frac{M}{2^{m+1}}, \tag{10.5}$$

$$B_1, B_2 \in \mathcal{G}_{m+1} \text{ implies } B_1 \cap B_2 = \emptyset, \tag{10.6}$$

$$\mathcal{G}_{m+1} \text{ is a maximal subset of } \mathcal{F}_m \text{ with respect to 10.5 and 10.6.}$$

Note it might be the case that $\mathcal{G}_{m+1} = \emptyset$ which happens if $\mathcal{F}_m = \emptyset$. Define

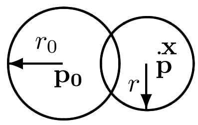
$$\mathcal{G} \equiv \cup_{k=1}^{\infty} \mathcal{G}_k.$$

Thus $\mathcal{G}$ is a collection of disjoint balls in $\mathcal{F}$. I must show $\{\widehat{B} : B \in \mathcal{G}\}$ covers A.

Let $\mathbf{x} \in B(\mathbf{p}, r) \in \mathcal{F}$ and let

$$\frac{M}{2^m} < r \leq \frac{M}{2^{m-1}}.$$

Then $B(\mathbf{p}, r)$ must intersect some set, $B(\mathbf{p}_0, r_0) \in \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ since otherwise, $\mathcal{G}_m$ would fail to be maximal. Then $r_0 > \frac{M}{2^m}$ because all balls in $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ satisfy this inequality.



Then for $\mathbf{x} \in B(\mathbf{p}, r)$, the following chain of inequalities holds because $r \leq \frac{M}{2^{m-1}}$ and $r_0 > \frac{M}{2^m}$

$$\begin{aligned} |\mathbf{x} - \mathbf{p}_0| &\leq |\mathbf{x} - \mathbf{p}| + |\mathbf{p} - \mathbf{p}_0| \leq r + r_0 + r \\ &\leq \frac{2M}{2^{m-1}} + r_0 = \frac{4M}{2^m} + r_0 < 5r_0. \end{aligned}$$

Thus $B(\mathbf{p}, r) \subseteq \widehat{B}(\mathbf{p}_0, r_0)$ and this proves the theorem.

10.3 The Vitali Covering Theorem (Elementary Version)

The proof given here is from Basic Analysis [35]. It first considers the case of open balls and then generalizes to balls which may be neither open nor closed or closed.

Lemma 10.12 *Let $\mathcal{F}$ be a countable collection of balls satisfying*

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0$$

and let $k \in (0, \infty)$. Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that

$$\textit{If } B(\mathbf{p}, r) \in \mathcal{G} \textit{ then } r > k, \tag{10.7}$$

$$\textit{If } B_1, B_2 \in \mathcal{G} \textit{ then } B_1 \cap B_2 = \emptyset, \tag{10.8}$$

$$\mathcal{G} \textit{ is maximal with respect to 10.7 and 10.8.} \tag{10.9}$$

Proof: If no ball of $\mathcal{F}$ has radius larger than k, let $\mathcal{G} = \emptyset$. Assume therefore, that some balls have radius larger than k. Let $\mathcal{F} \equiv \{B_i\}_{i=1}^{\infty}$. Now let B_{n_1} be the first ball in the list which has radius greater than k. If every ball having radius larger than k intersects this one, then stop. The maximal set is just B_{n_1}. Otherwise, let B_{n_2} be the next ball having radius larger than k which is disjoint from B_{n_1}. Continue this way obtaining $\{B_{n_i}\}_{i=1}^{\infty}$, a finite or infinite sequence of disjoint balls having radius larger than k. Then let $\mathcal{G} \equiv \{B_{n_i}\}$. To see that $\mathcal{G}$ is maximal with respect to 10.7 and 10.8, suppose $B \in \mathcal{F}$, B has radius larger than k, and $\mathcal{G} \cup \{B\}$ satisfies 10.7 and 10.8. Then at some point in the process, B would have been chosen because it would be the ball of radius larger than k which has the smallest index. Therefore, $B \in \mathcal{G}$ and this shows $\mathcal{G}$ is maximal with respect to 10.7 and 10.8.

For the next lemma, for an open ball, $B = B(\mathbf{x}, r)$, denote by $\widetilde{B}$ the open ball, $B(\mathbf{x}, 4r)$.

Lemma 10.13 *Let $\mathcal{F}$ be a collection of open balls, and let*

$$A \equiv \cup \{B : B \in \mathcal{F}\}.$$

Suppose

$$\infty > M \equiv \sup \{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0.$$

Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and

$$A \subseteq \cup\{\widetilde{B} : B \in \mathcal{G}\}.$$

Proof: Without loss of generality assume $\mathcal{F}$ is countable. This is because there is a countable subset of $\mathcal{F}$, $\mathcal{F}'$ such that $\cup \mathcal{F}' = A$. To see this, consider the set of balls having rational radii and centers having all components rational. This is a countable set of balls and you should verify that every open set is the union of balls of this form. Therefore, you can consider the subset of this set of balls consisting of those which are contained in some open set of $\mathcal{F}$, G so $\cup G = A$ and use the axiom of choice to define a subset of $\mathcal{F}$ consisting of a single set from $\mathcal{F}$ containing each set of G. Then this is $\mathcal{F}'$. The union of these sets equals A . Then consider $\mathcal{F}'$ instead of $\mathcal{F}$. Therefore, assume at the outset $\mathcal{F}$ is countable. By Lemma 10.12, there exists $\mathcal{G}_1 \subseteq \mathcal{F}$ which satisfies 10.7, 10.8, and 10.9 with $k = \frac{2M}{3}$.

Suppose $\mathcal{G}_1, \cdots, \mathcal{G}_{m-1}$ have been chosen for $m \geq 2$. Let

$$\mathcal{F}_m = \{B \in \mathcal{F} : B \subseteq \mathbb{R}^n \setminus \overbrace{\cup\{\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_{m-1}\}}^{\text{union of the balls in these } \mathcal{G}_j} \}$$

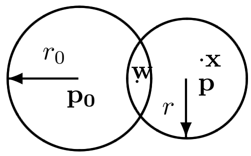
and using Lemma 10.12, let $\mathcal{G}_m$ be a maximal collection of disjoint balls from $\mathcal{F}_m$ with the property that each ball has radius larger than $\left(\frac{2}{3}\right)^m M$. Let $\mathcal{G} \equiv \cup_{k=1}^{\infty} \mathcal{G}_k$. Let $\mathbf{x} \in B(\mathbf{p}, r) \in \mathcal{F}$. Choose m such that

$$\left(\frac{2}{3}\right)^m M < r \leq \left(\frac{2}{3}\right)^{m-1} M$$

Then $B(\mathbf{p},r)$ must have nonempty intersection with some ball from $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ because if it didn't, then $\mathcal{G}_m$ would fail to be maximal. Denote by $B(\mathbf{p}_0, r_0)$ a ball in $\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m$ which has nonempty intersection with $B(\mathbf{p},r)$. Thus

$$r_0 > \left(\frac{2}{3}\right)^m M.$$

Consider the picture, in which $\mathbf{w} \in B(\mathbf{p}_0, r_0) \cap B(\mathbf{p},r)$.



Then

$$\begin{aligned} |\mathbf{x}-\mathbf{p}_0| &\leq |\mathbf{x}-\mathbf{p}| + |\mathbf{p}-\mathbf{w}| + \overbrace{|\mathbf{w}-\mathbf{p}_0|}^{<r_0} \\ &< r + r + r_0 \leq 2\overbrace{\left(\frac{2}{3}\right)^{m-1} M}^{<\frac{3}{2}r_0} + r_0 \\ &< 2\left(\frac{3}{2}\right) r_0 + r_0 = 4r_0. \end{aligned}$$

This proves the lemma since it shows $B(\mathbf{p},r) \subseteq B(\mathbf{p}_0, 4r_0)$.

With this Lemma consider a version of the Vitali covering theorem in which the balls do not have to be open. A ball centered at $\mathbf{x}$ of radius r will denote something which contains the open ball, $B(\mathbf{x},r)$ and is contained in the closed ball, $\overline{B(\mathbf{x},r)}$. Thus the balls could be open or they could contain some but not all of their boundary points.

Definition 10.14 *Let B be a ball centered at $\mathbf{x}$ having radius r. Denote by $\widehat{B}$ the open ball, $B(\mathbf{x}, 5r)$.*

Theorem 10.15 *(Vitali) Let $\mathcal{F}$ be a collection of balls, and let*

$$A \equiv \cup \{B : B \in \mathcal{F}\}.$$

Suppose

$$\infty > M \equiv \sup\{r : B(\mathbf{p},r) \in \mathcal{F}\} > 0.$$

Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and

$$A \subseteq \cup\{\widehat{B} : B \in \mathcal{G}\}.$$

Proof: For B one of these balls, say $\overline{B(\mathbf{x},r)} \supseteq B \supseteq B(\mathbf{x},r)$, denote by B_1, the ball $B\left(\mathbf{x}, \frac{5r}{4}\right)$. Let $\mathcal{F}_1 \equiv \{B_1 : B \in \mathcal{F}\}$ and let A_1 denote the union of the balls in $\mathcal{F}_1$. Apply Lemma 10.13 to $\mathcal{F}_1$ to obtain

$$A_1 \subseteq \cup\{\widetilde{B_1} : B_1 \in \mathcal{G}_1\}$$

where $\mathcal{G}_1$ consists of disjoint balls from $\mathcal{F}_1$. Now let $\mathcal{G} \equiv \{B \in \mathcal{F} : B_1 \in \mathcal{G}_1\}$. Thus $\mathcal{G}$ consists of disjoint balls from $\mathcal{F}$ because they are contained in the disjoint open balls, $\mathcal{G}_1$. Then

$$A \subseteq A_1 \subseteq \cup\{\widetilde{B_1} : B_1 \in \mathcal{G}_1\} = \cup\{\widehat{B} : B \in \mathcal{G}\}$$

because for $B_1 = B\left(\mathbf{x}, \frac{5r}{4}\right)$, it follows $\widetilde{B_1} = B(\mathbf{x}, 5r) = \widehat{B}$. This proves the theorem.

10.4 Vitali Coverings

There is another version of the Vitali covering theorem which is also of great importance. In this one, balls from the original set of balls almost cover the set,leaving out only a set of measure zero. It is like packing a truck with stuff. You keep trying to fill in the holes with smaller and smaller things so as to not waste space. It is remarkable that you can avoid wasting any space at all when you are dealing with balls of any sort provided you can use arbitrarily small balls.

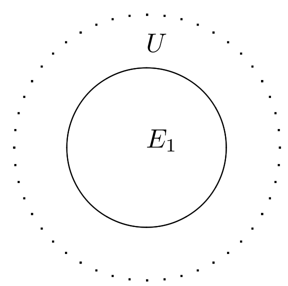
Definition 10.16 *Let $\mathcal{F}$ be a collection of balls that cover a set, E, which have the property that if $\mathbf{x} \in E$ and $\varepsilon > 0$, then there exists $B \in \mathcal{F}$, diameter of $B < \varepsilon$ and $\mathbf{x} \in B$. Such a collection covers E in the sense of Vitali.*

In the following covering theorem, $\overline{m_n}$ denotes the outer measure determined by n dimensional Lebesgue measure.

Theorem 10.17 *Let $E \subseteq \mathbb{R}^n$ and suppose $0 < \overline{m_n}(E) < \infty$ where $\overline{m_n}$ is the outer measure determined by m_n, n dimensional Lebesgue measure, and let $\mathcal{F}$ be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Proof: From the definition of outer measure there exists a Lebesgue measurable set, $E_1 \supseteq E$ such that $m_n(E_1) = \overline{m_n}(E)$. Now by outer regularity of Lebesgue measure, there exists U, an open set which satisfies

$$m_n(E_1) > (1 - 10^{-n})m_n(U),\ U \supseteq E_1.$$



Each point of E is contained in balls of $\mathcal{F}$ of arbitrarily small radii and so there exists a covering of E with balls of $\mathcal{F}$ which are themselves contained in U. Therefore, by the Vitali covering theorem, there exist disjoint balls, $\{B_i\}_{i=1}^{\infty} \subseteq \mathcal{F}$ such that

$$E \subseteq \cup_{j=1}^{\infty} \widehat{B}_j, \ B_j \subseteq U.$$

Therefore,

$$\begin{aligned} m_n(E_1) &= \overline{m_n}(E) \leq m_n\left(\cup_{j=1}^{\infty} \widehat{B}_j\right) \leq \sum_j m_n\left(\widehat{B}_j\right) \\ &= 5^n \sum_j m_n(B_j) = 5^n m_n\left(\cup_{j=1}^{\infty} B_j\right) \end{aligned}$$

Then

$$\begin{gathered} m_n(E_1) > (1-10^{-n}) m_n(U) \\ \geq (1-10^{-n})[m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) + m_n(\cup_{j=1}^{\infty} B_j)] \\ \geq (1-10^{-n})[m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) + 5^{-n} \overbrace{\overline{m_n}(E)}^{=m_n(E_1)}]. \end{gathered}$$

and so

$$\left(1-\left(1-10^{-n}\right)5^{-n}\right) m_n(E_1) \geq (1-10^{-n}) m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j)$$

which implies

$$m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) \leq \frac{(1-(1-10^{-n})\,5^{-n})}{(1-10^{-n})} m_n(E_1)$$

Now a short computation shows

$$0 < \frac{(1-(1-10^{-n})\,5^{-n})}{(1-10^{-n})} < 1$$

Hence, denoting by θ_n a number such that

$$\frac{(1-(1-10^{-n})\,5^{-n})}{(1-10^{-n})} < \theta_n < 1,$$

$$\overline{m_n}\left(E \setminus \cup_{j=1}^{\infty} B_j\right) \leq m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) < \theta_n m_n(E_1) = \theta_n \overline{m_n}(E)$$

Now pick N_1 large enough that

$$\theta_n \overline{m_n}(E) \geq m_n(E_1 \setminus \cup_{j=1}^{N_1} B_j) \geq \overline{m_n}(E \setminus \cup_{j=1}^{N_1} B_j) \tag{10.10}$$

Let $\mathcal{F}_1 = \{B \in \mathcal{F} : B_j \cap B = \emptyset, \ j = 1, \cdots, N_1\}$. If $E \setminus \cup_{j=1}^{N_1} B_j = \emptyset$, then $\mathcal{F}_1 = \emptyset$ and

$$\overline{m_n}\left(E \setminus \cup_{j=1}^{N_1} B_j\right) = 0$$

Therefore, in this case let $B_k = \emptyset$ for all $k > N_1$. Consider the case where

$$E \setminus \cup_{j=1}^{N_1} B_j \neq \emptyset.$$

In this case, $\mathcal{F}_1 \neq \emptyset$ and covers $E \setminus \cup_{j=1}^{N_1} B_j$ in the sense of Vitali. Repeat the same argument, letting $E \setminus \cup_{j=1}^{N_1} B_j$ play the role of E and letting $U \setminus \cup_{j=1}^{N_1} B_j$ play the role of U. (You pick a different E_1 whose measure equals the outer measure of $E \setminus \cup_{j=1}^{N_1} B_j$.) Then choosing B_j for $j = N_1 + 1, \cdots, N_2$ as in the above argument,

$$\theta_n \overline{m_n}(E \setminus \cup_{j=1}^{N_1} B_j) \geq \overline{m_n}(E \setminus \cup_{j=1}^{N_2} B_j)$$

and so from 10.10,

$$\theta_n^2 \overline{m_n}(E) \geq \overline{m_n}(E \setminus \cup_{j=1}^{N_2} B_j).$$

Continuing this way

$$\theta_n^k \overline{m_n}(E) \geq \overline{m_n}\left(E \setminus \cup_{j=1}^{N_k} B_j\right).$$

If it is ever the case that $E \setminus \cup_{j=1}^{N_k} B_j = \emptyset$, then, as in the above argument,

$$\overline{m_n}\left(E \setminus \cup_{j=1}^{N_k} B_j\right) = 0.$$

Otherwise, the process continues and

$$\overline{m_n}\left(E \setminus \cup_{j=1}^{\infty} B_j\right) \leq \overline{m_n}\left(E \setminus \cup_{j=1}^{N_k} B_j\right) \leq \theta_n^k \overline{m_n}(E)$$

for every $k \in \mathbb{N}$. Therefore, the conclusion holds in this case also. This proves the Theorem.

There is an obvious corollary which removes the assumption that $0 < \overline{m_n}(E)$.

Corollary 10.18 *Let $E \subseteq \mathbb{R}^n$ and suppose $\overline{m_n}(E) < \infty$ where $\overline{m_n}$ is the outer measure determined by m_n, n dimensional Lebesgue measure, and let $\mathcal{F}$, be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Proof: If $0 = \overline{m_n}(E)$ you simply pick any ball from $\mathcal{F}$ for your collection of disjoint balls.

It is also not hard to remove the assumption that $\overline{m_n}(E) < \infty$.

Corollary 10.19 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Proof: Let $R_m \equiv (-m, m)^n$ be the open rectangle having sides of length $2m$ which is centered at $\mathbf{0}$ and let $R_0 = \emptyset$. Let $H_m \equiv \overline{R_m} \setminus R_m$. Since both $\overline{R_m}$ and R_m have the same measure, $(2m)^n$, it follows $m_n(H_m) = 0$. Now for all $k \in \mathbb{N}$, $R_k \subseteq \overline{R_k} \subseteq R_{k+1}$. Consider the disjoint open sets, $U_k \equiv R_{k+1} \setminus \overline{R_k}$. Thus

$\mathbb{R}^n = \cup_{k=0}^{\infty} U_k \cup N$ where N is a set of measure zero equal to the union of the H_k. Let $\mathcal{F}_k$ denote those balls of $\mathcal{F}$ which are contained in U_k and let $E_k \equiv U_k \cap E$. Then from Theorem 10.17, there exists a sequence of disjoint balls, $D_k \equiv \left\{B_i^k\right\}_{i=1}^{\infty}$ of $\mathcal{F}_k$ such that $\overline{m_n}(E_k \setminus \cup_{j=1}^{\infty} B_j^k) = 0$. Letting $\{B_i\}_{i=1}^{\infty}$ be an enumeration of all the balls of $\cup_k D_k$, it follows that

$$\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) \leq m_n(N) + \sum_{k=1}^{\infty} \overline{m_n}(E_k \setminus \cup_{j=1}^{\infty} B_j^k) = 0.$$

Also, you don't have to assume the balls are closed.

Corollary 10.20 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of open balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Proof: Let $\overline{\mathcal{F}}$ be the collection of closures of balls in $\mathcal{F}$. Then $\overline{\mathcal{F}}$ covers E in the sense of Vitali and so from Corollary 10.19 there exists a sequence of disjoint closed balls from $\overline{\mathcal{F}}$ satisfying $\overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) = 0$. Now boundaries of the balls, B_i have measure zero and so $\{B_i\}$ is a sequence of disjoint open balls satisfying $\overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} B_i\right) = 0$. The reason for this is that

$$\left(E \setminus \cup_{i=1}^{\infty} B_i\right) \setminus \left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) \subseteq \cup_{i=1}^{\infty} \overline{B_i} \setminus \cup_{i=1}^{\infty} B_i \subseteq \cup_{i=1}^{\infty} \overline{B_i} \setminus B_i,$$

a set of measure zero. Therefore,

$$E \setminus \cup_{i=1}^{\infty} B_i \subseteq \left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) \cup \left(\cup_{i=1}^{\infty} \overline{B_i} \setminus B_i\right)$$

and so

$$\begin{aligned} \overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} B_i\right) &\leq \overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) + m_n\left(\cup_{i=1}^{\infty} \overline{B_i} \setminus B_i\right) \\ &= \overline{m_n}\left(E \setminus \cup_{i=1}^{\infty} \overline{B_i}\right) = 0. \end{aligned}$$

This implies you can fill up an open set with balls which cover the open set in the sense of Vitali.

Corollary 10.21 *Let $U \subseteq \mathbb{R}^n$ be an open set and let $\mathcal{F}$ be a collection of closed or even open balls of bounded radii contained in U such that $\mathcal{F}$ covers U in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(U \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

10.5 Change Of Variables For Linear Maps

To begin with certain kinds of functions map measurable sets to measurable sets. It will be assumed that U is an open set in $\mathbb{R}^n$ and that $\mathbf{h} : U \to \mathbb{R}^n$ satisfies

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in U, \tag{10.11}$$

Lemma 10.22 *Let* $\mathbf{h}$ *satisfy 10.11. If* $T \subseteq U$ *and* $m_n(T) = 0$*, then* $m_n(\mathbf{h}(T)) = 0$.

Proof: Let

$$T_k \equiv \{\mathbf{x} \in T : ||D\mathbf{h}(\mathbf{x})|| < k\}$$

and let $\varepsilon > 0$ be given. Now by outer regularity, there exists an open set, V, containing T_k which is contained in U such that $m_n(V) < \varepsilon$. Let $\mathbf{x} \in T_k$. Then by differentiability,

$$\mathbf{h}(\mathbf{x} + \mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v})$$

and so there exist arbitrarily small $r_{\mathbf{x}} < 1$ such that $B(\mathbf{x}, 5r_{\mathbf{x}}) \subseteq V$ and whenever $|\mathbf{v}| \leq r_{\mathbf{x}}, |o(\mathbf{v})| < k|\mathbf{v}|$. Thus

$$\mathbf{h}(B(\mathbf{x}, r_{\mathbf{x}})) \subseteq B(\mathbf{h}(\mathbf{x}), 2kr_{\mathbf{x}}).$$

From the Vitali covering theorem there exists a countable disjoint sequence of these sets, $\{B(\mathbf{x}_i, r_i)\}_{i=1}^{\infty}$ such that $\{B(\mathbf{x}_i, 5r_i)\}_{i=1}^{\infty} = \left\{\widehat{B}_i\right\}_{i=1}^{\infty}$ covers T_k Then letting $\overline{m_n}$ denote the outer measure determined by m_n,

$$\overline{m_n}(\mathbf{h}(T_k)) \leq \overline{m_n}\left(\mathbf{h}\left(\cup_{i=1}^{\infty}\widehat{B}_i\right)\right)$$

$$\leq \sum_{i=1}^{\infty} \overline{m_n}\left(\mathbf{h}\left(\widehat{B}_i\right)\right) \leq \sum_{i=1}^{\infty} m_n(B(\mathbf{h}(\mathbf{x}_i), 2kr_{\mathbf{x}_i}))$$

$$\begin{aligned} &= \sum_{i=1}^{\infty} m_n(B(\mathbf{x}_i, 2kr_{\mathbf{x}_i})) = (2k)^n \sum_{i=1}^{\infty} m_n(B(\mathbf{x}_i, r_{\mathbf{x}_i})) \\ &\leq (2k)^n m_n(V) \leq (2k)^n \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows $m_n(\mathbf{h}(T_k)) = 0$. Now

$$m_n(\mathbf{h}(T)) = \lim_{k \to \infty} m_n(\mathbf{h}(T_k)) = 0.$$

This proves the lemma.

Lemma 10.23 *Let* $\mathbf{h}$ *satisfy 10.11. If* S *is a Lebesgue measurable subset of* U*, then* $\mathbf{h}(S)$ *is Lebesgue measurable.*

Proof: Let $S_k = S \cap B(\mathbf{0}, k), k \in \mathbb{N}$. By inner regularity of Lebesgue measure, there exists a set, F, which is the countable union of compact sets and a set T with $m_n(T) = 0$ such that

$$F \cup T = S_k.$$

Then $\mathbf{h}(F) \subseteq \mathbf{h}(S_k) \subseteq \mathbf{h}(F) \cup \mathbf{h}(T)$. By continuity of $\mathbf{h}$, $\mathbf{h}(F)$ is a countable union of compact sets and so it is Borel. By Lemma 10.22, $m_n(\mathbf{h}(T)) = 0$ and so $\mathbf{h}(S_k)$ is Lebesgue measurable because of completeness of Lebesgue measure. Now $\mathbf{h}(S) = \cup_{k=1}^{\infty}\mathbf{h}(S_k)$ and so it is also true that $\mathbf{h}(S)$ is Lebesgue measurable. This proves the lemma.

In particular, this proves the following corollary.

Corollary 10.24 *Suppose A is an $n\times n$ matrix. Then if S is a Lebesgue measurable set, it follows AS is also a Lebesgue measurable set.*

Lemma 10.25 *Let R be unitary ($R^*R = RR^* = I$) and let V be a an open or closed set. Then $m_n(RV) = m_n(V)$.*

Proof: First assume V is a bounded open set. By Corollary 10.21 there is a disjoint sequence of closed balls, $\{B_i\}$ such that $U = \cup_{i=1}^{\infty} B_i \cup N$ where $m_n(N) = 0$. Denote by $\mathbf{x}_i$ the center of B_i and let r_i be the radius of B_i. Then by Lemma 10.22 $m_n(RV) = \sum_{i=1}^{\infty} m_n(RB_i)$. Now by invariance of translation of Lebesgue measure, this equals $\sum_{i=1}^{\infty} m_n(RB_i - R\mathbf{x}_i) = \sum_{i=1}^{\infty} m_n(RB(\mathbf{0}, r_i))$. Since R is unitary, it preserves all distances and so $RB(\mathbf{0}, r_i) = B(\mathbf{0}, r_i)$ and therefore,

$$m_n(RV) = \sum_{i=1}^{\infty} m_n(B(\mathbf{0}, r_i)) = \sum_{i=1}^{\infty} m_n(B_i) = m_n(V).$$

This proves the lemma in the case that V is bounded. Suppose now that V is just an open set. Let $V_k = V \cap B(\mathbf{0}, k)$. Then $m_n(RV_k) = m_n(V_k)$. Letting $k \to \infty$, this yields the desired conclusion. This proves the lemma in the case that V is open.

Suppose now that H is a closed and bounded set. Let $B(\mathbf{0},R) \supseteq H$. Then letting $B = B(\mathbf{0}, R)$ for short,

$$\begin{aligned} m_n(RH) &= m_n(RB) - m_n(R(B \setminus H)) \\ &= m_n(B) - m_n(B \setminus H) = m_n(H). \end{aligned}$$

In general, let $H_m = H \cap \overline{B(\mathbf{0},m)}$. Then from what was just shown, $m_n(RH_m) = m_n(H_m)$. Now let $m \to \infty$ to get the conclusion of the lemma in general. This proves the lemma.

Lemma 10.26 *Let E be Lebesgue measurable set in $\mathbb{R}^n$ and let R be unitary. Then $m_n(RE) = m_n(E)$.*

Proof: First suppose E is bounded. Then there exist sets, G and H such that $H \subseteq E \subseteq G$ and H is the countable union of closed sets while G is the countable intersection of open sets such that $m_n(G \setminus H) = 0$. By Lemma 10.25 applied to these sets whose union or intersection equals H or G respectively, it follows

$$m_n(RG) = m_n(G) = m_n(H) = m_n(RH).$$

Therefore,

$$m_n(H) = m_n(RH) \leq m_n(RE) \leq m_n(RG) = m_n(G) = m_n(E) = m_n(H).$$

In the general case, let $E_m = E \cap B(\mathbf{0}, m)$ and apply what was just shown and let $m \to \infty$.

Lemma 10.27 *Let V be an open or closed set in $\mathbb{R}^n$ and let A be an $n \times n$ matrix. Then $m_n(AV) = |\det(A)|\, m_n(V)$.*

Proof: Let RU be the right polar decomposition (Theorem 4.59 on Page 87) of A and let V be an open set. Then from Lemma 10.26,

$$m_n(AV) = m_n(RUV) = m_n(UV).$$

Now $U = Q^* DQ$ where D is a diagonal matrix such that $|\det(D)| = |\det(A)|$ and Q is unitary. Therefore,

$$m_n(AV) = m_n(Q^* DQV) = m_n(DQV).$$

Now QV is an open set and so by Corollary 10.6 on Page 270 and Lemma 10.25,

$$m_n(AV) = |\det(D)|\, m_n(QV) = |\det(D)|\, m_n(V) = |\det(A)|\, m_n(V).$$

This proves the lemma in case V is open.

Now let H be a closed set which is also bounded. First suppose $\det(A) = 0$. Then letting V be an open set containing H,

$$m_n(AH) \leq m_n(AV) = |\det(A)|\, m_n(V) = 0$$

which shows the desired equation is obvious in the case where $\det(A) = 0$. Therefore, assume A is one to one. Since H is bounded, $H \subseteq B(\mathbf{0}, R)$ for some $R > 0$. Then letting $B = B(\mathbf{0}, R)$ for short,

$$\begin{aligned} m_n(AH) &= m_n(AB) - m_n(A(B \setminus H)) \\ &= |\det(A)|\, m_n(B) - |\det(A)|\, m_n(B \setminus H) = |\det(A)|\, m_n(H). \end{aligned}$$

If H is not bounded, apply the result just obtained to $H_m \equiv H \cap \overline{B(\mathbf{0}, m)}$ and then let $m \to \infty$.

With this preparation, the main result is the following theorem.

Theorem 10.28 *Let E be Lebesgue measurable set in $\mathbb{R}^n$ and let A be an $n \times n$ matrix. Then $m_n(AE) = |\det(A)|\, m_n(E)$.*

Proof: First suppose E is bounded. Then there exist sets, G and H such that $H \subseteq E \subseteq G$ and H is the countable union of closed sets while G is the countable intersection of open sets such that $m_n(G \setminus H) = 0$. By Lemma 10.27 applied to these sets whose union or intersection equals H or G respectively, it follows

$$m_n(AG) = |\det(A)|\, m_n(G) = |\det(A)|\, m_n(H) = m_n(AH).$$

Therefore,

$$\begin{aligned} |\det(A)|\, m_n(E) &= |\det(A)|\, m_n(H) = m_n(AH) \leq m_n(AE) \\ &\leq m_n(AG) = |\det(A)|\, m_n(G) = |\det(A)|\, m_n(E). \end{aligned}$$

In the general case, let $E_m = E \cap B(\mathbf{0}, m)$ and apply what was just shown and let $m \to \infty$.

10.6 Change Of Variables For C^1 Functions

In this section theorems are proved which generalize the above to C^1 functions. More general versions can be seen in Kuttler [35], Kuttler [36], and Rudin [45]. There is also a very different approach to this theorem given in [35]. The more general version in [35] follows [45] and both are based on the Brouwer fixed point theorem and a very clever lemma presented in Rudin [45]. This same approach will be used later in this book to prove a different sort of change of variables theorem in which the functions are only Lipschitz. The proof will be based on a sequence of easy lemmas.

Lemma 10.29 *Let U and V be bounded open sets in $\mathbb{R}^n$ and let $\mathbf{h},\mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U)=V$. Also let $f\in C_c(V)$. Then*

$$\int_V f(\mathbf{y})\,dy=\int_U f(\mathbf{h}(\mathbf{x}))\left|\det(D\mathbf{h}(\mathbf{x}))\right|dx$$

Proof: Let $\mathbf{x}\in U$. By the assumption that $\mathbf{h}$ and $\mathbf{h}^{-1}$ are C^1,

$$\begin{aligned}\mathbf{h}(\mathbf{x}+\mathbf{v})-\mathbf{h}(\mathbf{x}) &= D\mathbf{h}(\mathbf{x})\mathbf{v}+\mathbf{o}(\mathbf{v})\\ &= D\mathbf{h}(\mathbf{x})\left(\mathbf{v}+D\mathbf{h}^{-1}(\mathbf{h}(\mathbf{x}))\mathbf{o}(\mathbf{v})\right)\\ &= D\mathbf{h}(\mathbf{x})(\mathbf{v}+\mathbf{o}(\mathbf{v}))\end{aligned}$$

and so if $r>0$ is small enough then $B(\mathbf{x},r)$ is contained in U and

$$\mathbf{h}(B(\mathbf{x},r))-\mathbf{h}(\mathbf{x})=\mathbf{h}(\mathbf{x}+B(\mathbf{0},r))-\mathbf{h}(\mathbf{x})\subseteq D\mathbf{h}(\mathbf{x})(B(\mathbf{0},(1+\varepsilon)r)). \tag{10.12}$$

Making r still smaller if necessary, one can also obtain

$$|f(\mathbf{y})-f(\mathbf{h}(\mathbf{x}))|<\varepsilon \tag{10.13}$$

for any $\mathbf{y}\in\mathbf{h}(B(\mathbf{x},r))$ and

$$|f(\mathbf{h}(x_1))\left|\det(D\mathbf{h}(\mathbf{x}_1))\right|-f(\mathbf{h}(\mathbf{x}))\left|\det(D\mathbf{h}(\mathbf{x}))\right||<\varepsilon \tag{10.14}$$

whenever $\mathbf{x}_1\in B(\mathbf{x},r)$. The collection of such balls is a Vitali cover of U. By Corollary 10.21 there is a sequence of disjoint closed balls $\{B_i\}$ such that $U=\cup_{i=1}^\infty B_i\cup N$ where $m_n(N)=0$. Denote by $\mathbf{x}_i$ the center of B_i and r_i the radius. Then by Lemma 10.22, the monotone convergence theorem, and 10.12 - 10.14,

$$\begin{gathered}\int_V f(\mathbf{y})\,dy=\sum_{i=1}^\infty\int_{\mathbf{h}(B_i)}f(\mathbf{y})\,dy\\ \leq\varepsilon m_n(V)+\sum_{i=1}^\infty\int_{\mathbf{h}(B_i)}f(\mathbf{h}(\mathbf{x}_i))\,dy\\ \leq\varepsilon m_n(V)+\sum_{i=1}^\infty f(\mathbf{h}(\mathbf{x}_i))\,m_n(\mathbf{h}(B_i))\\ \leq\varepsilon m_n(V)+\sum_{i=1}^\infty f(\mathbf{h}(\mathbf{x}_i))\,m_n(D\mathbf{h}(\mathbf{x}_i)(B(\mathbf{0},(1+\varepsilon)r_i)))\\ =\varepsilon m_n(V)+(1+\varepsilon)^n\sum_{i=1}^\infty\int_{B_i}f(\mathbf{h}(\mathbf{x}_i))\left|\det(D\mathbf{h}(\mathbf{x}_i))\right|dx\\ \leq\varepsilon m_n(V)+(1+\varepsilon)^n\sum_{i=1}^\infty\left(\int_{B_i}f(\mathbf{h}(\mathbf{x}))\left|\det(D\mathbf{h}(\mathbf{x}))\right|dx+\varepsilon m_n(B_i)\right)\\ \leq\varepsilon m_n(V)+(1+\varepsilon)^n\sum_{i=1}^\infty\int_{B_i}f(\mathbf{h}(\mathbf{x}))\left|\det(D\mathbf{h}(\mathbf{x}))\right|dx+(1+\varepsilon)^n\varepsilon m_n(U)\\ =\varepsilon m_n(V)+(1+\varepsilon)^n\int_U f(\mathbf{h}(\mathbf{x}))\left|\det(D\mathbf{h}(\mathbf{x}))\right|dx+(1+\varepsilon)^n\varepsilon m_n(U)\end{gathered}$$

Since $\varepsilon > 0$ is arbitrary, this shows

$$\int_V f(\mathbf{y})\,dy \leq \int_U f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx \tag{10.15}$$

whenever $f \in C_c(V)$. Now $\mathbf{x} \to f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|$ is in $C_c(U)$ and so using the same argument with U and V switching roles and replacing $\mathbf{h}$ with $\mathbf{h}^{-1}$,

$$\begin{aligned}&\int_U f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx\\ \leq\ &\int_V f\left(\mathbf{h}\left(\mathbf{h}^{-1}(\mathbf{y})\right)\right)\left|\det\left(D\mathbf{h}\left(\mathbf{h}^{-1}(\mathbf{y})\right)\right)\right|\left|\det\left(D\mathbf{h}^{-1}(\mathbf{y})\right)\right|dy\\ =\ &\int_V f(\mathbf{y})\,dy\end{aligned}$$

by the chain rule. This with 10.15 proves the lemma.

Corollary 10.30 *Let U and V be open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Also let $f \in C_c(V)$. Then*

$$\int_V f(\mathbf{y})\,dy = \int_U f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx$$

Proof: Choose m large enough that $\operatorname{spt}(f) \subseteq B(\mathbf{0},m) \cap V \equiv V_m$. Then let $\mathbf{h}^{-1}(V_m) = U_m$. From Lemma 10.29,

$$\begin{aligned}\int_V f(\mathbf{y})\,dy &= \int_{V_m} f(\mathbf{y})\,dy = \int_{U_m} f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx\\ &= \int_U f(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx.\end{aligned}$$

This proves the corollary.

Corollary 10.31 *Let U and V be open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Also let $E \subseteq V$ be measurable. Then*

$$\int_V \mathcal{X}_E(\mathbf{y})\,dy = \int_U \mathcal{X}_E(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx.$$

Proof: Let $E_m = E \cap V_m$ where V_m and U_m are as in Corollary 10.30. By regularity of the measure there exist sets, K_k, G_k such that $K_k \subseteq E_m \subseteq G_k$, G_k is open, K_k is compact, and $m_n(G_k \setminus K_k) < 2^{-k}$. Let $K_k \prec f_k \prec G_k$. Then $f_k(\mathbf{y}) \to \mathcal{X}_{E_m}(\mathbf{y})$ a.e. because if $\mathbf{y}$ is such that convergence fails, it must be the case that $\mathbf{y}$ is in $G_k \setminus K_k$ infinitely often and $\sum_k m_n(G_k \setminus K_k) < \infty$. Let $N = \cap_m \cup_{k=m}^{\infty} G_k \setminus K_k$, the set of $\mathbf{y}$ which is in infinitely many of the $G_k \setminus K_k$. Then $f_k(\mathbf{h}(\mathbf{x}))$ must converge to $\mathcal{X}_E(\mathbf{h}(\mathbf{x}))$ for all $\mathbf{x} \notin \mathbf{h}^{-1}(N)$, a set of measure zero by Lemma 10.22. By Corollary 10.30

$$\int_{V_m} f_k(\mathbf{y})\,dy = \int_{U_m} f_k(\mathbf{h}(\mathbf{x}))\,|\det(D\mathbf{h}(\mathbf{x}))|\,dx.$$

By the dominated convergence theorem using a dominating function, $\mathcal{X}_{V_m}$ in the integral on the left and $\mathcal{X}_{U_m} |\det(D\mathbf{h})|$ on the right,

$$\int_{V_m} \mathcal{X}_{E_m}(\mathbf{y})\, dy = \int_{U_m} \mathcal{X}_{E_m}(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|\, dx.$$

Therefore,

$$\begin{aligned}\int_V \mathcal{X}_{E_m}(\mathbf{y})\, dy &= \int_{V_m} \mathcal{X}_{E_m}(\mathbf{y})\, dy = \int_{U_m} \mathcal{X}_{E_m}(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|\, dx \\ &= \int_U \mathcal{X}_{E_m}(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|\, dx\end{aligned}$$

Let $m \to \infty$ and use the monotone convergence theorem to obtain the conclusion of the corollary.

With this corollary, the main theorem follows.

Theorem 10.32 *Let U and V be open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Then if g is a nonnegative Lebesgue measurable function,*

$$\int_V g(\mathbf{y})\, dy = \int_U g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|\, dx. \tag{10.16}$$

Proof: From Corollary 10.31, 10.16 holds for any nonnegative simple function in place of g. In general, let $\{s_k\}$ be an increasing sequence of simple functions which converges to g pointwise. Then from the monotone convergence theorem

$$\begin{aligned}\int_V g(\mathbf{y})\, dy &= \lim_{k\to\infty} \int_V s_k dy = \lim_{k\to\infty} \int_U s_k(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|\, dx \\ &= \int_U g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|\, dx.\end{aligned}$$

This proves the theorem.

This is a pretty good theorem but it isn't too hard to generalize it. In particular, it is not necessary to assume $\mathbf{h}^{-1}$ is C^1.

Lemma 10.33 *Suppose V is an $n-1$ dimensional subspace of $\mathbb{R}^n$ and K is a compact subset of V. Then letting*

$$K_\varepsilon \equiv \cup_{\mathbf{x}\in K} B(\mathbf{x},\varepsilon) = K + B(\mathbf{0},\varepsilon),$$

it follows that

$$m_n(K_\varepsilon) \leq 2^n \varepsilon (\operatorname{diam}(K) + \varepsilon)^{n-1}.$$

Proof: Let an orthonormal basis for V be $\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}\}$ and let

$$\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-1}, \mathbf{v}_n\}$$

be an orthonormal basis for $\mathbb{R}^n$. Now define a linear transformation, Q by $Q\mathbf{v}_i = \mathbf{e}_i$. Thus $QQ^* = Q^*Q = I$ and Q preserves all distances because

$$\left|Q\sum_i a_i \mathbf{e}_i\right|^2 = \left|\sum_i a_i \mathbf{v}_i\right|^2 = \sum_i |a_i|^2 = \left|\sum_i a_i \mathbf{e}_i\right|^2 .$$

Letting $\mathbf{k}_0 \in K$, it follows $K \subseteq B(\mathbf{k}_0, \text{diam}(K))$ and so,

$$QK \subseteq B^{n-1}(Q\mathbf{k}_0, \text{diam}(QK)) = B^{n-1}(Q\mathbf{k}_0, \text{diam}(K))$$

where B^{n-1} refers to the ball taken with respect to the usual norm in $\mathbb{R}^{n-1}$. Every point of K_ε is within ε of some point of K and so it follows that every point of QK_ε is within ε of some point of QK. Therefore,

$$QK_\varepsilon \subseteq B^{n-1}(Q\mathbf{k}_0, \text{diam}(QK) + \varepsilon) \times (-\varepsilon, \varepsilon),$$

To see this, let $\mathbf{x} \in QK_\varepsilon$. Then there exists $\mathbf{k} \in QK$ such that $|\mathbf{k} - \mathbf{x}| < \varepsilon$. Therefore, $|(x_1, \cdots, x_{n-1}) - (k_1, \cdots, k_{n-1})| < \varepsilon$ and $|x_n - k_n| < \varepsilon$ and so $\mathbf{x}$ is contained in the set on the right in the above inclusion because $k_n = 0$. However, the measure of the set on the right is smaller than

$$[2(\text{diam}(QK) + \varepsilon)]^{n-1}(2\varepsilon) = 2^n [(\text{diam}(K) + \varepsilon)]^{n-1} \varepsilon.$$

This proves the lemma.

Note this is a very sloppy estimate. You can certainly do much better but this estimate is sufficient to prove Sard's lemma which follows.

Definition 10.34 *In any metric space, if* $\mathbf{x}$ *is a point of the metric space and* S *is a nonempty subset,*

$$\text{dist}(\mathbf{x}, S) \equiv \inf\{d(x, s) : s \in S\}.$$

More generally, if T, S *are two nonempty sets,*

$$\text{dist}(S, T) \equiv \inf\{d(t, s) : s \in S, t \in T\}.$$

Lemma 10.35 *The function* $\mathbf{x} \to \text{dist}(\mathbf{x}, S)$ *is continuous.*

Proof: Let $\mathbf{x}, \mathbf{y}$ be given. Suppose $\text{dist}(\mathbf{x}, S) \geq \text{dist}(\mathbf{y}, S)$ and pick $\mathbf{s} \in S$ such that $\text{dist}(\mathbf{y}, S) + \varepsilon \geq d(\mathbf{y}, \mathbf{s})$. Then

$$\begin{aligned} 0 &\leq \text{dist}(\mathbf{x}, S) - \text{dist}(\mathbf{y}, S) \leq \text{dist}(\mathbf{x}, S) - (d(\mathbf{y}, \mathbf{s}) - \varepsilon) \\ &\leq d(\mathbf{x}, \mathbf{s}) - d(\mathbf{y}, \mathbf{s}) + \varepsilon \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{s}) - d(\mathbf{y}, \mathbf{s}) + \varepsilon = d(\mathbf{x}, \mathbf{y}) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows $|\text{dist}(\mathbf{x}, S) - \text{dist}(\mathbf{y}, S)| \leq d(\mathbf{x}, \mathbf{y})$. This proves the lemma.

Lemma 10.36 *Let* $\mathbf{h}$ *be a* C^1 *function defined on an open set,* U *and let* K *be a compact subset of* U. *Then if* $\varepsilon > 0$ *is given, there exits* $r_1 > 0$ *such that if* $|\mathbf{v}| \leq r_1$, *then for all* $\mathbf{x} \in K$,

$$|\mathbf{h}(\mathbf{x} + \mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}| < \varepsilon |\mathbf{v}|.$$

Proof: Let $0 < \delta < \text{dist}\left(K, U^C\right)$. Such a positive number exists because if there exists a sequence of points in $K, \{\mathbf{k}_k\}$ and points in $U^C, \{\mathbf{s}_k\}$ such that $|\mathbf{k}_k - \mathbf{s}_k| \to 0$, then you could take a subsequence, still denoted by k such that $\mathbf{k}_k \to \mathbf{k} \in K$ and then $\mathbf{s}_k \to \mathbf{k}$ also. But U^C is closed so $\mathbf{k} \in K \cap U^C$, a contradiction. Then

$$\begin{aligned}\frac{|\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}|}{|\mathbf{v}|} &\leq \frac{\left|\int_0^1 D\mathbf{h}(\mathbf{x}+t\mathbf{v})\mathbf{v}dt - D\mathbf{h}(\mathbf{x})\mathbf{v}\right|}{|\mathbf{v}|} \\ &\leq \frac{\int_0^1 |D\mathbf{h}(\mathbf{x}+t\mathbf{v})\mathbf{v} - D\mathbf{h}(\mathbf{x})\mathbf{v}|\, dt}{|\mathbf{v}|}.\end{aligned}$$

Now from uniform continuity of $D\mathbf{h}$ on the compact set, $\{\mathbf{x} : \text{dist}(\mathbf{x},K) \leq \delta\}$ it follows there exists $r_1 < \delta$ such that if $|\mathbf{v}| \leq r_1$, then $||D\mathbf{h}(\mathbf{x}+t\mathbf{v}) - D\mathbf{h}(\mathbf{x})|| < \varepsilon$ for every $\mathbf{x} \in K$. From the above formula, it follows that if $|\mathbf{v}| \leq r_1$,

$$\begin{aligned}\frac{|\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}|}{|\mathbf{v}|} &\leq \frac{\int_0^1 |D\mathbf{h}(\mathbf{x}+t\mathbf{v})\mathbf{v} - D\mathbf{h}(\mathbf{x})\mathbf{v}|\, dt}{|\mathbf{v}|} \\ &< \frac{\int_0^1 \varepsilon|\mathbf{v}|\, dt}{|\mathbf{v}|} = \varepsilon.\end{aligned}$$

This proves the lemma.

A different proof of the following is in [35]. See also [36].

Lemma 10.37 *(Sard) Let U be an open set in $\mathbb{R}^n$ and let $\mathbf{h} : U \to \mathbb{R}^n$ be C^1. Let*

$$Z \equiv \{\mathbf{x} \in U : \det D\mathbf{h}(\mathbf{x}) = 0\}.$$

Then $m_n(\mathbf{h}(Z)) = 0$.

Proof: Let $\{U_k\}_{k=1}^\infty$ be an increasing sequence of open sets whose closures are compact and whose union equals U and let $Z_k \equiv Z \cap \overline{U_k}$. To obtain such a sequence, let $U_k = \left\{\mathbf{x} \in U : \text{dist}\left(\mathbf{x},U^C\right) < \frac{1}{k}\right\} \cap B(\mathbf{0}, k)$. First it is shown that $\mathbf{h}(Z_k)$ has measure zero. Let W be an open set contained in U_{k+1} which contains Z_k and satisfies

$$m_n(Z_k) + \varepsilon > m_n(W)$$

where here and elsewhere, $\varepsilon < 1$. Let

$$r = \text{dist}\left(\overline{U_k}, U_{k+1}^C\right)$$

and let $r_1 > 0$ be a constant as in Lemma 10.36 such that whenever $\mathbf{x} \in \overline{U_k}$ and $0 < |\mathbf{v}| \leq r_1$,

$$|\mathbf{h}(\mathbf{x}+\mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}| < \varepsilon|\mathbf{v}|. \tag{10.17}$$

Now the closures of balls which are contained in W and which have the property that their diameters are less than r_1 yield a Vitali covering of W. Therefore, by Corollary 10.21 there is a disjoint sequence of these closed balls, $\left\{\widetilde{B}_i\right\}$ such that

$$W = \cup_{i=1}^\infty \widetilde{B}_i \cup N$$

where N is a set of measure zero. Denote by $\{B_i\}$ those closed balls in this sequence which have nonempty intersection with Z_k, let d_i be the diameter of B_i, and let $\mathbf{z}_i$ be a point in $B_i \cap Z_k$. Since $\mathbf{z}_i \in Z_k$, it follows $D\mathbf{h}(\mathbf{z}_i) B(\mathbf{0},d_i) = D_i$ where D_i is contained in a subspace, V which has dimension $n-1$ and the diameter of D_i is no larger than $2C_k d_i$ where

$$C_k \geq \max\{||D\mathbf{h}(\mathbf{x})|| : \mathbf{x} \in Z_k\}$$

Then by 10.17, if $\mathbf{z} \in B_i$,

$$\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{z}_i) \in D_i + B(\mathbf{0}, \varepsilon d_i) \subseteq \overline{D_i} + B(\mathbf{0},\varepsilon d_i).$$

Thus

$$\mathbf{h}(B_i) \subseteq \mathbf{h}(\mathbf{z}_i) + \overline{D_i} + B(\mathbf{0},\varepsilon d_i)$$

By Lemma 10.33

$$\begin{aligned} m_n(\mathbf{h}(B_i)) &\leq 2^n (2C_k d_i + \varepsilon d_i)^{n-1} \varepsilon d_i \\ &\leq d_i^n \left(2^n [2C_k + \varepsilon]^{n-1}\right) \varepsilon \\ &\leq C_{n,k} m_n(B_i) \varepsilon. \end{aligned}$$

Therefore, by Lemma 10.22

$$\begin{aligned} m_n(\mathbf{h}(Z_k)) &\leq m_n(W) = \sum_i m_n(\mathbf{h}(B_i)) \leq C_{n,k}\varepsilon \sum_i m_n(B_i) \\ &\leq \varepsilon C_{n,k} m_n(W) \leq \varepsilon C_{n,k}(m_n(Z_k) + \varepsilon) \end{aligned}$$

Since ε is arbitrary, this shows $m_n(\mathbf{h}(Z_k)) = 0$ and so $0 = \lim_{k\to\infty} m_n(\mathbf{h}(Z_k)) = m_n(\mathbf{h}(Z))$.

With this important lemma, here is a generalization of Theorem 10.32.

Theorem 10.38 *Let U be an open set and let $\mathbf{h}$ be a $1-1$, C^1 function with values in $\mathbb{R}^n$. Then if g is a nonnegative Lebesgue measurable function,*

$$\int_{\mathbf{h}(U)} g(\mathbf{y})\, dy = \int_U g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|\, dx. \tag{10.18}$$

Proof: Let $Z = \{\mathbf{x} : \det(D\mathbf{h}(\mathbf{x})) = 0\}$. Then by the inverse function theorem, $\mathbf{h}^{-1}$ is C^1 on $\mathbf{h}(U \setminus Z)$ and $\mathbf{h}(U \setminus Z)$ is an open set. Therefore, from Lemma 10.37 and Theorem 10.32,

$$\begin{aligned} \int_{\mathbf{h}(U)} g(\mathbf{y})\, dy &= \int_{\mathbf{h}(U\setminus Z)} g(\mathbf{y})\, dy = \int_{U\setminus Z} g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|\, dx \\ &= \int_U g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|\, dx. \end{aligned}$$

This proves the theorem.

Of course the next generalization considers the case when $\mathbf{h}$ is not even one to one.

10.7 Mappings Which Are Not One To One

Now suppose $\mathbf{h}$ is only C^1, not necessarily one to one. For

$$U_+ \equiv \{\mathbf{x} \in U : |\det D\mathbf{h}(x)| > 0\}$$

and Z the set where $|\det D\mathbf{h}(\mathbf{x})| = 0$, Lemma 10.37 implies $m_n(\mathbf{h}(Z)) = 0$. For $\mathbf{x} \in U_+$, the inverse function theorem implies there exists an open set $B_{\mathbf{x}}$ such that $\mathbf{x} \in B_{\mathbf{x}} \subseteq U_+$, $\mathbf{h}$ is one to one on $B_{\mathbf{x}}$.

Let $\{B_i\}$ be a countable subset of $\{B_{\mathbf{x}}\}_{\mathbf{x}\in U_+}$ such that $U_+ = \cup_{i=1}^{\infty} B_i$. Let $E_1 = B_1$. If $E_1, \cdots, E_k$ have been chosen, $E_{k+1} = B_{k+1} \setminus \cup_{i=1}^{k} E_i$. Thus

$$\cup_{i=1}^{\infty} E_i = U_+, \;\; \mathbf{h} \text{ is one to one on } E_i, \;\; E_i \cap E_j = \emptyset,$$

and each E_i is a Borel set contained in the open set B_i. Now define

$$n(\mathbf{y}) \equiv \sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) + \mathcal{X}_{\mathbf{h}(Z)}(\mathbf{y}).$$

The set, $\mathbf{h}(E_i), \mathbf{h}(Z)$ are measurable by Lemma 10.23. Thus $n(\cdot)$ is measurable.

Lemma 10.39 *Let $F \subseteq \mathbf{h}(U)$ be measurable. Then*

$$\int_{\mathbf{h}(U)} n(\mathbf{y})\mathcal{X}_F(\mathbf{y})dy = \int_U \mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx.$$

Proof: Using Lemma 10.37 and the Monotone Convergence Theorem or Fubini's Theorem,

$$\begin{aligned}
\int_{\mathbf{h}(U)} n(\mathbf{y})\mathcal{X}_F(\mathbf{y})dy &= \int_{\mathbf{h}(U)} \left(\sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) + \overbrace{\mathcal{X}_{\mathbf{h}(Z)}(\mathbf{y})}^{m_n(\mathbf{h}(Z))=0} \right) \mathcal{X}_F(\mathbf{y})dy \\
&= \sum_{i=1}^{\infty} \int_{\mathbf{h}(U)} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y})\mathcal{X}_F(\mathbf{y})dy \\
&= \sum_{i=1}^{\infty} \int_{\mathbf{h}(B_i)} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y})\mathcal{X}_F(\mathbf{y})dy \\
&= \sum_{i=1}^{\infty} \int_{B_i} \mathcal{X}_{E_i}(\mathbf{x})\mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx \\
&= \sum_{i=1}^{\infty} \int_{U} \mathcal{X}_{E_i}(\mathbf{x})\mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx \\
&= \int_{U} \sum_{i=1}^{\infty} \mathcal{X}_{E_i}(\mathbf{x})\mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx
\end{aligned}$$

$$= \int_{U_+} \mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx = \int_U \mathcal{X}_F(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx.$$

This proves the lemma.

Definition 10.40 *For $\mathbf{y} \in \mathbf{h}(U)$, define a function, $\#$, according to the formula*

$$\#(\mathbf{y}) \equiv \text{number of elements in } \mathbf{h}^{-1}(\mathbf{y}).$$

Observe that

$$\#(\mathbf{y}) = n(\mathbf{y}) \quad \text{a.e.} \tag{10.19}$$

because $n(\mathbf{y}) = \#(\mathbf{y})$ if $\mathbf{y} \notin \mathbf{h}(Z)$, a set of measure 0. Therefore, $\#$ is a measurable function.

Theorem 10.41 *Let $g \geq 0$, g measurable, and let $\mathbf{h}$ be $C^1(U)$. Then*

$$\int_{\mathbf{h}(U)} \#(\mathbf{y})g(\mathbf{y})dy = \int_U g(\mathbf{h}(\mathbf{x}))|\det D\mathbf{h}(\mathbf{x})|dx. \tag{10.20}$$

Proof: From 10.19 and Lemma 10.39, 10.20 holds for all g, a nonnegative simple function. Approximating an arbitrary measurable nonnegative function, g, with an increasing pointwise convergent sequence of simple functions and using the monotone convergence theorem, yields 10.20 for an arbitrary nonnegative measurable function, g. This proves the theorem.

10.8 Lebesgue Measure And Iterated Integrals

The following is the main result.

Theorem 10.42 *Let $f \geq 0$ and suppose f is a Lebesgue measurable function defined on $\mathbb{R}^n$ and $\int_{\mathbb{R}^n} f dm_n < \infty$. Then*

$$\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.$$

This will be accomplished by Fubini's theorem, Theorem 9.50 and the following lemma.

Lemma 10.43 *$\overline{m_k \times m_{n-k}} = m_n$ on the m_n measurable sets.*

Proof: First of all, let $R = \prod_{i=1}^n (a_i, b_i]$ be a measurable rectangle and let $R_k = \prod_{i=1}^k (a_i, b_i], R_{n-k} = \prod_{i=k+1}^n (a_i, b_i]$. Then by Fubini's theorem,

$$\begin{aligned}
\int \mathcal{X}_R d\left(\overline{m_k \times m_{n-k}}\right) &= \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} \mathcal{X}_{R_k} \mathcal{X}_{R_{n-k}} dm_k dm_{n-k} \\
&= \int_{\mathbb{R}^k} \mathcal{X}_{R_k} dm_k \int_{\mathbb{R}^{n-k}} \mathcal{X}_{R_{n-k}} dm_{n-k} \\
&= \int \mathcal{X}_R dm_n
\end{aligned}$$

and so $\overline{m_k \times m_{n-k}}$ and m_n agree on every half open rectangle. By Lemma 10.2 these two measures agree on every open set. Now if K is a compact set, then $K = \cap_{k=1}^{\infty} U_k$ where U_k is the open set, $K + B\left(\mathbf{0}, \frac{1}{k}\right)$. Another way of saying this is $U_k \equiv \left\{\mathbf{x} : \text{dist}(\mathbf{x}, K) < \frac{1}{k}\right\}$ which is obviously open because $\mathbf{x} \to \text{dist}(\mathbf{x}, K)$ is a continuous function. Since K is the countable intersection of these decreasing open sets, each of which has finite measure with respect to either of the two measures, it follows that $\overline{m_k \times m_{n-k}}$ and m_n agree on all the compact sets. Now let E be a bounded Lebesgue measurable set. Then there are sets, H and G such that H is a countable union of compact sets, G a countable intersection of open sets, $H \subseteq E \subseteq G$, and $m_n(G \setminus H) = 0$. Then from what was just shown about compact and open sets, the two measures agree on G and on H. Therefore,

$$\begin{aligned} m_n(H) &= \overline{m_k \times m_{n-k}}(H) \leq \overline{m_k \times m_{n-k}}(E) \\ &\leq \overline{m_k \times m_{n-k}}(G) = m_n(E) = m_n(H) \end{aligned}$$

By completeness of the measure space for $\overline{m_k \times m_{n-k}}$, it follows E is $\overline{m_k \times m_{n-k}}$ measurable and

$$\overline{m_k \times m_{n-k}}(E) = m_n(E).$$

This proves the lemma.

You could also show that the two σ algebras are the same. However, this is not needed for the lemma or the theorem.

Proof of Theorem 10.42: By the lemma and Fubini's theorem, Theorem 9.50,

$$\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^n} f d\left(\overline{m_k \times m_{n-k}}\right) = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.$$

Corollary 10.44 *Let f be a nonnegative real valued measurable function. Then*

$$\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.$$

Proof: Let $S_p \equiv \{\mathbf{x} \in \mathbb{R}^n : 0 \leq f(\mathbf{x}) \leq p\} \cap B(\mathbf{0}, p)$. Then $\int_{\mathbb{R}^n} f \mathcal{X}_{S_p} dm_n < \infty$. Therefore, from Theorem 10.42,

$$\int_{\mathbb{R}^n} f \mathcal{X}_{S_p} dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} \mathcal{X}_{S_p} f dm_{n-k} dm_k.$$

Now let $p \to \infty$ and use the Monotone convergence theorem and the Fubini Theorem 9.50 on Page 243.

Not surprisingly, the following corollary follows from this.

Corollary 10.45 *Let $f \in L^1(\mathbb{R}^n)$ where the measure is m_n. Then*

$$\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.$$

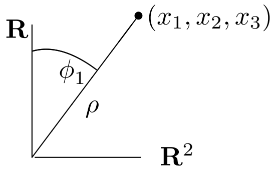
Proof: Apply Corollary 10.44 to the postive and negative parts of the real and imaginary parts of f.

10.9 Spherical Coordinates In Many Dimensions

Sometimes there is a need to deal with spherical coordinates in more than three dimensions. In this section, this concept is defined and formulas are derived for these coordinate systems. Recall polar coordinates are of the form

$$y_1 = \rho \cos \theta$$
$$y_2 = \rho \sin \theta$$

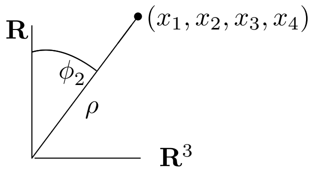
where $\rho > 0$ and $\theta \in [0, 2\pi)$. Here I am writing ρ in place of r to emphasize a pattern which is about to emerge. I will consider polar coordinates as spherical coordinates in two dimensions. I will also simply refer to such coordinate systems as polar coordinates regardless of the dimension. This is also the reason I am writing y_1 and y_2 instead of the more usual x and y. Now consider what happens when you go to three dimensions. The situation is depicted in the following picture.



From this picture, you see that $y_3 = \rho \cos \phi_1$. Also the distance between (y_1, y_2) and $(0, 0)$ is $\rho \sin (\phi_1)$. Therefore, using polar coordinates to write (y_1, y_2) in terms of θ and this distance,

$$y_1 = \rho \sin \phi_1 \cos \theta,$$
$$y_2 = \rho \sin \phi_1 \sin \theta,$$
$$y_3 = \rho \cos \phi_1.$$

where $\phi_1 \in [0, \pi]$. What was done is to replace ρ with $\rho \sin \phi_1$ and then to add in $y_3 = \rho \cos \phi_1$. Having done this, there is no reason to stop with three dimensions. Consider the following picture:



From this picture, you see that $y_4 = \rho \cos \phi_2$. Also the distance between (y_1, y_2, y_3) and $(0, 0, 0)$ is $\rho \sin (\phi_2)$. Therefore, using polar coordinates to write (y_1, y_2, y_3) in terms of θ, ϕ_1, and this distance,

$$\begin{aligned} y_1 &= \rho \sin \phi_2 \sin \phi_1 \cos \theta, \\ y_2 &= \rho \sin \phi_2 \sin \phi_1 \sin \theta, \\ y_3 &= \rho \sin \phi_2 \cos \phi_1, \\ y_4 &= \rho \cos \phi_2 \end{aligned}$$

where $\phi_2 \in [0, \pi]$.

Continuing this way, given spherical coordinates in $\mathbb{R}^n$, to get the spherical coordinates in $\mathbb{R}^{n+1}$, you let $y_{n+1} = \rho \cos \phi_{n-1}$ and then replace every occurance of ρ with $\rho \sin \phi_{n-1}$ to obtain $y_1 \cdots y_n$ in terms of $\phi_1, \phi_2, \cdots, \phi_{n-1}, \theta$, and ρ.

It is always the case that ρ measures the distance from the point in $\mathbb{R}^n$ to the origin in $\mathbb{R}^n$, $\mathbf{0}$. Each $\phi_i \in [0, \pi]$, and $\theta \in [0, 2\pi)$. It can be shown using math induction that these coordinates map $\prod_{i=1}^{n-2} [0, \pi] \times [0, 2\pi) \times (0, \infty)$ one to one onto $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

Theorem 10.46 *Let $\mathbf{y} = \mathbf{h}(\boldsymbol{\phi}, \theta, \rho)$ be the spherical coordinate transformations in $\mathbb{R}^n$. Then letting $A = \prod_{i=1}^{n-2} [0, \pi] \times [0, 2\pi)$, it follows $\mathbf{h}$ maps $A \times (0, \infty)$ one to one onto $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Also $|\det D\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)|$ will always be of the form*

$$|\det D\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)| = \rho^{n-1} \Phi(\boldsymbol{\phi}, \theta). \tag{10.21}$$

where Φ is a continuous function of $\boldsymbol{\phi}$ and θ.[1] *Furthermore whenever f is Lebesgue measurable and nonnegative,*

$$\int_{\mathbb{R}^n} f(\mathbf{y})\, dy = \int_0^\infty \rho^{n-1} \int_A f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)) \Phi(\boldsymbol{\phi}, \theta)\, d\boldsymbol{\phi}\, d\theta d\rho \tag{10.22}$$

where here $d\boldsymbol{\phi}\, d\theta$ denotes dm_{n-1} on A. The same formula holds if $f \in L^1(\mathbb{R}^n)$.

Proof: Formula 10.21 is obvious from the definition of the spherical coordinates. The first claim is also clear from the definition and math induction. It remains to verify 10.22. Let $A_0 \equiv \prod_{i=1}^{n-2} (0, \pi) \times (0, 2\pi)$. Then it is clear that $(A \setminus A_0) \times (0, \infty) \equiv N$ is a set of measure zero in $\mathbb{R}^n$. Therefore, from Lemma 10.22 it follows $\mathbf{h}(N)$ is also a set of measure zero. Therefore, using the change of variables theorem, Corollary 10.44, and Sard's lemma,

$$\begin{aligned} \int_{\mathbb{R}^n} f(\mathbf{y})\, dy &= \int_{\mathbb{R}^n \setminus \{\mathbf{0}\}} f(\mathbf{y})\, dy = \int_{\mathbb{R}^n \setminus (\{\mathbf{0}\} \cup \mathbf{h}(N))} f(\mathbf{y})\, dy \\ &= \int_{A_0 \times (0,\infty)} f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)) \rho^{n-1} \Phi(\boldsymbol{\phi}, \theta)\, dm_n \\ &= \int \mathcal{X}_{A \times (0,\infty)}(\boldsymbol{\phi}, \theta, \rho) f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)) \rho^{n-1} \Phi(\boldsymbol{\phi}, \theta)\, dm_n \\ &= \int_0^\infty \rho^{n-1} \left(\int_A f(\mathbf{h}(\boldsymbol{\phi}, \theta, \rho)) \Phi(\boldsymbol{\phi}, \theta)\, d\boldsymbol{\phi}\, d\theta \right) d\rho. \end{aligned}$$

[1] Actually it is only a function of the first but this is not important in what follows.

Now the claim about $f \in L^1$ follows routinely from considering the positive and negative parts of the real and imaginary parts of f in the usual way. This proves the theorem.

Notation 10.47 *Often this is written differently. Note that from the spherical coordinate formulas, $f(\mathbf{h}(\boldsymbol{\phi},\theta,\rho)) = f(\rho\boldsymbol{\omega})$ where $|\boldsymbol{\omega}| = 1$. Letting S^{n-1} denote the unit sphere, $\{\boldsymbol{\omega} \in \mathbb{R}^n : |\boldsymbol{\omega}| = 1\}$, the inside integral in the above formula is sometimes written as*

$$\int_{S^{n-1}} f(\rho\boldsymbol{\omega})\, d\sigma$$

where σ is a measure on S^{n-1}. See [35] for another description of this measure. It isn't an important issue here. Later in the book when integration on manifolds is discussed, more general considerations will be dealt with. Either 10.22 or the formula

$$\int_0^\infty \rho^{n-1}\left(\int_{S^{n-1}} f(\rho\boldsymbol{\omega})\, d\sigma\right) d\rho$$

will be referred to as polar coordinates and is very useful in establishing estimates. Here $\sigma(S^{n-1}) \equiv \int_A \Phi(\boldsymbol{\phi},\theta)\, d\boldsymbol{\phi}\, d\theta$.

Example 10.48 *For what values of s is the integral $\int_{B(\mathbf{0},R)} \left(1+|\mathbf{x}|^2\right)^s dy$ bounded independent of R? Here $B(\mathbf{0},R)$ is the ball, $\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq R\}$.*

I think you can see immediately that s must be negative but exactly how negative? It turns out it depends on n and using polar coordinates, you can find just exactly what is needed. From the polar coordinats formula above,

$$\begin{aligned}\int_{B(\mathbf{0},R)} \left(1+|\mathbf{x}|^2\right)^s dy &= \int_0^R \int_{S^{n-1}} \left(1+\rho^2\right)^s \rho^{n-1} d\sigma d\rho \\ &= C_n \int_0^R \left(1+\rho^2\right)^s \rho^{n-1} d\rho\end{aligned}$$

Now the very hard problem has been reduced to considering an easy one variable problem of finding when

$$\int_0^R \rho^{n-1}\left(1+\rho^2\right)^s d\rho$$

is bounded independent of R. You need $2s + (n-1) < -1$ so you need $s < -n/2$.

10.10 The Brouwer Fixed Point Theorem

This seems to be a good place to present a short proof of one of the most important of all fixed point theorems. There are many approaches to this but one of the easiest and shortest I have ever seen is the one in Dunford and Schwartz [18]. This is what is presented here. In Evans [21] there is a different proof which depends on integration theory. A good reference for an introduction to various kinds of fixed point theorems is the book by Smart [48]. This book also gives an entirely different approach to the Brouwer fixed point theorem.

The proof given here is based on the following lemma. Recall that the mixed partial derivatives of a C^2 function are equal. In the following lemma, and elsewhere, a comma followed by an index indicates the partial derivative with respect to the indicated variable. Thus, $f_{,j}$ will mean $\frac{\partial f}{\partial x_j}$. Also, write $D\mathbf{g}$ for the Jacobian matrix which is the matrix of $D\mathbf{g}$ taken with respect to the usual basis vectors in $\mathbb{R}^n$. Recall that for A an $n \times n$ matrix, $\operatorname{cof}(A)_{ij}$ is the determinant of the matrix which results from deleting the i^{th} row and the j^{th} column and multiplying by $(-1)^{i+j}$.

Lemma 10.49 *Let $\mathbf{g} : U \to \mathbb{R}^n$ be C^2 where U is an open subset of $\mathbb{R}^n$. Then*

$$\sum_{j=1}^{n} \operatorname{cof}(D\mathbf{g})_{ij,j} = 0,$$

where here $(D\mathbf{g})_{ij} \equiv g_{i,j} \equiv \frac{\partial g_i}{\partial x_j}$. Also, $\operatorname{cof}(D\mathbf{g})_{ij} = \frac{\partial \det(D\mathbf{g})}{\partial g_{i,j}}$.

Proof: From the cofactor expansion theorem,

$$\det(D\mathbf{g}) = \sum_{i=1}^{n} g_{i,j} \operatorname{cof}(D\mathbf{g})_{ij}$$

and so

$$\frac{\partial \det(D\mathbf{g})}{\partial g_{i,j}} = \operatorname{cof}(D\mathbf{g})_{ij} \tag{10.23}$$

which shows the last claim of the lemma. Also

$$\delta_{kj} \det(D\mathbf{g}) = \sum_{i} g_{i,k} (\operatorname{cof}(D\mathbf{g}))_{ij} \tag{10.24}$$

because if $k \neq j$ this is just the cofactor expansion of the determinant of a matrix in which the k^{th} and j^{th} columns are equal. Differentiate 10.24 with respect to x_j and sum on j. This yields

$$\sum_{r,s,j} \delta_{kj} \frac{\partial(\det D\mathbf{g})}{\partial g_{r,s}} g_{r,sj} = \sum_{ij} g_{i,kj} (\operatorname{cof}(D\mathbf{g}))_{ij} + \sum_{ij} g_{i,k} \operatorname{cof}(D\mathbf{g})_{ij,j}.$$

Hence, using $\delta_{kj} = 0$ if $j \neq k$ and 10.23,

$$\sum_{rs} (\operatorname{cof}(D\mathbf{g}))_{rs} g_{r,sk} = \sum_{rs} g_{r,ks} (\operatorname{cof}(D\mathbf{g}))_{rs} + \sum_{ij} g_{i,k} \operatorname{cof}(D\mathbf{g})_{ij,j}.$$

Subtracting the first sum on the right from both sides and using the equality of mixed partials,

$$\sum_{i} g_{i,k} \left(\sum_{j} (\operatorname{cof}(D\mathbf{g}))_{ij,j} \right) = 0.$$

If $\det(g_{i,k}) \neq 0$ so that $(g_{i,k})$ is invertible, this shows $\sum_j (\operatorname{cof}(D\mathbf{g}))_{ij,j} = 0$. If $\det(D\mathbf{g}) = 0$, let

$$g_k = g + \varepsilon_k I$$

where $\varepsilon_k \to 0$ and $\det(D\mathbf{g} + \varepsilon_k I) \equiv \det(D\mathbf{g}_k) \neq 0$. Then

$$\sum_j (\operatorname{cof}(D\mathbf{g}))_{ij,j} = \lim_{k\to\infty} \sum_j (\operatorname{cof}(D\mathbf{g}_k))_{ij,j} = 0$$

and this proves the lemma.

To prove the Brouwer fixed point theorem, first consider a version of it valid for C^2 mappings. This is the following lemma.

Lemma 10.50 *Let $B_r = \overline{B(\mathbf{0},r)}$ and suppose $\mathbf{g}$ is a C^2 function defined on $\mathbb{R}^n$ which maps B_r to B_r. Then $\mathbf{g}(\mathbf{x}) = \mathbf{x}$ for some $\mathbf{x} \in B_r$.*

Proof: Suppose not. Then $|\mathbf{g}(\mathbf{x}) - \mathbf{x}|$ must be bounded away from zero on B_r. Let $a(\mathbf{x})$ be the larger of the two roots of the equation,

$$|\mathbf{x}+a(\mathbf{x})(\mathbf{x}-\mathbf{g}(\mathbf{x}))|^2 = r^2. \tag{10.25}$$

Thus

$$a(\mathbf{x}) = \frac{-(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x}))) + \sqrt{(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x})))^2 + \left(r^2 - |\mathbf{x}|^2\right)|\mathbf{x}-\mathbf{g}(\mathbf{x})|^2}}{|\mathbf{x}-\mathbf{g}(\mathbf{x})|^2} \tag{10.26}$$

The expression under the square root sign is always nonnegative and it follows from the formula that $a(\mathbf{x}) \geq 0$. Therefore, $(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x}))) \geq 0$ for all $\mathbf{x} \in B_r$. The reason for this is that $a(\mathbf{x})$ is the larger zero of a polynomial of the form $p(z) = |\mathbf{x}|^2 + z^2|\mathbf{x} - g(\mathbf{x})|^2 - 2z(\mathbf{x},\mathbf{x}-\mathbf{g}(\mathbf{x}))$ and from the formula above, it is nonnegative. $-2(\mathbf{x},\mathbf{x}-\mathbf{g}(\mathbf{x}))$ is the slope of the tangent line to $p(z)$ at $z=0$. If $\mathbf{x} \neq \mathbf{0}$, then $|\mathbf{x}|^2 > 0$ and so this slope needs to be negative for the larger of the two zeros to be positive. If $\mathbf{x} = \mathbf{0}$, then $(\mathbf{x},\mathbf{x}-\mathbf{g}(\mathbf{x})) = 0$.

Now define for $t \in [0,1]$,

$$\mathbf{f}(t,\mathbf{x}) \equiv \mathbf{x}+ta(\mathbf{x})(\mathbf{x}-\mathbf{g}(\mathbf{x})).$$

The important properties of $\mathbf{f}(t,\mathbf{x})$ and $a(\mathbf{x})$ are that

$$a(\mathbf{x}) = 0 \text{ if } |\mathbf{x}| = r. \tag{10.27}$$

and

$$|\mathbf{f}(t,\mathbf{x})| = r \text{ for all } |\mathbf{x}| = r \tag{10.28}$$

These properties follow immediately from 10.26 and the above observation that for $\mathbf{x} \in B_r$, it follows $(\mathbf{x},(\mathbf{x}-\mathbf{g}(\mathbf{x}))) \geq 0$.

Also from 10.26, a is a C^2 function near B_r. This is obvious from 10.26 as long as $|\mathbf{x}| < r$. However, even if $|\mathbf{x}| = r$ it is still true. To show this, it suffices to verify the expression under the square root sign is positive. If this expression were not positive for some $|\mathbf{x}| = r$, then $(\mathbf{x}, (\mathbf{x} - \mathbf{g}(\mathbf{x}))) = 0$. Then also, since $\mathbf{g}(\mathbf{x}) \neq \mathbf{x}$,

$$\left| \frac{\mathbf{g}(\mathbf{x}) + \mathbf{x}}{2} \right| < r$$

and so

$$r^2 > \left(\mathbf{x}, \frac{\mathbf{g}(\mathbf{x}) + \mathbf{x}}{2} \right) = \frac{1}{2} (\mathbf{x}, \mathbf{g}(\mathbf{x})) + \frac{r^2}{2} = \frac{|\mathbf{x}|^2}{2} + \frac{r^2}{2} = r^2,$$

a contradiction. Therefore, the expression under the square root in 10.26 is always positive near B_r and so a is a C^2 function near B_r as claimed because the square root function is C^2 away from zero.

Now define

$$I(t) \equiv \int_{B_r} \det (D_2 \mathbf{f}(t, \mathbf{x}))\, dx.$$

Then

$$I(0) = \int_{B_r} dx = m_n(B_r) > 0. \tag{10.29}$$

Using the dominated convergence theorem one can differentiate $I(t)$ as follows.

$$\begin{aligned} I'(t) &= \int_{B_r} \sum_{ij} \frac{\partial \det (D_2 \mathbf{f}(t, \mathbf{x}))}{\partial f_{i,j}} \frac{\partial f_{i,j}}{\partial t} dx \\ &= \int_{B_r} \sum_{ij} \operatorname{cof} (D_2 \mathbf{f})_{ij} \frac{\partial (a(\mathbf{x}) (x_i - g_i(\mathbf{x})))}{\partial x_j} dx. \end{aligned}$$

Now from 10.27 $a(\mathbf{x}) = 0$ when $|\mathbf{x}| = r$ and so integration by parts and Lemma 10.49 yields

$$\begin{aligned} I'(t) &= \int_{B_r} \sum_{ij} \operatorname{cof} (D_2 \mathbf{f})_{ij} \frac{\partial (a(\mathbf{x}) (x_i - g_i(\mathbf{x})))}{\partial x_j} dx \\ &= -\int_{B_r} \sum_{ij} \operatorname{cof} (D_2 \mathbf{f})_{ij,j} a(\mathbf{x}) (x_i - g_i(\mathbf{x}))\, dx = 0. \end{aligned}$$

Therefore, $I(1) = I(0)$. However, from 10.25 it follows that for $t = 1$,

$$\sum_i f_i f_i = r^2$$

and so, $\sum_i f_{i,j} f_i = 0$ which implies since $|\mathbf{f}(1, \mathbf{x})| = r$ by 10.25, that $\det (f_{i,j}) = \det (D_2 \mathbf{f}(1, \mathbf{x})) = 0$ and so $I(1) = 0$, a contradiction to 10.29 since $I(1) = I(0)$. This proves the lemma.

The following theorem is the Brouwer fixed point theorem for a ball.

Theorem 10.51 *Let B_r be the above closed ball and let $\mathbf{f} : B_r \to B_r$ be continuous. Then there exists $\mathbf{x} \in B_r$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.*

Proof: Let $\mathbf{f}_k(\mathbf{x}) \equiv \frac{\mathbf{f}(\mathbf{x})}{1+k^{-1}}$. Thus $||\mathbf{f}_k - \mathbf{f}|| < \frac{r}{1+k}$ where

$$||\mathbf{h}|| \equiv \max\{|\mathbf{h}(\mathbf{x})| : \mathbf{x} \in B_r\}.$$

Using the Weierstrass approximation theorem, there exists a polynomial $\mathbf{g}_k$ such that $||\mathbf{g}_k - \mathbf{f}_k|| < \frac{r}{k+1}$. Then if $\mathbf{x} \in B_r$, it follows

$$\begin{aligned} |\mathbf{g}_k(\mathbf{x})| &\leq |\mathbf{g}_k(\mathbf{x}) - \mathbf{f}_k(\mathbf{x})| + |\mathbf{f}_k(\mathbf{x})| \\ &< \frac{r}{1+k} + \frac{kr}{1+k} = r \end{aligned}$$

and so $\mathbf{g}_k$ maps B_r to B_r. By Lemma 10.50 each of these $\mathbf{g}_k$ has a fixed point, $\mathbf{x}_k$ such that $\mathbf{g}_k(\mathbf{x}_k) = \mathbf{x}_k$. The sequence of points, $\{\mathbf{x}_k\}$ is contained in the compact set, B_r and so there exists a convergent subsequence still denoted by $\{\mathbf{x}_k\}$ which converges to a point, $\mathbf{x} \in B_r$. Then

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{x}| &\leq |\mathbf{f}(\mathbf{x}) - \mathbf{f}_k(\mathbf{x})| + |\mathbf{f}_k(\mathbf{x}) - \mathbf{f}_k(\mathbf{x}_k)| + \left|\mathbf{f}_k(\mathbf{x}_k) - \overbrace{\mathbf{g}_k(\mathbf{x}_k)}^{=\mathbf{x}_k}\right| + |\mathbf{x}_k - \mathbf{x}| \\ &\leq \frac{r}{1+k} + |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_k)| + \frac{r}{1+k} + |\mathbf{x}_k - \mathbf{x}|. \end{aligned}$$

Now let $k \to \infty$ in the right side to conclude $\mathbf{f}(\mathbf{x}) = \mathbf{x}$. This proves the theorem.

It is not surprising that the ball does not need to be centered at $\mathbf{0}$.

Corollary 10.52 *Let* $\mathbf{f} : \overline{B(\mathbf{a}, r)} \to \overline{B(\mathbf{a}, r)}$ *be continuous. Then there exists* $\mathbf{x} \in \overline{B(\mathbf{a}, r)}$ *such that* $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.

Proof: Let $\mathbf{g} : B_r \to B_r$ be defined by $\mathbf{g}(\mathbf{y}) \equiv \mathbf{f}(\mathbf{y} + \mathbf{a}) - \mathbf{a}$. Then $\mathbf{g}$ is a continuous map from B_r to B_r. Therefore, there exists $\mathbf{y} \in B_r$ such that $\mathbf{g}(\mathbf{y}) = \mathbf{y}$. Therefore, $\mathbf{f}(\mathbf{y} + \mathbf{a}) - \mathbf{a} = \mathbf{y}$ and so letting $\mathbf{x} = \mathbf{y} + \mathbf{a}$, $\mathbf{f}$ also has a fixed point as claimed.

10.11 The Brouwer Fixed Point Theorem Another Proof

This proof is also based on Lemma 10.49. I found this proof of the Brouwer fixed point theorem or one close to it in Evans [21]. It is even shorter than the proof just presented. I think it might be easier to remember also. It is also based on Lemma 10.49 which is stated next for convenience.

Lemma 10.53 *Let* $\mathbf{g} : U \to \mathbb{R}^n$ *be* C^2 *where* U *is an open subset of* $\mathbb{R}^n$. *Then*

$$\sum_{j=1}^{n} \operatorname{cof}(D\mathbf{g})_{ij,j} = 0,$$

where here $(D\mathbf{g})_{ij} \equiv g_{i,j} \equiv \frac{\partial g_i}{\partial x_j}$. *Also,* $\operatorname{cof}(D\mathbf{g})_{ij} = \frac{\partial \det(D\mathbf{g})}{\partial g_{i,j}}$.

Definition 10.54 *Let* $\mathbf{h}$ *be a function defined on an open set,* $U \subseteq \mathbb{R}^n$*. Then* $\mathbf{h} \in C^k\left(\overline{U}\right)$ *if there exists a function* $\mathbf{g}$ *defined on an open set,* W *containng* $\overline{U}$ *such that* $\mathbf{g} = \mathbf{h}$ *on* U *and* $\mathbf{g}$ *is* $C^k(W)$*.*

Lemma 10.55 *There does not exist* $\mathbf{h} \in C^2\left(\overline{B(\mathbf{0},R)}\right)$ *such that* $\mathbf{h} : \overline{B(\mathbf{0},R)} \to \partial B(\mathbf{0},R)$ *which also has the property that* $\mathbf{h}(\mathbf{x}) = \mathbf{x}$ *for all* $\mathbf{x} \in \partial B(\mathbf{0},R)$*. Such a function is called a retraction.*

Proof: Suppose such an $\mathbf{h}$ exists. Let $\lambda \in [0,1]$ and let $\mathbf{p}_\lambda(\mathbf{x}) \equiv \mathbf{x} + \lambda(\mathbf{h}(\mathbf{x}) - \mathbf{x})$. This function, $\mathbf{p}_\lambda$ is a homotopy of the identity map and the retraction, $\mathbf{h}$. Let

$$I(\lambda) \equiv \int_{B(\mathbf{0},R)} \det(D\mathbf{p}_\lambda(\mathbf{x}))\,dx.$$

Then using the dominated convergence theorem,

$$\begin{aligned} I'(\lambda) &= \int_{B(\mathbf{0},R)} \sum_{i.j} \frac{\partial \det(D\mathbf{p}_\lambda(\mathbf{x}))}{\partial p_{\lambda i,j}} \frac{\partial p_{\lambda ij}(\mathbf{x})}{\partial \lambda} \\ &= \int_{B(\mathbf{0},R)} \sum_i \sum_j \frac{\partial \det(D\mathbf{p}_\lambda(\mathbf{x}))}{\partial p_{\lambda i,j}} (h_i(\mathbf{x}) - x_i)_{,j}\,dx \\ &= \int_{B(\mathbf{0},R)} \sum_i \sum_j \operatorname{cof}(D\mathbf{p}_\lambda(\mathbf{x}))_{ij} (h_i(\mathbf{x}) - x_i)_{,j}\,dx \end{aligned}$$

Now by assumption, $h_i(\mathbf{x}) = x_i$ on $\partial B(\mathbf{0},R)$ and so one can integrate by parts and write

$$I'(\lambda) = -\sum_i \int_{B(\mathbf{0},R)} \sum_j \operatorname{cof}(D\mathbf{p}_\lambda(\mathbf{x}))_{ij,j} (h_i(\mathbf{x}) - x_i)\,dx = 0.$$

Therefore, $I(\lambda)$ equals a constant. However,

$$I(0) = m_n(B(\mathbf{0},R)) > 0$$

but

$$I(1) = \int_{B(\mathbf{0},1)} \det(D\mathbf{h}(\mathbf{x}))\,dm_n = \int_{\partial B(\mathbf{0},1)} \#(\mathbf{y})\,dm_n = 0$$

because from polar coordinates or other elementary reasoning, $m_n(\partial B(\mathbf{0},1)) = 0$. This proves the lemma.

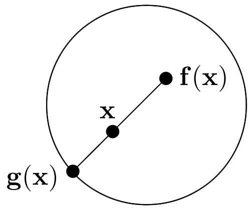
The following is the Brouwer fixed point theorem for C^2 maps.

Lemma 10.56 *If* $\mathbf{h} \in C^2\left(\overline{B(\mathbf{0},R)}\right)$ *and* $\mathbf{h} : \overline{B(\mathbf{0},R)} \to \overline{B(\mathbf{0},R)}$*, then* $\mathbf{h}$ *has a fixed point,* $\mathbf{x}$ *such that* $\mathbf{h}(\mathbf{x}) = \mathbf{x}$*.*

Proof: Suppose the lemma is not true. Then for all $\mathbf{x}, |\mathbf{x} - \mathbf{h}(\mathbf{x})| \neq 0$. Then define

$$\mathbf{g}(\mathbf{x}) = \mathbf{h}(\mathbf{x}) + \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|} t(\mathbf{x})$$

where $t(\mathbf{x})$ is nonnegative and is chosen such that $\mathbf{g}(\mathbf{x}) \in \partial B(\mathbf{0}, R)$. This mapping is illustrated in the following picture.



If $\mathbf{x} \rightarrow t(\mathbf{x})$ is C^2 near $\overline{B(\mathbf{0}, R)}$, it will follow $\mathbf{g}$ is a C^2 retraction onto $\partial B(\mathbf{0}, R)$ contrary to Lemma 10.55. Thus $t(\mathbf{x})$ is the nonnegative solution to

$$H(\mathbf{x}, \mathbf{t}) = |\mathbf{h}(\mathbf{x})|^2 + 2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|}\right) t + t^2 = R^2 \tag{10.30}$$

Then

$$H_t(\mathbf{x}, t) = 2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|}\right) + 2t.$$

If this is nonzero for all $\mathbf{x}$ near $\overline{B(\mathbf{0}, R)}$, it follows from the implicit function theorem that t is a C^2 function of $\mathbf{x}$. Then from 10.30

$$\begin{aligned} 2t &= -2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|}\right) \\ &\quad \pm\sqrt{4\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|}\right)^2 - 4\left(|\mathbf{h}(\mathbf{x})|^2 - R^2\right)} \end{aligned}$$

and so

$$\begin{aligned} H_t(\mathbf{x}, t) &= 2t + 2\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|}\right) \\ &= \pm\sqrt{4\left(R^2 - |\mathbf{h}(\mathbf{x})|^2\right) + 4\left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|}\right)^2} \end{aligned}$$

If $|\mathbf{h}(\mathbf{x})| < R$, this is nonzero. If $|\mathbf{h}(\mathbf{x})| = R$, then it is still nonzero unless

$$(\mathbf{h}(\mathbf{x}), \mathbf{x} - \mathbf{h}(\mathbf{x})) = 0.$$

But this cannot happen because the angle between $\mathbf{h}(\mathbf{x})$ and $\mathbf{x} - \mathbf{h}(\mathbf{x})$ cannot be $\pi/2$. Alternatively, if the above equals zero, you would need

$$(\mathbf{h}(\mathbf{x}), \mathbf{x}) = |\mathbf{h}(\mathbf{x})|^2 = R^2$$

which cannot happen unless $\mathbf{x} = \mathbf{h}(\mathbf{x})$ which is assumed not to happen. Therefore, $\mathbf{x} \rightarrow t(\mathbf{x})$ is C^2 near $\overline{B(\mathbf{0}, R)}$ and so $\mathbf{g}(\mathbf{x})$ given above contradicts Lemma 10.55. This proves the lemma.

Now it is easy to prove the Brouwer fixed point theorem.

Theorem 10.57 *Let* $\mathbf{f} : \overline{B(\mathbf{0}, R)} \to \overline{B(\mathbf{0}, R)}$ *be continuous. Then* $\mathbf{f}$ *has a fixed point.*

Proof: If this is not so, there exists $\varepsilon > 0$ such that for all $\mathbf{x} \in \overline{B(\mathbf{0}, R)}$,

$$|\mathbf{x} - \mathbf{f}(\mathbf{x})| > \varepsilon.$$

By the Weierstrass approximation theorem, there exists $\mathbf{h}$, a polynomial such that

$$\max\left\{|\mathbf{h}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| : \mathbf{x} \in \overline{B(\mathbf{0}, R)}\right\} < \frac{\varepsilon}{2}.$$

Then for all $\mathbf{x} \in \overline{B(\mathbf{0}, R)}$,

$$|\mathbf{x} - \mathbf{h}(\mathbf{x})| \geq |\mathbf{x} - \mathbf{f}(\mathbf{x})| - |\mathbf{h}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

contradicting Lemma 10.56. This proves the theorem.

Some Extension Theorems

11.1 Caratheodory Extension Theorem

The Caratheodory extension theorem is a fundamental result which makes possible the consideration of measures on infinite products among other things. The idea is that if a finite measure defined only on an algebra is trying to be a measure, then in fact it can be extended to a measure.

Definition 11.1 *Let $\mathcal{E}$ be an algebra of sets of Ω and let μ_0 be a finite measure on $\mathcal{E}$. This means μ_0 is finitely additive and if E_i, E are sets of $\mathcal{E}$ with the E_i disjoint and*

$$E = \cup_{i=1}^{\infty} E_i,$$

then

$$\mu_0 (E) = \sum_{i=1}^{\infty} \mu (E_i)$$

while $\mu_0 (\Omega) < \infty$.

In this definition, μ_0 is trying to be a measure and acts like one whenever possible. Under these conditions, μ_0 can be extended uniquely to a complete measure, μ, defined on a σ algebra of sets containing $\mathcal{E}$ such that μ agrees with μ_0 on $\mathcal{E}$. The following is the main result.

Theorem 11.2 *Let μ_0 be a measure on an algebra of sets, $\mathcal{E}$, which satisfies $\mu_0 (\Omega) < \infty$. Then there exists a complete measure space $(\Omega, \mathcal{S}, \mu)$ such that*

$$\mu (E) = \mu_0 (E)$$

for all $E \in \mathcal{E}$. Also if ν is any such measure which agrees with μ_0 on $\mathcal{E}$, then $\nu = \mu$ on $\sigma (\mathcal{E})$, the σ algebra generated by $\mathcal{E}$.

Proof: Define an outer measure as follows.

$$\mu (S) \equiv \inf \left\{ \sum_{i=1}^{\infty} \mu_0 (E_i) : S \subseteq \cup_{i=1}^{\infty} E_i, E_i \in \mathcal{E} \right\}$$

Claim 1: μ is an outer measure.

Proof of Claim 1: Let $S \subseteq \cup_{i=1}^{\infty} S_i$ and let $S_i \subseteq \cup_{j=1}^{\infty} E_{ij}$, where

$$\mu(S_i) + \frac{\varepsilon}{2^i} \geq \sum_{j=1}^{\infty} \mu(E_{ij}).$$

Then

$$\mu(S) \leq \sum_i \sum_j \mu(E_{ij}) = \sum_i \left(\mu(S_i) + \frac{\varepsilon}{2^i} \right) = \sum_i \mu(S_i) + \varepsilon.$$

Since ε is arbitrary, this shows μ is an outer measure as claimed.

By the Caratheodory procedure, there exists a unique σ algebra, $\mathcal{S}$, consisting of the μ measurable sets such that

$$(\Omega, \mathcal{S}, \mu)$$

is a complete measure space. It remains to show μ extends μ_0.

Claim 2: If $\mathcal{S}$ is the σ algebra of μ measurable sets, $\mathcal{S} \supseteq \mathcal{E}$ and $\mu = \mu_0$ on $\mathcal{E}$.

Proof of Claim 2: First observe that if $A \in \mathcal{E}$, then $\mu(A) \leq \mu_0(A)$ by definition. Letting

$$\mu(A) + \varepsilon > \sum_{i=1}^{\infty} \mu_0(E_i), \ \cup_{i=1}^{\infty} E_i \supseteq A, \ E_i \in \mathcal{E},$$

it follows

$$\mu(A) + \varepsilon > \sum_{i=1}^{\infty} \mu_0(E_i \cap A) \geq \mu_0(A)$$

since $A = \cup_{i=1}^{\infty} E_i \cap A$. Therefore, $\mu = \mu_0$ on $\mathcal{E}$.

Consider the assertion that $\mathcal{E} \subseteq \mathcal{S}$. Let $A \in \mathcal{E}$ and let $S \subseteq \Omega$ be any set. There exist sets $\{E_i\} \subseteq \mathcal{E}$ such that $\cup_{i=1}^{\infty} E_i \supseteq S$ but

$$\mu(S) + \varepsilon > \sum_{i=1}^{\infty} \mu(E_i).$$

Then

$$\begin{aligned} \mu(S) &\leq \mu(S \cap A) + \mu(S \setminus A) \\ &\leq \mu(\cup_{i=1}^{\infty} E_i \setminus A) + \mu(\cup_{i=1}^{\infty} (E_i \cap A)) \\ &\leq \sum_{i=1}^{\infty} \mu(E_i \setminus A) + \sum_{i=1}^{\infty} \mu(E_i \cap A) = \sum_{i=1}^{\infty} \mu(E_i) < \mu(S) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this shows $A \in \mathcal{S}$.

This has proved the existence part of the theorem. To verify uniqueness, Let

$$\mathcal{M} \equiv \{E \in \sigma(\mathcal{E}) : \mu(E) = \nu(E)\}.$$

Then $\mathcal{M}$ is given to contain $\mathcal{E}$ and is obviously a monotone class. Therefore by Theorem 9.57 on monotone classes, $\mathcal{M} = \sigma(\mathcal{E})$ and this proves the lemma.

The following lemma is also very significant.

Lemma 11.3 *Let M be a metric space with the closed balls compact and suppose μ is a measure defined on the Borel sets of M which is finite on compact sets. Then there exists a unique Radon measure, $\overline{\mu}$ which equals μ on the Borel sets. In particular μ must be both inner and outer regular on all Borel sets.*

Proof: Define a positive linear functional, $\Lambda(f) = \int f d\mu$. Let $\overline{\mu}$ be the Radon measure which comes from the Riesz representation theorem for positive linear functionals. Thus for all f continuous,

$$\int f d\mu = \int f d\overline{\mu}.$$

If V is an open set, let $\{f_n\}$ be a sequence of continuous functions which is increasing and converges to $\mathcal{X}_V$ pointwise. Then applying the monotone convergence theorem,

$$\int \mathcal{X}_V d\mu = \mu(V) = \int \mathcal{X}_V d\overline{\mu} = \overline{\mu}(V)$$

and so the two measures coincide on all open sets. Every compact set is a countable intersection of open sets and so the two measures coincide on all compact sets. Now let $B(a, n)$ be a ball of radius n and let E be a Borel set contained in this ball. Then by regularity of $\overline{\mu}$ there exist sets F, G such that G is a countable intersection of open sets and F is a countable union of compact sets such that $F \subseteq E \subseteq G$ and $\overline{\mu}(G \setminus F) = 0$. Now $\mu(G) = \overline{\mu}(G)$ and $\mu(F) = \overline{\mu}(F)$. Thus

$$\begin{aligned} \overline{\mu}(G \setminus F) + \overline{\mu}(F) &= \overline{\mu}(G) \\ &= \mu(G) = \mu(G \setminus F) + \mu(F) \end{aligned}$$

and so $\mu(G \setminus F) = \overline{\mu}(G \setminus F)$. It follows

$$\mu(E) = \mu(F) = \overline{\mu}(F) = \overline{\mu}(G) = \overline{\mu}(E).$$

If E is an arbitrary Borel set, then

$$\mu(E \cap B(a, n)) = \overline{\mu}(E \cap B(a, n))$$

and letting $n \to \infty$, this yields $\mu(E) = \overline{\mu}(E)$.

11.2 The Tychonoff Theorem

Sometimes it is necessary to consider infinite Cartesian products of topological spaces. When you have finitely many topological spaces in the product and each is compact, it can be shown that the Cartesian product is compact with the product topology. It turns out that the same thing holds for infinite products but you have to be careful how you define the topology. The first thing likely to come to mind by analogy with finite products is not the right way to do it.

First recall the Hausdorff maximal principle.

Theorem 11.4 *(Hausdorff maximal principle) Let $\mathcal{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.*

The main tool in the study of products of compact topological spaces is the Alexander subbasis theorem which is presented next. Recall a set is compact if every basic open cover admits a finite subcover. This was pretty easy to prove. However, there is a much smaller set of open sets called a subbasis which has this property. The proof of this result is much harder.

Definition 11.5 *$\mathcal{S} \subseteq \tau$ is called a subbasis for the topology τ if the set $\mathcal{B}$ of finite intersections of sets of $\mathcal{S}$ is a basis for the topology, τ.*

Theorem 11.6 *Let (X, τ) be a topological space and let $\mathcal{S} \subseteq \tau$ be a subbasis for τ. Then if $H \subseteq X$, H is compact if and only if every open cover of H consisting entirely of sets of $\mathcal{S}$ admits a finite subcover.*

Proof: The only if part is obvious because the subasic sets are themselves open.

By Lemma 6.56 on Page 6.56, if every basic open cover admits a finite subcover then the set in question is compact. Suppose then that H is a subset of X having the property that subbasic open covers admit finite subcovers. Is H compact? Assume this is not so. Then what was just observed about basic covers implies there exists a basic open cover of H, $\mathcal{O}$, which admits no finite subcover. Let $\mathcal{F}$ be defined as

$$\{\mathcal{O} : \mathcal{O} \text{ is a basic open cover of } H \text{ which admits no finite subcover}\}.$$

The assumption is that $\mathcal{F}$ is nonempty. Partially order $\mathcal{F}$ by set inclusion and use the Hausdorff maximal principle to obtain a maximal chain, $\mathcal{C}$, of such open covers and let

$$\mathcal{D} = \cup \mathcal{C}.$$

If $\mathcal{D}$ admits a finite subcover, then since $\mathcal{C}$ is a chain and the finite subcover has only finitely many sets, some element of $\mathcal{C}$ would also admit a finite subcover, contrary to the definition of $\mathcal{F}$. Therefore, $\mathcal{D}$ admits no finite subcover. If $\mathcal{D}' \supsetneq \mathcal{D}$ and $\mathcal{D}'$ is a basic open cover of H, then $\mathcal{D}'$ has a finite subcover of H since otherwise, $\mathcal{C}$ would fail to be a maximal chain, being properly contained in $\mathcal{C} \cup \{\mathcal{D}'\}$. Every set of $\mathcal{D}$ is of the form

$$U = \cap_{i=1}^{m} B_i, \ B_i \in \mathcal{S}$$

because they are all basic open sets. If it is the case that for all $U \in \mathcal{D}$ one of the B_i is found in $\mathcal{D}$, then replace each such U with the subbasic set from $\mathcal{D}$ containing it. But then this would be a subbasic open cover of H which by assumption would admit a finite subcover contrary to the properties of $\mathcal{D}$. Therefore, one of the sets of $\mathcal{D}$, denoted by U, has the property that

$$U = \cap_{i=1}^{m} B_i, \ B_i \in \mathcal{S}$$

and no B_i is in $\mathcal{D}$. Thus $\mathcal{D} \cup \{B_i\}$ admits a finite subcover, for each of the above B_i because it is strictly larger than $\mathcal{D}$. Let this finite subcover corresponding to B_i be denoted by

$$V_1^i, \cdot \cdot \cdot, V_{m_i}^i, B_i$$

Consider

$$\{U, V_j^i, \ j = 1, \cdot \cdot \cdot, m_i, \ i = 1, \cdot \cdot \cdot, m\}.$$

If $p \in H \setminus \cup\{V_j^i\}$, then $p \in B_i$ for each i and so $p \in U$. This is therefore a finite subcover of $\mathcal{D}$ contradicting the properties of $\mathcal{D}$. Therefore, $\mathcal{F}$ must be empty and by Lemma 6.56, this proves the theorem.

Let I be a set and suppose for each $i \in I$, (X_i, τ_i) is a nonempty topological space. The Cartesian product of the X_i, denoted by $\prod_{i \in I} X_i$, consists of the set of all choice functions defined on I which select a single element of each X_i. Thus $f \in \prod_{i \in I} X_i$ means for every $i \in I$, $f(i) \in X_i$. The axiom of choice says $\prod_{i \in I} X_i$ is nonempty. Let

$$P_j(A) = \prod_{i \in I} B_i$$

where $B_i = X_i$ if $i \neq j$ and $B_j = A$. A subbasis for a topology on the product space consists of all sets $P_j(A)$ where $A \in \tau_j$. (These sets have an open set from the topology of X_j in the j^{th} slot and the whole space in the other slots.) Thus a basis consists of finite intersections of these sets. Note that the intersection of two of these basic sets is another basic set and their union yields $\prod_{i \in I} X_i$. Therefore, they satisfy the condition needed for a collection of sets to serve as a basis for a topology. This topology is called the product topology and is denoted by $\prod \tau_i$.

It is tempting to define a basis for a topology to be sets of the form $\prod_{i \in I} A_i$ where A_i is open in X_i. This is not the same thing at all. Note that the basis just described has at most finitely many slots filled with an open set which is not the whole space. The thing just mentioned in which every slot may be filled by a proper open set is called the box topology and there exist people who are interested in it.

The Alexander subbasis theorem is used to prove the Tychonoff theorem which says that if each X_i is a compact topological space, then in the product topology, $\prod_{i \in I} X_i$ is also compact.

Theorem 11.7 *If* $(X_i \tau_i)$ *is compact, then so is* $(\prod_{i \in I} X_i, \prod \tau_i)$.

Proof: By the Alexander subbasis theorem, the theorem will be proved if every subbasic open cover admits a finite subcover. Therefore, let $\mathcal{O}$ be a subbasic open cover of $\prod_{i \in I} X_i$. Let

$$\mathcal{O}_j = \{Q \in \mathcal{O} : Q = P_j(A) \text{ for some } A \in \tau_j\}.$$

Thus $\mathcal{O}_j$ consists of those sets of $\mathcal{O}$ which have a possibly proper subset of X_i only in the slot $i = j$. Let

$$\pi_j \mathcal{O}_j = \{A : P_j(A) \in \mathcal{O}_j\}.$$

Thus $\pi_j \mathcal{O}_j$ picks out those proper open subsets of X_j which occur in $\mathcal{O}_j$.

If no $\pi_j \mathcal{O}_j$ covers X_j, then by the axiom of choice, there exists

$$f \in \prod_{i \in I} X_i \setminus \cup \pi_i \mathcal{O}_i$$

Therefore, $f(j) \notin \cup \pi_j \mathcal{O}_j$ for each $j \in I$. Now f is a point of $\prod_{i \in I} X_i$ and so $f \in P_k(A) \in \mathcal{O}$ for some k. However, this is a contradiction it was shown that $f(k)$ is not an element of A. (A is one of the sets whose union makes up $\cup \pi_k \mathcal{O}_k$.) This contradiction shows that for some j, $\pi_j \mathcal{O}_j$ covers X_j. Thus

$$X_j = \cup \pi_j \mathcal{O}_j$$

and so by compactness of X_j, there exist $A_1, \cdots, A_m$, sets in τ_j such that $X_j \subseteq \cup_{i=1}^m A_i$ and $P_j(A_i) \in \mathcal{O}$. Therefore, $\{P_j(A_i)\}_{i=1}^m$ covers $\prod_{i \in I} X_i$. By the Alexander subbasis theorem this proves $\prod_{i \in I} X_i$ is compact.

11.3 Kolmogorov Extension Theorem

Here M_t will be a metric space in which the closed balls are compact. (The case of interest is $\mathbb{R}^n$ but it is easier to write M.) Thus it is also a locally compact Hausdorff space. I will denote a totally ordered index set, and the interest will be in building a measure on the product space, $\prod_{t \in I} M_t$. The example of interest for I is $[0, \infty)$ but all that I will use is that I is totally ordered. By well ordering principle, you can always put an order on the index set. Also, I will denote M_t' as the one point compactification of M_t.

Let $J \subseteq I$. Then if $\mathbf{E} \equiv \prod_{t \in I} E_t$, define

$$\gamma_J \mathbf{E} \equiv \prod_{t \in I} F_t$$

where

$$F_t = \begin{cases} E_t \text{ if } t \in J \\ M_t' \text{ if } t \notin J \end{cases}$$

Thus $\gamma_J \mathbf{E}$ leaves alone E_t for $t \in J$ and changes the other E_t into M_t'. If $\gamma_J \mathbf{E} = \mathbf{E}$, then this means $E_t = M_t'$ for all $t \notin J$. Also define for J a subset of I,

$$\pi_J \mathbf{x} \equiv \prod_{t \in J} x_t$$

so π_J is a continuous mapping from $\prod_{t \in I} M_t'$ to $\prod_{t \in J} M_t'$.

$$\pi_J \mathbf{E} \equiv \prod_{t \in J} E_t.$$

Definition 11.8 *Now define for J a finite subset of I,*

$$\mathcal{R}_J \equiv \left\{ \mathbf{E} = \prod_{t\in I} E_t \ : \gamma_J \mathbf{E} = \mathbf{E},\ E_t \text{ a Borel set in } M_t' \right\}$$
$$\mathcal{R} \equiv \cup \{\mathcal{R}_J : J \subseteq I, \text{ and } J \text{ finite}\}$$

Thus $\mathcal{R}$ consists of those sets of $\prod_{t\in I} M_t'$ for which every slot is filled with M' except for a finite set, $J \subseteq I$ where the slots are filled with a Borel set, E_t. Define $\mathcal{E}$ as finite disjoint unions of sets of $\mathcal{R}$.

In fact $\mathcal{E}$ is an algebra of sets.

Lemma 11.9 *The sets, $\mathcal{E}$ defined above form an algebra of sets of $\prod_{t\in I} M_t'$.*

Proof: Clearly $\emptyset$ and $\prod_{t\in I} M_t'$ are both in $\mathcal{E}$. Suppose $\mathbf{A}, \mathbf{B} \in \mathcal{R}$. Then for some finite set, J,

$$\gamma_J \mathbf{A} = \mathbf{A},\ \gamma_J \mathbf{B} = \mathbf{B}.$$

Then

$$\gamma_J (\mathbf{A} \setminus \mathbf{B}) = \mathbf{A} \setminus \mathbf{B} \in \mathcal{E}, \gamma_J (\mathbf{A} \cap \mathbf{B}) = \mathbf{A} \cap \mathbf{B} \in \mathcal{R}.$$

By Lemma 9.54 on Page 246 this shows $\mathcal{E}$ is an algebra.

Here is a lemma which is useful although fussy.

Lemma 11.10 *Let J be a finite subset of I. Then $\mathbf{U}$ is a Borel set in $\prod_{t\in J} M_t$ if and only if there exists a Borel set, $\mathbf{U}'$ in $\prod_{t\in J} M_t'$ such that $\mathbf{U} = \mathbf{U}' \cap \prod_{t\in J} M_t$.*

Proof: First suppose Borel is replaced with open. If $\mathbf{U}$ is an open set in $\prod_{t\in J} M_t$, then it is open in $\prod_{t\in J} M_t'$ from the definition of the open sets of $\prod_{t\in J} M_t'$ which consist of open sets of $\prod_{t\in J} M_t$ along with complements of compact sets which have the point ∞ added in. Thus you can let $\mathbf{U}' = \mathbf{U}$ in this case. Next suppose $\mathbf{U} = \mathbf{U}' \cap \prod_{t\in J} M_t$ where $\mathbf{U}'$ is open in $\prod_{t\in J} M_t'$. I need show $\mathbf{U}$ is open in $\prod_{t\in J} M_t$. Letting $\mathbf{x} \in \mathbf{U}$, it follows $x_t \neq \infty$ for each t. It follows from $\mathbf{U}'$ open in $\prod_{t\in J} M_t'$ that $x_t \in V_t'$ where V_t' is open in M_t' and

$$\mathbf{x} \in \prod_{t\in J} V_t' \subseteq \mathbf{U}'$$

Letting $V_t = V_t' \setminus \{\infty\}$, it follows V_t is open in M_t and

$$\mathbf{x} \in \prod_{t\in J} V_t \subseteq \mathbf{U}' \cap \prod_{t\in J} M_t.$$

This shows that $\mathbf{U}$ is an open set in $\prod_{t\in J} M_t$. Now let

$$\mathcal{G} \equiv \left\{ \mathbf{F} \text{ Borel in } \prod_{t\in J} M_t' \text{ such that } \mathbf{F} \cap \prod_{t\in J} M_t \text{ is Borel in } \prod_{t\in J} M_t \right\}$$

then from what was just shown $\mathcal{G}$ contains the open sets. It is also clearly a σ algebra. Hence $\mathcal{G}$ equals the Borel sets of $\prod_{t\in J} M_t'$.

It only remains to verify that any Borel set in $\prod_{t\in J} M_t$ is the intersection of a Borel set of $\prod_{t\in J} M_t'$ with $\prod_{t\in J} M_t$. Let

$$\mathcal{H} \equiv \left\{ \mathbf{F} \text{ Borel in } \prod_{t\in J} M_t \text{ such that } \mathbf{F} = \mathbf{F}' \cap \prod_{t\in J} M_t, \mathbf{F}' \text{ Borel in } \prod_{t\in J} M_t' \right\}$$

From the first part of the argument, $\mathcal{H}$ contains the open sets. Now let $\{\mathbf{F}_n\}$ be a sequence in $\mathcal{H}$. Thus $\mathbf{F}_n = \mathbf{F}_n' \cap \prod_{t\in J} M_t$ where $\mathbf{F}_n'$ is Borel in $\prod_{t\in J} M_t'$. Then $\cup_n \mathbf{F}_n = \cup_n \mathbf{F}_n' \cap \prod_{t\in J} M_t$ and $\cup_n \mathbf{F}_n'$ is Borel in $\prod_{t\in J} M_t'$. Thus $\mathcal{H}$ is closed under countable intersections. Next let $\mathbf{F} \in \mathcal{H}$ and $\mathbf{F} = \mathbf{F}' \cap \prod_{t\in J} M_t$. Then $\mathbf{F}'^C \equiv \prod_{t\in J} M_t' \setminus \mathbf{F}'$ is Borel in $\prod_{t\in J} M_t'$ and $\left(\prod_{t\in J} M_t \setminus \mathbf{F}\right) = \mathbf{F}'^C \cap \prod_{t\in J} M_t$. Thus $\mathcal{H}$ is a σ algebra containing the open sets and so $\mathcal{H}$ equals the Borel sets in $\prod_{t\in J} M_t$. This proves this wretched little lemma.

With this preparation here is the Kolmogorov extension theorem. In the statement and proof of the theorem, F_i, G_i, and E_i will denote Borel sets. Any list of indices from I will always be assumed to be taken in order. Thus, if $J \subseteq I$ and $J = (t_1, \cdots, t_n)$, it will always be assumed $t_1 < t_2 < \cdots < t_n$.

Theorem 11.11 *(Kolmogorov extension theorem) For each finite set*

$$J = (t_1, \cdots, t_n) \subseteq I,$$

suppose there exists a Borel probability measure, $\nu_J = \nu_{t_1 \cdots t_n}$ *defined on the Borel sets of* $\prod_{t\in J} M_t$ *such that if*

$$(t_1, \cdots, t_n) \subseteq (s_1, \cdots, s_p),$$

then

$$\nu_{t_1 \cdots t_n}(F_{t_1} \times \cdots \times F_{t_n}) = \nu_{s_1 \cdots s_p}\left(G_{s_1} \times \cdots \times G_{s_p}\right) \tag{11.1}$$

where if $s_i = t_j$*, then* $G_{s_i} = F_{t_j}$ *and if* s_i *is not equal to any of the indices,* t_k*, then* $G_{s_i} = M_{s_i}$*. Then there exists a probability space,* $(\Omega, P, \mathcal{F})$ *and measurable functions,* $X_t : \Omega \to M_t$ *for each* $t \in I$ *such that for each* $(t_1 \cdots t_n) \subseteq I$*,*

$$\nu_{t_1 \cdots t_n}(F_{t_1} \times \cdots \times F_{t_n}) = P\left([X_{t_1} \in F_{t_1}] \cap \cdots \cap [X_{t_n} \in F_{t_n}]\right). \tag{11.2}$$

Proof: First of all, note that for J finite, $\prod_{t\in J} M_t$ is a metric space in which the closures of the balls are compact. Therefore, by Lemma 11.3 $\nu_{t_1 \cdots t_n}$ is both inner and outer regular. Also, it is convenient to extend each $\nu_{t_1 \cdots t_n}$ to the Borel sets of $\prod_{t\in J} M_t'$ where M_t' is the one point compactification of M_t as follows. For $\mathbf{F}$ a Borel set of $\prod_{t\in J} M_t'$,

$$\nu_{t_1 \cdots t_n}(\mathbf{F}) \equiv \nu_{t_1 \cdots t_n}\left(\mathbf{F} \cap \prod_{t\in J} M_t\right).$$

By Lemma 11.10 this is well defined.

Letting $\mathcal{E}$ be the algebra of sets defined in Definition 11.8 and suppose $\mathbf{E} = \mathbf{E}^1 \cup \mathbf{E}^2 \cup \cdots \cup \mathbf{E}^m$ where $\gamma_{J_k}\mathbf{E}^k = \mathbf{E}^k$. Thus $\mathbf{E} \in \mathcal{E}$. Let $\left(t_1^k \cdots t_{m_k}^k\right) = J_k$. Then letting

$$J = (s_1, \cdots, s_p) \supseteq \cup_{k=1}^m J_k$$

define

$$P_0(\mathbf{E}) \equiv \sum_{k=1}^m \nu_{s_1 \cdots s_p}\left(G_{s_1}^k \times \cdots \times G_{s_p}^k\right)$$

where $G_{s_i}^k = E_{t_j^k}^k$ in case $s_i = t_j^k$ and M'_{s_i} otherwise. By 11.1 this is well defined and equals

$$\sum_{k=1}^m \nu_{t_1^k \cdots t_{m_k}^k}\left(E_{t_1^k}^k \times \cdots \times E_{t_{m_k}^k}^k\right).$$

P_0 is clearly finitely additive because the ν_J are measures and one can pick J as large as desired. Also, from the definition,

$$P_0\left(\prod_{t\in I} M'_t\right) = \nu_{t_1}\left(M'_{t_1}\right) = 1.$$

Next I will show P_0 is a finite measure on $\mathcal{E}$. From this it is only a matter of using the Caratheodory extension theorem.

Claim: If $\mathbf{E}^n \downarrow \emptyset$, then $P_0(\mathbf{E}^n) \downarrow 0$.

Proof of the claim: If not, there exists a sequence such that although $\mathbf{E}^n \downarrow \emptyset, P_0(\mathbf{E}^n) \downarrow \varepsilon > 0$. Since each of the $\nu_{s_1 \cdots s_m}$ is inner regular, there exists a compact set, $\mathbf{K}^n \subseteq \pi_J(\mathbf{E}^n)$ for suitably large J such that if $\mathbf{K}^{n\prime} \subseteq \prod_{t\in I} M'_t$ is defined by $\gamma_J(\mathbf{K}^{n\prime}) = \mathbf{K}^{n\prime}$ and $\pi_J(\mathbf{K}^{n\prime}) = \mathbf{K}^n$ and $P_0(\mathbf{E}^n \setminus \mathbf{K}^{n\prime}) < \varepsilon/2^{n+1}$. (Less precisely, you get $\mathbf{K}^{n\prime}$ by filling in all the slots other than in J with the appropriate M'_t.) Thus by Tychonoff's theorem, $\mathbf{K}^{n\prime}$ is compact. The interesting thing about these $\mathbf{K}^{n\prime}$ is they have the finite intersection property. Here is why.

$$\begin{aligned} \varepsilon &\leq P_0\left(\cap_{k=1}^m \mathbf{K}^{k\prime}\right) + P_0\left(\mathbf{E}^m \setminus \cap_{k=1}^m \mathbf{K}^{k\prime}\right) \\ &\leq P_0\left(\cap_{k=1}^m \mathbf{K}^{k\prime}\right) + P_0\left(\cup_{k=1}^m \mathbf{E}^k \setminus \mathbf{K}^{k\prime}\right) \\ &< P_0\left(\cap_{k=1}^m \mathbf{K}^{k\prime}\right) + \sum_{k=1}^\infty \frac{\varepsilon}{2^{k+1}} < P_0\left(\cap_{k=1}^m \mathbf{K}^{k\prime}\right) + \varepsilon \end{aligned}$$

and so $P_0\left(\cap_{k=1}^m \mathbf{K}^{k\prime}\right) > 0$. Now this yields a contradiction because this finite intersection property implies the intersection of all the $\mathbf{K}^{n\prime}$ is nonempty contradicting $\mathbf{E}^n \downarrow \emptyset$ since each $\mathbf{K}^{n\prime}$ is contained in $\mathbf{E}^n$.

With the claim, it follows P_0 is a measure on $\mathcal{E}$. Here is why: If $\mathbf{E} = \cup_{k=1}^\infty \mathbf{E}^k$ where $\mathbf{E}, \mathbf{E}^k \in \mathcal{E}$, then $(\mathbf{E} \setminus \cup_{k=1}^n \mathbf{E}_k) \downarrow \emptyset$ and so

$$P_0\left(\cup_{k=1}^n \mathbf{E}_k\right) \to P_0(\mathbf{E}).$$

Hence if the $\mathbf{E}_k$ are disjoint, $P_0\left(\cup_{k=1}^n \mathbf{E}_k\right) = \sum_{k=1}^n P_0(\mathbf{E}_k) \to P_0(\mathbf{E})$.

Now to conclude the proof, apply the Caratheodory extension theorem to obtain P a probability measure which extends P_0 to $\sigma(\mathcal{E})$ the sigma algebra generated by $\mathcal{E}$. Let $\mathcal{S} \equiv \left\{\mathbf{E} \cap \prod_{t\in I} M_t : \mathbf{E} \in \sigma(\mathcal{E})\right\}$. It follows $\left(\prod_{t\in I} M_t, \mathcal{S}, P\right)$ is a probability measure space with the property that when $\gamma_J(\mathbf{E}) = \mathbf{E}$ for $J = (t_1 \cdots t_n)$ a finite subset of I, $P(\mathbf{E}) = P_0(\mathbf{E}) = \nu_{t_1\cdots t_n}(E_{t_1} \times \cdots \times E_{t_n})$.

For the last part, let $\left(\prod_{t\in I} M_t, \mathcal{S}, P\right)$ be the probability space and for $\mathbf{x} \in \prod_{t\in I} M_t$ let $X_t(\mathbf{x}) = x_t$, the t^{th} entry of $\mathbf{x}$. ($x_t = \pi_t \mathbf{x}$). It follows X_t is measurable because if U is open in M_t, then $X_t^{-1}(U)$ has a U in the t^{th} slot and M_s everywhere else for $s \neq t$ so this is actually in $\mathcal{E}$. Also, letting $(t_1 \cdots t_n)$ be a finite subset of I and $F_{t_1}, \cdots, F_{t_n}$ be Borel sets in $M_{t_1} \cdots M_{t_n}$ respectively,

$$P([X_{t_1} \in F_{t_1}] \cap [X_{t_2} \in F_{t_2}] \cap \cdots \cap [X_{t_n} \in F_{t_n}]) =$$

$$P((X_{t_1}, X_{t_2}, \cdots, X_{t_n}) \in F_{t_1} \times \cdots \times F_{t_n}) = P(F_{t_1} \times \cdots \times F_{t_n})$$
$$= \nu_{t_1\cdots t_n}(F_{t_1} \times \cdots \times F_{t_n})$$

This proves the theorem.

As a special case, you can obtain a version of product measure for possibly infinitely many factors. Suppose in the context of the above theorem that ν_t is a probability measure defined on the Borel sets of M_t and let the measures, $\nu_{t_1\cdots t_n}$ be defined on the Borel sets of $\prod_{i=1}^n M_{t_i}$ by

$$\nu_{t_1\cdots t_n}(\mathbf{E}) \equiv (\nu_{t_1} \times \cdots \times \nu_{t_n})(\mathbf{E}).$$

Then these measures satisfy the necessary consistency condition and so the Kolmogorov extension theorem given above can be applied to obtain a measure, P defined on a $\left(\prod_{t\in I} M_t, \mathcal{F}\right)$ and measurable functions $X_s : \prod_{t\in I} M_t \to M_s$ such that for F_{t_i} a Borel set in M_{t_i},

$$P\left((X_{t_1}, \cdots, X_{t_n}) \in \prod_{i=1}^n F_{t_i}\right) = \nu_{t_1\cdots t_n}(F_{t_1} \times \cdots \times F_{t_n})$$
$$= \nu_{t_1}(F_{t_1}) \cdots \nu_{t_n}(F_{t_n}). \tag{11.3}$$

In particular, $P(X_t \in F_t) = \nu_t(F_t)$. Then P in the resulting probability space,

$$\left(\prod_{t\in I} M_t, \mathcal{F}, P\right)$$

will be denoted as $\prod_{t\in I} \nu_t$. This proves the following theorem which describes an infinite product measure.

Theorem 11.12 *Let M_t for $t \in I$ be given as in Theorem 11.11 and let ν_t be a Borel probability measure defined on the Borel sets of M_t. Then there exists a measure, P and a σ algebra $\mathcal{F} \subseteq \mathcal{P}(\prod_t M_t)$ containing the algebra defined in Definition 11.8 such that $(\prod_t M_t, \mathcal{F}, P)$ is a probability space satisfying 11.3 whenever each F_{t_i} is a Borel set of M_{t_i}. This probability measure is sometimes denoted as $\prod_t \nu_t$.*

11.4 Exercises

1. Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ be two finite measure spaces. A subset of $X \times Y$ is called a measurable rectangle if it is of the form $A \times B$ where $A \in \mathcal{S}$ and $B \in \mathcal{F}$. A subset of $X \times Y$ is called an elementary set if it is a finite disjoint union of measurable rectangles. Denote this set of functions by $\mathcal{E}$. Show that $\mathcal{E}$ is an algebra of sets.

2. ↑For $A \in \sigma(\mathcal{E})$, the smallest σ algebra containing $\mathcal{E}$, show that $x \to \mathcal{X}_A(x, y)$ is μ measurable and that
$$y \to \int \mathcal{X}_A(x, y)\, d\mu$$
is λ measurable. Show similar assertions hold for $y \to \mathcal{X}_A(x, y)$ and
$$x \to \int \mathcal{X}_A(x, y)\, d\lambda$$
and that
$$\int \int \mathcal{X}_A(x, y)\, d\mu d\lambda = \int \int \mathcal{X}_A(x, y)\, d\lambda d\mu. \tag{11.4}$$
Hint: Let $\mathcal{M} \equiv \{A \in \sigma(\mathcal{E}) : 11.4 \text{ holds}\}$ along with all relevant measurability assertions. Show $\mathcal{M}$ contains $\mathcal{E}$ and is a monotone class. Then apply the Theorem 9.57.

3. ↑For $A \in \sigma(\mathcal{E})$ define $(\mu \times \lambda)(A) \equiv \int \int \mathcal{X}_A(x, y)\, d\mu d\lambda$. Show that $(\mu \times \lambda)$ is a measur on $\sigma(\mathcal{E})$ and that whenever $f \geq 0$ is measurable with respect to $\sigma(\mathcal{E})$,
$$\int_{X \times Y} f d(\mu \times \lambda) = \int \int f(x, y)\, d\mu d\lambda = \int \int f(x, y)\, d\lambda d\mu.$$
This is a common approach to Fubini's theorem.

4. ↑Generalize the above version of Fubini's theorem to the case where the measure spaces are only σ finite.

5. ↑Suppose now that μ and λ are both complete σ finite measures. Let $\overline{(\mu \times \lambda)}$ denote the completion of this measure. Let the larger measure space be $\left(X \times Y, \overline{\sigma(\mathcal{E})}, \overline{(\mu \times \lambda)}\right)$. Thus if $E \in \overline{\sigma(\mathcal{E})}$, it follows there exists a set $A \in \sigma(\mathcal{E})$ such that $E \cup N = A$ where $\overline{(\mu \times \lambda)}(N) = 0$. Now argue that for λ a.e. $y, x \to \mathcal{X}_N(x, y)$ is measurable because it is equal to zero μ a.e. and μ is complete. Therefore,
$$\int \int \mathcal{X}_N(x, y)\, d\mu d\lambda$$

makes sense and equals zero. Use to argue that for λ a.e. $y, x \rightarrow \mathcal{X}_E(x,y)$ is μ measurable and equals $\int \mathcal{X}_A(x,y)\, d\mu$. Then by completeness of $\lambda, y \rightarrow \int \mathcal{X}_E(x,y)\, d\mu$ is λ measurable and

$$\int \int \mathcal{X}_A(x,y)\, d\mu d\lambda = \int \int \mathcal{X}_E(x,y)\, d\mu d\lambda = \overline{(\mu \times \lambda)}(E).$$

Similarly

$$\int \int \mathcal{X}_E(x,y)\, d\lambda d\mu = \overline{(\mu \times \lambda)}(E).$$

Use this to give a generalization of the above Fubini theorem. Prove that if f is measurable with respect to the σ algebra, $\overline{\sigma(\mathcal{E})}$ and nonnegative, then

$$\int_{X \times Y} f d\overline{(\mu \times \lambda)} = \int \int f(x,y)\, d\mu d\lambda = \int \int f(x,y)\, d\lambda d\mu$$

where the iterated integrals make sense.

The L^p Spaces

12.1 Basic Inequalities And Properties

One of the main applications of the Lebesgue integral is to the study of various sorts of functions space. These are vector spaces whose elements are functions of various types. One of the most important examples of a function space is the space of measurable functions whose absolute values are p^{th} power integrable where $p \geq 1$. These spaces, referred to as L^p spaces, are very useful in applications. In the chapter $(\Omega, \mathcal{S}, \mu)$ will be a measure space.

Definition 12.1 *Let $1 \leq p < \infty$. Define*

$$L^p(\Omega) \equiv \{f \ : f \ \text{ is measurable and } \int_\Omega |f(\omega)|^p d\mu < \infty\}$$

In terms of the distribution function,

$$L^p(\Omega) = \{f \ : f \ \text{ is measurable and } \int_0^\infty pt^{p-1}\mu([|f| > t])\, dt < \infty\}$$

For each $p > 1$ define q by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Often one uses p' instead of q in this context.

$L^p(\Omega)$ is a vector space and has a norm. This is similar to the situation for $\mathbb{R}^n$ but the proof requires the following fundamental inequality. .

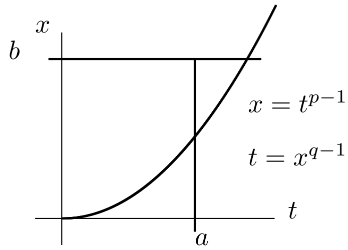
Theorem 12.2 *(Holder's inequality) If f and g are measurable functions, then if $p > 1$,*

$$\int |f|\,|g|\, d\mu \leq \left(\int |f|^p d\mu\right)^{\frac{1}{p}} \left(\int |g|^q d\mu\right)^{\frac{1}{q}}. \tag{12.1}$$

Proof: First here is a proof of Young's inequality .

Lemma 12.3 *If $p > 1$, and $0 \leq a, b$ then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.*

Proof: Consider the following picture:



From this picture, the sum of the area between the x axis and the curve added to the area between the t axis and the curve is at least as large as ab. Using beginning calculus, this is equivalent to the following inequality.

$$ab \leq \int_0^a t^{p-1}dt + \int_0^b x^{q-1}dx = \frac{a^p}{p} + \frac{b^q}{q}.$$

The above picture represents the situation which occurs when $p > 2$ because the graph of the function is concave up. If $2 \geq p > 1$ the graph would be concave down or a straight line. You should verify that the same argument holds in these cases just as well. In fact, the only thing which matters in the above inequality is that the function $x = t^{p-1}$ be strictly increasing.

Note equality occurs when $a^p = b^q$.

Here is an alternate proof.

Lemma 12.4 *For $a, b \geq 0$,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

and equality occurs when if and only if $a^p = b^q$.

Proof: If $b = 0$, the inequality is obvious. Fix $b > 0$ and consider

$$f(a) \equiv \frac{a^p}{p} + \frac{b^q}{q} - ab.$$

Then $f'(a) = a^{p-1} - b$. This is negative when $a < b^{1/(p-1)}$ and is positive when $a > b^{1/(p-1)}$. Therefore, f has a minimum when $a = b^{1/(p-1)}$. In other words, when $a^p = b^{p/(p-1)} = b^q$ since $1/p + 1/q = 1$. Thus the minimum value of f is

$$\frac{b^q}{p} + \frac{b^q}{q} - b^{1/(p-1)}b = b^q - b^q = 0.$$

It follows $f \geq 0$ and this yields the desired inequality.

Proof of Holder's inequality: If either $\int |f|^p d\mu$ or $\int |g|^p d\mu$ equals ∞, the inequality 12.1 is obviously valid because $\infty \geq$ anything. If either $\int |f|^p d\mu$ or

$\int |g|^p d\mu$ equals 0, then $f = 0$ a.e. or that $g = 0$ a.e. and so in this case the left side of the inequality equals 0 and so the inequality is therefore true. Therefore assume both $\int |f|^p d\mu$ and $\int |g|^p d\mu$ are less than ∞ and not equal to 0. Let

$$\left(\int |f|^p d\mu\right)^{1/p} = I(f)$$

and let $\left(\int |g|^p d\mu\right)^{1/q} = I(g)$. Then using the lemma,

$$\int \frac{|f|}{I(f)} \frac{|g|}{I(g)} d\mu \leq \frac{1}{p}\int \frac{|f|^p}{I(f)^p} d\mu + \frac{1}{q}\int \frac{|g|^q}{I(g)^q} d\mu = 1.$$

Hence,

$$\int |f| \, |g| \, d\mu \leq I(f) I(g) = \left(\int |f|^p d\mu\right)^{1/p} \left(\int |g|^q d\mu\right)^{1/q}.$$

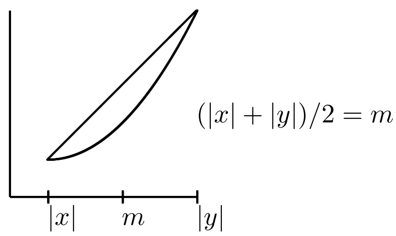
This proves Holder's inequality.

The following lemma will be needed.

Lemma 12.5 *Suppose $x, y \in \mathbb{C}$. Then*

$$|x + y|^p \leq 2^{p-1} \left(|x|^p + |y|^p\right).$$

Proof: The function $f(t) = t^p$ is concave up for $t \geq 0$ because $p > 1$. Therefore, the secant line joining two points on the graph of this function must lie above the graph of the function. This is illustrated in the following picture.



Now as shown above,

$$\left(\frac{|x| + |y|}{2}\right)^p \leq \frac{|x|^p + |y|^p}{2}$$

which implies

$$|x + y|^p \leq (|x| + |y|)^p \leq 2^{p-1} \left(|x|^p + |y|^p\right)$$

and this proves the lemma.

Note that if $y = \phi(x)$ is any function for which the graph of ϕ is concave up, you could get a similar inequality by the same argument.

Corollary 12.6 *(Minkowski inequality) Let* $1 \leq p < \infty$. *Then*

$$\left(\int |f+g|^p \, d\mu\right)^{1/p} \leq \left(\int |f|^p \, d\mu\right)^{1/p} + \left(\int |g|^p \, d\mu\right)^{1/p}. \tag{12.2}$$

Proof: If $p = 1$, this is obvious because it is just the triangle inequality. Let $p > 1$. Without loss of generality, assume

$$\left(\int |f|^p \, d\mu\right)^{1/p} + \left(\int |g|^p \, d\mu\right)^{1/p} < \infty$$

and $\left(\int |f+g|^p \, d\mu\right)^{1/p} \neq 0$ or there is nothing to prove. Therefore, using the above lemma,

$$\int |f+g|^p d\mu \leq 2^{p-1}\left(\int |f|^p + |g|^p d\mu\right) < \infty.$$

Now $|f(\omega) + g(\omega)|^p \leq |f(\omega) + g(\omega)|^{p-1}(|f(\omega)| + |g(\omega)|)$. Also, it follows from the definition of p and q that $p - 1 = \frac{p}{q}$. Therefore, using this and Holder's inequality,

$$\int |f+g|^p d\mu \leq$$

$$\begin{aligned}
& \int |f+g|^{p-1}|f|d\mu + \int |f+g|^{p-1}|g|d\mu \\
= & \int |f+g|^{\frac{p}{q}}|f|d\mu + \int |f+g|^{\frac{p}{q}}|g|d\mu \\
\leq & \left(\int |f+g|^p d\mu\right)^{\frac{1}{q}}\left(\int |f|^p d\mu\right)^{\frac{1}{p}} + \left(\int |f+g|^p d\mu\right)^{\frac{1}{q}}\left(\int |g|^p d\mu\right)^{\frac{1}{p}}.
\end{aligned}$$

Dividing both sides by $\left(\int |f+g|^p d\mu\right)^{\frac{1}{q}}$ yields 12.2. This proves the corollary.

The following follows immediately from the above.

Corollary 12.7 *Let* $f_i \in L^p(\Omega)$ *for* $i = 1, 2, \cdots, n$. *Then*

$$\left(\int \left|\sum_{i=1}^n f_i\right|^p d\mu\right)^{1/p} \leq \sum_{i=1}^n \left(\int |f_i|^p\right)^{1/p}.$$

This shows that if $f, g \in L^p$, then $f + g \in L^p$. Also, it is clear that if a is a constant and $f \in L^p$, then $af \in L^p$ because

$$\int |af|^p \, d\mu = |a|^p \int |f|^p \, d\mu < \infty.$$

Thus L^p is a vector space and

a.) $\left(\int |f|^p \, d\mu\right)^{1/p} \geq 0$, $\left(\int |f|^p \, d\mu\right)^{1/p} = 0$ if and only if $f = 0$ a.e.

b.) $\left(\int |af|^p \, d\mu\right)^{1/p} = |a| \left(\int |f|^p \, d\mu\right)^{1/p}$ if a is a scalar.

c.) $\left(\int |f+g|^p \, d\mu\right)^{1/p} \leq \left(\int |f|^p \, d\mu\right)^{1/p} + \left(\int |g|^p \, d\mu\right)^{1/p}$.

$f \to \left(\int |f|^p \, d\mu\right)^{1/p}$ would define a norm if $\left(\int |f|^p \, d\mu\right)^{1/p} = 0$ implied $f = 0$. Unfortunately, this is not so because if $f = 0$ a.e. but is nonzero on a set of measure zero, $\left(\int |f|^p \, d\mu\right)^{1/p} = 0$ and this is not allowed. However, all the other properties of a norm are available and so a little thing like a set of measure zero will not prevent the consideration of L^p as a normed vector space if two functions in L^p which differ only on a set of measure zero are considered the same. That is, an element of L^p is really an equivalence class of functions where two functions are equivalent if they are equal a.e. With this convention, here is a definition.

Definition 12.8 *Let $f \in L^p(\Omega)$. Define*

$$||f||_p \equiv ||f||_{L^p} \equiv \left(\int |f|^p \, d\mu\right)^{1/p}.$$

Then with this definition and using the convention that elements in L^p are considered to be the same if they differ only on a set of measure zero, $|| \, ||_p$ is a norm on $L^p(\Omega)$ because if $||f||_p = 0$ then $f = 0$ a.e. and so f is considered to be the zero function because it differs from 0 only on a set of measure zero.

The following is an important definition.

Definition 12.9 *A complete normed linear space is called a Banach*[1] *space.*

L^p is a Banach space. This is the next big theorem.

Theorem 12.10 *The following hold for $L^p(\Omega)$*

a.) $L^p(\Omega)$ is complete.

b.) If $\{f_n\}$ is a Cauchy sequence in $L^p(\Omega)$, then there exists $f \in L^p(\Omega)$ and a subsequence which converges a.e. to $f \in L^p(\Omega)$, and $||f_n - f||_p \to 0$.

Proof: Let $\{f_n\}$ be a Cauchy sequence in $L^p(\Omega)$. This means that for every $\varepsilon > 0$ there exists N such that if $n, m \geq N$, then $||f_n - f_m||_p < \varepsilon$. Now select a subsequence as follows. Let n_1 be such that $||f_n - f_m||_p < 2^{-1}$ whenever $n, m \geq n_1$.

[1] These spaces are named after Stefan Banach, 1892-1945. Banach spaces are the basic item of study in the subject of functional analysis and will be considered later in this book.

There is a recent biography of Banach, R. Katuża, *The Life of Stefan Banach,* (A. Kostant and W. Woyczyński, translators and editors) Birkhauser, Boston (1996). More information on Banach can also be found in a recent short article written by Douglas Henderson who is in the department of chemistry and biochemistry at BYU.

Banach was born in Austria, worked in Poland and died in the Ukraine but never moved. This is because borders kept changing. There is a rumor that he died in a German concentration camp which is apparently not true. It seems he died after the war of lung cancer.

He was an interesting character. He hated taking examinations so much that he did not receive his undergraduate university degree. Nevertheless, he did become a professor of mathematics due to his important research. He and some friends would meet in a cafe called the Scottish cafe where they wrote on the marble table tops until Banach's wife supplied them with a notebook which became the "Scotish notebook" and was eventually published.

Let n_2 be such that $n_2 > n_1$ and $||f_n - f_m||_p < 2^{-2}$ whenever $n, m \geq n_2$. If $n_1, \cdots, n_k$ have been chosen, let $n_{k+1} > n_k$ and whenever $n, m \geq n_{k+1}$, $||f_n - f_m||_p < 2^{-(k+1)}$. The subsequence just mentioned is $\{f_{n_k}\}$. Thus, $||f_{n_k} - f_{n_{k+1}}||_p < 2^{-k}$. Let

$$g_{k+1} = f_{n_{k+1}} - f_{n_k}.$$

Then by the corollary to Minkowski's inequality,

$$\infty > \sum_{k=1}^{\infty} ||g_{k+1}||_p \geq \sum_{k=1}^{m} ||g_{k+1}||_p \geq \left|\left|\sum_{k=1}^{m} |g_{k+1}|\right|\right|_p$$

for all m. It follows that

$$\int \left(\sum_{k=1}^{m} |g_{k+1}|\right)^p d\mu \leq \left(\sum_{k=1}^{\infty} ||g_{k+1}||_p\right)^p < \infty \tag{12.3}$$

for all m and so the monotone convergence theorem implies that the sum up to m in 12.3 can be replaced by a sum up to ∞. Thus,

$$\int \left(\sum_{k=1}^{\infty} |g_{k+1}|\right)^p d\mu < \infty$$

which requires

$$\sum_{k=1}^{\infty} |g_{k+1}(x)| < \infty \text{ a.e. } x.$$

Therefore, $\sum_{k=1}^{\infty} g_{k+1}(x)$ converges for a.e. x because the functions have values in a complete space, $\mathbb{C}$, and this shows the partial sums form a Cauchy sequence. Now let x be such that this sum is finite. Then define

$$f(x) \equiv f_{n_1}(x) + \sum_{k=1}^{\infty} g_{k+1}(x) = \lim_{m\to\infty} f_{n_m}(x)$$

since $\sum_{k=1}^{m} g_{k+1}(x) = f_{n_{m+1}}(x) - f_{n_1}(x)$. Therefore there exists a set, E having measure zero such that

$$\lim_{k\to\infty} f_{n_k}(x) = f(x)$$

for all $x \notin E$. Redefine f_{n_k} to equal 0 on E and let $f(x) = 0$ for $x \in E$. It then follows that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for all x. By Fatou's lemma, and the Minkowski inequality,

$$||f - f_{n_k}||_p = \left(\int |f - f_{n_k}|^p \, d\mu\right)^{1/p} \leq$$

$$\liminf_{m\to\infty} \left(\int |f_{n_m} - f_{n_k}|^p \, d\mu\right)^{1/p} = \liminf_{m\to\infty} ||f_{n_m} - f_{n_k}||_p \leq$$

$$\lim\inf_{m\to\infty}\sum_{j=k}^{m-1}\left|\left|f_{n_{j+1}}-f_{n_j}\right|\right|_p\leq\sum_{i=k}^{\infty}\left|\left|f_{n_{i+1}}-f_{n_i}\right|\right|_p\leq 2^{-(k-1)}. \tag{12.4}$$

Therefore, $f\in L^p(\Omega)$ because

$$||f||_p\leq||f-f_{n_k}||_p+||f_{n_k}||_p<\infty,$$

and $\lim_{k\to\infty}||f_{n_k}-f||_p=0$. This proves b.).

This has shown f_{n_k} converges to f in $L^p(\Omega)$. It follows the original Cauchy sequence also converges to f in $L^p(\Omega)$. This is a general fact that if a subsequence of a Cauchy sequence converges, then so does the original Cauchy sequence. You should give a proof of this. This proves the theorem.

In working with the L^p spaces, the following inequality also known as Minkowski's inequality is very useful. It is similar to the Minkowski inequality for sums. To see this, replace the integral, $\int_X$ with a finite summation sign and you will see the usual Minkowski inequality or rather the version of it given in Corollary 12.7.

To prove this theorem first consider a special case of it in which technical considerations which shed no light on the proof are excluded.

Lemma 12.11 *Let $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{F},\lambda)$ be finite complete measure spaces and let f be $\overline{\mu\times\lambda}$ measurable and uniformly bounded. Then the following inequality is valid for $p\geq 1$.*

$$\int_X\left(\int_Y|f(x,y)|^p\,d\lambda\right)^{\frac{1}{p}}d\mu\geq\left(\int_Y(\int_X|f(x,y)|\,d\mu)^pd\lambda\right)^{\frac{1}{p}}. \tag{12.5}$$

Proof: Since f is bounded and $\mu(X),\lambda(X)<\infty$,

$$\left(\int_Y(\int_X|f(x,y)|d\mu)^pd\lambda\right)^{\frac{1}{p}}<\infty.$$

Let

$$J(y)=\int_X|f(x,y)|d\mu.$$

Note there is no problem in writing this for a.e. y because f is product measurable. Then by Fubini's theorem,

$$\begin{aligned}\int_Y\left(\int_X|f(x,y)|d\mu\right)^p d\lambda &= \int_Y J(y)^{p-1}\int_X|f(x,y)|d\mu\,d\lambda\\ &= \int_X\int_Y J(y)^{p-1}|f(x,y)|d\lambda\,d\mu\end{aligned}$$

Now apply Holder's inequality in the last integral above and recall $p-1=\frac{p}{q}$. This yields

$$\begin{aligned}
&\int_Y \left(\int_X |f(x,y)|d\mu\right)^p d\lambda \\
\leq\ & \int_X \left(\int_Y J(y)^p d\lambda\right)^{\frac{1}{q}} \left(\int_Y |f(x,y)|^p d\lambda\right)^{\frac{1}{p}} d\mu \\
=\ & \left(\int_Y J(y)^p d\lambda\right)^{\frac{1}{q}} \int_X \left(\int_Y |f(x,y)|^p d\lambda\right)^{\frac{1}{p}} d\mu
\end{aligned}$$

$$= \left(\int_Y (\int_X |f(x,y)|d\mu)^p d\lambda\right)^{\frac{1}{q}} \int_X \left(\int_Y |f(x,y)|^p d\lambda\right)^{\frac{1}{p}} d\mu. \tag{12.6}$$

Therefore, dividing both sides by the first factor in the above expression,

$$\left(\int_Y \left(\int_X |f(x,y)|d\mu\right)^p d\lambda\right)^{\frac{1}{p}} \leq \int_X \left(\int_Y |f(x,y)|^p d\lambda\right)^{\frac{1}{p}} d\mu. \tag{12.7}$$

Note that 12.7 holds even if the first factor of 12.6 equals zero. This proves the lemma.

Now consider the case where f is not assumed to be bounded and where the measure spaces are σ finite.

Theorem 12.12 *Let $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{F},\lambda)$ be σ-finite measure spaces and let f be product measurable. Then the following inequality is valid for $p \geq 1$.*

$$\int_X \left(\int_Y |f(x,y)|^p \, d\lambda\right)^{\frac{1}{p}} d\mu \geq \left(\int_Y (\int_X |f(x,y)| \, d\mu)^p d\lambda\right)^{\frac{1}{p}}. \tag{12.8}$$

Proof: Since the two measure spaces are σ finite, there exist measurable sets, X_m and Y_k such that $X_m \subseteq X_{m+1}$ for all m, $Y_k \subseteq Y_{k+1}$ for all k, and $\mu(X_m), \lambda(Y_k) < \infty$. Now define

$$f_n(x,y) \equiv \begin{cases} f(x,y) \text{ if } |f(x,y)| \leq n \\ n \text{ if } |f(x,y)| > n. \end{cases}$$

Thus f_n is uniformly bounded and product measurable. By the above lemma,

$$\int_{X_m} \left(\int_{Y_k} |f_n(x,y)|^p \, d\lambda\right)^{\frac{1}{p}} d\mu \geq \left(\int_{Y_k} (\int_{X_m} |f_n(x,y)| \, d\mu)^p d\lambda\right)^{\frac{1}{p}}. \tag{12.9}$$

Now observe that $|f_n(x,y)|$ increases in n and the pointwise limit is $|f(x,y)|$. Therefore, using the monotone convergence theorem in 12.9 yields the same inequality with f replacing f_n. Next let $k \to \infty$ and use the monotone convergence theorem again to replace Y_k with Y. Finally let $m \to \infty$ in what is left to obtain 12.8. This proves the theorem.

Note that the proof of this theorem depends on two manipulations, the interchange of the order of integration and Holder's inequality. Note that there is nothing to check in the case of double sums. Thus if $a_{ij} \geq 0$, it is always the case that

$$\left(\sum_j \left(\sum_i a_{ij}\right)^p\right)^{1/p} \leq \sum_i \left(\sum_j a_{ij}^p\right)^{1/p}$$

because the integrals in this case are just sums and $(i, j) \to a_{ij}$ is measurable.

The L^p spaces have many important properties.

12.2 Density Considerations

Theorem 12.13 *Let $p \geq 1$ and let $(\Omega, \mathcal{S}, \mu)$ be a measure space. Then the simple functions are dense in $L^p(\Omega)$.*

Proof: Recall that a function, f, having values in $\mathbb{R}$ can be written in the form $f = f^+ - f^-$ where

$$f^+ = \max(0, f), \; f^- = \max(0, -f).$$

Therefore, an arbitrary complex valued function, f is of the form

$$f = \operatorname{Re} f^+ - \operatorname{Re} f^- + i\left(\operatorname{Im} f^+ - \operatorname{Im} f^-\right).$$

If each of these nonnegative functions is approximated by a simple function, it follows f is also approximated by a simple function. Therefore, there is no loss of generality in assuming at the outset that $f \geq 0$.

Since f is measurable, Theorem 8.27 implies there is an increasing sequence of simple functions, $\{s_n\}$, converging pointwise to $f(x)$. Now

$$|f(x) - s_n(x)| \leq |f(x)|.$$

By the Dominated Convergence theorem,

$$0 = \lim_{n\to\infty} \int |f(x) - s_n(x)|^p d\mu.$$

Thus simple functions are dense in L^p.

Recall that for Ω a topological space, $C_c(\Omega)$ is the space of continuous functions with compact support in Ω. Also recall the following definition.

Definition 12.14 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and suppose (Ω, τ) is also a topological space. Then $(\Omega, \mathcal{S}, \mu)$ is called a regular measure space if the σ algebra of Borel sets is contained in $\mathcal{S}$ and for all $E \in \mathcal{S}$,*

$$\mu(E) = \inf\{\mu(V) : V \supseteq E \text{ and } V \text{ open}\}$$

and if $\mu(E) < \infty$,

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact }\}$$

and $\mu(K) < \infty$ *for any compact set,* K.

For example Lebesgue measure is an example of such a measure.

Lemma 12.15 *Let* Ω *be a metric space in which the closed balls are compact and let* K *be a compact subset of* V*, an open set. Then there exists a continuous function* $f : \Omega \to [0,1]$ *such that* $f(x) = 1$ *for all* $x \in K$ *and* $\operatorname{spt}(f)$ *is a compact subset of* V*. That is,* $K \prec f \prec V$.

Proof: Let $K \subseteq W \subseteq \overline{W} \subseteq V$ and $\overline{W}$ is compact. To obtain this list of inclusions consider a point in K, x, and take $B(x, r_x)$ a ball containing x such that $\overline{B(x, r_x)}$ is a compact subset of V. Next use the fact that K is compact to obtain the existence of a list, $\{B(x_i, r_{x_i}/2)\}_{i=1}^m$ which covers K. Then let

$$W \equiv \cup_{i=1}^m B\left(x_i, \frac{r_{x_i}}{2}\right).$$

It follows since this is a finite union that

$$\overline{W} = \cup_{i=1}^m \overline{B\left(x_i, \frac{r_{x_i}}{2}\right)}$$

and so $\overline{W}$, being a finite union of compact sets is itself a compact set. Also, from the construction

$$\overline{W} \subseteq \cup_{i=1}^m B(x_i, r_{x_i}).$$

Define f by

$$f(x) = \frac{\operatorname{dist}(x, W^C)}{\operatorname{dist}(x, K) + \operatorname{dist}(x, W^C)}.$$

It is clear that f is continuous if the denominator is always nonzero. But this is clear because if $x \in W^C$ there must be a ball $B(x, r)$ such that this ball does not intersect K. Otherwise, x would be a limit point of K and since K is closed, $x \in K$. However, $x \notin K$ because $K \subseteq W$.

It is not necessary to be in a metric space to do this. You can accomplish the same thing using Urysohn's lemma.

Theorem 12.16 *Let* $(\Omega, \mathcal{S}, \mu)$ *be a regular measure space as in Definition 12.14 where the conclusion of Lemma 12.15 holds. Then* $C_c(\Omega)$ *is dense in* $L^p(\Omega)$.

Proof: First consider a measurable set, E where $\mu(E) < \infty$. Let $K \subseteq E \subseteq V$ where $\mu(V \setminus K) < \varepsilon$. Now let $K \prec h \prec V$. Then

$$\int |h - \mathcal{X}_E|^p \, d\mu \leq \int \mathcal{X}_{V \setminus K}^p d\mu = \mu(V \setminus K) < \varepsilon.$$

It follows that for each s a simple function in $L^p(\Omega)$, there exists $h \in C_c(\Omega)$ such that $||s-h||_p < \varepsilon$. This is because if

$$s(x) = \sum_{i=1}^{m} c_i \mathcal{X}_{E_i}(x)$$

is a simple function in L^p where the c_i are the distinct nonzero values of s each $\mu(E_i) < \infty$ since otherwise $s \notin L^p$ due to the inequality

$$\int |s|^p \, d\mu \geq |c_i|^p \mu(E_i).$$

By Theorem 12.13, simple functions are dense in $L^p(\Omega)$,and so this proves the Theorem.

12.3 Separability

Theorem 12.17 *For $p \geq 1$ and μ a Radon measure, $L^p(\mathbb{R}^n, \mu)$ is separable. Recall this means there exists a countable set, $\mathcal{D}$, such that if $f \in L^p(\mathbb{R}^n, \mu)$ and $\varepsilon > 0$, there exists $g \in \mathcal{D}$ such that $||f-g||_p < \varepsilon$.*

Proof: Let Q be all functions of the form $c\mathcal{X}_{[\mathbf{a},\mathbf{b})}$ where

$$[\mathbf{a}, \mathbf{b}) \equiv [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n),$$

and both a_i, b_i are rational, while c has rational real and imaginary parts. Let $\mathcal{D}$ be the set of all finite sums of functions in Q. Thus, $\mathcal{D}$ is countable. In fact $\mathcal{D}$ is dense in $L^p(\mathbb{R}^n, \mu)$. To prove this it is necessary to show that for every $f \in L^p(\mathbb{R}^n, \mu)$, there exists an element of $\mathcal{D}$, s such that $||s-f||_p < \varepsilon$. If it can be shown that for every $g \in C_c(\mathbb{R}^n)$ there exists $h \in \mathcal{D}$ such that $||g-h||_p < \varepsilon$, then this will suffice because if $f \in L^p(\mathbb{R}^n)$ is arbitrary, Theorem 12.16 implies there exists $g \in C_c(\mathbb{R}^n)$ such that $||f-g||_p \leq \frac{\varepsilon}{2}$ and then there would exist $h \in C_c(\mathbb{R}^n)$ such that $||h-g||_p < \frac{\varepsilon}{2}$. By the triangle inequality,

$$||f-h||_p \leq ||h-g||_p + ||g-f||_p < \varepsilon.$$

Therefore, assume at the outset that $f \in C_c(\mathbb{R}^n)$.

Let $\mathcal{P}_m$ consist of all sets of the form $[\mathbf{a}, \mathbf{b}) \equiv \prod_{i=1}^{n} [a_i, b_i)$ where $a_i = j2^{-m}$and $b_i = (j+1)2^{-m}$ for j an integer. Thus $\mathcal{P}_m$ consists of a tiling of $\mathbb{R}^n$ into half open rectangles having diameters $2^{-m}n^{\frac{1}{2}}$. There are countably many of these rectangles; so, let $\mathcal{P}_m = \{[\mathbf{a}_i, \mathbf{b}_i)\}_{i=1}^{\infty}$ and $\mathbb{R}^n = \cup_{i=1}^{\infty}[\mathbf{a}_i, \mathbf{b}_i)$. Let c_i^mbe complex numbers with rational real and imaginary parts satisfying

$$|f(\mathbf{a}_i) - c_i^m| < 2^{-m},$$

$$|c_i^m| \leq |f(\mathbf{a}_i)|. \tag{12.10}$$

Let

$$s_m(\mathbf{x}) = \sum_{i=1}^{\infty} c_i^m \mathcal{X}_{[\mathbf{a}_i,\mathbf{b}_i)}(\mathbf{x}).$$

Since $f(\mathbf{a}_i) = 0$ except for finitely many values of i, the above is a finite sum. Then 12.10 implies $s_m \in \mathcal{D}$. If s_m converges uniformly to f then it follows $||s_m - f||_p \to 0$ because $|s_m| \leq |f|$ and so

$$\begin{aligned} ||s_m - f||_p &= \left(\int |s_m - f|^p \, d\mu\right)^{1/p} \\ &= \left(\int_{\text{spt}(f)} |s_m - f|^p \, d\mu\right)^{1/p} \\ &\leq [\varepsilon m_n (\text{spt}(f))]^{1/p} \end{aligned}$$

whenever m is large enough.

Since $f \in C_c(\mathbb{R}^n)$ it follows that f is uniformly continuous and so given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon/2$. Now let m be large enough that every box in $\mathcal{P}_m$ has diameter less than δ and also that $2^{-m} < \varepsilon/2$. Then if $[\mathbf{a}_i, \mathbf{b}_i)$ is one of these boxes of $\mathcal{P}_m$, and $\mathbf{x} \in [\mathbf{a}_i, \mathbf{b}_i)$,

$$|f(\mathbf{x}) - f(\mathbf{a}_i)| < \varepsilon/2$$

and

$$|f(\mathbf{a}_i) - c_i^m| < 2^{-m} < \varepsilon/2.$$

Therefore, using the triangle inequality, it follows that

$$|f(\mathbf{x}) - c_i^m| = |s_m(\mathbf{x}) - f(\mathbf{x})| < \varepsilon$$

and since $\mathbf{x}$ is arbitrary, this establishes uniform convergence. This proves the theorem.

Here is an easier proof if you know the Weierstrass approximation theorem.

Theorem 12.18 *For $p \geq 1$ and μ a Radon measure, $L^p(\mathbb{R}^n, \mu)$ is separable. Recall this means there exists a countable set, $\mathcal{D}$, such that if $f \in L^p(\mathbb{R}^n, \mu)$ and $\varepsilon > 0$, there exists $g \in \mathcal{D}$ such that $||f - g||_p < \varepsilon$.*

Proof: Let $\mathcal{P}$ denote the set of all polynomials which have rational coefficients. Then $\mathcal{P}$ is countable. Let $\tau_k \in C_c\left((-(k+1), (k+1))^n\right)$ such that $\overline{[-k,k]^n} \prec \tau_k \prec (-(k+1), (k+1))^n$. Let $\mathcal{D}_k$ denote the functions which are of the form, $p\tau_k$ where $p \in \mathcal{P}$. Thus $\mathcal{D}_k$ is also countable. Let $\mathcal{D} \equiv \cup_{k=1}^{\infty} \mathcal{D}_k$. It follows each function in $\mathcal{D}$ is in $C_c(\mathbb{R}^n)$ and so it in $L^p(\mathbb{R}^n, \mu)$. Let $f \in L^p(\mathbb{R}^n, \mu)$. By regularity of μ there exists $g \in C_c(\mathbb{R}^n)$ such that $||f - g||_{L^p(\mathbb{R}^n,\mu)} < \frac{\varepsilon}{3}$. Let k be such that $\text{spt}(g) \subseteq (-k, k)^n$. Now by the Weierstrass approximation theorem there exists a polynomial q such that

$$\begin{aligned} ||g - q||_{[-(k+1),k+1]^n} &\equiv \sup\{|g(\mathbf{x}) - q(\mathbf{x})| : \mathbf{x} \in [-(k+1), (k+1)]^n\} \\ &< \frac{\varepsilon}{3\mu\left((-(k+1), k+1)^n\right)}. \end{aligned}$$

It follows

$$\begin{aligned} ||g-\tau_k q||_{[-(k+1),k+1]^n} &= ||\tau_k g-\tau_k q||_{[-(k+1),k+1]^n} \\ &< \frac{\varepsilon}{3\mu\left(\left(-\left(k+1\right),k+1\right)^n\right)}. \end{aligned}$$

Without loss of generality, it can be assumed this polynomial has all rational coefficients. Therefore, $\tau_k q \in \mathcal{D}$.

$$\begin{aligned} ||g-\tau_k q||_{L^p(\mathbb{R}^n)}^p &= \int_{(-(k+1),k+1)^n} |g(\mathbf{x})-\tau_k(\mathbf{x})\,q(\mathbf{x})|^p \, d\mu \\ &\leq \left(\frac{\varepsilon}{3\mu\left(\left(-\left(k+1\right),k+1\right)^n\right)}\right)^p \mu\left(\left(-\left(k+1\right),k+1\right)^n\right) \\ &< \left(\frac{\varepsilon}{3}\right)^p. \end{aligned}$$

It follows

$$||f-\tau_k q||_{L^p(\mathbb{R}^n,\mu)} \leq ||f-g||_{L^p(\mathbb{R}^n,\mu)} + ||g-\tau_k q||_{L^p(\mathbb{R}^n,\mu)}^p < \frac{\varepsilon}{3}+\frac{\varepsilon}{3} < \varepsilon.$$

This proves the theorem.

Corollary 12.19 *Let Ω be any μ measurable subset of $\mathbb{R}^n$ and let μ be a Radon measure. Then $L^p(\Omega,\mu)$ is separable. Here the σ algebra of measurable sets will consist of all intersections of measurable sets with Ω and the measure will be μ restricted to these sets.*

Proof: Let $\widetilde{\mathcal{D}}$ be the restrictions of $\mathcal{D}$ to Ω. If $f \in L^p(\Omega)$, let F be the zero extension of f to all of $\mathbb{R}^n$. Let $\varepsilon > 0$ be given. By Theorem 12.17 or 12.18 there exists $s \in \mathcal{D}$ such that $||F-s||_p < \varepsilon$. Thus

$$||s-f||_{L^p(\Omega,\mu)} \leq ||s-F||_{L^p(\mathbb{R}^n,\mu)} < \varepsilon$$

and so the countable set $\widetilde{\mathcal{D}}$ is dense in $L^p(\Omega)$.

12.4 Continuity Of Translation

Definition 12.20 *Let f be a function defined on $U \subseteq \mathbb{R}^n$ and let $\mathbf{w} \in \mathbb{R}^n$. Then $f_{\mathbf{w}}$ will be the function defined on $\mathbf{w}+U$ by*

$$f_{\mathbf{w}}(\mathbf{x}) = f(\mathbf{x}-\mathbf{w}).$$

Theorem 12.21 *(Continuity of translation in L^p) Let $f \in L^p(\mathbb{R}^n)$ with the measure being Lebesgue measure. Then*

$$\lim_{||\mathbf{w}||\to 0} ||f_{\mathbf{w}}-f||_p = 0.$$

Proof: Let $\varepsilon > 0$ be given and let $g \in C_c(\mathbb{R}^n)$ with $||g - f||_p < \frac{\varepsilon}{3}$. Since Lebesgue measure is translation invariant ($m_n(\mathbf{w} + E) = m_n(E)$),

$$||g_{\mathbf{w}} - f_{\mathbf{w}}||_p = ||g - f||_p < \frac{\varepsilon}{3}.$$

You can see this from looking at simple functions and passing to the limit or you could use the change of variables formula to verify it.

Therefore

$$\begin{aligned} ||f - f_{\mathbf{w}}||_p &\leq ||f - g||_p + ||g - g_{\mathbf{w}}||_p + ||g_{\mathbf{w}} - f_{\mathbf{w}}|| \\ &< \frac{2\varepsilon}{3} + ||g - g_{\mathbf{w}}||_p. \end{aligned} \tag{12.11}$$

But $\lim_{|\mathbf{w}|\to 0} g_{\mathbf{w}}(\mathbf{x}) = g(\mathbf{x})$ uniformly in $\mathbf{x}$ because g is uniformly continuous. Now let B be a large ball containing spt (g) and let δ_1 be small enough that $B(\mathbf{x}, \delta) \subseteq B$ whenever $\mathbf{x} \in$ spt (g). If $\varepsilon > 0$ is given there exists $\delta < \delta_1$ such that if $|\mathbf{w}| < \delta$, it follows that $|g(\mathbf{x} - \mathbf{w}) - g(\mathbf{x})| < \varepsilon/3\left(1 + m_n(B)^{1/p}\right)$. Therefore,

$$\begin{aligned} ||g - g_{\mathbf{w}}||_p &= \left(\int_B |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{w})|^p \, dm_n\right)^{1/p} \\ &\leq \varepsilon \frac{m_n(B)^{1/p}}{3\left(1 + m_n(B)^{1/p}\right)} < \frac{\varepsilon}{3}. \end{aligned}$$

Therefore, whenever $|\mathbf{w}| < \delta$, it follows $||g - g_{\mathbf{w}}||_p < \frac{\varepsilon}{3}$ and so from 12.11 $||f - f_{\mathbf{w}}||_p < \varepsilon$. This proves the theorem.

Part of the argument of this theorem is significant enough to be stated as a corollary.

Corollary 12.22 *Suppose $g \in C_c(\mathbb{R}^n)$ and μ is a Radon measure on $\mathbb{R}^n$. Then*

$$\lim_{\mathbf{w}\to\mathbf{0}} ||g - g_{\mathbf{w}}||_p = 0.$$

Proof: The proof of this follows from the last part of the above argument simply replacing m_n with μ. Translation invariance of the measure is not needed to draw this conclusion because of uniform continuity of g.

12.5 Mollifiers And Density Of Smooth Functions

Definition 12.23 *Let U be an open subset of $\mathbb{R}^n$. $C_c^\infty(U)$ is the vector space of all infinitely differentiable functions which equal zero for all $\mathbf{x}$ outside of some compact set contained in U. Similarly, $C_c^m(U)$ is the vector space of all functions which are m times continuously differentiable and whose support is a compact subset of U.*

Example 12.24 *Let* $U = B(\mathbf{z}, 2r)$

$$\psi(\mathbf{x}) = \begin{cases} \exp\left[\left(|\mathbf{x}-\mathbf{z}|^2 - r^2\right)^{-1}\right] \text{ if } |\mathbf{x}-\mathbf{z}| < r, \\ 0 \text{ if } |\mathbf{x}-\mathbf{z}| \geq r. \end{cases}$$

Then a little work shows $\psi \in C_c^\infty(U)$. *The following also is easily obtained.*

Lemma 12.25 *Let* U *be any open set. Then* $C_c^\infty(U) \neq \emptyset$.

Proof: Pick $\mathbf{z} \in U$ and let r be small enough that $B(\mathbf{z}, 2r) \subseteq U$. Then let $\psi \in C_c^\infty(B(\mathbf{z}, 2r)) \subseteq C_c^\infty(U)$ be the function of the above example.

Definition 12.26 *Let* $U = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\}$. *A sequence* $\{\psi_m\} \subseteq C_c^\infty(U)$ *is called a mollifier (sometimes an approximate identity) if*

$$\psi_m(\mathbf{x}) \geq 0,\ \psi_m(\mathbf{x}) = 0,\ \text{if } |\mathbf{x}| \geq \frac{1}{m},$$

and $\int \psi_m(\mathbf{x}) = 1$. *Sometimes it may be written as* $\{\psi_\varepsilon\}$ *where* ψ_ε *satisfies the above conditions except* $\psi_\varepsilon(\mathbf{x}) = 0$ *if* $|\mathbf{x}| \geq \varepsilon$. *In other words,* ε *takes the place of* $1/m$ *and in everything that follows* $\varepsilon \to 0$ *instead of* $m \to \infty$.

As before, $\int f(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y})$ will mean $\mathbf{x}$ is fixed and the function $\mathbf{y} \to f(\mathbf{x}, \mathbf{y})$ is being integrated. To make the notation more familiar, dx is written instead of $dm_n(x)$.

Example 12.27 *Let*

$$\psi \in C_c^\infty(B(0,1))\ \ (B(0,1) = \{\mathbf{x} : |\mathbf{x}| < 1\})$$

with $\psi(\mathbf{x}) \geq 0$ *and* $\int \psi dm = 1$. *Let* $\psi_m(\mathbf{x}) = c_m \psi(m\mathbf{x})$ *where* c_m *is chosen in such a way that* $\int \psi_m dm = 1$. *By the change of variables theorem* $c_m = m^n$.

Definition 12.28 *A function,* f, *is said to be in* $L^1_{loc}(\mathbb{R}^n, \mu)$ *if* f *is* μ *measurable and if* $|f|\mathcal{X}_K \in L^1(\mathbb{R}^n, \mu)$ *for every compact set,* K. *Here* μ *is a Radon measure on* $\mathbb{R}^n$. *Usually* $\mu = m_n$, *Lebesgue measure. When this is so, write* $L^1_{loc}(\mathbb{R}^n)$ *or* $L^p(\mathbb{R}^n)$, *etc. If* $f \in L^1_{loc}(\mathbb{R}^n, \mu)$, *and* $g \in C_c(\mathbb{R}^n)$,

$$f * g(\mathbf{x}) \equiv \int f(\mathbf{y}) g(\mathbf{x}-\mathbf{y}) d\mu.$$

The following lemma will be useful in what follows. It says that one of these very unregular functions in $L^1_{loc}(\mathbb{R}^n, \mu)$ is smoothed out by convolving with a mollifier.

Lemma 12.29 *Let* $f \in L^1_{loc}(\mathbb{R}^n, \mu)$, *and* $g \in C_c^\infty(\mathbb{R}^n)$. *Then* $f * g$ *is an infinitely differentiable function. Here* μ *is a Radon measure on* $\mathbb{R}^n$.

Proof: Consider the difference quotient for calculating a partial derivative of $f * g$.

$$\frac{f * g(\mathbf{x}+t\mathbf{e}_j) - f * g(\mathbf{x})}{t} = \int f(\mathbf{y}) \frac{g(\mathbf{x}+t\mathbf{e}_j - \mathbf{y}) - g(\mathbf{x}-\mathbf{y})}{t} d\mu(y).$$

Using the fact that $g \in C_c^\infty(\mathbb{R}^n)$, the quotient,

$$\frac{g(\mathbf{x}+t\mathbf{e}_j - \mathbf{y}) - g(\mathbf{x}-\mathbf{y})}{t},$$

is uniformly bounded. To see this easily, use Theorem 5.53 on Page 119 to get the existence of a constant, M depending on

$$\max\{||Dg(\mathbf{x})|| : \mathbf{x} \in \mathbb{R}^n\}$$

such that

$$|g(\mathbf{x}+t\mathbf{e}_j - \mathbf{y}) - g(\mathbf{x}-\mathbf{y})| \leq M|t|$$

for any choice of $\mathbf{x}$ and $\mathbf{y}$. Therefore, there exists a dominating function for the integrand of the above integral which is of the form $C|f(\mathbf{y})|\mathcal{X}_K$ where K is a compact set containing the support of g. It follows the limit of the difference quotient above passes inside the integral as $t \to 0$ and

$$\frac{\partial}{\partial x_j}(f * g)(\mathbf{x}) = \int f(\mathbf{y}) \frac{\partial}{\partial x_j} g(\mathbf{x}-\mathbf{y}) d\mu(y).$$

Now letting $\frac{\partial}{\partial x_j} g$ play the role of g in the above argument, partial derivatives of all orders exist. This proves the lemma.

Theorem 12.30 *Let K be a compact subset of an open set, U. Then there exists a function, $h \in C_c^\infty(U)$, such that $h(\mathbf{x}) = 1$ for all $\mathbf{x} \in K$ and $h(\mathbf{x}) \in [0,1]$ for all $\mathbf{x}$.*

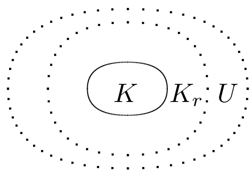
Proof: Let $r > 0$ be small enough that $K + B(\mathbf{0}, 3r) \subseteq U$. The symbol, $K + B(\mathbf{0}, 3r)$ means

$$\{\mathbf{k}+\mathbf{x} : \mathbf{k} \in K \text{ and } \mathbf{x} \in B(\mathbf{0}, 3r)\}.$$

Thus this is simply a way to write

$$\cup\{B(\mathbf{k}, 3r) : \mathbf{k} \in K\}.$$

Think of it as fattening up the set, K. Let $K_r = K + B(0, r)$. A picture of what is happening follows.



Consider $\mathcal{X}_{K_r} * \psi_m$where $\psi_m$is a mollifier. Let m be so large that $\frac{1}{m} < r$. Then from the definition of what is meant by a convolution, and using that ψ_m has support in $B\left(\mathbf{0}, \frac{1}{m}\right)$, $\mathcal{X}_{K_r} * \psi_m = 1$ on K and that its support is in $K + B(\mathbf{0}, 3r)$. Now using Lemma 12.29, $\mathcal{X}_{K_r} * \psi_m$ is also infinitely differentiable. Therefore, let $h = \mathcal{X}_{K_r} * \psi_m$.

The following corollary will be used later.

Corollary 12.31 *Let K be a compact set in $\mathbb{R}^n$ and let $\{U_i\}_{i=1}^{\infty}$ be an open cover of K. Then there exist functions, $\psi_k \in C_c^{\infty}(U_i)$ such that $\psi_i \prec U_i$ and*

$$\sum_{i=1}^{\infty} \psi_i(\mathbf{x}) = 1.$$

If K_1 is a compact subset of U_1 there exist such functions such that also $\psi_1(\mathbf{x}) = 1$ for all $\mathbf{x} \in K_1$.

Proof: This follows from a repeat of the proof of Theorem 9.18 on Page 220, replacing the lemma used in that proof with Theorem 12.30.

Theorem 12.32 *For each $p \geq 1$, $C_c^{\infty}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Here the measure is Lebesgue measure.*

Proof: Let $f \in L^p(\mathbb{R}^n)$ and let $\varepsilon > 0$ be given. Choose $g \in C_c(\mathbb{R}^n)$ such that $||f - g||_p < \frac{\varepsilon}{2}$. This can be done by using Theorem 12.16. Now let

$$g_m(\mathbf{x}) = g * \psi_m(\mathbf{x}) \equiv \int g(\mathbf{x} - \mathbf{y}) \psi_m(\mathbf{y}) \, dm_n(y) = \int g(\mathbf{y}) \psi_m(\mathbf{x} - \mathbf{y}) \, dm_n(y)$$

where $\{\psi_m\}$ is a mollifier. It follows from Lemma 12.29 $g_m \in C_c^{\infty}(\mathbb{R}^n)$. It vanishes if $\mathbf{x} \notin \operatorname{spt}(g) + B(0, \frac{1}{m})$.

$$\begin{aligned}
||g - g_m||_p &= \left(\int |g(\mathbf{x}) - \int g(\mathbf{x} - \mathbf{y}) \psi_m(\mathbf{y}) dm_n(\mathbf{y})|^p dm_n(\mathbf{x}) \right)^{\frac{1}{p}} \\
&\leq \left(\int (\int |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})| \psi_m(\mathbf{y}) dm_n(\mathbf{y}))^p dm_n(\mathbf{x}) \right)^{\frac{1}{p}} \\
&\leq \int \left(\int |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})|^p dm_n(\mathbf{x}) \right)^{\frac{1}{p}} \psi_m(\mathbf{y}) dm_n(\mathbf{y}) \\
&= \int_{B(0, \frac{1}{m})} ||g - g_{\mathbf{y}}||_p \psi_m(\mathbf{y}) dm_n(\mathbf{y}) < \frac{\varepsilon}{2}
\end{aligned}$$

whenever m is large enough. This follows from Corollary 12.22. Theorem 12.12 was used to obtain the third inequality. There is no measurability problem because the function

$$(\mathbf{x}, \mathbf{y}) \to |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})| \psi_m(\mathbf{y})$$

is continuous. Thus when m is large enough,

$$||f - g_m||_p \leq ||f - g||_p + ||g - g_m||_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the theorem.

This is a very remarkable result. Functions in $L^p(\mathbb{R}^n)$ don't need to be continuous anywhere and yet every such function is very close in the L^p norm to one which is infinitely differentiable having compact support.

Another thing should probably be mentioned. If you have had a course in complex analysis, you may be wondering whether these infinitely differentiable functions having compact support have anything to do with analytic functions which also have infinitely many derivatives. The answer is no! Recall that if an analytic function has a limit point in the set of zeros then it is identically equal to zero. Thus these functions in $C_c^\infty(\mathbb{R}^n)$ are not analytic. This is a strictly real analysis phenomenon and has absolutely nothing to do with the theory of functions of a complex variable.

12.6 Exercises

1. Let E be a Lebesgue measurable set in $\mathbb{R}$. Suppose $m(E) > 0$. Consider the set
$$E - E = \{x - y : x \in E, y \in E\}.$$
Show that $E - E$ contains an interval. **Hint:** Let
$$f(x) = \int \mathcal{X}_E(t)\mathcal{X}_E(x+t)dt.$$
Note f is continuous at 0 and $f(0) > 0$ and use continuity of translation in L^p.

2. Establish the inequality $||fg||_r \leq ||f||_p \, ||g||_q$ whenever $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

3. Let $(\Omega, \mathcal{S}, \mu)$ be counting measure on $\mathbb{N}$. Thus $\Omega = \mathbb{N}$ and $\mathcal{S} = \mathcal{P}(\mathbb{N})$ with $\mu(S) =$ number of things in S. Let $1 \leq p \leq q$. Show that in this case,
$$L^1(\mathbb{N}) \subseteq L^p(\mathbb{N}) \subseteq L^q(\mathbb{N}).$$
Hint: This is real easy if you consider what $\int_\Omega f d\mu$ equals. How are the norms related?

4. Consider the function, $f(x,y) = \frac{x^{p-1}}{py} + \frac{y^{q-1}}{qx}$ for $x, y > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show directly that $f(x,y) \geq 1$ for all such x, y and show this implies $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$.

5. Give an example of a sequence of functions in $L^p(\mathbb{R})$ which converges to zero in L^p but does not converge pointwise to 0. Does this contradict the proof of the theorem that L^p is complete?

6. Let K be a bounded subset of $L^p(\mathbb{R}^n)$ and suppose that there exists G such that $\overline{G}$ is compact with
$$\int_{\mathbb{R}^n \setminus \overline{G}} |u(\mathbf{x})|^p \, dx < \varepsilon^p$$

and for all $\varepsilon > 0$, there exist a $\delta > 0$ and such that if $|\mathbf{h}| < \delta$, then

$$\int |u(\mathbf{x}+\mathbf{h}) - u(\mathbf{x})|^p \, dx < \varepsilon^p$$

for all $u \in K$. Show that K is precompact in $L^p(\mathbb{R}^n)$. **Hint:** Let ϕ_k be a mollifier and consider

$$K_k \equiv \{u * \phi_k : u \in K\}.$$

Verify the conditions of the Ascoli Arzela theorem for these functions defined on $\overline{G}$ and show there is an ε net for each $\varepsilon > 0$. Can you modify this to let an arbitrary open set take the place of $\mathbb{R}^n$?

7. Let (Ω, d) be a metric space and suppose also that $(\Omega, \mathcal{S}, \mu)$ is a regular measure space such that $\mu(\Omega) < \infty$ and let $f \in L^1(\Omega)$ where f has complex values. Show that for every $\varepsilon > 0$, there exists an open set of measure less than ε, denoted here by V and a continuous function, g defined on Ω such that $f = g$ on V^C. Thus, aside from a set of small measure, f is continuous. If $|f(\omega)| \leq M$, show that it can be assumed that $|g(\omega)| \leq M$. This is called Lusin's theorem. **Hint:** Use Theorems 12.16 and 12.10 to obtain a sequence of functions in $C_c(\Omega), \{g_n\}$ which converges pointwise a.e. to f and then use Egoroff's theorem to obtain a small set, W of measure less than $\varepsilon/2$ such that convergence is uniform on W^C. Now let F be a closed subset of W^C such that $\mu(W^C \setminus F) < \varepsilon/2$. Let $V = F^C$. Thus $\mu(V) < \varepsilon$ and on $F = V^C$, the convergence of $\{g_n\}$ is uniform showing that the restriction of f to V^C is continuous. Now use the Tietze extension theorem.

8. Let $\phi_m \in C_c^\infty(\mathbb{R}^n), \phi_m(\mathbf{x}) \geq 0$,and $\int_{\mathbb{R}^n} \phi_m(\mathbf{y})dy = 1$ with

$$\lim_{m\to\infty} \sup\{|\mathbf{x}| : \mathbf{x} \in \operatorname{spt}(\phi_m)\} = 0.$$

Show if $f \in L^p(\mathbb{R}^n)$, $\lim_{m\to\infty} f * \phi_m = f$ in $L^p(\mathbb{R}^n)$.

9. Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. This means

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$

whenever $\lambda \in [0,1]$. Verify that if $x < y < z$, then $\frac{\phi(y)-\phi(x)}{y-x} \leq \frac{\phi(z)-\phi(y)}{z-y}$ and that $\frac{\phi(z)-\phi(x)}{z-x} \leq \frac{\phi(z)-\phi(y)}{z-y}$. Show if $s \in \mathbb{R}$ there exists λ such that $\phi(s) \leq \phi(t) + \lambda(s-t)$ for all t. Show that if ϕ is convex, then ϕ is continuous.

10. $\uparrow$ Prove Jensen's inequality. If $\phi : \mathbb{R} \to \mathbb{R}$ is convex, $\mu(\Omega) = 1$, and $f : \Omega \to \mathbb{R}$ is in $L^1(\Omega)$, then $\phi(\int_\Omega f\, du) \leq \int_\Omega \phi(f)d\mu$. **Hint:** Let $s = \int_\Omega f\, d\mu$ and use Problem 9.

11. Let $\frac{1}{p} + \frac{1}{p'} = 1,\ p > 1,$ let $f \in L^p(\mathbb{R}),\ g \in L^{p'}(\mathbb{R})$. Show $f * g$ is uniformly continuous on $\mathbb{R}$ and $|(f*g)(x)| \leq ||f||_{L^p}||g||_{L^{p'}}$. **Hint:** You need to consider why $f * g$ exists and then this follows from the definition of convolution and continuity of translation in L^p.

12. $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \Gamma(p) = \int_0^\infty e^{-t}t^{p-1}dt$ for $p,q > 0$. The first of these is called the beta function, while the second is the gamma function. Show a.) $\Gamma(p+1) = p\Gamma(p)$; b.) $\Gamma(p)\Gamma(q) = B(p,q)\Gamma(p+q)$.

13. Let $f \in C_c(0,\infty)$ and define $F(x) = \frac{1}{x}\int_0^x f(t)dt$. Show

$$||F||_{L^p(0,\infty)} \leq \frac{p}{p-1}||f||_{L^p(0,\infty)} \text{ whenever } p > 1.$$

Hint: Argue there is no loss of generality in assuming $f \geq 0$ and then assume this is so. Integrate $\int_0^\infty |F(x)|^p dx$ by parts as follows:

$$\int_0^\infty F^p dx = \overbrace{xF^p|_0^\infty}^{\text{show } = 0} - p\int_0^\infty xF^{p-1}F'dx.$$

Now show $xF' = f - F$ and use this in the last integral. Complete the argument by using Holder's inequality and $p - 1 = p/q$.

14. ↑ Now suppose $f \in L^p(0,\infty)$, $p > 1$, and f not necessarily in $C_c(0,\infty)$. Show that $F(x) = \frac{1}{x}\int_0^x f(t)dt$ still makes sense for each $x > 0$. Show the inequality of Problem 13 is still valid. This inequality is called Hardy's inequality. **Hint:** To show this, use the above inequality along with the density of $C_c(0,\infty)$ in $L^p(0,\infty)$.

15. Suppose $f, g \geq 0$. When does equality hold in Holder's inequality?

16. Prove Vitali's Convergence theorem: Let $\{f_n\}$ be uniformly integrable and complex valued, $\mu(\Omega) < \infty$, $f_n(x) \to f(x)$ a.e. where f is measurable. Then $f \in L^1$ and $\lim_{n\to\infty}\int_\Omega |f_n - f|d\mu = 0$. **Hint:** Use Egoroff's theorem to show $\{f_n\}$ is a Cauchy sequence in $L^1(\Omega)$. This yields a different and easier proof than what was done earlier. See Theorem 8.50 on Page 204.

17. ↑ Show the Vitali Convergence theorem implies the Dominated Convergence theorem for finite measure spaces but there exist examples where the Vitali convergence theorem works and the dominated convergence theorem does not.

18. ↑ Suppose $\mu(\Omega) < \infty$, $\{f_n\} \subseteq L^1(\Omega)$, and

$$\int_\Omega h(|f_n|)\,d\mu < C$$

for all n where h is a continuous, nonnegative function satisfying

$$\lim_{t\to\infty}\frac{h(t)}{t} = \infty.$$

Show $\{f_n\}$ is uniformly integrable. In applications, this often occurs in the form of a bound on $||f_n||_p$.

19. ↑ Sometimes, especially in books on probability, a different definition of uniform integrability is used than that presented here. A set of functions, $\mathfrak{S}$, defined on a finite measure space, $(\Omega, \mathcal{S}, \mu)$ is said to be uniformly integrable if for all $\varepsilon > 0$ there exists $\alpha > 0$ such that for all $f \in \mathfrak{S}$,

$$\int_{[|f| \geq \alpha]} |f| \, d\mu \leq \varepsilon.$$

Show that this definition is equivalent to the definition of uniform integrability given earlier in Definition 8.48 on Page 203 with the addition of the condition that there is a constant, $C < \infty$ such that

$$\int |f| \, d\mu \leq C$$

for all $f \in \mathfrak{S}$.

20. $f \in L^\infty(\Omega, \mu)$ if there exists a set of measure zero, E, and a constant $C < \infty$ such that $|f(x)| \leq C$ for all $x \notin E$.

$$||f||_\infty \equiv \inf\{C : |f(x)| \leq C \text{ a.e.}\}.$$

Show $|| \; ||_\infty$ is a norm on $L^\infty(\Omega, \mu)$ provided f and g are identified if $f(x) = g(x)$ a.e. Show $L^\infty(\Omega, \mu)$ is complete. **Hint:** You might want to show that $[|f| > ||f||_\infty]$ has measure zero so $||f||_\infty$ is the smallest number at least as large as $|f(x)|$ for a.e. x. Thus $||f||_\infty$ is one of the constants, C in the above.

21. Suppose $f \in L^\infty \cap L^1$. Show $\lim_{p \to \infty} ||f||_{L^p} = ||f||_\infty$. **Hint:**

$$(||f||_\infty - \varepsilon)^p \, \mu([|f| > ||f||_\infty - \varepsilon]) \leq \int_{[|f| > ||f||_\infty - \varepsilon]} |f|^p \, d\mu \leq$$

$$\int |f|^p \, d\mu = \int |f|^{p-1} |f| \, d\mu \leq ||f||_\infty^{p-1} \int |f| \, d\mu.$$

Now raise both ends to the $1/p$ power and take lim inf and lim sup as $p \to \infty$. You should get $||f||_\infty - \varepsilon \leq \liminf ||f||_p \leq \limsup ||f||_p \leq ||f||_\infty$

22. Suppose $\mu(\Omega) < \infty$. Show that if $1 \leq p < q$, then $L^q(\Omega) \subseteq L^p(\Omega)$. **Hint** Use Holder's inequality.

23. Show $L^1(\mathbb{R}) \nsubseteq L^2(\mathbb{R})$ and $L^2(\mathbb{R}) \nsubseteq L^1(\mathbb{R})$ if Lebesgue measure is used. **Hint:** Consider $1/\sqrt{x}$ and $1/x$.

24. Suppose that $\theta \in [0, 1]$ and $r, s, q > 0$ with

$$\frac{1}{q} = \frac{\theta}{r} + \frac{1 - \theta}{s}.$$

show that

$$\left(\int |f|^q d\mu\right)^{1/q} \leq \left(\left(\int |f|^r d\mu\right)^{1/r}\right)^\theta \left(\left(\int |f|^s d\mu\right)^{1/s}\right)^{1-\theta}.$$

If $q, r, s \geq 1$ this says that

$$||f||_q \leq ||f||_r^\theta ||f||_s^{1-\theta}.$$

Using this, show that

$$\ln\left(||f||_q\right) \leq \theta \ln\left(||f||_r\right) + (1-\theta)\ln\left(||f||_s\right).$$

Hint:

$$\int |f|^q d\mu = \int |f|^{q\theta} |f|^{q(1-\theta)} d\mu.$$

Now note that $1 = \frac{\theta q}{r} + \frac{q(1-\theta)}{s}$ and use Holder's inequality.

25. Suppose f is a function in $L^1(\mathbb{R})$ and f is infinitely differentiable. Is $f' \in L^1(\mathbb{R})$? **Hint:** What if $\phi \in C_c^\infty(0,1)$ and $f(x) = \phi(2^n(x-n))$ for $x \in (n, n+1)$, $f(x) = 0$ if $x < 0$?

Banach Spaces

13.1 Theorems Based On Baire Category

13.1.1 Baire Category Theorem

Some examples of Banach spaces that have been discussed up to now are $\mathbb{R}^n, \mathbb{C}^n$, and $L^p(\Omega)$. Theorems about general Banach spaces are proved in this chapter. The main theorems to be presented here are the uniform boundedness theorem, the open mapping theorem, the closed graph theorem, and the Hahn Banach Theorem. The first three of these theorems come from the Baire category theorem which is about to be presented. They are topological in nature. The Hahn Banach theorem has nothing to do with topology. Banach spaces are all normed linear spaces and as such, they are all metric spaces because a normed linear space may be considered as a metric space with $d(x,y) \equiv ||x-y||$. You can check that this satisfies all the axioms of a metric. As usual, if every Cauchy sequence converges, the metric space is called complete.

Definition 13.1 *A complete normed linear space is called a Banach space.*

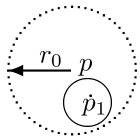
The following remarkable result is called the Baire category theorem. To get an idea of its meaning, imagine you draw a line in the plane. The complement of this line is an open set and is dense because every point, even those on the line, are limit points of this open set. Now draw another line. The complement of the two lines is still open and dense. Keep drawing lines and looking at the complements of the union of these lines. You always have an open set which is dense. Now what if there were countably many lines? The Baire category theorem implies the complement of the union of these lines is dense. In particular it is nonempty. Thus you cannot write the plane as a countable union of lines. This is a rather rough description of this very important theorem. The precise statement and proof follow.

Theorem 13.2 *Let (X,d) be a complete metric space and let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open subsets of X satisfying $\overline{U_n} = X$ (U_n is dense). Then $D \equiv \cap_{n=1}^{\infty} U_n$ is a dense subset of X.*

Proof: Let $p \in X$ and let $r_0 > 0$. I need to show $D \cap B(p, r_0) \neq \emptyset$. Since U_1 is dense, there exists $p_1 \in U_1 \cap B(p, r_0)$, an open set. Let $p_1 \in B(p_1, r_1) \subseteq \overline{B(p_1, r_1)} \subseteq U_1 \cap B(p, r_0)$ and $r_1 < 2^{-1}$. This is possible because $U_1 \cap B(p, r_0)$ is an open set and so there exists r_1 such that $B(p_1, 2r_1) \subseteq U_1 \cap B(p, r_0)$. But

$$B(p_1, r_1) \subseteq \overline{B(p_1, r_1)} \subseteq B(p_1, 2r_1)$$

because $\overline{B(p_1, r_1)} = \{x \in X : d(x, p) \leq r_1\}$. (Why?)



There exists $p_2 \in U_2 \cap B(p_1, r_1)$ because U_2 is dense. Let

$$p_2 \in B(p_2, r_2) \subseteq \overline{B(p_2, r_2)} \subseteq U_2 \cap B(p_1, r_1) \subseteq U_1 \cap U_2 \cap B(p, r_0).$$

and let $r_2 < 2^{-2}$. Continue in this way. Thus

$$r_n < 2^{-n},$$

$$\overline{B(p_n, r_n)} \subseteq U_1 \cap U_2 \cap ... \cap U_n \cap B(p, r_0),$$

$$\overline{B(p_n, r_n)} \subseteq B(p_{n-1}, r_{n-1}).$$

The sequence, $\{p_n\}$ is a Cauchy sequence because all terms of $\{p_k\}$ for $k \geq n$ are contained in $B(p_n, r_n)$, a set whose diameter is no larger than 2^{-n}. Since X is complete, there exists p_∞ such that

$$\lim_{n \to \infty} p_n = p_\infty.$$

Since all but finitely many terms of $\{p_n\}$ are in $\overline{B(p_m, r_m)}$, it follows that $p_\infty \in \overline{B(p_m, r_m)}$ for each m. Therefore,

$$p_\infty \in \cap_{m=1}^{\infty} \overline{B(p_m, r_m)} \subseteq \cap_{i=1}^{\infty} U_i \cap B(p, r_0).$$

This proves the theorem.

The following corollary is also called the Baire category theorem.

Corollary 13.3 *Let X be a complete metric space and suppose $X = \cup_{i=1}^{\infty} F_i$ where each F_i is a closed set. Then for some i, interior $F_i \neq \emptyset$.*

Proof: If all F_i has empty interior, then F_i^C would be a dense open set. Therefore, from Theorem 13.2, it would follow that

$$\emptyset = \left(\cup_{i=1}^{\infty} F_i\right)^C = \cap_{i=1}^{\infty} F_i^C \neq \emptyset.$$

The set D of Theorem 13.2 is called a G_δ set because it is the countable intersection of open sets. Thus D is a dense G_δ set.

Recall that a norm satisfies:

a.) $||x|| \geq 0$, $||x|| = 0$ if and only if $x = 0$.

b.) $||x + y|| \leq ||x|| + ||y||$.

c.) $||cx|| = |c|\ ||x||$ if c is a scalar and $x \in X$.

From the definition of continuity, it follows easily that a function is continuous if

$$\lim_{n\to\infty} x_n = x$$

implies

$$\lim_{n\to\infty} f(x_n) = f(x).$$

Theorem 13.4 *Let X and Y be two normed linear spaces and let $L : X \to Y$ be linear ($L(ax + by) = aL(x) + bL(y)$ for a, b scalars and $x, y \in X$). The following are equivalent*

a.) L is continuous at 0

b.) L is continuous

c.) There exists $K > 0$ such that $||Lx||_Y \leq K\ ||x||_X$ for all $x \in X$ (L is bounded).

Proof: a.)$\Rightarrow$b.) Let $x_n \to x$. It is necessary to show that $Lx_n \to Lx$. But $(x_n - x) \to 0$ and so from continuity at 0, it follows

$$L(x_n - x) = Lx_n - Lx \to 0$$

so $Lx_n \to Lx$. This shows a.) implies b.).

b.)$\Rightarrow$c.) Since L is continuous, L is continuous at 0. Hence $||Lx||_Y < 1$ whenever $||x||_X \leq \delta$ for some δ. Therefore, suppressing the subscript on the $||\ ||$,

$$||L\left(\frac{\delta x}{||x||}\right)|| \leq 1.$$

Hence

$$||Lx|| \leq \frac{1}{\delta}||x||.$$

c.)$\Rightarrow$a.) follows from the inequality given in c.).

Definition 13.5 *Let $L : X \to Y$ be linear and continuous where X and Y are normed linear spaces. Denote the set of all such continuous linear maps by $\mathcal{L}(X, Y)$ and define*

$$||L|| = \sup\{||Lx|| : ||x|| \leq 1\}. \tag{13.1}$$

This is called the operator norm.

Note that from Theorem 13.4 $||L||$ is well defined because of part c.) of that Theorem.

The next lemma follows immediately from the definition of the norm and the assumption that L is linear.

Lemma 13.6 *With $||L||$ defined in 13.1, $\mathcal{L}(X,Y)$ is a normed linear space. Also $||Lx|| \leq ||L|| \, ||x||$.*

Proof: Let $x \neq 0$ then $x/\,||x||$ has norm equal to 1 and so

$$\left|\left|L\left(\frac{x}{||x||}\right)\right|\right| \leq ||L||\,.$$

Therefore, multiplying both sides by $||x||$, $||Lx|| \leq ||L||\,||x||$. This is obviously a linear space. It remains to verify the operator norm really is a norm. First of all, if $||L|| = 0$, then $Lx = 0$ for all $||x|| \leq 1$. It follows that for any $x \neq 0, 0 = L\left(\frac{x}{||x||}\right)$ and so $Lx = 0$. Therefore, $L = 0$. Also, if c is a scalar,

$$||cL|| = \sup_{||x|| \leq 1} ||cL(x)|| = |c| \sup_{||x|| \leq 1} ||Lx|| = |c|\,||L||\,.$$

It remains to verify the triangle inequality. Let $L, M \in \mathcal{L}(X,Y)$.

$$\begin{aligned} ||L+M|| &\equiv \sup_{||x|| \leq 1} ||(L+M)(x)|| \leq \sup_{||x|| \leq 1} (||Lx|| + ||Mx||) \\ &\leq \sup_{||x|| \leq 1} ||Lx|| + \sup_{||x|| \leq 1} ||Mx|| = ||L|| + ||M||\,. \end{aligned}$$

This shows the operator norm is really a norm as hoped. This proves the lemma.

For example, consider the space of linear transformations defined on $\mathbb{R}^n$ having values in $\mathbb{R}^m$. The fact the transformation is linear automatically imparts continuity to it. You should give a proof of this fact. Recall that every such linear transformation can be realized in terms of matrix multiplication.

Thus, in finite dimensions the algebraic condition that an operator is linear is sufficient to imply the topological condition that the operator is continuous. The situation is not so simple in infinite dimensional spaces such as $C(X;\mathbb{R}^n)$. This explains the imposition of the topological condition of continuity as a criterion for membership in $\mathcal{L}(X,Y)$ in addition to the algebraic condition of linearity.

Theorem 13.7 *If Y is a Banach space, then $\mathcal{L}(X,Y)$ is also a Banach space.*

Proof: Let $\{L_n\}$ be a Cauchy sequence in $\mathcal{L}(X,Y)$ and let $x \in X$.

$$||L_n x - L_m x|| \leq ||x||\,||L_n - L_m||.$$

Thus $\{L_n x\}$ is a Cauchy sequence. Let

$$Lx = \lim_{n \to \infty} L_n x.$$

Then, clearly, L is linear because if x_1, x_2 are in X, and a, b are scalars, then

$$\begin{aligned} L(ax_1 + bx_2) &= \lim_{n\to\infty} L_n(ax_1 + bx_2) \\ &= \lim_{n\to\infty} (aL_nx_1 + bL_nx_2) \\ &= aLx_1 + bLx_2. \end{aligned}$$

Also L is continuous. To see this, note that $\{||L_n||\}$ is a Cauchy sequence of real numbers because $|||L_n|| - ||L_m||| \leq ||L_n - L_m||$. Hence there exists $K > \sup\{||L_n|| : n \in \mathbb{N}\}$. Thus, if $x \in X$,

$$||Lx|| = \lim_{n\to\infty} ||L_nx|| \leq K||x||.$$

This proves the theorem.

13.1.2 Uniform Boundedness Theorem

The next big result is sometimes called the Uniform Boundedness theorem, or the Banach-Steinhaus theorem. This is a very surprising theorem which implies that for a collection of bounded linear operators, if they are bounded pointwise, then they are also bounded uniformly. As an example of a situation in which pointwise bounded does not imply uniformly bounded, consider the functions $f_\alpha(x) \equiv \mathcal{X}_{(\alpha,1)}(x)\,x^{-1}$ for $\alpha \in (0,1)$. Clearly each function is bounded and the collection of functions is bounded at each point of $(0,1)$, but there is no bound for all these functions taken together. One problem is that $(0,1)$ is not a Banach space. Therefore, the functions cannot be linear.

Theorem 13.8 *Let X be a Banach space and let Y be a normed linear space. Let $\{L_\alpha\}_{\alpha\in\Lambda}$ be a collection of elements of $\mathcal{L}(X,Y)$. Then one of the following happens.*
a.) $\sup\{||L_\alpha|| : \alpha \in \Lambda\} < \infty$
b.) There exists a dense G_δ set, D, such that for all $x \in D$,

$$\sup\{||L_\alpha x||\ \alpha \in \Lambda\} = \infty.$$

Proof: For each $n \in \mathbb{N}$, define

$$U_n = \{x \in X : \sup\{||L_\alpha x|| : \alpha \in \Lambda\} > n\}.$$

Then U_n is an open set because if $x \in U_n$, then there exists $\alpha \in \Lambda$ such that

$$||L_\alpha x|| > n$$

But then, since L_α is continuous, this situation persists for all y sufficiently close to x, say for all $y \in B(x,\delta)$. Then $B(x,\delta) \subseteq U_n$ which shows U_n is open.

Case b.) is obtained from Theorem 13.2 if each U_n is dense.

The other case is that for some n, U_n is not dense. If this occurs, there exists x_0 and $r > 0$ such that for all $x \in B(x_0, r)$, $||L_\alpha x|| \leq n$ for all α. Now if $y \in$

$B(0,r)$, $x_0 + y \in B(x_0, r)$. Consequently, for all such y, $||L_\alpha(x_0 + y)|| \leq n$. This implies that for all $\alpha \in \Lambda$ and $||y|| < r$,

$$||L_\alpha y|| \leq n + ||L_\alpha(x_0)|| \leq 2n.$$

Therefore, if $||y|| \leq 1$, $\left|\left|\frac{r}{2}y\right|\right| < r$ and so for all α,

$$||L_\alpha\left(\frac{r}{2}y\right)|| \leq 2n.$$

Now multiplying by $r/2$ it follows that whenever $||y|| \leq 1$, $||L_\alpha(y)|| \leq 4n/r$. Hence case a.) holds.

13.1.3 Open Mapping Theorem

Another remarkable theorem which depends on the Baire category theorem is the open mapping theorem. Unlike Theorem 13.8 it requires both X and Y to be Banach spaces.

Theorem 13.9 *Let X and Y be Banach spaces, let $L \in \mathcal{L}(X,Y)$, and suppose L is onto. Then L maps open sets onto open sets.*

To aid in the proof, here is a lemma.

Lemma 13.10 *Let a and b be positive constants and suppose*

$$B(0,a) \subseteq \overline{L(B(0,b))}.$$

Then

$$\overline{L(B(0,b))} \subseteq L(B(0,2b)).$$

Proof of Lemma 13.10: Let $y \in \overline{L(B(0,b))}$. There exists $x_1 \in B(0,b)$ such that $||y - Lx_1|| < \frac{a}{2}$. Now this implies

$$2y - 2Lx_1 \in B(0,a) \subseteq \overline{L(B(0,b))}.$$

Thus $2y - 2Lx_1 \in \overline{L(B(0,b))}$ just like y was. Therefore, there exists $x_2 \in B(0,b)$ such that $||2y - 2Lx_1 - Lx_2|| < a/2$. Hence $||4y - 4Lx_1 - 2Lx_2|| < a$, and there exists $x_3 \in B(0,b)$ such that $||4y - 4Lx_1 - 2Lx_2 - Lx_3|| < a/2$. Continuing in this way, there exist $x_1, x_2, x_3, x_4, ...$ in $B(0,b)$ such that

$$||2^n y - \sum_{i=1}^{n} 2^{n-(i-1)} L(x_i)|| < a$$

which implies

$$||y - \sum_{i=1}^{n} 2^{-(i-1)} L(x_i)|| = ||y - L\left(\sum_{i=1}^{n} 2^{-(i-1)}(x_i)\right)|| < 2^{-n}a \tag{13.2}$$

Now consider the partial sums of the series, $\sum_{i=1}^{\infty} 2^{-(i-1)} x_i$.

$$||\sum_{i=m}^{n} 2^{-(i-1)} x_i|| \leq b \sum_{i=m}^{\infty} 2^{-(i-1)} = b\, 2^{-m+2}.$$

Therefore, these partial sums form a Cauchy sequence and so since X is complete, there exists $x = \sum_{i=1}^{\infty} 2^{-(i-1)} x_i$. Letting $n \to \infty$ in 13.2 yields $||y - Lx|| = 0$. Now

$$||x|| = \lim_{n\to\infty} ||\sum_{i=1}^{n} 2^{-(i-1)} x_i||$$

$$\leq \lim_{n\to\infty} \sum_{i=1}^{n} 2^{-(i-1)} ||x_i|| < \lim_{n\to\infty} \sum_{i=1}^{n} 2^{-(i-1)} b = 2b.$$

This proves the lemma.

Proof of Theorem 13.9: $Y = \cup_{n=1}^{\infty} \overline{L(B(0,n))}$. By Corollary 13.3, the set, $\overline{L(B(0,n_0))}$ has nonempty interior for some n_0. Thus $B(y,r) \subseteq \overline{L(B(0,n_0))}$ for some y and some $r > 0$. Since L is linear $B(-y,r) \subseteq \overline{L(B(0,n_0))}$ also. Here is why. If $z \in B(-y,r)$, then $-z \in B(y,r)$ and so there exists $x_n \in B(0,n_0)$ such that $Lx_n \to -z$. Therefore, $L(-x_n) \to z$ and $-x_n \in B(0,n_0)$ also. Therefore $z \in \overline{L(B(0,n_0))}$. Then it follows that

$$\begin{aligned} B(0,r) &\subseteq B(y,r) + B(-y,r) \\ &\equiv \{y_1 + y_2 : y_1 \in B(y,r) \text{ and } y_2 \in B(-y,r)\} \\ &\subseteq \overline{L(B(0,2n_0))} \end{aligned}$$

The reason for the last inclusion is that from the above, if $y_1 \in B(y,r)$ and $y_2 \in B(-y,r)$, there exists $x_n, z_n \in B(0,n_0)$ such that

$$Lx_n \to y_1,\ Lz_n \to y_2.$$

Therefore,

$$||x_n + z_n|| \leq 2n_0$$

and so $(y_1 + y_2) \in \overline{L(B(0,2n_0))}$.

By Lemma 13.10, $\overline{L(B(0,2n_0))} \subseteq L(B(0,4n_0))$ which shows

$$B(0,r) \subseteq L(B(0,4n_0)).$$

Letting $a = r(4n_0)^{-1}$, it follows, since L is linear, that $B(0,a) \subseteq L(B(0,1))$. It follows since L is linear,

$$L(B(0,r)) \supseteq B(0,ar). \tag{13.3}$$

Now let U be open in X and let $x + B(0,r) = B(x,r) \subseteq U$. Using 13.3,

$$L(U) \supseteq L(x + B(0,r))$$

$$= Lx + L(B(0,r)) \supseteq Lx + B(0,ar) = B(Lx,ar).$$

Hence

$$Lx \in B(Lx, ar) \subseteq L(U).$$

which shows that every point, $Lx \in LU$, is an interior point of LU and so LU is open. This proves the theorem.

This theorem is surprising because it implies that if $|\cdot|$ and $||\cdot||$ are two norms with respect to which a vector space X is a Banach space such that $|\cdot| \leq K\,||\cdot||$, then there exists a constant k, such that $||\cdot|| \leq k\,|\cdot|$. This can be useful because sometimes it is not clear how to compute k when all that is needed is its existence. To see the open mapping theorem implies this, consider the identity map $\operatorname{id} x = x$. Then $\operatorname{id} : (X, ||\cdot||) \to (X, |\cdot|)$ is continuous and onto. Hence id is an open map which implies id^{-1} is continuous. Theorem 13.4 gives the existence of the constant k.

13.1.4 Closed Graph Theorem

Definition 13.11 *Let $f : D \to E$. The set of all ordered pairs of the form $\{(x, f(x)) : x \in D\}$ is called the graph of f.*

Definition 13.12 *If X and Y are normed linear spaces, make $X \times Y$ into a normed linear space by using the norm $||(x, y)|| = \max(||x||, ||y||)$ along with component-wise addition and scalar multiplication. Thus $a(x, y) + b(z, w) \equiv (ax + bz, ay + bw)$.*

There are other ways to give a norm for $X \times Y$. For example, you could define $||(x, y)|| = ||x|| + ||y||$

Lemma 13.13 *The norm defined in Definition 13.12 on $X \times Y$ along with the definition of addition and scalar multiplication given there make $X \times Y$ into a normed linear space.*

Proof: The only axiom for a norm which is not obvious is the triangle inequality. Therefore, consider

$$\begin{aligned} ||(x_1, y_1) + (x_2, y_2)|| &= ||(x_1 + x_2, y_1 + y_2)|| \\ &= \max(||x_1 + x_2||, ||y_1 + y_2||) \\ &\leq \max(||x_1|| + ||x_2||, ||y_1|| + ||y_2||) \end{aligned}$$

$$\begin{aligned} &\leq \max(||x_1||, ||y_1||) + \max(||x_2||, ||y_2||) \\ &= ||(x_1, y_1)|| + ||(x_2, y_2)||. \end{aligned}$$

It is obvious $X \times Y$ is a vector space from the above definition. This proves the lemma.

Lemma 13.14 *If X and Y are Banach spaces, then $X \times Y$ with the norm and vector space operations defined in Definition 13.12 is also a Banach space.*

Proof: The only thing left to check is that the space is complete. But this follows from the simple observation that $\{(x_n, y_n)\}$ is a Cauchy sequence in $X \times Y$ if and only if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively. Thus if $\{(x_n, y_n)\}$ is a Cauchy sequence in $X \times Y$, it follows there exist x and y such that $x_n \to x$ and $y_n \to y$. But then from the definition of the norm, $(x_n, y_n) \to (x, y)$.

Lemma 13.15 *Every closed subspace of a Banach space is a Banach space.*

Proof: If $F \subseteq X$ where X is a Banach space and $\{x_n\}$ is a Cauchy sequence in F, then since X is complete, there exists a unique $x \in X$ such that $x_n \to x$. However this means $x \in \overline{F} = F$ since F is closed.

Definition 13.16 *Let X and Y be Banach spaces and let $D \subseteq X$ be a subspace. A linear map $L : D \to Y$ is said to be closed if its graph is a closed subspace of $X \times Y$. Equivalently, L is closed if $x_n \to x$ and $Lx_n \to y$ implies $x \in D$ and $y = Lx$.*

Note the distinction between closed and continuous. If the operator is closed the assertion that $y = Lx$ only follows if it is known that the sequence $\{Lx_n\}$ converges. In the case of a continuous operator, the convergence of $\{Lx_n\}$ follows from the assumption that $x_n \to x$. It is not always the case that a mapping which is closed is necessarily continuous. Consider the function $f(x) = \tan(x)$ if x is not an odd multiple of $\frac{\pi}{2}$ and $f(x) \equiv 0$ at every odd multiple of $\frac{\pi}{2}$. Then the graph is closed and the function is defined on $\mathbb{R}$ but it clearly fails to be continuous. Of course this function is not linear. You could also consider the map,

$$\frac{d}{dx} : \left\{ y \in C^1([0,1]) : y(0) = 0 \right\} \equiv D \to C([0,1]).$$

where the norm is the uniform norm on $C([0,1])$, $||y||_\infty$. If $y \in D$, then

$$y(x) = \int_0^x y'(t)\, dt.$$

Therefore, if $\frac{dy_n}{dx} \to f \in C([0,1])$ and if $y_n \to y$ in $C([0,1])$ it follows that

$$\begin{array}{ccc} y_n(x) & = & \int_0^x \frac{dy_n(t)}{dx} dt \\ \downarrow & & \downarrow \\ y(x) & = & \int_0^x f(t)\, dt \end{array}$$

and so by the fundamental theorem of calculus $f(x) = y'(x)$ and so the mapping is closed. It is obviously not continuous because it takes $y(x)$ and $y(x) + \frac{1}{n}\sin(nx)$ to two functions which are far from each other even though these two functions are very close in $C([0,1])$. Furthermore, it is not defined on the whole space, $C([0,1])$.

The next theorem, the closed graph theorem, gives conditions under which closed implies continuous.

Theorem 13.17 *Let X and Y be Banach spaces and suppose $L : X \to Y$ is closed and linear. Then L is continuous.*

Proof: Let G be the graph of L. $G = \{(x, Lx) : x \in X\}$. By Lemma 13.15 it follows that G is a Banach space. Define $P : G \to X$ by $P(x, Lx) = x$. P maps the Banach space G onto the Banach space X and is continuous and linear. By the open mapping theorem, P maps open sets onto open sets. Since P is also one to one, this says that P^{-1} is continuous. Thus $||P^{-1}x|| \leq K||x||$. Hence

$$||Lx|| \leq \max(||x||, ||Lx||) \leq K||x||$$

By Theorem 13.4 on Page 339, this shows L is continuous and proves the theorem.

The following corollary is quite useful. It shows how to obtain a new norm on the domain of a closed operator such that the domain with this new norm becomes a Banach space.

Corollary 13.18 *Let $L : D \subseteq X \to Y$ where X, Y are a Banach spaces, and L is a closed operator. Then define a new norm on D by*

$$||x||_D \equiv ||x||_X + ||Lx||_Y .$$

Then D with this new norm is a Banach space.

Proof: If $\{x_n\}$ is a Cauchy sequence in D with this new norm, it follows both $\{x_n\}$ and $\{Lx_n\}$ are Cauchy sequences and therefore, they converge. Since L is closed, $x_n \to x$ and $Lx_n \to Lx$ for some $x \in D$. Thus $||x_n - x||_D \to 0$.

13.2 Hahn Banach Theorem

The closed graph, open mapping, and uniform boundedness theorems are the three major topological theorems in functional analysis. The other major theorem is the Hahn-Banach theorem which has nothing to do with topology. Before presenting this theorem, here are some preliminaries about partially ordered sets.

Definition 13.19 *Let $\mathcal{F}$ be a nonempty set. $\mathcal{F}$ is called a partially ordered set if there is a relation, denoted here by $\leq$, such that*

$$x \leq x \text{ for all } x \in \mathcal{F}.$$

$$\text{If } x \leq y \text{ and } y \leq z \text{ then } x \leq z.$$

$\mathcal{C} \subseteq \mathcal{F}$ is said to be a chain if every two elements of $\mathcal{C}$ are related. This means that if $x, y \in \mathcal{C}$, then either $x \leq y$ or $y \leq x$. Sometimes a chain is called a totally ordered set. $\mathcal{C}$ is said to be a maximal chain if whenever $\mathcal{D}$ is a chain containing $\mathcal{C}$, $\mathcal{D} = \mathcal{C}$.

The most common example of a partially ordered set is the power set of a given set with $\subseteq$ being the relation. It is also helpful to visualize partially ordered sets as trees. Two points on the tree are related if they are on the same branch of the tree and one is higher than the other. Thus two points on different branches would not be related although they might both be larger than some point on the trunk. You might think of many other things which are best considered as partially ordered sets. Think of food for example. You might find it difficult to determine which of two favorite pies you like better although you may be able to say very easily that you would prefer either pie to a dish of lard topped with whipped cream and mustard. The following theorem is equivalent to the axiom of choice. For a discussion of this, see the appendix on the subject.

Theorem 13.20 *(Hausdorff Maximal Principle) Let $\mathcal{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.*

Definition 13.21 *Let X be a real vector space $\rho : X \to \mathbb{R}$ is called a gauge function if*

$$\rho(x+y) \leq \rho(x)+\rho(y),$$
$$\rho(ax) = a\rho(x) \text{ if } a \geq 0. \tag{13.4}$$

Suppose M is a subspace of X and $z \notin M$. Suppose also that f is a linear real-valued function having the property that $f(x) \leq \rho(x)$ for all $x \in M$. Consider the problem of extending f to $M \oplus \mathbb{R}z$ such that if F is the extended function, $F(y) \leq \rho(y)$ for all $y \in M \oplus \mathbb{R}z$ and F is linear. Since F is to be linear, it suffices to determine how to define $F(z)$. Letting $a > 0$, it is required to define $F(z)$ such that the following hold for all $x, y \in M$.

$$\overbrace{F(x)}^{f(x)} + aF(z) = F(x+az) \leq \rho(x+az),$$
$$\overbrace{F(y)}^{f(y)} - aF(z) = F(y-az) \leq \rho(y-az). \tag{13.5}$$

Now if these inequalities hold for all y/a, they hold for all y because M is given to be a subspace. Therefore, multiplying by a^{-1} 13.4 implies that what is needed is to choose $F(z)$ such that for all $x, y \in M$,

$$f(x) + F(z) \leq \rho(x+z), \ f(y) - \rho(y-z) \leq F(z)$$

and that if $F(z)$ can be chosen in this way, this will satisfy 13.5 for all x, y and the problem of extending f will be solved. Hence it is necessary to choose $F(z)$ such that for all $x, y \in M$

$$f(y) - \rho(y-z) \leq F(z) \leq \rho(x+z) - f(x). \tag{13.6}$$

Is there any such number between $f(y) - \rho(y-z)$ and $\rho(x+z) - f(x)$ for every pair $x, y \in M$? This is where $f(x) \leq \rho(x)$ on M and that f is linear is used. For $x, y \in M$,

$$\rho(x+z) - f(x) - [f(y) - \rho(y-z)]$$
$$= \rho(x+z) + \rho(y-z) - (f(x)+f(y))$$
$$\geq \rho(x+y) - f(x+y) \geq 0.$$

Therefore there exists a number between

$$\sup \{f(y) - \rho(y - z) : y \in M\}$$

and

$$\inf \{\rho(x + z) - f(x) : x \in M\}$$

Choose $F(z)$ to satisfy 13.6. This has proved the following lemma.

Lemma 13.22 *Let M be a subspace of X, a real linear space, and let ρ be a gauge function on X. Suppose $f : M \to \mathbb{R}$ is linear, $z \notin M$, and $f(x) \leq \rho(x)$ for all $x \in M$. Then f can be extended to $M \oplus \mathbb{R}z$ such that, if F is the extended function, F is linear and $F(x) \leq \rho(x)$ for all $x \in M \oplus \mathbb{R}z$.*

With this lemma, the Hahn Banach theorem can be proved.

Theorem 13.23 *(Hahn Banach theorem) Let X be a real vector space, let M be a subspace of X, let $f : M \to \mathbb{R}$ be linear, let ρ be a gauge function on X, and suppose $f(x) \leq \rho(x)$ for all $x \in M$. Then there exists a linear function, $F : X \to \mathbb{R}$, such that*

a.) $F(x) = f(x)$ for all $x \in M$

b.) $F(x) \leq \rho(x)$ for all $x \in X$.

Proof: Let $\mathcal{F} = \{(V, g) : V \supseteq M,\ V$ is a subspace of $X,\ g : V \to \mathbb{R}$ is linear, $g(x) = f(x)$ for all $x \in M$, and $g(x) \leq \rho(x)$ for $x \in V\}$. Then $(M, f) \in \mathcal{F}$ so $\mathcal{F} \neq \emptyset$. Define a partial order by the following rule.

$$(V, g) \leq (W, h)$$

means

$$V \subseteq W \text{ and } h(x) = g(x) \text{ if } x \in V.$$

By Theorem 13.20, there exists a maximal chain, $\mathcal{C} \subseteq \mathcal{F}$. Let $Y = \cup\{V : (V, g) \in \mathcal{C}\}$ and let $h : Y \to \mathbb{R}$ be defined by $h(x) = g(x)$ where $x \in V$ and $(V, g) \in \mathcal{C}$. This is well defined because if $x \in V_1$ and V_2 where (V_1, g_1) and (V_2, g_2) are both in the chain, then since $\mathcal{C}$ is a chain, the two element related. Therefore, $g_1(x) = g_2(x)$. Also h is linear because if $ax + by \in Y$, then $x \in V_1$ and $y \in V_2$ where (V_1, g_1) and (V_2, g_2) are elements of $\mathcal{C}$. Therefore, letting V denote the larger of the two V_i, and g be the function that goes with V, it follows $ax + by \in V$ where $(V, g) \in \mathcal{C}$. Therefore,

$$\begin{aligned} h(ax + by) &= g(ax + by) \\ &= ag(x) + bg(y) \\ &= ah(x) + bh(y). \end{aligned}$$

Also, $h(x) = g(x) \leq \rho(x)$ for any $x \in Y$ because for such x, $x \in V$ where $(V, g) \in \mathcal{C}$.

Is $Y = X$? If not, there exists $z \in X \setminus Y$ and there exists an extension of h to $Y \oplus \mathbb{R}z$ using Lemma 13.22. Letting $\overline{h}$ denote this extended function, contradicts the maximality of $\mathcal{C}$. Indeed, $\mathcal{C} \cup \{(Y \oplus \mathbb{R}z, \overline{h})\}$ would be a longer chain. This proves the Hahn Banach theorem.

This is the original version of the theorem. There is also a version of this theorem for complex vector spaces which is based on a trick.

Corollary 13.24 *(Hahn Banach) Let M be a subspace of a complex normed linear space, X, and suppose $f : M \to \mathbb{C}$ is linear and satisfies $|f(x)| \leq K||x||$ for all $x \in M$. Then there exists a linear function, F, defined on all of X such that $F(x) = f(x)$ for all $x \in M$ and $|F(x)| \leq K||x||$ for all x.*

Proof: First note $f(x) = \operatorname{Re} f(x) + i \operatorname{Im} f(x)$ and so

$$\operatorname{Re} f(ix) + i \operatorname{Im} f(ix) = f(ix) = if(x) = i \operatorname{Re} f(x) - \operatorname{Im} f(x).$$

Therefore, $\operatorname{Im} f(x) = -\operatorname{Re} f(ix)$, and

$$f(x) = \operatorname{Re} f(x) - i \operatorname{Re} f(ix).$$

This is important because it shows it is only necessary to consider $\operatorname{Re} f$ in understanding f. Now it happens that $\operatorname{Re} f$ is linear with respect to real scalars so the above version of the Hahn Banach theorem applies. This is shown next.

If c is a real scalar

$$\operatorname{Re} f(cx) - i \operatorname{Re} f(icx) = cf(x) = c \operatorname{Re} f(x) - ic \operatorname{Re} f(ix).$$

Thus $\operatorname{Re} f(cx) = c \operatorname{Re} f(x)$. Also,

$$\begin{aligned} \operatorname{Re} f(x+y) - i \operatorname{Re} f(i(x+y)) &= f(x+y) \\ &= f(x) + f(y) \end{aligned}$$

$$= \operatorname{Re} f(x) - i \operatorname{Re} f(ix) + \operatorname{Re} f(y) - i \operatorname{Re} f(iy).$$

Equating real parts, $\operatorname{Re} f(x+y) = \operatorname{Re} f(x) + \operatorname{Re} f(y)$. Thus $\operatorname{Re} f$ is linear with respect to real scalars as hoped.

Consider X as a real vector space and let $\rho(x) \equiv K||x||$. Then for all $x \in M$,

$$|\operatorname{Re} f(x)| \leq |f(x)| \leq K||x|| = \rho(x).$$

From Theorem 13.23, $\operatorname{Re} f$ may be extended to a function, h which satisfies

$$\begin{aligned} h(ax+by) &= ah(x) + bh(y) \text{ if } a, b \in \mathbb{R} \\ h(x) &\leq K||x|| \text{ for all } x \in X. \end{aligned}$$

Actually, $|h(x)| \leq K||x||$. The reason for this is that $h(-x) = -h(x) \leq K||-x|| = K||x||$ and therefore, $h(x) \geq -K||x||$. Let

$$F(x) \equiv h(x) - ih(ix).$$

By arguments similar to the above, F is linear.

$$\begin{aligned} F(ix) &= h(ix) - ih(-x) \\ &= ih(x) + h(ix) \\ &= i(h(x) - ih(ix)) = iF(x). \end{aligned}$$

If c is a real scalar,

$$\begin{aligned} F(cx) &= h(cx) - ih(icx) \\ &= ch(x) - cih(ix) = cF(x) \end{aligned}$$

Now

$$\begin{aligned} F(x+y) &= h(x+y) - ih(i(x+y)) \\ &= h(x) + h(y) - ih(ix) - ih(iy) \\ &= F(x) + F(y). \end{aligned}$$

Thus

$$\begin{aligned} F((a+ib)x) &= F(ax) + F(ibx) \\ &= aF(x) + ibF(x) \\ &= (a+ib)F(x). \end{aligned}$$

This shows F is linear as claimed.

Now $wF(x) = |F(x)|$ for some $|w| = 1$. Therefore

$$\begin{aligned} |F(x)| &= wF(x) = h(wx) - \overbrace{ih(iwx)}^{\text{must equal zero}} = h(wx) \\ &= |h(wx)| \leq K||wx|| = K||x||. \end{aligned}$$

This proves the corollary.

Definition 13.25 *Let X be a Banach space. Denote by X' the space of continuous linear functions which map X to the field of scalars. Thus $X' = \mathcal{L}(X, \mathbb{F})$. By Theorem 13.7 on Page 340, X' is a Banach space. Remember with the norm defined on $\mathcal{L}(X, \mathbb{F})$,*

$$||f|| = \sup\{|f(x)| : ||x|| \leq 1\}$$

X' is called the dual space.

Definition 13.26 *Let X and Y be Banach spaces and suppose $L \in \mathcal{L}(X, Y)$. Then define the adjoint map in $\mathcal{L}(Y', X')$, denoted by L^*, by*

$$L^* y^*(x) \equiv y^*(Lx)$$

for all $y^* \in Y'$.

The following diagram is a good one to help remember this definition.

$$\begin{array}{ccc} & L^* & \\ X' & \leftarrow & Y' \\ X & \underset{L}{\rightarrow} & Y \end{array}$$

This is a generalization of the adjoint of a linear transformation on an inner product space. Recall

$$(Ax, y) = (x, A^*y)$$

What is being done here is to generalize this algebraic concept to arbitrary Banach spaces. There are some issues which need to be discussed relative to the above definition. First of all, it must be shown that $L^*y^* \in X'$. Also, it will be useful to have the following lemma which is a useful application of the Hahn Banach theorem.

Lemma 13.27 *Let X be a normed linear space and let $x \in X$. Then there exists $x^* \in X'$ such that $||x^*|| = 1$ and $x^*(x) = ||x||$.*

Proof: Let $f : \mathbb{F}x \to \mathbb{F}$ be defined by $f(\alpha x) = \alpha||x||$. Then for $y = \alpha x \in \mathbb{F}x$,

$$|f(y)| = |f(\alpha x)| = |\alpha| \, ||x|| = |y| \, .$$

By the Hahn Banach theorem, there exists $x^* \in X'$ such that $x^*(\alpha x) = f(\alpha x)$ and $||x^*|| \leq 1$. Since $x^*(x) = ||x||$ it follows that $||x^*|| = 1$ because

$$||x^*|| \geq \left| x^* \left(\frac{x}{||x||} \right) \right| = \frac{||x||}{||x||} = 1.$$

This proves the lemma.

Theorem 13.28 *Let $L \in \mathcal{L}(X, Y)$ where X and Y are Banach spaces. Then*
a.) $L^* \in \mathcal{L}(Y', X')$ as claimed and $||L^*|| = ||L||$.
b.) If L maps one to one onto a closed subspace of Y, then L^* is onto.
c.) If L maps onto a dense subset of Y, then L^* is one to one.

Proof: It is routine to verify L^*y^* and L^* are both linear. This follows immediately from the definition. As usual, the interesting thing concerns continuity.

$$||L^*y^*|| = \sup_{||x|| \leq 1} |L^*y^*(x)| = \sup_{||x|| \leq 1} |y^*(Lx)| \leq ||y^*|| \, ||L|| \, .$$

Thus L^* is continuous as claimed and $||L^*|| \leq ||L||$.

By Lemma 13.27, there exists $y_x^* \in Y'$ such that $||y_x^*|| = 1$ and $y_x^*(Lx) = ||Lx||$.Therefore,

$$\begin{aligned} ||L^*|| &= \sup_{||y^*|| \leq 1} ||L^*y^*|| = \sup_{||y^*|| \leq 1} \sup_{||x|| \leq 1} |L^*y^*(x)| \\ &= \sup_{||y^*|| \leq 1} \sup_{||x|| \leq 1} |y^*(Lx)| = \sup_{||x|| \leq 1} \sup_{||y^*|| \leq 1} |y^*(Lx)| \\ &\geq \sup_{||x|| \leq 1} |y_x^*(Lx)| = \sup_{||x|| \leq 1} ||Lx|| = ||L|| \end{aligned}$$

showing that $||L^*|| \geq ||L||$ and this shows part a.).

If L is one to one and onto a closed subset of Y, then $L(X)$ being a closed subspace of a Banach space, is itself a Banach space and so the open mapping theorem implies $L^{-1} : L(X) \to X$ is continuous. Hence

$$||x|| = ||L^{-1}Lx|| \leq \left|\left|L^{-1}\right|\right| \, ||Lx||$$

Now let $x^* \in X'$ be given. Define $f \in \mathcal{L}(L(X), \mathbb{C})$ by $f(Lx) = x^*(x)$. The function, f is well defined because if $Lx_1 = Lx_2$, then since L is one to one, it follows $x_1 = x_2$ and so $f(L(x_1)) = x^*(x_1) = x^*(x_2) = f(L(x_1))$. Also, f is linear because

$$\begin{aligned} f(aL(x_1) + bL(x_2)) &= f(L(ax_1 + bx_2)) \\ &\equiv x^*(ax_1 + bx_2) \\ &= ax^*(x_1) + bx^*(x_2) \\ &= af(L(x_1)) + bf(L(x_2)). \end{aligned}$$

In addition to this,

$$|f(Lx)| = |x^*(x)| \leq ||x^*|| \, ||x|| \leq ||x^*|| \left|\left|L^{-1}\right|\right| \, ||Lx||$$

and so the norm of f on $L(X)$ is no larger than $||x^*|| \left|\left|L^{-1}\right|\right|$. By the Hahn Banach theorem, there exists an extension of f to an element $y^* \in Y'$ such that $||y^*|| \leq ||x^*|| \left|\left|L^{-1}\right|\right|$. Then

$$L^*y^*(x) = y^*(Lx) = f(Lx) = x^*(x)$$

so $L^*y^* = x^*$ because this holds for all x. Since x^* was arbitrary, this shows L^* is onto and proves b.).

Consider the last assertion. Suppose $L^*y^* = 0$. Is $y^* = 0$? In other words is $y^*(y) = 0$ for all $y \in Y$? Pick $y \in Y$. Since $L(X)$ is dense in Y, there exists a sequence, $\{Lx_n\}$ such that $Lx_n \to y$. But then by continuity of y^*, $y^*(y) = \lim_{n\to\infty} y^*(Lx_n) = \lim_{n\to\infty} L^*y^*(x_n) = 0$. Since $y^*(y) = 0$ for all y, this implies $y^* = 0$ and so L^* is one to one.

Corollary 13.29 *Suppose X and Y are Banach spaces, $L \in \mathcal{L}(X, Y)$, and L is one to one and onto. Then L^* is also one to one and onto.*

There exists a natural mapping, called the James map from a normed linear space, X, to the dual of the dual space which is described in the following definition.

Definition 13.30 *Define $J : X \to X''$ by $J(x)(x^*) = x^*(x)$.*

Theorem 13.31 *The map, J, has the following properties.*

a.) J is one to one and linear.

b.) $||Jx|| = ||x||$ and $||J|| = 1$.

c.) $J(X)$ is a closed subspace of X'' if X is complete.

Also if $x^* \in X'$,

$$||x^*|| = \sup\left\{|x^{**}(x^*)| : ||x^{**}|| \leq 1, \ x^{**} \in X''\right\}.$$

Proof:

$$\begin{aligned} J(ax+by)(x^*) &\equiv x^*(ax+by) \\ &= ax^*(x)+bx^*(y) \\ &= (aJ(x)+bJ(y))(x^*). \end{aligned}$$

Since this holds for all $x^* \in X'$, it follows that

$$J(ax+by) = aJ(x)+bJ(y)$$

and so J is linear. If $Jx = 0$, then by Lemma 13.27 there exists x^* such that $x^*(x) = ||x||$ and $||x^*|| = 1$. Then

$$0 = J(x)(x^*) = x^*(x) = ||x||.$$

This shows a.).

To show b.), let $x \in X$ and use Lemma 13.27 to obtain $x^* \in X'$ such that $x^*(x) = ||x||$ with $||x^*|| = 1$. Then

$$\begin{aligned} ||x|| &\geq \sup\{|y^*(x)| : ||y^*|| \leq 1\} \\ &= \sup\{|J(x)(y^*)| : ||y^*|| \leq 1\} = ||Jx|| \\ &\geq |J(x)(x^*)| = |x^*(x)| = ||x|| \end{aligned}$$

Therefore, $||Jx|| = ||x||$ as claimed. Therefore,

$$||J|| = \sup\{||Jx|| : ||x|| \leq 1\} = \sup\{||x|| : ||x|| \leq 1\} = 1.$$

This shows b.).

To verify c.), use b.). If $Jx_n \to y^{**} \in X''$ then by b.), x_n is a Cauchy sequence converging to some $x \in X$ because

$$||x_n - x_m|| = ||Jx_n - Jx_m||$$

and $\{Jx_n\}$ is a Cauchy sequence. Then $Jx = \lim_{n\to\infty} Jx_n = y^{**}$.

Finally, to show the assertion about the norm of x^*, use what was just shown applied to the James map from X' to X''' still referred to as J.

$$||x^*|| = \sup\{|x^*(x)| : ||x|| \leq 1\} = \sup\{|J(x)(x^*)| : ||Jx|| \leq 1\}$$

$$\leq \sup\{|x^{**}(x^*)| : ||x^{**}|| \leq 1\} = \sup\{|J(x^*)(x^{**})| : ||x^{**}|| \leq 1\}$$

$$\equiv ||Jx^*|| = ||x^*||.$$

This proves the theorem.

Definition 13.32 *When J maps X onto X'', X is called reflexive.*

It happens the L^p spaces are reflexive whenever $p > 1$. This is shown later.

13.3 Weak And Weak $*$ Topologies

13.3.1 Basic Definitions

Let X be a Banach space and let X' be its dual space.[1] For A' a finite subset of X', denote by $\rho_{A'}$ the function defined on X

$$\rho_{A'}(x) \equiv \max_{x^* \in A'} |x^*(x)| \tag{13.7}$$

and also let $B_{A'}(x,r)$ be defined by

$$B_{A'}(x,r) \equiv \{y \in X : \rho_{A'}(y-x) < r\} \tag{13.8}$$

Then certain things are obvious. First of all, if $a \in \mathbb{F}$ and $x, y \in X$,

$$\begin{aligned} \rho_{A'}(x+y) &\leq \rho_{A'}(x) + \rho_{A'}(y), \\ \rho_{A'}(ax) &= |a| \rho_{A'}(x). \end{aligned}$$

Similarly, letting A be a finite subset of X, denote by ρ_A the function defined on X'

$$\rho_A(x^*) \equiv \max_{x \in A} |x^*(x)| \tag{13.9}$$

and let $B_A(x^*,r)$ be defined by

$$B_A(x^*,r) \equiv \{y^* \in X' : \rho_A(y^* - x^*) < r\}. \tag{13.10}$$

It is also clear that

$$\begin{aligned} \rho_A(x^* + y^*) &\leq \rho(x^*) + \rho_A(y^*), \\ \rho_A(ax^*) &= |a| \rho_A(x^*). \end{aligned}$$

Lemma 13.33 *The sets, $B_{A'}(x,r)$ where A' is a finite subset of X' and $x \in X$ form a basis for a topology on X known as the weak topology. The sets $B_A(x^*,r)$ where A is a finite subset of X and $x^* \in X'$ form a basis for a topology on X' known as the weak $*$ topology.*

Proof: The two assertions are very similar. I will verify the one for the weak topology. The union of these sets, $B_{A'}(x,r)$ for $x \in X$ and $r > 0$ is all of X. Now suppose z is contained in the intersection of two of these sets. Say

$$z \in B_{A'}(x,r) \cap B_{A'_1}(x_1,r_1)$$

Then let $C' = A' \cup A'_1$ and let

$$0 < \delta \leq \min\Big(r - \rho_{A'}(z-x), r_1 - \rho_{A'_1}(z-x_1)\Big).$$

[1] Actually, all this works in much more general settings than this.

Consider $y \in B_{C'}(z, \delta)$. Then

$$r - \rho_{A'}(z - x) \geq \delta > \rho_{C'}(y - z) \geq \rho_{A'}(y - z)$$

and so

$$r > \rho_{A'}(y - z) + \rho_{A'}(z - x) \geq \rho_{A'}(y - x)$$

which shows $y \in B_{A'}(x, r)$. Similar reasoning shows $y \in B_{A_1'}(x_1, r_1)$ and so

$$B_{C'}(z, \delta) \subseteq B_{A'}(x, r) \cap B_{A_1'}(x_1, r_1).$$

Therefore, the weak topology consists of the union of all sets of the form $B_A(x, r)$.

13.3.2 Banach Alaoglu Theorem

Why does anyone care about these topologies? The short answer is that in the weak $*$ topology, closed unit ball in X' is compact. This is not true in the normal topology. This wonderful result is the Banach Alaoglu theorem. First recall the notion of the product topology, and the Tychonoff theorem, Theorem 11.7 on Page 307 which are stated here for convenience.

Definition 13.34 *Let I be a set and suppose for each $i \in I$, (X_i, τ_i) is a nonempty topological space. The Cartesian product of the X_i, denoted by $\prod_{i \in I} X_i$, consists of the set of all choice functions defined on I which select a single element of each X_i. Thus $f \in \prod_{i \in I} X_i$ means for every $i \in I$, $f(i) \in X_i$. The axiom of choice says $\prod_{i \in I} X_i$ is nonempty. Let*

$$P_j(A) = \prod_{i \in I} B_i$$

where $B_i = X_i$ if $i \neq j$ and $B_j = A$. A subbasis for a topology on the product space consists of all sets $P_j(A)$ where $A \in \tau_j$. (These sets have an open set from the topology of X_j in the j^{th} slot and the whole space in the other slots.) Thus a basis consists of finite intersections of these sets. Note that the intersection of two of these basic sets is another basic set and their union yields $\prod_{i \in I} X_i$. Therefore, they satisfy the condition needed for a collection of sets to serve as a basis for a topology. This topology is called the product topology and is denoted by $\prod \tau_i$.

Theorem 13.35 *If (X_i, τ_i) is compact, then so is $(\prod_{i \in I} X_i, \prod \tau_i)$.*

The Banach Alaoglu theorem is as follows.

Theorem 13.36 *Let B' be the closed unit ball in X'. Then B' is compact in the weak $*$ topology.*

Proof: By the Tychonoff theorem, Theorem 13.35

$$P \equiv \prod_{x \in X} \overline{B(0, ||x||)}$$

is compact in the product topology where the topology on $\overline{B(0,||x||)}$ is the usual topology of $\mathbb{F}$. Recall P is the set of functions which map a point, $x \in X$ to a point in $\overline{B(0,||x||)}$. Therefore, $B' \subseteq P$. Also the basic open sets in the weak $*$ topology on B' are obtained as the intersection of basic open sets in the product topology of P to B' and so it suffices to show B' is a closed subset of P. Suppose then that $f \in P \setminus B'$. It follows f cannot be linear. There are two ways this can happen. One way is that for some x, y

$$f(x+y) \neq f(x) + f(y)$$

for some $x, y \in X$. However, if g is close enough to f at the three points, $x+y, x$, and y, the above inequality will hold for g in place of f. In other words there is a basic open set containing f such that for all g in this basic open set, $g \notin B'$. A similar consideration applies in case $f(\lambda x) \neq \lambda f(x)$ for some scalar, λ and x. Since $P \setminus B'$ is open, it follows B' is a closed subset of P and is therefore, compact. This proves the theorem.

Sometimes one can consider the weak $*$ topology in terms of a metric space.

Theorem 13.37 *If $K \subseteq X'$ is compact in the weak $*$ topology and X is separable then there exists a metric, d, on K such that if τ_d is the topology on K induced by d and if τ is the topology on K induced by the weak $*$ topology of X', then $\tau = \tau_d$. Thus one can consider K with the weak $*$ topology as a metric space.*

Proof: Let $D = \{x_n\}$ be the dense countable subset. The metric is

$$d(f,g) \equiv \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_{x_n}(f-g)}{1+\rho_{x_n}(f-g)}$$

where $\rho_{x_n}(f) = |f(x_n)|$. Clearly $d(f,g) = d(g,f) \geq 0$. If $d(f,g) = 0$, then this requires $f(x_n) = g(x_n)$ for all $x_n \in D$. Since f and g are continuous and D is dense, this requires that $f(x) = g(x)$ for all x. It is routine to verify the triangle inequality from the easy to establish inequality,

$$\frac{x}{1+x} + \frac{y}{1+y} \geq \frac{x+y}{1+x+y},$$

valid whenever $x, y \geq 0$. Therefore this is a metric. Now for each n

$$g \to \frac{\rho_{x_n}(f-g)}{1+\rho_{x_n}(f-g)}$$

is a continuous function from (K,τ) to $[0,\infty)$ and also the above sum defining d converges uniformly. It follows

$$g \to d(f,g)$$

is continuous. Therefore, the ball with respect to d,

$$B_d(f,r) \equiv \{g \in K : d(g,f) < r\}$$

is open in τ which implies $\tau_d \subseteq \tau$.

Now suppose $U \in \tau$. Then $K \setminus U$ is closed in K. Hence, $K \setminus U$ is compact in τ because it is a closed subset of the compact set K. It follows that $K \setminus U$ is compact with respect to τ_d because $\tau_d \subseteq \tau$. But (K, τ_d) is a Hausdorff space and so $K \setminus U$ must be closed with respect to τ_d. This implies $U \in \tau_d$. Thus $\tau \subseteq \tau_d$ and this proves $\tau = \tau_d$. This proves the theorem.

The fact that this set with the weak $*$ topology can be considered a metric space is very significant because if a point is a limit point in a metric space, one can extract a convergent sequence.

Corollary 13.38 *If X is separable and $K \subseteq X'$ is compact in the weak $*$ topology, then K is sequentially compact. That is, if $\{f_n\}_{n=1}^{\infty} \subseteq K$, then there exists a subsequence f_{n_k} and $f \in K$ such that for all $x \in X$,*

$$\lim_{k \to \infty} f_{n_k}(x) = f(x).$$

Proof: By Theorem 13.37, K is a metric space for the metric described there and it is compact. Therefore by the characterization of compact metric spaces, Proposition 6.12 on Page 136, K is sequentially compact. This proves the corollary.

13.3.3 Eberlein Smulian Theorem

Next consider the weak topology. The most interesting results have to do with a reflexive Banach space. The following lemma ties together the weak and weak $*$ topologies in the case of a reflexive Banach space.

Lemma 13.39 *Let $J : X \to X''$ be the James map*

$$Jx(f) \equiv f(x)$$

and let X be reflexive so that J is onto. Then J is a homeomorphism of $(X,$ weak topology$)$ and $(X'',$ weak $*$ topology$)$.This means J is one to one, onto, and both J and J^{-1} are continuous.

Proof: Let $f \in X'$ and let

$$B_f(x, r) \equiv \{y : |f(x) - f(y)| < r\}.$$

Thus $B_f(x, r)$ is a subbasic set for the weak topology on X. Now by the definition of J,

$$\begin{aligned} y \in B_f(x, r) &\text{ if and only if } |Jy(f) - Jx(f)| < r \\ &\text{if and only if } Jy \in B_f(Jx, r) \equiv \\ &\{y^{**} \in X'' : |y^{**}(f) - J(x)(f)| < r\}, \end{aligned}$$

a subbasic set for the weak $*$ topology on X''. Since J^{-1} and J are one to one and onto and map subbasic sets to subbasic sets, it follows that J is a homeomorphism. This proves the Lemma.

The following is an easy corollary.

Corollary 13.40 *If X is a reflexive Banach space, then the closed unit ball is weakly compact.*

Proof: Let B be the closed unit ball. Then $B = J^{-1}(B^{**})$ where B^{**} is the unit ball in X'' which is compact in the weak $*$ topology. Therefore B is weakly compact because J^{-1} is continuous.

Corollary 13.41 *Let X be a reflexive Banach space. If $K \subseteq X$ is compact in the weak topology and X' is separable then there exists a metric d, on K such that if τ_d is the topology on K induced by d and if τ is the topology on K induced by the weak topology of X, then $\tau = \tau_d$. Thus one can consider K with the weak topology as a metric space.*

Proof: This follows from Theorem 13.37 and Lemma 13.39. Lemma 13.39 implies $J(K)$ is compact in X''. Then since X' is separable, there is a metric, d'' on $J(K)$ which delivers the weak $*$ topology. Let $d(x,y) \equiv d''(Jx, Jy)$. Then

$$(K, \tau_d) \xrightarrow{J} (J(K), \tau_{d''}) \xrightarrow{id} (J(K), \tau_{\text{weak } *}) \xrightarrow{J^{-1}} (K, \tau_{\text{weak}})$$

and all the maps are homeomorphisms.

Next is the Eberlein Smulian theorem which states that a Banach space is reflexive if and only if the closed unit ball is weakly sequentially compact. Actually, only half the theorem is proved here, the more useful only if part. The book by Yoshida [52] has the complete theorem discussed. First here is an interesting lemma for its own sake.

Lemma 13.42 *A closed subspace of a reflexive Banach space is reflexive.*

Proof: Let Y be the closed subspace of the reflexive space, X. Consider the following diagram

$$\begin{array}{ccc} Y'' & \overset{i^{**}}{\rightarrow}\text{1-1} & X'' \\ Y' & \overset{i^{*}}{\leftarrow}\text{onto} & X' \\ Y & \overset{i}{\rightarrow} & X \end{array}$$

This diagram follows from Theorem 13.28 on Page 351, the theorem on adjoints. Now let $y^{**} \in Y''$. Then $i^{**}y^{**} = J_X(y)$ because X is reflexive. I want to show that $y \in Y$. If it is not in Y then since Y is closed, there exists $x^* \in X'$ such that $x^*(y) \neq 0$ but $x^*(Y) = 0$. Then $i^*x^* = 0$. Hence

$$0 = y^{**}(i^*x^*) = i^{**}y^{**}(x^*) = J(y)(x^*) = x^*(y) \neq 0,$$

a contradiction. Hence $y \in Y$. Letting J_Y denote the James map from Y to Y'' and $x^* \in X'$,

$$\begin{aligned} y^{**}(i^*x^*) &= i^{**}y^{**}(x^*) = J_X(y)(x^*) \\ &= x^*(y) = x^*(iy) = i^*x^*(y) = J_Y(y)(i^*x^*) \end{aligned}$$

Since i^* is onto, this shows $y^{**} = J_Y(y)$ and this proves the lemma.

Theorem 13.43 (*Eberlein Smulian*) *The closed unit ball in a reflexive Banach space X, is weakly sequentially compact. By this is meant that if $\{x_n\}$ is contained in the closed unit ball, there exists a subsequence, $\{x_{n_k}\}$ and $x \in X$ such that for all $x^* \in X'$,*

$$x^*(x_{n_k}) \to x^*(x).$$

Proof: Let $\{x_n\} \subseteq B \equiv \overline{B(0,1)}$. Let Y be the closure of the linear span of $\{x_n\}$. Thus Y is a separable. It is reflexive because it is a closed subspace of a reflexive space so the above lemma applies. By the Banach Alaoglu theorem, the closed unit ball B^* in Y' is weak $*$ compact. Also by Theorem 13.37, B^* is a metric space with a suitable metric. Thus B^* is complete and totally bounded with respect to this metric and it follows that B^* with the weak $*$ topology is separable. This implies Y' is also separable in the weak $*$ topology. To see this, let $\{y_n^*\} \equiv D$ be a weak $*$ dense set in B^* and let $y^* \in Y'$. Let p be a large enough positive rational number that $y^*/p \in B^*$. Then if A is any finite set from Y, there exists $y_n^* \in D$ such that $\rho_A(y^*/p - y_n^*) < \frac{\varepsilon}{p}$. It follows $py_n^* \in B_A(y^*, \varepsilon)$ showing that rational multiples of D are weak $*$ dense in Y'. Since Y is reflexive, the weak and weak $*$ topologies on Y' coincide and so Y' is weakly separable. Since Y' is separable, Corollary 13.38 implies B^{**}, the closed unit ball in Y'' is weak $*$ sequentially compact. Then by Lemma 13.39 B, the unit ball in Y, is weakly sequentially compact. It follows there exists a subsequence x_{n_k}, of the sequence $\{x_n\}$ and a point $x \in Y$, such that for all $f \in Y'$,

$$f(x_{n_k}) \to f(x).$$

Now if $x^* \in X'$, and i is the inclusion map of Y into X,

$$x^*(x_{n_k}) = i^*x^*(x_{n_k}) \to i^*x^*(x) = x^*(x).$$

which shows x_{n_k} converges weakly and this shows the unit ball in X is weakly sequentially compact.

Corollary 13.44 *Let $\{x_n\}$ be any bounded sequence in a reflexive Banach space, X. Then there exists $x \in X$ and a subsequence, $\{x_{n_k}\}$ such that for all $x^* \in X'$,*

$$\lim_{k\to\infty} x^*(x_{n_k}) = x^*(x)$$

Proof: If a subsequence, x_{n_k} has $||x_{n_k}|| \to 0$, then the conclusion follows. Simply let $x = 0$. Suppose then that $||x_n||$ is bounded away from 0. That is, $||x_n|| \in [\delta, C]$. Take a subsequence such that $||x_{n_k}|| \to a$. Then consider $x_{n_k}/||x_{n_k}||$. By the Eberlein Smulian theorem, this subsequence has a further subsequence, $x_{n_{k_j}}/\left|\left|x_{n_{k_j}}\right|\right|$ which converges weakly to $x \in B$ where B is the closed unit ball. It follows from routine considerations that $x_{n_{k_j}} \to ax$ weakly. This proves the corollary.

13.4 Exercises

1. Is $\mathbb{N}$ a G_δ set? What about $\mathbb{Q}$? What about a countable dense subset of a complete metric space?

2. $\uparrow$ Let $f : \mathbb{R} \to \mathbb{C}$ be a function. Define the oscillation of a function in $B(x, r)$ by $\omega_r f(x) = \sup\{|f(z) - f(y)| : y, z \in B(x, r)\}$. Define the oscillation of the function at the point, x by $\omega f(x) = \lim_{r\to 0} \omega_r f(x)$. Show f is continuous at x if and only if $\omega f(x) = 0$. Then show the set of points where f is continuous is a G_δ set (try $U_n = \{x : \omega f(x) < \frac{1}{n}\}$). Does there exist a function continuous at only the rational numbers? Does there exist a function continuous at every irrational and discontinuous elsewhere? **Hint:** Suppose D is any countable set, $D = \{d_i\}_{i=1}^{\infty}$, and define the function, $f_n(x)$ to equal zero for every $x \notin \{d_1, \cdots, d_n\}$ and 2^{-n} for x in this finite set. Then consider $g(x) \equiv \sum_{n=1}^{\infty} f_n(x)$. Show that this series converges uniformly.

3. Let $f \in C([0, 1])$ and suppose $f'(x)$ exists. Show there exists a constant, K, such that $|f(x) - f(y)| \le K|x - y|$ for all $y \in [0, 1]$. Let $U_n = \{f \in C([0, 1])$ such that for each $x \in [0, 1]$ there exists $y \in [0, 1]$ such that $|f(x) - f(y)| > n|x - y|\}$. Show that U_n is open and dense in $C([0, 1])$ where for $f \in C([0, 1])$,
$$||f|| \equiv \sup\{|f(x)| : x \in [0, 1]\}.$$
Show that $\cap_n U_n$ is a dense G_δ set of nowhere differentiable continuous functions. Thus every continuous function is uniformly close to one which is nowhere differentiable.

4. $\uparrow$Suppose $f(x) = \sum_{k=1}^{\infty} u_k(x)$ where the convergence is uniform and each u_k is a polynomial. Is it reasonable to conclude that $f'(x) = \sum_{k=1}^{\infty} u_k'(x)$? The answer is no. Use Problem 3 and the Weierstrass approximation theorem do show this.

5. Let X be a normed linear space. We say $A \subseteq X$ is "weakly bounded" if for each $x^* \in X'$, $\sup\{|x^*(x)| : x \in A\} < \infty$, while A is bounded if $\sup\{||x|| : x \in A\} < \infty$. Show A is weakly bounded if and only if it is bounded.

6. Let X and Y be two Banach spaces. Define the norm
$$|||(x, y)||| \equiv ||x||_X + ||y||_Y.$$
Show this is a norm on $X \times Y$ which is equivalent to the norm given in the chapter for $X \times Y$. Can you do the same for the norm defined for $p > 1$ by
$$|(x, y)| \equiv (||x||_X^p + ||y||_Y^p)^{1/p}?$$

7. Let f be a 2π periodic locally integrable function on $\mathbb{R}$. The Fourier series for f is given by
$$\sum_{k=-\infty}^{\infty} a_k e^{ikx} \equiv \lim_{n\to\infty} \sum_{k=-n}^{n} a_k e^{ikx} \equiv \lim_{n\to\infty} S_n f(x)$$

where

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx} f(x)\,dx.$$

Show

$$S_n f(x) = \int_{-\pi}^{\pi} D_n(x-y) f(y)\,dy$$

where

$$D_n(t) = \frac{\sin((n+\frac{1}{2})t)}{2\pi\sin(\frac{t}{2})}.$$

Verify that $\int_{-\pi}^{\pi} D_n(t)\,dt = 1$. Also show that if $g \in L^1(\mathbb{R})$, then

$$\lim_{a\to\infty}\int_{\mathbb{R}} g(x)\sin(ax)\,dx = 0.$$

This last is called the Riemann Lebesgue lemma. **Hint:** For the last part, assume first that $g \in C_c^{\infty}(\mathbb{R})$ and integrate by parts. Then exploit density of the set of functions in $L^1(\mathbb{R})$.

8. ↑It turns out that the Fourier series sometimes converges to the function pointwise. Suppose f is 2π periodic and Holder continuous. That is $|f(x) - f(y)| \leq K|x-y|^{\theta}$ where $\theta \in (0,1]$. Show that if f is like this, then the Fourier series converges to f at every point. Next modify your argument to show that if at every point, x, $|f(x+) - f(y)| \leq K|x-y|^{\theta}$ for y close enough to x and larger than x and $|f(x-) - f(y)| \leq K|x-y|^{\theta}$ for every y close enough to x and smaller than x, then $S_n f(x) \to \frac{f(x+)+f(x-)}{2}$, the midpoint of the jump of the function. **Hint:** Use Problem 7.

9. ↑ Let $Y = \{f$ such that f is continuous, defined on $\mathbb{R}$, and 2π periodic$\}$. Define $||f||_Y = \sup\{|f(x)| : x \in [-\pi,\pi]\}$. Show that $(Y, ||\ ||_Y)$ is a Banach space. Let $x \in \mathbb{R}$ and define $L_n(f) = S_n f(x)$. Show $L_n \in Y'$ but $\lim_{n\to\infty}||L_n|| = \infty$. Show that for each $x \in \mathbb{R}$, there exists a dense G_δ subset of Y such that for f in this set, $|S_n f(x)|$ is unbounded. Finally, show there is a dense G_δ subset of Y having the property that $|S_n f(x)|$ is unbounded on the rational numbers. **Hint:** To do the first part, let $f(y)$ approximate $\operatorname{sgn}(D_n(x-y))$. Here $\operatorname{sgn} r = 1$ if $r > 0, -1$ if $r < 0$ and 0 if $r = 0$. This rules out one possibility of the uniform boundedness principle. After this, show the countable intersection of dense G_δ sets must also be a dense G_δ set.

10. Let $\alpha \in (0,1]$. Define, for X a compact subset of $\mathbb{R}^p$,

$$C^{\alpha}(X;\mathbb{R}^n) \equiv \{\mathbf{f} \in C(X;\mathbb{R}^n) : \rho_\alpha(\mathbf{f}) + ||\mathbf{f}|| \equiv ||\mathbf{f}||_\alpha < \infty\}$$

where

$$||\mathbf{f}|| \equiv \sup\{|\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X\}$$

and
$$\rho_{\alpha}(\mathbf{f}) \equiv \sup\{\frac{|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^{\alpha}} : \mathbf{x}, \mathbf{y} \in X, \ \mathbf{x} \neq \mathbf{y}\}.$$

Show that $(C^{\alpha}(X; \mathbb{R}^n), ||\cdot||_{\alpha})$ is a complete normed linear space. This is called a Holder space. What would this space consist of if $\alpha > 1$?

11. $\uparrow$Now recall Problem 10 about the Holder spaces. Let X be the Holder functions which are periodic of period 2π. Define $L_n f(x) = S_n f(x)$ where $L_n : X \to Y$ for Y given in Problem 9. Show $||L_n||$ is bounded independent of n. Conclude that $L_n f \to f$ in Y for all $f \in X$. In other words, for the Holder continuous and 2π periodic functions, the Fourier series converges to the function uniformly. **Hint:** $L_n f(x)$ is given by
$$L_n f(x) = \int_{-\pi}^{\pi} D_n(y) f(x-y)\, dy$$
where $f(x-y) = f(x) + g(x,y)$ where $|g(x,y)| \leq C|y|^{\alpha}$. Use the fact the Dirichlet kernel integrates to one to write
$$\left|\int_{-\pi}^{\pi} D_n(y) f(x-y)\, dy\right| \leq \overbrace{\left|\int_{-\pi}^{\pi} D_n(y) f(x)\, dy\right|}^{=|f(x)|}$$
$$+C\left|\int_{-\pi}^{\pi} \sin\left(\left(n+\frac{1}{2}\right)y\right)(g(x,y)/\sin(y/2))\, dy\right|$$
Show the functions, $y \to g(x,y)/\sin(y/2)$ are bounded in L^1 independent of x and get a uniform bound on $||L_n||$. Now use a similar argument to show $\{L_n f\}$ is equicontinuous in addition to being uniformly bounded. If $L_n f$ fails to converge to f uniformly, then there exists $\varepsilon > 0$ and a subsequence, n_k such that $||L_{n_k} f - f||_{\infty} \geq \varepsilon$ where this is the norm in Y or equivalently the sup norm on $[-\pi, \pi]$. By the Arzela Ascoli theorem, there is a further subsequence, $L_{n_{k_l}} f$ which converges uniformly on $[-\pi, \pi]$. But by Problem 8 $L_n f(x) \to f(x)$.

12. Let X be a normed linear space and let M be a convex open set containing 0. Define
$$\rho(x) = \inf\{t > 0 : \frac{x}{t} \in M\}.$$
Show ρ is a gauge function defined on X. This particular example is called a Minkowski functional. It is of fundamental importance in the study of locally convex topological vector spaces. A set, M, is convex if $\lambda x + (1-\lambda)y \in M$ whenever $\lambda \in [0,1]$ and $x, y \in M$.

13. $\uparrow$ The Hahn Banach theorem can be used to establish separation theorems. Let M be an open convex set containing 0. Let $x \notin M$. Show there exists $x^* \in X'$ such that $\operatorname{Re} x^*(x) \geq 1 > \operatorname{Re} x^*(y)$ for all $y \in M$. **Hint:** If $y \in M, \rho(y) < 1$.

Show this. If $x \notin M,\ \rho(x) \geq 1$. Try $f(\alpha x) = \alpha\rho(x)$ for $\alpha \in \mathbb{R}$. Then extend f to the whole space using the Hahn Banach theorem and call the result F, show F is continuous, then fix it so F is the real part of $x^* \in X'$.

14. A Banach space is said to be strictly convex if whenever $||x|| = ||y||$ and $x \neq y$, then
$$\left|\left|\frac{x+y}{2}\right|\right| < ||x||.$$
$F : X \to X'$ is said to be a duality map if it satisfies the following: a.) $||F(x)|| = ||x||$. b.) $F(x)(x) = ||x||^2$. Show that if X' is strictly convex, then such a duality map exists. The duality map is an attempt to duplicate some of the features of the Riesz map in Hilbert space which is discussed in the chapter on Hilbert space. **Hint:** For an arbitrary Banach space, let
$$F(x) \equiv \left\{x^* : ||x^*|| \leq ||x|| \text{ and } x^*(x) = ||x||^2\right\}$$
Show $F(x) \neq \emptyset$ by using the Hahn Banach theorem on $f(\alpha x) = \alpha||x||^2$. Next show $F(x)$ is closed and convex. Finally show that you can replace the inequality in the definition of $F(x)$ with an equal sign. Now use strict convexity to show there is only one element in $F(x)$.

15. Prove the following theorem which is an improved version of the open mapping theorem, [17]. Let X and Y be Banach spaces and let $A \in \mathcal{L}(X,Y)$. Then the following are equivalent.
$$AX = Y,$$
$$A \text{ is an open map.}$$
There exists a constant M such that for every $y \in Y$, there exists $x \in X$ with $y = Ax$ and
$$||x|| \leq M\,||y||.$$
Note this gives the equivalence between A being onto and A being an open map. The open mapping theorem says that if A is onto then it is open.

16. Suppose $D \subseteq X$ and D is dense in X. Suppose $L : D \to Y$ is linear and $||Lx|| \leq K||x||$ for all $x \in D$. Show there is a unique extension of L, $\widetilde{L}$, defined on all of X with $||\widetilde{L}x|| \leq K||x||$ and $\widetilde{L}$ is linear. You do not get uniqueness when you use the Hahn Banach theorem. Therefore, in the situation of this problem, it is better to use this result.

17. $\uparrow$ A Banach space is uniformly convex if whenever $||x_n||,\ ||y_n|| \leq 1$ and $||x_n + y_n|| \to 2$, it follows that $||x_n - y_n|| \to 0$. Show uniform convexity implies strict convexity (See Problem 14). **Hint:** Suppose it is not strictly convex. Then there exist $||x||$ and $||y||$ both equal to 1 and $\left|\left|\frac{x_n+y_n}{2}\right|\right| = 1$ consider $x_n \equiv x$ and $y_n \equiv y$, and use the conditions for uniform convexity to get a contradiction. It can be shown that L^p is uniformly convex whenever $\infty > p > 1$. See Hewitt and Stromberg [26] or Ray [43].

18. Show that a closed subspace of a reflexive Banach space is reflexive. **Hint:** The proof of this is an exercise in the use of the Hahn Banach theorem. Let Y be the closed subspace of the reflexive space X and let $y^{**} \in Y''$. Then $i^{**}y^{**} \in X''$ and so $i^{**}y^{**} = Jx$ for some $x \in X$ because X is reflexive. Now argue that $x \in Y$ as follows. If $x \notin Y$, then there exists x^* such that $x^*(Y) = 0$ but $x^*(x) \neq 0$. Thus, $i^*x^* = 0$. Use this to get a contradiction. When you know that $x = y \in Y$, the Hahn Banach theorem implies i^* is onto Y' and for all $x^* \in X'$,

$$y^{**}(i^*x^*) = i^{**}y^{**}(x^*) = Jx(x^*) = x^*(x) = x^*(iy) = i^*x^*(y).$$

19. We say that x_n converges weakly to x if for every $x^* \in X'$, $x^*(x_n) \to x^*(x)$. $x_n \rightharpoonup x$ denotes weak convergence. Show that if $||x_n - x|| \to 0$, then $x_n \rightharpoonup x$.

20. $\uparrow$ Show that if X is uniformly convex, then if $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, it follows $||x_n - x|| \to 0$. **Hint:** Use Lemma 13.27 to obtain $f \in X'$ with $||f|| = 1$ and $f(x) = ||x||$. See Problem 17 for the definition of uniform convexity. Now by the weak convergence, you can argue that if $x \neq 0$, $f(x_n/||x_n||) \to f(x/||x||)$. You also might try to show this in the special case where $||x_n|| = ||x|| = 1$.

21. Suppose $L \in \mathcal{L}(X,Y)$ and $M \in \mathcal{L}(Y,Z)$. Show $ML \in \mathcal{L}(X,Z)$ and that $(ML)^* = L^*M^*$.

22. Let X and Y be Banach spaces and suppose $f \in \mathcal{L}(X,Y)$ is compact. Recall this means that if B is a bounded set in X, then $f(B)$ has compact closure in Y. Show that f^* is also a compact map. **Hint:** Take a bounded subset of Y', S. You need to show $f^*(S)$ is totally bounded. You might consider using the Ascoli Arzela theorem on the functions of S applied to $f(B)$ where B is the closed unit ball in X.

Hilbert Spaces

14.1 Basic Theory

Definition 14.1 *Let X be a vector space. An inner product is a mapping from $X \times X$ to $\mathbb{C}$ if X is complex and from $X \times X$ to $\mathbb{R}$ if X is real, denoted by (x, y) which satisfies the following.*

$$(x, x) \geq 0, \ (x, x) = 0 \ \text{ if and only if } x = 0, \tag{14.1}$$

$$(x, y) = \overline{(y, x).} \tag{14.2}$$

For $a, b \in \mathbb{C}$ and $x, y, z \in X$,

$$(ax + by, z) = a(x, z) + b(y, z). \tag{14.3}$$

Note that 14.2 and 14.3 imply $(x, ay + bz) = \overline{a}(x, y) + \overline{b}(x, z)$. Such a vector space is called an inner product space.

The Cauchy Schwarz inequality is fundamental for the study of inner product spaces.

Theorem 14.2 *(Cauchy Schwarz) In any inner product space*

$$|(x, y)| \leq ||x|| \ ||y||.$$

Proof: Let $\omega \in \mathbb{C}$, $|\omega| = 1$, and $\overline{\omega}(x, y) = |(x, y)| = \text{Re}(x, y\omega)$. Let

$$F(t) = (x + ty\omega, x + t\omega y).$$

If $y = 0$ there is nothing to prove because

$$(x, 0) = (x, 0 + 0) = (x, 0) + (x, 0)$$

and so $(x, 0) = 0$. Thus, it can be assumed $y \neq 0$. Then from the axioms of the inner product,

$$F(t) = ||x||^2 + 2t \,\text{Re}(x, \omega y) + t^2 ||y||^2 \geq 0.$$

This yields

$$||x||^2 + 2t|(x,y)| + t^2||y||^2 \geq 0.$$

Since this inequality holds for all $t \in \mathbb{R}$, it follows from the quadratic formula that

$$4|(x,y)|^2 - 4||x||^2||y||^2 \leq 0.$$

This yields the conclusion and proves the theorem.

Proposition 14.3 *For an inner product space, $||x|| \equiv (x,x)^{1/2}$ does specify a norm.*

Proof: All the axioms are obvious except the triangle inequality. To verify this,

$$\begin{aligned} ||x+y||^2 &\equiv (x+y,x+y) \equiv ||x||^2 + ||y||^2 + 2\operatorname{Re}(x,y) \\ &\leq ||x||^2 + ||y||^2 + 2|(x,y)| \\ &\leq ||x||^2 + ||y||^2 + 2||x||\,||y|| = (||x|| + ||y||)^2. \end{aligned}$$

The following lemma is called the parallelogram identity.

Lemma 14.4 *In an inner product space,*

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2.$$

The proof, a straightforward application of the inner product axioms, is left to the reader.

Lemma 14.5 *For $x \in H$, an inner product space,*

$$||x|| = \sup_{||y|| \leq 1} |(x,y)| \tag{14.4}$$

Proof: By the Cauchy Schwarz inequality, if $x \neq 0$,

$$||x|| \geq \sup_{||y|| \leq 1} |(x,y)| \geq \left(x, \frac{x}{||x||}\right) = ||x||.$$

It is obvious that 14.4 holds in the case that $x = 0$.

Definition 14.6 *A Hilbert space is an inner product space which is complete. Thus a Hilbert space is a Banach space in which the norm comes from an inner product as described above.*

In Hilbert space, one can define a projection map onto closed convex nonempty sets.

Definition 14.7 *A set, K, is convex if whenever $\lambda \in [0,1]$ and $x, y \in K$, $\lambda x + (1-\lambda)y \in K$.*

Theorem 14.8 *Let K be a closed convex nonempty subset of a Hilbert space, H, and let $x \in H$. Then there exists a unique point $Px \in K$ such that $||Px - x|| \leq ||y - x||$ for all $y \in K$.*

Proof: Consider uniqueness. Suppose that z_1 and z_2 are two elements of K such that for $i = 1, 2$,

$$||z_i - x|| \leq ||y - x|| \tag{14.5}$$

for all $y \in K$. Also, note that since K is convex,

$$\frac{z_1 + z_2}{2} \in K.$$

Therefore, by the parallelogram identity,

$$\begin{aligned} ||z_1 - x||^2 &\leq ||\frac{z_1 + z_2}{2} - x||^2 = ||\frac{z_1 - x}{2} + \frac{z_2 - x}{2}||^2 \\ &= 2(||\frac{z_1 - x}{2}||^2 + ||\frac{z_2 - x}{2}||^2) - ||\frac{z_1 - z_2}{2}||^2 \\ &= \frac{1}{2}||z_1 - x||^2 + \frac{1}{2}||z_2 - x||^2 - ||\frac{z_1 - z_2}{2}||^2 \\ &\leq ||z_1 - x||^2 - ||\frac{z_1 - z_2}{2}||^2, \end{aligned}$$

where the last inequality holds because of 14.5 letting $z_i = z_2$ and $y = z_1$. Hence $z_1 = z_2$ and this shows uniqueness.

Now let $\lambda = \inf\{||x - y|| : y \in K\}$ and let y_n be a minimizing sequence. This means $\{y_n\} \subseteq K$ satisfies $\lim_{n\to\infty} ||x - y_n|| = \lambda$. Now the following follows from properties of the norm.

$$||y_n - x + y_m - x||^2 = 4(||\frac{y_n + y_m}{2} - x||^2)$$

Then by the parallelogram identity, and convexity of K, $\frac{y_n + y_m}{2} \in K$, and so

$$\begin{aligned} ||\,(y_n - x) - (y_m - x)\,||^2 &= 2(||y_n - x||^2 + ||y_m - x||^2) - \overbrace{4(||\frac{y_n + y_m}{2} - x||^2)}^{=||y_n - x + y_m - x||^2} \\ &\leq 2(||y_n - x||^2 + ||y_m - x||^2) - 4\lambda^2. \end{aligned}$$

Since $||x - y_n|| \to \lambda$, this shows $\{y_n - x\}$ is a Cauchy sequence. Thus also $\{y_n\}$ is a Cauchy sequence. Since H is complete, $y_n \to y$ for some $y \in H$ which must be in K because K is closed. Therefore

$$||x - y|| = \lim_{n\to\infty} ||x - y_n|| = \lambda.$$

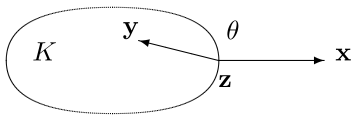
Let $Px = y$.

Corollary 14.9 *Let K be a closed, convex, nonempty subset of a Hilbert space, H, and let $x \in H$. Then for $z \in K$, $z = Px$ if and only if*

$$\operatorname{Re}(x - z, y - z) \leq 0 \tag{14.6}$$

for all $y \in K$.

Before proving this, consider what it says in the case where the Hilbert space is $\mathbb{R}^n$.



Condition 14.6 says the angle, θ, shown in the diagram is always obtuse. Remember from calculus, the sign of $\mathbf{x} \cdot \mathbf{y}$ is the same as the sign of the cosine of the included angle between $\mathbf{x}$ and $\mathbf{y}$. Thus, in finite dimensions, the conclusion of this corollary says that $z = Px$ exactly when the angle of the indicated angle is obtuse. Surely the picture suggests this is reasonable.

The inequality 14.6 is an example of a variational inequality and this corollary characterizes the projection of x onto K as the solution of this variational inequality.

Proof of Corollary: Let $z \in K$ and let $y \in K$ also. Since K is convex, it follows that if $t \in [0, 1]$,

$$z + t(y - z) = (1 - t)\, z + ty \in K.$$

Furthermore, every point of K can be written in this way. (Let $t = 1$ and $y \in K$.) Therefore, $z = Px$ if and only if for all $y \in K$ and $t \in [0, 1]$,

$$||x - (z + t(y - z))||^2 = ||(x - z) - t(y - z)||^2 \geq ||x - z||^2$$

for all $t \in [0, 1]$ and $y \in K$ if and only if for all $t \in [0, 1]$ and $y \in K$

$$||x - z||^2 + t^2 \, ||y - z||^2 - 2t \operatorname{Re}\,(x - z, y - z) \geq ||x - z||^2$$

If and only if for all $t \in [0, 1]$,

$$t^2 \, ||y - z||^2 - 2t \operatorname{Re}\,(x - z, y - z) \geq 0. \tag{14.7}$$

Now this is equivalent to 14.7 holding for all $t \in (0, 1)$. Therefore, dividing by $t \in (0, 1)$, 14.7 is equivalent to

$$t \, ||y - z||^2 - 2 \operatorname{Re}\,(x - z, y - z) \geq 0$$

for all $t \in (0, 1)$ which is equivalent to 14.6. This proves the corollary.

Corollary 14.10 *Let K be a nonempty convex closed subset of a Hilbert space, H. Then the projection map, P is continuous. In fact,*

$$|Px - Py| \leq |x - y| .$$

Proof: Let $x, x' \in H$. Then by Corollary 14.9,

$$\operatorname{Re}(x' - Px', Px - Px') \leq 0, \ \operatorname{Re}(x - Px, Px' - Px) \leq 0$$

Hence

$$\begin{aligned} 0 &\leq \operatorname{Re}(x - Px, Px - Px') - \operatorname{Re}(x' - Px', Px - Px') \\ &= \operatorname{Re}(x - x', Px - Px') - |Px - Px'|^2 \end{aligned}$$

and so

$$|Px - Px'|^2 \leq |x - x'| \, |Px - Px'| .$$

This proves the corollary.

The next corollary is a more general form for the Brouwer fixed point theorem.

Corollary 14.11 *Let $\mathbf{f} : K \to K$ where K is a convex compact subset of $\mathbb{R}^n$. Then $\mathbf{f}$ has a fixed point.*

Proof: Let $K \subseteq \overline{B(\mathbf{0}, R)}$ and let P be the projection map onto K. Then consider the map $\mathbf{f} \circ P$ which maps $\overline{B(\mathbf{0}, R)}$ to $\overline{B(\mathbf{0}, R)}$ and is continuous. By the Brouwer fixed point theorem for balls, this map has a fixed point. Thus there exists $\mathbf{x}$ such that

$$\mathbf{f} \circ P(\mathbf{x}) = \mathbf{x}$$

Now the equation also requires $\mathbf{x} \in K$ and so $P(\mathbf{x}) = \mathbf{x}$. Hence $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.

Definition 14.12 *Let H be a vector space and let U and V be subspaces. $U \oplus V = H$ if every element of H can be written as a sum of an element of U and an element of V in a unique way.*

The case where the closed convex set is a closed subspace is of special importance and in this case the above corollary implies the following.

Corollary 14.13 *Let K be a closed subspace of a Hilbert space, H, and let $x \in H$. Then for $z \in K$, $z = Px$ if and only if*

$$(x - z, y) = 0 \tag{14.8}$$

for all $y \in K$. Furthermore, $H = K \oplus K^{\perp}$ where

$$K^{\perp} \equiv \{x \in H : (x, k) = 0 \text{ for all } k \in K\}$$

and

$$||x||^2 = ||x - Px||^2 + ||Px||^2 . \tag{14.9}$$

Proof: Since K is a subspace, the condition 14.6 implies $\text{Re}(x - z, y) \leq 0$ for all $y \in K$. Replacing y with $-y$, it follows $\text{Re}(x - z, -y) \leq 0$ which implies $\text{Re}(x - z, y) \geq 0$ for all y. Therefore, $\text{Re}(x - z, y) = 0$ for all $y \in K$. Now let $|\alpha| = 1$ and $\alpha (x - z, y) = |(x - z, y)|$. Since K is a subspace, it follows $\overline{\alpha} y \in K$ for all $y \in K$. Therefore,

$$0 = \text{Re}(x - z, \overline{\alpha} y) = (x - z, \overline{\alpha} y) = \alpha (x - z, y) = |(x - z, y)|.$$

This shows that $z = Px$, if and only if 14.8.

For $x \in H$, $x = x - Px + Px$ and from what was just shown, $x - Px \in K^\perp$ and $Px \in K$. This shows that $K^\perp + K = H$. Is there only one way to write a given element of H as a sum of a vector in K with a vector in $K^\perp$? Suppose $y + z = y_1 + z_1$ where $z, z_1 \in K^\perp$ and $y, y_1 \in K$. Then $(y - y_1) = (z_1 - z)$ and so from what was just shown, $(y - y_1, y - y_1) = (y - y_1, z_1 - z) = 0$ which shows $y_1 = y$ and consequently $z_1 = z$. Finally, letting $z = Px$,

$$\begin{aligned} ||x||^2 &= (x - z + z, x - z + z) = ||x - z||^2 + (x - z, z) + (z, x - z) + ||z||^2 \\ &= ||x - z||^2 + ||z||^2 \end{aligned}$$

This proves the corollary.

The following theorem is called the Riesz representation theorem for the dual of a Hilbert space. If $z \in H$ then define an element $f \in H'$ by the rule $(x, z) \equiv f(x)$. It follows from the Cauchy Schwarz inequality and the properties of the inner product that $f \in H'$. The Riesz representation theorem says that all elements of H' are of this form.

Theorem 14.14 *Let H be a Hilbert space and let $f \in H'$. Then there exists a unique $z \in H$ such that*

$$f(x) = (x, z) \tag{14.10}$$

for all $x \in H$.

Proof: Letting $y, w \in H$ the assumption that f is linear implies

$$f(yf(w) - f(y)w) = f(w) f(y) - f(y) f(w) = 0$$

which shows that $yf(w) - f(y)w \in f^{-1}(0)$, which is a closed subspace of H since f is continuous. If $f^{-1}(0) = H$, then f is the zero map and $z = 0$ is the unique element of H which satisfies 14.10. If $f^{-1}(0) \neq H$, pick $u \notin f^{-1}(0)$ and let $w \equiv u - Pu \neq 0$. Thus Corollary 14.13 implies $(y, w) = 0$ for all $y \in f^{-1}(0)$. In particular, let $y = xf(w) - f(x)w$ where $x \in H$ is arbitrary. Therefore,

$$0 = (f(w)x - f(x)w, w) = f(w)(x, w) - f(x)||w||^2.$$

Thus, solving for $f(x)$ and using the properties of the inner product,

$$f(x) = (x, \frac{\overline{f(w)} w}{||w||^2})$$

Let $z = \overline{f(w)}w/||w||^2$. This proves the existence of z. If $f(x) = (x, z_i)$ $i = 1, 2$, for all $x \in H$, then for all $x \in H$, then $(x, z_1 - z_2) = 0$ which implies, upon taking $x = z_1 - z_2$ that $z_1 = z_2$. This proves the theorem.

If $R : H \to H'$ is defined by $Rx(y) \equiv (y, x)$, the Riesz representation theorem above states this map is onto. This map is called the Riesz map. It is routine to show R is linear and $|Rx| = |x|$.

14.2 Approximations In Hilbert Space

The Gram Schmidt process applies in any Hilbert space.

Theorem 14.15 *Let $\{x_1, \cdots, x_n\}$ be a basis for M a subspace of H a Hilbert space. Then there exists an orthonormal basis for M, $\{u_1, \cdots, u_n\}$ which has the property that for each $k \leq n$, span$(x_1, \cdots, x_k) = \text{span}(u_1, \cdots, u_k)$. Also if $\{x_1, \cdots, x_n\} \subseteq H$, then*

$$\text{span}(x_1, \cdots, x_n)$$

is a closed subspace.

Proof: Let $\{x_1, \cdots, x_n\}$ be a basis for M. Let $u_1 \equiv x_1/|x_1|$. Thus for $k = 1$, $\text{span}(u_1) = \text{span}(x_1)$ and $\{u_1\}$ is an orthonormal set. Now suppose for some $k < n$, $u_1, \cdots, u_k$ have been chosen such that $(u_j \cdot u_l) = \delta_{jl}$ and $\text{span}(x_1, \cdots, x_k) = \text{span}(u_1, \cdots, u_k)$. Then define

$$u_{k+1} \equiv \frac{x_{k+1} - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) u_j}{\left| x_{k+1} - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) u_j \right|}, \tag{14.11}$$

where the denominator is not equal to zero because the x_j form a basis and so

$$x_{k+1} \notin \text{span}(x_1, \cdots, x_k) = \text{span}(u_1, \cdots, u_k)$$

Thus by induction,

$$u_{k+1} \in \text{span}(u_1, \cdots, u_k, x_{k+1}) = \text{span}(x_1, \cdots, x_k, x_{k+1}).$$

Also, $x_{k+1} \in \text{span}(u_1, \cdots, u_k, u_{k+1})$ which is seen easily by solving 14.11 for x_{k+1} and it follows

$$\text{span}(x_1, \cdots, x_k, x_{k+1}) = \text{span}(u_1, \cdots, u_k, u_{k+1}).$$

If $l \leq k$,

$$\begin{aligned} (u_{k+1} \cdot u_l) &= C\left((x_{k+1} \cdot u_l) - \sum_{j=1}^{k} (x_{k+1} \cdot u_j)(u_j \cdot u_l) \right) \\ &= C\left((x_{k+1} \cdot u_l) - \sum_{j=1}^{k} (x_{k+1} \cdot u_j) \delta_{lj} \right) \\ &= C\left((x_{k+1} \cdot u_l) - (x_{k+1} \cdot u_l) \right) = 0. \end{aligned}$$

The vectors, $\{u_j\}_{j=1}^n$, generated in this way are therefore an orthonormal basis because each vector has unit length.

Consider the second claim about finite dimensional subspaces. Without loss of generality, assume $\{x_1, \cdots, x_n\}$ is linearly independent. If it is not, delete vectors until a linearly independent set is obtained. Then by the first part, $\text{span}(x_1, \cdots, x_n) = \text{span}(u_1, \cdots, u_n) \equiv M$ where the u_i are an orthonormal set of vectors. Suppose $\{y_k\} \subseteq M$ and $y_k \to y \in H$. Is $y \in M$? Let

$$y_k \equiv \sum_{j=1}^n c_j^k u_j$$

Then let $\mathbf{c}^k \equiv \left(c_1^k, \cdots, c_n^k\right)^T$. Then

$$\begin{aligned} \left|\mathbf{c}^k - \mathbf{c}^l\right|^2 &\equiv \sum_{j=1}^n \left|c_j^k - c_j^l\right|^2 = \left(\sum_{j=1}^n \left(c_j^k - c_j^l\right) u_j, \sum_{j=1}^n \left(c_j^k - c_j^l\right) u_j\right) \\ &= ||y_k - y_l||^2 \end{aligned}$$

which shows $\left\{\mathbf{c}^k\right\}$ is a Cauchy sequence in $\mathbb{F}^n$ and so it converges to $\mathbf{c} \in \mathbb{F}^n$. Thus

$$y = \lim_{k\to\infty} y_k = \lim_{k\to\infty} \sum_{j=1}^n c_j^k u_j = \sum_{j=1}^n c_j u_j \in M.$$

This completes the proof.

Theorem 14.16 *Let M be the span of $\{u_1, \cdots, u_n\}$ in a Hilbert space, H and let $y \in H$. Then Py is given by*

$$Py = \sum_{k=1}^n (y, u_k) u_k \tag{14.12}$$

and the distance is given by

$$\sqrt{|y|^2 - \sum_{k=1}^n |(y, u_k)|^2}. \tag{14.13}$$

Proof:

$$\begin{aligned} \left(y - \sum_{k=1}^n (y, u_k) u_k, u_p\right) &= (y, u_p) - \sum_{k=1}^n (y, u_k)(u_k, u_p) \\ &= (y, u_p) - (y, u_p) = 0 \end{aligned}$$

It follows that

$$\left(y - \sum_{k=1}^n (y, u_k) u_k, u\right) = 0$$

for all $u \in M$ and so by Corollary 14.13 this verifies 14.12.

The square of the distance, d is given by

$$\begin{aligned} d^2 &= \left(y - \sum_{k=1}^{n} (y, u_k) u_k, y - \sum_{k=1}^{n} (y, u_k) u_k \right) \\ &= |y|^2 - 2 \sum_{k=1}^{n} |(y, u_k)|^2 + \sum_{k=1}^{n} |(y, u_k)|^2 \end{aligned}$$

and this shows 14.13.

What if the subspace is the span of vectors which are not orthonormal? There is a very interesting formula for the distance between a point of a Hilbert space and a finite dimensional subspace spanned by an arbitrary basis.

Definition 14.17 *Let $\{x_1, \cdots, x_n\} \subseteq H$, a Hilbert space. Define*

$$\mathcal{G}(x_1, \cdots, x_n) \equiv \begin{pmatrix} (x_1, x_1) & \cdots & (x_1, x_n) \\ \vdots & & \vdots \\ (x_n, x_1) & \cdots & (x_n, x_n) \end{pmatrix} \tag{14.14}$$

Thus the ij^{th} entry of this matrix is (x_i, x_j). This is sometimes called the Gram matrix. Also define $G(x_1, \cdots, x_n)$ as the determinant of this matrix, also called the Gram determinant.

$$G(x_1, \cdots, x_n) \equiv \begin{vmatrix} (x_1, x_1) & \cdots & (x_1, x_n) \\ \vdots & & \vdots \\ (x_n, x_1) & \cdots & (x_n, x_n) \end{vmatrix} \tag{14.15}$$

The theorem is the following.

Theorem 14.18 *Let $M = \operatorname{span}(x_1, \cdots, x_n) \subseteq H$, a Real Hilbert space where $\{x_1, \cdots, x_n\}$ is a basis and let $y \in H$. Then letting d be the distance from y to M,*

$$d^2 = \frac{G(x_1, \cdots, x_n, y)}{G(x_1, \cdots, x_n)}. \tag{14.16}$$

Proof: By Theorem 14.15 M is a closed subspace of H. Let $\sum_{k=1}^{n} \alpha_k x_k$ be the element of M which is closest to y. Then by Corollary 14.13,

$$\left(y - \sum_{k=1}^{n} \alpha_k x_k, x_p \right) = 0$$

for each $p = 1, 2, \cdots, n$. This yields the system of equations,

$$(y, x_p) = \sum_{k=1}^{n} (x_p, x_k) \alpha_k, p = 1, 2, \cdots, n \tag{14.17}$$

Also by Corollary 14.13,

$$||y||^2 = \overbrace{\left|\left|y - \sum_{k=1}^{n} \alpha_k x_k\right|\right|^2}^{d^2} + \left|\left|\sum_{k=1}^{n} \alpha_k x_k\right|\right|^2$$

and so, using 14.17,

$$\begin{aligned} ||y||^2 &= d^2 + \sum_j \left(\sum_k \alpha_k (x_k, x_j) \right) \alpha_j \\ &= d^2 + \sum_j (y, x_j)\, \alpha_j \qquad (14.18) \\ &\equiv d^2 + \mathbf{y}_x^T \boldsymbol{\alpha} \qquad (14.19) \end{aligned}$$

in which

$$\mathbf{y}_x^T \equiv ((y, x_1), \cdots, (y, x_n)), \ \boldsymbol{\alpha}^T \equiv (\alpha_1, \cdots, \alpha_n).$$

Then 14.17 and 14.18 imply the following system

$$\begin{pmatrix} \mathcal{G}(x_1, \cdots, x_n) & \mathbf{0} \\ \mathbf{y}_x^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ d^2 \end{pmatrix} = \begin{pmatrix} \mathbf{y}_x \\ ||y||^2 \end{pmatrix}$$

By Cramer's rule,

$$\begin{aligned} d^2 &= \frac{\det \begin{pmatrix} \mathcal{G}(x_1, \cdots, x_n) & \mathbf{y}_x \\ \mathbf{y}_x^T & ||y||^2 \end{pmatrix}}{\det \begin{pmatrix} \mathcal{G}(x_1, \cdots, x_n) & \mathbf{0} \\ \mathbf{y}_x^T & 1 \end{pmatrix}} \\ &= \frac{\det \begin{pmatrix} \mathcal{G}(x_1, \cdots, x_n) & \mathbf{y}_x \\ \mathbf{y}_x^T & ||y||^2 \end{pmatrix}}{\det (\mathcal{G}(x_1, \cdots, x_n))} \\ &= \frac{\det (\mathcal{G}(x_1, \cdots, x_n, y))}{\det (\mathcal{G}(x_1, \cdots, x_n))} = \frac{G(x_1, \cdots, x_n, y)}{G(x_1, \cdots, x_n)} \end{aligned}$$

and this proves the theorem.

14.3 The Müntz Theorem

Recall the polynomials are dense in $C([0,1])$. This is a consequence of the Weierstrass approximation theorem. Now consider finite linear combinations of the functions, t^{p_k} where $\{p_0, p_1, p_2, \cdots\}$ is a sequence of nonnegative real numbers, $p_0 \equiv 0$. The Müntz theorem says this set, S of finite linear combinations is dense in $C([0,1])$ exactly when $\sum_{k=1}^{\infty} \frac{1}{p_k} = \infty$. There are two versions of this theorem, one for density of S in $L^2(0,1)$ and one for $C([0,1])$. The presentation follows Cheney [14].

Recall the Cauchy identity presented earlier, Theorem 4.46 on Page 76 which is stated here for convenience.

Theorem 14.19 *The following identity holds.*

$$\prod_{i,j}(a_i+b_j)\begin{vmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_n} \\ \vdots & & \vdots \\ \frac{1}{a_n+b_1} & \cdots & \frac{1}{a_n+b_n} \end{vmatrix} = \prod_{j<i}(a_i-a_j)(b_i-b_j). \tag{14.20}$$

Lemma 14.20 *Let $m, p_1, \cdots, p_n$ be distinct real numbers larger than $-1/2$. Thus the functions, $f_m(x) \equiv x^m, f_{p_j}(x) \equiv x^{p_j}$ are all in $L^2(0,1)$. Let $M = \operatorname{span}(f_{p_1}, \cdots, f_{p_n})$. Then the L^2 distance, d between f_m and M is*

$$d = \frac{1}{\sqrt{2m+1}}\prod_{j=1}^{n}\frac{|m-p_j|}{m+p_j+1}$$

Proof: By Theorem 14.18

$$d^2 = \frac{G(f_{p_1}, \cdots, f_{p_n}, f_m)}{G(f_{p_1}, \cdots, f_{p_n})}.$$

$$(f_{p_i}, f_{p_j}) = \int_0^1 x^{p_i}x^{p_j}\,dx = \frac{1}{1+p_i+p_j}$$

Therefore,

$$d^2 = \frac{\begin{vmatrix} \frac{1}{1+p_1+p_1} & \frac{1}{1+p_1+p_2} & \cdots & \frac{1}{1+p_1+p_n} & \frac{1}{1+m+p_1} \\ \frac{1}{1+p_2+p_1} & \frac{1}{1+p_2+p_2} & \cdots & \frac{1}{1+p_2+p_n} & \frac{1}{1+m+p_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{1+p_n+p_1} & \frac{1}{1+p_n+p_2} & \cdots & \frac{1}{1+p_n+p_n} & \frac{1}{1+p_n+m} \\ \frac{1}{1+m+p_1} & \frac{1}{1+m+p_2} & \cdots & \frac{1}{1+m+p_n} & \frac{1}{1+m+m} \end{vmatrix}}{\begin{vmatrix} \frac{1}{1+p_1+p_1} & \frac{1}{1+p_1+p_2} & \cdots & \frac{1}{1+p_1+p_n} \\ \frac{1}{1+p_2+p_1} & \frac{1}{1+p_2+p_2} & \cdots & \frac{1}{1+p_2+p_n} \\ \vdots & \vdots & & \vdots \\ \frac{1}{1+p_n+p_1} & \frac{1}{1+p_n+p_2} & \cdots & \frac{1}{1+p_n+p_n} \end{vmatrix}}$$

Now from the Cauchy identity, letting $a_i = p_i + \frac{1}{2}$ and $b_j = \frac{1}{2} + p_j$ with $p_{n+1} = m$, the numerator of the above equals

$$\frac{\prod_{j<i\leq n+1}(p_i-p_j)(p_i-p_j)}{\prod_{i,j\leq n+1}(p_i+p_j+1)}$$

$$\begin{aligned} &= \frac{\prod_{k=1}^{n}(m-p_k)^2\prod_{j<i\leq n}(p_i-p_j)^2}{\prod_{i=1}^{n}(m+p_i+1)\prod_{j=1}^{n}(m+p_j+1)\prod_{i,j\leq n}(p_i+p_j+1)(2m+1)} \\ &= \frac{\prod_{k=1}^{n}(m-p_k)^2\prod_{j<i\leq n}(p_i-p_j)^2}{\prod_{i=1}^{n}(m+p_i+1)^2\prod_{i,j\leq n}(p_i+p_j+1)(2m+1)} \end{aligned}$$

while the denominator equals

$$\frac{\prod_{j<i\leq n}(p_i - p_j)^2}{\prod_{i,j\leq n}(p_i + p_j + 1)}$$

Therefore,

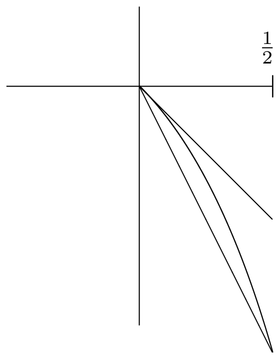
$$\begin{aligned} d^2 &= \frac{\left(\frac{\prod_{k=1}^n (m-p_k)^2 \prod_{j<i\leq n}(p_i-p_j)^2}{\prod_{i=1}^n (m+p_i+1)^2 \prod_{i,j\leq n}(p_i+p_j+1)(2m+1)}\right)}{\left(\frac{\prod_{j<i\leq n}(p_i-p_j)^2}{\prod_{i,j\leq n}(p_i+p_j+1)}\right)} \\ &= \frac{\prod_{k=1}^n (m-p_k)^2}{\prod_{i=1}^n (m+p_i+1)^2 (2m+1)} \end{aligned}$$

which shows

$$d = \frac{1}{\sqrt{2m+1}} \prod_{k=1}^n \frac{|m-p_k|}{m+p_k+1}.$$

and this proves the lemma.

The following lemma relates an infinite sum to a product. First consider the graph of $\ln(1-x)$ for $x \in \left[0, \frac{1}{2}\right]$. Here is a rough sketch with two lines, $y = -x$ which lies above the graph of $\ln(1-x)$ and $y = -2x$ which lies below.



Lemma 14.21 *Let* $a_n \neq 1, a_n > 0,$ *and* $\lim_{n\to\infty} a_n = 0$. *Then*

$$\prod_{k=1}^{\infty}(1-a_n) \equiv \lim_{n\to\infty}\prod_{k=1}^{n}(1-a_n) = 0$$

if and only if

$$\sum_{n=1}^{\infty} a_n = +\infty.$$

Proof: Without loss of generality, you can assume $a_n < 1/2$ because the two conditions are determined by the values of a_n for n large. By the above sketch the following is obtained.

$$\ln \prod_{k=1}^{n} (1 - a_k) = \sum_{k=1}^{n} \ln (1 - a_k) \in \left[-2 \sum_{k=1}^{n} a_k, -\sum_{k=1}^{n} a_k \right].$$

Therefore,

$$e^{-2 \sum_{k=1}^{n} a_k} \leq \prod_{k=1}^{n} (1 - a_k) \leq e^{-\sum_{k=1}^{n} a_k}$$

The conclusion follows.

The following is Müntz's first theorem.

Theorem 14.22 *Let $\{p_n\}$ be a sequence of real numbers larger than $-1/2$ such that $\lim_{n\to\infty} p_n = \infty$. Let S denote the set of finite linear combinations of the functions, $\{x^{p_1}, x^{p_2}, \cdots\}$. Then S is dense in $L^2(0,1)$ if and only if*

$$\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty.$$

Proof: The polynomials are dense in $L^2(0,1)$ and so S is dense in $L^2(0,1)$ if and only if for every $\varepsilon > 0$ there exists a function f from S such that for each integer $m \geq 0$, $\left(\int_0^1 |f(x) - x^m|^2 \, dx \right)^{1/2} < \varepsilon$. This happens if and only if for all n large enough, the distance in $L^2(0,1)$ between the function, $x \to x^m$ and $\operatorname{span}(x^{p_1}, x^{p_2}, \cdots, x^{p_n})$ is less than ε. However, from Lemma 14.20 this distance equals

$$\begin{aligned} & \frac{1}{\sqrt{2m+1}} \prod_{k=1}^{n} \frac{|m - p_k|}{m + p_k + 1} \\ = & \frac{1}{\sqrt{2m+1}} \prod_{k=1}^{n} 1 - \left(1 - \frac{|m - p_k|}{m + p_k + 1} \right) \end{aligned}$$

Thus S is dense if and only if

$$\prod_{k=1}^{\infty} \left(1 - \left(1 - \frac{|m - p_k|}{m + p_k + 1} \right) \right) = 0$$

which, by Lemma 14.21, happens if and only if

$$\sum_{k=1}^{\infty} \left(1 - \frac{|m - p_k|}{m + p_k + 1} \right) = +\infty$$

But this sum equals

$$\sum_{k=1}^{\infty} \left(\frac{m + p_k + 1 - |m - p_k|}{m + p_k + 1} \right)$$

which has the same convergence properties as $\sum \frac{1}{p_k}$ by the limit comparison test. This proves the theorem.

The following is Müntz's second theorem.

Theorem 14.23 *Let S be finite linear combinations of $\{1, x^{p_1}, x^{p_2}, \cdots\}$ where $p_j \geq 1$ and $\lim_{n\to\infty} p_n = \infty$. Then S is dense in $C([0,1])$ if and only if $\sum_{k=1}^{\infty} \frac{1}{p_k} = \infty$.*

Proof: If S is dense in $C([0,1])$ then S must also be dense in $L^2(0,1)$ and so by Theorem 14.22 $\sum_{k=1}^{\infty} \frac{1}{p_k} = \infty$.

Suppose then that $\sum_{k=1}^{\infty} \frac{1}{p_k} = \infty$ so that by Theorem 14.22, S is dense in $L^2(0,1)$. The theorem will be proved if it is shown that for all m a nonnegative integer,

$$\max\{|x^m - f(x)| : x \in [0,1]\} < \varepsilon$$

for some $f \in S$. This is true if $m = 0$ because $1 \in S$. Suppose then that $m > 0$. Let S' denote finite linear combinations of the functions

$$\left\{x^{p_1-1}, x^{p_2-1}, \cdots\right\}.$$

These functions are also dense in $L^2(0,1)$ because $\sum \frac{1}{p_k - 1} = \infty$ by the limit comparison test. Then by Theorem 14.22 there exists $f \in S'$ such that

$$\left(\int_0^1 \left|f(x) - mx^{m-1}\right|^2 dx\right)^{1/2} < \varepsilon.$$

Thus $F(x) \equiv \int_0^x f(t)\, dt \in S$ and

$$\begin{aligned} |F(x) - x^m| &= \left|\int_0^x \left(f(t) - mt^{m-1}\right) dt\right| \\ &\leq \int_0^x \left|f(t) - mt^{m-1}\right| dt \\ &\leq \left(\int_0^1 \left|f(t) - mt^{m-1}\right|^2 dt\right)^{1/2} \left(\int_0^1 dx\right)^{1/2} \\ &< \varepsilon \end{aligned}$$

and this proves the theorem.

14.4 Orthonormal Sets

The concept of an orthonormal set of vectors is a generalization of the notion of the standard basis vectors of $\mathbb{R}^n$ or $\mathbb{C}^n$.

Definition 14.24 *Let H be a Hilbert space. $S \subseteq H$ is called an orthonormal set if $||x|| = 1$ for all $x \in S$ and $(x, y) = 0$ if $x, y \in S$ and $x \neq y$. For any set, D,*

$$D^{\perp} \equiv \{x \in H : (x, d) = 0 \text{ for all } d \in D\}.$$

If S is a set, $\operatorname{span}(S)$ *is the set of all finite linear combinations of vectors from S.*

You should verify that $D^{\perp}$ is always a closed subspace of H.

Theorem 14.25 *In any separable Hilbert space, H, there exists a countable orthonormal set, $S = \{x_i\}$ such that the span of these vectors is dense in H. Furthermore, if* $\operatorname{span}(S)$ *is dense, then for $x \in H$,*

$$x = \sum_{i=1}^{\infty} (x, x_i)\, x_i \equiv \lim_{n \to \infty} \sum_{i=1}^{n} (x, x_i)\, x_i. \tag{14.21}$$

Proof: Let $\mathcal{F}$ denote the collection of all orthonormal subsets of H. $\mathcal{F}$ is nonempty because $\{x\} \in \mathcal{F}$ where $||x|| = 1$. The set, $\mathcal{F}$ is a partially ordered set with the order given by set inclusion. By the Hausdorff maximal theorem, there exists a maximal chain, $\mathfrak{C}$ in $\mathcal{F}$. Then let $S \equiv \cup \mathfrak{C}$. It follows S must be a maximal orthonormal set of vectors. Why? It remains to verify that S is countable $\operatorname{span}(S)$ is dense, and the condition, 14.21 holds. To see S is countable note that if $x, y \in S$, then

$$||x - y||^2 = ||x||^2 + ||y||^2 - 2\operatorname{Re}(x, y) = ||x||^2 + ||y||^2 = 2.$$

Therefore, the open sets, $B\left(x, \frac{1}{2}\right)$ for $x \in S$ are disjoint and cover S. Since H is assumed to be separable, there exists a point from a countable dense set in each of these disjoint balls showing there can only be countably many of the balls and that consequently, S is countable as claimed.

It remains to verify 14.21 and that $\operatorname{span}(S)$ is dense. If $\overline{\operatorname{span}(S)}$ is not dense, then $\overline{\operatorname{span}(S)}$ is a closed proper subspace of H and letting $y \notin \overline{\operatorname{span}(S)}$,

$$z \equiv \frac{y - Py}{||y - Py||} \in \operatorname{span}(S)^{\perp}.$$

But then $S \cup \{z\}$ would be a larger orthonormal set of vectors contradicting the maximality of S.

It remains to verify 14.21. Let $S = \{x_i\}_{i=1}^{\infty}$ and consider the problem of choosing the constants, c_k in such a way as to minimize the expression

$$\left|\left| x - \sum_{k=1}^{n} c_k x_k \right|\right|^2 =$$

$$||x||^2 + \sum_{k=1}^{n} |c_k|^2 - \sum_{k=1}^{n} \overline{c_k}\,(x, x_k) - \sum_{k=1}^{n} c_k \overline{(x, x_k)}.$$

This equals

$$||x||^2 + \sum_{k=1}^{n} |c_k - (x, x_k)|^2 - \sum_{k=1}^{n} |(x, x_k)|^2$$

and therefore, this minimum is achieved when $c_k = (x, x_k)$ and equals

$$||x||^2 - \sum_{k=1}^{n} |(x, x_k)|^2$$

Now since $\text{span}(S)$ is dense, there exists n large enough that for some choice of constants, c_k,

$$\left|\left|x - \sum_{k=1}^{n} c_k x_k\right|\right|^2 < \varepsilon.$$

However, from what was just shown,

$$\left|\left|x - \sum_{i=1}^{n} (x, x_i) x_i\right|\right|^2 \leq \left|\left|x - \sum_{k=1}^{n} c_k x_k\right|\right|^2 < \varepsilon$$

showing that $\lim_{n\to\infty} \sum_{i=1}^{n} (x, x_i) x_i = x$ as claimed. This proves the theorem.

The proof of this theorem contains the following corollary.

Corollary 14.26 *Let S be any orthonormal set of vectors and let*

$$\{x_1, \cdots, x_n\} \subseteq S.$$

Then if $x \in H$

$$\left|\left|x - \sum_{k=1}^{n} c_k x_k\right|\right|^2 \geq \left|\left|x - \sum_{i=1}^{n} (x, x_i) x_i\right|\right|^2$$

for all choices of constants, c_k. In addition to this, Bessel's inequality

$$||x||^2 \geq \sum_{k=1}^{n} |(x, x_k)|^2.$$

If S is countable and $\text{span}(S)$ *is dense, then letting $\{x_i\}_{i=1}^{\infty} = S$, 14.21 follows.*

14.5 Fourier Series, An Example

In this section consider the Hilbert space, $L^2(0, 2\pi)$ with the inner product,

$$(f, g) \equiv \int_0^{2\pi} f\overline{g}dm.$$

This is a Hilbert space because of the theorem which states the L^p spaces are complete, Theorem 12.10 on Page 319. An example of an orthonormal set of functions in $L^2(0, 2\pi)$ is

$$\phi_n(x) \equiv \frac{1}{\sqrt{2\pi}} e^{inx}$$

for n an integer. Is it true that the span of these functions is dense in $L^2(0, 2\pi)$?

Theorem 14.27 *Let $S = \{\phi_n\}_{n\in\mathbb{Z}}$. Then* $\text{span}(S)$ *is dense in $L^2(0, 2\pi)$.*

Proof: By regularity of Lebesgue measure, it follows from Theorem 12.16 that $C_c(0,2\pi)$ is dense in $L^2(0,2\pi)$. Therefore, it suffices to show that for $g \in C_c(0,2\pi)$, then for every $\varepsilon > 0$ there exists $h \in \text{span}(S)$ such that $||g-h||_{L^2(0,2\pi)} < \varepsilon$.

Let T denote the points of $\mathbb{C}$ which are of the form e^{it} for $t \in \mathbb{R}$. Let $\mathcal{A}$ denote the algebra of functions consisting of polynomials in z and $1/z$ for $z \in T$. Thus a typical such function would be one of the form

$$\sum_{k=-m}^{m} c_k z^k$$

for m chosen large enough. This algebra separates the points of T because it contains the function, $p(z) = z$. It annihilates no point of t because it contains the constant function 1. Furthermore, it has the property that for $f \in \mathcal{A}$, $\overline{f} \in \mathcal{A}$. By the Stone Weierstrass approximation theorem, Theorem 7.16 on Page 165, $\mathcal{A}$ is dense in $C(T)$. Now for $g \in C_c(0,2\pi)$, extend g to all of $\mathbb{R}$ to be 2π periodic. Then letting $G(e^{it}) \equiv g(t)$, it follows G is well defined and continuous on T. Therefore, there exists $H \in \mathcal{A}$ such that for all $t \in \mathbb{R}$,

$$\left|H(e^{it}) - G(e^{it})\right| < \varepsilon^2/2\pi.$$

Thus $H(e^{it})$ is of the form

$$H(e^{it}) = \sum_{k=-m}^{m} c_k \left(e^{it}\right)^k = \sum_{k=-m}^{m} c_k e^{ikt} \in \text{span}(S).$$

Let $h(t) = \sum_{k=-m}^{m} c_k e^{ikt}$. Then

$$\begin{aligned}
\left(\int_0^{2\pi} |g-h|^2 \, dx\right)^{1/2} &\leq \left(\int_0^{2\pi} \max\{|g(t)-h(t)| : t \in [0,2\pi]\} \, dx\right)^{1/2} \\
&= \left(\int_0^{2\pi} \max\left\{\left|G(e^{it}) - H(e^{it})\right| : t \in [0,2\pi]\right\} dx\right)^{1/2} \\
&< \left(\int_0^{2\pi} \frac{\varepsilon^2}{2\pi}\right)^{1/2} = \varepsilon.
\end{aligned}$$

This proves the theorem.

Corollary 14.28 *For $f \in L^2(0,2\pi)$,*

$$\lim_{m\to\infty} \left|\left| f - \sum_{k=-m}^{m} (f,\phi_k)\,\phi_k \right|\right|_{L^2(0,2\pi)}$$

Proof: This follows from Theorem 14.25 on Page 379.

14.6 Compact Operators

14.6.1 Compact Operators In Hilbert Space

Definition 14.29 *Let $A \in \mathcal{L}(H, H)$ where H is a Hilbert space. Then $|(Ax, y)| \leq ||A||\,||x||\,||y||$ and so the map, $x \to (Ax, y)$ is continuous and linear. By the Riesz representation theorem, there exists a unique element of H, denoted by A^*y such that*

$$(Ax, y) = (x, A^*y).$$

It is clear $y \to A^*y$ is linear and continuous. A^* is called the adjoint of A. A is a self adjoint operator if $A = A^*$. Thus for a self adjoint operator, $(Ax, y) = (x, Ay)$ for all $x, y \in H$. A is a compact operator if whenever $\{x_k\}$ is a bounded sequence, there exists a convergent subsequence of $\{Ax_k\}$. Equivalently, A maps bounded sets to sets whose closures are compact.

The big result is called the Hilbert Schmidt theorem. It is a generalization to arbitrary Hilbert spaces of standard finite dimensional results having to do with diagonalizing a symmetric matrix. There is another statement and proof of this theorem around Page 598.

Theorem 14.30 *Let A be a compact self adjoint operator defined on a Hilbert space, H. Then there exists a countable set of eigenvalues, $\{\lambda_i\}$ and an orthonormal set of eigenvectors, u_i satisfying*

$$\lambda_i \text{ is real, } |\lambda_n| \geq |\lambda_{n+1}|, \ Au_i = \lambda_i u_i, \tag{14.22}$$

and either

$$\lim_{n \to \infty} \lambda_n = 0, \tag{14.23}$$

or for some n,

$$\operatorname{span}(u_1, \cdots, u_n) = H. \tag{14.24}$$

In any case,

$$\operatorname{span}\left(\{u_i\}_{i=1}^{\infty}\right) \text{ is dense in } A(H). \tag{14.25}$$

and for all $x \in H$,

$$Ax = \sum_{k=1}^{\infty} \lambda_k (x, u_k) u_k. \tag{14.26}$$

This sequence of eigenvectors and eigenvalues also satisfies

$$|\lambda_n| = ||A_n||, \tag{14.27}$$

and

$$A_n : H_n \to H_n. \tag{14.28}$$

where $H \equiv H_1$ and $H_n \equiv \{u_1, \cdots, u_{n-1}\}^{\perp}$ and A_n is the restriction of A to H_n.

Proof: If $A = 0$ then pick $u \in H$ with $||u|| = 1$ and let $\lambda_1 = 0$. Since $A(H) = 0$ it follows the span of u is dense in $A(H)$ and this proves the theorem in this uninteresting case.

Assume from now on $A \neq 0$. Let λ_1 be real and $\lambda_1^2 \equiv ||A||^2$. From the definition of $||A||$ there exists $x_n, ||x_n|| = 1$, and $||Ax_n|| \to ||A|| = |\lambda_1|$. Now it is clear that A^2 is also a compact self adjoint operator. Consider

$$\left(\left(\lambda_1^2 - A^2\right) x_n, x_n\right) = \lambda_1^2 - ||Ax_n||^2 \to 0.$$

Since A is compact, there exists a subsequence of $\{x_n\}$ still denoted by $\{x_n\}$ such that Ax_n converges to some element of H. Thus since $\lambda_1^2 - A^2$ satisfies

$$\left(\left(\lambda_1^2 - A^2\right) y, y\right) \geq 0$$

in addition to being self adjoint, it follows $x, y \to \left(\left(\lambda_1^2 - A^2\right) x, y\right)$ satisfies all the axioms for an inner product except for the one which says that $(z, z) = 0$ only if $z = 0$. Therefore, the Cauchy Schwarz inequality may be used to write

$$\begin{aligned} \left|\left(\left(\lambda_1^2 - A^2\right) x_n, y\right)\right| &\leq \left(\left(\lambda_1^2 - A^2\right) y, y\right)^{1/2} \left(\left(\lambda_1^2 - A^2\right) x_n, x_n\right)^{1/2} \\ &\leq e_n ||y|| . \end{aligned}$$

where $e_n \to 0$ as $n \to \infty$. Therefore, taking the sup over all $||y|| \leq 1$,

$$\lim_{n\to\infty} \left|\left|\left(\lambda_1^2 - A^2\right) x_n\right|\right| = 0.$$

Since $A^2 x_n$ converges, it follows since $\lambda_1 \neq 0$ that $\{x_n\}$ is a Cauchy sequence converging to x with $||x|| = 1$. Therefore, $A^2 x_n \to A^2 x$ and so

$$\left|\left|\left(\lambda_1^2 - A^2\right) x\right|\right| = 0.$$

Now

$$(\lambda_1 I - A)(\lambda_1 I + A) x = (\lambda_1 I + A)(\lambda_1 I - A) x = 0.$$

If $(\lambda_1 I - A) x = 0$, let $u_1 \equiv x$. If $(\lambda_1 I - A) x = y \neq 0$, let $u_1 \equiv \frac{y}{||y||}$.

Suppose $\{u_1, \cdots, u_n\}$ is such that $Au_k = \lambda_k u_k$ and $|\lambda_k| \geq |\lambda_{k+1}|$, $|\lambda_k| = ||A_k||$ and $A_k : H_k \to H_k$ for $k \leq n$. If

$$\text{span}(u_1, \cdots, u_n) = H$$

this yields the conclusion of the theorem in the situation of 14.24. Therefore, assume the span of these vectors is always a proper subspace of H. It is shown next that $A_{n+1} : H_{n+1} \to H_{n+1}$. Let

$$y \in H_{n+1} \equiv \{u_1, \cdots, u_n\}^{\perp}$$

Then for $k \leq n$

$$(Ay, u_k) = (y, Au_k) = \lambda_k (y, u_k) = 0,$$

showing $A_{n+1} : H_{n+1} \to H_{n+1}$ as claimed. There are two cases. Either $\lambda_n = 0$ or it is not. In the case where $\lambda_n = 0$ it follows $A_n = 0$. Every element of H is the sum of one in $\text{span}(u_1, \cdots, u_n)$ and one in $\text{span}(u_1, \cdots, u_n)^{\perp}$. (note $\text{span}(u_1, \cdots, u_n)$ is a closed subspace.) Thus, if $x \in H$, $x = y + z$ where $y \in \text{span}(u_1, \cdots, u_n)$ and $z \in \text{span}(u_1, \cdots, u_n)^{\perp}$ and $Az = 0$. Say $y = \sum_{j=1}^{n} c_j u_j$. Then

$$\begin{aligned} Ax &= Ay = \sum_{j=1}^{n} c_j A u_j \\ &= \sum_{j=1}^{n} c_j \lambda_j u_j \in \text{span}(u_1, \cdots, u_n). \end{aligned}$$

The conclusion of the theorem holds in this case because the above equation holds if with $c_i = (x, u_i)$.

Now consider the case where $\lambda_n \neq 0$. In this case repeat the above argument used to find u_{n+1} and λ_{n+1} for the operator, A_{n+1}. This yields $u_{n+1} \in H_{n+1} \equiv \{u_1, \cdots, u_n\}^{\perp}$ such that

$$||u_{n+1}|| = 1, ||Au_{n+1}|| = |\lambda_{n+1}| = ||A_{n+1}|| \leq ||A_n|| = |\lambda_n|$$

and if it is ever the case that $\lambda_n = 0$, it follows from the above argument that the conclusion of the theorem is obtained.

I claim $\lim_{n\to\infty} \lambda_n = 0$. If this were not so, then for some $\varepsilon > 0$, $0 < \varepsilon = \lim_{n\to\infty} |\lambda_n|$ but then

$$\begin{aligned} ||Au_n - Au_m||^2 &= ||\lambda_n u_n - \lambda_m u_m||^2 \\ &= |\lambda_n|^2 + |\lambda_m|^2 \geq 2\varepsilon^2 \end{aligned}$$

and so there would not exist a convergent subsequence of $\{Au_k\}_{k=1}^{\infty}$ contrary to the assumption that A is compact. This verifies the claim that $\lim_{n\to\infty} \lambda_n = 0$.

It remains to verify that $\text{span}(\{u_i\})$ is dense in $A(H)$. If $w \in \text{span}(\{u_i\})^{\perp}$ then $w \in H_n$ for all n and so for all n,

$$||Aw|| \leq ||A_n|| \, ||w|| \leq |\lambda_n| \, ||w|| .$$

Therefore, $Aw = 0$. Now every vector from H can be written as a sum of one from

$$\text{span}(\{u_i\})^{\perp} = \overline{\text{span}(\{u_i\})}^{\perp}$$

and one from $\overline{\text{span}(\{u_i\})}$. Therefore, if $x \in H$, $x = y + w$ where $y \in \overline{\text{span}(\{u_i\})}$ and $w \in \overline{\text{span}(\{u_i\})}^{\perp}$. It follows $Aw = 0$. Also, since $y \in \overline{\text{span}(\{u_i\})}$, there exist constants, c_k and n such that

$$\left|\left| y - \sum_{k=1}^{n} c_k u_k \right|\right| < \varepsilon.$$

Therefore, from Corollary 14.26,

$$\left|\left|y-\sum_{k=1}^{n}(y,u_k)\,u_k\right|\right| = \left|\left|y-\sum_{k=1}^{n}(x,u_k)\,u_k\right|\right| < \varepsilon.$$

Therefore,

$$\begin{aligned} ||A||\,\varepsilon &> \left|\left|A\left(y-\sum_{k=1}^{n}(x,u_k)\,u_k\right)\right|\right| \\ &= \left|\left|Ax-\sum_{k=1}^{n}(x,u_k)\,\lambda_k u_k\right|\right|. \end{aligned}$$

Since ε is arbitrary, this shows span $(\{u_i\})$ is dense in $A(H)$ and also implies 14.26. This proves the theorem.

Define $v\otimes u\in\mathcal{L}(H,H)$ by

$$v\otimes u\,(x) = (x,u)\,v,$$

then 14.26 is of the form

$$A=\sum_{k=1}^{\infty}\lambda_k u_k\otimes u_k$$

This is the content of the following corollary.

Corollary 14.31 *The main conclusion of the above theorem can be written as*

$$A=\sum_{k=1}^{\infty}\lambda_k u_k\otimes u_k$$

where the convergence of the partial sums takes place in the operator norm.

Proof: Using 14.26

$$\begin{aligned} &\left|\left(\left(A-\sum_{k=1}^{n}\lambda_k u_k\otimes u_k\right)x,y\right)\right| \\ =\ &\left|\left(Ax-\sum_{k=1}^{n}\lambda_k\,(x,u_k)\,u_k,y\right)\right| \\ =\ &\left|\left(\sum_{k=n}^{\infty}\lambda_k\,(x,u_k)\,u_k,y\right)\right| \\ =\ &\left|\sum_{k=n}^{\infty}\lambda_k\,(x,u_k)\,(u_k,y)\right| \\ \leq\ &|\lambda_n|\left(\sum_{k=n}^{\infty}|(x,u_k)|^2\right)^{1/2}\left(\sum_{k=n}^{\infty}|(y,u_k)|^2\right)^{1/2} \\ \leq\ &|\lambda_n|\,||x||\,||y|| \end{aligned}$$

It follows
$$\left|\left|\left(A-\sum_{k=1}^{n}\lambda_k u_k\otimes u_k\right)(x)\right|\right|\leq|\lambda_n|\,||x||$$
and this proves the corollary.

Corollary 14.32 *Let A be a compact self adjoint operator defined on a separable Hilbert space, H. Then there exists a countable set of eigenvalues, $\{\lambda_i\}$ and an orthonormal set of eigenvectors, v_i satisfying*

$$Av_i=\lambda_i v_i, ||v_i||=1, \tag{14.29}$$

$$\operatorname{span}\left(\{v_i\}_{i=1}^{\infty}\right) \text{ is dense in } H. \tag{14.30}$$

Furthermore, if $\lambda_i\neq 0$, the space, $V_{\lambda_i}\equiv\{x\in H: Ax=\lambda_i x\}$ is finite dimensional.

Proof: In the proof of the above theorem, let $W\equiv\overline{\operatorname{span}\left(\{u_i\}\right)}^{\perp}$. By Theorem 14.25, there is an orthonormal set of vectors, $\{w_i\}_{i=1}^{\infty}$ whose span is dense in W. As shown in the proof of the above theorem, $Aw=0$ for all $w\in W$. Let $\{v_i\}_{i=1}^{\infty}=\{u_i\}_{i=1}^{\infty}\cup\{w_i\}_{i=1}^{\infty}$.

It remains to verify the space, V_{λ_i}, is finite dimensional. First observe that $A:V_{\lambda_i}\to V_{\lambda_i}$. Since A is continuous, it follows that $A:\overline{V_{\lambda_i}}\to\overline{V_{\lambda_i}}$. Thus A is a compact self adjoint operator on $\overline{V_{\lambda_i}}$ and by Theorem 14.30, 14.24 holds because the only eigenvalue is λ_i. This proves the corollary.

Note the last claim of this corollary holds independent of the separability of H. This proves the corollary.

Suppose $\lambda\notin\{\lambda_n\}$ and $\lambda\neq 0$. Then the above formula for A, 14.26, yields an interesting formula for $(A-\lambda I)^{-1}$. Note first that since $\lim_{n\to\infty}\lambda_n=0$, it follows that $\lambda_n^2/(\lambda_n-\lambda)^2$ must be bounded, say by a positive constant, M.

Corollary 14.33 *Let A be a compact self adjoint operator and let $\lambda\notin\{\lambda_n\}_{n=1}^{\infty}$ and $\lambda\neq 0$ where the λ_n are the eigenvalues of A. Then*

$$(A-\lambda I)^{-1}x=-\frac{1}{\lambda}x+\frac{1}{\lambda}\sum_{k=1}^{\infty}\frac{\lambda_k}{\lambda_k-\lambda}(x,u_k)\,u_k. \tag{14.31}$$

Proof: Let $m<n$. Then since the $\{u_k\}$ form an orthonormal set,

$$\begin{aligned}\left|\sum_{k=m}^{n}\frac{\lambda_k}{\lambda_k-\lambda}(x,u_k)\,u_k\right| &= \left(\sum_{k=m}^{n}\left(\frac{\lambda_k}{\lambda_k-\lambda}\right)^2|(x,u_k)|^2\right)^{1/2} \qquad (14.32)\\ &\leq M\left(\sum_{k=m}^{n}|(x,u_k)|^2\right)^{1/2}.\end{aligned}$$

But from Bessel's inequality,

$$\sum_{k=1}^{\infty}|(x,u_k)|^2\leq||x||^2$$

and so for m large enough, the first term in 14.32 is smaller than ε. This shows the infinite series in 14.31 converges. It is now routine to verify that the formula in 14.31 is the inverse.

14.6.2 Nuclear Operators

Definition 14.34 *A self adjoint operator $A \in \mathcal{L}(H,H)$ for H a separable Hilbert space is called a nuclear operator if for some complete orthonormal set, $\{e_k\}$,*

$$\sum_{k=1}^{\infty} |(Ae_k, e_k)| < \infty$$

To begin with here is an interesting lemma.

Lemma 14.35 *Suppose $\{A_n\}$ is a sequence of compact operators in $\mathcal{L}(X,Y)$ for two Banach spaces, X and Y and suppose $A \in \mathcal{L}(X,Y)$ and*

$$\lim_{n\to\infty} ||A - A_n|| = 0.$$

Then A is also compact.

Proof: Let B be a bounded set in X such that $||b|| \leq C$ for all $b \in B$. I need to verify AB is totally bounded. Suppose then it is not. Then there exists $\varepsilon > 0$ and a sequence, $\{Ab_i\}$ where $b_i \in B$ and

$$||Ab_i - Ab_j|| \geq \varepsilon$$

whenever $i \neq j$. Then let n be large enough that

$$||A - A_n|| \leq \frac{\varepsilon}{4C}.$$

Then

$$\begin{aligned} ||A_n b_i - A_n b_j|| &= ||Ab_i - Ab_j + (A_n - A) b_i - (A_n - A) b_j|| \\ &\geq ||Ab_i - Ab_j|| - ||(A_n - A) b_i|| - ||(A_n - A) b_j|| \\ &\geq ||Ab_i - Ab_j|| - \frac{\varepsilon}{4C} C - \frac{\varepsilon}{4C} C \geq \frac{\varepsilon}{2}, \end{aligned}$$

a contradiction to A_n being compact. This proves the lemma.

Then one can prove the following lemma. In this lemma, $A \geq 0$ will mean $(Ax, x) \geq 0$.

Lemma 14.36 *Let $A \geq 0$ be a nuclear operator defined on a separable Hilbert space, H. Then A is compact and also, whenever $\{e_k\}$ is a complete orthonormal set,*

$$A = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} (Ae_i, e_j)\, e_i \otimes e_j.$$

Proof: First consider the formula. Since A is given to be continuous,

$$Ax = A\left(\sum_{j=1}^{\infty} (x, e_j)\, e_j\right) = \sum_{j=1}^{\infty} (x, e_j)\, Ae_j,$$

the series converging because

$$x = \sum_{j=1}^{\infty} (x, e_j)\, e_j$$

Then also since A is self adjoint,

$$\begin{aligned}
\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} (Ae_i, e_j)\, e_i \otimes e_j\,(x) &\equiv \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} (Ae_i, e_j)\,(x, e_j)\, e_i \\
&= \sum_{j=1}^{\infty} (x, e_j) \sum_{i=1}^{\infty} (Ae_i, e_j)\, e_i \\
&= \sum_{j=1}^{\infty} (x, e_j) \sum_{i=1}^{\infty} (Ae_j, e_i)\, e_i \\
&= \sum_{j=1}^{\infty} (x, e_j)\, Ae_j
\end{aligned}$$

Next consider the claim that A is compact. Let $C_A \equiv \left(\sum_{j=1}^{\infty} |(Ae_j, e_j)|\right)^{1/2}$. Let A_n be defined by

$$A_n \equiv \sum_{j=1}^{\infty}\sum_{i=1}^{n} (Ae_i, e_j)\,(e_i \otimes e_j)\,.$$

Then A_n has values in $\operatorname{span}(e_1, \cdots, e_n)$ and so it must be a compact operator because bounded sets in a finite dimensional space must be precompact. Then

$$\begin{aligned}
|(Ax - A_n x, y)| &= \left|\sum_{j=1}^{\infty}\sum_{i=n+1}^{\infty} (Ae_i e_j)\,(y, e_j)\,(e_i, x)\right| \\
&= \left|\sum_{j=1}^{\infty} (y, e_j) \sum_{i=n+1}^{\infty} (Ae_i e_j)\,(e_i, x)\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \left|\sum_{j=1}^{\infty} |(y,e_j)| (Ae_j,e_j)^{1/2} \sum_{i=n+1}^{\infty} (Ae_ie_i)^{1/2} |(e_i,x)|\right| \\
&\leq \left(\sum_{j=1}^{\infty} |(y,e_j)|^2\right)^{1/2} \left(\sum_{j=1}^{\infty} |(Ae_j,e_j)|\right)^{1/2} \\
&\quad \cdot \left(\sum_{i=n+1}^{\infty} |(x,e_i)|^2\right)^{1/2} \left(\sum_{i=n+1}^{\infty} |(Ae_ie_i)|\right)^{1/2} \\
&\leq |y|\,|x|\, C_A \left(\sum_{i=n+1}^{\infty} |(Ae_i,e_i)|\right)^{1/2}
\end{aligned}$$

and this shows that if n is sufficiently large,

$$|((A - A_n)\,x, y)| \leq \varepsilon\,|x|\,|y|\,.$$

Therefore,

$$\lim_{n\to\infty} ||A - A_n|| = 0$$

and so A is the limit in operator norm of finite rank bounded linear operators, each of which is compact. Therefore, A is also compact.

Definition 14.37 *The trace of a nuclear operator $A \in \mathcal{L}(H,H)$ such that $A \geq 0$ is defined to equal*

$$\sum_{k=1}^{\infty} (Ae_k, e_k)$$

where $\{e_k\}$ is an orthonormal basis for the Hilbert space, H.

Theorem 14.38 *Definition 14.37 is well defined and equals $\sum_{j=1}^{\infty} \lambda_j$ where the λ_j are the eigenvalues of A.*

Proof: Suppose $\{u_k\}$ is some other orthonormal basis. Then

$$e_k = \sum_{j=1}^{\infty} u_j\,(e_k, u_j)$$

By Lemma 14.36 A is compact and so

$$A = \sum_{k=1}^{\infty} \lambda_k u_k \otimes u_k$$

where the u_k are the orthonormal eigenvectors of A which form a complete orthonormal set. Then

$$\begin{aligned}
\sum_{k=1}^{\infty}(Ae_k,e_k) &= \sum_{k=1}^{\infty}\left(A\left(\sum_{j=1}^{\infty}u_j(e_k,u_j)\right),\sum_{j=1}^{\infty}u_j(e_k,u_j)\right) \\
&= \sum_{k=1}^{\infty}\sum_{ij}(Au_j,u_i)(e_k,u_j)(u_i,e_k) \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(Au_j,u_j)\left|(e_k,u_j)\right|^2 \\
&= \sum_{j=1}^{\infty}(Au_j,u_j)\sum_{k=1}^{\infty}\left|(e_k,u_j)\right|^2=\sum_{j=1}^{\infty}(Au_j,u_j)\left|u_j\right|^2 \\
&= \sum_{j=1}^{\infty}(Au_j,u_j)=\sum_{j=1}^{\infty}\lambda_j
\end{aligned}$$

and this proves the theorem.

This is just like it is for a matrix. Recall the trace of a matrix is the sum of the eigenvalues.

It is also easy to see that in any separable Hilbert space, there exist nuclear operators. Let $\sum_{k=1}^{\infty}|\lambda_k|<\infty$. Then let $\{e_k\}$ be a complete orthonormal set of vectors. Let

$$A\equiv\sum_{k=1}^{\infty}\lambda_k e_k\otimes e_k.$$

It is not too hard to verify this works.

Much more can be said about nuclear operators.

14.6.3 Hilbert Schmidt Operators

Definition 14.39 *Let H and G be two separable Hilbert spaces and let T map H to G be linear. Then T is called a Hilbert Schmidt operator if there exists some orthonormal basis for H, $\{e_j\}$ such that*

$$\sum_j\left|\left|Te_j\right|\right|^2<\infty.$$

The collection of all such linear maps will be denoted by $\mathcal{L}_2(H,G)$.

Theorem 14.40 *$\mathcal{L}_2(H,G)\subseteq\mathcal{L}(H,G)$ and $\mathcal{L}_2(H,G)$ is a separable Hilbert space with norm given by*

$$||T||_{\mathcal{L}_2}\equiv\left(\sum_k\left|\left|Te_k\right|\right|^2\right)^{1/2}$$

where $\{e_k\}$ is some orthonormal basis for H. Also $\mathcal{L}_2(H,G) \subseteq \mathcal{L}(H,G)$ and

$$||T|| \leq ||T||_{\mathcal{L}_2}. \tag{14.33}$$

All Hilbert Schmidt opearators are compact. Also for $X \in H$ and $Y \in G, X \otimes Y \in \mathcal{L}_2(H,G)$ and

$$||X \otimes Y||_{\mathcal{L}_2} = ||X||_H \, ||Y||_G \tag{14.34}$$

Proof: First consider the norm. I need to verify the norm does not depend on the choice of orthonormal basis. Let $\{f_k\}$ be an orthonormal basis for G. Then for $\{e_k\}$ an orthonormal basis for H,

$$\begin{aligned} \sum_k ||Te_k||^2 &= \sum_k \sum_j |(Te_k, f_j)|^2 = \sum_k \sum_j |(e_k, T^* f_j)|^2 \\ &= \sum_j \sum_k |(e_k, T^* f_j)|^2 = \sum_j ||T^* f_j||^2 . \end{aligned}$$

The same result would be obtained for any other orthonormal basis $\{e'_j\}$ and this shows the norm is at least well defined. It is clear this does indeed satisfy the axioms of a norm.

Next I want to show $\mathcal{L}_2(H,G) \subseteq \mathcal{L}(H,G)$ and $||T|| \leq ||T||_{\mathcal{L}_2}$. Pick an orthonormal basis for $H, \{e_k\}$ and an orthonormal basis for $G, \{f_k\}$. Then letting

$$x = \sum_{k=1}^{n} x_k e_k,$$

$$Tx = T\left(\sum_{k=1}^{n} x_k e_k\right) = \sum_{k=1}^{n} x_k T(e_k)$$

where $x_k \equiv (x, e_k)$. Therefore using Minkowski's inequality,

$$\begin{aligned}
||Tx|| &= \left(\sum_{k=1}^{\infty} |(Tx, f_k)|^2\right)^{1/2} \\
&= \left(\sum_{k=1}^{\infty} \left|\left(\sum_{j=1}^{n} x_j T e_j, f_k\right)\right|^2\right)^{1/2} \\
&= \left(\sum_{k=1}^{\infty} \left|\sum_{j=1}^{n} (x_j T e_j, e_k)\right|^2\right)^{1/2} \\
&\leq \sum_{j=1}^{n} \left(\sum_{k=1}^{\infty} |(x_j T e_j, e_k)|^2\right)^{1/2} \\
&\leq \sum_{j} |x_j| \left(\sum_{k} |(T e_j, e_k)|^2\right)^{1/2} \\
&= \sum_{j} |x_j| \, ||T e_j|| \leq \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2} ||T||_{\mathcal{L}_2} \\
&= ||x|| \, ||T||_{\mathcal{L}_2}
\end{aligned}$$

Therefore, since finite sums of the form $\sum_{k=1}^{n} x_k e_k$ are dense in H, it follows $T \in \mathcal{L}(H, G)$ and $||T|| \leq ||T||_{\mathcal{L}_2}$ and this proves the above claims.

It only remains to verify $\mathcal{L}_2(H, G)$ is a separable Hilbert space. It is clear it is an inner product space because you only have to pick an orthonormal basis, $\{e_k\}$ and define the inner product as

$$(S, T) \equiv \sum_{k} (Se_k, Te_k).$$

The only remaining issue is the completeness. Suppose then that $\{T_n\}$ is a Cauchy sequence in $\mathcal{L}_2(H, G)$. Then from 14.33 $\{T_n\}$ is a Cauchy sequence in $\mathcal{L}(H, G)$ and so there exists a unique T such that $\lim_{n\to\infty} ||T_n - T|| = 0$. Then it only remains to verify $T \in \mathcal{L}_2(H, G)$. But by Fatou's lemma,

$$\begin{aligned}
\sum_{k} ||Te_k||^2 &\leq \lim \inf_{n\to\infty} \sum_{k} ||T_n e_k||^2 \\
&= \lim \inf_{n\to\infty} ||T_n||_{\mathcal{L}_2}^2 < \infty.
\end{aligned}$$

All that remains is to verify $\mathcal{L}_2(H, G)$ is separable and these Hilbert Schmidt operators are compact. I will show an orthonormal basis for $\mathcal{L}_2(H, G)$ is $\{f_j \otimes e_k\}$

where $\{f_k\}$ is an orthonormal basis for G and $\{e_k\}$ is an orthonormal basis for H. Here, for $f \in G$ and $e \in H$,

$$f \otimes e(x) \equiv (x, e) f.$$

I need to show $f_j \otimes e_k \in \mathcal{L}_2(H, G)$ and that it is an orthonormal basis for $\mathcal{L}_2(H, G)$ as claimed.

$$\sum_k ||f_j \otimes e_i(e_k)||^2 = \sum_k ||f_j \delta_{ik}||^2 = ||f_j||^2 = 1 < \infty$$

so each of these operators is in $\mathcal{L}_2(H, G)$. Next I show they are orthonormal.

$$\begin{aligned}(f_j \otimes e_k, f_s \otimes e_r) &= \sum_p (f_j \otimes e_k(e_p), f_s \otimes e_r(e_p)) \\ &= \sum_p \delta_{rp}\delta_{kp}(f_j, f_s) = \sum_p \delta_{rp}\delta_{kp}\delta_{js}\end{aligned}$$

If $j = s$ and $k = r$ this reduces to 1. Otherwise, this gives 0. Thus these operators are orthonormal. Now let $T \in \mathcal{L}_2(H, G)$. Consider

$$T_n \equiv \sum_{i=1}^n \sum_{j=1}^n (Te_i, f_j) f_j \otimes e_i$$

Then

$$\begin{aligned}T_n e_k &= \sum_{i=1}^n \sum_{j=1}^n (Te_i, f_j)(e_k, e_i) f_j \\ &= \sum_{j=1}^n (Te_k, f_j) f_j\end{aligned}$$

It follows

$$||T_n e_k|| \leq ||Te_k||$$

and

$$\lim_{n\to\infty} T_n e_k = Te_k.$$

Therefore, from the dominated convergence theorem,

$$\lim_{n\to\infty} ||T - T_n||_{\mathcal{L}_2}^2 \equiv \lim_{n\to\infty} \sum_k ||(T - T_n) e_k||^2 = 0.$$

Therefore, the linear combinations of the $f_j \otimes e_i$ are dense in $\mathcal{L}_2(H, G)$ and this proves completeness.

This also shows $\mathcal{L}_2(H, G)$ is separable. From 14.33 it also shows that every $T \in \mathcal{L}_2(H, G)$ is the limit in the operator norm of a sequence of compact operators. This follows because each of the $f_j \otimes e_i$ is easily seen to be a compact operator.

(This follows because each of the $f_j \otimes e_i$ is easily seen to be a compact operator because if $x_m \to x$ weakly, then

$$f_j \otimes e_i (x_m) = (x_m, e_i) f_j \to (x, e_i) f_j = f_j \otimes e_i (x)$$

and since if $\{x_m\}$ is any bounded sequence, there exists a subsequence, $\{x_{n_k}\}$ which converges weakly and by the above, $f_j \otimes e_i (x_{n_k}) \to f_j \otimes e_i (x)$ showing bounded sets are mapped to precompact sets.) Therefore, each $T \in \mathcal{L}_2 (H, G)$ must also be a compact operator. Here is why.

Let B be a bounded set in which $||x|| < M$ for all $x \in B$ and consider TB. I need to show TB is totally bounded. Let $\varepsilon > 0$ be given. Then let $||T_m - T|| < \frac{\varepsilon}{3M}$ where T_m is a compact operator like those described above and let $\{T_m x_j\}_{j=1}^N$ be an $\varepsilon/3$ net for $T_m (B)$. Then

$$||Tx_j - T_m x_j|| < \frac{\varepsilon}{3}$$

and so letting $x \in B$, pick x_j such that $||T_m x - T_m x_j|| < \varepsilon/3$. Then

$$\begin{aligned} ||Tx - Tx_j|| &\leq ||Tx - T_m x|| + ||T_m x - T_m x_j|| + ||T_m x_j - Tx_j|| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

showing $\{Tx_j\}_{j=1}^N$ is an ε net for TB.

Finally, consider 14.34. Let $\{e_k\}$ be an orthonormal basis for H and consider the following computation which establishes which establishes this equation.

$$\begin{aligned} ||Y \otimes X||_{\mathcal{L}_2}^2 &\equiv \sum_{k=1}^{\infty} ||Y \otimes X (e_k)||^2 \\ &= \sum_{k=1}^{\infty} ||(e_k, X) Y||^2 \\ &= ||Y||_G^2 \sum_{k=1}^{\infty} |(e_k, X)|^2 \\ &= ||Y||_G^2 ||X||_H^2 < \infty. \end{aligned} \tag{14.35}$$

This proves the theorem.

14.7 Compact Operators In Banach Space

In general for $A \in \mathcal{L}(X, Y)$ the following definition holds.

Definition 14.41 *Let $A \in \mathcal{L}(X, Y)$. Then A is compact if whenever $B \subseteq X$ is a bounded set, AB is precompact. Equivalently, if $\{x_n\}$ is a bounded sequence in X, then $\{Ax_n\}$ has a subsequence which converges in Y.*

An important result is the following theorem about the adjoint of a compact operator.

Theorem 14.42 *Let $A \in \mathcal{L}(X,Y)$ be compact. Then the adjoint operator, $A^* \in \mathcal{L}(Y', X')$ is also compact.*

Proof: Let $\{y_n^*\}$ be a bounded sequence in Y'. Let B be the closure of the unit ball in X. Then AB is precompact. Then it is clear that the functions $\{y_n^*\}$ are equicontinuous and uniformly bounded on the compact set, $\overline{A(B)}$. By the Ascoli Arzela theorem, there is a subsequence $\left\{y_{n_k}^*\right\}$ which converges uniformly to a continuous function, f on $\overline{A(B)}$. Now define g on AX by

$$g(Ax) = ||x|| f\left(A\left(\frac{x}{||x||}\right)\right), g(A0) = 0.$$

Thus for $x_1, x_2 \neq 0$, and a, b scalars,

$$\begin{aligned} g(aAx_1 + bAx_2) &\equiv ||ax_1 + bx_2|| f\left(\frac{A(ax_1 + bx_2)}{||ax_1 + bx_2||}\right) \\ &\equiv \lim_{k\to\infty} ||ax_1 + bx_2|| y_{n_k}^*\left(\frac{A(ax_1 + bx_2)}{||ax_1 + bx_2||}\right) \\ &= \lim_{k\to\infty} ay_{n_k}^*(Ax_1) + by_{n_k}^*(Ax_2) \\ &= a\lim_{k\to\infty} ||x_1|| y_{n_k}^*\left(\frac{Ax_1}{||x_1||}\right) + b\lim_{k\to\infty} ||x_2|| y_{n_k}^*\left(\frac{Ax_2}{||x_2||}\right) \\ &= a||x_1|| f\left(\frac{Ax_1}{||x_1||}\right) + b||x_2|| f\left(\frac{Ax_2}{||x_2||}\right) \\ &\equiv ag(Ax_1) + bg(Ax_2) \end{aligned}$$

showing that g is linear on AX. Also

$$|g(Ax)| = \lim_{k\to\infty}\left|||x|| y_{n_k}^*\left(A\left(\frac{x}{||x||}\right)\right)\right| \leq C||x||\left|\left|A\left(\frac{x}{||x||}\right)\right|\right| = C||Ax||$$

and so by the Hahn Banach theorem, there exists y^* extending g to all of Y having the same operator norm.

$$y^*(Ax) = \lim_{k\to\infty} ||x|| y_{n_k}^*\left(A\left(\frac{x}{||x||}\right)\right) = \lim_{k\to\infty} y_{n_k}^*(Ax)$$

Thus $A^*y_{n_k}^*(x) \to A^*y^*(x)$ for every x. In addition to this, for $x \in B$,

$$\begin{aligned} \left|\left|A^*y^*(x) - A^*y_{n_k}^*(x)\right|\right| &= \left|\left|y^*(Ax) - y_{n_k}^*(Ax)\right|\right| \\ &= \left|\left|g(Ax) - y_{n_k}^*(Ax)\right|\right| \\ &= \left|\left|||x|| f\left(A\left(\frac{x}{||x||}\right)\right) - ||x|| y_{n_k}^*\left(\frac{Ax}{||x||}\right)\right|\right| \\ &\leq \left|\left|f\left(A\left(\frac{x}{||x||}\right)\right) - y_{n_k}^*\left(\frac{Ax}{||x||}\right)\right|\right| \end{aligned}$$

and this is uniformly small for large k due to the uniform convergence of $y_{n_k}^*$ to f on $\overline{A(B)}$. Therefore, $\left|\left|A^* y^* - A^* y_{n_k}^*\right|\right| \to 0$.

14.8 The Fredholm Alternative

Recall that if A is an $n \times n$ matrix and if the only solution to the system, $A\mathbf{x} = 0$ is $\mathbf{x} = 0$ then for any $\mathbf{y} \in \mathbb{R}^n$ it follows that there exists a unique solution to the system $A\mathbf{x} = \mathbf{y}$. This holds because the first condition implies A is one to one and therefore, A^{-1} exists. Of course things are much harder in a general Banach space. Here is a simple example for a Hilbert space.

Example 14.43 *Let $L^2(\mathbb{N}; \mu) = H$ where μ is counting measure. Thus an element of H is a sequence, $\mathbf{a} = \{a_i\}_{i=1}^{\infty}$ having the property that*

$$||\mathbf{a}||_H \equiv \left(\sum_{k=1}^{\infty} |a_k|^2\right)^{1/2} < \infty.$$

Define $A : H \to H$ by

$$A\mathbf{a} \equiv \mathbf{b} \equiv \{0, a_1, a_2, \cdots\}.$$

Thus A slides the sequence to the right and puts a zero in the first slot. Clearly A is one to one and linear but it cannot be onto because it fails to yield $\mathbf{e}_1 \equiv \{1, 0, 0, \cdots\}$.

Notwithstanding the above example, there are theorems which are like the linear algebra theorem mentioned above which hold in an arbitrary Banach spaces in the case where the operator is compact. To begin with here is an interesting lemma.

Lemma 14.44 *Suppose $A \in \mathcal{L}(X, X)$ is compact for X a Banach space. Then $(I - A)(X)$ is a closed subspace of X.*

Proof: Suppose $(I - A)x_n \to y$. Let

$$\alpha_n \equiv \text{dist}(x_n, \ker(I - A))$$

and let $z_n \in \ker(I - A)$ be such that

$$\alpha_n \leq ||x_n - z_n|| \leq \left(1 + \frac{1}{n}\right)\alpha_n.$$

Thus $(I - A)(x_n - z_n) \to y$ because $(I - A)z_n = 0$.

Case 1: $\{x_n - z_n\}$ has a bounded subsequence.

If this is so, the compactness of A implies there exists a subsequence, still denoted by n such that $\{A(x_n - z_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. Since $(I - A)(x_n - z_n) \to y$, this implies $\{(x_n - z_n)\}$ is also a Cauchy sequence converging to a point, $x \in X$. Then, taking the limit as $n \to \infty$, $(I - A)x = y$ and so $y \in (I - A)(X)$.

Case 2: $\lim_{n \to \infty} ||x_n - z_n|| = \infty$. I will show this case cannot occur.

In this case, let $w_n \equiv \frac{x_n - z_n}{||x_n - z_n||}$. Thus $(I - A) w_n \to 0$ and w_n is bounded. Therefore, there exists a subsequence, still denoted by n such that $\{Aw_n\}$ is a Cauchy sequence. Now it follows

$$Aw_n - Aw_m + e_n - e_m = w_n - w_m$$

where $e_k \to 0$ as $k \to \infty$. This implies $\{w_n\}$ is a Cauchy sequence which must converge to some $w_\infty \in X$. Therefore, $(I - A) w_\infty = 0$ and so $w_\infty \in \ker (I - A)$. However, this is impossible because of the following argument. If $z \in \ker (I - A)$,

$$\begin{aligned} ||w_n - z|| &= \frac{1}{||x_n - z_n||} ||x_n - z_n - ||x_n - z_n|| \, z|| \\ &\geq \frac{1}{||x_n - z_n||} \alpha_n \geq \frac{\alpha_n}{\left(1 + \frac{1}{n}\right) \alpha_n} = \frac{n}{n+1}. \end{aligned}$$

Taking the limit, $||w_\infty - z|| \geq 1$. Since $z \in \ker (I - A)$ is arbitrary, this shows $\text{dist}\,(w_\infty, \ker (I - A)) \geq 1$.

Since Case 2 does not occur, this proves the lemma.

Theorem 14.45 *Let $A \in \mathcal{L}(X, X)$ be a compact operator and let $f \in X$. Then there exists a solution, x, to*

$$x - Ax = f \tag{14.36}$$

if and only if

$$x^* (f) = 0 \tag{14.37}$$

for all $x^* \in \ker (I - A^*)$.

Proof: Suppose x is a solution to 14.36 and let $x^* \in \ker (I - A^*)$. Then

$$x^* (f) = x^* ((I - A)(x)) = ((I - A^*) x^*)(x) = 0.$$

Next suppose $x^* (f) = 0$ for all $x^* \in \ker (I - A^*)$. I will show there exists x solving 14.36. By Lemma 14.44, $(I - A)(X)$ is a closed subspace of X. Is $f \in (I - A)(X)$? If not, then by the Hahn Banach theorem, there exists $x^* \in X'$ such that $x^* (f) \neq 0$ but $x^* ((I - A)(x)) = 0$ for all $x \in X$. However last statement says nothing more nor less than $(I - A^*) x^* = 0$. This is a contradiction because for such x^*, it is given that $x^* (f) = 0$. This proves the theorem.

The following corollary is called the Fredholm alternative.

Corollary 14.46 *Let $A \in \mathcal{L}(X, X)$ be a compact operator. Then there exists a solution to the equation*

$$x - Ax = f \tag{14.38}$$

for all $f \in X$ if and only if $(I - A^*)$ is one to one on X'.

Proof: Suppose $(I - A^*)$ is one to one first. Then if $x^* - A^*x^* = 0$ it follows $x^* = 0$ and so for any $f \in X$, $x^*(f) = 0$ for all $x^* \in \ker(I - A^*)$. By 14.45 there exists a solution to $(I - A)x = f$.

Now suppose there exists a solution, x, to $(I - A)x = f$ for every $f \in X$. If $(I - A^*)x^* = 0$, then for every $x \in X$,

$$(I - A^*)x^*(x) = x^*((I - A)(x)) = 0$$

Since $(I - A)$ is onto, this shows $x^* = 0$ and so $(I - A^*)$ is one to one as claimed. This proves the corollary.

The following is just an easier version of the above.

Corollary 14.47 *In the case where X is a Hilbert space, the conclusions of Corollary 14.46, Theorem 14.45, and Lemma 14.44 remain true if H' is replaced by H and the adjoint is understood in the usual manner for Hilbert space. That is*

$$(Ax, y)_H = (x, A^*y)_H$$

Representation Theorems

15.1 Radon Nikodym Theorem

This chapter is on various representation theorems. The first theorem, the Radon Nikodym Theorem, is a representation theorem for one measure in terms of another. The approach given here is due to Von Neumann and depends on the Riesz representation theorem for Hilbert space, Theorem 14.14 on Page 370.

Definition 15.1 *Let μ and λ be two measures defined on a σ-algebra, $\mathcal{S}$, of subsets of a set, Ω. λ is absolutely continuous with respect to μ,written as $\lambda \ll \mu$, if $\lambda(E) = 0$ whenever $\mu(E) = 0$.*

It is not hard to think of examples which should be like this. For example, suppose one measure is volume and the other is mass. If the volume of something is zero, it is reasonable to expect the mass of it should also be equal to zero. In this case, there is a function called the density which is integrated over volume to obtain mass. The Radon Nikodym theorem is an abstract version of this notion. Essentially, it gives the existence of the density function.

Theorem 15.2 *(Radon Nikodym) Let λ and μ be finite measures defined on a σ-algebra, $\mathcal{S}$, of subsets of Ω. Suppose $\lambda \ll \mu$. Then there exists a unique $f \in L^1(\Omega, \mu)$ such that $f(x) \geq 0$ and*

$$\lambda(E) = \int_E f \, d\mu.$$

If it is not necessarily the case that $\lambda \ll \mu$, there are two measures, $\lambda_\perp$ and $\lambda_{||}$ such that $\lambda = \lambda_\perp + \lambda_{||}, \lambda_{||} \ll \mu$ and there exists a set of μ measure zero, N such that for all E measurable, $\lambda_\perp(E) = \lambda(E \cap N) = \lambda_\perp(E \cap N)$. In this case the two mesures, $\lambda_\perp$ and $\lambda_{||}$ are unique and the representation of $\lambda = \lambda_\perp + \lambda_{||}$ is called the Lebesgue decomposition of λ. The measure $\lambda_{||}$ is the absolutely continuous part of λ and $\lambda_\perp$ is called the singular part of λ.

Proof: Let $\Lambda : L^2(\Omega, \mu + \lambda) \to \mathbb{C}$ be defined by

$$\Lambda g = \int_\Omega g \, d\lambda.$$

By Holder's inequality,

$$|\Lambda g| \leq \left(\int_{\Omega} 1^2 d\lambda\right)^{1/2} \left(\int_{\Omega} |g|^2 \, d\left(\lambda+\mu\right)\right)^{1/2} = \lambda\left(\Omega\right)^{1/2} ||g||_2$$

where $||g||_2$ is the L^2 norm of g taken with respect to $\mu+\lambda$. Therefore, since Λ is bounded, it follows from Theorem 13.4 on Page 339 that $\Lambda \in (L^2(\Omega,\mu+\lambda))'$, the dual space $L^2(\Omega,\mu+\lambda)$. By the Riesz representation theorem in Hilbert space, Theorem 14.14, there exists a unique $h \in L^2(\Omega,\mu+\lambda)$ with

$$\Lambda g = \int_{\Omega} g \, d\lambda = \int_{\Omega} hgd(\mu+\lambda). \tag{15.1}$$

The plan is to show h is real and nonnegative at least a.e. Therefore, consider the set where $\operatorname{Im} h$ is positive.

$$E = \left\{x \in \Omega : \operatorname{Im} h(x) > 0\right\},$$

Now let $g = \mathcal{X}_E$ and use 15.1 to get

$$\lambda(E) = \int_{E} (\operatorname{Re} h + i \ \operatorname{Im} h) d(\mu+\lambda). \tag{15.2}$$

Since the left side of 15.2 is real, this shows

$$\begin{aligned} 0 &= \int_{E} (\operatorname{Im} h)\, d(\mu+\lambda) \\ &\geq \int_{E_n} (\operatorname{Im} h)\, d(\mu+\lambda) \\ &\geq \frac{1}{n}\left(\mu+\lambda\right)\left(E_n\right) \end{aligned}$$

where

$$E_n \equiv \left\{x : \operatorname{Im} h\left(x\right) \geq \frac{1}{n}\right\}$$

Thus $\left(\mu+\lambda\right)\left(E_n\right) = 0$ and since $E = \cup_{n=1}^{\infty} E_n$, it follows $\left(\mu+\lambda\right)\left(E\right) = 0$. A similar argument shows that for

$$E = \{x \in \Omega : \operatorname{Im} h\left(x\right) < 0\},$$

$(\mu+\lambda)(E) = 0$. Thus there is no loss of generality in assuming h is real-valued.

The next task is to show h is nonnegative. This is done in the same manner as above. Define the set where it is negative and then show this set has measure zero.

Let $E \equiv \{x : h(x) < 0\}$ and let $E_n \equiv \{x : h(x) < -\frac{1}{n}\}$. Then let $g = \mathcal{X}_{E_n}$. Since $E = \cup_n E_n$, it follows that if $\left(\mu+\lambda\right)\left(E\right) > 0$ then this is also true for $\left(\mu+\lambda\right)\left(E_n\right)$ for all n large enough. Then from 15.2

$$\lambda(E_n) = \int_{E_n} h \, d(\mu+\lambda) \leq -\left(1/n\right)\left(\mu+\lambda\right)\left(E_n\right) < 0,$$

a contradiction. Thus it can be assumed $h \geq 0$.

At this point the argument splits into two cases.

Case Where $\lambda \ll \mu$.

In this case, $h < 1$. Let $E = [h \geq 1]$ and let $g = \mathcal{X}_E$. Then

$$\lambda(E) = \int_E h \, d(\mu + \lambda) \geq \mu(E) + \lambda(E).$$

Therefore $\mu(E) = 0$. Since $\lambda \ll \mu$, it follows that $\lambda(E) = 0$ also. Thus it can be assumed

$$0 \leq h(x) < 1$$

for all x.

From 15.1, whenever $g \in L^2(\Omega, \mu + \lambda)$,

$$\int_\Omega g(1-h)d\lambda = \int_\Omega hg d\mu. \tag{15.3}$$

Now let E be a measurable set and define

$$g(x) \equiv \sum_{i=0}^{n} h^i(x)\mathcal{X}_E(x)$$

in 15.3. This yields

$$\int_E (1 - h^{n+1}(x))d\lambda = \int_E \sum_{i=1}^{n+1} h^i(x)d\mu. \tag{15.4}$$

Let $f(x) = \sum_{i=1}^{\infty} h^i(x)$ and use the Monotone Convergence theorem in 15.4 to let $n \to \infty$ and conclude

$$\lambda(E) = \int_E f \, d\mu.$$

$f \in L^1(\Omega, \mu)$ because λ is finite.

The function, f is unique μ a.e. because, if g is another function which also serves to represent λ, consider for each $n \in \mathbb{N}$ the set,

$$E_n \equiv \left[f - g > \frac{1}{n} \right]$$

and conclude that

$$0 = \int_{E_n} (f - g) \, d\mu \geq \frac{1}{n} \mu(E_n).$$

Therefore, $\mu(E_n) = 0$. It follows that

$$\mu([f - g > 0]) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0$$

Similarly, the set where g is larger than f has measure zero. This proves the theorem.

Case where it is not necessarily true that $\lambda \ll \mu$.

In this case, let $N = [h \geq 1]$ and let $g = \mathcal{X}_N$. Then

$$\lambda(N) = \int_N h \, d(\mu + \lambda) \geq \mu(N) + \lambda(N).$$

and so $\mu(N) = 0$ and so $\mu(E) = \mu\left(E \cap N^C\right)$. Now define a measure, $\lambda_\perp$ by

$$\lambda_\perp(E) \equiv \lambda(E \cap N)$$

so

$$\lambda_\perp(E \cap N) \equiv \lambda(E \cap N \cap N) = \lambda(E \cap N) \equiv \lambda_\perp(E)$$

and let $\lambda_{||} \equiv \lambda - \lambda_\perp$. Thus,

$$\lambda_{||}(E) = \lambda(E) - \lambda_\perp(E) \equiv \lambda(E) - \lambda(E \cap N) = \lambda\left(E \cap N^C\right).$$

Suppose now that $\lambda_{||}(E) > 0$. It follows from the first part of the proof that since $h < 1$ on N^C

$$\begin{aligned} 0 &< \lambda_{||}(E) = \lambda\left(E \cap N^C\right) = \int_{E \cap N^C} h \, d(\mu + \lambda) \\ &< \mu\left(E \cap N^C\right) + \lambda\left(E \cap N^C\right) = \mu(E) + \lambda_{||}(E) \end{aligned}$$

which shows that $\mu(E) > 0$. Thus if $\mu(E) = 0$ it follows $\lambda_{||}(E) = 0$ and so $\lambda_{||} \ll \mu$.

It only remains to verify the two measures $\lambda_\perp$ and $\lambda_{||}$ are unique. Suppose then that ν_1 and ν_2 play the roles of $\lambda_\perp$ and $\lambda_{||}$ respectively. Let N_1 play the role of N in the definition of ν_1 and let g_1 play the role of g for ν_2. I will show that $g = g_1$ μ a.e. Let $E_k \equiv [g_1 - g > 1/k]$ for $k \in \mathbb{N}$. Then on observing that $\lambda_\perp - \nu_1 = \nu_2 - \lambda_{||}$

$$\begin{aligned} 0 &= (\lambda_\perp - \nu_1)\left(E_n \cap (N_1 \cup N)^C\right) = \int_{E_n \cap (N_1 \cup N)^C} (g_1 - g) \, d\mu \\ &\geq \frac{1}{k}\mu\left(E_k \cap (N_1 \cup N)^C\right) = \frac{1}{k}\mu(E_k). \end{aligned}$$

and so $\mu(E_k) = 0$. Therefore, $\mu([g_1 - g > 0]) = 0$ because $[g_1 - g > 0] = \cup_{k=1}^\infty E_k$. It follows $g_1 \leq g$ μ a.e. Similarly, $g \geq g_1$ μ a.e. Therefore, $\nu_2 = \lambda_{||}$ and so $\lambda_\perp = \nu_1$ also. This proves the theorem.

The f in the theorem for the absolutely continuous case is sometimes denoted by $\frac{d\lambda}{d\mu}$ and is called the Radon Nikodym derivative.

The next corollary is a useful generalization to σ finite measure spaces.

Corollary 15.3 *Suppose* $\lambda \ll \mu$ *and there exist sets* $S_n \in \mathcal{S}$ *with*

$$S_n \cap S_m = \emptyset, \ \cup_{n=1}^\infty S_n = \Omega,$$

and $\lambda(S_n)$, $\mu(S_n) < \infty$. *Then there exists* $f \geq 0$, *where* f *is* μ *measurable, and*

$$\lambda(E) = \int_E f\, d\mu$$

for all $E \in \mathcal{S}$. *The function* f *is* $\mu + \lambda$ *a.e. unique.*

Proof: Define the σ algebra of subsets of S_n,

$$\mathcal{S}_n \equiv \{E \cap S_n : E \in \mathcal{S}\}.$$

Then both λ, and μ are finite measures on $\mathcal{S}_n$, and $\lambda \ll \mu$. Thus, by Theorem 15.2, there exists a nonnegative $\mathcal{S}_n$ measurable function f_n,with $\lambda(E) = \int_E f_n d\mu$ for all $E \in \mathcal{S}_n$. Define $f(x) = f_n(x)$ for $x \in S_n$. Since the S_n are disjoint and their union is all of Ω, this defines f on all of Ω. The function, f is measurable because

$$f^{-1}((a, \infty]) = \cup_{n=1}^{\infty} f_n^{-1}((a, \infty]) \in \mathcal{S}.$$

Also, for $E \in \mathcal{S}$,

$$\begin{aligned} \lambda(E) &= \sum_{n=1}^{\infty} \lambda(E \cap S_n) = \sum_{n=1}^{\infty} \int \mathcal{X}_{E \cap S_n}(x) f_n(x) d\mu \\ &= \sum_{n=1}^{\infty} \int \mathcal{X}_{E \cap S_n}(x) f(x) d\mu \end{aligned}$$

By the monotone convergence theorem

$$\begin{aligned} \sum_{n=1}^{\infty} \int \mathcal{X}_{E \cap S_n}(x) f(x) d\mu &= \lim_{N \to \infty} \sum_{n=1}^{N} \int \mathcal{X}_{E \cap S_n}(x) f(x) d\mu \\ &= \lim_{N \to \infty} \int \sum_{n=1}^{N} \mathcal{X}_{E \cap S_n}(x) f(x) d\mu \\ &= \int \sum_{n=1}^{\infty} \mathcal{X}_{E \cap S_n}(x) f(x) d\mu = \int_E f\, d\mu. \end{aligned}$$

This proves the existence part of the corollary.

To see f is unique, suppose f_1 and f_2 both work and consider for $n \in \mathbb{N}$

$$E_k \equiv \left[f_1 - f_2 > \frac{1}{k} \right].$$

Then

$$0 = \lambda(E_k \cap S_n) - \lambda(E_k \cap S_n) = \int_{E_k \cap S_n} f_1(x) - f_2(x) d\mu.$$

Hence $\mu(E_k \cap S_n) = 0$ for all n so

$$\mu(E_k) = \lim_{n \to \infty} \mu(E \cap S_n) = 0.$$

Hence $\mu([f_1 - f_2 > 0]) \leq \sum_{k=1}^{\infty} \mu(E_k) = 0$. Therefore, $\lambda([f_1 - f_2 > 0]) = 0$ also. Similarly

$$(\mu + \lambda)([f_1 - f_2 < 0]) = 0.$$

This version of the Radon Nikodym theorem will suffice for most applications, but more general versions are available. To see one of these, one can read the treatment in Hewitt and Stromberg [26]. This involves the notion of decomposable measure spaces, a generalization of σ finite.

Not surprisingly, there is a simple generalization of the Lebesgue decomposition part of Theorem 15.2.

Corollary 15.4 *Let $(\Omega, \mathcal{S})$ be a set with a σ algebra of sets. Suppose λ and μ are two measures defined on the sets of $\mathcal{S}$ and suppose there exists a sequence of disjoint sets of $\mathcal{S}$, $\{\Omega_i\}_{i=1}^{\infty}$ such that $\lambda(\Omega_i), \mu(\Omega_i) < \infty$. Then there is a set of μ measure zero, N and measures $\lambda_\perp$ and $\lambda_{||}$ such that*

$$\lambda_\perp + \lambda_{||} = \lambda,\ \lambda_{||} \ll \mu, \lambda_\perp(E) = \lambda(E \cap N) = \lambda_\perp(E \cap N).$$

Proof: Let $\mathcal{S}_i \equiv \{E \cap \Omega_i : E \in \mathcal{S}\}$ and for $E \in \mathcal{S}_i$, let $\lambda^i(E) = \lambda(E)$ and $\mu^i(E) = \mu(E)$. Then by Theorem 15.2 there exist unique measures $\lambda_\perp^i$ and $\lambda_{||}^i$ such that $\lambda^i = \lambda_\perp^i + \lambda_{||}^i$, a set of μ^i measure zero, $N_i \in \mathcal{S}_i$ such that for all $E \in \mathcal{S}_i$, $\lambda_\perp^i(E) = \lambda^i(E \cap N_i)$ and $\lambda_{||}^i \ll \mu^i$. Define for $E \in \mathcal{S}$

$$\lambda_\perp(E) \equiv \sum_i \lambda_\perp^i(E \cap \Omega_i),\ \lambda_{||}(E) \equiv \sum_i \lambda_{||}^i(E \cap \Omega_i),\ N \equiv \cup_i N_i.$$

First observe that $\lambda_\perp$ and $\lambda_{||}$ are measures.

$$\begin{aligned}
\lambda_\perp\left(\cup_{j=1}^{\infty} E_j\right) &\equiv \sum_i \lambda_\perp^i\left(\cup_{j=1}^{\infty} E_j \cap \Omega_i\right) = \sum_i \sum_j \lambda_\perp^i(E_j \cap \Omega_i) \\
&= \sum_j \sum_i \lambda_\perp^i(E_j \cap \Omega_i) = \sum_j \sum_i \lambda(E_j \cap \Omega_i \cap N_i) \\
&= \sum_j \sum_i \lambda_\perp^i(E_j \cap \Omega_i) = \sum_j \lambda_\perp(E_j).
\end{aligned}$$

The argument for $\lambda_{||}$ is similar. Now

$$\mu(N) = \sum_i \mu(N \cap \Omega_i) = \sum_i \mu^i(N_i) = 0$$

and

$$\begin{aligned}
\lambda_\perp(E) &\equiv \sum_i \lambda_\perp^i(E \cap \Omega_i) = \sum_i \lambda^i(E \cap \Omega_i \cap N_i) \\
&= \sum_i \lambda(E \cap \Omega_i \cap N) = \lambda(E \cap N).
\end{aligned}$$

Also if $\mu(E) = 0$, then $\mu^i(E \cap \Omega_i) = 0$ and so $\lambda_{||}^i(E \cap \Omega_i) = 0$. Therefore,

$$\lambda_{||}(E) = \sum_i \lambda_{||}^i(E \cap \Omega_i) = 0.$$

The decomposition is unique because of the uniqueness of the $\lambda_{||}^i$ and $\lambda_{\perp}^i$ and the observation that some other decomposition must coincide with the given one on the Ω_i.

15.2 Vector Measures

The next topic will use the Radon Nikodym theorem. It is the topic of vector and complex measures. The main interest is in complex measures although a vector measure can have values in any topological vector space. Whole books have been written on this subject. See for example the book by Diestal and Uhl [16] titled Vector measures.

Definition 15.5 *Let $(V, ||\cdot||)$ be a normed linear space and let $(\Omega, \mathcal{S})$ be a measure space. A function $\mu : \mathcal{S} \to V$ is a vector measure if μ is countably additive. That is, if $\{E_i\}_{i=1}^{\infty}$ is a sequence of disjoint sets of $\mathcal{S}$,*

$$\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

Note that it makes sense to take finite sums because it is given that μ has values in a vector space in which vectors can be summed. In the above, $\mu(E_i)$ is a vector. It might be a point in $\mathbb{R}^n$ or in any other vector space. In many of the most important applications, it is a vector in some sort of function space which may be infinite dimensional. The infinite sum has the usual meaning. That is

$$\sum_{i=1}^{\infty} \mu(E_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i)$$

where the limit takes place relative to the norm on V.

Definition 15.6 *Let $(\Omega, \mathcal{S})$ be a measure space and let μ be a vector measure defined on $\mathcal{S}$. A subset, $\pi(E)$, of $\mathcal{S}$ is called a partition of E if $\pi(E)$ consists of finitely many disjoint sets of $\mathcal{S}$ and $\cup\pi(E) = E$. Let*

$$|\mu|(E) = \sup\{\sum_{F\in\pi(E)} ||\mu(F)|| : \pi(E) \textit{ is a partition of } E\}.$$

$|\mu|$ is called the total variation of μ.

The next theorem may seem a little surprising. It states that, if finite, the total variation is a nonnegative measure.

Theorem 15.7 *If* $|\mu|(\Omega) < \infty$*, then* $|\mu|$ *is a measure on* $\mathcal{S}$*. Even if* $|\mu|(\Omega) = \infty$*,* $|\mu|(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} |\mu|(E_i)$*. That is* $|\mu|$ *is subadditive and* $|\mu|(A) \leq |\mu|(B)$ *whenever* $A, B \in \mathcal{S}$ *with* $A \subseteq B$*.*

Proof: Consider the last claim. Let $a < |\mu|(A)$ and let $\pi(A)$ be a partition of A such that

$$a < \sum_{F \in \pi(A)} ||\mu(F)||.$$

Then $\pi(A) \cup \{B \setminus A\}$ is a partition of B and

$$|\mu|(B) \geq \sum_{F \in \pi(A)} ||\mu(F)|| + ||\mu(B \setminus A)|| > a.$$

Since this is true for all such a, it follows $|\mu|(B) \geq |\mu|(A)$ as claimed.

Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of disjoint sets of $\mathcal{S}$ and let $E_\infty = \cup_{j=1}^{\infty} E_j$. Then letting $a < |\mu|(E_\infty)$, it follows from the definition of total variation there exists a partition of E_∞, $\pi(E_\infty) = \{A_1, \cdots, A_n\}$ such that

$$a < \sum_{i=1}^{n} ||\mu(A_i)||.$$

Also,

$$A_i = \cup_{j=1}^{\infty} A_i \cap E_j$$

and so by the triangle inequality, $||\mu(A_i)|| \leq \sum_{j=1}^{\infty} ||\mu(A_i \cap E_j)||$. Therefore, by the above, and either Fubini's theorem or Lemma 8.21 on Page 184

$$\begin{aligned} a &< \sum_{i=1}^{n} \overbrace{\sum_{j=1}^{\infty} ||\mu(A_i \cap E_j)||}^{\geq ||\mu(A_i)||} \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{n} ||\mu(A_i \cap E_j)|| \\ &\leq \sum_{j=1}^{\infty} |\mu|(E_j) \end{aligned}$$

because $\{A_i \cap E_j\}_{i=1}^{n}$ is a partition of E_j.

Since a is arbitrary, this shows

$$|\mu|(\cup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} |\mu|(E_j).$$

If the sets, E_j are not disjoint, let $F_1 = E_1$ and if F_n has been chosen, let $F_{n+1} \equiv E_{n+1} \setminus \cup_{i=1}^{n} E_i$. Thus the sets, F_i are disjoint and $\cup_{i=1}^{\infty} F_i = \cup_{i=1}^{\infty} E_i$. Therefore,

$$|\mu|\left(\cup_{j=1}^{\infty} E_j\right) = |\mu|\left(\cup_{j=1}^{\infty} F_j\right) \leq \sum_{j=1}^{\infty} |\mu|(F_j) \leq \sum_{j=1}^{\infty} |\mu|(E_j)$$

and proves $|\mu|$ is always subadditive as claimed regarless of whether $|\mu|(\Omega) < \infty$.

Now suppose $|\mu|(\Omega) < \infty$ and let E_1 and E_2 be sets of $\mathcal{S}$ such that $E_1 \cap E_2 = \emptyset$ and let $\{A_1^i \cdots A_{n_i}^i\} = \pi(E_i)$, a partition of E_i which is chosen such that

$$|\mu|(E_i) - \varepsilon < \sum_{j=1}^{n_i} ||\mu(A_j^i)|| \ \ i = 1, 2.$$

Such a partition exists because of the definition of the total variation. Consider the sets which are contained in either of $\pi(E_1)$ or $\pi(E_2)$, it follows this collection of sets is a partition of $E_1 \cup E_2$ denoted by $\pi(E_1 \cup E_2)$. Then by the above inequality and the definition of total variation,

$$|\mu|(E_1 \cup E_2) \geq \sum_{F \in \pi(E_1 \cup E_2)} ||\mu(F)|| > |\mu|(E_1) + |\mu|(E_2) - 2\varepsilon,$$

which shows that since $\varepsilon > 0$ was arbitrary,

$$|\mu|(E_1 \cup E_2) \geq |\mu|(E_1) + |\mu|(E_2). \tag{15.5}$$

Then 15.5 implies that whenever the E_i are disjoint, $|\mu|(\cup_{j=1}^n E_j) \geq \sum_{j=1}^n |\mu|(E_j)$. Therefore,

$$\sum_{j=1}^{\infty} |\mu|(E_j) \geq |\mu|(\cup_{j=1}^{\infty} E_j) \geq |\mu|(\cup_{j=1}^n E_j) \geq \sum_{j=1}^n |\mu|(E_j).$$

Since n is arbitrary,

$$|\mu|(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} |\mu|(E_j)$$

which shows that $|\mu|$ is a measure as claimed. This proves the theorem.

In the case that μ is a complex measure, it is always the case that $|\mu|(\Omega) < \infty$.

Theorem 15.8 *Suppose μ is a complex measure on $(\Omega, \mathcal{S})$ where $\mathcal{S}$ is a σ algebra of subsets of Ω. That is, whenever, $\{E_i\}$ is a sequence of disjoint sets of $\mathcal{S}$,*

$$\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

Then $|\mu|(\Omega) < \infty$.

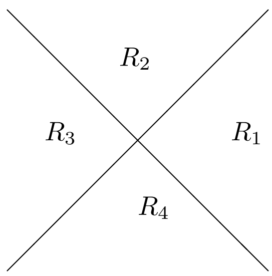
Proof: First here is a claim.

Claim: Suppose $|\mu|(E) = \infty$. Then there are subsets of E, A and B such that $E = A \cup B$, $|\mu(A)|, |\mu(B)| > 1$ and $|\mu|(B) = \infty$.

Proof of the claim: From the definition of $|\mu|$, there exists a partition of $E, \pi(E)$ such that

$$\sum_{F \in \pi(E)} |\mu(F)| > 20(1 + |\mu(E)|). \tag{15.6}$$

Here 20 is just a nice sized number. No effort is made to be delicate in this argument. Also note that $\mu(E) \in \mathbb{C}$ because it is given that μ is a complex measure. Consider the following picture consisting of two lines in the complex plane having slopes 1 and -1 which intersect at the origin, dividing the complex plane into four closed sets, R_1, R_2, R_3, and R_4 as shown.



Let π_i consist of those sets, A of $\pi(E)$ for which $\mu(A) \in R_i$. Thus, some sets, A of $\pi(E)$ could be in two of the π_i if $\mu(A)$ is on one of the intersecting lines. This is not important. The thing which is important is that if $\mu(A) \in R_1$ or R_3, then $\frac{\sqrt{2}}{2}|\mu(A)| \leq |\text{Re}(\mu(A))|$ and if $\mu(A) \in R_2$ or R_4 then $\frac{\sqrt{2}}{2}|\mu(A)| \leq |\text{Im}(\mu(A))|$ and $\text{Re}(z)$ has the same sign for z in R_1 and R_3 while $\text{Im}(z)$ has the same sign for z in R_2 or R_4. Then by 15.6, it follows that for some i,

$$\sum_{F\in\pi_i} |\mu(F)| > 5(1 + |\mu(E)|). \tag{15.7}$$

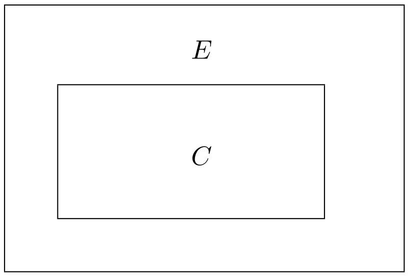
Suppose i equals 1 or 3. A similar argument using the imaginary part applies if i equals 2 or 4. Then,

$$\begin{aligned}\left|\sum_{F\in\pi_i}\mu(F)\right| &\geq \left|\sum_{F\in\pi_i}\text{Re}(\mu(F))\right| = \sum_{F\in\pi_i}|\text{Re}(\mu(F))| \\ &\geq \frac{\sqrt{2}}{2}\sum_{F\in\pi_i}|\mu(F)| > 5\frac{\sqrt{2}}{2}(1 + |\mu(E)|).\end{aligned}$$

Now letting C be the union of the sets in π_i,

$$|\mu(C)| = \left|\sum_{F\in\pi_i}\mu(F)\right| > \frac{5}{2}(1 + |\mu(E)|) > 1. \tag{15.8}$$

Define $D \equiv E \setminus C$.



Then $\mu(C) + \mu(E \setminus C) = \mu(E)$ and so

$$\begin{aligned} \frac{5}{2}(1 + |\mu(E)|) &< |\mu(C)| = |\mu(E) - \mu(E \setminus C)| \\ &= |\mu(E) - \mu(D)| \leq |\mu(E)| + |\mu(D)| \end{aligned}$$

and so

$$1 < \frac{5}{2} + \frac{3}{2}|\mu(E)| < |\mu(D)|.$$

Now since $|\mu|(E) = \infty$, it follows from Theorem 15.8 that $\infty = |\mu|(E) \leq |\mu|(C) + |\mu|(D)$ and so either $|\mu|(C) = \infty$ or $|\mu|(D) = \infty$. If $|\mu|(C) = \infty$, let $B = C$ and $A = D$. Otherwise, let $B = D$ and $A = C$. This proves the claim.

Now suppose $|\mu|(\Omega) = \infty$. Then from the claim, there exist A_1 and B_1 such that $|\mu|(B_1) = \infty, |\mu(B_1)|, |\mu(A_1)| > 1$, and $A_1 \cup B_1 = \Omega$. Let $B_1 \equiv \Omega \setminus A$ play the same role as Ω and obtain $A_2, B_2 \subseteq B_1$ such that $|\mu|(B_2) = \infty, |\mu(B_2)|, |\mu(A_2)| > 1$, and $A_2 \cup B_2 = B_1$. Continue in this way to obtain a sequence of disjoint sets, $\{A_i\}$ such that $|\mu(A_i)| > 1$. Then since μ is a measure,

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

but this is impossible because $\lim_{i \to \infty} \mu(A_i) \neq 0$. This proves the theorem.

Theorem 15.9 *Let $(\Omega, \mathcal{S})$ be a measure space and let $\lambda : \mathcal{S} \to \mathbb{C}$ be a complex vector measure. Thus $|\lambda|(\Omega) < \infty$. Let $\mu : \mathcal{S} \to [0, \mu(\Omega)]$ be a finite measure such that $\lambda \ll \mu$. Then there exists a unique $f \in L^1(\Omega)$ such that for all $E \in \mathcal{S}$,*

$$\int_E f d\mu = \lambda(E).$$

Proof: It is clear that $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$ are real-valued vector measures on $\mathcal{S}$. Since $|\lambda|(\Omega) < \infty$, it follows easily that $|\operatorname{Re}\lambda|(\Omega)$ and $|\operatorname{Im}\lambda|(\Omega) < \infty$. This is clear because

$$|\lambda(E)| \geq |\operatorname{Re}\lambda(E)|, |\operatorname{Im}\lambda(E)|.$$

Therefore, each of

$$\frac{|\operatorname{Re}\lambda| + \operatorname{Re}\lambda}{2}, \ \frac{|\operatorname{Re}\lambda| - \operatorname{Re}(\lambda)}{2}, \ \frac{|\operatorname{Im}\lambda| + \operatorname{Im}\lambda}{2}, \text{ and } \frac{|\operatorname{Im}\lambda| - \operatorname{Im}(\lambda)}{2}$$

are finite measures on $\mathcal{S}$. It is also clear that each of these finite measures are absolutely continuous with respect to μ and so there exist unique nonnegative functions in $L^1(\Omega)$, f_1, f_2, g_1, g_2 such that for all $E \in \mathcal{S}$,

$$\begin{aligned}
\frac{1}{2}(|\operatorname{Re}\lambda| + \operatorname{Re}\lambda)(E) &= \int_E f_1 d\mu, \\
\frac{1}{2}(|\operatorname{Re}\lambda| - \operatorname{Re}\lambda)(E) &= \int_E f_2 d\mu, \\
\frac{1}{2}(|\operatorname{Im}\lambda| + \operatorname{Im}\lambda)(E) &= \int_E g_1 d\mu, \\
\frac{1}{2}(|\operatorname{Im}\lambda| - \operatorname{Im}\lambda)(E) &= \int_E g_2 d\mu.
\end{aligned}$$

Now let $f = f_1 - f_2 + i(g_1 - g_2)$.

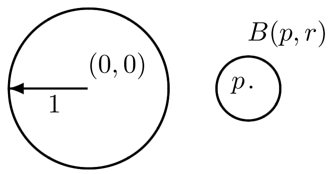
The following corollary is about representing a vector measure in terms of its total variation. It is like representing a complex number in the form $re^{i\theta}$. The proof requires the following lemma.

Lemma 15.10 *Suppose* $(\Omega, \mathcal{S}, \mu)$ *is a measure space and* f *is a function in* $L^1(\Omega, \mu)$ *with the property that*

$$|\int_E f \, d\mu| \leq \mu(E)$$

for all $E \in \mathcal{S}$. *Then* $|f| \leq 1$ *a.e.*

Proof of the lemma: Consider the following picture.



where $B(p,r) \cap B(0,1) = \emptyset$. Let $E = f^{-1}(B(p,r))$. In fact $\mu(E) = 0$. If $\mu(E) \neq 0$ then

$$\begin{aligned}
\left|\frac{1}{\mu(E)} \int_E f \, d\mu - p\right| &= \left|\frac{1}{\mu(E)} \int_E (f - p) d\mu\right| \\
&\leq \frac{1}{\mu(E)} \int_E |f - p| d\mu < r
\end{aligned}$$

because on E, $|f(x) - p| < r$. Hence

$$|\frac{1}{\mu(E)} \int_E f d\mu| > 1$$

because it is closer to p than r. (Refer to the picture.) However, this contradicts the assumption of the lemma. It follows $\mu(E) = 0$. Since the set of complex numbers, z such that $|z| > 1$ is an open set, it equals the union of countably many balls, $\{B_i\}_{i=1}^{\infty}$. Therefore,

$$\begin{aligned}\mu\left(f^{-1}(\{z \in \mathbb{C} : |z| > 1\}\right) &= \mu\left(\cup_{k=1}^{\infty} f^{-1}(B_k)\right) \\ &\leq \sum_{k=1}^{\infty} \mu\left(f^{-1}(B_k)\right) = 0.\end{aligned}$$

Thus $|f(x)| \leq 1$ a.e. as claimed. This proves the lemma.

Corollary 15.11 *Let λ be a complex vector measure with $|\lambda|(\Omega) < \infty$*[1] *Then there exists a unique $f \in L^1(\Omega)$ such that $\lambda(E) = \int_E f d|\lambda|$. Furthermore, $|f| = 1$ for $|\lambda|$ a.e. This is called the polar decomposition of λ.*

Proof: First note that $\lambda \ll |\lambda|$ and so such an L^1 function exists and is unique. It is required to show $|f| = 1$ a.e. If $|\lambda|(E) \neq 0$,

$$\left|\frac{\lambda(E)}{|\lambda|(E)}\right| = \left|\frac{1}{|\lambda|(E)} \int_E f \, d|\lambda|\right| \leq 1.$$

Therefore by Lemma 15.10, $|f| \leq 1$, $|\lambda|$ a.e. Now let

$$E_n = \left[|f| \leq 1 - \frac{1}{n}\right].$$

Let $\{F_1, \cdots, F_m\}$ be a partition of E_n. Then

$$\begin{aligned}\sum_{i=1}^{m} |\lambda(F_i)| &= \sum_{i=1}^{m} \left|\int_{F_i} f d|\lambda|\right| \leq \sum_{i=1}^{m} \int_{F_i} |f| \, d|\lambda| \\ &\leq \sum_{i=1}^{m} \int_{F_i} \left(1 - \frac{1}{n}\right) d|\lambda| = \sum_{i=1}^{m} \left(1 - \frac{1}{n}\right) |\lambda|(F_i) \\ &= |\lambda|(E_n)\left(1 - \frac{1}{n}\right).\end{aligned}$$

Then taking the supremum over all partitions,

$$|\lambda|(E_n) \leq \left(1 - \frac{1}{n}\right) |\lambda|(E_n)$$

which shows $|\lambda|(E_n) = 0$. Hence $|\lambda|([|f| < 1]) = 0$ because $[|f| < 1] = \cup_{n=1}^{\infty} E_n$.This proves Corollary 15.11.

[1] As proved above, the assumption that $|\lambda|(\Omega) < \infty$ is redundant.

Corollary 15.12 *Suppose $(\Omega, \mathcal{S})$ is a measure space and μ is a finite nonnegative measure on $\mathcal{S}$. Then for $h \in L^1(\mu)$, define a complex measure, λ by*

$$\lambda(E) \equiv \int_E h d\mu.$$

Then

$$|\lambda|(E) = \int_E |h| d\mu.$$

Furthermore, $|h| = \overline{g}h$ where $gd|\lambda|$ is the polar decomposition of λ,

$$\lambda(E) = \int_E gd|\lambda|$$

Proof: From Corollary 15.11 there exists g such that $|g| = 1, |\lambda|$ a.e. and for all $E \in \mathcal{S}$

$$\lambda(E) = \int_E gd|\lambda| = \int_E hd\mu.$$

Let s_n be a sequence of simple functions converging pointwise to $\overline{g}$. Then from the above,

$$\int_E gs_n d|\lambda| = \int_E s_n hd\mu.$$

Passing to the limit using the dominated convergence theorem,

$$\int_E d|\lambda| = \int_E \overline{g}hd\mu.$$

It follows $\overline{g}h \geq 0$ a.e. and $|\overline{g}| = 1$. Therefore, $|h| = |\overline{g}h| = \overline{g}h$. It follows from the above, that

$$|\lambda|(E) = \int_E d|\lambda| = \int_E \overline{g}hd\mu = \int_E d|\lambda| = \int_E |h| d\mu$$

and this proves the corollary.

15.3 Representation Theorems For The Dual Space Of L^p

Recall the concept of the dual space of a Banach space in the Chapter on Banach space starting on Page 337. The next topic deals with the dual space of L^p for $p \geq 1$ in the case where the measure space is σ finite or finite. In what follows $q = \infty$ if $p = 1$ and otherwise, $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 15.13 *(Riesz representation theorem) Let $p > 1$ and let $(\Omega, \mathcal{S}, \mu)$ be a finite measure space. If $\Lambda \in (L^p(\Omega))'$, then there exists a unique $h \in L^q(\Omega)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ such that*

$$\Lambda f = \int_\Omega hf d\mu.$$

This function satisfies $||h||_q = ||\Lambda||$ where $||\Lambda||$ is the operator norm of Λ.

Proof: (Uniqueness) If h_1 and h_2 both represent Λ, consider

$$f = |h_1 - h_2|^{q-2}(\overline{h_1} - \overline{h_2}),$$

where $\overline{h}$ denotes complex conjugation. By Holder's inequality, it is easy to see that $f \in L^p(\Omega)$. Thus

$$0 = \Lambda f - \Lambda f =$$

$$\int h_1|h_1 - h_2|^{q-2}(\overline{h_1} - \overline{h_2}) - h_2|h_1 - h_2|^{q-2}(\overline{h_1} - \overline{h_2})d\mu$$

$$= \int |h_1 - h_2|^q d\mu.$$

Therefore $h_1 = h_2$ and this proves uniqueness.

Now let $\lambda(E) = \Lambda(\mathcal{X}_E)$. Since this is a finite measure space $\mathcal{X}_E$ is an element of $L^p(\Omega)$ and so it makes sense to write $\Lambda(\mathcal{X}_E)$. In fact λ is a complex measure having finite total variation. Let $A_1, \cdots, A_n$ be a partition of Ω.

$$|\Lambda \mathcal{X}_{A_i}| = w_i(\Lambda \mathcal{X}_{A_i}) = \Lambda(w_i \mathcal{X}_{A_i})$$

for some $w_i \in \mathbb{C}$, $|w_i| = 1$. Thus

$$\sum_{i=1}^n |\lambda(A_i)| = \sum_{i=1}^n |\Lambda(\mathcal{X}_{A_i})| = \Lambda(\sum_{i=1}^n w_i \mathcal{X}_{A_i})$$

$$\leq ||\Lambda||(\int |\sum_{i=1}^n w_i \mathcal{X}_{A_i}|^p d\mu)^{\frac{1}{p}} = ||\Lambda||(\int_\Omega d\mu)^{\frac{1}{p}} = ||\Lambda|| \mu(\Omega)^{\frac{1}{p}}.$$

This is because if $x \in \Omega$, x is contained in exactly one of the A_i and so the absolute value of the sum in the first integral above is equal to 1. Therefore $|\lambda|(\Omega) < \infty$ because this was an arbitrary partition. Also, if $\{E_i\}_{i=1}^\infty$ is a sequence of disjoint sets of $\mathcal{S}$, let

$$F_n = \cup_{i=1}^n E_i, \ F = \cup_{i=1}^\infty E_i.$$

Then by the Dominated Convergence theorem,

$$||\mathcal{X}_{F_n} - \mathcal{X}_F||_p \to 0.$$

Therefore, by continuity of Λ,

$$\lambda(F) = \Lambda(\mathcal{X}_F) = \lim_{n\to\infty} \Lambda(\mathcal{X}_{F_n}) = \lim_{n\to\infty} \sum_{k=1}^n \Lambda(\mathcal{X}_{E_k}) = \sum_{k=1}^\infty \lambda(E_k).$$

This shows λ is a complex measure with $|\lambda|$ finite.

It is also clear from the definition of λ that $\lambda \ll \mu$. Therefore, by the Radon Nikodym theorem, there exists $h \in L^1(\Omega)$ with

$$\lambda(E) = \int_E h d\mu = \Lambda(\mathcal{X}_E).$$

Actually $h \in L^q$ and satisfies the other conditions above. Let $s = \sum_{i=1}^m c_i \mathcal{X}_{E_i}$ be a simple function. Then since Λ is linear,

$$\Lambda(s) = \sum_{i=1}^m c_i \Lambda(\mathcal{X}_{E_i}) = \sum_{i=1}^m c_i \int_{E_i} h d\mu = \int h s d\mu. \tag{15.9}$$

Claim: If f is uniformly bounded and measurable, then

$$\Lambda(f) = \int h f d\mu.$$

Proof of claim: Since f is bounded and measurable, there exists a sequence of simple functions, $\{s_n\}$ which converges to f pointwise and in $L^p(\Omega)$. This follows from Theorem 8.27 on Page 190 upon breaking f up into positive and negative parts of real and complex parts. In fact this theorem gives uniform convergence. Then

$$\Lambda(f) = \lim_{n\to\infty} \Lambda(s_n) = \lim_{n\to\infty} \int h s_n d\mu = \int h f d\mu,$$

the first equality holding because of continuity of Λ, the second following from 15.9 and the third holding by the dominated convergence theorem.

This is a very nice formula but it still has not been shown that $h \in L^q(\Omega)$.

Let $E_n = \{x : |h(x)| \leq n\}$. Thus $|h\mathcal{X}_{E_n}| \leq n$. Then

$$|h\mathcal{X}_{E_n}|^{q-2}(\overline{h}\mathcal{X}_{E_n}) \in L^p(\Omega).$$

By the claim, it follows that

$$||h\mathcal{X}_{E_n}||_q^q = \int h|h\mathcal{X}_{E_n}|^{q-2}(\overline{h}\mathcal{X}_{E_n})d\mu = \Lambda(|h\mathcal{X}_{E_n}|^{q-2}(\overline{h}\mathcal{X}_{E_n}))$$

$$\leq ||\Lambda|| \, \left|\left||h\mathcal{X}_{E_n}|^{q-2}(\overline{h}\mathcal{X}_{E_n})\right|\right|_p = ||\Lambda|| \, ||h\mathcal{X}_{E_n}||_q^{\frac{q}{p}},$$

the last equality holding because $q - 1 = q/p$ and so

$$\begin{aligned} \left(\int \left||h\mathcal{X}_{E_n}|^{q-2}(\overline{h}\mathcal{X}_{E_n})\right|^p d\mu\right)^{1/p} &= \left(\int \left(|h\mathcal{X}_{E_n}|^{q/p}\right)^p d\mu\right)^{1/p} \\ &= ||h\mathcal{X}_{E_n}||_q^{\frac{q}{p}} \end{aligned}$$

Therefore, since $q - \frac{q}{p} = 1$, it follows that

$$||h\mathcal{X}_{E_n}||_q \leq ||\Lambda||.$$

Letting $n \to \infty$, the Monotone Convergence theorem implies

$$||h||_q \leq ||\Lambda||. \tag{15.10}$$

Now that h has been shown to be in $L^q(\Omega)$, it follows from 15.9 and the density of the simple functions, Theorem 12.13 on Page 323, that

$$\Lambda f = \int h f d\mu$$

for all $f \in L^p(\Omega)$.

It only remains to verify the last claim.

$$||\Lambda|| = \sup\{\int hf : ||f||_p \leq 1\} \leq ||h||_q \leq ||\Lambda||$$

by 15.10, and Holder's inequality. This proves the theorem.

To represent elements of the dual space of $L^1(\Omega)$, another Banach space is needed.

Definition 15.14 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space. $L^\infty(\Omega)$ is the vector space of measurable functions such that for some $M > 0$, $|f(x)| \leq M$ for all x outside of some set of measure zero ($|f(x)| \leq M$ a.e.). Define $f = g$ when $f(x) = g(x)$ a.e. and $||f||_\infty \equiv \inf\{M : |f(x)| \leq M$ a.e.$\}$.*

Theorem 15.15 *$L^\infty(\Omega)$ is a Banach space.*

Proof: It is clear that $L^\infty(\Omega)$ is a vector space. Is $||\ ||_\infty$ a norm?

Claim: If $f \in L^\infty(\Omega)$, then $|f(x)| \leq ||f||_\infty$ a.e.

Proof of the claim: $\left\{x : |f(x)| \geq ||f||_\infty + n^{-1}\right\} \equiv E_n$ is a set of measure zero according to the definition of $||f||_\infty$. Furthermore, $\{x : |f(x)| > ||f||_\infty\} = \cup_n E_n$ and so it is also a set of measure zero. This verifies the claim.

Now if $||f||_\infty = 0$ it follows that $f(x) = 0$ a.e. Also if $f, g \in L^\infty(\Omega)$,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq ||f||_\infty + ||g||_\infty$$

a.e. and so $||f||_\infty + ||g||_\infty$ serves as one of the constants, M in the definition of $||f + g||_\infty$. Therefore,

$$||f + g||_\infty \leq ||f||_\infty + ||g||_\infty .$$

Next let c be a number. Then $|cf(x)| = |c| |f(x)| \leq |c| ||f||_\infty$ and so $||cf||_\infty \leq |c| ||f||_\infty$. Therefore since c is arbitrary, $||f||_\infty = ||c(1/c) f||_\infty \leq \left|\frac{1}{c}\right| ||cf||_\infty$ which implies $|c| ||f||_\infty \leq ||cf||_\infty$. Thus $||\ ||_\infty$ is a norm as claimed.

To verify completeness, let $\{f_n\}$ be a Cauchy sequence in $L^\infty(\Omega)$ and use the above claim to get the existence of a set of measure zero, E_{nm} such that for all $x \notin E_{nm}$,

$$|f_n(x) - f_m(x)| \leq ||f_n - f_m||_\infty$$

Let $E = \cup_{n,m} E_{nm}$. Thus $\mu(E) = 0$ and for each $x \notin E$, $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$. Let

$$f(x) = \begin{cases} 0 \text{ if } x \in E \\ \lim_{n\to\infty} f_n(x) \text{ if } x \notin E \end{cases} = \lim_{n\to\infty} \mathcal{X}_{E^C}(x) f_n(x).$$

Then f is clearly measurable because it is the limit of measurable functions. If

$$F_n = \{x : |f_n(x)| > ||f_n||_\infty\}$$

and $F = \cup_{n=1}^\infty F_n$, it follows $\mu(F) = 0$ and that for $x \notin F \cup E$,

$$|f(x)| \leq \lim \inf_{n\to\infty} |f_n(x)| \leq \lim \inf_{n\to\infty} ||f_n||_\infty < \infty$$

because $\{||f_n||_\infty\}$ is a Cauchy sequence. ($|||f_n||_\infty - ||f_m||_\infty| \leq ||f_n - f_m||_\infty$ by the triangle inequality.) Thus $f \in L^\infty(\Omega)$. Let n be large enough that whenever $m > n$,

$$||f_m - f_n||_\infty < \varepsilon.$$

Then, if $x \notin E$,

$$\begin{aligned} |f(x) - f_n(x)| &= \lim_{m\to\infty} |f_m(x) - f_n(x)| \\ &\leq \lim_{m\to\infty} \inf \; ||f_m - f_n||_\infty < \varepsilon. \end{aligned}$$

Hence $||f - f_n||_\infty < \varepsilon$ for all n large enough. This proves the theorem.

The next theorem is the Riesz representation theorem for $\left(L^1(\Omega)\right)'$.

Theorem 15.16 *(Riesz representation theorem) Let $(\Omega, \mathcal{S}, \mu)$ be a finite measure space. If $\Lambda \in (L^1(\Omega))'$, then there exists a unique $h \in L^\infty(\Omega)$ such that*

$$\Lambda(f) = \int_\Omega hf \, d\mu$$

for all $f \in L^1(\Omega)$. If h is the function in $L^\infty(\Omega)$ representing $\Lambda \in (L^1(\Omega))'$, then $||h||_\infty = ||\Lambda||$.

Proof: Just as in the proof of Theorem 15.13, there exists a unique $h \in L^1(\Omega)$ such that for all simple functions, s,

$$\Lambda(s) = \int hs \, d\mu. \tag{15.11}$$

To show $h \in L^\infty(\Omega)$, let $\varepsilon > 0$ be given and let

$$E = \{x : |h(x)| \geq ||\Lambda|| + \varepsilon\}.$$

Let $|k| = 1$ and $hk = |h|$. Since the measure space is finite, $k \in L^1(\Omega)$. As in Theorem 15.13 let $\{s_n\}$ be a sequence of simple functions converging to k in $L^1(\Omega)$, and pointwise. It follows from the construction in Theorem 8.27 on Page 190 that it can be assumed $|s_n| \leq 1$. Therefore

$$\Lambda(k\mathcal{X}_E) = \lim_{n\to\infty} \Lambda(s_n\mathcal{X}_E) = \lim_{n\to\infty} \int_E hs_n d\mu = \int_E hk d\mu$$

where the last equality holds by the Dominated Convergence theorem. Therefore,

$$\begin{aligned} ||\Lambda||\mu(E) &\geq |\Lambda(k\mathcal{X}_E)| = |\int_\Omega hk\mathcal{X}_E d\mu| = \int_E |h| d\mu \\ &\geq (||\Lambda|| + \varepsilon)\mu(E). \end{aligned}$$

It follows that $\mu(E) = 0$. Since $\varepsilon > 0$ was arbitrary, $||\Lambda|| \geq ||h||_\infty$. It was shown that $h \in L^\infty(\Omega)$, the density of the simple functions in $L^1(\Omega)$ and 15.11 imply

$$\Lambda f = \int_\Omega hf d\mu \ , \ ||\Lambda|| \geq ||h||_\infty. \tag{15.12}$$

This proves the existence part of the theorem. To verify uniqueness, suppose h_1 and h_2 both represent Λ and let $f \in L^1(\Omega)$ be such that $|f| \leq 1$ and $f(h_1 - h_2) = |h_1 - h_2|$. Then

$$0 = \Lambda f - \Lambda f = \int (h_1 - h_2) f d\mu = \int |h_1 - h_2| d\mu.$$

Thus $h_1 = h_2$. Finally,

$$||\Lambda|| = \sup\{|\int hf d\mu| : ||f||_1 \leq 1\} \leq ||h||_\infty \leq ||\Lambda||$$

by 15.12.

Next these results are extended to the σ finite case.

Lemma 15.17 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and suppose there exists a measurable function, r such that $r(x) > 0$ for all x, there exists M such that $|r(x)| < M$ for all x, and $\int r d\mu < \infty$. Then for*

$$\Lambda \in (L^p(\Omega, \mu))', \ p \geq 1,$$

there exists a unique $h \in L^{p'}(\Omega, \mu)$, $L^\infty(\Omega, \mu)$ if $p = 1$ such that

$$\Lambda f = \int hf d\mu.$$

Also $||h|| = ||\Lambda||$. ($||h|| = ||h||_{p'}$ if $p > 1$, $||h||_\infty$ if $p = 1$). Here

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Proof: Define a new measure $\widetilde{\mu}$, according to the rule

$$\widetilde{\mu}(E) \equiv \int_E r d\mu. \tag{15.13}$$

Thus $\widetilde{\mu}$ is a finite measure on $\mathcal{S}$. Now define a mapping, $\eta : L^p(\Omega, \mu) \to L^p(\Omega, \widetilde{\mu})$ by

$$\eta f = r^{-\frac{1}{p}} f.$$

Then

$$||\eta f||^p_{L^p(\widetilde{\mu})} = \int \left| r^{-\frac{1}{p}} f \right|^p r d\mu = ||f||^p_{L^p(\mu)}$$

and so η is one to one and in fact preserves norms. I claim that also η is onto. To see this, let $g \in L^p(\Omega, \widetilde{\mu})$ and consider the function, $r^{\frac{1}{p}} g$. Then

$$\int \left| r^{\frac{1}{p}} g \right|^p d\mu = \int |g|^p \, r d\mu = \int |g|^p \, d\widetilde{\mu} < \infty$$

Thus $r^{\frac{1}{p}} g \in L^p(\Omega, \mu)$ and $\eta\left(r^{\frac{1}{p}} g\right) = g$ showing that η is onto as claimed. Thus η is one to one, onto, and preserves norms. Consider the diagram below which is descriptive of the situation in which η^* must be one to one and onto.

$$\begin{array}{cccc} & & \eta^* & \\ h, L^{p'}(\widetilde{\mu}) & L^p(\widetilde{\mu})', \widetilde{\Lambda} & \rightarrow & L^p(\mu)', \Lambda \\ & & \eta & \\ & L^p(\widetilde{\mu}) & \leftarrow & L^p(\mu) \end{array}$$

Then for $\Lambda \in L^p(\mu)'$, there exists a unique $\widetilde{\Lambda} \in L^p(\widetilde{\mu})'$ such that $\eta^* \widetilde{\Lambda} = \Lambda$, $\left|\left|\widetilde{\Lambda}\right|\right| = ||\Lambda||$. By the Riesz representation theorem for finite measure spaces, there exists a unique $h \in L^{p'}(\widetilde{\mu})$ which represents $\widetilde{\Lambda}$ in the manner described in the Riesz representation theorem. Thus $||h||_{L^{p'}(\widetilde{\mu})} = \left|\left|\widetilde{\Lambda}\right|\right| = ||\Lambda||$ and for all $f \in L^p(\mu)$,

$$\begin{aligned} \Lambda(f) &= \eta^* \widetilde{\Lambda}(f) \equiv \widetilde{\Lambda}(\eta f) = \int h(\eta f) \, d\widetilde{\mu} = \int rh \left(f^{-\frac{1}{p}} f\right) d\mu \\ &= \int r^{\frac{1}{p'}} h f d\mu. \end{aligned}$$

Now

$$\int \left| r^{\frac{1}{p'}} h \right|^{p'} d\mu = \int |h|^{p'} \, r d\mu = ||h||^{p'}_{L^{p'}(\widetilde{\mu})} < \infty.$$

Thus $\left|\left| r^{\frac{1}{p'}} h \right|\right|_{L^{p'}(\mu)} = ||h||_{L^{p'}(\widetilde{\mu})} = \left|\left|\widetilde{\Lambda}\right|\right| = ||\Lambda||$ and represents Λ in the appropriate way. If $p = 1$, then $1/p' \equiv 0$. This proves the Lemma.

A situation in which the conditions of the lemma are satisfied is the case where the measure space is σ finite. In fact, you should show this is the only case in which the conditions of the above lemma hold.

Theorem 15.18 *(Riesz representation theorem) Let $(\Omega, \mathcal{S}, \mu)$ be σ finite and let*

$$\Lambda \in (L^p(\Omega, \mu))', \; p \geq 1.$$

Then there exists a unique $h \in L^q(\Omega, \mu)$, $L^\infty(\Omega, \mu)$ if $p = 1$ such that

$$\Lambda f = \int h f d\mu.$$

Also $||h|| = ||\Lambda||$. $(||h|| = ||h||_q$ *if* $p > 1,\ ||h||_\infty$ *if* $p = 1)$. *Here*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Proof: Let $\{\Omega_n\}$ be a sequence of disjoint elements of $\mathcal{S}$ having the property that

$$0 < \mu(\Omega_n) < \infty,\ \cup_{n=1}^{\infty} \Omega_n = \Omega.$$

Define

$$r(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \mathcal{X}_{\Omega_n}(x)\ \mu(\Omega_n)^{-1},\ \ \widetilde{\mu}(E) = \int_E r d\mu.$$

Thus

$$\int_\Omega r d\mu = \widetilde{\mu}(\Omega) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

so $\widetilde{\mu}$ is a finite measure. The above lemma gives the existence part of the conclusion of the theorem. Uniqueness is done as before.

With the Riesz representation theorem, it is easy to show that

$$L^p(\Omega),\ p > 1$$

is a reflexive Banach space. Recall Definition 13.32 on Page 353 for the definition.

Theorem 15.19 *For* $(\Omega, \mathcal{S}, \mu)$ *a* σ *finite measure space and* $p > 1,\ L^p(\Omega)$ *is reflexive.*

Proof: Let $\delta_r : (L^r(\Omega))' \to L^{r'}(\Omega)$ be defined for $\frac{1}{r} + \frac{1}{r'} = 1$ by

$$\int (\delta_r \Lambda) g\ d\mu = \Lambda g$$

for all $g \in L^r(\Omega)$. From Theorem 15.18 δ_r is one to one, onto, continuous and linear. By the open map theorem, δ_r^{-1} is also one to one, onto, and continuous ($\delta_r \Lambda$ equals the representor of Λ). Thus δ_r^* is also one to one, onto, and continuous by Corollary 13.29. Now observe that $J = \delta_p^* \circ \delta_q^{-1}$. To see this, let $z^* \in (L^q)'$, $y^* \in (L^p)'$,

$$\begin{aligned} \delta_p^* \circ \delta_q^{-1}(\delta_q z^*)(y^*) &= (\delta_p^* z^*)(y^*) \\ &= z^*(\delta_p y^*) \\ &= \int (\delta_q z^*)(\delta_p y^*) d\mu, \end{aligned}$$

$$\begin{aligned} J(\delta_q z^*)(y^*) &= y^*(\delta_q z^*) \\ &= \int (\delta_p y^*)(\delta_q z^*) d\mu. \end{aligned}$$

Therefore $\delta_p^* \circ \delta_q^{-1} = J$ on $\delta_q (L^q)' = L^p$. But the two δ maps are onto and so J is also onto.

15.4 The Dual Space Of $C(X)$

Consider the dual space of $C(X)$ where X is a compact Hausdorff space. It will turn out to be a space of measures. To show this, the following lemma will be convenient.

Lemma 15.20 *Suppose λ is a mapping which is defined on the positive continuous functions defined on X, some topological space which satisfies*

$$\lambda(af+bg) = a\lambda(f) + b\lambda(g) \tag{15.14}$$

whenever $a, b \geq 0$ and $f, g \geq 0$. Then there exists a unique extension of λ to all of $C(X)$, Λ such that whenever $f, g \in C(X)$ and $a, b \in \mathbb{C}$, it follows

$$\Lambda(af+bg) = a\Lambda(f) + b\Lambda(g).$$

Proof: Let $C(X;\mathbb{R})$ be the real-valued functions in $C(X)$ and define

$$\Lambda_R(f) = \lambda f^+ - \lambda f^-$$

for $f \in C(X;\mathbb{R})$. Use the identity

$$(f_1+f_2)^+ + f_1^- + f_2^- = f_1^+ + f_2^+ + (f_1+f_2)^-$$

and 15.14 to write

$$\lambda(f_1+f_2)^+ - \lambda(f_1+f_2)^- = \lambda f_1^+ - \lambda f_1^- + \lambda f_2^+ - \lambda f_2^-,$$

it follows that $\Lambda_R(f_1+f_2) = \Lambda_R(f_1) + \Lambda_R(f_2)$. To show that Λ_R is linear, it is necessary to verify that $\Lambda_R(cf) = c\Lambda_R(f)$ for all $c \in \mathbb{R}$. But

$$(cf)^{\pm} = cf^{\pm},$$

if $c \geq 0$ while

$$(cf)^+ = -c(f)^-,$$

if $c < 0$ and

$$(cf)^- = (-c)f^+,$$

if $c < 0$. Thus, if $c < 0$,

$$\Lambda_R(cf) = \lambda(cf)^+ - \lambda(cf)^- = \lambda\left((-c)\,f^-\right) - \lambda\left((-c)f^+\right)$$

$$= -c\lambda(f^-) + c\lambda(f^+) = c(\lambda(f^+) - \lambda(f^-)) = c\Lambda_R(f).$$

A similar formula holds more easily if $c \geq 0$. Now let

$$\Lambda f = \Lambda_R(\operatorname{Re} f) + i\Lambda_R(\operatorname{Im} f)$$

for arbitrary $f \in C(X)$. This is linear as desired. It is obvious that $\Lambda(f+g) = \Lambda(f) + \Lambda(g)$ from the fact that taking the real and imaginary parts are linear operations. The only thing to check is whether you can factor out a complex scalar.

$$\Lambda((a+ib)f) = \Lambda(af) + \Lambda(ibf)$$

$$\equiv \Lambda_R(a\operatorname{Re} f) + i\Lambda_R(a\operatorname{Im} f) + \Lambda_R(-b\operatorname{Im} f) + i\Lambda_R(b\operatorname{Re} f)$$

because $ibf = ib\operatorname{Re} f - b\operatorname{Im} f$ and so $\operatorname{Re}(ibf) = -b\operatorname{Im} f$ and $\operatorname{Im}(ibf) = b\operatorname{Re} f$. Therefore, the above equals

$$\begin{aligned} &= (a+ib)\Lambda_R(\operatorname{Re} f) + i(a+ib)\Lambda_R(\operatorname{Im} f) \\ &= (a+ib)(\Lambda_R(\operatorname{Re} f) + i\Lambda_R(\operatorname{Im} f)) = (a+ib)\Lambda f \end{aligned}$$

The extension is obviously unique. This proves the lemma.

Let $L \in C(X)'$. Also denote by $C^+(X)$ the set of nonnegative continuous functions defined on X. Define for $f \in C^+(X)$

$$\lambda(f) = \sup\{|Lg| : |g| \leq f\}.$$

Note that $\lambda(f) < \infty$ because $|Lg| \leq ||L||\ ||g|| \leq ||L||\ ||f||$ for $|g| \leq f$. Then the following lemma is important.

Lemma 15.21 *If* $c \geq 0$, $\lambda(cf) = c\lambda(f)$, $f_1 \leq f_2$ *implies* $\lambda f_1 \leq \lambda f_2$, *and*

$$\lambda(f_1+f_2) = \lambda(f_1) + \lambda(f_2).$$

Proof: The first two assertions are easy to see so consider the third. Let $|g_j| \leq f_j$ and let $\widetilde{g}_j = e^{i\theta_j} g_j$ where θ_j is chosen such that $e^{i\theta_j} Lg_j = |Lg_j|$. Thus $L\widetilde{g}_j = |Lg_j|$. Then

$$|\widetilde{g}_1 + \widetilde{g}_2| \leq f_1 + f_2.$$

Hence

$$\begin{gathered} |Lg_1| + |Lg_2| = L\widetilde{g}_1 + L\widetilde{g}_2 = \\ L(\widetilde{g}_1 + \widetilde{g}_2) = |L(\widetilde{g}_1 + \widetilde{g}_2)| \leq \lambda(f_1+f_2). \end{gathered} \tag{15.15}$$

Choose g_1 and g_2 such that $|Lg_i| + \varepsilon > \lambda(f_i)$. Then 15.15 shows

$$\lambda(f_1) + \lambda(f_2) - 2\varepsilon \leq \lambda(f_1+f_2).$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lambda(f_1) + \lambda(f_2) \leq \lambda(f_1+f_2). \tag{15.16}$$

Now let $|g| \leq f_1 + f_2$, $|Lg| \geq \lambda(f_1+f_2) - \varepsilon$. Let

$$h_i(x) = \begin{cases} \frac{f_i(x)g(x)}{f_1(x)+f_2(x)} \text{ if } f_1(x) + f_2(x) > 0, \\ 0 \text{ if } f_1(x) + f_2(x) = 0. \end{cases}$$

Then h_i is continuous and $h_1(x) + h_2(x) = g(x)$, $|h_i| \leq f_i$. Therefore,

$$\begin{aligned} -\varepsilon + \lambda(f_1 + f_2) &\leq |Lg| \leq |Lh_1 + Lh_2| \leq |Lh_1| + |Lh_2| \\ &\leq \lambda(f_1) + \lambda(f_2). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows with 15.16 that

$$\lambda(f_1 + f_2) \leq \lambda(f_1) + \lambda(f_2) \leq \lambda(f_1 + f_2)$$

which proves the lemma.

Let Λ be defined in Lemma 15.20. Then Λ is linear by this lemma. Also, if $f \geq 0$,

$$\Lambda f = \Lambda_R f = \lambda(f) \geq 0.$$

Therefore, Λ is a positive linear functional on $C(X)$ $(= C_c(X)$ since X is compact). By Theorem 9.21 on Page 221, there exists a unique Radon measure μ such that

$$\Lambda f = \int_X f \, d\mu$$

for all $f \in C(X)$. Thus $\Lambda(1) = \mu(X)$. What follows is the Riesz representation theorem for $C(X)'$.

Theorem 15.22 *Let $L \in (C(X))'$. Then there exists a Radon measure μ and a function $\sigma \in L^\infty(X, \mu)$ such that*

$$L(f) = \int_X f \, \sigma \, d\mu.$$

Proof: Let $f \in C(X)$. Then there exists a unique Radon measure μ such that

$$|Lf| \leq \Lambda(|f|) = \int_X |f| d\mu = ||f||_1.$$

Since μ is a Radon measure, $C(X)$ is dense in $L^1(X, \mu)$. Therefore L extends uniquely to an element of $(L^1(X, \mu))'$. By the Riesz representation theorem for L^1, there exists a unique $\sigma \in L^\infty(X, \mu)$ such that

$$Lf = \int_X f \, \sigma \, d\mu$$

for all $f \in C(X)$.

15.5 The Dual Space Of $C_0(X)$

It is possible to give a simple generalization of the above theorem. For X a locally compact Hausdorff space, $\widetilde{X}$ denotes the one point compactification of X. Thus, $\widetilde{X} = X \cup \{\infty\}$ and the topology of $\widetilde{X}$ consists of the usual topology of X along with all complements of compact sets which are defined as the open sets containing ∞. Also $C_0(X)$ will denote the space of continuous functions, f, defined on X such that in the topology of $\widetilde{X}$, $\lim_{x\to\infty} f(x) = 0$. For this space of functions, $||f||_0 \equiv \sup\{|f(x)| : x \in X\}$ is a norm which makes this into a Banach space. Then the generalization is the following corollary.

Corollary 15.23 *Let $L \in (C_0(X))'$ where X is a locally compact Hausdorff space. Then there exists $\sigma \in L^\infty(X,\mu)$ for μ a finite Radon measure such that for all $f \in C_0(X)$,*

$$L(f) = \int_X f\sigma d\mu.$$

Proof: Let

$$\widetilde{D} \equiv \left\{f \in C\left(\widetilde{X}\right) : f(\infty) = 0\right\}.$$

Thus $\widetilde{D}$ is a closed subspace of the Banach space $C\left(\widetilde{X}\right)$. Let $\theta : C_0(X) \to \widetilde{D}$ be defined by

$$\theta f(x) = \begin{cases} f(x) \text{ if } x \in X, \\ 0 \text{ if } x = \infty. \end{cases}$$

Then θ is an isometry of $C_0(X)$ and $\widetilde{D}$. $(||\theta u|| = ||u||.)$The following diagram is obtained.

$$\begin{array}{ccccc} C_0(X)' & \overset{\theta^*}{\leftarrow} & \left(\widetilde{D}\right)' & \overset{i^*}{\leftarrow} & C\left(\widetilde{X}\right)' \\ & & & & \\ C_0(X) & \underset{\theta}{\rightarrow} & \widetilde{D} & \underset{i}{\rightarrow} & C\left(\widetilde{X}\right) \end{array}$$

By the Hahn Banach theorem, there exists $L_1 \in C\left(\widetilde{X}\right)'$ such that $\theta^* i^* L_1 = L$. Now apply Theorem 15.22 to get the existence of a finite Radon measure, μ_1, on $\widetilde{X}$ and a function $\sigma \in L^\infty\left(\widetilde{X}, \mu_1\right)$, such that

$$L_1 g = \int_{\widetilde{X}} g\sigma d\mu_1.$$

Letting the σ algebra of μ_1 measurable sets be denoted by $\mathcal{S}_1$, define

$$\mathcal{S} \equiv \{E \setminus \{\infty\} : E \in \mathcal{S}_1\}$$

and let μ be the restriction of μ_1 to $\mathcal{S}$. If $f \in C_0(X)$,

$$Lf = \theta^* i^* L_1 f \equiv L_1 i\theta f = L_1 \theta f = \int_{\widetilde{X}} \theta f \sigma d\mu_1 = \int_X f\sigma d\mu.$$

This proves the corollary.

15.6 More Attractive Formulations

In this section, Corollary 15.23 will be refined and placed in an arguably more attractive form. The measures involved will always be complex Borel measures defined on a σ algebra of subsets of X, a locally compact Hausdorff space.

Definition 15.24 *Let λ be a complex measure. Then $\int f d\lambda \equiv \int fhd\,|\lambda|$ where $hd\,|\lambda|$ is the polar decomposition of λ described above. The complex measure, λ is called regular if $|\lambda|$ is regular.*

The following lemma says that the difference of regular complex measures is also regular.

Lemma 15.25 *Suppose $\lambda_i, i = 1, 2$ is a complex Borel measure with total variation finite[2] defined on X, a locally compact Hausdorf space. Then $\lambda_1 - \lambda_2$ is also a regular measure on the Borel sets.*

Proof: Let E be a Borel set. That way it is in the σ algebras associated with both λ_i. Then by regularity of λ_i, there exist K and V compact and open respectively such that $K \subseteq E \subseteq V$ and $|\lambda_i|(V \setminus K) < \varepsilon/2$. Therefore,

$$\begin{aligned}\sum_{A\in\pi(V\setminus K)} |(\lambda_1 - \lambda_2)(A)| &= \sum_{A\in\pi(V\setminus K)} |\lambda_1(A) - \lambda_2(A)| \\ &\leq \sum_{A\in\pi(V\setminus K)} |\lambda_1(A)| + |\lambda_2(A)| \\ &\leq |\lambda_1|(V \setminus K) + |\lambda_2|(V \setminus K) < \varepsilon.\end{aligned}$$

Therefore, $|\lambda_1 - \lambda_2|(V \setminus K) \leq \varepsilon$ and this shows $\lambda_1 - \lambda_2$ is regular as claimed.

Theorem 15.26 *Let $L \in C_0(X)'$ Then there exists a unique complex measure, λ with $|\lambda|$ regular and Borel, such that for all $f \in C_0(X)$,*

$$L(f) = \int_X f d\lambda.$$

Furthermore, $||L|| = |\lambda|(X)$.

Proof: By Corollary 15.23 there exists $\sigma \in L^\infty(X, \mu)$ where μ is a Radon measure such that for all $f \in C_0(X)$,

$$L(f) = \int_X f\sigma d\mu.$$

Let a complex Borel measure, λ be given by

$$\lambda(E) \equiv \int_E \sigma d\mu.$$

[2]Recall this is automatic for a complex measure.

This is a well defined complex measure because μ is a finite measure. By Corollary 15.12

$$|\lambda|(E) = \int_E |\sigma| \, d\mu \tag{15.17}$$

and $\sigma = g|\sigma|$ where $gd|\lambda|$ is the polar decomposition for λ. Therefore, for $f \in C_0(X)$,

$$L(f) = \int_X f\sigma d\mu = \int_X fg|\sigma| \, d\mu = \int_X fgd|\lambda| \equiv \int_X fd\lambda. \tag{15.18}$$

From 15.17 and the regularity of μ, it follows that $|\lambda|$ is also regular.

What of the claim about $||L||$? By the regularity of $|\lambda|$, it follows that $C_0(X)$ (In fact, $C_c(X)$) is dense in $L^1(X, |\lambda|)$. Since $|\lambda|$ is finite, $g \in L^1(X, |\lambda|)$. Therefore, there exists a sequence of functions in $C_0(X), \{f_n\}$ such that $f_n \to \overline{g}$ in $L^1(X, |\lambda|)$. Therefore, there exists a subsequence, still denoted by $\{f_n\}$ such that $f_n(x) \to \overline{g}(x)$ $|\lambda|$ a.e. also. But since $|\overline{g}(x)| = 1$ a.e. it follows that $h_n(x) \equiv \frac{f_n(x)}{|f_n(x)| + \frac{1}{n}}$ also converges pointwise $|\lambda|$ a.e. Then from the dominated convergence theorem and 15.18

$$||L|| \geq \lim_{n\to\infty} \int_X h_n gd|\lambda| = |\lambda|(X).$$

Also, if $||f||_{C_0(X)} \leq 1$, then

$$|L(f)| = \left| \int_X fgd|\lambda| \right| \leq \int_X |f| \, d|\lambda| \leq |\lambda|(X) \, ||f||_{C_0(X)}$$

and so $||L|| \leq |\lambda|(X)$. This proves everything but uniqueness.

Suppose λ and λ_1 both work. Then for all $f \in C_0(X)$,

$$0 = \int_X fd(\lambda - \lambda_1) = \int_X fhd|\lambda - \lambda_1|$$

where $hd|\lambda - \lambda_1|$ is the polar decomposition for $\lambda - \lambda_1$. By Lemma 15.25 $\lambda - \lambda_1$ is regular and so, as above, there exists $\{f_n\}$ such that $|f_n| \leq 1$ and $f_n \to \overline{h}$ pointwise. Therefore, $\int_X d|\lambda - \lambda_1| = 0$ so $\lambda = \lambda_1$. This proves the theorem.

15.7 Exercises

1. Suppose μ is a vector measure having values in $\mathbb{R}^n$ or $\mathbb{C}^n$. Can you show that $|\mu|$ must be finite? **Hint:** You might define for each $\mathbf{e}_i$, one of the standard basis vectors, the real or complex measure, $\mu_{\mathbf{e}_i}$ given by $\mu_{\mathbf{e}_i}(E) \equiv \mathbf{e}_i \cdot \mu(E)$. Why would this approach not yield anything for an infinite dimensional normed linear space in place of $\mathbb{R}^n$?

2. The Riesz representation theorem of the L^p spaces can be used to prove a very interesting inequality. Let $r, p, q \in (1, \infty)$ satisfy

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Then

$$\frac{1}{q} = 1 + \frac{1}{r} - \frac{1}{p} > \frac{1}{r}$$

and so $r > q$. Let $\theta \in (0,1)$ be chosen so that $\theta r = q$. Then also we have

$$\frac{1}{r} = \left(\overbrace{1 - \frac{1}{p'}}^{1/p+1/p'=1} \right) + \frac{1}{q} - 1 = \frac{1}{q} - \frac{1}{p'}$$

and so

$$\frac{\theta}{q} = \frac{1}{q} - \frac{1}{p'}$$

which implies $p'(1-\theta) = q$. Now let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $f, g \geq 0$. Justify the steps in the following argument using what was just shown that $\theta r = q$ and $p'(1-\theta) = q$. Let

$$h \in L^{r'}(\mathbb{R}^n) . \left(\frac{1}{r} + \frac{1}{r'} = 1 \right)$$

$$\int f * g(\mathbf{x}) |h(\mathbf{x})| \, dx = \int \int f(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) |h(\mathbf{x})| \, dx dy.$$

$$\leq \int \int |f(\mathbf{y})| |g(\mathbf{x}-\mathbf{y})|^{\theta} |g(\mathbf{x}-\mathbf{y})|^{1-\theta} |h(\mathbf{x})| \, dy dx$$

$$\leq \int \left(\int \left(|g(\mathbf{x}-\mathbf{y})|^{1-\theta} |h(\mathbf{x})| \right)^{r'} dx \right)^{1/r'} \cdot$$

$$\left(\int \left(|f(\mathbf{y})| |g(\mathbf{x}-\mathbf{y})|^{\theta} \right)^{r} dx \right)^{1/r} dy$$

$$\leq \left[\int \left(\int \left(|g(\mathbf{x}-\mathbf{y})|^{1-\theta} |h(\mathbf{x})| \right)^{r'} dx \right)^{p'/r'} dy \right]^{1/p'} \cdot$$

$$\left[\int \left(\int \left(|f(\mathbf{y})| |g(\mathbf{x}-\mathbf{y})|^{\theta} \right)^{r} dx \right)^{p/r} dy \right]^{1/p}$$

$$\leq \left[\int \left(\int \left(|g(\mathbf{x}-\mathbf{y})|^{1-\theta} |h(\mathbf{x})| \right)^{p'} dy \right)^{r'/p'} dx \right]^{1/r'} \cdot$$

$$\left[\int |f(\mathbf{y})|^{p} \left(\int |g(\mathbf{x}-\mathbf{y})|^{\theta r} dx \right)^{p/r} dy \right]^{1/p}$$

$$= \left[\int |h(\mathbf{x})|^{r'} \left(\int |g(\mathbf{x}-\mathbf{y})|^{(1-\theta)p'} dy \right)^{r'/p'} dx \right]^{1/r'} ||g||_q^{q/r} ||f||_p$$

$$= ||g||_q^{q/r} ||g||_q^{q/p'} ||f||_p ||h||_{r'} = ||g||_q ||f||_p ||h||_{r'} . \tag{15.19}$$

Young's inequality says that

$$||f * g||_r \leq ||g||_q ||f||_p . \tag{15.20}$$

Therefore $||f * g||_r \leq ||g||_q ||f||_p$. How does this inequality follow from the above computation? Does 15.19 continue to hold if r, p, q are only assumed to be in $[1, \infty]$? Explain. Does 15.20 hold even if r, p, and q are only assumed to lie in $[1, \infty]$?

3. Show that in a reflexive Banach space, weak and weak $*$ convergence are the same.

4. Suppose $(\Omega, \mu, \mathcal{S})$ is a finite measure space and that $\{f_n\}$ is a sequence of functions which converge weakly to 0 in $L^p(\Omega)$. Suppose also that $f_n(x) \to 0$ a.e. Show that then $f_n \to 0$ in $L^{p-\varepsilon}(\Omega)$ for every $p > \varepsilon > 0$.

5. Give an example of a sequence of functions in $L^\infty(-\pi, \pi)$ which converges weak $*$ to zero but which does not converge pointwise a.e. to zero.

Integrals And Derivatives

16.1 The Fundamental Theorem Of Calculus

The version of the fundamental theorem of calculus found in Calculus has already been referred to frequently. It says that if f is a Riemann integrable function, the function

$$x \to \int_a^x f(t)\, dt,$$

has a derivative at every point where f is continuous. It is natural to ask what occurs for f in L^1. It is an amazing fact that the same result is obtained asside from a set of measure zero even though f, being only in L^1 may fail to be continuous anywhere. Proofs of this result are based on some form of the Vitali covering theorem presented above. In what follows, the measure space is $(\mathbb{R}^n, \mathcal{S}, m)$ where m is n-dimensional Lebesgue measure although the same theorems can be proved for arbitrary Radon measures [36]. To save notation, m is written in place of m_n.

By Lemma 9.7 on Page 214 and the completeness of m, the Lebesgue measurable sets are exactly those measurable in the sense of Caratheodory. Also, to save on notation m is also the name of the outer measure defined on all of $\mathcal{P}(\mathbb{R}^n)$ which is determined by m_n. Recall

$$B(\mathbf{p}, r) = \{\mathbf{x} : |\mathbf{x} - \mathbf{p}| < r\}. \tag{16.1}$$

Also define the following.

$$\text{If } B = B(\mathbf{p}, r), \text{ then } \widehat{B} = B(\mathbf{p}, 5r). \tag{16.2}$$

The first version of the Vitali covering theorem presented above will now be used to establish the fundamental theorem of calculus. The space of locally integrable functions is the most general one for which the maximal function defined below makes sense.

Definition 16.1 *$f \in L^1_{loc}(\mathbb{R}^n)$ means $f\mathcal{X}_{B(0,R)} \in L^1(\mathbb{R}^n)$ for all $R > 0$. For $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy Littlewood Maximal Function, Mf, is defined by*

$$Mf(\mathbf{x}) \equiv \sup_{r>0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y})| dy.$$

Theorem 16.2 *If $f \in L^1(\mathbb{R}^n)$, then for $\alpha > 0$,*

$$\overline{m}([Mf > \alpha]) \leq \frac{5^n}{\alpha} ||f||_1.$$

(Here and elsewhere, $[Mf > \alpha] \equiv \{\mathbf{x} \in \mathbb{R}^n : Mf(\mathbf{x}) > \alpha\}$ with other occurrences of $[\ ]$ being defined similarly.)

Proof: Let $S \equiv [Mf > \alpha]$. For $\mathbf{x} \in S$, choose $r_{\mathbf{x}} > 0$ with

$$\frac{1}{m(B(\mathbf{x}, r_{\mathbf{x}}))} \int_{B(\mathbf{x}, r_{\mathbf{x}})} |f| \, dm > \alpha.$$

The $r_{\mathbf{x}}$ are all bounded because

$$m(B(\mathbf{x}, r_{\mathbf{x}})) < \frac{1}{\alpha} \int_{B(\mathbf{x}, r_{\mathbf{x}})} |f| \, dm < \frac{1}{\alpha} ||f||_1.$$

By the Vitali covering theorem, there are disjoint balls $B(\mathbf{x}_i, r_i)$ such that

$$S \subseteq \cup_{i=1}^{\infty} B(\mathbf{x}_i, 5r_i)$$

and

$$\frac{1}{m(B(\mathbf{x}_i, r_i))} \int_{B(\mathbf{x}_i, r_i)} |f| \, dm > \alpha.$$

Therefore

$$\begin{aligned}
\overline{m}(S) &\leq \sum_{i=1}^{\infty} m(B(\mathbf{x}_i, 5r_i)) = 5^n \sum_{i=1}^{\infty} m(B(\mathbf{x}_i, r_i)) \\
&\leq \frac{5^n}{\alpha} \sum_{i=1}^{\infty} \int_{B(\mathbf{x}_i, r_i)} |f| \, dm \\
&\leq \frac{5^n}{\alpha} \int_{\mathbb{R}^n} |f| \, dm,
\end{aligned}$$

the last inequality being valid because the balls $B(\mathbf{x}_i, r_i)$ are disjoint. This proves the theorem.

Note that at this point it is unknown whether S is measurable. This is why $\overline{m}(S)$ and not $m(S)$ is written.

The following is the fundamental theorem of calculus from elementary calculus.

Lemma 16.3 *Suppose g is a continuous function. Then for all $\mathbf{x}$,*

$$\lim_{r \to 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} g(\mathbf{y}) dy = g(\mathbf{x}).$$

Proof: Note that

$$g(\mathbf{x}) = \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} g(\mathbf{x})\, dy$$

and so

$$\left| g(\mathbf{x}) - \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} g(\mathbf{y}) dy \right|$$

$$= \left| \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} (g(\mathbf{y}) - g(\mathbf{x}))\, dy \right|$$

$$\leq \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |g(\mathbf{y}) - g(\mathbf{x})|\, dy.$$

Now by continuity of g at $\mathbf{x}$, there exists $r > 0$ such that if $|\mathbf{x} - \mathbf{y}| < r$, $|g(\mathbf{y}) - g(\mathbf{x})| < \varepsilon$. For such r, the last expression is less than

$$\frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} \varepsilon dy < \varepsilon.$$

This proves the lemma.

Definition 16.4 *Let $f \in L^1(\mathbb{R}^k, m)$. A point, $\mathbf{x} \in \mathbb{R}^k$ is said to be a Lebesgue point if*

$$\limsup_{r \to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\, dm = 0.$$

Note that if $\mathbf{x}$ is a Lebesgue point, then

$$\lim_{r \to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} f(\mathbf{y})\, dm = f(\mathbf{x}).$$

and so the symmetric derivative exists at all Lebesgue points.

Theorem 16.5 *(Fundamental Theorem of Calculus) Let $f \in L^1(\mathbb{R}^k)$. Then there exists a set of measure $0, N$, such that if $\mathbf{x} \notin N$, then*

$$\lim_{r \to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})| dy = 0.$$

Proof: Let $\lambda > 0$ and let $\varepsilon > 0$. By density of $C_c(\mathbb{R}^k)$ in $L^1(\mathbb{R}^k, m)$ there exists $g \in C_c(\mathbb{R}^k)$ such that $||g - f||_{L^1(\mathbb{R}^k)} < \varepsilon$. Now since g is continuous,

$$\limsup_{r \to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\, dm$$

$$= \limsup_{r \to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\, dm$$

$$- \lim_{r \to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |g(\mathbf{y}) - g(\mathbf{x})|\, dm$$

$$\begin{aligned}
&= \limsup_{r\to 0}\left(\frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)}|f(\mathbf{y})-f(\mathbf{x})|-|g(\mathbf{y})-g(\mathbf{x})|\,dm\right)\\
&\leq \limsup_{r\to 0}\left(\frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)}||f(\mathbf{y})-f(\mathbf{x})|-|g(\mathbf{y})-g(\mathbf{x})||\,dm\right)\\
&\leq \limsup_{r\to 0}\left(\frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)}|f(\mathbf{y})-g(\mathbf{y})-(f(\mathbf{x})-g(\mathbf{x}))|\,dm\right)\\
&\leq \limsup_{r\to 0}\left(\frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)}|f(\mathbf{y})-g(\mathbf{y})|\,dm\right)+|f(\mathbf{x})-g(\mathbf{x})|\\
&\leq M([f-g])(\mathbf{x})+|f(\mathbf{x})-g(\mathbf{x})|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left[\mathbf{x}:\limsup_{r\to 0}\frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)}|f(\mathbf{y})-f(\mathbf{x})|\,dm>\lambda\right]\\
\subseteq\ &\left[M([f-g])>\frac{\lambda}{2}\right]\cup\left[|f-g|>\frac{\lambda}{2}\right]
\end{aligned}$$

Now

$$\begin{aligned}
\varepsilon &> \int|f-g|\,dm\geq\int_{\left[|f-g|>\frac{\lambda}{2}\right]}|f-g|\,dm\\
&\geq \frac{\lambda}{2}m\left(\left[|f-g|>\frac{\lambda}{2}\right]\right)
\end{aligned}$$

This along with the weak estimate of Theorem 16.2 implies

$$\begin{aligned}
&m\left(\left[\mathbf{x}:\limsup_{r\to 0}\frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)}|f(\mathbf{y})-f(\mathbf{x})|\,dm>\lambda\right]\right)\\
<\ &\left(\frac{2}{\lambda}5^k+\frac{2}{\lambda}\right)||f-g||_{L^1(\mathbb{R}^k)}\\
<\ &\left(\frac{2}{\lambda}5^k+\frac{2}{\lambda}\right)\varepsilon.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, it follows

$$m_n\left(\left[\mathbf{x}:\limsup_{r\to 0}\frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)}|f(\mathbf{y})-f(\mathbf{x})|\,dm>\lambda\right]\right)=0.$$

Now let

$$N=\left[\mathbf{x}:\limsup_{r\to 0}\frac{1}{m(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)}|f(\mathbf{y})-f(\mathbf{x})|\,dm>0\right]$$

and

$$N_n = \left[\mathbf{x} : \limsup_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\, dm > \frac{1}{n}\right]$$

It was just shown that $m(N_n) = 0$. Also, $N = \cup_{n=1}^{\infty} N_n$. Therefore, $m(N) = 0$ also. It follows that for $\mathbf{x} \notin N$,

$$\limsup_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\, dm = 0$$

and this proves a.e. point is a Lebesgue point.

Of course it is sufficient to assume f is only in $L^1_{loc}\left(\mathbb{R}^k\right)$.

Corollary 16.6 *(Fundamental Theorem of Calculus) Let $f \in L^1_{loc}(\mathbb{R}^k)$. Then there exists a set of measure $0, N$, such that if $\mathbf{x} \notin N$, then*

$$\lim_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})| dy = 0.$$

Proof: Consider $B(\mathbf{0},n)$ where n is a positive integer. Then $f_n \equiv f\mathcal{X}_{B(\mathbf{0},n)} \in L^1\left(\mathbb{R}^k\right)$ and so there exists a set of measure 0, N_n such that if $\mathbf{x} \in B(\mathbf{0},n) \setminus N_n$, then

$$\begin{aligned} & \lim_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f_n(\mathbf{y}) - f_n(\mathbf{x})| dy \\ = & \lim_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})| dy = 0. \end{aligned}$$

Let $N = \cup_{n=1}^{\infty} N_n$. Then if $\mathbf{x} \notin N$, the above equation holds.

Corollary 16.7 *If $f \in L^1_{loc}(\mathbb{R}^n)$, then*

$$\lim_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} f(\mathbf{y}) dy = f(\mathbf{x}) \quad \text{a.e. } \mathbf{x}. \tag{16.3}$$

Proof:

$$\begin{aligned} & \left| \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} f(\mathbf{y}) dy - f(\mathbf{x}) \right| \\ \leq & \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\, dy \end{aligned}$$

and the last integral converges to 0 a.e. $\mathbf{x}$.

Definition 16.8 *For N the set of Theorem 16.5 or Corollary 16.6, N^C is called the Lebesgue set or the set of Lebesgue points.*

The next corollary is a one dimensional version of what was just presented.

Corollary 16.9 *Let $f \in L^1(\mathbb{R})$ and let*

$$F(x) = \int_{-\infty}^{x} f(t)dt.$$

Then for a.e. x, $F'(x) = f(x)$.

Proof: For $h > 0$

$$\frac{1}{h}\int_{x}^{x+h} |f(y) - f(x)|dy \leq 2(\frac{1}{2h})\int_{x-h}^{x+h} |f(y) - f(x)|dy$$

By Theorem 16.5, this converges to 0 a.e. Similarly

$$\frac{1}{h}\int_{x-h}^{x} |f(y) - f(x)|dy$$

converges to 0 a.e. x.

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \leq \frac{1}{h}\int_{x}^{x+h} |f(y) - f(x)|dy \tag{16.4}$$

and

$$\left|\frac{F(x) - F(x-h)}{h} - f(x)\right| \leq \frac{1}{h}\int_{x-h}^{x} |f(y) - f(x)|dy. \tag{16.5}$$

Now the expression on the right in 16.4 and 16.5 converges to zero for a.e. x. Therefore, by 16.4, for a.e. x the derivative from the right exists and equals $f(x)$ while from 16.5 the derivative from the left exists and equals $f(x)$ a.e. It follows

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x) \text{ a.e. } x$$

This proves the corollary.

16.2 Absolutely Continuous Functions

Definition 16.10 *Let $[a, b]$ be a closed and bounded interval and let $f : [a, b] \to \mathbb{R}$. Then f is said to be absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\sum_{i=1}^{m} |y_i - x_i| < \delta$, then $\sum_{i=1}^{m} |f(y_i) - f(x_i)| < \varepsilon$.*

Definition 16.11 *A finite subset, P of $[a, b]$ is called a partition of $[x, y] \subseteq [a, b]$ if $P = \{x_0, x_1, \cdots, x_n\}$ where*

$$x = x_0 < x_1 < \cdots, < x_n = y.$$

For $f : [a, b] \to \mathbb{R}$ and $P = \{x_0, x_1, \cdots, x_n\}$ define

$$V_P[x, y] \equiv \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|.$$

Denoting by $\mathcal{P}[x,y]$ the set of all partitions of $[x,y]$ define

$$V[x,y] \equiv \sup_{P \in \mathcal{P}[x,y]} V_P[x,y].$$

For simplicity, $V[a,x]$ will be denoted by $V(x)$. It is called the total variation of the function, f.

There are some simple facts about the total variation of an absolutely continuous function, f which are contained in the next lemma.

Lemma 16.12 *Let f be an absolutely continuous function defined on $[a,b]$ and let V be its total variation function as described above. Then V is an increasing bounded function. Also if P and Q are two partitions of $[x,y]$ with $P \subseteq Q$, then $V_P[x,y] \leq V_Q[x,y]$ and if $[x,y] \subseteq [z,w]$,*

$$V[x,y] \leq V[z,w] \tag{16.6}$$

If $P = \{x_0, x_1, \cdots, x_n\}$ is a partition of $[x,y]$, then

$$V[x,y] = \sum_{i=1}^{n} V[x_i, x_{i-1}]. \tag{16.7}$$

Also if $y > x$,

$$V(y) - V(x) \geq |f(y) - f(x)| \tag{16.8}$$

and the function, $x \to V(x) - f(x)$ is increasing. The total variation function, V is absolutely continuous.

Proof: The claim that V is increasing is obvious as is the next claim about $P \subseteq Q$ leading to $V_P[x,y] \leq V_Q[x,y]$. To verify this, simply add in one point at a time and verify that from the triangle inequality, the sum involved gets no smaller. The claim that V is increasing consistent with set inclusion of intervals is also clearly true and follows directly from the definition.

Now let $t < V[x,y]$ where $P_0 = \{x_0, x_1, \cdots, x_n\}$ is a partition of $[x,y]$. There exists a partition, P of $[x,y]$ such that $t < V_P[x,y]$. Without loss of generality it can be assumed that $\{x_0, x_1, \cdots, x_n\} \subseteq P$ since if not, you can simply add in the points of P_0 and the resulting sum for the total variation will get no smaller. Let P_i be those points of P which are contained in $[x_{i-1}, x_i]$. Then

$$t < V_p[x,y] = \sum_{i=1}^{n} V_{P_i}[x_{i-1}, x_i] \leq \sum_{i=1}^{n} V[x_{i-1}, x_i].$$

Since $t < V[x,y]$ is arbitrary,

$$V[x,y] \leq \sum_{i=1}^{n} V[x_i, x_{i-1}] \tag{16.9}$$

Note that 16.9 does not depend on f being absolutely continuous. Suppose now that f is absolutely continuous. Let δ correspond to $\varepsilon = 1$. Then if $[x, y]$ is an interval of length no larger than δ, the definition of absolute continuity implies

$$V[x, y] < 1.$$

Then from 16.9

$$V[a, n\delta] \leq \sum_{i=1}^{n} V[a + (i-1)\delta, a + i\delta] < \sum_{i=1}^{n} 1 = n.$$

Thus V is bounded on $[a, b]$. Now let P_i be a partition of $[x_{i-1}, x_i]$ such that

$$V_{P_i}[x_{i-1}, x_i] > V[x_{i-1}, x_i] - \frac{\varepsilon}{n}$$

Then letting $P = \cup P_i$,

$$-\varepsilon + \sum_{i=1}^{n} V[x_{i-1}, x_i] < \sum_{i=1}^{n} V_{P_i}[x_{i-1}, x_i] = V_P[x, y] \leq V[x, y].$$

Since ε is arbitrary, 16.7 follows from this and 16.9.

Now let $x < y$

$$\begin{aligned} V(y) - f(y) - (V(x) - f(x)) &= V(y) - V(x) - (f(y) - f(x)) \\ &\geq V(y) - V(x) - |f(y) - f(x)| \geq 0. \end{aligned}$$

It only remains to verify that V is absolutely continuous.

Let $\varepsilon > 0$ be given and let δ correspond to $\varepsilon/2$ in the definition of absolute continuity applied to f. Suppose $\sum_{i=1}^{n} |y_i - x_i| < \delta$ and consider $\sum_{i=1}^{n} |V(y_i) - V(x_i)|$. By 16.9 this last equals $\sum_{i=1}^{n} V[x_i, y_i]$. Now let P_i be a partition of $[x_i, y_i]$ such that $V_{P_i}[x_i, y_i] + \frac{\varepsilon}{2n} > V[x_i, y_i]$. Then by the definition of absolute continuity,

$$\begin{aligned} \sum_{i=1}^{n} |V(y_i) - V(x_i)| &= \sum_{i=1}^{n} V[x_i, y_i] \\ &\leq \sum_{i=1}^{n} V_{P_i}[x_i, y_i] + \eta < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

and shows V is absolutely continuous as claimed.

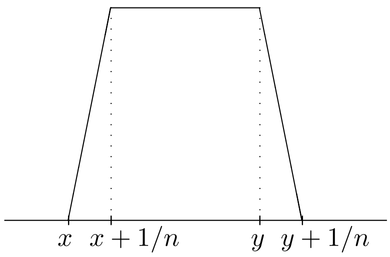
Lemma 16.13 *Suppose $f : [a, b] \to \mathbb{R}$ is absolutely continuous and increasing. Then f' exists a.e., is in $L^1([a, b])$, and*

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

Proof: Define L, a positive linear functional on $C([a,b])$ by

$$Lg \equiv \int_a^b g df$$

where this integral is the Riemann Stieltjes integral with respect to the integrating function, f. By the Riesz representation theorem for positive linear functionals, there exists a unique Radon measure, μ such that $Lg = \int g d\mu$. Now consider the following picture for $g_n \in C([a,b])$ in which g_n equals 1 for x between $x + 1/n$ and y.



Then $g_n(t) \to \mathcal{X}_{(x,y]}(t)$ pointwise. Therefore, by the dominated convergence theorem,

$$\mu((x,y]) = \lim_{n\to\infty} \int g_n d\mu.$$

However,

$$\begin{aligned} & \left(f(y) - f\left(x + \frac{1}{n}\right)\right) \\ \leq & \int g_n d\mu = \int_a^b g_n df \leq \left(f\left(y + \frac{1}{n}\right) - f(y)\right) \\ & + \left(f(y) - f\left(x + \frac{1}{n}\right)\right) + \left(f\left(x + \frac{1}{n}\right) - f(x)\right) \end{aligned}$$

and so as $n \to \infty$ the continuity of f implies

$$\mu((x,y]) = f(y) - f(x).$$

Similarly, $\mu(x,y) = f(y) - f(y)$ and $\mu([x,y]) = f(y) - f(x)$, the argument used to establish this being very similar to the above. It follows in particular that

$$f(x) - f(a) = \int_{[a,x]} d\mu.$$

Note that up till now, no referrence has been made to the absolute continuity of f. Any increasing continuous function would be fine.

Now if E is a Borel set such that $m(E) = 0$, Then the outer regularity of m implies there exists an open set, V containing E such that $m(V) < \delta$ where δ corresponds to ε in the definition of absolute continuity of f. Then letting $\{I_k\}$ be the connected components of V it follows $E \subseteq \cup_{k=1}^{\infty} I_k$ with $\sum_k m(I_k) = m(V) < \delta$. Therefore, from absolute continuity of f, it follows that for $I_k = (a_k, b_k)$ and each n

$$\mu\left(\cup_{k=1}^{n} I_k\right) = \sum_{k=1}^{n} \mu(I_k) = \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

and so letting $n \to \infty$,

$$\mu(E) \leq \mu(V) = \sum_{k=1}^{\infty} |f(b_k) - f(a_k)| \leq \varepsilon.$$

Since ε is arbitrary, it follows $\mu(E) = 0$. Therefore, $\mu \ll m$ and so by the Radon Nikodym theorem there exists a unique $h \in L^1([a,b])$ such that

$$\mu(E) = \int_E h dm.$$

In particular,

$$\mu([a,x]) = f(x) - f(a) = \int_{[a,x]} h dm.$$

From the fundamental theorem of calculus $f'(x) = h(x)$ at every Lebesgue point of h. Therefore, writing in usual notation,

$$f(x) = f(a) + \int_a^x f'(t)\, dt$$

as claimed. This proves the lemma.

With the above lemmas, the following is the main theorem about absolutely continuous functions.

Theorem 16.14 *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous if and only if $f'(x)$ exists a.e., $f' \in L^1([a,b])$ and*

$$f(x) = f(a) + \int_a^x f'(t)\, dt.$$

Proof: Suppose first that f is absolutely continuous. By Lemma 16.12 the total variation function, V is absolutely continuous and $f(x) = V(x) - (V(x) - f(x))$ where both V and $V - f$ are increasing and absolutely continuous. By Lemma 16.13

$$\begin{aligned} f(x) - f(a) &= V(x) - V(a) - [(V(x) - f(x)) - (V(a) - f(a))] \\ &= \int_a^x V'(t)\, dt - \int_a^x (V - f)'(t)\, dt. \end{aligned}$$

Now f' exists and is in L^1 becasue $f = V - (V - f)$ and V and $V - f$ have derivatives in L^1. Therefore, $(V - f)' = V' - f'$ and so the above reduces to

$$f(x) - f(a) = \int_a^x f'(t)\,dt.$$

This proves one half of the theorem.

Now suppose $f' \in L^1$ and $f(x) = f(a) + \int_a^x f'(t)\,dt$. It is necessary to verify that f is absolutely continuous. But this follows easily from Lemma 8.49 on Page 203 which implies that a single function, f' is uniformly integrable. This lemma implies that if $\sum_i |y_i - x_i|$ is sufficiently small then

$$\sum_i \left| \int_{x_i}^{y_i} f'(t)\,dt \right| = \sum_i |f(y_i) - f(x_i)| < \varepsilon.$$

16.3 Differentiation Of Measures With Respect To Lebesgue Measure

Recall the Vitali covering theorem in Corollary 10.20 on Page 279.

Corollary 16.15 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of open balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m}(E \setminus \cup_{j=1}^\infty B_j) = 0$.*

Definition 16.16 *Let μ be a Radon mesure defined on $\mathbb{R}^n$. Then*

$$\frac{d\mu}{dm}(\mathbf{x}) \equiv \lim_{r \to 0} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}$$

whenever this limit exists.

It turns out this limit exists for m a.e. $\mathbf{x}$. To verify this here is another definition.

Definition 16.17 *Let $f(r)$ be a function having values in $[-\infty, \infty]$. Then*

$$\begin{aligned} \limsup_{r \to 0+} f(r) &\equiv \lim_{r \to 0} (\sup \{f(t) : t \in [0, r]\}) \\ \liminf_{r \to 0+} f(r) &\equiv \lim_{r \to 0} (\inf \{f(t) : t \in [0, r]\}) \end{aligned}$$

This is well defined because the function $r \to \inf \{f(t) : t \in [0, r]\}$ is increasing and $r \to \sup \{f(t) : t \in [0, r]\}$ is decreasing. Also note that $\lim_{r \to 0+} f(r)$ exists if and only if

$$\limsup_{r \to 0+} f(r) = \liminf_{r \to 0+} f(r)$$

and if this happens

$$\lim_{r \to 0+} f(r) = \liminf_{r \to 0+} f(r) = \limsup_{r \to 0+} f(r).$$

The claims made in the above definition follow immediately from the definition of what is meant by a limit in $[-\infty,\infty]$ and are left for the reader.

Theorem 16.18 *Let μ be a Borel measure on $\mathbb{R}^n$ then $\frac{d\mu}{dm}(\mathbf{x})$ exists in $[-\infty,\infty]$ m a.e.*

Proof: Let $p<q$ and let p,q be rational numbers. Define

$$\begin{aligned} N_{pq}(M) &\equiv \left\{\mathbf{x}\in\mathbb{R}^n \text{ such that } \lim\sup_{r\to 0+}\frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))}>q\right. \\ &\left.>p>\lim\inf_{r\to 0+}\frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))}\right\}\cap B(\mathbf{0},M), \\ N_{pq} &\equiv \left\{\mathbf{x}\in\mathbb{R}^n \text{ such that } \lim\sup_{r\to 0+}\frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))}>q\right. \\ &\left.>p>\lim\inf_{r\to 0+}\frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))}\right\}, \\ N &\equiv \left\{\mathbf{x}\in\mathbb{R}^n \text{ such that } \lim\sup_{r\to 0+}\frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))}>\right. \\ &\left.\lim\inf_{r\to 0+}\frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))}\right\}. \end{aligned}$$

I will show $\overline{m}(N_{pq}(M))=0$. Use outer regularity to obtain an open set, V containing $N_{pq}(M)$ such that

$$\overline{m}(N_{pq}(M))+\varepsilon>m(V).$$

From the definition of $N_{pq}(M)$, it follows that for each $\mathbf{x}\in N_{pq}(M)$ there exist arbitrarily small $r>0$ such that

$$\frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))}<p.$$

Only consider those r which are small enough to be contained in $B(\mathbf{0},M)$ so that the collection of such balls has bounded radii. This is a Vitali cover of $N_{pq}(M)$ and so by Corollary 16.15 there exists a sequence of disjoint balls of this sort, $\{B_i\}_{i=1}^{\infty}$ such that

$$\mu(B_i)<pm(B_i),\ \overline{m}(N_{pq}(M)\setminus\cup_{i=1}^{\infty}B_i)=0. \tag{16.10}$$

Now for $\mathbf{x}\in N_{pq}(M)\cap(\cup_{i=1}^{\infty}B_i)$ (most of $N_{pq}(M)$), there exist arbitrarily small balls, $B(\mathbf{x},r)$, such that $B(\mathbf{x},r)$ is contained in some set of $\{B_i\}_{i=1}^{\infty}$ and

$$\frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))}>q.$$

This is a Vitali cover of $N_{pq}(M)\cap(\cup_{i=1}^{\infty}B_i)$ and so there exists a sequence of disjoint balls of this sort, $\left\{B_j'\right\}_{j=1}^{\infty}$ such that

$$\overline{m}\left((N_{pq}(M)\cap(\cup_{i=1}^{\infty}B_i))\setminus\cup_{j=1}^{\infty}B_j'\right)=0,\ \mu\left(B_j'\right)>qm\left(B_j'\right). \tag{16.11}$$

It follows from 16.10 and 16.11 that

$$\overline{m}\left(N_{pq}\left(M\right)\right) \leq \overline{m}\left(\left(N_{pq}\left(M\right) \cap \left(\cup_{i=1}^{\infty} B_i\right)\right)\right) \leq m\left(\cup_{j=1}^{\infty} B_j'\right) \tag{16.12}$$

Therefore,

$$\begin{aligned}
\sum_j \mu\left(B_j'\right) &> q \sum_j m\left(B_j'\right) \geq q\overline{m}\left(N_{pq}\left(M\right) \cap \left(\cup_i B_i\right)\right) = q\overline{m}\left(N_{pq}\left(M\right)\right) \\
&\geq p\overline{m}\left(N_{pq}\left(M\right)\right) \geq p\left(m\left(V\right) - \varepsilon\right) \geq p \sum_i m\left(B_i\right) - p\varepsilon \\
&\geq \sum_i \mu\left(B_i\right) - p\varepsilon \geq \sum_j \mu\left(B_j'\right) - p\varepsilon.
\end{aligned}$$

It follows

$$p\varepsilon \geq \left(q - p\right)\overline{m}\left(N_{pq}\left(M\right)\right)$$

Since ε is arbitrary, $m\left(N_{pq}\left(M\right)\right) = 0$. Now $N_{pq} \subseteq \cup_{M=1}^{\infty} N_{pq}\left(M\right)$ and so $m\left(N_{pq}\right) = 0$. Now

$$N = \cup_{p.q \in \mathbb{Q}} N_{pq}$$

and since this is a countable union of sets of measure zero, $m\left(N\right) = 0$ also. This proves the theorem.

From Theorem 15.8 on Page 407 it follows that if μ is a complex measure then $|\mu|$ is a finite measure. This makes possible the following definition.

Definition 16.19 *Let μ be a real measure. Define the following measures. For E a measurable set,*

$$\begin{aligned}
\mu^+\left(E\right) &\equiv \frac{1}{2}\left(|\mu| + \mu\right)\left(E\right), \\
\mu^-\left(E\right) &\equiv \frac{1}{2}\left(|\mu| - \mu\right)\left(E\right).
\end{aligned}$$

These are measures thanks to Theorem 15.7 on Page 406 and $\mu^+ - \mu^- = \mu$. These measures have values in $[0, \infty)$. They are called the positive and negative parts of μ respectively. For μ a complex measure, define $\operatorname{Re}\mu$ *and* $\operatorname{Im}\mu$ *by*

$$\begin{aligned}
\operatorname{Re}\mu\left(E\right) &\equiv \frac{1}{2}\left(\mu\left(E\right) + \overline{\mu\left(E\right)}\right) \\
\operatorname{Im}\mu\left(E\right) &\equiv \frac{1}{2i}\left(\mu\left(E\right) - \overline{\mu\left(E\right)}\right)
\end{aligned}$$

Then $\operatorname{Re}\mu$ *and* $\operatorname{Im}\mu$ *are both real measures. Thus for μ a complex measure,*

$$\begin{aligned}
\mu &= \operatorname{Re}\mu^+ - \operatorname{Re}\mu^- + i\left(\operatorname{Im}\mu^+ - \operatorname{Im}\mu^-\right) \\
&= \nu_1 - \nu_1 + i\left(\nu_3 - \nu_4\right)
\end{aligned}$$

where each ν_i is a real measure having values in $[0, \infty)$.

Then there is an obvious corollary to Theorem 16.18.

Corollary 16.20 *Let μ be a complex Borel measure on $\mathbb{R}^n$. Then $\frac{d\mu}{dm}(\mathbf{x})$ exists a.e.*

Proof: Letting ν_i be defined in Definition 16.19. By Theorem 16.18, for m a.e. $\mathbf{x}$, $\frac{d\nu_i}{dm}(\mathbf{x})$ exists. This proves the corollary because μ is just a finite sum of these ν_i.

Theorem 15.2 on Page 399, the Radon Nikodym theorem, implies that if you have two finite measures, μ and λ, you can write λ as the sum of a measure absolutely continuous with respect to μ and one which is singular to μ in a unique way. The next topic is related to this. It has to do with the differentiation of a measure which is singular with respect to Lebesgue measure.

Theorem 16.21 *Let μ be a Radon measure on $\mathbb{R}^n$ and suppose there exists a μ measurable set, N such that for all Borel sets, E, $\mu(E) = \mu(E \cap N)$ where $\overline{m}(N) = 0$. Then*

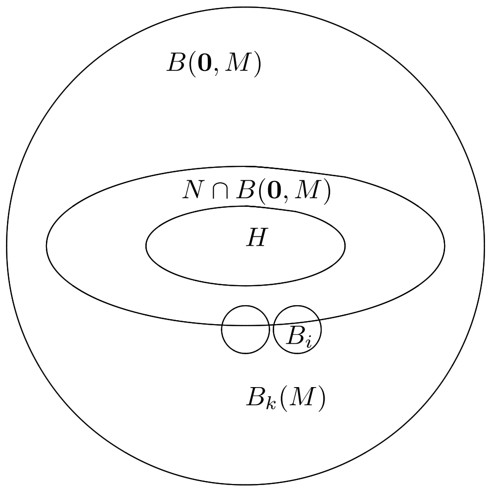
$$\frac{d\mu}{dm}(\mathbf{x}) = 0 \ m \ a.e.$$

Proof: For $k \in \mathbb{N}$, let

$$\begin{aligned} B_k(M) &\equiv \left\{\mathbf{x} \in N^C : \lim\sup_{r\to 0+} \frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))} > \frac{1}{k}\right\} \cap B(\mathbf{0},M), \\ B_k &\equiv \left\{\mathbf{x} \in N^C : \lim\sup_{r\to 0+} \frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))} > \frac{1}{k}\right\}, \\ B &\equiv \left\{\mathbf{x} \in N^C : \lim\sup_{r\to 0+} \frac{\mu(B(\mathbf{x},r))}{m(B(\mathbf{x},r))} > 0\right\}. \end{aligned}$$

Let $\varepsilon > 0$. Since μ is regular, there exists H, a compact set such that $H \subseteq N \cap B(\mathbf{0},M)$ and

$$\mu(N \cap B(\mathbf{0},M) \setminus H) < \varepsilon.$$



For each $\mathbf{x} \in B_k(M)$, there exist arbitrarily small $r > 0$ such that $B(\mathbf{x}, r) \subseteq B(\mathbf{0}, M) \setminus H$ and

$$\frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > \frac{1}{k}. \tag{16.13}$$

Two such balls are illustrated in the above picture. This is a Vitali cover of $B_k(M)$ and so there exists a sequence of disjoint balls of this sort, $\{B_i\}_{i=1}^{\infty}$ such that $\overline{m}(B_k(M) \setminus \cup_i B_i) = 0$. Therefore,

$$\begin{aligned}
\overline{m}(B_k(M)) &\leq \overline{m}(B_k(M) \cap (\cup_i B_i)) \leq \sum_i \overline{m}(B_i) \leq k \sum_i \mu(B_i) \\
&= k \sum_i \mu(B_i \cap N) = k \sum_i \mu(B_i \cap N \cap B(\mathbf{0}, M)) \\
&\leq k\mu(N \cap B(\mathbf{0}, M) \setminus H) < \varepsilon k
\end{aligned}$$

Since ε was arbitrary, this shows $\overline{m}(B_k(M)) = 0$.

Therefore,

$$\overline{m}(B_k) \leq \sum_{M=1}^{\infty} \overline{m}(B_k(M)) = 0$$

and $\overline{m}(B) \leq \sum_k \overline{m}(B_k) = 0$. Since $\overline{m}(N) = 0$, this proves the theorem.

It is easy to obtain a different version of the above theorem. This is done with the aid of the following lemma.

Lemma 16.22 *Suppose μ is a Borel measure on $\mathbb{R}^n$ having values in $[0, \infty)$. Then there exists a Radon measure, μ_1 such that $\mu_1 = \mu$ on all Borel sets.*

Proof: By assumption, $\mu(\mathbb{R}^n) < \infty$ and so it is possible to define a positive linear functional, L on $C_c(\mathbb{R}^n)$ by

$$Lf \equiv \int f d\mu.$$

By the Riesz representation theorem for positive linear functionals of this sort, there exists a unique Radon measure, μ_1 such that for all $f \in C_c(\mathbb{R}^n)$,

$$\int f d\mu_1 = Lf = \int f d\mu.$$

Now let V be an open set and let $K_k \equiv \left\{\mathbf{x} \in V : \text{dist}\left(\mathbf{x}, V^C\right) \leq 1/k\right\} \cap \overline{B(\mathbf{0},k)}$. Then $\{K_k\}$ is an incresing sequence of compact sets whose union is V. Let $K_k \prec f_k \prec V$. Then $f_k(\mathbf{x}) \to \mathcal{X}_V(\mathbf{x})$ for every $\mathbf{x}$. Therefore,

$$\mu_1(V) = \lim_{k\to\infty} \int f_k d\mu_1 = \lim_{k\to\infty} \int f_k d\mu = \mu(V)$$

and so $\mu = \mu_1$ on open sets. Now if K is a compact set, let

$$V_k \equiv \{\mathbf{x} \in \mathbb{R}^n : \text{dist}(\mathbf{x}, K) < 1/k\}.$$

Then V_k is an open set and $\cap_k V_k = K$. Letting $K \prec f_k \prec V_k$, it follows that $f_k(\mathbf{x}) \to \mathcal{X}_K(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Therefore, by the dominated convergence theorem with a dominating function, $\mathcal{X}_{\mathbb{R}^n}$

$$\mu_1(K) = \lim_{k\to\infty} \int f_k d\mu_1 = \lim_{k\to\infty} \int f_k d\mu = \mu(K)$$

and so μ and μ_1 are equal on all compact sets. It follows $\mu = \mu_1$ on all countable unions of compact sets and countable intersections of open sets.

Now let E be a Borel set. By regularity of μ_1, there exist sets, H and G such that H is the countable union of an increasing sequence of compact sets, G is the countable intersection of a decreasing sequence of open sets, $H \subseteq E \subseteq G$, and $\mu_1(H) = \mu_1(G) = \mu_1(E)$. Therefore,

$$\mu_1(H) = \mu(H) \leq \mu(E) \leq \mu(G) = \mu_1(G) = \mu_1(E) = \mu_1(H).$$

therefore, $\mu(E) = \mu_1(E)$ and this proves the lemma.

Corollary 16.23 *Suppose μ is a complex Borel measure defined on $\mathbb{R}^n$ for which there exists a μ measurable set, N such that for all Borel sets, E, $\mu(E) = \mu(E \cap N)$ where $\overline{m}(N) = 0$. Then*

$$\frac{d\mu}{dm}(\mathbf{x}) = 0 \ m \ a.e.$$

Proof: Each of $\operatorname{Re}\mu^+, \operatorname{Re}\mu^-, \operatorname{Im}\mu^+$, and $\operatorname{Im}\mu^-$ are real measures having values in $[0,\infty)$ and so by Lemma 16.22 each is a Radon measure having the same property that μ has in terms of being supported on a set of m measure zero. Therefore, for ν equal to any of these, $\frac{d\nu}{dm}(\mathbf{x}) = 0$ m a.e. This proves the corollary.

16.4 Exercises

1. Suppose A and B are sets of positive Lebesgue measure in $\mathbb{R}^n$. Show that $A - B$ must contain $B(\mathbf{c}, \varepsilon)$ for some $\mathbf{c} \in \mathbb{R}^n$ and $\varepsilon > 0$.

$$A - B \equiv \{\mathbf{a} - \mathbf{b} : \mathbf{a} \in A \text{ and } \mathbf{b} \in B\}.$$

Hint: First assume both sets are bounded. This creates no loss of generality. Next there exist $\mathbf{a}_0 \in A$, $\mathbf{b}_0 \in B$ and $\delta > 0$ such that

$$\int_{B(a_0,\delta)} \mathcal{X}_A(t)\, dt > \frac{3}{4} m(B(\mathbf{a}_0, \delta)), \int_{B(b_0,\delta)} \mathcal{X}_B(t)\, dt > \frac{3}{4} m(B(\mathbf{b}_0, \delta)).$$

Now explain why this implies

$$m(A - \mathbf{a}_0 \cap B(\mathbf{0},\delta)) > \frac{3}{4} m(B(\mathbf{0}, \delta))$$

and

$$m(B - \mathbf{b}_0 \cap B(\mathbf{0},\delta)) > \frac{3}{4} m(B(\mathbf{0}, \delta)).$$

Explain why

$$m\left(\left(A-\mathbf{a}_0\right)\cap\left(B-\mathbf{b}_0\right)\right)>\frac{1}{2}m\left(B\left(\mathbf{0},\delta\right)\right)>0.$$

Let

$$f\left(\mathbf{x}\right)\equiv\int\mathcal{X}_{A-\mathbf{a}_0}\left(\mathbf{x}+\mathbf{t}\right)\mathcal{X}_{B-\mathbf{b}_0}\left(\mathbf{t}\right)dt.$$

Explain why $f\left(\mathbf{0}\right)>0$. Next explain why f is continuous and why $f\left(\mathbf{x}\right)>0$ for all $\mathbf{x}\in B\left(\mathbf{0},\varepsilon\right)$ for some $\varepsilon>0$. Thus if $|\mathbf{x}|<\varepsilon$, there exists $\mathbf{t}$ such that $\mathbf{x}+\mathbf{t}\in A-\mathbf{a}_0$ and $\mathbf{t}\in B-\mathbf{b}_0$. Subtract these.

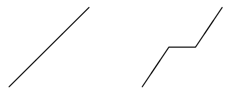
2. Show Mf is Borel measurable by verifying that $[Mf>\lambda]\equiv E_\lambda$ is actually an open set. **Hint:** If $\mathbf{x}\in E_\lambda$ then for some r, $\int_{B(\mathbf{x},r)}|f|\,dm>\lambda m\left(B\left(\mathbf{x},r\right)\right).$ Then for δ a small enough positive number, $\int_{B(\mathbf{x},r)}|f|\,dm>\lambda m\left(B\left(\mathbf{x},r+2\delta\right)\right).$ Now pick $\mathbf{y}\in B\left(\mathbf{x},\delta\right)$ and argue that $B\left(\mathbf{y},\delta+r\right)\supseteq B\left(\mathbf{x},r\right)$. Therefore show that,

$$\int_{B(\mathbf{y},\delta+r)}|f|\,dm>\int_{B(\mathbf{x},\mathbf{r})}|f|\,dm>\lambda B\left(\mathbf{x},r+2\delta\right)\geq\lambda m\left(B\left(\mathbf{y},r+\delta\right)\right).$$

Thus $B\left(\mathbf{x},\delta\right)\subseteq E_\lambda$.

3. Consider the following nested sequence of compact sets, $\{P_n\}$.Let $P_1=[0,1]$, $P_2=\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]$, etc. To go from P_n to P_{n+1}, delete the open interval which is the middle third of each closed interval in P_n. Let $P=\cap_{n=1}^{\infty}P_n$. By the finite intersection property of compact sets, $P\neq\emptyset$. Show $m(P)=0$. If you feel ambitious also show there is a one to one onto mapping of $[0,1]$ to P. The set P is called the Cantor set. Thus, although P has measure zero, it has the same number of points in it as $[0,1]$ in the sense that there is a one to one and onto mapping from one to the other. **Hint:** There are various ways of doing this last part but the most enlightenment is obtained by exploiting the topological properties of the Cantor set rather than some silly representation in terms of sums of powers of two and three. All you need to do is use the Schroder Bernstein theorem and show there is an onto map from the Cantor set to $[0,1]$. If you do this right and remember the theorems about characterizations of compact metric spaces, Proposition 6.12 on Page 136, you may get a pretty good idea why every compact metric space is the continuous image of the Cantor set.

4. Consider the sequence of functions defined in the following way. Let $f_1(x)=x$ on $[0,1]$. To get from f_n to f_{n+1}, let $f_{n+1}=f_n$ on all intervals where f_n is constant. If f_n is nonconstant on $[a,b]$, let $f_{n+1}(a)=f_n(a)$, $f_{n+1}(b)=f_n(b)$, f_{n+1} is piecewise linear and equal to $\frac{1}{2}(f_n(a)+f_n(b))$ on the middle third of $[a,b]$. Sketch a few of these and you will see the pattern. The process of modifying a nonconstant section of the graph of this function is illustrated in the following picture.



Show $\{f_n\}$ converges uniformly on $[0,1]$. If $f(x) = \lim_{n\to\infty} f_n(x)$, show that $f(0) = 0,\ f(1) = 1,\ f$ is continuous, and $f'(x) = 0$ for all $x \notin P$ where P is the Cantor set of Problem 3. This function is called the Cantor function.It is a very important example to remember. Note it has derivative equal to zero a.e. and yet it succeeds in climbing from 0 to 1. Explain why this interesting function is not absolutely continuous although it is continuous. **Hint:** This isn't too hard if you focus on getting a careful estimate on the difference between two successive functions in the list considering only a typical small interval in which the change takes place. The above picture should be helpful.

5. A function, $f : [a,b] \to \mathbb{R}$ is Lipschitz if $|f(x) - f(y)| \le K|x-y|$. Show that every Lipschitz function is absolutely continuous. Thus every Lipschitz function is differentiable a.e., $f' \in L^1$, and $f(y) - f(x) = \int_x^y f'(t)\,dt$.

6. Suppose f, g are both absolutely continuous on $[a,b]$. Show the product of these functions is also absolutely continuous. Explain why $(fg)' = f'g + g'f$ and show the usual integration by parts formula
$$f(b)\,g(b) - f(a)\,g(a) - \int_a^b fg'dt = \int_a^b f'gdt.$$

7. In Problem 4 f' failed to give the expected result for $\int_a^b f'dx$ [1] but at least $f' \in L^1$. Suppose f' exists for f a continuous function defined on $[a,b]$. Does it follow that f' is measurable? Can you conclude $f' \in L^1([a,b])$?

8. A sequence of sets, $\{E_i\}$ containing the point $\mathbf{x}$ is said to shrink to $\mathbf{x}$ nicely if there exists a sequence of positive numbers, $\{r_i\}$ and a positive constant, α such that $r_i \to 0$ and
$$m(E_i) \ge \alpha m(B(\mathbf{x}, r_i)),\ E_i \subseteq B(\mathbf{x}, r_i).$$
Show the above theorems about differentiation of measures with respect to Lebesgue measure all have a version valid for E_i replacing $B(\mathbf{x}, r)$.

9. Suppose $F(x) = \int_a^x f(t)\,dt$. Using the concept of nicely shrinking sets in Problem 8 show $F'(x) = f(x)$ a.e.

10. A random variable, X is a measurable real valued function defined on a measure space, $(\Omega, \mathcal{S}, P)$ where P is just a measure with $P(\Omega) = 1$ called a probability measure. The distribution function for X is the function, $F(x) \equiv P([X \le x])$ in words, $F(x)$ is the probability that X has values no larger than x. Show that F is a right continuous increasing function with the property that $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

[1] In this example, you only know that f' exists a.e.

11. Suppose F is an increasing right continuous function.

    (a) Show that $Lf \equiv \int_a^b f dF$ is a well defined positive linear functional on $C_c(\mathbb{R})$ where here $[a,b]$ is a closed interval containing the support of $f \in C_c(\mathbb{R})$.

    (b) Using the Riesz representation theorem for positive linear functionals on $C_c(\mathbb{R})$, let μ denote the Radon measure determined by L. Show that $\mu((a,b]) = F(b) - F(a)$ and $\mu(\{b\}) = F(b) - F(b-)$ where $F(b-) \equiv \lim_{x\to b-} F(x)$.

    (c) Review Corollary 15.4 on Page 404 at this point. Show that the conditions of this corollary hold for μ and m. Consider $\mu_\perp + \mu_{||}$, the Lebesgue decomposition of μ where $\mu_{||} \ll m$ and there exists a set of m measure zero, N such that $\mu_\perp(E) = \mu_\perp(E \cap N)$. Show $\mu((0,x]) = \mu_\perp((0,x]) + \int_0^x h(t)\,dt$ for some $h \in L^1(m)$. Using Theorem 16.21 show $h(x) = F'(x)$ m a.e. Explain why $F(x) = F(0) + S(x) + \int_0^x F'(t)\,dt$ for some function, $S(x)$ which is increasing but has $S'(x) = 0$ a.e. Note this shows in particular that a right continuous increasing function has a derivative a.e.

12. Suppose now that G is just an increasing function defined on $\mathbb{R}$. Show that $G'(x)$ exists a.e. **Hint:** You can mimic the proof of Theorem 16.18. The Dini derivates are defined as

$$D_+G(x) \equiv \liminf_{h\to 0+} \frac{G(x+h)-G(x)}{h},$$
$$D^+G(x) \equiv \limsup_{h\to 0+} \frac{G(x+h)-G(x)}{h}$$

$$D_-G(x) \equiv \liminf_{h\to 0+} \frac{G(x)-G(x-h)}{h},$$
$$D^-G(x) \equiv \limsup_{h\to 0+} \frac{G(x)-G(x-h)}{h}.$$

When $D_+G(x) = D^+G(x)$ the derivative from the right exists and when $D^-G(x) = D_-G(x)$, then the derivative from the left exists. Let (a,b) be an open interval and let

$$N_{pq} \equiv \left\{x \in (a,b) : D^+G(x) > q > p > D_+G(x)\right\}.$$

Let $V \subseteq (a,b)$ be an open set containing N_{pq} such that $n(V) < m(N_{pq}) + \varepsilon$. Show using a Vitali covering theorem there is a disjoint sequence of intervals contained in V, $\{(x_i, x_i + h_i)\}_{i=1}^\infty$ such that

$$\frac{G(x_i+h_i) - G(x_i)}{h_i} < p.$$

Next show there is a disjoint sequence of intervals $\left\{\left(x_i', x_j' + h_j'\right)\right\}_{j=1}^{\infty}$ such that each of these is contained in one of the former intervals and

$$\frac{G\left(x_j' + h_j'\right) - G\left(x_j'\right)}{h_j'} > q,\ \sum_j h_j' \geq m\left(N_{pq}\right).$$

Then

$$\begin{aligned} qm\left(N_{pq}\right) &\leq q\sum_j h_j' \leq \sum_j G\left(x_j' + h_j'\right) - G\left(x_j'\right) \leq \sum_i G\left(x_i + h_i\right) - G\left(x_i\right) \\ &\leq p\sum_i h_i \leq pm\left(V\right) \leq p\left(m\left(N_{pq}\right) + \varepsilon\right). \end{aligned}$$

Since ε was arbitrary, this shows $m\left(N_{pq}\right) = 0$. Taking a union of all N_{pq} for p, q rational, shows the derivative from the right exists a.e. Do a similar argument to show the derivative from the left exists a.e. and then show the derivative from the left equals the derivative from the right a.e. using a simlar argument. Thus $G'\left(x\right)$ exists on $\left(a, b\right)$ a.e. and so it exists a.e. on $\mathbb{R}$ because $\left(a, b\right)$ was arbitrary.

Hausdorff Measure

17.1 Definition Of Hausdorff Measures

This chapter is on Hausdorff measures. First I will discuss some outer measures. In all that is done here, $\alpha(n)$ will be the volume of the ball in $\mathbb{R}^n$ which has radius 1.

Definition 17.1 *For a set, E, denote by $r(E)$ the number which is half the diameter of E. Thus*

$$r(E) \equiv \frac{1}{2} \sup \{|\mathbf{x}-\mathbf{y}| : \mathbf{x},\mathbf{y} \in E\} \equiv \frac{1}{2} \operatorname{diam}(E)$$

Let $E \subseteq \mathbb{R}^n$.

$$\mathcal{H}_\delta^s(E) \equiv \inf\{\sum_{j=1}^{\infty} \beta(s)(r(C_j))^s : E \subseteq \cup_{j=1}^{\infty} C_j, diam(C_j) \leq \delta\}$$

$$\mathcal{H}^s(E) \equiv \lim_{\delta \to 0} \mathcal{H}_\delta^s(E).$$

In the above definition, $\beta(s)$ is an appropriate positive constant depending on s. Later I will tell what this constant is but it is not important for now.

Lemma 17.2 *$\mathcal{H}^s$ and $\mathcal{H}_\delta^s$ are outer measures.*

Proof: It is clear that $\mathcal{H}^s(\emptyset) = 0$ and if $A \subseteq B$, then $\mathcal{H}^s(A) \leq \mathcal{H}^s(B)$ with similar assertions valid for $\mathcal{H}_\delta^s$. Suppose $E = \cup_{i=1}^{\infty} E_i$ and $\mathcal{H}_\delta^s(E_i) < \infty$ for each i. Let $\{C_j^i\}_{j=1}^{\infty}$ be a covering of E_i with

$$\sum_{j=1}^{\infty} \beta(s)(r(C_j^i))^s - \varepsilon/2^i < \mathcal{H}_\delta^s(E_i)$$

and $\operatorname{diam}(C_j^i) \leq \delta$. Then

$$\begin{aligned}\mathcal{H}_\delta^s(E) &\leq \sum_{i=1}^\infty \sum_{j=1}^\infty \beta(s)(r(C_j^i))^s \\ &\leq \sum_{i=1}^\infty \mathcal{H}_\delta^s(E_i) + \varepsilon/2^i \\ &\leq \varepsilon + \sum_{i=1}^\infty \mathcal{H}_\delta^s(E_i).\end{aligned}$$

It follows that since $\varepsilon > 0$ is arbitrary,

$$\mathcal{H}_\delta^s(E) \leq \sum_{i=1}^\infty \mathcal{H}_\delta^s(E_i)$$

which shows $\mathcal{H}_\delta^s$ is an outer measure. Now notice that $\mathcal{H}_\delta^s(E)$ is increasing as $\delta \to 0$. Picking a sequence δ_k decreasing to 0, the monotone convergence theorem implies

$$\mathcal{H}^s(E) \leq \sum_{i=1}^\infty \mathcal{H}^s(E_i).$$

This proves the lemma.

The outer measure $\mathcal{H}^s$ is called s dimensional Hausdorff measure when restricted to the σ algebra of $\mathcal{H}^s$ measurable sets.

Next I will show the σ algebra of $\mathcal{H}^s$ measurable sets includes the Borel sets. This is done by the following very interesting condition known as Caratheodory's criterion.

17.1.1 Properties Of Hausdorff Measure

Definition 17.3 *For two sets, A, B in a metric space, we define*

$$\operatorname{dist}(A, B) \equiv \inf \{d(x, y) : x \in A, y \in B\}.$$

Theorem 17.4 *Let μ be an outer measure on the subsets of (X, d), a metric space. If*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever $\operatorname{dist}(A, B) > 0$, *then the σ algebra of measurable sets contains the Borel sets.*

Proof: It suffices to show that closed sets are in $\mathcal{S}$, the σ-algebra of measurable sets, because then the open sets are also in $\mathcal{S}$ and consequently $\mathcal{S}$ contains the Borel sets. Let K be closed and let S be a subset of Ω. Is $\mu(S) \geq \mu(S \cap K) + \mu(S \setminus K)$? It suffices to assume $\mu(S) < \infty$. Let

$$K_n \equiv \{x : \operatorname{dist}(x, K) \leq \frac{1}{n}\}$$

By Lemma 6.7 on Page 135, $x \to \operatorname{dist}(x, K)$ is continuous and so K_n is closed. By the assumption of the theorem,

$$\mu(S) \geq \mu((S \cap K) \cup (S \setminus K_n)) = \mu(S \cap K) + \mu(S \setminus K_n) \tag{17.1}$$

since $S \cap K$ and $S \setminus K_n$ are a positive distance apart. Now

$$\mu(S \setminus K_n) \leq \mu(S \setminus K) \leq \mu(S \setminus K_n) + \mu((K_n \setminus K) \cap S). \tag{17.2}$$

If $\lim_{n\to\infty} \mu((K_n \setminus K) \cap S) = 0$ then the theorem will be proved because this limit along with 17.2 implies $\lim_{n\to\infty} \mu(S \setminus K_n) = \mu(S \setminus K)$ and then taking a limit in 17.1, $\mu(S) \geq \mu(S \cap K) + \mu(S \setminus K)$ as desired. Therefore, it suffices to establish this limit.

Since K is closed, a point, $x \notin K$ must be at a positive distance from K and so

$$K_n \setminus K = \cup_{k=n}^{\infty} K_k \setminus K_{k+1}.$$

Therefore

$$\mu(S \cap (K_n \setminus K)) \leq \sum_{k=n}^{\infty} \mu(S \cap (K_k \setminus K_{k+1})). \tag{17.3}$$

If

$$\sum_{k=1}^{\infty} \mu(S \cap (K_k \setminus K_{k+1})) < \infty, \tag{17.4}$$

then $\mu(S \cap (K_n \setminus K)) \to 0$ because it is dominated by the tail of a convergent series so it suffices to show 17.4.

$$\sum_{k=1}^{M} \mu(S \cap (K_k \setminus K_{k+1})) =$$

$$\sum_{k \text{ even},\, k \leq M} \mu(S \cap (K_k \setminus K_{k+1})) + \sum_{k \text{ odd},\, k \leq M} \mu(S \cap (K_k \setminus K_{k+1})). \tag{17.5}$$

By the construction, the distance between any pair of sets, $S \cap (K_k \setminus K_{k+1})$ for different even values of k is positive and the distance between any pair of sets, $S \cap (K_k \setminus K_{k+1})$ for different odd values of k is positive. Therefore,

$$\sum_{k \text{ even},\, k \leq M} \mu(S \cap (K_k \setminus K_{k+1})) + \sum_{k \text{ odd},\, k \leq M} \mu(S \cap (K_k \setminus K_{k+1})) \leq$$

$$\mu(\bigcup_{k \text{ even}} S \cap (K_k \setminus K_{k+1})) + \mu(\bigcup_{k \text{ odd}} S \cap (K_k \setminus K_{k+1})) \leq 2\mu(S) < \infty$$

and so for all M, $\sum_{k=1}^{M} \mu(S \cap (K_k \setminus K_{k+1})) \leq 2\mu(S)$ showing 17.4 and proving the theorem.

With the above theorem, the following theorem is easy to obtain. This property is sometimes called Borel regularity.

Theorem 17.5 *The σ algebra of $\mathcal{H}^s$ measurable sets contains the Borel sets and $\mathcal{H}^s$ has the property that for all $E \subseteq \mathbb{R}^n$, there exists a Borel set $F \supseteq E$ such that $\mathcal{H}^s(F) = \mathcal{H}^s(E)$.*

Proof: Let $\text{dist}(A, B) = 2\delta_0 > 0$. Is it the case that

$$\mathcal{H}^s(A) + \mathcal{H}^s(B) = \mathcal{H}^s(A \cup B)?$$

This is what is needed to use Caratheodory's criterion.

Let $\{C_j\}_{j=1}^{\infty}$be a covering of $A \cup B$ such that $\text{diam}(C_j) \leq \delta < \delta_0$ for each j and

$$\mathcal{H}_\delta^s(A \cup B) + \varepsilon > \sum_{j=1}^{\infty} \beta(s)(r(C_j))^s.$$

Thus

$$\mathcal{H}_\delta^s(A \cup \dot{B}) + \varepsilon > \sum_{j \in J_1} \beta(s)(r(C_j))^s + \sum_{j \in J_2} \beta(s)(r(C_j))^s$$

where

$$J_1 = \{j : C_j \cap A \neq \emptyset\},\ J_2 = \{j : C_j \cap B \neq \emptyset\}.$$

Recall $\text{dist}(A, B) = 2\delta_0,\ J_1 \cap J_2 = \emptyset$. It follows

$$\mathcal{H}_\delta^s(A \cup B) + \varepsilon > \mathcal{H}_\delta^s(A) + \mathcal{H}_\delta^s(B).$$

Letting $\delta \to 0$, and noting $\varepsilon > 0$ was arbitrary, yields

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

Equality holds because $\mathcal{H}^s$ is an outer measure. By Caratheodory's criterion, $\mathcal{H}^s$ is a Borel measure.

To verify the second assertion, note first there is no loss of generality in letting $\mathcal{H}^s(E) < \infty$. Let

$$E \subseteq \cup_{j=1}^{\infty} C_j,\ \text{diam}(C_j) < \delta,$$

and

$$\mathcal{H}_\delta^s(E) + \delta > \sum_{j=1}^{\infty} \beta(s)(r(C_j))^s.$$

Let

$$F_\delta = \cup_{j=1}^{\infty} \overline{C_j}.$$

Thus $F_\delta \supseteq E$ and

$$\begin{aligned} \mathcal{H}_\delta^s(E) &\leq \mathcal{H}_\delta^s(F_\delta) \leq \sum_{j=1}^{\infty} \beta(s)(r(\overline{C_j}))^s \\ &= \sum_{j=1}^{\infty} \beta(s)(r(C_j))^s < \delta + \mathcal{H}_\delta^s(E). \end{aligned}$$

Let $\delta_k \to 0$ and let $F = \cap_{k=1}^{\infty} F_{\delta_k}$. Then $F \supseteq E$ and

$$\mathcal{H}^s_{\delta_k}(E) \leq \mathcal{H}^s_{\delta_k}(F) \leq \mathcal{H}^s_{\delta_k}(F_\delta) \leq \delta_k + \mathcal{H}^s_{\delta_k}(E).$$

Letting $k \to \infty$,

$$\mathcal{H}^s(E) \leq \mathcal{H}^s(F) \leq \mathcal{H}^s(E)$$

and this proves the theorem.

A measure satisfying the conclusion of Theorem 17.5 is sometimes called a Borel regular measure.

17.1.2 $\mathcal{H}^n$ And m_n

Next I will compare $\mathcal{H}^n$ and m_n. To do this, recall the following covering theorem which is a summary of Corollaries 10.20 and 10.19 found on Page 279.

Theorem 17.6 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$.*

Lemma 17.7 *There exists a constant, k such that $\mathcal{H}^n(E) \leq k m_n(E)$ for all E Borel. Also, if $Q_0 \equiv [0,1)^n$, the unit cube, then $\mathcal{H}^n([0,1)^n) > 0$.*

Proof: First let U be an open set and letting $\delta > 0$, consider all balls, B contained in U which have diameters less than δ. This is a Vitali covering of U and therefore by Theorem 17.6, there exists $\{B_i\}$, a sequence of disjoint balls of radii less than δ contained in U such that $\cup_{i=1}^{\infty} B_i$ differs from U by a set of Lebesgue measure zero. Let $\alpha(n)$ be the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Then

$$\begin{aligned}
\mathcal{H}^n_\delta(U) &\leq \sum_{i=1}^{\infty} \beta(n) r(B_i)^n = \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^{\infty} \alpha(n) r(B_i)^n \\
&= \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^{\infty} m_n(B_i) = \frac{\beta(n)}{\alpha(n)} m_n(U) \equiv k m_n(U).
\end{aligned}$$

Now letting E be Borel, it follows from the outer regularity of m_n there exists a decreasing sequence of open sets, $\{V_i\}$ containing E such such that $m_n(V_i) \to m_n(E)$. Then from the above,

$$\mathcal{H}^n_\delta(E) \leq \lim_{i\to\infty} \mathcal{H}^n_\delta(V_i) \leq \lim_{i\to\infty} k m_n(V_i) = k m_n(E).$$

Since $\delta > 0$ is arbitrary, it follows that also

$$\mathcal{H}^n(E) \leq k m_n(E).$$

This proves the first part of the lemma.

To verify the second part, note that it is obvious $\mathcal{H}^n_\delta$ and $\mathcal{H}^n$ are translation invariant because diameters of sets do not change when translated. Therefore, if

$\mathcal{H}^n([0,1)^n) = 0$, it follows $\mathcal{H}^n(\mathbb{R}^n) = 0$ because $\mathbb{R}^n$ is the countable union of translates of $Q_0 \equiv [0,1)^n$. Since each $\mathcal{H}_\delta^n$ is no larger than $\mathcal{H}^n$, the same must hold for $\mathcal{H}_\delta^n$. Therefore, there exists a sequence of sets, $\{C_i\}$ each having diameter less than δ such that

$$1 > \sum_{i=1}^{\infty} \beta(n) r(C_i)^n .$$

Now let B_i be a ball having radius equal to $\operatorname{diam}(C_i) = 2r(C_i)$ which contains C_i. It follows

$$m_n(B_i) = \alpha(n) 2^n r(C_i)^n = \frac{\alpha(n) 2^n}{\beta(n)} \beta(n) r(C_i)^n$$

which implies

$$1 > \sum_{i=1}^{\infty} \beta(n) r(C_i)^n = \sum_{i=1}^{\infty} \frac{\beta(n)}{\alpha(n) 2^n} m_n(B_i) = \infty,$$

a contradiction. This proves the lemma.

Theorem 17.8 *If $\beta(n) \equiv \alpha(n)$, then $\mathcal{H}^n = m_n$ on all Lebesgue measurable sets.*

Proof: First I will show $\mathcal{H}^n$ is a positive multiple of m_n. Let

$$k = \frac{m_n(Q_0)}{\mathcal{H}^n(Q_0)}$$

I will show $k\mathcal{H}^n(E) = m_n(E)$. When this is done, it will follow that by adjusting $\beta(n)$ the multiple can be taken to be 1. I will only need to show that the right value for $\beta(n)$ is $\alpha(n)$. Recall Lemma 10.2 on Page 267 which is listed here for convenience.

Lemma 17.9 *Every open set in $\mathbb{R}^n$ is the countable disjoint union of half open boxes of the form*

$$\prod_{i=1}^{n} (a_i, a_i + 2^{-k}]$$

where $a_i = l2^{-k}$ for some integers, l, k. The sides of these boxes are of equal length. One could also have half open boxes of the form

$$\prod_{i=1}^{n} [a_i, a_i + 2^{-k})$$

and the conclusion would be unchanged.

Let $Q = \prod_{i=1}^{n} [a_i, a_i + 2^{-k})$ be one of the half open boxes just mentioned in the above lemma. By translation invariance, of $\mathcal{H}^n$ and m_n

$$\left(2^k\right)^n \mathcal{H}^n(Q) = \mathcal{H}^n(Q_0) = \frac{1}{k} m_n(Q_0) = \frac{1}{k} \left(2^k\right)^n m_n(Q).$$

Therefore, $k\mathcal{H}^n(Q) = m_n(Q)$. It follows from Lemma 10.2 on Page 267 stated above that $k\mathcal{H}^n(U) = m_n(U)$ for all open sets. It follows immediately, since every compact set is the countable intersection of open sets that $k\mathcal{H}^n = m_n$ on compact sets. Therefore, they are also equal on all closed sets because every closed set is the countable union of compact sets. Now let F be an arbitrary Lebesgue measurable set. I will show that F is $\mathcal{H}^n$ measurable and that $k\mathcal{H}^n(F) = m_n(F)$. Let $F_l = B(\mathbf{0}, l) \cap F$. Then there exists H a countable union of compact sets and G a countable intersection of open sets such that

$$H \subseteq F_l \subseteq G \tag{17.6}$$

and

$$m_n(G \setminus H) = k\mathcal{H}^n(G \setminus H) = 0. \tag{17.7}$$

To do this, let $\{G_i\}$ be a decreasing sequence of bounded open sets containing F_l and let $\{H_i\}$ be an increasing sequence of compact sets contained in F_l such that

$$k\mathcal{H}^n(G_i \setminus H_i) = m_n(G_i \setminus H_i) < 2^{-i}$$

Then letting $G = \cap_i G_i$ and $H = \cup_i H_i$ this establishes 17.6 and 17.7. Then by completeness of $\mathcal{H}^n$ it follows F_l is $\mathcal{H}^n$ measurable and

$$k\mathcal{H}^n(F_l) = k\mathcal{H}^n(H) = m_n(H) = m_n(F_l).$$

Now taking $l \to \infty$, it follows F is $\mathcal{H}^n$ measurable and $k\mathcal{H}^n(F) = m_n(F)$. Therefore, adjusting $\beta(n)$ it can be assumed the constant, k is 1.

It only remains to show that the proper determination of $\beta(n)$ is $\alpha(n)$. By the Vitali covering theorem, there exists a sequence of disjoint balls, $\{B_i\}$ such that $B(\mathbf{0}, 1) = (\cup_{i=1}^{\infty} B_i) \cup N$ where $m_n(N) = 0$. Then $\mathcal{H}_\delta^n(N) = 0$ can be concluded because $\mathcal{H}_\delta^n \leq \mathcal{H}^n$and $\mathcal{H}^n(N) = 0$. Therefore,

$$\begin{aligned}
\mathcal{H}_\delta^n(B(\mathbf{0}, 1)) &= \mathcal{H}_\delta^n(\cup_i B_i) \leq \sum_{i=1}^{\infty} \beta(n) r(B_i)^n \\
&= \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^{\infty} \alpha(n) r(B_i)^n = \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^{\infty} m_n(B_i) \\
&= \frac{\beta(n)}{\alpha(n)} m_n(\cup_i B_i) = \frac{\beta(n)}{\alpha(n)} m_n(B(\mathbf{0}, 1)) = \frac{\beta(n)}{\alpha(n)} \mathcal{H}^n(B(\mathbf{0}, 1))
\end{aligned}$$

Taking the limit as $\delta \to 0$,

$$\mathcal{H}^n(B(\mathbf{0}, 1)) \leq \frac{\beta(n)}{\alpha(n)} \mathcal{H}^n(B(\mathbf{0}, 1))$$

and so $\alpha(n) \leq \beta(n)$.

Also

$$\begin{aligned}
\mathcal{H}^n(B(\mathbf{0},1)) &\geq \mathcal{H}_\delta^n(B(\mathbf{0},1)) = \mathcal{H}_\delta^n(\cup_i B_i) = \sum_{i=1}^\infty \mathcal{H}_\delta^n(B_i) \\
&\geq \sum_{i=1}^\infty \beta(n) r(B_i)^n = \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^\infty \alpha(n) r(B_i)^n \\
&= \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^\infty m_n(B_i) = \frac{\beta(n)}{\alpha(n)} m_n(B(\mathbf{0},1)) \\
&= \frac{\beta(n)}{\alpha(n)} \mathcal{H}^n(B(\mathbf{0},1))
\end{aligned}$$

which shows $\alpha(n) \geq \beta(n)$ and so the two are equal. This proves the theorem.

This gives another way to think of Lebesgue measure which is a particularly nice way because it is coordinate free, depending only on the notion of distance.

For $s < n$, note that $\mathcal{H}^s$ is not a Radon measure because it will not generally be finite on compact sets. For example, let $n = 2$ and consider $\mathcal{H}^1(L)$ where L is a line segment joining $(0,0)$ to $(1,0)$. Then $\mathcal{H}^1(L)$ is no smaller than $\mathcal{H}^1(L)$ when L is considered a subset of $\mathbb{R}^1, n = 1$. Thus by what was just shown, $\mathcal{H}^1(L) \geq 1$. Hence $\mathcal{H}^1([0,1] \times [0,1]) = \infty$. The situation is this: L is a one-dimensional object inside $\mathbb{R}^2$ and $\mathcal{H}^1$ is giving a one-dimensional measure of this object. In fact, Hausdorff measures can make such heuristic remarks as these precise. Define the Hausdorff dimension of a set, A, as

$$\dim(A) = \inf\{s : \mathcal{H}^s(A) = 0\}$$

17.1.3 A Formula For $\alpha(n)$

What is $\alpha(n)$? Recall the gamma function which makes sense for all $p > 0$.

$$\Gamma(p) \equiv \int_0^\infty e^{-t} t^{p-1} dt.$$

Lemma 17.10 *The following identities hold.*

$$p\Gamma(p) = \Gamma(p+1),$$

$$\Gamma(p)\Gamma(q) = \left(\int_0^1 x^{p-1}(1-x)^{q-1} dx\right) \Gamma(p+q),$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Proof: Using integration by parts,

$$\begin{aligned}
\Gamma(p+1) &= \int_0^\infty e^{-t} t^p dt = -e^{-t} t^p|_0^\infty + p \int_0^\infty e^{-t} t^{p-1} dt \\
&= p\Gamma(p)
\end{aligned}$$

Next

$$\begin{aligned}
\Gamma(p)\Gamma(q) &= \int_0^\infty e^{-t}t^{p-1}dt\int_0^\infty e^{-s}s^{q-1}ds \\
&= \int_0^\infty\int_0^\infty e^{-(t+s)}t^{p-1}s^{q-1}dtds \\
&= \int_0^\infty\int_s^\infty e^{-u}(u-s)^{p-1}s^{q-1}duds \\
&= \int_0^\infty\int_0^u e^{-u}(u-s)^{p-1}s^{q-1}dsdu \\
&= \int_0^\infty\int_0^1 e^{-u}(u-ux)^{p-1}(ux)^{q-1}udxdu \\
&= \int_0^\infty\int_0^1 e^{-u}u^{p+q-1}(1-x)^{p-1}x^{q-1}dxdu \\
&= \Gamma(p+q)\left(\int_0^1 x^{p-1}(1-x)^{q-1}dx\right).
\end{aligned}$$

It remains to find $\Gamma\left(\frac{1}{2}\right)$.

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}t^{-1/2}dt = \int_0^\infty e^{-u^2}\frac{1}{u}2udu = 2\int_0^\infty e^{-u^2}du$$

Now

$$\begin{aligned}
\left(\int_0^\infty e^{-x^2}dx\right)^2 &= \int_0^\infty e^{-x^2}dx\int_0^\infty e^{-y^2}dy = \int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dxdy \\
&= \int_0^\infty\int_0^{\pi/2} e^{-r^2}rd\theta dr = \frac{1}{4}\pi
\end{aligned}$$

and so

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}du = \sqrt{\pi}$$

This proves the lemma.

Next let n be a positive integer.

Theorem 17.11 $\alpha(n) = \pi^{n/2}(\Gamma(n/2+1))^{-1}$ *where* $\Gamma(s)$ *is the gamma function*

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt.$$

Proof: First let $n = 1$.

$$\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Thus
$$\pi^{1/2}(\Gamma(1/2+1))^{-1} = \frac{2}{\sqrt{\pi}}\sqrt{\pi} = 2 = \alpha(1).$$
and this shows the theorem is true if $n = 1$.

Assume the theorem is true for n and let B_{n+1} be the unit ball in $\mathbb{R}^{n+1}$. Then by the result in $\mathbb{R}^n$,
$$m_{n+1}(B_{n+1}) = \int_{-1}^{1} \alpha(n)(1-x_{n+1}^2)^{n/2} dx_{n+1}$$
$$= 2\alpha(n)\int_0^1 (1-t^2)^{n/2} dt.$$
Doing an integration by parts and using Lemma 17.10
$$\begin{aligned}
&= 2\alpha(n)n\int_0^1 t^2(1-t^2)^{(n-2)/2}dt \\
&= 2\alpha(n)n\frac{1}{2}\int_0^1 u^{1/2}(1-u)^{n/2-1}du \\
&= n\alpha(n)\int_0^1 u^{3/2-1}(1-u)^{n/2-1}du \\
&= n\alpha(n)\Gamma(3/2)\Gamma(n/2)(\Gamma((n+3)/2))^{-1} \\
&= n\pi^{n/2}(\Gamma(n/2+1))^{-1}(\Gamma((n+3)/2))^{-1}\Gamma(3/2)\Gamma(n/2) \\
&= n\pi^{n/2}(\Gamma(n/2)(n/2))^{-1}(\Gamma((n+1)/2+1))^{-1}\Gamma(3/2)\Gamma(n/2) \\
&= 2\pi^{n/2}\Gamma(3/2)(\Gamma((n+1)/2+1))^{-1} \\
&= \pi^{(n+1)/2}(\Gamma((n+1)/2+1))^{-1}.
\end{aligned}$$
This proves the theorem.

From now on, in the definition of Hausdorff measure, it will always be the case that $\beta(s) = \alpha(s)$. As shown above, this is the right thing to have $\beta(s)$ equal to if s is a positive integer because this yields the important result that Hausdorff measure is the same as Lebesgue measure. Note the formula, $\pi^{s/2}(\Gamma(s/2+1))^{-1}$ makes sense for any $s \geq 0$.

17.1.4 Hausdorff Measure And Linear Transformations

Hausdorff measure makes possible a unified development of n dimensional area. As in the case of Lebesgue measure, the first step in this is to understand basic considerations related to linear transformations. Recall that for $L \in \mathcal{L}\left(\mathbb{R}^k, \mathbb{R}^l\right), L^*$ is defined by
$$(L\mathbf{u}, \mathbf{v}) = (\mathbf{u}, L^*\mathbf{v}).$$
Also recall Theorem 4.59 on Page 87 which is stated here for convenience. This theorem says you can write a linear transformation as the composition of two linear transformations, one which preserves length and the other which distorts. The one

which distorts is the one which will have a nontrivial interaction with Hausdorff measure while the one which preserves lengths does not change Hausdorff measure. These ideas are behind the following theorems and lemmas.

Theorem 17.12 *Let F be an $n \times m$ matrix where $m \geq n$. Then there exists an $m \times n$ matrix R and a $n \times n$ matrix U such that*

$$F = RU,\ U = U^*,$$

all eigenvalues of U are non negative,

$$U^2 = F^* F,\ R^* R = I,$$

and $|R\mathbf{x}| = |\mathbf{x}|$.

Lemma 17.13 *Let $R \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $n \leq m$, and $R^* R = I$. Then if $A \subseteq \mathbb{R}^n$,*

$$\mathcal{H}^n(RA) = \mathcal{H}^n(A).$$

In fact, if $P : \mathbb{R}^n \to \mathbb{R}^m$ satisfies $|P\mathbf{x} - P\mathbf{y}| = |\mathbf{x} - \mathbf{y}|$, then

$$\mathcal{H}^n(PA) = \mathcal{H}^n(A).$$

Proof: Note that

$$|R(\mathbf{x} - \mathbf{y})|^2 = (R(\mathbf{x} - \mathbf{y}), R(\mathbf{x} - \mathbf{y})) = (R^* R(\mathbf{x} - \mathbf{y}), \mathbf{x} - \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2$$

Thus R preserves lengths.

Now let P be an arbitrary mapping which preserves lengths and let A be bounded, $P(A) \subseteq \cup_{j=1}^{\infty} C_j$, $\text{diam}(C_j) \leq \delta$, and

$$\mathcal{H}_\delta^n(PA) + \varepsilon > \sum_{j=1}^{\infty} \alpha(n)(r(C_j))^n.$$

Since P preserves lengths, it follows P is one to one on $P(\mathbb{R}^n)$ and P^{-1} also preserves lengths on $P(\mathbb{R}^n)$. Replacing each C_j with $C_j \cap (PA)$,

$$\begin{aligned}\mathcal{H}_\delta^n(PA) + \varepsilon &> \sum_{j=1}^{\infty} \alpha(n) r(C_j \cap (PA))^n \\ &= \sum_{j=1}^{\infty} \alpha(n) r\left(P^{-1}(C_j \cap (PA))\right)^n \\ &\geq \mathcal{H}_\delta^n(A).\end{aligned}$$

Thus $\mathcal{H}_\delta^n(PA) \geq \mathcal{H}_\delta^n(A)$.

Now let $A \subseteq \cup_{j=1}^{\infty} C_j$, $\text{diam}(C_j) \leq \delta$, and

$$\mathcal{H}_\delta^n(A) + \varepsilon \geq \sum_{j=1}^{\infty} \alpha(n)(r(C_j))^n$$

Then

$$\begin{aligned}\mathcal{H}_\delta^n(A)+\varepsilon &\geq \sum_{j=1}^{\infty}\alpha(n)\left(r\left(C_j\right)\right)^n \\ &= \sum_{j=1}^{\infty}\alpha(n)\left(r\left(PC_j\right)\right)^n \\ &\geq \mathcal{H}_\delta^n(PA).\end{aligned}$$

Hence $\mathcal{H}_\delta^n(PA) = \mathcal{H}_\delta^n(A)$. Letting $\delta \to 0$ yields the desired conclusion in the case where A is bounded. For the general case, let $A_r = A \cap B(0,r)$. Then $\mathcal{H}^n(PA_r) = \mathcal{H}^n(A_r)$. Now let $r \to \infty$. This proves the lemma.

Lemma 17.14 *Let $F \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m), n \leq m$, and let $F = RU$ where R and U are described in Theorem 4.59 on Page 87. Then if $A \subseteq \mathbb{R}^n$ is Lebesgue measurable,*

$$\mathcal{H}^n(FA) = \det(U) m_n(A).$$

Proof: Using Theorem 10.28 on Page 282 and Theorem 17.8,

$$\mathcal{H}^n(FA) = \mathcal{H}^n(RUA)$$

$$= \mathcal{H}^n(UA) = m_n(UA) = \det(U) m_n(A).$$

Definition 17.15 *Define J to equal* $\det(U)$. *Thus*

$$J = \det((F^*F)^{1/2}) = (\det(F^*F))^{1/2}.$$

17.2 The Area Formula

17.2.1 Preliminary Results

It was shown in Lemma 17.14 that

$$\mathcal{H}^n(FA) = \det(U) m_n(A)$$

where $F = RU$ with R preserving distances and U a symmetric matrix having all positive eigenvalues. The area formula gives a generalization of this simple relationship to the case where F is replaced by a nonlinear mapping, $\mathbf{h}$. It contains as a special case the earlier change of variables formula. There are two parts to this development. The first part is to generalize Lemma 17.14 to the case of nonlinear maps. When this is done, the area formula can be presented.

In this section, U will be an open set in $\mathbb{R}^n$ on which $\mathbf{h}$ is defined and $A \subseteq U$ will be a Lebesgue measurable set. Assume $m \geq n$ and

$$\mathbf{h} : U \to \mathbb{R}^m \text{ is continuous,} \tag{17.8}$$

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in A, \tag{17.9}$$

Also assume that for every $\mathbf{x} \in A$, there exists $R_{\mathbf{x}}$ and $L_{\mathbf{x}}$ such that for all $\mathbf{y}, \mathbf{z} \in B(\mathbf{x}, R_{\mathbf{x}})$,

$$|\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y})| \leq L_{\mathbf{x}} |\mathbf{x} - \mathbf{y}| \tag{17.10}$$

This last condition is weaker than saying $\mathbf{h}$ is Lipschitz. Instead, it is an assumption that $\mathbf{h}$ is locally Lipshitz, the Lipschitz constant depending on the point considered. An interesting case in which this would hold would be when $\mathbf{h}$ is differentiable on U and $||D\mathbf{h}(\mathbf{x})||$ is uniformly bounded near each point $\mathbf{x}$. Actually, it is the case that 17.10 will suffice to obtain 17.9 on all but a subset of measure zero of A but this has not been shown yet. Also, the condition 17.8 is redundant because you can simply replace U with the union of the sets $B(\mathbf{x}, R_{\mathbf{x}})$ for $\mathbf{x} \in A$. I think it is easiest to retain this condition because differentiability is defined for functions whose domains are open sets. To make this more formal, here is a definition.

Definition 17.16 *Let $\mathbf{h}$ be defined in some open set containing a set, A. Then $\mathbf{h}$ is locally Lipschitz on A if for every $\mathbf{x} \in A$ there exits $R_{\mathbf{x}} > 0$ and a constant, $L_{\mathbf{x}}$ such that whenever $\mathbf{y}, \mathbf{z} \in B(\mathbf{x}, R_{\mathbf{x}})$,*

$$|\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y})| \leq L_{\mathbf{x}} |\mathbf{z} - \mathbf{y}|.$$

Lemma 17.17 *If $T \subseteq A$ and $m_n(T) = 0$, then $\mathcal{H}^n(\mathbf{h}(T)) = 0$.*

Proof: Let

$$T_k \equiv \{\mathbf{x} \in T : ||D\mathbf{h}(\mathbf{x})|| < k\}.$$

Thus $T = \cup_k T_k$. I will show $\mathbf{h}(T_k)$ has $\mathcal{H}^n$ measure zero and then it will follow that

$$\mathbf{h}(T) = \cup_{k=1}^{\infty} \mathbf{h}(T_k)$$

must also have measure zero.

Let $\varepsilon > 0$ be given. By outer regularity, there exists an open set, V, containing T_k which is contained in U such that $m_n(V) < \frac{\varepsilon}{k^n 6^n}$. For $\mathbf{x} \in T_k$ it follows from differentiability,

$$\mathbf{h}(\mathbf{x} + \mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v})$$

and so whenever $r_{\mathbf{x}}$ is small enough, $B(\mathbf{x}, 5r_{\mathbf{x}}) \subseteq V$ and whenever $|\mathbf{v}| < 5r_{\mathbf{x}}$, $|o(\mathbf{v})| < kr_{\mathbf{x}}$. Therefore, if $|\mathbf{v}| < 5r_{\mathbf{x}}$,

$$D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v}) \in B(\mathbf{0}, 5kr_{\mathbf{x}}) + B(\mathbf{0}, kr_{\mathbf{x}}) \subseteq B(\mathbf{0}, 6kr_{\mathbf{x}})$$

and so

$$\mathbf{h}(B(\mathbf{x}, 5r_{\mathbf{x}})) \subseteq B(\mathbf{h}(\mathbf{x}), 6kr_{\mathbf{x}}).$$

Letting $\delta > 0$ be given, the Vitali covering theorem implies there exists a sequence of disjoint balls $\{B_i\}$, $B_i = B(\mathbf{x}_i, r_{\mathbf{x}_i})$, which are contained in V such that the sequence of enlarged balls, $\left\{\widehat{B}_i\right\}$, having the same center but 5 times the radius, covers T_k and $6kr_{\mathbf{x}_i} < \delta$. Then

$$\mathcal{H}_\delta^n(\mathbf{h}(T_k)) \leq \mathcal{H}_\delta^n\left(\mathbf{h}\left(\cup_{i=1}^{\infty} \widehat{B}_i\right)\right)$$

$$\leq \sum_{i=1}^{\infty} \mathcal{H}_\delta^n \left(\mathbf{h} \left(\widehat{B}_i \right) \right) \leq \sum_{i=1}^{\infty} \mathcal{H}_\delta^n \left(B \left(\mathbf{h} \left(\mathbf{x}_i \right), 6kr_{\mathbf{x}_i} \right) \right)$$

$$\begin{aligned} &\leq \sum_{i=1}^{\infty} \alpha(n) \left(6kr_{\mathbf{x}_i}\right)^n = (6k)^n \sum_{i=1}^{\infty} \alpha(n) r_{\mathbf{x}_i}^n \\ &= (6k)^n \sum_{i=1}^{\infty} m_n \left(B \left(\mathbf{x}_i, r_{\mathbf{x}_i} \right) \right) \\ &\leq (6k)^n m_n(V) \leq (6k)^n \frac{\varepsilon}{k^n 6^n} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows $\mathcal{H}_\delta^n (\mathbf{h}(T_k)) = 0$. Since δ is arbitrary, this implies $\mathcal{H}^n (\mathbf{h}(T_k)) = 0$. Now

$$\mathcal{H}^n (\mathbf{h}(T)) = \lim_{k \to \infty} \mathcal{H}^n (\mathbf{h}(T_k)) = 0.$$

This proves the lemma.

Lemma 17.18 *If S is a Lebesgue measurable subset of A, then $\mathbf{h}(S)$ is $\mathcal{H}^n$ measurable.*

Proof: Let $S_k = S \cap B(\mathbf{0}, k), k \in \mathbb{N}$. By inner regularity of Lebesgue measure, there exists a set, F, which is the countable union of compact sets and a set T with $m_n(T) = 0$ such that

$$F \cup T = S_k.$$

Then $\mathbf{h}(F) \subseteq \mathbf{h}(S_k) \subseteq \mathbf{h}(F) \cup \mathbf{h}(T)$. By continuity of $\mathbf{h}$, $\mathbf{h}(F)$ is a countable union of compact sets and so it is Borel. By Lemma 17.17, $\mathcal{H}^n(\mathbf{h}(T)) = 0$ and so $\mathbf{h}(S_k)$ is $\mathcal{H}^n$ measurable because of completeness of Hausdorff measure, which comes from $\mathcal{H}^n$ being obtained from an outer measure. Now $\mathbf{h}(S) = \cup_{k=1}^{\infty} \mathbf{h}(S_k)$ and so it is also true that $\mathbf{h}(S)$ is $\mathcal{H}^n$ measurable. This proves the lemma.

The following lemma, depending on the Brouwer fixed point theorem and found in Rudin [45], will be important for the following arguments. The idea is that if a continuous function mapping a ball in $\mathbb{R}^k$ to $\mathbb{R}^k$ doesn't move any point very much, then the image of the ball must contain a slightly smaller ball.

Lemma 17.19 *Let $B = B(\mathbf{0}, r)$, a ball in $\mathbb{R}^k$ and let $\mathbf{F} : \overline{B} \to \mathbb{R}^k$ be continuous and suppose for some $\varepsilon < 1$,*

$$|\mathbf{F}(\mathbf{v}) - \mathbf{v}| < \varepsilon r \tag{17.11}$$

for all $\mathbf{v} \in \overline{B}$. Then

$$\mathbf{F}(B) \supseteq B(\mathbf{0}, r(1 - \varepsilon)).$$

Proof: Suppose $\mathbf{a} \in B(\mathbf{0}, r(1-\varepsilon)) \setminus \mathbf{F}(B)$.

I claim that $\mathbf{a} \neq \mathbf{F}(\mathbf{v})$ for all $\mathbf{v} \in \overline{B}$. Here is why. By assumption, if $\mathbf{F}(\mathbf{v}) = \mathbf{a}$, then $|\mathbf{v}| = r$ and so

$$|\mathbf{F}(\mathbf{v}) - \mathbf{v}| = |\mathbf{a} - \mathbf{v}| \geq |\mathbf{v}| - |\mathbf{a}| > r - r(1-\varepsilon) = r\varepsilon,$$

a contradiction to 17.11.

Now letting $\mathbf{G} : \overline{B} \to \overline{B}$, be defined by

$$\mathbf{G}(\mathbf{v}) \equiv \frac{r(\mathbf{a}-\mathbf{F}(\mathbf{v}))}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|},$$

it follows $\mathbf{G}$ is continuous. Then by the Brouwer fixed point theorem, $\mathbf{G}(\mathbf{v}) = \mathbf{v}$ for some $\mathbf{v} \in \overline{B}$. Using the formula for $\mathbf{G}$, it follows $|\mathbf{v}| = r$. Taking the inner product with $\mathbf{v}$,

$$\begin{aligned}
(\mathbf{G}(\mathbf{v}), \mathbf{v}) &= |\mathbf{v}|^2 = r^2 = \frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}(\mathbf{a}-\mathbf{F}(\mathbf{v}), \mathbf{v}) \\
&= \frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}(\mathbf{a}-\mathbf{v}+\mathbf{v}-\mathbf{F}(\mathbf{v}), \mathbf{v}) \\
&= \frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}[(\mathbf{a}-\mathbf{v}, \mathbf{v}) + (\mathbf{v}-\mathbf{F}(\mathbf{v}), \mathbf{v})] \\
&= \frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}\left[(\mathbf{a}, \mathbf{v}) - |\mathbf{v}|^2 + (\mathbf{v}-\mathbf{F}(\mathbf{v}), \mathbf{v})\right] \\
&\leq \frac{r}{|\mathbf{a}-\mathbf{F}(\mathbf{v})|}\left[r^2(1-\varepsilon) - r^2 + r^2\varepsilon\right] = 0,
\end{aligned}$$

a contradiction to $|\mathbf{v}| = r$. Therefore, $B(\mathbf{0}, r(1-\varepsilon)) \setminus \mathbf{F}(B) = \emptyset$ and this proves the lemma.

By Theorem 4.59 on Page 87, when $D\mathbf{h}(\mathbf{x})$ exists,

$$D\mathbf{h}(\mathbf{x}) = R(\mathbf{x})\, U(\mathbf{x})$$

where $(U(\mathbf{x})\mathbf{u}, \mathbf{v}) = (U(\mathbf{x})\mathbf{v}, \mathbf{u}), (U(\mathbf{x})\mathbf{u}, \mathbf{u}) \geq 0$ and $R^*R = I$.

Lemma 17.20 *In this situation,* $|R^*\mathbf{u}| \leq |\mathbf{u}|$.

Proof: First note that

$$\begin{aligned}
(\mathbf{u} - RR^*\mathbf{u}, RR^*\,\mathbf{u}) &= (\mathbf{u}, RR^*\,\mathbf{u}) - |RR^*\mathbf{u}|^2 \\
&= |R^*\mathbf{u}|^2 - |R^*\mathbf{u}|^2 = 0,
\end{aligned}$$

and so

$$\begin{aligned}
|\mathbf{u}|^2 &= |\mathbf{u} - RR^*\mathbf{u} + RR^*\mathbf{u}|^2 \\
&= |\mathbf{u} - RR^*\mathbf{u}|^2 + |RR^*\mathbf{u}|^2 \\
&= |\mathbf{u} - RR^*\mathbf{u}|^2 + |R^*\mathbf{u}|^2.
\end{aligned}$$

This proves the lemma.

Lemma 17.21 *If* $|P\mathbf{x} - P\mathbf{y}| \leq L|\mathbf{x}-\mathbf{y}|$, *then for* E *a set,*

$$\mathcal{H}^n(PE) \leq L^n \mathcal{H}^n(E).$$

Proof: Without loss of generality, assume $\mathcal{H}^n(E) < \infty$. Let $\delta > 0$ and let $\{C_i\}_{i=1}^{\infty}$ be a covering of E such that $\operatorname{diam}(C_i) \leq \delta$ for each i and

$$\sum_{i=1}^{\infty} \alpha(n) r(C_i)^n \leq \mathcal{H}_\delta^n(E) + \varepsilon.$$

Then $\{PC_i\}_{i=1}^{\infty}$ is a covering of PE such that $\operatorname{diam}(PC_i) \leq L\delta$. Therefore,

$$\begin{aligned}
\mathcal{H}_{L\delta}^n(PE) &\leq \sum_{i=1}^{\infty} \alpha(n) r(PC_i)^n \\
&\leq L^n \sum_{i=1}^{\infty} \alpha(n) r(C_i)^n \leq L^n \mathcal{H}_\delta^n(E) + L^n\varepsilon \\
&\leq \mathcal{H}^n(E) + \varepsilon.
\end{aligned}$$

Letting $\delta \to 0$,

$$\mathcal{H}^n(PE) \leq L^n \mathcal{H}^n(E) + L^n\varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this proves the Lemma.

Then the following corollary follows from Lemma 17.20.

Corollary 17.22 *Let $T \subseteq \mathbb{R}^m$. Then*

$$\mathcal{H}^n(T) \geq \mathcal{H}^n(RR^*T) = \mathcal{H}^n(R^*T).$$

Definition 17.23 *Let E be a Lebesgue measurable set. $\mathbf{x} \in E$ is a point of density if*

$$\lim_{r\to 0} \frac{m_n(E \cap B(\mathbf{x}, r))}{m_n(B(\mathbf{x}, r))} = 1.$$

Recall that from the fundamental theorem of calculus applied to $\mathcal{X}_E$ almost every point of E is a point of density.

Lemma 17.24 *Let $\mathbf{x} \in A$ be a point where $U(\mathbf{x})^{-1}$ exists. Then if $\varepsilon \in (0,1)$ the following hold for all r small enough.*

$$\mathbf{h}(B(\mathbf{x}, r)) \subseteq \mathbf{h}(\mathbf{x}) + R(\mathbf{x}) U(\mathbf{x}) B(\mathbf{0}, r(1+\varepsilon)), \tag{17.12}$$

$$\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) \leq m_n(U(\mathbf{x}) B(\mathbf{0}, r(1+\varepsilon))). \tag{17.13}$$

$$R^*(\mathbf{x}) \mathbf{h}(B(\mathbf{x}, r)) \supseteq R^*(\mathbf{x}) \mathbf{h}(\mathbf{x}) + U(\mathbf{x}) B(\mathbf{0}, r(1-\varepsilon)), \tag{17.14}$$

$$\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) \geq m_n(U(\mathbf{x}) B(\mathbf{0}, r(1-\varepsilon))), \tag{17.15}$$

If $\mathbf{x}$ is also a point of density of A, then

$$\lim_{r\to 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))} = 1. \tag{17.16}$$

Proof: Since $D\mathbf{h}(\mathbf{x})$ exists,

$$\mathbf{h}(\mathbf{x}+\mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(|\mathbf{v}|). \tag{17.17}$$

Consequently, when r is small enough, 17.12 holds.

Using the fact $R(\mathbf{x})$ preserves all distances, and Theorem 17.8 which says $\mathcal{H}^n = m_n$ on the Borel sets of $\mathbb{R}^n$ implies,

$$\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r))) \leq \mathcal{H}^n(R(\mathbf{x})U(\mathbf{x})B(\mathbf{0},r(1+\varepsilon)))$$

$$= \mathcal{H}^n(U(\mathbf{x})B(\mathbf{0},r(1+\varepsilon))) = m_n(U(\mathbf{x})B(\mathbf{0},r(1+\varepsilon)))$$

which shows 17.13.

From 17.17,

$$R^*(\mathbf{x})\mathbf{h}(\mathbf{x}+\mathbf{v}) = R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) + U(\mathbf{x})(\mathbf{v}+o(|\mathbf{v}|)).$$

Thus, from the assumption that $U(\mathbf{x})^{-1}$ exists and letting $\mathbf{F}(\mathbf{v})$ be given by

$$\mathbf{F}(\mathbf{v}) \equiv U(\mathbf{x})^{-1}R^*(\mathbf{x})\mathbf{h}(\mathbf{x}+\mathbf{v}) - U(\mathbf{x})^{-1}R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) \tag{17.18}$$

It follows

$$\mathbf{F}(\mathbf{v}) - \mathbf{v} = o(|\mathbf{v}|)$$

and so Lemma 17.19 implies that for all r small enough,

$$\begin{aligned}\mathbf{F}(B(\mathbf{0},r)) &\equiv U(\mathbf{x})^{-1}R^*(\mathbf{x})\mathbf{h}(\mathbf{x}+B(\mathbf{0},r)) - U(\mathbf{x})^{-1}R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) \\ &\supseteq B(\mathbf{0},(1-\varepsilon)r).\end{aligned}$$

Therefore,

$$R^*(\mathbf{x})\mathbf{h}(B(\mathbf{x},r)) \supseteq R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) + U(\mathbf{x})B(\mathbf{0},(1-\varepsilon)r)$$

which proves 17.14. Therefore,

$$R(\mathbf{x})R^*(\mathbf{x})\mathbf{h}(B(\mathbf{x},r)) \supseteq$$

$$R(\mathbf{x})R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) + R(\mathbf{x})U(\mathbf{x})B(\mathbf{0},r(1-\varepsilon)).$$

From Lemma 17.22, this implies

$$\begin{aligned}\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r))) &\geq \mathcal{H}^n(R^*(\mathbf{x})\mathbf{h}(B(\mathbf{x},r))) \\ &= \mathcal{H}^n(R(\mathbf{x})R^*(\mathbf{x})\mathbf{h}(B(\mathbf{x},r)))\end{aligned}$$

$$\geq \mathcal{H}^n(R(\mathbf{x})U(\mathbf{x})B(\mathbf{0},r(1-\varepsilon)))$$

$$= \mathcal{H}^n(U(\mathbf{x})B(\mathbf{0},r(1-\varepsilon))) = m_n(U(\mathbf{x})B(\mathbf{0},r(1-\varepsilon)))$$

which shows 17.15.

Now suppose that $\mathbf{x}$ is also a point of density of A. Then whenever r is small enough,

$$\frac{m_n\left(A\cap B\left(\mathbf{x},r\right)\right)}{m_n\left(B\left(\mathbf{x},r\right)\right)}>1-\varepsilon.$$

Consequently, for such r,

$$\begin{aligned}1 &= \frac{m_n\left(A\cap B\left(\mathbf{x},r\right)\right)}{m_n\left(B\left(\mathbf{x},r\right)\right)}+\frac{m_n\left(B\left(\mathbf{x},r\right)\setminus A\right)}{m_n\left(B\left(\mathbf{x},r\right)\right)}\\ &> 1-\varepsilon+\frac{m_n\left(B\left(\mathbf{x},r\right)\setminus A\right)}{\alpha\left(n\right)r^n}\end{aligned}$$

and so

$$m_n\left(B\left(\mathbf{x},r\right)\setminus A\right)<\varepsilon\alpha\left(n\right)r^n. \tag{17.19}$$

Also,

$$\mathbf{h}\left(B\left(\mathbf{x},r\right)\cap A\right)\cup\mathbf{h}\left(B\left(\mathbf{x},r\right)\setminus A\right)=\mathbf{h}\left(B\left(\mathbf{x},r\right)\right)$$

and so

$$\begin{aligned}&\mathcal{H}^n\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\cap A\right)\right)+\mathcal{H}^n\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\setminus A\right)\right)\\ \geq\ &\mathcal{H}^n\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\right)\right)\end{aligned}$$

Then also letting r also be smaller than $R_{\mathbf{x}}$ mentioned in 17.10, it follows from Lemmas 17.21, 17.18, and 17.19, 17.14 that

$$\begin{aligned}1 &\geq \frac{\mathcal{H}^n\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\cap A\right)\right)}{\mathcal{H}^n\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\right)\right)}\\ &\geq \frac{\mathcal{H}^n\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\right)\right)-\mathcal{H}^n\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\setminus A\right)\right)}{\mathcal{H}^n\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\right)\right)}\\ &\geq 1-\frac{L_{\mathbf{x}}^n m_n\left(B\left(\mathbf{x},r\right)\setminus A\right)}{\mathcal{H}^n\left(R^*\left(\mathbf{x}\right)\mathbf{h}\left(B\left(\mathbf{x},r\right)\right)\right)}\\ &\geq 1-\frac{L_{\mathbf{x}}^n m_n\left(B\left(\mathbf{x},r\right)\setminus A\right)}{m_n\left(U\left(\mathbf{x}\right)B\left(\mathbf{0},r\left(1-\varepsilon\right)\right)\right)}\\ &\geq 1-\frac{L_{\mathbf{x}}^n\varepsilon\alpha\left(n\right)r^n}{\det\left(U\left(\mathbf{x}\right)\right)\alpha\left(n\right)r^n\left(1-\varepsilon\right)^n}\\ &= 1-g\left(\varepsilon\right)\end{aligned}$$

where $\lim_{\varepsilon\to 0}g\left(\varepsilon\right)=0$. Since ε is arbitrary, this proves 17.16.

The next theorem is the generalization of Lemma 17.14 to nonlinear maps.

Theorem 17.25 *Let $\mathbf{h}:U\to\mathbb{R}^m$ where U is an open set in $\mathbb{R}^n$ for $n\leq m$ and suppose $\mathbf{h}$ is locally Lipschitz at every point of a Lebesgue measurable subset, A of U. Also suppose that for every $\mathbf{x}\in A$, $D\mathbf{h}\left(\mathbf{x}\right)$ exists. Then for $\mathbf{x}\in A$,*

$$J\left(\mathbf{x}\right)=\lim_{r\to 0}\frac{\mathcal{H}^n\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\right)\right)}{m_n\left(B\left(\mathbf{x},r\right)\right)}, \tag{17.20}$$

where $J\left(\mathbf{x}\right)\equiv\det\left(U\left(\mathbf{x}\right)\right)=\det\left(D\mathbf{h}\left(\mathbf{x}\right)^*D\mathbf{h}\left(\mathbf{x}\right)\right)^{1/2}$.

Proof: Suppose first that $U(\mathbf{x})^{-1}$ exists. Using 17.15, 17.13 and the change of variables formula for linear maps,

$$\begin{aligned} & J(\mathbf{x})(1-\varepsilon)^n \\ = & \frac{m_n(U(\mathbf{x})B(\mathbf{0},r(1-\varepsilon)))}{m_n(B(\mathbf{x},r))} \leq \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)))}{m_n(B(\mathbf{x},r))} \\ \leq & \frac{m_n(U(\mathbf{x})B(\mathbf{0},r(1+\varepsilon)))}{m_n(B(\mathbf{x},r))} = J(\mathbf{x})(1+\varepsilon)^n \end{aligned}$$

whenever r is small enough. It follows that since $\varepsilon > 0$ is arbitrary, 17.20 holds.

Now suppose $U(\mathbf{x})^{-1}$ does not exist. The first part shows that the conclusion of the theorem holds when $J(\mathbf{x}) \neq 0$. I will apply this to a modified function. Let

$$\mathbf{k} : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$$

be defined as

$$\mathbf{k}(\mathbf{x}) \equiv \begin{pmatrix} \mathbf{h}(\mathbf{x}) \\ \varepsilon\mathbf{x} \end{pmatrix}.$$

Then

$$D\mathbf{k}(\mathbf{x})^* D\mathbf{k}(\mathbf{x}) = D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}) + \varepsilon^2 I_n$$

and so

$$\begin{aligned} J\mathbf{k}(\mathbf{x})^2 &\equiv \det\left(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}) + \varepsilon^2 I_n\right) \\ &= \det\left(Q^* DQ + \varepsilon^2 I_n\right) \end{aligned}$$

where D is a diagonal matrix having the nonnegative eigenvalues of $D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})$ down the main diagonal. Thus, since one of these eigenvalues equals 0, letting λ_i^2 denote the i^{th} eigenvalue, there exists a constant, C independent of ε such that

$$0 < J\mathbf{k}(\mathbf{x})^2 = \prod_{i=1}^n \left(\lambda_i^2 + \varepsilon^2\right) \leq C^2\varepsilon^2. \tag{17.21}$$

Therefore, what was just shown applies to $\mathbf{k}$.

Let

$$T \equiv \left\{(\mathbf{h}(\mathbf{w}), \mathbf{0})^T : \mathbf{w} \in B(\mathbf{x},r)\right\},$$

$$\begin{aligned} T_\varepsilon &\equiv \left\{(\mathbf{h}(\mathbf{w}), \varepsilon\mathbf{w})^T : \mathbf{w} \in B(\mathbf{x},r)\right\} \\ &\equiv \mathbf{k}(B(\mathbf{x},r)), \end{aligned}$$

then

$$T = PT_\varepsilon$$

where P is the projection map defined by

$$P\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix}.$$

Since P decreases distances, it follows from Lemma 17.21

$$\begin{aligned}\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r))) &= \mathcal{H}^n(T) = \mathcal{H}^n(PT_\varepsilon)\\ &\leq \mathcal{H}^n(T_\varepsilon) = \mathcal{H}^n(\mathbf{k}(B(\mathbf{x},r))).\end{aligned}$$

It follows from 17.21 and the first part of the proof applied to $\mathbf{k}$ that

$$\begin{aligned}C\varepsilon &\geq J\mathbf{k}(\mathbf{x}) = \lim_{r\to 0}\frac{\mathcal{H}^n(\mathbf{k}(B(\mathbf{x},r)))}{m_n(B(\mathbf{x},r))}\\ &\geq \limsup_{r\to 0}\frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)))}{m_n(B(\mathbf{x},r))}.\end{aligned}$$

Since ε is arbitrary, this establishes 17.20 in the case where $U(\mathbf{x})^{-1}$ does not exist and completes the proof of the theorem.

Define the following set for future reference.

$$S \equiv \{\mathbf{x} \in A : U(\mathbf{x})^{-1} \text{ does not exist}\} \tag{17.22}$$

17.2.2 The Area Formula

Assume $\mathbf{h} : A \to \mathbb{R}^m$ is one to one in addition to 17.8 - 17.10. Since $\mathbf{h}$ is one to one, Lemma 17.18 implies one can define a measure, ν, on the $\sigma-$ algebra of Lebesgue measurable sets as follows.

$$\nu(E) \equiv \mathcal{H}^n(\mathbf{h}(E\cap A)).$$

By Lemma 17.18, this is a measure and $\nu \ll m$. Therefore by the corollary to the Radon Nikodym theorem, Corollary 15.3 on Page 402, there exists $f \in L^1_{loc}(\mathbb{R}^n)$, $f \geq 0$, $f(\mathbf{x}) = 0$ if $\mathbf{x} \notin A$, and

$$\nu(E) = \int_E f\,dm = \int_{A\cap E} f\,dm.$$

What is f? I will show that $f(\mathbf{x}) = J(\mathbf{x}) = \det(U(\mathbf{x}))$ a.e. Define

$$E \equiv \{\mathbf{x} \in A : \mathbf{x} \text{ is not a point of density of } A\} \cup$$

$$\{\mathbf{x} \in A : \mathbf{x} \text{ is not a Lebesgue point of } f\}.$$

Then E is a set of measure zero and if $\mathbf{x} \in (A \setminus E)$, Lemma 17.24 and Theorem 17.25 imply

$$\begin{aligned}f(\mathbf{x}) &= \lim_{r\to 0}\frac{1}{m_n(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} f(\mathbf{y})\,dm\\ &= \lim_{r\to 0}\frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)\cap A))}{m_n(B(\mathbf{x},r))}\\ &= \lim_{r\to 0}\frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)\cap A))}{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)))}\frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)))}{m_n(B(\mathbf{x},r))}\\ &= J(\mathbf{x}).\end{aligned}$$

Therefore, $f(\mathbf{x}) = J(\mathbf{x})$ a.e., whenever $\mathbf{x} \in A \setminus E$.

Now let F be a Borel set in $\mathbb{R}^m$. Recall this implies F is $\mathcal{H}^n$ measurable. Then

$$\begin{aligned}
\int_{\mathbf{h}(A)} \mathcal{X}_F(\mathbf{y})\, d\mathcal{H}^n &= \int \mathcal{X}_{F\cap\mathbf{h}(A)}(\mathbf{y})\, d\mathcal{H}^n \\
&= \mathcal{H}^n\left(\mathbf{h}\left(\mathbf{h}^{-1}(F)\cap A\right)\right) \\
&= \nu\left(\mathbf{h}^{-1}(F)\right) = \int \mathcal{X}_{A\cap\mathbf{h}^{-1}(F)}(\mathbf{x})\, J(\mathbf{x})\, dm \\
&= \int_A \mathcal{X}_F(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm. \qquad (17.23)
\end{aligned}$$

Note there are no measurability questions in the above formula because $\mathbf{h}^{-1}(F)$ is a Borel set due to the continuity of $\mathbf{h}$. The Borel measurability of $J(\mathbf{x})$ also follows from the observation that $\mathbf{h}$ is continuous and therefore, the partial derivatives are Borel measurable, being the limit of continuous functions. Then $J(\mathbf{x})$ is just a continuous function of these partial derivatives. However, things are not so clear if E is only assumed $\mathcal{H}^n$ measurable. Is there a similar formula for F only $\mathcal{H}^n$ measurable?

First consider the case where E is only $\mathcal{H}^n$ measurable but

$$\mathcal{H}^n(E\cap\mathbf{h}(A)) = 0.$$

By Theorem 17.5 on Page 452, there exists a Borel set $F \supseteq E\cap\mathbf{h}(A)$ such that

$$\mathcal{H}^n(F) = \mathcal{H}^n(E\cap\mathbf{h}(A)) = 0.$$

Then from 17.23,

$$\mathcal{X}_{A\cap\mathbf{h}^{-1}(F)}(\mathbf{x})\, J(\mathbf{x}) = 0 \text{ a.e.}$$

But

$$0 \le \mathcal{X}_{A\cap\mathbf{h}^{-1}(E)}(\mathbf{x})\, J(\mathbf{x}) \le \mathcal{X}_{A\cap\mathbf{h}^{-1}(F)}(\mathbf{x})\, J(\mathbf{x}) \qquad (17.24)$$

which shows the two functions in 17.24 are equal a.e. Therefore $\mathcal{X}_{A\cap\mathbf{h}^{-1}(E)}(\mathbf{x})\, J(\mathbf{x})$ is Lebesgue measurable and so from 17.23,

$$0 = \int \mathcal{X}_{E\cap\mathbf{h}(A)}(\mathbf{y})\, d\mathcal{H}^n = \int \mathcal{X}_{F\cap\mathbf{h}(A)}(\mathbf{y})\, d\mathcal{H}^n$$

$$= \int \mathcal{X}_{A\cap\mathbf{h}^{-1}(F)}(\mathbf{x})\, J(\mathbf{x})\, dm_n = \int \mathcal{X}_{A\cap\mathbf{h}^{-1}(E)}(\mathbf{x})\, J(\mathbf{x})\, dm_n, \qquad (17.25)$$

which shows 17.23 holds in this case where

$$\mathcal{H}^n(E\cap\mathbf{h}(A)) = 0.$$

Now let $A_R \equiv A\cap B(\mathbf{0},R)$ where R is large enough that $A_R \neq \emptyset$ and let E be $\mathcal{H}^n$ measurable. By Theorem 17.5, there exists $F \supseteq E\cap\mathbf{h}(A_R)$ such that F is Borel and

$$\mathcal{H}^n(F\setminus(E\cap\mathbf{h}(A_R))) = 0. \qquad (17.26)$$

Then

$$(E\cap \mathbf{h}(A_R))\cup (F\setminus (E\cap \mathbf{h}(A_R))\cap \mathbf{h}(A_R)) = F\cap \mathbf{h}(A_R)$$

and so

$$\mathcal{X}_{A_R\cap \mathbf{h}^{-1}(F)}J = \mathcal{X}_{A_R\cap \mathbf{h}^{-1}(E)}J + \mathcal{X}_{A_R\cap \mathbf{h}^{-1}(F\setminus (E\cap \mathbf{h}(A_R)))}J$$

where from 17.26 and 17.25, the second function on the right of the equal sign is Lebesgue measurable and equals zero a.e. Therefore, the first function on the right of the equal sign is also Lebesgue measurable and equals the function on the left a.e. Thus,

$$\int \mathcal{X}_{E\cap \mathbf{h}(A_R)}(\mathbf{y})\, d\mathcal{H}^n = \int \mathcal{X}_{F\cap \mathbf{h}(A_R)}(\mathbf{y})\, d\mathcal{H}^n$$

$$= \int \mathcal{X}_{A_R\cap \mathbf{h}^{-1}(F)}(\mathbf{x})\, J(\mathbf{x})\, dm_n = \int \mathcal{X}_{A_R\cap \mathbf{h}^{-1}(E)}(\mathbf{x})\, J(\mathbf{x})\, dm_n. \tag{17.27}$$

Letting $R\to\infty$ yields 17.27 with A replacing A_R and the function

$$\mathbf{x}\to \mathcal{X}_{A_R\cap \mathbf{h}^{-1}(E)}(\mathbf{x})\, J(\mathbf{x})$$

is Lebesgue measurable. Writing this in a more familiar form yields

$$\int_{\mathbf{h}(A)} \mathcal{X}_E(\mathbf{y})\, d\mathcal{H}^n = \int_A \mathcal{X}_E(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm_n. \tag{17.28}$$

From this, it follows that if s is a nonnegative $\mathcal{H}^n$ measurable simple function, 17.28 continues to be valid with s in place of $\mathcal{X}_E$. Then approximating an arbitrary nonnegative $\mathcal{H}^n$ measurable function, g, by an increasing sequence of simple functions, it follows that 17.28 holds with g in place of $\mathcal{X}_E$ and there are no measurability problems because $\mathbf{x}\to g(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})$ is Lebesgue measurable. This proves the area formula.

Theorem 17.26 *Let $g:\mathbf{h}(A)\to[0,\infty]$ be $\mathcal{H}^n$ measurable where $\mathbf{h}$ is a continuous function and A is a Lebesgue measurable set which satisfies 17.8 - 17.10. That is, U is an open set in $\mathbb{R}^n$ on which $\mathbf{h}$ is defined and $A\subseteq U$ is a Lebesgue measurable set, $m\geq n$, and*

$$\mathbf{h}: A\to\mathbb{R}^m \text{ is continuous,} \tag{17.29}$$

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x}\in A, \tag{17.30}$$

Also assume that for every $\mathbf{x}\in A$, there exists $R_{\mathbf{x}}$ and $L_{\mathbf{x}}$ such that for all $\mathbf{y},\mathbf{z}\in B(\mathbf{x},R_{\mathbf{x}})$,

$$|\mathbf{h}(\mathbf{z})-\mathbf{h}(\mathbf{y})|\leq L_{\mathbf{x}}|\mathbf{x}-\mathbf{y}| \tag{17.31}$$

Then

$$\mathbf{x}\to (g\circ \mathbf{h})(\mathbf{x})\, J(\mathbf{x})$$

is Lebesgue measurable and

$$\int_{\mathbf{h}(A)} g(\mathbf{y})\, d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm$$

where $J(\mathbf{x}) = \det(U(\mathbf{x})) = \det\left(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})\right)^{1/2}$.

17.3 The Area Formula Alternate Version

17.3.1 Preliminary Results

It was shown in Lemma 17.14 that

$$\mathcal{H}^n(FA) = \det(U)m_n(A)$$

where $F = RU$ with R preserving distances and U a symmetric matrix having all positive eigenvalues. The area formula gives a generalization of this simple relationship to the case where F is replaced by a nonlinear mapping, $\mathbf{h}$. It contains as a special case the earlier change of variables formula. There are two parts to this development. The first part is to generalize Lemma 17.14 to the case of nonlinear maps. When this is done, the area formula can be presented.

In this section, U will be an open set in $\mathbb{R}^n$ on which $\mathbf{h}$ is defined and $A \subseteq U$ will be a Lebesgue measurable set. Assume $m \geq n$ and

$$\mathbf{h} : U \to \mathbb{R}^m \text{ is continuous,} \tag{17.32}$$

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in A, \tag{17.33}$$

$$\mathcal{H}^n(\mathbf{h}(U \setminus A)) = 0 \tag{17.34}$$

These conditions are different than the ones considered earlier. Here no Lipschitz assumption is needed on $\mathbf{h}$. In this sense, these conditions are more general than those considered earlier. However, they are not really more general because of 17.34 which says that A is essentially an open set. This was not necessary earlier. The area formula which results from the above conditions is a generalization of the change of variables formula given in Rudin [45] and the proof is essentially the same as the proof given in this book with modifications to account for the Hausdorff measure.

Lemma 17.27 *If $T \subseteq A$ and $m_n(T) = 0$, then $\mathcal{H}^n(\mathbf{h}(T)) = 0$.*

Proof: Let

$$T_k \equiv \{\mathbf{x} \in T : ||D\mathbf{h}(\mathbf{x})|| < k\}.$$

Thus $T = \cup_k T_k$. I will show $\mathbf{h}(T_k)$ has $\mathcal{H}^n$ measure zero and then it will follow that

$$\mathbf{h}(T) = \cup_{k=1}^{\infty} \mathbf{h}(T_k)$$

must also have measure zero.

Let $\varepsilon > 0$ be given. By outer regularity, there exists an open set, V, containing T_k which is contained in U such that $m_n(V) < \frac{\varepsilon}{k^n 6^n}$. For $\mathbf{x} \in T_k$ it follows from differentiability,

$$\mathbf{h}(\mathbf{x} + \mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v})$$

and so whenever $r_{\mathbf{x}}$ is small enough, $B(\mathbf{x},5r_{\mathbf{x}}) \subseteq V$ and whenever $|\mathbf{v}| < 5r_{\mathbf{x}}$, $|o(\mathbf{v})| < kr_{\mathbf{x}}$. Therefore, if $|\mathbf{v}| < 5r_{\mathbf{x}}$,

$$D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v}) \in B(\mathbf{0}, 5kr_{\mathbf{x}}) + B(\mathbf{0},kr_{\mathbf{x}}) \subseteq B(\mathbf{0}, 6kr_{\mathbf{x}})$$

and so

$$\mathbf{h}\left(B\left(\mathbf{x},5r_{\mathbf{x}}\right)\right)\subseteq B\left(\mathbf{h}\left(\mathbf{x}\right),6kr_{\mathbf{x}}\right).$$

Letting $\delta>0$ be given, the Vitali covering theorem implies there exists a sequence of disjoint balls $\{B_i\}$, $B_i=B\left(\mathbf{x}_i,r_{\mathbf{x}_i}\right)$, which are contained in V such that the sequence of enlarged balls, $\left\{\widehat{B}_i\right\}$, having the same center but 5 times the radius, covers T_k and $6kr_{\mathbf{x}_i}<\delta$. Then

$$\mathcal{H}_\delta^n\left(\mathbf{h}\left(T_k\right)\right)\leq\mathcal{H}_\delta^n\left(\mathbf{h}\left(\cup_{i=1}^\infty\widehat{B}_i\right)\right)$$

$$\leq\sum_{i=1}^\infty\mathcal{H}_\delta^n\left(\mathbf{h}\left(\widehat{B}_i\right)\right)\leq\sum_{i=1}^\infty\mathcal{H}_\delta^n\left(B\left(\mathbf{h}\left(\mathbf{x}_i\right),6kr_{\mathbf{x}_i}\right)\right)$$

$$\begin{aligned}&\leq\sum_{i=1}^\infty\alpha\left(n\right)\left(6kr_{\mathbf{x}_i}\right)^n=\left(6k\right)^n\sum_{i=1}^\infty\alpha\left(n\right)r_{\mathbf{x}_i}^n\\&=\left(6k\right)^n\sum_{i=1}^\infty m_n\left(B\left(\mathbf{x}_i,r_{\mathbf{x}_i}\right)\right)\\&\leq\left(6k\right)^n m_n\left(V\right)\leq\left(6k\right)^n\frac{\varepsilon}{k^n6^n}=\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, this shows $\mathcal{H}_\delta^n\left(\mathbf{h}\left(T_k\right)\right)=0$. Since δ is arbitrary, this implies $\mathcal{H}^n\left(\mathbf{h}\left(T_k\right)\right)=0$. Now

$$\mathcal{H}^n\left(\mathbf{h}\left(T\right)\right)=\lim_{k\to\infty}\mathcal{H}^n\left(\mathbf{h}\left(T_k\right)\right)=0.$$

This proves the lemma.

Lemma 17.28 *If S is a Lebesgue measurable subset of A, then $\mathbf{h}(S)$ is $\mathcal{H}^n$ measurable.*

Proof: Let $S_k=S\cap B\left(\mathbf{0},k\right),k\in\mathbb{N}$. By inner regularity of Lebesgue measure, there exists a set, F, which is the countable union of compact sets and a set T with $m_n\left(T\right)=0$ such that

$$F\cup T=S_k.$$

Then $\mathbf{h}\left(F\right)\subseteq\mathbf{h}\left(S_k\right)\subseteq\mathbf{h}\left(F\right)\cup\mathbf{h}\left(T\right)$. By continuity of $\mathbf{h}$, $\mathbf{h}\left(F\right)$ is a countable union of compact sets and so it is Borel. By Lemma 17.27, $\mathcal{H}^n\left(\mathbf{h}\left(T\right)\right)=0$ and so $\mathbf{h}\left(S_k\right)$ is $\mathcal{H}^n$ measurable because of completeness of Hausdorff measure, which comes from $\mathcal{H}^n$ being obtained from an outer measure. Now $\mathbf{h}\left(S\right)=\cup_{k=1}^\infty\mathbf{h}\left(S_k\right)$ and so it is also true that $\mathbf{h}\left(S\right)$ is $\mathcal{H}^n$ measurable. This proves the lemma.

The following lemma, depending on the Brouwer fixed point theorem and found in Rudin [45], will be important for the following arguments. The idea is that if a continuous function mapping a ball in $\mathbb{R}^k$ to $\mathbb{R}^k$ doesn't move any point very much, then the image of the ball must contain a slightly smaller ball.

Lemma 17.29 *Let $B = B(\mathbf{0}, r)$, a ball in $\mathbb{R}^k$ and let $\mathbf{F} : \overline{B} \to \mathbb{R}^k$ be continuous and suppose for some $\varepsilon < 1$,*

$$|\mathbf{F}(\mathbf{v}) - \mathbf{v}| < \varepsilon r \tag{17.35}$$

for all $\mathbf{v} \in \overline{B}$. Then

$$\mathbf{F}(B) \supseteq B(\mathbf{0}, r(1-\varepsilon)).$$

Proof: Suppose $\mathbf{a} \in B(\mathbf{0}, r(1-\varepsilon)) \setminus \mathbf{F}(B)$.

I claim that $\mathbf{a} \neq \mathbf{F}(\mathbf{v})$ for all $\mathbf{v} \in \overline{B}$. Here is why. By assumption, if $\mathbf{F}(\mathbf{v}) = \mathbf{a}$, then $|\mathbf{v}| = r$ and so

$$|\mathbf{F}(\mathbf{v}) - \mathbf{v}| = |\mathbf{a} - \mathbf{v}| \geq |\mathbf{v}| - |\mathbf{a}| > r - r(1-\varepsilon) = r\varepsilon,$$

a contradiction to 17.35.

Now letting $\mathbf{G} : \overline{B} \to \overline{B}$, be defined by

$$\mathbf{G}(\mathbf{v}) \equiv \frac{r(\mathbf{a} - \mathbf{F}(\mathbf{v}))}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|},$$

it follows $\mathbf{G}$ is continuous. Then by the Brouwer fixed point theorem, $\mathbf{G}(\mathbf{v}) = \mathbf{v}$ for some $\mathbf{v} \in \overline{B}$. Using the formula for $\mathbf{G}$, it follows $|\mathbf{v}| = r$. Taking the inner product with $\mathbf{v}$,

$$\begin{aligned}
(\mathbf{G}(\mathbf{v}), \mathbf{v}) &= |\mathbf{v}|^2 = r^2 = \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|}(\mathbf{a} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \\
&= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|}(\mathbf{a} - \mathbf{v} + \mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \\
&= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|}[(\mathbf{a} - \mathbf{v}, \mathbf{v}) + (\mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v})] \\
&= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|}\left[(\mathbf{a}, \mathbf{v}) - |\mathbf{v}|^2 + (\mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v})\right] \\
&\leq \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|}\left[r^2(1-\varepsilon) - r^2 + r^2\varepsilon\right] = 0,
\end{aligned}$$

a contradiction to $|\mathbf{v}| = r$. Therefore, $B(\mathbf{0}, r(1-\varepsilon)) \setminus \mathbf{F}(B) = \emptyset$ and this proves the lemma.

By Theorem 4.59 on Page 87, when $D\mathbf{h}(\mathbf{x})$ exists,

$$D\mathbf{h}(\mathbf{x}) = R(\mathbf{x})U(\mathbf{x})$$

where $(U(\mathbf{x})\mathbf{u}, \mathbf{v}) = (U(\mathbf{x})\mathbf{v}, \mathbf{u}), (U(\mathbf{x})\mathbf{u}, \mathbf{u}) \geq 0$ and $R^*R = I$.

Lemma 17.30 *In this situation, $|R^*\mathbf{u}| \leq |\mathbf{u}|$.*

Proof: First note that

$$\begin{aligned}
(\mathbf{u} - RR^*\mathbf{u}, RR^*\mathbf{u}) &= (\mathbf{u}, RR^*\mathbf{u}) - |RR^*\mathbf{u}|^2 \\
&= |R^*\mathbf{u}|^2 - |R^*\mathbf{u}|^2 = 0,
\end{aligned}$$

and so

$$\begin{aligned}|\mathbf{u}|^2 &= |\mathbf{u}-RR^*\mathbf{u}+RR^*\mathbf{u}|^2 \\ &= |\mathbf{u}-RR^*\mathbf{u}|^2+|RR^*\mathbf{u}|^2 \\ &= |\mathbf{u}-RR^*\mathbf{u}|^2+|R^*\mathbf{u}|^2.\end{aligned}$$

This proves the lemma.

Lemma 17.31 *If* $|P\mathbf{x}-P\mathbf{y}|\leq L\,|\mathbf{x}-\mathbf{y}|$, *then for* E *a set,*

$$\mathcal{H}^n(PE)\leq L^n\mathcal{H}^n(E).$$

Proof: Without loss of generality, assume $\mathcal{H}^n(E)<\infty$. Let $\delta>0$ and let $\{C_i\}_{i=1}^\infty$ be a covering of E such that $\operatorname{diam}(C_i)\leq\delta$ for each i and

$$\sum_{i=1}^\infty\alpha(n)\,r(C_i)^n\leq\mathcal{H}_\delta^n(E)+\varepsilon.$$

Then $\{PC_i\}_{i=1}^\infty$ is a covering of PE such that $\operatorname{diam}(PC_i)\leq L\delta$. Therefore,

$$\begin{aligned}\mathcal{H}_{L\delta}^n(PE) &\leq \sum_{i=1}^\infty\alpha(n)\,r(PC_i)^n \\ &\leq L^n\sum_{i=1}^\infty\alpha(n)\,r(C_i)^n\leq L^n\mathcal{H}_\delta^n(E)+L^n\varepsilon \\ &\leq \mathcal{H}^n(E)+\varepsilon.\end{aligned}$$

Letting $\delta\to0$,

$$\mathcal{H}^n(PE)\leq L^n\mathcal{H}^n(E)+L^n\varepsilon$$

and since $\varepsilon>0$ is arbitrary, this proves the Lemma.

Then the following corollary follows from Lemma 17.30.

Corollary 17.32 *Let* $T\subseteq\mathbb{R}^m$. *Then*

$$\mathcal{H}^n(T)\geq\mathcal{H}^n(RR^*T)=\mathcal{H}^n(R^*T).$$

Definition 17.33 *Let* E *be a Lebesgue measurable set.* $\mathbf{x}\in E$ *is a point of density if*

$$\lim_{r\to0}\frac{m_n(E\cap B(\mathbf{x},r))}{m_n(B(\mathbf{x},r))}=1.$$

Recall that from the fundamental theorem of calculus applied to $\mathcal{X}_E$ almost every point of E is a point of density.

Lemma 17.34 *Let* $\mathbf{x} \in A$ *be a point where* $U(\mathbf{x})^{-1}$ *exists. Then if* $\varepsilon \in (0,1)$ *the following hold for all* r *small enough.*

$$\mathbf{h}(B(\mathbf{x},r)) \subseteq \mathbf{h}(\mathbf{x}) + R(\mathbf{x}) U(\mathbf{x}) B(\mathbf{0}, r(1+\varepsilon)), \tag{17.36}$$

$$\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r))) \leq m_n(U(\mathbf{x}) B(\mathbf{0}, r(1+\varepsilon))). \tag{17.37}$$

$$R^*(\mathbf{x}) \mathbf{h}(B(\mathbf{x},r)) \supseteq R^*(\mathbf{x}) \mathbf{h}(\mathbf{x}) + U(\mathbf{x}) B(\mathbf{0}, r(1-\varepsilon)), \tag{17.38}$$

$$\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r))) \geq m_n(U(\mathbf{x}) B(\mathbf{0}, r(1-\varepsilon))), \tag{17.39}$$

If $\mathbf{x}$ *is a point of* A*, then*

$$\lim_{r \to 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r) \cap A))}{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)))} = 1. \tag{17.40}$$

Proof: Since $D\mathbf{h}(\mathbf{x})$ exists,

$$\mathbf{h}(\mathbf{x}+\mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(|\mathbf{v}|). \tag{17.41}$$

Consequently, when r is small enough, 17.36 holds.

Using the fact $R(\mathbf{x})$ preserves all distances, and Theorem 17.8 which says $\mathcal{H}^n = m_n$ on the Borel sets of $\mathbb{R}^n$,this implies,

$$\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r))) \leq \mathcal{H}^n(R(\mathbf{x}) U(\mathbf{x}) B(\mathbf{0}, r(1+\varepsilon)))$$

$$= \mathcal{H}^n(U(\mathbf{x}) B(\mathbf{0}, r(1+\varepsilon))) = m_n(U(\mathbf{x}) B(\mathbf{0}, r(1+\varepsilon)))$$

which shows 17.37.

From 17.41,

$$R^*(\mathbf{x}) \mathbf{h}(\mathbf{x}+\mathbf{v}) = R^*(\mathbf{x}) \mathbf{h}(\mathbf{x}) + U(\mathbf{x})(\mathbf{v} + o(|\mathbf{v}|)).$$

Thus, from the assumption that $U(\mathbf{x})^{-1}$ exists and letting $\mathbf{F}(\mathbf{v})$ be given by

$$\mathbf{F}(\mathbf{v}) \equiv U(\mathbf{x})^{-1} R^*(\mathbf{x}) \mathbf{h}(\mathbf{x}+\mathbf{v}) - U(\mathbf{x})^{-1} R^*(\mathbf{x}) \mathbf{h}(\mathbf{x}) \tag{17.42}$$

Since $\mathbf{h}$ is continuous near A, it follows

$$\mathbf{F}(\mathbf{v}) - \mathbf{v} = o(|\mathbf{v}|)$$

and so Lemma 17.29 implies that for all r small enough,

$$\begin{aligned} \mathbf{F}(B(\mathbf{0},r)) &\equiv U(\mathbf{x})^{-1} R^*(\mathbf{x}) \mathbf{h}(\mathbf{x}+B(\mathbf{0},r)) - U(\mathbf{x})^{-1} R^*(\mathbf{x}) \mathbf{h}(\mathbf{x}) \\ &\supseteq B(\mathbf{0}, (1-\varepsilon) r). \end{aligned}$$

Therefore,

$$R^*(\mathbf{x}) \mathbf{h}(B(\mathbf{x},r)) \supseteq R^*(\mathbf{x}) \mathbf{h}(\mathbf{x}) + U(\mathbf{x}) B(\mathbf{0}, (1-\varepsilon) r)$$

which proves 17.38. Therefore,

$$R(\mathbf{x})\,R^*(\mathbf{x})\,\mathbf{h}(B(\mathbf{x},r)) \supseteq$$

$$R(\mathbf{x})\,R^*(\mathbf{x})\,\mathbf{h}(\mathbf{x}) + R(\mathbf{x})\,U(\mathbf{x})\,B(\mathbf{0},r(1-\varepsilon)).$$

From Lemma 17.32, this implies

$$\begin{aligned}\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r))) &\geq \mathcal{H}^n(R^*(\mathbf{x})\,\mathbf{h}(B(\mathbf{x},r))) \\ &= \mathcal{H}^n(R(\mathbf{x})\,R^*(\mathbf{x})\,\mathbf{h}(B(\mathbf{x},r)))\end{aligned}$$

$$\geq \mathcal{H}^n(R(\mathbf{x})\,U(\mathbf{x})\,B(\mathbf{0},r(1-\varepsilon)))$$

$$= \mathcal{H}^n(U(\mathbf{x})\,B(\mathbf{0},r(1-\varepsilon))) = m_n(U(\mathbf{x})\,B(\mathbf{0},r(1-\varepsilon)))$$

which shows 17.39.

Let $\mathbf{x} \in A$. Choosing r small enough that $B(\mathbf{x},r) \subseteq U$,

$$\mathbf{h}(B(\mathbf{x},r) \cap A) \cup \mathbf{h}(B(\mathbf{x},r) \setminus A) = \mathbf{h}(B(\mathbf{x},r))$$

and so

$$\begin{aligned}&\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r) \cap A)) + \mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r) \setminus A)) \\ \geq\ &\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)))\end{aligned}$$

Now by assumption 17.34, $\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r) \setminus A)) = 0$ and so for all r small enough,

$$\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r) \cap A)) = \mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)))\,.$$

This establishes 17.40.

The next theorem is the generalization of Lemma 17.14 to nonlinear maps.

Theorem 17.35 *Let $\mathbf{h} : U \to \mathbb{R}^m$ where U is an open set in $\mathbb{R}^n, n \leq m, \mathbf{h}$ is continuous on $U, \mathbf{h}$ is differentiable on $A \subseteq U$, and $\mathcal{H}^n(U \setminus A) = 0$. Then for $\mathbf{x} \in A$,*

$$J(\mathbf{x}) = \lim_{r \to 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)))}{m_n(B(\mathbf{x},r))}, \tag{17.43}$$

where $J(\mathbf{x}) \equiv \det(U(\mathbf{x})) = \det\left(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})\right)^{1/2}$.

Proof: Suppose first that $U(\mathbf{x})^{-1}$ exists. Using 17.39, 17.37 and the change of variables formula for linear maps,

$$\begin{aligned}&J(\mathbf{x})(1-\varepsilon)^n \\ =\ &\frac{m_n(U(\mathbf{x})\,B(\mathbf{0},r(1-\varepsilon)))}{m_n(B(\mathbf{x},r))} \leq \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)))}{m_n(B(\mathbf{x},r))} \\ \leq\ &\frac{m_n(U(\mathbf{x})\,B(\mathbf{0},r(1+\varepsilon)))}{m_n(B(\mathbf{x},r))} = J(\mathbf{x})(1+\varepsilon)^n\end{aligned}$$

whenever r is small enough. It follows that since $\varepsilon > 0$ is arbitrary, 17.43 holds.

Now suppose $U(\mathbf{x})^{-1}$ does not exist. The first part shows that the conclusion of the theorem holds when $J(\mathbf{x}) \neq 0$. I will apply this to a modified function. Let

$$\mathbf{k} : \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$$

be defined as

$$\mathbf{k}(\mathbf{x}) \equiv \begin{pmatrix} \mathbf{h}(\mathbf{x}) \\ \varepsilon \mathbf{x} \end{pmatrix}.$$

Then

$$D\mathbf{k}(\mathbf{x})^* D\mathbf{k}(\mathbf{x}) = D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}) + \varepsilon^2 I_n$$

and so

$$\begin{aligned} J\mathbf{k}(\mathbf{x})^2 &\equiv \det\left(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}) + \varepsilon^2 I_n\right) \\ &= \det\left(Q^* D Q + \varepsilon^2 I_n\right) \end{aligned}$$

where D is a diagonal matrix having the nonnegative eigenvalues of $D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})$ down the main diagonal. Thus, since one of these eigenvalues equals 0, letting λ_i^2 denote the i^{th} eigenvalue, there exists a constant, C independent of ε such that

$$0 < J\mathbf{k}(\mathbf{x})^2 = \prod_{i=1}^n \left(\lambda_i^2 + \varepsilon^2\right) \le C^2 \varepsilon^2. \tag{17.44}$$

Therefore, what was just shown applies to $\mathbf{k}$.

Let

$$T \equiv \left\{ (\mathbf{h}(\mathbf{w}), \mathbf{0})^T : \mathbf{w} \in B(\mathbf{x}, r) \right\},$$

$$\begin{aligned} T_\varepsilon &\equiv \left\{ (\mathbf{h}(\mathbf{w}), \varepsilon \mathbf{w})^T : \mathbf{w} \in B(\mathbf{x}, r) \right\} \\ &\equiv \mathbf{k}(B(\mathbf{x}, r)), \end{aligned}$$

then

$$T = P T_\varepsilon$$

where P is the projection map defined by

$$P \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix}.$$

Since P decreases distances, it follows from Lemma 17.31

$$\begin{aligned} \mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) &= \mathcal{H}^n(T) = \mathcal{H}^n(P T_\varepsilon) \\ &\le \mathcal{H}^n(T_\varepsilon) = \mathcal{H}^n(\mathbf{k}(B(\mathbf{x}, r))). \end{aligned}$$

It follows from 17.44 and the first part of the proof applied to $\mathbf{k}$ that

$$\begin{aligned} C\varepsilon &\geq J\mathbf{k}(\mathbf{x}) = \lim_{r\to 0} \frac{\mathcal{H}^n(\mathbf{k}(B(\mathbf{x},r)))}{m_n(B(\mathbf{x},r))} \\ &\geq \limsup_{r\to 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)))}{m_n(B(\mathbf{x},r))}. \end{aligned}$$

Since ε is arbitrary, this establishes 17.43 in the case where $U(\mathbf{x})^{-1}$ does not exist and completes the proof of the theorem.

17.3.2 The Area Formula

Assume $\mathbf{h} : A \to \mathbb{R}^m$ is one to one in addition to 17.32 - 17.34. Since $\mathbf{h}$ is one to one on A, Lemma 17.28 implies one can define a measure, ν, on the $\sigma-$ algebra of Lebesgue measurable sets as follows.

$$\nu(E) \equiv \mathcal{H}^n(\mathbf{h}(E\cap A)).$$

By Lemma 17.28, this is a measure and $\nu \ll m$. Therefore by the corollary to the Radon Nikodym theorem, Corollary 15.3 on Page 402, there exists $f \in L^1_{loc}(\mathbb{R}^n)$, $f \geq 0$, $f(\mathbf{x}) = 0$ if $\mathbf{x} \notin A$, and

$$\nu(E) = \int_E f dm_n = \int_{A\cap E} f dm_n.$$

What is f? I will show that $f(\mathbf{x}) = J(\mathbf{x}) = \det(U(\mathbf{x}))$ a.e. Let $\mathbf{x}$ be a Lebesgue point of f. Then by Lemma 17.34 and Theorem 17.35

$$\begin{aligned} f(\mathbf{x}) &= \lim_{r\to 0} \frac{1}{m_n(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} f(\mathbf{y})\, dm \\ &= \lim_{r\to 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)\cap A))}{m_n(B(\mathbf{x},r))} \\ &= \lim_{r\to 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)\cap A))}{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)))} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x},r)))}{m_n(B(\mathbf{x},r))} \\ &= J(\mathbf{x}). \end{aligned}$$

Therefore, $f(\mathbf{x}) = J(\mathbf{x})$ a.e., whenever $\mathbf{x}$ is a Lebesgue point of f.

Now let F be a Borel set in $\mathbb{R}^m$. Recall this implies F is $\mathcal{H}^n$ measurable. Then

$$\begin{aligned} \int_{\mathbf{h}(A)} \mathcal{X}_F(\mathbf{y})\, d\mathcal{H}^n &= \int \mathcal{X}_{F\cap \mathbf{h}(A)}(\mathbf{y})\, d\mathcal{H}^n \\ &= \mathcal{H}^n\left(\mathbf{h}\left(\mathbf{h}^{-1}(F)\cap A\right)\right) \\ &= \nu\left(\mathbf{h}^{-1}(F)\right) = \int \mathcal{X}_{A\cap \mathbf{h}^{-1}(F)}(\mathbf{x})\, J(\mathbf{x})\, dm_n \\ &= \int_A \mathcal{X}_F(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm_n. \end{aligned} \tag{17.45}$$

Note there are no measurability questions in the above formula because $\mathbf{h}^{-1}(F)$ is a Borel set due to the continuity of $\mathbf{h}$. The Borel measurability of $J(\mathbf{x})$ also follows from the observation that $\mathbf{h}$ is continuous and therefore, the partial derivatives are Borel measurable, being the limit of continuous functions. Then $J(\mathbf{x})$ is just a continuous function of these partial derivatives. However, things are not so clear if E is only assumed $\mathcal{H}^n$ measurable. Is there a similar formula for F only $\mathcal{H}^n$ measurable?

First consider the case where E is only $\mathcal{H}^n$ measurable but

$$\mathcal{H}^n(E \cap \mathbf{h}(A)) = 0.$$

By Theorem 17.5 on Page 452, there exists a Borel set $F \supseteq E \cap \mathbf{h}(A)$ such that

$$\mathcal{H}^n(F) = \mathcal{H}^n(E \cap \mathbf{h}(A)) = 0.$$

Then from 17.45,

$$\mathcal{X}_{A\cap\mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) = 0 \text{ a.e.}$$

But

$$0 \leq \mathcal{X}_{A\cap\mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x}) \leq \mathcal{X}_{A\cap\mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) \tag{17.46}$$

which shows the two functions in 17.46 are equal a.e. Therefore $\mathcal{X}_{A\cap\mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x})$ is Lebesgue measurable and so from 17.45,

$$0 = \int \mathcal{X}_{E\cap\mathbf{h}(A)}(\mathbf{y})\, d\mathcal{H}^n = \int \mathcal{X}_{F\cap\mathbf{h}(A)}(\mathbf{y})\, d\mathcal{H}^n$$

$$= \int \mathcal{X}_{A\cap\mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x})\, dm_n = \int \mathcal{X}_{A\cap\mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x})\, dm_n, \tag{17.47}$$

which shows 17.45 holds in this case where

$$\mathcal{H}^n(E \cap \mathbf{h}(A)) = 0.$$

Now let $A_R \equiv A \cap B(\mathbf{0},R)$ where R is large enough that $A_R \neq \emptyset$ and let E be $\mathcal{H}^n$ measurable. By Theorem 17.5, there exists $F \supseteq E \cap \mathbf{h}(A_R)$ such that F is Borel and

$$\mathcal{H}^n(F \setminus (E \cap \mathbf{h}(A_R))) = 0. \tag{17.48}$$

Then

$$(E \cap \mathbf{h}(A_R)) \cup (F \setminus (E \cap \mathbf{h}(A_R)) \cap \mathbf{h}(A_R)) = F \cap \mathbf{h}(A_R)$$

and so

$$\mathcal{X}_{A_R\cap\mathbf{h}^{-1}(F)} J = \mathcal{X}_{A_R\cap\mathbf{h}^{-1}(E)} J + \mathcal{X}_{A_R\cap\mathbf{h}^{-1}(F\setminus(E\cap\mathbf{h}(A_R)))} J$$

where from 17.48 and 17.47, the second function on the right of the equal sign is Lebesgue measurable and equals zero a.e. Therefore, the first function on the right of the equal sign is also Lebesgue measurable and equals the function on the left a.e. Thus,

$$\int \mathcal{X}_{E\cap\mathbf{h}(A_R)}(\mathbf{y})\, d\mathcal{H}^n = \int \mathcal{X}_{F\cap\mathbf{h}(A_R)}(\mathbf{y})\, d\mathcal{H}^n$$

$$= \int \mathcal{X}_{A_R \cap \mathbf{h}^{-1}(F)} (\mathbf{x}) J (\mathbf{x}) dm_n = \int \mathcal{X}_{A_R \cap \mathbf{h}^{-1}(E)} (\mathbf{x}) J (\mathbf{x}) dm_n. \tag{17.49}$$

Letting $R \to \infty$ yields 17.49 with A replacing A_R and the function

$$\mathbf{x} \to \mathcal{X}_{A_R \cap \mathbf{h}^{-1}(E)} (\mathbf{x}) J (\mathbf{x})$$

is Lebesgue measurable. Writing this in a more familiar form yields

$$\int_{\mathbf{h}(A)} \mathcal{X}_E (\mathbf{y}) d\mathcal{H}^n = \int_A \mathcal{X}_E (\mathbf{h}(\mathbf{x})) J (\mathbf{x}) dm_n. \tag{17.50}$$

From this, it follows that if s is a nonnegative $\mathcal{H}^n$ measurable simple function, 17.50 continues to be valid with s in place of $\mathcal{X}_E$. Then approximating an arbitrary nonnegative $\mathcal{H}^n$ measurable function, g, by an increasing sequence of simple functions, it follows that 17.50 holds with g in place of $\mathcal{X}_E$ and there are no measurability problems because $\mathbf{x} \to g(\mathbf{h}(\mathbf{x})) J(\mathbf{x})$ is Lebesgue measurable. This proves the area formula.

Theorem 17.36 *Let $g : \mathbf{h}(A) \to [0, \infty]$ be $\mathcal{H}^n$ measurable where $\mathbf{h}$ is a continuous function and A is a Lebesgue measurable set which satisfies 17.32 - 17.34. That is, U is an open set in $\mathbb{R}^n$ on which $\mathbf{h}$ is defined and continuous and $A \subseteq U$ is a Lebesgue measurable set, $m \geq n$, and*

$$\mathbf{h} : U \to \mathbb{R}^m \text{ is continuous, } \mathbf{h} \text{ one to one on } A, \tag{17.51}$$

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in A, \tag{17.52}$$

$$\mathcal{H}^n (U \setminus A) = 0, \tag{17.53}$$

Then

$$\mathbf{x} \to (g \circ \mathbf{h})(\mathbf{x}) J(\mathbf{x})$$

is Lebesgue measurable and

$$\int_{\mathbf{h}(A)} g(\mathbf{y}) d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm$$

where $J(\mathbf{x}) = \det(U(\mathbf{x})) = \det\left(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})\right)^{1/2}$.

17.4 The Divergence Theorem

As an important application of the area formula I will give a general version of the divergence theorem. It will always be assumed $n \geq 2$. Actually it is not necessary to make this assumption but what results in the case where $n = 1$ is nothing more than the fundamental theorem of calculus and the considerations necessary to draw this conclusion seem unneccessarily tedious. You have to consider $\mathcal{H}^0$, zero dimensional Hausdorff measure. It is left as an exercise but I will not present it.

It will be convenient to have some lemmas and theorems in hand before beginning the proof. First recall the Tietze extension theorem on Page 146. It is stated next for convenience.

Theorem 17.37 *Let M be a closed nonempty subset of a metric space (X,d) and let $f : M \to [a,b]$ be continuous at every point of M. Then there exists a function, g continuous on all of X which coincides with f on M such that $g(X) \subseteq [a,b]$.*

The next topic needed is the concept of an infinitely differentiable partition of unity.

Definition 17.38 *Let $\mathfrak{C}$ be a set whose elements are subsets of $\mathbb{R}^n$.*[1] *Then $\mathfrak{C}$ is said to be locally finite if for every $\mathbf{x} \in \mathbb{R}^n$, there exists an open set, $U_{\mathbf{x}}$ containing $\mathbf{x}$ such that $U_{\mathbf{x}}$ has nonempty intersection with only finitely many sets of $\mathfrak{C}$.*

Lemma 17.39 *Let $\mathfrak{C}$ be a set whose elements are open subsets of $\mathbb{R}^n$ and suppose $\cup\mathfrak{C} \supseteq H$, a closed set. Then there exists a countable list of open sets, $\{U_i\}_{i=1}^{\infty}$ such that each U_i is bounded, each U_i is a subset of some set of $\mathfrak{C}$, and $\cup_{i=1}^{\infty} U_i \supseteq H$.*

Proof: Let $W_k \equiv B(\mathbf{0},k), W_0 = W_{-1} = \emptyset$. For each $\mathbf{x} \in H \cap \overline{W_k}$ there exists an open set, $U_{\mathbf{x}}$ such that $U_{\mathbf{x}}$ is a subset of some set of $\mathfrak{C}$ and $U_{\mathbf{x}} \subseteq W_{k+1} \setminus \overline{W_{k-1}}$. Then since $H \cap \overline{W_k}$ is compact, there exist finitely many of these sets, $\{U_i^k\}_{i=1}^{m(k)}$ whose union contains $H \cap \overline{W_k}$. If $H \cap \overline{W_k} = \emptyset$, let $m(k) = 0$ and there are no such sets obtained.The desired countable list of open sets is $\cup_{k=1}^{\infty} \{U_i^k\}_{i=1}^{m(k)}$. Each open set in this list is bounded. Furthermore, if $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{x} \in W_k$ where k is the first positive integer with $\mathbf{x} \in W_k$. Then $W_k \setminus \overline{W_{k-1}}$ is an open set containing $\mathbf{x}$ and this open set can have nonempty intersection only with with a set of $\{U_i^k\}_{i=1}^{m(k)} \cup \{U_i^{k-1}\}_{i=1}^{m(k-1)}$, a finite list of sets. Therefore, $\cup_{k=1}^{\infty} \{U_i^k\}_{i=1}^{m(k)}$ is locally finite.

The set, $\{U_i\}_{i=1}^{\infty}$ is said to be a locally finite cover of H. The following lemma gives some important reasons why a locally finite list of sets is so significant. First of all consider the rational numbers, $\{r_i\}_{i=1}^{\infty}$ each rational number is a closed set.

$$\mathbb{Q} = \{r_i\}_{i=1}^{\infty} = \cup_{i=1}^{\infty} \overline{\{r_i\}} \neq \overline{\cup_{i=1}^{\infty} \{r_i\}} = \mathbb{R}$$

The set of rational numbers is definitely not locally finite.

Lemma 17.40 *Let $\mathfrak{C}$ be locally finite. Then*

$$\overline{\cup\mathfrak{C}} = \cup\left\{\overline{H} : H \in \mathfrak{C}\right\}.$$

Next suppose the elements of $\mathfrak{C}$ are open sets and that for each $U \in \mathfrak{C}$, there exists a differentiable function, ψ_U having $\operatorname{spt}(\psi_U) \subseteq U$. Then you can define the following finite sum for each $\mathbf{x} \in \mathbb{R}^n$

$$f(\mathbf{x}) \equiv \sum \{\psi_U(\mathbf{x}) : \mathbf{x} \in U \in \mathfrak{C}\}.$$

Furthermore, f is also a differentiable function[2] *and*

$$Df(\mathbf{x}) = \sum \{D\psi_U(\mathbf{x}) : \mathbf{x} \in U \in \mathfrak{C}\}.$$

[1] The definition applies with no change to a general topological space in place of $\mathbb{R}^n$.

[2] If each ψ_U were only continuous, one could conclude f is continuous. Here the main interest is differentiable.

Proof: Let $\mathbf{p}$ be a limit point of $\cup\mathfrak{C}$ and let W be an open set which intersects only finitely many sets of $\mathfrak{C}$. Then $\mathbf{p}$ must be a limit point of one of these sets. It follows $\mathbf{p} \in \cup\left\{\overline{H} : H \in \mathfrak{C}\right\}$ and so $\overline{\cup\mathfrak{C}} \subseteq \cup\left\{\overline{H} : H \in \mathfrak{C}\right\}$. The inclusion in the other direction is obvious.

Now consider the second assertion. Letting $\mathbf{x} \in \mathbb{R}^n$, there exists an open set, W intersecting only finitely many open sets of $\mathfrak{C}$, $U_1, U_2, \cdots, U_m$. Then for all $\mathbf{y} \in W$,

$$f(\mathbf{y}) = \sum_{i=1}^{m} \psi_{U_i}(\mathbf{y})$$

and so the desired result is obvious. It merely says that a finite sum of differentiable functions is differentiable. Recall the following definition.

Definition 17.41 *Let K be a closed subset of an open set, U. $K \prec f \prec U$ if f is continuous, has values in $[0,1]$, equals 1 on K, and has compact support contained in U.*

Lemma 17.42 *Let U be a bounded open set and let K be a closed subset of U. Then there exist an open set, W, such that $W \subseteq \overline{W} \subseteq U$ and a function, $f \in C_c^\infty(U)$ such that $K \prec f \prec U$.*

Proof: The set, K is compact so is at a positive distance from U^C. Let

$$W \equiv \left\{\mathbf{x} : \operatorname{dist}(\mathbf{x}, K) < 3^{-1} \operatorname{dist}\left(K, U^C\right)\right\}.$$

Also let

$$W_1 \equiv \left\{\mathbf{x} : \operatorname{dist}(\mathbf{x}, K) < 2^{-1} \operatorname{dist}\left(K, U^C\right)\right\}$$

Then it is clear

$$K \subseteq W \subseteq \overline{W} \subseteq W_1 \subseteq \overline{W_1} \subseteq U$$

Now consider the function,

$$h(\mathbf{x}) \equiv \frac{\operatorname{dist}\left(\mathbf{x}, W_1^C\right)}{\operatorname{dist}\left(\mathbf{x}, W_1^C\right) + \operatorname{dist}\left(\mathbf{x}, \overline{W}\right)}$$

Since $\overline{W}$ is compact it is at a positive distance from W_1^C and so h is a well defined continuous function which has compact support contained in $\overline{W}_1$, equals 1 on W, and has values in $[0,1]$. Now let ϕ_k be a mollifier. Letting

$$k^{-1} < \min\left(\operatorname{dist}\left(K, W^C\right), 2^{-1} \operatorname{dist}\left(\overline{W}_1, U^C\right)\right),$$

it follows that for such k,the function, $h * \phi_k \in C_c^\infty(U)$, has values in $[0,1]$, and equals 1 on K. Let $f = h * \phi_k$.

The above lemma is used repeatedly in the following.

Lemma 17.43 *Let K be a closed set and let $\{V_i\}_{i=1}^{\infty}$ be a locally finite list of bounded open sets whose union contains K. Then there exist functions, $\psi_i \in C_c^{\infty}(V_i)$ such that for all $\mathbf{x} \in K$,*

$$1 = \sum_{i=1}^{\infty} \psi_i(\mathbf{x})$$

and the function $f(\mathbf{x})$ given by

$$f(\mathbf{x}) = \sum_{i=1}^{\infty} \psi_i(\mathbf{x})$$

is in $C^{\infty}(\mathbb{R}^n)$.

Proof: Let $K_1 = K \setminus \cup_{i=2}^{\infty} V_i$. Thus K_1 is compact because $K_1 \subseteq V_1$. Let

$$K_1 \subseteq W_1 \subseteq \overline{W}_1 \subseteq V_1$$

Thus $W_1, V_2, \cdots, V_n$ covers K and $\overline{W}_1 \subseteq V_1$. Suppose $W_1, \cdots, W_r$ have been defined such that $\overline{W_i} \subseteq V_i$ for each i, and $W_1, \cdots, W_r, V_{r+1}, \cdots, V_n$ covers K. Then let

$$K_{r+1} \equiv K \setminus \left(\left(\cup_{i=r+2}^{\infty} V_i\right) \cup \left(\cup_{j=1}^{r} W_j\right)\right).$$

It follows K_{r+1} is compact because $K_{r+1} \subseteq V_{r+1}$. Let W_{r+1} satisfy

$$K_{r+1} \subseteq W_{r+1} \subseteq \overline{W}_{r+1} \subseteq V_{r+1}$$

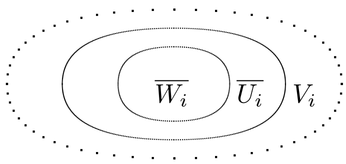
Continuing this way defines a sequence of open sets, $\{W_i\}_{i=1}^{\infty}$ with the property

$$\overline{W_i} \subseteq V_i, \ K \subseteq \cup_{i=1}^{\infty} W_i.$$

Note $\{W_i\}_{i=1}^{\infty}$ is locally finite because the original list, $\{V_i\}_{i=1}^{\infty}$ was locally finite. Now let U_i be open sets which satisfy

$$\overline{W}_i \subseteq U_i \subseteq \overline{U}_i \subseteq V_i.$$

Similarly, $\{U_i\}_{i=1}^{\infty}$ is locally finite.



Since the set, $\{W_i\}_{i=1}^{\infty}$ is locally finite, it follows $\overline{\cup_{i=1}^{\infty} W_i} = \cup_{i=1}^{\infty} \overline{W_i}$ and so it is possible to define ϕ_i and γ, infinitely differentiable functions having compact support such that

$$\overline{U}_i \prec \phi_i \prec V_i, \ \cup_{i=1}^{\infty} \overline{W}_i \prec \gamma \prec \cup_{i=1}^{\infty} U_i.$$

Now define

$$\psi_i(\mathbf{x}) = \begin{cases} \gamma(\mathbf{x})\phi_i(\mathbf{x})/\sum_{j=1}^{\infty}\phi_j(\mathbf{x}) \text{ if } \sum_{j=1}^{\infty}\phi_j(\mathbf{x}) \neq 0, \\ 0 \text{ if } \sum_{j=1}^{\infty}\phi_j(\mathbf{x}) = 0. \end{cases}$$

If $\mathbf{x}$ is such that $\sum_{j=1}^{\infty}\phi_j(\mathbf{x}) = 0$, then $\mathbf{x} \notin \cup_{i=1}^{\infty}\overline{U_i}$ because ϕ_i equals one on $\overline{U_i}$. Consequently $\gamma(\mathbf{y}) = 0$ for all $\mathbf{y}$ near $\mathbf{x}$ thanks to the fact that $\cup_{i=1}^{\infty}\overline{U_i}$ is closed and so $\psi_i(\mathbf{y}) = 0$ for all $\mathbf{y}$ near $\mathbf{x}$. Hence ψ_i is infinitely differentiable at such $\mathbf{x}$. If $\sum_{j=1}^{\infty}\phi_j(\mathbf{x}) \neq 0$, this situation persists near $\mathbf{x}$ because each ϕ_j is continuous and so ψ_i is infinitely differentiable at such points also thanks to Lemma 17.40. Therefore ψ_i is infinitely differentiable. If $\mathbf{x} \in K$, then $\gamma(\mathbf{x}) = 1$ and so $\sum_{j=1}^{\infty}\psi_j(\mathbf{x}) = 1$. Clearly $0 \leq \psi_i(\mathbf{x}) \leq 1$ and $\text{spt}(\psi_j) \subseteq V_j$. This proves the theorem.

The functions, $\{\psi_i\}$ are called a C^{∞} partition of unity.

The method of proof of this lemma easily implies the following useful corollary.

Corollary 17.44 *If H is a compact subset of V_i for some V_i there exists a partition of unity such that $\psi_i(x) = 1$ for all $x \in H$ in addition to the conclusion of Lemma 39.6.*

Proof: Keep V_i the same but replace V_j with $\widetilde{V_j} \equiv V_j \setminus H$. Now in the proof above, applied to this modified collection of open sets, if $j \neq i, \phi_j(x) = 0$ whenever $x \in H$. Therefore, $\psi_i(x) = 1$ on H.

Lemma 17.45 *Let Ω be a metric space with the closed balls compact and suppose μ is a measure defined on the Borel sets of Ω which is finite on compact sets. Then there exists a unique Radon measure, $\overline{\mu}$ which equals μ on the Borel sets. In particular μ must be both inner and outer regular on all Borel sets.*

Proof: Define a positive linear functional, $\Lambda(f) = \int f d\mu$. Let $\overline{\mu}$ be the Radon measure which comes from the Riesz representation theorem for positive linear functionals. Thus for all f continuous,

$$\int f d\mu = \int f d\overline{\mu}.$$

If V is an open set, let $\{f_n\}$ be a sequence of continuous functions which is increasing and converges to $\mathcal{X}_V$ pointwise. Then applying the monotone convergence theorem,

$$\int \mathcal{X}_V d\mu = \mu(V) = \int \mathcal{X}_V d\overline{\mu} = \overline{\mu}(V)$$

and so the two measures coincide on all open sets. Every compact set is a countable intersection of open sets and so the two measures coincide on all compact sets. Now let $B(a, n)$ be a ball of radius n and let E be a Borel set contained in this ball. Then by regularity of $\overline{\mu}$ there exist sets F, G such that G is a countable intersection of open sets and F is a countable union of compact sets such that $F \subseteq E \subseteq G$ and $\overline{\mu}(G \setminus F) = 0$. Now $\mu(G) = \overline{\mu}(G)$ and $\mu(F) = \overline{\mu}(F)$. Thus

$$\begin{aligned} \overline{\mu}(G \setminus F) + \overline{\mu}(F) &= \overline{\mu}(G) \\ &= \mu(G) = \mu(G \setminus F) + \mu(F) \end{aligned}$$

and so $\mu(G \setminus F) = \overline{\mu}(G \setminus F)$. Thus

$$\mu(E) = \mu(F) = \overline{\mu}(F) = \overline{\mu}(G) = \overline{\mu}(E).$$

If E is an arbitrary Borel set, then

$$\mu(E \cap B(a,n)) = \overline{\mu}(E \cap B(a,n))$$

and letting $n \to \infty$, this yields $\mu(E) = \overline{\mu}(E)$.

One more lemma will be useful.

Lemma 17.46 *Let V be a bounded open set and let X be the closed subspace of $C(\overline{V})$, the space of continuous functions defined on $\overline{V}$, which is given by the following.*

$$X = \{u \in C(\overline{V}) : u(\mathbf{x}) = 0 \text{ on } \partial V\}.$$

Then $C_c^\infty(V)$ is dense in X with respect to the norm given by

$$||u|| = \max\left\{|u(x)| : x \in \overline{V}\right\}$$

Proof: Let $O \subseteq \overline{O} \subseteq W \subseteq \overline{W} \subseteq V$ be such that $\text{dist}\left(\overline{O}, V^C\right) < \eta$ and let $\psi_\delta(\cdot)$ be a mollifier. Let $u \in X$ and consider $\mathcal{X}_W u * \psi_\delta$. Let $\varepsilon > 0$ be given and let η be small enough that $|u(\mathbf{x})| < \varepsilon/2$ whenever $\mathbf{x} \in V \setminus \overline{O}$. Then if δ is small enough $|\mathcal{X}_W u * \psi_\delta(\mathbf{x}) - u(\mathbf{x})| < \varepsilon$ for all $\mathbf{x} \in \overline{O}$ and $\mathcal{X}_W u * \psi_\delta$ is in $C_c^\infty(V)$. For $\mathbf{x} \in V \setminus \overline{O}$, $|\mathcal{X}_W u * \psi_\delta(\mathbf{x})| \leq \varepsilon/2$ and so for such $\mathbf{x}$,

$$|\mathcal{X}_W u * \psi_\delta(\mathbf{x}) - u(\mathbf{x})| \leq \varepsilon.$$

This proves the lemma since ε was arbitrary.

Definition 17.47 *A bounded open set, $U \subseteq \mathbb{R}^n$ is said to have a Lipschitz boundary and to lie on one side of its boundary if the following conditions hold. There exist open boxes, $Q_1, \cdots, Q_N$,*

$$Q_i = \prod_{j=1}^{n} \left(a_j^i, b_j^i\right)$$

such that $\partial U \equiv \overline{U} \setminus U$ is contained in their union. Also, for each Q_i, there exists k and a Lipschitz function, g_i such that $U \cap Q_i$ is of the form

$$\left\{\mathbf{x} : (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \prod_{j=1}^{k-1} \left(a_j^i, b_j^i\right) \times \prod_{j=k+1}^{n} \left(a_j^i, b_j^i\right) \text{ and } a_k^i < x_k < g_i(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n)\right\} \quad (17.54)$$

or else of the form

$$\left\{ \mathbf{x} : (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \prod_{j=1}^{k-1} \left(a_j^i, b_j^i\right) \times \right.$$

$$\left. \prod_{j=k+1}^{n} \left(a_j^i, b_j^i\right) \text{ and } g_i (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) < x_k < b_j^i \right\}. \quad (17.55)$$

The function, g_i has a derivative on $A_i \subseteq \prod_{j=1}^{k-1} \left(a_j^i, b_j^i\right) \times \prod_{j=k+1}^{n} \left(a_j^i, b_j^i\right)$ where

$$m_{n-1} \left(\prod_{j=1}^{k-1} \left(a_j^i, b_j^i\right) \times \prod_{j=k+1}^{n} \left(a_j^i, b_j^i\right) \setminus A_i \right) = 0.$$

Also, there exists an open set, Q_0 such that $Q_0 \subseteq \overline{Q_0} \subseteq U$ and $\overline{U} \subseteq Q_0 \cup Q_1 \cup \cdots \cup Q_N$.

Note that since there are only finitely many Q_i and each g_i is Lipschitz, it follows from an application of Lemma 17.21 that $\mathcal{H}^{n-1}(\partial U) < \infty$. Also from Lemma 17.45 $\mathcal{H}^{n-1}$ is inner and outer regular on ∂U.

Lemma 17.48 *Suppose U is a bounded open set as described above. Then there exists a unique function in $L^\infty\left(\partial U, \mathcal{H}^{n-1}\right)^n$, $\mathbf{n}(\mathbf{y})$ for $\mathbf{y} \in \partial U$ such that $|\mathbf{n}(\mathbf{y})| = 1, \mathbf{n}$ is $\mathcal{H}^{n-1}$ measurable, (meaning each component of $\mathbf{n}$ is $\mathcal{H}^{n-1}$ measurable) and for every $\mathbf{w} \in \mathbb{R}^n$ satisfying $|\mathbf{w}| = 1$, and for every $f \in C_c^1(\mathbb{R}^n)$,*

$$\lim_{t \to 0} \int_U \frac{f(\mathbf{x} + t\mathbf{w}) - f(\mathbf{x})}{t} dx = \int_{\partial U} f(\mathbf{n} \cdot \mathbf{w}) \, d\mathcal{H}^{n-1}$$

Proof: Let $\overline{U} \subseteq V \subseteq \overline{V} \subseteq \cup_{i=0}^N Q_i$ and let $\{\psi_i\}_{i=0}^N$ be a C^∞ partition of unity on $\overline{V}$ such that $\operatorname{spt}(\psi_i) \subseteq Q_i$. Then for all t small enough and $\mathbf{x} \in U$,

$$\frac{f(\mathbf{x} + t\mathbf{w}) - f(\mathbf{x})}{t} = \frac{1}{t} \sum_{i=0}^{N} \psi_i f(\mathbf{x} + t\mathbf{w}) - \psi_i f(\mathbf{x}).$$

Thus using the dominated convergence theorem,

$$\begin{aligned} & \lim_{t \to 0} \int_U \frac{f(\mathbf{x} + t\mathbf{w}) - f(\mathbf{x})}{t} dx \\ = & \lim_{t \to 0} \int_U \left(\frac{1}{t} \sum_{i=0}^{N} \psi_i f(\mathbf{x} + t\mathbf{w}) - \psi_i f(\mathbf{x}) \right) dx \\ = & \int_U \sum_{i=0}^{N} \sum_{j=1}^{n} D_j (\psi_i f(\mathbf{x})) \, w_j dx \end{aligned}$$

$$= \int_U \sum_{j=1}^n D_j (\psi_0 f(\mathbf{x})) w_j dx + \sum_{i=1}^N \int_U \sum_{j=1}^n D_j (\psi_i f(\mathbf{x})) w_j dx \tag{17.56}$$

Since $\text{spt}(\psi_0) \subseteq Q_0$, it follows the first term in the above equals zero. In the second term, fix i. Without loss of generality, suppose the k in the above definition equals n and 17.54 holds. This just makes things a little easier to write. Thus g_i is a function of

$$(x_1, \cdots, x_{n-1}) \in \prod_{j=1}^{n-1} (a_j^i, b_j^i) \equiv B_i$$

Then

$$\int_U \sum_{j=1}^n D_j (\psi_i f(\mathbf{x})) w_j dx$$

$$= \int_{B_i} \int_{a_n^i}^{g_i(x_1,\cdots,x_{n-1})} \sum_{j=1}^n D_j (\psi_i f(\mathbf{x})) w_j dx_n dx_1 \cdots dx_{n-1}$$

$$= \int_{B_i} \int_{-\infty}^{g_i(x_1,\cdots,x_{n-1})} \sum_{j=1}^n D_j (\psi_i f(\mathbf{x})) w_j dx_n dx_1 \cdots dx_{n-1}$$

Letting $x_n = y + g_i(x_1, \cdots, x_{n-1})$ and changing the variable, this equals

$$= \int_{B_i} \int_{-\infty}^0 \sum_{j=1}^n D_j (\psi_i f(x_1, \cdots, x_{n-1}, y + g_i(x_1, \cdots, x_{n-1}))) \cdot w_j dy dx_1 \cdots dx_{n-1}$$

$$= \int_{A_i} \int_{-\infty}^0 \sum_{j=1}^n D_j (\psi_i f(x_1, \cdots, x_{n-1}, y + g_i(x_1, \cdots, x_{n-1}))) \cdot w_j dy dx_1 \cdots dx_{n-1}$$

$$= \int_{A_i} \int_{-\infty}^0 \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} (\psi_i f(x_1, \cdots, x_{n-1}, y + g_i(x_1, \cdots, x_{n-1}))) w_j - D_n (\psi_i f)(x_1, \cdots, x_{n-1}, y + g_i(x_1, \cdots, x_{n-1})) \cdot g_{i,j}(x_1, \cdots, x_{n-1}) w_j dy dx_1 \cdots dx_{n-1}$$
$$+ \int_{A_i} \int_{-\infty}^0 D_n (\psi_i f(x_1, \cdots, x_{n-1}, y + g_i(x_1, \cdots, x_{n-1}))) \cdot w_n dy dx_1 \cdots dx_{n-1} \tag{17.57}$$

Consider the term

$$\int_{A_i} \int_{-\infty}^0 \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} (\psi_i f(x_1, \cdots, x_{n-1}, y + g_i(x_1, \cdots, x_{n-1}))) \cdot w_j dy dx_1 \cdots dx_{n-1}.$$

This equals

$$\int_{B_i}\int_{-\infty}^{0}\sum_{j=1}^{n-1}\frac{\partial}{\partial x_j}\left(\psi_i f\left(x_1,\cdots,x_{n-1},y+g_i\left(x_1,\cdots,x_{n-1}\right)\right)\right)\cdot$$
$$w_j dy dx_1\cdots dx_{n-1},$$

and now interchanging the order of integration and using the fact that $\operatorname{spt}(\psi_i)\subseteq Q_i$, it follows this term equals zero. (The reason this is valid is that

$$x_j\to\psi_i f\left(x_1,\cdots,x_{n-1},y+g_i\left(x_1,\cdots,x_{n-1}\right)\right)$$

is the composition of Lipschitz functions and is therefore Lipschitz. Therefore, this function is absolutely continuous and can be recovered by integrating its derivative.)

Then, changing the variable back to x_n it follows 17.57 reduces to

$$= \int_{A_i}\int_{-\infty}^{g_i(x_1,\cdots,x_{n-1})} -D_n\left(\psi_i f\right)\left(x_1,\cdots,x_{n-1},x_n\right)\cdot$$
$$g_{i,j}\left(x_1,\cdots,x_{n-1}\right)w_j dx_n dx_1\cdots dx_{n-1} \tag{17.58}$$
$$+\int_{A_i}\int_{-\infty}^{g_i(x_1,\cdots,x_{n-1})} D_n\left(\psi_i f\left(x_1,\cdots,x_{n-1},x_n\right)\right)w_n dx_n dx_1\cdots dx_{n-1} \tag{17.59}$$

Doing the integrals, this reduces further to

$$\int_{A_i}\left(\psi_i f\right)\left(x_1,\cdots,x_{n-1},x_n\right)\mathbf{N}_i\left(x_1,\cdots,x_{n-1},g_i\left(x_1,\cdots,x_{n-1}\right)\right)\cdot\mathbf{w}dm_{n-1} \tag{17.60}$$

where $\mathbf{N}_i\left(x_1,\cdots,x_{n-1},g_i\left(x_1,\cdots,x_{n-1}\right)\right)$ is given by

$$\left(-g_{i,1}\left(x_1,\cdots,x_{n-1}\right),-g_{i,2}\left(x_1,\cdots,x_{n-1}\right),\cdots,-g_{i,n-1}\left(x_1,\cdots,x_{n-1}\right),1\right). \tag{17.61}$$

At this point I need a technical lemma which will allow the use of the area formula. The part of the boundary of U which is contained in Q_i is the image of the map, $\mathbf{h}_i\left(x_1,\cdots,x_{n-1}\right)$ given by $\left(x_1,\cdots,x_{n-1},g_i\left(x_1,\cdots,x_{n-1}\right)\right)$ for $\left(x_1,\cdots,x_{n-1}\right)\in A_i$. I need a formula for

$$\det\left(D\mathbf{h}_i\left(x_1,\cdots,x_{n-1}\right)^* D\mathbf{h}_i\left(x_1,\cdots,x_{n-1}\right)\right)^{1/2}.$$

To avoid interupting the argument, I will state the lemma here and prove it later.

Lemma 17.49

$$\det\left(D\mathbf{h}_i\left(x_1,\cdots,x_{n-1}\right)^* D\mathbf{h}_i\left(x_1,\cdots,x_{n-1}\right)\right)^{1/2}$$
$$=\sqrt{1+\sum_{j-1}^{n-1}g_{i,j}\left(x_1,\cdots,x_{n-1}\right)^2}\equiv J_i\left(x_1,\cdots,x_{n-1}\right).$$

For

$$\mathbf{y} = (x_1, \cdots, x_{n-1}, g_i(x_1, \cdots, x_{n-1})) \in \partial U \cap Q_i$$

define It follows if $\mathbf{n}$ is defined by

$$\mathbf{n}_i(\mathbf{y}) = \frac{1}{J_i(x_1, \cdots, x_{n-1})} \mathbf{N}_i(\mathbf{y})$$

it follows from the description of $J_i(x_1, \cdots, x_{n-1})$ given in the above lemma, that $\mathbf{n}_i$ is a unit vector. All components of $\mathbf{n}_i$ are continuous functions of limits of continuous functions. Therefore, $\mathbf{n}_i$ is Borel measurable and so it is $\mathcal{H}^{n-1}$ measurable. Now 17.60 reduces to

$$\int_{A_i} (\psi_i f)(x_1, \cdots, x_{n-1}, g_i(x_1, \cdots, x_{n-1})) \times$$
$$\mathbf{n}_i(x_1, \cdots, x_{n-1}, g_i(x_1, \cdots, x_{n-1})) \cdot \mathbf{w} J_i(x_1, \cdots, x_{n-1})\, dm_{n-1}.$$

By the area formula this equals

$$\int_{\mathbf{h}(A_i)} \psi_i f(\mathbf{y})\, \mathbf{n}_i(\mathbf{y}) \cdot \mathbf{w}\, d\mathcal{H}^{n-1}.$$

Now by Lemma 17.21 and the equality of m_{n-1} and $\mathcal{H}^{n-1}$ on $\mathbb{R}^{n-1}$, the above integral equals

$$\int_{\partial U \cap Q_i} \psi_i f(\mathbf{y})\, \mathbf{n}_i(\mathbf{y}) \cdot \mathbf{w}\, d\mathcal{H}^{n-1} = \int_{\partial U} \psi_i f(\mathbf{y})\, \mathbf{n}_i(\mathbf{y}) \cdot \mathbf{w}\, d\mathcal{H}^{n-1}.$$

Returning to 17.56 similar arguments apply to the other terms and therefore,

$$\begin{aligned}
&\lim_{t\to 0} \int_U \frac{f(\mathbf{x} + t\mathbf{w}) - f(\mathbf{x})}{t} dm_n \\
= &\sum_{i=1}^N \int_{\partial U} \psi_i f(\mathbf{y})\, \mathbf{n}_i(\mathbf{y}) \cdot \mathbf{w}\, d\mathcal{H}^{n-1} \\
= &\int_{\partial U} f(\mathbf{y}) \sum_{i=1}^N \psi_i(\mathbf{y})\, \mathbf{n}_i(\mathbf{y}) \cdot \mathbf{w}\, d\mathcal{H}^{n-1} \\
= &\int_{\partial U} f(\mathbf{y})\, \mathbf{n}(\mathbf{y}) \cdot \mathbf{w}\, d\mathcal{H}^{n-1} \qquad (17.62)
\end{aligned}$$

Then let $\mathbf{n}(\mathbf{y}) \equiv \sum_{i=1}^N \psi_i(\mathbf{y})\, \mathbf{n}_i(\mathbf{y})$.

I need to show first there is no other $\mathbf{n}$ which satisfies 17.62 and then I need to show that $|\mathbf{n}(\mathbf{y})| = 1$. Note that it is clear $|\mathbf{n}(\mathbf{y})| \leq 1$ because each $\mathbf{n}_i$ is a unit vector and this is just a convex combination of these. Suppose then that $\mathbf{n}_1 \in L^\infty(\partial U, \mathcal{H}^{n-1})$ also works in 17.62. Then for all $f \in C_c^1(\mathbb{R}^n)$,

$$\int_{\partial U} f(\mathbf{y})\, \mathbf{n}(\mathbf{y}) \cdot \mathbf{w}\, d\mathcal{H}^{n-1} = \int_{\partial U} f(\mathbf{y})\, \mathbf{n}_1(\mathbf{y}) \cdot \mathbf{w}\, d\mathcal{H}^{n-1}.$$

Suppose $h \in C(\partial U)$. Then by the Tietze extension theorem, there exists $f \in C_c(\mathbb{R}^n)$ such that the restriction of f to ∂U equals h. Now by Lemma 17.46 applied to a bounded open set containing the support of f, there exists a sequence $\{f_m\}$ of functions in $C_c^1(\mathbb{R}^n)$ converging uniformly to f. Therefore,

$$\begin{aligned}
&\int_{\partial U} h(\mathbf{y})\,\mathbf{n}(\mathbf{y})\cdot\mathbf{w}\,d\mathcal{H}^{n-1} \\
=& \lim_{m\to\infty}\int_{\partial U} f_m(\mathbf{y})\,\mathbf{n}(\mathbf{y})\cdot\mathbf{w}\,d\mathcal{H}^{n-1} \\
=& \lim_{m\to\infty}\int_{\partial U} f_m(\mathbf{y})\,\mathbf{n}_i(\mathbf{y})\cdot\mathbf{w}\,d\mathcal{H}^{n-1} \\
=& \int_{\partial U} h(\mathbf{y})\,\mathbf{n}_i(\mathbf{y})\cdot\mathbf{w}\,d\mathcal{H}^{n-1}.
\end{aligned}$$

Now $\mathcal{H}^{n-1}$ is a Radon measure on ∂U and so the continuous functions on ∂U are dense in $L^1\left(\partial U, \mathcal{H}^{n-1}\right)$. It follows $\mathbf{n}\cdot\mathbf{w} = \mathbf{n}_i\cdot\mathbf{w}$ a.e. Now let $\{\mathbf{w}_m\}_{m=1}^{\infty}$ be a countable dense subset of the unit sphere. From what was just shown, $\mathbf{n}\cdot\mathbf{w}_m = \mathbf{n}_i\cdot\mathbf{w}_m$ except for a set of measure zero, N_m. Letting $N = \cup_m N_m$, it follows that for $\mathbf{y}\notin N$, $\mathbf{n}(\mathbf{y})\cdot\mathbf{w}_m = \mathbf{n}_i(\mathbf{y})\cdot\mathbf{w}_m$ for all m. Since the set is dense, it follows $\mathbf{n}(\mathbf{y})\cdot\mathbf{w} = \mathbf{n}_i(\mathbf{y})\cdot\mathbf{w}$ for all $\mathbf{y}\notin N$ and for all $\mathbf{w}$ a unit vector. Therefore, $\mathbf{n}(\mathbf{y}) = \mathbf{n}_i(\mathbf{y})$ for all $\mathbf{y}\notin N$ and this shows $\mathbf{n}$ is unique. In particular, although it appears to depend on the partition of unity $\{\psi_i\}$ from its definition, this is not the case.

It only remains to verify $|\mathbf{n}(\mathbf{y})| = 1$ a.e. I will do this by showing how to compute $\mathbf{n}$. In particular, I will show that $\mathbf{n} = \mathbf{n}_i$ a.e. on $\partial U\cap Q_i$. Let $W \subseteq \overline{W} \subseteq Q_i\cap\partial U$ where W is open in ∂U. Let O be an open set such that $O\cap\partial U = W$ and $\overline{O}\subseteq Q_i$. Using Corollary 17.44 there exists a C^∞ partition of unity $\{\psi_m\}$ such that $\psi_i = 1$ on $\overline{O}$. Therefore, if $m\neq i, \psi_m = 0$ on $\overline{O}$. Then if $f\in C_c^1(O)$,

$$\begin{aligned}
&\int_W f\mathbf{w}\cdot\mathbf{n}\,d\mathcal{H}^{n-1} \\
=& \int_{\partial U} f\mathbf{w}\cdot\mathbf{n}\,d\mathcal{H}^{n-1} = \int_U \nabla f\cdot\mathbf{w}\,dm_n \\
=& \int_U \nabla(\psi_i f)\cdot\mathbf{w}\,dm_n
\end{aligned}$$

which by the first part of the argument given above equals

$$\int_W \psi_i f\mathbf{n}_i\cdot\mathbf{w}\,d\mathcal{H}^{n-1} = \int_W f\mathbf{w}\cdot\mathbf{n}_i\,d\mathcal{H}^{n-1}.$$

Thus for all $f\in C_c^1(O)$,

$$\int_W f\mathbf{w}\cdot\mathbf{n}\,d\mathcal{H}^{n-1} = \int_W f\mathbf{w}\cdot\mathbf{n}_i\,d\mathcal{H}^{n-1} \tag{17.63}$$

Since $C_c^1(O)$ is dense in $C_c(O)$, the above equation is also true for all $f \in C_c(O)$. Now letting $h \in C_c(W)$, the Tietze extension theorem implies there exists $f_1 \in C(\overline{O})$ whose restriction to $\overline{W}$ equals h. Let f be defined by

$$f_1(\mathbf{x}) \frac{\operatorname{dist}(\mathbf{x}, O^C)}{\operatorname{dist}(\mathbf{x}, \operatorname{spt}(h)) + \operatorname{dist}(\mathbf{x}, O^C)} = f(\mathbf{x}).$$

Then $f = h$ on W and so this has shown that for all $h \in C_c(W)$, 17.63 holds for h in place of f. But as observed earlier, $\mathcal{H}^{n-1}$ is outer and inner regular on ∂U and so $C_c(W)$ is dense in $L^1(W, \mathcal{H}^{n-1})$ which implies $\mathbf{w} \cdot \mathbf{n}(\mathbf{y}) = \mathbf{w} \cdot \mathbf{n}_i(\mathbf{y})$ for a.e. $\mathbf{y}$. Considering a countable dense subset of the unit sphere as above, this implies $\mathbf{n}(\mathbf{y}) = \mathbf{n}_i(\mathbf{y})$ a.e. $\mathbf{y}$. This proves $|\mathbf{n}(\mathbf{y})| = 1$ a.e. and in fact $\mathbf{n}(\mathbf{y})$ can be computed by using the formula for $\mathbf{n}_i(\mathbf{y})$. This proves the lemma.

It remains to prove Lemma 17.49.

Proof of Lemma 17.49: Let $\mathbf{h}(\mathbf{x}) = (x_1, \cdots, x_{n-1}, g(x_1, \cdots, x_{n-1}))^T$

$$D\mathbf{h}(\mathbf{x}) = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & 1 \\ g_{,x_1} & \cdots & g_{,x_{n-1}} \end{pmatrix}$$

Therefore,

$$J(\mathbf{x}) = \left(\det\left(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})\right)\right)^{1/2}.$$

Therefore, $J(\mathbf{x})$ is the square root of the determinant of the following $n \times n$ matrix.

$$\begin{pmatrix} 1 + (g_{,x_1})^2 & g_{,x_1} g_{,x_2} & \cdots & g_{,x_1} g_{,x_{n-1}} \\ g_{,x_2} g_{,x_1} & 1 + (g_{,x_2})^2 & \cdots & g_{,x_2} g_{,x_{n-1}} \\ \vdots & & \ddots & \vdots \\ g_{,x_{n-1}} g_{,x_1} & g_{,x_{n-1}} g_{,x_2} & \cdots & 1 + (g_{,x_{n-1}})^2 \end{pmatrix}. \tag{17.64}$$

I need to show the determinant of the above matrix equals

$$1 + \sum_{i=1}^{n-1} (g_{,x_i}(\mathbf{x}))^2.$$

This is implied by the following claim. To simplify the notation I will replace $n-1$ with n.

Claim: Let $a_1, \cdots, a_n$ be real numbers and let $A(a_1, \cdots, a_n)$ be the matrix which has $1 + a_i^2$ in the ii^{th} slot and $a_i a_j$ in the ij^{th} slot when $i \neq j$. Then

$$\det A = 1 + \sum_{i=1}^{n} a_i^2.$$

Proof of the claim: The matrix, $A(a_1,\cdots,a_n)$ is of the form

$$A(a_1,\cdots,a_n)=\begin{pmatrix} 1+a_1^2 & a_1a_2 & \cdots & a_1a_n \\ a_1a_2 & 1+a_2^2 & & a_2a_n \\ \vdots & & \ddots & \vdots \\ a_1a_n & a_2a_n & \cdots & 1+a_n^2 \end{pmatrix}$$

Now consider the product of a matrix and its transpose, B^TB below.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & & 0 & a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & a_n \\ -a_1 & -a_2 & \cdots & -a_n & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & -a_n \\ a_1 & a_2 & \cdots & a_n & 1 \end{pmatrix} \tag{17.65}$$

This product equals a matrix of the form

$$\begin{pmatrix} A(a_1,\cdots,a_n) & \mathbf{0} \\ \mathbf{0} & 1+\sum_{i=1}^n a_i^2 \end{pmatrix}$$

Therefore, $\left(1+\sum_{i=1}^n a_i^2\right)\det(A(a_1,\cdots,a_n)) = \det(B)^2 = \det\left(B^T\right)^2$. However, using row operations,

$$\begin{aligned} \det B^T &= \det\begin{pmatrix} 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & & 0 & a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & a_n \\ 0 & 0 & \cdots & 0 & 1+\sum_{i=1}^n a_i^2 \end{pmatrix} \\ &= 1+\sum_{i=1}^n a_i^2 \end{aligned}$$

and therefore,

$$\left(1+\sum_{i=1}^n a_i^2\right)\det(A(a_1,\cdots,a_n)) = \left(1+\sum_{i=1}^n a_i^2\right)^2$$

which shows $\det(A(a_1,\cdots,a_n)) = \left(1+\sum_{i=1}^n a_i^2\right)$. This proves the claim.

Now the above lemma implies the divergence theorem.

Theorem 17.50 *Let U be a bounded open set with a Lipschitz boundary which lies on one side of its boundary. Then if $f\in C_c^1(\mathbb{R}^n)$,*

$$\int_U f_{,k}(\mathbf{x})\,dm_n = \int_{\partial U} f n_k d\mathcal{H}^{n-1} \tag{17.66}$$

where $\mathbf{n} = (n_1, \cdots, n_n)$ *is the* $\mathcal{H}^{n-1}$ *measurable unit vector of Lemma 17.48. Also, if* $\mathbf{F}$ *is a vector field such that each component is in* $C_c^1(\mathbb{R}^n)$ *, then*

$$\int_U \nabla \cdot \mathbf{F}(\mathbf{x})\, dm_n = \int_{\partial U} \mathbf{F} \cdot \mathbf{n} d\mathcal{H}^{n-1}. \tag{17.67}$$

Proof: To obtain 17.66 apply Lemma 17.48 to $\mathbf{w} = \mathbf{e}_k$. Then to obtain 17.67 from this,

$$\begin{aligned} &\int_U \nabla \cdot \mathbf{F}(\mathbf{x})\, dm_n \\ = &\sum_{j=1}^n \int_U F_{j,j} dm_n = \sum_{j=1}^n \int_{\partial U} F_j n_j d\mathcal{H}^{n-1} \\ = &\int_{\partial U} \sum_{j=1}^n F_j n_j d\mathcal{H}^{n-1} = \int_{\partial U} \mathbf{F} \cdot \mathbf{n} d\mathcal{H}^{n-1}. \end{aligned}$$

This proves the theorem.

What is the geometric significance of the vector, $\mathbf{n}$? Recall that in the part of the boundary contained in Q_i, this vector points in the same direction as the vector

$$\mathbf{N}_i(x_1, \cdots, x_{n-1}, g_i(x_1, \cdots, x_{n-1}))$$

given by

$$(-g_{i,1}(x_1, \cdots, x_{n-1}), -g_{i,2}(x_1, \cdots, x_{n-1}), \cdots, -g_{i,n-1}(x_1, \cdots, x_{n-1}), 1) \tag{17.68}$$

in the case where $k = n$. This vector is the gradient of the function,

$$x_n - g_i(x_1, \cdots, x_{n-1})$$

and so is perpendicular to the level surface given by

$$x_n - g_i(x_1, \cdots, x_{n-1}) = 0$$

in the case where g_i is C^1. It also points away from U so the vector $\mathbf{n}$ is the unit outer normal. The other cases work similarly.

The divergence theorem is valid in situations more general than for Lipschitz boundaries. What you need is essentially the ability to say that the functions, g_i above can be differentiated a.e. and more importantly that these functions can be recovered by integrating their partial derivatives. In other words, you need absolute continuity in each variable. Later in the chapter on weak derivatives, examples of such functions which are more general than Lipschitz functions will be discussed. However, the Lipschitz functions are pretty general and will suffice for now.

Differentiation With Respect To General Radon Measures

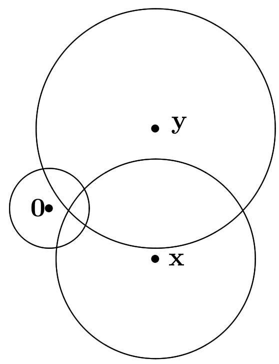
This is a brief chapter on certain important topics on the differentiation theory for general Radon measures. For different proofs and some results which are not discussed here, a good source is [20] which is where I first read some of these things.

18.1 Besicovitch Covering Theorem

The fundamental theorem of calculus presented above for Lebesgue measures can be generalized to arbitrary Radon measures. It turns out that the same approach works if a different covering theorem is employed instead of the Vitali theorem. This covering theorem is the Besicovitch covering theorem of this section. It is necessary because for a general Radon measure μ, it is no longer the case that the measure is translation invariant. This implies there is no way to estimate $\mu\left(\widehat{B}\right)$ in terms of $\mu(B)$ and thus the Vitali covering theorem is of no use. In the Besicovitch covering theorem the balls in the covering are not enlarged as they are in the Vitali theorem. In this theorem they can also be either open or closed or neither open nor closed. The balls can also be taken with respect to any norm on $\mathbb{R}^n$. The notation, $B(\mathbf{x},r)$ in the above argument will denote any set which satisfies

$$\{\mathbf{y} : ||\mathbf{y}-\mathbf{x}|| < r\} \subseteq B(\mathbf{x},r) \subseteq \overline{\{\mathbf{y} : ||\mathbf{y}-\mathbf{x}|| < r\}}$$

and the norm, $||\cdot||$ is just some norm on $\mathbb{R}^n$. The following picture is a distorted picture of the situation described in the following lemma.



Lemma 18.1 *Let* $10 \leq r_x \leq r_y$ *and suppose* $B(\mathbf{x}, r_x)$ *and* $B(\mathbf{y}, r_y)$ *both have nonempty intersection with* $B(\mathbf{0}, 1)$ *but neither of these balls contains* $\mathbf{0}$. *Suppose also that*

$$||\mathbf{x} - \mathbf{y}|| \geq r_y$$

so that neither ball contains both centers in its interior. Then

$$\left|\left|\frac{\mathbf{x}}{||\mathbf{x}||} - \frac{\mathbf{y}}{||\mathbf{y}||}\right|\right| \geq \frac{4}{5}.$$

Proof: By hypothesis,

$$||\mathbf{x}|| \geq r_x \geq ||\mathbf{x}|| - 1,\ ||\mathbf{y}|| \geq r_y \geq ||\mathbf{y}|| - 1.$$

Then

$$\left|\left|\frac{\mathbf{x}}{||\mathbf{x}||} - \frac{\mathbf{y}}{||\mathbf{y}||}\right|\right| = \left|\left|\frac{\mathbf{x}\,||\mathbf{y}|| - ||\mathbf{x}||\,\mathbf{y}}{||\mathbf{x}||\,||\mathbf{y}||}\right|\right|$$

$$= \left|\left|\frac{\mathbf{x}\,||\mathbf{y}|| - \mathbf{y}\,||\mathbf{y}|| + \mathbf{y}\,||\mathbf{y}|| - ||\mathbf{x}||\,\mathbf{y}}{||\mathbf{x}||\,||\mathbf{y}||}\right|\right|$$

$$\geq \frac{||\mathbf{x} - \mathbf{y}||}{||\mathbf{x}||} - \frac{|||\mathbf{y}|| - ||\mathbf{x}|||}{||\mathbf{x}||} \tag{18.1}$$

Now there are two cases.

First suppose $||\mathbf{y}|| \geq ||\mathbf{x}||$. Then the above is larger than

$$\begin{aligned} &\geq \frac{r_y}{||\mathbf{x}||} - \frac{||\mathbf{y}||}{||\mathbf{x}||} + 1 \geq \frac{r_y}{||\mathbf{x}||} - \frac{(r_y + 1)}{||\mathbf{x}||} + 1 \\ &= 1 - \frac{1}{||\mathbf{x}||} \geq 1 - \frac{1}{r_x} \geq 1 - \frac{1}{10} = \frac{9}{10}. \end{aligned}$$

Next suppose $||\mathbf{x}|| \geq ||\mathbf{y}||$. Then 18.1 is at least as large as

$$\begin{aligned}
& \frac{r_y}{||\mathbf{x}||} - \frac{||\mathbf{x}|| - ||\mathbf{y}||}{||\mathbf{x}||} \\
= & \frac{r_y}{||\mathbf{x}||} - 1 + \frac{||\mathbf{y}||}{||\mathbf{x}||} \\
\geq & \frac{2r_y}{||\mathbf{x}||} - 1 \geq \frac{2r_y}{r_x + 1} - 1 \\
\geq & \frac{2r_x}{r_x + 1} - 1 \geq \frac{20}{10+1} - 1 \\
= & .818\,18
\end{aligned}$$

This proves the lemma.

Lemma 18.2 *There exists L_n depending on dimension, n, such that for $\{\mathbf{x}_k\}_{k=1}^m$ distinct points on $\partial B(\mathbf{0}, 1)$, if $m \geq L_n$, then the distance between some pair of points of $\{\mathbf{x}_k\}_{k=1}^m$ is less than $4/5$.*

Proof: Let $\{\mathbf{z}_j\}_{j=1}^{L_n - 1}$ be a $1/3$ net on $\partial B(\mathbf{0}, 1)$. Then for $m \geq L_n$, if $\{\mathbf{x}_k\}_{k=1}^m$ is a set of m distinct points on $\partial B(\mathbf{0}, 1)$, there must exist $\mathbf{x}_i$ and $\mathbf{x}_j$ for $i \neq j$ such that both $\mathbf{x}_i$ and $\mathbf{x}_j$ are contained in some $B(\mathbf{z}_k, 1/3)$. This follows from the pigeon hole principle. There are more $\mathbf{x}_i$ than there are $B(\mathbf{z}_k, 1/3)$ and so one of these must have more than one $\mathbf{x}_k$ in it. But then

$$||\mathbf{x}_i - \mathbf{x}_j|| \leq ||\mathbf{x}_i - \mathbf{z}_k|| + ||\mathbf{z}_k - \mathbf{x}_j|| \leq \frac{2}{3} < \frac{4}{5}$$

This proves the lemma.

Corollary 18.3 *Let $B_0 = B(\mathbf{0},1)$ and let $B_j = B(\mathbf{x}_j, r_j)$ for $j = 1, \cdots, K$ such that $r_j \geq 10, \mathbf{0} \notin B_j$ for all $j > 0$, $B_j \cap B_0 \neq \emptyset$, and for all $i \neq j$,*

$$||\mathbf{x}_i - \mathbf{x}_j|| \geq \max(r_i, r_j).$$

That is, no B_j contains two centers in its interior. Then $K \leq L_n$, the constant of the above lemma.

Proof: By Lemma 18.2, if $K > L_n$, there exist two of the centers, $\mathbf{x}_i$ and $\mathbf{x}_j$ such that $\left|\left|\frac{\mathbf{x}_i}{||\mathbf{x}_i||} - \frac{\mathbf{x}_j}{||\mathbf{x}_j||}\right|\right| < 4/5$. By Lemma 18.1, $||\mathbf{x}_i - \mathbf{x}_j|| < \max(r_i, r_j)$ contrary to the assumptions of the corollary. Hence $K \leq L_n$ as claimed.

Theorem 18.4 *There exists a constant N_n, depending only on n with the following property. If $\mathcal{F}$ is any collection of nonempty balls in $\mathbb{R}^n$ with*

$$\sup\{\text{diam}(B) : B \in \mathcal{F}\} < D < \infty$$

and if A is the set of centers of the balls in $\mathcal{F}$, then there exist subsets of $\mathcal{F}$, $\mathcal{G}_1, \cdots, \mathcal{G}_{N_n}$, such that each $\mathcal{G}_i$ is a countable collection of disjoint balls from $\mathcal{F}$ and

$$A \subseteq \cup_{i=1}^{N_n} \{B : B \in \mathcal{G}_i\}.$$

Lemma 18.5 *In the situation of Theorem 18.4, suppose the set of centers A is bounded. Then there exists a sequence of balls from $\mathcal{F}$, $\{B_j\}_{j=1}^J$ where $J \leq \infty$ such that*

$$r(B_1) \geq \frac{3}{4} \sup\{r(B) : B \in \mathcal{F}\} \tag{18.2}$$

and if

$$A_m \equiv A \setminus (\cup_{i=1}^m B_i) \neq \emptyset, \tag{18.3}$$

then

$$r(B_{m+1}) \geq \frac{3}{4} \sup\{r : B(\mathbf{a}, r) \in \mathcal{F}, \mathbf{a} \in A_m\}. \tag{18.4}$$

Letting $B_j = B(\mathbf{a}_j, r_j)$, this sequence satisfies

$$A \subseteq \cup_{i=1}^J B_i,\ r(B_k) \leq \frac{4}{3} r(B_j) \text{ for } j < k,\ \{B(\mathbf{a}_j, r_j/3)\}_{j=1}^J \text{ are disjoint.} \tag{18.5}$$

Proof: Pick B_1 satisfying 18.2. If $B_1, \cdots, B_m$ have been chosen, and A_m is given in 18.3, then if it equals $\emptyset$, it follows $A \subseteq \cup_{i=1}^m B_i$. Set $J = m$. If $A_m \neq \emptyset$, pick B_{m+1} to satisfy 18.4. This defines the desired sequence. It remains to verify the claims in 18.5. Consider the second claim. Letting $A_0 \equiv A$, $A_k \subseteq A_{j-1}$ and so

$$\sup\{r : B(\mathbf{a}, r) \in \mathcal{F}, \mathbf{a} \in A_{j-1}\} \geq r_k.$$

Hence $r_j \geq (3/4) r_k$. This proves the second claim of 18.5.

Consider the third claim of 18.5. Suppose to the contrary that $\mathbf{x} \in B(\mathbf{a}_j, r_j/3) \cap B(\mathbf{a}_i, r_i/3)$ where $i < j$. Then

$$\begin{aligned} ||\mathbf{a}_i - \mathbf{a}_j|| &\leq ||\mathbf{a}_i - \mathbf{x}|| + ||\mathbf{x} - \mathbf{a}_j|| \\ &< \frac{1}{3}(r_i + r_j) \leq \frac{1}{3}\left(r_i + \frac{4}{3} r_i\right) \\ &= \frac{7}{9} r_i < r_i \end{aligned}$$

contrary to the construction which requires $\mathbf{a}_j \notin B(\mathbf{a}_i, r_i)$.

Finally consider the first claim of 18.5. It is true if $J < \infty$. This follows from the construction. If $J = \infty$, then since A is bounded and the balls, $B(\mathbf{a}_j, r_j/3)$ are disjoint, it must be the case that $\lim_{i\to\infty} r_i = 0$. Suppose $J = \infty$ so that $A_m \neq \emptyset$ for all m. If $\mathbf{a}_0$ fails to be covered, then $\mathbf{a}_0 \in A_k$ for all k. Let $\mathbf{a}_0 \in B(\mathbf{a}_0, r_0) \in \mathcal{F}$ for some ball $B(\mathbf{a}_0, r_0)$. Then for i large enough, $r_i < \frac{1}{10} r_0$ and so since $\mathbf{a}_0 \in A_{i-1}$,

$$\frac{3}{4} r_0 \leq \frac{3}{4} \sup\{r : B(\mathbf{a}, r) \in \mathcal{F}, \mathbf{a} \in A_{i-1}\} \leq r_i < \frac{1}{10} r_0,$$

a contradiction. This proves the lemma.

Lemma 18.6 *There exists a constant M_n depending only on n such that for each $1 \leq k \leq J$, M_n exceeds the number of sets B_j for $j < k$ which have nonempty intersection with B_k.*

Proof: These sets B_j which intersect B_k are of two types. Either they have large radius, $r_j > 10r_k$, or they have small radius, $r_j \leq 10r_k$. In this argument let $card(S)$ denote the number of elements in the set S. Define for fixed k,

$$I \equiv \{j : 1 \leq j < k,\ B_j \cap B_k \neq \emptyset,\ r_j \leq 10r_k\},$$

$$K \equiv \{j : 1 \leq j < k,\ B_j \cap B_k \neq \emptyset,\ r_j > 10r_k\}.$$

Claim 1: $B\left(\mathbf{a}_j, \frac{r_j}{3}\right) \subseteq B(\mathbf{a}_k, 15r_k)$ for $j \in I$.

Proof: Let $j \in I$. Then $B_j \cap B_k \neq \emptyset$ and $r_j \leq 10r_k$. Now if

$$\mathbf{x} \in B\left(\mathbf{a}_j, \frac{r_j}{3}\right),$$

then since $r_j \leq 10r_k$,

$$\begin{aligned} ||\mathbf{x} - \mathbf{a}_k|| &\leq ||\mathbf{x} - \mathbf{a}_j|| + ||\mathbf{a}_j - \mathbf{a}_k|| \leq \frac{r_j}{3} + r_j + r_k = \\ \frac{4}{3}r_j + r_k &\leq \frac{4}{3}(10r_k) + r_k = \frac{43}{3}r_k < 15r_k. \end{aligned}$$

Therefore, $B\left(\mathbf{a}_j, \frac{r_j}{3}\right) \subseteq B(\mathbf{a}_k, 15r_k)$.

Claim 2: $card(I) \leq 60^n$.

Proof: Recall $r(B_k) \leq \frac{4}{3}r(B_j)$. Then letting $\alpha(n)r^n$ be the Lebesgue measure of the n dimensional ball of radius r, (Note this $\alpha(n)$ depends on the norm used.)

$$\begin{aligned} \alpha(n)15^n r_k^n &\equiv m_n(B(\mathbf{a}_k, 15r_k)) \geq \sum_{j \in I} m_n\left(B\left(\mathbf{a}_j, \frac{r_j}{3}\right)\right) \\ &= \sum_{j \in I} \alpha(n)\left(\frac{r_j}{3}\right)^n \geq \sum_{j \in I} \alpha(n)\left(\frac{r_k}{4}\right)^n \left(\text{since } r_k \leq \frac{4}{3}r_j\right) \\ &= card(I)\,\alpha(n)\left(\frac{r_k}{4}\right)^n \end{aligned}$$

and so it follows $card(I) \leq 60^n$ as claimed.

Claim 3: $card(K) \leq L_n$ where L_n is the constant of Corollary 18.3.

Proof: Consider $\{B_j : j \in K\}$ and B_k. Let $\mathbf{f}(\mathbf{x}) \equiv r_k^{-1}(\mathbf{x} - \mathbf{x}_k)$. Then $\mathbf{f}(B_k) = B(\mathbf{0}, 1)$ and

$$\mathbf{f}(B_j) = r_k^{-1}B(\mathbf{x}_j - \mathbf{x}_k, r_j) = B\left(\frac{\mathbf{x}_j - \mathbf{x}_k}{r_k}, r_j/r_k\right).$$

Then $r_j/r_k \geq 10$ because $j \in K$. None of the balls, $\mathbf{f}(B_j)$ contain $\mathbf{0}$ but all these balls intersect $B(\mathbf{0}, 1)$ and as just noted, each of these balls has radius ≥ 10 and none of them contains two centers on its interior. By Corollary 18.3, it follows there are no more than L_n of them. This proves the claim. A constant which will satisfy the desired conditions is

$$M_n \equiv L_n + 60^n + 1.$$

This completes the proof of Lemma 18.6.

Next subdivide the balls $\{B_i\}_{i=1}^J$ into M_n subsets $\mathcal{G}_1, \cdots, \mathcal{G}_{M_n}$ each of which consists of disjoint balls. This is done in the following way. Let $B_1 \in \mathcal{G}_1$. If $B_1, \cdots, B_k$ have each been assigned to one of the sets $\mathcal{G}_1, \cdots, \mathcal{G}_{M_n}$, let $B_{k+1} \in \mathcal{G}_r$ where r is the smallest index having the property that B_{k+1} does not intersect any of the balls already in $\mathcal{G}_r$. There must exist such an index $r \in \{1, \cdots, M_n\}$ because otherwise $B_{k+1} \cap B_j \neq \emptyset$ for at least M_n values of $j < k+1$ contradicting Lemma 18.6. By Lemma 18.5

$$A \subseteq \cup_{i=1}^{M_n} \{B : B \in \mathcal{G}_i\} = \cup_{j=1}^{J} B_j.$$

This proves Theorem 18.4 in the case where A is bounded.

To complete the proof of this theorem, the restriction that A is bounded must be removed. Define

$$A_l \equiv A \cap \{\mathbf{x} \in \mathbb{R}^n : 10\,(l-1)\,D \leq ||\mathbf{x}|| < 10lD\}\,,\ l = 1, 2, \cdots$$

and

$$\mathcal{F}_l = \{B\,(\mathbf{a},r) : B\,(\mathbf{a},r) \in \mathcal{F} \text{ and } \mathbf{a} \in A_l\}.$$

Then since D is an upper bound for all the diameters of these balls,

$$(\cup \mathcal{F}_l) \cap (\cup \mathcal{F}_m) = \emptyset \tag{18.6}$$

whenever $m \geq l+2$. Therefore, applying what was just shown to the pair $(A_l, \mathcal{F}_l)$, there exist subsets of $\mathcal{F}_l$, $\mathcal{G}_1^l \cdots \mathcal{G}_{M_n}^l$ such that each $\mathcal{G}_i^l$ is a countable collection of disjoint balls of $\mathcal{F}_l \subseteq \mathcal{F}$ and

$$A_l \subseteq \cup_{i=1}^{M_n} \left\{B : B \in \mathcal{G}_i^l\right\}.$$

Now let $\mathcal{G}_j \equiv \cup_{l=1}^{\infty} \mathcal{G}_j^{2l-1}$ for $1 \leq j \leq M_n$ and for $1 \leq j \leq M_n$, let $\mathcal{G}_{j+M_n} \equiv \cup_{l=1}^{\infty} \mathcal{G}_j^{2l}$. Thus, letting $N_n \equiv 2M_n$,

$$A = \cup_{l=1}^{\infty} A_{2l} \cup \cup_{l=1}^{\infty} A_{2l-1} \subseteq \cup_{j=1}^{N_n} \{B : B \in \mathcal{G}_j\}$$

and by 18.6, each $\mathcal{G}_j$ is a countable set of disjoint balls of $\mathcal{F}$. This proves the Besicovitch covering theorem.

18.2 Fundamental Theorem Of Calculus For Radon Measures

In this section the Besicovitch covering theorem will be used to give a generalization of the Lebesgue differentiation theorem to general Radon measures. In what follows, μ will be a Radon measure,

$$Z \equiv \{\mathbf{x} \in \mathbb{R}^n : \mu\,(B\,(\mathbf{x},r)) = 0 \text{ for some } r > 0\},$$

$$⨍_{B(\mathbf{x},r)} f d\mu \equiv \begin{cases} 0 \text{ if } \mathbf{x} \in Z, \\ \frac{1}{\mu(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} f d\mu \text{ if } \mathbf{x} \notin Z, \end{cases}$$

and the maximal function $Mf : \mathbb{R}^n \to [0, \infty]$ is given by

$$Mf(\mathbf{x}) \equiv \sup_{r \leq 1} \fint_{B(\mathbf{x},r)} |f| \, d\mu.$$

Lemma 18.7 *Z is measurable and $\mu(Z) = 0$.*

Proof: For each $\mathbf{x} \in Z$, there exists a ball $B(\mathbf{x},r)$ with $\mu(B(\mathbf{x},r)) = 0$. Let $\mathcal{C}$ be the collection of these balls. Since $\mathbb{R}^n$ has a countable basis, a countable subset, $\widetilde{\mathcal{C}}$, of $\mathcal{C}$ also covers Z. Let

$$\widetilde{\mathcal{C}} = \{B_i\}_{i=1}^{\infty}.$$

Then letting $\overline{\mu}$ denote the outer measure determined by μ,

$$\overline{\mu}(Z) \leq \sum_{i=1}^{\infty} \overline{\mu}(B_i) = \sum_{i=1}^{\infty} \mu(B_i) = 0$$

Therefore, Z is measurable and has measure zero as claimed.

Theorem 18.8 *Let μ be a Radon measure and let $f \in L^1(\mathbb{R}^n, \mu)$. Then*

$$\lim_{r \to 0} \fint_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})| \, d\mu(y) = 0$$

for μ a.e. $\mathbf{x} \in \mathbb{R}^n$.

Proof: First consider the following claim which is a weak type estimate of the same sort used when differentiating with respect to Lebesgue measure.

Claim 1:

$$\overline{\mu}([Mf > \varepsilon]) \leq N_n \varepsilon^{-1} \|f\|_1$$

Proof: First note $A \cap Z = \emptyset$. For each $\mathbf{x} \in A$ there exists a ball $B_{\mathbf{x}} = B(\mathbf{x},r_{\mathbf{x}})$ with $r_{\mathbf{x}} \leq 1$ and

$$\mu(B_{\mathbf{x}})^{-1} \int_{B(\mathbf{x},r_{\mathbf{x}})} |f| \, d\mu > \varepsilon.$$

Let $\mathcal{F}$ be this collection of balls so that A is the set of centers of balls of $\mathcal{F}$. By the Besicovitch covering theorem,

$$A \subseteq \cup_{i=1}^{N_n} \{B : B \in \mathcal{G}_i\}$$

where $\mathcal{G}_i$ is a collection of disjoint balls of $\mathcal{F}$. Now for some i,

$$\overline{\mu}(A) / N_n \leq \mu(\cup \{B : B \in \mathcal{G}_i\})$$

because if this is not so, then

$$\overline{\mu}(A) \leq \sum_{i=1}^{N_n} \mu(\cup \{B : B \in \mathcal{G}_i\}) < \sum_{i=1}^{N_n} \frac{\overline{\mu}(A)}{N_n} = \overline{\mu}(A),$$

a contradiction. Therefore for this i,

$$\begin{aligned}\frac{\overline{\mu}(A)}{N_n} &\leq \mu\left(\cup\{B : B\in\mathcal{G}_i\}\right) = \sum_{B\in\mathcal{G}_i}\mu(B) \leq \sum_{B\in\mathcal{G}_i}\varepsilon^{-1}\int_B |f|\,d\mu \\ &\leq \varepsilon^{-1}\int_{\mathbb{R}^n}|f|\,d\mu = \varepsilon^{-1}\|f\|_1 .\end{aligned}$$

This shows Claim 1.

Claim 2: If g is any continuous function defined on $\mathbb{R}^n$, then

$$\lim_{r\to 0} ⨍_{B(\mathbf{x},r)} |g(\mathbf{y}) - g(\mathbf{x})|\,d\mu(y) = 0$$

and if $\mathbf{x}\notin Z$,

$$\lim_{r\to 0}\frac{1}{\mu(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} g(\mathbf{y})\,d\mu(y) = g(\mathbf{x}). \tag{18.7}$$

Proof: If $\mathbf{x}\in Z$ there is nothing to prove. If $\mathbf{x}\notin Z$, then since g is continuous at $\mathbf{x}$, whenever r is small enough,

$$⨍_{B(\mathbf{x},r)} |g(\mathbf{y}) - g(\mathbf{x})|\,d\mu(y)$$

$$\begin{aligned} &= \frac{1}{\mu(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} |g(\mathbf{y}) - g(\mathbf{x})|\,d\mu(y) \\ &\leq \frac{1}{\mu(B(\mathbf{x},r))}\int_{B(\mathbf{x},r)} \varepsilon\,d\mu(y) = \varepsilon.\end{aligned}$$

18.7 follows from the above and the triangle inequality. This proves the claim.

Now let $g\in C_c(\mathbb{R}^n)$ and $\mathbf{x}\notin Z$. Then from the above observations about continuous functions,

$$\overline{\mu}\left(\left[\mathbf{x} : \limsup_{r\to 0} ⨍_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})|\,d\mu(y) > \varepsilon\right]\right) \tag{18.8}$$

$$\begin{aligned} &\leq \overline{\mu}\left(\left[\mathbf{x} : \limsup_{r\to 0} ⨍_{B(\mathbf{x},r)} |f(\mathbf{y}) - g(\mathbf{y})|\,d\mu(y) > \frac{\varepsilon}{2}\right]\right) \\ &\quad + \overline{\mu}\left(\left[\mathbf{x} : |g(\mathbf{x}) - f(\mathbf{x})| > \frac{\varepsilon}{2}\right]\right). \\ &\quad \leq \overline{\mu}\left(\left[M(f-g) > \frac{\varepsilon}{2}\right]\right) + \overline{\mu}\left(\left[|f-g| > \frac{\varepsilon}{2}\right]\right) \end{aligned} \tag{18.9}$$

Now

$$\int_{\left[|f-g|>\frac{\varepsilon}{2}\right]} |f-g|\,d\mu \geq \frac{\varepsilon}{2}\overline{\mu}\left(\left[|f-g| > \frac{\varepsilon}{2}\right]\right)$$

and so from Claim 1 18.9 and hence 18.8 is dominated by

$$\left(\frac{2}{\varepsilon}+\frac{N_n}{\varepsilon}\right)||f-g||_{L^1(\mathbb{R}^n,\mu)}.$$

But by regularity of Radon measures, $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n,\mu)$ and so since g in the above is arbitrary, this shows 18.8 equals 0. Now

$$\begin{aligned}
&\overline{\mu}\left(\left[\mathbf{x}:\limsup_{r\to 0}\fint_{B(\mathbf{x},r)}|f(\mathbf{y})-f(\mathbf{x})|\,d\mu(y)>0\right]\right)\\
=&\ \overline{\mu}\left(\cup_{m=1}^{\infty}\left(\left[\mathbf{x}:\limsup_{r\to 0}\fint_{B(\mathbf{x},r)}|f(\mathbf{y})-f(\mathbf{x})|\,d\mu(y)>\frac{1}{m}\right]\right)\right)\\
\leq&\ \sum_{m=1}^{\infty}\overline{\mu}\left(\left[\mathbf{x}:\limsup_{r\to 0}\fint_{B(\mathbf{x},r)}|f(\mathbf{y})-f(\mathbf{x})|\,d\mu(y)>\frac{1}{m}\right]\right)=0.
\end{aligned}$$

By Lemma 18.7 the set Z is a set of measure zero and so if

$$\mathbf{x}\notin\left[\limsup_{r\to 0}\fint_{B(\cdot,r)}|f(\mathbf{y})-f(\cdot)|\,d\mu(y)>0\right]\cup Z$$

the above has shown

$$\begin{aligned}
0 &\leq \liminf_{r\to 0}\fint_{B(\mathbf{x},r)}|f(\mathbf{y})-f(\mathbf{x})|\,d\mu(y)\\
&\leq \limsup_{r\to 0}\fint_{B(\mathbf{x},r)}|f(\mathbf{y})-f(\mathbf{x})|\,d\mu(y)=0
\end{aligned}$$

which proves the theorem.

The following corollary is the main result referred to as the Lebesgue Besicovitch Differentiation theorem.

Corollary 18.9 *If $f\in L^1_{loc}(\mathbb{R}^n,\mu)$,*

$$\lim_{r\to 0}\fint_{B(\mathbf{x},r)}|f(\mathbf{y})-f(\mathbf{x})|\,d\mu(y)=0\ \mu\ a.e.\ \mathbf{x}. \tag{18.10}$$

Proof: If f is replaced by $f\mathcal{X}_{B(\mathbf{0},k)}$ then the conclusion 18.10 holds for all $\mathbf{x}\notin F_k$ where F_k is a set of μ measure 0. Letting $k=1,2,\cdots$, and $F\equiv\cup_{k=1}^{\infty}F_k$, it follows that F is a set of measure zero and for any $\mathbf{x}\notin F$, and $k\in\{1,2,\cdots\}$, 18.10 holds if f is replaced by $f\mathcal{X}_{B(\mathbf{0},k)}$. Picking any such $\mathbf{x}$, and letting $k>|\mathbf{x}|+1$, this shows

$$\begin{aligned}
&\lim_{r\to 0}\fint_{B(\mathbf{x},r)}|f(\mathbf{y})-f(\mathbf{x})|\,d\mu(y)\\
=&\lim_{r\to 0}\fint_{B(\mathbf{x},r)}\left|f\mathcal{X}_{B(\mathbf{0},k)}(\mathbf{y})-f\mathcal{X}_{B(\mathbf{0},k)}(\mathbf{x})\right|d\mu(y)=0.
\end{aligned}$$

This proves the corollary.

18.3 Slicing Measures

Let μ be a finite Radon measure. I will show here that a formula of the following form holds.

$$\mu(F) = \int_F d\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \mathcal{X}_F(\mathbf{x}, \mathbf{y})\, d\nu_{\mathbf{x}}(y)\, d\alpha(x)$$

where $\alpha(E) = \mu(E \times \mathbb{R}^m)$. When this is done, the measures, $\nu_{\mathbf{x}}$, are called slicing measures and this shows that an integral with respect to μ can be written as an iterated integral in terms of the measure α and the slicing measures, $\nu_{\mathbf{x}}$. This is like going backwards in the construction of product measure. One starts with a measure, μ, defined on the Cartesian product and produces α and an infinite family of slicing measures from it whereas in the construction of product measure, one starts with two measures and obtains a new measure on a σ algebra of subsets of the Cartesian product of two spaces. First here are two technical lemmas.

Lemma 18.10 *The space $C_c(\mathbb{R}^m)$ with the norm*

$$||f|| \equiv \sup\{|f(\mathbf{y})| : \mathbf{y} \in \mathbb{R}^m\}$$

is separable.

Proof: Let $\mathcal{D}_l$ consist of all functions which are of the form

$$\sum_{|\alpha| \leq N} a_\alpha \mathbf{y}^\alpha \left(\text{dist}\left(\mathbf{y}, B(\mathbf{0}, l+1)^C\right)\right)^{n_\alpha}$$

where $a_\alpha \in \mathbb{Q}$, α is a multi-index, and n_α is a positive integer. Then $\mathcal{D}_l$ is countable, separates the points of $\overline{B(\mathbf{0}, l)}$ and annihilates no point of $\overline{B(\mathbf{0}, l)}$. By the Stone Weierstrass theorem $\mathcal{D}_l$ is dense in the space $C\left(\overline{B(\mathbf{0}, l)}\right)$ and so $\cup\{\mathcal{D}_l : l \in \mathbb{N}\}$ is a countable dense subset of $C_c(\mathbb{R}^m)$.

From the regularity of Radon measures, the following lemma follows.

Lemma 18.11 *If μ and ν are two Radon measures defined on σ algebras, $\mathcal{S}_\mu$ and $\mathcal{S}_\nu$, of subsets of $\mathbb{R}^n$ and if $\mu(V) = \nu(V)$ for all V open, then $\mu = \nu$ and $\mathcal{S}_\mu = \mathcal{S}_\nu$.*

Proof: Every compact set is a countable intersection of open sets so the two measures agree on every compact set. Hence it is routine that the two measures agree on every G_δ and F_σ set. (Recall G_δ sets are countable intersections of open sets and F_σ sets are countable unions of closed sets.) Now suppose $E \in \mathcal{S}_\nu$ is a bounded set. Then by regularity of ν there exists G a G_δ set and F, an F_σ set such that $F \subseteq E \subseteq G$ and $\nu(G \setminus F) = 0$. Then it is also true that $\mu(G \setminus F) = 0$. Hence $E = F \cup (E \setminus F)$ and $E \setminus F$ is a subset of $G \setminus F$, a set of μ measure zero. By completeness of μ, it follows $E \in \mathcal{S}_\mu$ and

$$\mu(E) = \mu(F) = \nu(F) = \nu(E).$$

If $E \in \mathcal{S}_\nu$ not necessarily bounded, let $E_m = E \cap B(0, m)$ and then $E_m \in \mathcal{S}_\mu$ and $\mu(E_m) = \nu(E_m)$. Letting $m \to \infty, E \in \mathcal{S}_\mu$ and $\mu(E) = \nu(E)$. Similarly, $\mathcal{S}_\mu \subseteq \mathcal{S}_\nu$ and the two measures are equal on $\mathcal{S}_\mu$.

The main result in the section is the following theorem.

Theorem 18.12 *Let μ be a finite Radon measure on $\mathbb{R}^{n+m}$ defined on a σ algebra, $\mathcal{F}$. Then there exists a unique finite Radon measure, α, defined on a σ algebra, $\mathcal{S}$, of sets of $\mathbb{R}^n$ which satisfies*

$$\alpha(E) = \mu(E \times \mathbb{R}^m) \tag{18.11}$$

for all E Borel. There also exists a Borel set of α measure zero, N, such that for each $\mathbf{x} \notin N$, there exists a Radon probability measure $\nu_{\mathbf{x}}$ such that if f is a nonnegative μ measurable function or a μ measurable function in $L^1(\mu)$,

$$\mathbf{y} \to f(\mathbf{x}, \mathbf{y}) \text{ is } \nu_{\mathbf{x}} \text{ measurable } \alpha \text{ a.e.}$$

$$\mathbf{x} \to \int_{\mathbb{R}^m} f(\mathbf{x}, \mathbf{y}) \, d\nu_{\mathbf{x}}(y) \text{ is } \alpha \text{ measurable} \tag{18.12}$$

and

$$\int_{\mathbb{R}^{n+m}} f(\mathbf{x}, \mathbf{y}) \, d\mu = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(\mathbf{x}, \mathbf{y}) \, d\nu_{\mathbf{x}}(y) \right) d\alpha(x). \tag{18.13}$$

If $\widehat{\nu}_{\mathbf{x}}$ is any other collection of Radon measures satisfying 18.12 and 18.13, then $\widehat{\nu}_{\mathbf{x}} = \nu_{\mathbf{x}}$ for α a.e. $\mathbf{x}$.

Proof: First consider the uniqueness of α. Suppose α_1 is another Radon measure satisfying 18.11. Then in particular, α_1 and α agree on open sets and so the two measures are the same by Lemma 18.11.

To establish the existence of α, define α_0 on Borel sets by

$$\alpha_0(E) = \mu(E \times \mathbb{R}^m).$$

Thus α_0 is a finite Borel measure and so it is finite on compact sets. Lemma 11.3 on Page 11.3 implies the existence of the Radon measure α extending α_0.

Next consider the uniqueness of $\nu_{\mathbf{x}}$. Suppose $\nu_{\mathbf{x}}$ and $\widehat{\nu}_{\mathbf{x}}$ satisfy all conclusions of the theorem with exceptional sets denoted by N and $\widehat{N}$ respectively. Then, enlarging N and $\widehat{N}$, one may also assume, using Lemma 18.7, that for $\mathbf{x} \notin N \cup \widehat{N}$, $\alpha(B(\mathbf{x},r)) > 0$ whenever $r > 0$. Now let

$$A = \prod_{i=1}^{m} (a_i, b_i]$$

where a_i and b_i are rational. Thus there are countably many such sets. Then from the conclusion of the theorem, if $\mathbf{x}_0 \notin N \cup \widehat{N}$,

$$\frac{1}{\alpha(B(\mathbf{x}_0, r))} \int_{B(\mathbf{x}_0,r)} \int_{\mathbb{R}^m} \mathcal{X}_A(\mathbf{y}) \, d\nu_{\mathbf{x}}(y) \, d\alpha$$

$$= \frac{1}{\alpha(B(\mathbf{x}_0, r))} \int_{B(\mathbf{x}_0,r)} \int_{\mathbb{R}^m} \mathcal{X}_A(\mathbf{y}) \, d\widehat{\nu}_{\mathbf{x}}(y) \, d\alpha,$$

and by the Lebesgue Besicovitch Differentiation theorem, there exists a set of α measure zero, E_A, such that if $\mathbf{x}_0 \notin E_A \cup N \cup \widehat{N}$, then the limit in the above exists as $r \to 0$ and yields

$$\nu_{\mathbf{x}_0}(A) = \widehat{\nu}_{\mathbf{x}_0}(A).$$

Letting E denote the union of all the sets E_A for A as described above, it follows that E is a set of measure zero and if $\mathbf{x}_0 \notin E \cup N \cup \widehat{N}$ then $\nu_{\mathbf{x}_0}(A) = \widehat{\nu}_{\mathbf{x}_0}(A)$ for all such sets A. But every open set can be written as a disjoint union of sets of this form and so for all such $\mathbf{x}_0$, $\nu_{\mathbf{x}_0}(V) = \widehat{\nu}_{\mathbf{x}_0}(V)$ for all V open. By Lemma 18.11 this shows the two measures are equal and proves the uniqueness assertion for $\nu_{\mathbf{x}}$. It remains to show the existence of the measures $\nu_{\mathbf{x}}$. This will be done with the aid of the following lemma. The idea is to define a positive linear functional which will yield the desired measure, $\nu_{\mathbf{x}}$ and this lemma will help in making this definition.

Lemma 18.13 *There exists a set N of α measure 0, independent of $f \in C_c(\mathbb{R}^{n+m})$ such that if $\mathbf{x} \notin N$, $\alpha(B(\mathbf{x},r)) > 0$ for all $r > 0$ and*

$$\lim_{r\to 0} \frac{1}{\alpha(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)\times\mathbb{R}^m} f d\mu \ = g_f(\mathbf{x})$$

where g_f is a α measurable function with the property that

$$\int_{\mathbb{R}^n} g_f(\mathbf{x})\, d\alpha = \int_{\mathbb{R}^n\times\mathbb{R}^m} f d\mu.$$

Proof: Let $f \in C_c(\mathbb{R}^{n+m})$ and let

$$\eta_f(E) \equiv \int_{E\times\mathbb{R}^m} f d\mu.$$

Then η_f is a finite measure defined on the Borel sets with $\eta_f \ll \alpha$. By the Radon Nikodym theorem, there exists a Borel measurable function $\widetilde{g}_f$ such that for all Borel E,

$$\eta_f(E) \equiv \int_{E\times\mathbb{R}^m} f d\mu = \int_E \widetilde{g}_f d\alpha. \tag{18.14}$$

By the theory of differentiation for Radon measures, there exists a set of α measure zero, N_f such that if $\mathbf{x} \notin N_f$, then $\alpha(B(\mathbf{x},r)) > 0$ for all $r > 0$ and

$$\lim_{r\to 0} \frac{1}{\alpha(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)\times\mathbb{R}^m} f d\mu = \lim_{r\to 0} \fint_{B(\mathbf{x},r)} \widetilde{g}_f d\alpha = \widetilde{g}_f(\mathbf{x}).$$

Let $\mathcal{D}$ be a countable dense subset of $C_c(\mathbb{R}^{n+m})$ and let

$$N \equiv \cup\{N_f : f \in \mathcal{D}\}.$$

Then if $f \in C_c(\mathbb{R}^{n+m})$ is arbitrary, and $\mathbf{x} \notin N$, referring to 18.14, it follows there exists $g \in \mathcal{D}$ close enough to f such that for all r_1, r_2,

$$\left| \frac{1}{\alpha(B(\mathbf{x},r_1))} \int_{B(\mathbf{x},r_1)\times\mathbb{R}^m} f d\mu - \frac{1}{\alpha(B(\mathbf{x},r_2))} \int_{B(\mathbf{x},r_2)\times\mathbb{R}^m} f d\mu \right| <$$

$$\frac{\varepsilon}{2}+\left|\frac{1}{\alpha\left(B\left(\mathbf{x},r_{1}\right)\right)}\int_{B\left(\mathbf{x},r_{1}\right)\times\mathbb{R}^{m}}gd\mu-\frac{1}{\alpha\left(B\left(\mathbf{x},r_{2}\right)\right)}\int_{B\left(\mathbf{x},r_{2}\right)\times\mathbb{R}^{m}}gd\mu\right|$$

Therefore, taking r_i small enough, the right side is less than ε. Since ε is arbitrary, this shows the limit as $r \to 0$ exists. Define this limit which exists for all $f \in C_c\left(\mathbb{R}^{n+m}\right)$ and $\mathbf{x} \notin N$ as $g_f\left(\mathbf{x}\right)$. By the first part of the argument, $g_f\left(\mathbf{x}\right) = \widetilde{g}_f\left(\mathbf{x}\right)$ a.e. Thus, $g_f\left(\mathbf{x}\right)$ is α measurable because it equals a Borel measurable function α a.e. The final formula follows from

$$\int_{\mathbb{R}^{n}}g_{f}\left(\mathbf{x}\right)d\alpha=\int_{\mathbb{R}^{n}}\widetilde{g}_{f}\left(\mathbf{x}\right)d\alpha\equiv\eta_{f}\left(\mathbb{R}^{n}\right)\equiv\int_{\mathbb{R}^{n}\times\mathbb{R}^{m}}fd\mu.$$

This proves the lemma.

Continuing with the proof of the theorem, let $\mathbf{x} \notin N$ and let $f\left(\mathbf{z},\mathbf{y}\right) \equiv \psi\left(\mathbf{z}\right)\phi\left(\mathbf{y}\right)$ where ψ and ϕ are continuous functions with compact support in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Suppose first that $\psi\left(\mathbf{x}\right) = 1$. Then define a positive linear functional

$$L_{\mathbf{x}}\phi\equiv\lim_{r\to0}\frac{1}{\alpha\left(B\left(\mathbf{x},r\right)\right)}\int_{B\left(\mathbf{x},r\right)\times\mathbb{R}^{m}}\psi\left(\mathbf{z}\right)\phi\left(\mathbf{y}\right)d\mu\left(z,y\right).$$

This functional may appear to depend on the choice of ψ satisfying $\psi\left(\mathbf{x}\right) = 1$ but this is not the case because all such $\psi' s$ used in the definition of $L_{\mathbf{x}}$ are continuous.

Let $\nu_{\mathbf{x}}$ be the Radon measure representing $L_{\mathbf{x}}$. Thus replacing an arbitrary $\psi \in C_c\left(\mathbb{R}^n\right)$ with $\frac{\psi}{\psi(\mathbf{x})}$, in the case when $\psi\left(\mathbf{x}\right) \neq 0$,

$$\begin{aligned}\psi\left(\mathbf{x}\right)L_{\mathbf{x}}\left(\phi\right) &= \psi\left(\mathbf{x}\right)\int_{\mathbb{R}^{m}}\phi\left(\mathbf{y}\right)d\nu_{\mathbf{x}}\left(y\right)\\ &= \lim_{r\to0}\frac{\psi\left(\mathbf{x}\right)}{\alpha\left(B\left(\mathbf{x},r\right)\right)}\int_{B\left(\mathbf{x},r\right)\times\mathbb{R}^{m}}\frac{\psi}{\psi\left(\mathbf{x}\right)}\phi d\mu\\ &= \lim_{r\to0}\frac{1}{\alpha\left(B\left(\mathbf{x},r\right)\right)}\int_{B\left(\mathbf{x},r\right)\times\mathbb{R}^{m}}\psi\phi d\mu\end{aligned}$$

By Lemma 18.13,

$$\int_{\mathbb{R}^{n}}\int_{\mathbb{R}^{m}}\psi\phi d\nu_{\mathbf{x}}d\alpha=\int_{\mathbb{R}^{n}}g_{\psi\phi}d\alpha=\int_{\mathbb{R}^{n}\times\mathbb{R}^{m}}\psi\phi d\mu.$$

Letting ψ_k, ϕ_k increase to 1 pointwise, the monotone convergence theorem implies

$$\int_{\mathbb{R}^{n}}\nu_{\mathbf{x}}\left(\mathbb{R}^{m}\right)d\alpha=\int_{\mathbb{R}^{n}\times\mathbb{R}^{m}}d\mu=\mu\left(\mathbb{R}^{n}\times\mathbb{R}^{m}\right)<\infty \tag{18.15}$$

showing that $\mathbf{x} \to \nu_{\mathbf{x}}\left(\mathbb{R}^m\right)$ is a function in $L^1\left(\alpha\right)$. In particular, this function is finite α a.e. Summarizing, the above has shown that whenever $\psi \in C_c\left(\mathbb{R}^n\right)$ and $\phi \in C_c\left(\mathbb{R}^m\right)$

$$\int_{\mathbb{R}^{n}}\int_{\mathbb{R}^{m}}\psi\phi d\nu_{\mathbf{x}}d\alpha=\int_{\mathbb{R}^{n}\times\mathbb{R}^{m}}\psi\phi d\mu. \tag{18.16}$$

Also $\nu_{\mathbf{x}}$ is a finite measure α *a.e.* and $\nu_{\mathbf{x}}$ and α are Radon measures.

Next it is shown that $\psi\phi$ can be replaced by $\mathcal{X}_E$ where E is an arbitrary μ measurable set. To do so, let

$$R_1 \equiv \prod_{i=1}^{n}(a_i, b_i],\ R_2 \equiv \prod_{i=1}^{m}(c_i, d_i] \tag{18.17}$$

and let ψ_k be a sequence of functions in $C_c(\mathbb{R}^n)$ which is bounded, piecewise linear in each variable, and converging pointwise to $\mathcal{X}_{R_1}$. Also let ϕ_k be a similar sequence converging pointwise to $\mathcal{X}_{R_2}$. Then by the dominated convergence theorem,

$$\int_{\mathbb{R}^n\times\mathbb{R}^m} \mathcal{X}_{R_1}(\mathbf{x})\,\mathcal{X}_{R_2}(\mathbf{y})\,d\mu = \lim_{k\to\infty}\int_{\mathbb{R}^n\times\mathbb{R}^m}\psi_k(\mathbf{x})\,\phi_k(\mathbf{y})\,d\mu$$

$$= \lim_{k\to\infty}\int_{\mathbb{R}^n}\psi_k(\mathbf{x})\left(\int_{\mathbb{R}^m}\phi_k(\mathbf{y})\,d\nu_{\mathbf{x}}\right)d\alpha. \tag{18.18}$$

Since $\nu_{\mathbf{x}}$ is finite α a.e., it follows that for α a.e. $\mathbf{x}$,

$$\lim_{k\to\infty}\int_{\mathbb{R}^m}\phi_k(\mathbf{y})\,d\nu_{\mathbf{x}} = \int_{\mathbb{R}^m}\mathcal{X}_{R_2}(\mathbf{y})\,d\nu_{\mathbf{x}}.$$

Since the ϕ_k are uniformly bounded, 18.15 implies the existence of a dominating function for the integrand in 18.18. Therefore, one can take the limit inside the integrals and obtain

$$\int_{\mathbb{R}^n\times\mathbb{R}^m}\mathcal{X}_{R_1}(\mathbf{x})\,\mathcal{X}_{R_2}(\mathbf{y})\,d\mu = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\mathcal{X}_{R_1}(\mathbf{x})\,\mathcal{X}_{R_2}(\mathbf{y})\,d\nu_{\mathbf{x}}d\alpha$$

Every open set, V in $\mathbb{R}^{n+m}$ is a countable disjoint union of such half open rectangles and so the monotone convergence theorem implies for all V open in $\mathbb{R}^{n+m}$,

$$\int_{\mathbb{R}^n\times\mathbb{R}^m}\mathcal{X}_V d\mu = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\mathcal{X}_V d\nu_{\mathbf{x}}d\alpha. \tag{18.19}$$

Since every compact set is the countable intersection of open sets, the above formula holds for V replaced with K where K is compact. Then it follows from the dominated convergence and monotone convergence theorems that whenever H is either a G_δ (countable intersection of open sets) or a F_σ (countable union of closed sets)

$$\int_{\mathbb{R}^n\times\mathbb{R}^m}\mathcal{X}_H d\mu = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\mathcal{X}_H d\nu_{\mathbf{x}}d\alpha.$$

Now let E be μ measurable. Using the regularity of μ there exists F, G such that F is F_σ, G is $G_\delta, \mu(G\setminus F) = 0$, and $F \subseteq E \subseteq G$. Also a routine application of the dominated convergence theorem and 18.19 shows

$$\int_{\mathbb{R}^n\times\mathbb{R}^m}\mathcal{X}_{(G\setminus F)}d\mu = \int_{\mathbb{R}^n}\int_{\mathbb{R}^m}\mathcal{X}_{(G\setminus F)}d\nu_{\mathbf{x}}d\alpha$$

and $(G \setminus F)_{\mathbf{x}} \equiv \{\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in G \setminus F\}$ is $\nu_{\mathbf{x}}$ measurable for α a.e. $\mathbf{x}$, wherever $\nu_{\mathbf{x}}$ is a finite measure, and for α a.e. $\mathbf{x}$, $\nu_{\mathbf{x}} (G \setminus F)_{\mathbf{x}} = 0$. Therefore, for α a.e. $\mathbf{x}$, $E_{\mathbf{x}}$ is $\nu_{\mathbf{x}}$ measurable because $E_{\mathbf{x}} = F_{\mathbf{x}} + S_{\mathbf{x}}$ where $S_{\mathbf{x}} \subseteq (G \setminus F)_{\mathbf{x}}$, a set of $\nu_{\mathbf{x}}$ measure zero and $F_{\mathbf{x}}$ is an F_σ set which is measurable because $\nu_{\mathbf{x}}$ is a Radon measure coming as it does from a positive linear functional. Therefore,

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^m} \mathcal{X}_E d\mu &= \int_{\mathbb{R}^n \times \mathbb{R}^m} \mathcal{X}_F d\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \mathcal{X}_F d\nu_{\mathbf{x}} d\alpha \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \mathcal{X}_E d\nu_{\mathbf{x}} d\alpha. \end{aligned} \tag{18.20}$$

It follows from 18.20 that one can replace $\mathcal{X}_E$ with an arbitrary nonnegative μ measurable simple function, s. Letting f be a nonnegative μ measurable function, it follows there is an increasing sequence of nonnegative simple functions converging to f pointwise and so by the monotone convergence theorem,

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f d\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f d\nu_{\mathbf{x}} d\alpha$$

where $\mathbf{y} \to \mathbf{f}(\mathbf{x}, \mathbf{y})$ is $\nu_{\mathbf{x}}$ measurable for α a.e. $\mathbf{x}$ and $\mathbf{x} \to \int_{\mathbb{R}^m} f d\nu_{\mathbf{x}}$ is α measurable so the iterated integral makes sense.

To see $\nu_{\mathbf{x}}$ is a probability measure for a.e. $\mathbf{x}$,

$$\begin{aligned} & \frac{1}{\alpha (B(\mathbf{x}, r))} \int_{B(\mathbf{x},r)} \int_{\mathbb{R}^m} d\nu_{\mathbf{x}} d\alpha \\ = & \frac{1}{\alpha (B(\mathbf{x}, r))} \mu (B(\mathbf{x}, r) \times \mathbb{R}^m) = 1 \end{aligned}$$

and so, using the fundamental theorem of calculus it follows that upon passing to a limit as $r \to 0$, it follows that for α a.e. $\mathbf{x}$

$$\nu_{\mathbf{x}} (\mathbb{R}^m) = \int_{\mathbb{R}^m} d\nu_{\mathbf{x}} = 1$$

Due to the regularity of the measures, all sets of measure zero may be taken to be Borel. In the case of $f \in L^1(\mu)$, one applies the above to the positive and negative parts of the real and imaginary parts. This proves the theorem.

18.4 Vitali Coverings

There is another covering theorem which may also be referred to as the Besicovitch covering theorem. As before, the balls can be taken with respect to any norm on $\mathbb{R}^n$. At first, the balls will be closed but this assumption will be removed.

Definition 18.14 *A collection of balls, $\mathcal{F}$ covers a set, E in the sense of Vitali if whenever $\mathbf{x} \in E$ and $\varepsilon > 0$, there exists a ball $B \in \mathcal{F}$ whose center is $\mathbf{x}$ having diameter less than ε.*

I will give a proof of the following theorem.

Theorem 18.15 *Let μ be a Radon measure on $\mathbb{R}^n$ and let E be a set with $\overline{\mu}(E) < \infty$. Where $\overline{\mu}$ is the outer measure determined by μ. Suppose $\mathcal{F}$ is a collection of closed balls which cover E in the sense of Vitali. Then there exists a sequence of disjoint balls, $\{B_i\} \subseteq \mathcal{F}$ such that*

$$\overline{\mu}\left(E \setminus \cup_{j=1}^{\infty} B_j\right) = 0.$$

Proof: Let N_n be the constant of the Besicovitch covering theorem. Choose $r > 0$ such that

$$(1-r)^{-1}\left(1 - \frac{1}{2N_n + 2}\right) \equiv \lambda < 1.$$

If $\overline{\mu}(E) = 0$, there is nothing to prove so assume $\overline{\mu}(E) > 0$. Let U_1 be an open set containing E with $(1-r)\mu(U_1) < \overline{\mu}(E)$ and $2\overline{\mu}(E) > \mu(U_1)$, and let $\mathcal{F}_1$ be those sets of $\mathcal{F}$ which are contained in U_1 whose centers are in E. Thus $\mathcal{F}_1$ is also a Vitali cover of E. Now by the Besicovitch covering theorem proved earlier, there exist balls, B, of $\mathcal{F}_1$ such that

$$E \subseteq \cup_{i=1}^{N_n} \{B : B \in \mathcal{G}_i\}$$

where $\mathcal{G}_i$ consists of a collection of disjoint balls of $\mathcal{F}_1$. Therefore,

$$\overline{\mu}(E) \leq \sum_{i=1}^{N_n} \sum_{B \in \mathcal{G}_i} \mu(B)$$

and so, for some $i \leq N_n$,

$$(N_n + 1) \sum_{B \in \mathcal{G}_i} \mu(B) > \overline{\mu}(E).$$

It follows there exists a finite set of balls of $\mathcal{G}_i$, $\{B_1, \cdots, B_{m_1}\}$ such that

$$(N_n + 1) \sum_{i=1}^{m_1} \mu(B_i) > \overline{\mu}(E) \tag{18.21}$$

and so

$$(2N_n + 2) \sum_{i=1}^{m_1} \mu(B_i) > 2\overline{\mu}(E) > \mu(U_1).$$

Since $2\overline{\mu}(E) \geq \mu(U_1)$, 18.21 implies

$$\frac{\mu(U_1)}{2N_2 + 2} \leq \frac{2\overline{\mu}(E)}{2N_2 + 2} = \frac{\overline{\mu}(E)}{N_2 + 1} < \sum_{i=1}^{m_1} \mu(B_i).$$

Also U_1 was chosen such that $(1-r)\mu(U_1) < \overline{\mu}(E)$, and so

$$\lambda\overline{\mu}(E) \geq \lambda(1-r)\mu(U_1) = \left(1 - \frac{1}{2N_n + 2}\right)\mu(U_1)$$

$$\geq \mu(U_1) - \sum_{i=1}^{m_1} \mu(B_i) = \mu(U_1) - \mu\left(\cup_{j=1}^{m_1} B_j\right)$$
$$= \mu\left(U_1 \setminus \cup_{j=1}^{m_1} B_j\right) \geq \overline{\mu}\left(E \setminus \cup_{j=1}^{m_1} B_j\right).$$

Since the balls are closed, you can consider the sets of $\mathcal{F}$ which have empty intersection with $\cup_{j=1}^{m_1} B_j$ and this new collection of sets will be a Vitali cover of $E \setminus \cup_{j=1}^{m_1} B_j$. Letting this collection of balls play the role of $\mathcal{F}$ in the above argument and letting $E \setminus \cup_{j=1}^{m_1} B_j$ play the role of E, repeat the above argument and obtain disjoint sets of $\mathcal{F}$,

$$\{B_{m_1+1}, \cdots, B_{m_2}\},$$

such that

$$\lambda \overline{\mu}\left(E \setminus \cup_{j=1}^{m_1} B_j\right) > \overline{\mu}\left(\left(E \setminus \cup_{j=1}^{m_1} B_j\right) \setminus \cup_{j=m_1+1}^{m_2} B_j\right) = \overline{\mu}\left(E \setminus \cup_{j=1}^{m_2} B_j\right),$$

and so

$$\lambda^2 \overline{\mu}\ (E) > \overline{\mu}\ \left(E \setminus \cup_{j=1}^{m_2} B_j\right).$$

Continuing in this way, yields a sequence of disjoint balls $\{B_i\}$ contained in $\mathcal{F}$ and

$$\overline{\mu}\ \left(E \setminus \cup_{j=1}^{\infty} B_j\right) \leq \overline{\mu}\ \left(E \setminus \cup_{j=1}^{m_k} B_j\right) < \lambda^k \overline{\mu}\ (E\ )$$

for all k. Therefore, $\overline{\mu}\ \left(E \setminus \cup_{j=1}^{\infty} B_j\right) = 0$ and this proves the Theorem.

It is not necessary to assume $\overline{\mu}(E) < \infty$.

Corollary 18.16 *Let μ be a Radon measure on $\mathbb{R}^n$. Letting $\overline{\mu}$ be the outer measure determined by μ, suppose $\mathcal{F}$ is a collection of closed balls which cover E in the sense of Vitali. Then there exists a sequence of disjoint balls, $\{B_i\} \subseteq \mathcal{F}$ such that*

$$\overline{\mu}\left(E \setminus \cup_{j=1}^{\infty} B_j\right) = 0.$$

Proof: Since μ is a Radon measure it is finite on compact sets. Therefore, there are at most countably many numbers, $\{b_i\}_{i=1}^{\infty}$ such that $\mu(\partial B(\mathbf{0}, b_i)) > 0$. It follows there exists an increasing sequence of positive numbers, $\{r_i\}_{i=1}^{\infty}$ such that $\lim_{i\to\infty} r_i = \infty$ and $\mu(\partial B(\mathbf{0}, r_i)) = 0$. Now let

$$\begin{aligned} D_1 &\equiv \{\mathbf{x} : ||\mathbf{x}|| < r_1\}, D_2 \equiv \{\mathbf{x} : r_1 < ||\mathbf{x}|| < r_2\}, \\ \cdots, D_m &\equiv \{\mathbf{x} : r_{m-1} < ||\mathbf{x}|| < r_m\}, \cdots. \end{aligned}$$

Let $\mathcal{F}_m$ denote those closed balls of $\mathcal{F}$ which are contained in D_m. Then letting E_m denote $E \cap D_m$, $\mathcal{F}_m$ is a Vitali cover of $E_m, \overline{\mu}(E_m) < \infty$, and so by Theorem 18.15, there exists a countable sequence of balls from $\mathcal{F}_m$ $\left\{B_j^m\right\}_{j=1}^{\infty}$, such that $\overline{\mu}\left(E_m \setminus \cup_{j=1}^{\infty} B_j^m\right) = 0$. Then consider the countable collection of balls, $\left\{B_j^m\right\}_{j,m=1}^{\infty}$.

$$\begin{aligned} \overline{\mu}\left(E \setminus \cup_{m=1}^{\infty} \cup_{j=1}^{\infty} B_j^m\right) &\leq \overline{\mu}\left(\cup_{j=1}^{\infty} \partial B(\mathbf{0}, r_i)\right) + \\ + \sum_{m=1}^{\infty} \overline{\mu}\left(E_m \setminus \cup_{j=1}^{\infty} B_j^m\right) &= 0 \end{aligned}$$

This proves the corollary.

You don't need to assume the balls are closed. In fact, the balls can be open closed or anything in between and the same conclusion can be drawn.

Corollary 18.17 *Let μ be a Radon measure on $\mathbb{R}^n$. Letting $\overline{\mu}$ be the outer measure determined by μ, suppose $\mathcal{F}$ is a collection of balls which cover E in the sense of Vitali, open closed or neither. Then there exists a sequence of disjoint balls, $\{B_i\} \subseteq \mathcal{F}$ such that*

$$\overline{\mu}\left(E \setminus \cup_{j=1}^{\infty} B_j\right) = 0.$$

Proof: Let $\mathbf{x} \in E$. Thus $\mathbf{x}$ is the center of arbitrarily small balls from $\mathcal{F}$. Since μ is a Radon measure, at most countably many radii, r of these balls can have the property that $\mu(\partial B(\mathbf{0}, r)) = 0$. Let $\mathcal{F}'$ denote the closures of the balls of $\mathcal{F}$, $\overline{B(\mathbf{x}, r)}$ with the property that $\mu(\partial B(\mathbf{x}, r)) = 0$. Since for each $\mathbf{x} \in E$ there are only countably many exceptions, $\mathcal{F}'$ is still a Vitali cover of E. Therefore, by Corollary 18.16 there is a disjoint sequence of these balls of $\mathcal{F}'$, $\left\{\overline{B_i}\right\}_{i=1}^{\infty}$ for which

$$\overline{\mu}\left(E \setminus \cup_{j=1}^{\infty} \overline{B_j}\right) = 0$$

However, since their boundaries have μ measure zero, it follows

$$\overline{\mu}\left(E \setminus \cup_{j=1}^{\infty} B_j\right) = 0.$$

This proves the corollary.

18.5 Differentiation Of Radon Measures

This section is a generalization of earlier material in which a measure was differentiated with respect to Lebesgue measure. Here an arbitrary Radon measure will be differentiated with respect to another arbitrary Radon measure. In this section, $B(\mathbf{x}, r)$ will denote a ball with center $\mathbf{x}$ and radius r. Also, let λ and μ be Radon measures and as above, Z will denote a μ measure zero set off of which $\mu(B(\mathbf{x}, r)) > 0$ for all $r > 0$.

Definition 18.18 *For $\mathbf{x} \notin Z$, define the upper and lower symmetric derivatives as*

$$\overline{D}_{\mu}\lambda(\mathbf{x}) \equiv \limsup_{r\to 0} \frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))},\ \underline{D}_{\mu}\lambda(\mathbf{x}) \equiv \liminf_{r\to 0} \frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))}.$$

respectively. Also define

$$D_{\mu}\lambda(\mathbf{x}) \equiv \overline{D}_{\mu}\lambda(\mathbf{x}) = \underline{D}_{\mu}\lambda(\mathbf{x})$$

in the case when both the upper and lower derivatives are equal.

Lemma 18.19 *Let λ and μ be Radon measures. If A is a bounded subset of $\left\{\mathbf{x} \notin Z : \overline{D}_{\mu}\lambda(\mathbf{x}) \geq a\right\}$, then*

$$\overline{\lambda}(A) \geq a\overline{\mu}(A)$$

and if A is a bounded subset of $\left\{\mathbf{x} \notin Z : \underline{D}_{\mu}\lambda(\mathbf{x}) \leq a\right\}$, then

$$\overline{\lambda}(A) \leq a\overline{\mu}(A)$$

Proof: Suppose first that A is a bounded subset of $\left\{\mathbf{x} \notin Z : \overline{D}_{\mu}\lambda(\mathbf{x}) \geq a\right\}$, let $\varepsilon > 0$, and let V be a bounded open set with $V \supseteq A$ and $\lambda(V) - \varepsilon < \overline{\lambda}(A), \mu(V) - \varepsilon < \overline{\mu}(A)$. Then if $\mathbf{x} \in A$,

$$\frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))} > a - \varepsilon, \; B(\mathbf{x}, r) \subseteq V,$$

for infinitely many values of r which are arbitrarily small. Thus the collection of such balls constitutes a Vitali cover for A. By Corollary 18.17 there is a disjoint sequence of these balls $\{B_i\}$ such that

$$\overline{\mu}(A \setminus \cup_{i=1}^{\infty} B_i) = 0. \tag{18.22}$$

Therefore,

$$(a - \varepsilon)\sum_{i=1}^{\infty}\mu(B_i) < \sum_{i=1}^{\infty}\lambda(B_i) \leq \lambda(V) < \varepsilon + \overline{\lambda}(A)$$

and so

$$\begin{aligned} a\sum_{i=1}^{\infty}\mu(B_i) &\leq \varepsilon + \varepsilon\mu(V) + \overline{\lambda}(A) \\ &\leq \varepsilon + \varepsilon(\overline{\mu}(A) + \varepsilon) + \overline{\lambda}(A) \end{aligned} \tag{18.23}$$

Now

$$\overline{\mu}(A \setminus \cup_{i=1}^{\infty} B_i) + \overline{\mu}(\cup_{i=1}^{\infty} B_i) \geq \overline{\mu}(A)$$

and so by 18.22 and the fact the B_i are disjoint,

$$\begin{aligned} a\overline{\mu}(A) &\leq a\overline{\mu}(\cup_{i=1}^{\infty} B_i) = a\sum_{i=1}^{\infty}\mu(B_i) \\ &\leq \varepsilon + \varepsilon(\overline{\mu}(A) + \varepsilon) + \overline{\lambda}(A). \end{aligned} \tag{18.24}$$

Hence $a\overline{\mu}(A) \leq \overline{\lambda}(A)$ since $\varepsilon > 0$ was arbitrary.

Now suppose A is a bounded subset of $\left\{\mathbf{x} \notin Z : \underline{D}_{\mu}\lambda(\mathbf{x}) \leq a\right\}$ and let V be a bounded open set containing A with $\mu(V) - \varepsilon < \overline{\mu}(A)$. Then if $\mathbf{x} \in A$,

$$\frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))} < a + \varepsilon, \; B(\mathbf{x}, r) \subseteq V$$

for values of r which are arbitrarily small. Therefore, by Corollary 18.17 again, there exists a disjoint sequence of these balls, $\{B_i\}$ satisfying this time,

$$\overline{\lambda}\left(A \setminus \cup_{i=1}^{\infty} B_i\right) = 0.$$

Then by arguments similar to the above,

$$\overline{\lambda}(A) \leq \sum_{i=1}^{\infty} \lambda(B_i) < (a+\varepsilon)\,\mu(V) < (a+\varepsilon)\left(\overline{\mu}(A)+\varepsilon\right).$$

Since ε was arbitrary, this proves the lemma.

Theorem 18.20 *There exists a set of measure zero, N containing Z such that for $\mathbf{x} \notin N$, $D_\mu\lambda(\mathbf{x})$ exists and also $\mathcal{X}_{N^C}(\cdot)\,D_\mu\lambda(\cdot)$ is a μ measurable function. Furthermore, $D_\mu\lambda(\mathbf{x}) < \infty$ μ a.e.*

Proof: First I show $D_\mu\lambda(\mathbf{x})$ exists a.e. Let $0 \leq a < b < \infty$ and let A be any bounded subset of

$$N(a,b) \equiv \left\{\mathbf{x} \notin Z : \overline{D}_\mu\lambda(\mathbf{x}) > b > a > \underline{D}_\mu\lambda(\mathbf{x})\right\}.$$

By Lemma 18.19,

$$a\overline{\mu}(A) \geq \overline{\lambda}(A) \geq b\overline{\mu}(A)$$

and so $\mu(A) = 0$ and A is μ measurable. It follows $\mu(N(a,b)) = 0$ because

$$\mu(N(a,b)) \leq \sum_{m=1}^{\infty} \mu\left(N(a,b) \cap B(\mathbf{0},m)\right) = 0.$$

Define

$$N_0 \equiv \left\{\mathbf{x} \notin Z : \overline{D}_\mu\lambda(\mathbf{x}) > \underline{D}_\mu\lambda(\mathbf{x})\right\}.$$

Thus $\mu(N_0) = 0$ because

$$N_0 \subseteq \cup\left\{N(a,b) : 0 \leq a < b, \text{ and } a,b \in \mathbb{Q}\right\}$$

Therefore, N_0 is also μ measurable and has μ measure zero. Letting $N \equiv N_0 \cup Z$, it follows $D_\mu\lambda(\mathbf{x})$ exists on N^C. It remains to verify $\mathcal{X}_{N^C}(\cdot)\,D_\mu\lambda(\cdot)$ is finite a.e. and is μ measurable.

Let

$$I = \{\mathbf{x} : D_\mu\lambda(\mathbf{x}) = \infty\}.$$

Then by Lemma 18.19

$$\overline{\lambda}\left(I \cap B(\mathbf{0},m)\right) \geq a\overline{\mu}\left(I \cap B(\mathbf{0},m)\right)$$

for all a and since λ is finite on bounded sets, the above implies $\overline{\mu}\left(I \cap B(\mathbf{0},m)\right) = 0$ for each m which implies that I is μ measurable and has μ measure zero since

$$I = \cup_{m=1}^{\infty} I_m.$$

Letting η be an arbitrary Radon measure, let $r > 0$, and suppose $\eta(\partial B(\mathbf{x}, r)) = 0$. (Since η is finite on every ball, there are only countably many r such that $\eta(\partial B(\mathbf{x}, r)) > 0$.) and let V be an open set containing $\overline{B(\mathbf{x}, r)}$. Then whenever $\mathbf{y}$ is close enough to $\mathbf{x}$, it follows that $\overline{B(\mathbf{y}, r)}$ is also a subset of V. Since V is an arbitrary open set containing $\overline{B(\mathbf{x}, r)}$, it follows

$$\eta(B(\mathbf{x}, r)) = \eta\left(\overline{B(\mathbf{x}, r)}\right) \geq \limsup_{\mathbf{y} \to \mathbf{x}} \eta(B(\mathbf{y}, r))$$

and so $\mathbf{y} \to \eta(B(\mathbf{y}, r))$ an upper semicontinuous real valued function of $\mathbf{x}$, one which satisfies

$$f(\mathbf{x}) \geq \limsup_{n \to \infty} f(\mathbf{x}_n)$$

whenever $\mathbf{x}_n \to \mathbf{x}$. Now it is routine to verify that a function f is upper semicontinuous if and only if $f^{-1}([-\infty, a))$ is open for all $a \in \mathbb{R}$. Therefore, $f^{-1}([-\infty, a))$ is a Borel set for all $a \in \mathbb{R}$ and so f is Borel measurable by Lemma 8.6. Now the measurability of $\mathcal{X}_{N^C}(\cdot) D_\mu \lambda(\cdot)$ follows from

$$\mathcal{X}_{N^C}(\mathbf{x}) D_\mu \lambda(\mathbf{x}) = \lim_{r_i \to 0} \frac{\lambda(B(\mathbf{x}, r_i))}{\mu(B(\mathbf{x}, r_i))} \mathcal{X}_{N^C}(\mathbf{x})$$

where r_i is such that $\partial B(\mathbf{x}, r_i)$ has μ and λ measure zero.

18.6 The Radon Nikodym Theorem For Radon Measures

The above theory can be used to give an alternate treatment of the Radon Nikodym theorem which exhibits the Radon Nikodym derivative as a specific limit.

Theorem 18.21 *Let λ and μ be Radon measures and suppose $\lambda \ll \mu$. Then for all E a μ measurable set,*

$$\lambda(E) = \int_E (D_\mu \lambda) \, d\mu.$$

Proof: Let $t > 1$ and let E be a μ measurable set which is bounded and a subset of N^C where N is the exceptional set of μ measure zero in Theorem 18.20 off of which $\mu(B(\mathbf{x},r)) > 0$ for all $r > 0$ and $D_\mu \lambda(\mathbf{x})$ exists. Consider

$$E_m \equiv E \cap \left\{\mathbf{x} \in N^C : t^m \leq D_\mu \lambda(\mathbf{x}) < t^{m+1}\right\}$$

for $m \in \mathbb{Z}$, the integers. First note that

$$E \cap \left\{\mathbf{x} \in N^C : \ D_\mu \lambda(\mathbf{x}) = 0 \ \right\}$$

has λ measure zero because by Lemma 18.19,

$$\lambda\left(E \cap \left\{\mathbf{x} \in N^C : \ D_\mu \lambda(\mathbf{x}) = 0 \ \right\}\right) \leq a\mu(E \ )$$

for all $a > 0$ and $\mu(E\ )$ is finite due to the assumption that E is bounded and μ is a Radon measure. Therefore, by Lemma 18.19,

$$\lambda(E) = \sum_{m\in\mathbb{Z}} \lambda(E_m) \leq \sum_{m\in\mathbb{Z}} t^{m+1}\mu(E_m) = t\sum_{m\in\mathbb{Z}} t^m \mu(E_m)$$

$$\leq t\sum_{m\in\mathbb{Z}} \int_{E_m} D_\mu\lambda(\mathbf{x})\,d\mu = t\int_E D_\mu\lambda(\mathbf{x})\,d\mu.$$

Also by this same lemma,

$$\lambda(E) = \sum_{m\in\mathbb{Z}} \lambda(E_m) \geq \sum_{m\in\mathbb{Z}} t^m\mu(E_m) = t^{-1}\sum_{m\in\mathbb{Z}} t^{m+1}\mu(E_m)$$

$$\geq t^{-1}\sum_{m\in\mathbb{Z}} \int_{E_m} D_\mu\lambda(\mathbf{x})\,d\mu = t^{-1}\int_E D_\mu\lambda(\mathbf{x})\,d\mu.$$

Thus,

$$t\int_E D_\mu\lambda(\mathbf{x})\,d\mu \geq \lambda(E) \geq t^{-1}\int_E D_\mu\lambda(\mathbf{x})\,d\mu$$

and letting $t \to 1$, it follows

$$\lambda(E) = \int_E D_\mu\lambda(\mathbf{x})\,d\mu. \tag{18.25}$$

Now if E is an arbitrary measurable set, contained in N^C, this formula holds with E replaced with $E \cap B(\mathbf{0},k)$. Letting $k \to \infty$ and using the monotone convergence theorem, the above formula holds for all $E \subseteq N^C$. Since N is a set of μ measure zero, it follows N is also a set of λ measure zero due to the assumption of absolute continuity. Therefore 18.25 continues to hold for arbitrary μ measurable sets, E. This proves the theorem.

Fourier Transforms

19.1 An Algebra Of Special Functions

First recall the following definition of a polynomial.

Definition 19.1 *$\alpha = (\alpha_1, \cdots, \alpha_n)$ for $\alpha_1 \cdots \alpha_n$ positive integers is called a multi-index. For α a multi-index, $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n$ and if $\mathbf{x} \in \mathbb{R}^n$,*

$$\mathbf{x} = (x_1, \cdots, x_n),$$

and f a function, define

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

A polynomial in n variables of degree m is a function of the form

$$p(\mathbf{x}) = \sum_{|\alpha| \leq m} a_\alpha \mathbf{x}^\alpha.$$

Here α is a multi-index as just described and $a_\alpha \in \mathbb{C}$. Also define for $\alpha = (\alpha_1, \cdots, \alpha_n)$ a multi-index

$$D^\alpha f(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Definition 19.2 *Define $\mathcal{G}_1$ to be the functions of the form $p(\mathbf{x}) e^{-a|\mathbf{x}|^2}$ where $a > 0$ and $p(\mathbf{x})$ is a polynomial. Let $\mathcal{G}$ be all finite sums of functions in $\mathcal{G}_1$. Thus $\mathcal{G}$ is an algebra of functions which has the property that if $f \in \mathcal{G}$ then $\overline{f} \in \mathcal{G}$.*

It is always assumed, unless stated otherwise that the measure will be Lebesgue measure.

Lemma 19.3 *$\mathcal{G}$ is dense in $C_0(\mathbb{R}^n)$ with respect to the norm,*

$$||f||_\infty \equiv \sup \{|f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\}$$

Proof: By the Weierstrass approximation theorem, it suffices to show $\mathcal{G}$ separates the points and annihilates no point. It was already observed in the above definition that $\overline{f} \in \mathcal{G}$ whenever $f \in \mathcal{G}$. If $\mathbf{y}_1 \neq \mathbf{y}_2$ suppose first that $|\mathbf{y}_1| \neq |\mathbf{y}_2|$. Then in this case, you can let $f(\mathbf{x}) \equiv e^{-|\mathbf{x}|^2}$ and $f \in \mathcal{G}$ and $f(\mathbf{y}_1) \neq f(\mathbf{y}_2)$. If $|\mathbf{y}_1| = |\mathbf{y}_2|$, then suppose $y_{1k} \neq y_{2k}$. This must happen for some k because $\mathbf{y}_1 \neq \mathbf{y}_2$. Then let $f(\mathbf{x}) \equiv x_k e^{-|\mathbf{x}|^2}$. Thus $\mathcal{G}$ separates points. Now $e^{-|\mathbf{x}|^2}$ is never equal to zero and so $\mathcal{G}$ annihilates no point of $\mathbb{R}^n$. This proves the lemma.

These functions are clearly quite specialized. Therefore, the following theorem is somewhat surprising.

Theorem 19.4 *For each $p \geq 1, p < \infty, \mathcal{G}$ is dense in $L^p(\mathbb{R}^n)$.*

Proof: Let $f \in L^p(\mathbb{R}^n)$. Then there exists $g \in C_c(\mathbb{R}^n)$ such that $||f - g||_p < \varepsilon$. Now let $b > 0$ be large enough that

$$\int_{\mathbb{R}^n} \left(e^{-b|\mathbf{x}|^2}\right)^p dx < \varepsilon^p.$$

Then $\mathbf{x} \to g(\mathbf{x}) e^{b|\mathbf{x}|^2}$ is in $C_c(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n)$. Therefore, from Lemma 19.3 there exists $\psi \in \mathcal{G}$ such that

$$\left|\left|ge^{b|\cdot|^2} - \psi\right|\right|_\infty < 1$$

Therefore, letting $\phi(\mathbf{x}) \equiv e^{-b|\mathbf{x}|^2}\psi(\mathbf{x})$ it follows that $\phi \in \mathcal{G}$ and for all $\mathbf{x} \in \mathbb{R}^n$,

$$|g(\mathbf{x}) - \phi(\mathbf{x})| < e^{-b|\mathbf{x}|^2}$$

Therefore,

$$\left(\int_{\mathbb{R}^n} |g(\mathbf{x}) - \phi(\mathbf{x})|^p dx\right)^{1/p} \leq \left(\int_{\mathbb{R}^n} \left(e^{-b|\mathbf{x}|^2}\right)^p dx\right)^{1/p} < \varepsilon.$$

It follows

$$||f - \phi||_p \leq ||f - g||_p + ||g - \phi||_p < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves the theorem.

The following lemma is also interesting even if it is obvious.

Lemma 19.5 *For $\psi \in \mathcal{G}$, p a polynomial, and α, β multiindices, $D^\alpha\psi \in \mathcal{G}$ and $p\psi \in \mathcal{G}$. Also*

$$\sup\{|\mathbf{x}^\beta D^\alpha \psi(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\} < \infty$$

19.2 Fourier Transforms Of Functions In $\mathcal{G}$

Definition 19.6 *For $\psi \in \mathcal{G}$ Define the Fourier transform, F and the inverse Fourier transform, F^{-1} by*

$$F\psi(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}}\psi(\mathbf{x})dx,$$

$$F^{-1}\psi(\mathbf{t}) \equiv (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}\psi(\mathbf{x})dx.$$

where $\mathbf{t}\cdot\mathbf{x} \equiv \sum_{i=1}^{n} t_i x_i$*.Note there is no problem with this definition because* ψ *is in* $L^1(\mathbb{R}^n)$ *and therefore,*

$$\left|e^{i\mathbf{t}\cdot\mathbf{x}}\psi(\mathbf{x})\right| \leq |\psi(\mathbf{x})|,$$

an integrable function.

One reason for using the functions, $\mathcal{G}$ is that it is very easy to compute the Fourier transform of these functions. The first thing to do is to verify F and F^{-1} map $\mathcal{G}$ to $\mathcal{G}$ and that $F^{-1}\circ F(\psi) = \psi$.

Lemma 19.7 *The following formulas are true*

$$\int_{\mathbb{R}} e^{-c(x+it)^2}dx = \int_{\mathbb{R}} e^{-c(x-it)^2}dx = \frac{\sqrt{\pi}}{\sqrt{c}}, \tag{19.1}$$

$$\int_{\mathbb{R}^n} e^{-c(\mathbf{x}+i\mathbf{t})\cdot(\mathbf{x}+i\mathbf{t})}dx = \int_{\mathbb{R}^n} e^{-c(\mathbf{x}-i\mathbf{t})\cdot(\mathbf{x}-i\mathbf{t})}dx = \left(\frac{\sqrt{\pi}}{\sqrt{c}}\right)^n, \tag{19.2}$$

$$\int_{\mathbb{R}} e^{-ct^2}e^{-ist}dt = \int_{\mathbb{R}} e^{-ct^2}e^{ist}dt = e^{-\frac{s^2}{4c}}\frac{\sqrt{\pi}}{\sqrt{c}}, \tag{19.3}$$

$$\int_{\mathbb{R}^n} e^{-c|\mathbf{t}|^2}e^{-i\mathbf{s}\cdot\mathbf{t}}dt = \int_{\mathbb{R}^n} e^{-c|\mathbf{t}|^2}e^{i\mathbf{s}\cdot\mathbf{t}}dt = e^{-\frac{|\mathbf{s}|^2}{4c}}\left(\frac{\sqrt{\pi}}{\sqrt{c}}\right)^n. \tag{19.4}$$

Proof: Consider the first one. Simple manipulations yield

$$H(t) \equiv \int_{\mathbb{R}} e^{-c(x+it)^2}dx = e^{ct^2}\int_{\mathbb{R}} e^{-cx^2}\cos(2cxt)\,dx.$$

Now using the dominated convergence theorem to justify passing derivatives inside the integral where necessary and using integration by parts,

$$\begin{aligned} H'(t) &= 2cte^{ct^2}\int_{\mathbb{R}} e^{-cx^2}\cos(2cxt)\,dx - e^{ct^2}\int_{\mathbb{R}} e^{-cx^2}\sin(2cxt)\,2xcdx \\ &= 2ctH(t) - e^{ct^2}2ct\int_{\mathbb{R}} e^{-cx^2}\cos(2cxt)\,dx = 2ct(H(t)-H(t)) = 0 \end{aligned}$$

and so $H(t) = H(0) = \int_{\mathbb{R}} e^{-cx^2}dx \equiv I$. Thus

$$I^2 = \int_{\mathbb{R}^2} e^{-c\left(x^2+y^2\right)}dxdy = \int_0^\infty\int_0^{2\pi} e^{-cr^2}rd\theta dr = \frac{\pi}{c}.$$

Therefore, $I = \sqrt{\pi}/\sqrt{c}$. Since the sign of t is unimportant, this proves 19.1. This also proves 19.2 after writing as iterated integrals.

Consider 19.3.

$$\begin{aligned}\int_{\mathbb{R}} e^{-ct^2} e^{ist} dt &= \int_{\mathbb{R}} e^{-c\left(t^2 - \frac{ist}{c} + \left(\frac{is}{2c}\right)^2\right)} e^{-\frac{s^2}{4c}} dt \\ &= e^{-\frac{s^2}{4c}} \int_{\mathbb{R}} e^{-c\left(t - \frac{is}{2c}\right)^2} dt = e^{-\frac{s^2}{4c}} \frac{\sqrt{\pi}}{\sqrt{c}}.\end{aligned}$$

Changing the variable $t \to -t$ gives the other part of 19.3.

Finally 19.4 follows from using iterated integrals.

With these formulas, it is easy to verify F, F^{-1} map $\mathcal{G}$ to $\mathcal{G}$ and $F \circ F^{-1} = F^{-1} \circ F = id$.

Theorem 19.8 *Each of F and F^{-1} map $\mathcal{G}$ to $\mathcal{G}$. Also $F^{-1} \circ F(\psi) = \psi$ and $F \circ F^{-1}(\psi) = \psi$.*

Proof: The first claim will be shown if it is shown that $F\psi \in \mathcal{G}$ for $\psi(\mathbf{x}) = \mathbf{x}^\alpha e^{-b|\mathbf{x}|^2}$ because an arbitrary function of $\mathcal{G}$ is a finite sum of scalar multiples of functions such as ψ. Using Lemma 19.7,

$$\begin{aligned}F\psi(\mathbf{t}) &\equiv \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} \mathbf{x}^\alpha e^{-b|\mathbf{x}|^2} dx \\ &= \left(\frac{1}{2\pi}\right)^{n/2} (i)^{-|\alpha|} D_t^\alpha \left(\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} e^{-b|\mathbf{x}|^2} dx\right) \\ &= \left(\frac{1}{2\pi}\right)^{n/2} (i)^{-|\alpha|} D_t^\alpha \left(e^{-\frac{|\mathbf{t}|^2}{2b}} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n\right)\end{aligned}$$

and this is clearly in $\mathcal{G}$ because it equals a polynomial times $e^{-\frac{|\mathbf{t}|^2}{2b}}$. It remains to verify the other assertion. As in the first case, it suffices to consider $\psi(\mathbf{x}) = \mathbf{x}^\alpha e^{-b|\mathbf{x}|^2}$. Using Lemma 19.7 and ordinary integration by parts on the iterated integrals, $\int_{\mathbb{R}^n} e^{-c|\mathbf{t}|^2} e^{i\mathbf{s}\cdot\mathbf{t}} dt = e^{-\frac{|\mathbf{s}|^2}{2c}} \left(\frac{\sqrt{\pi}}{\sqrt{c}}\right)^n$,

$$\begin{aligned}&F^{-1} \circ F(\psi)(\mathbf{s}) \\ \equiv\ & \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{t}} \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} \mathbf{x}^\alpha e^{-b|\mathbf{x}|^2} dx dt \\ =\ & \left(\frac{1}{2\pi}\right)^{n} \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{t}} (-i)^{-|\alpha|} D_t^\alpha \left(\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} e^{-b|\mathbf{x}|^2} dx dt\right) \\ =\ & \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{t}} \left(\frac{1}{2\pi}\right)^{n/2} (-i)^{-|\alpha|} D_t^\alpha \left(e^{-\frac{|\mathbf{t}|^2}{4b}} \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n\right) dt\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2\pi}\right)^n \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n (-i)^{-|\alpha|} \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{t}} D_t^\alpha \left(e^{-\frac{|\mathbf{t}|^2}{4b}}\right) dt \\
&= \left(\frac{1}{2\pi}\right)^n \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n (-i)^{-|\alpha|} (-1)^{|\alpha|} \mathbf{s}^\alpha (i)^{|\alpha|} \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{t}} e^{-\frac{|\mathbf{t}|^2}{4b}} dt \\
&= \left(\frac{1}{2\pi}\right)^n \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n \mathbf{s}^\alpha \int_{\mathbb{R}^n} e^{i\mathbf{s}\cdot\mathbf{t}} e^{-\frac{|\mathbf{t}|^2}{4b}} dt \\
&= \left(\frac{1}{2\pi}\right)^n \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n \mathbf{s}^\alpha e^{-\frac{|\mathbf{s}|^2}{4(1/(4b))}} \left(\frac{\sqrt{\pi}}{\sqrt{1/(4b)}}\right)^n \\
&= \left(\frac{1}{2\pi}\right)^n \left(\frac{\sqrt{\pi}}{\sqrt{b}}\right)^n \mathbf{s}^\alpha e^{-b|\mathbf{s}|^2} \left(\sqrt{\pi} 2\sqrt{b}\right)^n = \mathbf{s}^\alpha e^{-b|\mathbf{s}|^2} = \psi(\mathbf{s}).
\end{aligned}
$$

This little computation proves the theorem. The other case is entirely similar.

19.3 Fourier Transforms Of Just About Anything

Definition 19.9 *Let $\mathcal{G}^*$ denote the vector space of linear functions defined on $\mathcal{G}$ which have values in $\mathbb{C}$. Thus $T \in \mathcal{G}^*$ means $T : \mathcal{G} \to \mathbb{C}$ and T is linear,*

$$T(a\psi + b\phi) = aT(\psi) + bT(\phi) \text{ for all } a, b \in \mathbb{C}, \quad \psi, \phi \in \mathcal{G}$$

Let $\psi \in \mathcal{G}$. Then define $T_\psi \in \mathcal{G}^*$ by

$$T_\psi(\phi) \equiv \int_{\mathbb{R}^n} \psi(\mathbf{x}) \phi(\mathbf{x}) \, dx$$

Lemma 19.10 *The following is obtained for all $\phi, \psi \in \mathcal{G}$.*

$$T_{F\psi}(\phi) = T_\psi(F\phi), \ T_{F^{-1}\psi}(\phi) = T_\psi\left(F^{-1}\phi\right)$$

Also if $\psi \in \mathcal{G}$ and $T_\psi = 0$, then $\psi = 0$.

Proof:

$$
\begin{aligned}
T_{F\psi}(\phi) &\equiv \int_{\mathbb{R}^n} F\psi(\mathbf{t}) \phi(\mathbf{t}) \, dt \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} \psi(\mathbf{x}) dx \phi(\mathbf{t}) \, dt \\
&= \int_{\mathbb{R}^n} \psi(\mathbf{x}) \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} \phi(\mathbf{t}) \, dt dx \\
&= \int_{\mathbb{R}^n} \psi(\mathbf{x}) F\phi(\mathbf{x}) \, dx \equiv T_\psi(F\phi)
\end{aligned}
$$

The other claim is similar.

Suppose now $T_\psi = 0$. Then

$$\int_{\mathbb{R}^n} \psi\phi dx = 0$$

for all $\phi \in \mathcal{G}$. Therefore, this is true for $\phi = \psi$ and so $\psi = 0$. This proves the lemma.

From now on regard $\mathcal{G} \subseteq \mathcal{G}^*$ and for $\psi \in \mathcal{G}$ write $\psi(\phi)$ instead of $T_\psi(\phi)$. It was just shown that with this interpretation[1],

$$F\psi(\phi) = \psi(F(\phi)),\ F^{-1}\psi(\phi) = \psi\left(F^{-1}\phi\right).$$

This lemma suggests a way to define the Fourier transform of something in $\mathcal{G}^*$.

Definition 19.11 *For $T \in \mathcal{G}^*$, define $FT, F^{-1}T \in \mathcal{G}^*$ by*

$$FT(\phi) \equiv T(F\phi),\ F^{-1}T(\phi) \equiv T\left(F^{-1}\phi\right)$$

Lemma 19.12 *F and F^{-1} are both one to one, onto, and are inverses of each other.*

Proof: First note F and F^{-1} are both linear. This follows directly from the definition. Suppose now $FT = 0$. Then $FT(\phi) = T(F\phi) = 0$ for all $\phi \in \mathcal{G}$. But F and F^{-1} map $\mathcal{G}$ onto $\mathcal{G}$ because if $\psi \in \mathcal{G}$, then $\psi = F\left(F^{-1}(\psi)\right)$. Therefore, $T = 0$ and so F is one to one. Similarly F^{-1} is one to one. Now

$$F^{-1}(FT)(\phi) \equiv (FT)\left(F^{-1}\phi\right) \equiv T\left(F\left(F^{-1}(\phi)\right)\right) = T\phi.$$

Therefore, $F^{-1} \circ F(T) = T$. Similarly, $F \circ F^{-1}(T) = T$. Thus both F and F^{-1} are one to one and onto and are inverses of each other as suggested by the notation. This proves the lemma.

Probably the most interesting things in $\mathcal{G}^*$ are functions of various kinds. The following lemma has to do with this situation.

Lemma 19.13 *If $f \in L^1_{loc}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f\phi dx = 0$ for all $\phi \in C_c(\mathbb{R}^n)$, then $f = 0$ a.e.*

Proof: First suppose $f \geq 0$. Let

$$E \equiv \{\mathbf{x} : f(\mathbf{x}) \geq r\},\ E_R \equiv E \cap B(\mathbf{0},R).$$

Let K_m be an increasing sequence of compact sets and let V_m be a decreasing sequence of open sets satisfying

$$K_m \subseteq E_R \subseteq V_m,\ m_n(V_m) \leq m_n(K_m) + 2^{-m}, V_1 \subseteq B(\mathbf{0},R).$$

[1]This is not all that different from what was done with the derivative. Remember when you consider the derivative of a function of one variable, in elementary courses you think of it as a number but thinking of it as a linear transformation acting on $\mathbb{R}$ is better because this leads to the concept of a derivative which generalizes to functions of many variables. So it is here. You can think of $\psi \in \mathcal{G}$ as simply an element of $\mathcal{G}$ but it is better to think of it as an element of $\mathcal{G}^*$ as just described.

Therefore,

$$m_n(V_m \setminus K_m) \leq 2^{-m}.$$

Let

$$\phi_m \in C_c(V_m),\ K_m \prec \phi_m \prec V_m.$$

Then $\phi_m(\mathbf{x}) \to \mathcal{X}_{E_R}(\mathbf{x})$ a.e. because the set where $\phi_m(\mathbf{x})$ fails to converge to this set is contained in the set of all $\mathbf{x}$ which are in infinitely many of the sets $V_m \setminus K_m$. This set has measure zero because

$$\sum_{m=1}^{\infty} m_n(V_m \setminus K_m) < \infty$$

and so, by the dominated convergence theorem,

$$0 = \lim_{m\to\infty} \int_{\mathbb{R}^n} f\phi_m dx = \lim_{m\to\infty} \int_{V_1} f\phi_m dx = \int_{E_R} f dx \geq rm(E_R).$$

Thus, $m_n(E_R) = 0$ and therefore $m_n(E) = \lim_{R\to\infty} m_n(E_R) = 0$. Since $r > 0$ is arbitrary, it follows

$$m_n([f > 0]) = \cup_{k=1}^{\infty} m_n\left(\left[f > k^{-1}\right]\right) = 0.$$

Now suppose f has values in $\mathbb{R}$. Let $E_+ = [f \geq 0]$ and $E_- = [f < 0]$. Thus these are two measurable sets. As in the first part, let K_m and V_m be sequences of compact and open sets such that $K_m \subseteq E_+ \cap B(\mathbf{0}, R) \subseteq V_m \subseteq B(\mathbf{0}, R)$ and let $K_m \prec \phi_m \prec V_m$ with $m_n(V_m \setminus K_m) < 2^{-m}$. Thus $\phi_m \in C_c(\mathbb{R}^n)$ and the sequence converges pointwise to $\mathcal{X}_{E_+\cap B(\mathbf{0},R)}$. Then by the dominated convergence theorem, if ψ is any function in $C_c(\mathbb{R}^n)$

$$0 = \int f\phi_m\psi dm_n \to \int f\psi\mathcal{X}_{E_+\cap B(\mathbf{0},R)} dm_n.$$

Hence, letting $R \to \infty$,

$$\int f\psi\mathcal{X}_{E_+} dm_n = \int f_+\psi dm_n = 0$$

Since ψ is arbitrary, the first part of the argument applies to f_+ and implies $f_+ = 0$. Similarly $f_- = 0$. Finally, if f is complcx valued, the assumptions mean

$$\int \operatorname{Re}(f)\phi dm_n = 0,\ \int \operatorname{Im}(f)\phi dm_n = 0$$

for all $\phi \in C_c(\mathbb{R}^n)$ and so both $\operatorname{Re}(f), \operatorname{Im}(f)$ equal zero a.e. This proves the lemma.

Corollary 19.14 *Let $f \in L^1(\mathbb{R}^n)$ and suppose*

$$\int_{\mathbb{R}^n} f(\mathbf{x})\phi(\mathbf{x})dx = 0$$

for all $\phi \in \mathcal{G}$. Then $f = 0$ a.e.

Proof: Let $\psi \in C_c(\mathbb{R}^n)$. Then by the Stone Weierstrass approximation theorem, there exists a sequence of functions, $\{\phi_k\} \subseteq \mathcal{G}$ such that $\phi_k \to \psi$ uniformly. Then by the dominated convergence theorem,

$$\int f\psi dx = \lim_{k\to\infty} \int f\phi_k dx = 0.$$

By Lemma 19.13 $f = 0$.

The next theorem is the main result of this sort.

Theorem 19.15 *Let $f \in L^p(\mathbb{R}^n), p \geq 1$, or suppose f is measurable and has polynomial growth,*

$$|f(\mathbf{x})| \leq K\left(1 + |\mathbf{x}|^2\right)^m$$

for some $m \in \mathbb{N}$. Then if

$$\int f\psi dx = 0$$

for all $\psi \in \mathcal{G}$ then it follows $f = 0$.

Proof: The case where $f \in L^1(\mathbb{R}^n)$ was dealt with in Corollary 19.14. Suppose $f \in L^p(\mathbb{R}^n)$ for $p > 1$. Then by Holder's inequality and the density of $\mathcal{G}$ in $L^{p'}(\mathbb{R}^n)$, it follows that $\int fgdx = 0$ for all $g \in L^{p'}(\mathbb{R}^n)$. By the Riesz representation theorem, $f = 0$.

It remains to consider the case where f has polynomial growth. Thus $\mathbf{x} \to f(\mathbf{x}) e^{-|\mathbf{x}|^2} \in L^1(\mathbb{R}^n)$. Therefore, for all $\psi \in \mathcal{G}$,

$$0 = \int f(\mathbf{x}) e^{-|\mathbf{x}|^2} \psi(\mathbf{x}) dx$$

because $e^{-|\mathbf{x}|^2}\psi(\mathbf{x}) \in \mathcal{G}$. Therefore, by the first part, $f(\mathbf{x}) e^{-|\mathbf{x}|^2} = 0$ a.e.

The following theorem shows that you can consider most functions you are likely to encounter as elements of $\mathcal{G}^*$.

Theorem 19.16 *Let f be a measurable function with polynomial growth,*

$$|f(\mathbf{x})| \leq C\left(1 + |\mathbf{x}|^2\right)^N \quad \textit{for some } N,$$

or let $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty]$. Then $f \in \mathcal{G}^*$ if

$$f(\phi) \equiv \int f\phi dx.$$

Proof: Let f have polynomial growth first. Then the above integral is clearly well defined and so in this case, $f \in \mathcal{G}^*$.

Next suppose $f \in L^p(\mathbb{R}^n)$ with $\infty > p \geq 1$. Then it is clear again that the above integral is well defined because of the fact that ϕ is a sum of polynomials times exponentials of the form $e^{-c|\mathbf{x}|^2}$ and these are in $L^{p'}(\mathbb{R}^n)$. Also $\phi \to f(\phi)$ is clearly linear in both cases. This proves the theorem.

This has shown that for nearly any reasonable function, you can define its Fourier transform as described above. Also you should note that $\mathcal{G}^*$ includes $C_0(\mathbb{R}^n)'$, the space of complex measures whose total variation are Radon measures. It is especially interesting when the Fourier transform yields another function of some sort.

19.3.1 Fourier Transforms Of Functions In $L^1(\mathbb{R}^n)$

First suppose $f \in L^1(\mathbb{R}^n)$.

Theorem 19.17 *Let $f \in L^1(\mathbb{R}^n)$. Then $Ff(\phi) = \int_{\mathbb{R}^n} g\phi dt$ where*

$$g(\mathbf{t}) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x})\, dx$$

and $F^{-1}f(\phi) = \int_{\mathbb{R}^n} g\phi dt$ where $g(\mathbf{t}) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x})\, dx$. In short,

$$Ff(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x}) dx,$$

$$F^{-1}f(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x}) dx.$$

Proof: From the definition and Fubini's theorem,

$$\begin{aligned} Ff(\phi) &\equiv \int_{\mathbb{R}^n} f(\mathbf{t})\, F\phi(\mathbf{t})\, dt = \int_{\mathbb{R}^n} f(\mathbf{t}) \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} \phi(\mathbf{x})\, dxdt \\ &= \int_{\mathbb{R}^n} \left(\left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\mathbf{t})\, e^{-i\mathbf{t}\cdot\mathbf{x}} dt \right) \phi(\mathbf{x})\, dx. \end{aligned}$$

Since $\phi \in \mathcal{G}$ is arbitrary, it follows from Theorem 19.15 that $Ff(\mathbf{x})$ is given by the claimed formula. The case of F^{-1} is identical.

Here are interesting properties of these Fourier transforms of functions in L^1.

Theorem 19.18 *If $f \in L^1(\mathbb{R}^n)$ and $||f_k - f||_1 \to 0$, then Ff_k and $F^{-1}f_k$ converge uniformly to Ff and $F^{-1}f$ respectively. If $f \in L^1(\mathbb{R}^n)$, then $F^{-1}f$ and Ff are both continuous and bounded. Also,*

$$\lim_{|\mathbf{x}|\to\infty} F^{-1}f(\mathbf{x}) = \lim_{|\mathbf{x}|\to\infty} Ff(\mathbf{x}) = 0. \tag{19.5}$$

Furthermore, for $f \in L^1(\mathbb{R}^n)$ both Ff and $F^{-1}f$ are uniformly continuous.

Proof: The first claim follows from the following inequality.

$$\begin{aligned}
|Ff_k(\mathbf{t}) - Ff(\mathbf{t})| &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left|e^{-i\mathbf{t}\cdot\mathbf{x}} f_k(\mathbf{x}) - e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x})\right| dx \\
&= (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f_k(\mathbf{x}) - f(\mathbf{x})| dx \\
&= (2\pi)^{-n/2} ||f - f_k||_1 .
\end{aligned}$$

which a similar argument holding for F^{-1}.

Now consider the second claim of the theorem.

$$|Ff(\mathbf{t}) - Ff(\mathbf{t}')| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} \left|e^{-i\mathbf{t}\cdot\mathbf{x}} - e^{-i\mathbf{t}'\cdot\mathbf{x}}\right| |f(\mathbf{x})| dx$$

The integrand is bounded by $2|f(\mathbf{x})|$, a function in $L^1(\mathbb{R}^n)$ and converges to 0 as $\mathbf{t}' \to \mathbf{t}$ and so the dominated convergence theorem implies Ff is continuous. To see $Ff(\mathbf{t})$ is uniformly bounded,

$$|Ff(\mathbf{t})| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(\mathbf{x})| dx < \infty.$$

A similar argument gives the same conclusions for F^{-1}.

It remains to verify 19.5 and the claim that Ff and $F^{-1}f$ are uniformly continuous.

$$|Ff(\mathbf{t})| \leq \left|(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x}) dx\right|$$

Now let $\varepsilon > 0$ be given and let $g \in C_c^\infty(\mathbb{R}^n)$ such that $(2\pi)^{-n/2} ||g - f||_1 < \varepsilon/2$. Then

$$\begin{aligned}
|Ff(\mathbf{t})| &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(\mathbf{x}) - g(\mathbf{x})| dx \\
&\quad + \left|(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} g(\mathbf{x}) dx\right| \\
&\leq \varepsilon/2 + \left|(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} g(\mathbf{x}) dx\right| .
\end{aligned}$$

Now integrating by parts, it follows that for $||\mathbf{t}||_\infty \equiv \max\{|t_j| : j = 1, \cdots, n\} > 0$

$$|Ff(\mathbf{t})| \leq \varepsilon/2 + (2\pi)^{-n/2} \left|\frac{1}{||\mathbf{t}||_\infty} \int_{\mathbb{R}^n} \sum_{j=1}^n \left|\frac{\partial g(\mathbf{x})}{\partial x_j}\right| dx\right| \tag{19.6}$$

and this last expression converges to zero as $||\mathbf{t}||_\infty \to \infty$. The reason for this is that if $t_j \neq 0$, integration by parts with respect to x_j gives

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} g(\mathbf{x}) dx = (2\pi)^{-n/2} \frac{1}{-it_j} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} \frac{\partial g(\mathbf{x})}{\partial x_j} dx.$$

Therefore, choose the j for which $||\mathbf{t}||_\infty = |t_j|$ and the result of 19.6 holds. Therefore, from 19.6, if $||\mathbf{t}||_\infty$ is large enough, $|Ff(\mathbf{t})| < \varepsilon$. Similarly, $\lim_{||\mathbf{t}||\to\infty} F^{-1}(\mathbf{t}) = 0$. Consider the claim about uniform continuity. Let $\varepsilon > 0$ be given. Then there exists R such that if $||\mathbf{t}||_\infty > R$, then $|Ff(\mathbf{t})| < \frac{\varepsilon}{2}$. Since Ff is continuous, it is uniformly continuous on the compact set, $[-R-1, R+1]^n$. Therefore, there exists δ_1 such that if $||\mathbf{t}-\mathbf{t}'||_\infty < \delta_1$ for $\mathbf{t}', \mathbf{t} \in [-R-1, R+1]^n$, then

$$|Ff(\mathbf{t}) - Ff(\mathbf{t}')| < \varepsilon/2. \tag{19.7}$$

Now let $0 < \delta < \min(\delta_1, 1)$ and suppose $||\mathbf{t}-\mathbf{t}'||_\infty < \delta$. If both $\mathbf{t}, \mathbf{t}'$ are contained in $[-R, R]^n$, then 19.7 holds. If $\mathbf{t} \in [-R, R]^n$ and $\mathbf{t}' \notin [-R, R]^n$, then both are contained in $[-R-1, R+1]^n$ and so this verifies 19.7 in this case. The other case is that neither point is in $[-R, R]^n$ and in this case,

$$\begin{aligned} |Ff(\mathbf{t}) - Ff(\mathbf{t}')| &\leq |Ff(\mathbf{t})| + |Ff(\mathbf{t}')| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves the theorem.

There is a very interesting relation between the Fourier transform and convolutions.

Theorem 19.19 *Let $f, g \in L^1(\mathbb{R}^n)$. Then $f*g \in L^1$ and $F(f*g) = (2\pi)^{n/2} FfFg$.*

Proof: Consider

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})|\, dydx.$$

The function, $(\mathbf{x}, \mathbf{y}) \to |f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})|$ is Lebesgue measurable and so by Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})|\, dydx &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})|\, dxdy \\ &= ||f||_1\, ||g||_1 < \infty. \end{aligned}$$

It follows that for a.e. $\mathbf{x}$, $\int_{\mathbb{R}^n} |f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})|\, dy < \infty$ and for each of these values of $\mathbf{x}$, it follows that $\int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})\, dy$ exists and equals a function of $\mathbf{x}$ which is in $L^1(\mathbb{R}^n)$, $f * g(\mathbf{x})$. Now

$$\begin{aligned} & F(f*g)(\mathbf{t}) \\ \equiv\ & (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f*g(\mathbf{x})\, dx \\ =\ & (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} \int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})\, dydx \\ =\ & (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{y}} g(\mathbf{y}) \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot(\mathbf{x}-\mathbf{y})} f(\mathbf{x}-\mathbf{y})\, dxdy \\ =\ & (2\pi)^{n/2} Ff(\mathbf{t})\, Fg(\mathbf{t}). \end{aligned}$$

There are many other considerations involving Fourier transforms of functions in $L^1(\mathbb{R}^n)$.

19.3.2 Fourier Transforms Of Functions In $L^2(\mathbb{R}^n)$

Consider Ff and $F^{-1}f$ for $f \in L^2(\mathbb{R}^n)$. First note that the formula given for Ff and $F^{-1}f$ when $f \in L^1(\mathbb{R}^n)$ will not work for $f \in L^2(\mathbb{R}^n)$ unless f is also in $L^1(\mathbb{R}^n)$. Recall that $\overline{a+ib} = a - ib$.

Theorem 19.20 *For $\phi \in \mathcal{G}$, $||F\phi||_2 = ||F^{-1}\phi||_2 = ||\phi||_2$.*

Proof: First note that for $\psi \in \mathcal{G}$,

$$F(\overline{\psi}) = \overline{F^{-1}(\psi)}\ ,\ F^{-1}(\overline{\psi}) = \overline{F(\psi)}. \tag{19.8}$$

This follows from the definition. For example,

$$\begin{aligned} F\overline{\psi}(\mathbf{t}) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}}\overline{\psi}(\mathbf{x})\,dx \\ &= \overline{(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}\psi(\mathbf{x})\,dx} \end{aligned}$$

Let $\phi, \psi \in \mathcal{G}$. It was shown above that

$$\int_{\mathbb{R}^n}(F\phi)\psi(\mathbf{t})dt = \int_{\mathbb{R}^n}\phi(F\psi)dx.$$

Similarly,

$$\int_{\mathbb{R}^n}\phi(F^{-1}\psi)dx = \int_{\mathbb{R}^n}(F^{-1}\phi)\psi dt. \tag{19.9}$$

Now, 19.8 - 19.9 imply

$$\begin{aligned} \int_{\mathbb{R}^n}|\phi|^2dx &= \int_{\mathbb{R}^n}\phi\overline{F^{-1}(F\phi)}dx \\ &= \int_{\mathbb{R}^n}\phi F(\overline{F\phi})dx \\ &= \int_{\mathbb{R}^n}F\phi(\overline{F\phi})dx \\ &= \int_{\mathbb{R}^n}|F\phi|^2dx. \end{aligned}$$

Similarly

$$||\phi||_2 = ||F^{-1}\phi||_2.$$

This proves the theorem.

Lemma 19.21 *Let $f \in L^2(\mathbb{R}^n)$ and let $\phi_k \to f$ in $L^2(\mathbb{R}^n)$ where $\phi_k \in \mathcal{G}$. (Such a sequence exists because of density of $\mathcal{G}$ in $L^2(\mathbb{R}^n)$.) Then Ff and $F^{-1}f$ are both in $L^2(\mathbb{R}^n)$ and the following limits take place in L^2.*

$$\lim_{k\to\infty}F(\phi_k) = F(f)\,,\ \lim_{k\to\infty}F^{-1}(\phi_k) = F^{-1}(f)\,.$$

Proof: Let $\psi \in \mathcal{G}$ be given. Then

$$\begin{aligned} Ff(\psi) &\equiv f(F\psi) \equiv \int_{\mathbb{R}^n} f(\mathbf{x}) F\psi(\mathbf{x}) \, dx \\ &= \lim_{k\to\infty} \int_{\mathbb{R}^n} \phi_k(\mathbf{x}) F\psi(\mathbf{x}) \, dx = \lim_{k\to\infty} \int_{\mathbb{R}^n} F\phi_k(\mathbf{x}) \psi(\mathbf{x}) \, dx. \end{aligned}$$

Also by Theorem 19.20 $\{F\phi_k\}_{k=1}^{\infty}$ is Cauchy in $L^2(\mathbb{R}^n)$ and so it converges to some $h \in L^2(\mathbb{R}^n)$. Therefore, from the above,

$$Ff(\psi) = \int_{\mathbb{R}^n} h(\mathbf{x}) \psi(\mathbf{x})$$

which shows that $F(f) \in L^2(\mathbb{R}^n)$ and $h = F(f)$. The case of F^{-1} is entirely similar. This proves the lemma.

Since Ff and $F^{-1}f$ are in $L^2(\mathbb{R}^n)$, this also proves the following theorem.

Theorem 19.22 *If $f \in L^2(\mathbb{R}^n)$, Ff and $F^{-1}f$ are the unique elements of $L^2(\mathbb{R}^n)$ such that for all $\phi \in \mathcal{G}$,*

$$\int_{\mathbb{R}^n} Ff(\mathbf{x})\phi(\mathbf{x})dx = \int_{\mathbb{R}^n} f(\mathbf{x})F\phi(\mathbf{x})dx, \tag{19.10}$$

$$\int_{\mathbb{R}^n} F^{-1}f(\mathbf{x})\phi(\mathbf{x})dx = \int_{\mathbb{R}^n} f(\mathbf{x})F^{-1}\phi(\mathbf{x})dx. \tag{19.11}$$

Theorem 19.23 *(Plancherel)*

$$||f||_2 = ||Ff||_2 = ||F^{-1}f||_2. \tag{19.12}$$

Proof: Use the density of $\mathcal{G}$ in $L^2(\mathbb{R}^n)$ to obtain a sequence, $\{\phi_k\}$ converging to f in $L^2(\mathbb{R}^n)$. Then by Lemma 19.21

$$||Ff||_2 = \lim_{k\to\infty} ||F\phi_k||_2 = \lim_{k\to\infty} ||\phi_k||_2 = ||f||_2 .$$

Similarly,

$$||f||_2 = ||F^{-1}f||_2.$$

This proves the theorem.

The following corollary is a simple generalization of this. To prove this corollary, use the following simple lemma which comes as a consequence of the Cauchy Schwarz inequality.

Lemma 19.24 *Suppose $f_k \to f$ in $L^2(\mathbb{R}^n)$ and $g_k \to g$ in $L^2(\mathbb{R}^n)$. Then*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} f_k g_k dx = \int_{\mathbb{R}^n} fg dx$$

Proof:

$$\left|\int_{\mathbb{R}^n} f_k g_k dx - \int_{\mathbb{R}^n} fg dx\right| \leq \left|\int_{\mathbb{R}^n} f_k g_k dx - \int_{\mathbb{R}^n} f_k g dx\right| +$$
$$\left|\int_{\mathbb{R}^n} f_k g dx - \int_{\mathbb{R}^n} fg dx\right|$$
$$\leq \|f_k\|_2 \|g - g_k\|_2 + \|g\|_2 \|f_k - f\|_2 .$$

Now $\|f_k\|_2$ is a Cauchy sequence and so it is bounded independent of k. Therefore, the above expression is smaller than ε whenever k is large enough. This proves the lemma.

Corollary 19.25 *For $f, g \in L^2(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} f\overline{g} dx = \int_{\mathbb{R}^n} Ff\, \overline{Fg} dx = \int_{\mathbb{R}^n} F^{-1}f\, \overline{F^{-1}g} dx.$$

Proof: First note the above formula is obvious if $f, g \in \mathcal{G}$. To see this, note

$$\begin{aligned}
\int_{\mathbb{R}^n} Ff\, \overline{Fg} dx &= \int_{\mathbb{R}^n} Ff(x) \overline{\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{x}\cdot\mathbf{t}} g(t)\, dt} dx \\
&= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\mathbf{x}\cdot\mathbf{t}} Ff(x)\, dx \overline{g(t)} dt \\
&= \int_{\mathbb{R}^n} \left(F^{-1} \circ F\right) f(t) \overline{g(t)} dt \\
&= \int_{\mathbb{R}^n} f(t) \overline{g(t)} dt.
\end{aligned}$$

The formula with F^{-1} is exactly similar.

Now to verify the corollary, let $\phi_k \to f$ in $L^2(\mathbb{R}^n)$ and let $\psi_k \to g$ in $L^2(\mathbb{R}^n)$. Then by Lemma 19.21

$$\begin{aligned}
\int_{\mathbb{R}^n} Ff\, \overline{Fg} dx &= \lim_{k\to\infty} \int_{\mathbb{R}^n} F\phi_k\, \overline{F\psi_k} dx \\
&= \lim_{k\to\infty} \int_{\mathbb{R}^n} \phi_k \overline{\psi_k} dx \\
&= \int_{\mathbb{R}^n} f\overline{g} dx
\end{aligned}$$

A similar argument holds for F^{-1}.This proves the corollary.

How does one compute Ff and $F^{-1}f$?

Theorem 19.26 *For $f \in L^2(\mathbb{R}^n)$, let $f_r = f\mathcal{X}_{E_r}$ where E_r is a bounded measurable set with $E_r \uparrow \mathbb{R}^n$. Then the following limits hold in $L^2(\mathbb{R}^n)$.*

$$Ff = \lim_{r\to\infty} Ff_r\ ,\ F^{-1}f = \lim_{r\to\infty} F^{-1}f_r.$$

Proof: $||f-f_r||_2 \to 0$ and so $||Ff-Ff_r||_2 \to 0$ and $||F^{-1}f-F^{-1}f_r||_2 \to 0$ by Plancherel's Theorem. This proves the theorem.

What are Ff_r and $F^{-1}f_r$? Let $\phi \in \mathcal{G}$

$$\begin{aligned}\int_{\mathbb{R}^n} Ff_r\phi dx &= \int_{\mathbb{R}^n} f_r F\phi dx \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f_r(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{y}}\phi(\mathbf{y})dydx \\ &= \int_{\mathbb{R}^n} [(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_r(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{y}}dx]\phi(\mathbf{y})dy.\end{aligned}$$

Since this holds for all $\phi \in \mathcal{G}$, a dense subset of $L^2(\mathbb{R}^n)$, it follows that

$$Ff_r(\mathbf{y}) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_r(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{y}}dx.$$

Similarly

$$F^{-1}f_r(\mathbf{y}) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_r(\mathbf{x})e^{i\mathbf{x}\cdot\mathbf{y}}dx.$$

This shows that to take the Fourier transform of a function in $L^2(\mathbb{R}^n)$, it suffices to take the limit as $r \to \infty$ in $L^2(\mathbb{R}^n)$ of $(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_r(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{y}}dx$. A similar procedure works for the inverse Fourier transform.

Note this reduces to the earlier definition in case $f \in L^1(\mathbb{R}^n)$. Now consider the convolution of a function in L^2 with one in L^1.

Theorem 19.27 *Let $h \in L^2(\mathbb{R}^n)$ and let $f \in L^1(\mathbb{R}^n)$. Then $h * f \in L^2(\mathbb{R}^n)$,*

$$F^{-1}(h*f) = (2\pi)^{n/2} F^{-1}hF^{-1}f,$$

$$F(h*f) = (2\pi)^{n/2} FhFf,$$

and

$$||h*f||_2 \le ||h||_2 \, ||f||_1 . \tag{19.13}$$

Proof: An application of Minkowski's inequality yields

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} |h(\mathbf{x}-\mathbf{y})| \, |f(\mathbf{y})| \, dy\right)^2 dx\right)^{1/2} \le ||f||_1 \, ||h||_2 . \tag{19.14}$$

Hence $\int |h(\mathbf{x}-\mathbf{y})| \, |f(\mathbf{y})| \, dy < \infty$ a.e. $\mathbf{x}$ and

$$\mathbf{x} \to \int h(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) \, dy$$

is in $L^2(\mathbb{R}^n)$. Let $E_r \uparrow \mathbb{R}^n$, $m(E_r) < \infty$. Thus,

$$h_r \equiv \mathcal{X}_{E_r} h \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),$$

and letting $\phi \in \mathcal{G}$,

$$\int F(h_r * f)(\phi)\,dx$$

$$\begin{aligned}
&\equiv \int (h_r * f)(F\phi)\,dx \\
&= (2\pi)^{-n/2} \int\int\int h_r(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) e^{-i\mathbf{x}\cdot\mathbf{t}} \phi(\mathbf{t})\,dtdydx \\
&= (2\pi)^{-n/2} \int\int \left(\int h_r(\mathbf{x}-\mathbf{y}) e^{-i(\mathbf{x}-\mathbf{y})\cdot\mathbf{t}} dx\right) f(\mathbf{y}) e^{-i\mathbf{y}\cdot\mathbf{t}} dy\phi(\mathbf{t})\,dt \\
&= \int (2\pi)^{n/2} Fh_r(\mathbf{t}) Ff(\mathbf{t}) \phi(\mathbf{t})\,dt.
\end{aligned}$$

Since ϕ is arbitrary and $\mathcal{G}$ is dense in $L^2(\mathbb{R}^n)$,

$$F(h_r * f) = (2\pi)^{n/2} Fh_r Ff.$$

Now by Minkowski's Inequality, $h_r * f \to h * f$ in $L^2(\mathbb{R}^n)$ and also it is clear that $h_r \to h$ in $L^2(\mathbb{R}^n)$; so, by Plancherel's theorem, you may take the limit in the above and conclude

$$F(h * f) = (2\pi)^{n/2} FhFf.$$

The assertion for F^{-1} is similar and 19.13 follows from 19.14.

19.3.3 The Schwartz Class

The problem with $\mathcal{G}$ is that it does not contain $C_c^\infty(\mathbb{R}^n)$. I have used it in presenting the Fourier transform because the functions in $\mathcal{G}$ have a very specific form which made some technical details work out easier than in any other approach I have seen. The Schwartz class is a larger class of functions which does contain $C_c^\infty(\mathbb{R}^n)$ and also has the same nice properties as $\mathcal{G}$. The functions in the Schwartz class are infinitely differentiable and they vanish very rapidly as $|\mathbf{x}| \to \infty$ along with all their partial derivatives. This is the description of these functions, not a specific form involving polynomials times $e^{-\alpha|\mathbf{x}|^2}$. To describe this precisely requires some notation.

Definition 19.28 *$f \in \mathfrak{S}$, the Schwartz class, if $f \in C^\infty(\mathbb{R}^n)$ and for all positive integers N,*

$$\rho_N(f) < \infty$$

where

$$\rho_N(f) = \sup\{(1+|\mathbf{x}|^2)^N |D^\alpha f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\ ,\ |\alpha| \le N\}.$$

Thus $f \in \mathfrak{S}$ if and only if $f \in C^\infty(\mathbb{R}^n)$ and

$$\sup\{|\mathbf{x}^\beta D^\alpha f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\} < \infty \tag{19.15}$$

for all multi indices α and β.

Also note that if $f \in \mathfrak{S}$, then $p(f) \in \mathfrak{S}$ for any polynomial, p with $p(0) = 0$ and that

$$\mathfrak{S} \subseteq L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

for any $p \geq 1$. To see this assertion about the $p(f)$, it suffices to consider the case of the product of two elements of the Schwartz class. If $f, g \in \mathfrak{S}$, then $D^\alpha (fg)$ is a finite sum of derivatives of f times derivatives of g. Therefore, $\rho_N (fg) < \infty$ for all N. You may wonder about examples of things in $\mathfrak{S}$. Clearly any function in $C_c^\infty (\mathbb{R}^n)$ is in $\mathfrak{S}$. However there are other functions in $\mathfrak{S}$. For example $e^{-|\mathbf{x}|^2}$ is in $\mathfrak{S}$ as you can verify for yourself and so is any function from $\mathcal{G}$. Note also that the density of $C_c(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ shows that $\mathfrak{S}$ is dense in $L^p(\mathbb{R}^n)$ for every p.

Recall the Fourier transform of a function in $L^1(\mathbb{R}^n)$ is given by

$$Ff(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x}) dx.$$

Therefore, this gives the Fourier transform for $f \in \mathfrak{S}$. The nice property which $\mathfrak{S}$ has in common with $\mathcal{G}$ is that the Fourier transform and its inverse map $\mathfrak{S}$ one to one onto $\mathfrak{S}$. This means I could have presented the whole of the above theory as well as what follows in terms of $\mathfrak{S}$ and its algebraic dual, $\mathfrak{S}^*$ rather than in terms of $\mathcal{G}$ and $\mathcal{G}^*$. However, it is more technical. Nevertheless, letting $\mathfrak{S}$ play the role of $\mathcal{G}$ in the above is convenient in certain applications because it is easier to reduce to $\mathfrak{S}$ than $\mathcal{G}$. I will make use of this simple observation whenever it will simplify a presentation. The fundamental result which makes it possible is the following.

Theorem 19.29 *If $f \in \mathfrak{S}$, then Ff and $F^{-1}f$ are also in $\mathfrak{S}$.*

Proof: To begin with, let $\alpha = \mathbf{e}_j = (0, 0, \cdots, 1, 0, \cdots, 0)$, the 1 in the j^{th} slot.

$$\frac{F^{-1}f(\mathbf{t} + h\mathbf{e}_j) - F^{-1}f(\mathbf{t})}{h} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x}) (\frac{e^{ihx_j} - 1}{h}) dx. \tag{19.16}$$

Consider the integrand in 19.16.

$$\begin{aligned} \left| e^{i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x}) (\frac{e^{ihx_j} - 1}{h}) \right| &= |f(\mathbf{x})| \left| (\frac{e^{i(h/2)x_j} - e^{-i(h/2)x_j}}{h}) \right| \\ &= |f(\mathbf{x})| \left| \frac{i \sin((h/2) x_j)}{(h/2)} \right| \\ &\leq |f(\mathbf{x})| |x_j| \end{aligned}$$

and this is a function in $L^1(\mathbb{R}^n)$ because $f \in \mathfrak{S}$. Therefore by the Dominated Convergence Theorem,

$$\begin{aligned} \frac{\partial F^{-1} f(\mathbf{t})}{\partial t_j} &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}} i x_j f(\mathbf{x}) dx \\ &= i(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}} \mathbf{x}^{\mathbf{e}_j} f(\mathbf{x}) dx. \end{aligned}$$

Now $\mathbf{x}^{\mathbf{e}_j}f(\mathbf{x}) \in \mathfrak{S}$ and so one can continue in this way and take derivatives indefinitely. Thus $F^{-1}f \in C^{\infty}(\mathbb{R}^n)$ and from the above argument,

$$D^{\alpha}F^{-1}f(\mathbf{t}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}(i\mathbf{x})^{\alpha}f(\mathbf{x})dx.$$

To complete showing $F^{-1}f \in \mathfrak{S}$,

$$\mathbf{t}^{\beta}D^{\alpha}F^{-1}f(\mathbf{t}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}}\mathbf{t}^{\beta}(i\mathbf{x})^{a}f(\mathbf{x})dx.$$

Integrate this integral by parts to get

$$\mathbf{t}^{\beta}D^{\alpha}F^{-1}f(\mathbf{t}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} i^{|\beta|}e^{i\mathbf{t}\cdot\mathbf{x}}D^{\beta}((i\mathbf{x})^{a}f(\mathbf{x}))dx. \tag{19.17}$$

Here is how this is done.

$$\begin{aligned}\int_{\mathbb{R}} e^{it_jx_j}t_j^{\beta_j}(i\mathbf{x})^{\alpha}f(\mathbf{x})dx_j &= \frac{e^{it_jx_j}}{it_j}t_j^{\beta_j}(i\mathbf{x})^{\alpha}f(\mathbf{x})\,|_{-\infty}^{\infty} + \\ & \quad i\int_{\mathbb{R}} e^{it_jx_j}t_j^{\beta_j-1}D^{\mathbf{e}_j}((i\mathbf{x})^{\alpha}f(\mathbf{x}))dx_j\end{aligned}$$

where the boundary term vanishes because $f \in \mathfrak{S}$. Returning to 19.17, use the fact that $|e^{ia}| = 1$ to conclude

$$|\mathbf{t}^{\beta}D^{\alpha}F^{-1}f(\mathbf{t})| \leq C\int_{\mathbb{R}^n} |D^{\beta}((i\mathbf{x})^{a}f(\mathbf{x}))|dx < \infty.$$

It follows $F^{-1}f \in \mathfrak{S}$. Similarly $Ff \in \mathfrak{S}$ whenever $f \in \mathfrak{S}$.

Theorem 19.30 *Let $\psi \in \mathfrak{S}$. Then $(F \circ F^{-1})(\psi) = \psi$ and $(F^{-1} \circ F)(\psi) = \psi$ whenever $\psi \in \mathfrak{S}$. Also F and F^{-1} map $\mathfrak{S}$ one to one and onto $\mathfrak{S}$.*

Proof: The first claim follows from the fact that F and F^{-1} are inverses of each other which was established above. For the second, let $\psi \in \mathfrak{S}$. Then $\psi = F\left(F^{-1}\psi\right)$. Thus F maps $\mathfrak{S}$ onto $\mathfrak{S}$. If $F\psi = 0$, then do F^{-1} to both sides to conclude $\psi = 0$. Thus F is one to one and onto. Similarly, F^{-1} is one to one and onto.

Note the above equations involving F and F^{-1} hold pointwise everywhere because $F\psi$ and $F^{-1}\psi$ are continuous.

19.3.4 Convolution

To begin with it is necessary to discuss the meaning of ϕf where $f \in \mathcal{G}^*$ and $\phi \in \mathcal{G}$. What should it mean? First suppose $f \in L^p(\mathbb{R}^n)$ or measurable with polynomial growth. Then ϕf also has these properties. Hence, it should be the case that $\phi f(\psi) = \int_{\mathbb{R}^n} \phi f\psi dx = \int_{\mathbb{R}^n} f(\phi\psi)\,dx$. This motivates the following definition.

Definition 19.31 *Let $T \in \mathcal{G}^*$ and let $\phi \in \mathcal{G}$. Then $\phi T \equiv T\phi \in \mathcal{G}^*$ will be defined by*

$$\phi T(\psi) \equiv T(\phi\psi).$$

The next topic is that of convolution. It was just shown that

$$F(f * \phi) = (2\pi)^{n/2} F\phi F f,\ F^{-1}(f * \phi) = (2\pi)^{n/2} F^{-1}\phi F^{-1} f$$

whenever $f \in L^2(\mathbb{R}^n)$ and $\phi \in \mathcal{G}$ so the same definition is retained in the general case because it makes perfect sense and agrees with the earlier definition.

Definition 19.32 *Let $f \in \mathcal{G}^*$ and let $\phi \in \mathcal{G}$. Then define the convolution of f with an element of $\mathcal{G}$ as follows.*

$$f * \phi \equiv (2\pi)^{n/2} F^{-1}(F\phi F f) \in \mathcal{G}^*$$

There is an obvious question. With this definition, is it true that $F^{-1}(f * \phi) = (2\pi)^{n/2} F^{-1}\phi F^{-1} f$ as it was earlier?

Theorem 19.33 *Let $f \in \mathcal{G}^*$ and let $\phi \in \mathcal{G}$.*

$$F(f * \phi) = (2\pi)^{n/2} F\phi F f, \tag{19.18}$$

$$F^{-1}(f * \phi) = (2\pi)^{n/2} F^{-1}\phi F^{-1} f. \tag{19.19}$$

Proof: Note that 19.18 follows from Definition 19.32 and both assertions hold for $f \in \mathcal{G}$. Consider 19.19. Here is a simple formula involving a pair of functions in $\mathcal{G}$.

$$\left(\psi * F^{-1}F^{-1}\phi\right)(\mathbf{x})$$

$$\begin{aligned}
&= \left(\int\int\int \psi(\mathbf{x}-\mathbf{y})\, e^{i\mathbf{y}\cdot\mathbf{y}_1} e^{i\mathbf{y}_1\cdot\mathbf{z}} \phi(\mathbf{z})\, dz dy_1 dy\right)(2\pi)^n \\
&= \left(\int\int\int \psi(\mathbf{x}-\mathbf{y})\, e^{-i\mathbf{y}\cdot\tilde{\mathbf{y}}_1} e^{-i\tilde{\mathbf{y}}_1\cdot\mathbf{z}} \phi(\mathbf{z})\, dz d\tilde{y}_1 dy\right)(2\pi)^n \\
&= (\psi * FF\phi)(\mathbf{x}).
\end{aligned}$$

Now for $\psi \in \mathcal{G}$,

$$(2\pi)^{n/2} F\left(F^{-1}\phi F^{-1} f\right)(\psi) \equiv (2\pi)^{n/2}\left(F^{-1}\phi F^{-1} f\right)(F\psi) \equiv$$

$$(2\pi)^{n/2} F^{-1} f\left(F^{-1}\phi F\psi\right) \equiv (2\pi)^{n/2} f\left(F^{-1}\left(F^{-1}\phi F\psi\right)\right) =$$

$$f\left((2\pi)^{n/2} F^{-1}\left(\left(FF^{-1}F^{-1}\phi\right)(F\psi)\right)\right) \equiv$$

$$f\left(\psi * F^{-1}F^{-1}\phi\right) = f(\psi * FF\phi) \tag{19.20}$$

Also

$$(2\pi)^{n/2} F^{-1} (F\phi F f) (\psi) \equiv (2\pi)^{n/2} (F\phi F f) \left(F^{-1}\psi\right) \equiv$$
$$(2\pi)^{n/2} F f \left(F\phi F^{-1}\psi\right) \equiv (2\pi)^{n/2} f \left(F \left(F\phi F^{-1}\psi\right)\right) =$$
$$= f \left(F \left((2\pi)^{n/2} \left(F\phi F^{-1}\psi\right)\right)\right)$$
$$= f \left(F \left((2\pi)^{n/2} \left(F^{-1}FF\phi F^{-1}\psi\right)\right)\right) = f \left(F \left(F^{-1} (FF\phi * \psi)\right)\right)$$
$$f (FF\phi * \psi) = f (\psi * FF\phi) . \tag{19.21}$$

The last line follows from the following.

$$\begin{aligned} \int FF\phi (\mathbf{x}-\mathbf{y}) \psi (\mathbf{y}) dy &= \int F\phi (\mathbf{x}-\mathbf{y}) F\psi (\mathbf{y}) dy \\ &= \int F\psi (\mathbf{x}-\mathbf{y}) F\phi (\mathbf{y}) dy \\ &= \int \psi (\mathbf{x}-\mathbf{y}) FF\phi (y) dy. \end{aligned}$$

From 19.21 and 19.20 , since ψ was arbitrary,

$$(2\pi)^{n/2} F \left(F^{-1}\phi F^{-1} f\right) = (2\pi)^{n/2} F^{-1} (F\phi F f) \equiv f * \phi$$

which shows 19.19.

19.4 Exercises

1. For $f \in L^1 (\mathbb{R}^n)$, show that if $F^{-1} f \in L^1$ or $Ff \in L^1$, then f equals a continuous bounded function a.e.
2. Suppose $f, g \in L^1(\mathbb{R})$ and $Ff = Fg$. Show $f = g$ a.e.
3. Show that if $f \in L^1 (\mathbb{R}^n)$, then $\lim_{|\mathbf{x}|\to\infty} Ff (\mathbf{x}) = 0$.
4. $\uparrow$ Suppose $f * f = f$ or $f * f = 0$ and $f \in L^1(\mathbb{R})$. Show $f = 0$.
5. For this problem define $\int_a^\infty f (t) dt \equiv \lim_{r\to\infty} \int_a^r f (t) dt$. Note this coincides with the Lebesgue integral when $f \in L^1 (a,\infty)$. Show
   (a) $\int_0^\infty \frac{\sin(u)}{u} du = \frac{\pi}{2}$
   (b) $\lim_{r\to\infty} \int_\delta^\infty \frac{\sin(ru)}{u} du = 0$ whenever $\delta > 0$.
   (c) If $f \in L^1 (\mathbb{R})$, then $\lim_{r\to\infty} \int_{\mathbb{R}} \sin (ru) f (u) du = 0$.

   Hint: For the first two, use $\frac{1}{u} = \int_0^\infty e^{-ut} dt$ and apply Fubini's theorem to $\int_0^R \sin u \int_{\mathbb{R}} e^{-ut} dt du$. For the last part, first establish it for $f \in C_c^\infty (\mathbb{R})$ and then use the density of this set in $L^1 (\mathbb{R})$ to obtain the result. This is sometimes called the Riemann Lebesgue lemma.

6. $\uparrow$Suppose that $g \in L^1(\mathbb{R})$ and that at some $x > 0$, g is locally Holder continuous from the right and from the left. This means

$$\lim_{r \to 0+} g(x+r) \equiv g(x+)$$

exists,

$$\lim_{r \to 0+} g(x-r) \equiv g(x-)$$

exists and there exist constants $K, \delta > 0$ and $r \in (0,1]$ such that for $|x-y| < \delta$,

$$|g(x+) - g(y)| < K|x-y|^r$$

for $y > x$ and

$$|g(x-) - g(y)| < K|x-y|^r$$

for $y < x$. Show that under these conditions,

$$\lim_{r \to \infty} \frac{2}{\pi} \int_0^\infty \frac{\sin(ur)}{u} \left(\frac{g(x-u) + g(x+u)}{2} \right) du$$

$$= \frac{g(x+) + g(x-)}{2}.$$

7. $\uparrow$ Let $g \in L^1(\mathbb{R})$ and suppose g is locally Holder continuous from the right and from the left at x. Show that then

$$\lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} e^{ixt} \int_{-\infty}^{\infty} e^{-ity} g(y)\, dy dt = \frac{g(x+) + g(x-)}{2}.$$

This is very interesting. If $g \in L^2(\mathbb{R})$, this shows $F^{-1}(Fg)(x) = \frac{g(x+)+g(x-)}{2}$, the midpoint of the jump in g at the point, x. In particular, if $g \in \mathcal{G}$, $F^{-1}(Fg) = g$. **Hint:** Show the left side of the above equation reduces to

$$\frac{2}{\pi} \int_0^\infty \frac{\sin(ur)}{u} \left(\frac{g(x-u) + g(x+u)}{2} \right) du$$

and then use Problem 6 to obtain the result.

8. $\uparrow$ A measurable function g defined on $(0,\infty)$ has exponential growth if $|g(t)| \leq Ce^{\eta t}$ for some η. For $\operatorname{Re}(s) > \eta$, define the Laplace Transform by

$$Lg(s) \equiv \int_0^\infty e^{-su} g(u)\, du.$$

Assume that g has exponential growth as above and is Holder continuous from the right and from the left at t. Pick $\gamma > \eta$. Show that

$$\lim_{R \to \infty} \frac{1}{2\pi} \int_{-R}^{R} e^{\gamma t} e^{iyt} Lg(\gamma + iy)\, dy = \frac{g(t+) + g(t-)}{2}.$$

This formula is sometimes written in the form

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}Lg\left(s\right)ds$$

and is called the complex inversion integral for Laplace transforms. It can be used to find inverse Laplace transforms. **Hint:**

$$\frac{1}{2\pi}\int_{-R}^{R} e^{\gamma t}e^{iyt}Lg\left(\gamma+iy\right)dy=$$

$$\frac{1}{2\pi}\int_{-R}^{R} e^{\gamma t}e^{iyt}\int_{0}^{\infty} e^{-(\gamma+iy)u}g\left(u\right)dudy.$$

Now use Fubini's theorem and do the integral from $-R$ to R to get this equal to

$$\frac{e^{\gamma t}}{\pi}\int_{-\infty}^{\infty} e^{-\gamma u}\overline{g}\left(u\right)\frac{\sin\left(R\left(t-u\right)\right)}{t-u}du$$

where $\overline{g}$ is the zero extension of g off $[0,\infty)$. Then this equals

$$\frac{e^{\gamma t}}{\pi}\int_{-\infty}^{\infty} e^{-\gamma(t-u)}\overline{g}\left(t-u\right)\frac{\sin\left(Ru\right)}{u}du$$

which equals

$$\frac{2e^{\gamma t}}{\pi}\int_{0}^{\infty}\frac{\overline{g}\left(t-u\right)e^{-\gamma(t-u)}+\overline{g}\left(t+u\right)e^{-\gamma(t+u)}}{2}\frac{\sin\left(Ru\right)}{u}du$$

and then apply the result of Problem 6.

9. Suppose $f\in\mathfrak{S}$. Show $F(f_{x_j})(\mathbf{t})=it_jFf(\mathbf{t})$.

10. Let $f\in\mathfrak{S}$ and let k be a positive integer.

$$||f||_{k,2}\equiv(||f||_2^2+\sum_{|\alpha|\leq k}||D^\alpha f||_2^2)^{1/2}.$$

One could also define

$$|||f|||_{k,2}\equiv(\int_{R^n}|Ff(\mathbf{x})|^2(1+|\mathbf{x}|^2)^k dx)^{1/2}.$$

Show both $||\ ||_{k,2}$ and $|||\ |||_{k,2}$ are norms on $\mathfrak{S}$ and that they are equivalent. These are Sobolev space norms. For which values of k does the second norm make sense? How about the first norm?

11. ↑ Define $H^k(\mathbb{R}^n), k\geq 0$ by $f\in L^2(\mathbb{R}^n)$ such that

$$(\int|Ff(\mathbf{x})|^2(1+|\mathbf{x}|^2)^k dx)^{\frac{1}{2}}<\infty,$$

$$|||f|||_{k,2} \equiv (\int |Ff(\mathbf{x})|^2(1+|\mathbf{x}|^2)^k dx)^{\frac{1}{2}}.$$

Show $H^k(\mathbb{R}^n)$ is a Banach space, and that if k is a positive integer, $H^k(\mathbb{R}^n)$ $=\{ f \in L^2(\mathbb{R}^n) :$ there exists $\{u_j\} \subseteq \mathcal{G}$ with $||u_j - f||_2 \to 0$ and $\{u_j\}$ is a Cauchy sequence in $|| \ ||_{k,2}$ of Problem 10$\}$. This is one way to define Sobolev Spaces. **Hint:** One way to do the second part of this is to define a new measure, μ by

$$\mu(E) \equiv \int_E \left(1+|\mathbf{x}|^2\right)^k dx.$$

Then show μ is a Radon measure and show there exists $\{g_m\}$ such that $g_m \in \mathcal{G}$ and $g_m \to Ff$ in $L^2(\mu)$. Thus $g_m = Ff_m,\ f_m \in \mathcal{G}$ because F maps $\mathcal{G}$ onto $\mathcal{G}$. Then by Problem 10, $\{f_m\}$ is Cauchy in the norm $|| \ ||_{k,2}$.

12. $\uparrow$ If $2k > n$, show that if $f \in H^k(\mathbb{R}^n)$, then f equals a bounded continuous function a.e. **Hint:** Show that for k this large, $Ff \in L^1(\mathbb{R}^n)$, and then use Problem 1. To do this, write

$$|Ff(\mathbf{x})| = |Ff(\mathbf{x})|(1+|\mathbf{x}|^2)^{\frac{k}{2}}(1+|\mathbf{x}|^2)^{\frac{-k}{2}},$$

So

$$\int |Ff(\mathbf{x})|dx = \int |Ff(\mathbf{x})|(1+|\mathbf{x}|^2)^{\frac{k}{2}}(1+|\mathbf{x}|^2)^{\frac{-k}{2}} dx.$$

Use the Cauchy Schwarz inequality. This is an example of a Sobolev imbedding Theorem.

13. Let $u \in \mathcal{G}$. Then $Fu \in \mathcal{G}$ and so, in particular, it makes sense to form the integral,

$$\int_{\mathbb{R}} Fu(\mathbf{x}', x_n)\, dx_n$$

where $(\mathbf{x}', x_n) = \mathbf{x} \in \mathbb{R}^n$. For $u \in \mathcal{G}$, define $\gamma u(\mathbf{x}') \equiv u(\mathbf{x}', 0)$. Find a constant such that $F(\gamma u)(\mathbf{x}')$ equals this constant times the above integral. **Hint:** By the dominated convergence theorem

$$\int_{\mathbb{R}} Fu(\mathbf{x}', x_n)\, dx_n = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} e^{-(\varepsilon x_n)^2} Fu(\mathbf{x}', x_n)\, dx_n.$$

Now use the definition of the Fourier transform and Fubini's theorem as required in order to obtain the desired relationship.

14. Recall the Fourier series of a function in $L^2(-\pi, \pi)$ converges to the function in $L^2(-\pi, \pi)$. Prove a similar theorem with $L^2(-\pi, \pi)$ replaced by $L^2(-m\pi, m\pi)$ and the functions

$$\left\{(2\pi)^{-(1/2)} e^{inx}\right\}_{n \in \mathbb{Z}}$$

used in the Fourier series replaced with

$$\left\{ (2m\pi)^{-(1/2)} e^{i\frac{n}{m}x} \right\}_{n\in\mathbb{Z}}$$

Now suppose f is a function in $L^2(\mathbb{R})$ satisfying $Ff(t) = 0$ if $|t| > m\pi$. Show that if this is so, then

$$f(x) = \frac{1}{\pi} \sum_{n\in\mathbb{Z}} f\left(\frac{-n}{m}\right) \frac{\sin(\pi(mx+n))}{mx+n}.$$

Here m is a positive integer. This is sometimes called the Shannon sampling theorem.**Hint:** First note that since $Ff \in L^2$ and is zero off a finite interval, it follows $Ff \in L^1$. Also

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-m\pi}^{m\pi} e^{itx} Ff(x)\, dx$$

and you can conclude from this that f has all derivatives and they are all bounded. Thus f is a very nice function. You can replace Ff with its Fourier series. Then consider carefully the Fourier coefficient of Ff. Argue it equals $f\left(\frac{-n}{m}\right)$ or at least an appropriate constant times this. When you get this the rest will fall quickly into place if you use Ff is zero off $[-m\pi, m\pi]$.

Fourier Analysis In $\mathbb{R}^n$ An Introduction

The purpose of this chapter is to present some of the most important theorems on Fourier analysis in $\mathbb{R}^n$. These theorems are the Marcinkiewicz interpolation theorem, the Calderon Zygmund decomposition, and Mihlin's theorem. They are all fundamental results whose proofs depend on the methods of real analysis.

20.1 The Marcinkiewicz Interpolation Theorem

Let $(\Omega, \mu, \mathcal{S})$ be a measure space.

Definition 20.1 *$L^p(\Omega) + L^1(\Omega)$ will denote the space of measurable functions, f, such that f is the sum of a function in $L^p(\Omega)$ and $L^1(\Omega)$. Also, if $T : L^p(\Omega) + L^1(\Omega) \to$ space of measurable functions, T is subadditive if*

$$|T(f+g)(x)| \leq |Tf(x)| + |Tg(x)|.$$

T is of type (p,p) if there exists a constant independent of $f \in L^p(\Omega)$ such that

$$||Tf||_p \leq A \|f\|_p, \ f \in L^p(\Omega).$$

T is weak type (p,p) if there exists a constant A independent of f such that

$$\mu([x : |Tf(x)| > \alpha]) \leq \left(\frac{A}{\alpha} ||f||_p\right)^p, \ f \in L^p(\Omega).$$

The following lemma involves writing a function as a sum of a functions whose values are small and one whose values are large.

Lemma 20.2 *If $p \in [1, r]$, then $L^p(\Omega) \subseteq L^1(\Omega) + L^r(\Omega)$.*

Proof: Let $\lambda > 0$ and let $f \in L^p(\Omega)$

$$f_1(x) \equiv \begin{cases} f(x) \text{ if } |f(x)| \leq \lambda \\ 0 \text{ if } |f(x)| > \lambda \end{cases}, \quad f_2(x) \equiv \begin{cases} f(x) \text{ if } |f(x)| > \lambda \\ 0 \text{ if } |f(x)| \leq \lambda \end{cases}.$$

Thus $f(x) = f_1(x) + f_2(x)$.

$$\int |f_1(x)|^r \, d\mu = \int_{[|f| \leq \lambda]} |f(x)|^r \, d\mu \leq \lambda^{r-p} \int_{[|f| \leq \lambda]} |f(x)|^p \, d\mu < \infty.$$

Therefore, $f_1 \in L^r(\Omega)$.

$$\int |f_2(x)| \, d\mu = \int_{[|f| > \lambda]} |f(x)| \, d\mu \leq \mu[|f| > \lambda]^{1/p'} \left(\int |f|^p \, d\mu \right)^{1/p} < \infty.$$

This proves the lemma since $f = f_1 + f_2$, $f_1 \in L^r$ and $f_2 \in L^1$.

For f a function having nonnegative real values, $\alpha \to \mu([f > \alpha])$ is called the distribution function.

Lemma 20.3 *Let $\phi(0) = 0$, ϕ is strictly increasing, and C^1. Let $f : \Omega \to [0, \infty)$ be measurable. Then*

$$\int_\Omega (\phi \circ f) \, d\mu = \int_0^\infty \phi'(\alpha) \, \mu[f > \alpha] \, d\alpha. \tag{20.1}$$

Proof: First suppose

$$f = \sum_{i=1}^m a_i \mathcal{X}_{E_i}$$

where $a_i > 0$ and the a_i are all distinct nonzero values of f, the sets, E_i being disjoint. Thus,

$$\int_\Omega (\phi \circ f) \, d\mu = \sum_{i=1}^m \phi(a_i) \, \mu(E_i).$$

Suppose without loss of generality $a_1 < a_2 < \cdots < a_m$. Observe

$$\alpha \to \mu([f > \alpha])$$

is constant on the intervals $[0, a_1), [a_1, a_2), \cdots$. For example, on $[a_i, a_{i+1})$, this function has the value

$$\sum_{j=i+1}^m \mu(E_j).$$

The function equals zero on $[a_m, \infty)$. Therefore,

$$\alpha \to \phi'(\alpha) \, \mu([|f| > \alpha])$$

is Lebesgue measurable and letting $a_0 = 0$, the second integral in 20.1 equals

$$\begin{aligned}\int_0^\infty \phi'(\alpha)\,\mu([f>\alpha])\,d\alpha &= \sum_{i=1}^m \int_{a_{i-1}}^{a_i} \phi'(\alpha)\,\mu([f>\alpha])\,d\alpha \\ &= \sum_{i=1}^m \sum_{j=i}^m \mu(E_j) \int_{a_{i-1}}^{a_i} \phi'(\alpha)\,d\alpha \\ &= \sum_{j=1}^m \sum_{i=1}^j \mu(E_j)\left(\phi(a_i) - \phi(a_{i-1})\right) \\ &= \sum_{j=1}^m \mu(E_j)\,\phi(a_j) = \int_\Omega (\phi\circ f)\,d\mu\end{aligned}$$

and so this establishes 20.1 in the case when f is a nonnegative simple function. Since every measurable nonnegative function may be written as the pointwise limit of such simple functions, the desired result will follow by the Monotone convergence theorem and the next claim.

Claim: If $f_n \uparrow f$, then for each $\alpha > 0$,

$$\mu([f>\alpha]) = \lim_{n\to\infty} \mu([f_n>\alpha]).$$

Proof of the claim: $[f_n > \alpha] \uparrow [f > \alpha]$ because if $f(x) > \alpha$ then for large enough n, $f_n(x) > \alpha$ and so

$$\mu([f_n>\alpha]) \uparrow \mu([f>\alpha]).$$

This proves the lemma. (Note the importance of the strict inequality in $[f > \alpha]$ in proving the claim.)

The next theorem is the main result in this section. It is called the Marcinkiewicz interpolation theorem.

Theorem 20.4 *Let $(\Omega, \mu, \mathcal{S})$ be a σ finite measure space, $1 < r < \infty$, and let*

$$T : L^1(\Omega) + L^r(\Omega) \to \text{ space of measurable functions}$$

be subadditive, weak (r, r), and weak $(1, 1)$. Then T is of type (p, p) for every $p \in (1, r)$ and

$$||Tf||_p \leq A_p\,||f||_p$$

where the constant A_p depends only on p and the constants in the definition of weak $(1, 1)$ and weak (r, r).

Proof: Let $\alpha > 0$ and let f_1 and f_2 be defined as in Lemma 20.2,

$$f_1(x) \equiv \begin{cases} f(x) \text{ if } |f(x)| \leq \alpha \\ 0 \text{ if } |f(x)| > \alpha \end{cases}, \quad f_2(x) \equiv \begin{cases} f(x) \text{ if } |f(x)| > \alpha \\ 0 \text{ if } |f(x)| \leq \alpha \end{cases}.$$

Thus $f = f_1 + f_2$ where $f_1 \in L^r$ and $f_2 \in L^1$. Since T is subadditive ,

$$[|Tf| > \alpha] \subseteq [|Tf_1| > \alpha/2] \cup [|Tf_2| > \alpha/2] .$$

Let $p \in (1, r)$. By Lemma 20.3,

$$\int |Tf|^p \, d\mu \leq p \int_0^\infty \alpha^{p-1} \mu \left([|Tf_1| > \alpha/2] \right) d\alpha +$$

$$+ p \int_0^\infty \alpha^{p-1} \mu \left([|Tf_2| > \alpha/2] \right) d\alpha.$$

Therefore, since T is weak $(1, 1)$ and weak (r, r),

$$\int |Tf|^p \, d\mu \leq p \int_0^\infty \alpha^{p-1} \left(\frac{2A_r}{\alpha} ||f_1||_r \right)^r d\alpha + p \int_0^\infty \alpha^{p-1} \frac{2A_1}{\alpha} ||f_2||_1 \, d\alpha. \quad (20.2)$$

Therefore, the right side of 20.2 equals

$$p (2A_r)^r \int_0^\infty \alpha^{p-1-r} \int_\Omega |f_1|^r \, d\mu d\alpha + 2A_1 p \int_0^\infty \alpha^{p-2} \int_\Omega |f_2| \, d\mu d\alpha =$$

$$p (2A_r)^r \int_\Omega \int_0^\infty \alpha^{p-1-r} |f_1|^r \, d\alpha d\mu + 2A_1 p \int_\Omega \int_0^\infty \alpha^{p-2} |f_2| \, d\alpha d\mu.$$

Now $f_1(x) = 0$ unless $|f_1(x)| \leq \alpha$ and $f_2(x) = 0$ unless $|f_2(x)| > \alpha$ so this equals

$$p (2A_r)^r \int_\Omega |f(x)|^r \int_{|f(x)|}^\infty \alpha^{p-1-r} d\alpha d\mu + 2A_1 p \int_\Omega |f(x)| \int_0^{|f(x)|} \alpha^{p-2} d\alpha d\mu$$

which equals

$$\frac{2^r A_r^r p}{r - p} \int_\Omega |f(x)|^p \, d\mu + \frac{2pA_1}{p-1} \int_\Omega |f(x)|^p \, d\mu$$

$$\leq \max \left(\frac{2^r A_r^r p}{r - p}, \frac{2pA_1}{p-1} \right) ||f||_{L^p(\Omega)}^p$$

and this proves the theorem.

20.2 The Calderon Zygmund Decomposition

For a given nonnegative integrable function, $\mathbb{R}^n$ can be decomposed into a set where the function is small and a set which is the union of disjoint cubes on which the average of the function is under some control. The measure in this section will always be Lebesgue measure on $\mathbb{R}^n$. This theorem depends on the Lebesgue theory of differentiation.

Theorem 20.5 *Let $f \geq 0$, $\int f dx < \infty$, and let α be a positive constant. Then there exist sets F and Ω such that*

$$\mathbb{R}^n = F \cup \Omega,\ F \cap \Omega = \emptyset \tag{20.3}$$

$$f(x) \leq \alpha \ a.e. \ on \ F \tag{20.4}$$

$\Omega = \cup_{k=1}^{\infty} Q_k$ where the interiors of the cubes are disjoint and for each cube, Q_k,

$$\alpha < \frac{1}{m(Q_k)} \int_{Q_k} f(x)\, dx \leq 2^n \alpha. \tag{20.5}$$

Proof: Let S_0 be a tiling of $\mathbb{R}^n$ into cubes having sides of length M where M is chosen large enough that if Q is one of these cubes, then

$$\frac{1}{m(Q)} \int_Q f dm \leq \alpha. \tag{20.6}$$

Suppose $S_0, \cdot \cdot \cdot, S_m$ have been chosen. To get S_{m+1}, replace each cube of S_m by the 2^n cubes obtained by bisecting the sides. Then S_{m+1} consists of exactly those cubes of S_m for which 20.6 holds and let T_{m+1} consist of the bisected cubes from S_m for which 20.6 does not hold. Now define

$$F \equiv \{\mathbf{x} : \mathbf{x} \text{ is contained in some cube from } S_m \text{ for all } m\},$$

$$\Omega \equiv \mathbb{R}^n \setminus F = \cup_{m=1}^{\infty} \cup \{Q : Q \in T_m\}$$

Note that the cubes from T_m have pair wise disjoint interiors and also the interiors of cubes from T_m have empty intersections with the interiors of cubes of T_k if $k \neq m$.

Let $\mathbf{x}$ be a point of Ω and let $\mathbf{x}$ be in a cube of T_m such that m is the first index for which this happens. Let Q be the cube in S_{m-1} containing $\mathbf{x}$ and let Q^* be the cube in the bisection of Q which contains $\mathbf{x}$. Therefore 20.6 does not hold for Q^*. Thus

$$\alpha < \frac{1}{m(Q^*)} \int_{Q^*} f dx \leq \frac{m(Q)}{m(Q^*)} \overbrace{\frac{1}{m(Q)} \int_Q f dx}^{\leq \alpha} \leq 2^n \alpha$$

which shows Ω is the union of cubes having disjoint interiors for which 20.5 holds.

Now a.e. point of F is a Lebesgue point of f. Let $\mathbf{x}$ be such a point of F and suppose $\mathbf{x} \in Q_k$ for $Q_k \in S_k$. Let $d_k \equiv$ diameter of Q_k. Thus $d_k \to 0$.

$$\frac{1}{m(Q_k)} \int_{Q_k} |f(\mathbf{y}) - f(\mathbf{x})|\, dy \leq \frac{1}{m(Q_k)} \int_{B(\mathbf{x},d_k)} |f(\mathbf{y}) - f(\mathbf{x})|\, dy$$

$$= \frac{m(B(\mathbf{x},d_k))}{m(Q_k)} \frac{1}{m(B(\mathbf{x},d_k))} \int_{B(\mathbf{x},d_k)} |f(\mathbf{x}) - f(\mathbf{y})|\, dy$$

$$\leq K_n \frac{1}{m(B(\mathbf{x},d_k))} \int_{B(\mathbf{x},d_k)} |f(\mathbf{x}) - f(\mathbf{y})|\, dy$$

where K_n is a constant which depends on n and measures the ratio of the volume of a ball with diamiter $2d$ and a cube with diameter d. The last expression converges to 0 because $\mathbf{x}$ is a Lebesgue point. Hence

$$f(\mathbf{x}) = \lim_{k\to\infty} \frac{1}{m(Q_k)} \int_{Q_k} f(\mathbf{y})\, dy \leq \alpha$$

and this shows $f(\mathbf{x}) \leq \alpha$ a.e. on F. This proves the theorem.

20.3 Mihlin's Theorem

In this section, the Marcinkiewicz interpolation theorem and Calderon Zygmund decomposition will be used to establish a remarkable theorem of Mihlin, a generalization of Plancherel's theorem to the L^p spaces. It is of fundamental importance in the study of elliptic partial differential equations and can also be used to give proofs for the theory of singular integrals. Mihlin's theorem involves a conclusion which is of the form

$$\left|\left|F^{-1}\rho * \phi\right|\right|_p \leq A_p \,||\phi||_p \tag{20.7}$$

for $p > 1$ and $\phi \in \mathcal{G}$. Thus $F^{-1}\rho*$ extends to a continuous linear map defined on L^p because of the density of $\mathcal{G}$. It is proved by showing various weak type estimates and then applying the Marcinkiewicz Interpolation Theorem to get an estimate like the above.

Recall that by Corollary 19.27, if $f \in L^2(\mathbb{R}^n)$ and if $\phi \in \mathcal{G}$, then $f * \phi \in L^2(\mathbb{R}^n)$ and

$$F(f * \phi)(\mathbf{x}) = (2\pi)^{n/2} F\phi(\mathbf{x}) Ff(\mathbf{x}).$$

The next lemma is essentially a weak $(1,1)$ estimate. The inequality 20.7 is established under the condition, 20.8 and then it is shown there exist conditions which are easier to verify which imply condition 20.8. I think the approach used here is due to Hormander [29] and is found in Berg and Lofstrom [8]. For many more references and generalizations, you might look in Triebel [50]. A different proof based on singular integrals is in Stein [49]. Functions, ρ which yield an inequality of the sort in 20.7 are called L^p multipliers.

Lemma 20.6 *Suppose $\rho \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and suppose also there exists a constant C_1 such that*

$$\int_{|\mathbf{x}|\geq 2|\mathbf{y}|} \left|F^{-1}\rho(\mathbf{x}-\mathbf{y}) - F^{-1}\rho(\mathbf{x})\right| dx \leq C_1. \tag{20.8}$$

Then there exists a constant A depending only on $C_1, ||\rho||_\infty$, and n such that

$$m\left(\left[\mathbf{x} : \left|F^{-1}\rho*\phi(\mathbf{x})\right| > \alpha\right]\right) \leq \frac{A}{\alpha} ||\phi||_1$$

for all $\phi \in \mathcal{G}$.

Proof: Let $\phi \in \mathcal{G}$ and use the Calderon decomposition to write $\mathbb{R}^n = E \cup \Omega$ where Ω is a union of cubes, $\{Q_i\}$ with disjoint interiors such that

$$\alpha m(Q_i) \leq \int_{Q_i} |\phi(\mathbf{x})| \, dx \leq 2^n \alpha m(Q_i), \ |\phi(\mathbf{x})| \leq \alpha \text{ a.e. on } E. \tag{20.9}$$

The proof is accomplished by writing ϕ as the sum of a good function and a bad function and establishing a similar weak inequality for these two functions separately. Then this information is used to obtain the desired conclusion.

$$g(\mathbf{x}) = \begin{cases} \phi(\mathbf{x}) \text{ if } \mathbf{x} \in E \\ \frac{1}{m(Q_i)} \int_{Q_i} \phi(\mathbf{x}) \, dx \text{ if } \mathbf{x} \in Q_i \subseteq \Omega \end{cases}, \ g(\mathbf{x}) + b(\mathbf{x}) = \phi(\mathbf{x}). \tag{20.10}$$

Thus

$$\int_{Q_i} b(\mathbf{x}) \, dx = \int_{Q_i} (\phi(\mathbf{x}) - g(\mathbf{x})) \, dx = \int_{Q_i} \phi(\mathbf{x}) \, dx - \int_{Q_i} \phi(\mathbf{x}) \, dx = 0, \tag{20.11}$$

$$b(\mathbf{x}) = 0 \text{ if } \mathbf{x} \notin \Omega. \tag{20.12}$$

Claim:

$$||g||_2^2 \leq \alpha (1 + 4^n) ||\phi||_1, \ ||g||_1 \leq ||\phi||_1. \tag{20.13}$$

Proof of claim:

$$||g||_2^2 = ||g||_{L^2(E)}^2 + ||g||_{L^2(\Omega)}^2.$$

Thus

$$\begin{aligned} ||g||_{L^2(\Omega)}^2 &= \sum_i \int_{Q_i} |g(x)|^2 \, dx \\ &\leq \sum_i \int_{Q_i} \left(\frac{1}{m(Q_i)} \int_{Q_i} |\phi(y)| \, dy \right)^2 dx \\ &\leq \sum_i \int_{Q_i} (2^n \alpha)^2 \, dx \leq 4^n \alpha^2 \sum_i m(Q_i) \\ &\leq 4^n \alpha^2 \frac{1}{\alpha} \sum_i \int_{Q_i} |\phi(x)| \, dx \leq 4^n \alpha \, ||\phi||_1. \end{aligned}$$

$$||g||_{L^2(E)}^2 = \int_E |\phi(x)|^2 \, dx \leq \alpha \int_E |\phi(x)| \, dx = \alpha \, ||\phi||_1.$$

Now consider the second of the inequalities in 20.13.

$$\begin{aligned} ||g||_1 &= \int_E |g(\mathbf{x})| \, dx + \int_\Omega |g(\mathbf{x})| \, dx \\ &= \int_E |\phi(\mathbf{x})| \, dx + \sum_i \int_{Q_i} |g| \, dx \\ &\leq \int_E |\phi(\mathbf{x})| \, dx + \sum_i \int_{Q_i} \frac{1}{m(Q_i)} \int_{Q_i} |\phi(\mathbf{x})| \, dm(x) \, dm \\ &= \int_E |\phi(\mathbf{x})| \, dx + \sum_i \int_{Q_i} |\phi(\mathbf{x})| \, dm(x) = ||\phi||_1 \end{aligned}$$

This proves the claim. From the claim, it follows that $b \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

Because of 20.13, $g \in L^1(\mathbb{R}^n)$ and so $F^{-1}\rho * g \in L^2(\mathbb{R}^n)$. (Since $\rho \in L^2$, it follows $F^{-1}\rho \in L^2$ and so this convolution is indeed in L^2.) By Plancherel's theorem,

$$\left|\left|F^{-1}\rho * g\right|\right|_2 = \left|\left|F\left(F^{-1}\rho * g\right)\right|\right|_2.$$

By Corollary 19.27 on Page 531, the expression on the right equals

$$(2\pi)^{n/2}\,||\rho F g||_2$$

and so

$$\left|\left|F^{-1}\rho * g\right|\right|_2 = (2\pi)^{n/2}\,||\rho F g||_2 \leq C_n\,||\rho||_\infty\,||g||_2.$$

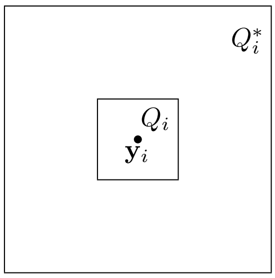
From this and 20.13

$$m\left(\left[\left|F^{-1}\rho * g\right| \geq \alpha/2\right]\right)$$

$$\leq \frac{C_n\,||\rho||_\infty^2}{\alpha^2}\alpha\left(1+4^n\right)||\phi||_1 = C_n\alpha^{-1}\,||\phi||_1. \tag{20.14}$$

This is what is wanted so far as g is concerned. Next it is required to estimate $m\left(\left[\left|F^{-1}\rho * b\right| \geq \alpha/2\right]\right)$.

If Q is one of the cubes whose union is Ω, let Q^* be the cube with the same center as Q but whose sides are $2\sqrt{n}$ times as long.



Let

$$\Omega^* \equiv \cup_{i=1}^\infty Q_i^*$$

and let

$$E^* \equiv \mathbb{R}^n \setminus \Omega^*.$$

Thus $E^* \subseteq E$. Let $\mathbf{x} \in E^*$. Then because of 20.11,

$$\int_{Q_i} F^{-1}\rho(\mathbf{x}-\mathbf{y})\,b(\mathbf{y})\,dy$$

$$= \int_{Q_i}\left[F^{-1}\rho(\mathbf{x}-\mathbf{y}) - F^{-1}\rho(\mathbf{x}-\mathbf{y}_i)\right]b(\mathbf{y})\,dy, \tag{20.15}$$

where $\mathbf{y}_i$ is the center of Q_i. Consequently if the sides of Q_i have length $2t/\sqrt{n}$, 20.15 implies

$$\int_{E^*}\left|\int_{Q_i} F^{-1}\rho(\mathbf{x}-\mathbf{y})\,b(\mathbf{y})\,dy\right|dx \leq \tag{20.16}$$

$$\int_{E^*}\int_{Q_i}\left|F^{-1}\rho(\mathbf{x}-\mathbf{y})-F^{-1}\rho(\mathbf{x}-\mathbf{y}_i)\right||b(\mathbf{y})|\,dydx$$

$$= \int_{Q_i}\int_{E^*}\left|F^{-1}\rho(\mathbf{x}-\mathbf{y})-F^{-1}\rho(\mathbf{x}-\mathbf{y}_i)\right|dx\,|b(\mathbf{y})|\,dy \tag{20.17}$$

$$\leq \int_{Q_i}\int_{|\mathbf{x}-\mathbf{y}_i|\geq 2t}\left|F^{-1}\rho(\mathbf{x}-\mathbf{y})-F^{-1}\rho(\mathbf{x}-\mathbf{y}_i)\right|dx\,|b(\mathbf{y})|\,dy \tag{20.18}$$

since if $\mathbf{x}\in E^*$, then $|\mathbf{x}-\mathbf{y}_i|\geq 2t$. Now for $\mathbf{y}\in Q_i$,

$$|\mathbf{y}-\mathbf{y}_i|\leq\left(\sum_{j=1}^{n}\left(\frac{t}{\sqrt{n}}\right)^2\right)^{1/2}=t.$$

From 20.8 and the change of variables $\mathbf{u}=\mathbf{x}-\mathbf{y}_i$ 20.16 - 20.18 imply

$$\int_{E^*}\left|\int_{Q_i}F^{-1}\rho(\mathbf{x}-\mathbf{y})\,b(\mathbf{y})\,dy\right|dx\leq C_1\int_{Q_i}|b(\mathbf{y})|\,dy. \tag{20.19}$$

Now from 20.19, and the fact that $b=0$ off Ω,

$$\begin{aligned}
\int_{E^*}\left|F^{-1}\rho*b(\mathbf{x})\right|dx &= \int_{E^*}\left|\int_{\mathbb{R}^n}F^{-1}\rho(\mathbf{x}-\mathbf{y})\,b(\mathbf{y})\,dy\right|dx\\
&= \int_{E^*}\left|\sum_{i=1}^{\infty}\int_{Q_i}F^{-1}\rho(\mathbf{x}-\mathbf{y})\,b(\mathbf{y})\,dy\right|dx\\
&\leq \int_{E^*}\sum_{i=1}^{\infty}\left|\int_{Q_i}F^{-1}\rho(\mathbf{x}-\mathbf{y})\,b(\mathbf{y})\,dy\right|dx\\
&= \sum_{i=1}^{\infty}\int_{E^*}\left|\int_{Q_i}F^{-1}\rho(\mathbf{x}-\mathbf{y})\,b(\mathbf{y})\,dy\right|dx\\
&\leq \sum_{i=1}^{\infty}C_1\int_{Q_i}|b(\mathbf{y})|\,dy=C_1\,||b||_1.
\end{aligned}$$

Thus, by 20.13,

$$\begin{aligned}
\int_{E^*}\left|F^{-1}\rho*b(\mathbf{x})\right|dx &\leq C_1\,||b||_1\\
&\leq C_1\left[||\phi||_1+||g||_1\right]\\
&\leq C_1\left[||\phi||_1+||\phi||_1\right]\\
&\leq 2C_1\,||\phi||_1\,.
\end{aligned}$$

Consequently,

$$m\left(\left[\left|F^{-1}\rho*b\right|\geq\frac{\alpha}{2}\right]\cap E^*\right)\leq\frac{4C_1}{\alpha}\,||\phi||_1\,.$$

From 20.10, 20.14, and 20.9,

$$m\left[\left|F^{-1}\rho * \phi\right| > \alpha\right] \leq m\left[\left|F^{-1}\rho * g\right| \geq \frac{\alpha}{2}\right] + m\left[\left|F^{-1}\rho * b\right| \geq \frac{\alpha}{2}\right]$$

$$\leq \frac{C_n}{\alpha}||\phi||_1 + m\left(\left[\left|F^{-1}\rho * b\right| \geq \frac{\alpha}{2}\right] \cap E^*\right) + m(\Omega^*)$$

$$\leq \frac{C_n}{\alpha}||\phi||_1 + \frac{4C_1}{\alpha}||\phi||_1 + C_n m(\Omega) \leq \frac{A}{\alpha}||\phi||_1$$

because

$$m(\Omega) \leq \alpha^{-1}||\phi||_1$$

by 20.9. This proves the lemma.

The next lemma extends this lemma by giving a weak $(2,2)$ estimate and a $(2,2)$ estimate.

Lemma 20.7 *Suppose $\rho \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and suppose also that there exists a constant C_1 such that*

$$\int_{|\mathbf{x}|>2|\mathbf{y}|} \left|F^{-1}\rho(\mathbf{x}-\mathbf{y}) - F^{-1}\rho(\mathbf{x})\right| dx \leq C_1. \tag{20.20}$$

Then $F^{-1}\rho *$ maps $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ to measurable functions and there exists a constant A depending only on $C_1, n, ||\rho||_\infty$ such that

$$m\left(\left[\left|F^{-1}\rho * f\right| > \alpha\right]\right) \leq A\frac{||f||_1}{\alpha} \text{ if } f \in L^1(\mathbb{R}^n), \tag{20.21}$$

$$m\left(\left[\left|F^{-1}\rho * f\right| > \alpha\right]\right) \leq \left(A\frac{||f||_2}{\alpha}\right)^2 \text{ if } f \in L^2(\mathbb{R}^n). \tag{20.22}$$

Thus, $F^{-1}\rho *$ is weak type $(1,1)$ and weak type $(2,2)$. Also

$$\left|\left|F^{-1}\rho * f\right|\right|_2 \leq A\,||f||_2 \text{ if } f \in L^2(\mathbb{R}^n). \tag{20.23}$$

Proof: By Plancherel's theorem $F^{-1}\rho$ is in $L^2(\mathbb{R}^n)$. If $f \in L^1(\mathbb{R}^n)$, then by Minkowski's inequality,

$$F^{-1}\rho * f \in L^2(\mathbb{R}^n).$$

Now let $g \in L^2(\mathbb{R}^n)$. By Holder's inequality,

$$\int \left|F^{-1}\rho(\mathbf{x}-\mathbf{y})\right| |g(\mathbf{y})|\, dy \leq \left(\int \left|F^{-1}\rho(\mathbf{x}-\mathbf{y})\right|^2 dy\right)^{1/2} \left(\int |g(\mathbf{y})|^2 dy\right)^{1/2} < \infty$$

and so the following is well defined a.e.

$$F^{-1}\rho * g(\mathbf{x}) \equiv \int F^{-1}\rho(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})\, dy$$

also,

$$\begin{aligned}\left|F^{-1}\rho * g(\mathbf{x}) - F^{-1}\rho * g(\mathbf{x}')\right| &\leq \int \left|F^{-1}\rho(\mathbf{x}-\mathbf{y}) - F^{-1}\rho(\mathbf{x}'-\mathbf{y})\right| |g(\mathbf{y})| \, dy \\ &\leq \left|\left|F^{-1}\rho - F^{-1}\rho_{\mathbf{x}'-\mathbf{x}}\right|\right| \, ||g||_{l^2}\end{aligned}$$

and by continuity of translation in $L^2(\mathbb{R}^n)$, this shows $\mathbf{x} \to F^{-1}\rho * g(\mathbf{x})$ is continuous. Therefore, $F^{-1}\rho*$ maps $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ to the space of measurable functions. (Continuous functions are measurable.) It is clear that $F^{-1}\rho*$ is subadditive.

If $\phi \in \mathcal{G}$, Plancherel's theorem implies as before,

$$\left|\left|F^{-1}\rho * \phi\right|\right|_2 = \left|\left|F\left(F^{-1}\rho * \phi\right)\right|\right|_2 =$$

$$(2\pi)^{n/2} ||\rho F\phi||_2 \leq (2\pi)^{n/2} ||\rho||_\infty ||\phi||_2 . \tag{20.24}$$

Now let $f \in L^2(\mathbb{R}^n)$ and let $\phi_k \in \mathcal{G}$, with

$$||\phi_k - f||_2 \to 0.$$

Then by Holder's inequality,

$$\int F^{-1}\rho(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) \, dy = \lim_{k\to\infty} \int F^{-1}\rho(\mathbf{x}-\mathbf{y}) \phi_k(\mathbf{y}) \, dy$$

and so by Fatou's lemma, Plancherel's theorem, and 20.24,

$$\left|\left|F^{-1}\rho * f\right|\right|_2 = \left(\int \left|\int F^{-1}\rho(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) \, dy\right|^2 dx\right)^{1/2} \leq$$

$$\leq \lim\inf_{k\to\infty} \left(\int \left|\int F^{-1}\rho(\mathbf{x}-\mathbf{y}) \phi_k(\mathbf{y}) \, dy\right|^2 dx\right)^{1/2} = \lim\inf_{k\to\infty} \left|\left|F^{-1}\rho * \phi_k\right|\right|_2$$

$$\leq ||\rho||_\infty (2\pi)^{n/2} \lim\inf_{k\to\infty} ||\phi_k||_2 = ||\rho||_\infty (2\pi)^{n/2} ||f||_2 .$$

Thus, 20.23 holds with $A = ||\rho||_\infty (2\pi)^{n/2}$. Consequently,

$$A ||f||_2 \geq \left(\int_{[|F^{-1}\rho * f| > \alpha]} \left|F^{-1}\rho * f(\mathbf{x})\right|^2 dx\right)^{1/2}$$

$$\geq \alpha m\left(\left[\left|F^{-1}\rho * f\right| > \alpha\right]\right)^{1/2}$$

and so 20.22 follows.

It remains to prove 20.21 which holds for all $f \in \mathcal{G}$ by Lemma 20.6. Let $f \in L^1(\mathbb{R}^n)$ and let $\phi_k \to f$ in $L^1(\mathbb{R}^n), \phi_k \in \mathcal{G}$. Without loss of generality, assume that both f and $F^{-1}\rho$ are Borel measurable. Therefore, by Minkowski's inequality, and Plancherel's theorem,

$$\left|\left|F^{-1}\rho * \phi_k - F^{-1}\rho * f\right|\right|_2$$

$$\begin{aligned}
&\leq \left(\int \left|\int F^{-1}\rho(\mathbf{x}-\mathbf{y})(\phi_k(\mathbf{y})-f(\mathbf{y}))\,dy\right|^2 dx\right)^{1/2} \\
&\leq ||\phi_k - f||_1 \, ||\rho||_2
\end{aligned}$$

which shows that $F^{-1}\rho * \phi_k$ converges to $F^{-1}\rho * f$ in $L^2(\mathbb{R}^n)$. Therefore, there exists a subsequence such that the convergence is pointwise a.e. Then, denoting the subsequence by k,

$$\mathcal{X}_{[|F^{-1}\rho * f|>\alpha]}(\mathbf{x}) \leq \lim\inf_{k\to\infty} \mathcal{X}_{[|F^{-1}\rho*\phi_k|>\alpha]}(\mathbf{x}) \text{ a.e. } \mathbf{x}.$$

Thus by Lemma 20.6 and Fatou's lemma, there exists a constant, A, depending on C_1, n, and $||\rho||_\infty$ such that

$$\begin{aligned}
m\left(\left[\left|F^{-1}\rho * f\right| > \alpha\right]\right) &\leq \lim\inf_{k\to\infty} m\left(\left[\left|F^{-1}\rho * \phi_k\right| > \alpha\right]\right) \\
&\leq \lim\inf_{k\to\infty} A\frac{||\phi_k||_1}{\alpha} = A\frac{||f||_1}{\alpha}.
\end{aligned}$$

This shows 20.21 and proves the lemma.

Theorem 20.8 *Let $\rho \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and suppose*

$$\int_{|\mathbf{x}|\geq 2|\mathbf{y}|} \left|F^{-1}\rho(\mathbf{x}-\mathbf{y}) - F^{-1}\rho(\mathbf{x})\right| dx \leq C_1.$$

Then for each $p \in (1,\infty)$, there exists a constant, A_p, depending only on

$$p, n, ||\rho||_\infty,$$

and C_1 such that for all $\phi \in \mathcal{G}$,

$$\left|\left|F^{-1}\rho * \phi\right|\right|_p \leq A_p \, ||\phi||_p.$$

Proof: From Lemma 20.7, $F^{-1}\rho *$ is weak $(1,1)$, weak $(2,2)$, and maps

$$L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$$

to measurable functions. Therefore, by the Marcinkiewicz interpolation theorem, there exists a constant A_p depending only on p, C_1, n, and $||\rho||_\infty$ for $p \in (1,2]$, such that for $f \in L^p(\mathbb{R}^n)$, and $p \in (1,2]$,

$$\left|\left|F^{-1}\rho * f\right|\right|_p \leq A_p \, ||f||_p.$$

Thus the theorem is proved for these values of p. Now suppose $p > 2$. Then $p' < 2$ where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

By Plancherel's theorem and Theorem 19.33,

$$\begin{aligned}\int F^{-1}\rho * \phi (\mathbf{x}) \psi (\mathbf{x}) dx &= (2\pi)^{n/2} \int \rho (\mathbf{x}) F\phi (\mathbf{x}) F\psi (\mathbf{x}) dx \\ &= \int F \left(F^{-1}\rho * \psi\right) F\phi dx \\ &= \int \left(F^{-1}\rho * \psi\right) (\phi) dx.\end{aligned}$$

Thus by the case for $p \in (1,2)$ and Holder's inequality,

$$\begin{aligned}\left|\int F^{-1}\rho * \phi (\mathbf{x}) \psi (\mathbf{x}) dx\right| &= \left|\int \left(F^{-1}\rho * \psi\right) (\phi) dx\right| \\ &\leq \left|\left|F^{-1}\rho * \psi\right|\right|_{p'} ||\phi||_p \\ &\leq A_{p'} ||\psi||_{p'} ||\phi||_p .\end{aligned}$$

Letting $L\psi \equiv \int F^{-1}\rho * \phi (\mathbf{x}) \psi (\mathbf{x}) dx$, this shows $L \in L^{p'} (\mathbb{R}^n)'$ and $||L||_{\left(L^{p'}\right)'} \leq A_{p'} ||\phi||_p$which implies by the Riesz representation theorem that $F^{-1}\rho*\phi$ represents L and

$$||L||_{\left(L^{p'}\right)'} = \left|\left|F^{-1}\rho * \phi\right|\right|_{L^p} \leq A_{p'} ||\phi||_p$$

Since $p' = p/(p-1)$, this proves the theorem.

It is possible to give verifiable conditions on ρ which imply 20.20. The condition on ρ which is presented here is the existence of a constant, C_0 such that

$$C_0 \geq \sup\{|\mathbf{x}|^{|\alpha|} |D^\alpha \rho (\mathbf{x})| : |\alpha| \leq L,\ \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}\}, L > n/2. \tag{20.25}$$

$$\rho \in C^L (\mathbb{R}^n \setminus \{\mathbf{0}\}) \text{ where } L \text{ is an integer.}$$

Here α is a multi-index and $|\alpha| = \sum_{i=1}^n \alpha_i$. The condition says roughly that ρ is pretty smooth away from $\mathbf{0}$ and all the partial derivatives vanish pretty fast as $|\mathbf{x}| \to \infty$. Also recall the notation

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

where $\alpha = (\alpha_1 \cdots \alpha_n)$. For more general conditions, see [29].

Lemma 20.9 *Let 20.25 hold and suppose $\psi \in C_c^\infty (\mathbb{R}^n \setminus \{\mathbf{0}\})$. Then for each α, $|\alpha| \leq L$, there exists a constant $C \equiv C(\alpha, n, \psi)$ independent of k such that*

$$\sup_{\mathbf{x} \in \mathbb{R}^n} |\mathbf{x}|^{|\alpha|} \left|D^\alpha \left(\rho (\mathbf{x}) \psi \left(2^k \mathbf{x}\right)\right)\right| \leq CC_0.$$

Proof:

$$|\mathbf{x}|^{|\alpha|} \left|D^\alpha \left(\rho (\mathbf{x}) \psi \left(2^k \mathbf{x}\right)\right)\right| \leq |\mathbf{x}|^{|\alpha|} \sum_{\beta+\gamma=\alpha} \left|D^\beta \rho (\mathbf{x})\right| 2^{k|\gamma|} \left|D^\gamma \psi \left(2^k \mathbf{x}\right)\right|$$

$$= \sum_{\beta+\gamma=\alpha} |\mathbf{x}|^{|\beta|} \left|D^{\beta}\rho(\mathbf{x})\right| \left|2^k\mathbf{x}\right|^{|\gamma|} \left|D^{\gamma}\psi\left(2^k\mathbf{x}\right)\right|$$

$$\leq C_0 C(\alpha, n) \sum_{|\gamma|\leq|\alpha|} \sup\{|\mathbf{z}|^{|\gamma|} |D^{\gamma}\psi(\mathbf{z})| : \mathbf{z}\in\mathbb{R}^n\} = C_0 C(\alpha, n, \psi)$$

and this proves the lemma.

Lemma 20.10 *There exists*

$$\phi \in C_c^{\infty}\left(\left[\mathbf{x} : 4^{-1} < |\mathbf{x}| < 4\right]\right),\ \phi(\mathbf{x}) \geq 0,$$

and

$$\sum_{k=-\infty}^{\infty} \phi\left(2^k\mathbf{x}\right) = 1$$

for each $\mathbf{x} \neq \mathbf{0}$.

Proof: Let

$$\psi \geq 0,\ \psi = 1 \text{ on } \left[2^{-1} \leq |\mathbf{x}| \leq 2\right],$$

$$\operatorname{spt}(\psi) \subseteq \left[4^{-1} < |\mathbf{x}| < 4\right].$$

Consider

$$g(\mathbf{x}) = \sum_{k=-\infty}^{\infty} \psi\left(2^k\mathbf{x}\right).$$

Then for each $\mathbf{x}$, only finitely many terms are not equal to 0. Also, $g(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$. To verify this last claim, note that for some k an integer, $|\mathbf{x}| \in \left[2^l, 2^{l+2}\right]$. Therefore, choose k an integer such that $2^k|\mathbf{x}| \in \left[2^{-1}, 2\right]$. For example, let $k = -l-1$. This works because $2^k|\mathbf{x}| \in \left[2^l 2^k, 2^{l+2}2^k\right] = \left[2^{l-l-1}, 2^{l+2-l-1}\right] = \left[2^{-1}, 2\right]$. Therefore, for this value of k, $\psi\left(2^k\mathbf{x}\right) = 1$ so $g(\mathbf{x}) > 0$.

Now notice that

$$g(2^r\mathbf{x}) = \sum_{k=-\infty}^{\infty} \psi\left(2^k 2^r\mathbf{x}\right) = \sum_{k=-\infty}^{\infty} \psi\left(2^k\mathbf{x}\right) = g(\mathbf{x}).$$

Let $\phi(\mathbf{x}) \equiv \psi(\mathbf{x})\, g(\mathbf{x})^{-1}$. Then

$$\sum_{k=-\infty}^{\infty} \phi\left(2^k\mathbf{x}\right) = \sum_{k=-\infty}^{\infty} \frac{\psi\left(2^k\mathbf{x}\right)}{g\left(2^k\mathbf{x}\right)} = g(\mathbf{x})^{-1} \sum_{k=-\infty}^{\infty} \psi\left(2^k\mathbf{x}\right) = 1$$

for each $\mathbf{x} \neq \mathbf{0}$. This proves the lemma.

Now define

$$\rho_m(\mathbf{x}) \equiv \sum_{k=-m}^{m} \rho(\mathbf{x})\,\phi\left(2^k\mathbf{x}\right),\ \gamma_k(\mathbf{x}) \equiv \rho(\mathbf{x})\,\phi\left(2^k\mathbf{x}\right).$$

Let $t > 0$ and let $|\mathbf{y}| \leq t$. Consider the problem of estimating

$$\int_{|\mathbf{x}|\geq 2t} \left| F^{-1}\gamma_k(\mathbf{x}-\mathbf{y}) - F^{-1}\gamma_k(\mathbf{x}) \right| dx. \tag{20.26}$$

In the following estimates, $C(a, b, \cdot\cdot\cdot, d)$ will denote a generic constant depending only on the indicated objects, $a, b, \cdot\cdot\cdot, d$. For the first estimate, note that since $|\mathbf{y}| \leq t$, 20.26 is no larger than

$$2\int_{|\mathbf{x}|\geq t} \left| F^{-1}\gamma_k(\mathbf{x}) \right| dx = 2\int_{|\mathbf{x}|\geq t} \left| F^{-1}\gamma_k(\mathbf{x}) \right| |\mathbf{x}|^{-L} |\mathbf{x}|^{L} dx$$

$$\leq 2\left(\int_{|\mathbf{x}|\geq t} |\mathbf{x}|^{-2L} dx\right)^{1/2} \left(\int_{|\mathbf{x}|\geq t} |\mathbf{x}|^{2L} \left| F^{-1}\gamma_k(\mathbf{x}) \right|^2 dx\right)^{1/2}$$

Using spherical coordinates and Plancherel's theorem,

$$\begin{aligned} &\leq C(n, L)\, t^{n/2-L} \left(\int |\mathbf{x}|^{2L} \left| F^{-1}\gamma_k(\mathbf{x}) \right|^2 dx\right)^{1/2} \\ &\leq C(n, L)\, t^{n/2-L} \left(\int \sum_{j=1}^n |x_j|^{2L} \left| F^{-1}\gamma_k(\mathbf{x}) \right|^2 dx\right)^{1/2} \\ &\leq C(n, L)\, t^{n/2-L} \left(\sum_{j=1}^n \int \left| F^{-1}D_j^L\gamma_k(\mathbf{x}) \right|^2 dx\right)^{1/2} \\ &= C(n, L)\, t^{n/2-L} \left(\sum_{j=1}^n \int_{S_k} \left| D_j^L\gamma_k(\mathbf{x}) \right|^2 dx\right)^{1/2} \end{aligned} \tag{20.27}$$

where

$$S_k \equiv \left[\mathbf{x} : 2^{-2-k} < |\mathbf{x}| < 2^{2-k}\right], \tag{20.28}$$

a set containing the support of γ_k. Now from the definition of γ_k,

$$\left| D_j^L\gamma_k(\mathbf{z}) \right| = \left| D_j^L \left(\rho(\mathbf{z})\, \phi\left(2^k\mathbf{z}\right)\right) \right|.$$

By Lemma 20.9, this is no larger than

$$C(L, n, \phi)\, C_0\, |\mathbf{z}|^{-L}. \tag{20.29}$$

It follows, using polar coordinates, that the last expression in 20.27 is no larger than

$$C(n, L, \phi, C_0)\, t^{n/2-L} \left(\int_{S_k} |\mathbf{z}|^{-2L} dz\right)^{1/2} \leq C(n, L, \phi, C_0)\, t^{n/2-L}. \tag{20.30}$$

$$\left(\int_{2^{-2-k}}^{2^{2-k}} \rho^{n-1-2L} d\rho\right)^{1/2} \leq C(n, L, \phi, C_0)\, t^{n/2-L} 2^{k(L-n/2)}.$$

Now estimate 20.26 in another way. The support of γ_k is in S_k, a bounded set, and so $F^{-1}\gamma_k$ is differentiable. Therefore,

$$\int_{|\mathbf{x}|\geq 2t} \left| F^{-1}\gamma_k(\mathbf{x}-\mathbf{y}) - F^{-1}\gamma_k(\mathbf{x}) \right| dx =$$

$$\int_{|\mathbf{x}|\geq 2t}\left|\int_0^1\sum_{j=1}^n D_jF^{-1}\gamma_k(\mathbf{x}-s\mathbf{y})\,y_j ds\right|dx$$

$$\leq \quad t\int_{|\mathbf{x}|\geq 2t}\int_0^1\sum_{j=1}^n \left|D_jF^{-1}\gamma_k(\mathbf{x}-s\mathbf{y})\right|dsdx$$
$$\leq \quad t\int\sum_{j=1}^n\left|D_jF^{-1}\gamma_k(\mathbf{x})\right|dx$$

$$\leq \quad t\sum_{j=1}^n\left(\int\left(1+\left|2^{-k}\mathbf{x}\right|^2\right)^{-L}dx\right)^{1/2}$$
$$\cdot\left(\int\left(1+\left|2^{-k}\mathbf{x}\right|^2\right)^{L}\left|D_jF^{-1}\gamma_k(\mathbf{x})\right|^2dx\right)^{1/2}$$

$$\leq C(n,L)\,t2^{kn/2}\sum_{j=1}^n\left(\int\left(1+\left|2^{-k}\mathbf{x}\right|^2\right)^{L}\left|D_jF^{-1}\gamma_k(\mathbf{x})\right|^2dx\right)^{1/2}. \tag{20.31}$$

Now consider the j^{th} term in the last sum in 20.31.

$$\begin{array}{c}\int\left(1+\left|2^{-k}\mathbf{x}\right|^2\right)^{L}\left|D_jF^{-1}\gamma_k(\mathbf{x})\right|^2dx\leq\\ C(n,L)\int\sum_{|\alpha|\leq L}2^{-2k|\alpha|}\mathbf{x}^{2\alpha}\left|D_jF^{-1}\gamma_k(\mathbf{x})\right|^2dx\\ =C(n,L)\sum_{|\alpha|\leq L}2^{-2k|\alpha|}\int\mathbf{x}^{2\alpha}\left|F^{-1}(\pi_j\gamma_k)(\mathbf{x})\right|^2dx\end{array} \tag{20.32}$$

where $\pi_j(\mathbf{z})\equiv z_j$. This last assertion follows from

$$D_j\int e^{-i\mathbf{x}\cdot\mathbf{y}}\gamma_k(\mathbf{y})\,dy=\int(-i)\,e^{-i\mathbf{x}\cdot\mathbf{y}}y_j\gamma_k(\mathbf{y})\,dy.$$

Therefore, a similar computation and Plancherel's theorem implies 20.32 equals

$$=C(n,L)\sum_{|\alpha|\leq L}2^{-2k|\alpha|}\int\left|F^{-1}D^\alpha(\pi_j\gamma_k)(\mathbf{x})\right|^2dx$$

$$=C(n,L)\sum_{|\alpha|\leq L}2^{-2k|\alpha|}\int_{S_k}\left|D^\alpha(z_j\gamma_k(\mathbf{z}))\right|^2dz \tag{20.33}$$

where S_k is given in 20.28. Now

$$\begin{aligned}\left|D^\alpha(z_j\gamma_k(\mathbf{z}))\right| &= 2^{-k}\left|D^\alpha\left(\rho(\mathbf{z})\,z_j2^k\phi\left(2^k\mathbf{z}\right)\right)\right|\\ &= 2^{-k}\left|D^\alpha\left(\rho(\mathbf{z})\,\psi_j\left(2^k\mathbf{z}\right)\right)\right|\end{aligned}$$

where $\psi_j(\mathbf{z}) \equiv z_j \phi(\mathbf{z})$. By Lemma 20.9, this is dominated by

$$2^{-k} C(\alpha, n, \phi, j, C_0) |\mathbf{z}|^{-|\alpha|}.$$

Therefore, 20.33 is dominated by

$$\begin{aligned}
& C(L, n, \phi, j, C_0) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} \int_{S_k} 2^{-2k} |\mathbf{z}|^{-2|\alpha|} dz \\
\leq \quad & C(L, n, \phi, j, C_0) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} 2^{-2k} \left(2^{-2-k}\right)^{(-2|\alpha|)} \left(2^{2-k}\right)^n \\
\leq \quad & C(L, n, \phi, j, C_0) \sum_{|\alpha| \leq L} 2^{-kn-2k}
\end{aligned}$$

$$\leq C(L, n, \phi, j, C_0) 2^{-kn} 2^{-2k}.$$

It follows that 20.31 is no larger than

$$C(L, n, \phi, C_0) t 2^{kn/2} 2^{-kn/2} 2^{-k} = C(L, n, \phi, C_0) t 2^{-k}. \tag{20.34}$$

It follows from 20.34 and 20.30 that if $|\mathbf{y}| \leq t$,

$$\int_{|\mathbf{x}| \geq 2t} \left| F^{-1}\gamma_k(\mathbf{x} - \mathbf{y}) - F^{-1}\gamma_k(\mathbf{x}) \right| dx \leq$$

$$C(L, n, \phi, C_0) \min\left(t 2^{-k}, \left(2^{-k} t\right)^{n/2 - L}\right).$$

With this inequality, the next lemma which is the desired result can be obtained.

Lemma 20.11 *There exists a constant depending only on the indicated objects, $C_1 = C(L, n, \phi, C_0)$ such that when $|\mathbf{y}| \leq t$,*

$$\int_{|\mathbf{x}| \geq 2t} \left| F^{-1}\rho(\mathbf{x} - \mathbf{y}) - F^{-1}\rho(\mathbf{x}) \right| dx \leq C_1$$

$$\int_{|\mathbf{x}| \geq 2t} \left| F^{-1}\rho_m(\mathbf{x} - \mathbf{y}) - F^{-1}\rho_m(\mathbf{x}) \right| dx \leq C_1. \tag{20.35}$$

Proof: $F^{-1}\rho = \lim_{m \to \infty} F^{-1}\rho_m$ in $L^2(\mathbb{R}^n)$. Let $m_k \to \infty$ be such that convergence is pointwise a.e. Then if $|\mathbf{y}| \leq t$, Fatou's lemma implies

$$\int_{|\mathbf{x}| \geq 2t} \left| F^{-1}\rho(\mathbf{x} - \mathbf{y}) - F^{-1}\rho(\mathbf{x}) \right| dx \leq$$

$$\liminf_{l \to \infty} \int_{|\mathbf{x}| \geq 2t} \left| F^{-1}\rho_{m_l}(\mathbf{x} - \mathbf{y}) - F^{-1}\rho_{m_l}(\mathbf{x}) \right| dx$$

$$\leq \liminf_{l\to\infty} \sum_{k=-m_l}^{m_l} \int_{|\mathbf{x}|\geq 2t} \left|F^{-1}\gamma_k(\mathbf{x}-\mathbf{y}) - F^{-1}\gamma_k(\mathbf{x})\right| dx$$

$$\leq C(L,n,\phi,C_0) \sum_{k=-\infty}^{\infty} \min\left(t2^{-k}, \left(2^{-k}t\right)^{n/2-L}\right). \tag{20.36}$$

Now consider the sum in 20.36,

$$\sum_{k=-\infty}^{\infty} \min\left(t2^{-k}, \left(2^{-k}t\right)^{n/2-L}\right). \tag{20.37}$$

$t2^j = \min\left(t2^j, \left(2^j t\right)^{n/2-L}\right)$ exactly when $t2^j \leq 1$. This occurs if and only if $j \leq -\ln(t)/\ln(2)$. Therefore 20.37 is no larger than

$$\sum_{j\leq -\ln(t)/\ln(2)} 2^j t + \sum_{j\geq -\ln(t)/\ln(2)} \left(2^j t\right)^{n/2-L}.$$

Letting $a = L - n/2$, this equals

$$\begin{aligned}
& t \sum_{k\geq \ln(t)/\ln(2)} 2^{-k} + t^{-\alpha} \sum_{j\geq -\ln(t)/\ln(2)} \left(2^{-a}\right)^j \\
\leq\ & 2t\left(\frac{1}{2}\right)^{\ln(t)/\ln(2)} + t^{-a}\left(\frac{1}{2^a}\right)^{-\ln(t)/\ln(2)} \\
=\ & 2t\left(\frac{1}{2}\right)^{\log_2(t)} + t^{-a}\left(\frac{1}{2^a}\right)^{-\log_2(t)} \\
=\ & 2+1=3.
\end{aligned}$$

Similarly, 20.35 holds. This proves the lemma.

Now it is possible to prove Mihlin's theorem.

Theorem 20.12 *(Mihlin's theorem) Suppose ρ satisfies*

$$C_0 \geq \sup\{|\mathbf{x}|^{|\alpha|} |D^\alpha \rho(\mathbf{x})| : |\alpha| \leq L,\ \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}\},$$

where L is an integer greater than $n/2$ and $\rho \in C^L(\mathbb{R}^n \setminus \{\mathbf{0}\})$. Then for every $p > 1$, there exists a constant A_p depending only on p, C_0, ϕ, n, and L, such that for all $\psi \in \mathcal{G}$,

$$\left|\left|F^{-1}\rho * \psi\right|\right|_p \leq A_p \,||\psi||_p.$$

Proof: Since ρ_m satisfies 20.35, and is obviously in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, Theorem 20.8 implies there exists a constant A_p depending only on $p, n, ||\rho_m||_\infty$, and C_1 such that for all $\psi \in \mathcal{G}$ and $p \in (1,\infty)$,

$$\left|\left|F^{-1}\rho_m * \psi\right|\right|_p \leq A_p \,||\psi||_p.$$

Now $||\rho_m||_\infty \leq ||\rho||_\infty$ because

$$|\rho_m(\mathbf{x})| \leq |\rho(\mathbf{x})| \sum_{k=-m}^{m} \phi\left(2^k \mathbf{x}\right) \leq |\rho(\mathbf{x})|. \tag{20.38}$$

Therefore, since $C_1 = C_1(L, n, \phi, C_0)$ and $C_0 \geq ||\rho||_\infty$,

$$\left|\left|F^{-1}\rho_m * \psi\right|\right|_p \leq A_p(L, n, \phi, C_0, p)\, ||\psi||_p .$$

In particular, A_p does not depend on m. Now, by 20.38, the observation that $\rho \in L^\infty(\mathbb{R}^n)$, $\lim_{m\to\infty} \rho_m(\mathbf{y}) = \rho(\mathbf{y})$ and the dominated convergence theorem, it follows that for $\theta \in \mathcal{G}$.

$$\left|\left(F^{-1}\rho * \psi\right)(\theta)\right| \equiv \left|(2\pi)^{n/2} \int \rho(\mathbf{x})\, F\psi(\mathbf{x})\, F^{-1}\theta(\mathbf{x})\, dx\right|$$

$$= \lim_{m\to\infty} \left|\left(F^{-1}\rho_m * \psi\right)(\theta)\right| \leq \lim_{m\to\infty} \sup \left|\left|F^{-1}\rho_m * \psi\right|\right|_p ||\theta||_{p'}$$
$$\leq A_p(L, n, \phi, C_0, p)\, ||\psi||_p\, ||\theta||_{p'} .$$

Hence $F^{-1}\rho * \psi \in L^p(\mathbb{R}^n)$ and $\left|\left|F^{-1}\rho * \psi\right|\right|_p \leq A_p ||\psi||_p$. This proves the theorem.

20.4 Singular Integrals

If $K \in L^1(\mathbb{R}^n)$ then when $p > 1$,

$$||K * f||_p \leq ||f||_p .$$

It turns out that some meaning can be assigned to $K * f$ for some functions K which are not in L^1. This involves assuming a certain form for K and exploiting cancellation. The resulting theory of singular integrals is very useful. To illustrate, an application will be given to the Helmholtz decomposition of vector fields in the next section. Like Mihlin's theorem, the theory presented here rests on Theorem 20.8, restated here for convenience.

Theorem 20.13 *Let $\rho \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and suppose*

$$\int_{|\mathbf{x}| \geq 2|\mathbf{y}|} \left|F^{-1}\rho(\mathbf{x}-\mathbf{y}) - F^{-1}\rho(\mathbf{x})\right| dx \leq C_1.$$

Then for each $p \in (1, \infty)$, there exists a constant, A_p, depending only on

$$p, n, ||\rho||_\infty ,$$

and C_1 such that for all $\phi \in \mathcal{G}$,

$$\left|\left|F^{-1}\rho * \phi\right|\right|_p \leq A_p ||\phi||_p .$$

Lemma 20.14 *Suppose*

$$K \in L^2(\mathbb{R}^n), \ ||FK||_\infty \leq B < \infty, \tag{20.39}$$

and

$$\int_{|\mathbf{x}|>2|\mathbf{y}|} |K(\mathbf{x}-\mathbf{y}) - K(\mathbf{x})| \, dx \leq B.$$

Then for all $p > 1$, there exists a constant, $A(p, n, B)$, depending only on the indicated quantities such that

$$||K * f||_p \leq A(p, n, B) \, ||f||_p$$

for all $f \in \mathcal{G}$.

Proof: Let $FK = \rho$ so $F^{-1}\rho = K$. Then from 20.39 $\rho \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $K = F^{-1}\rho$. By Theorem 20.8 listed above,

$$||K * f||_p = \left|\left|F^{-1}\rho * f\right|\right|_p \leq A(p, n, B) \, ||f||_p$$

for all $f \in \mathcal{G}$. This proves the lemma.

The next lemma provides a situation in which the above conditions hold.

Lemma 20.15 *Suppose*

$$|K(\mathbf{x})| \leq B \, |\mathbf{x}|^{-n}, \tag{20.40}$$

$$\int_{a<|\mathbf{x}|<b} K(\mathbf{x}) \, dx = 0, \tag{20.41}$$

$$\int_{|\mathbf{x}|>2|\mathbf{y}|} |K(\mathbf{x}-\mathbf{y}) - K(\mathbf{x})| \, dx \leq B. \tag{20.42}$$

Define

$$K_\varepsilon(\mathbf{x}) = \begin{cases} K(\mathbf{x}) \ \text{if } |\mathbf{x}| \geq \varepsilon, \\ 0 \ \text{if } |\mathbf{x}| < \varepsilon. \end{cases} \tag{20.43}$$

Then there exists a constant $C(n)$ such that

$$\int_{|\mathbf{x}|>2|\mathbf{y}|} |K_\varepsilon(\mathbf{x}-\mathbf{y}) - K_\varepsilon(\mathbf{x})| \, dx \leq C(n) \, B \tag{20.44}$$

and

$$||FK_\varepsilon||_\infty \leq C(n) \, B. \tag{20.45}$$

Proof: In the argument, $C(n)$ will denote a generic constant depending only on n. Consider 20.44 first. The integral is broken up according to whether $|\mathbf{x}|, |\mathbf{x}-\mathbf{y}| > \varepsilon$.

$\lvert\mathbf{x}\rvert$	$> \varepsilon$	$> \varepsilon$	$< \varepsilon$	$< \varepsilon$
$\lvert\mathbf{x}-\mathbf{y}\rvert$	$> \varepsilon$	$< \varepsilon$	$< \varepsilon$	$> \varepsilon$

$$\int_{|\mathbf{x}|\geq 2|\mathbf{y}|} |K_\varepsilon(\mathbf{x}-\mathbf{y}) - K_\varepsilon(\mathbf{x})|\,dx = \int_{|\mathbf{x}|\geq 2|\mathbf{y}|,|\mathbf{x}-\mathbf{y}|>\varepsilon,|\mathbf{x}|<\varepsilon} |K_\varepsilon(\mathbf{x}-\mathbf{y}) - K_\varepsilon(\mathbf{x})|\,dx + \\ + \int_{|\mathbf{x}|\geq 2|\mathbf{y}|,|\mathbf{x}-\mathbf{y}|<\varepsilon,|\mathbf{x}|\geq\varepsilon} |K_\varepsilon(\mathbf{x}-\mathbf{y}) - K_\varepsilon(\mathbf{x})|\,dx + \\ \int_{|\mathbf{x}|\geq 2|\mathbf{y}|,|\mathbf{x}-\mathbf{y}|>\varepsilon,|\mathbf{x}|>\varepsilon} |K_\varepsilon(\mathbf{x}-\mathbf{y}) - K_\varepsilon(\mathbf{x})|\,dx + \\ + \int_{|\mathbf{x}|\geq 2|\mathbf{y}|,|\mathbf{x}-\mathbf{y}|<\varepsilon,|\mathbf{x}|<\varepsilon} |K_\varepsilon(\mathbf{x}-\mathbf{y}) - K_\varepsilon(\mathbf{x})|\,dx. \tag{20.46}$$

Now consider the terms in the above expression. The last integral in 20.46 equals 0 from the definition of K_ε. The third integral on the right is no larger than B by the definition of K_ε and 20.42. Consider the second integral on the right. This integral is no larger than

$$\int_{|\mathbf{x}|\geq 2|\mathbf{y}|,|\mathbf{x}|\geq\varepsilon,|\mathbf{x}-\mathbf{y}|<\varepsilon} B\,|\mathbf{x}|^{-n}\,dx.$$

Now $|\mathbf{x}| \leq |\mathbf{y}| + \varepsilon \leq |\mathbf{x}|/2 + \varepsilon$ and so $|\mathbf{x}| < 2\varepsilon$. Thus this is no larger than

$$\int_{\varepsilon\leq|\mathbf{x}|\leq 2\varepsilon} B\,|\mathbf{x}|^{-n}\,dx = B\int_{S^{n-1}}\int_\varepsilon^{2\varepsilon} \rho^{n-1}\frac{1}{\rho^n}\,d\rho\,d\sigma \leq BC(n)\ln 2 = C(n)\,B.$$

It remains to estimate the first integral on the right in 20.46. This integral is bounded by

$$\int_{|\mathbf{x}|\geq 2|\mathbf{y}|,|\mathbf{x}-\mathbf{y}|>\varepsilon,|\mathbf{x}|<\varepsilon} B\,|\mathbf{x}-\mathbf{y}|^{-n}\,dx$$

In the integral above, $|\mathbf{x}| < \varepsilon$ and so $|\mathbf{x}-\mathbf{y}| - |\mathbf{y}| < \varepsilon$. Therefore, $|\mathbf{x}-\mathbf{y}| < \varepsilon + |\mathbf{y}| < \varepsilon + |\mathbf{x}|/2 < \varepsilon + \varepsilon/2 = (3/2)\,\varepsilon$. Hence $\varepsilon \leq |\mathbf{x}-\mathbf{y}| \leq (3/2)\,|\mathbf{x}-\mathbf{y}|$. Therefore, the above integral is no larger than

$$\int_\varepsilon^{(3/2)\varepsilon} B\,|\mathbf{z}|^{-n}\,dz = B\int_{S^{n-1}}\int_\varepsilon^{(3/2)\varepsilon} \rho^{-1}\,d\rho\,d\sigma = BC(n)\ln(3/2).$$

This establishes 20.44.

Now it remains to show 20.45, a statement about the Fourier transforms of K_ε. Fix ε and let $\mathbf{y} \neq \mathbf{0}$ also be given.

$$K_{\varepsilon R}(\mathbf{y}) \equiv \begin{cases} K_\varepsilon(\mathbf{y}) \text{ if } |\mathbf{y}| < R, \\ 0 \text{ if } |\mathbf{y}| \geq R \end{cases}$$

where $R > \frac{3\pi}{|\mathbf{y}|}$. (The 3 here isn't important. It just needs to be larger than 1.) Then

$$|FK_{\varepsilon R}(\mathbf{y})| \leq \left|\int_{0<|\mathbf{x}|<3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x})\,e^{-i\mathbf{x}\cdot\mathbf{y}}\,dx\right| + \left|\int_{3\pi|\mathbf{y}|^{-1}<|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x})\,e^{-i\mathbf{x}\cdot\mathbf{y}}\,dx\right|$$

$$= \mathbf{A} + \mathbf{B}. \tag{20.47}$$

Consider $\mathbf{A}$. By 20.41

$$\int_{\varepsilon<|\mathbf{x}|<3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x})\, dx = 0$$

and so

$$\mathbf{A} = \left| \int_{\varepsilon<|\mathbf{x}|<3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x}) \left(e^{-i\mathbf{x}\cdot\mathbf{y}} - 1\right) dx \right|$$

Now

$$\left|e^{-i\mathbf{x}\cdot\mathbf{y}} - 1\right| = \left|2 - 2\cos(\mathbf{x}\cdot\mathbf{y})\right|^{1/2} \leq 2\left|\mathbf{x}\cdot\mathbf{y}\right| \leq 2\left|\mathbf{x}\right|\left|\mathbf{y}\right|$$

so, using polar coordinates, this expression is no larger than

$$2B \int_{\varepsilon<|\mathbf{x}|<3\pi|\mathbf{y}|^{-1}} |\mathbf{x}|^{-n}\, |\mathbf{x}|\, |\mathbf{y}|\, dx \leq C(n)\, B\, |\mathbf{y}| \int_\varepsilon^{3\pi/|\mathbf{y}|} d\rho \leq BC(n).$$

Next, consider $\mathbf{B}$. This estimate is based on the trick which follows. Let

$$\mathbf{z} \equiv \mathbf{y}\pi / |\mathbf{y}|^2$$

so that

$$|\mathbf{z}| = \pi / |\mathbf{y}|\,,\ \mathbf{z}\cdot\mathbf{y} = \pi.$$

Then

$$\begin{aligned}\int_{3\pi|\mathbf{y}|^{-1}<|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx &= \tfrac{1}{2} \int_{3\pi|\mathbf{y}|^{-1}<|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx \\ &- \tfrac{1}{2} \int_{3\pi|\mathbf{y}|^{-1}<|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x})\, e^{-i(\mathbf{x}+\mathbf{z})\cdot\mathbf{y}} dx.\end{aligned} \tag{20.48}$$

Here is why. Note in the second of these integrals,

$$\begin{aligned} & -\frac{1}{2} \int_{3\pi|\mathbf{y}|^{-1}<|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x})\, e^{-i(\mathbf{x}+\mathbf{z})\cdot\mathbf{y}} dx \\ = & -\frac{1}{2} \int_{3\pi|\mathbf{y}|^{-1}<|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x})\, e^{-i\mathbf{x}\cdot\mathbf{y}} e^{-i\mathbf{z}\cdot\mathbf{y}} dx \\ = & -\frac{1}{2} \int_{3\pi|\mathbf{y}|^{-1}<|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x})\, e^{-i\mathbf{x}\cdot\mathbf{y}} e^{-i\pi} dx \\ = & \ \frac{1}{2} \int_{3\pi|\mathbf{y}|^{-1}<|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx. \end{aligned}$$

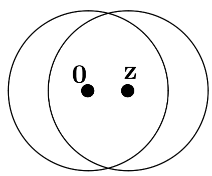
Then changing the variables in 20.48,,

$$\int_{3\pi|\mathbf{y}|^{-1}<|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx$$
$$= \frac{1}{2}\int_{3\pi|\mathbf{y}|^{-1}<|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx - \frac{1}{2}\int_{3\pi|\mathbf{y}|^{-1}<|\mathbf{x}-\mathbf{z}|\leq R} K_\varepsilon(\mathbf{x}-\mathbf{z})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx.$$

Thus

$$\int_{3\pi|\mathbf{y}|^{-1}<|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx = \frac{1}{2}\int_{|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx - \frac{1}{2}\int_{|\mathbf{x}-\mathbf{z}|\leq R} K_\varepsilon(\mathbf{x}-\mathbf{z})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx + \frac{1}{2}\int_{|\mathbf{x}-\mathbf{z}|\leq 3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x}-\mathbf{z})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx - \frac{1}{2}\int_{|\mathbf{x}|\leq 3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx. \tag{20.49}$$

Since $|\mathbf{z}| = \pi/|\mathbf{y}|$, it follows $|\mathbf{z}| = \frac{\pi}{|\mathbf{y}|} < \frac{3\pi}{|\mathbf{y}|} < R$ and so the following picture describes the situation. In this picture, the radius of each ball equals either R or $3\pi|\mathbf{y}|^{-1}$ and each integral above is taken over one of the two balls in the picture, either the one centered at $\mathbf{0}$ or the one centered at $\mathbf{z}$.



To begin with, consider the integrals which involve $K_\varepsilon(\mathbf{x}-\mathbf{z})$.

$$\int_{|\mathbf{x}-\mathbf{z}|\leq R} K_\varepsilon(\mathbf{x}-\mathbf{z})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx = \int_{|\mathbf{x}|\leq R} K_\varepsilon(\mathbf{x}-\mathbf{z})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx - \int_{|\mathbf{x}-\mathbf{z}|>R,|\mathbf{x}|<R} K_\varepsilon(\mathbf{x}-\mathbf{z})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx + \int_{|\mathbf{x}-\mathbf{z}|<R,|\mathbf{x}|>R} K_\varepsilon(\mathbf{x}-\mathbf{z})\, e^{-i\mathbf{x}\cdot\mathbf{y}} dx. \tag{20.50}$$

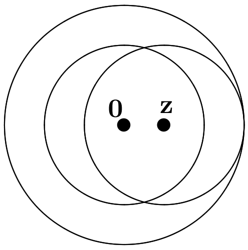
Look at the picture. Similarly,

$$\begin{aligned} &\int_{|\mathbf{x}-\mathbf{z}|\leq 3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x}-\mathbf{z})\, e^{-i\mathbf{x}\cdot\mathbf{y}}dx \\ &= \int_{|\mathbf{x}|\leq 3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x}-\mathbf{z})\, e^{-i\mathbf{x}\cdot\mathbf{y}}dx \\ &- \int_{|\mathbf{x}-\mathbf{z}|>3\pi|\mathbf{y}|^{-1},|\mathbf{x}|<3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x}-\mathbf{z})\, e^{-i\mathbf{x}\cdot\mathbf{y}}dx + \\ &\int_{|\mathbf{x}-\mathbf{z}|<3\pi|\mathbf{y}|^{-1},|\mathbf{x}|>3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x}-\mathbf{z})\, e^{-i\mathbf{x}\cdot\mathbf{y}}dx. \end{aligned} \tag{20.51}$$

The last integral in 20.50 is taken over a set that is contained in

$$B(\mathbf{0}, R+|\mathbf{z}|) \setminus B(\mathbf{0}, R)$$

illustrated in the following picture as the region between the small ball centered at $\mathbf{0}$ and the big ball which surrounds the two small balls



and so this integral is dominated by

$$B\left(\frac{1}{(R-|\mathbf{z}|)^n}\right)\alpha(n)\left((R+|\mathbf{z}|)^n - R^n\right),$$

an expression which converges to 0 as $R\to\infty$. Similarly, the second integral on the right in 20.50 converges to zero as $R\to\infty$. Now consider the last two integrals in 20.51. Letting $3\pi|\mathbf{y}|^{-1}$ play the role of R and using $|\mathbf{z}| = \pi/|\mathbf{y}|$, these are each dominated by an expression of the form

$$B\left(\frac{1}{\left(3\pi|\mathbf{y}|^{-1}-|\mathbf{z}|\right)^n}\right)\alpha(n)\left(\left(3\pi|\mathbf{y}|^{-1}+|\mathbf{z}|\right)^n - \left(3\pi|\mathbf{y}|^{-1}\right)^n\right)$$

$$= B\left(\frac{1}{\left(3\pi|\mathbf{y}|^{-1}-\pi|\mathbf{y}|^{-1}\right)^n}\right)\alpha(n)\cdot \left(\left(3\pi|\mathbf{y}|^{-1}+\pi|\mathbf{y}|^{-1}\right)^n - \left(3\pi|\mathbf{y}|^{-1}\right)^n\right)$$

$$= \alpha(n) B \frac{|\mathbf{y}|^n}{(2\pi)^n} \frac{1}{|\mathbf{y}|^n} ((4\pi)^n - (3\pi)^n) = C(n) B.$$

Returning to 20.49, the terms involving $\mathbf{x} - \mathbf{y}$ have now been estimated. Thus, collecting the terms which have not yet been estimated along with those that have,

$$\mathbf{B} = \left| \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{y}} dx \right|$$

$$\leq \frac{1}{2} \left| \int_{|\mathbf{x}|<R} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{y}} dx - \int_{|\mathbf{x}|<R} K_\varepsilon(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x}\cdot\mathbf{y}} dx \right.$$

$$\left. + \int_{|\mathbf{x}|<3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x}\cdot\mathbf{y}} dx - \int_{|\mathbf{x}|<3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{y}} dx \right|$$

$$+ C(n) B + g(R)$$

where $g(R) \to 0$ as $R \to \infty$. Using $|\mathbf{z}| = \pi / |\mathbf{y}|$ again,

$$\mathbf{B} \leq \frac{1}{2} \int_{3|\mathbf{z}|<|\mathbf{x}|<R} |K_\varepsilon(\mathbf{x}) - K_\varepsilon(\mathbf{x}-\mathbf{z})| dx + C(n) B + g(R).$$

But the integral in the above is dominated by $C(n) B$ by 20.44 which was established earlier. Therefore, from 20.47,

$$|FK_{\varepsilon R}| \leq C(n) B + g(R)$$

where $g(R) \to 0$.

Now $K_{\varepsilon R} \to K_\varepsilon$ in $L^2(\mathbb{R}^n)$ because

$$\begin{aligned} ||K_{\varepsilon R} - K_\varepsilon||_{L^2(\mathbb{R}^n)} &\leq B \int_{|\mathbf{x}|>R} \frac{1}{|\mathbf{x}|^{2n}} dx \\ &= B \int_{S^{n-1}} \int_R^\infty \frac{1}{\rho^{n+1}} d\rho d\sigma, \end{aligned}$$

which converges to 0 as $R \to \infty$ and so $FK_{\varepsilon R} \to FK_\varepsilon$ in $L^2(\mathbb{R}^n)$ by Plancherel's theorem. Therefore, by taking a subsequence, still denoted by R, $FK_{\varepsilon R}(\mathbf{y}) \to FK_\varepsilon(\mathbf{y})$ a.e. which shows

$$|FK_\varepsilon(\mathbf{y})| \leq C(n) B \text{ a.e.}$$

This proves the lemma.

Corollary 20.16 *Suppose 20.40 - 20.42 hold. Then if $g \in C_c^1(\mathbb{R}^n)$, $K_\varepsilon * g$ converges uniformly and in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$.*

Proof:

$$K_\varepsilon * g(\mathbf{x}) \equiv \int K_\varepsilon(\mathbf{y}) g(\mathbf{x}-\mathbf{y})\,dy.$$

Let $0 < \eta < \varepsilon$. Then since $g \in C_c^1(\mathbb{R}^n)$, there exists a constant, K such that $K|\mathbf{u}-\mathbf{v}| \geq |g(\mathbf{u}) - g(\mathbf{v})|$ for all $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$.

$$\begin{aligned} |K_\varepsilon * g(\mathbf{x}) - K_\eta * g(\mathbf{x})| &\leq BK \int_{\eta<|\mathbf{y}|<\varepsilon} \frac{1}{|\mathbf{y}|^n} |\mathbf{y}|\,dy \\ &= BK \int_{S^{n-1}} \int_\eta^\varepsilon d\rho d\sigma = C_n |\varepsilon - \eta|. \end{aligned}$$

This proves the corollary.

Theorem 20.17 *Suppose 20.40 - 20.42. Then for K_ε given by 20.43 and $p > 1$, there exists a constant $A(p,n,B)$ such that for all $f \in L^p(\mathbb{R}^n)$,*

$$||K_\varepsilon * f||_p \leq A(p,n,B)\,||f||_p. \tag{20.52}$$

Also, for each $f \in L^p(\mathbb{R}^n)$,

$$Tf \equiv \lim_{\varepsilon\to 0} K_\varepsilon * f \tag{20.53}$$

exists in $L^p(\mathbb{R}^n)$ and for all $f \in L^p(\mathbb{R}^n)$,

$$||Tf||_p \leq A(p,n,B)\,||f||_p. \tag{20.54}$$

Thus T is a linear and continuous map defined on $L^p(\mathbb{R}^n)$ for each $p > 1$.

Proof: From 20.40 it follows $K_\varepsilon \in L^{p'}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ where, as usual, $1/p + 1/p' = 1$. By continuity of translation in $L^{p'}(\mathbb{R}^n)$, $x \to K_\varepsilon * f(x)$ is a continuous function.By Lemma 20.15, $||FK_\varepsilon||_\infty \leq C(n)B$ for all ε. Therefore, by Lemma 20.14,

$$||K_\varepsilon * g||_p \leq A(p,n,B)\,||g||_p$$

for all $g \in \mathcal{G}$. Now let $f \in L^p(\mathbb{R}^n)$ and $g_k \to f$ in $L^p(\mathbb{R}^n)$ where $g_k \in \mathcal{G}$. Then

$$\begin{aligned} |K_\varepsilon * f(\mathbf{x}) - K_\varepsilon * g_k(\mathbf{x})| &\leq \int |K_\varepsilon(\mathbf{x}-\mathbf{y})|\,|g_k(\mathbf{y}) - f(\mathbf{y})|\,dy \\ &\leq ||K_\varepsilon||_{p'}\,||g_k - f||_p \end{aligned}$$

which shows that $K_\varepsilon * g_k(\mathbf{x}) \to K_\varepsilon * f(\mathbf{x})$ pointwise and so by Fatou's lemma,

$$\begin{aligned} ||K_\varepsilon * f||_p &\leq \lim\inf_{k\to\infty} ||K_\varepsilon * g_k||_p \leq \lim\inf_{k\to\infty} A(p,n,B)\,||g_k||_p \\ &= A(p,n,B)\,||f||_p. \end{aligned}$$

This verifies 20.52.

To verify 20.53, let $\delta > 0$ be given and let

$$f \in L^p(\mathbb{R}^n), g \in C_c^\infty(\mathbb{R}^n).$$

$$\begin{aligned} ||K_\varepsilon * f - K_\eta * f||_p &\leq ||K_\varepsilon * (f-g)||_p + ||K_\varepsilon * g - K_\eta * g||_p \\ &\quad + ||K_\eta * (f-g)||_p \\ &\leq 2A(p,n,B)\,||f-g||_p + ||K_\varepsilon * g - K_\eta * g||_p . \end{aligned}$$

Choose g such that $2A(p,n,B)\,||f-g||_p \leq \delta/2$. Then if ε, η are small enough, Corollary 20.16 implies the last term is also less than $\delta/2$. Thus, $\lim_{\varepsilon\to 0} K_\varepsilon * f$ exists in $L^p(\mathbb{R}^n)$. Let Tf be the element of $L^p(\mathbb{R}^n)$ to which it converges. Then 20.54 follows and T is obviously linear because

$$\begin{aligned} T(af+bg) &= \lim_{\varepsilon\to 0} K_\varepsilon * (af+bg) = \lim_{\varepsilon\to 0} (aK_\varepsilon * f + bK_\varepsilon * g) \\ &= aTf + bTg. \end{aligned}$$

This proves the theorem.

When do conditions 20.40-20.42 hold? It turns out this happens for K given by the following.

$$K(\mathbf{x}) \equiv \frac{\Omega(\mathbf{x})}{|\mathbf{x}|^n}, \tag{20.55}$$

where

$$\Omega(\lambda\mathbf{x}) = \Omega(\mathbf{x}) \text{ for all } \lambda > 0, \tag{20.56}$$

$$\Omega \text{ is Lipschitz on } S^{n-1},$$

$$\int_{S^{n-1}} \Omega(\mathbf{x})\, d\sigma = 0. \tag{20.57}$$

Theorem 20.18 *For K given by 20.55 - 20.57, it follows there exists a constant B such that*

$$|K(\mathbf{x})| \leq B\,|\mathbf{x}|^{-n}, \tag{20.58}$$

$$\int_{a<|\mathbf{x}|<b} K(\mathbf{x})\, dx = 0, \tag{20.59}$$

$$\int_{|\mathbf{x}|>2|\mathbf{y}|} |K(\mathbf{x}-\mathbf{y}) - K(\mathbf{x})|\, dx \leq B. \tag{20.60}$$

Consequently, the conclusions of Theorem 20.17 hold also.

Proof: 20.58 is obvious. To verify 20.59,

$$\begin{aligned} \int_{a<|\mathbf{x}|<b} K(\mathbf{x})\, dx &= \int_a^b \int_{S^{n-1}} \frac{\Omega(\rho\mathbf{w})}{\rho^n} \rho^{n-1} d\sigma d\rho \\ &= \int_a^b \frac{1}{\rho} \int_{S^{n-1}} \Omega(\mathbf{w})\, d\sigma d\rho = 0. \end{aligned}$$

It remains to show 20.60.

$$\begin{aligned} K(\mathbf{x}-\mathbf{y}) - K(\mathbf{x}) &= |\mathbf{x}-\mathbf{y}|^{-n} \left(\Omega\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} \right) - \Omega\left(\frac{\mathbf{x}}{|\mathbf{x}|} \right) \right) \\ &\quad + \Omega(\mathbf{x}) \left(\frac{1}{|\mathbf{x}-\mathbf{y}|^n} - \frac{1}{|\mathbf{x}|^n} \right) \end{aligned} \tag{20.61}$$

where 20.56 was used to write $\Omega\left(\frac{\mathbf{z}}{|\mathbf{z}|}\right) = \Omega(\mathbf{z})$. The first group of terms in 20.61 is dominated by

$$|\mathbf{x}-\mathbf{y}|^{-n} \operatorname{Lip}(\Omega) \left| \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} - \frac{\mathbf{x}}{|\mathbf{x}|} \right|$$

and an estimate is required for $|\mathbf{x}| > 2|\mathbf{y}|$. Since $|\mathbf{x}| > 2|\mathbf{y}|$,

$$|\mathbf{x}-\mathbf{y}|^{-n} \leq (|\mathbf{x}| - |\mathbf{y}|)^{-n} \leq \frac{2^n}{|\mathbf{x}|^n}.$$

Also

$$\left| \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|} - \frac{\mathbf{x}}{|\mathbf{x}|} \right| = \left| \frac{(\mathbf{x}-\mathbf{y})|\mathbf{x}| - \mathbf{x}|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}||\mathbf{x}-\mathbf{y}|} \right|$$

$$\leq \left| \frac{(\mathbf{x}-\mathbf{y})|\mathbf{x}| - \mathbf{x}|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}|(|\mathbf{x}| - |\mathbf{y}|)} \right| \leq \left| \frac{(\mathbf{x}-\mathbf{y})|\mathbf{x}| - \mathbf{x}|\mathbf{x}-\mathbf{y}|}{|\mathbf{x}|(|\mathbf{x}|/2)} \right|$$

$$= \frac{2}{|\mathbf{x}|^2} \left| \mathbf{x}|\mathbf{x}| - \mathbf{y}|\mathbf{x}| - \mathbf{x}|\mathbf{x}-\mathbf{y}| \right| = \frac{2}{|\mathbf{x}|^2} \left| \mathbf{x}(|\mathbf{x}| - |\mathbf{x}-\mathbf{y}|) - \mathbf{y}|\mathbf{x}| \right|$$

$$\leq \frac{2}{|\mathbf{x}|^2} |\mathbf{x}| \left| |\mathbf{x}| - |\mathbf{x}-\mathbf{y}| \right| + |\mathbf{y}||\mathbf{x}| \leq \frac{2}{|\mathbf{x}|^2} (|\mathbf{x}| |\mathbf{x} - (\mathbf{x}-\mathbf{y})| + |\mathbf{y}||\mathbf{x}|)$$

$$\leq \frac{4}{|\mathbf{x}|^2} |\mathbf{x}||\mathbf{y}| = 4\frac{|\mathbf{y}|}{|\mathbf{x}|}.$$

Therefore,

$$\int_{|\mathbf{x}|>2|\mathbf{y}|} |\mathbf{x}-\mathbf{y}|^{-n} \left| \Omega\left(\frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\right) - \Omega\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \right| dx$$

$$\leq 4(2^n) \int_{|\mathbf{x}|>2|\mathbf{y}|} \frac{1}{|\mathbf{x}|^n} \frac{|\mathbf{y}|}{|\mathbf{x}|} dx \operatorname{Lip}(\Omega)$$

$$= C(n, \operatorname{Lip}\Omega) \int_{|\mathbf{x}|>2|\mathbf{y}|} \frac{|\mathbf{y}|}{|\mathbf{x}|^{n+1}} dx$$

$$= C(n, \operatorname{Lip}\Omega) \int_{|\mathbf{u}|>2} \frac{1}{|\mathbf{u}|^{n+1}} du. \tag{20.62}$$

It remains to consider the second group of terms in 20.61 when $|\mathbf{x}| > 2|\mathbf{y}|$.

$$\left| \frac{1}{|\mathbf{x}-\mathbf{y}|^n} - \frac{1}{|\mathbf{x}|^n} \right| = \left| \frac{|\mathbf{x}|^n - |\mathbf{x}-\mathbf{y}|^n}{|\mathbf{x}-\mathbf{y}|^n |\mathbf{x}|^n} \right|$$

$$\leq \frac{2^n}{|\mathbf{x}|^{2n}} \left| |\mathbf{x}|^n - |\mathbf{x}-\mathbf{y}|^n \right|$$

$$\leq \frac{2^n}{|\mathbf{x}|^{2n}} |\mathbf{y}| \Big[|\mathbf{x}|^{n-1} + |\mathbf{x}|^{n-2} |\mathbf{x}-\mathbf{y}| +$$

$$\cdots + |\mathbf{x}||\mathbf{x}-\mathbf{y}|^{n-2} + |\mathbf{x}-\mathbf{y}|^{n-1} \Big]$$

$$\leq \frac{2^n |\mathbf{y}| C(n) |\mathbf{x}|^{n-1}}{|\mathbf{x}|^{2n}} = \frac{C(n) 2^n |\mathbf{y}|}{|\mathbf{x}|^{n+1}}.$$

Thus

$$\int_{|\mathbf{x}|>2|\mathbf{y}|} \left| \Omega(\mathbf{x}) \left(\frac{1}{|\mathbf{x}-\mathbf{y}|^n} - \frac{1}{|\mathbf{x}|^n} \right) \right| dx$$

$$\leq C(n) \int_{|\mathbf{x}|>2|\mathbf{y}|} \frac{|\mathbf{y}|}{|\mathbf{x}|^{n+1}} dx$$

$$\leq C(n) \int_{|\mathbf{u}|>2} \frac{1}{|\mathbf{u}|^{n+1}} du. \tag{20.63}$$

From 20.62 and 20.63,

$$\int_{|\mathbf{x}|>2|\mathbf{y}|} |K(\mathbf{x}-\mathbf{y}) - K(\mathbf{x})| \, dx \leq C(n, \operatorname{Lip}\Omega).$$

This proves the theorem.

20.5 Helmholtz Decompositions

It turns out that every vector field which has its components in L^p can be written as a sum of a gradient and a vector field which has zero divergence. This is a very remarkable result, especially when applied to vector fields which are only in L^p. Recall that for u a function of n variables, $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$.

Definition 20.19 *Define*

$$\Phi(\mathbf{y}) \equiv \begin{cases} -\frac{1}{a_1} \ln |\mathbf{y}|, & \text{if } n = 2, \\ \frac{1}{(n-2)a_{n-1}} |\mathbf{y}|^{2-n}, & \text{if } n > 2. \end{cases}$$

where a_k denotes the area of the unit sphere, S^k.

Then it is routine to verify $\Delta\Phi = 0$ away from 0. In fact, if $n > 2$,

$$\Phi_{,ii}(\mathbf{y}) = C_n \left[\frac{1}{|\mathbf{y}|^n} - n \frac{y_i^2}{|\mathbf{y}|^{n+2}} \right], \ \Phi_{,ij}(\mathbf{y}) = C_n \frac{y_i y_j}{|\mathbf{y}|^{n+2}}, \tag{20.64}$$

while if $n = 2$,

$$\Phi_{,22}(\mathbf{y}) = C_2 \frac{y_1^2 - y_2^2}{(y_1^2 + y_2^2)^2}, \ \Phi_{,11}(\mathbf{y}) = C_2 \frac{y_2^2 - y_1^2}{(y_1^2 + y_2^2)^2},$$

$$\Phi_{,ij}(\mathbf{y}) = C_2 \frac{y_1 y_2}{(y_1^2 + y_2^2)^2}.$$

Also,

$$\nabla\Phi(\mathbf{y}) = \frac{-\mathbf{y}}{a_{n-1} |\mathbf{y}|^n}. \tag{20.65}$$

In the above the subscripts following a comma denote partial derivatives.

Lemma 20.20 *For* $n \geq 2$
$$\Phi_{,ij}(\mathbf{y}) = \frac{\Omega_{ij}(\mathbf{y})}{|\mathbf{y}|^n}$$
where
$$\Omega_{ij} \text{ is Lipschitz continuous on } S^{n-1}, \tag{20.66}$$
$$\Omega_{ij}(\lambda \mathbf{y}) = \Omega_{ij}(\mathbf{y}), \tag{20.67}$$
for all $\lambda > 0$*, and*
$$\int_{S^{n-1}} \Omega_{ij}(\mathbf{y})\, d\sigma = 0. \tag{20.68}$$

Proof:

Proof: The case $n = 2$ is left to the reader. 20.66 and 20.67 are obvious from the above descriptions. It remains to verify 20.68. If $n \geq 3$ and $i \neq j$, then this formula is also clear from 20.64. Thus consider the case when $n \geq 3$ and $i = j$. By symmetry,
$$I \equiv \int_{S^{n-1}} 1 - ny_i^2 d\sigma = \int_{S^{n-1}} 1 - ny_j^2 d\sigma.$$
Hence
$$\begin{aligned} nI &= \sum_{i=1}^n \int_{S^{n-1}} 1 - ny_i^2 d\sigma = \int_{S^{n-1}} \left(n - n\sum_i y_i^2\right) d\sigma \\ &= \int_{S^{n-1}} (n-n)\, d\sigma = 0. \end{aligned}$$
This proves the lemma.

Let U be a bounded open set locally on one side of its boundary having Lipschitz boundary so the divergence theorem holds and let $B = B(\mathbf{0}, R)$ where
$$B \supseteq U - U \equiv \{\mathbf{x} - \mathbf{y} : \mathbf{x} \in U, \mathbf{y} \in U\}$$
Let $f \in C_c^\infty(U)$ and define for $\mathbf{x} \in U$,
$$u(\mathbf{x}) \equiv \int_B \Phi(\mathbf{y}) f(\mathbf{x} - \mathbf{y})\, dy = \int_U \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, dy.$$
Let $h(\mathbf{y}) = f(\mathbf{x} - \mathbf{y})$. Then since Φ is in $L^1(B)$,
$$\Delta u(\mathbf{x}) = \int_B \Phi(\mathbf{y}) \Delta f(\mathbf{x} - \mathbf{y})\, dy = \int_B \Phi(\mathbf{y}) \Delta h(\mathbf{y})\, dy$$
$$= \int_{B \setminus B(\mathbf{0},\varepsilon)} \nabla \cdot (\nabla h(\mathbf{y}) \Phi(\mathbf{y})) - \nabla \Phi(\mathbf{y}) \cdot \nabla h(\mathbf{y})\, dy$$
$$+ \int_{B(\mathbf{0},\varepsilon)} \Phi(\mathbf{y}) \Delta h(\mathbf{y})\, dy.$$

The last term converges to 0 as $\varepsilon \to 0$ because Φ is in L^1 and Δh is bounded. Since $\operatorname{spt}(h) \subseteq B$, the divergence theorem implies

$$\Delta u(\mathbf{x}) = -\int_{\partial B(\mathbf{0},\varepsilon)} \Phi(\mathbf{y}) \nabla h(\mathbf{y}) \cdot \mathbf{n} d\sigma - \int_{B \setminus B(\mathbf{0},\varepsilon)} \nabla \Phi(\mathbf{y}) \cdot \nabla h(\mathbf{y}) \, dy + e(\varepsilon) \quad (20.69)$$

where here and below, $e(\varepsilon) \to 0$ as $\varepsilon \to 0$. The first term in 20.69 converges to 0 as $\varepsilon \to 0$ because

$$\left| \int_{\partial B(\mathbf{0},\varepsilon)} \Phi(\mathbf{y}) \nabla h(\mathbf{y}) \cdot \mathbf{n} d\sigma \right| \leq \begin{cases} C_{nh} \frac{1}{\varepsilon^{n-2}} \varepsilon^{n-1} = C_{nh} \varepsilon \text{ if } n > 2 \\ C_h (\ln \varepsilon) \varepsilon \text{ if } n = 2 \end{cases}$$

and since $\Delta \Phi(\mathbf{y}) = 0$,

$$\nabla \Phi(\mathbf{y}) \cdot \nabla h(\mathbf{y}) = \nabla \cdot (\nabla \Phi(\mathbf{y}) h(\mathbf{y})).$$

Consequently

$$\Delta u(\mathbf{x}) = -\int_{B \setminus B(\mathbf{0},\varepsilon)} \nabla \cdot (\nabla \Phi(\mathbf{y}) h(\mathbf{y})) \, dy + e(\varepsilon).$$

Thus, by the divergence theorem, 20.65, and the definition of h above,

$$\begin{aligned} \Delta u(\mathbf{x}) &= \int_{\partial B(\mathbf{0},\varepsilon)} f(\mathbf{x} - \mathbf{y}) \nabla \Phi(\mathbf{y}) \cdot \mathbf{n} d\sigma + e(\varepsilon) \\ &= \int_{\partial B(\mathbf{0},\varepsilon)} f(\mathbf{x} - \mathbf{y}) \left(-\frac{\mathbf{y}}{a_{n-1} |\mathbf{y}|^n} \right) \cdot \left(-\frac{\mathbf{y}}{|\mathbf{y}|} \right) d\sigma + e(\varepsilon) \\ &= -\left(\int_{\partial B(\mathbf{0},\varepsilon)} f(\mathbf{x} - \mathbf{y}) \, d\sigma(y) \right) \frac{1}{a_{n-1} \varepsilon^{n-1}} + e(\varepsilon). \end{aligned}$$

Letting $\varepsilon \to 0$,

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}).$$

This proves the following lemma.

Lemma 20.21 *Let U be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary and let $B \supseteq U - U$ where $B = B(\mathbf{0},R)$. Let $f \in C_c^\infty(U)$. Then for $\mathbf{x} \in U$,*

$$\int_B \Phi(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) \, dy = \int_U \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) \, dy,$$

and it follows that if u is given by one of the above formulas, then for all $x \in U$,

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}).$$

Theorem 20.22 *Let $f \in L^p(U)$. Then there exists $u \in L^p(U)$ whose weak derivatives are also in $L^p(U)$ such that in the sense of weak derivatives,*

$$-\Delta u = f.$$

It is given by

$$u(\mathbf{x}) = \int_B \Phi(\mathbf{y}) \widetilde{f}(\mathbf{x}-\mathbf{y})\,dy = \int_U \Phi(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\,dy \tag{20.70}$$

where $\widetilde{f}$ *denotes the zero extension of* f *off of* U.

Proof: Let $f \in L^p(U)$ and let $f_k \in C_c^\infty(U)$, $||f_k - f||_{L^p(U)} \to 0$, and let u_k be given by 20.70 with f_k in place of f. Then by Minkowski's inequality,

$$\begin{aligned} ||u-u_k||_{L^p(U)} &= \left(\int_U \left(\int_B \Phi(\mathbf{y}) \left|\widetilde{f}(\mathbf{x}-\mathbf{y}) - f_k(\mathbf{x}-\mathbf{y})\right| dy\right)^p dx\right)^{1/p} \\ &\leq \left(\int_B |\Phi(\mathbf{y})| \left(\int_U \left|\widetilde{f}(\mathbf{x}-\mathbf{y}) - f_k(\mathbf{x}-\mathbf{y})\right|^p dx\right)^{1/p} dy\right) \\ &\leq \int_B |\Phi(\mathbf{y})|\,dy\,||f-f_k||_{L^p(U)} = C(B)\,||f-f_k||_{L^p(U)} \end{aligned}$$

and so $u_k \to u$ in $L^p(U)$. Also

$$u_{k,i}(\mathbf{x}) = \int_U \Phi_{,i}(\mathbf{x}-\mathbf{y}) f_k(\mathbf{y})\,dy = \int_B f_k(\mathbf{x}-\mathbf{y}) \Phi_{,i}(\mathbf{y})\,dy.$$

Now let

$$w_i \equiv \int_B \widetilde{f}(\mathbf{x}-\mathbf{y}) \Phi_{,i}(\mathbf{y})\,dy. \tag{20.71}$$

and since $\Phi_{,i} \in L^1(B)$, it follows from Minkowski's inequality that

$$\begin{aligned} & ||u_{k,i} - w_i||_{L^p(U)} \\ \leq\ & \left(\int_U \left(\int_B \left|f_k(\mathbf{x}-\mathbf{y}) - \widetilde{f}(\mathbf{x}-\mathbf{y})\right| |\Phi_{,i}(\mathbf{y})|\,dy\right)^p dx\right)^{1/p} \\ \leq\ & \int_B |\Phi_{,i}(\mathbf{y})| \left(\int_U \left|f_k(\mathbf{x}-\mathbf{y}) - \widetilde{f}(\mathbf{x}-\mathbf{y})\right|^p dx\right)^{1/p} dy \\ \leq\ & C(B)\,||f_k - f||_{L^p(U)} \end{aligned}$$

and so $u_{k,i} \to w_i$ in $L^p(U)$.

Now let $\phi \in C_c^\infty(U)$. Then

$$\int_U w_i \phi\,dx = -\lim_{k\to\infty} \int_U u_k \phi_{,i}\,dx = -\int_U u\phi_{,i}\,dx.$$

Thus $u_{,i} = w_i \in L^p(\mathbb{R}^n)$ and so if $\phi \in C_c^\infty(U)$,

$$\int_U f\phi\,dx = \lim_{k\to\infty} \int_U f_k \phi\,dx = \lim_{k\to\infty} \int_U \nabla u_k \cdot \nabla\phi\,dx = \int_U \nabla u \cdot \nabla \phi\,dx$$

and so $-\Delta u = f$ as claimed. This proves the theorem.

One could also ask whether the second weak partial derivatives of u are in $L^p(U)$. This is where the theory singular integrals is used. Recall from 20.70 and 20.71 along with the argument of the above lemma, that if u is given by 20.70, then $u_{,i}$ is given by 20.71 which equals

$$\int_U \Phi_{,i}(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\, dy.$$

Lemma 20.23 *Let $f \in L^p(U)$ and let*

$$w_i(\mathbf{x}) \equiv \int_U \Phi_{,i}(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\, dy.$$

Then $w_{i,j} \in L^p(U)$ for each $j = 1 \cdots n$ and the map $f \to w_{i,j}$ is continuous and linear on $L^p(U)$.

Proof: First let $f \in C_c^\infty(U)$. For such f,

$$\begin{aligned} w_i(\mathbf{x}) &= \int_U \Phi_{,i}(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\, dy = \int_{\mathbb{R}^n} \Phi_{,i}(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\, dy \\ &= \int_{\mathbb{R}^n} \Phi_{,i}(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\, dy = \int_B \Phi_{,i}(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\, dy \end{aligned}$$

and

$$\begin{aligned} w_{i,j}(\mathbf{x}) &= \int_B \Phi_{,i}(\mathbf{y}) f_{,j}(\mathbf{x}-\mathbf{y})\, dy \\ &= \int_{B\setminus B(\mathbf{0},\varepsilon)} \Phi_{,i}(\mathbf{y}) f_{,j}(\mathbf{x}-\mathbf{y})\, dy + \int_{B(\mathbf{0},\varepsilon)} \Phi_{,i}(\mathbf{y}) f_{,j}(\mathbf{x}-\mathbf{y})\, dy. \end{aligned}$$

The second term converges to 0 because $f_{,j}$ is bounded and by 20.65, $\Phi_{,i} \in L^1_{loc}$. Thus

$$\begin{aligned} w_{i,j}(\mathbf{x}) &= \int_{B\setminus B(\mathbf{0},\varepsilon)} \Phi_{,i}(\mathbf{y}) f_{,j}(\mathbf{x}-\mathbf{y})\, dy + e(\varepsilon) \\ &= \int_{B\setminus B(\mathbf{0},\varepsilon)} -\left(\Phi_{,i}(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\right)_{,j} + \Phi_{,ij}(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\, dy + e(\varepsilon) \end{aligned}$$

where $e(\varepsilon) \to 0$ as $\varepsilon \to 0$. Using the divergence theorem, this yields

$$w_{i,j}(\mathbf{x}) = \int_{\partial B(\mathbf{0},\varepsilon)} \Phi_{,i}(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\, n_j d\sigma + \int_{B\setminus B(\mathbf{0},\varepsilon)} \Phi_{,ij}(\mathbf{y}) f(\mathbf{x}-\mathbf{y})\, dy + e(\varepsilon).$$

Consider the first term on the right. This term equals, after letting $\mathbf{y} = \varepsilon\mathbf{z}$,

$$\begin{aligned} \varepsilon^{n-1} \int_{\partial B(\mathbf{0},1)} \Phi_{,i}(\varepsilon\mathbf{z}) f(\mathbf{x}-\varepsilon\mathbf{z})\, n_j d\sigma &= C_n \varepsilon^{n-1} \int_{\partial B(\mathbf{0},1)} \varepsilon^{1-n} z_i z_j f(\mathbf{x}-\varepsilon\mathbf{z})\, d\sigma(z) \\ &= C_n \int_{\partial B(\mathbf{0},1)} z_i z_j f(\mathbf{x}-\varepsilon\mathbf{z})\, d\sigma(z) \end{aligned}$$

and this converges to 0 if $i \neq j$ and it converges to

$$C_n f(\mathbf{x}) \int_{\partial B(\mathbf{0},1)} z_i^2 d\sigma(z)$$

if $i = j$. Thus

$$w_{i,j}(\mathbf{x}) = C_n \delta_{ij} f(\mathbf{x}) + \int_{B \setminus B(\mathbf{0},\varepsilon)} \Phi_{,ij}(\mathbf{y}) f(\mathbf{x}-\mathbf{y}) dy + e(\varepsilon).$$

Letting

$$\Phi_{ij}^{\varepsilon} \equiv \begin{cases} 0 \text{ if } |\mathbf{y}| < \varepsilon, \\ \Phi_{,ij}(\mathbf{y}) \text{ if } |\mathbf{y}| \geq \varepsilon, \end{cases}$$

it follows

$$w_{i,j}(\mathbf{x}) = C_n \delta_{ij} f(\mathbf{x}) + \Phi_{ij}^{\varepsilon} * \widetilde{f}(\mathbf{x}) + e(\varepsilon).$$

By the theory of singular integrals, there exists a continuous linear map, $K_{ij} \in \mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ such that

$$K_{ij} f \equiv \lim_{\varepsilon \to 0} \Phi_{ij}^{\varepsilon} * f.$$

Therefore, letting $\varepsilon \to 0$,

$$w_{i,j} = C_n \delta_{ij} f + K_{ij} \widetilde{f}$$

whenever $f \in C_c^{\infty}(U)$.

Now let $f \in L^p(U)$, let

$$||f_k - f||_{L^p(U)} \to 0,$$

where $f_k \in C_c^{\infty}(U)$, and let

$$w_i^k(\mathbf{x}) = \int_U \Phi_{,i}(\mathbf{x}-\mathbf{y}) f_k(\mathbf{y}) dy.$$

Then it follows as before that $w_i^k \to w_i$ in $L^p(U)$ and

$$w_{i,j}^k = C_n \delta_{ij} f_k + K_{ij} \widetilde{f}_k.$$

Now let $\phi \in C_c^{\infty}(U)$.

$$\begin{aligned} w_{i,j}(\phi) &\equiv -\int_U w_i \phi_{,j} dx = -\lim_{k\to\infty} \int_U w_i^k \phi_{,j} dx \\ &= \lim_{k\to\infty} \int_U w_{i,j}^k \phi dx = \lim_{k\to\infty} \int_U \left(C_n \delta_{ij} \widetilde{f}_k + K_{ij} \widetilde{f}_k \right) \phi dx \\ &= \int_U \left(C_n \delta_{ij} \widetilde{f} + K_{ij} \widetilde{f} \right) \phi dx. \end{aligned}$$

It follows

$$w_{i,j} = C_n \delta_{ij} \widetilde{f} + K_{ij} \widetilde{f}$$

and this proves the lemma.

Corollary 20.24 *In the situation of Theorem 20.22, all weak derivatives of u of order 2 are in $L^p(U)$ and also $f \to u_{,ij}$ is a continuous map.*

Proof:

$$u_{,i}(\mathbf{x}) = \int_U \Phi_{,i}(\mathbf{x}-\mathbf{y}) f(\mathbf{y}) \, dy$$

and so $u_{,ij} \in L^p(U)$ and $f \to u_{,ij}$ is continuous by Lemma 20.23.

With this preparation, it is possible to consider the Helmholtz decomposition. Let $\mathbf{F} \in L^p(U;\mathbb{R}^n)$ and define

$$\phi(\mathbf{x}) \equiv \int_U \nabla\Phi(\mathbf{x}-\mathbf{y}) \cdot \mathbf{F}(\mathbf{y}) \, dy. \tag{20.72}$$

Then by Lemma 20.23,

$$\phi_{,j} = C_n \widetilde{F}_j + \sum_i K_{ij} \widetilde{F}_i \in L^p(\mathbb{R}^n)$$

and the mapping $\mathbf{F} \to \nabla\phi$ is continuous from $L^p(U;\mathbb{R}^n)$ to $L^p(U;\mathbb{R}^n)$.

Now suppose $\mathbf{F} \in C_c^\infty(U;\mathbb{R}^n)$. Then

$$\begin{aligned}
\phi(\mathbf{x}) &= \int_U \sum_{i=1}^n -\frac{\partial}{\partial y^i}(\Phi(\mathbf{x}-\mathbf{y}) F_i(\mathbf{y})) + \Phi(\mathbf{x}-\mathbf{y}) \nabla\cdot\mathbf{F}(\mathbf{y}) \, dy \\
&= \int_U \Phi(\mathbf{x}-\mathbf{y}) \nabla\cdot\mathbf{F}(\mathbf{y}) \, dy
\end{aligned}$$

and so by Lemma 20.21,

$$\nabla\cdot\nabla\phi = \Delta\phi = -\nabla\cdot\mathbf{F}.$$

This continues to hold in the sense of weak derivatives if $\mathbf{F}$ is only in $L^p(U;\mathbb{R}^n)$ because by Minkowski's inequality and 20.72 the map $\mathbf{F} \to \phi$ is continuous. Also note that for $\mathbf{F} \in C_c^\infty(U;\mathbb{R}^n)$,

$$\phi(\mathbf{x}) = \int_B \Phi(\mathbf{y}) \nabla\cdot\mathbf{F}(\mathbf{x}-\mathbf{y}) \, dy.$$

Next define $\pi : L^p(U;\mathbb{R}^n) \to L^p(U;\mathbb{R}^n)$ by

$$\pi\mathbf{F} = -\nabla\phi, \ \phi(\mathbf{x}) = \int_U \nabla\Phi(\mathbf{x}-\mathbf{y}) \cdot \mathbf{F}(\mathbf{y}) \, dy.$$

It was already shown that π is continuous, linear, and $\nabla\cdot\pi\mathbf{F} = \nabla\cdot\mathbf{F}$. It is also true that π is a projection. To see this, let $\mathbf{F} \in C_c^\infty(U;\mathbb{R}^n)$. Then for B large enough,

$$\begin{aligned}
\pi^2\mathbf{F}(\mathbf{x}) &= -\nabla \int_B \Phi(\mathbf{z}) \nabla\cdot\pi\mathbf{F}(\mathbf{x}-\mathbf{z}) \, dz \\
&= -\nabla \int_B \Phi(\mathbf{z}) \nabla\cdot\nabla \int_B \Phi(\mathbf{w}) \nabla\cdot\mathbf{F}(\mathbf{x}-\mathbf{z}-\mathbf{w}) \, dwdz \\
&= -\nabla \int_B \Phi(\mathbf{z}) \nabla\cdot\mathbf{F}(\mathbf{x}-\mathbf{z}) \, dz = \pi\mathbf{F}(\mathbf{x}).
\end{aligned}$$

Since π is continuous and $C_c^\infty(U;\mathbb{R}^n)$ is dense in $L^p(U;\mathbb{R}^n)$, $\pi^2\mathbf{F} = \pi\mathbf{F}$ for all $\mathbf{F} \in L^p(U;\mathbb{R}^n)$. This proves the following theorem which is the Helmholtz decomposition.

Theorem 20.25 *There exists a continuous projection*

$$\pi : L^p(U;\mathbb{R}^n) \to L^p(U;\mathbb{R}^n)$$

such that $\pi\mathbf{F}$ *is a gradient and*

$$\nabla \cdot (\mathbf{F} - \pi\mathbf{F}) = 0$$

in the sense of weak derivatives.

Note this theorem shows that any L^p vector field is the sum of a gradient and a part which is divergence free. $\mathbf{F} = \mathbf{F} - \pi\mathbf{F} + \pi\mathbf{F}$.

The Bochner Integral

21.1 Strong And Weak Measurability

In this chapter $(\Omega, \mathcal{S}, \mu)$ will be a σ finite measure space and X will be a Banach space which contains the values of either a function or a measure. The Banach space will be either a real or a complex Banach space but the field of scalars does not matter and so it is denoted by $\mathbb{F}$ with the understanding that $\mathbb{F} = \mathbb{C}$ unless otherwise stated. The theory presented here includes the case where $X = \mathbb{R}^n$ or $\mathbb{C}^n$ but it does not include the situation where f could have values in a space like $[0, \infty]$. To begin with here is a definition.

Definition 21.1 *A function, $x : \Omega \to X$, for X a Banach space, is a simple function if it is of the form*

$$x(s) = \sum_{i=1}^{n} a_i \mathcal{X}_{B_i}(s)$$

where $B_i \in \mathcal{S}$ and $\mu(B_i) < \infty$ for each i. A function x from Ω to X is said to be strongly measurable if there exists a sequence of simple functions $\{x_n\}$ converging pointwise to x. The function x is said to be weakly measurable if, for each $f \in X'$,

$$f \circ x$$

is a scalar valued measurable function.

Earlier, a function was measurable if inverse images of open sets were measurable. Something similar holds here. The difference is that another condition needs to hold.

Theorem 21.2 *x is strongly measurable if and only if $x^{-1}(U)$ is measurable for all U open in X and $x(\Omega)$ is separable.*

Proof: Suppose first $x^{-1}(U)$ is measurable for all U open in X and $x(\Omega)$ is separable. Let $\{a_n\}_{n=1}^{\infty}$ be the dense subset of $x(\Omega)$. It follows $x^{-1}(B)$ is measurable for all B Borel because

$$\{B : x^{-1}(B) \text{ is measurable}\}$$

is a σ algebra containing the open sets. Let

$$U_k^n \equiv \{z \in X : ||z - a_k|| \leq \min\{||z - a_l||_{l=1}^n\}\}.$$

In words, U_k^m is the set of points of X which are as close to a_k as they are to any of the a_l for $l \leq n$.

$$B_k^n \equiv x^{-1}\left(U_k^n\right), \ D_k^n \equiv B_k^n \setminus \left(\cup_{i=1}^{k-1} B_i^n\right), \ D_1^n \equiv B_1^n,$$

and

$$x_n\left(s\right) \equiv \sum_{k=1}^n a_k \mathcal{X}_{D_k^n}\left(s\right).$$

Thus $x_n(s)$ is a closest approximation to $x(s)$ from $\{a_k\}_{k=1}^n$ and so $x_n(s) \to x(s)$ because $\{a_n\}_{n=1}^\infty$ is dense in $x(\Omega)$. Furthermore, x_n is measurable because each D_k^n is measurable.

Since $(\Omega, \mathcal{S}, \mu)$ is σ finite, there exists $\Omega_n \uparrow \Omega$ with $\mu(\Omega_n) < \infty$. Let

$$y_n\left(s\right) \equiv \mathcal{X}_{\Omega_n}\left(s\right) x_n\left(s\right).$$

Then $y_n(s) \to x(s)$ for each s because for any s, $s \in \Omega_n$ if n is large enough. Also y_n is a simple function because it equals 0 off a set of finite measure.

Now suppose that x is strongly measurable. Then some sequence of simple functions, $\{x_n\}$, converges pointwise to x. Then $x_n^{-1}(W)$ is measurable for every open set W because it is just a finite union of measurable sets. Thus, $x_n^{-1}(W)$ is measurable for every Borel set W. This follows by considering

$$\left\{W : x_n^{-1}\left(W\right) \text{ is measurable}\right\}$$

and observing this is a σ algebra which contains the open sets. Since X is a metric space, it follows that if U is an open set in X , there exists a sequence of open sets, $\{V_n\}$ which satisfies

$$\overline{V}_n \subseteq U, \ \overline{V}_n \subseteq V_{n+1}, \ U = \cup_{n=1}^\infty V_n.$$

Then

$$x^{-1}\left(V_m\right) \subseteq \bigcup_{n<\infty} \bigcap_{k \geq n} x_k^{-1}\left(V_m\right) \subseteq x^{-1}\left(\overline{V}_m\right).$$

This implies

$$x^{-1}\left(U\right) = \bigcup_{m<\infty} x^{-1}\left(V_m\right)$$

$$\subseteq \bigcup_{m<\infty} \bigcup_{n<\infty} \bigcap_{k\geq n} x_k^{-1}\left(V_m\right) \subseteq \bigcup_{m<\infty} x^{-1}\left(\overline{V}_m\right) \subseteq x^{-1}\left(U\right).$$

Since

$$x^{-1}\left(U\right) = \bigcup_{m<\infty} \bigcup_{n<\infty} \bigcap_{k\geq n} x_k^{-1}\left(V_m\right),$$

it follows that $x^{-1}(U)$ is measurable for every open U. It remains to show $x(\Omega)$ is separable. Let

$$D \equiv \text{ all values of the simple functions } x_n$$

which converge to x pointwise. Then D is clearly countable and dense in $\overline{D}$, a set which contains $x(\Omega)$.

Claim: $x(\Omega)$ is separable.

Proof of claim: For $n \in \mathbb{N}$, let $\mathcal{B}_n \equiv \left\{B(d,r) : 0 < r < \frac{1}{n},\ r \text{ rational}, d \in D\right\}$. Thus $\mathcal{B}_n$ is countable. Let $z \in \overline{D}$. Consider $B\left(z, \frac{1}{n}\right)$. Then there exists $d \in D \cap B\left(z, \frac{1}{3n}\right)$. Now pick $r \in \mathbb{Q} \cap \left(\frac{1}{3n}, \frac{1}{n}\right)$ so that $B(d,r) \in \mathcal{B}_n$. Now $z \in B(d,r)$ and so this shows that $x(\Omega) \subseteq \overline{D} \subseteq \cup \mathcal{B}_n$ for each n. Now let $\mathcal{B}_n'$ denote those sets of $\mathcal{B}_n$ which have nonempty intersection with $x(\Omega)$. Say $\mathcal{B}_n' = \{B_k^n\}_{n,k=1}^{\infty}$. By the axiom of choice, there exists $x_k^n \in B_k^n \cap x(\Omega)$. Then if $z \in x(\Omega)$, z is contained in some set of $\mathcal{B}_n'$ which also contains a point of $\{x_k^n\}_{n,k=1}^{\infty}$. Therefore, z is at least as close as $2/n$ to some point of $\{x_k^n\}_{n,k=1}^{\infty}$ which shows $\{x_k^n\}_{n,k=1}^{\infty}$ is a countable dense subset of $x(\Omega)$. Therefore $x(\Omega)$ is separable. This proves the theorem.

The last part also shows that a subset of a separable metric space is also separable. Therefore, the following simple corollary is obtained.

Corollary 21.3 *If X is a separable Banach space then x is strongly measurable if and only if $x^{-1}(U)$ is measurable for all U open in X.*

The next lemma is interesting for its own sake. Roughly it says that if a Banach space is separable, then the unit ball in the dual space is weak $*$ separable. This will be used to prove Pettis's theorem, one of the major theorems in this subject which relates weak measurability to strong measurability.

Lemma 21.4 *If X is a separable Banach space with B' the closed unit ball in X', then there exists a sequence $\{f_n\}_{n=1}^{\infty} \equiv D' \subseteq B'$ with the property that for every $x \in X$,*

$$||x|| = \sup_{f \in D'} |f(x)|$$

Proof: Let $\{a_k\}$ be a countable dense set in X and consider the mapping

$$\phi_n : B' \to \mathbb{F}^n$$

given by

$$\phi_n(f) \equiv (f(a_1), \cdots, f(a_n)).$$

Then $\phi_n(B')$ is contained in a compact subset of $\mathbb{F}^n$ because $|f(a_k)| \leq ||a_k||$. Therefore, there exists a countable dense subset of $\phi_n(B')$, $\{\phi_n(f_k^n)\}_{k=1}^{\infty}$. Let $D_n' \equiv \{f_k^n\}_{k=1}^{\infty}$. Let

$$D' \equiv \cup_{k=1}^{\infty} D_k'.$$

It remains to show this works. Letting $x \in X$ and $\varepsilon > 0$ be given, there exists a_m such that $||a_m - x|| < \varepsilon$. Then by the usual argument involving the Hahn Banach theorem, there exists $f_x \in B'$ such that $||x|| = f_x(x)$. Letting $n > m$, let $g \in B'$ be one of the f_k^n with $\{\phi_n(f_k^n)\}_{k=1}^{\infty}$ a dense subset of $\phi_n(B')$ such that

$$|g(a_m) - f_x(a_m)| < \varepsilon.$$

Then

$$\begin{aligned} ||x|| &= |f_x(x)| = |f_x(a_m) + f_x(x) - f_x(a_m)| \\ &\leq |f_x(a_m)| + \varepsilon \leq |g(a_m)| + 2\varepsilon \leq |g(x)| + 3\varepsilon \end{aligned}$$

and so since $\varepsilon > 0$ is arbitrary,

$$||x|| \leq \sup_{g \in D'} |g(x)| \leq ||x||$$

and this proves the lemma.

The next theorem is one of the most important results in the subject. It is due to Pettis and appeared in 1938.

Theorem 21.5 *If x has values in a separable Banach space, X, and if x is weakly measurable, then x is strongly measurable.*

Proof: It is necessary to show $x^{-1}(U)$ is measurable whenever U is open. Since every open set is a countable union of balls, it suffices to show $x^{-1}(B(a,r))$ is measurable for any ball, $B(a,r)$. Since every open ball is the countable union of closed balls, it suffices to verify $x^{-1}\left(\overline{B(a,r)}\right)$ is measurable. From Lemma 21.4

$$\begin{aligned} x^{-1}\left(\overline{B(a,r)}\right) &= \{s : ||x(s) - a|| \leq r\} \\ &= \left\{s : \sup_{f \in D'} |f(x(s) - a)| \leq r\right\} \\ &= \cap_{f \in D'} \{s : |f(x(s) - a)| \leq r\} \\ &= \cap_{f \in D'} \{s : |f(x(s)) - f(a)| \leq r\} \\ &= \cap_{f \in D'} (f \circ x)^{-1} \overline{B(f(a), r)} \end{aligned}$$

which equals a countable union of measurable sets because it is assumed that $f \circ x$ is measurable for all $f \in X'$. This proves the theorem.

The same method of proof yields the following interesting corollary.

Corollary 21.6 *Let X be a separable Banach space and let $\mathcal{B}(X)$ denote the σ algebra of Borel sets. Then $\mathcal{B}(X) = \mathcal{F}$ where $\mathcal{F}$ is the smallest σ algebra of subsets of X which has the property that every function, $x^* \in X'$ is $\mathcal{F}$ measurable.*

Proof: First I need to show $\mathcal{F}$ contains open balls because then $\mathcal{F}$ will contain the open sets, since every open set is a countable union of open balls, which will imply $\mathcal{F} \supseteq \mathcal{B}(X)$. As noted above, it suffices to verify $\mathcal{F}$ contains the closed balls because every open ball is a countable union of closed balls. Let D' be those functionals in B' defined in Lemma 21.4. Then

$$\begin{aligned}\{x : ||x-a|| \leq r\} &= \left\{x : \sup_{x^* \in D'} |x^*(x-a)| \leq r\right\} \\ &= \cap_{x^* \in D'} \{x : |x^*(x-a)| \leq r\} \\ &= \cap_{x^* \in D'} \{x : |x^*(x) - x^*(a)| \leq r\} \\ &= \cap_{x^* \in D'} x^{*-1}\left(\overline{B(x^*(a), r)}\right)\end{aligned}$$

which is measurable because this is a countable intersection of measurable sets. Thus $\mathcal{F}$ contains open sets so $\mathcal{F} \supseteq \mathcal{B}(X)$.

To show the other direction for the inclusion, note that each x^* is $\mathcal{B}(X)$ measurable because x^{*-1} (open set) $=$ open set. Therefore, $\mathcal{B}(X)$ is a σ algebra with respect to which each x^* is measurable and $\mathcal{F}$ is the smallest of these so $\mathcal{B}(X) \supseteq \mathcal{F}$. This proves the corollary.

It is important to verify the limit of strongly measurable functions is itself strongly measurable. This happens under very general conditions. Suppose X is any separable metric space and let τ denote the open sets of X. Then it is routine to see that

$$\tau \text{ has a countable basis, } \mathcal{B}. \tag{21.1}$$

Whenever $U \in \mathcal{B}$, there exists a sequence of open sets, $\{V_m\}_{m=1}^\infty$, such that

$$\cdots V_m \subseteq \overline{V}_m \subseteq V_{m+1} \subseteq \cdots, \; U = \bigcup_{m=1}^\infty V_m. \tag{21.2}$$

Theorem 21.7 *Let f_n and f be functions mapping Ω to X where $\mathcal{F}$ is a σ algebra of measurable sets of Ω and (X, τ) is a topological space satisfying 21.1 - 21.2. Then if f_n is measurable, and $f(\omega) = \lim_{n\to\infty} f_n(\omega)$, it follows that f is also measurable. (Pointwise limits of measurable functions are measurable.)*

Proof: Let $\mathcal{B}$ be the countable basis of 21.1 and let $U \in \mathcal{B}$. Let $\{V_m\}$ be the sequence of 21.2. Since f is the pointwise limit of f_n,

$$f^{-1}(V_m) \subseteq \{\omega : f_k(\omega) \in V_m \text{ for all } k \text{ large enough}\} \subseteq f^{-1}(\overline{V_m}).$$

Therefore,

$$\begin{aligned} f^{-1}(U) = \cup_{m=1}^\infty f^{-1}(V_m) &\subseteq \cup_{m=1}^\infty \cup_{n=1}^\infty \cap_{k=n}^\infty f_k^{-1}(V_m) \\ &\subseteq \cup_{m=1}^\infty f^{-1}(\overline{V_m}) = f^{-1}(U).\end{aligned}$$

It follows $f^{-1}(U) \in \mathcal{F}$ because it equals the expression in the middle which is measurable. Now let $W \in \tau$. Since $\mathcal{B}$ is countable, $W = \cup_{n=1}^\infty U_n$ for some sets $U_n \in \mathcal{B}$. Hence

$$f^{-1}(W) = \cup_{n=1}^\infty f^{-1}(U_n) \in \mathcal{F}.$$

This proves the theorem.

Corollary 21.8 *x is strongly measurable if and only if $x(\Omega)$ is separable and x is weakly measurable.*

Proof: Strong measurability clearly implies weak measurability. If $x_n(s) \to x(s)$ where x_n is simple, then $f(x_n(s)) \to f(x(s))$ for all $f \in X'$. Hence $f \circ x$ is measurable by Theorem 21.7 because it is the limit of a sequence of measurable functions. Let D denote the set of all values of x_n. Then $\overline{D}$ is a separable set containing $x(\Omega)$. Thus $\overline{D}$ is a separable metric space. Therefore $x(\Omega)$ is separable also by the last part of the proof of Theorem 21.2.

Now suppose D is a countable dense subset of $x(\Omega)$ and x is weakly measurable. Let Z be the subset consisting of all finite linear combinations of D with the scalars coming from the set of rational points of $\mathbb{F}$. Thus, Z is countable. Letting $Y = \overline{Z}$, Y is a separable Banach space containing $x(\Omega)$. If $f \in Y'$, f can be extended to an element of X' by the Hahn Banach theorem. Therefore, x is a weakly measurable Y valued function. Now use Theorem 21.5 to conclude x is strongly measurable. This proves the corollary.

Weakly measurable as defined above means $s \to x^*(x(s))$ is measurable for every $x^* \in X'$. The next lemma ties this to the usual version of measurability in which a function is measurable when inverse images of open sets are measurable.

Lemma 21.9 *Let X be a Banach space and let $x : (\Omega, \mathcal{F}) \to K \subseteq X$ where K is weakly compact and X' is separable. Then x is weakly measurable if and only if $x^{-1}(U) \in \mathcal{F}$ whenever U is a weakly open set.*

Proof: By Corollary 13.41 on Page 358, there exists a metric d, such that the metric space topology with respect to d coincides with the weak topology. Since K is compact, it follows that K is also separable. Hence it is completely separable and so there exists a countable basis of open sets, $\mathcal{B}$ for the weak topology on K. It follows that if U is any weakly open set, covered by basic sets of the form $B_A(x, r)$ where A is a finite subset of X', there exists a countable collection of these sets of the form $B_A(x, r)$ which covers U.

Suppose now that x is weakly measurable. To show $x^{-1}(U) \in \mathcal{F}$ whenever U is weakly open, it suffices to verify $x^{-1}(B_A(z, r)) \in \mathcal{F}$ for any set, $B_A(z, r)$. Let $A = \{x_1^*, \cdots, x_m^*\}$. Then

$$\begin{aligned} x^{-1}(B_A(z,r)) &= \{s \in \Omega : \rho_A(x(s) - z) < r\} \\ &\equiv \left\{s \in \Omega : \max_{x^* \in A} |x^*(x(s) - z)| < r\right\} \\ &= \cup_{i=1}^m \{s \in \Omega : |x_i^*(x(s) - z)| < r\} \\ &= \cup_{i=1}^m \{s \in \Omega : |x_i^*(x(s)) - x_i^*(z)| < r\} \end{aligned}$$

which is measurable because each $x_i^* \circ x$ is given to be measurable.

Next suppose $x^{-1}(U) \in \mathcal{F}$ whenever U is weakly open. Then in particular this holds when $U = B_{x^*}(z, r)$ for arbitrary x^*. Hence

$$\{s \in \Omega : x(s) \in B_{x^*}(z, r)\} \in \mathcal{F}.$$

But this says the same as

$$\{s \in \Omega : |x^* (x(s)) - x^* (z)| < r\} \in \mathcal{F}$$

Since $x^* (z)$ can be a completely arbitrary element of $\mathbb{F}$, it follows $x^* \circ x$ is an $\mathbb{F}$ valued measurable function. In other words, x is weakly measurable according to the former definition. This proves the lemma.

One can also define weak $*$ measurability and prove a theorem just like the Pettis theorem above. The next lemma is the analogue of Lemma 21.4.

Lemma 21.10 *Let B be the closed unit ball in X. If X' is separable, there exists a sequence $\{x_m\}_{m=1}^{\infty} \equiv D \subseteq B$ with the property that for all $y^* \in X'$,*

$$||y^*|| = \sup_{x \in D} |y^* (x)| .$$

Proof: Let

$$\{x_k^*\}_{k=1}^{\infty}$$

be the dense subspace of X'. Define $\phi_n : B \to \mathbb{F}^n$ by

$$\phi_n (x) \equiv (x_1^* (x), \cdots, x_n^* (x)).$$

Then $|x_k^* (x)| \leq ||x_k^*||$ and so $\phi_n (B)$ is contained in a compact subset of $\mathbb{F}^n$. Therefore, there exists a countable set, $D_n \subseteq B$ such that $\phi_n (D_n)$ is dense in $\phi_n (B)$. Let

$$D \equiv \cup_{n=1}^{\infty} D_n.$$

It remains to verify this works. Let $y^* \in X'$. Then there exists y such that

$$|y^* (y)| > ||y^*|| - \varepsilon.$$

By density, there exists one of the x_k^* from the countable dense subset of X' such that also

$$|x_k^* (y)| > ||y^*|| - \varepsilon, \; ||x_k^* - y^*|| < \varepsilon.$$

Now $x_k^* (y) \in \phi_k (B)$ and so there exists $x \in D_k \subseteq D$ such that

$$|x_k^* (x)| > ||y^*|| - \varepsilon.$$

Then since $||x_k^* - y^*|| < \varepsilon$, this implies

$$|y^* (x)| \geq ||y^*|| - 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$||y^*|| \leq \sup_{x \in D} |y^* (x)| \leq ||y^*||$$

and this proves the lemma.

The next theorem is another version of the Pettis theorem. First here is a definition.

Definition 21.11 *A function y having values in X' is weak $*$ measurable, when for each $x \in X$, $y(\cdot)(x)$ is a measurable scalar valued function.*

Theorem 21.12 *If X' is separable and $y : \Omega \to X'$ is weak $*$ measurable, then y is strongly measurable.*

Proof: It is necessary to show $y^{-1}(B(a^*, r))$ is measurable. This will suffice because the separability of X' implies every open set is the countable union of such balls of the form $B(a^*, r)$. It also suffices to verify inverse images of closed balls are measurable because every open ball is the countable union of closed balls. From Lemma 21.10,

$$\begin{aligned} y^{-1}\left(\overline{B(a^*, r)}\right) &= \{s : \|y(s) - a^*\| \le r\} \\ &= \left\{s : \sup_{x \in D} |(y(s) - a^*)(x)| \le r\right\} \\ &= \left\{s : \sup_{x \in D} |y(s)(x) - a^*(x)| \le r\right\} \\ &= \cap_{x \in D} y(\cdot)(x)^{-1}\left(\overline{B(a^*(x), r)}\right) \end{aligned}$$

which is a countable intersection of measurable sets by hypothesis. This proves the theorem.

The following are interesting consequences of the theory developed so far and are of interest independent of the theory of integration of vector valued functions.

Theorem 21.13 *If X' is separable, then so is X.*

Proof: Let $D = \{x_m\} \subseteq B$, the unit ball of X, be the sequence promised by Lemma 21.10. Let V be all finite linear combinations of elements of $\{x_m\}$ with rational scalars. Thus $\overline{V}$ is a separable subspace of X. The claim is that $\overline{V} = X$. If not, there exists

$$x_0 \in X \setminus \overline{V}.$$

But by the Hahn Banach theorem there exists $x_0^* \in X'$ satisfying $x_0^*(x_0) \neq 0$, but $x_0^*(v) = 0$ for every $v \in \overline{V}$. Hence

$$\|x_0^*\| = \sup_{x \in D} |x_0^*(x)| = 0,$$

a contradiction. This proves the theorem.

Corollary 21.14 *If X is reflexive, then X is separable if and only if X' is separable.*

Proof: From the above theorem, if X' is separable, then so is X. Now suppose X is separable with a dense subset equal to D. Then since X is reflexive, $J(D)$ is dense in X'' where J is the James map satisfying $Jx(x^*) \equiv x^*(x)$. Then since X'' is separable, it follows from the above theorem that X' is also separable.

21.2 The Bochner Integral

21.2.1 Definition And Basic Properties

Definition 21.15 *Let $a_k \in X$, a Banach space and let*

$$x(s) = \sum_{k=1}^{n} a_k \mathcal{X}_{E_k}(s) \tag{21.3}$$

where for each k, E_k is measurable and $\mu(E_k) < \infty$. Then define

$$\int_\Omega x(s)\, d\mu \equiv \sum_{k=1}^{n} a_k \mu(E_k).$$

Proposition 21.16 *Definition 21.15 is well defined.*

Proof: It suffices to verify that if

$$\sum_{k=1}^{n} a_k \mathcal{X}_{E_k}(s) = 0,$$

then

$$\sum_{k=1}^{n} a_k \mu(E_k) = 0.$$

Let $f \in X'$. Then

$$f\left(\sum_{k=1}^{n} a_k \mathcal{X}_{E_k}(s)\right) = \sum_{k=1}^{n} f(a_k) \mathcal{X}_{E_k}(s) = 0$$

and, therefore,

$$0 = \int_\Omega \left(\sum_{k=1}^{n} f(a_k) \mathcal{X}_{E_k}(s)\right) d\mu = \sum_{k=1}^{n} f(a_k) \mu(E_k) = f\left(\sum_{k=1}^{n} a_k \mu(E_k)\right).$$

Since $f \in X'$ is arbitrary, and X' separates the points of X, it follows that

$$\sum_{k=1}^{n} a_k \mu(E_k) = 0$$

as claimed. This proves the proposition.

It follows easily from this proposition that $\int_\Omega d\mu$ is well defined and linear on simple functions.

Definition 21.17 *A strongly measurable function x is Bochner integrable if there exists a sequence of simple functions x_n converging to x pointwise and satisfying*

$$\int_\Omega ||x_n(s) - x_m(s)||\, d\mu \to 0 \text{ as } m, n \to \infty. \tag{21.4}$$

If x is Bochner integrable, define

$$\int_\Omega x(s)\, d\mu \equiv \lim_{n\to\infty} \int_\Omega x_n(s)\, d\mu. \tag{21.5}$$

Theorem 21.18 *The Bochner integral is well defined and if x is Bochner integrable and $f \in X'$,*

$$f\left(\int_\Omega x(s)\, d\mu\right) = \int_\Omega f(x(s))\, d\mu \tag{21.6}$$

and

$$\left|\left|\int_\Omega x(s)\, d\mu\right|\right| \le \int_\Omega ||x(s)||\, d\mu. \tag{21.7}$$

Also, the Bochner integral is linear. That is, if a, b are scalars and x, y are two Bochner integrable functions, then

$$\int_\Omega (ax(s) + by(s))\, d\mu = a\int_\Omega x(s)\, d\mu + b\int_\Omega y(s)\, d\mu \tag{21.8}$$

Proof: First it is shown that the triangle inequality holds on simple functions and that the limit in 21.5 exists. Thus, if x is given by 21.3 with the E_k disjoint,

$$\left|\left|\int_\Omega x(s)\, d\mu\right|\right|$$

$$\begin{aligned}
&= \left|\left|\int_\Omega \sum_{k=1}^n a_k \mathcal{X}_{E_k}(s)\, d\mu\right|\right| = \left|\left|\sum_{k=1}^n a_k \mu(E_k)\right|\right| \\
&\le \sum_{k=1}^n ||a_k||\, \mu(E_k) = \int_\Omega \sum_{k=1}^n ||a_k||\, \mathcal{X}_{E_k}(s)\, d\mu = \int_\Omega ||x(s)||\, d\mu
\end{aligned}$$

which shows the triangle inequality holds on simple functions. This implies

$$\begin{aligned}
\left|\left|\int_\Omega x_n(s)\, d\mu - \int_\Omega x_m(s)\, d\mu\right|\right| &= \left|\left|\int_\Omega (x_n(s) - x_m(s))\, d\mu\right|\right| \\
&\le \int_\Omega ||x_n(s) - x_m(s)||\, d\mu
\end{aligned}$$

which verifies the existence of the limit in 21.5. This completes the first part of the argument.

Next it is shown the integral does not depend on the choice of the sequence satisfying 21.4 so that the integral is well defined. Suppose y_n, x_n both satisfy 21.4 and converge to x pointwise. By Fatou's lemma,

$$\left|\left|\int_\Omega y_n d\mu - \int_\Omega x_m d\mu\right|\right| \leq \int_\Omega ||y_n - x||\, d\mu + \int_\Omega ||x - x_m||\, d\mu$$

$$\leq \lim\inf_{k\to\infty} \int_\Omega ||y_n - y_k||\, d\mu + \lim\inf_{k\to\infty} \int_\Omega ||x_k - x_m||$$

$$\leq \varepsilon/2 + \varepsilon/2$$

if m and n are chosen large enough. Since ε is arbitrary, this shows the limit is the same for both sequences and demonstrates the Bochner integral is well defined.

It remains to verify the triangle inequality on Bochner integral functions and the claim about passing a continuous linear functional inside the integral. Let x be Bochner integrable and let x_n be a sequence which satisfies the conditions of the definition. Define

$$y_n(s) \equiv \begin{cases} x_n(s) \text{ if } ||x_n(s)|| \leq 2\,||x(s)||, \\ 0 \text{ if } ||x_n(s)|| > 2\,||x(s)||. \end{cases} \tag{21.9}$$

If $x(s) = 0$ then $y_n(s) = 0$ for all n. If $||x(s)|| > 0$ then for all n large enough,

$$y_n(s) = x_n(s).$$

Thus, $y_n(s) \to x(s)$ and

$$||y_n(s)|| \leq 2\,||x(s)||. \tag{21.10}$$

By Fatou's lemma,

$$\int_\Omega ||x||\, d\mu \leq \lim\inf_{n\to\infty} \int_\Omega ||x_n||\, d\mu. \tag{21.11}$$

Also from 21.4 and the triangle inequality on simple functions, $\left\{\int_\Omega ||x_n||\, d\mu\right\}_{n=1}^\infty$ is a Cauchy sequence and so it must be bounded. Therefore, by 21.10, 21.11, and the dominated convergence theorem,

$$0 = \lim_{n,m\to\infty} \int_\Omega ||y_n - y_m||\, d\mu \tag{21.12}$$

and it follows x_n can be replaced with y_n in Definition 21.17.

From Definition 21.15,

$$f\left(\int_\Omega y_n d\mu\right) = \int_\Omega f(y_n)\, d\mu.$$

Thus,

$$f\left(\int_\Omega x d\mu\right) = \lim_{n\to\infty} f\left(\int_\Omega y_n d\mu\right) = \lim_{n\to\infty} \int_\Omega f(y_n)\, d\mu = \int_\Omega f(x)\, d\mu,$$

the last equation holding from the dominated convergence theorem and 21.10 and 21.11. This shows 21.6. To verify 21.7,

$$\left|\left|\int_\Omega x(s)\,d\mu\right|\right| = \lim_{n\to\infty}\left|\left|\int_\Omega y_n(s)\,d\mu\right|\right|$$

$$\leq \lim_{n\to\infty}\int_\Omega ||y_n(s)||\,d\mu = \int_\Omega ||x(s)||\,d\mu$$

where the last equation follows from the dominated convergence theorem and 21.10, 21.11.

It remains to verify 21.8. Let $f \in X'$. Then from 21.6

$$\begin{aligned} f\left(\int_\Omega (ax(s)+by(s))\,d\mu\right) &= \int_\Omega (af(x(s))+bf(y(s)))\,d\mu \\ &= a\int_\Omega f(x(s))\,d\mu + b\int_\Omega f(y(s))\,d\mu \\ &= f\left(a\int_\Omega x(s)\,d\mu + b\int_\Omega y(s)\,d\mu\right). \end{aligned}$$

Since X' separates the points of X,it follows

$$\int_\Omega (ax(s)+by(s))\,d\mu = a\int_\Omega x(s)\,d\mu + b\int_\Omega y(s)\,d\mu$$

and this proves 21.8. This proves the theorem.

Theorem 21.19 *An X valued function, x, is Bochner integrable if and only if x is strongly measurable and*

$$\int_\Omega ||x(s)||\,d\mu < \infty. \tag{21.13}$$

In this case there exists a sequence of simple functions $\{y_n\}$ satisfying 21.4, $y_n(s)$ converging pointwise to $x(s)$,

$$||y_n(s)|| \leq 2\,||x(s)|| \tag{21.14}$$

and

$$\lim_{n\to\infty}\int_\Omega ||x(s)-y_n(s)||\,d\mu = 0. \tag{21.15}$$

Proof: Suppose x is strongly measurable and condition 21.13 holds. Since x is strongly measurable, there exists a sequence of simple functions, $\{x_n\}$ converging pointwise to x. As before, let

$$y_n(s) = \begin{cases} x_n(s) \text{ if } ||x_n(s)|| \leq 2\,||x(s)||, \\ 0 \text{ if } ||x_n(s)|| > 2\,||x(s)||. \end{cases} \tag{21.16}$$

Then 21.14 holds for y_n and $y_n(s) \to x(s)$. Also

$$0 = \lim_{m,n\to\infty} \int_\Omega ||y_n(s) - y_m(s)||\, d\mu$$

since otherwise, there would exist $\varepsilon > 0$ and $N_\varepsilon \to \infty$ as $\varepsilon \to 0$ and $n_\varepsilon, m_\varepsilon > N_\varepsilon$ such that

$$\int_\Omega ||y_{n_\varepsilon}(s) - y_{m_\varepsilon}(s)||\, d\mu \geq \varepsilon.$$

But then taking a limit as $\varepsilon \to 0$ and using the dominated convergence theorem and 21.14 and 21.13, this would imply $0 \geq \varepsilon$. Therefore, x is Bochner integrable. 21.15 follows from the dominated convergence theorem and 21.14.

Now suppose x is Bochner integrable. Then it is strongly measurable and there exists a sequence of simple functions $\{x_n\}$ such that $x_n(s)$ converges pointwise to x and

$$\lim_{m,n\to\infty} \int_\Omega ||x_n(s) - x_m(s)||\, d\mu = 0.$$

Therefore, as before, since $\left\{\int_\Omega x_n d\mu\right\}_{n=1}^\infty$ is a Cauchy sequence, it follows

$$\left\{\int_\Omega ||x_n||\, d\mu\right\}_{n=1}^\infty$$

is also a Cauchy sequence because

$$\begin{aligned}\left|\int_\Omega ||x_n||\, d\mu - \int_\Omega ||x_m||\, d\mu\right| &\leq \int_\Omega \big|||x_n|| - ||x_m||\big|\, d\mu \\ &\leq \int_\Omega ||x_n - x_m||\, d\mu.\end{aligned}$$

Thus

$$\int_\Omega ||x||\, d\mu \leq \lim\inf_{n\to\infty} \int_\Omega ||x_n||\, d\mu < \infty$$

Using 21.16 it follows y_n satisfies 21.14, converges pointwise to x and then from the dominated convergence theorem 21.15 holds. This proves the theorem.

21.2.2 Taking A Closed Operator Out Of The Integral

Now let X and Y be separable Banach spaces and suppose $A : D(A) \subseteq X \to Y$ be a closed operator. Recall this means that the graph of A,

$$G(A) \equiv \{(x, Ax) : x \in D(A)\}$$

is a closed subset of $X \times Y$ with respect to the product topology obtained from the norm

$$||(x, y)|| = \max(||x||, ||y||).$$

Thus also $G(A)$ is a separable Banach space with the above norm. You can also consider $D(A)$ as a separable Banach space having the norm

$$||x||_{D(A)} \equiv \max(||x||, ||Ax||) \tag{21.17}$$

which is isometric to $G(A)$ with the mapping, $\theta x \equiv (x, Ax)$.

Lemma 21.20 *A closed subspace of a reflexive Banach space is reflexive.*

Proof: Consider the following diagram in which Y is a closed subspace of the reflexive space, X.

$$\begin{array}{ccc} Y'' & \overset{i^{**}\ \text{1-1}}{\rightarrow} & X'' \\ Y' & \overset{i^{*}\ \text{onto}}{\leftarrow} & X' \\ Y & \overset{i}{\rightarrow} & X \end{array}$$

This diagram follows from theorems on adjoints presented earlier.

Now let $y^{**} \in Y''$. Then $i^{**}y^{**} = Jy$ because X is reflexive. I want to show that $y \in Y$. If it is not in Y then there exists $x^* \in X'$ such that $x^*(y) \neq 0$ but $x^*(Y) = 0$. Then $i^*x^* = 0$. Hence

$$0 = y^{**}(i^*x^*) = i^{**}y^{**}(x^*) = J(y)(x^*) = x^*(y) \neq 0,$$

a contradiction. Hence $y \in Y$. Since i^* is onto, I want to show that for all $x^* \in X'$,

$$i^*x^*(y) = y^{**}(i^*x^*)$$

because this will imply $y^{**} = J_Y(y)$ where J_Y is the James map from Y to Y'. However, the above is equivalent to the following holding for all $x^* \in X'$.

$$i^*x^*(y) = J_Y(y)(i^*x^*) = i^{**}J_Y(y)(x^*) = i^{**}y^{**}(x^*)$$

Since i^{**} is onto, it follows $J_Y(y) = y^{**}$ and this proves it.

Lemma 21.21 *Suppose V and W are reflexive Banach spaces and that V is a dense subset of W in the topology of W. Then i^*W' is a dense subset of V' where here i is the inclusion map of V into W.*

Proof: First note that i^* is one to one. If $i^*w^* = 0$ for $w^* \in W'$, then this means that for all $v \in V$,

$$i^*w(v) = w^*(v) = 0$$

and since V is dense in W, this shows $w^* = 0$.

Consider the following diagram

$$\begin{array}{ccc} V'' & \overset{i^{**}}{\rightarrow} & W'' \\ V' & \overset{i^{*}}{\leftarrow} & W' \\ V & \overset{i}{\rightarrow} & W \end{array}$$

in which i is the inclusion map. Next suppose i^*W' is not dense in V'. Then there exists $v^{**} \in V''$ such that $v^{**} \neq 0$ but $v^{**}(i^*W') = 0$. It follows from V being reflexive, that $v^{**} = Jv_0$ where J is the James map from V to $V''$for some $v_0 \in V$. Thus for every w^*W',

$$\begin{aligned} 0 &= v^{**}(i^*w^*) = i^{**}v^{**}(w^*) \\ &= i^{**}Jv_0(w^*) = Jv_0(i^*w^*) \\ &= i^*w^*(v_0) = w^*(v_0) \end{aligned}$$

and since W' separates the points of W, it follows $v_0 = 0$ which contradicts $v^{**} \neq 0$. This proves the lemma.

Note that in the proof, only V reflexive was used.

This lemma implies an easy corollary.

Corollary 21.22 *Let E and F be reflexive Banach spaces and let A be a closed operator, $A : D(A) \subseteq E \to F$. Suppose also that $D(A)$ is dense in E. Then making $D(A)$ into a Banach space by using the above graph norm given in 21.17, it follows that $D(A)$ is a Banach space and i^*E' is a dense subspace of $D(A)'$.*

Proof: First note that $E \times F$ is a reflexive Banach space and $\mathcal{G}(A)$ is a closed subspace of $E \times F$ so it is also a reflexive Banach space. Now $D(A)$ is isometric to $\mathcal{G}(A)$ and so it follows $D(A)$ is a dense subspace of E which is reflexive. Therefore, from Lemma 21.21 the conclusion follows.

With this preparation, here is another interesting theorem. This one is about taking outside the integral a closed linear operator as opposed to a continuous linear operator.

Theorem 21.23 *Let X, Y be separable Banach spaces and let $A : D(A) \subseteq X \to Y$ be a closed operator where $D(A)$ is a dense subset of X. Suppose also that i^*X' is a dense subspace of $D(A)'$ where $D(A)$ is a Banach space having the graph norm described in 21.17. Suppose that $(\Omega, \mathcal{F}, \mu)$ is a σ finite measure space and $x : \Omega \to X$ is strongly measurable and it happens that $x(s) \in D(A)$ for all $s \in \Omega$. Then x is strongly measurable as a mapping into $D(A)$. Also Ax is strongly measurable as a map into Y and if*

$$\int_\Omega ||x(s)||\, d\mu, \int_\Omega ||Ax(s)||\, d\mu < \infty, \tag{21.18}$$

then

$$\int_\Omega x(s)\, d\mu \in D(A) \tag{21.19}$$

and

$$A\int_\Omega x(s)\, d\mu = \int_\Omega Ax(s)\, d\mu. \tag{21.20}$$

Proof: First of all, consider the assertion that x is strongly measurable into $D(A)$. Letting $f \in D(A)'$ be given, there exists a sequence, $\{g_n\} \subseteq i^*X'$ such that $g_n \to f$ in $D(A)'$. Therefore,

$$s \to g_n(x(s))$$

is measurable by assumption and

$$g_n(x(s)) \to f(x(s))$$

which shows that $s \to f(x(s))$ is measurable. By the Pettis theorem, it follows

$$s \to x(s)$$

is strongly measurable as a map into $D(A)$.

It follows from Theorem 21.19 there exists a sequence of simple functions, $\{x_n\}$ of the form

$$x_n(s) = \sum_{k=1}^{m_n} a_k^n \mathcal{X}_{E_k^n}(s), x_n(s) \in D(A),$$

which converges strongly and pointwise to $x(s)$ in $D(A)$. Thus

$$x_n(s) \to x(s), Ax_n(s) \to Ax(s),$$

which shows $s \to Ax(s)$ is stongly measurable in Y as claimed.

It remains to verify the assertions about the integral. 21.18 implies x is Bochner integrable as a function having values in $D(A)$ with the norm on $D(A)$ described above. Therefore, by Theorem 21.19 there exists a sequence of simple functions $\{y_n\}$ having values in $D(A)$,

$$\lim_{m,n\to\infty} \int_\Omega ||y_n - y_m||_{D(A)} \, d\mu = 0,$$

$y_n(s)$ converging pointwise to $x(s)$,

$$||y_n(s)||_{D(A)} \leq 2\,||x(s)||_{D(A)}$$

and

$$\lim_{n\to\infty} \int_\Omega ||x(s) - y_n(s)||_{D(A)} \, ds = 0.$$

Therefore,

$$\int_\Omega y_n(s)\, d\mu \in D(A), \int_\Omega y_n(s)\, d\mu \to \int_\Omega x(s)\, d\mu \text{ in } X,$$

and since y_n is a simple function and A is linear,

$$A\int_\Omega y_n(s)\, d\mu = \int_\Omega Ay_n(s)\, d\mu \to \int_\Omega Ax(s)\, d\mu \text{ in } Y.$$

It follows, since A is a closed operator, that

$$\int_\Omega x(s)\, d\mu \in D(A)$$

and

$$A\int_\Omega x(s)\, d\mu = \int_\Omega Ax(s)\, d\mu.$$

This proves the theorem.

Here is another version of this theorem which has different hypotheses.

Theorem 21.24 *Let X and Y be separable Banach spaces and let $A : D(A) \subseteq X \to Y$ be a closed operator. Also let $(\Omega, \mathcal{F}, \mu)$ be a σ finite measure space and let $x : \Omega \to X$ be Bochner integrable such that $x(s) \in D(A)$ for all s. Also suppose Ax is Bochner integrable. Then*

$$\int Axd\mu = A\int xd\mu$$

and $\int xd\mu \in D(A)$.

Proof: Consider the graph of A,

$$G(A) \equiv \{(x, Ax) : x \in D(A)\} \subseteq X \times Y.$$

Then since A is closed, $G(A)$ is a closed separable Banach space with the norm $||(x, y)|| \equiv \max(||x||, ||y||)$. Therefore, for $g^* \in G(A)'$, one can apply the Hahn Banach theorem and obtain $(x^*, y^*) \in (X \times Y)'$ such that $g^*(x, Ax) = (x^*(x), y^*(Ax))$. Now it follows from the assumptions that $s \to (x^*(x(s)), y^*(Ax(s)))$ is measurable with values in $G(A)$. It is also separably valued because this is true of $G(A)$. By the Pettis theorem, $s \to (x(s), A(x(s)))$ must be strongly measurable. Also $\int ||x(s)|| + ||A(x(s))|| d\mu < \infty$ by assumption and so there exists a sequence of simple functions having values in $G(A)$, $\{(x_n(s), Ax_n(s))\}$ which converges to $(x(s), A(s))$ pointwise such that $\int ||(x_n, Ax_n) - (x, Ax)|| d\mu \to 0$ in $G(A)$. Now for simple functions is it routine to verify that

$$\begin{aligned}\int (x_n, Ax_n)\, d\mu &= \left(\int x_n d\mu, \int Ax_n d\mu\right) \\ &= \left(\int x_n d\mu, A\int x_n d\mu\right)\end{aligned}$$

Also

$$\begin{aligned}\left|\left|\int x_n d\mu - \int xd\mu\right|\right| &\leq \int ||x_n - x||\, d\mu \\ &\leq \int ||(x_n, Ax_n) - (x, Ax)||\, d\mu\end{aligned}$$

which converges to 0. Also

$$\begin{aligned}\left|\left|\int Ax_n d\mu - \int Axd\mu\right|\right| &= \left|\left|A\int x_n d\mu - \int Axd\mu\right|\right| \\ &\leq \int ||Ax_n - Ax||\, d\mu \\ &\leq \int ||(x_n, Ax_n) - (x, Ax)||\, d\mu\end{aligned}$$

and this converges to 0. Therefore, $\int x_n d\mu \to \int xd\mu$ and $A\int x_n d\mu \to \int Axd\mu$. Since each $\int x_n d\mu \in D(A)$, and A is closed, this implies $\int xd\mu \in D(A)$ and $A\int xd\mu = \int Axd\mu$. This proves the theorem.

21.3 Operator Valued Functions

Consider the case where $A(s) \in \mathcal{L}(X, Y)$ for X and Y separable Banach spaces. With the operator norm $\mathcal{L}(X, Y)$ is a Banach space and so if A is strongly measurable, the Bochner integral can be defined as before. However, it is also possible to define the Bochner integral of such operator valued functions for more general situations. In this section, $(\Omega, \mathcal{F}, \mu)$ will be a σ finite measure space as usual.

Lemma 21.25 *Let $x \in X$ and suppose A is strongly measurable. Then*

$$s \rightarrow A(s)\, x$$

is strongly measurable as a map into Y.

Proof: Since A is assumed to be strongly measurable, it is the pointwise limit of simple functions of the form

$$A_n(s) \equiv \sum_{k=1}^{m_n} A_k^n \mathcal{X}_{E_k^n}(s)$$

where A_k^n is in $\mathcal{L}(X, Y)$. It follows $A_n(s)\, x \rightarrow A(s)\, x$ for each s and so, since $s \rightarrow A_n(s)\, x$ is a simple Y valued function, $s \rightarrow A(s)\, x$ must be strongly measurable.

Definition 21.26 *Suppose $A(s) \in \mathcal{L}(X, Y)$ for each $s \in \Omega$ where X, Y are separable Banach spaces. Suppose also that for each $x \in X$,*

$$s \rightarrow A(s)\, x \text{ is strongly measurable} \tag{21.21}$$

and there exists C such that for each $x \in X$,

$$\int_{\Omega} ||A(s)\, x||\, d\mu < C\, ||x|| \tag{21.22}$$

Then $\int_{\Omega} A(s)\, d\mu \in \mathcal{L}(X, Y)$ is defined by the following formula.

$$\left(\int_{\Omega} A(s)\, d\mu\right)(x) \equiv \int_{\Omega} A(s)\, x d\mu \tag{21.23}$$

Lemma 21.27 *The above definition is well defined. Furthermore, if 21.21 holds then $s \rightarrow ||A(s)||$ is measurable and if 21.22 holds, then*

$$\left|\left|\int_{\Omega} A(s)\, d\mu\right|\right| \leq \int_{\Omega} ||A(s)||\, d\mu.$$

Proof: It is clear that in case $s \rightarrow A(s)\, x$ is measurable for all $x \in X$ there exists a unique $\Psi \in \mathcal{L}(X, Y)$ such that

$$\Psi(x) = \int_{\Omega} A(s)\, x d\mu.$$

This is because $x \to \int_\Omega A(s)\,x d\mu$ is linear and continuous. Thus $\Psi = \int_\Omega A(s)\,d\mu$ and the definition is well defined.

Now consider the assertion about $s \to ||A(s)||$. Let $D' \subseteq B'$ the closed unit ball in Y' be such that D' is countable and

$$||y|| = \sup_{y^* \in D'} |y^*(y)|.$$

Also let D be a countable dense subset of B, the unit ball of X. Then

$$\begin{aligned}\{s : ||A(s)|| > \alpha\} &= \left\{s : \sup_{x \in D} ||A(s)\,x|| > \alpha\right\} \\ &= \cup_{x \in D} \{s : ||A(s)\,x|| > \alpha\} \\ &= \cup_{x \in D} \left(\cup_{y^* \in D'} \{|y^*(A(s)\,x)| > \alpha\}\right)\end{aligned}$$

and this is measurable because $s \to A(s)\,x$ is strongly, hence weakly measurable.

Now suppose 21.22 holds. Then for all x,

$$\int_\Omega ||A(s)\,x||\,d\mu < C\,||x||.$$

It follows that for $||x|| \leq 1$,

$$\begin{aligned}\left|\left|\left(\int_\Omega A(s)\,d\mu\right)(x)\right|\right| &= \left|\left|\int_\Omega A(s)\,x d\mu\right|\right| \\ &\leq \int_\Omega ||A(s)\,x||\,d\mu \\ &\leq \int_\Omega ||A(s)||\,d\mu\end{aligned}$$

and so

$$\left|\left|\int_\Omega A(s)\,d\mu\right|\right| \leq \int_\Omega ||A(s)||\,d\mu.$$

This proves the lemma.

Now it is interesting to consider the case where $A(s) \in \mathcal{L}(H,H)$ where $s \to A(s)\,x$ is strongly measurable and $A(s)$ is compact and self adjoint. Recall the Kuratowski measurable selection theorem, Theorem 8.11 on Page 176 listed here for convenience.

Theorem 21.28 *Let E be a compact metric space and let $(\Omega, \mathcal{F})$ be a measure space. Suppose $\psi : E \times \Omega \to \mathbb{R}$ has the property that $x \to \psi(x,\omega)$ is continuous and $\omega \to \psi(x,\omega)$ is measurable. Then there exists a measurable function, f having values in E such that*

$$\psi(f(\omega),\omega) = \sup_{x \in E} \psi(x,\omega).$$

Furthermore, $\omega \to \psi(f(\omega),\omega)$ is measurable.

21.3.1 Review Of Hilbert Schmidt Theorem

Here I will give a proof of the Hilbert Schmidt theorem which will generalize to a result about measurable operators. Recall the following.

Definition 21.29 *Define* $v \otimes u \in \mathcal{L}(H, H)$ *by*

$$v \otimes u(x) = (x, u) v.$$

$A \in \mathcal{L}(H, H)$ *is a compact operator if whenever* $\{x_k\}$ *is a bounded sequence, there exists a convergent subsequence of* $\{Ax_k\}$*. Equivalently,* A *maps bounded sets to sets whose closures are compact or to use other terminology,* A *maps bounded sets to sets which are precompact.*

Lemma 21.30 *Let* H *be a separable Hilbert space and suppose* $A \in \mathcal{L}(H, H)$ *is a compact operator. Let* B *denote the closed unit ball in* H*. Then* A *is continuous as a map from* B *with the weak topology into* H *with the strong topology. For* $u, v \in H$*,* $v \otimes u : H \to H$ *is a compact operator. If* A *is self adjoint and compact, the function*

$$x \to (Ax, x)$$

is continuous on B *with respect to the weak topology on* B*. The function,*

$$x \to (v \otimes u(x), x)$$

is continuous and the operator $u \otimes u$ *is self adjoint.*

Proof: Since H is separable, it follows from Corollary 13.41 on Page 358 that B can be considered as a metric space. Therefore, showing continuity reduces to showing convergent sequences are taken to convergent sequences. Let $x_n \to x$ weakly in B. Suppose Ax_ndoes not converge to Ax. Then there exists a subsequence, still denoted by $\{x_n\}$ such that

$$||Ax_n - Ax|| \geq \varepsilon > 0 \tag{21.24}$$

for all n. Then since A maps bounded sets to compact sets, there is a further subsequence, still denoted by $\{x_n\}$ such that Ax_n converges to some $y \in H$. Therefore,

$$\begin{aligned}(y, w) &= \lim_{n\to\infty} (Ax_n, w) = \lim_{n\to\infty} (x_n, A^* w) \\ &= (x, A^* w) = (Ax, w)\end{aligned}$$

which shows $Ax = y$ since w is arbitrary. However, this contradicts 21.24.

Next consider the claim about $v \otimes u$. Letting $\{x_n\}$ be a bounded sequence,

$$v \otimes u(x_n) = (x_n, u) v.$$

There exists a weakly convergent subsequence of $\{x_n\}$ say $\{x_{n_k}\}$ converging weakly to $x \in H$. Therefore,

$$||v \otimes u(x_{n_k}) - v \otimes u(x)|| = ||(x_{n_k}, u) - (x, u)|| \, ||v||$$

which converges to 0. Thus $v \otimes u$ is compact as claimed. It takes bounded sets to precompact sets.

To verify the assertion about $x \to (Ax, x)$, let $x_n \to x$ weakly. Then

$$\begin{aligned} & |(Ax_n, x_n) - (Ax, x)| \\ \leq & |(Ax_n, x_n) - (Ax, x_n)| + |(Ax, x_n) - (Ax, x)| \\ \leq & |(Ax_n, x_n) - (Ax, x_n)| + |(Ax_n, x) - (Ax, x)| \\ \leq & ||Ax_n - Ax|| \, ||x_n|| + ||Ax_n - Ax|| \, ||x|| \leq 2 \, ||Ax_n - Ax|| \end{aligned}$$

which converges to 0.

$$\begin{aligned} & |(v \otimes u (x_n), x_n) - (v \otimes u (x), x)| \\ = & |(x_n, u)(v, x_n) - (x, u)(v, x)| \end{aligned}$$

and this converges to 0 by weak convergence. It follows from the definition that $u \otimes u$ is self adjoint. This proves the lemma.

Observation 21.31 *Note that if A is any self adjoint operator,*

$$\overline{(Ax, x)} = (x, Ax) = (Ax, x).$$

so (Ax, x) is real valued.

Lemma 21.32 *Let $A \in \mathcal{L}(H, H)$ and suppose it is self adjoint and compact. Let B denote the closed unit ball in H. Let $e \in B$ be such that*

$$|(Ae, e)| = \max_{x \in B} |(Ax, x)|.$$

Then letting $\lambda = (Ae, e)$, it follows $Ae = \lambda e$. If $\lambda \neq 0$, then $||e|| = 1$ and if $\lambda = 0$, it can be assumed $e = 0$ so it is still the case $Ae = \lambda e$.

Proof: From the above observation, (Ax, x) is always real and since A is compact, $|(Ax, x)|$ achieves a maximum at e. It remains to verify e is an eigenvector. Note that $||e|| = 1$ whenever $\lambda \neq 0$ since otherwise $|(Ae, e)|$ could be made larger by replacing e with $e / ||e||$.

Suppose $\lambda = (Ae, e) > 0$. Then it is easy to verify that $\lambda I - A$ is a nonnegative $(((\lambda I - A) x, x) \geq 0$ for all x.) and self adjoint operator. Therefore, the Cauchy Schwarz inequality can be applied to write

$$((\lambda I - A) e, x) \leq ((\lambda I - A) e, e)^{1/2} ((\lambda I - A) x, x)^{1/2} = 0$$

Since this is true for all x it follows $Ae = \lambda e$.

Next suppose $\lambda = (Ae, e) < 0$. Then $-\lambda = (-Ae, e)$ and the previous result can be applied to $-A$ and $-\lambda$. Thus $-\lambda e = -Ae$ and so $Ae = \lambda e$.

Finally consider the case where $\lambda = 0$. Then $0 = (A0, 0)$ and so it suffices to take $e = 0$ as claimed. This proves the lemma.

With these lemmas here is a major theorem, the Hilbert Schmidt theorem.

Theorem 21.33 *Let $A \in \mathcal{L}(H,H)$ be a compact self adjoint operator on a Hilbert space. Then there exist real numbers $\{\lambda_k\}_{k=1}^{\infty}$ and vectors $\{e_k\}_{k=1}^{\infty}$ such that*

$$\begin{aligned} ||e_k|| &= 1 \text{ if } \lambda_k \neq 0, \\ ||e_k|| &= 0 \text{ if } \lambda_k = 0, \end{aligned}$$

$$(e_k, e_j)_H = 0 \text{ if } k \neq j,$$

$$Ae_k = \lambda_k e_k,$$

$$|\lambda_n| \geq |\lambda_{n+1}| \text{ for all } n,$$

$$\lim_{n\to\infty} \lambda_n = 0,$$

$$\lim_{n\to\infty} \left|\left| A - \sum_{k=1}^{n} \lambda_k (e_k \otimes e_k) \right|\right|_{\mathcal{L}(H,H)} = 0. \tag{21.25}$$

Proof: This is done by considering a sequence of compact self adjoint operators, $A, A_1, A_2, \cdots$. Here is how these are defined. Using Lemma 21.32 let e_1, λ_1 be given by that lemma such that

$$|(Ae_1, e_1)| = \max_{x\in B} |(Ax, x)|, \ \lambda_1 = (Ae_1, e_1).$$

Then by that lemma, $Ae_1 = \lambda_1 e_1$ and $||e_1|| = 1$ if $\lambda_1 \neq 0$ while $e_1 = 0$ if $\lambda_1 = 0$.

If A_n has been obtained, use Lemma 21.32 to obtain e_{n+1} and λ_{n+1} such that

$$|(A_n e_{n+1}, e_{n+1})| = \max_{x\in B} |(A_n x, x)|, \ \lambda_{n+1} = (A_n e_{n+1}, e_{n+1}).$$

By that lemma again, $A_n e_{n+1} = \lambda_{n+1} e_{n+1}$ and $||e_{n+1}|| = 1$ if $\lambda_{n+1} \neq 0$ while $e_{n+1} = 0$ if $\lambda_{n+1} = 0$. Then

$$A_{n+1} \equiv A_n - \lambda_{n+1} e_{n+1} \otimes e_{n+1}$$

Thus

$$A_n = A - \sum_{k=1}^{n} \lambda_k e_k \otimes e_k. \tag{21.26}$$

Claim 1: If $k < n+1$ then $(e_{n+1}, e_k) = 0$. Also $Ae_k = \lambda_k e_k$ for all k.

Proof of claim: From the above,

$$\lambda_{n+1} e_{n+1} = A_n e_{n+1} = Ae_{n+1} - \sum_{k=1}^{n} \lambda_k (e_{n+1}, e_k) e_k.$$

If $\lambda_{n+1} = 0$, then $(e_{n+1}, e_k) = 0$ because $e_{n+1} = 0$. If $\lambda_{n+1} \neq 0$, then from the above and an induction hypothesis

$$\begin{aligned} \lambda_{n+1} (e_{n+1}, e_j) &= (Ae_{n+1}, e_j) - \sum_{k=1}^{n} \lambda_k (e_{n+1}, e_k)(e_k, e_j) \\ &= (e_{n+1}, Ae_j) - \sum_{k=1}^{n} \lambda_k (e_{n+1}, e_k)(e_k, e_j) \\ &= \lambda_j (e_{n+1}, e_j) - \lambda_j (e_{n+1}, e_j) = 0. \end{aligned}$$

To verify the second part of this claim,

$$\lambda_{n+1}e_{n+1} = A_n e_{n+1} = Ae_{n+1} - \sum_{k=1}^{n} \lambda_k e_k \left(e_{n+1}, e_k\right) = Ae_{n+1}$$

This proves the claim.

Claim 2: $|\lambda_n| \geq |\lambda_{n+1}|$.

Proof of claim: From 21.26 and the definition of A_n and $e_k \otimes e_k$,

$$\begin{aligned} \lambda_{n+1} &= (A_n e_{n+1}, e_{n+1}) \\ &= (A_{n-1}e_{n+1}, e_{n+1}) - \lambda_n \left|(e_n, e_{n+1})\right|^2 \\ &= (A_{n-1}e_{n+1}, e_{n+1}) \end{aligned}$$

By the previous claim. Therefore,

$$|\lambda_{n+1}| = |(A_{n-1}e_{n+1}, e_{n+1})| \leq |(A_{n-1}e_n, e_n)| = |\lambda_n|$$

by the definition of $|\lambda_n|$. (e_n makes $|(A_{n-1}x, x)|$ as large as possible, not necessarily e_{n+1}.)

Claim 3: $\lim_{n\to\infty} \lambda_n = 0$.

Proof of claim: If for some $n, \lambda_n = 0$, then $\lambda_k = 0$ for all $k > n$ by claim 2. Assume then that $\lambda_k \neq 0$ for any k. Then if $\lim_{k\to\infty} |\lambda_k| = \varepsilon > 0$, contrary to the claim, $||e_k|| = 1$ for all k and

$$\begin{aligned} ||Ae_n - Ae_m||^2 &= ||\lambda_n e_n - \lambda_m e_m||^2 \\ &= \lambda_n^2 + \lambda_m^2 \geq 2\varepsilon^2 \end{aligned}$$

which shows there is no Cauchy subsequence of $\{Ae_n\}_{n=1}^{\infty}$, which contradicts the compactness of A. This proves the claim.

Claim 4: $||A_n|| \to 0$

Proof of claim: Let $x, y \in B$

$$\begin{aligned} |\lambda_{n+1}| &\geq \left|\left(A_n \frac{x+y}{2}, \frac{x+y}{2}\right)\right| \\ &= \left|\frac{1}{4}(A_n x, x) + \frac{1}{4}(A_n y, y) + \frac{1}{2}(A_n x, y)\right| \\ &\geq \frac{1}{2}|(A_n x, y)| - \frac{1}{4}|(A_n x, x) + (A_n y, y)| \\ &\geq \frac{1}{2}|(A_n x, y)| - \frac{1}{4}(|(A_n x, x)| + |(A_n y, y)|) \\ &\geq \frac{1}{2}|(A_n x, y)| - \frac{1}{2}|\lambda_{n+1}| \end{aligned}$$

and so

$$3|\lambda_{n+1}| \geq |(A_n x, y)|.$$

It follows $||A_n|| \leq 3|\lambda_{n+1}|$. This proves the claim.

By 21.26 this proves 21.25 and completes the proof.

21.3.2 Measurable Compact Operators

Here the operators will be of the form $A(s)$ where $s \in \Omega$ and $s \to A(s)x$ is strongly measurable and $A(s)$ is a compact operator in $\mathcal{L}(H,H)$.

Theorem 21.34 *Let $A(s) \in \mathcal{L}(H,H)$ be a compact self adjoint operator and H is a separable Hilbert space such that $s \to A(s)x$ is strongly measurable. Then there exist real numbers $\{\lambda_k(s)\}_{k=1}^{\infty}$ and vectors $\{e_k(s)\}_{k=1}^{\infty}$ such that*

$$\begin{aligned}||e_k(s)|| &= 1 \text{ if } \lambda_k \neq 0, \\ ||e_k(s)|| &= 0 \text{ if } \lambda_k = 0,\end{aligned}$$

$$(e_k(s), e_j(s))_H = 0 \text{ if } k \neq j,$$

$$A(s)e_k(s) = \lambda_k(s)e_k(s),$$

$$|\lambda_n(s)| \geq |\lambda_{n+1}(s)| \text{ for all } n,$$

$$\lim_{n\to\infty} \lambda_n(s) = 0,$$

$$\lim_{n\to\infty} \left|\left| A(s) - \sum_{k=1}^{n} \lambda_k(s)(e_k(s) \otimes e_k(s)) \right|\right|_{\mathcal{L}(H,H)} = 0.$$

The function $s \to \lambda_j(s)$ is measurable and $s \to e_j(s)$ is strongly measurable.

Proof: It is simply a repeat of the above proof of the Hilbert Schmidt theorem except at every step when the e_k and λ_k are defined, you use the Kuratowski measurable selection theorem, Theorem 21.28 on Page 595 to obtain $\lambda_k(s)$ is measurable and that $s \to e_k(s)$ is also measurable.

When you consider $\max_{x\in B} |(A_n(s)x, x)|$, let $\psi(x,s) = |(A_n(s)x,x)|$. Then ψ is continuous in x by Lemma 21.30 on Page 596 and it is measurable in s by assumption. Therefore, by the Kuratowski theorem, $e_k(s)$ is measurable in the sense that inverse images of weakly open sets in B are measurable. However, by Lemma 21.9 on Page 582 this is the same as weakly measurable. Since H is separable, this implies $s \to e_k(s)$ is also strongly measurable. The measurability of λ_k and e_k is the only new thing here and so this completes the proof.

21.4 Fubini's Theorem For Bochner Integrals

Now suppose $(\Omega_1, \mathcal{F}, \mu)$ and $(\Omega_2, \mathcal{S}, \lambda)$ are two σ finite measure spaces. Recall the notion of product measure. There was a σ algebra, denoted by $\mathcal{F} \times \mathcal{S}$ which is the smallest σ algebra containing the elementary sets, (finite disjoint unions of measurable rectangles) and a measure, denoted by $\mu \times \lambda$ defined on this σ algebra such that for $E \in \mathcal{F} \times \mathcal{S}$,

$$s_1 \to \lambda(E_{s_1}), \; (E_{s_1} \equiv \{s_2 : (s_1, s_2) \in E\})$$

is μ measurable and

$$s_2 \to \mu(E_{s_2}), \ (E_{s_2} \equiv \{s_1 : (s_1, s_2) \in E\})$$

is λ measurable. In terms of nonnegative functions which are $\mathcal{F} \times \mathcal{S}$ measurable,

$$\begin{aligned} s_1 &\to f(s_1, s_2) \text{ is } \mu \text{ measurable,} \\ s_2 &\to f(s_1, s_2) \text{ is } \lambda \text{ measurable,} \\ s_1 &\to \int_{\Omega_2} f(s_1, s_2)\, d\lambda \text{ is } \mu \text{ measurable,} \\ s_2 &\to \int_{\Omega_1} f(s_1, s_2)\, d\mu \text{ is } \lambda \text{ measurable,} \end{aligned}$$

and the conclusion of Fubini's theorem holds.

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f d(\mu \times \lambda) &= \int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2)\, d\lambda d\mu \\ &= \int_{\Omega_2} \int_{\Omega_1} f(s_1, s_2)\, d\mu d\lambda. \end{aligned}$$

The following theorem is the version of Fubini's theorem valid for Bochner integrable functions.

Theorem 21.35 *Let $f : \Omega_1 \times \Omega_2 \to X$ be strongly measurable with respect to $\mu \times \lambda$ and suppose*

$$\int_{\Omega_1 \times \Omega_2} ||f(s_1, s_2)||\, d(\mu \times \lambda) < \infty. \tag{21.27}$$

Then there exist a set of μ measure zero, N and a set of λ measure zero, M such that the following formula holds with all integrals making sense.

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f(s_1, s_2)\, d(\mu \times \lambda) &= \int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2)\, \mathcal{X}_N(s_1)\, d\lambda d\mu \\ &= \int_{\Omega_2} \int_{\Omega_1} f(s_1, s_2)\, \mathcal{X}_M(s_2)\, d\mu d\lambda. \end{aligned}$$

Proof: First note that from 21.27 and the usual Fubini theorem for nonnegative valued functions,

$$\int_{\Omega_1 \times \Omega_2} ||f(s_1, s_2)||\, d(\mu \times \lambda) = \int_{\Omega_1} \int_{\Omega_2} ||f(s_1, s_2)||\, d\lambda d\mu$$

and so

$$\int_{\Omega_2} ||f(s_1, s_2)||\, d\lambda < \infty \tag{21.28}$$

for μ a.e. s_1. Say for all $s_1 \notin N$ where $\mu(N) = 0$.

Let $\phi \in X'$. Then $\phi \circ f$ is $\mathcal{F} \times \mathcal{S}$ measurable and

$$\begin{aligned} & \int_{\Omega_1 \times \Omega_2} |\phi \circ f(s_1, s_2)| \, d(\mu \times \lambda) \\ \leq & \int_{\Omega_1 \times \Omega_2} ||\phi|| \, ||f(s_1, s_2)|| \, d(\mu \times \lambda) < \infty \end{aligned}$$

and so from the usual Fubini theorem for complex valued functions,

$$\int_{\Omega_1 \times \Omega_2} \phi \circ f(s_1, s_2) \, d(\mu \times \lambda) = \int_{\Omega_1} \int_{\Omega_2} \phi \circ f(s_1, s_2) \, d\lambda d\mu. \tag{21.29}$$

Now also if you fix s_2, it follows from the definition of strongly measurable and the properties of product measure mentioned above that

$$s_1 \to f(s_1, s_2)$$

is strongly measurable. Also, by 21.28

$$\int_{\Omega_2} ||f(s_1, s_2)|| \, d\lambda < \infty$$

for $s_1 \notin N$. Therefore, by Theorem 21.19 $s_2 \to f(s_1, s_2) \mathcal{X}_{N^C}(s_1)$ is Bochner integrable. By 21.29 and 21.6

$$\begin{aligned} & \int_{\Omega_1 \times \Omega_2} \phi \circ f(s_1, s_2) \, d(\mu \times \lambda) \\ = & \int_{\Omega_1} \int_{\Omega_2} \phi \circ f(s_1, s_2) \, d\lambda d\mu \\ = & \int_{\Omega_1} \int_{\Omega_2} \phi(f(s_1, s_2) \mathcal{X}_{N^C}(s_1)) \, d\lambda d\mu \\ = & \int_{\Omega_1} \phi\left(\int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^C}(s_1) \, d\lambda\right) d\mu. \end{aligned} \tag{21.30}$$

Each iterated integral makes sense and

$$\begin{aligned} s_1 \to & \int_{\Omega_2} \phi(f(s_1, s_2) \mathcal{X}_{N^C}(s_1)) \, d\lambda \\ = & \, \phi\left(\int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^C}(s_1) \, d\lambda\right) \end{aligned} \tag{21.31}$$

is μ measurable because

$$\begin{aligned} (s_1, s_2) \to & \, \phi(f(s_1, s_2) \mathcal{X}_{N^C}(s_1)) \\ = & \, \phi(f(s_1, s_2)) \mathcal{X}_{N^C}(s_1) \end{aligned}$$

is product measurable. Now consider the function,

$$s_1 \rightarrow \int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^C}(s_1) \, d\lambda. \tag{21.32}$$

I want to show this is also Bochner integrable with respect to μ so I can factor out ϕ once again. It's measurability follows from the Pettis theorem and the above observation 21.31. Also,

$$\begin{aligned}
& \int_{\Omega_1} \left|\left| \int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^C}(s_1) \, d\lambda \right|\right| d\mu \\
\leq & \int_{\Omega_1} \int_{\Omega_2} ||f(s_1, s_2)|| \, d\lambda d\mu \\
= & \int_{\Omega_1 \times \Omega_2} ||f(s_1, s_2)|| \, d(\mu \times \lambda) < \infty.
\end{aligned}$$

Therefore, the function in 21.32 is indeed Bochner integrable and so in 21.30 the ϕ can be taken outside the last integral. Thus,

$$\begin{aligned}
& \phi \left(\int_{\Omega_1 \times \Omega_2} f(s_1, s_2) \, d(\mu \times \lambda) \right) \\
= & \int_{\Omega_1 \times \Omega_2} \phi \circ f(s_1, s_2) \, d(\mu \times \lambda) \\
= & \int_{\Omega_1} \int_{\Omega_2} \phi \circ f(s_1, s_2) \, d\lambda d\mu \\
= & \int_{\Omega_1} \phi \left(\int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^C}(s_1) \, d\lambda \right) d\mu \\
= & \phi \left(\int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^C}(s_1) \, d\lambda d\mu \right).
\end{aligned}$$

Since X' separates the points,

$$\int_{\Omega_1 \times \Omega_2} f(s_1, s_2) \, d(\mu \times \lambda) = \int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^C}(s_1) \, d\lambda d\mu.$$

The other formula follows from similar reasoning. This proves the theorem.

21.5 The Spaces $L^p(\Omega; X)$

Definition 21.36 *$x \in L^p(\Omega; X)$ for $p \in [1, \infty)$ if x is strongly measurable and*

$$\int_{\Omega} ||x(s)||^p \, d\mu < \infty$$

Also

$$||x||_{L^p(\Omega; X)} \equiv ||x||_p \equiv \left(\int_{\Omega} ||x(s)||^p \, d\mu \right)^{1/p}. \tag{21.33}$$

As in the case of scalar valued functions, two functions in $L^p(\Omega; X)$ are considered equal if they are equal a.e. With this convention, and using the same arguments found in the presentation of scalar valued functions it is clear that $L^p(\Omega; X)$ is a normed linear space with the norm given by 21.33. In fact, $L^p(\Omega; X)$ is a Banach space. This is the main contribution of the next theorem.

Lemma 21.37 *If x_n is a Cauchy sequence in $L^p(\Omega; X)$ satisfying*

$$\sum_{n=1}^{\infty} ||x_{n+1} - x_n||_p < \infty,$$

then there exists $x \in L^p(\Omega; X)$ such that $x_n(s) \to x(s)$ a.e. and

$$||x - x_n||_p \to 0.$$

Proof: Let

$$g_N(s) \equiv \sum_{n=1}^{N} ||x_{n+1}(s) - x_n(s)||_X$$

Then by the triangle inequality,

$$\begin{aligned}\left(\int_\Omega g_N(s)^p \, d\mu\right)^{1/p} &\leq \sum_{n=1}^{N} \left(\int_\Omega ||x_{n+1}(s) - x_n(s)||^p \, d\mu\right)^{1/p} \\ &\leq \sum_{n=1}^{\infty} ||x_{n+1} - x_n||_p < \infty.\end{aligned}$$

Let

$$g(s) = \lim_{N\to\infty} g_N(s) = \sum_{n=1}^{\infty} ||x_{n+1}(s) - x_n(s)||_X .$$

By the monotone convergence theorem,

$$\left(\int_\Omega g(s)^p \, d\mu\right)^{1/p} = \lim_{N\to\infty} \left(\int_\Omega g_N(s)^p \, d\mu\right)^{1/p} < \infty.$$

Therefore, there exists a set of measure 0, E, such that for $s \notin E$, $g(s) < \infty$. Hence, for $s \notin E$,

$$\lim_{N\to\infty} x_{N+1}(s)$$

exists because

$$x_{N+1}(s) = x_{N+1}(s) - x_1(s) + x_1(s) = \sum_{n=1}^{N} (x_{n+1}(s) - x_n(s)) + x_1(s).$$

Thus, if $N > M$, and s is a point where $g(s) < \infty$,

$$\begin{aligned} ||x_{N+1}(s) - x_{M+1}(s)||_X &\leq \sum_{n=M+1}^{N} ||x_{n+1}(s) - x_n(s)||_X \\ &\leq \sum_{n=M+1}^{\infty} ||x_{n+1}(s) - x_n(s)||_X \end{aligned}$$

which shows that $\{x_{N+1}(s)\}_{N=1}^{\infty}$ is a Cauchy sequence. Now let

$$x(s) \equiv \begin{cases} \lim_{N\to\infty} x_N(s) \text{ if } s \notin E, \\ 0 \text{ if } s \in E. \end{cases}$$

By Theorem 21.2, $x_n(\Omega)$ is separable for each n. Therefore, $x(\Omega)$ is also separable. Also, if $f \in X'$, then

$$f(x(s)) = \lim_{N\to\infty} f(x_N(s))$$

if $s \notin E$ and $f(x(s)) = 0$ if $s \in E$. Therefore, $f \circ x$ is measurable because it is the limit of the measurable functions,

$$f \circ x_N \mathcal{X}_{E^C}.$$

Since x is weakly measurable and $x(\Omega)$ is separable, Corollary 21.8 shows that x is strongly measurable. By Fatou's lemma,

$$\int_{\Omega} ||x(s) - x_N(s)||^p \, d\mu \leq \lim\inf_{M\to\infty} \int_{\Omega} ||x_M(s) - x_N(s)||^p \, d\mu.$$

But if N and M are large enough with $M > N$,

$$\begin{aligned} \left(\int_{\Omega} ||x_M(s) - x_N(s)||^p \, d\mu \right)^{1/p} &\leq \sum_{n=N}^{M} ||x_{n+1} - x_n||_p \\ &\leq \sum_{n=N}^{\infty} ||x_{n+1} - x_n||_p < \varepsilon \end{aligned}$$

and this shows, since ε is arbitrary, that

$$\lim_{N\to\infty} \int_{\Omega} ||x(s) - x_N(s)||^p \, d\mu = 0.$$

It remains to show $x \in L^p(\Omega; X)$. This follows from the above and the triangle inequality. Thus, for N large enough,

$$\left(\int_{\Omega} ||x(s)||^p \, d\mu \right)^{1/p}$$

$$\leq \left(\int_\Omega ||x_N(s)||^p \, d\mu\right)^{1/p} + \left(\int_\Omega ||x(s) - x_N(s)||^p \, d\mu\right)^{1/p}$$
$$\leq \left(\int_\Omega ||x_N(s)||^p \, d\mu\right)^{1/p} + \varepsilon < \infty.$$

This proves the lemma.

Theorem 21.38 *$L^p(\Omega; X)$ is complete. Also every Cauchy sequence has a subsequence which converges pointwise.*

Proof: If $\{x_n\}$ is Cauchy in $L^p(\Omega; X)$, extract a subsequence $\{x_{n_k}\}$ satisfying

$$\left|\left|x_{n_{k+1}} - x_{n_k}\right|\right|_p \leq 2^{-k}$$

and apply Lemma 21.37. The pointwise convergence of this subsequence was established in the proof of this lemma. This proves the theorem because if a subsequence of a Cauchy sequence converges, then the Cauchy sequence must also converge.

Observation 21.39 *If the measure space is Lebesgue measure then you have continuity of translation in $L^p(\mathbb{R}^n; X)$ in the usual way. More generally, for μ a Radon measure on Ω a locally compact Hausdorff space, $C_c(\Omega; X)$ is dense in $L^p(\Omega; X)$. Here $C_c(\Omega; X)$ is the space of continuous X valued functions which have compact support in Ω. The proof of this little observation follows immediately from approximating with simple functions and then applying the appropriate considerations to the simple functions.*

Clearly Fatou's lemma and the monotone convergence theorem make no sense for functions with values in a Banach space but the dominated convergence theorem holds in this setting.

Theorem 21.40 *If x is strongly measurable and $x_n(s) \to x(s)$ a.e. with*

$$||x_n(s)|| \leq g(s) \text{ a.e.}$$

where $g \in L^1(\Omega)$, then x is Bochner integrable and

$$\int_\Omega x(s) \, d\mu = \lim_{n\to\infty} \int_\Omega x_n(s) \, d\mu.$$

Proof: $||x_n(s) - x(s)|| \leq 2g(s)$ a.e. so by the usual dominated convergence theorem,

$$0 = \lim_{n\to\infty} \int_\Omega ||x_n(s) - x(s)|| \, d\mu.$$

Also,

$$\int_\Omega ||x_n(s) - x_m(s)|| \, d\mu$$

$$\leq \int_\Omega ||x_n(s) - x(s)||\,d\mu + \int_\Omega ||x_m(s) - x(s)||\,d\mu,$$

and so $\{x_n\}$ is a Cauchy sequence in $L^1(\Omega;X)$. Therefore, by Theorem 21.38, there exists $y \in L^1(\Omega;X)$ and a subsequence $x_{n'}$ satisfying

$$x_{n'}(s) \to y(s) \text{ a.e. and in } L^1(\Omega;X).$$

But $x(s) = \lim_{n'\to\infty} x_{n'}(s)$ a.e. and so $x(s) = y(s)$ a.e. Hence

$$\int_\Omega ||x(s)||\,d\mu = \int_\Omega ||y(s)||\,d\mu < \infty$$

which shows that x is Bochner integrable. Finally, since the integral is linear,

$$\begin{aligned}\left|\left|\int_\Omega x(s)\,d\mu - \int_\Omega x_n(s)\,d\mu\right|\right| &= \left|\left|\int_\Omega (x(s) - x_n(s))\,d\mu\right|\right| \\ &\leq \int_\Omega ||x_n(s) - x(s)||\,d\mu,\end{aligned}$$

and this last integral converges to 0. This proves the theorem.

The following theorem is interesting.

Theorem 21.41 *Let $1 \leq p < \infty$ and let $p < r \leq \infty$. Then $L^r([0,T],X)$ is a Borel subset of $L^p([0,T];X)$. Letting $C([0,T];X)$ denote the functions having values in X which are continuous, $C([0,T];X)$ is also a Borel subset of $L^p([0,T];X)$. Here the measure is ordinary one dimensional Lebesgue measure on $[0,T]$.*

Proof: First consider the claim about $L^r([0,T];X)$. Let

$$B_M \equiv \left\{x \in L^p([0,T];X) : ||x||_{L^r([0,T];X)} \leq M\right\}.$$

Then B_M is a closed subset of $L^p([0,T];X)$. Here is why. If $\{x_n\}$ is a sequence of elements of B_M and $x_n \to x$ in $L^p([0,T];X)$, then passing to a subsequence, still denoted by x_n, it can be assumed $x_n(s) \to x(s)$ a.e. Hence Fatou's lemma can be applied to conclude

$$\int_0^T ||x(s)||^r\,ds \leq \lim\inf_{n\to\infty} \int_0^T ||x_n(s)||^r\,ds \leq M^r < \infty.$$

Now $\cup_{M=1}^\infty B_M = L^r([0,T];X)$. Note this did not depend on the measure space used. It would have been equally valid on any measure space.

Consider now $C([0,T];X)$. The norm on this space is the usual norm, $||\cdot||_\infty$. The argument above shows $||\cdot||_\infty$ is a Borel measurable function on $L^p([0,T];X)$. This is because $B_M \equiv \{x \in L^p([0,T];X) : ||x||_\infty \leq M\}$ is a closed, hence Borel subset of $L^p([0,T];X)$. Now let $\theta \in \mathcal{L}(L^p([0,T];X), L^p(\mathbb{R};X))$ such that $\theta(x(t)) = x(t)$ for all $t \in [0,T]$ and also $\theta \in \mathcal{L}(C([0,T];X), BC(\mathbb{R};X))$ where $BC(\mathbb{R};X)$ denotes the bounded continuous functions with a norm given by

$$||x|| \equiv \sup_{t\in\mathbb{R}} ||x(t)||,$$

and θx has compact support.

For example, you could define

$$\widetilde{x}(t) \equiv \begin{cases} x(t) \text{ if } t \in [0,T] \\ x(2T-t) \text{ if } t \in [T,2T] \\ x(-t) \text{ if } t \in [-T,0] \\ 0 \text{ if } t \notin [-T,2T] \end{cases}$$

and let $\Phi \in C_c^\infty(-T,2T)$ such that $\Phi(t) = 1$ for $t \in [0,T]$. Then you could let

$$\theta x(t) \equiv \Phi(t)\widetilde{x}(t).$$

Then let $\{\phi_n\}$ be a mollifier and define

$$\psi_n x(t) \equiv \phi_n * \theta x(t).$$

It follows $\psi_n x$ is uniformly continuous because

$$\begin{aligned} & ||\psi_n x(t) - \psi_n x(t')||_X \\ \leq & \int_{\mathbb{R}} |\phi_n(t'-s) - \phi_n(t-s)| \, ||\theta x(s)||_X \, ds \\ \leq & C\,||x||_p \left(\int_{\mathbb{R}} |\phi_n(t'-s) - \phi_n(t-s)|^{p'} ds \right)^{1/p'} \end{aligned}$$

Also for $x \in C([0,T];X)$, it follows from usual mollifier arguments that

$$||\psi_n x - x||_{L^\infty([0,T];X)} \to 0.$$

Here is why. For $t \in [0,T]$,

$$\begin{aligned} ||\psi_n x(t) - x(t)||_X & \leq \int_{\mathbb{R}} \phi_n(s) \, ||\theta x(t-s) - \theta x(t)|| \, ds \\ & \leq C_\theta \int_{-1/n}^{1/n} \phi_n(s) \, ds \varepsilon = C_\theta \varepsilon \end{aligned}$$

provided n is large enough due to the compact support and consequent uniform continuity of θx.

If $||\psi_n x - x||_{L^\infty([0,T];X)} \to 0$, then $\{\psi_n x\}$ must be a Cauchy sequence in $C([0,T];X)$ and this requires that x equals a continuous function a.e. Thus $C([0,T];X)$ consists exactly of those functions, x of $L^p([0,T];X)$ such that $||\psi_n x - x||_\infty \to 0$. It follows

$$C([0,T];X) = \\ \cap_{n=1}^\infty \cup_{m=1}^\infty \cap_{k=m}^\infty \left\{ x \in L^p([0,T];X) : ||\psi_k x - x||_\infty \leq \frac{1}{n} \right\}. \tag{21.34}$$

It only remains to show

$$S \equiv \{x \in L^p([0,T];X) : ||\psi_k x - x||_\infty \leq \alpha\}$$

is a Borel set. Suppose then that $x_n \in S$ and $x_n \to x$ in $L^p([0,T];X)$. Then there exists a subsequence, still denoted by n such that $x_n \to x$ pointwise a.e. as well as in L^p. There exists a set of measure 0 such that for all n, and t not in this set,

$$\begin{aligned} \|\psi_k x_n(t) - x_n(t)\| &\equiv \left\| \int_{-1/k}^{1/k} \phi_k(s)(\theta x_n(t-s))\,ds - x_n(t) \right\| \le \alpha \\ x_n(t) &\to x(t). \end{aligned}$$

Then

$$\begin{aligned} &\|\psi_k x_n(t) - x_n(t) - (\psi_k x(t) - x(t))\| \\ \le\ &\|x_n(t) - x(t)\|_X + \left\| \int_{-1/k}^{1/k} \phi_k(s)(\theta x_n(t-s) - \theta x(t-s))\,ds \right\| \\ \le\ &\|x_n(t) - x(t)\|_X + C_{k,\theta}\|x_n - x\|_{L^p(0,T;X)} \end{aligned}$$

which converges to 0 as $n \to \infty$. It follows that for a.e. t,

$$\|\psi_k x(t) - x(t)\| \le \alpha.$$

Thus S is closed and so the set in 21.34 is a Borel set. This proves the theorem.

As in the scalar case, the following lemma holds in this more general context.

Lemma 21.42 *Let (Ω,μ) be a regular measure space where Ω is a locally compact Hausdorff space. Then $C_c(\Omega;X)$ the space of continuous functions having compact support and values in X is dense in $L^p(0,T;X)$ for all $p \in [0,\infty)$. For any σ finite measure space, the simple functions are dense in $L^p(0,T;X)$.*

Proof: First is it shown the simple functions are dense in $L^p(0,T;X)$. Let $f \in L^p(0,T;X)$ and let $\{x_n\}$ denote a sequence of simple functions which converge to f pointwise which also have the property that

$$\|x_n(s)\| \le 2\|f(s)\|$$

Then

$$\int_\Omega \|x_n(s) - f(s)\|^p\,d\mu \to 0$$

from the dominated convergence theorem. Therefore, the simple functions are indeed dense in $L^p(0,T;X)$.

Next suppose (Ω,μ) is a regular measure space. If $x(s) \equiv \sum_i a_i \mathcal{X}_{E_i}(s)$ is a simple function, then by regularity, there exist compact sets, K_i and open sets, V_i such that $K_i \subseteq E_i \subseteq V_i$ and $\mu(V_i \setminus K_i)^{1/p} < \varepsilon / \sum_i \|a_i\|$. Let $K_i \prec h_i \prec V_i$. Then consider

$$\sum_i a_i h_i \in C_c(\Omega).$$

By the triangle inequality,

$$\begin{aligned}
&\left(\int_{\Omega}\left|\left|\sum_{i}a_{i}h_{i}\left(s\right)-a_{i}\mathcal{X}_{E_{i}}\left(s\right)\right|\right|^{p}d\mu\right)^{1/p}\\
\leq&\sum_{i}\left(\int_{\Omega}\left|\left|a_{i}\left(h_{i}\left(s\right)-\mathcal{X}_{E_{i}}\left(s\right)\right)\right|\right|^{p}d\mu\right)^{1/p}\\
\leq&\sum_{i}\left(\int_{\Omega}\left|\left|a_{i}\right|\right|^{p}\left|h_{i}\left(s\right)-\mathcal{X}_{E_{i}}\left(s\right)\right|^{p}d\mu\right)^{1/p}\\
\leq&\sum_{i}\left|\left|a_{i}\right|\right|\left(\int_{V_{i}\setminus K_{i}}d\mu\right)^{1/p}\\
\leq&\sum_{i}\left|\left|a_{i}\right|\right|\mu\left(V_{i}\setminus K_{i}\right)^{1/p}<\varepsilon
\end{aligned}$$

Since ε is arbitrary, this and the first part of the lemma shows $C_c\left(\Omega;X\right)$ is dense in $L^p\left(\Omega;X\right)$.

21.6 Measurable Representatives

In this section consider the special case where $X=L^1\left(B,\nu\right)$ where $\left(B,\mathcal{F},\nu\right)$ is a σ finite measure space and $x\in L^1\left(\Omega;X\right)$. Thus for each $s\in\Omega,\ x\left(s\right)\in L^1\left(B,\nu\right)$. In general, the map

$$\left(s,t\right)\rightarrow x\left(s\right)\left(t\right)$$

will not be measurable, but one can obtain a measurable representative. This is important because it allows the use of Fubini's theorem on the measurable representative.

By Theorem 21.19, there exists a sequence of simple functions, $\{x_n\}$, of the form

$$x_n\left(s\right)=\sum_{k=1}^{m}a_k\mathcal{X}_{E_k}\left(s\right) \tag{21.35}$$

where $a_k\in L^1\left(B,\nu\right)$ which satisfy the conditions of Definition 21.17 and

$$\left|\left|x_n-x\right|\right|_1\rightarrow 0. \tag{21.36}$$

Because of the form of x_n given in 21.35, if

$$x_n\left(s,t\right)\equiv x_n\left(s\right)\left(t\right),$$

then x_n is measurable.

$$\int_{\Omega}\int_{B}\left|x_n\left(s,t\right)-x_m\left(s,t\right)\right|d\nu d\mu\leq\int_{\Omega}\int_{B}\left|x_n\left(s,t\right)-x\left(s\right)\left(t\right)\right|d\nu d\mu$$

$$+\int_{\Omega}\int_{B}|x_m(s,t)-x(s)(t)|\,d\nu d\mu. \tag{21.37}$$

It follows from 21.37 and 21.36 that $\{x_n\}$ is a Cauchy sequence in $L^1(\Omega\times B)$. Therefore, there exists $y\in L^1(\Omega\times B)$ and a subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, such that

$$\lim_{n\to\infty}x_n(s,t)=y(s,t)\text{ a.e.}$$

and

$$\lim_{n\to\infty}||x_n-y||_1=0.$$

It follows that

$$\begin{aligned}\int_{\Omega}\int_{B}|y(s,t)-x(s)(t)|\,d\nu d\mu &\leq \int_{\Omega}\int_{B}|y(s,t)-x_n(s,t)|\,d\nu d\mu \\ &+\int_{\Omega}\int_{B}|x(s)(t)-x_n(s,t)|\,d\nu d\mu.\end{aligned} \tag{21.38}$$

Since $\lim_{n\to\infty}||x_n-x||_1=0$, it follows from 21.38 that $y=x$ in $L^1(\Omega;X)$. Thus, for a.e. s,

$$y(s,\cdot)=x(s)\text{ in }X=L^1(B).$$

Now $\int_{\Omega}x(s)\,d\mu\in X=L^1(B,\nu)$ so it makes sense to ask for $\left(\int_{\Omega}x(s)\,d\mu\right)(t)$, at least a.e. To find what this is, note

$$\left|\left|\int_{\Omega}x_n(s)\,d\mu-\int_{\Omega}x(s)\,d\mu\right|\right|_X\leq\int_{\Omega}||x_n(s)-x(s)||_X\,d\mu.$$

Therefore, since the right side converges to 0,

$$\lim_{n\to\infty}\left|\left|\int_{\Omega}x_n(s)\,d\mu-\int_{\Omega}x(s)\,d\mu\right|\right|_X=$$

$$\lim_{n\to\infty}\int_B\left|\left(\int_{\Omega}x_n(s)\,d\mu\right)(t)-\left(\int_{\Omega}x(s)\,d\mu\right)(t)\right|d\nu=0.$$

But

$$\left(\int_{\Omega}x_n(s)\,d\mu\right)(t)=\int_{\Omega}x_n(s,t)\,d\mu\text{ a.e. }t.$$

Therefore

$$\lim_{n\to\infty}\int_B\left|\int_{\Omega}x_n(s,t)\,d\mu-\left(\int_{\Omega}x(s)\,d\mu\right)(t)\right|d\nu=0. \tag{21.39}$$

Also, since $x_n\to y$ in $L^1(\Omega\times B)$,

$$0=\lim_{n\to\infty}\int_B\int_{\Omega}|x_n(s,t)-y(s,t)|\,d\mu d\nu\geq$$

$$\lim_{n\to\infty}\int_B\left|\int_{\Omega}x_n(s,t)\,d\mu-\int_{\Omega}y(s,t)\,d\mu\right|d\nu. \tag{21.40}$$

From 21.39 and 21.40

$$\int_{\Omega} y(s,t)\,d\mu = \left(\int_{\Omega} x(s)\,d\mu\right)(t) \text{ a.e. } t.$$

This proves the following theorem.

Theorem 21.43 *Let $X = L^1(B)$ where $(B, \mathcal{F}, \nu)$ is a σ finite measure space and let $x \in L^1(\Omega; X)$. Then there exists a measurable representative, $y \in L^1(\Omega \times B)$, such that*

$$x(s) = y(s,\cdot) \text{ a.e. } s \text{ in } \Omega,$$

and

$$\int_{\Omega} y(s,t)\,d\mu = \left(\int_{\Omega} x(s)\,d\mu\right)(t) \text{ a.e. } t.$$

21.7 Vector Measures

There is also a concept of vector measures.

Definition 21.44 *Let $(\Omega, \mathcal{S})$ be a set and a σ algebra of subsets of Ω. A mapping*

$$F : \mathcal{S} \to X$$

is said to be a vector measure if

$$F(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} F(E_i)$$

whenever $\{E_i\}_{i=1}^{\infty}$ is a sequence of disjoint elements of $\mathcal{S}$. For F a vector measure,

$$|F|(A) \equiv \sup\{\sum_{F \in \pi(A)} ||\mu(F)|| : \pi(A) \text{ is a partition of } A\}.$$

This is the same definition that was given in the case where F would have values in $\mathbb{C}$, the only difference being the fact that now F has values in a general Banach space X as the vector space of values of the vector measure. Recall that a partition of A is a finite set, $\{F_1, \cdots, F_m\} \subseteq \mathcal{S}$ such that $\cup_{i=1}^{m} F_i = A$. The same theorem about $|F|$ proved in the case of complex valued measures holds in this context with the same proof. For completeness, it is included here.

Theorem 21.45 *If $|F|(\Omega) < \infty$, then $|F|$ is a measure on $\mathcal{S}$.*

Proof: Let E_1 and E_2 be sets of $\mathcal{S}$ such that $E_1 \cap E_2 = \emptyset$ and let $\{A_1^i \cdots A_{n_i}^i\} = \pi(E_i)$, a partition of E_i which is chosen such that

$$|F|(E_i) - \varepsilon < \sum_{j=1}^{n_i} ||F(A_j^i)|| \quad i = 1, 2.$$

Consider the sets which are contained in either of $\pi(E_1)$ or $\pi(E_2)$, it follows this collection of sets is a partition of $E_1 \cup E_2$ which is denoted here by $\pi(E_1 \cup E_2)$. Then by the above inequality and the definition of total variation,

$$|F|(E_1 \cup E_2) \geq \sum_{F \in \pi(E_1 \cup E_2)} ||F(F)|| > |F|(E_1) + |F|(E_2) - 2\varepsilon,$$

which shows that since $\varepsilon > 0$ was arbitrary,

$$|F|(E_1 \cup E_2) \geq |F|(E_1) + |F|(E_2). \tag{21.41}$$

Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of disjoint sets of $\mathcal{S}$ and let $E_\infty = \cup_{j=1}^{\infty} E_j$. Then by the definition of total variation there exists a partition of E_∞, $\pi(E_\infty) = \{A_1, \cdot \cdot \cdot, A_n\}$ such that

$$|F|(E_\infty) - \varepsilon < \sum_{i=1}^{n} ||F(A_i)||.$$

Also,

$$A_i = \cup_{j=1}^{\infty} A_i \cap E_j$$

and so by the triangle inequality, $||F(A_i)|| \leq \sum_{j=1}^{\infty} ||F(A_i \cap E_j)||$. Therefore, by the above,

$$\begin{aligned} |F|(E_\infty) - \varepsilon &< \sum_{i=1}^{n} \overbrace{\sum_{j=1}^{\infty} ||F(A_i \cap E_j)||}^{\geq ||F(A_i)||} \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^{n} ||F(A_i \cap E_j)|| \\ &\leq \sum_{j=1}^{\infty} |F|(E_j) \end{aligned}$$

because $\{A_i \cap E_j\}_{i=1}^{n}$ is a partition of E_j.

Since $\varepsilon > 0$ is arbitrary, this shows

$$|F|(\cup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} |F|(E_j).$$

Also, 21.41 implies that whenever the E_i are disjoint, $|F|(\cup_{j=1}^{n} E_j) \geq \sum_{j=1}^{n} |F|(E_j)$. Therefore,

$$\sum_{j=1}^{\infty} |F|(E_j) \geq |F|(\cup_{j=1}^{\infty} E_j) \geq |F|(\cup_{j=1}^{n} E_j) \geq \sum_{j=1}^{n} |F|(E_j).$$

Since n is arbitrary,

$$|F|(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} |F|(E_j)$$

which shows that $|F|$ is a measure as claimed. This proves the theorem.

Definition 21.46 *A Banach space is said to have the Radon Nikodym property if whenever*

$$(\Omega, \mathcal{S}, \mu) \text{ is a finite measure space}$$

$$F : \mathcal{S} \to X \text{ is a vector measure with } |F|(\Omega) < \infty$$

$$F \ll \mu$$

then one may conclude there exists $g \in L^1(\Omega; X)$ *such that*

$$F(E) = \int_E g(s)\, d\mu$$

for all $E \in \mathcal{S}$.

Some Banach spaces have the Radon Nikodym property and some don't. No attempt is made to give a complete answer to the question of which Banach spaces have this property but the next theorem gives examples of many spaces which do.

Theorem 21.47 *Suppose* X' *is a separable dual space. Then* X' *has the Radon Nikodym property.*

Proof: Let $F \ll \mu$ and let $|F|(\Omega) < \infty$ for $F : \mathcal{S} \to X'$, a vector measure. Pick $x \in X$ and consider the map

$$E \to F(E)(x)$$

for $E \in \mathcal{S}$. This defines a complex measure which is absolutely continuous with respect to $|F|$. Therefore, by the Radon Nikodym theorem, there exists $f_x \in L^1(\Omega, |F|)$ such that

$$F(E)(x) = \int_E f_x(s)\, d|F|. \tag{21.42}$$

Claim: $|f_x(s)| \le ||x||$ for $|F|$ a.e. s.

Proof of claim: Consider the closed ball in $\mathbb{F}$, $\overline{B(0, ||x||)}$ and let $B \equiv B(p, r)$ be an open ball contained in its complement. Let $f_x^{-1}(B) \equiv E \in \mathcal{S}$. I want to argue that $|F|(E) = 0$ so suppose $|F|(E) > 0$. then

$$|F|(E)\, ||x|| \ge ||F(E)||\, ||x|| \ge |F(E)(x)|$$

and so from 21.42,

$$\frac{1}{|F|(E)} \left| \int_E f_x(s)\, d|F| \right| \le ||x||. \tag{21.43}$$

But on $E, |f_x(s) - p| < r$ and so

$$\left| \frac{1}{|F|(E)} \int_E f_x(s)\, d|F| - p \right| < r$$

which contradicts 21.43 because $B(p, r)$ was given to have empty intersection with $\overline{B(0, ||x||)}$. Therefore, $|F|(E) = 0$ as hoped. Now $\mathbb{F} \setminus \overline{B(0, ||x||)}$ can be covered by countably many such balls and so $|F|\left(\mathbb{F} \setminus \overline{B(0, ||x||)}\right) = 0$.

Denote the exceptional set of measure zero by N_x. By Theorem 21.13, X is separable. Letting D be a dense, countable subset of X, define

$$N_1 \equiv \cup_{x\in D} N_x.$$

Thus

$$|F|(N_1) = 0.$$

For any $E \in \mathcal{S}$, $x, y \in D$, and $a, b \in \mathbb{F}$,

$$\int_E f_{ax+by}(s)\, d|F| = F(E)(ax+by) = aF(E)(x) + bF(E)(y)$$

$$= \int_E (af_x(s) + bf_y(s))\, d|F|. \tag{21.44}$$

Since 21.44 holds for all $E \in \mathcal{S}$, it follows

$$f_{ax+by}(s) = af_x(s) + bf_y(s)$$

for $|F|$ a.e. s and $x, y \in D$. Let $\tilde{D}$ consist of all finite linear combinations of the form $\sum_{i=1}^m a_i x_i$ where a_i is a rational point of $\mathbb{F}$ and $x_i \in D$. If

$$\sum_{i=1}^m a_i x_i \in \tilde{D},$$

the above argument implies

$$f_{\sum_{i=1}^m a_i x_i}(s) = \sum_{i=1}^m a_i f_{x_i}(s) \text{ a.e.}$$

Since $\tilde{D}$ is countable, there exists a set, N_2, with

$$|F|(N_2) = 0$$

such that for $s \notin N_2$,

$$f_{\sum_{i=1}^m a_i x_i}(s) = \sum_{i=1}^m a_i f_{x_i}(s) \tag{21.45}$$

whenever $\sum_{i=1}^m a_i x_i \in \tilde{D}$. Let

$$N = N_1 \cup N_2$$

and let

$$\tilde{h}_x(s) \equiv \mathcal{X}_{N^C}(s) f_x(s)$$

for all $x \in \tilde{D}$. Now for $x \in X$ define

$$h_x(s) \equiv \lim_{x' \to x} \{\tilde{h}_{x'}(s) : x' \in \tilde{D}\}.$$

This is well defined because if x' and y' are elements of $\tilde{D}$, the above claim and 21.45 imply

$$\left|\tilde{h}_{x'}(s)-\tilde{h}_{y'}(s)\right|=\left|\tilde{h}_{(x'-y')}(s)\right|\leq ||x'-y'||.$$

Using 21.45, the dominated convergence theorem may be applied to conclude that for $x_n \to x$, with $x_n \in \tilde{D}$,

$$\int_E h_x(s)\,d|F|=\lim_{n\to\infty}\int_E \tilde{h}_{x_n}(s)\,d|F|=\lim_{n\to\infty}F(E)(x_n)=F(E)(x). \qquad (21.46)$$

It follows from the density of $\tilde{D}$ that for all $x,y\in X$ and $a,b\in\mathbb{F}$,

$$|h_x(s)|\leq ||x||\,,\ h_{ax+by}(s)=ah_x(s)+bh_y(s), \qquad (21.47)$$

for all s because if $s\in N$, both sides of the equation in 21.47 equal 0.

Let $\theta(s)$ be given by

$$\theta(s)(x)=h_x(s).$$

By 21.47 it follows that $\theta(s)\in X'$ for each s. Also

$$\theta(s)(x)=h_x(s)\in L^1(\Omega)$$

so $\theta(\cdot)$ is weak $*$ measurable. Since X' is separable, Theorem 21.12 implies that θ is strongly measurable. Furthermore, by 21.47,

$$||\theta(s)||\equiv\sup_{||x||\leq 1}|\theta(s)(x)|\leq\sup_{||x||\leq 1}|h_x(s)|\leq 1.$$

Therefore,

$$\int_\Omega ||\theta(s)||\,d|F|<\infty$$

so $\theta\in L^1(\Omega;X')$. By 21.6, if $E\in\mathcal{S}$,

$$\int_E h_x(s)\,d|F|=\int_E\theta(s)(x)\,d|F|=\left(\int_E\theta(s)\,d|F|\right)(x). \qquad (21.48)$$

From 21.46 and 21.48,

$$\left(\int_E\theta(s)\,d|F|\right)(x)=F(E)(x)$$

for all $x\in X$ and therefore,

$$\int_E\theta(s)\,d|F|=F(E).$$

Finally, since $F\ll\mu$, $|F|\ll\mu$ also and so there exists $k\in L^1(\Omega)$ such that

$$|F|(E)=\int_E k(s)\,d\mu$$

for all $E \in \mathcal{S}$, by the Radon Nikodym Theorem. It follows

$$F(E) = \int_E \theta(s)\, d|F| = \int_E \theta(s)\, k(s)\, d\mu.$$

Letting $g(s) = \theta(s)\, k(s)$, this has proved the theorem.

Corollary 21.48 *Any separable reflexive Banach space has the Radon Nikodym property.*

It is not necessary to assume separability in the above corollary. For the proof of a more general result, consult *Vector Measures* by Diestal and Uhl, [16].

21.8 The Riesz Representation Theorem

The Riesz representation theorem for the spaces $L^p(\Omega; X)$ holds under certain conditions. The proof follows the proofs given earlier for scalar valued functions.

Definition 21.49 *If X and Y are two Banach spaces, X is isometric to Y if there exists $\theta \in \mathcal{L}(X, Y)$ such that*

$$||\theta x||_Y = ||x||_X .$$

This will be written as $X \cong Y$. The map θ is called an isometry.

The next theorem says that $L^{p'}(\Omega; X')$ is always isometric to a subspace of $(L^p(\Omega; X))'$ for any Banach space, X.

Theorem 21.50 *Let X be any Banach space and let $(\Omega, \mathcal{S}, \mu)$ be a finite measure space. Let $p \geq 1$ and let $1/p + 1/p' = 1$.(If $p = 1$, $p' \equiv \infty$.) Then $L^{p'}(\Omega; X')$ is isometric to a subspace of $(L^p(\Omega; X))'$. Also, for $g \in L^{p'}(\Omega; X')$,*

$$\sup_{||f||_p \leq 1} \left| \int_\Omega g(s)(f(s))\, d\mu \right| = ||g||_{p'} .$$

Proof: First observe that for $f \in L^p(\Omega; X)$ and $g \in L^{p'}(\Omega; X')$,

$$s \to g(s)(f(s))$$

is a function in $L^1(\Omega)$. (To obtain measurability, write f as a limit of simple functions. Holder's inequality then yields the function is in $L^1(\Omega)$.) Define

$$\theta : L^{p'}(\Omega; X') \to (L^p(\Omega; X))'$$

by

$$\theta g(f) \equiv \int_\Omega g(s)(f(s))\, d\mu.$$

Holder's inequality implies

$$||\theta g|| \leq ||g||_{p'} \tag{21.49}$$

and it is also clear that θ is linear. Next it is required to show

$$||\theta g|| = ||g||.$$

This will first be verified for simple functions. Let

$$g(s) = \sum_{i=1}^{m} c_i \mathcal{X}_{E_i}(s)$$

where $c_i \in X'$, the E_i are disjoint and

$$\cup_{i=1}^{m} E_i = \Omega.$$

Then $||g|| \in L^{p'}(\Omega)$. Let $\varepsilon > 0$ be given. By the scalar Riesz representation theorem, there exists $h \in L^p(\Omega)$ such that $||h||_p = 1$ and

$$\int_\Omega ||g(s)||_{X'} h(s)\, d\mu \geq ||g||_{L^{p'}(\Omega;X')} - \varepsilon.$$

Now let d_i be chosen such that

$$c_i(d_i) \geq ||c_i||_{X'} - \varepsilon / ||h||_{L^1(\Omega)}$$

and $||d_i||_X \leq 1$. Let

$$f(s) \equiv \sum_{i=1}^{m} d_i h(s) \mathcal{X}_{E_i}(s).$$

Thus $f \in L^p(\Omega; X)$ and $||f||_{L^p(\Omega;X)} \leq 1$. This follows from

$$\begin{aligned} ||f||_p^p &= \int_\Omega \sum_{i=1}^{m} ||d_i||_X^p |h(s)|^p \mathcal{X}_{E_i}(s)\, d\mu \\ &= \sum_{i=1}^{m} \left(\int_{E_i} |h(s)^p|\, d\mu \right) ||d_i||_X^p \leq \int_\Omega |h|^p\, d\mu = 1. \end{aligned}$$

Also

$$||\theta g|| \geq |\theta g(f)| = \left| \int_\Omega g(s)(f(s))\, d\mu \right| \geq$$

$$\left| \int_\Omega \sum_{i=1}^{m} \left(||c_i||_{X'} - \varepsilon / ||h||_{L^1(\Omega)} \right) h(s) \mathcal{X}_{E_i}(s)\, d\mu \right|$$

$$\geq \left| \int_\Omega ||g(s)||_{X'} h(s)\, d\mu \right| - \varepsilon \left| \int_\Omega h(s) / ||h||_{L^1(\Omega)}\, d\mu \right|$$

$$\geq ||g||_{L^{p'}(\Omega;X')} - 2\varepsilon.$$

Since ε was arbitrary,

$$||\theta g|| \geq ||g|| \tag{21.50}$$

and from 21.49 this shows equality holds in 21.50 whenever g is a simple function.

In general, let $g \in L^{p'}(\Omega; X')$ and let g_n be a sequence of simple functions converging to g in $L^{p'}(\Omega; X')$. Then

$$||\theta g|| = \lim_{n\to\infty} ||\theta g_n|| = \lim_{n\to\infty} ||g_n|| = ||g||.$$

This proves the theorem and shows θ is the desired isometry.

Theorem 21.51 *If X is a Banach space and X' has the Radon Nikodym property, then if $(\Omega, \mathcal{S}, \mu)$ is a finite measure space,*

$$(L^p(\Omega; X))' \cong L^{p'}(\Omega; X')$$

and in fact the mapping θ of Theorem 21.50 is onto.

Proof: Let $l \in (L^p(\Omega; X))'$ and define $F(E) \in X'$ by

$$F(E)(x) \equiv l(\mathcal{X}_E(\cdot)x).$$

Lemma 21.52 *F defined above is a vector measure with values in X' and $|F|(\Omega) < \infty$.*

Proof of the lemma: Clearly $F(E)$ is linear. Also

$$||F(E)|| = \sup_{||x||\leq 1} ||F(E)(x)||$$

$$\leq ||l|| \sup_{||x||\leq 1} ||\mathcal{X}_E(\cdot)x||_{L^p(\Omega;X)} \leq ||l|| \mu(E)^{1/p}.$$

Let $\{E_i\}_{i=1}^\infty$ be a sequence of disjoint elements of $\mathcal{S}$ and let $E = \cup_{n<\infty} E_n$.

$$\begin{aligned} \left| F(E)(x) - \sum_{k=1}^n F(E_k)(x) \right| &= \left| l(\mathcal{X}_E(\cdot)x) - \sum_{i=1}^n l(\mathcal{X}_{E_i}(\cdot)x) \right| \\ &\leq ||l|| \left|\left| \mathcal{X}_E(\cdot)x - \sum_{i=1}^n \mathcal{X}_{E_i}(\cdot)x \right|\right|_{L^p(\Omega;X)} \\ &\leq ||l|| \mu\left(\bigcup_{k>n} E_k \right)^{1/p} ||x||. \end{aligned} \tag{21.51}$$

Since $\mu(\Omega) < \infty$,

$$\lim_{n\to\infty} \mu\left(\bigcup_{k>n} E_k \right)^{1/p} = 0$$

and so inequality 21.51 shows that

$$\lim_{n\to\infty}\left|\left|F(E)-\sum_{k=1}^{n}F(E_k)\right|\right|_{X'}=0.$$

To show $|F|(\Omega)<\infty$, let $\varepsilon>0$ be given, let $\{H_1,\cdots,H_n\}$ be a partition of Ω, and let $||x_i||\leq 1$ be chosen in such a way that

$$F(H_i)(x_i)>||F(H_i)||-\varepsilon/n.$$

Thus

$$-\varepsilon+\sum_{i=1}^{n}||F(H_i)||<\sum_{i=1}^{n}l(\mathcal{X}_{H_i}(\cdot)x_i)\leq||l||\left|\left|\sum_{i=1}^{n}\mathcal{X}_{H_i}(\cdot)x_i\right|\right|_{L^p(\Omega;X)}$$

$$\leq||l||\left(\int_\Omega\sum_{i=1}^{n}\mathcal{X}_{H_i}(s)\,d\mu\right)^{1/p}=||l||\,\mu(\Omega)^{1/p}.$$

Since $\varepsilon>0$ was arbitrary,

$$\sum_{i=1}^{n}||F(H_i)||<||l||\,\mu(\Omega)^{1/p}.$$

Since the partition was arbitrary, this shows $|F|(\Omega)\leq||l||\,\mu(\Omega)^{1/p}$ and this proves the lemma.

Continuing with the proof of Theorem 21.51, note that

$$F\ll\mu.$$

Since X' has the Radon Nikodym property, there exists $g\in L^1(\Omega;X')$ such that

$$F(E)=\int_E g(s)\,d\mu.$$

Also, from the definition of $F(E)$,

$$l\left(\sum_{i=1}^{n}x_i\mathcal{X}_{E_i}(\cdot)\right)=\sum_{i=1}^{n}l(\mathcal{X}_{E_i}(\cdot)x_i)$$

$$=\sum_{i=1}^{n}F(E_i)(x_i)=\sum_{i=1}^{n}\int_{E_i}g(s)(x_i)\,d\mu. \tag{21.52}$$

It follows from 21.52 that whenever h is a simple function,

$$l(h)=\int_\Omega g(s)(h(s))\,d\mu. \tag{21.53}$$

Let

$$G_n \equiv \{s : ||g(s)||_{X'} \leq n\}$$

and let

$$j : L^p(G_n; X) \to L^p(\Omega; X)$$

be given by

$$jh(s) = \begin{cases} h(s) \text{ if } s \in G_n, \\ 0 \text{ if } s \notin G_n. \end{cases}$$

Letting h be a simple function in $L^p(G_n; X)$,

$$j^* l(h) = l(jh) = \int_{G_n} g(s)(h(s))\, d\mu. \tag{21.54}$$

Since the simple functions are dense in $L^p(G_n; X)$, and $g \in L^{p'}(G_n; X')$, it follows 21.54 holds for all $h \in L^p(G_n; X)$. By Theorem 21.50,

$$||g||_{L^{p'}(G_n;X')} = ||j^* l||_{(L^p(G_n;X))'} \leq ||l||_{(L^p(\Omega;X))'}.$$

By the monotone convergence theorem,

$$||g||_{L^{p'}(\Omega;X')} = \lim_{n\to\infty} ||g||_{L^{p'}(G_n;X')} \leq ||l||_{(L^p(\Omega;X))'}.$$

Therefore $g \in L^{p'}(\Omega; X')$ and since simple functions are dense in $L^p(\Omega; X)$, 21.53 holds for all $h \in L^p(\Omega; X)$. Thus $l = \theta g$ and the theorem is proved because, by Theorem 21.50, $||l|| = ||g||$ and the mapping θ is onto because l was arbitrary.

Corollary 21.53 *If X' is separable, then*

$$(L^p(\Omega; X))' \cong L^{p'}(\Omega; X').$$

Corollary 21.54 *If X is separable and reflexive, then*

$$(L^p(\Omega; X))' \cong L^{p'}(\Omega; X').$$

Corollary 21.55 *If X is separable and reflexive, then if $p \in (1, \infty)$, then $L^p(\Omega; X)$ is reflexive.*

Proof: This is just like the scalar valued case.

21.9 Exercises

1. Show $L^1(\mathbb{R})$ is not reflexive. **Hint:** $L^1(\mathbb{R})$ is separable. What about $L^\infty(\mathbb{R})$?
2. If $f \in L^1(\mathbb{R}^n; X)$ for X a Banach space, does the usual fundamental theorem of calculus work? That is, can you say $\lim_{r\to 0} \frac{1}{m(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} f(t)\, dm = f(x)$ a.e.?
3. Does the Vitali convergence theorem hold for Bochner integrable functions? If so, give a statement of the appropriate theorem and a proof.

Part III

Complex Analysis

The Complex Numbers

The reader is presumed familiar with the algebraic properties of complex numbers, including the operation of conjugation. Here a short review of the distance in $\mathbb{C}$ is presented.

The length of a complex number, referred to as the modulus of z and denoted by $|z|$ is given by

$$|z| \equiv \left(x^2 + y^2\right)^{1/2} = (z\overline{z})^{1/2},$$

Then $\mathbb{C}$ is a metric space with the distance between two complex numbers, z and w defined as

$$d(z, w) \equiv |z - w|.$$

This metric on $\mathbb{C}$ is the same as the usual metric of $\mathbb{R}^2$. A sequence, $z_n \to z$ if and only if $x_n \to x$ in $\mathbb{R}$ and $y_n \to y$ in $\mathbb{R}$ where $z = x + iy$ and $z_n = x_n + iy_n$. For example if $z_n = \frac{n}{n+1} + i\frac{1}{n}$, then $z_n \to 1 + 0i = 1$.

Definition 22.1 *A sequence of complex numbers, $\{z_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists N such that $n, m > N$ implies $|z_n - z_m| < \varepsilon$.*

This is the usual definition of Cauchy sequence. There are no new ideas here.

Proposition 22.2 *The complex numbers with the norm just mentioned forms a complete normed linear space.*

Proof: Let $\{z_n\}$ be a Cauchy sequence of complex numbers with $z_n = x_n + iy_n$. Then $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences of real numbers and so they converge to real numbers, x and y respectively. Thus $z_n = x_n + iy_n \to x + iy$. $\mathbb{C}$ is a linear space with the field of scalars equal to $\mathbb{C}$. It only remains to verify that $|\ |$ satisfies the axioms of a norm which are:

$$|z + w| \leq |z| + |w|$$

$$|z| \geq 0 \text{ for all } z$$

$$|z| = 0 \text{ if and only if } z = 0$$

$$|\alpha z| = |\alpha|\,|z|.$$

The only one of these axioms of a norm which is not completely obvious is the first one, the triangle inequality. Let $z = x + iy$ and $w = u + iv$

$$\begin{aligned} |z+w|^2 &= (z+w)(\overline{z}+\overline{w}) = |z|^2 + |w|^2 + 2\operatorname{Re}(z\overline{w}) \\ &\leq |z|^2 + |w|^2 + 2|(z\overline{w})| = (|z|+|w|)^2 \end{aligned}$$

and this verifies the triangle inequality.

Definition 22.3 *An infinite sum of complex numbers is defined as the limit of the sequence of partial sums. Thus,*

$$\sum_{k=1}^{\infty} a_k \equiv \lim_{n\to\infty} \sum_{k=1}^{n} a_k.$$

Just as in the case of sums of real numbers, an infinite sum converges if and only if the sequence of partial sums is a Cauchy sequence.

From now on, when f is a function of a complex variable, it will be assumed that f has values in X, a complex Banach space. Usually in complex analysis courses, f has values in $\mathbb{C}$ but there are many important theorems which don't require this so I will leave it fairly general for a while. Later the functions will have values in $\mathbb{C}$. If you are only interested in this case, think $\mathbb{C}$ whenever you see X.

Definition 22.4 *A sequence of functions of a complex variable, $\{f_n\}$ converges uniformly to a function, g for $z \in S$ if for every $\varepsilon > 0$ there exists N_ε such that if $n > N_\varepsilon$, then*

$$||f_n(z) - g(z)|| < \varepsilon$$

for all $z \in S$. The infinite sum $\sum_{k=1}^{\infty} f_n$ converges uniformly on S if the partial sums converge uniformly on S. Here $||\cdot||$ refers to the norm in X, the Banach space in which f has its values.

The following proposition is also a routine application of the above definition. Neither the definition nor this proposition say anything new.

Proposition 22.5 *A sequence of functions, $\{f_n\}$ defined on a set S, converges uniformly to some function, g if and only if for all $\varepsilon > 0$ there exists N_ε such that whenever $m, n > N_\varepsilon$,*

$$||f_n - f_m||_\infty < \varepsilon.$$

Here $||f||_\infty \equiv \sup\{||f(z)|| : z \in S\}$.

Just as in the case of functions of a real variable, one of the important theorems is the Weierstrass M test. Again, there is nothing new here. It is just a review of earlier material.

Theorem 22.6 *Let $\{f_n\}$ be a sequence of complex valued functions defined on $S \subseteq \mathbb{C}$. Suppose there exists M_n such that $||f_n||_\infty < M_n$ and $\sum M_n$ converges. Then $\sum f_n$ converges uniformly on S.*

Proof: Let $z \in S$. Then letting $m < n$

$$\left|\left|\sum_{k=1}^{n} f_k(z) - \sum_{k=1}^{m} f_k(z)\right|\right| \leq \sum_{k=m+1}^{n} ||f_k(z)|| \leq \sum_{k=m+1}^{\infty} M_k < \varepsilon$$

whenever m is large enough. Therefore, the sequence of partial sums is uniformly Cauchy on S and therefore, converges uniformly to $\sum_{k=1}^{\infty} f_k(z)$ on S.

22.1 The Extended Complex Plane

The set of complex numbers has already been considered along with the topology of $\mathbb{C}$ which is nothing but the topology of $\mathbb{R}^2$. Thus, for $z_n = x_n + iy_n$, $z_n \to z \equiv x + iy$ if and only if $x_n \to x$ and $y_n \to y$. The norm in $\mathbb{C}$ is given by

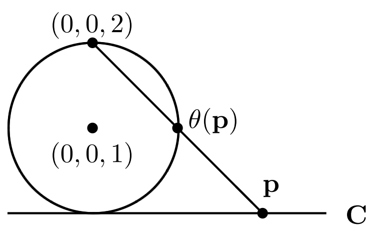
$$|x + iy| \equiv ((x + iy)(x - iy))^{1/2} = \left(x^2 + y^2\right)^{1/2}$$

which is just the usual norm in $\mathbb{R}^2$ identifying (x, y) with $x + iy$. Therefore, $\mathbb{C}$ is a complete metric space topologically like $\mathbb{R}^2$ and so the Heine Borel theorem that compact sets are those which are closed and bounded is valid. Thus, as far as topology is concerned, there is nothing new about $\mathbb{C}$.

The extended complex plane, denoted by $\widehat{\mathbb{C}}$, consists of the complex plane, $\mathbb{C}$ along with another point not in $\mathbb{C}$ known as ∞. For example, ∞ could be any point in $\mathbb{R}^3$. A sequence of complex numbers, z_n, converges to ∞ if, whenever K is a compact set in $\mathbb{C}$, there exists a number, N such that for all $n > N$, $z_n \notin K$. Since compact sets in $\mathbb{C}$ are closed and bounded, this is equivalent to saying that for all $R > 0$, there exists N such that if $n > N$, then $z_n \notin B(0, R)$ which is the same as saying $\lim_{n\to\infty} |z_n| = \infty$ where this last symbol has the same meaning as it does in calculus.

A geometric way of understanding this in terms of more familiar objects involves a concept known as the Riemann sphere.

Consider the unit sphere, S^2 given by $(z - 1)^2 + y^2 + x^2 = 1$. Define a map from the complex plane to the surface of this sphere as follows. Extend a line from the point, p in the complex plane to the point $(0, 0, 2)$ on the top of this sphere and let $\theta(p)$ denote the point of this sphere which the line intersects. Define $\theta(\infty) \equiv (0, 0, 2)$.



Then θ^{-1} is sometimes called sterographic projection. The mapping θ is clearly continuous because it takes converging sequences, to converging sequences. Furthermore, it is clear that θ^{-1} is also continuous. In terms of the extended complex plane, $\widehat{\mathbb{C}}$, a sequence, z_n converges to ∞ if and only if θz_n converges to $(0,0,2)$ and a sequence, z_n converges to $z \in \mathbb{C}$ if and only if $\theta(z_n) \to \theta(z)$.

In fact this makes it easy to define a metric on $\widehat{\mathbb{C}}$.

Definition 22.7 *Let $z, w \in \widehat{\mathbb{C}}$ including possibly $w = \infty$. Then let $d(x, w) \equiv |\theta(z) - \theta(w)|$ where this last distance is the usual distance measured in $\mathbb{R}^3$.*

Theorem 22.8 *$\left(\widehat{\mathbb{C}}, d\right)$ is a compact, hence complete metric space.*

Proof: Suppose $\{z_n\}$ is a sequence in $\widehat{\mathbb{C}}$. This means $\{\theta(z_n)\}$ is a sequence in S^2 which is compact. Therefore, there exists a subsequence, $\{\theta z_{n_k}\}$ and a point, $z \in S^2$ such that $\theta z_{n_k} \to \theta z$ in S^2 which implies immediately that $d(z_{n_k}, z) \to 0$. A compact metric space must be complete.

22.2 Exercises

1. Prove the root test for series of complex numbers. If $a_k \in \mathbb{C}$ and $r \equiv \limsup_{n\to\infty} |a_n|^{1/n}$ then
$$\sum_{k=0}^{\infty} a_k \begin{cases} \text{converges absolutely if } r < 1 \\ \text{diverges if } r > 1 \\ \text{test fails if } r = 1. \end{cases}$$

2. Does $\lim_{n\to\infty} n\left(\frac{2+i}{3}\right)^n$ exist? Tell why and find the limit if it does exist.

3. Let $A_0 = 0$ and let $A_n \equiv \sum_{k=1}^{n} a_k$ if $n > 0$. Prove the partial summation formula,
$$\sum_{k=p}^{q} a_k b_k = A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}).$$
Now using this formula, suppose $\{b_n\}$ is a sequence of real numbers which converges to 0 and is decreasing. Determine those values of ω such that $|\omega| = 1$ and $\sum_{k=1}^{\infty} b_k \omega^k$ converges.

4. Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be given by $f(x+iy) = u(x,y) + iv(x,y)$. Show f is continuous on U if and only if $u : U \to \mathbb{R}$ and $v : U \to \mathbb{R}$ are both continuous.

Riemann Stieltjes Integrals

In the theory of functions of a complex variable, the most important results are those involving contour integration. I will base this on the notion of Riemann Stieltjes integrals as in [13], [39], and [27]. The Riemann Stieltjes integral is a generalization of the usual Riemann integral and requires the concept of a function of bounded variation.

Definition 23.1 *Let $\gamma : [a, b] \to \mathbb{C}$ be a function. Then γ is of bounded variation if*

$$\sup\left\{\sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| : a = t_0 < \cdots < t_n = b\right\} \equiv V(\gamma, [a, b]) < \infty$$

where the sums are taken over all possible lists, $\{a = t_0 < \cdots < t_n = b\}$.

The idea is that it makes sense to talk of the length of the curve $\gamma([a, b])$, defined as $V(\gamma, [a, b])$. For this reason, in the case that γ is continuous, such an image of a bounded variation function is called a rectifiable curve.

Definition 23.2 *Let $\gamma : [a, b] \to \mathbb{C}$ be of bounded variation and let $f : [a, b] \to X$. Letting $\mathcal{P} \equiv \{t_0, \cdots, t_n\}$ where $a = t_0 < t_1 < \cdots < t_n = b$, define*

$$||\mathcal{P}|| \equiv \max\{|t_j - t_{j-1}| : j = 1, \cdots, n\}$$

and the Riemann Steiltjes sum by

$$S(\mathcal{P}) \equiv \sum_{j=1}^{n} f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1}))$$

where $\tau_j \in [t_{j-1}, t_j]$. (Note this notation is a little sloppy because it does not identify the specific point, τ_j used. It is understood that this point is arbitrary.) Define $\int_\gamma f d\gamma$ as the unique number which satisfies the following condition. For all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $||\mathcal{P}|| \leq \delta$, then

$$\left|\int_\gamma f d\gamma - S(\mathcal{P})\right| < \varepsilon.$$

Sometimes this is written as

$$\int_\gamma f d\gamma \equiv \lim_{||\mathcal{P}|| \to 0} S(\mathcal{P}).$$

The set of points in the curve, $\gamma([a,b])$ *will be denoted sometimes by* γ^*.

Then γ^* is a set of points in $\mathbb{C}$ and as t moves from a to b, $\gamma(t)$ moves from $\gamma(a)$ to $\gamma(b)$. Thus γ^* has a first point and a last point. If $\phi : [c,d] \to [a,b]$ is a continuous nondecreasing function, then $\gamma \circ \phi : [c,d] \to \mathbb{C}$ is also of bounded variation and yields the same set of points in $\mathbb{C}$ with the same first and last points.

Theorem 23.3 *Let* ϕ *and* γ *be as just described. Then assuming that*

$$\int_\gamma f d\gamma$$

exists, so does

$$\int_{\gamma \circ \phi} f d(\gamma \circ \phi)$$

and

$$\int_\gamma f d\gamma = \int_{\gamma \circ \phi} f d(\gamma \circ \phi). \tag{23.1}$$

Proof: There exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a,b]$ such that $||\mathcal{P}|| < \delta$, then

$$\left| \int_\gamma f d\gamma - S(\mathcal{P}) \right| < \varepsilon.$$

By continuity of ϕ, there exists $\sigma > 0$ such that if $\mathcal{Q}$ is a partition of $[c,d]$ with $||\mathcal{Q}|| < \sigma$, $\mathcal{Q} = \{s_0, \cdots, s_n\}$, then $|\phi(s_j) - \phi(s_{j-1})| < \delta$. Thus letting $\mathcal{P}$ denote the points in $[a,b]$ given by $\phi(s_j)$ for $s_j \in \mathcal{Q}$, it follows that $||\mathcal{P}|| < \delta$ and so

$$\left| \int_\gamma f d\gamma - \sum_{j=1}^{n} f(\gamma(\phi(\tau_j)))(\gamma(\phi(s_j)) - \gamma(\phi(s_{j-1}))) \right| < \varepsilon$$

where $\tau_j \in [s_{j-1}, s_j]$. Therefore, from the definition 23.1 holds and

$$\int_{\gamma \circ \phi} f d(\gamma \circ \phi)$$

exists.

This theorem shows that $\int_\gamma f d\gamma$ is independent of the particular γ used in its computation to the extent that if ϕ is any nondecreasing function from another interval, $[c,d]$, mapping to $[a,b]$, then the same value is obtained by replacing γ with $\gamma \circ \phi$.

The fundamental result in this subject is the following theorem.

Theorem 23.4 *Let $f : \gamma^* \to X$ be continuous and let $\gamma : [a, b] \to \mathbb{C}$ be continuous and of bounded variation. Then $\int_\gamma f d\gamma$ exists. Also letting $\delta_m > 0$ be such that $|t - s| < \delta_m$ implies $||f(\gamma(t)) - f(\gamma(s))|| < \frac{1}{m}$,*

$$\left| \int_\gamma f d\gamma - S(\mathcal{P}) \right| \leq \frac{2V(\gamma, [a, b])}{m}$$

whenever $||\mathcal{P}|| < \delta_m$.

Proof: The function, $f \circ \gamma$, is uniformly continuous because it is defined on a compact set. Therefore, there exists a decreasing sequence of positive numbers, $\{\delta_m\}$ such that if $|s - t| < \delta_m$, then

$$|f(\gamma(t)) - f(\gamma(s))| < \frac{1}{m}.$$

Let

$$F_m \equiv \overline{\{S(\mathcal{P}) : ||\mathcal{P}|| < \delta_m\}}.$$

Thus F_m is a closed set. (The symbol, $S(\mathcal{P})$ in the above definition, means to include all sums corresponding to $\mathcal{P}$ for any choice of τ_j.) It is shown that

$$\text{diam}(F_m) \leq \frac{2V(\gamma, [a, b])}{m} \tag{23.2}$$

and then it will follow there exists a unique point, $I \in \cap_{m=1}^\infty F_m$. This is because X is complete. It will then follow $I = \int_\gamma f(t) \, d\gamma(t)$. To verify 23.2, it suffices to verify that whenever $\mathcal{P}$ and $\mathcal{Q}$ are partitions satisfying $||\mathcal{P}|| < \delta_m$ and $||\mathcal{Q}|| < \delta_m$,

$$|S(\mathcal{P}) - S(\mathcal{Q})| \leq \frac{2}{m} V(\gamma, [a, b]). \tag{23.3}$$

Suppose $||\mathcal{P}|| < \delta_m$ and $\mathcal{Q} \supseteq \mathcal{P}$. Then also $||\mathcal{Q}|| < \delta_m$. To begin with, suppose that $\mathcal{P} \equiv \{t_0, \cdots, t_p, \cdots, t_n\}$ and $\mathcal{Q} \equiv \{t_0, \cdots, t_{p-1}, t^*, t_p, \cdots, t_n\}$. Thus $\mathcal{Q}$ contains only one more point than $\mathcal{P}$. Letting $S(\mathcal{Q})$ and $S(\mathcal{P})$ be Riemann Steiltjes sums,

$$S(\mathcal{Q}) \equiv \sum_{j=1}^{p-1} f(\gamma(\sigma_j)) (\gamma(t_j) - \gamma(t_{j-1})) + f(\gamma(\sigma_*)) (\gamma(t^*) - \gamma(t_{p-1}))$$

$$+ f(\gamma(\sigma^*)) (\gamma(t_p) - \gamma(t^*)) + \sum_{j=p+1}^{n} f(\gamma(\sigma_j)) (\gamma(t_j) - \gamma(t_{j-1})),$$

$$S(\mathcal{P}) \equiv \sum_{j=1}^{p-1} f(\gamma(\tau_j)) (\gamma(t_j) - \gamma(t_{j-1})) +$$

$$\overbrace{f(\gamma(\tau_p)) (\gamma(t^*) - \gamma(t_{p-1})) + f(\gamma(\tau_p)) (\gamma(t_p) - \gamma(t^*))}^{= f(\gamma(\tau_p))(\gamma(t_p) - \gamma(t_{p-1}))}$$

$$+\sum_{j=p+1}^{n} f\left(\gamma\left(\tau_j\right)\right)\left(\gamma\left(t_j\right)-\gamma\left(t_{j-1}\right)\right).$$

Therefore,

$$\left|S\left(\mathcal{P}\right)-S\left(\mathcal{Q}\right)\right| \leq \sum_{j=1}^{p-1}\frac{1}{m}\left|\gamma\left(t_j\right)-\gamma\left(t_{j-1}\right)\right|+\frac{1}{m}\left|\gamma\left(t^*\right)-\gamma\left(t_{p-1}\right)\right|+$$

$$\frac{1}{m}\left|\gamma\left(t_p\right)-\gamma\left(t^*\right)\right|+\sum_{j=p+1}^{n}\frac{1}{m}\left|\gamma\left(t_j\right)-\gamma\left(t_{j-1}\right)\right| \leq \frac{1}{m}V\left(\gamma,[a,b]\right). \tag{23.4}$$

Clearly the extreme inequalities would be valid in 23.4 if $\mathcal{Q}$ had more than one extra point. You simply do the above trick more than one time. Let $S\left(\mathcal{P}\right)$ and $S\left(\mathcal{Q}\right)$ be Riemann Steiltjes sums for which $||\mathcal{P}||$ and $||\mathcal{Q}||$ are less than δ_m and let $\mathcal{R} \equiv \mathcal{P} \cup \mathcal{Q}$. Then from what was just observed,

$$\left|S\left(\mathcal{P}\right)-S\left(\mathcal{Q}\right)\right| \leq \left|S\left(\mathcal{P}\right)-S\left(\mathcal{R}\right)\right|+\left|S\left(\mathcal{R}\right)-S\left(\mathcal{Q}\right)\right| \leq \frac{2}{m}V\left(\gamma,[a,b]\right).$$

and this shows 23.3 which proves 23.2. Therefore, there exists a unique complex number, $I \in \cap_{m=1}^{\infty} F_m$ which satisfies the definition of $\int_\gamma f d\gamma$. This proves the theorem.

The following theorem follows easily from the above definitions and theorem.

Theorem 23.5 *Let $f \in C\left(\gamma^*\right)$ and let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and continuous. Let*

$$M \geq \max\left\{||f \circ \gamma\left(t\right)|| : t \in [a,b]\right\}. \tag{23.5}$$

Then

$$\left|\left|\int_\gamma f d\gamma\right|\right| \leq MV\left(\gamma,[a,b]\right). \tag{23.6}$$

Also if $\{f_n\}$ is a sequence of functions of $C\left(\gamma^*\right)$ which is converging uniformly to the function, f on γ^*, then

$$\lim_{n\to\infty}\int_\gamma f_n d\gamma = \int_\gamma f d\gamma. \tag{23.7}$$

Proof: Let 23.5 hold. From the proof of the above theorem, when $||\mathcal{P}|| < \delta_m$,

$$\left|\left|\int_\gamma f d\gamma - S\left(\mathcal{P}\right)\right|\right| \leq \frac{2}{m}V\left(\gamma,[a,b]\right)$$

and so

$$\left|\left|\int_\gamma f d\gamma\right|\right| \leq ||S\left(\mathcal{P}\right)|| + \frac{2}{m}V\left(\gamma,[a,b]\right)$$

$$\leq \sum_{j=1}^{n} M\left|\gamma\left(t_j\right)-\gamma\left(t_{j-1}\right)\right|+\frac{2}{m} V(\gamma,[a, b])$$
$$\leq MV(\gamma,[a, b])+\frac{2}{m} V(\gamma,[a, b]) .$$

This proves 23.6 since m is arbitrary. To verify 23.7 use the above inequality to write

$$\left|\left|\int_{\gamma} f d\gamma-\int_{\gamma} f_n d\gamma\right|\right|=\left|\left|\int_{\gamma}\left(f-f_n\right) d\gamma(t)\right|\right|$$
$$\leq \max \left\{\left|\left|f \circ \gamma(t)-f_n \circ \gamma(t)\right|\right|: t \in[a, b]\right\} V(\gamma,[a, b]) .$$

Since the convergence is assumed to be uniform, this proves 23.7.

It turns out to be much easier to evaluate such integrals in the case where γ is also $C^1([a, b])$. The following theorem about approximation will be very useful but first here is an easy lemma.

Lemma 23.6 *Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be in $C^1([a, b])$. Then $V(\gamma,[a, b])<\infty$ so γ is of bounded variation.*

Proof: This follows from the following

$$\begin{aligned}
\sum_{j=1}^{n}\left|\gamma\left(t_j\right)-\gamma\left(t_{j-1}\right)\right| &= \sum_{j=1}^{n}\left|\int_{t_{j-1}}^{t_j} \gamma'(s) ds\right| \\
&\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j}\left|\gamma'(s)\right| ds \\
&\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j}\left|\left|\gamma'\right|\right|_{\infty} ds \\
&= \left|\left|\gamma'\right|\right|_{\infty}(b-a) .
\end{aligned}$$

Therefore it follows $V(\gamma,[a, b]) \leq\left|\left|\gamma'\right|\right|_{\infty}(b-a)$. Here $||\gamma||_{\infty}=\max \{|\gamma(t)|: t \in[a, b]\}$.

Theorem 23.7 *Let $\gamma:[a, b] \rightarrow \mathbb{C}$ be continuous and of bounded variation. Let Ω be an open set containing γ^* and let $f: \Omega \times K \rightarrow X$ be continuous for K a compact set in $\mathbb{C}$, and let $\varepsilon>0$ be given. Then there exists $\eta:[a, b] \rightarrow \mathbb{C}$ such that $\eta(a)=\gamma(a)$, $\gamma(b)=\eta(b)$, $\eta \in C^1([a, b])$, and*

$$||\gamma-\eta||<\varepsilon, \tag{23.8}$$

$$\left|\int_{\gamma} f(\cdot, z) d\gamma-\int_{\eta} f(\cdot, z) d\eta\right|<\varepsilon, \tag{23.9}$$

$$V(\eta,[a, b]) \leq V(\gamma,[a, b]), \tag{23.10}$$

where $||\gamma-\eta|| \equiv \max \{|\gamma(t)-\eta(t)|: t \in[a, b]\}$.

Proof: Extend γ to be defined on all $\mathbb{R}$ according to $\gamma(t) = \gamma(a)$ if $t < a$ and $\gamma(t) = \gamma(b)$ if $t > b$. Now define

$$\gamma_h(t) \equiv \frac{1}{2h} \int_{-2h+t+\frac{2h}{(b-a)}(t-a)}^{t+\frac{2h}{(b-a)}(t-a)} \gamma(s)\, ds.$$

where the integral is defined in the obvious way. That is,

$$\int_a^b \alpha(t) + i\beta(t)\, dt \equiv \int_a^b \alpha(t)\, dt + i \int_a^b \beta(t)\, dt.$$

Therefore,

$$\gamma_h(b) = \frac{1}{2h} \int_b^{b+2h} \gamma(s)\, ds = \gamma(b),$$

$$\gamma_h(a) = \frac{1}{2h} \int_{a-2h}^{a} \gamma(s)\, ds = \gamma(a).$$

Also, because of continuity of γ and the fundamental theorem of calculus,

$$\gamma_h'(t) = \frac{1}{2h} \left\{ \gamma\left(t + \frac{2h}{b-a}(t-a)\right)\left(1 + \frac{2h}{b-a}\right) - \right.$$

$$\left. \gamma\left(-2h + t + \frac{2h}{b-a}(t-a)\right)\left(1 + \frac{2h}{b-a}\right) \right\}$$

and so $\gamma_h \in C^1([a,b])$. The following lemma is significant.

Lemma 23.8 $V(\gamma_h, [a,b]) \leq V(\gamma, [a,b])$.

Proof: Let $a = t_0 < t_1 < \cdots < t_n = b$. Then using the definition of γ_h and changing the variables to make all integrals over $[0, 2h]$,

$$\sum_{j=1}^{n} |\gamma_h(t_j) - \gamma_h(t_{j-1})| =$$

$$\sum_{j=1}^{n} \left| \frac{1}{2h} \int_0^{2h} \left[\gamma\left(s - 2h + t_j + \frac{2h}{b-a}(t_j - a)\right) - \right.\right.$$

$$\left.\left. \gamma\left(s - 2h + t_{j-1} + \frac{2h}{b-a}(t_{j-1} - a)\right)\right] \right|$$

$$\leq \frac{1}{2h} \int_0^{2h} \sum_{j=1}^{n} \left| \gamma\left(s - 2h + t_j + \frac{2h}{b-a}(t_j - a)\right) - \right.$$

$$\left. \gamma\left(s - 2h + t_{j-1} + \frac{2h}{b-a}(t_{j-1} - a)\right) \right| ds.$$

For a given $s \in [0, 2h]$, the points, $s - 2h + t_j + \frac{2h}{b-a}(t_j - a)$ for $j = 1, \cdots, n$ form an increasing list of points in the interval $[a - 2h, b + 2h]$ and so the integrand is bounded above by $V(\gamma, [a - 2h, b + 2h]) = V(\gamma, [a, b])$. It follows

$$\sum_{j=1}^{n} |\gamma_h(t_j) - \gamma_h(t_{j-1})| \leq V(\gamma, [a, b])$$

which proves the lemma.

With this lemma the proof of the theorem can be completed without too much trouble. Let H be an open set containing γ^* such that $\overline{H}$ is a compact subset of Ω. Let $0 < \varepsilon < \text{dist}\left(\gamma^*, H^C\right)$. Then there exists δ_1 such that if $h < \delta_1$, then for all t,

$$\begin{aligned} |\gamma(t) - \gamma_h(t)| &\leq \frac{1}{2h} \int_{-2h+t+\frac{2h}{(b-a)}(t-a)}^{t+\frac{2h}{(b-a)}(t-a)} |\gamma(s) - \gamma(t)|\, ds \\ &< \frac{1}{2h} \int_{-2h+t+\frac{2h}{(b-a)}(t-a)}^{t+\frac{2h}{(b-a)}(t-a)} \varepsilon ds = \varepsilon \end{aligned} \tag{23.11}$$

due to the uniform continuity of γ. This proves 23.8.

From 23.2 and the above lemma, there exists δ_2 such that if $||\mathcal{P}|| < \delta_2$, then for all $z \in K$,

$$\left|\left|\int_\gamma f(\cdot, z)\, d\gamma(t) - S(\mathcal{P})\right|\right| < \frac{\varepsilon}{3}, \left|\left|\int_{\gamma_h} f(\cdot, z)\, d\gamma_h(t) - S_h(\mathcal{P})\right|\right| < \frac{\varepsilon}{3}$$

for all h. Here $S(\mathcal{P})$ is a Riemann Steiltjes sum of the form

$$\sum_{i=1}^{n} f(\gamma(\tau_i), z)(\gamma(t_i) - \gamma(t_{i-1}))$$

and $S_h(\mathcal{P})$ is a similar Riemann Steiltjes sum taken with respect to γ_h instead of γ. Because of 23.11 $\gamma_h(t)$ has values in $H \subseteq \Omega$. Therefore, fix the partition, $\mathcal{P}$, and choose h small enough that in addition to this, the following inequality is valid for all $z \in K$.

$$|S(\mathcal{P}) - S_h(\mathcal{P})| < \frac{\varepsilon}{3}$$

This is possible because of 23.11 and the uniform continuity of f on $\overline{H} \times K$. It follows

$$\left|\left|\int_\gamma f(\cdot, z)\, d\gamma(t) - \int_{\gamma_h} f(\cdot, z)\, d\gamma_h(t)\right|\right| \leq$$

$$\left|\left|\int_\gamma f(\cdot, z)\, d\gamma(t) - S(\mathcal{P})\right|\right| + ||S(\mathcal{P}) - S_h(\mathcal{P})||$$

$$+ \left|\left|S_h(\mathcal{P}) - \int_{\gamma_h} f(\cdot, z)\, d\gamma_h(t)\right|\right| < \varepsilon.$$

Formula 23.10 follows from the lemma. This proves the theorem.

Of course the same result is obtained without the explicit dependence of f on z.

This is a very useful theorem because if γ is $C^1([a,b])$, it is easy to calculate $\int_\gamma f d\gamma$ and the above theorem allows a reduction to the case where γ is C^1. The next theorem shows how easy it is to compute these integrals in the case where γ is C^1. First note that if f is continuous and $\gamma \in C^1([a,b])$, then by Lemma 23.6 and the fundamental existence theorem, Theorem 23.4, $\int_\gamma f d\gamma$ exists.

Theorem 23.9 *If $f : \gamma^* \to X$ is continuous and $\gamma : [a,b] \to \mathbb{C}$ is in $C^1([a,b])$, then*

$$\int_\gamma f d\gamma = \int_a^b f(\gamma(t))\gamma'(t)\,dt. \tag{23.12}$$

Proof: Let $\mathcal{P}$ be a partition of $[a,b]$, $\mathcal{P} = \{t_0, \cdots, t_n\}$ and $||\mathcal{P}||$ is small enough that whenever $|t-s| < ||\mathcal{P}||$,

$$|f(\gamma(t)) - f(\gamma(s))| < \varepsilon \tag{23.13}$$

and

$$\left|\left|\int_\gamma f d\gamma - \sum_{j=1}^n f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1}))\right|\right| < \varepsilon.$$

Now

$$\sum_{j=1}^n f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1})) = \int_a^b \sum_{j=1}^n f(\gamma(\tau_j))\mathcal{X}_{[t_{j-1},t_j]}(s)\gamma'(s)\,ds$$

where here

$$\mathcal{X}_{[a,b]}(s) \equiv \begin{cases} 1 \text{ if } s \in [a,b] \\ 0 \text{ if } s \notin [a,b] \end{cases}.$$

Also,

$$\int_a^b f(\gamma(s))\gamma'(s)\,ds = \int_a^b \sum_{j=1}^n f(\gamma(s))\mathcal{X}_{[t_{j-1},t_j]}(s)\gamma'(s)\,ds$$

and thanks to 23.13,

$$\left|\left|\overbrace{\int_a^b \sum_{j=1}^n f(\gamma(\tau_j))\mathcal{X}_{[t_{j-1},t_j]}(s)\gamma'(s)\,ds}^{=\sum_{j=1}^n f(\gamma(\tau_j))(\gamma(t_j)-\gamma(t_{j-1}))} - \overbrace{\int_a^b \sum_{j=1}^n f(\gamma(s))\mathcal{X}_{[t_{j-1},t_j]}(s)\gamma'(s)\,ds}^{=\int_a^b f(\gamma(s))\gamma'(s)ds}\right|\right|$$

$$\begin{aligned} &\leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} ||f(\gamma(\tau_j)) - f(\gamma(s))||\,|\gamma'(s)|\,ds \leq ||\gamma'||_\infty \sum_j \varepsilon(t_j - t_{j-1}) \\ &= \varepsilon\,||\gamma'||_\infty (b-a). \end{aligned}$$

It follows that

$$\left|\left|\int_\gamma f d\gamma - \int_a^b f(\gamma(s))\gamma'(s)\,ds\right|\right| \leq \left|\left|\int_\gamma f d\gamma - \sum_{j=1}^n f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1}))\right|\right|$$

$$+\left|\left|\sum_{j=1}^n f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1})) - \int_a^b f(\gamma(s))\gamma'(s)\,ds\right|\right| \leq \varepsilon\,||\gamma'||_\infty (b-a) + \varepsilon.$$

Since ε is arbitrary, this verifies 23.12.

Definition 23.10 *Let Ω be an open subset of $\mathbb{C}$ and let $\gamma : [a,b] \to \Omega$ be a continuous function with bounded variation $f : \Omega \to X$ be a continuous function. Then the following notation is more customary.*

$$\int_\gamma f(z)\,dz \equiv \int_\gamma f d\gamma.$$

The expression, $\int_\gamma f(z)\,dz$, is called a contour integral and γ is referred to as the contour. A function $f : \Omega \to X$ for Ω an open set in $\mathbb{C}$ has a primitive if there exists a function, F, the primitive, such that $F'(z) = f(z)$. Thus F is just an antiderivative. Also if $\gamma_k : [a_k, b_k] \to \mathbb{C}$ is continuous and of bounded variation, for $k = 1, \cdot\cdot\cdot, m$ and $\gamma_k(b_k) = \gamma_{k+1}(a_k)$, define

$$\int_{\sum_{k=1}^m \gamma_k} f(z)\,dz \equiv \sum_{k=1}^m \int_{\gamma_k} f(z)\,dz. \tag{23.14}$$

In addition to this, for $\gamma : [a,b] \to \mathbb{C}$, define $-\gamma : [a,b] \to \mathbb{C}$ by $-\gamma(t) \equiv \gamma(b+a-t)$. Thus γ simply traces out the points of γ^* in the opposite order.

The following lemma is useful and follows quickly from Theorem 23.3.

Lemma 23.11 *In the above definition, there exists a continuous bounded variation function, γ defined on some closed interval, $[c,d]$, such that $\gamma([c,d]) = \cup_{k=1}^m \gamma_k([a_k,b_k])$ and $\gamma(c) = \gamma_1(a_1)$ while $\gamma(d) = \gamma_m(b_m)$. Furthermore,*

$$\int_\gamma f(z)\,dz = \sum_{k=1}^m \int_{\gamma_k} f(z)\,dz.$$

If $\gamma : [a,b] \to \mathbb{C}$ is of bounded variation and continuous, then

$$\int_\gamma f(z)\,dz = -\int_{-\gamma} f(z)\,dz.$$

Re stating Theorem 23.7 with the new notation in the above definition,

Theorem 23.12 *Let K be a compact set in $\mathbb{C}$ and let $f : \Omega \times K \to X$ be continuous for Ω an open set in $\mathbb{C}$. Also let $\gamma : [a,b] \to \Omega$ be continuous with bounded variation. Then if $r > 0$ is given, there exists $\eta : [a,b] \to \Omega$ such that $\eta(a) = \gamma(a)$, $\eta(b) = \gamma(b)$, η is $C^1([a,b])$, and*

$$\left| \int_\gamma f(z,w)\, dz - \int_\eta f(z,w)\, dz \right| < r,\ ||\eta - \gamma|| < r.$$

It will be very important to consider which functions have primitives. It turns out, it is not enough for f to be continuous in order to possess a primitive. This is in stark contrast to the situation for functions of a real variable in which the fundamental theorem of calculus will deliver a primitive for any continuous function. The reason for the interest in such functions is the following theorem and its corollary.

Theorem 23.13 *Let $\gamma : [a,b] \to \mathbb{C}$ be continuous and of bounded variation. Also suppose $F'(z) = f(z)$ for all $z \in \Omega$, an open set containing γ^* and f is continuous on Ω. Then*

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof: By Theorem 23.12 there exists $\eta \in C^1([a,b])$ such that $\gamma(a) = \eta(a)$, and $\gamma(b) = \eta(b)$ such that

$$\left|\left| \int_\gamma f(z)\, dz - \int_\eta f(z)\, dz \right|\right| < \varepsilon.$$

Then since η is in $C^1([a,b])$,

$$\begin{aligned} \int_\eta f(z)\, dz &= \int_a^b f(\eta(t))\, \eta'(t)\, dt = \int_a^b \frac{dF(\eta(t))}{dt} dt \\ &= F(\eta(b)) - F(\eta(a)) = F(\gamma(b)) - F(\gamma(a)). \end{aligned}$$

Therefore,

$$\left|\left| (F(\gamma(b)) - F(\gamma(a))) - \int_\gamma f(z)\, dz \right|\right| < \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this proves the theorem.

Corollary 23.14 *If $\gamma : [a,b] \to \mathbb{C}$ is continuous, has bounded variation, is a closed curve, $\gamma(a) = \gamma(b)$, and $\gamma^* \subseteq \Omega$ where Ω is an open set on which $F'(z) = f(z)$, then*

$$\int_\gamma f(z)\, dz = 0.$$

23.1 Exercises

1. Let $\gamma : [a, b] \to \mathbb{R}$ be increasing. Show $V(\gamma, [a, b]) = \gamma(b) - \gamma(a)$.

2. Suppose $\gamma : [a, b] \to \mathbb{C}$ satisfies a Lipschitz condition, $|\gamma(t) - \gamma(s)| \leq K|s - t|$. Show γ is of bounded variation and that $V(\gamma, [a, b]) \leq K|b - a|$.

3. $\gamma : [c_0, c_m] \to \mathbb{C}$ is piecewise smooth if there exist numbers, $c_k, k = 1, \cdots, m$ such that $c_0 < c_1 < \cdots < c_{m-1} < c_m$ such that γ is continuous and $\gamma : [c_k, c_{k+1}] \to \mathbb{C}$ is C^1. Show that such piecewise smooth functions are of bounded variation and give an estimate for $V(\gamma, [c_0, c_m])$.

4. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = r(\cos mt + i \sin mt)$ for m an integer. Find $\int_\gamma \frac{dz}{z}$.

5. Show that if $\gamma : [a, b] \to \mathbb{C}$ then there exists an increasing function $h : [0, 1] \to [a, b]$ such that $\gamma \circ h([0, 1]) = \gamma^*$.

6. Let $\gamma : [a, b] \to \mathbb{C}$ be an arbitrary continuous curve having bounded variation and let f, g have continuous derivatives on some open set containing γ^*. Prove the usual integration by parts formula.
$$\int_\gamma f g' dz = f(\gamma(b)) g(\gamma(b)) - f(\gamma(a)) g(\gamma(a)) - \int_\gamma f' g dz.$$

7. Let $f(z) \equiv |z|^{-(1/2)} e^{-i\frac{\theta}{2}}$ where $z = |z| e^{i\theta}$. This function is called the principle branch of $z^{-(1/2)}$. Find $\int_\gamma f(z)\, dz$ where γ is the semicircle in the upper half plane which goes from $(1, 0)$ to $(-1, 0)$ in the counter clockwise direction. Next do the integral in which γ goes in the clockwise direction along the semicircle in the lower half plane.

8. Prove an open set, U is connected if and only if for every two points in U, there exists a C^1 curve having values in U which joins them.

9. Let $\mathcal{P}, \mathcal{Q}$ be two partitions of $[a, b]$ with $\mathcal{P} \subseteq \mathcal{Q}$. Each of these partitions can be used to form an approximation to $V(\gamma, [a, b])$ as described above. Recall the total variation was the supremum of sums of a certain form determined by a partition. How is the sum associated with $\mathcal{P}$ related to the sum associated with $\mathcal{Q}$? Explain.

10. Consider the curve,
$$\gamma(t) = \begin{cases} t + it^2 \sin\left(\frac{1}{t}\right) \text{ if } t \in (0, 1] \\ 0 \text{ if } t = 0 \end{cases}.$$
Is γ a continuous curve having bounded variation? What if the t^2 is replaced with t? Is the resulting curve continuous? Is it a bounded variation curve?

11. Suppose $\gamma : [a, b] \to \mathbb{R}$ is given by $\gamma(t) = t$. What is $\int_\gamma f(t)\, d\gamma$? Explain.

Fundamentals Of Complex Analysis

24.1 Analytic Functions

Definition 24.1 *Let* Ω *be an open set in* $\mathbb{C}$ *and let* $f : \Omega \to X$. *Then* f *is analytic on* Ω *if for every* $z \in \Omega$,

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \equiv f'(z)$$

exists and is a continuous function of $z \in \Omega$. *Here* $h \in \mathbb{C}$.

Note that if f is analytic, it must be the case that f is continuous. It is more common to not include the requirement that f' is continuous but it is shown later that the continuity of f' follows.

What are some examples of analytic functions? In the case where $X = \mathbb{C}$, the simplest example is any polynomial. Thus

$$p(z) \equiv \sum_{k=0}^{n} a_k z^k$$

is an analytic function and

$$p'(z) = \sum_{k=1}^{n} a_k k z^{k-1}.$$

More generally, power series are analytic. This will be shown soon but first here is an important definition and a convergence theorem called the root test.

Definition 24.2 *Let* $\{a_k\}$ *be a sequence in* X. *Then* $\sum_{k=1}^{\infty} a_k \equiv \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ *whenever this limit exists. When the limit exists, the series is said to converge.*

Theorem 24.3 *Consider* $\sum_{k=1}^{\infty} a_k$ *and let* $\rho \equiv \limsup_{k\to\infty} ||a_k||^{1/k}$ *. Then if* $\rho < 1$*, the series converges absolutely and if* $\rho > 1$ *the series diverges spectacularly in the sense that* $\lim_{k\to\infty} a_k \neq 0$*. If* $\rho = 1$ *the test fails. Also* $\sum_{k=1}^{\infty} a_k (z-a)^k$ *converges on some disk* $B(a,R)$ *. It converges absolutely if* $|z-a| < R$ *and uniformly on* $B(a,r_1)$ *whenever* $r_1 < R$*. The function* $f(z) = \sum_{k=1}^{\infty} a_k (z-a)^k$ *is continuous on* $B(a,R)$ *.*

Proof: Suppose $\rho < 1$. Then there exists $r \in (\rho, 1)$. Therefore, $||a_k|| \leq r^k$ for all k large enough and so by a comparison test, $\sum_k ||a_k||$ converges because the partial sums are bounded above. Therefore, the partial sums of the original series form a Cauchy sequence in X and so they also converge due to completeness of X.

Now suppose $\rho > 1$. Then letting $\rho > r > 1$, it follows $||a_k||^{1/k} \geq r$ infinitely often. Thus $||a_k|| \geq r^k$ infinitely often. Thus there exists a subsequence for which $||a_{n_k}||$ converges to ∞. Therefore, the series cannot converge.

Now consider $\sum_{k=1}^{\infty} a_k (z-a)^k$. This series converges absolutely if

$$\limsup_{k\to\infty} ||a_k||^{1/k} |z-a| < 1$$

which is the same as saying $|z-a| < 1/\rho$ where $\rho \equiv \limsup_{k\to\infty} ||a_k||^{1/k}$. Let $R = 1/\rho$.

Now suppose $r_1 < R$. Consider $|z-a| \leq r_1$. Then for such z,

$$||a_k|| \, |z-a|^k \leq ||a_k|| \, r_1^k$$

and

$$\limsup_{k\to\infty} \left(||a_k|| \, r_1^k\right)^{1/k} = \limsup_{k\to\infty} ||a_k||^{1/k} r_1 = \frac{r_1}{R} < 1$$

so $\sum_k ||a_k|| \, r_1^k$ converges. By the Weierstrass M test, $\sum_{k=1}^{\infty} a_k (z-a)^k$ converges uniformly for $|z-a| \leq r_1$. Therefore, f is continuous on $B(a,R)$ as claimed because it is the uniform limit of continuous functions, the partial sums of the infinite series.

What if $\rho = 0$? In this case,

$$\limsup_{k\to\infty} ||a_k||^{1/k} |z-a| = 0 \cdot |z-a| = 0$$

and so $R = \infty$ and the series, $\sum ||a_k|| \, |z-a|^k$ converges everywhere.

What if $\rho = \infty$? Then in this case, the series converges only at $z = a$ because if $z \neq a$,

$$\limsup_{k\to\infty} ||a_k||^{1/k} |z-a| = \infty.$$

Theorem 24.4 *Let* $f(z) \equiv \sum_{k=1}^{\infty} a_k (z-a)^k$ *be given in Theorem 24.3 where* $R > 0$*. Then* f *is analytic on* $B(a,R)$ *. So are all its derivatives.*

Proof: Consider $g(z) = \sum_{k=2}^{\infty} a_k k (z-a)^{k-1}$ on $B(a,R)$ where $R = \rho^{-1}$ as above. Let $r_1 < r < R$. Then letting $|z-a| < r_1$ and $h < r - r_1$,

$$\begin{aligned}
&\left|\left|\frac{f(z+h)-f(z)}{h} - g(z)\right|\right| \\
\leq\ & \sum_{k=2}^{\infty} ||a_k|| \left|\frac{(z+h-a)^k - (z-a)^k}{h} - k(z-a)^{k-1}\right| \\
\leq\ & \sum_{k=2}^{\infty} ||a_k|| \left|\frac{1}{h}\left(\sum_{i=0}^{k}\binom{k}{i}(z-a)^{k-i}h^i - (z-a)^k\right) - k(z-a)^{k-1}\right| \\
=\ & \sum_{k=2}^{\infty} ||a_k|| \left|\frac{1}{h}\left(\sum_{i=1}^{k}\binom{k}{i}(z-a)^{k-i}h^i\right) - k(z-a)^{k-1}\right| \\
\leq\ & \sum_{k=2}^{\infty} ||a_k|| \left|\left(\sum_{i=2}^{k}\binom{k}{i}(z-a)^{k-i}h^{i-1}\right)\right| \\
\leq\ & |h| \sum_{k=2}^{\infty} ||a_k|| \left(\sum_{i=0}^{k-2}\binom{k}{i+2}|z-a|^{k-2-i}|h|^i\right) \\
=\ & |h| \sum_{k=2}^{\infty} ||a_k|| \left(\sum_{i=0}^{k-2}\binom{k-2}{i}\frac{k(k-1)}{(i+2)(i+1)}|z-a|^{k-2-i}|h|^i\right) \\
\leq\ & |h| \sum_{k=2}^{\infty} ||a_k|| \frac{k(k-1)}{2}\left(\sum_{i=0}^{k-2}\binom{k-2}{i}|z-a|^{k-2-i}|h|^i\right) \\
=\ & |h| \sum_{k=2}^{\infty} ||a_k|| \frac{k(k-1)}{2}(|z-a|+|h|)^{k-2} < |h| \sum_{k=2}^{\infty} ||a_k|| \frac{k(k-1)}{2} r^{k-2}.
\end{aligned}$$

Then

$$\limsup_{k\to\infty}\left(||a_k|| \frac{k(k-1)}{2} r^{k-2}\right)^{1/k} = \rho r < 1$$

and so

$$\left|\left|\frac{f(z+h)-f(z)}{h} - g(z)\right|\right| \leq C|h|.$$

therefore, $g(z) = f'(z)$. Now by Theorem 24.3 it also follows that f' is continuous. Since $r_1 < R$ was arbitrary, this shows that $f'(z)$ is given by the differentiated series above for $|z-a| < R$. Now a repeat of the argument shows all the derivatives of f exist and are continuous on $B(a,R)$.

24.1.1 Cauchy Riemann Equations

Next consider the very important Cauchy Riemann equations which give conditions under which complex valued functions of a complex variable are analytic.

Theorem 24.5 *Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be a function, such that for $z = x + iy \in \Omega$,*

$$f(z) = u(x, y) + iv(x, y).$$

Then f is analytic if and only if u, v are $C^1(\Omega)$ and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Furthermore,

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y).$$

Proof: Suppose f is analytic first. Then letting $t \in \mathbb{R}$,

$$f'(z) = \lim_{t \to 0} \frac{f(z+t) - f(z)}{t} =$$

$$\lim_{t \to 0}\left(\frac{u(x+t, y) + iv(x+t, y)}{t} - \frac{u(x, y) + iv(x, y)}{t}\right)$$

$$= \frac{\partial u(x, y)}{\partial x} + i\frac{\partial v(x, y)}{\partial x}.$$

But also

$$f'(z) = \lim_{t \to 0} \frac{f(z+it) - f(z)}{it} =$$

$$\lim_{t \to 0}\left(\frac{u(x, y+t) + iv(x, y+t)}{it} - \frac{u(x, y) + iv(x, y)}{it}\right)$$

$$\frac{1}{i}\left(\frac{\partial u(x, y)}{\partial y} + i\frac{\partial v(x, y)}{\partial y}\right)$$

$$= \frac{\partial v(x, y)}{\partial y} - i\frac{\partial u(x, y)}{\partial y}.$$

This verifies the Cauchy Riemann equations. We are assuming that $z \to f'(z)$ is continuous. Therefore, the partial derivatives of u and v are also continuous. To see this, note that from the formulas for $f'(z)$ given above, and letting $z_1 = x_1 + iy_1$

$$\left|\frac{\partial v(x, y)}{\partial y} - \frac{\partial v(x_1, y_1)}{\partial y}\right| \leq |f'(z) - f'(z_1)|,$$

showing that $(x, y) \to \frac{\partial v(x,y)}{\partial y}$ is continuous since $(x_1, y_1) \to (x, y)$ if and only if $z_1 \to z$. The other cases are similar.

Now suppose the Cauchy Riemann equations hold and the functions, u and v are $C^1(\Omega)$. Then letting $h = h_1 + ih_2$,

$$f(z+h) - f(z) = u(x+h_1, y+h_2)$$

$$+iv(x+h_1,y+h_2)-(u(x,y)+iv(x,y))$$

We know u and v are both differentiable and so

$$f(z+h)-f(z)=\frac{\partial u}{\partial x}(x,y)h_1+\frac{\partial u}{\partial y}(x,y)h_2+$$

$$i\left(\frac{\partial v}{\partial x}(x,y)h_1+\frac{\partial v}{\partial y}(x,y)h_2\right)+o(h).$$

Dividing by h and using the Cauchy Riemann equations,

$$\frac{f(z+h)-f(z)}{h}=\frac{\frac{\partial u}{\partial x}(x,y)h_1+i\frac{\partial v}{\partial y}(x,y)h_2}{h}+$$

$$\frac{i\frac{\partial v}{\partial x}(x,y)h_1+\frac{\partial u}{\partial y}(x,y)h_2}{h}+\frac{o(h)}{h}$$

$$=\frac{\partial u}{\partial x}(x,y)\frac{h_1+ih_2}{h}+i\frac{\partial v}{\partial x}(x,y)\frac{h_1+ih_2}{h}+\frac{o(h)}{h}$$

Taking the limit as $h\to 0$,

$$f'(z)=\frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y).$$

It follows from this formula and the assumption that u,v are $C^1(\Omega)$ that f' is continuous.

It is routine to verify that all the usual rules of derivatives hold for analytic functions. In particular, the product rule, the chain rule, and quotient rule.

24.1.2 An Important Example

An important example of an analytic function is $e^z\equiv\exp(z)\equiv e^x(\cos y+i\sin y)$ where $z=x+iy$. You can verify that this function satisfies the Cauchy Riemann equations and that all the partial derivatives are continuous. Also from the above discussion, $(e^z)'=e^x\cos(y)+ie^x\sin y=e^z$. Later I will show that e^z is given by the usual power series. An important property of this function is that it can be used to parameterize the circle centered at z_0 having radius r.

Lemma 24.6 *Let γ denote the closed curve which is a circle of radius r centered at z_0. Then a parameterization this curve is $\gamma(t)=z_0+re^{it}$ where $t\in[0,2\pi]$.*

Proof: $|\gamma(t)-z_0|^2=\left|re^{it}re^{-it}\right|=r^2$. Also, you can see from the definition of the sine and cosine that the point described in this way moves counter clockwise over this circle.

24.2 Exercises

1. Verify all the usual rules of differentiation including the product and chain rules.

2. Suppose f and $f' : U \to \mathbb{C}$ are analytic and $f(z) = u(x, y) + iv(x, y)$. Verify $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. This partial differential equation satisfied by the real and imaginary parts of an analytic function is called Laplace's equation. We say these functions satisfying Laplace's equation are harmonic functions. If u is a harmonic function defined on $B(0, r)$ show that $v(x, y) \equiv \int_0^y u_x(x, t)\,dt - \int_0^x u_y(t, 0)\,dt$ is such that $u + iv$ is analytic.

3. Let $f : U \to \mathbb{C}$ be analytic and $f(z) = u(x, y) + iv(x, y)$. Show u, v and uv are all harmonic although it can happen that u^2 is not. Recall that a function, w is harmonic if $w_{xx} + w_{yy} = 0$.

4. Define a function $f(z) \equiv \overline{z} \equiv x - iy$ where $z = x + iy$. Is f analytic?

5. If $f(z) = u(x, y) + iv(x, y)$ and f is analytic, verify that
$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = |f'(z)|^2.$$

6. Show that if $u(x, y) + iv(x, y) = f(z)$ is analytic, then $\nabla u \cdot \nabla v = 0$. Recall
$$\nabla u(x, y) = \langle u_x(x, y), u_y(x, y) \rangle.$$

7. Show that every polynomial is analytic.

8. If $\gamma(t) = x(t) + iy(t)$ is a C^1 curve having values in U, an open set of $\mathbb{C}$, and if $f : U \to \mathbb{C}$ is analytic, we can consider $f \circ \gamma$, another C^1 curve having values in $\mathbb{C}$. Also, $\gamma'(t)$ and $(f \circ \gamma)'(t)$ are complex numbers so these can be considered as vectors in $\mathbb{R}^2$ as follows. The complex number, $x + iy$ corresponds to the vector, $\langle x, y \rangle$. Suppose that γ and η are two such C^1 curves having values in U and that $\gamma(t_0) = \eta(s_0) = z$ and suppose that $f : U \to \mathbb{C}$ is analytic. Show that the angle between $(f \circ \gamma)'(t_0)$ and $(f \circ \eta)'(s_0)$ is the same as the angle between $\gamma'(t_0)$ and $\eta'(s_0)$ assuming that $f'(z) \neq 0$. Thus analytic mappings preserve angles at points where the derivative is nonzero. Such mappings are called isogonal. . **Hint:** To make this easy to show, first observe that $\langle x, y \rangle \cdot \langle a, b \rangle = \frac{1}{2}(z\overline{w} + \overline{z}w)$ where $z = x + iy$ and $w = a + ib$.

9. Analytic functions are even better than what is described in Problem 8. In addition to preserving angles, they also preserve orientation. To verify this show that if $z = x + iy$ and $w = a + ib$ are two complex numbers, then $\langle x, y, 0 \rangle$ and $\langle a, b, 0 \rangle$ are two vectors in $\mathbb{R}^3$. Recall that the cross product, $\langle x, y, 0 \rangle \times \langle a, b, 0 \rangle$, yields a vector normal to the two given vectors such that the triple, $\langle x, y, 0 \rangle$, $\langle a, b, 0 \rangle$, and $\langle x, y, 0 \rangle \times \langle a, b, 0 \rangle$ satisfies the right hand rule

and has magnitude equal to the product of the sine of the included angle times the product of the two norms of the vectors. In this case, the cross product either points in the direction of the positive z axis or in the direction of the negative z axis. Thus, either the vectors $\langle x, y, 0\rangle, \langle a, b, 0\rangle, \mathbf{k}$ form a right handed system or the vectors $\langle a, b, 0\rangle, \langle x, y, 0\rangle, \mathbf{k}$ form a right handed system. These are the two possible orientations. Show that in the situation of Problem 8 the orientation of $\gamma'(t_0), \eta'(s_0), \mathbf{k}$ is the same as the orientation of the vectors $(f \circ \gamma)'(t_0), (f \circ \eta)'(s_0), \mathbf{k}$. Such mappings are called conformal. If f is analytic and $f'(z) \neq 0$, then we know from this problem and the above that f is a conformal map. **Hint:** You can do this by verifying that $(f \circ \gamma)'(t_0) \times (f \circ \eta)'(s_0) = |f'(\gamma(t_0))|^2 \gamma'(t_0) \times \eta'(s_0)$. To make the verification easier, you might first establish the following simple formula for the cross product where here $x + iy = z$ and $a + ib = w$.

$$(x, y, 0) \times (a, b, 0) = \operatorname{Re}(zi\overline{w})\,\mathbf{k}.$$

10. Write the Cauchy Riemann equations in terms of polar coordinates. Recall the polar coordinates are given by

$$x = r\cos\theta,\ y = r\sin\theta.$$

This means, letting $u(x, y) = u(r, \theta), v(x, y) = v(r, \theta)$, write the Cauchy Riemann equations in terms of r and θ. You should eventually show the Cauchy Riemann equations are equivalent to

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

11. Show that a real valued analytic function must be constant.

24.3 Cauchy's Formula For A Disk

The Cauchy integral formula is the most important theorem in complex analysis. It will be established for a disk in this chapter and later will be generalized to much more general situations but the version given here will suffice to prove many interesting theorems needed in the later development of the theory. The following are some advanced calculus results.

Lemma 24.7 *Let $f : [a, b] \to \mathbb{C}$. Then $f'(t)$ exists if and only if $\operatorname{Re} f'(t)$ and $\operatorname{Im} f'(t)$ exist. Furthermore,*

$$f'(t) = \operatorname{Re} f'(t) + i\operatorname{Im} f'(t).$$

Proof: The if part of the equivalence is obvious.

Now suppose $f'(t)$ exists. Let both t and $t + h$ be contained in $[a, b]$

$$\left|\frac{\operatorname{Re} f(t+h) - \operatorname{Re} f(t)}{h} - \operatorname{Re}(f'(t))\right| \leq \left|\frac{f(t+h) - f(t)}{h} - f'(t)\right|$$

and this converges to zero as $h \to 0$. Therefore, $\operatorname{Re} f'(t) = \operatorname{Re}(f'(t))$. Similarly, $\operatorname{Im} f'(t) = \operatorname{Im}(f'(t))$.

Lemma 24.8 *If $g : [a,b] \to \mathbb{C}$ and g is continuous on $[a,b]$ and differentiable on (a,b) with $g'(t) = 0$, then $g(t)$ is a constant.*

Proof: From the above lemma, you can apply the mean value theorem to the real and imaginary parts of g.

Applying the above lemma to the components yields the following lemma.

Lemma 24.9 *If $g : [a,b] \to \mathbb{C}^n = X$ and g is continuous on $[a,b]$ and differentiable on (a,b) with $g'(t) = 0$, then $g(t)$ is a constant.*

If you want to have X be a complex Banach space, the result is still true.

Lemma 24.10 *If $g : [a,b] \to X$ and g is continuous on $[a,b]$ and differentiable on (a,b) with $g'(t) = 0$, then $g(t)$ is a constant.*

Proof: Let $\Lambda \in X'$. Then $\Lambda g : [a,b] \to \mathbb{C}$. Therefore, from Lemma 24.8, for each $\Lambda \in X', \Lambda g(s) = \Lambda g(t)$ and since X' separates the points, it follows $g(s) = g(t)$ so g is constant.

Lemma 24.11 *Let $\phi : [a,b] \times [c,d] \to \mathbb{R}$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s,t)\,ds. \tag{24.1}$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s,t)}{\partial t} ds. \tag{24.2}$$

Proof: The first claim follows from the uniform continuity of ϕ on $[a,b] \times [c,d]$, which uniform continuity results from the set being compact. To establish 24.2, let t and $t+h$ be contained in $[c,d]$ and form, using the mean value theorem,

$$\begin{aligned} \frac{g(t+h) - g(t)}{h} &= \frac{1}{h} \int_a^b [\phi(s,t+h) - \phi(s,t)]\,ds \\ &= \frac{1}{h} \int_a^b \frac{\partial \phi(s,t+\theta h)}{\partial t} h ds \\ &= \int_a^b \frac{\partial \phi(s,t+\theta h)}{\partial t} ds, \end{aligned}$$

where θ may depend on s but is some number between 0 and 1. Then by the uniform continuity of $\frac{\partial \phi}{\partial t}$, it follows that 24.2 holds.

Corollary 24.12 *Let $\phi : [a,b] \times [c,d] \to \mathbb{C}$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s,t)\, ds. \tag{24.3}$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s,t)}{\partial t} ds. \tag{24.4}$$

Proof: Apply Lemma 24.11 to the real and imaginary parts of ϕ.

Applying the above corollary to the components, you can also have the same result for ϕ having values in $\mathbb{C}^n$.

Corollary 24.13 *Let $\phi : [a,b] \times [c,d] \to \mathbb{C}^n$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s,t)\, ds. \tag{24.5}$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s,t)}{\partial t} ds. \tag{24.6}$$

If you want to consider ϕ having values in X, a complex Banach space a similar result holds.

Corollary 24.14 *Let $\phi : [a,b] \times [c,d] \to X$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s,t)\, ds. \tag{24.7}$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s,t)}{\partial t} ds. \tag{24.8}$$

Proof: Let $\Lambda \in X'$. Then $\Lambda\phi : [a,b] \times [c,d] \to \mathbb{C}$ is continuous and $\frac{\partial \Lambda\phi}{\partial t}$ exists and is continuous on $[a,b] \times [c,d]$. Therefore, from 24.8,

$$\Lambda(g'(t)) = (\Lambda g)'(t) = \int_a^b \frac{\partial \Lambda\phi(s,t)}{\partial t} ds = \Lambda \int_a^b \frac{\partial \phi(s,t)}{\partial t} ds$$

and since X' separates the points, it follows 24.8 holds.

The following is Cauchy's integral formula for a disk.

Theorem 24.15 *Let $f : \Omega \to X$ be analytic on the open set, Ω and let*

$$\overline{B(z_0, r)} \subseteq \Omega.$$

Let $\gamma(t) \equiv z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then if $z \in B(z_0, r)$,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw. \tag{24.9}$$

Proof: Consider for $\alpha \in [0, 1]$,

$$g(\alpha) \equiv \int_0^{2\pi} \frac{f\left(z + \alpha\left(z_0 + re^{it} - z\right)\right)}{re^{it} + z_0 - z} rie^{it} dt.$$

If α equals one, this reduces to the integral in 24.9. The idea is to show g is a constant and that $g(0) = f(z) 2\pi i$. First consider the claim about $g(0)$.

$$\begin{aligned} g(0) &= \left(\int_0^{2\pi} \frac{re^{it}}{re^{it} + z_0 - z} dt\right) if(z) \\ &= if(z) \left(\int_0^{2\pi} \frac{1}{1 - \frac{z - z_0}{re^{it}}} dt\right) \\ &= if(z) \int_0^{2\pi} \sum_{n=0}^{\infty} r^{-n} e^{-int} (z - z_0)^n dt \end{aligned}$$

because $\left|\frac{z-z_0}{re^{it}}\right| < 1$. Since this sum converges uniformly you can interchange the sum and the integral to obtain

$$\begin{aligned} g(0) &= if(z) \sum_{n=0}^{\infty} r^{-n} (z - z_0)^n \int_0^{2\pi} e^{-int} dt \\ &= 2\pi i f(z) \end{aligned}$$

because $\int_0^{2\pi} e^{-int} dt = 0$ if $n > 0$.

Next consider the claim that g is constant. By Corollary 24.13, for $\alpha \in (0, 1)$,

$$\begin{aligned} g'(\alpha) &= \int_0^{2\pi} \frac{f'\left(z + \alpha\left(z_0 + re^{it} - z\right)\right)\left(re^{it} + z_0 - z\right)}{re^{it} + z_0 - z} rie^{it} dt \\ &= \int_0^{2\pi} f'\left(z + \alpha\left(z_0 + re^{it} - z\right)\right) rie^{it} dt \\ &= \int_0^{2\pi} \frac{d}{dt}\left(f\left(z + \alpha\left(z_0 + re^{it} - z\right)\right) \frac{1}{\alpha}\right) dt \\ &= f\left(z + \alpha\left(z_0 + re^{i2\pi} - z\right)\right) \frac{1}{\alpha} - f\left(z + \alpha\left(z_0 + re^{0} - z\right)\right) \frac{1}{\alpha} = 0. \end{aligned}$$

Now g is continuous on $[0, 1]$ and $g'(t) = 0$ on $(0, 1)$ so by Lemma 24.9, g equals a constant. This constant can only be $g(0) = 2\pi i f(z)$. Thus,

$$g(1) = \int_\gamma \frac{f(w)}{w - z} dw = g(0) = 2\pi i f(z).$$

This proves the theorem.

This is a very significant theorem. A few applications are given next.

Theorem 24.16 *Let $f : \Omega \to X$ be analytic where Ω is an open set in $\mathbb{C}$. Then f has infinitely many derivatives on Ω. Furthermore, for all $z \in B(z_0, r)$,*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}dw \tag{24.10}$$

where $\gamma(t) \equiv z_0 + re^{it}, t \in [0, 2\pi]$ for r small enough that $B(z_0, r) \subseteq \Omega$.

Proof: Let $z \in B(z_0, r) \subseteq \Omega$ and let $\overline{B(z_0, r)} \subseteq \Omega$. Then, letting $\gamma(t) \equiv z_0 + re^{it}, t \in [0, 2\pi]$, and h small enough,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw,\ f(z+h) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z-h}dw$$

Now

$$\frac{1}{w-z-h} - \frac{1}{w-z} = \frac{h}{(-w+z+h)(-w+z)}$$

and so

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{1}{2\pi hi}\int_\gamma \frac{hf(w)}{(-w+z+h)(-w+z)}dw \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(-w+z+h)(-w+z)}dw.\end{aligned}$$

Now for all h sufficiently small, there exists a constant C independent of such h such that

$$\begin{aligned}&\left|\frac{1}{(-w+z+h)(-w+z)} - \frac{1}{(-w+z)(-w+z)}\right| \\ =& \left|\frac{h}{(w-z-h)(w-z)^2}\right| \leq C|h|\end{aligned}$$

and so, the integrand converges uniformly as $h \to 0$ to

$$= \frac{f(w)}{(w-z)^2}$$

Therefore, the limit as $h \to 0$ may be taken inside the integral to obtain

$$f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z)^2}dw.$$

Continuing in this way, yields 24.10.

This is a very remarkable result. It shows the existence of one continuous derivative implies the existence of all derivatives, in contrast to the theory of functions of a real variable. Actually, more than what is stated in the theorem was shown. The above proof establishes the following corollary.

Corollary 24.17 *Suppose f is continuous on $\partial B(z_0, r)$ and suppose that for all $z \in B(z_0, r)$,*

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw,$$

where $\gamma(t) \equiv z_0 + re^{it}, t \in [0, 2\pi]$. Then f is analytic on $B(z_0, r)$ and in fact has infinitely many derivatives on $B(z_0, r)$.

Another application is the following lemma.

Lemma 24.18 *Let $\gamma(t) = z_0 + re^{it}$, for $t \in [0, 2\pi]$, suppose $f_n \to f$ uniformly on $\overline{B(z_0, r)}$, and suppose*

$$f_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(w)}{w - z} dw \tag{24.11}$$

for $z \in B(z_0, r)$. Then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw, \tag{24.12}$$

implying that f is analytic on $B(z_0, r)$.

Proof: From 24.11 and the uniform convergence of f_n to f on $\gamma([0, 2\pi])$, the integrals in 24.11 converge to

$$\frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z} dw.$$

Therefore, the formula 24.12 follows.

Uniform convergence on a closed disk of the analytic functions implies the target function is also analytic. This is amazing. Think of the Weierstrass approximation theorem for polynomials. You can obtain a continuous nowhere differentiable function as the uniform limit of polynomials.

The conclusions of the following proposition have all been obtained earlier in Theorem 24.4 but they can be obtained more easily if you use the above theorem and lemmas.

Proposition 24.19 *Let $\{a_n\}$ denote a sequence in X. Then there exists $R \in [0, \infty]$ such that*

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

converges absolutely if $|z - z_0| < R$, diverges if $|z - z_0| > R$ and converges uniformly on $B(z_0, r)$ for all $r < R$. Furthermore, if $R > 0$, the function,

$$f(z) \equiv \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

is analytic on $B(z_0, R)$.

Proof: The assertions about absolute convergence are routine from the root test if

$$R \equiv \left(\limsup_{n \to \infty} |a_n|^{1/n} \right)^{-1}$$

with $R = \infty$ if the quantity in parenthesis equals zero. The root test can be used to verify absolute convergence which then implies convergence by completeness of X.

The assertion about uniform convergence follows from the Weierstrass M test and $M_n \equiv |a_n| r^n$. ($\sum_{n=0}^{\infty} |a_n| r^n < \infty$ by the root test). It only remains to verify the assertion about $f(z)$ being analytic in the case where $R > 0$.

Let $0 < r < R$ and define $f_n(z) \equiv \sum_{k=0}^{n} a_k (z - z_0)^k$. Then f_n is a polynomial and so it is analytic. Thus, by the Cauchy integral formula above,

$$f_n(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(w)}{w - z} dw$$

where $\gamma(t) = z_0 + re^{it}$, for $t \in [0, 2\pi]$. By Lemma 24.18 and the first part of this proposition involving uniform convergence,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw.$$

Therefore, f is analytic on $B(z_0, r)$ by Corollary 24.17. Since $r < R$ is arbitrary, this shows f is analytic on $B(z_0, R)$.

This proposition shows that all functions having values in X which are given as power series are analytic on their circle of convergence, the set of complex numbers, z, such that $|z - z_0| < R$. In fact, every analytic function can be realized as a power series.

Theorem 24.20 *If $f : \Omega \to X$ is analytic and if $B(z_0, r) \subseteq \Omega$, then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{24.13}$$

for all $|z - z_0| < r$. Furthermore,

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{24.14}$$

Proof: Consider $|z - z_0| < r$ and let $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. Then for $w \in \gamma([0, 2\pi])$,

$$\left| \frac{z - z_0}{w - z_0} \right| < 1$$

and so, by the Cauchy integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)\left(1-\frac{z-z_0}{w-z_0}\right)}dw \\ &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n dw. \end{aligned}$$

Since the series converges uniformly, you can interchange the integral and the sum to obtain

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\right)(z-z_0)^n \\ &\equiv \sum_{n=0}^{\infty} a_n (z-z_0)^n \end{aligned}$$

By Theorem 24.16, 24.14 holds.

Note that this also implies that if a function is analytic on an open set, then all of its derivatives are also analytic. This follows from Theorem 24.4 which says that a function given by a power series has all derivatives on the disk of convergence.

24.4 Exercises

1. Show that if $|e_k| \leq \varepsilon$, then $\left|\sum_{k=m}^{\infty} e_k\left(r^k - r^{k+1}\right)\right| < \varepsilon$ if $0 \leq r < 1$. **Hint:** Let $|\theta| = 1$ and verify that

$$\theta\sum_{k=m}^{\infty} e_k\left(r^k - r^{k+1}\right) = \left|\sum_{k=m}^{\infty} e_k\left(r^k - r^{k+1}\right)\right| = \sum_{k=m}^{\infty} \operatorname{Re}(\theta e_k)\left(r^k - r^{k+1}\right)$$

where $-\varepsilon < \operatorname{Re}(\theta e_k) < \varepsilon$.

2. Abel's theorem says that if $\sum_{n=0}^{\infty} a_n (z-a)^n$ has radius of convergence equal to 1 and if $A = \sum_{n=0}^{\infty} a_n$, then $\lim_{r\to 1-}\sum_{n=0}^{\infty} a_n r^n = A$. **Hint:** Show $\sum_{k=0}^{\infty} a_k r^k = \sum_{k=0}^{\infty} A_k\left(r^k - r^{k+1}\right)$ where A_k denotes the k^{th} partial sum of $\sum a_j$. Thus

$$\sum_{k=0}^{\infty} a_k r^k = \sum_{k=m+1}^{\infty} A_k\left(r^k - r^{k+1}\right) + \sum_{k=0}^{m} A_k\left(r^k - r^{k+1}\right),$$

where $|A_k - A| < \varepsilon$ for all $k \geq m$. In the first sum, write $A_k = A + e_k$ and use Problem 1. Use this theorem to verify that $\arctan(1) = \sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1}$.

3. Find the integrals using the Cauchy integral formula.

   (a) $\int_\gamma \frac{\sin z}{z-i} dz$ where $\gamma(t) = 2e^{it} : t \in [0, 2\pi]$.

   (b) $\int_\gamma \frac{1}{z-a} dz$ where $\gamma(t) = a + re^{it} : t \in [0, 2\pi]$

   (c) $\int_\gamma \frac{\cos z}{z^2} dz$ where $\gamma(t) = e^{it} : t \in [0, 2\pi]$

   (d) $\int_\gamma \frac{\log(z)}{z^n} dz$ where $\gamma(t) = 1 + \frac{1}{2}e^{it} : t \in [0, 2\pi]$ and $n = 0, 1, 2$. In this problem, $\log(z) \equiv \ln|z| + i \arg(z)$ where $\arg(z) \in (-\pi, \pi)$ and $z = |z| e^{i \arg(z)}$. Thus $e^{\log(z)} = z$ and $\log(z)' = \frac{1}{z}$.

4. Let $\gamma(t) = 4e^{it} : t \in [0, 2\pi]$ and find $\int_\gamma \frac{z^2+4}{z(z^2+1)} dz$.

5. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $|z| < R$. Show that then

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

   for all $r \in [0, R)$. **Hint:** Let

$$f_n(z) \equiv \sum_{k=0}^{n} a_k z^k,$$

   show

$$\frac{1}{2\pi} \int_0^{2\pi} \left| f_n\left(re^{i\theta}\right) \right|^2 d\theta = \sum_{k=0}^{n} |a_k|^2 r^{2k}$$

   and then take limits as $n \to \infty$ using uniform convergence.

6. The Cauchy integral formula, marvelous as it is, can actually be improved upon. The Cauchy integral formula involves representing f by the values of f on the boundary of the disk, $B(a, r)$. It is possible to represent f by using only the values of $\operatorname{Re} f$ on the boundary. This leads to the Schwarz formula . Supply the details in the following outline.

   Suppose f is analytic on $|z| < R$ and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{24.15}$$

   with the series converging uniformly on $|z| = R$. Then letting $|w| = R$,

$$2u(w) = f(w) + \overline{f(w)}$$

   and so

$$2u(w) = \sum_{k=0}^{\infty} a_k w^k + \sum_{k=0}^{\infty} \overline{a_k} (\overline{w})^k. \tag{24.16}$$

Now letting $\gamma(t) = Re^{it},\ t \in [0, 2\pi]$

$$\begin{aligned}\int_\gamma \frac{2u(w)}{w} dw &= (a_0 + \overline{a_0}) \int_\gamma \frac{1}{w} dw \\ &= 2\pi i (a_0 + \overline{a_0}).\end{aligned}$$

Thus, multiplying 24.16 by w^{-1},

$$\frac{1}{\pi i} \int_\gamma \frac{u(w)}{w} dw = a_0 + \overline{a_0}.$$

Now multiply 24.16 by $w^{-(n+1)}$ and integrate again to obtain

$$a_n = \frac{1}{\pi i} \int_\gamma \frac{u(w)}{w^{n+1}} dw.$$

Using these formulas for a_n in 24.15, we can interchange the sum and the integral (Why can we do this?) to write the following for $|z| < R$.

$$\begin{aligned} f(z) &= \frac{1}{\pi i} \int_\gamma \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^{k+1} u(w)\, dw - \overline{a_0} \\ &= \frac{1}{\pi i} \int_\gamma \frac{u(w)}{w - z} dw - \overline{a_0}, \end{aligned}$$

which is the Schwarz formula. Now $\operatorname{Re} a_0 = \frac{1}{2\pi i} \int_\gamma \frac{u(w)}{w} dw$ and $\overline{a_0} = \operatorname{Re} a_0 - i \operatorname{Im} a_0$. Therefore, we can also write the Schwarz formula as

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{u(w)(w+z)}{(w-z)\, w} dw + i \operatorname{Im} a_0. \tag{24.17}$$

7. Take the real parts of the second form of the Schwarz formula to derive the Poisson formula for a disk,

$$u\left(re^{i\alpha}\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u\left(Re^{i\theta}\right)\left(R^2 - r^2\right)}{R^2 + r^2 - 2Rr\cos(\theta - \alpha)} d\theta. \tag{24.18}$$

8. Suppose that $u(w)$ is a given real continuous function defined on $\partial B(0, R)$ and define $f(z)$ for $|z| < R$ by 24.17. Show that f, so defined is analytic. Explain why u given in 24.18 is harmonic. Show that

$$\lim_{r \to R-} u\left(re^{i\alpha}\right) = u\left(Re^{i\alpha}\right).$$

Thus u is a harmonic function which approaches a given function on the boundary and is therefore, a solution to the Dirichlet problem.

9. Suppose $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ for all $|z - z_0| < R$. Show that $f'(z) = \sum_{k=0}^{\infty} a_k k (z - z_0)^{k-1}$ for all $|z - z_0| < R$. **Hint:** Let $f_n(z)$ be a partial sum of f. Show that f_n' converges uniformly to some function, g on $|z - z_0| \leq r$ for any $r < R$. Now use the Cauchy integral formula for a function and its derivative to identify g with f'.

10. Use Problem 9 to find the exact value of $\sum_{k=0}^{\infty} k^2 \left(\frac{1}{3}\right)^k$.

11. Prove the binomial formula,
$$(1+z)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$$
where
$$\binom{\alpha}{n} \equiv \frac{\alpha \cdots (\alpha - n + 1)}{n!}.$$
Can this be used to give a proof of the binomial formula,
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k?$$
Explain.

12. Suppose f is analytic on $B(z_0, r)$ and continuous on $\overline{B(z_0, r)}$ and $|f(z)| \leq M$ on $\overline{B(z_0, r)}$. Show that then $\left|f^{(n)}(a)\right| \leq \frac{Mn!}{r^n}$.

24.5 Zeros Of An Analytic Function

In this section we give a very surprising property of analytic functions which is in stark contrast to what takes place for functions of a real variable.

Definition 24.21 *A region is a connected open set.*

It turns out the zeros of an analytic function which is not constant on some region cannot have a limit point. This is also a good time to define the order of a zero.

Definition 24.22 *Suppose f is an analytic function defined near a point, α where $f(\alpha) = 0$. Thus α is a zero of the function, f. The zero is of order m if $f(z) = (z - \alpha)^m g(z)$ where g is an analytic function which is not equal to zero at α.*

Theorem 24.23 *Let Ω be a connected open set (region) and let $f : \Omega \to X$ be analytic. Then the following are equivalent.*

1. *$f(z) = 0$ for all $z \in \Omega$*
2. *There exists $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all n.*

3. There exists $z_0 \in \Omega$ *which is a limit point of the set,*

$$Z \equiv \{z \in \Omega : f(z) = 0\}.$$

Proof: It is clear the first condition implies the second two. Suppose the third holds. Then for z near z_0

$$f(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

where $k \geq 1$ since z_0 is a zero of f. Suppose $k < \infty$. Then,

$$f(z) = (z - z_0)^k g(z)$$

where $g(z_0) \neq 0$. Letting $z_n \to z_0$ where $z_n \in Z, z_n \neq z_0$, it follows

$$0 = (z_n - z_0)^k g(z_n)$$

which implies $g(z_n) = 0$. Then by continuity of g, we see that $g(z_0) = 0$ also, contrary to the choice of k. Therefore, k cannot be less than ∞ and so z_0 is a point satisfying the second condition.

Now suppose the second condition and let

$$S \equiv \left\{z \in \Omega : f^{(n)}(z) = 0 \text{ for all } n\right\}.$$

It is clear that S is a closed set which by assumption is nonempty. However, this set is also open. To see this, let $z \in S$. Then for all w close enough to z,

$$f(w) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!} (w - z)^k = 0.$$

Thus f is identically equal to zero near $z \in S$. Therefore, all points near z are contained in S also, showing that S is an open set. Now $\Omega = S \cup (\Omega \setminus S)$, the union of two disjoint open sets, S being nonempty. It follows the other open set, $\Omega \setminus S$, must be empty because Ω is connected. Therefore, the first condition is verified. This proves the theorem. (See the following diagram.)

$$\begin{array}{ccccc} & & 1.) & & \\ & \swarrow\nearrow & & \searrow & \\ 2.) & & \longleftarrow & & 3.) \end{array}$$

Note how radically different this is from the theory of functions of a real variable. Consider, for example the function

$$f(x) \equiv \begin{cases} x^2 \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

which has a derivative for all $x \in \mathbb{R}$ and for which 0 is a limit point of the set, Z, even though f is not identically equal to zero.

Here is a very important application called Euler's formula. Recall that

$$e^z \equiv e^x (\cos(y) + i \sin(y)) \tag{24.19}$$

Is it also true that $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$?

Theorem 24.24 *(Euler's Formula) Let $z = x + iy$. Then*

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Proof: It was already observed that e^z given by 24.19 is analytic. So is $\exp(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{k!}$. In fact the power series converges for all $z \in \mathbb{C}$. Furthermore the two functions, e^z and $\exp(z)$ agree on the real line which is a set which contains a limit point. Therefore, they agree for all values of $z \in \mathbb{C}$.

This formula shows the famous two identities,

$$e^{i\pi} = -1 \text{ and } e^{2\pi i} = 1.$$

24.6 Liouville's Theorem

The following theorem pertains to functions which are analytic on all of $\mathbb{C}$, "entire" functions.

Definition 24.25 *A function, $f : \mathbb{C} \to \mathbb{C}$ or more generally, $f : \mathbb{C} \to X$ is entire means it is analytic on $\mathbb{C}$.*

Theorem 24.26 *(Liouville's theorem) If f is a bounded entire function having values in X, then f is a constant.*

Proof: Since f is entire, pick any $z \in \mathbb{C}$ and write

$$f'(z) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{(w-z)^2} dw$$

where $\gamma_R(t) = z + Re^{it}$ for $t \in [0, 2\pi]$. Therefore,

$$||f'(z)|| \leq C \frac{1}{R}$$

where C is some constant depending on the assumed bound on f. Since R is arbitrary, let $R \to \infty$ to obtain $f'(z) = 0$ for any $z \in \mathbb{C}$. It follows from this that f is constant for if z_j $j = 1, 2$ are two complex numbers, let $h(t) = f(z_1 + t(z_2 - z_1))$ for $t \in [0, 1]$. Then $h'(t) = f'(z_1 + t(z_2 - z_1))(z_2 - z_1) = 0$. By Lemmas 24.8 - 24.10 h is a constant on $[0, 1]$ which implies $f(z_1) = f(z_2)$.

With Liouville's theorem it becomes possible to give an easy proof of the fundamental theorem of algebra. It is ironic that all the best proofs of this theorem in algebra come from the subjects of analysis or topology. Out of all the proofs that have been given of this very important theorem, the following one based on Liouville's theorem is the easiest.

Theorem 24.27 *(Fundamental theorem of Algebra) Let*

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial where $n \geq 1$ *and each coefficient is a complex number. Then there exists* $z_0 \in \mathbb{C}$ *such that* $p(z_0) = 0$.

Proof: Suppose not. Then $p(z)^{-1}$ is an entire function. Also

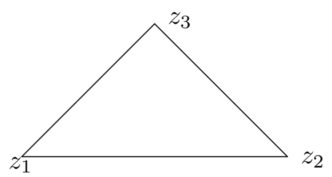
$$|p(z)| \geq |z|^n - \left(|a_{n-1}|\,|z|^{n-1} + \cdots + |a_1|\,|z| + |a_0|\right)$$

and so $\lim_{|z|\to\infty} |p(z)| = \infty$ which implies $\lim_{|z|\to\infty} \left|p(z)^{-1}\right| = 0$. It follows that, since $p(z)^{-1}$ is bounded for z in any bounded set, we must have that $p(z)^{-1}$ is a bounded entire function. But then it must be constant. However since $p(z)^{-1} \to 0$ as $|z| \to \infty$, this constant can only be 0. However, $\frac{1}{p(z)}$ is never equal to zero. This proves the theorem.

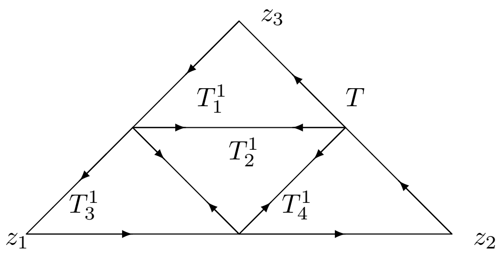
24.7 The General Cauchy Integral Formula

24.7.1 The Cauchy Goursat Theorem

This section gives a fundamental theorem which is essential to the development which follows and is closely related to the question of when a function has a primitive. First of all, if you have two points in $\mathbb{C}$, z_1 and z_2, you can consider $\gamma(t) \equiv z_1 + t(z_2 - z_1)$ for $t \in [0,1]$ to obtain a continuous bounded variation curve from z_1 to z_2. More generally, if $z_1, \cdots, z_m$ are points in $\mathbb{C}$ you can obtain a continuous bounded variation curve from z_1 to z_m which consists of first going from z_1 to z_2 and then from z_2 to z_3 and so on, till in the end one goes from z_{m-1} to z_m. We denote this piecewise linear curve as $\gamma(z_1, \cdots, z_m)$. Now let T be a triangle with vertices z_1, z_2 and z_3 encountered in the counter clockwise direction as shown.



Denote by $\int_{\partial T} f(z)\,dz$, the expression, $\int_{\gamma(z_1,z_2,z_3,z_1)} f(z)\,dz$. Consider the following picture.



By Lemma 23.11

$$\int_{\partial T} f(z)\,dz = \sum_{k=1}^{4} \int_{\partial T_k^1} f(z)\,dz. \tag{24.20}$$

On the "inside lines" the integrals cancel as claimed in Lemma 23.11 because there are two integrals going in opposite directions for each of these inside lines.

Theorem 24.28 *(Cauchy Goursat) Let $f : \Omega \to X$ have the property that $f'(z)$ exists for all $z \in \Omega$ and let T be a triangle contained in Ω. Then*

$$\int_{\partial T} f(w)\,dw = 0.$$

Proof: Suppose not. Then

$$\left|\left|\int_{\partial T} f(w)\,dw\right|\right| = \alpha \neq 0.$$

From 24.20 it follows

$$\alpha \leq \sum_{k=1}^{4} \left|\left|\int_{\partial T_k^1} f(w)\,dw\right|\right|$$

and so for at least one of these T_k^1, denoted from now on as T_1,

$$\left|\left|\int_{\partial T_1} f(w)\,dw\right|\right| \geq \frac{\alpha}{4}.$$

Now let T_1 play the same role as T, subdivide as in the above picture, and obtain T_2 such that

$$\left|\left|\int_{\partial T_2} f(w)\,dw\right|\right| \geq \frac{\alpha}{4^2}.$$

Continue in this way, obtaining a sequence of triangles,

$$T_k \supseteq T_{k+1}, \operatorname{diam}(T_k) \leq \operatorname{diam}(T)\,2^{-k},$$

and

$$\left|\left|\int_{\partial T_k} f(w)\,dw\right|\right| \geq \frac{\alpha}{4^k}.$$

Then let $z \in \cap_{k=1}^{\infty} T_k$ and note that by assumption, $f'(z)$ exists. Therefore, for all k large enough,

$$\int_{\partial T_k} f(w)\, dw = \int_{\partial T_k} f(z) + f'(z)(w-z) + g(w)\, dw$$

where $||g(w)|| < \varepsilon |w-z|$. Now observe that $w \to f(z) + f'(z)(w-z)$ has a primitive, namely,

$$F(w) = f(z) w + f'(z)(w-z)^2/2.$$

Therefore, by Corollary 23.14.

$$\int_{\partial T_k} f(w)\, dw = \int_{\partial T_k} g(w)\, dw.$$

From the definition, of the integral,

$$\begin{aligned} \frac{\alpha}{4^k} &\leq \left|\left|\int_{\partial T_k} g(w)\, dw\right|\right| \leq \varepsilon \operatorname{diam}(T_k)(\text{length of } \partial T_k) \\ &\leq \varepsilon 2^{-k} (\text{length of } T) \operatorname{diam}(T) 2^{-k}, \end{aligned}$$

and so

$$\alpha \leq \varepsilon (\text{length of } T) \operatorname{diam}(T).$$

Since ε is arbitrary, this shows $\alpha = 0$, a contradiction. Thus $\int_{\partial T} f(w)\, dw = 0$ as claimed.

This fundamental result yields the following important theorem.

Theorem 24.29 *(Morera[1]) Let Ω be an open set and let $f'(z)$ exist for all $z \in \Omega$. Let $D \equiv \overline{B(z_0, r)} \subseteq \Omega$. Then there exists $\varepsilon > 0$ such that f has a primitive on $B(z_0, r+\varepsilon)$.*

Proof: Choose $\varepsilon > 0$ small enough that $B(z_0, r+\varepsilon) \subseteq \Omega$. Then for $w \in B(z_0, r+\varepsilon)$, define

$$F(w) \equiv \int_{\gamma(z_0, w)} f(u)\, du.$$

Then by the Cauchy Goursat theorem, and $w \in B(z_0, r+\varepsilon)$, it follows that for $|h|$ small enough,

$$\begin{aligned} \frac{F(w+h) - F(w)}{h} &= \frac{1}{h} \int_{\gamma(w, w+h)} f(u)\, du \\ = \frac{1}{h} \int_0^1 f(w+th)\, h dt &= \int_0^1 f(w+th)\, dt \end{aligned}$$

which converges to $f(w)$ due to the continuity of f at w. This proves the theorem.

The following is a slight generalization of the above theorem which is also referred to as Morera's theorem.

[1] Giancinto Morera 1856-1909. This theorem or one like it dates from around 1886

Corollary 24.30 *Let Ω be an open set and suppose that whenever*

$$\gamma(z_1, z_2, z_3, z_1)$$

is a closed curve bounding a triangle T, which is contained in Ω, and f is a continuous function defined on Ω, it follows that

$$\int_{\gamma(z_1,z_2,z_3,z_1)} f(z)\,dz = 0,$$

then f is analytic on Ω.

Proof: As in the proof of Morera's theorem, let $\overline{B(z_0, r)} \subseteq \Omega$ and use the given condition to construct a primitive, F for f on $B(z_0, r)$. Then F is analytic and so by Theorem 24.16, it follows that F and hence f have infinitely many derivatives, implying that f is analytic on $B(z_0, r)$. Since z_0 is arbitrary, this shows f is analytic on Ω.

24.7.2 A Redundant Assumption

Earlier in the definition of analytic, it was assumed the derivative is continuous. This assumption is **redundant.**

Theorem 24.31 *Let Ω be an open set in $\mathbb{C}$ and suppose $f : \Omega \to X$ has the property that $f'(z)$ exists for each $z \in \Omega$. Then f is analytic on Ω.*

Proof: Let $z_0 \in \Omega$ and let $B(z_0, r) \subseteq \Omega$. By Morera's theorem f has a primitive, F on $B(z_0, r)$. It follows that F is analytic because it has a derivative, f, and this derivative is continuous. Therefore, by Theorem 24.16 F has infinitely many derivatives on $B(z_0, r)$ implying that f also has infinitely many derivatives on $B(z_0, r)$. Thus f is analytic as claimed.

It follows a function is analytic on an open set, Ω if and only if $f'(z)$ exists for $z \in \Omega$. This is because it was just shown the derivative, if it exists, is automatically continuous.

The same proof used to prove Theorem 24.29 implies the following corollary.

Corollary 24.32 *Let Ω be a convex open set and suppose that $f'(z)$ exists for all $z \in \Omega$. Then f has a primitive on Ω.*

Note that this implies that if Ω is a convex open set on which $f'(z)$ exists and if $\gamma : [a, b] \to \Omega$ is a closed, continuous curve having bounded variation, then letting F be a primitive of f Theorem 23.13 implies

$$\int_\gamma f(z)\,dz = F(\gamma(b)) - F(\gamma(a)) = 0.$$

Notice how different this is from the situation of a function of a real variable! It is possible for a function of a real variable to have a derivative everywhere and yet the derivative can be discontinuous. A simple example is the following.

$$f(x) \equiv \begin{cases} x^2 \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}.$$

Then $f'(x)$ exists for all $x \in \mathbb{R}$. Indeed, if $x \neq 0$, the derivative equals $2x \sin \frac{1}{x} - \cos \frac{1}{x}$ which has no limit as $x \to 0$. However, from the definition of the derivative of a function of one variable, $f'(0) = 0$.

24.7.3 Classification Of Isolated Singularities

First some notation.

Definition 24.33 *Let $B'(a,r) \equiv \{z \in \mathbb{C}$ such that $0 < |z-a| < r\}$. Thus this is the usual ball without the center. A function is said to have an isolated singularity at the point $a \in \mathbb{C}$ if f is analytic on $B'(a,r)$ for some $r > 0$.*

It turns out isolated singularities can be neatly classified into three types, removable singularities, poles, and essential singularities. The next theorem deals with the case of a removable singularity.

Definition 24.34 *An isolated singularity of f is said to be removable if there exists an analytic function, g analytic at a and near a such that $f = g$ at all points near a.*

Theorem 24.35 *Let $f : B'(a,r) \to X$ be analytic. Thus f has an isolated singularity at a. Suppose also that*

$$\lim_{z \to a} f(z)(z-a) = 0.$$

Then there exists a unique analytic function, $g : B(a,r) \to X$ such that $g = f$ on $B'(a,r)$. Thus the singularity at a is removable.

Proof: Let $h(z) \equiv (z-a)^2 f(z), h(a) \equiv 0$. Then h is analytic on $B(a,r)$ because it is easy to see that $h'(a) = 0$. It follows h is given by a power series,

$$h(z) = \sum_{k=2}^{\infty} a_k (z-a)^k$$

where $a_0 = a_1 = 0$ because of the observation above that $h'(a) = h(a) = 0$. It follows that for $|z-a| > 0$

$$f(z) = \sum_{k=2}^{\infty} a_k (z-a)^{k-2} \equiv g(z).$$

This proves the theorem.

What of the other case where the singularity is not removable? This situation is dealt with by the amazing Casorati Weierstrass theorem.

Theorem 24.36 *(Casorati Weierstrass) Let a be an isolated singularity and suppose for some $r > 0, f(B'(a,r))$ is not dense in $\mathbb{C}$. Then either a is a removable singularity or there exist finitely many $b_1, \cdots, b_M$ for some finite number, M such that for z near a,*

$$f(z) = g(z) + \sum_{k=1}^{M} \frac{b_k}{(z-a)^k} \tag{24.21}$$

where $g(z)$ is analytic near a.

Proof: Suppose $B(z_0, \delta)$ has no points of $f(B'(a,r))$. Such a ball must exist if $f(B'(a,r))$ is not dense. Then for $z \in B'(a,r), |f(z) - z_0| \geq \delta > 0$. It follows from Theorem 24.35 that $\frac{1}{f(z)-z_0}$ has a removable singularity at a. Hence, there exists h an analytic function such that for z near a,

$$h(z) = \frac{1}{f(z) - z_0}. \tag{24.22}$$

There are two cases. First suppose $h(a) = 0$. Then $\sum_{k=1}^{\infty} a_k (z-a)^k = \frac{1}{f(z)-z_0}$ for z near a. If all the $a_k = 0$, this would be a contradiction because then the left side would equal zero for z near a but the right side could not equal zero. Therefore, there is a first m such that $a_m \neq 0$. Hence there exists an analytic function, $k(z)$ which is not equal to zero in some ball, $B(a, \varepsilon)$ such that

$$k(z)(z-a)^m = \frac{1}{f(z) - z_0}.$$

Hence, taking both sides to the -1 power,

$$f(z) - z_0 = \frac{1}{(z-a)^m} \sum_{k=0}^{\infty} b_k (z-a)^k$$

and so 24.21 holds.

The other case is that $h(a) \neq 0$. In this case, raise both sides of 24.22 to the -1 power and obtain

$$f(z) - z_0 = h(z)^{-1},$$

a function analytic near a. Therefore, the singularity is removable. This proves the theorem.

This theorem is the basis for the following definition which classifies isolated singularities.

Definition 24.37 *Let a be an isolated singularity of a complex valued function, f. When 24.21 holds for z near a, then a is called a pole. The order of the pole in 24.21 is M. If for every $r > 0, f(B'(a,r))$ is dense in $\mathbb{C}$ then a is called an essential singularity.*

In terms of the above definition, isolated singularities are either removable, a pole, or essential. There are no other possibilities.

Theorem 24.38 *Suppose $f : \Omega \to \mathbb{C}$ has an isolated singularity at $a \in \Omega$. Then a is a pole if and only if*

$$\lim_{z \to a} d(f(z), \infty) = 0$$

in $\widehat{\mathbb{C}}$.

Proof: Suppose first f has a pole at a. Then by definition, $f(z) = g(z) + \sum_{k=1}^{M} \frac{b_k}{(z-a)^k}$ for z near a where g is analytic. Then

$$\begin{aligned} |f(z)| &\geq \frac{|b_M|}{|z-a|^M} - |g(z)| - \sum_{k=1}^{M-1} \frac{|b_k|}{|z-a|^k} \\ &= \frac{1}{|z-a|^M}\left(|b_M| - \left(|g(z)||z-a|^M + \sum_{k=1}^{M-1} |b_k||z-a|^{M-k}\right)\right). \end{aligned}$$

Now $\lim_{z\to a}\left(|g(z)||z-a|^M + \sum_{k=1}^{M-1}|b_k||z-a|^{M-k}\right) = 0$ and so the above inequality proves $\lim_{z\to a}|f(z)| = \infty$. Referring to the diagram on Page 628, you see this is the same as saying

$$\lim_{z\to a}|\theta f(z) - (0,0,2)| = \lim_{z\to a}|\theta f(z) - \theta(\infty)| = \lim_{z\to a} d(f(z), \infty) = 0$$

Conversely, suppose $\lim_{z\to a} d(f(z), \infty) = 0$. Then from the diagram on Page 628, it follows $\lim_{z\to a}|f(z)| = \infty$ and in particular, a cannot be either removable or an essential singularity by the Casorati Weierstrass theorem, Theorem 24.36. The only case remaining is that a is a pole. This proves the theorem.

Definition 24.39 *Let $f : \Omega \to \mathbb{C}$ where Ω is an open subset of $\mathbb{C}$. Then f is called meromorphic if all singularities are isolated and are either poles or removable and this set of singularities has no limit point. It is convenient to regard meromorphic functions as having values in $\widehat{\mathbb{C}}$ where if a is a pole, $f(a) \equiv \infty$. From now on, this will be assumed when a meromorphic function is being considered.*

The usefulness of the above convention about $f(a) \equiv \infty$ at a pole is made clear in the following theorem.

Theorem 24.40 *Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \to \widehat{\mathbb{C}}$ be meromorphic. Then f is continuous with respect to the metric, d on $\widehat{\mathbb{C}}$.*

Proof: Let $z_n \to z$ where $z \in \Omega$. Then if z is a pole, it follows from Theorem 24.38 that

$$d(f(z_n), \infty) \equiv d(f(z_n), f(z)) \to 0.$$

If z is not a pole, then $f(z_n) \to f(z)$ in $\mathbb{C}$ which implies $|\theta(f(z_n)) - \theta(f(z))| = d(f(z_n), f(z)) \to 0$. Recall that θ is continuous on $\mathbb{C}$.

24.7.4 The Cauchy Integral Formula

This section presents the general version of the Cauchy integral formula valid for arbitrary closed rectifiable curves. The key idea in this development is the notion of the winding number. This is the number also called the index, defined in the following theorem. This winding number, along with the earlier results, especially Liouville's theorem, yields an extremely general Cauchy integral formula.

Definition 24.41 *Let $\gamma : [a,b] \to \mathbb{C}$ and suppose $z \notin \gamma^*$. The winding number, $n(\gamma, z)$ is defined by*

$$n(\gamma, z) \equiv \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z}.$$

The main interest is in the case where γ is closed curve. However, the same notation will be used for any such curve.

Theorem 24.42 *Let $\gamma : [a,b] \to \mathbb{C}$ be continuous and have bounded variation with $\gamma(a) = \gamma(b)$. Also suppose that $z \notin \gamma^*$. Define*

$$n(\gamma, z) \equiv \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z}. \tag{24.23}$$

Then $n(\gamma, \cdot)$ is continuous and integer valued. Furthermore, there exists a sequence, $\eta_k : [a,b] \to \mathbb{C}$ such that η_k is $C^1([a,b])$,

$$||\eta_k - \gamma|| < \frac{1}{k}, \eta_k(a) = \eta_k(b) = \gamma(a) = \gamma(b),$$

and $n(\eta_k, z) = n(\gamma, z)$ for all k large enough. Also $n(\gamma, \cdot)$ is constant on every connected component of $\mathbb{C} \setminus \gamma^*$ and equals zero on the unbounded component of $\mathbb{C} \setminus \gamma^*$.

Proof: First consider the assertion about continuity.

$$\begin{aligned} |n(\gamma, z) - n(\gamma, z_1)| &\leq C \left| \int_\gamma \left(\frac{1}{w - z} - \frac{1}{w - z_1} \right) dw \right| \\ &\leq \widetilde{C} \,(\text{Length of } \gamma)\, |z_1 - z| \end{aligned}$$

whenever z_1 is close enough to z. This proves the continuity assertion. Note this did not depend on γ being closed.

Next it is shown that for a closed curve the winding number equals an integer. To do so, use Theorem 23.12 to obtain η_k, a function in $C^1([a,b])$ such that $z \notin \eta_k([a,b])$ for all k large enough, $\eta_k(x) = \gamma(x)$ for $x = a, b$, and

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z} - \frac{1}{2\pi i} \int_{\eta_k} \frac{dw}{w - z} \right| < \frac{1}{k}, \; ||\eta_k - \gamma|| < \frac{1}{k}.$$

It is shown that each of $\frac{1}{2\pi i} \int_{\eta_k} \frac{dw}{w-z}$ is an integer. To simplify the notation, write η instead of η_k.

$$\int_\eta \frac{dw}{w - z} = \int_a^b \frac{\eta'(s)\, ds}{\eta(s) - z}.$$

Define
$$g(t) \equiv \int_a^t \frac{\eta'(s)\,ds}{\eta(s)-z}. \tag{24.24}$$
Then
$$\begin{aligned}\left(e^{-g(t)}(\eta(t)-z)\right)' &= e^{-g(t)}\eta'(t) - e^{-g(t)}g'(t)(\eta(t)-z)\\ &= e^{-g(t)}\eta'(t) - e^{-g(t)}\eta'(t) = 0.\end{aligned}$$
It follows that $e^{-g(t)}(\eta(t)-z)$ equals a constant. In particular, using the fact that $\eta(a)=\eta(b)$,
$$e^{-g(b)}(\eta(b)-z) = e^{-g(a)}(\eta(a)-z) = (\eta(a)-z) = (\eta(b)-z)$$
and so $e^{-g(b)}=1$. This happens if and only if $-g(b)=2m\pi i$ for some integer m. Therefore, 24.24 implies
$$2m\pi i = \int_a^b \frac{\eta'(s)\,ds}{\eta(s)-z} = \int_\eta \frac{dw}{w-z}.$$
Therefore, $\frac{1}{2\pi i}\int_{\eta_k}\frac{dw}{w-z}$ is a sequence of integers converging to $\frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z} \equiv n(\gamma,z)$ and so $n(\gamma,z)$ must also be an integer and $n(\eta_k,z)=n(\gamma,z)$ for all k large enough.

Since $n(\gamma,\cdot)$ is continuous and integer valued, it follows from Corollary 6.67 on Page 155 that it must be constant on every connected component of $\mathbb{C}\setminus\gamma^*$. It is clear that $n(\gamma,z)$ equals zero on the unbounded component because from the formula,
$$\lim_{z\to\infty}|n(\gamma,z)| \le \lim_{z\to\infty} V(\gamma,[a,b])\left(\frac{1}{|z|-c}\right)$$
where $c \ge \max\{|w| : w\in\gamma^*\}$ .This proves the theorem.

Corollary 24.43 *Suppose $\gamma:[a,b]\to\mathbb{C}$ is a continuous bounded variation curve and $n(\gamma,z)$ is an integer where $z\notin\gamma^*$. Then $\gamma(a)=\gamma(b)$. Also $z\to n(\gamma,z)$ for $z\notin\gamma^*$ is continuous.*

Proof: Letting η be a C^1 curve for which $\eta(a)=\gamma(a)$ and $\eta(b)=\gamma(b)$ and which is close enough to γ that $n(\eta,z)=n(\gamma,z)$, the argument is similar to the above. Let
$$g(t) \equiv \int_a^t \frac{\eta'(s)\,ds}{\eta(s)-z}. \tag{24.25}$$
Then
$$\begin{aligned}\left(e^{-g(t)}(\eta(t)-z)\right)' &= e^{-g(t)}\eta'(t) - e^{-g(t)}g'(t)(\eta(t)-z)\\ &= e^{-g(t)}\eta'(t) - e^{-g(t)}\eta'(t) = 0.\end{aligned}$$
Hence
$$e^{-g(t)}(\eta(t)-z) = c \neq 0. \tag{24.26}$$

By assumption

$$g(b) = \int_{\eta} \frac{1}{w-z} dw = 2\pi i m$$

for some integer, m. Therefore, from 24.26

$$1 = e^{2\pi m i} = \frac{\eta(b) - z}{c}.$$

Thus $c = \eta(b) - z$ and letting $t = a$ in 24.26,

$$1 = \frac{\eta(a) - z}{\eta(b) - z}$$

which shows $\eta(a) = \eta(b)$. This proves the corollary since the assertion about continuity was already observed.

It is a good idea to consider a simple case to get an idea of what the winding number is measuring. To do so, consider $\gamma : [a, b] \to \mathbb{C}$ such that γ is continuous, closed and bounded variation. Suppose also that γ is one to one on (a, b). Such a curve is called a simple closed curve. It can be shown that such a simple closed curve divides the plane into exactly two components, an "inside" bounded component and an "outside" unbounded component. This is called the Jordan Curve theorem or the Jordan separation theorem. This is a difficult theorem which requires some very hard topology such as homology theory or degree theory. It won't be used here beyond making reference to it. For now, it suffices to simply assume that γ is such that this result holds. This will usually be obvious anyway. Also suppose that it is possible to change the parameter to be in $[0, 2\pi]$, in such a way that $\gamma(t) + \lambda\left(z + re^{it} - \gamma(t)\right) - z \neq 0$ for all $t \in [0, 2\pi]$ and $\lambda \in [0, 1]$. (As t goes from 0 to 2π the point $\gamma(t)$ traces the curve $\gamma([0, 2\pi])$ in the counter clockwise direction.) Suppose $z \in D$, the inside of the simple closed curve and consider the curve $\delta(t) = z + re^{it}$ for $t \in [0, 2\pi]$ where r is chosen small enough that $\overline{B(z, r)} \subseteq D$. Then it happens that $n(\delta, z) = n(\gamma, z)$.

Proposition 24.44 *Under the above conditions,*

$$n(\delta, z) = n(\gamma, z)$$

and $n(\delta, z) = 1$.

Proof: By changing the parameter, assume that $[a, b] = [0, 2\pi]$. From Theorem 24.42 it suffices to assume also that γ is C^1. Define $h_\lambda(t) \equiv \gamma(t) + \lambda\left(z + re^{it} - \gamma(t)\right)$ for $\lambda \in [0, 1]$. (This function is called a homotopy of the curves γ and δ.) Note that for each $\lambda \in [0, 1]$, $t \to h_\lambda(t)$ is a closed C^1 curve. Also,

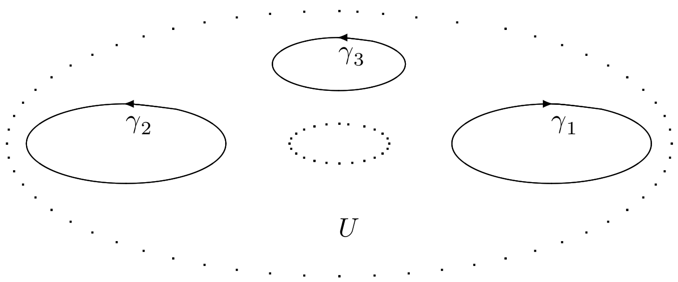
$$\frac{1}{2\pi i} \int_{h_\lambda} \frac{1}{w-z} dw = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma'(t) + \lambda\left(rie^{it} - \gamma'(t)\right)}{\gamma(t) + \lambda\left(z + re^{it} - \gamma(t)\right) - z} dt.$$

This number is an integer and it is routine to verify that it is a continuous function of λ. When $\lambda = 0$ it equals $n(\gamma, z)$ and when $\lambda = 1$ it equals $n(\delta, z)$. Therefore, $n(\delta, z) = n(\gamma, z)$. It only remains to compute $n(\delta, z)$.

$$n(\delta, z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{rie^{it}}{re^{it}} dt = 1.$$

This proves the proposition.

Now if γ was not one to one but caused the point, $\gamma(t)$ to travel around γ^* twice, you could modify the above argument to have the parameter interval, $[0, 4\pi]$ and still find $n(\delta, z) = n(\gamma, z)$ only this time, $n(\delta, z) = 2$. Thus the winding number is just what its name suggests. It measures the number of times the curve winds around the point. One might ask why bother with the winding number if this is all it does. The reason is that the notion of counting the number of times a curve winds around a point is rather vague. The winding number is precise. It is also the natural thing to consider in the general Cauchy integral formula presented below. Consider a situation typified by the following picture in which Ω is the open set between the dotted curves and γ_j are closed rectifiable curves in Ω.



The following theorem is the general Cauchy integral formula.

Definition 24.45 *Let $\{\gamma_k\}_{k=1}^n$ be continuous oriented curves having bounded variation. Then this is called a cycle if whenever, $z \notin \cup_{k=1}^n \gamma_k^*$, $\sum_{k=1}^n n(\gamma_k, z)$ is an integer.*

By Theorem 24.42 if each γ_k is a closed curve, then $\{\gamma_k\}_{k=1}^n$ is a cycle.

Theorem 24.46 *Let Ω be an open subset of the plane and let $f : \Omega \to X$ be analytic. If $\gamma_k : [a_k, b_k] \to \Omega$, $k = 1, \cdots, m$ are continuous curves having bounded variation such that for all $z \notin \cup_{k=1}^m \gamma_k([a_k, b_k])$*

$$\sum_{k=1}^m n(\gamma_k, z) \text{ equals an integer}$$

and for all $z \notin \Omega$,

$$\sum_{k=1}^m n(\gamma_k, z) = 0.$$

Then for all $z \in \Omega \setminus \cup_{k=1}^{m} \gamma_k ([a_k, b_k])$,

$$f(z) \sum_{k=1}^{m} n(\gamma_k, z) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} dw.$$

Proof: Let ϕ be defined on $\Omega \times \Omega$ by

$$\phi(z, w) \equiv \begin{cases} \frac{f(w)-f(z)}{w-z} \text{ if } w \neq z \\ f'(z) \text{ if } w = z \end{cases}.$$

Then ϕ is analytic as a function of both z and w and is continuous in $\Omega \times \Omega$. This is easily seen using Theorem 24.35. Consider the case of $w \to \phi(z, w)$.

$$\lim_{w \to z} (w - z)(\phi(z, w) - \phi(z, z)) = \lim_{w \to z} \left(\frac{f(w) - f(z)}{w - z} - f'(z) \right) = 0.$$

Thus $w \to \phi(z, w)$ has a removable singularity at z. The case of $z \to \phi(z, w)$ is similar.

Define

$$h(z) \equiv \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \phi(z, w)\, dw.$$

Is h is analytic on Ω? To show this is the case, verify

$$\int_{\partial T} h(z)\, dz = 0$$

for every triangle, T, contained in Ω and apply Corollary 24.30. To do this, use Theorem 23.12 to obtain for each k, a sequence of functions, $\eta_{kn} \in C^1([a_k, b_k])$ such that

$$\eta_{kn}(x) = \gamma_k(x) \text{ for } x \in \{a_k, b_k\}$$

and

$$\eta_{kn}([a_k, b_k]) \subseteq \Omega, \ \ ||\eta_{kn} - \gamma_k|| < \frac{1}{n},$$

$$\left|\left| \int_{\eta_{kn}} \phi(z, w)\, dw - \int_{\gamma_k} \phi(z, w)\, dw \right|\right| < \frac{1}{n}, \tag{24.27}$$

for all $z \in T$. Then applying Fubini's theorem,

$$\int_{\partial T} \int_{\eta_{kn}} \phi(z, w)\, dw dz = \int_{\eta_{kn}} \int_{\partial T} \phi(z, w)\, dz dw = 0$$

because ϕ is given to be analytic. By 24.27,

$$\int_{\partial T} \int_{\gamma_k} \phi(z, w)\, dw dz = \lim_{n \to \infty} \int_{\partial T} \int_{\eta_{kn}} \phi(z, w)\, dw dz = 0$$

and so h is analytic on Ω as claimed.

Now let H denote the set,

$$H \equiv \left\{ z \in \mathbb{C} \setminus \cup_{k=1}^{m} \gamma_k \left([a_k, b_k] \right) : \sum_{k=1}^{m} n \left(\gamma_k, z \right) = 0 \right\}.$$

H is an open set because $z \to \sum_{k=1}^{m} n(\gamma_k, z)$ is integer valued by assumption and continuous. Define

$$g(z) \equiv \begin{cases} h(z) \text{ if } z \in \Omega \\ \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{w-z} dw \text{ if } z \in H \end{cases}. \tag{24.28}$$

Why is $g(z)$ well defined? For $z \in \Omega \cap H$, $z \notin \cup_{k=1}^{m} \gamma_k ([a_k, b_k])$ and so

$$\begin{aligned} g(z) &= \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \phi(z,w)\, dw = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w) - f(z)}{w-z} dw \\ &= \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(z)}{w-z} dw \\ &= \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{w-z} dw \end{aligned}$$

because $z \in H$. This shows $g(z)$ is well defined. Also, g is analytic on Ω because it equals h there. It is routine to verify that g is analytic on H also because of the second line of 24.28. By assumption, $\Omega^C \subseteq H$ because it is assumed that $\sum_k n(\gamma_k, z) = 0$ for $z \notin \Omega$ and so $\Omega \cup H = \mathbb{C}$ showing that g is an entire function.

Now note that $\sum_{k=1}^{m} n(\gamma_k, z) = 0$ for all z contained in the unbounded component of $\mathbb{C} \setminus \cup_{k=1}^{m} \gamma_k ([a_k, b_k])$ which component contains $B(0,r)^C$ for r large enough. It follows that for $|z| > r$, it must be the case that $z \in H$ and so for such z, the bottom description of $g(z)$ found in 24.28 is valid. Therefore, it follows

$$\lim_{|z| \to \infty} ||g(z)|| = 0$$

and so g is bounded and entire. By Liouville's theorem, g is a constant. Hence, from the above equation, the constant can only equal zero.

For $z \in \Omega \setminus \cup_{k=1}^{m} \gamma_k ([a_k, b_k])$,

$$0 = h(z) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \phi(z,w)\, dw = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w) - f(z)}{w-z} dw =$$

$$\frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} \frac{f(w)}{w-z} dw - f(z) \sum_{k=1}^{m} n(\gamma_k, z).$$

This proves the theorem.

Corollary 24.47 *Let Ω be an open set and let $\gamma_k : [a_k, b_k] \to \Omega$, $k = 1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^{m} n(\gamma_k, z) = 0$$

for all $z \notin \Omega$. Then if $f : \Omega \to \mathbb{C}$ is analytic,

$$\sum_{k=1}^{m} \int_{\gamma_k} f(w)\, dw = 0.$$

Proof: This follows from Theorem 24.46 as follows. Let

$$g(w) = f(w)(w - z)$$

where $z \in \Omega \setminus \cup_{k=1}^{m} \gamma_k([a_k, b_k])$. Then by this theorem,

$$0 = 0 \sum_{k=1}^{m} n(\gamma_k, z) = g(z) \sum_{k=1}^{m} n(\gamma_k, z) =$$

$$\sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{g(w)}{w - z} dw = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} f(w)\, dw.$$

Another simple corollary to the above theorem is Cauchy's theorem for a simply connected region.

Definition 24.48 *An open set, $\Omega \subseteq \mathbb{C}$ is a region if it is open and connected. A region, Ω is simply connected if $\widehat{\mathbb{C}} \setminus \Omega$ is connected where $\widehat{\mathbb{C}}$ is the extended complex plane. In the future, the term simply connected open set will be an open set which is connected and $\widehat{\mathbb{C}} \setminus \Omega$ is connected .*

Corollary 24.49 *Let $\gamma : [a, b] \to \Omega$ be a continuous closed curve of bounded variation where Ω is a simply connected region in $\mathbb{C}$ and let $f : \Omega \to X$ be analytic. Then*

$$\int_{\gamma} f(w)\, dw = 0.$$

Proof: Let D denote the unbounded component of $\widehat{\mathbb{C}} \setminus \gamma^*$. Thus $\infty \in \widehat{\mathbb{C}} \setminus \gamma^*$. Then the connected set, $\widehat{\mathbb{C}} \setminus \Omega$ is contained in D since every point of $\widehat{\mathbb{C}} \setminus \Omega$ must be in some component of $\widehat{\mathbb{C}} \setminus \gamma^*$ and ∞ is contained in both $\widehat{\mathbb{C}} \setminus \Omega$ and D. Thus D must be the component that contains $\widehat{\mathbb{C}} \setminus \Omega$. It follows that $n(\gamma, \cdot)$ must be constant on $\widehat{\mathbb{C}} \setminus \Omega$, its value being its value on D. However, for $z \in D$,

$$n(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - z} dw$$

and so $\lim_{|z|\to\infty} n(\gamma, z) = 0$ showing $n(\gamma, z) = 0$ on D. Therefore this verifies the hypothesis of Theorem 24.46. Let $z \in \Omega \cap D$ and define

$$g(w) \equiv f(w)(w - z).$$

Thus g is analytic on Ω and by Theorem 24.46,

$$0 = n(z, \gamma) g(z) = \frac{1}{2\pi i} \int_\gamma \frac{g(w)}{w - z} dw = \frac{1}{2\pi i} \int_\gamma f(w) dw.$$

This proves the corollary.

The following is a very significant result which will be used later.

Corollary 24.50 *Suppose Ω is a simply connected open set and $f : \Omega \to X$ is analytic. Then f has a primitive, F, on Ω. Recall this means there exists F such that $F'(z) = f(z)$ for all $z \in \Omega$.*

Proof: Pick a point, $z_0 \in \Omega$ and let V denote those points, z of Ω for which there exists a curve, $\gamma : [a, b] \to \Omega$ such that γ is continuous, of bounded variation, $\gamma(a) = z_0$, and $\gamma(b) = z$. Then it is easy to verify that V is both open and closed in Ω and therefore, $V = \Omega$ because Ω is connected. Denote by $\gamma_{z_0,z}$ such a curve from z_0 to z and define

$$F(z) \equiv \int_{\gamma_{z_0,z}} f(w) dw.$$

Then F is well defined because if $\gamma_j, j = 1, 2$ are two such curves, it follows from Corollary 24.49 that

$$\int_{\gamma_1} f(w) dw + \int_{-\gamma_2} f(w) dw = 0,$$

implying that

$$\int_{\gamma_1} f(w) dw = \int_{\gamma_2} f(w) dw.$$

Now this function, F is a primitive because, thanks to Corollary 24.49

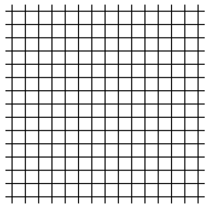
$$\begin{aligned} (F(z+h) - F(z)) h^{-1} &= \frac{1}{h} \int_{\gamma_{z,z+h}} f(w) dw \\ &= \frac{1}{h} \int_0^1 f(z + th) h dt \end{aligned}$$

and so, taking the limit as $h \to 0$, $F'(z) = f(z)$.

24.7.5 An Example Of A Cycle

The next theorem deals with the existence of a cycle with nice properties. Basically, you go around the compact subset of an open set with suitable contours while staying in the open set. The method involves the following simple concept.

Definition 24.51 *A tiling of $\mathbb{R}^2 = \mathbb{C}$ is the union of infinitely many equally spaced vertical and horizontal lines. You can think of the small squares which result as tiles. To tile the plane or $\mathbb{R}^2 = \mathbb{C}$ means to consider such a union of horizontal and vertical lines. It is like graph paper. See the picture below for a representation of part of a tiling of $\mathbb{C}$.*



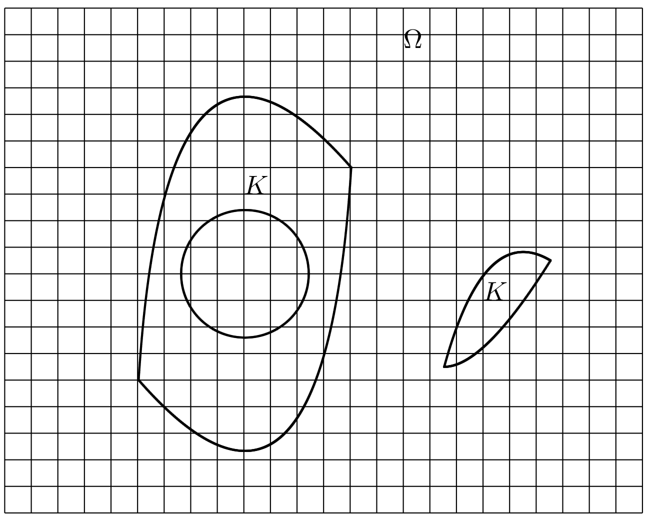
Theorem 24.52 *Let K be a compact subset of an open set, Ω. Then there exist continuous, closed, bounded variation oriented curves $\{\Gamma_j\}_{j=1}^m$ for which $\Gamma_j^* \cap K = \emptyset$ for each j, $\Gamma_j^* \subseteq \Omega$, and for all $p \in K$,*

$$\sum_{k=1}^{m} n(\Gamma_k, p) = 1.$$

while for all $z \notin \Omega$

$$\sum_{k=1}^{m} n(\Gamma_k, z) = 0.$$

Proof: Let $\delta = \text{dist}\left(K, \Omega^C\right)$. Since K is compact, $\delta > 0$. Now tile the plane with squares, each of which has diameter less than $\delta/2$.

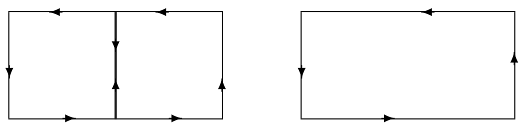


Let S denote the set of all the closed squares in this tiling which have nonempty intersection with K.Thus, all the squares of S are contained in Ω. First suppose p is a point of K which is in the interior of one of these squares in the tiling. Denote by ∂S_k the boundary of S_k one of the squares in S, oriented in the counter clockwise direction and S_m denote the square of S which contains the point, p in its interior. Let the edges of the square, S_j be $\left\{\gamma_k^j\right\}_{k=1}^4$. Thus a short computation shows $n\left(\partial S_m, p\right) = 1$ but $n\left(\partial S_j, p\right) = 0$ for all $j \neq m$. The reason for this is that for z in S_j, the values $\{z - p : z \in S_j\}$ lie in an open square, Q which is located at a positive distance from 0. Then $\widehat{\mathbb{C}} \setminus Q$ is connected and $1/\left(z-p\right)$ is analytic on Q. It follows from Corollary 24.50 that this function has a primitive on Q and so

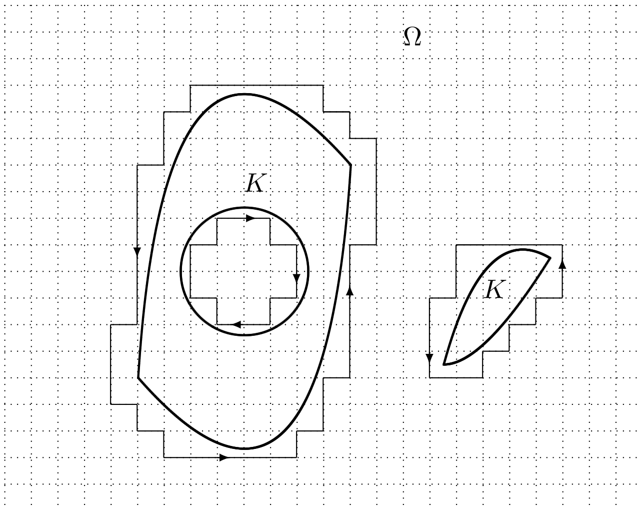
$$\int_{\partial S_j} \frac{1}{z-p} dz = 0.$$

Similarly, if $z \notin \Omega, n\left(\partial S_j, z\right) = 0$. On the other hand, a direct computation will verify that $n\left(p, \partial S_m\right) = 1$. Thus $1 = \sum_{j,k} n\left(p, \gamma_k^j\right) = \sum_{S_j \in S} n\left(p, \partial S_j\right)$ and if $z \notin \Omega$, $0 = \sum_{j,k} n\left(z, \gamma_k^j\right) = \sum_{S_j \in S} n\left(z, \partial S_j\right)$.

If γ_k^{j*} coincides with γ_l^{l*}, then the contour integrals taken over this edge are taken in opposite directions and so the edge the two squares have in common can be deleted without changing $\sum_{j,k} n\left(z, \gamma_k^j\right)$ for any z not on any of the lines in the tiling. For example, see the picture,



From the construction, if any of the γ_k^{j*} contains a point of K then this point is on one of the four edges of S_j and at this point, there is at least one edge of some S_l which also contains this point. As just discussed, this shared edge can be deleted without changing $\sum_{i,j} n\left(z, \gamma_k^j\right)$. Delete the edges of the S_k which intersect K but not the endpoints of these edges. That is, delete the open edges. When this is done, delete all isolated points. Let the resulting oriented curves be denoted by $\{\gamma_k\}_{k=1}^m$. Note that you might have $\gamma_k^* = \gamma_l^*$. The construction is illustrated in the following picture.



Then as explained above, $\sum_{k=1}^{m} n(p, \gamma_k) = 1$. It remains to prove the claim about the closed curves.

Each orientation on an edge corresponds to a direction of motion over that edge. Call such a motion over the edge a route. Initially, every vertex, (corner of a square in S) has the property there are the same number of routes to and from that vertex. When an open edge whose closure contains a point of K is deleted, every vertex either remains unchanged as to the number of routes to and from that vertex or it loses both a route away and a route to. Thus the property of having the same number of routes to and from each vertex is preserved by deleting these open edges.. The isolated points which result lose all routes to and from. It follows that upon removing the isolated points you can begin at any of the remaining vertices and follow the routes leading out from this and successive vertices according to orientation and eventually return to that end. Otherwise, there would be a vertex which would have only one route leading to it which does not happen. Now if you have used all the routes out of this vertex, pick another vertex and do the same process. Otherwise, pick an unused route out of the vertex and follow it to return. Continue this way till all routes are used exactly once, resulting in closed oriented curves, Γ_k. Then

$$\sum_k n(\Gamma_k, p) = \sum_j n(\gamma_k, p) = 1.$$

In case $p \in K$ is on some line of the tiling, it is not on any of the Γ_k because $\Gamma_k^* \cap K = \emptyset$ and so the continuity of $z \to n(\Gamma_k, z)$ yields the desired result in this case also. This proves the lemma.

24.8 Exercises

1. If U is simply connected, f is analytic on U and f has no zeros in U, show there exists an analytic function, F, defined on U such that $e^F = f$.

2. Let f be defined and analytic near the point $a \in \mathbb{C}$. Show that then $f(z) = \sum_{k=0}^{\infty} b_k (z-a)^k$ whenever $|z-a| < R$ where R is the distance between a and the nearest point where f fails to have a derivative. The number R, is called the radius of convergence and the power series is said to be expanded about a.

3. Find the radius of convergence of the function $\frac{1}{1+z^2}$ expanded about $a = 2$. Note there is nothing wrong with the function, $\frac{1}{1+x^2}$ when considered as a function of a real variable, x for any value of x. However, if you insist on using power series, you find there is a limitation on the values of x for which the power series converges due to the presence in the complex plane of a point, i, where the function fails to have a derivative.

4. Suppose f is analytic on all of $\mathbb{C}$ and satisfies $|f(z)| < A + B|z|^{1/2}$. Show f is constant.

5. What if you defined an open set, U to be simply connected if $\mathbb{C} \setminus U$ is connected. Would it amount to the same thing? **Hint:** Consider the outside of $B(0,1)$.

6. Let $\gamma(t) = e^{it} : t \in [0, 2\pi]$. Find $\int_\gamma \frac{1}{z^n} dz$ for $n = 1, 2, \cdots$.

7. Show $i \int_0^{2\pi} (2\cos\theta)^{2n} d\theta = \int_\gamma \left(z + \frac{1}{z}\right)^{2n} \left(\frac{1}{z}\right) dz$ where $\gamma(t) = e^{it} : t \in [0, 2\pi]$. Then evaluate this integral using the binomial theorem and the previous problem.

8. Suppose that for some constants $a, b \neq 0$, $a, b \in \mathbb{R}$, $f(z + ib) = f(z)$ for all $z \in \mathbb{C}$ and $f(z + a) = f(z)$ for all $z \in \mathbb{C}$. If f is analytic, show that f must be constant. Can you generalize this? **Hint:** This uses Liouville's theorem.

9. Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic for $z \in U$, an open set. Let $g(z) = u^*(x, y) + iv^*(x, y)$ where
$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} = Q \begin{pmatrix} u \\ v \end{pmatrix}$$
where Q is a unitary matrix. That is $QQ^* = Q^*Q = I$. When will g be analytic?

10. Suppose f is analytic on an open set, U, except for $\gamma^* \subset U$ where γ is a one to one continuous function having bounded variation, but it is known that f is continuous on γ^*. Show that in fact f is analytic on γ^* also. **Hint:** Pick a point on γ^*, say $\gamma(t_0)$ and suppose for now that $t_0 \in (a, b)$. Pick $r > 0$ such that $B = B(\gamma(t_0), r) \subseteq U$. Then show there exists $t_1 < t_0$ and $t_2 > t_0$ such

that $\gamma([t_1,t_2]) \subseteq \overline{B}$ and $\gamma(t_i) \notin B$. Thus $\gamma([t_1,t_2])$ is a path across B going through the center of B which divides B into two open sets, B_1, and B_2 along with γ^*. Let the boundary of B_k consist of $\gamma([t_1,t_2])$ and a circular arc, C_k. Now letting $z \in B_k$, the line integral of $\frac{f(w)}{w-z}$ over γ^* in two different directions cancels. Therefore, if $z \in B_k$, you can argue that $f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw$. By continuity, this continues to hold for $z \in \gamma((t_1,t_2))$. Therefore, f must be analytic on $\gamma((t_1,t_1))$ also. This shows that f must be analytic on $\gamma((a,b))$. To get the endpoints, simply extend γ to have the same properties but defined on $[a-\varepsilon, b+\varepsilon]$ and repeat the above argument or else do this at the beginning and note that you get $[a,b] \subseteq (a-\varepsilon, b+\varepsilon)$.

11. Let U be an open set contained in the upper half plane and suppose that there are finitely many line segments on the x axis which are contained in the boundary of U. Now suppose that f is defined, real, and continuous on these line segments and is defined and analytic on U. Now let $\widetilde{U}$ denote the reflection of U across the x axis. Show that it is possible to extend f to a function, g defined on all of

$$W \equiv \widetilde{U} \cup U \cup \{\text{the line segments mentioned earlier}\}$$

such that g is analytic in W. **Hint:** For $z \in \widetilde{U}$, the reflection of U across the x axis, let $g(z) \equiv \overline{f(\overline{z})}$. Show that g is analytic on $\widetilde{U} \cup U$ and continuous on the line segments. Then use Problem 10 or Morera's theorem to argue that g is analytic on the line segments also. The result of this problem is know as the Schwarz reflection principle.

12. Show that rotations and translations of analytic functions yield analytic functions and use this observation to generalize the Schwarz reflection principle to situations in which the line segments are part of a line which is not the x axis. Thus, give a version which involves reflection about an arbitrary line.

The Open Mapping Theorem

25.1 A Local Representation

The open mapping theorem, is an even more surprising result than the theorem about the zeros of an analytic function. The following proof of this important theorem uses an interesting local representation of the analytic function.

Theorem 25.1 *(Open mapping theorem) Let Ω be a region in $\mathbb{C}$ and suppose $f : \Omega \to \mathbb{C}$ is analytic. Then $f(\Omega)$ is either a point or a region. In the case where $f(\Omega)$ is a region, it follows that for each $z_0 \in \Omega$, there exists an open set, V containing z_0 and $m \in \mathbb{N}$ such that for all $z \in V$,*

$$f(z) = f(z_0) + \phi(z)^m \tag{25.1}$$

where $\phi : V \to B(0,\delta)$ is one to one, analytic and onto, $\phi(z_0) = 0$, $\phi'(z) \neq 0$ on V and ϕ^{-1} analytic on $B(0,\delta)$. If f is one to one then $m = 1$ for each z_0 and $f^{-1} : f(\Omega) \to \Omega$ is analytic.

Proof: Suppose $f(\Omega)$ is not a point. Then if $z_0 \in \Omega$ it follows there exists $r > 0$ such that $f(z) \neq f(z_0)$ for all $z \in B(z_0, r) \setminus \{z_0\}$. Otherwise, z_0 would be a limit point of the set,

$$\{z \in \Omega : f(z) - f(z_0) = 0\}$$

which would imply from Theorem 24.23 that $f(z) = f(z_0)$ for all $z \in \Omega$. Therefore, making r smaller if necessary and using the power series of f,

$$f(z) = f(z_0) + (z - z_0)^m g(z) \ \left(\overset{?}{=} \left((z - z_0) g(z)^{1/m}\right)^m\right)$$

for all $z \in B(z_0, r)$, where $g(z) \neq 0$ on $B(z_0, r)$. As implied in the above formula, one wonders if you can take the m^{th} root of $g(z)$.

$\frac{g'}{g}$ is an analytic function on $B(z_0, r)$ and so by Corollary 24.32 it has a primitive on $B(z_0, r)$, h. Therefore by the product rule and the chain rule, $\left(ge^{-h}\right)' = 0$ and so there exists a constant, $C = e^{a+ib}$ such that on $B(z_0, r)$,

$$ge^{-h} = e^{a+ib}.$$

Therefore,

$$g(z) = e^{h(z)+a+ib}$$

and so, modifying h by adding in the constant, $a+ib$, $g(z) = e^{h(z)}$ where $h'(z) = \frac{g'(z)}{g(z)}$ on $B(z_0, r)$. Letting

$$\phi(z) = (z - z_0) e^{\frac{h(z)}{m}}$$

implies formula 25.1 is valid on $B(z_0, r)$. Now

$$\phi'(z_0) = e^{\frac{h(z_0)}{m}} \neq 0.$$

Shrinking r if necessary you can assume $\phi'(z) \neq 0$ on $B(z_0, r)$. Is there an open set, V contained in $B(z_0, r)$ such that ϕ maps V onto $B(0, \delta)$ for some $\delta > 0$?

Let $\phi(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. Consider the mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

where u, v are C^1 because ϕ is given to be analytic. The Jacobian of this map at $(x, y) \in B(z_0, r)$ is

$$\begin{vmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{vmatrix} = \begin{vmatrix} u_x(x, y) & -v_x(x, y) \\ v_x(x, y) & u_x(x, y) \end{vmatrix}$$

$$= u_x(x, y)^2 + v_x(x, y)^2 = \left|\phi'(z)\right|^2 \neq 0.$$

This follows from a use of the Cauchy Riemann equations. Also

$$\begin{pmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore, by the inverse function theorem there exists an open set, V, containing z_0 and $\delta > 0$ such that $(u, v)^T$ maps V one to one onto $B(0, \delta)$. Thus ϕ is one to one onto $B(0, \delta)$ as claimed. Applying the same argument to other points, z of V and using the fact that $\phi'(z) \neq 0$ at these points, it follows ϕ maps open sets to open sets. In other words, ϕ^{-1} is continuous.

It also follows that ϕ^m maps V onto $B(0, \delta^m)$. Therefore, the formula 25.1 implies that f maps the open set, V, containing z_0 to an open set. This shows $f(\Omega)$ is an open set because z_0 was arbitrary. It is connected because f is continuous and Ω is connected. Thus $f(\Omega)$ is a region. It remains to verify that ϕ^{-1} is analytic on $B(0, \delta)$. Since ϕ^{-1} is continuous,

$$\lim_{\phi(z_1) \to \phi(z)} \frac{\phi^{-1}(\phi(z_1)) - \phi^{-1}(\phi(z))}{\phi(z_1) - \phi(z)} = \lim_{z_1 \to z} \frac{z_1 - z}{\phi(z_1) - \phi(z)} = \frac{1}{\phi'(z)}.$$

Therefore, ϕ^{-1} is analytic as claimed.

It only remains to verify the assertion about the case where f is one to one. If $m > 1$, then $e^{\frac{2\pi i}{m}} \neq 1$ and so for $z_1 \in V$,

$$e^{\frac{2\pi i}{m}} \phi(z_1) \neq \phi(z_1). \tag{25.2}$$

But $e^{\frac{2\pi i}{m}} \phi(z_1) \in B(0,\delta)$ and so there exists $z_2 \neq z_1$(since ϕ is one to one) such that $\phi(z_2) = e^{\frac{2\pi i}{m}} \phi(z_1)$. But then

$$\phi(z_2)^m = \left(e^{\frac{2\pi i}{m}} \phi(z_1)\right)^m = \phi(z_1)^m$$

implying $f(z_2) = f(z_1)$ contradicting the assumption that f is one to one. Thus $m = 1$ and $f'(z) = \phi'(z) \neq 0$ on V. Since f maps open sets to open sets, it follows that f^{-1} is continuous and so

$$\begin{aligned}
\left(f^{-1}\right)'(f(z)) &= \lim_{f(z_1)\to f(z)} \frac{f^{-1}(f(z_1)) - f^{-1}(f(z))}{f(z_1) - f(z)} \\
&= \lim_{z_1\to z} \frac{z_1 - z}{f(z_1) - f(z)} = \frac{1}{f'(z)}.
\end{aligned}$$

This proves the theorem.

One does not have to look very far to find that this sort of thing does not hold for functions mapping $\mathbb{R}$ to $\mathbb{R}$. Take for example, the function $f(x) = x^2$. Then $f(\mathbb{R})$ is neither a point nor a region. In fact $f(\mathbb{R})$ fails to be open.

Corollary 25.2 *Suppose in the situation of Theorem 25.1 $m > 1$ for the local representation of f given in this theorem. Then there exists $\delta > 0$ such that if $w \in B(f(z_0), \delta) = f(V)$ for V an open set containing z_0, then $f^{-1}(w)$ consists of m distinct points in V. (f is m to one on V)*

Proof: Let $w \in B(f(z_0), \delta)$. Then $w = f(\widehat{z})$ where $\widehat{z} \in V$. Thus $f(\widehat{z}) = f(z_0) + \phi(\widehat{z})^m$. Consider the m distinct numbers, $\left\{e^{\frac{2k\pi i}{m}} \phi(\widehat{z})\right\}_{k=1}^m$. Then each of these numbers is in $B(0,\delta)$ and so since ϕ maps V one to one onto $B(0,\delta)$, there are m distinct numbers in V, $\{z_k\}_{k=1}^m$ such that $\phi(z_k) = e^{\frac{2k\pi i}{m}} \phi(\widehat{z})$. Then

$$\begin{aligned}
f(z_k) &= f(z_0) + \phi(z_k)^m = f(z_0) + \left(e^{\frac{2k\pi i}{m}} \phi(\widehat{z})\right)^m \\
&= f(z_0) + e^{2k\pi i} \phi(\widehat{z})^m = f(z_0) + \phi(\widehat{z})^m = f(\widehat{z}) = w
\end{aligned}$$

This proves the corollary.

25.2 Branches Of The Logarithm

The argument used in to prove the next theorem was used in the proof of the open mapping theorem. It is a very important result and deserves to be stated as a theorem.

Theorem 25.3 *Let Ω be a simply connected region and suppose $f : \Omega \to \mathbb{C}$ is analytic and nonzero on Ω. Then there exists an analytic function, g such that $e^{g(z)} = f(z)$ for all $z \in \Omega$.*

Proof: The function, f'/f is analytic on Ω and so by Corollary 24.50 there is a primitive for f'/f, denoted as g_1. Then

$$\left(e^{-g_1} f\right)' = -\frac{f'}{f} e^{-g_1} f + e^{-g_1} f' = 0$$

and so since Ω is connected, it follows $e^{-g_1} f$ equals a constant, e^{a+ib}. Therefore, $f(z) = e^{g_1(z)+a+ib}$. Define $g(z) \equiv g_1(z) + a + ib$.

The function, g in the above theorem is called a branch of the logarithm of f and is written as $\log(f(z))$.

Definition 25.4 *Let ρ be a ray starting at 0. Thus ρ is a straight line of infinite length extending in one direction with its initial point at 0.*

A special case of the above theorem is the following.

Theorem 25.5 *Let ρ be a ray starting at 0. Then there exists an analytic function, $L(z)$ defined on $\mathbb{C} \setminus \rho$ such that*

$$e^{L(z)} = z.$$

This function, L is called a branch of the logarithm. This branch of the logarithm satisfies the usual formula for logarithms, $L(zw) = L(z) + L(w)$ provided $zw \notin \rho$.

Proof: $\mathbb{C} \setminus \rho$ is a simply connected region because its complement with respect to $\widehat{\mathbb{C}}$ is connected. Furthermore, the function, $f(z) = z$ is not equal to zero on $\mathbb{C} \setminus \rho$. Therefore, by Theorem 25.3 there exists an analytic function $L(z)$ such that $e^{L(z)} = f(z) = z$. Now consider the problem of finding a description of $L(z)$. Each $z \in \mathbb{C} \setminus \rho$ can be written in a unique way in the form

$$z = |z| e^{i \arg_\theta(z)}$$

where $\arg_\theta(z)$ is the angle in $(\theta, \theta + 2\pi)$ associated with z. (You could of course have considered this to be the angle in $(\theta - 2\pi, \theta)$ associated with z or in infinitely many other open intervals of length 2π. The description of the log is not unique.) Then letting $L(z) = a + ib$

$$z = |z| e^{i \arg_\theta(z)} = e^{L(z)} = e^a e^{ib}$$

and so you can let $L(z) = \ln|z| + i \arg_\theta(z)$.

Does $L(z)$ satisfy the usual properties of the logarithm? That is, for $z, w \in \mathbb{C} \setminus \rho$, is $L(zw) = L(z) + L(w)$? This follows from the usual rules of exponents. You know $e^{z+w} = e^z e^w$. (You can verify this directly or you can reduce to the case where z, w are real. If z is a fixed real number, then the equation holds for all real w. Therefore, it must also hold for all complex w because the real line contains a limit point. Now for this fixed w, the equation holds for all z real. Therefore, by similar reasoning, it holds for all complex z.)

Now suppose $z, w \in \mathbb{C} \setminus \rho$ and $zw \notin \rho$. Then

$$e^{L(zw)} = zw,\ e^{L(z)+L(w)} = e^{L(z)}e^{L(w)} = zw$$

and so $L(zw) = L(z) + L(w)$ as claimed. This proves the theorem.

In the case where the ray is the negative real axis, it is called the principal branch of the logarithm. Thus $\arg(z)$ is a number between $-\pi$ and π.

Definition 25.6 *Let* log *denote the branch of the logarithm which corresponds to the ray for $\theta = \pi$. That is, the ray is the negative real axis. Sometimes this is called the principal branch of the logarithm.*

25.3 Maximum Modulus Theorem

Here is another very significant theorem known as the maximum modulus theorem which follows immediately from the open mapping theorem.

Theorem 25.7 *(maximum modulus theorem) Let Ω be a bounded region and let $f : \Omega \to \mathbb{C}$ be analytic and $f : \overline{\Omega} \to \mathbb{C}$ continuous. Then if $z \in \Omega$,*

$$|f(z)| \leq \max\{|f(w)| : w \in \partial\Omega\}. \tag{25.3}$$

If equality is achieved for any $z \in \Omega$, then f is a constant.

Proof: Suppose f is not a constant. Then $f(\Omega)$ is a region and so if $z \in \Omega$, there exists $r > 0$ such that $B(f(z), r) \subseteq f(\Omega)$. It follows there exists $z_1 \in \Omega$ with $|f(z_1)| > |f(z)|$. Hence $\max\left\{|f(w)| : w \in \overline{\Omega}\right\}$ is not achieved at any interior point of Ω. Therefore, the point at which the maximum is achieved must lie on the boundary of Ω and so

$$\max\{|f(w)| : w \in \partial\Omega\} = \max\left\{|f(w)| : w \in \overline{\Omega}\right\} > |f(z)|$$

for all $z \in \Omega$ or else f is a constant. This proves the theorem.

You can remove the assumption that Ω is bounded and give a slightly different version.

Theorem 25.8 *Let $f : \Omega \to \mathbb{C}$ be analytic on a region, Ω and suppose $\overline{B(a, r)} \subseteq \Omega$. Then*

$$|f(a)| \leq \max\left\{\left|f\left(a + re^{i\theta}\right)\right| : \theta \in [0, 2\pi]\right\}.$$

Equality occurs for some $r > 0$ and $a \in \Omega$ if and only if f is constant in Ω hence equality occurs for all such a, r.

Proof: The claimed inequality holds by Theorem 25.7. Suppose equality in the above is achieved for some $\overline{B(a,r)} \subseteq \Omega$. Then by Theorem 25.7 f is equal to a constant, w on $B(a,r)$. Therefore, the function, $f(\cdot) - w$ has a zero set which has a limit point in Ω and so by Theorem 24.23 $f(z) = w$ for all $z \in \Omega$.

Conversely, if f is constant, then the equality in the above inequality is achieved for all $\overline{B(a,r)} \subseteq \Omega$.

Next is yet another version of the maximum modulus principle which is in Conway [13]. Let Ω be an open set.

Definition 25.9 *Define $\partial_\infty \Omega$ to equal $\partial \Omega$ in the case where Ω is bounded and $\partial\Omega \cup \{\infty\}$ in the case where Ω is not bounded.*

Definition 25.10 *Let f be a complex valued function defined on a set $S \subseteq \mathbb{C}$ and let a be a limit point of S.*

$$\limsup_{z\to a} |f(z)| \equiv \lim_{r\to 0} \{\sup |f(w)| : w \in B'(a,r) \cap S\}.$$

The limit exists because $\{\sup |f(w)| : w \in B'(a,r) \cap S\}$ is decreasing in r. In case $a = \infty$,

$$\limsup_{z\to \infty} |f(z)| \equiv \lim_{r\to \infty} \{\sup |f(w)| : |w| > r, w \in S\}$$

Note that if $\limsup_{z\to a} |f(z)| \leq M$ and $\delta > 0$, then there exists $r > 0$ such that if $z \in B'(a,r) \cap S$, then $|f(z)| < M + \delta$. If $a = \infty$, there exists $r > 0$ such that if $|z| > r$ and $z \in S$, then $|f(z)| < M + \delta$.

Theorem 25.11 *Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \to \mathbb{C}$ be analytic. Suppose also that for every $a \in \partial_\infty \Omega$,*

$$\limsup_{z\to a} |f(z)| \leq M < \infty.$$

Then in fact $|f(z)| \leq M$ for all $z \in \Omega$.

Proof: Let $\delta > 0$ and let $H \equiv \{z \in \Omega : |f(z)| > M + \delta\}$. Suppose $H \neq \emptyset$. Then H is an open subset of Ω. I claim that H is actually bounded. If Ω is bounded, there is nothing to show so assume Ω is unbounded. Then the condition involving the lim sup implies there exists $r > 0$ such that if $|z| > r$ and $z \in \Omega$, then $|f(z)| \leq M + \delta/2$. It follows H is contained in $\overline{B(0,r)}$ and so it is bounded. Now consider the components of Ω. One of these components contains points from H. Let this component be denoted as V and let $H_V \equiv H \cap V$. Thus H_V is a bounded open subset of V. Let U be a component of H_V. First suppose $\overline{U} \subseteq V$. In this case, it follows that on $\partial U, |f(z)| = M + \delta$ and so by Theorem 25.7 $|f(z)| \leq M + \delta$ for all $z \in U$ contradicting the definition of H. Next suppose ∂U contains a point of $\partial V, a$. Then in this case, a violates the condition on lim sup. Either way you get a contradiction. Hence $H = \emptyset$ as claimed. Since $\delta > 0$ is arbitrary, this shows $|f(z)| \leq M$.

25.4 Extensions Of Maximum Modulus Theorem

25.4.1 Phragmên Lindelöf Theorem

This theorem is an extension of Theorem 25.11. It uses a growth condition near the extended boundary to conclude that f is bounded. I will present the version found in Conway [13]. It seems to be more of a method than an actual theorem. There are several versions of it.

Theorem 25.12 *Let Ω be a simply connected region in $\mathbb{C}$ and suppose f is analytic on Ω. Also suppose there exists a function, ϕ which is nonzero and uniformly bounded on Ω. Let M be a positive number. Now suppose $\partial_\infty \Omega = A \cup B$ such that for every $a \in A$, $\limsup_{z\to a} |f(z)| \leq M$ and for every $b \in B$, and $\eta > 0$, $\limsup_{z \to b} |f(z)| |\phi(z)|^\eta \leq M$. Then $|f(z)| \leq M$ for all $z \in \Omega$.*

Proof: By Theorem 25.3 there exists $\log(\phi(z))$ analytic on Ω. Now define $g(z) \equiv \exp(\eta \log(\phi(z)))$ so that $g(z) = \phi(z)^\eta$. Now also

$$|g(z)| = |\exp(\eta \log(\phi(z)))| = |\exp(\eta \ln |\phi(z)|)| = |\phi(z)|^\eta .$$

Let $m \geq |\phi(z)|$ for all $z \in \Omega$. Define $F(z) \equiv f(z) g(z) m^{-\eta}$. Thus F is analytic and for $b \in B$,

$$\limsup_{z \to b} |F(z)| = \limsup_{z \to b} |f(z)| |\phi(z)|^\eta m^{-\eta} \leq M m^{-\eta}$$

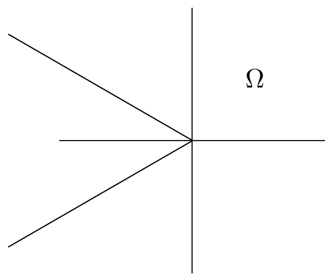
while for $a \in A$,

$$\limsup_{z \to a} |F(z)| \leq M.$$

Therefore, for $\alpha \in \partial_\infty \Omega$, $\limsup_{z \to \alpha} |F(z)| \leq \max(M, M\eta^{-\eta})$ and so by Theorem 25.11, $|f(z)| \leq \left(\frac{m^\eta}{|\phi(z)|^\eta}\right) \max(M, M\eta^{-\eta})$. Now let $\eta \to 0$ to obtain $|f(z)| \leq M$.

In applications, it is often the case that $B = \{\infty\}$.

Now here is an interesting case of this theorem. It involves a particular form for Ω, in this case $\Omega = \left\{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2a}\right\}$ where $a \geq \frac{1}{2}$.



Then $\partial \Omega$ equals the two slanted lines. Also on Ω you can define a logarithm, $\log(z) = \ln |z| + i \arg(z)$ where $\arg(z)$ is the angle associated with z between $-\pi$

and π. Therefore, if c is a real number you can define z^c for such z in the usual way:

$$\begin{aligned} z^c &\equiv \exp(c\log(z)) = \exp(c[\ln|z| + i\arg(z)]) \\ &= |z|^c \exp(ic\arg(z)) = |z|^c (\cos(c\arg(z)) + i\sin(c\arg(z))) . \end{aligned}$$

If $|c| < a$, then $|c\arg(z)| < \frac{\pi}{2}$ and so $\cos(c\arg(z)) > 0$. Therefore, for such c,

$$\begin{aligned} |\exp(-(z^c))| &= |\exp(-|z|^c (\cos(c\arg(z)) + i\sin(c\arg(z))))| \\ &= |\exp(-|z|^c (\cos(c\arg(z))))| \end{aligned}$$

which is bounded since $\cos(c\arg(z)) > 0$.

Corollary 25.13 *Let* $\Omega = \left\{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2a}\right\}$ *where* $a \geq \frac{1}{2}$ *and suppose* f *is analytic on* Ω *and satisfies* $\limsup_{z\to a} |f(z)| \leq M$ *on* $\partial\Omega$ *and suppose there are positive constants,* P, b *where* $b < a$ *and*

$$|f(z)| \leq P\exp\left(|z|^b\right)$$

for all $|z|$ *large enough. Then* $|f(z)| \leq M$ *for all* $z \in \Omega$.

Proof: Let $b < c < a$ and let $\phi(z) \equiv \exp(-(z^c))$. Then as discussed above, $\phi(z) \neq 0$ on Ω and $|\phi(z)|$ is bounded on Ω. Now

$$|\phi(z)|^\eta = |\exp(-|z|^c \eta(\cos(c\arg(z))))|$$

$$\limsup_{z\to\infty} |f(z)| |\phi(z)|^\eta \leq \limsup_{z\to\infty} \frac{P\exp\left(|z|^b\right)}{|\exp(|z|^c \eta(\cos(c\arg(z))))|} = 0 \leq M$$

and so by Theorem 25.12 $|f(z)| \leq M$.

The following is another interesting case. This case is presented in Rudin [45]

Corollary 25.14 *Let* Ω *be the open set consisting of* $\{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ *and suppose* f *is analytic on* Ω *, continuous on* $\overline{\Omega}$*, and bounded on* Ω*. Suppose also that* $f(z) \leq 1$ *on the two lines* $\operatorname{Re} z = a$ *and* $\operatorname{Re} z = b$*. Then* $|f(z)| \leq 1$ *for all* $z \in \Omega$.

Proof: This time let $\phi(z) = \frac{1}{1+z-a}$. Thus $|\phi(z)| \leq 1$ because $\operatorname{Re}(z-a) > 0$ and $\phi(z) \neq 0$ for all $z \in \Omega$. Also, $\limsup_{z\to\infty} |\phi(z)|^\eta = 0$ for every $\eta > 0$. Therefore, if a is a point of the sides of Ω, $\limsup_{z\to a} |f(z)| \leq 1$ while $\limsup_{z\to\infty} |f(z)| |\phi(z)|^\eta = 0 \leq 1$ and so by Theorem 25.12, $|f(z)| \leq 1$ on Ω.

This corollary yields an interesting conclusion.

Corollary 25.15 *Let* Ω *be the open set consisting of* $\{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ *and suppose* f *is analytic on* Ω *, continuous on* $\overline{\Omega}$*, and bounded on* Ω*. Define*

$$M(x) \equiv \sup\{|f(z)| : \operatorname{Re} z = x\}$$

Then for $x \in (a, b)$.

$$M(x) \leq M(a)^{\frac{b-x}{b-a}} M(b)^{\frac{x-a}{b-a}} .$$

Proof: Let $\varepsilon > 0$ and define

$$g(z) \equiv (M(a)+\varepsilon)^{\frac{b-z}{b-a}} (M(b)+\varepsilon)^{\frac{z-a}{b-a}}$$

where for $M > 0$ and $z \in \mathbb{C}$, $M^z \equiv \exp(z \ln(M))$. Thus $g \neq 0$ and so f/g is analytic on Ω and continuous on $\overline{\Omega}$. Also on the left side,

$$\left|\frac{f(a+iy)}{g(a+iy)}\right| = \left|\frac{f(a+iy)}{(M(a)+\varepsilon)^{\frac{b-a-iy}{b-a}}}\right| = \left|\frac{f(a+iy)}{(M(a)+\varepsilon)^{\frac{b-a}{b-a}}}\right| \leq 1$$

while on the right side a similar computation shows $\left|\frac{f}{g}\right| \leq 1$ also. Therefore, by Corollary 25.14 $|f/g| \leq 1$ on Ω. Therefore, letting $x+iy=z$,

$$|f(z)| \leq \left|(M(a)+\varepsilon)^{\frac{b-z}{b-a}} (M(b)+\varepsilon)^{\frac{z-a}{b-a}}\right| = \left|(M(a)+\varepsilon)^{\frac{b-x}{b-a}} (M(b)+\varepsilon)^{\frac{x-a}{b-a}}\right|$$

and so

$$M(x) \leq (M(a)+\varepsilon)^{\frac{b-x}{b-a}} (M(b)+\varepsilon)^{\frac{x-a}{b-a}}.$$

Since $\varepsilon > 0$ is arbitrary, it yields the conclusion of the corollary.

Another way of saying this is that $x \to \ln(M(x))$ is a convex function.

This corollary has an interesting application known as the Hadamard three circles theorem.

25.4.2 Hadamard Three Circles Theorem

Let $0 < R_1 < R_2$ and suppose f is analytic on $\{z \in \mathbb{C} : R_1 < |z| < R_2\}$. Then letting $R_1 < a < b < R_2$, note that $g(z) \equiv \exp(z)$ maps the strip $\{z \in \mathbb{C} : \ln a < \operatorname{Re} z < b\}$ onto $\{z \in \mathbb{C} : a < |z| < b\}$ and that in fact, g maps the line $\ln r + iy$ onto the circle $re^{i\theta}$. Now let $M(x)$ be defined as above and m be defined by

$$m(r) \equiv \max_{\theta} \left|f\left(re^{i\theta}\right)\right|.$$

Then for $a < r < b$, Corollary 25.15 implies

$$\begin{aligned} m(r) &= \sup_y \left|f\left(e^{\ln r+iy}\right)\right| = M(\ln r) \leq M(\ln a)^{\frac{\ln b-\ln r}{\ln b-\ln a}} M(\ln b)^{\frac{\ln r-\ln a}{\ln b-\ln a}} \\ &= m(a)^{\ln(b/r)/\ln(b/a)} m(b)^{\ln(r/a)/\ln(b/a)} \end{aligned}$$

and so

$$m(r)^{\ln(b/a)} \leq m(a)^{\ln(b/r)} m(b)^{\ln(r/a)}.$$

Taking logarithms, this yields

$$\ln\left(\frac{b}{a}\right) \ln(m(r)) \leq \ln\left(\frac{b}{r}\right) \ln(m(a)) + \ln\left(\frac{r}{a}\right) \ln(m(b))$$

which says the same as $r \to \ln(m(r))$ is a convex function of $\ln r$.

The next example, also in Rudin [45] is very dramatic. An unbelievably weak assumption is made on the growth of the function and still you get a uniform bound in the conclusion.

Corollary 25.16 *Let* $\Omega = \left\{z \in \mathbb{C} : |\text{Im}(z)| < \frac{\pi}{2}\right\}$. *Suppose* f *is analytic on* Ω, *continuous on* $\overline{\Omega}$, *and there exist constants,* $\alpha < 1$ *and* $A < \infty$ *such that*

$$|f(z)| \leq \exp(A \exp(\alpha |x|)) \text{ for } z = x + iy$$

and

$$\left|f\left(x \pm i\frac{\pi}{2}\right)\right| \leq 1$$

for all $x \in \mathbb{R}$. *Then* $|f(z)| \leq 1$ *on* Ω.

Proof: This time let $\phi(z) = [\exp(A\exp(\beta z))\exp(A\exp(-\beta z))]^{-1}$ where $\alpha < \beta < 1$. Then $\phi(z) \neq 0$ on Ω and for $\eta > 0$

$$|\phi(z)|^{\eta} = \frac{1}{|\exp(\eta A\exp(\beta z))\exp(\eta A\exp(-\beta z))|}$$

Now

$$\begin{aligned} & \exp(\eta A\exp(\beta z))\exp(\eta A\exp(-\beta z)) \\ = & \exp(\eta A(\exp(\beta z) + \exp(-\beta z))) \\ = & \exp\left[\eta A\left(\cos(\beta y)\left(e^{\beta x} + e^{-\beta x}\right) + i\sin(\beta y)\left(e^{\beta x} - e^{-\beta x}\right)\right)\right] \end{aligned}$$

and so

$$|\phi(z)|^{\eta} = \frac{1}{\exp[\eta A(\cos(\beta y)(e^{\beta x} + e^{-\beta x}))]}$$

Now $\cos\beta y > 0$ because $\beta < 1$ and $|y| < \frac{\pi}{2}$. Therefore,

$$\limsup_{z \to \infty} |f(z)| |\phi(z)|^{\eta} \leq 0 \leq 1$$

and so by Theorem 25.12, $|f(z)| \leq 1$.

25.4.3 Schwarz's Lemma

This interesting lemma comes from the maximum modulus theorem. It will be used later as part of the proof of the Riemann mapping theorem.

Lemma 25.17 *Suppose* $F : B(0,1) \to B(0,1)$, F *is analytic, and* $F(0) = 0$. *Then for all* $z \in B(0,1)$,

$$|F(z)| \leq |z|, \tag{25.4}$$

and

$$|F'(0)| \leq 1. \tag{25.5}$$

If equality holds in 25.5 then there exists $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$ *and*

$$F(z) = \lambda z. \tag{25.6}$$

Proof: First note that by assumption, $F(z)/z$ has a removable singularity at 0 if its value at 0 is defined to be $F'(0)$. By the maximum modulus theorem, if $|z| < r < 1$,

$$\left|\frac{F(z)}{z}\right| \leq \max_{t\in[0,2\pi]} \frac{\left|F\left(re^{it}\right)\right|}{r} \leq \frac{1}{r}.$$

Then letting $r \to 1$,

$$\left|\frac{F(z)}{z}\right| \leq 1$$

this shows 25.4 and it also verifies 25.5 on taking the limit as $z \to 0$. If equality holds in 25.5, then $|F(z)/z|$ achieves a maximum at an interior point so $F(z)/z$ equals a constant, λ by the maximum modulus theorem. Since $F(z) = \lambda z$, it follows $F'(0) = \lambda$ and so $|\lambda| = 1$.

Rudin [45] gives a memorable description of what this lemma says. It says that if an analytic function maps the unit ball to itself, keeping 0 fixed, then it must do one of two things, either be a rotation or move all points closer to 0. (This second part follows in case $|F'(0)| < 1$ because in this case, you must have $|F(z)| \neq |z|$ and so by 25.4, $|F(z)| < |z|$)

25.4.4 One To One Analytic Maps On The Unit Ball

The transformation in the next lemma is of fundamental importance.

Lemma 25.18 *Let $\alpha \in B(0,1)$ and define*

$$\phi_\alpha(z) \equiv \frac{z-\alpha}{1-\overline{\alpha}z}.$$

Then $\phi_\alpha : B(0,1) \to B(0,1)$, $\phi_\alpha : \partial B(0,1) \to \partial B(0,1)$, and is one to one and onto. Also $\phi_{-\alpha} = \phi_\alpha^{-1}$. Also

$$\phi_\alpha'(0) = 1 - |\alpha|^2,\ \phi'(\alpha) = \frac{1}{1-|\alpha|^2}.$$

Proof: First of all, for $|z| < 1/|\alpha|$,

$$\phi_\alpha \circ \phi_{-\alpha}(z) \equiv \frac{\left(\frac{z+\alpha}{1+\overline{\alpha}z}\right) - \alpha}{1 - \overline{\alpha}\left(\frac{z+\alpha}{1+\overline{\alpha}z}\right)} = z$$

after a few computations. If I show that ϕ_α maps $B(0,1)$ to $B(0,1)$ for all $|\alpha| < 1$, this will have shown that ϕ_α is one to one and onto $B(0,1)$.

Consider $\left|\phi_\alpha\left(e^{i\theta}\right)\right|$. This yields

$$\left|\frac{e^{i\theta}-\alpha}{1-\overline{\alpha}e^{i\theta}}\right| = \left|\frac{1-\alpha e^{-i\theta}}{1-\overline{\alpha}e^{i\theta}}\right| = 1$$

where the first equality is obtained by multiplying by $\left|e^{-i\theta}\right| = 1$. Therefore, ϕ_α maps $\partial B(0,1)$ one to one and onto $\partial B(0,1)$. Now notice that ϕ_α is analytic on $\overline{B(0,1)}$ because the only singularity, a pole is at $z = 1/\overline{\alpha}$. By the maximum modulus theorem, it follows

$$|\phi_\alpha(z)| < 1$$

whenever $|z| < 1$. The same is true of $\phi_{-\alpha}$.

It only remains to verify the assertions about the derivatives. Long division gives $\phi_\alpha(z) = (-\overline{\alpha})^{-1} + \left(\frac{-\alpha + (\overline{\alpha})^{-1}}{1 - \overline{\alpha}z}\right)$ and so

$$\begin{aligned}
\phi_\alpha'(z) &= (-1)(1 - \overline{\alpha}z)^{-2}\left(-\alpha + (\overline{\alpha})^{-1}\right)(-\overline{\alpha}) \\
&= \overline{\alpha}(1 - \overline{\alpha}z)^{-2}\left(-\alpha + (\overline{\alpha})^{-1}\right) \\
&= (1 - \overline{\alpha}z)^{-2}\left(-|\alpha|^2 + 1\right)
\end{aligned}$$

Hence the two formulas follow. This proves the lemma.

One reason these mappings are so important is the following theorem.

Theorem 25.19 *Suppose f is an analytic function defined on $B(0,1)$ and f maps $B(0,1)$ one to one and onto $B(0,1)$. Then there exists θ such that*

$$f(z) = e^{i\theta}\phi_\alpha(z)$$

for some $\alpha \in B(0,1)$.

Proof: Let $f(\alpha) = 0$. Then $h(z) \equiv f \circ \phi_{-\alpha}(z)$ maps $B(0,1)$ one to one and onto $B(0,1)$ and has the property that $h(0) = 0$. Therefore, by the Schwarz lemma,

$$|h(z)| \leq |z|.$$

but it is also the case that $h^{-1}(0) = 0$ and h^{-1} maps $B(0,1)$ to $B(0,1)$. Therefore, the same inequality holds for h^{-1}. Therefore,

$$|z| = \left|h^{-1}(h(z))\right| \leq |h(z)|$$

and so $|h(z)| = |z|$. By the Schwarz lemma again, $h(z) \equiv f\left(\phi_{-\alpha}(z)\right) = e^{i\theta}z$. Letting $z = \phi_\alpha$, you get $f(z) = e^{i\theta}\phi_\alpha(z)$.

25.5 Exercises

1. Consider the function, $g(z) = \frac{z-i}{z+i}$. Show this is analytic on the upper half plane, $P+$ and maps the upper half plane one to one and onto $B(0,1)$. **Hint:** First show g maps the real axis to $\partial B(0,1)$. This is really easy because you end up looking at a complex number divided by its conjugate. Thus $|g(z)| = 1$ for z on $\partial(P+)$. Now show that $\limsup_{z\to\infty}|g(z)| = 1$. Then apply a version of the maximum modulus theorem. You might note that $g(z) = 1 + \frac{-2i}{z+i}$. This will show $|g(z)| \leq 1$. Next pick $w \in B(0,1)$ and solve $g(z) = w$. You just have to show there exists a unique solution and its imaginary part is positive.

2. Does there exist an entire function f which maps $\mathbb{C}$ onto the upper half plane?

3. Letting g be the function of Problem 1 show that $\left(g^{-1}\right)'(0) = 2$. Also note that $g^{-1}(0) = i$. Now suppose f is an analytic function defined on the upper half plane which has the property that $|f(z)| \leq 1$ and $f(i) = \beta$ where $|\beta| < 1$. Find an upper bound to $|f'(i)|$. Also find all functions, f which satisfy the condition, $f(i) = \beta, |f(z)| \leq 1$, and achieve this maximum value. **Hint:** You could consider the function, $h(z) \equiv \phi_\beta \circ f \circ g^{-1}(z)$ and check the conditions for the Schwarz lemma for this function, h.

4. This and the next two problems follow a presentation of an interesting topic in Rudin [45]. Let ϕ_α be given in Lemma 25.18. Suppose f is an analytic function defined on $B(0,1)$ which satisfies $|f(z)| \leq 1$. Suppose also there are $\alpha, \beta \in B(0,1)$ and it is required $f(\alpha) = \beta$. If f is such a function, show that $|f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}$. **Hint:** To show this consider $g = \phi_\beta \circ f \circ \phi_{-\alpha}$. Show $g(0) = 0$ and $|g(z)| \leq 1$ on $B(0,1)$. Now use Lemma 25.17.

5. In Problem 4 show there exists a function, f analytic on $B(0,1)$ such that $f(\alpha) = \beta$, $|f(z)| \leq 0$, and $|f'(\alpha)| = \frac{1-|\beta|^2}{1-|\alpha|^2}$. **Hint:** You do this by choosing g in the above problem such that equality holds in Lemma 25.17. Thus you need $g(z) = \lambda z$ where $|\lambda| = 1$ and solve $g = \phi_\beta \circ f \circ \phi_{-\alpha}$ for f.

6. Suppose that $f : B(0,1) \to B(0,1)$ and that f is analytic, one to one, and onto with $f(\alpha) = 0$. Show there exists $\lambda, |\lambda| = 1$ such that $f(z) = \lambda\phi_\alpha(z)$. This gives a different way to look at Theorem 25.19. **Hint:** Let $g = f^{-1}$. Then $g'(0) f'(\alpha) = 1$. However, $f(\alpha) = 0$ and $g(0) = \alpha$. From Problem 4 with $\beta = 0$, you can conclude an inequality for $|f'(\alpha)|$ and another one for $|g'(0)|$. Then use the fact that the product of these two equals 1 which comes from the chain rule to conclude that equality must take place. Now use Problem 5 to obtain the form of f.

7. In Corollary 25.16 show that it is essential that $\alpha < 1$. That is, show there exists an example where the conclusion is not satisfied with a slightly weaker growth condition. **Hint:** Consider $\exp(\exp(z))$.

8. Suppose $\{f_n\}$ is a sequence of functions which are analytic on Ω, a bounded region such that each f_n is also continuous on $\overline{\Omega}$. Suppose that $\{f_n\}$ converges uniformly on $\partial\Omega$. Show that then $\{f_n\}$ converges uniformly on $\overline{\Omega}$ and that the function to which the sequence converges is analytic on Ω and continuous on $\overline{\Omega}$.

9. Suppose Ω is a bounded region and there exists a point $z_0 \in \Omega$ such that $|f(z_0)| = \min\left\{|f(z)| : z \in \overline{\Omega}\right\}$. Can you conclude f must equal a constant?

10. Suppose f is continuous on $\overline{B(a,r)}$ and analytic on $B(a,r)$ and that f is not constant. Suppose also $|f(z)| = C \neq 0$ for all $|z - a| = r$. Show that there exists $\alpha \in B(a,r)$ such that $f(\alpha) = 0$. **Hint:** If not, consider f/C and C/f. Both would be analytic on $B(a,r)$ and are equal to 1 on the boundary.

11. Suppose f is analytic on $B(0,1)$ but for every $a \in \partial B(0,1)$, $\lim_{z \to a} |f(z)| = \infty$. Show there exists a sequence, $\{z_n\} \subseteq B(0,1)$ such that $\lim_{n \to \infty} |z_n| = 1$ and $f(z_n) = 0$.

25.6 Counting Zeros

The above proof of the open mapping theorem relies on the very important inverse function theorem from real analysis. There are other approaches to this important theorem which do not rely on the big theorems from real analysis and are more oriented toward the use of the Cauchy integral formula and specialized techniques from complex analysis. One of these approaches is given next which involves the notion of "counting zeros". The next theorem is the one about counting zeros. It will also be used later in the proof of the Riemann mapping theorem.

Theorem 25.20 *Let Ω be an open set in $\mathbb{C}$ and let $\gamma : [a,b] \to \Omega$ be closed, continuous, bounded variation, and $n(\gamma, z) = 0$ for all $z \notin \Omega$. Suppose also that f is analytic on Ω having zeros $a_1, \cdots, a_m$ where the zeros are repeated according to multiplicity, and suppose that none of these zeros are on γ^*. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^m n(\gamma, a_k).$$

Proof: Let $f(z) = \prod_{j=1}^m (z - a_j) g(z)$ where $g(z) \neq 0$ on Ω. Hence

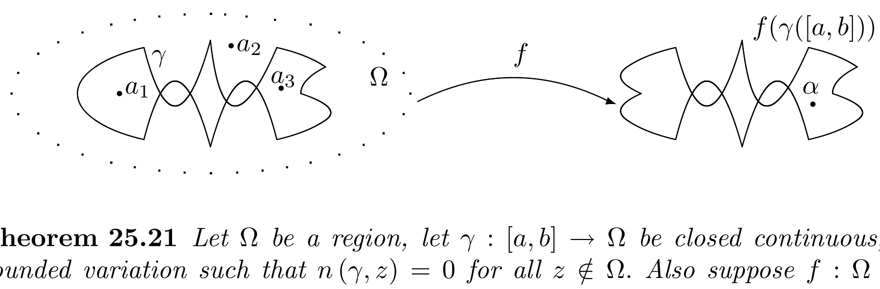
$$\frac{f'(z)}{f(z)} = \sum_{j=1}^m \frac{1}{z - a_j} + \frac{g'(z)}{g(z)}$$

and so

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{j=1}^m n(\gamma, a_j) + \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} dz.$$

But the function, $z \to \frac{g'(z)}{g(z)}$ is analytic and so by Corollary 24.47, the last integral in the above expression equals 0. Therefore, this proves the theorem.

The following picture is descriptive of the situation described in the next theorem.



Theorem 25.21 *Let Ω be a region, let $\gamma : [a,b] \to \Omega$ be closed continuous, and bounded variation such that $n(\gamma, z) = 0$ for all $z \notin \Omega$. Also suppose $f : \Omega \to \mathbb{C}$*

is analytic and that $\alpha \notin f(\gamma^*)$. *Then* $f \circ \gamma : [a,b] \to \mathbb{C}$ *is continuous, closed, and bounded variation. Also suppose* $\{a_1, \cdots, a_m\} = f^{-1}(\alpha)$ *where these points are counted according to their multiplicities as zeros of the function* $f - \alpha$ *Then*

$$n(f \circ \gamma, \alpha) = \sum_{k=1}^{m} n(\gamma, a_k).$$

Proof: It is clear that $f \circ \gamma$ is continuous. It only remains to verify that it is of bounded variation. Suppose first that $\gamma^* \subseteq B \subseteq \overline{B} \subseteq \Omega$ where B is a ball. Then

$$\begin{aligned} &|f(\gamma(t)) - f(\gamma(s))| = \\ &\left|\int_0^1 f'(\gamma(s) + \lambda(\gamma(t) - \gamma(s)))(\gamma(t) - \gamma(s))\, d\lambda\right| \\ \leq\ & C|\gamma(t) - \gamma(s)| \end{aligned}$$

where $C \geq \max\left\{|f'(z)| : z \in \overline{B}\right\}$. Hence, in this case,

$$V(f \circ \gamma, [a,b]) \leq CV(\gamma, [a,b]).$$

Now let ε denote the distance between γ^* and $\mathbb{C} \setminus \Omega$. Since γ^* is compact, $\varepsilon > 0$. By uniform continuity there exists $\delta = \frac{b-a}{p}$ for p a positive integer such that if $|s - t| < \delta$, then $|\gamma(s) - \gamma(t)| < \frac{\varepsilon}{2}$. Then

$$\gamma([t, t+\delta]) \subseteq \overline{B\left(\gamma(t), \frac{\varepsilon}{2}\right)} \subseteq \Omega.$$

Let $C \geq \max\left\{|f'(z)| : z \in \cup_{j=1}^{p} \overline{B\left(\gamma(t_j), \frac{\varepsilon}{2}\right)}\right\}$ where $t_j \equiv \frac{j}{p}(b-a) + a$. Then from what was just shown,

$$\begin{aligned} V(f \circ \gamma, [a,b]) &\leq \sum_{j=0}^{p-1} V(f \circ \gamma, [t_j, t_{j+1}]) \\ &\leq C\sum_{j=0}^{p-1} V(\gamma, [t_j, t_{j+1}]) < \infty \end{aligned}$$

showing that $f \circ \gamma$ is bounded variation as claimed. Now from Theorem 24.42 there exists $\eta \in C^1([a,b])$ such that

$$\eta(a) = \gamma(a) = \gamma(b) = \eta(b),\ \eta([a,b]) \subseteq \Omega,$$

and

$$n(\eta, a_k) = n(\gamma, a_k),\ n(f \circ \gamma, \alpha) = n(f \circ \eta, \alpha) \tag{25.7}$$

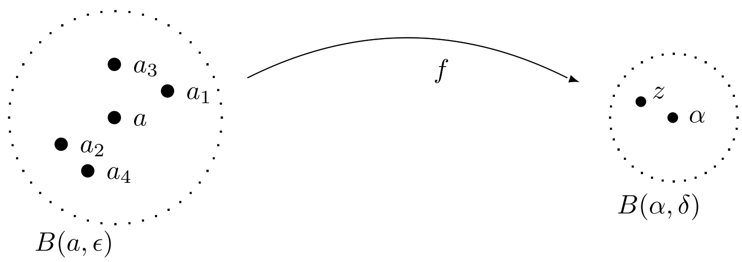
for $k = 1, \cdots, m$. Then

$$n(f \circ \gamma, \alpha) = n(f \circ \eta, \alpha)$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{f\circ\eta}\frac{dw}{w-\alpha} \\
&= \frac{1}{2\pi i}\int_a^b \frac{f'(\eta(t))}{f(\eta(t))-\alpha}\eta'(t)\,dt \\
&= \frac{1}{2\pi i}\int_\eta \frac{f'(z)}{f(z)-\alpha}dz \\
&= \sum_{k=1}^m n(\eta, a_k)
\end{aligned}
$$

By Theorem 25.20. By 25.7, this equals $\sum_{k=1}^m n(\gamma, a_k)$ which proves the theorem.

The next theorem is incredible and is very interesting for its own sake. The following picture is descriptive of the situation of this theorem.



Theorem 25.22 *Let $f : B(a,R) \to \mathbb{C}$ be analytic and let*

$$f(z) - \alpha = (z-a)^m g(z), \ \infty > m \geq 1$$

where $g(z) \neq 0$ in $B(a,R)$. ($f(z)-\alpha$ has a zero of order m at $z=a$.) Then there exist $\varepsilon, \delta > 0$ with the property that for each z satisfying $0 < |z-\alpha| < \delta$, there exist points,

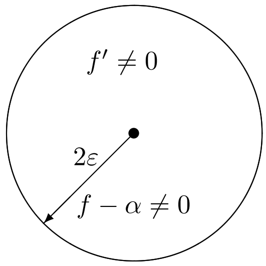
$$\{a_1, \cdots, a_m\} \subseteq B(a,\varepsilon),$$

such that

$$f^{-1}(z) \cap B(a,\varepsilon) = \{a_1, \cdots, a_m\}$$

and each a_k is a zero of order 1 for the function $f(\cdot) - z$.

Proof: By Theorem 24.23 f is not constant on $B(a,R)$ because it has a zero of order m. Therefore, using this theorem again, there exists $\varepsilon > 0$ such that $\overline{B(a,2\varepsilon)} \subseteq B(a,R)$ and there are no solutions to the equation $f(z) - \alpha = 0$ for $z \in \overline{B(a,2\varepsilon)}$ except a. Also assume ε is small enough that for $0 < |z-a| \leq 2\varepsilon$, $f'(z) \neq 0$. This can be done since otherwise, a would be a limit point of a sequence of points, z_n, having $f'(z_n) = 0$ which would imply, by Theorem 24.23 that $f' = 0$ on $B(a,R)$, contradicting the assumption that $f - \alpha$ has a zero of order m and is therefore not constant. Thus the situation is described by the following picture.



Now pick $\gamma(t) = a + \varepsilon e^{it}, t \in [0, 2\pi]$. Then $\alpha \notin f(\gamma^*)$ so there exists $\delta > 0$ with

$$B(\alpha, \delta) \cap f(\gamma^*) = \emptyset. \tag{25.8}$$

Therefore, $B(\alpha, \delta)$ is contained on one component of $\mathbb{C} \setminus f(\gamma([0, 2\pi]))$. Therefore, $n(f \circ \gamma, \alpha) = n(f \circ \gamma, z)$ for all $z \in B(\alpha, \delta)$. Now consider f restricted to $B(a, 2\varepsilon)$. For $z \in B(\alpha, \delta)$, $f^{-1}(z)$ must consist of a finite set of points because $f'(w) \neq 0$ for all w in $\overline{B(a, 2\varepsilon)} \setminus \{a\}$ implying that the zeros of $f(\cdot) - z$ in $\overline{B(a, 2\varepsilon)}$ have no limit point. Since $\overline{B(a, 2\varepsilon)}$ is compact, this means there are only finitely many. By Theorem 25.21,

$$n(f \circ \gamma, z) = \sum_{k=1}^{p} n(\gamma, a_k) \tag{25.9}$$

where $\{a_1, \cdots, a_p\} = f^{-1}(z)$. Each point, a_k of $f^{-1}(z)$ is either inside the circle traced out by γ, yielding $n(\gamma, a_k) = 1$, or it is outside this circle yielding $n(\gamma, a_k) = 0$ because of 25.8. It follows the sum in 25.9 reduces to the number of points of $f^{-1}(z)$ which are contained in $B(a, \varepsilon)$. Thus, letting those points in $f^{-1}(z)$ which are contained in $B(a, \varepsilon)$ be denoted by $\{a_1, \cdots, a_r\}$

$$n(f \circ \gamma, \alpha) = n(f \circ \gamma, z) = r.$$

Also, by Theorem 25.20, $m = n(f \circ \gamma, \alpha)$ because a is a zero of $f - \alpha$ of order m. Therefore, for $z \in B(\alpha, \delta)$

$$m = n(f \circ \gamma, \alpha) = n(f \circ \gamma, z) = r$$

It is required to show $r = m$, the order of the zero of $f - \alpha$. Therefore, $r = m$. Each of these a_k is a zero of order 1 of the function $f(\cdot) - z$ because $f'(a_k) \neq 0$. This proves the theorem.

This is a very fascinating result partly because it implies that for values of f near a value, α, at which $f(\cdot) - \alpha$ has a zero of order m for $m > 1$, the inverse image of these values includes at least m points, not just one. Thus the topological properties of the inverse image changes radically. This theorem also shows that $f(B(a, \varepsilon)) \supseteq B(\alpha, \delta)$.

Theorem 25.23 *(open mapping theorem) Let Ω be a region and $f : \Omega \to \mathbb{C}$ be analytic. Then $f(\Omega)$ is either a point or a region. If f is one to one, then $f^{-1} : f(\Omega) \to \Omega$ is analytic.*

Proof: If f is not constant, then for every $\alpha \in f(\Omega)$, it follows from Theorem 24.23 that $f(\cdot) - \alpha$ has a zero of order $m < \infty$ and so from Theorem 25.22 for each $a \in \Omega$ there exist $\varepsilon, \delta > 0$ such that $f(B(a, \varepsilon)) \supseteq B(\alpha, \delta)$ which clearly implies that f maps open sets to open sets. Therefore, $f(\Omega)$ is open, connected because f is continuous. If f is one to one, Theorem 25.22 implies that for every $\alpha \in f(\Omega)$ the zero of $f(\cdot) - \alpha$ is of order 1. Otherwise, that theorem implies that for z near α, there are m points which f maps to z contradicting the assumption that f is one to one. Therefore, $f'(z) \neq 0$ and since f^{-1} is continuous, due to f being an open map, it follows

$$\begin{aligned}\left(f^{-1}\right)'(f(z)) &= \lim_{f(z_1) \to f(z)} \frac{f^{-1}(f(z_1)) - f^{-1}(f(z))}{f(z_1) - f(z)} \\ &= \lim_{z_1 \to z} \frac{z_1 - z}{f(z_1) - f(z)} = \frac{1}{f'(z)}.\end{aligned}$$

This proves the theorem.

25.7 An Application To Linear Algebra

Gerschgorin's theorem gives a convenient way to estimate eigenvalues of a matrix from easy to obtain information. For A an $n \times n$ matrix, denote by $\sigma(A)$ the collection of all eigenvalues of A.

Theorem 25.24 *Let A be an $n \times n$ matrix. Consider the n Gerschgorin discs defined as*

$$D_i \equiv \left\{ \lambda \in \mathbb{C} : |\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}.$$

Then every eigenvalue is contained in some Gerschgorin disc.

This theorem says to add up the absolute values of the entries of the i^{th} row which are off the main diagonal and form the disc centered at a_{ii} having this radius. The union of these discs contains $\sigma(A)$.

Proof: Suppose $A\mathbf{x} = \lambda \mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$. Then for $A = (a_{ij})$

$$\sum_{j \neq i} a_{ij} x_j = (\lambda - a_{ii}) x_i.$$

Therefore, if we pick k such that $|x_k| \geq |x_j|$ for all x_j, it follows that $|x_k| \neq 0$ since $|\mathbf{x}| \neq 0$ and

$$|x_k| \sum_{j \neq k} |a_{kj}| \geq \sum_{j \neq k} |a_{kj}| |x_j| \geq |\lambda - a_{kk}| |x_k|.$$

Now dividing by $|x_k|$ we see that λ is contained in the k^{th} Gerschgorin disc.

More can be said using the theory about counting zeros. To begin with the distance between two $n \times n$ matrices, $A = (a_{ij})$ and $B = (b_{ij})$ as follows.

$$||A - B||^2 \equiv \sum_{ij} |a_{ij} - b_{ij}|^2 .$$

Thus two matrices are close if and only if their corresponding entries are close.

Let A be an $n \times n$ matrix. Recall the eigenvalues of A are given by the zeros of the polynomial, $p_A(z) = \det(zI - A)$ where I is the $n \times n$ identity. Then small changes in A will produce small changes in $p_A(z)$ and $p_A'(z)$. Let γ_k denote a very small closed circle which winds around z_k, one of the eigenvalues of A, in the counter clockwise direction so that $n(\gamma_k, z_k) = 1$. This circle is to enclose only z_k and is to have no other eigenvalue on it. Then apply Theorem 25.20. According to this theorem

$$\frac{1}{2\pi i} \int_\gamma \frac{p_A'(z)}{p_A(z)} dz$$

is always an integer equal to the multiplicity of z_k as a root of $p_A(t)$. Therefore, small changes in A result in no change to the above contour integral because it must be an integer and small changes in A result in small changes in the integral. Therefore whenever every entry of the matrix B is close enough to the corresponding entry of the matrix A, the two matrices have the same number of zeros inside γ_k under the usual convention that zeros are to be counted according to multiplicity. By making the radius of the small circle equal to ε where ε is less than the minimum distance between any two distinct eigenvalues of A, this shows that if B is close enough to A, every eigenvalue of B is closer than ε to some eigenvalue of A. The next theorem is about continuous dependence of eigenvalues.

Theorem 25.25 *If λ is an eigenvalue of A, then if $||B - A||$ is small enough, some eigenvalue of B will be within ε of λ.*

Consider the situation that $A(t)$ is an $n \times n$ matrix and that $t \to A(t)$ is continuous for $t \in [0, 1]$.

Lemma 25.26 *Let $\lambda(t) \in \sigma(A(t))$ for $t < 1$ and let $\Sigma_t = \cup_{s \geq t} \sigma(A(s))$. Also let K_t be the connected component of $\lambda(t)$ in Σ_t. Then there exists $\eta > 0$ such that $K_t \cap \sigma(A(s)) \neq \emptyset$ for all $s \in [t, t + \eta]$.*

Proof: Denote by $D(\lambda(t), \delta)$ the disc centered at $\lambda(t)$ having radius $\delta > 0$, with other occurrences of this notation being defined similarly. Thus

$$D(\lambda(t), \delta) \equiv \{z \in \mathbb{C} : |\lambda(t) - z| \leq \delta\} .$$

Suppose $\delta > 0$ is small enough that $\lambda(t)$ is the only element of $\sigma(A(t))$ contained in $D(\lambda(t), \delta)$ and that $p_{A(t)}$ has no zeroes on the boundary of this disc. Then by continuity, and the above discussion and theorem, there exists $\eta > 0, t + \eta < 1$, such that for $s \in [t, t + \eta]$, $p_{A(s)}$ also has no zeroes on the boundary of this disc and that

$A(s)$ has the same number of eigenvalues, counted according to multiplicity, in the disc as $A(t)$. Thus $\sigma(A(s)) \cap D(\lambda(t), \delta) \neq \emptyset$ for all $s \in [t, t+\eta]$. Now let

$$H = \bigcup_{s \in [t, t+\eta]} \sigma(A(s)) \cap D(\lambda(t), \delta).$$

I will show H is connected. Suppose not. Then $H = P \cup Q$ where P, Q are separated and $\lambda(t) \in P$. Let

$$s_0 \equiv \inf\{s : \lambda(s) \in Q \text{ for some } \lambda(s) \in \sigma(A(s))\}.$$

There exists $\lambda(s_0) \in \sigma(A(s_0)) \cap D(\lambda(t), \delta)$. If $\lambda(s_0) \notin Q$, then from the above discussion there are

$$\lambda(s) \in \sigma(A(s)) \cap Q$$

for $s > s_0$ arbitrarily close to $\lambda(s_0)$. Therefore, $\lambda(s_0) \in Q$ which shows that $s_0 > t$ because $\lambda(t)$ is the only element of $\sigma(A(t))$ in $D(\lambda(t), \delta)$ and $\lambda(t) \in P$. Now let $s_n \uparrow s_0$. Then $\lambda(s_n) \in P$ for any

$$\lambda(s_n) \in \sigma(A(s_n)) \cap D(\lambda(t), \delta)$$

and from the above discussion, for some choice of $s_n \to s_0$, $\lambda(s_n) \to \lambda(s_0)$ which contradicts P and Q separated and nonempty. Since P is nonempty, this shows $Q = \emptyset$. Therefore, H is connected as claimed. But $K_t \supseteq H$ and so $K_t \cap \sigma(A(s)) \neq \emptyset$ for all $s \in [t, t+\eta]$. This proves the lemma.

The following is the necessary theorem.

Theorem 25.27 *Suppose $A(t)$ is an $n \times n$ matrix and that $t \to A(t)$ is continuous for $t \in [0,1]$. Let $\lambda(0) \in \sigma(A(0))$ and define $\Sigma \equiv \cup_{t \in [0,1]} \sigma(A(t))$. Let $K_{\lambda(0)} = K_0$ denote the connected component of $\lambda(0)$ in Σ. Then $K_0 \cap \sigma(A(t)) \neq \emptyset$ for all $t \in [0,1]$.*

Proof: Let $S \equiv \{t \in [0,1] : K_0 \cap \sigma(A(s)) \neq \emptyset \text{ for all } s \in [0,t]\}$. Then $0 \in S$. Let $t_0 = \sup(S)$. Say $\sigma(A(t_0)) = \lambda_1(t_0), \cdots, \lambda_r(t_0)$. I claim at least one of these is a limit point of K_0 and consequently must be in K_0 which will show that S has a last point. Why is this claim true? Let $s_n \uparrow t_0$ so $s_n \in S$. Now let the discs, $D(\lambda_i(t_0), \delta), i = 1, \cdots, r$ be disjoint with $p_{A(t_0)}$ having no zeroes on γ_i the boundary of $D(\lambda_i(t_0), \delta)$. Then for n large enough it follows from Theorem 25.20 and the discussion following it that $\sigma(A(s_n))$ is contained in $\cup_{i=1}^r D(\lambda_i(t_0), \delta)$. Therefore, $K_0 \cap (\sigma(A(t_0)) + D(0, \delta)) \neq \emptyset$ for all δ small enough. This requires at least one of the $\lambda_i(t_0)$ to be in $\overline{K_0}$. Therefore, $t_0 \in S$ and S has a last point.

Now by Lemma 25.26, if $t_0 < 1$, then $K_0 \cup K_t$ would be a strictly larger connected set containing $\lambda(0)$. (The reason this would be strictly larger is that $K_0 \cap \sigma(A(s)) = \emptyset$ for some $s \in (t, t+\eta)$ while $K_t \cap \sigma(A(s)) \neq \emptyset$ for all $s \in [t, t+\eta]$.) Therefore, $t_0 = 1$ and this proves the theorem.

The following is an interesting corollary of the Gerschgorin theorem.

Corollary 25.28 *Suppose one of the Gerschgorin discs, D_i is disjoint from the union of the others. Then D_i contains an eigenvalue of A. Also, if there are n disjoint Gerschgorin discs, then each one contains an eigenvalue of A.*

Proof: Denote by $A(t)$ the matrix $\left(a_{ij}^t\right)$ where if $i \neq j$, $a_{ij}^t = ta_{ij}$ and $a_{ii}^t = a_{ii}$. Thus to get $A(t)$ we multiply all non diagonal terms by t. Let $t \in [0,1]$. Then $A(0) = diag(a_{11}, \cdots, a_{nn})$ and $A(1) = A$. Furthermore, the map, $t \to A(t)$ is continuous. Denote by D_j^t the Gerschgorin disc obtained from the j^{th} row for the matrix, $A(t)$. Then it is clear that $D_j^t \subseteq D_j$ the j^{th} Gerschgorin disc for A. Then a_{ii} is the eigenvalue for $A(0)$ which is contained in the disc, consisting of the single point a_{ii} which is contained in D_i. Letting K be the connected component in Σ for Σ defined in Theorem 25.27 which is determined by a_{ii}, it follows by Gerschgorin's theorem that $K \cap \sigma(A(t)) \subseteq \cup_{j=1}^n D_j^t \subseteq \cup_{j=1}^n D_j = D_i \cup (\cup_{j \neq i} D_j)$ and also, since K is connected, there are no points of K in both D_i and $(\cup_{j \neq i} D_j)$. Since at least one point of K is in D_i,(a_{ii}) it follows all of K must be contained in D_i. Now by Theorem 25.27 this shows there are points of $K \cap \sigma(A)$ in D_i. The last assertion follows immediately.

Actually, this can be improved slightly. It involves the following lemma.

Lemma 25.29 *In the situation of Theorem 25.27 suppose $\lambda(0) = K_0 \cap \sigma(A(0))$ and that $\lambda(0)$ is a simple root of the characteristic equation of $A(0)$. Then for all $t \in [0,1]$,*

$$\sigma(A(t)) \cap K_0 = \lambda(t)$$

where $\lambda(t)$ is a simple root of the characteristic equation of $A(t)$.

Proof: Let $S \equiv$

$$\{t \in [0,1] : K_0 \cap \sigma(A(s)) = \lambda(s), \text{ a simple eigenvalue for all } s \in [0,t]\}.$$

Then $0 \in S$ so it is nonempty. Let $t_0 = \sup(S)$ and suppose $\lambda_1 \neq \lambda_2$ are two elements of $\sigma(A(t_0)) \cap K_0$. Then choosing $\eta > 0$ small enough, and letting D_i be disjoint discs containing λ_i respectively, similar arguments to those of Lemma 25.26 imply

$$H_i \equiv \cup_{s \in [t_0 - \eta, t_0]} \sigma(A(s)) \cap D_i$$

is a connected and nonempty set for $i = 1, 2$ which would require that $H_i \subseteq K_0$. But then there would be two different eigenvalues of $A(s)$ contained in K_0, contrary to the definition of t_0. Therefore, there is at most one eigenvalue, $\lambda(t_0) \in K_0 \cap \sigma(A(t_0))$. The possibility that it could be a repeated root of the characteristic equation must be ruled out. Suppose then that $\lambda(t_0)$ is a repeated root of the characteristic equation. As before, choose a small disc, D centered at $\lambda(t_0)$ and η small enough that

$$H \equiv \cup_{s \in [t_0 - \eta, t_0]} \sigma(A(s)) \cap D$$

is a nonempty connected set containing either multiple eigenvalues of $A(s)$ or else a single repeated root to the characteristic equation of $A(s)$. But since H is connected and contains $\lambda(t_0)$ it must be contained in K_0 which contradicts the condition for

$s \in S$ for all these $s \in [t_0 - \eta, t_0]$. Therefore, $t_0 \in S$ as hoped. If $t_0 < 1$, there exists a small disc centered at $\lambda(t_0)$ and $\eta > 0$ such that for all $s \in [t_0, t_0 + \eta]$, $A(s)$ has only simple eigenvalues in D and the only eigenvalues of $A(s)$ which could be in K_0 are in D. (This last assertion follows from noting that $\lambda(t_0)$ is the only eigenvalue of $A(t_0)$ in K_0 and so the others are at a positive distance from K_0. For s close enough to t_0, the eigenvalues of $A(s)$ are either close to these eigenvalues of $A(t_0)$ at a positive distance from K_0 or they are close to the eigenvalue, $\lambda(t_0)$ in which case it can be assumed they are in D.) But this shows that t_0 is not really an upper bound to S. Therefore, $t_0 = 1$ and the lemma is proved.

With this lemma, the conclusion of the above corollary can be improved.

Corollary 25.30 *Suppose one of the Gerschgorin discs, D_i is disjoint from the union of the others. Then D_i contains exactly one eigenvalue of A and this eigenvalue is a simple root to the characteristic polynomial of A.*

Proof: In the proof of Corollary 25.28, first note that a_{ii} is a simple root of $A(0)$ since otherwise the i^{th} Gerschgorin disc would not be disjoint from the others. Also, K, the connected component determined by a_{ii} must be contained in D_i because it is connected and by Gerschgorin's theorem above, $K \cap \sigma(A(t))$ must be contained in the union of the Gerschgorin discs. Since all the other eigenvalues of $A(0)$, the a_{jj}, are outside D_i, it follows that $K \cap \sigma(A(0)) = a_{ii}$. Therefore, by Lemma 25.29, $K \cap \sigma(A(1)) = K \cap \sigma(A)$ consists of a single simple eigenvalue. This proves the corollary.

Example 25.31 *Consider the matrix,*

$$\begin{pmatrix} 5 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The Gerschgorin discs are $D(5,1), D(1,2)$, and $D(0,1)$. Then $D(5,1)$ is disjoint from the other discs. Therefore, there should be an eigenvalue in $D(5,1)$. The actual eigenvalues are not easy to find. They are the roots of the characteristic equation, $t^3 - 6t^2 + 3t + 5 = 0$. The numerical values of these are $-.669\,66, 1.423\,1$, and $5.246\,55$, verifying the predictions of Gerschgorin's theorem.

25.8 Exercises

1. Use Theorem 25.20 to give an alternate proof of the fundamental theorem of algebra. **Hint:** Take a contour of the form $\gamma_r = re^{it}$ where $t \in [0, 2\pi]$. Consider $\int_{\gamma_r} \frac{p'(z)}{p(z)} dz$ and consider the limit as $r \to \infty$.

2. Let M be an $n \times n$ matrix. Recall that the eigenvalues of M are given by the zeros of the polynomial, $p_M(z) = \det(M - zI)$ where I is the $n \times n$ identity. Formulate a theorem which describes how the eigenvalues depend on small

changes in M. **Hint:** You could define a norm on the space of $n \times n$ matrices as $||M|| \equiv tr\,(MM^*)^{1/2}$ where M^* is the conjugate transpose of M. Thus

$$||M|| = \left(\sum_{j,k} |M_{jk}|^2\right)^{1/2}.$$

Argue that small changes will produce small changes in $p_M(z)$. Then apply Theorem 25.20 using γ_k a very small circle surrounding z_k, the k^{th} eigenvalue.

3. Suppose that two analytic functions defined on a region are equal on some set, S which contains a limit point. (Recall p is a limit point of S if every open set which contains p, also contains infinitely many points of S.) Show the two functions coincide. We defined $e^z \equiv e^x(\cos y + i \sin y)$ earlier and we showed that e^z, defined this way was analytic on $\mathbb{C}$. Is there any other way to define e^z on all of $\mathbb{C}$ such that the function coincides with e^x on the real axis?

4. You know various identities for real valued functions. For example $\cosh^2 x - \sinh^2 x = 1$. If you define $\cosh z \equiv \frac{e^z + e^{-z}}{2}$ and $\sinh z \equiv \frac{e^z - e^{-z}}{2}$, does it follow that
$$\cosh^2 z - \sinh^2 z = 1$$
for all $z \in \mathbb{C}$? What about
$$\sin(z + w) = \sin z \cos w + \cos z \sin w?$$
Can you verify these sorts of identities just from your knowledge about what happens for real arguments?

5. Was it necessary that U be a region in Theorem 24.23? Would the same conclusion hold if U were only assumed to be an open set? Why? What about the open mapping theorem? Would it hold if U were not a region?

6. Let $f : U \to \mathbb{C}$ be analytic and one to one. Show that $f'(z) \neq 0$ for all $z \in U$. Does this hold for a function of a real variable?

7. We say a real valued function, u is subharmonic if $u_{xx} + u_{yy} \geq 0$. Show that if u is subharmonic on a bounded region, (open connected set) U, and continuous on $\overline{U}$ and $u \leq m$ on ∂U, then $u \leq m$ on U. **Hint:** If not, u achieves its maximum at $(x_0, y_0) \in U$. Let $u(x_0, y_0) > m + \delta$ where $\delta > 0$. Now consider $u_\varepsilon(x, y) = \varepsilon x^2 + u(x, y)$ where ε is small enough that $0 < \varepsilon x^2 < \delta$ for all $(x, y) \in U$. Show that u_ε also achieves its maximum at some point of U and that therefore, $u_{\varepsilon xx} + u_{\varepsilon yy} \leq 0$ at that point implying that $u_{xx} + u_{yy} \leq -\varepsilon$, a contradiction.

8. If u is harmonic on some region, U, show that u coincides locally with the real part of an analytic function and that therefore, u has infinitely many

derivatives on U. **Hint:** Consider the case where $0 \in U$. You can always reduce to this case by a suitable translation. Now let $B(0,r) \subseteq U$ and use the Schwarz formula to obtain an analytic function whose real part coincides with u on $\partial B(0,r)$. Then use Problem 7.

9. Show the solution to the Dirichlet problem of Problem 8 on Page 656 is unique. You need to formulate this precisely and then prove uniqueness.

Residues

Definition 26.1 *The residue of f at an isolated singularity α which is a pole, written* $\operatorname{res}(f,\alpha)$ *is the coefficient of* $(z-\alpha)^{-1}$ *where*

$$f(z) = g(z) + \sum_{k=1}^{m} \frac{b_k}{(z-\alpha)^k}.$$

Thus $\operatorname{res}(f,\alpha) = b_1$ *in the above.*

At this point, recall Corollary 24.47 which is stated here for convenience.

Corollary 26.2 *Let Ω be an open set and let $\gamma_k : [a_k, b_k] \to \Omega$, $k = 1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^{m} n(\gamma_k, z) = 0$$

for all $z \notin \Omega$. Then if $f : \Omega \to \mathbb{C}$ is analytic,

$$\sum_{k=1}^{m} \int_{\gamma_k} f(w)\, dw = 0.$$

The following theorem is called the residue theorem. Note the resemblance to Corollary 24.47.

Theorem 26.3 *Let Ω be an open set and let $\gamma_k : [a_k, b_k] \to \Omega$, $k = 1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^{m} n(\gamma_k, z) = 0$$

for all $z \notin \Omega$. Then if $f : \Omega \to \widehat{\mathbb{C}}$ is meromorphic such that no γ_k^* contains any poles of f,

$$\frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} f(w)\, dw = \sum_{\alpha \in A} \operatorname{res}(f,\alpha) \sum_{k=1}^{m} n(\gamma_k, \alpha) \tag{26.1}$$

where here A denotes the set of poles of f in Ω. The sum on the right is a finite sum.

Proof: First note that there are at most finitely many α which are not in the unbounded component of $\mathbb{C} \setminus \cup_{k=1}^{m} \gamma_k ([a_k, b_k])$. Thus there exists a finite set, $\{\alpha_1, \cdots, \alpha_N\} \subseteq A$ such that these are the only possibilities for which $\sum_{k=1}^{n} n(\gamma_k, \alpha)$ might not equal zero. Therefore, 26.1 reduces to

$$\frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\gamma_k} f(w)\, dw = \sum_{j=1}^{N} \text{res}(f, \alpha_j) \sum_{k=1}^{n} n(\gamma_k, \alpha_j)$$

and it is this last equation which is established. Near α_j,

$$f(z) = g_j(z) + \sum_{r=1}^{m_j} \frac{b_r^j}{(z - \alpha_j)^r} \equiv g_j(z) + Q_j(z).$$

where g_j is analytic at and near α_j. Now define

$$G(z) \equiv f(z) - \sum_{j=1}^{N} Q_j(z).$$

It follows that $G(z)$ has a removable singularity at each α_j. Therefore, by Corollary 24.47,

$$0 = \sum_{k=1}^{m} \int_{\gamma_k} G(z)\, dz = \sum_{k=1}^{m} \int_{\gamma_k} f(z)\, dz - \sum_{j=1}^{N} \sum_{k=1}^{m} \int_{\gamma_k} Q_j(z)\, dz.$$

Now

$$\begin{aligned} \sum_{k=1}^{m} \int_{\gamma_k} Q_j(z)\, dz &= \sum_{k=1}^{m} \int_{\gamma_k} \left(\frac{b_1^j}{(z - \alpha_j)} + \sum_{r=2}^{m_j} \frac{b_r^j}{(z - \alpha_j)^r} \right) dz \\ &= \sum_{k=1}^{m} \int_{\gamma_k} \frac{b_1^j}{(z - \alpha_j)} dz \equiv \sum_{k=1}^{m} n(\gamma_k, \alpha_j) \text{res}(f, \alpha_j)(2\pi i). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k=1}^{m} \int_{\gamma_k} f(z)\, dz &= \sum_{j=1}^{N} \sum_{k=1}^{m} \int_{\gamma_k} Q_j(z)\, dz \\ &= \sum_{j=1}^{N} \sum_{k=1}^{m} n(\gamma_k, \alpha_j) \text{res}(f, \alpha_j)(2\pi i) \\ &= 2\pi i \sum_{j=1}^{N} \text{res}(f, \alpha_j) \sum_{k=1}^{m} n(\gamma_k, \alpha_j) \\ &= (2\pi i) \sum_{\alpha \in A} \text{res}(f, \alpha) \sum_{k=1}^{m} n(\gamma_k, \alpha) \end{aligned}$$

which proves the theorem.

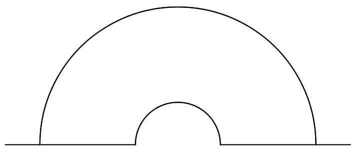
The following is an important example. This example can also be done by real variable methods and there are some who think that real variable methods are always to be preferred to complex variable methods. However, I will use the above theorem to work this example.

Example 26.4 *Find* $\lim_{R\to\infty}\int_{-R}^{R}\frac{\sin(x)}{x}dx$

Things are easier if you write it as

$$\lim_{R\to\infty}\frac{1}{i}\left(\int_{-R}^{-R^{-1}}\frac{e^{ix}}{x}dx+\int_{R^{-1}}^{R}\frac{e^{ix}}{x}dx\right).$$

This gives the same answer because $\cos(x)/x$ is odd. Consider the following contour in which the orientation involves counterclockwise motion exactly once around.



Denote by $\gamma_{R^{-1}}$ the little circle and γ_R the big one. Then on the inside of this contour there are no singularities of e^{iz}/z and it is contained in an open set with the property that the winding number with respect to this contour about any point not in the open set equals zero. By Theorem 24.22

$$\frac{1}{i}\left(\int_{-R}^{-R^{-1}}\frac{e^{ix}}{x}dx+\int_{\gamma_{R^{-1}}}\frac{e^{iz}}{z}dz+\int_{R^{-1}}^{R}\frac{e^{ix}}{x}dx+\int_{\gamma_R}\frac{e^{iz}}{z}dz\right)=0 \tag{26.2}$$

Now

$$\left|\int_{\gamma_R}\frac{e^{iz}}{z}dz\right|=\left|\int_0^{\pi}e^{R(i\cos\theta-\sin\theta)}id\theta\right|\leq\int_0^{\pi}e^{-R\sin\theta}d\theta$$

and this last integral converges to 0 by the dominated convergence theorem. Now consider the other circle. By the dominated convergence theorem again,

$$\int_{\gamma_{R^{-1}}}\frac{e^{iz}}{z}dz=\int_{\pi}^{0}e^{R^{-1}(i\cos\theta-\sin\theta)}id\theta\to-i\pi$$

as $R \to \infty$. Then passing to the limit in 26.2,

$$\begin{aligned}
& \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin(x)}{x} dx \\
= & \lim_{R\to\infty} \frac{1}{i} \left(\int_{-R}^{-R^{-1}} \frac{e^{ix}}{x} dx + \int_{R^{-1}}^{R} \frac{e^{ix}}{x} dx \right) \\
= & \lim_{R\to\infty} \frac{1}{i} \left(- \int_{\gamma_{R^{-1}}} \frac{e^{iz}}{z} dz - \int_{\gamma_R} \frac{e^{iz}}{z} dz \right) = \frac{-1}{i} (-i\pi) = \pi.
\end{aligned}$$

Example 26.5 *Find* $\lim_{R\to\infty} \int_{-R}^{R} e^{ixt} \frac{\sin x}{x} dx$. *Note this is essentially finding the inverse Fourier transform of the function,* $\sin(x)/x$.

This equals

$$\begin{aligned}
& \lim_{R\to\infty} \int_{-R}^{R} (\cos(xt) + i \sin(xt)) \frac{\sin(x)}{x} dx \\
= & \lim_{R\to\infty} \int_{-R}^{R} \cos(xt) \frac{\sin(x)}{x} dx \\
= & \lim_{R\to\infty} \int_{-R}^{R} \cos(xt) \frac{\sin(x)}{x} dx \\
= & \lim_{R\to\infty} \frac{1}{2} \int_{-R}^{R} \frac{\sin(x(t+1)) + \sin(x(1-t))}{x} dx
\end{aligned}$$

Let $t \neq 1, -1$. Then changing variables yields

$$\lim_{R\to\infty} \left(\frac{1}{2} \int_{-R(1+t)}^{R(1+t)} \frac{\sin(u)}{u} du + \frac{1}{2} \int_{-R(1-t)}^{R(1-t)} \frac{\sin(u)}{u} du \right).$$

In case $|t| < 1$ Example 26.4 implies this limit is π. However, if $t > 1$ the limit equals 0 and this is also the case if $t < -1$. Summarizing,

$$\lim_{R\to\infty} \int_{-R}^{R} e^{ixt} \frac{\sin x}{x} dx = \begin{cases} \pi \text{ if } |t| < 1 \\ 0 \text{ if } |t| > 1 \end{cases}.$$

26.1 Rouche's Theorem And The Argument Principle

26.1.1 Argument Principle

A simple closed curve is just one which is homeomorphic to the unit circle. The Jordan Curve theorem states that every simple closed curve in the plane divides the plane into exactly two connected components, one bounded and the other unbounded. This is a very hard theorem to prove. However, in most applications the conclusion is obvious. Nevertheless, to avoid using this big topological result and to attain some extra generality, I will state the following theorem in terms of the winding number to avoid using it. This theorem is called the argument principle. First recall that f has a zero of order m at α if $f(z) = g(z)(z-\alpha)^m$ where g is an analytic function which is not equal to zero at α. This is equivalent to having $f(z) = \sum_{k=m}^{\infty} a_k (z-\alpha)^k$ for z near α where $a_m \neq 0$. Also recall that f has a pole of order m at α if for z near α, $f(z)$ is of the form

$$f(z) = h(z) + \sum_{k=1}^{m} \frac{b_k}{(z-\alpha)^k} \tag{26.3}$$

where $b_m \neq 0$ and h is a function analytic near α.

Theorem 26.6 *(argument principle) Let f be meromorphic in Ω. Also suppose γ^* is a closed bounded variation curve containing none of the poles or zeros of f with the property that for all $z \notin \Omega, n(\gamma, z) = 0$ and for all $z \in \Omega, n(\gamma, z)$ either equals 0 or 1. Now let $\{p_1, \cdots, p_m\}$ and $\{z_1, \cdots, z_n\}$ be respectively the poles and zeros for which the winding number of γ about these points equals 1. Let z_k be a zero of order r_k and let p_k be a pole of order l_k. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{n} r_k - \sum_{k=1}^{m} l_k$$

Proof: This theorem follows from computing the residues of f'/f. It has residues at poles and zeros. I will do this now. First suppose f has a pole of order p at α. Then f has the form given in 26.3. Therefore,

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{h'(z) - \sum_{k=1}^{p} \frac{k b_k}{(z-\alpha)^{k+1}}}{h(z) + \sum_{k=1}^{p} \frac{b_k}{(z-\alpha)^k}} \\ &= \frac{h'(z)(z-\alpha)^p - \sum_{k=1}^{p-1} k b_k (z-\alpha)^{-k-1+p} - \frac{p b_p}{(z-\alpha)}}{h(z)(z-\alpha)^p + \sum_{k=1}^{p-1} b_k (z-\alpha)^{p-k} + b_p} \end{aligned}$$

This is of the form

$$= \frac{b_p}{s(z) + b_p} \frac{r(z) - \frac{p b_p}{(z-\alpha)}}{b_p} = \frac{b_p}{s(z) + b_p} \left(\frac{r(z)}{b_p} - \frac{p}{(z-\alpha)} \right)$$

where $s(\alpha) = r(\alpha) = 0$. From this, it is clear $\operatorname{res}(f'/f, \alpha) = -p$, the order of the pole.

Next suppose f has a zero of order p at α. Then

$$\frac{f'(z)}{f(z)} = \frac{\sum_{k=p}^{\infty} a_k k (z-\alpha)^{k-1}}{\sum_{k=p}^{\infty} a_k (z-\alpha)^k} = \frac{\sum_{k=p}^{\infty} a_k k (z-\alpha)^{k-1-p}}{\sum_{k=p}^{\infty} a_k (z-\alpha)^{k-p}}$$

and from this it is clear $\operatorname{res}(f'/f) = p$, the order of the zero. The conclusion of this theorem now follows from Theorem 26.3.

One can also generalize the theorem to the case where there are many closed curves involved. This is proved in the same way as the above.

Theorem 26.7 *(argument principle) Let f be meromorphic in Ω and let $\gamma_k : [a_k, b_k] \to \Omega$, $k = 1, \cdots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^{m} n(\gamma_k, z) = 0$$

and for all $z \notin \Omega$ and for $z \in \Omega$, $\sum_{k=1}^{m} n(\gamma_k, z)$ either equals 0 or 1. Now let $\{p_1, \cdots, p_m\}$ and $\{z_1, \cdots, z_n\}$ be respectively the poles and zeros for which the above sum of winding numbers equals 1. Let z_k be a zero of order r_k and let p_k be a pole of order l_k. Then

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}dz = \sum_{k=1}^{n} r_k - \sum_{k=1}^{m} l_k$$

There is also a simple extension of this important principle which I found in [27].

Theorem 26.8 *(argument principle) Let f be meromorphic in Ω. Also suppose γ^* is a closed bounded variation curve with the property that for all $z \notin \Omega, n(\gamma, z) = 0$ and for all $z \in \Omega, n(\gamma, z)$ either equals 0 or 1. Now let $\{p_1, \cdots, p_m\}$ and $\{z_1, \cdots, z_n\}$ be respectively the poles and zeros for which the winding number of γ about these points equals 1 listed according to multiplicity. Thus if there is a pole of order m there will be this value repeated m times in the list for the poles. Also let $g(z)$ be an analytic function. Then*

$$\frac{1}{2\pi i}\int_{\gamma} g(z)\frac{f'(z)}{f(z)}dz = \sum_{k=1}^{n} g(z_k) - \sum_{k=1}^{m} g(p_k)$$

Proof: This theorem follows from computing the residues of $g(f'/f)$. It has residues at poles and zeros. I will do this now. First suppose f has a pole of order m at α. Then f has the form given in 26.3. Therefore,

$$\begin{aligned} & g(z)\frac{f'(z)}{f(z)} \\ = \ & \frac{g(z)\left(h'(z) - \sum_{k=1}^{m} \frac{kb_k}{(z-\alpha)^{k+1}}\right)}{h(z) + \sum_{k=1}^{m} \frac{b_k}{(z-\alpha)^k}} \\ = \ & g(z)\frac{h'(z)(z-\alpha)^m - \sum_{k=1}^{m-1} kb_k (z-\alpha)^{-k-1+m} - \frac{mb_m}{(z-\alpha)}}{h(z)(z-\alpha)^m + \sum_{k=1}^{m-1} b_k (z-\alpha)^{m-k} + b_m} \end{aligned}$$

From this, it is clear $\text{res}(g(f'/f), \alpha) = -mg(\alpha)$, where m is the order of the pole. Thus α would have been listed m times in the list of poles. Hence the residue at this point is equivalent to adding $-g(\alpha)$ m times.

Next suppose f has a zero of order m at α. Then

$$g(z)\frac{f'(z)}{f(z)} = g(z)\frac{\sum_{k=m}^{\infty} a_k k (z-\alpha)^{k-1}}{\sum_{k=m}^{\infty} a_k (z-\alpha)^{k}} = g(z)\frac{\sum_{k=m}^{\infty} a_k k (z-\alpha)^{k-1-m}}{\sum_{k=m}^{\infty} a_k (z-\alpha)^{k-m}}$$

and from this it is clear $\operatorname{res}(g(f'/f)) = g(\alpha)m$, where m is the order of the zero. The conclusion of this theorem now follows from the residue theorem, Theorem 26.3.

The way people usually apply these theorems is to suppose γ^* is a simple closed bounded variation curve, often a circle. Thus it has an inside and an outside, the outside being the unbounded component of $\mathbb{C}\setminus\gamma^*$. The orientation of the curve is such that you go around it once in the counterclockwise direction. Then letting r_k and l_k be as described, the conclusion of the theorem follows. In applications, this is likely the way it will be.

26.1.2 Rouche's Theorem

With the argument principle, it is possible to prove Rouche's theorem . In the argument principle, denote by Z_f the quantity $\sum_{k=1}^{m} r_k$ and by P_f the quantity $\sum_{k=1}^{n} l_k$. Thus Z_f is the number of zeros of f counted according to the order of the zero with a similar definition holding for P_f. Thus the conclusion of the argument principle is.

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = Z_f - P_f$$

Rouche's theorem allows the comparison of $Z_h - P_h$ for $h = f, g$. It is a wonderful and amazing result.

Theorem 26.9 *(Rouche's theorem)Let f, g be meromorphic in an open set Ω. Also suppose γ^* is a closed bounded variation curve with the property that for all $z \notin \Omega, n(\gamma, z) = 0$, no zeros or poles are on γ^*, and for all $z \in \Omega, n(\gamma, z)$ either equals 0 or 1. Let Z_f and P_f denote respectively the numbers of zeros and poles of f, which have the property that the winding number equals 1, counted according to order, with Z_g and P_g being defined similarly. Also suppose that for $z \in \gamma^*$*

$$|f(z) + g(z)| < |f(z)| + |g(z)|. \tag{26.4}$$

Then

$$Z_f - P_f = Z_g - P_g.$$

Proof: From the hypotheses,

$$\left|1 + \frac{f(z)}{g(z)}\right| < 1 + \left|\frac{f(z)}{g(z)}\right|$$

which shows that for all $z \in \gamma^*$,

$$\frac{f(z)}{g(z)} \in \mathbb{C}\setminus[0,\infty).$$

Letting l denote a branch of the logarithm defined on $\mathbb{C} \setminus [0, \infty)$, it follows that $l\left(\frac{f(z)}{g(z)}\right)$ is a primitive for the function,

$$\frac{(f/g)'}{(f/g)} = \frac{f'}{f} - \frac{g'}{g}.$$

Therefore, by the argument principle,

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_\gamma \frac{(f/g)'}{(f/g)} dz = \frac{1}{2\pi i} \int_\gamma \left(\frac{f'}{f} - \frac{g'}{g} \right) dz \\ &= Z_f - P_f - (Z_g - P_g). \end{aligned}$$

This proves the theorem.

Often another condition other than 26.4 is used.

Corollary 26.10 *In the situation of Theorem 26.9 change 26.4 to the condition,*

$$|f(z) - g(z)| < |f(z)|$$

for $z \in \gamma^*$. Then the conclusion is the same.

Proof: The new condition implies $\left|1 - \frac{g}{f}(z)\right| < \left|\frac{g(z)}{f(z)}\right|$ on γ^*. Therefore, $\frac{g(z)}{f(z)} \notin (-\infty, 0]$ and so you can do the same argument with a branch of the logarithm.

26.1.3 A Different Formulation

In [47] I found this modification for Rouche's theorem concerned with the counting of zeros of analytic functions. This is a very useful form of Rouche's theorem because it makes no mention of a contour.

Theorem 26.11 *Let Ω be a bounded open set and suppose f, g are continuous on $\overline{\Omega}$ and analytic on Ω. Also suppose $|f(z)| < |g(z)|$ on $\partial\Omega$. Then g and $f + g$ have the same number of zeros in Ω provided each zero is counted according to multiplicity.*

Proof: Let $K = \left\{z \in \overline{\Omega} : |f(z)| \geq |g(z)|\right\}$. Then letting $\lambda \in [0, 1]$, if $z \notin K$, then $|f(z)| < |g(z)|$ and so

$$0 < |g(z)| - |f(z)| \leq |g(z)| - \lambda |f(z)| \leq |g(z) + \lambda f(z)|$$

which shows that all zeros of $g + \lambda f$ are contained in K which must be a compact subset of Ω due to the assumption that $|f(z)| < |g(z)|$ on $\partial\Omega$. By Theorem 24.52 on Page 675 there exists a cycle, $\{\gamma_k\}_{k=1}^n$ such that $\cup_{k=1}^n \gamma_k^* \subseteq \Omega \setminus K$, $\sum_{k=1}^n n(\gamma_k, z) = 1$ for every $z \in K$ and $\sum_{k=1}^n n(\gamma_k, z) = 0$ for all $z \notin \Omega$. Then as above, it follows from the residue theorem or more directly, Theorem 26.7,

$$\sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{\lambda f'(z) + g'(z)}{\lambda f(z) + g(z)} dz = \sum_{j=1}^p m_j$$

where m_j is the order of the j^{th} zero of $\lambda f + g$ in K, hence in Ω. However,

$$\lambda \to \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{\lambda f'(z) + g'(z)}{\lambda f(z) + g(z)} dz$$

is integer valued and continuous so it gives the same value when $\lambda = 0$ as when $\lambda = 1$. When $\lambda = 0$ this gives the number of zeros of g in Ω and when $\lambda = 1$ it is the number of zeros of $f + g$. This proves the theorem.

Here is another formulation of this theorem.

Corollary 26.12 *Let Ω be a bounded open set and suppose f, g are continuous on $\overline{\Omega}$ and analytic on Ω. Also suppose $|f(z) - g(z)| < |g(z)|$ on $\partial\Omega$. Then f and g have the same number of zeros in Ω provided each zero is counted according to multiplicity.*

Proof: You let $f - g$ play the role of f in Theorem 26.11. Thus $f - g + g = f$ and g have the same number of zeros. Alternatively, you can give a proof of this directly as follows.

Let $K = \{z \in \Omega : |f(z) - g(z)| \geq |g(z)|\}$. Then if $g(z) + \lambda(f(z) - g(z)) = 0$ it follows

$$\begin{aligned} 0 &= |g(z) + \lambda(f(z) - g(z))| \geq |g(z)| - \lambda|f(z) - g(z)| \\ &\geq |g(z)| - |f(z) - g(z)| \end{aligned}$$

and so $z \in K$. Thus all zeros of $g(z) + \lambda(f(z) - g(z))$ are contained in K. By Theorem 24.52 on Page 675 there exists a cycle, $\{\gamma_k\}_{k=1}^n$ such that $\cup_{k=1}^n \gamma_k^* \subseteq \Omega \setminus K, \sum_{k=1}^n n(\gamma_k, z) = 1$ for every $z \in K$ and $\sum_{k=1}^n n(\gamma_k, z) = 0$ for all $z \notin \Omega$. Then by Theorem 26.7,

$$\sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{\lambda(f'(z) - g'(z)) + g'(z)}{\lambda(f(z) - g(z)) + g(z)} dz = \sum_{j=1}^{p} m_j$$

where m_j is the order of the j^{th} zero of $\lambda(f - g) + g$ in K, hence in Ω. The left side is continuous as a function of λ and so the number of zeros of g corresponding to $\lambda = 0$ equals the number of zeros of f corresponding to $\lambda = 1$. This proves the corollary.

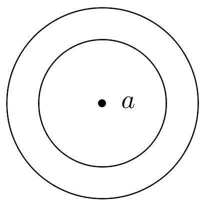
26.2 Singularities And The Laurent Series

26.2.1 What Is An Annulus?

In general, when you consider singularities, isolated or not, the fundamental tool is the Laurent series. This series is important for many other reasons also. In particular, it is fundamental to the spectral theory of various operators in functional analysis and is one way to obtain relationships between algebraic and analytical conditions essential in various convergence theorems. A Laurent series lives on an annulus. In all this f has values in X where X is a complex Banach space. If you like, let $X = \mathbb{C}$.

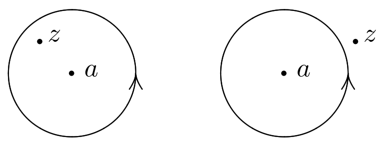
Definition 26.13 *Define* $\text{ann}\,(a, R_1, R_2) \equiv \{z : R_1 < |z-a| < R_2\}$.

Thus $\text{ann}\,(a, 0, R)$ would denote the punctured ball, $B(a, R) \setminus \{0\}$ and when $R_1 > 0$, the annulus looks like the following.



The annulus is the stuff between the two circles.

Here is an important lemma which is concerned with the situation described in the following picture.



Lemma 26.14 *Let* $\gamma_r(t) \equiv a + re^{it}$ *for* $t \in [0, 2\pi]$ *and let* $|z-a| < r$. *Then* $n(\gamma_r, z) = 1$. *If* $|z-a| > r$, *then* $n(\gamma_r, z) = 0$.

Proof: For the first claim, consider for $t \in [0, 1]$,

$$f(t) \equiv n(\gamma_r, a + t(z-a)).$$

Then from properties of the winding number derived earlier, $f(t) \in \mathbb{Z}$, f is continuous, and $f(0) = 1$. Therefore, $f(t) = 1$ for all $t \in [0, 1]$. This proves the first claim because $f(1) = n(\gamma_r, z)$.

For the second claim,

$$\begin{aligned} n(\gamma_r, z) &= \frac{1}{2\pi i}\int_{\gamma_r} \frac{1}{w-z}\,dw \\ &= \frac{1}{2\pi i}\int_{\gamma_r} \frac{1}{w-a-(z-a)}\,dw \\ &= \frac{1}{2\pi i}\frac{-1}{z-a}\int_{\gamma_r} \frac{1}{1-\left(\frac{w-a}{z-a}\right)}\,dw \\ &= \frac{-1}{2\pi i(z-a)}\int_{\gamma_r} \sum_{k=0}^{\infty}\left(\frac{w-a}{z-a}\right)^k dw. \end{aligned}$$

The series converges uniformly for $w \in \gamma_r$ because

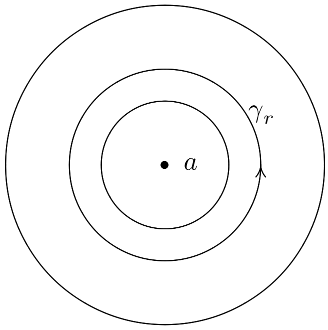
$$\left|\frac{w-a}{z-a}\right| = \frac{r}{r+c}$$

for some $c > 0$ due to the assumption that $|z-a| > r$. Therefore, the sum and the integral can be interchanged to give

$$n(\gamma_r, z) = \frac{-1}{2\pi i (z-a)} \sum_{k=0}^{\infty} \int_{\gamma_r} \left(\frac{w-a}{z-a}\right)^k dw = 0$$

because $w \rightarrow \left(\frac{w-a}{z-a}\right)^k$ has an antiderivative. This proves the lemma.

Now consider the following picture which pertains to the next lemma.



Lemma 26.15 *Let g be analytic on* $\text{ann}(a, R_1, R_2)$. *Then if $\gamma_r(t) \equiv a + re^{it}$ for $t \in [0, 2\pi]$ and $r \in (R_1, R_2)$, then $\int_{\gamma_r} g(z)\, dz$ is independent of r.*

Proof: Let $R_1 < r_1 < r_2 < R_2$ and denote by $-\gamma_r(t)$ the curve, $-\gamma_r(t) \equiv a + re^{i(2\pi - t)}$ for $t \in [0, 2\pi]$. Then if $z \in \overline{B(a, R_1)}$, Lemma 26.14 implies both $n(\gamma_{r_2}, z)$ and $n(\gamma_{r_1}, z) = 1$ and so

$$n(-\gamma_{r_1}, z) + n(\gamma_{r_2}, z) = -1 + 1 = 0.$$

Also if $z \notin B(a, R_2)$, then Lemma 26.14 implies $n\left(\gamma_{r_j}, z\right) = 0$ for $j = 1, 2$. Therefore, whenever $z \notin \text{ann}(a, R_1, R_2)$, the sum of the winding numbers equals zero. Therefore, by Theorem 24.46 applied to the function, $f(w) = g(z)(w - z)$ and $z \in \text{ann}(a, R_1, R_2) \setminus \cup_{j=1}^{2} \gamma_{r_j}([0, 2\pi])$,

$$\begin{aligned} f(z)\left(n(\gamma_{r_2}, z) + n(-\gamma_{r_1}, z)\right) = 0\left(n(\gamma_{r_2}, z) + n(-\gamma_{r_1}, z)\right) = \\ \frac{1}{2\pi i} \int_{\gamma_{r_2}} \frac{g(w)(w-z)}{w-z} dw - \frac{1}{2\pi i} \int_{\gamma_{r_1}} \frac{g(w)(w-z)}{w-z} dw \\ = \frac{1}{2\pi i} \int_{\gamma_{r_2}} g(w)\, dw - \frac{1}{2\pi i} \int_{\gamma_{r_1}} g(w)\, dw \end{aligned}$$

which proves the desired result.

26.2.2 The Laurent Series

The Laurent series is like a power series except it allows for negative exponents. First here is a definition of what is meant by the convergence of such a series.

Definition 26.16 *$\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ converges if both the series,*

$$\sum_{n=0}^{\infty} a_n (z-a)^n \text{ and } \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$$

converge. When this is the case, the symbol, $\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ is defined as

$$\sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}.$$

Lemma 26.17 *Suppose*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

for all $|z-a| \in (R_1, R_2)$. Then both $\sum_{n=0}^{\infty} a_n (z-a)^n$ and $\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$ converge absolutely and uniformly on $\{z : r_1 \le |z-a| \le r_2\}$ for any $r_1 < r_2$ satisfying $R_1 < r_1 < r_2 < R_2$.

Proof: Let $R_1 < |w-a| = r_1 - \delta < r_1$. Then $\sum_{n=1}^{\infty} a_{-n} (w-a)^{-n}$ converges and so

$$\lim_{n\to\infty} |a_{-n}| |w-a|^{-n} = \lim_{n\to\infty} |a_{-n}| (r_1-\delta)^{-n} = 0$$

which implies that for all n sufficiently large,

$$|a_{-n}| (r_1-\delta)^{-n} < 1.$$

Therefore,

$$\sum_{n=1}^{\infty} |a_{-n}| |z-a|^{-n} = \sum_{n=1}^{\infty} |a_{-n}| (r_1-\delta)^{-n} (r_1-\delta)^n |z-a|^{-n}.$$

Now for $|z-a| \ge r_1$,

$$|z-a|^{-n} \le \frac{1}{r_1^n}$$

and so for all sufficiently large n

$$|a_{-n}| |z-a|^{-n} \le \frac{(r_1-\delta)^n}{r_1^n}.$$

Therefore, by the Weierstrass M test, the series, $\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$ converges absolutely and uniformly on the set

$$\{z \in \mathbb{C} : |z-a| \ge r_1\}.$$

Similar reasoning shows the series, $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges uniformly on the set

$$\{z \in \mathbb{C} : |z-a| \leq r_2\} .$$

This proves the Lemma.

Theorem 26.18 *Let f be analytic on $\operatorname{ann}(a, R_1, R_2)$. Then there exist numbers, $a_n \in \mathbb{C}$ such that for all $z \in \operatorname{ann}(a, R_1, R_2)$,*

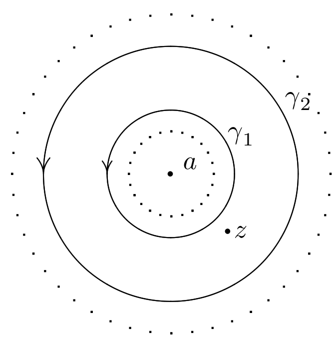
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n , \tag{26.5}$$

where the series converges absolutely and uniformly on $\overline{\operatorname{ann}(a, r_1, r_2)}$ whenever $R_1 < r_1 < r_2 < R_2$. Also

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{(w-a)^{n+1}} dw \tag{26.6}$$

where $\gamma(t) = a + re^{it}, t \in [0, 2\pi]$ for any $r \in (R_1, R_2)$. Furthermore the series is unique in the sense that if 26.5 holds for $z \in \operatorname{ann}(a, R_1, R_2)$, then a_n is given in 26.6.

Proof: Let $R_1 < r_1 < r_2 < R_2$ and define $\gamma_1(t) \equiv a + (r_1 - \varepsilon) e^{it}$ and $\gamma_2(t) \equiv a + (r_2 + \varepsilon) e^{it}$ for $t \in [0, 2\pi]$ and ε chosen small enough that $R_1 < r_1 - \varepsilon < r_2 + \varepsilon < R_2$.



Then using Lemma 26.14, if $z \notin \operatorname{ann}(a, R_1, R_2)$ then

$$n(-\gamma_1, z) + n(\gamma_2, z) = 0$$

and if $z \in \operatorname{ann}(a, r_1, r_2)$,

$$n(-\gamma_1, z) + n(\gamma_2, z) = 1.$$

Therefore, by Theorem 24.46, for $z \in \text{ann}(a, r_1, r_2)$

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\left[\int_{-\gamma_1} \frac{f(w)}{w-z}dw + \int_{\gamma_2} \frac{f(w)}{w-z}dw\right] \\ &= \frac{1}{2\pi i}\left[\int_{\gamma_1} \frac{f(w)}{(z-a)\left[1-\frac{w-a}{z-a}\right]}dw + \int_{\gamma_2} \frac{f(w)}{(w-a)\left[1-\frac{z-a}{w-a}\right]}dw\right] \\ &= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(w)}{w-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{w-a}\right)^n dw + \\ &\quad \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(w)}{(z-a)}\sum_{n=0}^{\infty}\left(\frac{w-a}{z-a}\right)^n dw. \end{aligned} \tag{26.7}$$

From the formula 26.7, it follows that for $z \in \overline{\text{ann}(a, r_1, r_2)}$, the terms in the first sum are bounded by an expression of the form $C\left(\frac{r_2}{r_2+\varepsilon}\right)^n$ while those in the second are bounded by one of the form $C\left(\frac{r_1-\varepsilon}{r_1}\right)^n$ and so by the Weierstrass M test, the convergence is uniform and so the integrals and the sums in the above formula may be interchanged and after renaming the variable of summation, this yields

$$\begin{aligned} f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_2}\frac{f(w)}{(w-a)^{n+1}}dw\right)(z-a)^n + \\ \sum_{n=-\infty}^{-1}\left(\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(w)}{(w-a)^{n+1}}\right)(z-a)^n. \end{aligned} \tag{26.8}$$

Therefore, by Lemma 26.15, for any $r \in (R_1, R_2)$,

$$\begin{aligned} f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_r}\frac{f(w)}{(w-a)^{n+1}}dw\right)(z-a)^n + \\ \sum_{n=-\infty}^{-1}\left(\frac{1}{2\pi i}\int_{\gamma_r}\frac{f(w)}{(w-a)^{n+1}}\right)(z-a)^n. \end{aligned} \tag{26.9}$$

and so

$$f(z) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{2\pi i}\int_{\gamma_r}\frac{f(w)}{(w-a)^{n+1}}dw\right)(z-a)^n.$$

where $r \in (R_1, R_2)$ is arbitrary. This proves the existence part of the theorem. It remains to characterize a_n.

If $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ on $\text{ann}(a, R_1, R_2)$ let

$$f_n(z) \equiv \sum_{k=-n}^{n} a_k (z-a)^k. \tag{26.10}$$

This function is analytic in $\text{ann}(a, R_1, R_2)$ and so from the above argument,

$$f_n(z) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw \right) (z-a)^k . \tag{26.11}$$

Also if $k > n$ or if $k < -n$,

$$\left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw \right) = 0.$$

and so

$$f_n(z) = \sum_{k=-n}^{n} \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw \right) (z-a)^k$$

which implies from 26.10 that for each $k \in [-n, n]$,

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw = a_k$$

However, from the uniform convergence of the series,

$$\sum_{n=0}^{\infty} a_n (w-a)^n$$

and

$$\sum_{n=1}^{\infty} a_{-n} (w-a)^{-n}$$

ensured by Lemma 26.17 which allows the interchange of sums and integrals, if $k \in [-n, n]$,

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{k+1}} dw \\
= &\frac{1}{2\pi i} \int_{\gamma_r} \frac{\sum_{m=0}^{\infty} a_m (w-a)^m + \sum_{m=1}^{\infty} a_{-m} (w-a)^{-m}}{(w-a)^{k+1}} dw \\
= &\sum_{m=0}^{\infty} a_m \frac{1}{2\pi i} \int_{\gamma_r} (w-a)^{m-(k+1)} dw \\
&+ \sum_{m=1}^{\infty} a_{-m} \int_{\gamma_r} (w-a)^{-m-(k+1)} dw \\
= &\sum_{m=0}^{n} a_m \frac{1}{2\pi i} \int_{\gamma_r} (w-a)^{m-(k+1)} dw \\
&+ \sum_{m=1}^{n} a_{-m} \int_{\gamma_r} (w-a)^{-m-(k+1)} dw \\
= &\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw
\end{aligned}$$

because if $l > n$ or $l < -n$,

$$\int_{\gamma_r} \frac{a_l (w-a)^l}{(w-a)^{k+1}} dw = 0$$

for all $k \in [-n, n]$. Therefore,

$$a_k = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{k+1}} dw$$

and so this establishes uniqueness. This proves the theorem.

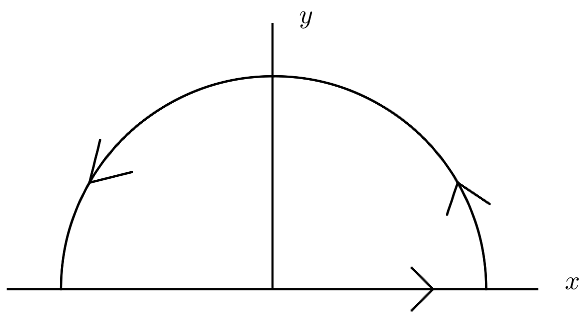
26.2.3 Contour Integrals And Evaluation Of Integrals

Here are some examples of hard integrals which can be evaluated by using residues. This will be done by integrating over various closed curves having bounded variation.

Example 26.19 *The first example we consider is the following integral.*

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$$

One could imagine evaluating this integral by the method of partial fractions and it should work out by that method. However, we will consider the evaluation of this integral by the method of residues instead. To do so, consider the following picture.



Let $\gamma_r(t) = re^{it}, t \in [0, \pi]$ and let $\sigma_r(t) = t : t \in [-r, r]$. Thus γ_r parameterizes the top curve and σ_r parameterizes the straight line from $-r$ to r along the x axis. Denoting by Γ_r the closed curve traced out by these two, we see from simple estimates that

$$\lim_{r\to\infty} \int_{\gamma_r} \frac{1}{1+z^4} dz = 0.$$

This follows from the following estimate.

$$\left|\int_{\gamma_r} \frac{1}{1+z^4} dz\right| \leq \frac{1}{r^4-1}\pi r.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \lim_{r\to\infty} \int_{\Gamma_r} \frac{1}{1+z^4} dz.$$

We compute $\int_{\Gamma_r} \frac{1}{1+z^4} dz$ using the method of residues. The only residues of the integrand are located at points, z where $1+z^4=0$. These points are

$$\begin{aligned} z &= -\frac{1}{2}\sqrt{2}-\frac{1}{2}i\sqrt{2}, z = \frac{1}{2}\sqrt{2}-\frac{1}{2}i\sqrt{2}, \\ z &= \frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}, z = -\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2} \end{aligned}$$

and it is only the last two which are found in the inside of Γ_r. Therefore, we need to calculate the residues at these points. Clearly this function has a pole of order one at each of these points and so we may calculate the residue at α in this list by evaluating

$$\lim_{z\to\alpha} (z-\alpha) \frac{1}{1+z^4}$$

Thus

$$\begin{aligned} &\operatorname{Res}\left(f, \frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right) \\ = &\lim_{z\to\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}} \left(z-\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)\right)\frac{1}{1+z^4} \\ = &-\frac{1}{8}\sqrt{2}-\frac{1}{8}i\sqrt{2} \end{aligned}$$

Similarly we may find the other residue in the same way

$$\begin{aligned} &\operatorname{Res}\left(f, -\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right) \\ = &\lim_{z\to-\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}} \left(z-\left(-\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right)\right)\frac{1}{1+z^4} \\ = &-\frac{1}{8}i\sqrt{2}+\frac{1}{8}\sqrt{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\Gamma_r} \frac{1}{1+z^4} dz &= 2\pi i\left(-\frac{1}{8}i\sqrt{2}+\frac{1}{8}\sqrt{2}+\left(-\frac{1}{8}\sqrt{2}-\frac{1}{8}i\sqrt{2}\right)\right) \\ &= \frac{1}{2}\pi\sqrt{2}. \end{aligned}$$

Thus, taking the limit we obtain $\frac{1}{2}\pi\sqrt{2} = \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$.

Obviously many different variations of this are possible. The main idea being that the integral over the semicircle converges to zero as $r \to \infty$.

Sometimes we don't blow up the curves and take limits. Sometimes the problem of interest reduces directly to a complex integral over a closed curve. Here is an example of this.

Example 26.20 *The integral is*

$$\int_0^\pi \frac{\cos\theta}{2+\cos\theta} d\theta$$

This integrand is even and so it equals

$$\frac{1}{2}\int_{-\pi}^{\pi} \frac{\cos\theta}{2+\cos\theta} d\theta.$$

For z on the unit circle, $z = e^{i\theta}$, $\overline{z} = \frac{1}{z}$ and therefore, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$. Thus $dz = ie^{i\theta} d\theta$ and so $d\theta = \frac{dz}{iz}$. Note this is proceeding formally to get a complex integral which reduces to the one of interest. It follows that a complex integral which reduces to the one desired is

$$\frac{1}{2i}\int_\gamma \frac{\frac{1}{2}\left(z+\frac{1}{z}\right)}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{z} = \frac{1}{2i}\int_\gamma \frac{z^2+1}{z\left(4z+z^2+1\right)} dz$$

where γ is the unit circle. Now the integrand has poles of order 1 at those points where $z\left(4z+z^2+1\right) = 0$. These points are

$$0, -2+\sqrt{3}, -2-\sqrt{3}.$$

Only the first two are inside the unit circle. It is also clear the function has simple poles at these points. Therefore,

$$\text{Res}\,(f,0) = \lim_{z\to 0} z\left(\frac{z^2+1}{z\left(4z+z^2+1\right)}\right) = 1.$$

$$\text{Res}\left(f, -2+\sqrt{3}\right) =$$

$$\lim_{z\to -2+\sqrt{3}} \left(z - \left(-2+\sqrt{3}\right)\right) \frac{z^2+1}{z\left(4z+z^2+1\right)} = -\frac{2}{3}\sqrt{3}.$$

It follows

$$\begin{aligned}
\int_0^\pi \frac{\cos\theta}{2+\cos\theta} d\theta &= \frac{1}{2i}\int_\gamma \frac{z^2+1}{z\left(4z+z^2+1\right)} dz \\
&= \frac{1}{2i} 2\pi i \left(1 - \frac{2}{3}\sqrt{3}\right) \\
&= \pi\left(1 - \frac{2}{3}\sqrt{3}\right).
\end{aligned}$$

Other rational functions of the trig functions will work out by this method also.

Sometimes you have to be clever about which version of an analytic function that reduces to a real function you should use. The following is such an example.

Example 26.21 *The integral here is*

$$\int_0^\infty \frac{\ln x}{1+x^4}dx.$$

The same curve used in the integral involving $\frac{\sin x}{x}$ earlier will create problems with the log since the usual version of the log is not defined on the negative real axis. This does not need to be of concern however. Simply use another branch of the logarithm. Leave out the ray from 0 along the negative y axis and use Theorem 25.5 to define $L(z)$ on this set. Thus $L(z) = \ln|z| + i\arg_1(z)$ where $\arg_1(z)$ will be the angle, θ, between $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$ such that $z = |z|e^{i\theta}$. Now the only singularities contained in this curve are

$$\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}, -\frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}$$

and the integrand, f has simple poles at these points. Thus using the same procedure as in the other examples,

$$\operatorname{Res}\left(f, \frac{1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right) =$$

$$\frac{1}{32}\sqrt{2}\pi - \frac{1}{32}i\sqrt{2}\pi$$

and

$$\operatorname{Res}\left(f, \frac{-1}{2}\sqrt{2}+\frac{1}{2}i\sqrt{2}\right) =$$

$$\frac{3}{32}\sqrt{2}\pi + \frac{3}{32}i\sqrt{2}\pi.$$

Consider the integral along the small semicircle of radius r. This reduces to

$$\int_\pi^0 \frac{\ln|r| + it}{1+(re^{it})^4}\left(rie^{it}\right)dt$$

which clearly converges to zero as $r \to 0$ because $r\ln r \to 0$. Therefore, taking the limit as $r \to 0$,

$$\int_{\text{large semicircle}} \frac{L(z)}{1+z^4}dz + \lim_{r\to 0+}\int_{-R}^{-r}\frac{\ln(-t)+i\pi}{1+t^4}dt +$$

$$\lim_{r\to 0+}\int_r^R \frac{\ln t}{1+t^4}dt = 2\pi i\left(\frac{3}{32}\sqrt{2}\pi + \frac{3}{32}i\sqrt{2}\pi + \frac{1}{32}\sqrt{2}\pi - \frac{1}{32}i\sqrt{2}\pi\right).$$

Observing that $\int_{\text{large semicircle}} \frac{L(z)}{1+z^4} dz \to 0$ as $R \to \infty$,

$$e(R) + 2 \lim_{r \to 0+} \int_r^R \frac{\ln t}{1+t^4} dt + i\pi \int_{-\infty}^0 \frac{1}{1+t^4} dt = \left(-\frac{1}{8} + \frac{1}{4} i\right) \pi^2 \sqrt{2}$$

where $e(R) \to 0$ as $R \to \infty$. From an earlier example this becomes

$$e(R) + 2 \lim_{r \to 0+} \int_r^R \frac{\ln t}{1+t^4} dt + i\pi \left(\frac{\sqrt{2}}{4} \pi\right) = \left(-\frac{1}{8} + \frac{1}{4} i\right) \pi^2 \sqrt{2}.$$

Now letting $r \to 0+$ and $R \to \infty$,

$$\begin{aligned} 2 \int_0^\infty \frac{\ln t}{1+t^4} dt &= \left(-\frac{1}{8} + \frac{1}{4} i\right) \pi^2 \sqrt{2} - i\pi \left(\frac{\sqrt{2}}{4} \pi\right) \\ &= -\frac{1}{8} \sqrt{2} \pi^2, \end{aligned}$$

and so

$$\int_0^\infty \frac{\ln t}{1+t^4} dt = -\frac{1}{16} \sqrt{2} \pi^2,$$

which is probably not the first thing you would thing of. You might try to imagine how this could be obtained using elementary techniques.

The next example illustrates the use of what is referred to as a branch cut. It includes many examples.

Example 26.22 *Mellin transformations are of the form*

$$\int_0^\infty f(x) x^\alpha \frac{dx}{x}.$$

Sometimes it is possible to evaluate such a transform in terms of the constant, α.

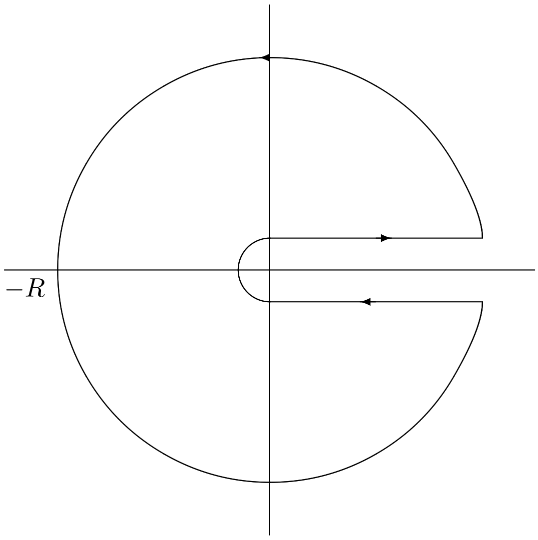
Assume f is an analytic function except at isolated singularities, none of which are on $(0, \infty)$. Also assume that f has the growth conditions,

$$|f(z)| \le \frac{C}{|z|^b}, b > \alpha$$

for all large $|z|$ and assume that

$$|f(z)| \le \frac{C'}{|z|^{b_1}}, b_1 < \alpha$$

for all $|z|$ sufficiently small. It turns out there exists an explicit formula for this Mellin transformation under these conditions. Consider the following contour.



In this contour the small semicircle in the center has radius ε which will converge to 0. Denote by γ_R the large circular path which starts at the upper edge of the slot and continues to the lower edge. Denote by γ_ε the small semicircular contour and denote by $\gamma_{\varepsilon R+}$ the straight part of the contour from 0 to R which provides the top edge of the slot. Finally denote by $\gamma_{\varepsilon R-}$ the straight part of the contour from R to 0 which provides the bottom edge of the slot. The interesting aspect of this problem is the definition of $f(z) z^{\alpha-1}$. Let

$$z^{\alpha-1} \equiv e^{(\ln|z|+i\arg(z))(\alpha-1)} = e^{(\alpha-1)\log(z)}$$

where $\arg(z)$ is the angle of z in $(0, 2\pi)$. Thus you use a branch of the logarithm which is defined on $\mathbb{C}\setminus(0,\infty)$. Then it is routine to verify from the assumed estimates that

$$\lim_{R\to\infty} \int_{\gamma_R} f(z) z^{\alpha-1} dz = 0$$

and

$$\lim_{\varepsilon\to 0+} \int_{\gamma_\varepsilon} f(z) z^{\alpha-1} dz = 0.$$

Also, it is routine to verify

$$\lim_{\varepsilon\to 0+} \int_{\gamma_{\varepsilon R+}} f(z) z^{\alpha-1} dz = \int_0^R f(x) x^{\alpha-1} dx$$

and

$$\lim_{\varepsilon\to 0+} \int_{\gamma_{\varepsilon R-}} f(z) z^{\alpha-1} dz = -e^{i2\pi(\alpha-1)} \int_0^R f(x) x^{\alpha-1} dx.$$

Therefore, letting Σ_R denote the sum of the residues of $f(z) z^{\alpha-1}$ which are contained in the disk of radius R except for the possible residue at 0,

$$e(R) + \left(1 - e^{i2\pi(\alpha-1)}\right) \int_0^R f(x) x^{\alpha-1} dx = 2\pi i \Sigma_R$$

where $e(R) \to 0$ as $R \to \infty$. Now letting $R \to \infty$,

$$\lim_{R\to\infty} \int_0^R f(x) x^{\alpha-1} dx = \frac{2\pi i}{1 - e^{i2\pi(\alpha-1)}} \Sigma = \frac{\pi e^{-\pi i \alpha}}{\sin(\pi\alpha)} \Sigma$$

where Σ denotes the sum of all the residues of $f(z) z^{\alpha-1}$ except for the residue at 0.

The next example is similar to the one on the Mellin transform. In fact it is a Mellin transform but is worked out independently of the above to emphasize a slightly more informal technique related to the contour.

Example 26.23 $\int_0^\infty \frac{x^{p-1}}{1+x} dx$, $p \in (0,1)$.

Since the exponent of x in the numerator is larger than -1. The integral does converge. However, the techniques of real analysis don't tell us what it converges to. The contour to be used is as follows: From $(\varepsilon, 0)$ to $(r, 0)$ along the x axis and then from $(r, 0)$ to $(r, 0)$ counter clockwise along the circle of radius r, then from $(r, 0)$ to $(\varepsilon, 0)$ along the x axis and from $(\varepsilon, 0)$ to $(\varepsilon, 0)$, clockwise along the circle of radius ε. You should draw a picture of this contour. The interesting thing about this is that z^{p-1} cannot be defined all the way around 0. Therefore, use a branch of z^{p-1} corresponding to the branch of the logarithm obtained by deleting the positive x axis. Thus

$$z^{p-1} = e^{(\ln|z| + iA(z))(p-1)}$$

where $z = |z| e^{iA(z)}$ and $A(z) \in (0, 2\pi)$. Along the integral which goes in the positive direction on the x axis, let $A(z) = 0$ while on the one which goes in the negative direction, take $A(z) = 2\pi$. This is the appropriate choice obtained by replacing the line from $(\varepsilon, 0)$ to $(r, 0)$ with two lines having a small gap joined by a circle of radius ε and then taking a limit as the gap closes. You should verify that the two integrals taken along the circles of radius ε and r converge to 0 as $\varepsilon \to 0$ and as $r \to \infty$. Therefore, taking the limit,

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx + \int_\infty^0 \frac{x^{p-1}}{1+x} \left(e^{2\pi i(p-1)}\right) dx = 2\pi i \operatorname{Res}(f, -1).$$

Calculating the residue of the integrand at -1, and simplifying the above expression,

$$\left(1 - e^{2\pi i(p-1)}\right) \int_0^\infty \frac{x^{p-1}}{1+x} dx = 2\pi i e^{(p-1)i\pi}.$$

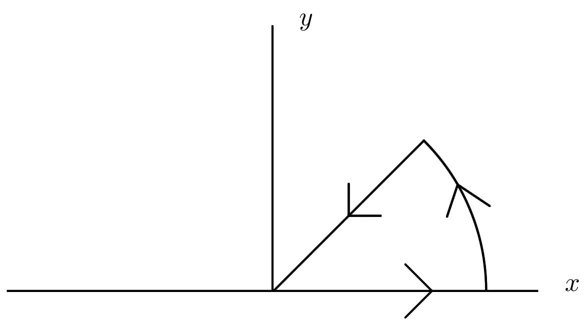
Upon simplification

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}.$$

Example 26.24 *The Fresnel integrals are*

$$\int_0^\infty \cos\left(x^2\right) dx, \int_0^\infty \sin\left(x^2\right) dx.$$

To evaluate these integrals consider $f(z) = e^{iz^2}$ on the curve which goes from the origin to the point r on the x axis and from this point to the point $r\left(\frac{1+i}{\sqrt{2}}\right)$ along a circle of radius r, and from there back to the origin as illustrated in the following picture.



Thus the curve to integrate over is shaped like a slice of pie. Denote by γ_r the curved part. Since f is analytic,

$$\begin{aligned} 0 &= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{ix^2} dx - \int_0^r e^{i\left(t\left(\frac{1+i}{\sqrt{2}}\right)\right)^2} \left(\frac{1+i}{\sqrt{2}}\right) dt \\ &= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{ix^2} dx - \int_0^r e^{-t^2} \left(\frac{1+i}{\sqrt{2}}\right) dt \\ &= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{ix^2} dx - \frac{\sqrt{\pi}}{2} \left(\frac{1+i}{\sqrt{2}}\right) + e(r) \end{aligned}$$

where $e(r) \to 0$ as $r \to \infty$. Here we used the fact that $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$. Now consider the first of these integrals.

$$\begin{aligned} \left| \int_{\gamma_r} e^{iz^2} dz \right| &= \left| \int_0^{\frac{\pi}{4}} e^{i\left(re^{it}\right)^2} rie^{it} dt \right| \\ &\leq r \int_0^{\frac{\pi}{4}} e^{-r^2 \sin 2t} dt \\ &= \frac{r}{2} \int_0^1 \frac{e^{-r^2 u}}{\sqrt{1-u^2}} du \end{aligned}$$

$$\leq \frac{r}{2} \int_0^{r^{-(3/2)}} \frac{1}{\sqrt{1-u^2}} du + \frac{r}{2} \left(\int_0^1 \frac{1}{\sqrt{1-u^2}} \right) e^{-\left(r^{1/2}\right)}$$

which converges to zero as $r \to \infty$. Therefore, taking the limit as $r \to \infty$,

$$\frac{\sqrt{\pi}}{2}\left(\frac{1+i}{\sqrt{2}}\right) = \int_0^\infty e^{ix^2}dx$$

and so

$$\int_0^\infty \sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}} = \int_0^\infty \cos x^2 dx.$$

The following example is one of the most interesting. By an auspicious choice of the contour it is possible to obtain a very interesting formula for $\cot \pi z$ known as the Mittag- Leffler expansion of $\cot \pi z$.

Example 26.25 *Let γ_N be the contour which goes from $-N-\frac{1}{2}-Ni$ horizontally to $N+\frac{1}{2}-Ni$ and from there, vertically to $N+\frac{1}{2}+Ni$ and then horizontally to $-N-\frac{1}{2}+Ni$ and finally vertically to $-N-\frac{1}{2}-Ni$. Thus the contour is a large rectangle and the direction of integration is in the counter clockwise direction. Consider the following integral.*

$$I_N \equiv \int_{\gamma_N} \frac{\pi \cos \pi z}{\sin \pi z\left(\alpha^2 - z^2\right)} dz$$

where $\alpha \in \mathbb{R}$ is not an integer. This will be used to verify the formula of Mittag Leffler,

$$\frac{1}{\alpha^2} + \sum_{n=1}^{\infty} \frac{2}{\alpha^2 - n^2} = \frac{\pi \cot \pi\alpha}{\alpha}. \tag{26.12}$$

You should verify that $\cot \pi z$ is bounded on this contour and that therefore, $I_N \to 0$ as $N \to \infty$. Now you compute the residues of the integrand at $\pm\alpha$ and at n where $|n| < N + \frac{1}{2}$ for n an integer. These are the only singularities of the integrand in this contour and therefore, you can evaluate I_N by using these. It is left as an exercise to calculate these residues and find that the residue at $\pm\alpha$ is

$$\frac{-\pi \cos \pi\alpha}{2\alpha \sin \pi\alpha}$$

while the residue at n is

$$\frac{1}{\alpha^2 - n^2}.$$

Therefore,

$$0 = \lim_{N\to\infty} I_N = \lim_{N\to\infty} 2\pi i\left[\sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} - \frac{\pi \cot \pi\alpha}{\alpha}\right]$$

which establishes the following formula of Mittag Leffler.

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} = \frac{\pi \cot \pi\alpha}{\alpha}.$$

Writing this in a slightly nicer form, yields 26.12.

26.3 The Spectral Radius Of A Bounded Linear Transformation

As a very important application of the theory of Laurent series, I will give a short description of the spectral radius. This is a fundamental result which must be understood in order to prove convergence of various important numerical methods such as the Gauss Seidel or Jacobi methods.

Definition 26.26 *Let X be a complex Banach space and let $A \in \mathcal{L}(X, X)$. Then*

$$r(A) \equiv \left\{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in \mathcal{L}(X, X)\right\}$$

This is called the resolvent set. The spectrum of A, denoted by $\sigma(A)$ is defined as all the complex numbers which are not in the resolvent set. Thus

$$\sigma(A) \equiv \mathbb{C} \setminus r(A)$$

Lemma 26.27 *$\lambda \in r(A)$ if and only if $\lambda I - A$ is one to one and onto X. Also if $|\lambda| > ||A||$, then $\lambda \in \sigma(A)$. If the Neumann series,*

$$\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^k$$

converges, then

$$\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^k = (\lambda I - A)^{-1}.$$

Proof: Note that to be in $r(A), \lambda I - A$ must be one to one and map X onto X since otherwise, $(\lambda I - A)^{-1} \notin \mathcal{L}(X, X)$.

By the open mapping theorem, if these two algebraic conditions hold, then $(\lambda I - A)^{-1}$ is continuous and so this proves the first part of the lemma. Now suppose $|\lambda| > ||A||$. Consider the Neumann series

$$\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^k.$$

By the root test, Theorem 24.3 on Page 642 this series converges to an element of $\mathcal{L}(X, X)$ denoted here by B. Now suppose the series converges. Letting $B_n \equiv \frac{1}{\lambda}\sum_{k=0}^{n}\left(\frac{A}{\lambda}\right)^k$,

$$\begin{aligned}
(\lambda I - A) B_n &= B_n(\lambda I - A) = \sum_{k=0}^{n}\left(\frac{A}{\lambda}\right)^k - \sum_{k=0}^{n}\left(\frac{A}{\lambda}\right)^{k+1} \\
&= I - \left(\frac{A}{\lambda}\right)^{n+1} \to I
\end{aligned}$$

as $n \to \infty$ because the convergence of the series requires the n^{th} term to converge to 0. Therefore,

$$(\lambda I - A) B = B (\lambda I - A) = I$$

which shows $\lambda I - A$ is both one to one and onto and the Neumann series converges to $(\lambda I - A)^{-1}$. This proves the lemma.

This lemma also shows that $\sigma(A)$ is bounded. In fact, $\sigma(A)$ is closed.

Lemma 26.28 *$r(A)$ is open. In fact, if $\lambda \in r(A)$ and $|\mu - \lambda| < \left|\left|(\lambda I - A)^{-1}\right|\right|^{-1}$, then $\mu \in r(A)$.*

Proof: First note

$$(\mu I - A) = \left(I - (\lambda - \mu)(\lambda I - A)^{-1}\right)(\lambda I - A) \tag{26.13}$$

$$= (\lambda I - A)\left(I - (\lambda - \mu)(\lambda I - A)^{-1}\right) \tag{26.14}$$

Also from the assumption about $|\lambda - \mu|$,

$$\left|\left|(\lambda - \mu)(\lambda I - A)^{-1}\right|\right| \leq |\lambda - \mu| \left|\left|(\lambda I - A)^{-1}\right|\right| < 1$$

and so by the root test,

$$\sum_{k=0}^{\infty} \left((\lambda - \mu)(\lambda I - A)^{-1}\right)^k$$

converges to an element of $\mathcal{L}(X, X)$. As in Lemma 26.27,

$$\sum_{k=0}^{\infty} \left((\lambda - \mu)(\lambda I - A)^{-1}\right)^k = \left(I - (\lambda - \mu)(\lambda I - A)^{-1}\right)^{-1}.$$

Therefore, from 26.13,

$$(\mu I - A)^{-1} = (\lambda I - A)^{-1} \left(I - (\lambda - \mu)(\lambda I - A)^{-1}\right)^{-1}.$$

This proves the lemma.

Corollary 26.29 *$\sigma(A)$ is a compact set.*

Proof: Lemma 26.27 shows $\sigma(A)$ is bounded and Lemma 26.28 shows it is closed.

Definition 26.30 *The spectral radius, denoted by $\rho(A)$ is defined by*

$$\rho(A) \equiv \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

Since $\sigma(A)$ is compact, this maximum exists. Note from Lemma 26.27, $\rho(A) \leq ||A||$.

There is a simple formula for the spectral radius.

Lemma 26.31 *If* $|\lambda| > \rho(A)$, *then the Neumann series,*

$$\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^k$$

converges.

Proof: This follows directly from Theorem 26.18 on Page 717 and the observation above that $\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^k = (\lambda I - A)^{-1}$ for all $|\lambda| > ||A||$. Thus the analytic function, $\lambda \to (\lambda I - A)^{-1}$ has a Laurent expansion on $|\lambda| > \rho(A)$ by Theorem 26.18 and it must coincide with $\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^k$ on $|\lambda| > ||A||$ so the Laurent expansion of $\lambda \to (\lambda I - A)^{-1}$ must equal $\frac{1}{\lambda}\sum_{k=0}^{\infty}\left(\frac{A}{\lambda}\right)^k$ on $|\lambda| > \rho(A)$. This proves the lemma.

The theorem on the spectral radius follows. It is due to Gelfand.

Theorem 26.32 $\rho(A) = \lim_{n\to\infty} ||A^n||^{1/n}$.

Proof: If

$$|\lambda| < \limsup_{n\to\infty} ||A^n||^{1/n}$$

then by the root test, the Neumann series does not converge and so by Lemma 26.31 $|\lambda| \leq \rho(A)$. Thus

$$\rho(A) \geq \limsup_{n\to\infty} ||A^n||^{1/n}.$$

Now let p be a positive integer. Then $\lambda \in \sigma(A)$ implies $\lambda^p \in \sigma(A^p)$ because

$$\begin{aligned}\lambda^p I - A^p &= (\lambda I - A)\left(\lambda^{p-1} + \lambda^{p-2}A + \cdots + A^{p-1}\right) \\ &= \left(\lambda^{p-1} + \lambda^{p-2}A + \cdots + A^{p-1}\right)(\lambda I - A)\end{aligned}$$

It follows from Lemma 26.27 applied to A^pthat for $\lambda \in \sigma(A), |\lambda^p| \leq ||A^p||$ and so $|\lambda| \leq ||A^p||^{1/p}$. Therefore, $\rho(A) \leq ||A^p||^{1/p}$ and since p is arbitrary,

$$\liminf_{p\to\infty} ||A^p||^{1/p} \geq \rho(A) \geq \limsup_{n\to\infty} ||A^n||^{1/n}.$$

This proves the theorem.

26.4 Exercises

1. Example 26.19 found the integral of a rational function of a certain sort. The technique used in this example typically works for rational functions of the form $\frac{f(x)}{g(x)}$ where $\deg(g(x)) \geq \deg f(x) + 2$ provided the rational function has no poles on the real axis. State and prove a theorem based on these observations.

2. Fill in the missing details of Example 26.25 about $I_N \to 0$. Note how important it was that the contour was chosen just right for this to happen. Also verify the claims about the residues.

3. Suppose f has a pole of order m at $z = a$. Define $g(z)$ by
$$g(z) = (z-a)^m f(z).$$
Show
$$\operatorname{Res}(f, a) = \frac{1}{(m-1)!} g^{(m-1)}(a).$$
Hint: Use the Laurent series.

4. Give a proof of Theorem 26.6. **Hint:** Let p be a pole. Show that near p, a pole of order m,
$$\frac{f'(z)}{f(z)} = \frac{-m + \sum_{k=1}^{\infty} b_k (z-p)^k}{(z-p) + \sum_{k=2}^{\infty} c_k (z-p)^k}$$
Show that $\operatorname{Res}(f, p) = -m$. Carry out a similar procedure for the zeros.

5. Use Rouche's theorem to prove the fundamental theorem of algebra which says that if $p(z) = z^n + a_{n-1} z^{n-1} \cdots + a_1 z + a_0$, then p has n zeros in $\mathbb{C}$. **Hint:** Let $q(z) = -z^n$ and let γ be a large circle, $\gamma(t) = re^{it}$ for r sufficiently large.

6. Consider the two polynomials $z^5 + 3z^2 - 1$ and $z^5 + 3z^2$. Show that on $|z| = 1$, the conditions for Rouche's theorem hold. Now use Rouche's theorem to verify that $z^5 + 3z^2 - 1$ must have two zeros in $|z| < 1$.

7. Consider the polynomial, $z^{11} + 7z^5 + 3z^2 - 17$. Use Rouche's theorem to find a bound on the zeros of this polynomial. In other words, find r such that if z is a zero of the polynomial, $|z| < r$. Try to make r fairly small if possible.

8. Verify that $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$. **Hint:** Use polar coordinates.

9. Use the contour described in Example 26.19 to compute the exact values of the following improper integrals.

   (a) $\int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2} dx$

   (b) $\int_0^{\infty} \frac{x^2}{(x^2+a^2)^2} dx$

   (c) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, a, b > 0$

10. Evaluate the following improper integrals.

   (a) $\int_0^{\infty} \frac{\cos ax}{(x^2+b^2)^2} dx$

(b) $\int_0^\infty \frac{x \sin x}{(x^2+a^2)^2} dx$

11. Find the Cauchy principle value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{(x^2+1)(x-1)} dx$$

defined as

$$\lim_{\varepsilon \to 0+} \left(\int_{-\infty}^{1-\varepsilon} \frac{\sin x}{(x^2+1)(x-1)} dx + \int_{1+\varepsilon}^{\infty} \frac{\sin x}{(x^2+1)(x-1)} dx \right).$$

12. Find a formula for the integral $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^{n+1}}$ where n is a nonnegative integer.

13. Find $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx$.

14. If $m < n$ for m and n integers, show

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}} dx = \frac{\pi}{2n} \frac{1}{\sin\left(\frac{2m+1}{2n}\pi\right)}.$$

15. Find $\int_{-\infty}^{\infty} \frac{1}{(1+x^4)^2} dx$.

16. Find $\int_0^\infty \frac{\ln(x)}{1+x^2} dx = 0$.

17. Suppose f has an isolated singularity at α. Show the singularity is essential if and only if the principal part of the Laurent series of f has infinitely many terms. That is, show $f(z) = \sum_{k=0}^{\infty} a_k (z-\alpha)^k + \sum_{k=1}^{\infty} \frac{b_k}{(z-\alpha)^k}$ where infinitely many of the b_k are nonzero.

18. Suppose Ω is a bounded open set and f_n is analytic on Ω and continuous on $\overline{\Omega}$. Suppose also that $f_n \to f$ uniformly on $\overline{\Omega}$ and that $f \neq 0$ on $\partial\Omega$. Show that for all n large enough, f_n and f have the same number of zeros on Ω provided the zeros are counted according to multiplicity.

Complex Mappings

27.1 Conformal Maps

If $\gamma(t) = x(t) + iy(t)$ is a C^1 curve having values in U, an open set of $\mathbb{C}$, and if $f : U \to \mathbb{C}$ is analytic, consider $f \circ \gamma$, another C^1 curve having values in $\mathbb{C}$. Also, $\gamma'(t)$ and $(f \circ \gamma)'(t)$ are complex numbers so these can be considered as vectors in $\mathbb{R}^2$ as follows. The complex number, $x + iy$ corresponds to the vector, (x, y). Suppose that γ and η are two such C^1 curves having values in U and that $\gamma(t_0) = \eta(s_0) = z$ and suppose that $f : U \to \mathbb{C}$ is analytic. What can be said about the angle between $(f \circ \gamma)'(t_0)$ and $(f \circ \eta)'(s_0)$? It turns out this angle is the same as the angle between $\gamma'(t_0)$ and $\eta'(s_0)$ assuming that $f'(z) \neq 0$. To see this, note $(x, y) \cdot (a, b) = \frac{1}{2}(z\overline{w} + \overline{z}w)$ where $z = x + iy$ and $w = a + ib$. Therefore, letting θ be the cosine between the two vectors, $(f \circ \gamma)'(t_0)$ and $(f \circ \eta)'(s_0)$, it follows from calculus that

$$
\begin{aligned}
& \cos\theta \\
= & \frac{(f \circ \gamma)'(t_0) \cdot (f \circ \eta)'(s_0)}{\left|(f \circ \eta)'(s_0)\right| \left|(f \circ \gamma)'(t_0)\right|} \\
= & \frac{1}{2} \frac{f'(\gamma(t_0))\gamma'(t_0)\overline{f'(\eta(s_0))\eta'(s_0)} + \overline{f'(\gamma(t_0))\gamma'(t_0)} f'(\eta(s_0))\eta'(s_0)}{|f'(\gamma(t_0))| \, |f'(\eta(s_0))|} \\
= & \frac{1}{2} \frac{f'(z)\overline{f'(z)}\gamma'(t_0)\overline{\eta'(s_0)} + \overline{f'(z)} f'(z)\overline{\gamma'(t_0)}\eta'(s_0)}{|f'(z)| \, |f'(z)|} \\
= & \frac{1}{2} \frac{\gamma'(t_0)\overline{\eta'(s_0)} + \eta'(s_0)\overline{\gamma'(t_0)}}{1}
\end{aligned}
$$

which equals the angle between the vectors, $\gamma'(t_0)$ and $\eta'(t_0)$. Thus analytic mappings preserve angles at points where the derivative is nonzero. Such mappings are called isogonal. .

Actually, they also preserve orientations. If $z = x + iy$ and $w = a + ib$ are two complex numbers, then $(x, y, 0)$ and $(a, b, 0)$ are two vectors in $\mathbb{R}^3$. Recall that the cross product, $(x, y, 0) \times (a, b, 0)$, yields a vector normal to the two given vectors such that the triple, $(x, y, 0)$, $(a, b, 0)$, and $(x, y, 0) \times (a, b, 0)$ satisfies the right hand rule and has magnitude equal to the product of the sine of the included angle times the product of the two norms of the vectors. In this case, the cross product will produce a vector which is a multiple of $\mathbf{k}$, the unit vector in the direction of the z axis. In fact, you can verify by computing both sides that, letting $z = x + iy$ and $w = a + ib$,

$$(x, y, 0) \times (a, b, 0) = \text{Re}\,(zi\overline{w})\,\mathbf{k}.$$

Therefore, in the above situation,

$$\begin{aligned} & (f \circ \gamma)'(t_0) \times (f \circ \eta)'(s_0) \\ = \;& \text{Re}\left(f'(\gamma(t_0))\gamma'(t_0)\, i\overline{f'(\eta(s_0))\eta'(s_0)}\right)\mathbf{k} \\ = \;& |f'(z)|^2\,\text{Re}\left(\gamma'(t_0)\, i\overline{\eta'(s_0)}\right)\mathbf{k} \end{aligned}$$

which shows that the orientation of $\gamma'(t_0)$, $\eta'(s_0)$ is the same as the orientation of $(f \circ \gamma)'(t_0), (f \circ \eta)'(s_0)$. Mappings which preserve both orientation and angles are called conformal mappings and this has shown that analytic functions are conformal mappings if the derivative does not vanish.

27.2 Fractional Linear Transformations

27.2.1 Circles And Lines

These mappings map lines and circles to either lines or circles.

Definition 27.1 *A fractional linear transformation is a function of the form*

$$f(z) = \frac{az + b}{cz + d} \tag{27.1}$$

where $ad - bc \neq 0$.

Note that if $c = 0$, this reduces to a linear transformation $(a/d)\,z + (b/d)$. Special cases of these are defined as follows.

$$\text{dilations: } z \to \delta z,\ \delta \neq 0,\ \text{inversions: } z \to \frac{1}{z},$$

$$\text{translations: } z \to z + \rho.$$

The next lemma is the key to understanding fractional linear transformations.

Lemma 27.2 *The fractional linear transformation, 27.1 can be written as a finite composition of dilations, inversions, and translations.*

Proof: Let

$$S_1(z) = z + \frac{d}{c}, S_2(z) = \frac{1}{z}, S_3(z) = \frac{(bc - ad)}{c^2} z$$

and

$$S_4(z) = z + \frac{a}{c}$$

in the case where $c \neq 0$. Then $f(z)$ given in 27.1 is of the form

$$f(z) = S_4 \circ S_3 \circ S_2 \circ S_1.$$

Here is why.

$$S_2(S_1(z)) = S_2\left(z + \frac{d}{c}\right) \equiv \frac{1}{z + \frac{d}{c}} = \frac{c}{zc + d}.$$

Now consider

$$S_3\left(\frac{c}{zc+d}\right) \equiv \frac{(bc - ad)}{c^2}\left(\frac{c}{zc+d}\right) = \frac{bc - ad}{c(zc+d)}.$$

Finally, consider

$$S_4\left(\frac{bc-ad}{c(zc+d)}\right) \equiv \frac{bc-ad}{c(zc+d)} + \frac{a}{c} = \frac{b+az}{zc+d}.$$

In case that $c = 0$, $f(z) = \frac{a}{d}z + \frac{b}{d}$ which is a translation composed with a dilation. Because of the assumption that $ad - bc \neq 0$, it follows that since $c = 0$, both a and $d \neq 0$. This proves the lemma.

This lemma implies the following corollary.

Corollary 27.3 *Fractional linear transformations map circles and lines to circles or lines.*

Proof: It is obvious that dilations and translations map circles to circles and lines to lines. What of inversions? If inversions have this property, the above lemma implies a general fractional linear transformation has this property as well.

Note that all circles and lines may be put in the form

$$\alpha\left(x^2 + y^2\right) - 2ax - 2by = r^2 - \left(a^2 + b^2\right)$$

where $\alpha = 1$ gives a circle centered at (a, b) with radius r and $\alpha = 0$ gives a line. In terms of complex variables you may therefore consider all possible circles and lines in the form

$$\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma = 0, \tag{27.2}$$

To see this let $\beta = \beta_1 + i\beta_2$ where $\beta_1 \equiv -a$ and $\beta_2 \equiv b$. Note that even if α is not 0 or 1 the expression still corresponds to either a circle or a line because you can divide by α if $\alpha \neq 0$. Now I verify that replacing z with $\frac{1}{z}$ results in an expression of the form in 27.2. Thus, let $w = \frac{1}{z}$ where z satisfies 27.2. Then

$$\left(\alpha + \beta\overline{w} + \overline{\beta}w + \gamma w\overline{w}\right) = \frac{1}{z\overline{z}}\left(\alpha z\overline{z} + \beta z + \overline{\beta}\overline{z} + \gamma\right) = 0$$

and so w also satisfies a relation like 27.2. One simply switches α with γ and β with $\overline{\beta}$. Note the situation is slightly different than with dilations and translations. In the case of an inversion, a circle becomes either a line or a circle and similarly, a line becomes either a circle or a line. This proves the corollary.

The next example is quite important.

Example 27.4 *Consider the fractional linear transformation,* $w = \frac{z-i}{z+i}$.

First consider what this mapping does to the points of the form $z = x + i0$. Substituting into the expression for w,

$$w = \frac{x-i}{x+i} = \frac{x^2 - 1 - 2xi}{x^2+1},$$

a point on the unit circle. Thus this transformation maps the real axis to the unit circle.

The upper half plane is composed of points of the form $x + iy$ where $y > 0$. Substituting in to the transformation,

$$w = \frac{x + i(y-1)}{x + i(y+1)},$$

which is seen to be a point on the interior of the unit disk because $|y-1| < |y+1|$ which implies $|x + i(y+1)| > |x + i(y-1)|$. Therefore, this transformation maps the upper half plane to the interior of the unit disk.

One might wonder whether the mapping is one to one and onto. The mapping is clearly one to one because it has an inverse, $z = -i\frac{w+1}{w-1}$ for all w in the interior of the unit disk. Also, a short computation verifies that z so defined is in the upper half plane. Therefore, this transformation maps $\{z \in \mathbb{C} \text{ such that } \operatorname{Im} z > 0\}$ one to one and onto the unit disk $\{z \in \mathbb{C} \text{ such that } |z| < 1\}$.

A fancy way to do part of this is to use Theorem 25.11. $\limsup_{z\to a} \left|\frac{z-i}{z+i}\right| \leq 1$ whenever a is the real axis or ∞. Therefore, $\left|\frac{z-i}{z+i}\right| \leq 1$. This is a little shorter.

27.2.2 Three Points To Three Points

There is a simple procedure for determining fractional linear transformations which map a given set of three points to another set of three points. The problem is as follows: There are three distinct points in the extended complex plane, z_1, z_2, and z_3 and it is desired to find a fractional linear transformation such that $z_i \to w_i$ for $i = 1, 2, 3$ where here w_1, w_2, and w_3 are three distinct points in the extended complex plane. Then the procedure says that to find the desired fractional linear transformation solve the following equation for w.

$$\frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

The result will be a fractional linear transformation with the desired properties. If any of the points equals ∞, then the quotient containing this point should be adjusted.

Why should this procedure work? Here is a heuristic argument to indicate why you would expect this to happen rather than a rigorous proof. The reader may want to tighten the argument to give a proof. First suppose $z = z_1$. Then the right side equals zero and so the left side also must equal zero. However, this requires $w = w_1$. Next suppose $z = z_2$. Then the right side equals 1. To get a 1 on the left, you need $w = w_2$. Finally suppose $z = z_3$. Then the right side involves division by 0. To get the same bad behavior, on the left, you need $w = w_3$.

Example 27.5 *Let* $\operatorname{Im} \xi > 0$ *and consider the fractional linear transformation which takes* ξ *to 0,* $\overline{\xi}$ *to* ∞ *and* 0 *to* $\xi/\overline{\xi},$.

The equation for w is

$$\frac{w-0}{w-\left(\xi/\overline{\xi}\right)} = \frac{z-\xi}{z-0} \cdot \frac{\overline{\xi}-0}{\overline{\xi}-\xi}$$

After some computations,

$$w = \frac{z-\xi}{z-\overline{\xi}}.$$

Note that this has the property that $\frac{x-\xi}{x-\overline{\xi}}$ is always a point on the unit circle because it is a complex number divided by its conjugate. Therefore, this fractional linear transformation maps the real line to the unit circle. It also takes the point, ξ to 0 and so it must map the upper half plane to the unit disk. You can verify the mapping is onto as well.

Example 27.6 *Let* $z_1 = 0, z_2 = 1,$ *and* $z_3 = 2$ *and let* $w_1 = 0, w_2 = i,$ *and* $w_3 = 2i.$

Then the equation to solve is

$$\frac{w}{w-2i} \cdot \frac{-i}{i} = \frac{z}{z-2} \cdot \frac{-1}{1}$$

Solving this yields $w = iz$ which clearly works.

27.3 Riemann Mapping Theorem

From the open mapping theorem analytic functions map regions to other regions or else to single points. The Riemann mapping theorem states that for every simply connected region, Ω which is not equal to all of $\mathbb{C}$ there exists an analytic function, f such that $f(\Omega) = B(0,1)$ and in addition to this, f is one to one. The proof involves several ideas which have been developed up to now. The proof is based on the following important theorem, a case of Montel's theorem. Before, beginning, note that the Riemann mapping theorem is a classic example of a major existence theorem. In mathematics there are two sorts of questions, those related to whether something exists and those involving methods for finding it. The real questions are often related to questions of existence. There is a long and involved history for proofs of this theorem. The first proofs were based on the Dirichlet principle and turned out to be incorrect, thanks to Weierstrass who pointed out the errors. For more on the history of this theorem, see Hille [27].

The following theorem is really wonderful. It is about the existence of a subsequence having certain salubrious properties. It is this wonderful result which will give the existence of the mapping desired. The other parts of the argument are technical details to set things up and use this theorem.

27.3.1 Montel's Theorem

Theorem 27.7 *Let Ω be an open set in $\mathbb{C}$ and let $\mathcal{F}$ denote a set of analytic functions mapping Ω to $B(0,M) \subseteq \mathbb{C}$. Then there exists a sequence of functions from $\mathcal{F}$, $\{f_n\}_{n=1}^{\infty}$ and an analytic function, f such that $f_n^{(k)}$ converges uniformly to $f^{(k)}$ on every compact subset of Ω.*

Proof: First note there exists a sequence of compact sets, K_n such that $K_n \subseteq \operatorname{int} K_{n+1} \subseteq \Omega$ for all n where here $\operatorname{int} K$ denotes the interior of the set K, the union of all open sets contained in K and $\cup_{n=1}^{\infty} K_n = \Omega$. In fact, you can verify that $\overline{B(0,n)} \cap \left\{z \in \Omega : \operatorname{dist}\left(z, \Omega^C\right) \leq \frac{1}{n}\right\}$ works for K_n. Then there exist positive numbers, δ_n such that if $z \in K_n$, then $\overline{B(z,\delta_n)} \subseteq \operatorname{int} K_{n+1}$. Now denote by $\mathcal{F}_n$ the set of restrictions of functions of $\mathcal{F}$ to K_n. Then let $z \in K_n$ and let $\gamma(t) \equiv z + \delta_n e^{it}, t \in [0, 2\pi]$. It follows that for $z_1 \in B(z,\delta_n)$, and $f \in \mathcal{F}$,

$$\begin{aligned}
|f(z) - f(z_1)| &= \left|\frac{1}{2\pi i}\int_\gamma f(w)\left(\frac{1}{w-z} - \frac{1}{w-z_1}\right)dw\right| \\
&\leq \frac{1}{2\pi}\left|\int_\gamma f(w)\frac{z-z_1}{(w-z)(w-z_1)}dw\right|
\end{aligned}$$

Letting $|z_1 - z| < \frac{\delta_n}{2}$,

$$\begin{aligned}
|f(z) - f(z_1)| &\leq \frac{M}{2\pi}2\pi\delta_n\frac{|z-z_1|}{\delta_n^2/2} \\
&\leq 2M\frac{|z-z_1|}{\delta_n}.
\end{aligned}$$

It follows that $\mathcal{F}_n$ is equicontinuous and uniformly bounded so by the Arzela Ascoli theorem there exists a sequence, $\{f_{nk}\}_{k=1}^{\infty} \subseteq \mathcal{F}$ which converges uniformly on K_n. Let $\{f_{1k}\}_{k=1}^{\infty}$ converge uniformly on K_1. Then use the Arzela Ascoli theorem applied to this sequence to get a subsequence, denoted by $\{f_{2k}\}_{k=1}^{\infty}$ which also converges uniformly on K_2. Continue in this way to obtain $\{f_{nk}\}_{k=1}^{\infty}$ which converges uniformly on $K_1, \cdots, K_n$. Now the sequence $\{f_{nn}\}_{n=m}^{\infty}$ is a subsequence of $\{f_{mk}\}_{k=1}^{\infty}$ and so it converges uniformly on K_m for all m. Denoting f_{nn} by f_n for short, this is the sequence of functions promised by the theorem. It is clear $\{f_n\}_{n=1}^{\infty}$ converges uniformly on every compact subset of Ω because every such set is contained in K_m for all m large enough. Let $f(z)$ be the point to which $f_n(z)$ converges. Then f is a continuous function defined on Ω. Is f is analytic? Yes it is by Lemma 24.18. Alternatively, you could let $T \subseteq \Omega$ be a triangle. Then

$$\int_{\partial T} f(z)\,dz = \lim_{n\to\infty} \int_{\partial T} f_n(z)\,dz = 0.$$

Therefore, by Morera's theorem, f is analytic.

As for the uniform convergence of the derivatives of f, recall Theorem 24.52 about the existence of a cycle. Let K be a compact subset of $\operatorname{int}(K_n)$ and let $\{\gamma_k\}_{k=1}^{m}$ be closed oriented curves contained in

$$\operatorname{int}(K_n) \setminus K$$

such that $\sum_{k=1}^{m} n(\gamma_k, z) = 1$ for every $z \in K$. Also let η denote the distance between $\cup_j \gamma_j^*$ and K. Then for $z \in K$,

$$\begin{aligned}\left|f^{(k)}(z) - f_n^{(k)}(z)\right| &= \left|\frac{k!}{2\pi i}\sum_{j=1}^{m}\int_{\gamma_j}\frac{f(w)-f_n(w)}{(w-z)^{k+1}}dw\right| \\ &\leq \frac{k!}{2\pi}\,||f_k - f||_{K_n}\sum_{j=1}^{m}(\text{length of }\gamma_k)\,\frac{1}{\eta^{k+1}}.\end{aligned}$$

where here $||f_k - f||_{K_n} \equiv \sup\{|f_k(z) - f(z)| : z \in K_n\}$. Thus you get uniform convergence of the derivatives.

Since the family, $\mathcal{F}$ satisfies the conclusion of Theorem 27.7 it is known as a normal family of functions. More generally,

Definition 27.8 *Let $\mathcal{F}$ denote a collection of functions which are analytic on Ω, a region. Then $\mathcal{F}$ is normal if every sequence contained in $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets of Ω.*

The following result is about a certain class of fractional linear transformations. Recall Lemma 25.18 which is listed here for convenience.

Lemma 27.9 *For $\alpha \in B(0,1)$, let*

$$\phi_\alpha(z) \equiv \frac{z-\alpha}{1-\overline{\alpha}z}.$$

Then ϕ_α maps $B(0,1)$ one to one and onto $B(0,1)$, $\phi_\alpha^{-1} = \phi_{-\alpha}$, and

$$\phi_\alpha'(\alpha) = \frac{1}{1-|\alpha|^2}.$$

The next lemma, known as Schwarz's lemma is interesting for its own sake but will also be an important part of the proof of the Riemann mapping theorem. It was stated and proved earlier but for convenience it is given again here.

Lemma 27.10 *Suppose* $F : B(0,1) \to B(0,1)$, *F is analytic, and* $F(0) = 0$. *Then for all* $z \in B(0,1)$,

$$|F(z)| \leq |z|, \tag{27.3}$$

and

$$|F'(0)| \leq 1. \tag{27.4}$$

If equality holds in 27.4 then there exists $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$ *and*

$$F(z) = \lambda z. \tag{27.5}$$

Proof: First note that by assumption, $F(z)/z$ has a removable singularity at 0 if its value at 0 is defined to be $F'(0)$. By the maximum modulus theorem, if $|z| < r < 1$,

$$\left|\frac{F(z)}{z}\right| \leq \max_{t\in[0,2\pi]} \frac{\left|F\left(re^{it}\right)\right|}{r} \leq \frac{1}{r}.$$

Then letting $r \to 1$,

$$\left|\frac{F(z)}{z}\right| \leq 1$$

this shows 27.3 and it also verifies 27.4 on taking the limit as $z \to 0$. If equality holds in 27.4, then $|F(z)/z|$ achieves a maximum at an interior point so $F(z)/z$ equals a constant, λ by the maximum modulus theorem. Since $F(z) = \lambda z$, it follows $F'(0) = \lambda$ and so $|\lambda| = 1$. This proves the lemma.

Definition 27.11 *A region,* Ω *has the square root property if whenever* $f, \frac{1}{f} : \Omega \to \mathbb{C}$ *are both analytic*[1], *it follows there exists* $\phi : \Omega \to \mathbb{C}$ *such that* ϕ *is analytic and* $f(z) = \phi^2(z)$.

The next theorem will turn out to be equivalent to the Riemann mapping theorem.

27.3.2 Regions With Square Root Property

Theorem 27.12 *Let* $\Omega \neq \mathbb{C}$ *for* Ω *a region and suppose* Ω *has the square root property. Then for* $z_0 \in \Omega$ *there exists* $h : \Omega \to B(0,1)$ *such that* h *is one to one, onto, analytic, and* $h(z_0) = 0$.

Proof: Define $\mathcal{F}$ to be the set of functions, f such that $f : \Omega \to B(0,1)$ is one to one and analytic. The first task is to show $\mathcal{F}$ is nonempty. Then, using Montel's theorem it will be shown there is a function in $\mathcal{F}$, h, such that $|h'(z_0)| \geq |\psi'(z_0)|$

[1] This implies f has no zero on Ω.

for all $\psi \in \mathcal{F}$. When this has been done it will be shown that h is actually onto. This will prove the theorem.

Claim 1: $\mathcal{F}$ is nonempty.

Proof of Claim 1: Since $\Omega \neq \mathbb{C}$ it follows there exists $\xi \notin \Omega$. Then it follows $z-\xi$ and $\frac{1}{z-\xi}$ are both analytic on Ω. Since Ω has the square root property, there exists an analytic function, $\phi : \Omega \to \mathbb{C}$ such that $\phi^2(z) = z-\xi$ for all $z \in \Omega$, $\phi(z) = \sqrt{z-\xi}$. Since $z-\xi$ is not constant, neither is ϕ and it follows from the open mapping theorem that $\phi(\Omega)$ is a region. Note also that ϕ is one to one because if $\phi(z_1) = \phi(z_2)$, then you can square both sides and conclude $z_1 - \xi = z_2 - \xi$ implying $z_1 = z_2$.

Now pick $a \in \phi(\Omega)$. Thus $\sqrt{z_a - \xi} = a$. I claim there exists a positive lower bound to $\left|\sqrt{z-\xi}+a\right|$ for $z \in \Omega$. If not, there exists a sequence, $\{z_n\} \subseteq \Omega$ such that

$$\sqrt{z_n-\xi}+a = \sqrt{z_n-\xi}+\sqrt{z_a-\xi} \equiv \varepsilon_n \to 0.$$

Then

$$\sqrt{z_n-\xi} = \left(\varepsilon_n - \sqrt{z_a-\xi}\right) \tag{27.6}$$

and squaring both sides,

$$z_n - \xi = \varepsilon_n^2 + z_a - \xi - 2\varepsilon_n\sqrt{z_a-\xi}.$$

Consequently, $(z_n - z_a) = \varepsilon_n^2 - 2\varepsilon_n\sqrt{z_a-\xi}$ which converges to 0. Taking the limit in 27.6, it follows $2\sqrt{z_a-\xi} = 0$ and so $\xi = z_a$, a contradiction to $\xi \notin \Omega$. Choose $r > 0$ such that for all $z \in \Omega$, $\left|\sqrt{z-\xi}+a\right| > r > 0$. Then consider

$$\psi(z) \equiv \frac{r}{\sqrt{z-\xi}+a}. \tag{27.7}$$

This is one to one, analytic, and maps Ω into $B(0,1)$ ($\left|\sqrt{z-\xi}+a\right| > r$). Thus $\mathcal{F}$ is not empty and this proves the claim.

Claim 2: Let $z_0 \in \Omega$. There exists a finite positive real number, η, defined by

$$\eta \equiv \sup\left\{\left|\psi'(z_0)\right| : \psi \in \mathcal{F}\right\} \tag{27.8}$$

and an analytic function, $h \in \mathcal{F}$ such that $|h'(z_0)| = \eta$. Furthermore, $h(z_0) = 0$.

Proof of Claim 2: First you show $\eta < \infty$. Let $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ and r is small enough that $B(z_0, r) \subseteq \Omega$. Then for $\psi \in \mathcal{F}$, the Cauchy integral formula for the derivative implies

$$\psi'(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{\psi(w)}{(w-z_0)^2}dw$$

and so $\left|\psi'(z_0)\right| \leq (1/2\pi)\, 2\pi r\left(1/r^2\right) = 1/r$. Therefore, $\eta < \infty$ as desired. For ψ defined above in 27.7

$$\psi'(z_0) = \frac{-r\phi'(z_0)}{(\phi(z_0)+a)^2} = \frac{-r(1/2)\left(\sqrt{z_0-\xi}\right)^{-1}}{(\phi(z_0)+a)^2} \neq 0.$$

Therefore, $\eta > 0$. It remains to verify the existence of the function, h.

By Theorem 27.7, there exists a sequence, $\{\psi_n\}$, of functions in $\mathcal{F}$ and an analytic function, h, such that

$$\left|\psi_n'(z_0)\right| \to \eta \tag{27.9}$$

and

$$\psi_n \to h, \psi_n' \to h', \tag{27.10}$$

uniformly on all compact subsets of Ω. It follows

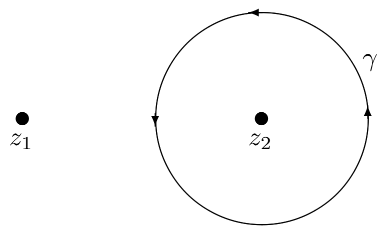
$$|h'(z_0)| = \lim_{n\to\infty} \left|\psi_n'(z_0)\right| = \eta \tag{27.11}$$

and for all $z \in \Omega$,

$$|h(z)| = \lim_{n\to\infty} |\psi_n(z)| \leq 1. \tag{27.12}$$

By 27.11, h is not a constant. Therefore, in fact, $|h(z)| < 1$ for all $z \in \Omega$ in 27.12 by the open mapping theorem.

Next it must be shown that h is one to one in order to conclude $h \in \mathcal{F}$. Pick $z_1 \in \Omega$ and suppose z_2 is another point of Ω. Since the zeros of $h - h(z_1)$ have no limit point, there exists a circular contour bounding a circle which contains z_2 but not z_1 such that γ^* contains no zeros of $h - h(z_1)$.



Using the theorem on counting zeros, Theorem 25.20, and the fact that ψ_n is one to one,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_\gamma \frac{\psi_n'(w)}{\psi_n(w) - \psi_n(z_1)} dw \\ &= \frac{1}{2\pi i} \int_\gamma \frac{h'(w)}{h(w) - h(z_1)} dw, \end{aligned}$$

which shows that $h - h(z_1)$ has no zeros in $B(z_2, r)$. In particular z_2 is not a zero of $h - h(z_1)$. This shows that h is one to one since $z_2 \neq z_1$ was arbitrary. Therefore, $h \in \mathcal{F}$. It only remains to verify that $h(z_0) = 0$.

If $h(z_0) \neq 0$,consider $\phi_{h(z_0)} \circ h$ where ϕ_α is the fractional linear transformation defined in Lemma 27.9. By this lemma it follows $\phi_{h(z_0)} \circ h \in \mathcal{F}$. Now using the chain rule,

$$\begin{aligned}\left|\left(\phi_{h(z_0)} \circ h\right)'(z_0)\right| &= \left|\phi'_{h(z_0)}(h(z_0))\right| |h'(z_0)| \\ &= \left|\frac{1}{1-|h(z_0)|^2}\right| |h'(z_0)| \\ &= \left|\frac{1}{1-|h(z_0)|^2}\right| \eta > \eta\end{aligned}$$

Contradicting the definition of η. This proves Claim 2.

Claim 3: The function, h just obtained maps Ω onto $B(0,1)$.

Proof of Claim 3: To show h is onto, use the fractional linear transformation of Lemma 27.9. Suppose h is not onto. Then there exists $\alpha \in B(0,1) \setminus h(\Omega)$. Then $0 \neq \phi_\alpha \circ h(z)$ for all $z \in \Omega$ because

$$\phi_\alpha \circ h(z) = \frac{h(z)-\alpha}{1-\overline{\alpha}h(z)}$$

and it is assumed $\alpha \notin h(\Omega)$. Therefore, since Ω has the square root property, you can consider an analytic function $z \to \sqrt{\phi_\alpha \circ h(z)}$. This function is one to one because both ϕ_α and h are. Also, the values of this function are in $B(0,1)$ by Lemma 27.9 so it is in $\mathcal{F}$.

Now let

$$\psi \equiv \phi_{\sqrt{\phi_\alpha \circ h(z_0)}} \circ \sqrt{\phi_\alpha \circ h}. \tag{27.13}$$

Thus

$$\psi(z_0) = \phi_{\sqrt{\phi_\alpha \circ h(z_0)}} \circ \sqrt{\phi_\alpha \circ h(z_0)} = 0$$

and ψ is a one to one mapping of Ω into $B(0,1)$ so ψ is also in $\mathcal{F}$. Therefore,

$$\left|\psi'(z_0)\right| \leq \eta, \ \left|\left(\sqrt{\phi_\alpha \circ h}\right)'(z_0)\right| \leq \eta. \tag{27.14}$$

Define $s(w) \equiv w^2$. Then using Lemma 27.9, in particular, the description of $\phi_\alpha^{-1} = \phi_{-\alpha}$, you can solve 27.13 for h to obtain

$$\begin{aligned}h(z) &= \phi_{-\alpha} \circ s \circ \phi_{-\sqrt{\phi_\alpha \circ h(z_0)}} \circ \psi \\ &= \left(\overbrace{\phi_{-\alpha} \circ s \circ \phi_{-\sqrt{\phi_\alpha \circ h(z_0)}}}^{\equiv F} \circ \psi\right)(z) \\ &= (F \circ \psi)(z) \end{aligned} \tag{27.15}$$

Now

$$F(0) = \phi_{-\alpha} \circ s \circ \phi_{-\sqrt{\phi_\alpha \circ h(z_0)}}(0) = \phi_\alpha^{-1}(\phi_\alpha \circ h(z_0)) = h(z_0) = 0$$

and F maps $B(0,1)$ into $B(0,1)$. Also, F is not one to one because it maps $B(0,1)$ to $B(0,1)$ and has s in its definition. Thus there exists $z_1 \in B(0,1)$ such that $\phi_{-\sqrt{\phi_\alpha \circ h(z_0)}}(z_1) = -\frac{1}{2}$ and another point $z_2 \in B(0,1)$ such that $\phi_{-\sqrt{\phi_\alpha \circ h(z_0)}}(z_2) = \frac{1}{2}$. However, thanks to $s, F(z_1) = F(z_2)$.

Since $F(0) = h(z_0) = 0$, you can apply the Schwarz lemma to F. Since F is not one to one, it can't be true that $F(z) = \lambda z$ for $|\lambda| = 1$ and so by the Schwarz lemma it must be the case that $|F'(0)| < 1$. But this implies from 27.15 and 27.14 that

$$\begin{aligned} \eta &= |h'(z_0)| = |F'(\psi(z_0))| \left|\psi'(z_0)\right| \\ &= |F'(0)| \left|\psi'(z_0)\right| < \left|\psi'(z_0)\right| \leq \eta, \end{aligned}$$

a contradiction. This proves the theorem.

The following lemma yields the usual form of the Riemann mapping theorem.

Lemma 27.13 *Let Ω be a simply connected region with $\Omega \neq \mathbb{C}$. Then Ω has the square root property.*

Proof: Let f and $\frac{1}{f}$ both be analytic on Ω. Then $\frac{f'}{f}$ is analytic on Ω so by Corollary 24.50, there exists $\widetilde{F}$, analytic on Ω such that $\widetilde{F}' = \frac{f'}{f}$ on Ω. Then $\left(fe^{-\widetilde{F}}\right)' = 0$ and so $f(z) = Ce^{\widetilde{F}} = e^{a+ib}e^{\widetilde{F}}$. Now let $F = \widetilde{F} + a + ib$. Then F is still a primitive of f'/f and $f(z) = e^{F(z)}$. Now let $\phi(z) \equiv e^{\frac{1}{2}F(z)}$. Then ϕ is the desired square root and so Ω has the square root property.

Corollary 27.14 *(Riemann mapping theorem) Let Ω be a simply connected region with $\Omega \neq \mathbb{C}$ and let $z_0 \in \Omega$. Then there exists a function, $f : \Omega \to B(0,1)$ such that f is one to one, analytic, and onto with $f(z_0) = 0$. Furthermore, f^{-1} is also analytic.*

Proof: From Theorem 27.12 and Lemma 27.13 there exists a function, $f : \Omega \to B(0,1)$ which is one to one, onto, and analytic such that $f(z_0) = 0$. The assertion that f^{-1} is analytic follows from the open mapping theorem.

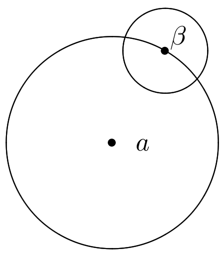
27.4 Analytic Continuation

27.4.1 Regular And Singular Points

Given a function which is analytic on some set, can you extend it to an analytic function defined on a larger set? Sometimes you can do this. It was done in the proof of the Cauchy integral formula. There are also reflection theorems like those discussed in the exercises starting with Problem 10 on Page 678. Here I will give a systematic way of extending an analytic function to a larger set. I will emphasize simply connected regions. The subject of analytic continuation is much larger than the introduction given here. A good source for much more on this is found in Alfors

[2]. The approach given here is suggested by Rudin [45] and avoids many of the standard technicalities.

Definition 27.15 *Let f be analytic on $B(a,r)$ and let $\beta \in \partial B(a,r)$. Then β is called a regular point of f if there exists some $\delta > 0$ and a function, g analytic on $B(\beta,\delta)$ such that $g = f$ on $B(\beta,\delta) \cap B(a,r)$. Those points of $\partial B(a,r)$ which are not regular are called singular.*



Theorem 27.16 *Suppose f is analytic on $B(a,r)$ and the power series*

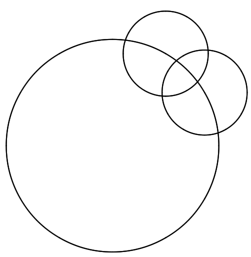
$$f(z) = \sum_{k=0}^{\infty} a_k (z-a)^k$$

has radius of convergence r. Then there exists a singular point on $\partial B(a,r)$.

Proof: If not, then for every $z \in \partial B(a,r)$ there exists $\delta_z > 0$ and g_z analytic on $B(z,\delta_z)$ such that $g_z = f$ on $B(z,\delta_z) \cap B(a,r)$. Since $\partial B(a,r)$ is compact, there exist $z_1, \cdots, z_n$, points in $\partial B(a,r)$ such that $\{B(z_k,\delta_{z_k})\}_{k=1}^{n}$ covers $\partial B(a,r)$. Now define

$$g(z) \equiv \begin{cases} f(z) \text{ if } z \in B(a,r) \\ g_{z_k}(z) \text{ if } z \in B(z_k,\delta_{z_k}) \end{cases}$$

Is this well defined? If $z \in B(z_i,\delta_{z_i}) \cap B(z_j,\delta_{z_j})$, is $g_{z_i}(z) = g_{z_j}(z)$? Consider the following picture representing this situation.



You see that if $z \in B(z_i,\delta_{z_i}) \cap B(z_j,\delta_{z_j})$ then $I \equiv B(z_i,\delta_{z_i}) \cap B(z_j,\delta_{z_j}) \cap B(a,r)$ is a nonempty open set. Both g_{z_i} and g_{z_j} equal f on I. Therefore, they must be equal on $B(z_i,\delta_{z_i}) \cap B(z_j,\delta_{z_j})$ because I has a limit point. Therefore, g is well defined and analytic on an open set containing $\overline{B(a,r)}$. Since g agrees with f on $B(a,r)$, the power series for g is the same as the power series for f and converges on a ball which is larger than $B(a,r)$ contrary to the assumption that the radius of convergence of the above power series equals r. This proves the theorem.

27.4.2 Continuation Along A Curve

Next I will describe what is meant by continuation along a curve. The following definition is standard and is found in Rudin [45].

Definition 27.17 *A function element is an ordered pair, (f,D) where D is an open ball and f is analytic on D. (f_0,D_0) and (f_1,D_1) are direct continuations of each other if $D_1\cap D_0\neq\emptyset$ and $f_0=f_1$ on $D_1\cap D_0$. In this case I will write $(f_0,D_0)\sim(f_1,D_1)$. A chain is a finite sequence, of disks, $\{D_0,\cdots,D_n\}$ such that $D_{i-1}\cap D_i\neq\emptyset$. If (f_0,D_0) is a given function element and there exist function elements, (f_i,D_i) such that $\{D_0,\cdots,D_n\}$ is a chain and $(f_{j-1},D_{j-1})\sim(f_j,D_j)$ then (f_n,D_n) is called the analytic continuation of (f_0,D_0) along the chain $\{D_0,\cdots,D_n\}$. Now suppose γ is an oriented curve with parameter interval $[a,b]$ and there exists a chain, $\{D_0,\cdots,D_n\}$ such that $\gamma^*\subseteq\cup_{k=1}^n D_k,\gamma(a)$ is the center of $D_0,\gamma(b)$ is the center of D_n, and there is an increasing list of numbers in $[a,b]$, $a=s_0<s_1\cdots<s_n=b$ such that $\gamma([s_i,s_{i+1}])\subseteq D_i$ and (f_n,D_n) is an analytic continuation of (f_0,D_0) along the chain. Then (f_n,D_n) is called an analytic continuation of (f_0,D_0) along the curve γ. (γ will always be a continuous curve. Nothing more is needed.)*

In the above situation it does not follow that if $D_n\cap D_0\neq\emptyset$, that $f_n=f_0$! However, there are some cases where this will happen. This is the monodromy theorem which follows. This is as far as I will go on the subject of analytic continuation. For more on this subject including a development of the concept of Riemann surfaces, see Alfors [2].

Lemma 27.18 *Suppose $(f,B(0,r))$ for $r<1$ is a function element and $(f,B(0,r))$ can be analytically continued along every curve in $B(0,1)$ that starts at 0. Then there exists an analytic function, g defined on $B(0,1)$ such that $g=f$ on $B(0,r)$.*

Proof: Let

$$\begin{aligned} R &= \sup\{r_1\geq r \text{ such that there exists } g_{r_1} \\ &\qquad \text{analytic on } B(0,r_1) \text{ which agrees with } f \text{ on } B(0,r).\} \end{aligned}$$

Define $g_R(z)\equiv g_{r_1}(z)$ where $|z|<r_1$. This is well defined because if you use r_1 and r_2, both g_{r_1} and g_{r_2} agree with f on $B(0,r)$, a set with a limit point and so the two functions agree at every point in both $B(0,r_1)$ and $B(0,r_2)$. Thus g_R is analytic on $B(0,R)$. If $R<1$, then by the assumption there are no singular points on $B(0,R)$ and so Theorem 27.16 implies the radius of convergence of the power series for g_R is larger than R contradicting the choice of R. Therefore, $R=1$ and this proves the lemma. Let $g=g_R$.

The following theorem is the main result in this subject, the monodromy theorem.

Theorem 27.19 *Let Ω be a simply connected proper subset of $\mathbb{C}$ and suppose $(f, B(a,r))$ is a function element with $B(a,r) \subseteq \Omega$. Suppose also that this function element can be analytically continued along every curve through a. Then there exists G analytic on Ω such that G agrees with f on $B(a,r)$.*

Proof: By the Riemann mapping theorem, there exists $h : \Omega \to B(0,1)$ which is analytic, one to one and onto such that $f(a) = 0$. Since h is an open map, there exists $\delta > 0$ such that

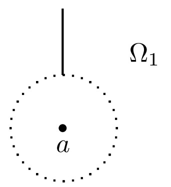
$$B(0,\delta) \subseteq h(B(a,r)).$$

It follows $f \circ h^{-1}$ can be analytically continued along every curve through 0. By Lemma 27.18 there exists g analytic on $B(0,1)$ which agrees with $f \circ h^{-1}$ on $B(0,\delta)$. Define $G(z) \equiv g(h(z))$. For $z = h^{-1}(w)$, it follows $G\left(h^{-1}(w)\right) = g(w)$. If $w \in B(0,\delta)$, then $G\left(h^{-1}(w)\right) = f \circ h^{-1}(w)$ and so $G = f$ on $h^{-1}(B(0,\delta))$, an open set contained in $B(a,r)$. Therefore, $G = f$ on $B(a,r)$ because $h^{-1}(B(0,\delta))$ has a limit point. This proves the theorem.

Actually, you sometimes want to consider the case where $\Omega = \mathbb{C}$. This requires a small modification to obtain from the above theorem.

Corollary 27.20 *Suppose $(f, B(a,r))$ is a function element with $B(a,r) \subseteq \mathbb{C}$. Suppose also that this function element can be analytically continued along every curve through a. Then there exists G analytic on $\mathbb{C}$ such that G agrees with f on $B(a,r)$.*

Proof: Let $\Omega_1 \equiv \{z \in \mathbb{C} : a + it : t > a\}$ and $\Omega_2 \equiv \{z \in \mathbb{C} : a - it : t > a\}$. Here is a picture of Ω_1.



A picture of Ω_2 is similar except the line extends down from the boundary of $B(a,r)$.

Thus $B(a,r) \subseteq \Omega_i$ and Ω_i is simply connected and proper. By Theorem 27.19 there exist analytic functions, G_i analytic on Ω_i such that $G_i = f$ on $B(a,r)$. Thus $G_1 = G_2$ on $B(a,r)$, a set with a limit point. Therefore, $G_1 = G_2$ on $\Omega_1 \cap \Omega_2$. Now let $G(z) = G_i(z)$ where $z \in \Omega_i$. This is well defined and analytic on $\mathbb{C}$. This proves the corollary.

27.5 The Picard Theorems

The Picard theorem says that if f is an entire function and there are two complex numbers not contained in $f(\mathbb{C})$, then f is constant. This is certainly one of the most amazing things which could be imagined. However, this is only the little

Picard theorem. The big Picard theorem is even more incredible. This one asserts that to be non constant the entire function must take every value of $\mathbb{C}$ but two infinitely many times! I will begin with the little Picard theorem. The method of proof I will use is the one found in Saks and Zygmund [47], Conway [13] and Hille [27]. This is not the way Picard did it in 1879. That approach is very different and is presented at the end of the material on elliptic functions. This approach is much more recent dating it appears from around 1924.

Lemma 27.21 *Let f be analytic on a region containing $\overline{B(0,r)}$ and suppose*

$$|f'(0)| = b > 0, f(0) = 0,$$

and $|f(z)| \leq M$ for all $z \in \overline{B(0,r)}$. Then $f(B(0,r)) \supseteq B\left(0, \frac{r^2b^2}{6M}\right)$.

Proof: By assumption,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \ |z| \leq r. \tag{27.16}$$

Then by the Cauchy integral formula for the derivative,

$$a_k = \frac{1}{2\pi i}\int_{\partial B(0,r)} \frac{f(w)}{w^{k+1}} dw$$

where the integral is in the counter clockwise direction. Therefore,

$$|a_k| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\left|f\left(re^{i\theta}\right)\right|}{r^k} d\theta \leq \frac{M}{r^k}.$$

In particular, $br \leq M$. Therefore, from 27.16

$$\begin{aligned} |f(z)| &\geq b|z| - \sum_{k=2}^{\infty} \frac{M}{r^k}|z|^k = b|z| - \frac{M\left(\frac{|z|}{r}\right)^2}{1-\frac{|z|}{r}} \\ &= b|z| - \frac{M|z|^2}{r^2 - r|z|} \end{aligned}$$

Suppose $|z| = \frac{r^2b}{4M} < r$. Then this is no larger than

$$\frac{1}{4}b^2r^2\frac{3M-br}{M(4M-br)} \geq \frac{1}{4}b^2r^2\frac{3M-M}{M(4M-M)} = \frac{r^2b^2}{6M}.$$

Let $|w| < \frac{r^2b}{4M}$. Then for $|z| = \frac{r^2b}{4M}$ and the above,

$$|w| = |(f(z)-w) - f(z)| < \frac{r^2b}{4M} \leq |f(z)|$$

and so by Rouche's theorem, $z \to f(z) - w$ and $z \to f(z)$ have the same number of zeros in $B\left(0, \frac{r^2b}{4M}\right)$. But f has at least one zero in this ball and so this shows there exists at least one $z \in B\left(0, \frac{r^2b}{4M}\right)$ such that $f(z) - w = 0$. This proves the lemma.

27.5.1 Two Competing Lemmas

Lemma 27.21 is a really nice lemma but there is something even better, Bloch's lemma. This lemma does not depend on the bound of f. Like the above two lemmas it is interesting for its own sake and in addition is the key to a fairly short proof of Picard's theorem. It features the number $\frac{1}{24}$. The best constant is not currently known.

Lemma 27.22 *Let f be analytic on an open set containing $\overline{B(0,R)}$ and suppose $|f'(0)| > 0$. Then there exists $a \in B(0,R)$ such that*

$$f(B(0,R)) \supseteq B\left(f(a), \frac{|f'(0)| R}{24}\right).$$

Proof: Let $K(\rho) \equiv \max\{|f'(z)| : |z| = \rho\}$. For simplicity, let $C_\rho \equiv \{z : |z| = \rho\}$.

Claim: K is continuous from the left.

Proof of claim: Let $z_\rho \in C_\rho$ such that $|f'(z_\rho)| = K(\rho)$. Then by the maximum modulus theorem, if $\lambda \in (0,1)$,

$$|f'(\lambda z_\rho)| \leq K(\lambda\rho) \leq K(\rho) = |f'(z_\rho)|.$$

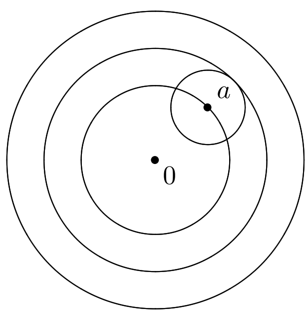
Letting $\lambda \to 1$ yields the claim.

Let ρ_0 be the largest such that $(R - \rho_0) K(\rho_0) = R|f'(0)|$. (Note $(R-0)K(0) = R|f'(0)|$.) Thus $\rho_0 < R$ because $(R-R)K(R) = 0$. Let $|a| = \rho_0$ such that $|f'(a)| = K(\rho_0)$. Thus

$$|f'(a)|(R - \rho_0) = |f'(0)| R \tag{27.17}$$

Now let $r = \frac{R-\rho_0}{2}$. From 27.17,

$$|f'(a)| r = \frac{1}{2}|f'(0)| R,\ B(a,r) \subseteq B(0,\rho_0 + r) \subseteq B(0,R). \tag{27.18}$$



Therefore, if $z \in B(a,r)$, it follows from the maximum modulus theorem and the definition of ρ_0 that

$$\begin{aligned} |f'(z)| &\leq K(\rho_0 + r) < \frac{R|f'(0)|}{R-\rho_0-r} = \frac{2R|f'(0)|}{R-\rho_0} \\ &= \frac{2R|f'(0)|}{2r} = \frac{R|f'(0)|}{r} \end{aligned} \tag{27.19}$$

Let $g(z) = f(a+z) - f(a)$ where $z \in B(0,r)$. Then $|g'(0)| = |f'(a)| > 0$ and for $z \in \overline{B(0,r)}$,

$$|g(z)| \leq \left| \int_{\gamma(a,z)} g'(w)\,dw \right| \leq |z-a| \frac{R|f'(0)|}{r} = R|f'(0)|.$$

By Lemma 27.21 and 27.18,

$$\begin{aligned} g(B(0,r)) &\supseteq B\left(0, \frac{r^2|f'(a)|^2}{6R|f'(0)|}\right) \\ &= B\left(0, \frac{r^2\left(\frac{1}{2r}|f'(0)|R\right)^2}{6R|f'(0)|}\right) = B\left(0, \frac{|f'(0)|R}{24}\right) \end{aligned}$$

Now $g(B(0,r)) = f(B(a,r)) - f(a)$ and so this implies

$$f(B(0,R)) \supseteq f(B(a,r)) \supseteq B\left(f(a), \frac{|f'(0)|R}{24}\right).$$

This proves the lemma.

Here is a slightly more general version which allows the center of the open set to be arbitrary.

Lemma 27.23 *Let f be analytic on an open set containing $\overline{B(z_0,R)}$ and suppose $|f'(z_0)| > 0$. Then there exists $a \in B(z_0,R)$ such that*

$$f(B(z_0,R)) \supseteq B\left(f(a), \frac{|f'(z_0)|R}{24}\right).$$

Proof: You look at $g(z) \equiv f(z_0+z) - f(z_0)$ for $z \in B(0,R)$. Then $g'(0) = f'(z_0)$ and so by Lemma 27.22 there exists $a_1 \in B(0,R)$ such that

$$g(B(0,R)) \supseteq B\left(g(a_1), \frac{|f'(z_0)|R}{24}\right).$$

Now $g(B(0,R)) = f(B(z_0,R)) - f(z_0)$ and $g(a_1) = f(a) - f(z_0)$ for some $a \in B(z_0,R)$ and so

$$\begin{aligned} f(B(z_0,R)) - f(z_0) &\supseteq B\left(g(a_1), \frac{|f'(z_0)|R}{24}\right) \\ &= B\left(f(a) - f(z_0), \frac{|f'(z_0)|R}{24}\right) \end{aligned}$$

which implies

$$f(B(z_0, R)) \supseteq B\left(f(a), \frac{|f'(z_0)| R}{24}\right)$$

as claimed. This proves the lemma.

No attempt was made to find the best number to multiply by $R|f'(z_0)|$. A discussion of this is given in Conway [13]. See also [27]. Much larger numbers than $1/24$ are available and there is a conjecture due to Alfors about the best value. The conjecture is that $1/24$ can be replaced with

$$\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\left(1+\sqrt{3}\right)^{1/2}\Gamma\left(\frac{1}{4}\right)} \approx .471\,86$$

You can see there is quite a gap between the constant for which this lemma is proved above and what is thought to be the best constant.

Bloch's lemma above gives the existence of a ball of a certain size inside the image of a ball. By contrast the next lemma leads to conditions under which the values of a function do not contain a ball of certain radius. It concerns analytic functions which do not achieve the values 0 and 1.

Lemma 27.24 *Let $\mathcal{F}$ denote the set of functions, f defined on Ω, a simply connected region which do not achieve the values 0 and 1. Then for each such function, it is possible to define a function analytic on Ω, $H(z)$ by the formula*

$$H(z) \equiv \log\left[\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right].$$

There exists a constant C independent of $f \in \mathcal{F}$ such that $H(\Omega)$ does not contain any ball of radius C.

Proof: Let $f \in \mathcal{F}$. Then since f does not take the value 0, there exists g_1 a primitive of f'/f. Thus

$$\frac{d}{dz}\left(e^{-g_1} f\right) = 0$$

so there exists a, b such that $f(z) e^{-g_1(z)} = e^{a+bi}$. Letting $g(z) = g_1(z) + a + ib$, it follows $e^{g(z)} = f(z)$. Let $\log(f(z)) = g(z)$. Then for $n \in \mathbb{Z}$, the integers,

$$\frac{\log(f(z))}{2\pi i}, \frac{\log(f(z))}{2\pi i} - 1 \neq n$$

because if equality held, then $f(z) = 1$ which does not happen. It follows $\frac{\log(f(z))}{2\pi i}$ and $\frac{\log(f(z))}{2\pi i} - 1$ are never equal to zero. Therefore, using the same reasoning, you can define a logarithm of these two quantities and therefore, a square root. Hence there exists a function analytic on Ω,

$$\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}. \tag{27.20}$$

For n a positive integer, this function cannot equal $\sqrt{n} \pm \sqrt{n-1}$ because if it did, then

$$\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right) = \sqrt{n} \pm \sqrt{n-1} \tag{27.21}$$

and you could take reciprocals of both sides to obtain

$$\left(\sqrt{\frac{\log(f(z))}{2\pi i}} + \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right) = \sqrt{n} \mp \sqrt{n-1}. \tag{27.22}$$

Then adding 27.21 and 27.22

$$2\sqrt{\frac{\log(f(z))}{2\pi i}} = 2\sqrt{n}$$

which contradicts the above observation that $\frac{\log(f(z))}{2\pi i}$ is not equal to an integer.

Also, the function of 27.20 is never equal to zero. Therefore, you can define the logarithm of this function also. It follows

$$H(z) \equiv \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right) \neq \ln\left(\sqrt{n} \pm \sqrt{n-1}\right) + 2m\pi i$$

where m is an arbitrary integer and n is a positive integer. Now

$$\lim_{n\to\infty} \ln\left(\sqrt{n} + \sqrt{n-1}\right) = \infty$$

and $\lim_{n\to\infty} \ln\left(\sqrt{n} - \sqrt{n-1}\right) = -\infty$ and so $\mathbb{C}$ is covered by rectangles having vertices at points $\ln\left(\sqrt{n} \pm \sqrt{n-1}\right) + 2m\pi i$ as described above. Each of these rectangles has height equal to 2π and a short computation shows their widths are bounded. Therefore, there exists C independent of $f \in \mathcal{F}$ such that C is larger than the diameter of all these rectangles. Hence $H(\Omega)$ cannot contain any ball of radius larger than C.

27.5.2 The Little Picard Theorem

Now here is the little Picard theorem. It is easy to prove from the above.

Theorem 27.25 *If h is an entire function which omits two values then h is a constant.*

Proof: Suppose the two values omitted are a and b and that h is not constant. Let $f(z) = (h(z) - a)/(b - a)$. Then f omits the two values 0 and 1. Let H be defined in Lemma 27.24. Then $H(z)$ is clearly not of the form $az+b$ because then it would have values equal to the vertices $\ln\left(\sqrt{n} \pm \sqrt{n-1}\right) + 2m\pi i$ or else be constant neither of which happen if h is not constant. Therefore, by Liouville's theorem, H' must be unbounded. Pick ξ such that $|H'(\xi)| > 24C$ where C is such that $H(\mathbb{C})$

contains no balls of radius larger than C. But by Lemma 27.23 $H(B(\xi,1))$ must contain a ball of radius $\frac{|H'(\xi)|}{24} > \frac{24C}{24} = C$, a contradiction. This proves Picard's theorem.

The following is another formulation of this theorem.

Corollary 27.26 *If f is a meromophic function defined on $\mathbb{C}$ which omits three distinct values, a, b, c, then f is a constant.*

Proof: Let $\phi(z) \equiv \frac{z-a}{z-c}\frac{b-c}{b-a}$. Then $\phi(c) = \infty, \phi(a) = 0$, and $\phi(b) = 1$. Now consider the function, $h = \phi \circ f$. Then h misses the three points $\infty, 0$, and 1. Since h is meromorphic and does not have ∞ in its values, it must actually be analytic. Thus h is an entire function which misses the two values 0 and 1. Therefore, h is constant by Theorem 27.25.

27.5.3 Schottky's Theorem

Lemma 27.27 *Let f be analytic on an open set containing $\overline{B(0,R)}$ and suppose that f does not take on either of the two values 0 or 1. Also suppose $|f(0)| \leq \beta$. Then letting $\theta \in (0,1)$, it follows*

$$|f(z)| \leq M(\beta,\theta)$$

for all $z \in B(0,\theta R)$, where $M(\beta,\theta)$ is a function of only the two variables β, θ. (In particular, there is no dependence on R.)

Proof: Consider the function, $H(z)$ used in Lemma 27.24 given by

$$H(z) \equiv \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right). \tag{27.23}$$

You notice there are two explicit uses of logarithms. Consider first the logarithm inside the radicals. Choose this logarithm such that

$$\log(f(0)) = \ln|f(0)| + i\arg(f(0)),\ \arg(f(0)) \in (-\pi,\pi]. \tag{27.24}$$

You can do this because

$$e^{\log(f(0))} = f(0) = e^{\ln|f(0)|}e^{i\alpha} = e^{\ln|f(0)|+i\alpha}$$

and by replacing α with $\alpha + 2m\pi$ for a suitable integer, m it follows the above equation still holds. Therefore, you can assume 27.24. Similar reasoning applies to the logarithm on the outside of the parenthesis. It can be assumed $H(0)$ equals

$$\ln\left|\sqrt{\frac{\log(f(0))}{2\pi i}} - \sqrt{\frac{\log(f(0))}{2\pi i} - 1}\right| + i\arg\left(\sqrt{\frac{\log(f(0))}{2\pi i}} - \sqrt{\frac{\log(f(0))}{2\pi i} - 1}\right) \tag{27.25}$$

where the imaginary part is no larger than π in absolute value.

Now if $\xi \in B(0,R)$ is a point where $H'(\xi) \neq 0$, then by Lemma 27.22

$$H(B(\xi, R-|\xi|)) \supseteq B\left(H(a), \frac{|H'(\xi)|(R-|\xi|)}{24}\right)$$

where a is some point in $B(\xi, R-|\xi|)$. But by Lemma 27.24 $H(B(\xi, R-|\xi|))$ contains no balls of radius C where C depended only on the maximum diameters of those rectangles having vertices $\ln\left(\sqrt{n} \pm \sqrt{n-1}\right) + 2m\pi i$ for n a positive integer and m an integer. Therefore,

$$\frac{|H'(\xi)|(R-|\xi|)}{24} < C$$

and consequently

$$|H'(\xi)| < \frac{24C}{R-|\xi|}.$$

Even if $H'(\xi) = 0$, this inequality still holds. Therefore, if $z \in B(0,R)$ and $\gamma(0,z)$ is the straight segment from 0 to z,

$$\begin{aligned} |H(z) - H(0)| &= \left|\int_{\gamma(0,z)} H'(w)\,dw\right| = \left|\int_0^1 H'(tz)\,zdt\right| \\ &\leq \int_0^1 |H'(tz)\,z|\,dt \leq \int_0^1 \frac{24C}{R-t|z|}|z|\,dt \\ &= 24C \ln\left(\frac{R}{R-|z|}\right). \end{aligned}$$

Therefore, for $z \in \partial B(0, \theta R)$,

$$|H(z)| \leq |H(0)| + 24C \ln\left(\frac{1}{1-\theta}\right). \tag{27.26}$$

By the maximum modulus theorem, the above inequality holds for all $|z| < \theta R$ also.

Next I will use 27.23 to get an inequality for $|f(z)|$ in terms of $|H(z)|$. From 27.23,

$$H(z) = \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right)$$

and so

$$\begin{aligned} 2H(z) &= \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right)^2 \\ -2H(z) &= \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right)^{-2} \\ &= \log\left(\sqrt{\frac{\log(f(z))}{2\pi i}} + \sqrt{\frac{\log(f(z))}{2\pi i} - 1}\right)^2 \end{aligned}$$

Therefore,

$$\begin{aligned}
&\left(\sqrt{\frac{\log(f(z))}{2\pi i}}+\sqrt{\frac{\log(f(z))}{2\pi i}-1}\right)^2 \\
&+\left(\sqrt{\frac{\log(f(z))}{2\pi i}}-\sqrt{\frac{\log(f(z))}{2\pi i}-1}\right)^2 \\
=\ & \exp(2H(z))+\exp(-2H(z))
\end{aligned}$$

and

$$\left(\frac{\log(f(z))}{\pi i}-1\right)=\frac{1}{2}\left(\exp(2H(z))+\exp(-2H(z))\right).$$

Thus

$$\log(f(z))=\pi i+\frac{\pi i}{2}\left(\exp(2H(z))+\exp(-2H(z))\right)$$

which shows

$$\begin{aligned}
|f(z)| &= \left|\exp\left[\frac{\pi i}{2}\left(\exp(2H(z))+\exp(-2H(z))\right)\right]\right| \\
&\leq \exp\left|\frac{\pi i}{2}\left(\exp(2H(z))+\exp(-2H(z))\right)\right| \\
&\leq \exp\left|\frac{\pi}{2}\left(|\exp(2H(z))|+|\exp(-2H(z))|\right)\right| \\
&\leq \exp\left|\frac{\pi}{2}\left(\exp(2|H(z)|)+\exp(|-2H(z)|)\right)\right| \\
&= \exp(\pi\exp 2|H(z)|).
\end{aligned}$$

Now from 27.26 this is dominated by

$$\begin{aligned}
&\exp\left(\pi\exp 2\left(|H(0)|+24C\ln\left(\frac{1}{1-\theta}\right)\right)\right) \\
=\ &\exp\left(\pi\exp(2|H(0)|)\exp\left(48C\ln\left(\frac{1}{1-\theta}\right)\right)\right) \qquad (27.27)
\end{aligned}$$

Consider $\exp(2|H(0)|)$. I want to obtain an inequality for this which involves β. This is where I will use the convention about the logarithms discussed above. From 27.25,

$$2|H(0)|=2\left|\log\left(\sqrt{\frac{\log(f(0))}{2\pi i}}-\sqrt{\frac{\log(f(0))}{2\pi i}-1}\right)\right|$$

$$\begin{aligned}
&\leq 2\left(\left(\ln\left|\sqrt{\frac{\log(f(0))}{2\pi i}}-\sqrt{\frac{\log(f(0))}{2\pi i}-1}\right|\right)^2+\pi^2\right)^{1/2}\\
&\leq 2\left(\left|\ln\left(\left|\sqrt{\frac{\log(f(0))}{2\pi i}}\right|+\left|\sqrt{\frac{\log(f(0))}{2\pi i}-1}\right|\right)\right|^2+\pi^2\right)^{1/2}\\
&\leq 2\left|\ln\left(\left|\sqrt{\frac{\log(f(0))}{2\pi i}}\right|+\left|\sqrt{\frac{\log(f(0))}{2\pi i}-1}\right|\right)\right|+2\pi\\
&\leq \ln\left(2\left(\left|\frac{\log(f(0))}{2\pi i}\right|+\left|\frac{\log(f(0))}{2\pi i}-1\right|\right)\right)+2\pi\\
&= \ln\left(\left(\left|\frac{\log(f(0))}{\pi i}\right|+\left|\frac{\log(f(0))}{\pi i}-2\right|\right)\right)+2\pi \qquad (27.28)
\end{aligned}$$

Consider $\left|\frac{\log(f(0))}{\pi i}\right|$

$$\frac{\log(f(0))}{\pi i}=-\frac{\ln|f(0)|}{\pi}i+\frac{\arg(f(0))}{\pi}$$

and so

$$\begin{aligned}
\left|\frac{\log(f(0))}{\pi i}\right| &= \left(\left|\frac{\ln|f(0)|}{\pi}\right|^2+\left(\frac{\arg(f(0))}{\pi}\right)^2\right)^{1/2}\\
&\leq \left(\left|\frac{\ln\beta}{\pi}\right|^2+\left(\frac{\pi}{\pi}\right)^2\right)^{1/2}\\
&= \left(\left|\frac{\ln\beta}{\pi}\right|^2+1\right)^{1/2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left|\frac{\log(f(0))}{\pi i}-2\right| &\leq \left(\left|\frac{\ln\beta}{\pi}\right|^2+(2+1)^2\right)^{1/2}\\
&= \left(\left|\frac{\ln\beta}{\pi}\right|^2+9\right)^{1/2}
\end{aligned}$$

It follows from 27.28 that

$$2|H(0)|\leq\ln\left(2\left(\left|\frac{\ln\beta}{\pi}\right|^2+9\right)^{1/2}\right)+2\pi.$$

Hence from 27.27

$$|f(z)|\leq$$

$$\exp\left(\pi\exp\left(\ln\left(2\left(\left|\frac{\ln\beta}{\pi}\right|^2+9\right)^{1/2}\right)+2\pi\right)\exp\left(48C\ln\left(\frac{1}{1-\theta}\right)\right)\right)$$

and so, letting $M(\beta,\theta)$ be given by the above expression on the right, the lemma is proved.

The following theorem will be referred to as Schottky's theorem. It looks just like the above lemma except it is only assumed that f is analytic on $B(0,R)$ rather than on an open set containing $\overline{B(0,R)}$. Also, the case of an arbitrary center is included along with arbitrary points which are not attained as values of the function.

Theorem 27.28 *Let f be analytic on $B(z_0,R)$ and suppose that f does not take on either of the two distinct values a or b. Also suppose $|f(z_0)|\leq\beta$. Then letting $\theta\in(0,1)$, it follows*

$$|f(z)|\leq M(a,b,\beta,\theta)$$

for all $z\in B(z_0,\theta R)$, where $M(a,b,\beta,\theta)$ is a function of only the variables β,θ,a,b. (In particular, there is no dependence on R.)

Proof: First you can reduce to the case where the two values are 0 and 1 by considering

$$h(z)\equiv\frac{f(z)-a}{b-a}.$$

If there exists an estimate of the desired sort for h, then there exists such an estimate for f. Of course here the function, M would depend on a and b. Therefore, there is no loss of generality in assuming the points which are missed are 0 and 1.

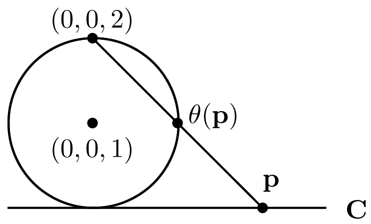
Apply Lemma 27.27 to $B(0,R_1)$ for the function, $g(z)\equiv f(z_0+z)$ and $R_1<R$. Then if $\beta\geq|f(z_0)|=|g(0)|$, it follows $|g(z)|=|f(z_0+z)|\leq M(\beta,\theta)$ for every $z\in B(0,\theta R_1)$. Now let $\theta\in(0,1)$ and choose $R_1<R$ large enough that $\theta R=\theta_1R_1$ where $\theta_1\in(0,1)$. Then if $|z-z_0|<\theta R$, it follows

$$|f(z)|\leq M(\beta,\theta_1).$$

Now let $R_1\to R$ so $\theta_1\to\theta$.

27.5.4 A Brief Review

First recall the definition of the metric on $\widehat{\mathbb{C}}$. For convenience it is listed here again. Consider the unit sphere, S^2 given by $(z-1)^2+y^2+x^2=1$. Define a map from the complex plane to the surface of this sphere as follows. Extend a line from the point, p in the complex plane to the point $(0,0,2)$ on the top of this sphere and let $\theta(p)$ denote the point of this sphere which the line intersects. Define $\theta(\infty)\equiv(0,0,2)$.



Then θ^{-1} is sometimes called sterographic projection. The mapping θ is clearly continuous because it takes converging sequences, to converging sequences. Furthermore, it is clear that θ^{-1} is also continuous. In terms of the extended complex plane, $\widehat{\mathbb{C}}$, a sequence, z_n converges to ∞ if and only if θz_n converges to $(0,0,2)$ and a sequence, z_n converges to $z \in \mathbb{C}$ if and only if $\theta(z_n) \to \theta(z)$.

In fact this makes it easy to define a metric on $\widehat{\mathbb{C}}$.

Definition 27.29 *Let $z, w \in \widehat{\mathbb{C}}$. Then let $d(x,y) \equiv |\theta(z) - \theta(w)|$ where this last distance is the usual distance measured in $\mathbb{R}^3$.*

Theorem 27.30 $\left(\widehat{\mathbb{C}}, d\right)$ *is a compact, hence complete metric space.*

Proof: Suppose $\{z_n\}$ is a sequence in $\widehat{\mathbb{C}}$. This means $\{\theta(z_n)\}$ is a sequence in S^2 which is compact. Therefore, there exists a subsequence, $\{\theta z_{n_k}\}$ and a point, $z \in S^2$ such that $\theta z_{n_k} \to \theta z$ in S^2 which implies immediately that $d(z_{n_k}, z) \to 0$. A compact metric space must be complete.

Also recall the interesting fact that meromorphic functions are continuous with values in $\widehat{\mathbb{C}}$ which is reviewed here for convenience. It came from the theory of classification of isolated singularities.

Theorem 27.31 *Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \to \widehat{\mathbb{C}}$ be meromorphic. Then f is continuous with respect to the metric, d on $\widehat{\mathbb{C}}$.*

Proof: Let $z_n \to z$ where $z \in \Omega$. Then if z is a pole, it follows from Theorem 24.38 that

$$d(f(z_n), \infty) \equiv d(f(z_n), f(z)) \to 0.$$

If z is not a pole, then $f(z_n) \to f(z)$ in $\mathbb{C}$ which implies $|\theta(f(z_n)) - \theta(f(z))| = d(f(z_n), f(z)) \to 0$. Recall that θ is continuous on $\mathbb{C}$.

The fundamental result behind all the theory about to be presented is the Ascoli Arzela theorem also listed here for convenience.

Definition 27.32 *Let (X,d) be a complete metric space. Then it is said to be locally compact if $\overline{B(x,r)}$ is compact for each $r > 0$.*

Thus if you have a locally compact metric space, then if $\{a_n\}$ is a bounded sequence, it must have a convergent subsequence.

Let K be a compact subset of $\mathbb{R}^n$ and consider the continuous functions which have values in a locally compact metric space, (X,d) where d denotes the metric on X. Denote this space as $C(K,X)$.

Definition 27.33 *For $f, g \in C(K, X)$, where K is a compact subset of $\mathbb{R}^n$ and X is a locally compact complete metric space define*

$$\rho_K(f, g) \equiv \sup \{d(f(\mathbf{x}), g(\mathbf{x})) : \mathbf{x} \in K\}.$$

The Ascoli Arzela theorem, Theorem 6.24 is a major result which tells which subsets of $C(K, X)$ are sequentially compact.

Definition 27.34 *Let $A \subseteq C(K, X)$ for K a compact subset of $\mathbb{R}^n$. Then A is said to be uniformly equicontinuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $\mathbf{x}, \mathbf{y} \in K$ with $|\mathbf{x} - \mathbf{y}| < \delta$ and $f \in A$,*

$$d(f(\mathbf{x}), f(\mathbf{y})) < \varepsilon.$$

The set, A is said to be uniformly bounded if for some $M < \infty$, and $a \in X$,

$$f(\mathbf{x}) \in B(a, M)$$

for all $f \in A$ and $\mathbf{x} \in K$.

The Ascoli Arzela theorem follows.

Theorem 27.35 *Suppose K is a nonempty compact subset of $\mathbb{R}^n$ and $A \subseteq C(K, X)$, is uniformly bounded and uniformly equicontinuous where X is a locally compact complete metric space. Then if $\{f_k\} \subseteq A$, there exists a function, $f \in C(K, X)$ and a subsequence, f_{k_l} such that*

$$\lim_{l \to \infty} \rho_K(f_{k_l}, f) = 0.$$

In the cases of interest here, $X = \widehat{\mathbb{C}}$ with the metric defined above.

27.5.5 Montel's Theorem

The following lemma is another version of Montel's theorem. It is this which will make possible a proof of the big Picard theorem.

Lemma 27.36 *Let Ω be a region and let $\mathcal{F}$ be a set of functions analytic on Ω none of which achieve the two distinct values, a and b. If $\{f_n\} \subseteq \mathcal{F}$ then one of the following hold: Either there exists a function, f analytic on Ω and a subsequence, $\{f_{n_k}\}$ such that for any compact subset, K of Ω,*

$$\lim_{k \to \infty} ||f_{n_k} - f||_{K,\infty} = 0. \tag{27.29}$$

or there exists a subsequence $\{f_{n_k}\}$ such that for all compact subsets K,

$$\lim_{k \to \infty} \rho_K(f_{n_k}, \infty) = 0. \tag{27.30}$$

Proof: Let $B(z_0, 2R) \subseteq \Omega$. There are two cases to consider. The first case is that there exists a subsequence, n_k such that $\{f_{n_k}(z_0)\}$ is bounded. The second case is that $\lim_{n\to\infty} |f_{n_k}(z_0)| = \infty$.

Consider the first case. By Theorem 27.28 $\{f_{n_k}(z)\}$ is uniformly bounded on $\overline{B(z_0, R)}$ because by this theorem, and letting $\theta = 1/2$ applied to $B(z_0, 2R)$, it follows $|f_{n_k}(z)| \leq M\left(a, b, \frac{1}{2}, \beta\right)$ where β is an upper bound to the numbers, $|f_{n_k}(z_0)|$. The Cauchy integral formula implies the existence of a uniform bound on the $\left\{f'_{n_k}\right\}$ which implies the functions are equicontinuous and uniformly bounded. Therefore, by the Ascoli Arzela theorem there exists a further subsequence which converges uniformly on $\overline{B(z_0, R)}$ to a function, f analytic on $B(z_0, R)$. Thus denoting this subsequence by $\{f_{n_k}\}$ to save on notation,

$$\lim_{k\to\infty} ||f_{n_k} - f||_{\overline{B(z_0,R)},\infty} = 0. \tag{27.31}$$

Consider the second case. In this case, it follows $\{1/f_n(z_0)\}$ is bounded on $\overline{B(z_0, R)}$ and so by the same argument just given $\{1/f_n(z)\}$ is uniformly bounded on $\overline{B(z_0, R)}$. Therefore, a subsequence converges uniformly on $\overline{B(z_0, R)}$. But $\{1/f_n(z)\}$ converges to 0 and so this requires that $\{1/f_n(z)\}$ must converge uniformly to 0. Therefore,

$$\lim_{k\to\infty} \rho_{\overline{B(z_0,R)}}(f_{n_k}, \infty) = 0. \tag{27.32}$$

Now let $\{D_k\}$ denote a countable set of closed balls, $D_k = \overline{B(z_k, R_k)}$ such that $B(z_k, 2R_k) \subseteq \Omega$ and $\cup_{k=1}^{\infty} \text{int}(D_k) = \Omega$. Using a Cantor diagonal process, there exists a subsequence, $\{f_{n_k}\}$ of $\{f_n\}$ such that for each D_j, one of the above two alternatives holds. That is, either

$$\lim_{k\to\infty} ||f_{n_k} - g_j||_{D_j,\infty} = 0 \tag{27.33}$$

or,

$$\lim_{k\to\infty} \rho_{D_j}(f_{n_k}, \infty). \tag{27.34}$$

Let $A = \{\cup \text{int}(D_j) : 27.33 \text{ holds}\}$, $B = \{\cup \text{int}(D_j) : 27.34 \text{ holds}\}$. Note that the balls whose union is A cannot intersect any of the balls whose union is B. Therefore, one of A or B must be empty since otherwise, Ω would not be connected.

If K is any compact subset of Ω, it follows K must be a subset of some finite collection of the D_j. Therefore, one of the alternatives in the lemma must hold. That the limit function, f must be analytic follows easily in the same way as the proof in Theorem 27.7 on Page 740. You could also use Morera's theorem. This proves the lemma.

27.5.6 The Great Big Picard Theorem

The next theorem is the main result which the above lemmas lead to. It is the Big Picard theorem, also called the Great Picard theorem.Recall $B'(a, r)$ is the deleted ball consisting of all the points of the ball except the center.

Theorem 27.37 *Suppose f has an isolated essential singularity at 0. Then for every $R > 0$, and $\beta \in \mathbb{C}$, $f^{-1}(\beta) \cap B'(0,R)$ is an infinite set except for one possible exceptional β.*

Proof: Suppose this is not true. Then there exists $R_1 > 0$ and two points, α and β such that $f^{-1}(\beta) \cap B'(0,R_1)$ and $f^{-1}(\alpha) \cap B'(0,R_1)$ are both finite sets. Then shrinking R_1 and calling the result R, there exists $B(0,R)$ such that

$$f^{-1}(\beta) \cap B'(0,R) = \emptyset,\ f^{-1}(\alpha) \cap B'(0,R) = \emptyset.$$

Now let A_0 denote the annulus $\left\{z \in \mathbb{C} : \frac{R}{2^2} < |z| < \frac{3R}{2^2}\right\}$ and let A_n denote the annulus $\left\{z \in \mathbb{C} : \frac{R}{2^{2+n}} < |z| < \frac{3R}{2^{2+n}}\right\}$. The reason for the 3 is to insure that $A_n \cap A_{n+1} \neq \emptyset$. This follows from the observation that $3R/2^{2+1+n} > R/2^{2+n}$. Now define a set of functions on A_0 as follows:

$$f_n(z) \equiv f\left(\frac{z}{2^n}\right).$$

By the choice of R, this set of functions missed the two points α and β. Therefore, by Lemma 27.36 there exists a subsequence such that one of the two options presented there holds.

First suppose $\lim_{k\to\infty} ||f_{n_k} - f||_{K,\infty} = 0$ for all K a compact subset of A_0 and f is analytic on A_0. In particular, this happens for γ_0 the circular contour having radius $R/2$. Thus f_{n_k} must be bounded on this contour. But this says the same thing as $f(z/2^{n_k})$ is bounded for $|z| = R/2$, this holding for each $k = 1, 2, \cdots$. Thus there exists a constant, M such that on each of a shrinking sequence of concentric circles whose radii converge to 0, $|f(z)| \leq M$. By the maximum modulus theorem, $|f(z)| \leq M$ at every point between successive circles in this sequence. Therefore, $|f(z)| \leq M$ in $B'(0,R)$ contradicting the Weierstrass Casorati theorem.

The other option which might hold from Lemma 27.36 is that $\lim_{k\to\infty} \rho_K(f_{n_k}, \infty) = 0$ for all K compact subset of A_0. Since f has an essential singularity at 0 the zeros of f in $B(0,R)$ are isolated. Therefore, for all k large enough, f_{n_k} has no zeros for $|z| < 3R/2^2$. This is because the values of f_{n_k} are the values of f on A_{n_k}, a small anulus which avoids all the zeros of f whenever k is large enough. Only consider k this large. Then use the above argument on the analytic functions $1/f_{n_k}$. By the assumption that $\lim_{k\to\infty} \rho_K(f_{n_k}, \infty) = 0$, it follows $\lim_{k\to\infty} ||1/f_{n_k} - 0||_{K,\infty} = 0$ and so as above, there exists a shrinking sequence of concentric circles whose radii converge to 0 and a constant, M such that for z on any of these circles, $|1/f(z)| \leq M$. This implies that on some deleted ball, $B'(0,r)$ where $r \leq R$, $|f(z)| \geq 1/M$ which again violates the Weierstrass Casorati theorem. This proves the theorem.

As a simple corollary, here is what this remarkable theorem says about entire functions.

Corollary 27.38 *Suppose f is entire and nonconstant and not a polynomial. Then f assumes every complex value infinitely many times with the possible exception of one.*

Proof: Since f is entire, $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Define for $z \neq 0$,

$$g(z) \equiv f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n.$$

Thus 0 is an isolated essential singular point of g. By the big Picard theorem, Theorem 27.37 it follows g takes every complex number but possibly one an infinite number of times. This proves the corollary.

Note the difference between this and the little Picard theorem which says that an entire function which is not constant must achieve every value but two.

27.6 Exercises

1. Prove that in Theorem 27.7 it suffices to assume $\mathcal{F}$ is uniformly bounded on each compact subset of Ω.

2. Find conditions on a, b, c, d such that the fractional linear transformation, $\frac{az+b}{cz+d}$ maps the upper half plane onto the upper half plane.

3. Let D be a simply connected region which is a proper subset of $\mathbb{C}$. Does there exist an entire function, f which maps $\mathbb{C}$ onto D? Why or why not?

4. Verify the conclusion of Theorem 27.7 involving the higher order derivatives.

5. What if $\Omega = \mathbb{C}$? Does there exist an analytic function, f mapping Ω one to one and onto $B(0,1)$? Explain why or why not. Was $\Omega \neq \mathbb{C}$ used in the proof of the Riemann mapping theorem?

6. Verify that $|\phi_\alpha(z)| = 1$ if $|z| = 1$. Apply the maximum modulus theorem to conclude that $|\phi_\alpha(z)| \leq 1$ for all $|z| < 1$.

7. Suppose that $|f(z)| \leq 1$ for $|z| = 1$ and $f(\alpha) = 0$ for $|\alpha| < 1$. Show that $|f(z)| \leq |\phi_\alpha(z)|$ for all $z \in B(0,1)$. **Hint:** Consider $\frac{f(z)(1-\overline{\alpha}z)}{z-\alpha}$ which has a removable singularity at α. Show the modulus of this function is bounded by 1 on $|z| = 1$. Then apply the maximum modulus theorem.

8. Let U and V be open subsets of $\mathbb{C}$ and suppose $u : U \to \mathbb{R}$ is harmonic while h is an analytic map which takes V one to one onto U. Show that $u \circ h$ is harmonic on V.

9. Show that for a harmonic function, u defined on $B(0,R)$, there exists an analytic function, $h = u + iv$ where

$$v(x,y) \equiv \int_0^y u_x(x,t)\,dt - \int_0^x u_y(t,0)\,dt.$$

10. Suppose Ω is a simply connected region and u is a real valued function defined on Ω such that u is harmonic. Show there exists an analytic function, f such that $u = \operatorname{Re} f$. Show this is not true if Ω is not a simply connected region. **Hint:** You might use the Riemann mapping theorem and Problems 8 and 9. For the second part it might be good to try something like $u(x, y) = \ln\left(x^2 + y^2\right)$ on the annulus $1 < |z| < 2$.

11. Show that $w = \frac{1+z}{1-z}$ maps $\{z \in \mathbb{C} : \operatorname{Im} z > 0 \text{ and } |z| < 1\}$ to the first quadrant, $\{z = x + iy : x, y > 0\}$.

12. Let $f(z) = \frac{az+b}{cz+d}$ and let $g(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$. Show that $f \circ g(z)$ equals the quotient of two expressions, the numerator being the top entry in the vector

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

and the denominator being the bottom entry. Show that if you define

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \equiv \frac{az+b}{cz+d},$$

then $\phi(AB) = \phi(A) \circ \phi(B)$. Find an easy way to find the inverse of $f(z) = \frac{az+b}{cz+d}$ and give a condition on the a, b, c, d which insures this function has an inverse.

13. The modular group[2] is the set of fractional linear transformations, $\frac{az+b}{cz+d}$ such that a, b, c, d are integers and $ad - bc = 1$. Using Problem 12 or brute force show this modular group is really a group with the group operation being composition. Also show the inverse of $\frac{az+b}{cz+d}$ is $\frac{dz-b}{-cz+a}$.

14. Let Ω be a region and suppose f is analytic on Ω and that the functions f_n are also analytic on Ω and converge to f uniformly on compact subsets of Ω. Suppose f is one to one. Can it be concluded that for an arbitrary compact set, $K \subseteq \Omega$ that f_n is one to one for all n large enough?

15. The Vitali theorem says that if Ω is a region and $\{f_n\}$ is a uniformly bounded sequence of functions which converges pointwise on a set, $S \subseteq \Omega$ which has a limit point in Ω, then in fact, $\{f_n\}$ must converge uniformly on compact subsets of Ω to an analytic function. Prove this theorem. **Hint:** If the sequence fails to converge, show you can get two different subsequences converging uniformly on compact sets to different functions. Then argue these two functions coincide on S.

16. Does there exist a function analytic on $B(0, 1)$ which maps $B(0, 1)$ onto $B'(0, 1)$, the open unit ball in which 0 has been deleted?

[2]This is the terminology used in Rudin's book Real and Complex Analysis.

Approximation By Rational Functions

28.1 Runge's Theorem

Consider the function, $\frac{1}{z} = f(z)$ for z defined on $\Omega \equiv B(0,1) \setminus \{0\} = B'(0,1)$. Clearly f is analytic on Ω. Suppose you could approximate f uniformly by polynomials on $\overline{\text{ann}\left(0, \frac{1}{2}, \frac{3}{4}\right)}$, a compact subset of Ω. Then, there would exist a suitable polynomial $p(z)$, such that $\left|\frac{1}{2\pi i}\int_{\gamma} f(z) - p(z)\,dz\right| < \frac{1}{10}$ where here γ is a circle of radius $\frac{2}{3}$. However, this is impossible because $\frac{1}{2\pi i}\int_{\gamma} f(z)\,dz = 1$ while $\frac{1}{2\pi i}\int_{\gamma} p(z)\,dz = 0$. This shows you can't expect to be able to uniformly approximate analytic functions on compact sets using polynomials. This is just horrible! In real variables, you can approximate any **continuous function** on a compact set with a polynomial. However, that is just the way it is. It turns out that the ability to approximate an analytic function on Ω with polynomials is dependent on Ω being simply connected.

All these theorems work for f having values in a complex Banach space. However, I will present them in the context of functions which have values in $\mathbb{C}$. The changes necessary to obtain the extra generality are very minor.

Definition 28.1 *Approximation will be taken with respect to the following norm.*

$$||f - g||_{K,\infty} \equiv \sup\{||f(z) - g(z)|| : z \in K\}$$

28.1.1 Approximation With Rational Functions

It turns out you can approximate analytic functions by rational functions, quotients of polynomials. The resulting theorem is one of the most profound theorems in complex analysis. The basic idea is simple. The Riemann sums for the Cauchy integral formula are rational functions. The idea used to implement this observation is that if you have a compact subset, K of an open set, Ω there exists a cycle composed of closed oriented curves $\left\{\gamma_j\right\}_{j=1}^{n}$ which are contained in $\Omega \setminus K$ such that for every $z \in K, \sum_{k=1}^{n} n(\gamma_k, z) = 1$. One more ingredient is needed and this is a theorem which lets you keep the approximation but move the poles.

To begin with, consider the part about the cycle of closed oriented curves. Recall Theorem 24.52 which is stated for convenience.

Theorem 28.2 *Let K be a compact subset of an open set, Ω. Then there exist continuous, closed, bounded variation oriented curves $\{\gamma_j\}_{j=1}^{m}$ for which $\gamma_j^* \cap K = \emptyset$ for each j, $\gamma_j^* \subseteq \Omega$, and for all $p \in K$,*

$$\sum_{k=1}^{m} n(p, \gamma_k) = 1.$$

and

$$\sum_{k=1}^{m} n(z, \gamma_k) = 0$$

for all $z \notin \Omega$.

This theorem implies the following.

Theorem 28.3 *Let $K \subseteq \Omega$ where K is compact and Ω is open. Then there exist oriented closed curves, γ_k such that $\gamma_k^* \cap K = \emptyset$ but $\gamma_k^* \subseteq \Omega$, such that for all $z \in K$,*

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^{p} \int_{\gamma_k} \frac{f(w)}{w - z} dw. \tag{28.1}$$

Proof: This follows from Theorem 24.52 and the Cauchy integral formula. As shown in the proof, you can assume the γ_k are linear mappings but this is not important.

Next I will show how the Cauchy integral formula leads to approximation by rational functions, quotients of polynomials.

Lemma 28.4 *Let K be a compact subset of an open set, Ω and let f be analytic on Ω. Then there exists a rational function, Q whose poles are not in K such that*

$$||Q - f||_{K,\infty} < \varepsilon.$$

Proof: By Theorem 28.3 there are oriented curves, γ_k described there such that for all $z \in K$,

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^{p} \int_{\gamma_k} \frac{f(w)}{w - z} dw. \tag{28.2}$$

Defining $g(w, z) \equiv \frac{f(w)}{w-z}$ for $(w, z) \in \cup_{k=1}^{p} \gamma_k^* \times K$, it follows since the distance between K and $\cup_k \gamma_k^*$ is positive that g is uniformly continuous and so there exists a $\delta > 0$ such that if $||\mathcal{P}|| < \delta$, then for all $z \in K$,

$$\left| f(z) - \frac{1}{2\pi i} \sum_{k=1}^{p} \sum_{j=1}^{n} \frac{f(\gamma_k(\tau_j))(\gamma_k(t_i) - \gamma_k(t_{i-1}))}{\gamma_k(\tau_j) - z} \right| < \frac{\varepsilon}{2}.$$

The complicated expression is obtained by replacing each integral in 28.2 with a Riemann sum. Simplifying the appearance of this, it follows there exists a rational function of the form

$$R(z) = \sum_{k=1}^{M} \frac{A_k}{w_k - z}$$

where the w_k are elements of components of $\mathbb{C} \setminus K$ and A_k are complex numbers or in the case where f has values in X, these would be elements of X such that

$$||R - f||_{K,\infty} < \frac{\varepsilon}{2}.$$

This proves the lemma.

28.1.2 Moving The Poles And Keeping The Approximation

Lemma 28.4 is a nice lemma but needs refining. In this lemma, the Riemann sum handed you the poles. It is much better if you can pick the poles. The following theorem from advanced calculus, called Merten's theorem, will be used

28.1.3 Merten's Theorem.

Theorem 28.5 *Suppose $\sum_{i=r}^{\infty} a_i$ and $\sum_{j=r}^{\infty} b_j$ both converge absolutely*[1]*. Then*

$$\left(\sum_{i=r}^{\infty} a_i\right)\left(\sum_{j=r}^{\infty} b_j\right) = \sum_{n=r}^{\infty} c_n$$

where

$$c_n = \sum_{k=r}^{n} a_k b_{n-k+r}.$$

Proof: Let $p_{nk} = 1$ if $r \leq k \leq n$ and $p_{nk} = 0$ if $k > n$. Then

$$c_n = \sum_{k=r}^{\infty} p_{nk} a_k b_{n-k+r}.$$

[1] Actually, it is only necessary to assume one of the series converges and the other converges absolutely. This is known as Merten's theorem and may be read in the 1974 book by Apostol listed in the bibliography.

Also,

$$\begin{aligned}\sum_{k=r}^{\infty}\sum_{n=r}^{\infty}p_{nk}\left|a_k\right|\left|b_{n-k+r}\right| &= \sum_{k=r}^{\infty}\left|a_k\right|\sum_{n=r}^{\infty}p_{nk}\left|b_{n-k+r}\right| \\ &= \sum_{k=r}^{\infty}\left|a_k\right|\sum_{n=k}^{\infty}\left|b_{n-k+r}\right| \\ &= \sum_{k=r}^{\infty}\left|a_k\right|\sum_{n=k}^{\infty}\left|b_{n-(k-r)}\right| \\ &= \sum_{k=r}^{\infty}\left|a_k\right|\sum_{m=r}^{\infty}\left|b_m\right| < \infty.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=r}^{\infty}c_n &= \sum_{n=r}^{\infty}\sum_{k=r}^{n}a_k b_{n-k+r} = \sum_{n=r}^{\infty}\sum_{k=r}^{\infty}p_{nk}a_k b_{n-k+r} \\ &= \sum_{k=r}^{\infty}a_k\sum_{n=r}^{\infty}p_{nk}b_{n-k+r} = \sum_{k=r}^{\infty}a_k\sum_{n=k}^{\infty}b_{n-k+r} \\ &= \sum_{k=r}^{\infty}a_k\sum_{m=r}^{\infty}b_m\end{aligned}$$

and this proves the theorem.

It follows that $\sum_{n=r}^{\infty}c_n$ converges absolutely. Also, you can see by induction that you can multiply any number of absolutely convergent series together and obtain a series which is absolutely convergent. Next, here are some similar results related to Merten's theorem.

Lemma 28.6 *Let $\sum_{n=0}^{\infty}a_n(z)$ and $\sum_{n=0}^{\infty}b_n(z)$ be two convergent series for $z \in K$ which satisfy the conditions of the Weierstrass M test. Thus there exist positive constants, A_n and B_n such that $|a_n(z)| \leq A_n, |b_n(z)| \leq B_n$ for all $z \in K$ and $\sum_{n=0}^{\infty}A_n < \infty, \sum_{n=0}^{\infty}B_n < \infty$. Then defining the Cauchy product,*

$$c_n(z) \equiv \sum_{k-0}^{n}a_{n-k}(z)\,b_k(z),$$

it follows $\sum_{n=0}^{\infty}c_n(z)$ also converges absolutely and uniformly on K because $c_n(z)$ satisfies the conditions of the Weierstrass M test. Therefore,

$$\sum_{n=0}^{\infty}c_n(z) = \left(\sum_{k=0}^{\infty}a_k(z)\right)\left(\sum_{n=0}^{\infty}b_n(z)\right). \tag{28.3}$$

Proof:

$$|c_n(z)| \leq \sum_{k=0}^{n}|a_{n-k}(z)|\,|b_k(z)| \leq \sum_{k=0}^{n}A_{n-k}B_k.$$

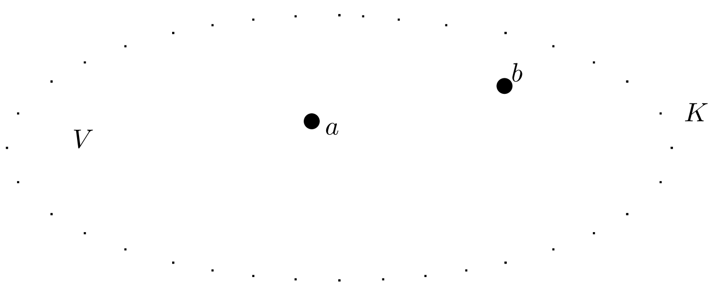
Also,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A_{n-k}B_k = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} A_{n-k}B_k = \sum_{k=0}^{\infty} B_k \sum_{n=0}^{\infty} A_n < \infty.$$

The claim of 28.3 follows from Merten's theorem. This proves the lemma.

Corollary 28.7 *Let P be a polynomial and let $\sum_{n=0}^{\infty} a_n(z)$ converge uniformly and absolutely on K such that the a_n satisfy the conditions of the Weierstrass M test. Then there exists a series for $P\left(\sum_{n=0}^{\infty} a_n(z)\right), \sum_{n=0}^{\infty} c_n(z)$, which also converges absolutely and uniformly for $z \in K$ because $c_n(z)$ also satisfies the conditions of the Weierstrass M test.*

The following picture is descriptive of the following lemma. This lemma says that if you have a rational function with one pole off a compact set, then you can approximate on the compact set with another rational function which has a different pole.



Lemma 28.8 *Let R be a rational function which has a pole only at $a \in V$, a component of $\mathbb{C} \setminus K$ where K is a compact set. Suppose $b \in V$. Then for $\varepsilon > 0$ given, there exists a rational function, Q, having a pole only at b such that*

$$||R - Q||_{K,\infty} < \varepsilon. \tag{28.4}$$

If it happens that V is unbounded, then there exists a polynomial, P such that

$$||R - P||_{K,\infty} < \varepsilon. \tag{28.5}$$

Proof: Say that $b \in V$ satisfies $\mathcal{P}$ if for all $\varepsilon > 0$ there exists a rational function, Q_b, having a pole only at b such that

$$||R - Q_b||_{K,\infty} < \varepsilon$$

Now define a set,

$$S \equiv \{b \in V : b \text{ satisfies } \mathcal{P}\}.$$

Observe that $S \neq \emptyset$ because $a \in S$.

I claim S is open. Suppose $b_1 \in S$. Then there exists a $\delta > 0$ such that

$$\left|\frac{b_1 - b}{z - b}\right| < \frac{1}{2} \tag{28.6}$$

for all $z \in K$ whenever $b \in B(b_1, \delta)$. In fact, it suffices to take $|b - b_1| < \operatorname{dist}(b_1, K)/4$ because then

$$\begin{aligned}\left|\frac{b_1 - b}{z - b}\right| &< \left|\frac{\operatorname{dist}(b_1, K)/4}{z - b}\right| \leq \frac{\operatorname{dist}(b_1, K)/4}{|z - b_1| - |b_1 - b|} \\ &\leq \frac{\operatorname{dist}(b_1, K)/4}{\operatorname{dist}(b_1, K) - \operatorname{dist}(b_1, K)/4} \leq \frac{1}{3} < \frac{1}{2}.\end{aligned}$$

Since b_1 satisfies $\mathcal{P}$, there exists a rational function Q_{b_1} with the desired properties. It is shown next that you can approximate Q_{b_1} with Q_b thus yielding an approximation to R by the use of the triangle inequality,

$$||R - Q_{b_1}||_{K,\infty} + ||Q_{b_1} - Q_b||_{K,\infty} \geq ||R - Q_b||_{K,\infty}.$$

Since Q_{b_1} has poles only at b_1, it follows it is a sum of functions of the form $\frac{\alpha_n}{(z-b_1)^n}$. Therefore, it suffices to consider the terms of Q_{b_1} or that Q_{b_1} is of the special form

$$Q_{b_1}(z) = \frac{1}{(z - b_1)^n}.$$

However,

$$\frac{1}{(z - b_1)^n} = \frac{1}{(z - b)^n \left(1 - \frac{b_1 - b}{z - b}\right)^n}$$

Now from the choice of b_1, the series

$$\sum_{k=0}^{\infty} \left(\frac{b_1 - b}{z - b}\right)^k = \frac{1}{\left(1 - \frac{b_1 - b}{z - b}\right)}$$

converges absolutely independent of the choice of $z \in K$ because

$$\left|\left(\frac{b_1 - b}{z - b}\right)^k\right| < \frac{1}{2^k}.$$

By Corollary 28.7 the same is true of the series for $\frac{1}{\left(1 - \frac{b_1 - b}{z - b}\right)^n}$. Thus a suitable partial sum can be made uniformly on K as close as desired to $\frac{1}{(z-b_1)^n}$. This shows that b satisfies $\mathcal{P}$ whenever b is close enough to b_1 verifying that S is open.

Next it is shown S is closed in V. Let $b_n \in S$ and suppose $b_n \to b \in V$. Then since $b_n \in S$, there exists a rational function, Q_{b_n} such that

$$||Q_{b_n} - R||_{K,\infty} < \frac{\varepsilon}{2}.$$

Then for all n large enough,

$$\frac{1}{2}\operatorname{dist}(b,K)\geq|b_n-b|$$

and so for all n large enough,

$$\left|\frac{b-b_n}{z-b_n}\right|<\frac{1}{2},$$

for all $z\in K$. Pick such a b_n. As before, it suffices to assume Q_{b_n}, is of the form $\frac{1}{(z-b_n)^n}$. Then

$$Q_{b_n}(z)=\frac{1}{(z-b_n)^n}=\frac{1}{(z-b)^n\left(1-\frac{b_n-b}{z-b}\right)^n}$$

and because of the estimate, there exists M such that for all $z\in K$

$$\left|\frac{1}{\left(1-\frac{b_n-b}{z-b}\right)^n}-\sum_{k=0}^{M}a_k\left(\frac{b_n-b}{z-b}\right)^k\right|<\frac{\varepsilon(\operatorname{dist}(b,K))^n}{2}. \tag{28.7}$$

Therefore, for all $z\in K$

$$\begin{aligned}\left|Q_{b_n}(z)-\frac{1}{(z-b)^n}\sum_{k=0}^{M}a_k\left(\frac{b_n-b}{z-b}\right)^k\right| &= \\ \left|\frac{1}{(z-b)^n\left(1-\frac{b_n-b}{z-b}\right)^n}-\frac{1}{(z-b)^n}\sum_{k=0}^{M}a_k\left(\frac{b_n-b}{z-b}\right)^k\right| &\leq \\ \frac{\varepsilon(\operatorname{dist}(b,K))^n}{2}\frac{1}{\operatorname{dist}(b,K)^n} &= \frac{\varepsilon}{2}\end{aligned}$$

and so, letting $Q_b(z)=\frac{1}{(z-b)^n}\sum_{k=0}^{M}a_k\left(\frac{b_n-b}{z-b}\right)^k$,

$$\begin{aligned}||R-Q_b||_{K,\infty} &\leq ||R-Q_{b_n}||_{K,\infty}+||Q_{b_n}-Q_b||_{K,\infty} \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

showing that $b\in S$. Since S is both open and closed in V it follows that, since $S\neq\emptyset$, $S=V$. Otherwise V would fail to be connected.

It remains to consider the case where V is unbounded. Pick $b\in V$ large enough that

$$\left|\frac{z}{b}\right|<\frac{1}{2} \tag{28.8}$$

for all $z\in K$. From what was just shown, there exists a rational function, Q_b having a pole only at b such that $||Q_b-R||_{K,\infty}<\frac{\varepsilon}{2}$. It suffices to assume that Q_b is of the form

$$\begin{aligned} Q_b(z) &= \frac{p(z)}{(z-b)^n} = p(z)(-1)^n \frac{1}{b^n} \frac{1}{\left(1-\frac{z}{b}\right)^n} \\ &= p(z)(-1)^n \frac{1}{b^n} \left(\sum_{k=0}^{\infty} \left(\frac{z}{b}\right)^k \right)^n \end{aligned}$$

Then by an application of Corollary 28.7 there exists a partial sum of the power series for Q_b which is uniformly close to Q_b on K. Therefore, you can approximate Q_b and therefore also R uniformly on K by a polynomial consisting of a partial sum of the above infinite sum. This proves the theorem.

If f is a polynomial, then f has a pole at ∞. This will be discussed more later.

28.1.4 Runge's Theorem

Now what follows is the first form of Runge's theorem.

Theorem 28.9 *Let K be a compact subset of an open set, Ω and let $\{b_j\}$ be a set which consists of one point from each component of $\widehat{\mathbb{C}} \setminus K$. Let f be analytic on Ω. Then for each $\varepsilon > 0$, there exists a rational function, Q whose poles are all contained in the set, $\{b_j\}$ such that*

$$||Q - f||_{K,\infty} < \varepsilon. \tag{28.9}$$

If $\widehat{\mathbb{C}} \setminus K$ has only one component, then Q may be taken to be a polynomial.

Proof: By Lemma 28.4 there exists a rational function of the form

$$R(z) = \sum_{k=1}^{M} \frac{A_k}{w_k - z}$$

where the w_k are elements of components of $\mathbb{C} \setminus K$ and A_k are complex numbers such that

$$||R - f||_{K,\infty} < \frac{\varepsilon}{2}.$$

Consider the rational function, $R_k(z) \equiv \frac{A_k}{w_k - z}$ where $w_k \in V_j$, one of the components of $\mathbb{C} \setminus K$, the given point of V_j being b_j. By Lemma 28.8, there exists a function, Q_k which is either a rational function having its only pole at b_j or a polynomial, depending on whether V_j is bounded such that

$$||R_k - Q_k||_{K,\infty} < \frac{\varepsilon}{2M}.$$

Letting $Q(z) \equiv \sum_{k=1}^{M} Q_k(z)$,

$$||R - Q||_{K,\infty} < \frac{\varepsilon}{2}.$$

It follows
$$||f-Q||_{K,\infty} \leq ||f-R||_{K,\infty} + ||R-Q||_{K,\infty} < \varepsilon.$$
In the case of only one component of $\mathbb{C}\setminus K$, this component is the unbounded component and so you can take Q to be a polynomial. This proves the theorem.

The next version of Runge's theorem concerns the case where the given points are contained in $\widehat{\mathbb{C}}\setminus\Omega$ for Ω an open set rather than a compact set. Note that here there could be uncountably many components of $\widehat{\mathbb{C}}\setminus\Omega$ because the components are no longer open sets. An easy example of this phenomenon in one dimension is where $\Omega = [0,1]\setminus P$ for P the Cantor set. Then you can show that $\mathbb{R}\setminus\Omega$ has uncountably many components. Nevertheless, Runge's theorem will follow from Theorem 28.9 with the aid of the following interesting lemma.

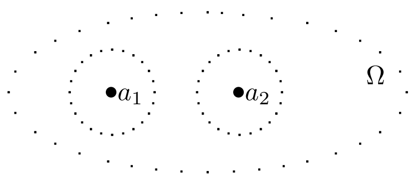
Lemma 28.10 *Let Ω be an open set in $\mathbb{C}$. Then there exists a sequence of compact sets, $\{K_n\}$ such that*
$$\Omega = \cup_{k=1}^{\infty} K_n,\ \cdots, K_n \subseteq \operatorname{int} K_{n+1} \cdots, \tag{28.10}$$
and for any $K \subseteq \Omega$,
$$K \subseteq K_n, \tag{28.11}$$
for all n sufficiently large, and every component of $\widehat{\mathbb{C}}\setminus K_n$ contains a component of $\widehat{\mathbb{C}}\setminus\Omega$.

Proof: Let
$$V_n \equiv \{z : |z| > n\} \cup \bigcup_{z\notin\Omega} B\left(z, \frac{1}{n}\right).$$
Thus $\{z : |z| > n\}$ contains the point, ∞. Now let
$$K_n \equiv \widehat{\mathbb{C}}\setminus V_n = \mathbb{C}\setminus V_n \subseteq \Omega.$$
You should verify that 28.10 and 28.11 hold. It remains to show that every component of $\widehat{\mathbb{C}}\setminus K_n$ contains a component of $\widehat{\mathbb{C}}\setminus\Omega$. Let D be a component of $\widehat{\mathbb{C}}\setminus K_n \equiv V_n$.

If $\infty \notin D$, then D contains no point of $\{z : |z| > n\}$ because this set is connected and D is a component. (If it did contain a point of this set, it would have to contain the whole set.) Therefore, $D \subseteq \bigcup_{z\notin\Omega} B\left(z, \frac{1}{n}\right)$ and so D contains some point of $B\left(z, \frac{1}{n}\right)$ for some $z \notin \Omega$. Therefore, since this ball is connected, it follows D must contain the whole ball and consequently D contains some point of Ω^C. (The point z at the center of the ball will do.) Since D contains $z \notin \Omega$, it must contain the component, H_z, determined by this point. The reason for this is that
$$H_z \subseteq \widehat{\mathbb{C}}\setminus\Omega \subseteq \widehat{\mathbb{C}}\setminus K_n$$
and H_z is connected. Therefore, H_z can only have points in one component of $\widehat{\mathbb{C}}\setminus K_n$. Since it has a point in D, it must therefore, be totally contained in D. This verifies the desired condition in the case where $\infty \notin D$.

Now suppose that $\infty \in D$. $\infty \notin \Omega$ because Ω is given to be a set in $\mathbb{C}$. Letting H_∞ denote the component of $\widehat{\mathbb{C}} \setminus \Omega$ determined by ∞, it follows both D and H_∞ contain ∞. Therefore, the connected set, H_∞ cannot have any points in another component of $\widehat{\mathbb{C}} \setminus K_n$ and it is a set which is contained in $\widehat{\mathbb{C}} \setminus K_n$ so it must be contained in D. This proves the lemma.

The following picture is a very simple example of the sort of thing considered by Runge's theorem. The picture is of a region which has a couple of holes.



However, there could be many more holes than two. In fact, there could be infinitely many. Nor does it follow that the components of the complement of Ω need to have any interior points. Therefore, the picture is certainly not representative.

Theorem 28.11 *(Runge) Let Ω be an open set, and let A be a set which has one point in each component of $\widehat{\mathbb{C}} \setminus \Omega$ and let f be analytic on Ω. Then there exists a sequence of rational functions, $\{R_n\}$ having poles only in A such that R_n converges uniformly to f on compact subsets of Ω.*

Proof: Let K_n be the compact sets of Lemma 28.10 where each component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus \Omega$. It follows each component of $\widehat{\mathbb{C}} \setminus K_n$ contains a point of A. Therefore, by Theorem 28.9 there exists R_n a rational function with poles only in A such that

$$||R_n - f||_{K_n,\infty} < \frac{1}{n}.$$

It follows, since a given compact set, K is a subset of K_n for all n large enough, that $R_n \to f$ uniformly on K. This proves the theorem.

Corollary 28.12 *Let Ω be simply connected and f analytic on Ω. Then there exists a sequence of polynomials, $\{p_n\}$ such that $p_n \to f$ uniformly on compact sets of Ω.*

Proof: By definition of what is meant by simply connected, $\widehat{\mathbb{C}} \setminus \Omega$ is connected and so there are no bounded components of $\widehat{\mathbb{C}} \setminus \Omega$. Therefore, in the proof of Theorem 28.11 when you use Theorem 28.9, you can always have R_n be a polynomial by Lemma 28.8.

28.2 The Mittag-Leffler Theorem

28.2.1 A Proof From Runge's Theorem

This theorem is fairly easy to prove once you have Theorem 28.9. Given a set of complex numbers, does there exist a meromorphic function having its poles equal to this set of numbers? The Mittag-Leffler theorem provides a very satisfactory answer to this question. Actually, it says somewhat more. You can specify, not just the location of the pole but also the kind of singularity the meromorphic function is to have at that pole.

Theorem 28.13 *Let $P \equiv \{z_k\}_{k=1}^{\infty}$ be a set of points in an open subset of $\mathbb{C}$, Ω. Suppose also that $P \subseteq \Omega \subseteq \mathbb{C}$. For each z_k, denote by $S_k(z)$ a function of the form*

$$S_k(z) = \sum_{j=1}^{m_k} \frac{a_j^k}{(z - z_k)^j}.$$

Then there exists a meromorphic function, Q defined on Ω such that the poles of Q are the points, $\{z_k\}_{k=1}^{\infty}$ and the singular part of the Laurent expansion of Q at z_k equals $S_k(z)$. In other words, for z near $z_k, Q(z) = g_k(z) + S_k(z)$ for some function, g_k analytic near z_k.

Proof: Let $\{K_n\}$ denote the sequence of compact sets described in Lemma 28.10. Thus $\cup_{n=1}^{\infty} K_n = \Omega$, $K_n \subseteq \text{int}(K_{n+1}) \subseteq K_{n+1} \cdots$, and the components of $\widehat{\mathbb{C}} \setminus K_n$ contain the components of $\widehat{\mathbb{C}} \setminus \Omega$. Renumbering if necessary, you can assume each $K_n \neq \emptyset$. Also let $K_0 = \emptyset$. Let $P_m \equiv P \cap (K_m \setminus K_{m-1})$ and consider the rational function, R_m defined by

$$R_m(z) \equiv \sum_{z_k \in K_m \setminus K_{m-1}} S_k(z).$$

Since each K_m is compact, it follows P_m is finite and so the above really is a rational function. Now for $m > 1$,this rational function is analytic on some open set containing K_{m-1}. There exists a set of points, A one point in each component of $\widehat{\mathbb{C}} \setminus \Omega$. Consider $\widehat{\mathbb{C}} \setminus K_{m-1}$. Each of its components contains a component of $\widehat{\mathbb{C}} \setminus \Omega$ and so for each of these components of $\widehat{\mathbb{C}} \setminus K_{m-1}$, there exists a point of A which is contained in it. Denote the resulting set of points by A'. By Theorem 28.9 there exists a rational function, Q_m whose poles are all contained in the set, $A' \subseteq \Omega^C$ such that

$$||R_m - Q_m||_{K_{m-1},\infty} < \frac{1}{2^m}.$$

The meromorphic function is

$$Q(z) \equiv R_1(z) + \sum_{k=2}^{\infty} (R_k(z) - Q_k(z)).$$

It remains to verify this function works. First consider K_1. Then on K_1, the above sum converges uniformly. Furthermore, the terms of the sum are analytic in some open set containing K_1. Therefore, the infinite sum is analytic on this open set and so for $z \in K_1$ The function, f is the sum of a rational function, R_1, having poles at

P_1 with the specified singular terms and an analytic function. Therefore, Q works on K_1. Now consider K_m for $m > 1$. Then

$$Q(z) = R_1(z) + \sum_{k=2}^{m+1} (R_k(z) - Q_k(z)) + \sum_{k=m+2}^{\infty} (R_k(z) - Q_k(z)).$$

As before, the infinite sum converges uniformly on K_{m+1} and hence on some open set, O containing K_m. Therefore, this infinite sum equals a function which is analytic on O. Also,

$$R_1(z) + \sum_{k=2}^{m+1} (R_k(z) - Q_k(z))$$

is a rational function having poles at $\cup_{k=1}^{m} P_k$ with the specified singularities because the poles of each Q_k are not in Ω. It follows this function is meromorphic because it is analytic except for the points in P. It also has the property of retaining the specified singular behavior.

28.2.2 A Direct Proof Without Runge's Theorem

There is a direct proof of this important theorem which is not dependent on Runge's theorem in the case where $\Omega = \mathbb{C}$. I think it is arguably easier to understand and the Mittag-Leffler theorem is very important so I will give this proof here.

Theorem 28.14 *Let $P \equiv \{z_k\}_{k=1}^{\infty}$ be a set of points in $\mathbb{C}$ which satisfies $\lim_{n\to\infty} |z_n| = \infty$. For each z_k, denote by $S_k(z)$ a polynomial in $\frac{1}{z-z_k}$ which is of the form*

$$S_k(z) = \sum_{j=1}^{m_k} \frac{a_j^k}{(z-z_k)^j}.$$

Then there exists a meromorphic function, Q defined on $\mathbb{C}$ such that the poles of Q are the points, $\{z_k\}_{k=1}^{\infty}$ and the singular part of the Laurent expansion of Q at z_k equals $S_k(z)$. In other words, for z near z_k,

$$Q(z) = g_k(z) + S_k(z)$$

for some function, g_k analytic in some open set containing z_k.

Proof: First consider the case where none of the $z_k = 0$. Letting

$$K_k \equiv \{z : |z| \leq |z_k|/2\},$$

there exists a power series for $\frac{1}{z-z_k}$ which converges uniformly and absolutely on this set. Here is why:

$$\frac{1}{z-z_k} = \left(\frac{-1}{1-\frac{z}{z_k}}\right)\frac{1}{z_k} = \frac{-1}{z_k}\sum_{l=0}^{\infty}\left(\frac{z}{z_k}\right)^l$$

and the Weierstrass M test can be applied because

$$\left|\frac{z}{z_k}\right| < \frac{1}{2}$$

on this set. Therefore, by Corollary 28.7, $S_k(z)$, being a polynomial in $\frac{1}{z-z_k}$, has a power series which converges uniformly to $S_k(z)$ on K_k. Therefore, there exists a polynomial, $P_k(z)$ such that

$$||P_k - S_k||_{\overline{B(0,|z_k|/2)},\infty} < \frac{1}{2^k}.$$

Let

$$Q(z) \equiv \sum_{k=1}^{\infty} (S_k(z) - P_k(z)). \tag{28.12}$$

Consider $z \in K_m$ and let N be large enough that if $k > N$, then $|z_k| > 2|z|$

$$Q(z) = \sum_{k=1}^{N} (S_k(z) - P_k(z)) + \sum_{k=N+1}^{\infty} (S_k(z) - P_k(z)).$$

On K_m, the second sum converges uniformly to a function analytic on $\text{int}(K_m)$ (interior of K_m) while the first is a rational function having poles at $z_1, \cdots, z_N$. Since any compact set is contained in K_m for large enough m, this shows $Q(z)$ is meromorphic as claimed and has poles with the given singularities.

Now consider the case where the poles are at $\{z_k\}_{k=0}^{\infty}$ with $z_0 = 0$. Everything is similar in this case. Let

$$Q(z) \equiv S_0(z) + \sum_{k=1}^{\infty} (S_k(z) - P_k(z)).$$

The series converges uniformly on every compact set because of the assumption that $\lim_{n\to\infty} |z_n| = \infty$ which implies that any compact set is contained in K_k for k large enough. Choose N such that $z \in \text{int}(K_N)$ and $z_n \notin K_N$ for all $n \geq N+1$. Then

$$Q(z) = S_0(z) + \sum_{k=1}^{N} (S_k(z) - P_k(z)) + \sum_{k=N+1}^{\infty} (S_k(z) - P_k(z)).$$

The last sum is analytic on $\text{int}(K_N)$ because each function in the sum is analytic due to the fact that none of its poles are in K_N. Also, $S_0(z) + \sum_{k=1}^{N} (S_k(z) - P_k(z))$ is a finite sum of rational functions so it is a rational function and P_k is a polynomial so z_m is a pole of this function with the correct singularity whenever $z_m \in \text{int}(K_N)$.

28.2.3 Functions Meromorphic On $\widehat{\mathbb{C}}$

Sometimes it is useful to think of isolated singular points at ∞.

Definition 28.15 *Suppose f is analytic on $\{z \in \mathbb{C} : |z| > r\}$. Then f is said to have a removable singularity at ∞ if the function, $g(z) \equiv f\left(\frac{1}{z}\right)$ has a removable singularity at 0. f is said to have a pole at ∞ if the function, $g(z) = f\left(\frac{1}{z}\right)$ has a pole at 0. Then f is said to be meromorphic on $\widehat{\mathbb{C}}$ if all its singularities are isolated and either poles or removable.*

So what is f like for these cases? First suppose f has a removable singularity at ∞. Then $zg(z)$ converges to 0 as $z \to 0$. It follows $g(z)$ must be analytic near 0 and so can be given as a power series. Thus $f(z)$ is of the form $f(z) = g\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n$. Next suppose f has a pole at ∞. This means $g(z)$ has a pole at 0 so $g(z)$ is of the form $g(z) = \sum_{k=1}^{m} \frac{b_k}{z^k} + h(z)$ where $h(z)$ is analytic near 0. Thus in the case of a pole at ∞, $f(z)$ is of the form $f(z) = g\left(\frac{1}{z}\right) = \sum_{k=1}^{m} b_k z^k + \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n$.

It turns out that the functions which are meromorphic on $\widehat{\mathbb{C}}$ are all rational functions. To see this suppose f is meromorphic on $\widehat{\mathbb{C}}$ and note that there exists $r > 0$ such that $f(z)$ is analytic for $|z| > r$. This is required if ∞ is to be isolated. Therefore, there are only finitely many poles of f for $|z| \leq r, \{a_1, \cdots, a_m\}$, because by assumption, these poles are isolated and this is a compact set. Let the singular part of f at a_k be denoted by $S_k(z)$. Then $f(z) - \sum_{k=1}^{m} S_k(z)$ is analytic on all of $\mathbb{C}$. Therefore, it is bounded on $|z| \leq r$. In one case, f has a removable singularity at ∞. In this case, f is bounded as $z \to \infty$ and $\sum_k S_k$ also converges to 0 as $z \to \infty$. Therefore, by Liouville's theorem, $f(z) - \sum_{k=1}^{m} S_k(z)$ equals a constant and so $f - \sum_k S_k$ is a constant. Thus f is a rational function. In the other case that f has a pole at ∞, $f(z) - \sum_{k=1}^{m} S_k(z) - \sum_{k=1}^{m} b_k z^k = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n - \sum_{k=1}^{m} S_k(z)$. Now $f(z) - \sum_{k=1}^{m} S_k(z) - \sum_{k=1}^{m} b_k z^k$ is analytic on $\mathbb{C}$ and so is bounded on $|z| \leq r$. But now $\sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n - \sum_{k=1}^{m} S_k(z)$ converges to 0 as $z \to \infty$ and so by Liouville's theorem, $f(z) - \sum_{k=1}^{m} S_k(z) - \sum_{k=1}^{m} b_k z^k$ must equal a constant and again, $f(z)$ equals a rational function.

28.2.4 A Great And Glorious Theorem About Simply Connected Regions

Here is given a laundry list of properties which are equivalent to an open set being simply connected. Recall Definition 24.48 on Page 673 which said that an open set, Ω is simply connected means $\widehat{\mathbb{C}} \setminus \Omega$ is connected. Recall also that this is not the same thing at all as saying $\mathbb{C} \setminus \Omega$ is connected. Consider the outside of a disk for example. I will continue to use this definition for simply connected because it is the most convenient one for complex analysis. However, there are many other equivalent conditions. First here is an interesting lemma which is interesting for its own sake. Recall $n(p, \gamma)$ means the winding number of γ about p. Now recall Theorem 24.52 implies the following lemma in which B^C is playing the role of Ω in Theorem 24.52.

Lemma 28.16 *Let K be a compact subset of B^C, the complement of a closed set. Then there exist continuous, closed, bounded variation oriented curves $\{\Gamma_j\}_{j=1}^m$ for which $\Gamma_j^* \cap K = \emptyset$ for each j, $\Gamma_j^* \subseteq \Omega$, and for all $p \in K$,*

$$\sum_{k=1}^m n(\Gamma_k, p) = 1.$$

while for all $z \in B$

$$\sum_{k=1}^m n(\Gamma_k, z) = 0.$$

Definition 28.17 *Let γ be a closed curve in an open set, $\Omega, \gamma : [a,b] \to \Omega$. Then γ is said to be homotopic to a point, p in Ω if there exists a continuous function, $H : [0,1] \times [a,b] \to \Omega$ such that $H(0,t) = p, H(\alpha, a) = H(\alpha, b)$, and $H(1,t) = \gamma(t)$. This function, H is called a homotopy.*

Lemma 28.18 *Suppose γ is a closed continuous bounded variation curve in an open set, Ω which is homotopic to a point. Then if $a \notin \Omega$, it follows $n(a, \gamma) = 0$.*

Proof: Let H be the homotopy described above. The problem with this is that it is not known that $H(\alpha, \cdot)$ is of bounded variation. There is no reason it should be. Therefore, it might not make sense to take the integral which defines the winding number. There are various ways around this. Extend H as follows. $H(\alpha, t) = H(\alpha, a)$ for $t < a, H(\alpha, t) = H(\alpha, b)$ for $t > b$. Let $\varepsilon > 0$.

$$H_\varepsilon(\alpha, t) \equiv \frac{1}{2\varepsilon} \int_{-2\varepsilon + t + \frac{2\varepsilon}{(b-a)}(t-a)}^{t + \frac{2\varepsilon}{(b-a)}(t-a)} H(\alpha, s)\, ds,\ H_\varepsilon(0,t) = p.$$

Thus $H_\varepsilon(\alpha, \cdot)$ is a closed curve which has bounded variation and when $\alpha = 1$, this converges to γ uniformly on $[a,b]$. Therefore, for ε small enough, $n(a, H_\varepsilon(1, \cdot)) = n(a, \gamma)$ because they are both integers and as $\varepsilon \to 0, n(a, H_\varepsilon(1, \cdot)) \to n(a, \gamma)$. Also, $H_\varepsilon(\alpha, t) \to H(\alpha, t)$ uniformly on $[0,1] \times [a,b]$ because of uniform continuity of H. Therefore, for small enough ε, you can also assume $H_\varepsilon(\alpha, t) \in \Omega$ for all α, t. Now $\alpha \to n(a, H_\varepsilon(\alpha, \cdot))$ is continuous. Hence it must be constant because the winding number is integer valued. But

$$\lim_{\alpha \to 0} \frac{1}{2\pi i} \int_{H_\varepsilon(\alpha, \cdot)} \frac{1}{z-a} dz = 0$$

because the length of $H_\varepsilon(\alpha, \cdot)$ converges to 0 and the integrand is bounded because $a \notin \Omega$. Therefore, the constant can only equal 0. This proves the lemma.

Now it is time for the great and glorious theorem on simply connected regions. The following equivalence of properties is taken from Rudin [45]. There is a slightly different list in Conway [13] and a shorter list in Ash [6].

Theorem 28.19 *The following are equivalent for an open set, Ω.*

1. *Ω is homeomorphic to the unit disk,* $B(0,1)$.

2. *Every closed curve contained in Ω is homotopic to a point in Ω.*

3. *If $z \notin \Omega$, and if γ is a closed bounded variation continuous curve in Ω, then* $n(\gamma, z) = 0$.

4. *Ω is simply connected, ($\widehat{\mathbb{C}} \setminus \Omega$ is connected and Ω is connected.)*

5. *Every function analytic on Ω can be uniformly approximated by polynomials on compact subsets.*

6. *For every f analytic on Ω and every closed continuous bounded variation curve, γ,*
$$\int_\gamma f(z)\, dz = 0.$$

7. *Every function analytic on Ω has a primitive on Ω.*

8. *If $f, 1/f$ are both analytic on Ω, then there exists an analytic, g on Ω such that* $f = \exp(g)$.

9. *If $f, 1/f$ are both analytic on Ω, then there exists ϕ analytic on Ω such that* $f = \phi^2$.

Proof: 1⇒2**.** Assume 1 and let γ be a closed curve in Ω. Let h be the homeomorphism, $h : B(0,1) \to \Omega$. Let $H(\alpha, t) = h\left(\alpha\left(h^{-1}\gamma(t)\right)\right)$. This works.

2⇒3 This is Lemma 28.18.

3⇒4. Suppose 3 but 4 fails to hold. Then if $\widehat{\mathbb{C}} \setminus \Omega$ is not connected, there exist disjoint nonempty sets, A and B such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. It follows each of these sets must be closed because neither can have a limit point in Ω nor in the other. Also, one and only one of them contains ∞. Let this set be B. Thus A is a closed set which must also be bounded. Otherwise, there would exist a sequence of points in A, $\{a_n\}$ such that $\lim_{n\to\infty} a_n = \infty$ which would contradict the requirement that no limit points of A can be in B. Therefore, A is a compact set contained in the open set, $B^C \equiv \{z \in \mathbb{C} : z \notin B\}$. Pick $p \in A$. By Lemma 28.16 there exist continuous bounded variation closed curves $\{\Gamma_k\}_{k=1}^m$ which are contained in B^C, do not intersect A and such that
$$1 = \sum_{k=1}^m n(p, \Gamma_k)$$
However, if these curves do not intersect A and they also do not intersect B then they must be all contained in Ω. Since $p \notin \Omega$, it follows by 3 that for each k, $n(p, \Gamma_k) = 0$, a contradiction.

4⇒5 This is Corollary 28.12 on Page 776.

5⇒6 Every polynomial has a primitive and so the integral over any closed bounded variation curve of a polynomial equals 0. Let f be analytic on Ω. Then let $\{f_n\}$ be a sequence of polynomials converging uniformly to f on γ^*. Then

$$0=\lim_{n\to\infty}\int_\gamma f_n(z)\,dz=\int_\gamma f(z)\,dz.$$

6⇒7 Pick $z_0\in\Omega$. Letting $\gamma(z_0,z)$ be a bounded variation continuous curve joining z_0 to z in Ω, you define a primitive for f as follows.

$$F(z)=\int_{\gamma(z_0,z)} f(w)\,dw.$$

This is well defined by 6 and is easily seen to be a primitive. You just write the difference quotient and take a limit using 6.

$$\begin{aligned}\lim_{w\to0}\frac{F(z+w)-F(z)}{w} &= \lim_{w\to0}\frac{1}{w}\left(\int_{\gamma(z_0,z+w)} f(u)\,du-\int_{\gamma(z_0,z)} f(u)\,du\right)\\ &= \lim_{w\to0}\frac{1}{w}\int_{\gamma(z,z+w)} f(u)\,du\\ &= \lim_{w\to0}\frac{1}{w}\int_0^1 f(z+tw)\,wdt=f(z).\end{aligned}$$

7⇒8 Suppose then that $f,1/f$ are both analytic. Then f'/f is analytic and so it has a primitive by 7. Let this primitive be g_1. Then

$$\begin{aligned}\left(e^{-g_1}f\right)' &= e^{-g_1}(-g_1')f+e^{-g_1}f'\\ &= -e^{-g_1}\left(\frac{f'}{f}\right)f+e^{-g_1}f'=0.\end{aligned}$$

Therefore, since Ω is connected, it follows $e^{-g_1}f$ must equal a constant. (Why?) Let the constant be e^{a+ibi}. Then $f(z)=e^{g_1(z)}e^{a+ib}$. Therefore, you let $g(z)=g_1(z)+a+ib$.

8⇒9 Suppose then that $f,1/f$ are both analytic on Ω. Then by 8 $f(z)=e^{g(z)}$. Let $\phi(z)\equiv e^{g(z)/2}$.

9⇒1 There are two cases. First suppose $\Omega=\mathbb{C}$. This satisfies condition 9 because if $f,1/f$ are both analytic, then the same argument involved in 8⇒9 gives the existence of a square root. A homeomorphism is $h(z)\equiv\frac{z}{\sqrt{1+|z|^2}}$. It obviously maps onto $B(0,1)$ and is continuous. To see it is 1 - 1 consider the case of z_1 and z_2 having different arguments. Then $h(z_1)\neq h(z_2)$. If $z_2=tz_1$ for a positive $t\neq1$, then it is also clear $h(z_1)\neq h(z_2)$. To show h^{-1} is continuous, note that if you have an open set in $\mathbb{C}$ and a point in this open set, you can get a small open set containing this point by allowing the modulus and the argument to lie in some open interval. Reasoning this way, you can verify h maps open sets to open sets. In the case where $\Omega\neq\mathbb{C}$, there exists a one to one analytic map which maps Ω onto $B(0,1)$ by the Riemann mapping theorem. This proves the theorem.

28.3 Exercises

1. Let $a \in \mathbb{C}$. Show there exists a sequence of polynomials, $\{p_n\}$ such that $p_n(a) = 1$ but $p_n(z) \to 0$ for all $z \neq a$.

2. Let l be a line in $\mathbb{C}$. Show there exists a sequence of polynomials $\{p_n\}$ such that $p_n(z) \to 1$ on one side of this line and $p_n(z) \to -1$ on the other side of the line. **Hint:** The complement of this line is simply connected.

3. Suppose Ω is a simply connected region, f is analytic on Ω, $f \neq 0$ on Ω, and $n \in \mathbb{N}$. Show that there exists an analytic function, g such that $g(z)^n = f(z)$ for all $z \in \Omega$. That is, you can take the n^{th} root of $f(z)$. If Ω is a region which contains 0, is it possible to find $g(z)$ such that g is analytic on Ω and $g(z)^2 = z$?

4. Suppose Ω is a region (connected open set) and f is an analytic function defined on Ω such that $f(z) \neq 0$ for any $z \in \Omega$. Suppose also that for every positive integer, n there exists an analytic function, g_n defined on Ω such that $g_n^n(z) = f(z)$. Show that then it is possible to define an analytic function, L on $f(\Omega)$ such that $e^{L(f(z))} = f(z)$ for all $z \in \Omega$.

5. You know that $\phi(z) \equiv \frac{z-i}{z+i}$ maps the upper half plane onto the unit ball. Its inverse, $\psi(z) = i\frac{1+z}{1-z}$ maps the unit ball onto the upper half plane. Also for z in the upper half plane, you can define a square root as follows. If $z = |z| e^{i\theta}$ where $\theta \in (0, \pi)$, let $z^{1/2} \equiv |z|^{1/2} e^{i\theta/2}$ so the square root maps the upper half plane to the first quadrant. Now consider
$$z \to \exp\left(-i \log\left[i\left(\frac{1+z}{1-z}\right)\right]^{1/2}\right). \tag{28.13}$$
Show this is an analytic function which maps the unit ball onto an annulus. Is it possible to find a one to one analytic map which does this?

Infinite Products

The Mittag-Leffler theorem gives existence of a meromorphic function which has specified singular part at various poles. It would be interesting to do something similar to zeros of an analytic function. That is, given the order of the zero at various points, does there exist an analytic function which has these points as zeros with the specified orders? You know that if you have the zeros of the polynomial, you can factor it. Can you do something similar with analytic functions which are just limits of polynomials? These questions involve the concept of an infinite product.

Definition 29.1 $\prod_{n=1}^{\infty}(1+u_n) \equiv \lim_{n\to\infty}\prod_{k=1}^{n}(1+u_k)$ *whenever this limit exists. If $u_n = u_n(z)$ for $z \in H$, we say the infinite product converges uniformly on H if the partial products, $\prod_{k=1}^{n}(1+u_k(z))$ converge uniformly on H.*

The main theorem is the following.

Theorem 29.2 *Let $H \subseteq \mathbb{C}$ and suppose that $\sum_{n=1}^{\infty}|u_n(z)|$ converges uniformly on H where $u_n(z)$ bounded on H. Then*

$$P(z) \equiv \prod_{n=1}^{\infty}(1+u_n(z))$$

converges uniformly on H. If $(n_1, n_2, \cdots)$ is any permutation of $(1, 2, \cdots)$, then for all $z \in H$,

$$P(z) = \prod_{k=1}^{\infty}(1+u_{n_k}(z))$$

and P has a zero at z_0 if and only if $u_n(z_0) = -1$ for some n.

Proof: First a simple estimate:

$$\begin{aligned}
& \prod_{k=m}^{n} \left(1+\left|u_k\left(z\right)\right|\right) \\
= & \exp\left(\ln\left(\prod_{k=m}^{n} \left(1+\left|u_k\left(z\right)\right|\right)\right)\right) = \exp\left(\sum_{k=m}^{n} \ln\left(1+\left|u_k\left(z\right)\right|\right)\right) \\
\leq & \exp\left(\sum_{k=m}^{\infty} \left|u_k\left(z\right)\right|\right) < e
\end{aligned}$$

for all $z \in H$ provided m is large enough. Since $\sum_{k=1}^{\infty} \left|u_k\left(z\right)\right|$ converges uniformly on H, $\left|u_k\left(z\right)\right| < \frac{1}{2}$ for all $z \in H$ provided k is large enough. Thus you can take $\log\left(1+u_k\left(z\right)\right)$. Pick N_0 such that for $n > m \geq N_0$,

$$\left|u_m\left(z\right)\right| < \frac{1}{2}, \prod_{k=m}^{n} \left(1+\left|u_k\left(z\right)\right|\right) < e. \tag{29.1}$$

Now having picked N_0, the assumption the u_n are bounded on H implies there exists a constant, C, independent of $z \in H$ such that for all $z \in H$,

$$\prod_{k=1}^{N_0} \left(1+\left|u_k\left(z\right)\right|\right) < C. \tag{29.2}$$

Let $N_0 < M < N$. Then

$$\begin{aligned}
& \left|\prod_{k=1}^{N} \left(1+u_k\left(z\right)\right) - \prod_{k=1}^{M} \left(1+u_k\left(z\right)\right)\right| \\
\leq & \prod_{k=1}^{N_0} \left(1+\left|u_k\left(z\right)\right|\right) \left|\prod_{k=N_0+1}^{N} \left(1+u_k\left(z\right)\right) - \prod_{k=N_0+1}^{M} \left(1+u_k\left(z\right)\right)\right| \\
\leq & C\left|\prod_{k=N_0+1}^{N} \left(1+u_k\left(z\right)\right) - \prod_{k=N_0+1}^{M} \left(1+u_k\left(z\right)\right)\right| \\
\leq & C\left(\prod_{k=N_0+1}^{M} \left(1+\left|u_k\left(z\right)\right|\right)\right)\left|\prod_{k=M+1}^{N} \left(1+u_k\left(z\right)\right) - 1\right| \\
\leq & Ce\left|\prod_{k=M+1}^{N} \left(1+\left|u_k\left(z\right)\right|\right) - 1\right|.
\end{aligned}$$

Since $1 \leq \prod_{k=M+1}^{N}\left(1+\left|u_k(z)\right|\right) \leq e$, it follows the term on the far right is dominated by

$$\begin{aligned}
& Ce^2\left|\ln\left(\prod_{k=M+1}^{N}\left(1+\left|u_k(z)\right|\right)\right)-\ln 1\right| \\
\leq\ & Ce^2\sum_{k=M+1}^{N}\ln\left(1+\left|u_k(z)\right|\right) \\
\leq\ & Ce^2\sum_{k=M+1}^{N}\left|u_k(z)\right| < \varepsilon
\end{aligned}$$

uniformly in $z \in H$ provided M is large enough. This follows from the simple observation that if $1 < x < e$, then $x-1 \leq e(\ln x - \ln 1)$. Therefore, $\left\{\prod_{k=1}^{m}\left(1+u_k(z)\right)\right\}_{m=1}^{\infty}$ is uniformly Cauchy on H and therefore, converges uniformly on H. Let $P(z)$ denote the function it converges to.

What about the permutations? Let $\{n_1, n_2, \cdots\}$ be a permutation of the indices. Let $\varepsilon > 0$ be given and let N_0 be such that if $n > N_0$,

$$\left|\prod_{k=1}^{n}\left(1+u_k(z)\right)-P(z)\right| < \varepsilon$$

for all $z \in H$. Let $\{1, 2, \cdots, n\} \subseteq \left\{n_1, n_2, \cdots, n_{p(n)}\right\}$ where $p(n)$ is an increasing sequence. Then from 29.1 and 29.2,

$$\begin{aligned}
& \left|P(z)-\prod_{k=1}^{p(n)}\left(1+u_{n_k}(z)\right)\right| \\
\leq\ & \left|P(z)-\prod_{k=1}^{n}\left(1+u_k(z)\right)\right|+\left|\prod_{k=1}^{n}\left(1+u_k(z)\right)-\prod_{k=1}^{p(n)}\left(1+u_{n_k}(z)\right)\right| \\
\leq\ & \varepsilon+\left|\prod_{k=1}^{n}\left(1+u_k(z)\right)-\prod_{k=1}^{p(n)}\left(1+u_{n_k}(z)\right)\right| \\
\leq\ & \varepsilon+\left|\prod_{k=1}^{n}\left(1+\left|u_k(z)\right|\right)\right|\left|1-\prod_{n_k>n}\left(1+u_{n_k}(z)\right)\right| \\
\leq\ & \varepsilon+\left|\prod_{k=1}^{N_0}\left(1+\left|u_k(z)\right|\right)\right|\left|\prod_{k=N_0+1}^{n}\left(1+\left|u_k(z)\right|\right)\right|\left|1-\prod_{n_k>n}\left(1+u_{n_k}(z)\right)\right| \\
\leq\ & \varepsilon+Ce\left|\prod_{n_k>n}\left(1+\left|u_{n_k}(z)\right|\right)-1\right| \leq \varepsilon+Ce\left|\prod_{k=n+1}^{M(p(n))}\left(1+\left|u_{n_k}(z)\right|\right)-1\right|
\end{aligned}$$

where $M(p(n))$ is the largest index in the permuted list, $\{n_1, n_2, \cdots, n_{p(n)}\}$. then from 29.1, this last term is dominated by

$$\varepsilon + Ce^2 \left| \ln \left(\prod_{k=n+1}^{M(p(n))} (1 + |u_{n_k}(z)|) \right) - \ln 1 \right|$$

$$\leq \varepsilon + Ce^2 \sum_{k=n+1}^{\infty} \ln(1 + |u_{n_k}|) \leq \varepsilon + Ce^2 \sum_{k=n+1}^{\infty} |u_{n_k}| < 2\varepsilon$$

for all n large enough uniformly in $z \in H$. Therefore, $\left| P(z) - \prod_{k=1}^{p(n)} (1 + u_{n_k}(z)) \right| < 2\varepsilon$ whenever n is large enough. This proves the part about the permutation.

It remains to verify the assertion about the points, z_0, where $P(z_0) = 0$. Obviously, if $u_n(z_0) = -1$, then $P(z_0) = 0$. Suppose then that $P(z_0) = 0$ and $M > N_0$. Then

$$\left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right| =$$

$$\left| \prod_{k=1}^{M} (1 + u_k(z_0)) - \prod_{k=1}^{\infty} (1 + u_k(z_0)) \right|$$

$$\leq \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right| \left| 1 - \prod_{k=M+1}^{\infty} (1 + u_k(z_0)) \right|$$

$$\leq \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right| \left| \prod_{k=M+1}^{\infty} (1 + |u_k(z_0)|) - 1 \right|$$

$$\leq e \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right| \left| \ln \prod_{k=M+1}^{\infty} (1 + |u_k(z_0)|) - \ln 1 \right|$$

$$\leq e \left(\sum_{k=M+1}^{\infty} \ln(1 + |u_k(z)|) \right) \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right|$$

$$\leq e \sum_{k=M+1}^{\infty} |u_k(z)| \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right|$$

$$\leq \frac{1}{2} \left| \prod_{k=1}^{M} (1 + u_k(z_0)) \right|$$

whenever M is large enough. Therefore, for such M,

$$\prod_{k=1}^{M} (1 + u_k(z_0)) = 0$$

and so $u_k(z_0) = -1$ for some $k \leq M$. This proves the theorem.

29.1 Analytic Function With Prescribed Zeros

Suppose you are given complex numbers, $\{z_n\}$ and you want to find an analytic function, f such that these numbers are the zeros of f. How can you do it? The problem is easy if there are only finitely many of these zeros, $\{z_1, z_2, \cdots, z_m\}$. You just write $(z - z_1)(z - z_2) \cdots (z - z_m)$. Now if none of the $z_k = 0$ you could also write it at $\prod_{k=1}^{m} \left(1 - \frac{z}{z_k}\right)$ and this might have a better chance of success in the case of infinitely many prescribed zeros. However, you would need to verify something like $\sum_{n=1}^{\infty} \left|\frac{z}{z_n}\right| < \infty$ which might not be so. The way around this is to adjust the product, making it $\prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{g_k(z)}$ where $g_k(z)$ is some analytic function. Recall also that for $|x| < 1$, $\ln\left((1-x)^{-1}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n}$. If you had x/x_n small and real, then $1 = (1 - x/x_n) \exp\left(\ln\left((1 - x/x_n)^{-1}\right)\right)$ and $\prod_{k=1}^{\infty} 1$ of course converges but loses all the information about zeros. However, this is why it is not too unreasonable to consider factors of the form

$$\left(1 - \frac{z}{z_k}\right) e^{\sum_{k=1}^{p_k} \left(\frac{z}{z_k}\right)^k \frac{1}{k}}$$

where p_k is suitably chosen.

First here are some estimates.

Lemma 29.3 *For $z \in \mathbb{C}$,*

$$|e^z - 1| \leq |z| e^{|z|}, \tag{29.3}$$

and if $|z| \leq 1/2$,

$$\left|\sum_{k=m}^{\infty} \frac{z^k}{k}\right| \leq \frac{1}{m} \frac{|z|^m}{1 - |z|} \leq \frac{2}{m} |z|^m \leq \frac{1}{m} \frac{1}{2^{m-1}}. \tag{29.4}$$

Proof: Consider 29.3.

$$|e^z - 1| = \left|\sum_{k=1}^{\infty} \frac{z^k}{k!}\right| \leq \sum_{k=1}^{\infty} \frac{|z|^k}{k!} = e^{|z|} - 1 \leq |z| e^{|z|}$$

the last inequality holding by the mean value theorem. Now consider 29.4.

$$\begin{aligned} \left|\sum_{k=m}^{\infty} \frac{z^k}{k}\right| &\leq \sum_{k=m}^{\infty} \frac{|z|^k}{k} \leq \frac{1}{m} \sum_{k=m}^{\infty} |z|^k \\ &= \frac{1}{m} \frac{|z|^m}{1 - |z|} \leq \frac{2}{m} |z|^m \leq \frac{1}{m} \frac{1}{2^{m-1}}. \end{aligned}$$

This proves the lemma.

The functions, E_p in the next definition are called the elementary factors.

Definition 29.4 *Let $E_0(z) \equiv 1 - z$ and for $p \geq 1$,*

$$E_p(z) \equiv (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right)$$

In terms of this new symbol, here is another estimate. A sharper inequality is available in Rudin [45] but it is more difficult to obtain.

Corollary 29.5 *For E_p defined above and $|z| \leq 1/2$,*

$$|E_p(z) - 1| \leq 3 |z|^{p+1}.$$

Proof: From elementary calculus, $\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ for all $|x| < 1$. Therefore, for $|z| < 1$,

$$\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \ \log\left((1 - z)^{-1}\right) = \sum_{n=1}^{\infty} \frac{z^n}{n},$$

because the function $\log(1 - z)$ and the analytic function, $-\sum_{n=1}^{\infty} \frac{z^n}{n}$ both are equal to $\ln(1 - x)$ on the real line segment $(-1, 1)$, a set which has a limit point. Therefore, using Lemma 29.3,

$$\begin{aligned}
& |E_p(z) - 1| \\
= & \left|(1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) - 1\right| \\
= & \left|(1 - z) \exp\left(\log\left((1 - z)^{-1}\right) - \sum_{n=p+1}^{\infty} \frac{z^n}{n}\right) - 1\right| \\
= & \left|\exp\left(-\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right) - 1\right| \\
\leq & \left|-\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right| e^{\left|-\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right|} \\
\leq & \frac{1}{p+1} \cdot 2 \cdot e^{1/(p+1)} |z|^{p+1} . \leq 3 |z|^{p+1}
\end{aligned}$$

This proves the corollary.

With this estimate, it is easy to prove the Weierstrass product formula.

Theorem 29.6 *Let $\{z_n\}$ be a sequence of nonzero complex numbers which have no limit point in $\mathbb{C}$ and let $\{p_n\}$ be a sequence of nonnegative integers such that*

$$\sum_{n=1}^{\infty} \left(\frac{R}{|z_n|}\right)^{p_n+1} < \infty \tag{29.5}$$

for all $R \in \mathbb{R}$. *Then*

$$P(z) \equiv \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

is analytic on $\mathbb{C}$ *and has a zero at each point,* z_n *and at no others. If* w *occurs* m *times in* $\{z_n\}$, *then* P *has a zero of order* m *at* w.

Proof: Since $\{z_n\}$ has no limit point, it follows $\lim_{n\to\infty} |z_n| = \infty$. Therefore, if $p_n = n - 1$ the condition, 29.5 holds for this choice of p_n. Now by Theorem 29.2, the infinite product in this theorem will converge uniformly on $|z| \leq R$ if the same is true of the sum,

$$\sum_{n=1}^{\infty} \left| E_{p_n}\left(\frac{z}{z_n}\right) - 1 \right|. \tag{29.6}$$

But by Corollary 29.5 the n^{th} term of this sum satisfies

$$\left| E_{p_n}\left(\frac{z}{z_n}\right) - 1 \right| \leq 3 \left| \frac{z}{z_n} \right|^{p_n+1}.$$

Since $|z_n| \to \infty$, there exists N such that for $n > N, |z_n| > 2R$. Therefore, for $|z| < R$ and letting $0 < a = \min\{|z_n| : n \leq N\}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \left| E_{p_n}\left(\frac{z}{z_n}\right) - 1 \right| &\leq 3 \sum_{n=1}^{N} \left| \frac{R}{a} \right|^{p_n+1} \\ +3 \sum_{n=N}^{\infty} \left(\frac{R}{2R} \right)^{p_n+1} &< \infty. \end{aligned}$$

By the Weierstrass M test, the series in 29.6 converges uniformly for $|z| < R$ and so the same is true of the infinite product. It follows from Lemma 24.18 on Page 652 that $P(z)$ is analytic on $|z| < R$ because it is a uniform limit of analytic functions.

Also by Theorem 29.2 the zeros of the analytic $P(z)$ are exactly the points, $\{z_n\}$, listed according to multiplicity. That is, if z_n is a zero of order m, then if it is listed m times in the formula for $P(z)$, then it is a zero of order m for P. This proves the theorem.

The following corollary is an easy consequence and includes the case where there is a zero at 0.

Corollary 29.7 *Let* $\{z_n\}$ *be a sequence of nonzero complex numbers which have no limit point and let* $\{p_n\}$ *be a sequence of nonnegative integers such that*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|} \right)^{1+p_n} < \infty \tag{29.7}$$

for all $r \in \mathbb{R}$. *Then*

$$P(z) \equiv z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

is analytic Ω *and has a zero at each point,* z_n *and at no others along with a zero of order* m *at* 0. *If* w *occurs* m *times in* $\{z_n\}$, *then* P *has a zero of order* m *at* w.

The above theory can be generalized to include the case of an arbitrary open set. First, here is a lemma.

Lemma 29.8 *Let* Ω *be an open set. Also let* $\{z_n\}$ *be a sequence of points in* Ω *which is bounded and which has no point repeated more than finitely many times such that* $\{z_n\}$ *has no limit point in* Ω. *Then there exist* $\{w_n\} \subseteq \partial\Omega$ *such that* $\lim_{n\to\infty} |z_n - w_n| = 0$.

Proof: Since $\partial\Omega$ is closed, there exists $w_n \in \partial\Omega$ such that $\operatorname{dist}(z_n, \partial\Omega) = |z_n - w_n|$. Now if there is a subsequence, $\{z_{n_k}\}$ such that $|z_{n_k} - w_{n_k}| \geq \varepsilon$ for all k, then $\{z_{n_k}\}$ must possess a limit point because it is a bounded infinite set of points. However, this limit point can only be in Ω because $\{z_{n_k}\}$ is bounded away from $\partial\Omega$. This is a contradiction. Therefore, $\lim_{n\to\infty} |z_n - w_n| = 0$. This proves the lemma.

Corollary 29.9 *Let* $\{z_n\}$ *be a sequence of complex numbers contained in* Ω, *an open subset of* $\mathbb{C}$ *which has no limit point in* Ω. *Suppose each* z_n *is repeated no more than finitely many times. Then there exists a function* f *which is analytic on* Ω *whose zeros are exactly* $\{z_n\}$. *If* $w \in \{z_n\}$ *and* w *is listed* m *times, then* w *is a zero of order* m *of* f.

Proof: There is nothing to prove if $\{z_n\}$ is finite. You just let $f(z) = \prod_{j=1}^{m} (z - z_j)$ where $\{z_n\} = \{z_1, \cdots, z_m\}$.

Pick $w \in \Omega \setminus \{z_n\}_{n=1}^{\infty}$ and let $h(z) \equiv \frac{1}{z-w}$. Since w is not a limit point of $\{z_n\}$, there exists $r > 0$ such that $B(w, r)$ contains no points of $\{z_n\}$. Let $\Omega_1 \equiv \Omega \setminus \{w\}$. Now h is not constant and so $h(\Omega_1)$ is an open set by the open mapping theorem. In fact, h maps each component of Ω to a region. $|z_n - w| > r$ for all z_n and so $|h(z_n)| < r^{-1}$. Thus the sequence, $\{h(z_n)\}$ is a bounded sequence in the open set $h(\Omega_1)$. It has no limit point in $h(\Omega_1)$ because this is true of $\{z_n\}$ and Ω_1. By Lemma 29.8 there exist $w_n \in \partial(h(\Omega_1))$ such that $\lim_{n\to\infty} |w_n - h(z_n)| = 0$. Consider for $z \in \Omega_1$

$$f(z) \equiv \prod_{n=1}^{\infty} E_n\left(\frac{h(z_n) - w_n}{h(z) - w_n}\right). \tag{29.8}$$

Letting K be a compact subset of Ω_1, $h(K)$ is a compact subset of $h(\Omega_1)$ and so if $z \in K$, then $|h(z) - w_n|$ is bounded below by a positive constant. Therefore, there exists N large enough that for all $z \in K$ and $n \geq N$,

$$\left|\frac{h(z_n) - w_n}{h(z) - w_n}\right| < \frac{1}{2}$$

and so by Corollary 29.5, for all $z \in K$ and $n \geq N$,

$$\left|E_n\left(\frac{h(z_n) - w_n}{h(z) - w_n}\right) - 1\right| \leq 3\left(\frac{1}{2}\right)^n. \tag{29.9}$$

Therefore,

$$\sum_{n=1}^{\infty}\left|E_n\left(\frac{h(z_n)-w_n}{h(z)-w_n}\right)-1\right|$$

converges uniformly for $z\in K$. This implies $\prod_{n=1}^{\infty}E_n\left(\frac{h(z_n)-w_n}{h(z)-w_n}\right)$ also converges uniformly for $z\in K$ by Theorem 29.2. Since K is arbitrary, this shows f defined in 29.8 is analytic on Ω_1.

Also if z_n is listed m times so it is a zero of multiplicity m and w_n is the point from $\partial(h(\Omega_1))$ closest to $h(z_n)$, then there are m factors in 29.8 which are of the form

$$\begin{aligned}E_n\left(\frac{h(z_n)-w_n}{h(z)-w_n}\right) &= \left(1-\frac{h(z_n)-w_n}{h(z)-w_n}\right)e^{g_n(z)}\\ &= \left(\frac{h(z)-h(z_n)}{h(z)-w_n}\right)e^{g_n(z)}\\ &= \frac{z_n-z}{(z-w)(z_n-w)}\left(\frac{1}{h(z)-w_n}\right)e^{g_n(z)}\\ &= (z-z_n)G_n(z) \qquad (29.10)\end{aligned}$$

where G_n is an analytic function which is not zero at and near z_n. Therefore, f has a zero of order m at z_n. This proves the theorem except for the point, w which has been left out of Ω_1. It is necessary to show f is analytic at this point also and right now, f is not even defined at w.

The $\{w_n\}$ are bounded because $\{h(z_n)\}$ is bounded and $\lim_{n\to\infty}|w_n-h(z_n)|=0$ which implies $|w_n-h(z_n)|\leq C$ for some constant, C. Therefore, there exists $\delta>0$ such that if $z\in B'(w,\delta)$, then for all n,

$$\left|\frac{h(z_n)-w}{\left(\frac{1}{z-w}\right)-w_n}\right|=\left|\frac{h(z_n)-w_n}{h(z)-w_n}\right|<\frac{1}{2}.$$

Thus 29.9 holds for all $z\in B'(w,\delta)$ and n so by Theorem 29.2, the infinite product in 29.8 converges uniformly on $B'(w,\delta)$. This implies f is bounded in $B'(w,\delta)$ and so w is a removable singularity and f can be extended to w such that the result is analytic. It only remains to verify $f(w)\neq 0$. After all, this would not do because it would be another zero other than those in the given list. By 29.10, a partial product is of the form

$$\prod_{n=1}^{N}\left(\frac{h(z)-h(z_n)}{h(z)-w_n}\right)e^{g_n(z)} \qquad (29.11)$$

where

$$g_n(z)\equiv\left(\frac{h(z_n)-w_n}{h(z)-w_n}+\frac{1}{2}\left(\frac{h(z_n)-w_n}{h(z)-w_n}\right)^2+\cdots+\frac{1}{n}\left(\frac{h(z_n)-w_n}{h(z)-w_n}\right)^n\right)$$

Each of the quotients in the definition of $g_n(z)$ converges to 0 as $z \to w$ and so the partial product of 29.11 converges to 1 as $z \to w$ because $\left(\frac{h(z)-h(z_n)}{h(z)-w_n}\right) \to 1$ as $z \to w$.

If $f(w) = 0$, then if z is close enough to w, it follows $|f(z)| < \frac{1}{2}$. Also, by the uniform convergence on $B'(w, \delta)$, it follows that for some N, the partial product up to N must also be less than $1/2$ in absolute value for all z close enough to w and as noted above, this does not occur because such partial products converge to 1 as $z \to w$. Hence $f(w) \neq 0$. This proves the corollary.

Recall the definition of a meromorphic function on Page 666. It was a function which is analytic everywhere except at a countable set of isolated points at which the function has a pole. It is clear that the quotient of two analytic functions yields a meromorphic function but is this the only way it can happen?

Theorem 29.10 *Suppose Q is a meromorphic function on an open set, Ω. Then there exist analytic functions on Ω, $f(z)$ and $g(z)$ such that $Q(z) = f(z)/g(z)$ for all z not in the set of poles of Q.*

Proof: Let Q have a pole of order $m(z)$ at z. Then by Corollary 29.9 there exists an analytic function, g which has a zero of order $m(z)$ at every $z \in \Omega$. It follows gQ has a removable singularity at the poles of Q. Therefore, there is an analytic function, f such that $f(z) = g(z)Q(z)$. This proves the theorem.

Corollary 29.11 *Suppose Ω is a region and Q is a meromorphic function defined on Ω such that the set, $\{z \in \Omega : Q(z) = c\}$ has a limit point in Ω. Then $Q(z) = c$ for all $z \in \Omega$.*

Proof: From Theorem 29.10 there are analytic functions, f, g such that $Q = \frac{f}{g}$. Therefore, the zero set of the function, $f(z) - cg(z)$ has a limit point in Ω and so $f(z) - cg(z) = 0$ for all $z \in \Omega$. This proves the corollary.

29.2 Factoring A Given Analytic Function

The next theorem is the Weierstrass factorization theorem which can be used to factor a given analytic function f. If f has a zero of order m when $z = 0$, then you could factor out a z^m and from there consider the factorization of what remains when you have factored out the z^m. Therefore, the following is the main thing of interest.

Theorem 29.12 *Let f be analytic on $\mathbb{C}$, $f(0) \neq 0$, and let the zeros of f, be $\{z_k\}$, listed according to order. (Thus if z is a zero of order m, it will be listed m times in the list, $\{z_k\}$.) Choosing nonnegative integers, p_n such that for all $r > 0$,*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{p_n+1} < \infty,$$

There exists an entire function, g such that

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right). \tag{29.12}$$

Note that $e^{g(z)} \neq 0$ for any z and this is the interesting thing about this function.

Proof: $\{z_n\}$ cannot have a limit point because if there were a limit point of this sequence, it would follow from Theorem 24.23 that $f(z) = 0$ for all z, contradicting the hypothesis that $f(0) \neq 0$. Hence $\lim_{n\to\infty} |z_n| = \infty$ and so

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{1+n-1} = \sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{n} < \infty$$

by the root test. Therefore, by Theorem 29.6

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

a function analytic on $\mathbb{C}$ by picking $p_n = n-1$ or perhaps some other choice. ($p_n = n-1$ works but there might be another choice that would work.) Then f/P has only removable singularities in $\mathbb{C}$ and no zeros thanks to Theorem 29.6. Thus, letting $h(z) = f(z)/P(z)$, Corollary 24.50 implies that h'/h has a primitive, $\widetilde{g}$. Then

$$\left(he^{-\widetilde{g}}\right)' = 0$$

and so

$$h(z) = e^{a+ib}e^{\widetilde{g}(z)}$$

for some constants, a, b. Therefore, letting $g(z) = \widetilde{g}(z) + a + ib$, $h(z) = e^{g(z)}$ and thus 29.12 holds. This proves the theorem.

Corollary 29.13 *Let f be analytic on $\mathbb{C}$, f has a zero of order m at* 0, *and let the other zeros of f be $\{z_k\}$, listed according to order. (Thus if z is a zero of order l, it will be listed l times in the list, $\{z_k\}$.) Also let*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|}\right)^{1+p_n} < \infty \tag{29.13}$$

for any choice of $r > 0$. Then there exists an entire function, g such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right). \tag{29.14}$$

Proof: Since f has a zero of order m at 0, it follows from Theorem 24.23 that $\{z_k\}$ cannot have a limit point in $\mathbb{C}$ and so you can apply Theorem 29.12 to the function, $f(z)/z^m$ which has a removable singularity at 0. This proves the corollary.

29.2.1 Factoring Some Special Analytic Functions

Factoring a polynomial is in general a hard task. It is true it is easy to prove the factors exist but finding them is another matter. Corollary 29.13 gives the existence of factors of a certain form but it does not tell how to find them. This should not be surprising. You can't expect things to get easier when you go from polynomials to analytic functions. Nevertheless, it is possible to factor some popular analytic functions. These factorizations are based on the following Mitag-Leffler expansions. By an auspicious choice of the contour and the method of residues it is possible to obtain a very interesting formula for $\cot \pi z$.

Example 29.14 *Let γ_N be the contour which goes from $-N-\frac{1}{2}-Ni$ horizontally to $N+\frac{1}{2}-Ni$ and from there, vertically to $N+\frac{1}{2}+Ni$ and then horizontally to $-N-\frac{1}{2}+Ni$ and finally vertically to $-N-\frac{1}{2}-Ni$. Thus the contour is a large rectangle and the direction of integration is in the counter clockwise direction. Consider the integral*

$$I_N \equiv \int_{\gamma_N} \frac{\pi \cos \pi z}{\sin \pi z \left(\alpha^2 - z^2\right)} dz$$

where $\alpha \in \mathbb{R}$ is not an integer. This will be used to verify the formula of Mittag-Leffler,

$$\frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2} = \pi \cot \pi\alpha. \tag{29.15}$$

First you show that $\cot \pi z$ is bounded on this contour. This is easy using the formula for $\cot(z) = \frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$. Therefore, $I_N \to 0$ as $N \to \infty$ because the integrand is of order $1/N^2$ while the diameter of γ_N is of order N. Next you compute the residues of the integrand at $\pm\alpha$ and at n where $|n| < N + \frac{1}{2}$ for n an integer. These are the only singularities of the integrand in this contour and therefore, using the residue theorem, you can evaluate I_N by using these. You can calculate these residues and find that the residue at $\pm\alpha$ is

$$\frac{-\pi \cos \pi\alpha}{2\alpha \sin \pi\alpha}$$

while the residue at n is

$$\frac{1}{\alpha^2 - n^2}.$$

Therefore

$$0 = \lim_{N\to\infty} I_N = \lim_{N\to\infty} 2\pi i \left[\sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} - \frac{\pi \cot \pi\alpha}{\alpha}\right]$$

which establishes the following formula of Mittag Leffler.

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{\alpha^2 - n^2} = \frac{\pi \cot \pi\alpha}{\alpha}.$$

Writing this in a slightly nicer form, you obtain 29.15.

This is a very interesting formula. This will be used to factor $\sin(\pi z)$. The zeros of this function are at the integers. Therefore, considering 29.13 you can pick $p_n = 1$ in the Weierstrass factorization formula. Therefore, by Corollary 29.13 there exists an analytic function $g(z)$ such that

$$\sin(\pi z) = z e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n} \tag{29.16}$$

where the z_n are the nonzero integers. Remember you can permute the factors in these products. Therefore, this can be written more conveniently as

$$\sin(\pi z) = z e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right)$$

and it is necessary to find $g(z)$. Differentiating both sides of 29.16

$$\begin{aligned} \pi \cos(\pi z) &= e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right) + z g'(z) e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right) \\ &\quad + z e^{g(z)} \sum_{n=1}^{\infty} - \left(\frac{2z}{n^2}\right) \prod_{k \neq n} \left(1 - \left(\frac{z}{k}\right)^2\right) \end{aligned}$$

Now divide both sides by $\sin(\pi z)$ to obtain

$$\begin{aligned} \pi \cot(\pi z) &= \frac{1}{z} + g'(z) - \sum_{n=1}^{\infty} \frac{2z/n^2}{(1 - z^2/n^2)} \\ &= \frac{1}{z} + g'(z) + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \end{aligned}$$

By 29.15, this yields $g'(z) = 0$ for z not an integer and so $g(z) = c$, a constant. So far this yields

$$\sin(\pi z) = z e^{c} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right)$$

and it only remains to find c. Divide both sides by πz and take a limit as $z \to 0$. Using the power series of $\sin(\pi z)$, this yields

$$1 = \frac{e^c}{\pi}$$

and so $c = \ln \pi$. Therefore,

$$\sin(\pi z) = z\pi \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right). \tag{29.17}$$

Example 29.15 *Find an interesting formula for* $\tan(\pi z)$.

This is easy to obtain from the formula for $\cot(\pi z)$.

$$\cot\left(\pi\left(z+\frac{1}{2}\right)\right)=-\tan \pi z$$

for z real and therefore, this formula holds for z complex also. Therefore, for $z+\frac{1}{2}$ not an integer

$$\pi\cot\left(\pi\left(z+\frac{1}{2}\right)\right)=\frac{2}{2z+1}+\sum_{n=1}^{\infty}\frac{2z+1}{\left(\frac{2z+1}{2}\right)^2-n^2}$$

29.3 The Existence Of An Analytic Function With Given Values

The Weierstrass product formula, Theorem 29.6, along with the Mittag-Leffler theorem, Theorem 28.13 can be used to obtain an analytic function which has given values on a countable set of points, having no limit point. This is clearly an amazing result and indicates how potent these theorems are. In fact, you can show that it isn't just the values of the function which may be specified at the points in this countable set of points but the derivatives up to any finite order.

Theorem 29.16 *Let* $P\equiv\{z_k\}_{k=1}^{\infty}$ *be a set of points in* $\mathbb{C}$,*which has no limit point. For each* z_k, *consider*

$$\sum_{j=0}^{m_k} a_j^k (z-z_k)^j . \tag{29.18}$$

Then there exists an analytic function defined on $\mathbb{C}$ *such that the Taylor series of* f *at* z_k *has the first* m_k *terms given by 29.18.*[1]

Proof: By the Weierstrass product theorem, Theorem 29.6, there exists an analytic function, f defined on all of Ω such that f has a zero of order m_k+1 at z_k. Consider this z_k Thus for z near z_k,

$$f(z)=\sum_{j=m_k+1}^{\infty} c_j (z-z_k)^j$$

where $c_{m_k+1}\neq 0$. You choose $b_1, b_2, \cdots, b_{m_k+1}$ such that

$$f(z)\left(\sum_{l=1}^{m_k+1}\frac{b_l}{(z-z_k)^k}\right)=\sum_{j=0}^{m_k} a_j^k (z-z_k)^j+\sum_{k=m_k+1}^{\infty} c_j^k (z-z_k)^j .$$

[1] This says you can specify the first m_k derivatives of the function at the point z_k.

Thus you need

$$\sum_{l=1}^{m_k+1}\sum_{j=m_k+1}^{\infty} c_j b_l (z-z_k)^{j-l} = \sum_{r=0}^{m_k} a_r^k (z-z_k)^r + \text{Higher order terms.}$$

It follows you need to solve the following system of equations for $b_1, \cdots, b_{m_k+1}$.

$$\begin{gathered} c_{m_k+1} b_{m_k+1} = a_0^k \\ c_{m_k+2} b_{m_k+1} + c_{m_k+1} b_{m_k} = a_1^k \\ c_{m_k+3} b_{m_k+1} + c_{m_k+2} b_{m_k} + c_{m_k+1} b_{m_k-1} = a_2^k \\ \vdots \\ c_{m_k+m_k+1} b_{m_k+1} + c_{m_k+m_k} b_{m_k} + \cdots + c_{m_k+1} b_1 = a_{m_k}^k \end{gathered}$$

Since $c_{m_k+1} \neq 0$, it follows there exists a unique solution to the above system. You first solve for b_{m_k+1} in the top. Then, having found it, you go to the next and use $c_{m_k+1} \neq 0$ again to find b_{m_k} and continue in this manner. Let $S_k(z)$ be determined in this manner for each z_k. By the Mittag-Leffler theorem, there exists a Meromorphic function, g such that g has exactly the singularities, $S_k(z)$. Therefore, $f(z) g(z)$ has removable singularities at each z_k and for z near z_k, the first m_k terms of fg are as prescribed. This proves the theorem.

Corollary 29.17 *Let $P \equiv \{z_k\}_{k=1}^{\infty}$ be a set of points in Ω, an open set such that P has no limit points in Ω. For each z_k, consider*

$$\sum_{j=0}^{m_k} a_j^k (z-z_k)^j . \tag{29.19}$$

Then there exists an analytic function defined on Ω such that the Taylor series of f at z_k has the first m_k terms given by 29.19.

Proof: The proof is identical to the above except you use the versions of the Mittag-Leffler theorem and Weierstrass product which pertain to open sets.

Definition 29.18 *Denote by $H(\Omega)$ the analytic functions defined on Ω, an open subset of $\mathbb{C}$. Then $H(\Omega)$ is a commutative ring[2] with the usual operations of addition and multiplication. A set, $I \subseteq H(\Omega)$ is called a finitely generated ideal of the ring if I is of the form*

$$\left\{ \sum_{k=1}^{n} g_k f_k : f_k \in H(\Omega) \text{ for } k = 1, 2, \cdots, n \right\}$$

where $g_1, \cdots, g_n$ are given functions in $H(\Omega)$. This ideal is also denoted as $[g_1, \cdots, g_n]$ and is called the ideal generated by the functions, $\{g_1, \cdots, g_n\}$. Since there are finitely many of these functions it is called a finitely generated ideal. A principal ideal is one which is generated by a single function. An example of such a thing is $[1] = H(\Omega)$.

[2]It is not a field because you can't divide two analytic functions and get another one.

Then there is the following interesting theorem.

Theorem 29.19 *Every finitely generated ideal in $H(\Omega)$ for Ω a connected open set (region) is a principal ideal.*

Proof: Let $I = [g_1, \cdots, g_n]$ be a finitely generated ideal as described above. Then if any of the functions has no zeros, this ideal would consist of $H(\Omega)$ because then $g_i^{-1} \in H(\Omega)$ and so $1 \in I$. It follows all the functions have zeros. If any of the functions has a zero of infinite order, then the function equals zero on Ω because Ω is connected and can be deleted from the list. Similarly, if the zeros of any of these functions have a limit point in Ω, then the function equals zero and can be deleted from the list. Thus, without loss of generality, all zeros are of finite order and there are no limit points of the zeros in Ω. Let $m(g_i, z)$ denote the order of the zero of g_i at z. If g_i has no zero at z, then $m(g_i, z) = 0$.

I claim that if no point of Ω is a zero of all the g_i, then the conclusion of the theorem is true and in fact $[g_1, \cdots, g_n] = [1] = H(\Omega)$. The claim is obvious if $n = 1$ because this assumption that no point is a zero of all the functions implies $g \neq 0$ and so g^{-1} is analytic. Hence $1 \in [g_1]$. Suppose it is true for $n - 1$ and consider $[g_1, \cdots, g_n]$ where no point of Ω is a zero of all the g_i. Even though this may be true of $\{g_1, \cdots, g_n\}$, it may not be true of $\{g_1, \cdots, g_{n-1}\}$. By Corollary 29.9 there exists ϕ, a function analytic on Ω such that $m(\phi, z) = \min\{m(g_i, z), i = 1, 2, \cdots, n-1\}$. Thus the functions $\{g_1/\phi, \cdots, g_{n-1}/\phi\}$ .are all analytic. Could they all equal zero at some point, z? If so, pick i where $m(\phi, z) = m(g_i, z)$. Thus g_i/ϕ is not equal to zero at z after all and so these functions are analytic there is no point of Ω which is a zero of all of them. By induction, $[g_1/\phi, \cdots, g_{n-1}/\phi] = H(\Omega)$. (Also there are no new zeros obtained in this way.)

Now this means there exist functions $f_i \in H(\Omega)$ such that

$$\sum_{i=1}^{n} f_i \left(\frac{g_i}{\phi}\right) = 1$$

and so $\phi = \sum_{i=1}^{n} f_i g_i$. Therefore, $[\phi] \subseteq [g_1, \cdots, g_{n-1}]$. On the other hand, if $\sum_{k=1}^{n-1} h_k g_k \in [g_1, \cdots, g_{n-1}]$ you could define $h \equiv \sum_{k=1}^{n-1} h_k (g_k/\phi)$, an analytic function with the property that $h\phi = \sum_{k=1}^{n-1} h_k g_k$ which shows $[\phi] = [g_1, \cdots, g_{n-1}]$. Therefore,

$$[g_1, \cdots, g_n] = [\phi, g_n]$$

Now ϕ has no zeros in common with g_n because the zeros of ϕ are contained in the set of zeros for $g_1, \cdots, g_{n-1}$. Now consider a zero, α of ϕ. It is not a zero of g_n and so near α, these functions have the form

$$\phi(z) = \sum_{k=m}^{\infty} a_k (z-\alpha)^k, \; g_n(z) = \sum_{k=0}^{\infty} b_k (z-\alpha)^k, \; b_0 \neq 0.$$

I want to determine coefficients for an analytic function, h such that

$$m(1 - hg_n, \alpha) \geq m(\phi, \alpha). \tag{29.20}$$

Let

$$h(z) = \sum_{k=0}^{\infty} c_k (z-\alpha)^k$$

and the c_k must be determined. Using Merten's theorem, the power series for $1-hg_n$ is of the form

$$1 - b_0c_0 - \sum_{j=1}^{\infty} \left(\sum_{r=0}^{j} b_{j-r}c_r \right) (z-\alpha)^j .$$

First determine c_0 such that $1 - c_0b_0 = 0$. This is no problem because $b_0 \neq 0$. Next you need to get the coefficients of $(z-\alpha)$ to equal zero. This requires

$$b_1c_0 + b_0c_1 = 0.$$

Again, there is no problem because $b_0 \neq 0$. In fact, $c_1 = (-b_1c_0/b_0)$. Next consider the second order terms if $m \geq 2$.

$$b_2c_0 + b_1c_1 + b_0c_2 = 0$$

Again there is no problem in solving, this time for c_2 because $b_0 \neq 0$. Continuing this way, you see that in every step, the c_k which needs to be solved for is multiplied by $b_0 \neq 0$. Therefore, by Corollary 29.9 there exists an analytic function, h satisfying 29.20. Therefore, $(1-hg_n)/\phi$ has a removable singularity at every zero of ϕ and so may be considered an analytic function. Therefore,

$$1 = \frac{1-hg_n}{\phi}\phi + hg_n \in [\phi, g_n] = [g_1 \cdots g_n]$$

which shows $[g_1 \cdots g_n] = H(\Omega) = [1]$. It follows the claim is established.

Now suppose $\{g_1 \cdots g_n\}$ are just elements of $H(\Omega)$. As explained above, it can be assumed they all have zeros of finite order and the zeros have no limit point in Ω since if these occur, you can delete the function from the list. By Corollary 29.9 there exists $\phi \in H(\Omega)$ such that $m(\phi, z) \leq \min\{m(g_i, z) : i = 1, \cdots, n\}$. Then g_k/ϕ has a removable singularity at each zero of g_k and so can be regarded as an analytic function. Also, as before, there is no point which is a zero of each g_k/ϕ and so by the first part of this argument, $[g_1/\phi \cdots g_n/\phi] = H(\Omega)$. As in the first part of the argument, this implies $[g_1 \cdots g_n] = [\phi]$ which proves the theorem. $[g_1 \cdots g_n]$ is a principal ideal as claimed.

The following corollary follows from the above theorem. You don't need to assume Ω is connected.

Corollary 29.20 *Every finitely generated ideal in $H(\Omega)$ for Ω an open set is a principal ideal.*

Proof: Let $[g_1, \cdots, g_n]$ be a finitely generated ideal in $H(\Omega)$. Let $\{U_k\}$ be the components of Ω. Then applying the above to each component, there exists $h_k \in H(U_k)$ such that restricting each g_i to U_k, $[g_1, \cdots, g_n] = [h_k]$. Then let $h(z) = h_k(z)$ for $z \in U_k$. This is an analytic function which works.

29.4 Jensen's Formula

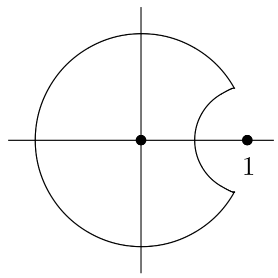
This interesting formula relates the zeros of an analytic function to an integral. The proof given here follows Alfors, [2]. First, here is a technical lemma.

Lemma 29.21

$$\int_{-\pi}^{\pi} \ln\left|1 - e^{i\theta}\right| d\theta = 0.$$

Proof: First note that the only problem with the integrand occurs when $\theta = 0$. However, this is an integrable singularity so the integral will end up making sense. Letting $z = e^{i\theta}$, you could get the above integral as a limit as $\varepsilon \to 0$ of the following contour integral where γ_ε is the contour shown in the following picture with the radius of the big circle equal to 1 and the radius of the little circle equal to ε..

$$\int_{\gamma_\varepsilon} \frac{\ln|1 - z|}{iz} dz.$$



On the indicated contour, $1-z$ lies in the half plane $\operatorname{Re} z > 0$ and so $\log(1 - z) = \ln|1 - z| + i \arg(1 - z)$. The above integral equals

$$\int_{\gamma_\varepsilon} \frac{\log(1 - z)}{iz} dz - \int_{\gamma_\varepsilon} \frac{\arg(1 - z)}{z} dz$$

The first of these integrals equals zero because the integrand has a removable singularity at 0. The second equals

$$i \int_{-\pi}^{-\eta_\varepsilon} \arg\left(1 - e^{i\theta}\right) d\theta + i \int_{\eta_\varepsilon}^{\pi} \arg\left(1 - e^{i\theta}\right) d\theta$$
$$+\varepsilon i \int_{-\frac{\pi}{2}-\lambda_\varepsilon}^{-\pi} \theta d\theta + \varepsilon i \int_{\pi}^{\frac{\pi}{2}-\lambda_\varepsilon} \theta d\theta$$

where $\eta_\varepsilon, \lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. The last two terms converge to 0 as $\varepsilon \to 0$ while the first two add to zero. To see this, change the variable in the first integral and then recall that when you multiply complex numbers you add the arguments. Thus you end up integrating arg (real valued function) which equals zero.

In this material on Jensen's equation, ε will denote a small positive number. Its value is not important as long as it is positive. Therefore, it may change from place to place. Now suppose f is analytic on $B(0,r+\varepsilon)$, and f has no zeros on $\overline{B(0,r)}$. Then you can define a branch of the logarithm which makes sense for complex numbers near $f(z)$. Thus $z \to \log(f(z))$ is analytic on $B(0,r+\varepsilon)$. Therefore, its real part, $u(x,y) \equiv \ln|f(x+iy)|$ must be harmonic. Consider the following lemma.

Lemma 29.22 *Let u be harmonic on $B(0,r+\varepsilon)$. Then*

$$u(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(re^{i\theta}\right) d\theta.$$

Proof: For a harmonic function, u defined on $B(0,r+\varepsilon)$, there exists an analytic function, $h = u + iv$ where

$$v(x,y) \equiv \int_0^y u_x(x,t)\,dt - \int_0^x u_y(t,0)\,dt.$$

By the Cauchy integral theorem,

$$h(0) = \frac{1}{2\pi i}\int_{\gamma_r} \frac{h(z)}{z}dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} h\left(re^{i\theta}\right) d\theta.$$

Therefore, considering the real part of h,

$$u(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u\left(re^{i\theta}\right) d\theta.$$

This proves the lemma.

Now this shows the following corollary.

Corollary 29.23 *Suppose f is analytic on $B(0,r+\varepsilon)$ and has no zeros on $\overline{B(0,r)}$. Then*

$$\ln|f(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|f\left(re^{i\theta}\right)\right| \tag{29.21}$$

What if f has some zeros on $|z| = r$ but none on $B(0,r)$? It turns out 29.21 is still valid. Suppose the zeros are at $\left\{re^{i\theta_k}\right\}_{k=1}^m$, listed according to multiplicity. Then let

$$g(z) = \frac{f(z)}{\prod_{k=1}^m \left(z - re^{i\theta_k}\right)}.$$

It follows g is analytic on $B(0, r+\varepsilon)$ but has no zeros in $\overline{B(0,r)}$. Then 29.21 holds for g in place of f. Thus

$$\begin{aligned}
&\ln|f(0)| - \sum_{k=1}^{m} \ln|r| \\
= &\ \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{m}\ln\left|re^{i\theta} - re^{i\theta_k}\right| d\theta \\
= &\ \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{m}\ln\left|e^{i\theta} - e^{i\theta_k}\right| d\theta - \sum_{k=1}^{m}\ln|r| \\
= &\ \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta - \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{m}\ln\left|e^{i\theta} - 1\right| d\theta - \sum_{k=1}^{m}\ln|r|
\end{aligned}$$

Therefore, 29.21 will continue to hold exactly when $\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{m}\ln\left|e^{i\theta}-1\right| d\theta = 0$. But this is the content of Lemma 29.21. This proves the following lemma.

Lemma 29.24 *Suppose f is analytic on $B(0, r+\varepsilon)$ and has no zeros on $B(0,r)$. Then*

$$\ln|f(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|f\left(re^{i\theta}\right)\right| \tag{29.22}$$

With this preparation, it is now not too hard to prove Jensen's formula. Suppose there are n zeros of f in $B(0,r)$, $\{a_k\}_{k=1}^n$, listed according to multiplicity, none equal to zero. Let

$$F(z) \equiv f(z)\prod_{i=1}^{n}\frac{r^2 - \overline{a_i}z}{r(z-a_i)}.$$

Then F is analytic on $B(0, r+\varepsilon)$ and has no zeros in $B(0,r)$. The reason for this is that $f(z)/\prod_{i=1}^{n} r(z-a_i)$ has no zeros there and $r^2 - \overline{a_i}z$ cannot equal zero if $|z| < r$ because if this expression equals zero, then

$$|z| = \frac{r^2}{|a_i|} > r.$$

The other interesting thing about $F(z)$ is that when $z = re^{i\theta}$,

$$\begin{aligned}
F\left(re^{i\theta}\right) &= f\left(re^{i\theta}\right)\prod_{i=1}^{n}\frac{r^2 - \overline{a_i}re^{i\theta}}{r\left(re^{i\theta} - a_i\right)} \\
&= f\left(re^{i\theta}\right)\prod_{i=1}^{n}\frac{r - \overline{a_i}e^{i\theta}}{\left(re^{i\theta} - a_i\right)} = f\left(re^{i\theta}\right)e^{i\theta}\prod_{i=1}^{n}\frac{re^{-i\theta} - \overline{a_i}}{re^{i\theta} - a_i}
\end{aligned}$$

so $\left|F\left(re^{i\theta}\right)\right| = \left|f\left(re^{i\theta}\right)\right|$.

Theorem 29.25 *Let f be analytic on $B(0, r+\varepsilon)$ and suppose $f(0) \neq 0$. If the zeros of f in $B(0,r)$ are $\{a_k\}_{k=1}^n$, listed according to multiplicity, then*

$$\ln|f(0)| = -\sum_{i=1}^{n} \ln\left(\frac{r}{|a_i|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta.$$

Proof: From the above discussion and Lemma 29.24,

$$\ln|F(0)| = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta$$

But $F(0) = f(0)\prod_{i=1}^n \frac{r}{a_i}$ and so $\ln|F(0)| = \ln|f(0)| + \sum_{i=1}^n \ln\left|\frac{r}{a_i}\right|$. Therefore,

$$\ln|f(0)| = -\sum_{i=1}^{n} \ln\left|\frac{r}{a_i}\right| + \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta$$

as claimed.

Written in terms of exponentials this is

$$|f(0)|\prod_{k=1}^{n}\left|\frac{r}{a_k}\right| = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta\right).$$

29.5 Blaschke Products

The Blaschke[3] product is a way to produce a function which is bounded and analytic on $B(0,1)$ which also has given zeros in $B(0,1)$. The interesting thing here is that there may be infinitely many of these zeros. Thus, unlike the above case of Jensen's inequality, the function is not analytic on $\overline{B(0,1)}$. Recall for purposes of comparison, Liouville's theorem which says bounded entire functions are constant. The Blaschke product gives examples of bounded functions on $B(0,1)$ which are definitely not constant.

Theorem 29.26 *Let $\{\alpha_n\}$ be a sequence of nonzero points in $B(0,1)$ with the property that*

$$\sum_{n=1}^{\infty}(1-|\alpha_n|) < \infty.$$

Then for $k \geq 0$, an integer

$$B(z) \equiv z^k \prod_{k=1}^{\infty} \frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n}$$

is a bounded function which is analytic on $B(0,1)$ which has zeros only at 0 if $k > 0$ and at the α_n.

[3] Wilhelm Blaschke, 1915

Proof: From Theorem 29.2 the above product will converge uniformly on $B(0,r)$ for $r<1$ to an analytic function if

$$\sum_{k=1}^{\infty}\left|\frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n}-1\right|$$

converges uniformly on $B(0,r)$. But for $|z|<r$,

$$\begin{aligned}
&\left|\frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n}-1\right| \\
=\ &\left|\frac{\alpha_n - z}{1-\overline{\alpha_n}z}\frac{|\alpha_n|}{\alpha_n}-\frac{\alpha_n(1-\overline{\alpha_n}z)}{\alpha_n(1-\overline{\alpha_n}z)}\right| \\
=\ &\left|\frac{|\alpha_n|\alpha_n - |\alpha_n|z-\alpha_n+|\alpha_n|^2 z}{(1-\overline{\alpha_n}z)\alpha_n}\right| \\
=\ &\left|\frac{|\alpha_n|\alpha_n - \alpha_n - |\alpha_n|z+|\alpha_n|^2 z}{(1-\overline{\alpha_n}z)\alpha_n}\right| \\
=\ &||\alpha_n|-1|\left|\frac{\alpha_n + z|\alpha_n|}{(1-\overline{\alpha_n}z)\alpha_n}\right| \\
=\ &||\alpha_n|-1|\left|\frac{1+z(|\alpha_n|/\alpha_n)}{(1-\overline{\alpha_n}z)}\right| \\
\leq\ &||\alpha_n|-1|\left|\frac{1+|z|}{1-|z|}\right| \leq ||\alpha_n|-1|\left|\frac{1+r}{1-r}\right|
\end{aligned}$$

and so the assumption on the sum gives uniform convergence of the product on $B(0,r)$ to an analytic function. Since $r<1$ is arbitrary, this shows $B(z)$ is analytic on $B(0,1)$ and has the specified zeros because the only place the factors equal zero are at the α_n or 0.

Now consider the factors in the product. The claim is that they are all no larger in absolute value than 1. This is very easy to see from the maximum modulus theorem. Let $|\alpha|<1$ and $\phi(z)=\frac{\alpha-z}{1-\overline{\alpha}z}$. Then ϕ is analytic near $B(0,1)$ because its only pole is $1/\overline{\alpha}$. Consider $z=e^{i\theta}$. Then

$$\left|\phi\left(e^{i\theta}\right)\right| = \left|\frac{\alpha-e^{i\theta}}{1-\overline{\alpha}e^{i\theta}}\right| = \left|\frac{1-\alpha e^{-i\theta}}{1-\overline{\alpha}e^{i\theta}}\right| = 1.$$

Thus the modulus of $\phi(z)$ equals 1 on $\partial B(0,1)$. Therefore, by the maximum modulus theorem, $|\phi(z)|<1$ if $|z|<1$. This proves the claim that the terms in the product are no larger than 1 and shows the function determined by the Blaschke product is bounded. This proves the theorem.

Note in the conditions for this theorem the one for the sum, $\sum_{n=1}^{\infty}(1-|\alpha_n|)<\infty$. The Blaschke product gives an analytic function, whose absolute value is bounded by 1 and which has the α_n as zeros. What if you had a bounded function, analytic on $B(0,1)$ which had zeros at $\{\alpha_k\}$? Could you conclude the condition on the sum?

The answer is yes. In fact, you can get by with less than the assumption that f is bounded but this will not be presented here. See Rudin [45]. This theorem is an exciting use of Jensen's equation.

Theorem 29.27 *Suppose f is an analytic function on $B(0,1), f(0) \neq 0$, and $|f(z)| \leq M$ for all $z \in B(0,1)$. Suppose also that the zeros of f are $\{\alpha_k\}_{k=1}^{\infty}$, listed according to multiplicity. Then $\sum_{k=1}^{\infty}(1-|\alpha_k|) < \infty$.*

Proof: If there are only finitely many zeros, there is nothing to prove so assume there are infinitely many. Also let the zeros be listed such that $|\alpha_n| \leq |\alpha_{n+1}| \cdots$ Let $n(r)$ denote the number of zeros in $B(0,r)$. By Jensen's formula,

$$\ln|f(0)| + \sum_{i=1}^{n(r)} \ln r - \ln|\alpha_i| = \frac{1}{2\pi}\int_0^{2\pi} \ln\left|f\left(re^{i\theta}\right)\right| d\theta \leq \ln(M).$$

Therefore, by the mean value theorem,

$$\sum_{i=1}^{n(r)} \frac{1}{r}(r-|\alpha_i|) \leq \sum_{i=1}^{n(r)} \ln r - \ln|\alpha_i| \leq \ln(M) - \ln|f(0)|$$

As $r \to 1-, n(r) \to \infty$, and so an application of Fatous lemma yields

$$\sum_{i=1}^{\infty}(1-|\alpha_i|) \leq \lim \inf_{r\to 1-} \sum_{i=1}^{n(r)} \frac{1}{r}(r-|\alpha_i|) \leq \ln(M) - \ln|f(0)|.$$

This proves the theorem.

You don't need the assumption that $f(0) \neq 0$.

Corollary 29.28 *Suppose f is an analytic function on $B(0,1)$ and $|f(z)| \leq M$ for all $z \in B(0,1)$. Suppose also that the nonzero zeros[4] of f are $\{\alpha_k\}_{k=1}^{\infty}$, listed according to multiplicity. Then $\sum_{k=1}^{\infty}(1-|\alpha_k|) < \infty$.*

Proof: Suppose f has a zero of order m at 0. Then consider the analytic function, $g(z) \equiv f(z)/z^m$ which has the same zeros except for 0. The argument goes the same way except here you use g instead of f and only consider $r > r_0 > 0$.

[4]This is a fun thing to say: nonzero zeros.

Thus from Jensen's equation,

$$\begin{aligned}
& \ln |g(0)| + \sum_{i=1}^{n(r)} \ln r - \ln |\alpha_i| \\
= & \frac{1}{2\pi} \int_0^{2\pi} \ln \left| g\left(re^{i\theta}\right)\right| d\theta \\
= & \frac{1}{2\pi} \int_0^{2\pi} \ln \left| f\left(re^{i\theta}\right)\right| d\theta - \frac{1}{2\pi} \int_0^{2\pi} m \ln (r) \\
\leq & M + \frac{1}{2\pi} \int_0^{2\pi} m \ln \left(r^{-1}\right) \\
\leq & M + m \ln \left(\frac{1}{r_0}\right).
\end{aligned}$$

Now the rest of the argument is the same.

An interesting restatement yields the following amazing result.

Corollary 29.29 *Suppose f is analytic and bounded on $B(0,1)$ having zeros $\{\alpha_n\}$. Then if $\sum_{k=1}^{\infty}(1-|\alpha_n|)=\infty$, it follows f is identically equal to zero.*

29.5.1 The Müntz-Szasz Theorem Again

Corollary 29.29 makes possible an easy proof of a remarkable theorem named above which yields a wonderful generalization of the Weierstrass approximation theorem. In what follows $b>0$. The Weierstrass approximation theorem states that linear combinations of $1, t, t^2, t^3, \cdots$ (polynomials) are dense in $C([0,b])$. Let $\lambda_1<\lambda_2<\lambda_3<\cdots$ be an increasing list of positive real numbers. This theorem tells when linear combinations of $1, t^{\lambda_1}, t^{\lambda_2}, \cdots$ are dense in $C([0,b])$. The proof which follows is like the one given in Rudin [45]. There is a much longer one in Cheney [14] which discusses more aspects of the subject. See also Page 377 where the version given in Cheney is presented. This other approach is much more elementary and does not depend in any way on the theory of functions of a complex variable. There are those of us who automatically prefer real variable techniques. Nevertheless, this proof by Rudin is a very nice and insightful application of the preceding material. Cheney refers to the theorem as the second Müntz theorem. I guess Szasz must also have been involved.

Theorem 29.30 *Let $\lambda_1<\lambda_2<\lambda_3<\cdots$ be an increasing list of positive real numbers and let $a>0$. If*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n}=\infty, \tag{29.23}$$

then linear combinations of $1, t^{\lambda_1}, t^{\lambda_2}, \cdots$ are dense in $C([0,b])$.

Proof: Let X denote the closure of linear combinations of $\left\{1, t^{\lambda_1}, t^{\lambda_2}, \cdots\right\}$ in $C([0,b])$. If $X \neq C([0,b])$, then letting $f \in C([0,b]) \setminus X$, define $\Lambda \in C([0,b])'$ as follows. First let $\Lambda_0 : X + \mathbb{C}f$ be given by $\Lambda_0(g + \alpha f) = \alpha ||f||_\infty$. Then

$$\begin{aligned}\sup_{||g+\alpha f|| \leq 1} |\Lambda_0(g+\alpha f)| &= \sup_{||g+\alpha f|| \leq 1} |\alpha| \, ||f||_\infty \\ &= \sup_{||g/\alpha + f|| \leq \frac{1}{|\alpha|}} |\alpha| \, ||f||_\infty \\ &= \sup_{||g+f|| \leq \frac{1}{|\alpha|}} |\alpha| \, ||f||_\infty\end{aligned}$$

Now $\operatorname{dist}(f, X) > 0$ because X is closed. Therefore, there exists a lower bound, $\eta > 0$ to $||g+f||$ for $g \in X$. Therefore, the above is no larger than

$$\sup_{|\alpha| \leq \frac{1}{\eta}} |\alpha| \, ||f||_\infty = \left(\frac{1}{\eta}\right) ||f||_\infty$$

which shows that $||\Lambda_0|| \leq \left(\frac{1}{\eta}\right) ||f||_\infty$. By the Hahn Banach theorem Λ_0 can be extended to $\Lambda \in C([0,b])'$ which has the property that $\Lambda(X) = 0$ but $\Lambda(f) = ||f|| \neq 0$. By the Weierstrass approximation theorem, Theorem 7.6 or one of its cases, there exists a polynomial, p such that $\Lambda(p) \neq 0$. Therefore, if it can be shown that whenever $\Lambda(X) = 0$, it is the case that $\Lambda(p) = 0$ for all polynomials, it must be the case that X is dense in $C([0,b])$.

By the Riesz representation theorem the elements of $C([0,b])'$ are complex measures. Suppose then that for μ a complex measure it follows that for all t^{λ_k},

$$\int_{[0,b]} t^{\lambda_k} d\mu = 0.$$

I want to show that then

$$\int_{[0,b]} t^k d\mu = 0$$

for all positive integers. It suffices to modify μ is necessary to have $\mu(\{0\}) = 0$ since this will not change any of the above integrals. Let $\mu_1(E) = \mu(E \cap (0,b])$ and use μ_1. I will continue using the symbol, μ.

For $\operatorname{Re}(z) > 0$, define

$$F(z) \equiv \int_{[0,b]} t^z d\mu = \int_{(0,b]} t^z d\mu$$

The function $t^z = \exp(z \ln(t))$ is analytic. I claim that $F(z)$ is also analytic for $\operatorname{Re} z > 0$. Apply Morea's theorem. Let T be a triangle in $\operatorname{Re} z > 0$. Then

$$\int_{\partial T} F(z) \, dz = \int_{\partial T} \int_{(0,b]} e^{(z \ln(t))} \xi d|\mu| \, dz$$

Now $\int_{\partial T}$ can be split into three integrals over intervals of $\mathbb{R}$ and so this integral is essentially a Lebesgue integral taken with respect to Lebesgue measure. Furthermore, $e^{(z\ln(t))}$ is a continuous function of the two variables and ξ is a function of only the one variable, t. Thus the integrand is product measurable. The iterated integral is also absolutely integrable because $\left|e^{(z\ln(t))}\right| \leq e^{x\ln t} \leq e^{x\ln b}$ where $x+iy=z$ and x is given to be positive. Thus the integrand is actually bounded. Therefore, you can apply Fubini's theorem and write

$$\begin{aligned}\int_{\partial T} F(z)\,dz &= \int_{\partial T}\int_{(0,b]} e^{(z\ln(t))}\xi d|\mu|\,dz \\ &= \int_{(0,b]}\xi\int_{\partial T} e^{(z\ln(t))}dzd|\mu| = 0.\end{aligned}$$

By Morea's theorem, F is analytic on $\operatorname{Re} z>0$ which is given to have zeros at the λ_k.

Now let $\phi(z)=\frac{1+z}{1-z}$. Then ϕ maps $B(0,1)$ one to one onto $\operatorname{Re} z>0$. To see this let $0<r<1$.

$$\phi\left(re^{i\theta}\right)=\frac{1+re^{i\theta}}{1-re^{i\theta}}=\frac{1-r^2+i2r\sin\theta}{1+r^2-2r\cos\theta}$$

and so $\operatorname{Re}\phi\left(re^{i\theta}\right)>0$. Now the inverse of ϕ is $\phi^{-1}(z)=\frac{z-1}{z+1}$. For $\operatorname{Re} z>0$,

$$\left|\phi^{-1}(z)\right|^2=\frac{z-1}{z+1}\cdot\frac{\overline{z}-1}{\overline{z}+1}=\frac{|z|^2-2\operatorname{Re} z+1}{|z|^2+2\operatorname{Re} z+1}<1.$$

Consider $F\circ\phi$, an analytic function defined on $B(0,1)$. This function is given to have zeros at z_n where $\phi(z_n)=\frac{1+z_n}{1-z_n}=\lambda_n$. This reduces to $z_n=\frac{-1+\lambda_n}{1+\lambda_n}$. Now

$$1-|z_n|\geq\frac{c}{1+\lambda_n}$$

for a positive constant, c. It is given that $\sum\frac{1}{\lambda_n}=\infty$. so it follows $\sum(1-|z_n|)=\infty$ also. Therefore, by Corollary 29.29, $F\circ\phi=0$. It follows $F=0$ also. In particular, $F(k)$ for k a positive integer equals zero. This has shown that if $\Lambda\in C([0,b])'$ and Λ sends 1 and all the t^{λ_n} to 0, then Λ sends 1 and all t^k for k a positive integer to zero. As explained above, X is dense in $C((0,b])$.

The converse of this theorem is also true and is proved in Rudin [45].

29.6 Exercises

1. Suppose f is an entire function with $f(0)=1$. Let

$$M(r)=\max\{|f(z)|:|z|=r\}.$$

Use Jensen's equation to establish the following inequality.

$$M(2r)\geq 2^{n(r)}$$

where $n(r)$ is the number of zeros of f in $\overline{B(0,r)}$.

2. The version of the Blaschke product presented above is that found in most complex variable texts. However, there is another one in [37]. Instead of $\frac{\alpha_n - z}{1-\overline{\alpha_n} z}\frac{|\alpha_n|}{\alpha_n}$ you use
$$\frac{\alpha_n - z}{\frac{1}{\overline{\alpha_n}} - z}$$
Prove a version of Theorem 29.26 using this modification.

3. The Weierstrass approximation theorem holds for polynomials of n variables on any compact subset of $\mathbb{R}^n$. Give a multidimensional version of the Müntz-Szasz theorem which will generalize the Weierstrass approximation theorem for n dimensions. You might just pick a compact subset of $\mathbb{R}^n$ in which all components are positive. You have to do something like this because otherwise, t^λ might not be defined.

4. Show $\cos(\pi z) = \prod_{k=1}^{\infty}\left(1 - \frac{4z^2}{(2k-1)^2}\right)$.

5. Recall $\sin(\pi z) = z\pi \prod_{n=1}^{\infty}\left(1 - \left(\frac{z}{n}\right)^2\right)$. Use this to derive Wallis product, $\frac{\pi}{2} = \prod_{k=1}^{\infty}\frac{4k^2}{(2k-1)(2k+1)}$.

6. The order of an entire function, f is defined as
$$\inf\left\{a \geq 0 : |f(z)| \leq e^{|z|^a} \text{ for all large enough } |z|\right\}$$
If no such a exists, the function is said to be of infinite order. Show the order of an entire function is also equal to $\limsup_{r\to\infty}\frac{\ln(\ln(M(r)))}{\ln(r)}$ where $M(r) \equiv \max\{|f(z)| : |z| = r\}$.

7. Suppose Ω is a simply connected region and let f be meromorphic on Ω. Suppose also that the set, $S \equiv \{z \in \Omega : f(z) = c\}$ has a limit point in Ω. Can you conclude $f(z) = c$ for all $z \in \Omega$?

8. This and the next collection of problems are dealing with the gamma function. Show that
$$\left|\left(1+\frac{z}{n}\right)e^{\frac{-z}{n}} - 1\right| \leq \frac{C(z)}{n^2}$$
and therefore,
$$\sum_{n=1}^{\infty}\left|\left(1+\frac{z}{n}\right)e^{\frac{-z}{n}} - 1\right| < \infty$$
with the convergence uniform on compact sets.

9. ↑ Show $\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{\frac{-z}{n}}$ converges to an analytic function on $\mathbb{C}$ which has zeros only at the negative integers and that therefore,
$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{\frac{z}{n}}$$

is a meromorphic function (Analytic except for poles) having simple poles at the negative integers.

10. ↑Show there exists γ such that if

$$\Gamma(z) \equiv \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}},$$

then $\Gamma(1) = 1$. Thus Γ is a meromorphic function having simple poles at the negative integers. **Hint:** $\prod_{n=1}^{\infty} (1+n) e^{-1/n} = c = e^{\gamma}$.

11. ↑Now show that

$$\gamma = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln n\right]$$

12. ↑Justify the following argument leading to Gauss's formula

$$\Gamma(z) = \lim_{n\to\infty} \left(\prod_{k=1}^{n} \left(\frac{k}{k+z}\right) e^{\frac{z}{k}}\right) \frac{e^{-\gamma z}}{z}$$

$$\begin{aligned}
&= \lim_{n\to\infty} \left(\frac{n!}{(1+z)(2+z)\cdots(n+z)} e^{z\left(\sum_{k=1}^{n} \frac{1}{k}\right)}\right) \frac{e^{-\gamma z}}{z} \\
&= \lim_{n\to\infty} \frac{n!}{(1+z)(2+z)\cdots(n+z)} e^{z\left(\sum_{k=1}^{n} \frac{1}{k}\right)} e^{-z\left[\sum_{k=1}^{n} \frac{1}{k} - \ln n\right]} \\
&= \lim_{n\to\infty} \frac{n! n^z}{(1+z)(2+z)\cdots(n+z)}.
\end{aligned}$$

13. ↑ Verify from the Gauss formula above that $\Gamma(z+1) = \Gamma(z) z$ and that for n a nonnegative integer, $\Gamma(n+1) = n!$.

14. ↑ The usual definition of the gamma function for positive x is

$$\Gamma_1(x) \equiv \int_0^{\infty} e^{-t} t^{x-1} dt.$$

Show $\left(1 - \frac{t}{n}\right)^n \leq e^{-t}$ for $t \in [0, n]$. Then show

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1} dt = \frac{n! n^x}{x(x+1)\cdots(x+n)}.$$

Use the first part to conclude that

$$\Gamma_1(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\cdots(x+n)} = \Gamma(x).$$

Hint: To show $\left(1 - \frac{t}{n}\right)^n \leq e^{-t}$ for $t \in [0, n]$, verify this is equivalent to showing $(1-u)^n \leq e^{-nu}$ for $u \in [0, 1]$.

15. ↑Show $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$. whenever $\operatorname{Re} z > 0$. **Hint:** You have already shown that this is true for positive real numbers. Verify this formula for $\operatorname{Re} z > 0$ yields an analytic function.

16. ↑Show $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Then find $\Gamma\left(\frac{5}{2}\right)$.

17. Show that $\int_{-\infty}^{\infty} e^{\frac{-s^2}{2}}ds = \sqrt{2\pi}$. **Hint:** Denote this integral by I and observe that $I^2 = \int_{\mathbb{R}^2} e^{-\left(x^2+y^2\right)/2}dxdy$. Then change variables to polar coordinates, $x = r\cos(\theta)$, $y = r\sin\theta$.

18. ↑ Now that you know what the gamma function is, consider in the formula for $\Gamma(\alpha+1)$ the following change of variables. $t = \alpha + \alpha^{1/2}s$. Then in terms of the new variable, s, the formula for $\Gamma(\alpha+1)$ is

$$e^{-\alpha}\alpha^{\alpha+\frac{1}{2}}\int_{-\sqrt{\alpha}}^{\infty} e^{-\sqrt{\alpha}s}\left(1+\frac{s}{\sqrt{\alpha}}\right)^{\alpha}ds$$

$$= e^{-\alpha}\alpha^{\alpha+\frac{1}{2}}\int_{-\sqrt{\alpha}}^{\infty} e^{\alpha\left[\ln\left(1+\frac{s}{\sqrt{\alpha}}\right)-\frac{s}{\sqrt{\alpha}}\right]}ds$$

Show the integrand converges to $e^{-\frac{s^2}{2}}$. Show that then

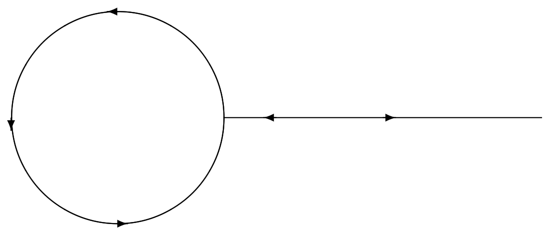
$$\lim_{\alpha\to\infty}\frac{\Gamma(\alpha+1)}{e^{-\alpha}\alpha^{\alpha+(1/2)}} = \int_{-\infty}^{\infty} e^{\frac{-s^2}{2}}ds = \sqrt{2\pi}.$$

Hint: You will need to obtain a dominating function for the integral so that you can use the dominated convergence theorem. You might try considering $s \in (-\sqrt{\alpha}, \sqrt{\alpha})$ first and consider something like $e^{1-\left(s^2/4\right)}$ on this interval. Then look for another function for $s > \sqrt{\alpha}$. This formula is known as Stirling's formula.

19. This and the next several problems develop the zeta function and give a relation between the zeta and the gamma function. Define for $0 < r < 2\pi$

$$\begin{aligned} I_r(z) \equiv & \int_0^{2\pi}\frac{e^{(z-1)(\ln r+i\theta)}}{e^{re^{i\theta}}-1}ire^{i\theta}d\theta + \int_r^{\infty}\frac{e^{(z-1)(\ln t+2\pi i)}}{e^t-1}dt \quad (29.24)\\ & + \int_{\infty}^{r}\frac{e^{(z-1)\ln t}}{e^t-1}dt \end{aligned}$$

Show that I_r is an entire function. The reason $0 < r < 2\pi$ is that this prevents $e^{re^{i\theta}} - 1$ from equaling zero. The above is just a precise description of the contour integral, $\int_\gamma \frac{w^{z-1}}{e^w-1}dw$ where γ is the contour shown below.

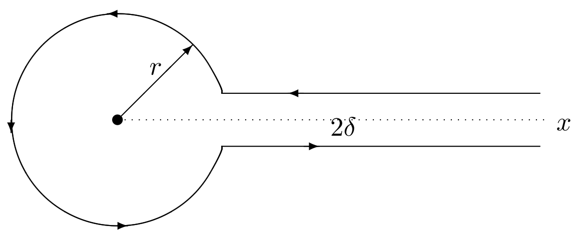


in which on the integrals along the real line, the argument is different in going from r to ∞ than it is in going from ∞ to r. Now I have not defined such contour integrals over contours which have infinite length and so have chosen to simply write out explicitly what is involved. You have to work with these integrals given above anyway but the contour integral just mentioned is the motivation for them. **Hint:** You may want to use convergence theorems from real analysis if it makes this more convenient but you might not have to.

20. $\uparrow$ In the context of Problem 19 define for small $\delta > 0$

$$I_{r\delta}(z) \equiv \int_{\gamma_{r,\delta}} \frac{w^{z-1}}{e^w - 1} dw$$

where $\gamma_{r\delta}$ is shown below.

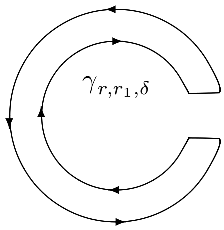


Show that $\lim_{\delta \to 0} I_{r\delta}(z) = I_r(z)$. **Hint:** Use the dominated convergence theorem if it makes this go easier. This is not a hard problem if you use these theorems but you can probably do it without them with more work.

21. $\uparrow$ In the context of Problem 20 show that for $r_1 < r$, $I_{r\delta}(z) - I_{r_1\delta}(z)$ is a contour integral,

$$\int_{\gamma_{r,r_1,\delta}} \frac{w^{z-1}}{e^w - 1} dw$$

where the oriented contour is shown below.



In this contour integral, w^{z-1} denotes $e^{(z-1)\log(w)}$ where $\log(w) = \ln|w| + i\arg(w)$ for $\arg(w) \in (0, 2\pi)$. Explain why this integral equals zero. From Problem 20 it follows that $I_r = I_{r_1}$. Therefore, you can define an entire function, $I(z) \equiv I_r(z)$ for all r positive but sufficiently small. **Hint:** Remember the Cauchy integral formula for analytic functions defined on simply connected regions. You could argue there is a simply connected region containing $\gamma_{r,r_1,\delta}$.

22. ↑ In case $\operatorname{Re} z > 1$, you can get an interesting formula for $I(z)$ by taking the limit as $r \to 0$. Recall that

$$\begin{aligned} I_r(z) &\equiv \int_0^{2\pi} \frac{e^{(z-1)(\ln r + i\theta)}}{e^{re^{i\theta}} - 1} ire^{i\theta} d\theta + \int_r^\infty \frac{e^{(z-1)(\ln t + 2\pi i)}}{e^t - 1} dt \qquad (29.25) \\ &+ \int_\infty^r \frac{e^{(z-1)\ln t}}{e^t - 1} dt \end{aligned}$$

and now it is desired to take a limit in the case where $\operatorname{Re} z > 1$. Show the first integral above converges to 0 as $r \to 0$. Next argue the sum of the two last integrals converges to

$$\left(e^{(z-1)2\pi i} - 1\right) \int_0^\infty \frac{e^{(z-1)\ln(t)}}{e^t - 1} dt.$$

Thus

$$I(z) = \left(e^{z2\pi i} - 1\right) \int_0^\infty \frac{e^{(z-1)\ln(t)}}{e^t - 1} dt \qquad (29.26)$$

when $\operatorname{Re} z > 1$.

23. ↑ So what does all this have to do with the zeta function and the gamma function? The zeta function is defined for $\operatorname{Re} z > 1$ by

$$\sum_{n=1}^\infty \frac{1}{n^z} \equiv \zeta(z).$$

By Problem 15, whenever $\operatorname{Re} z > 0$,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Change the variable and conclude

$$\Gamma(z)\frac{1}{n^z}=\int_0^\infty e^{-ns}s^{z-1}ds.$$

Therefore, for $\operatorname{Re} z>1$,

$$\zeta(z)\Gamma(z)=\sum_{n=1}^\infty\int_0^\infty e^{-ns}s^{z-1}ds.$$

Now show that you can interchange the order of the sum and the integral. This is possibly most easily done by using Fubini's theorem. Show that $\sum_{n=1}^\infty\int_0^\infty\left|e^{-ns}s^{z-1}\right|ds<\infty$ and then use Fubini's theorem. I think you could do it other ways though. It is possible to do it without any reference to Lebesgue integration. Thus

$$\begin{aligned}\zeta(z)\Gamma(z) &= \int_0^\infty s^{z-1}\sum_{n=1}^\infty e^{-ns}ds\\ &= \int_0^\infty\frac{s^{z-1}e^{-s}}{1-e^{-s}}ds=\int_0^\infty\frac{s^{z-1}}{e^s-1}ds\end{aligned}$$

By 29.26,

$$\begin{aligned}I(z) &= \left(e^{z2\pi i}-1\right)\int_0^\infty\frac{e^{(z-1)\ln(t)}}{e^t-1}dt\\ &= \left(e^{z2\pi i}-1\right)\zeta(z)\Gamma(z)\\ &= \left(e^{2\pi iz}-1\right)\zeta(z)\Gamma(z)\end{aligned}$$

whenever $\operatorname{Re} z>1$.

24. ↑ Now show there exists an entire function, $h(z)$ such that

$$\zeta(z)=\frac{1}{z-1}+h(z)$$

for $\operatorname{Re} z>1$. Conclude $\zeta(z)$ extends to a meromorphic function defined on all of $\mathbb{C}$ which has a simple pole at $z=1$, namely, the right side of the above formula. **Hint:** Use Problem 10 to observe that $\Gamma(z)$ is never equal to zero but has simple poles at every nonnegative integer. Then for $\operatorname{Re} z>1$,

$$\zeta(z)\equiv\frac{I(z)}{(e^{2\pi iz}-1)\Gamma(z)}.$$

By 29.26 ζ has no poles for $\operatorname{Re} z>1$. The right side of the above equation is defined for all z. There are no poles except possibly when z is a nonnegative integer. However, these points are not poles either because of Problem 10 which states that Γ has simple poles at these points thus cancelling the simple zeros of $\left(e^{2\pi iz}-1\right)$. The only remaining possibility for a pole for ζ is at $z=1$. Show it has a simple pole at this point. You can use the formula for $I(z)$

$$\begin{aligned} I(z) &\equiv \int_0^{2\pi} \frac{e^{(z-1)(\ln r+i\theta)}}{e^{re^{i\theta}}-1} ire^{i\theta} d\theta + \int_r^{\infty} \frac{e^{(z-1)(\ln t+2\pi i)}}{e^t-1} dt \quad (29.27) \\ &+ \int_{\infty}^{r} \frac{e^{(z-1)\ln t}}{e^t-1} dt \end{aligned}$$

Thus $I(1)$ is given by

$$I(1) \equiv \int_0^{2\pi} \frac{1}{e^{re^{i\theta}}-1} ire^{i\theta} d\theta + \int_r^{\infty} \frac{1}{e^t-1} dt + \int_{\infty}^{r} \frac{1}{e^t-1} dt$$

$= \int_{\gamma_r} \frac{dw}{e^w-1}$ where γ_r is the circle of radius r. This contour integral equals $2\pi i$ by the residue theorem. Therefore,

$$\frac{I(z)}{\left(e^{2\pi iz}-1\right)\Gamma(z)} = \frac{1}{z-1} + h(z)$$

where $h(z)$ is an entire function. People worry a lot about where the zeros of ζ are located. In particular, the zeros for $\operatorname{Re} z \in (0,1)$ are of special interest. The Riemann hypothesis says they are all on the line $\operatorname{Re} z = 1/2$. This is a good problem for you to do next.

25. There is an important relation between prime numbers and the zeta function due to Euler. Let $\{p_n\}_{n=1}^{\infty}$ be the prime numbers. Then for $\operatorname{Re} z > 1$,

$$\prod_{n=1}^{\infty} \frac{1}{1-p_n^{-z}} = \zeta(z).$$

To see this, consider a partial product.

$$\prod_{n=1}^{N} \frac{1}{1-p_n^{-z}} = \prod_{n=1}^{N} \sum_{j_n=1}^{\infty} \left(\frac{1}{p_n^z}\right)^{j_n}.$$

Let S_N denote all positive integers which use only $p_1, \cdots, p_N$ in their prime factorization. Then the above equals $\sum_{n\in S_N} \frac{1}{n^z}$. Letting $N \to \infty$ and using the fact that $\operatorname{Re} z > 1$ so that the order in which you sum is not important (See Theorem 30.1 or recall advanced calculus.) you obtain the desired equation. Show $\sum_{n=1}^{\infty} \frac{1}{p_n} = \infty$.

Elliptic Functions

This chapter is to give a short introduction to elliptic functions. There is much more available. There are books written on elliptic functions. What I am presenting here follows Alfors [2] although the material is found in many books on complex analysis. Hille, [27] has a much more extensive treatment than what I will attempt here. There are also many references and historical notes available in the book by Hille. Another good source for more having much the same emphasis as what is presented here is in the book by Saks and Zygmund [47]. This is a very interesting subject because it has considerable overlap with algebra.

Before beginning, recall that an absolutely convergent series can be summed in any order and you always get the same answer. The easy way to see this is to think of the series as a Lebesgue integral with respect to counting measure and apply convergence theorems as needed. The following theorem provides the necessary results.

Theorem 30.1 *Suppose $\sum_{n=1}^{\infty} |a_n| < \infty$ and let $\theta, \phi : \mathbb{N} \to \mathbb{N}$ be one to one and onto mappings. Then $\sum_{n=1}^{\infty} a_{\phi(n)}$ and $\sum_{n=1}^{\infty} a_{\theta(n)}$ both converge and the two sums are equal.*

Proof: By the monotone convergence theorem,

$$\sum_{n=1}^{\infty} |a_n| = \lim_{n\to\infty} \sum_{k=1}^{n} \left|a_{\phi(k)}\right| = \lim_{n\to\infty} \sum_{k=1}^{n} \left|a_{\theta(k)}\right|$$

but these last two equal $\sum_{k=1}^{\infty} \left|a_{\phi(k)}\right|$ and $\sum_{k=1}^{\infty} \left|a_{\theta(k)}\right|$ respectively. Therefore, $\sum_{k=1}^{\infty} a_{\theta(k)}$ and $\sum_{k=1}^{\infty} a_{\phi(k)}$ exist ($n \to a_{\theta(n)}$ is in L^1 with respect to counting measure.) It remains to show the two are equal. There exists M such that if $n > M$ then

$$\sum_{k=n+1}^{\infty} \left|a_{\theta(k)}\right| < \varepsilon, \quad \sum_{k=n+1}^{\infty} \left|a_{\phi(k)}\right| < \varepsilon$$

$$\left|\sum_{k=1}^{\infty} a_{\phi(k)} - \sum_{k=1}^{n} a_{\phi(k)}\right| < \varepsilon, \quad \left|\sum_{k=1}^{\infty} a_{\theta(k)} - \sum_{k=1}^{n} a_{\theta(k)}\right| < \varepsilon$$

Pick such an n denoted by n_1. Then pick $n_2 > n_1 > M$ such that

$$\{\theta(1), \cdots, \theta(n_1)\} \subseteq \{\phi(1), \cdots, \phi(n_2)\}.$$

Then

$$\sum_{k=1}^{n_2} a_{\phi(k)} = \sum_{k=1}^{n_1} a_{\theta(k)} + \sum_{\phi(k) \notin \{\theta(1),\cdots,\theta(n_1)\}} a_{\phi(k)}.$$

Therefore,

$$\left|\sum_{k=1}^{n_2} a_{\phi(k)} - \sum_{k=1}^{n_1} a_{\theta(k)}\right| = \left|\sum_{\phi(k) \notin \{\theta(1),\cdots,\theta(n_1)\}, k \leq n_2} a_{\phi(k)}\right|$$

Now all of these $\phi(k)$ in the last sum are contained in $\{\theta(n_1+1), \cdots\}$ and so the last sum above is dominated by

$$\leq \sum_{k=n_1+1}^{\infty} \left|a_{\theta(k)}\right| < \varepsilon.$$

Therefore,

$$\begin{aligned}
\left|\sum_{k=1}^{\infty} a_{\phi(k)} - \sum_{k=1}^{\infty} a_{\theta(k)}\right| &\leq \left|\sum_{k=1}^{\infty} a_{\phi(k)} - \sum_{k=1}^{n_2} a_{\phi(k)}\right| \\
&\quad + \left|\sum_{k=1}^{n_2} a_{\phi(k)} - \sum_{k=1}^{n_1} a_{\theta(k)}\right| \\
+ \left|\sum_{k=1}^{n_1} a_{\theta(k)} - \sum_{k=1}^{\infty} a_{\theta(k)}\right| &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon
\end{aligned}$$

and since ε is arbitrary, it follows $\sum_{k=1}^{\infty} a_{\phi(k)} = \sum_{k=1}^{\infty} a_{\theta(k)}$ as claimed. This proves the theorem.

30.1 Periodic Functions

Definition 30.2 *A function defined on $\mathbb{C}$ is said to be periodic if there exists w such that $f(z+w) = f(z)$ for all $z \in \mathbb{C}$. Denote by M the set of all periods. Thus if $w_1, w_2 \in M$ and $a, b \in \mathbb{Z}$, then $aw_1 + bw_2 \in M$. For this reason M is called the module of periods.*[1] *In all which follows it is assumed f is meromorphic.*

Theorem 30.3 *Let f be a meromorphic function and let M be the module of periods. Then if M has a limit point, then f equals a constant. If this does not happen then either there exists $w_1 \in M$ such that $\mathbb{Z}w_1 = M$ or there exist $w_1, w_2 \in M$ such that $M = \{aw_1 + bw_2 : a, b \in \mathbb{Z}\}$ and w_1/w_2 is not real. Also if $\tau = w_2/w_1$,*

$$|\tau| \geq 1, \ \frac{-1}{2} \leq \operatorname{Re} \tau \leq \frac{1}{2}.$$

[1] A module is like a vector space except instead of a field of scalars, you have a ring of scalars.

Proof: Suppose f is meromorphic and M has a limit point, w_0. By Theorem 29.10 on Page 794 there exist analytic functions, p, q such that $f(z) = \frac{p(z)}{q(z)}$. Now pick z_0 such that z_0 is not a pole of f. Then letting $w_n \to w_0$ where $\{w_n\} \subseteq M$, $f(z_0 + w_n) = f(z_0)$. Therefore, $p(z_0 + w_n) = f(z_0) q(z_0 + w_n)$ and so the analytic function, $p(z) - f(z_0) q(z)$ has a zero set which has a limit point. Therefore, this function is identically equal to zero because of Theorem 24.23 on Page 657. Thus f equals a constant as claimed.

This has shown that if f is not constant, then M is discreet. Therefore, there exists $w_1 \in M$ such that $|w_1| = \min\{|w| : w \in M\}$. Suppose first that every element of M is a real multiple of w_1. Thus, if $w \in M$, it follows there exists a real number, x such that $w = xw_1$. Then there exist positive integers, $k, k+1$ such that $k \leq x < k+1$. If $x > k$, then $w - kw_1 = (x-k) w_1$ is a period having smaller absolute value than $|w_1|$ which would be a contradiction. Hence, $x = k$ and so $M = \mathbb{Z}w_1$.

Now suppose there exists $w_2 \in M$ which is not a real multiple of w_1. You can let w_2 be the element of M having this property which has smallest absolute value. Now let $w \in M$. Since w_1 and w_2 point in different directions, it follows $w = xw_1 + yw_2$ for some real numbers, x, y. Let $|m - x| \leq \frac{1}{2}$ and $|n - y| \leq \frac{1}{2}$ where m, n are integers. Therefore,

$$w = mw_1 + nw_2 + (x-m) w_1 + (y-n) w_2$$

and so

$$w - mw_1 - nw_2 = (x-m) w_1 + (y-n) w_2 \tag{30.1}$$

Now since $w_2/w_1 \notin \mathbb{R}$,

$$\begin{aligned} |(x-m) w_1 + (y-n) w_2| &< |(x-m) w_1| + |(y-n) w_2| \\ &= \frac{1}{2}|w_1| + \frac{1}{2}|w_2|. \end{aligned}$$

Therefore, from 30.1,

$$\begin{aligned} |w - mw_1 - nw_2| &= |(x-m) w_1 + (y-n) w_2| \\ &< \frac{1}{2}|w_1| + \frac{1}{2}|w_2| \leq |w_2| \end{aligned}$$

and so the period, $w - mw_1 - nw_2$ cannot be a non real multiple of w_1 because w_2 is the one which has smallest absolute value and this period has smaller absolute value than w_2. Therefore, the ratio $w - mw_1 - nw_2/w_1$ must be a real number, x. Thus

$$w - mw_1 - nw_2 = xw_1$$

Since w_1 has minimal absolute value of all periods, it follows $|x| \geq 1$. Let $k \leq x < k+1$ for some integer, k. If $x > k$, then

$$w - mw_1 - nw_2 - kw_1 = (x-k) w_1$$

which would contradict the choice of w_1 as being the period having minimal absolute value because the expression on the left in the above is a period and it equals something which has absolute value less than $|w_1|$. Therefore, $x = k$ and w is an integer linear combination of w_1 and w_2. It only remains to verify the claim about τ.

From the construction, $|w_1| \leq |w_2|$ and $|w_2| \leq |w_1 - w_2|, |w_2| \leq |w_1 + w_2|$. Therefore,

$$|\tau| \geq 1, \ |\tau| \leq |1 - \tau|, \ |\tau| \leq |1 + \tau|.$$

The last two of these inequalities imply $-1/2 \leq \operatorname{Re} \tau \leq 1/2$.

This proves the theorem.

Definition 30.4 *For f a meromorphic function which has the last of the above alternatives holding in which $M = \{aw_1 + bw_2 : a, b \in \mathbb{Z}\}$, the function, f is called elliptic. This is also called doubly periodic.*

Theorem 30.5 *Suppose f is an elliptic function which has no poles. Then f is constant.*

Proof: Since f has no poles it is analytic. Now consider the parallelograms determined by the vertices, $mw_1 + nw_2$ for $m, n \in \mathbb{Z}$. By periodicity of f it must be bounded because its values are identical on each of these parallelograms. Therefore, it equals a constant by Liouville's theorem.

Definition 30.6 *Define P_a to be the parallelogram determined by the points*

$$a + mw_1 + nw_2, a + (m+1) w_1 + nw_2, a + mw_1 + (n+1) w_2,$$
$$a + (m+1) w_1 + (n+1) w_2$$

Such P_a will be referred to as a period parallelogram. The sum of the orders of the poles in a period parallelogram which contains no poles or zeros of f on its boundary is called the order of the function. This is well defined because of the periodic property of f.

Theorem 30.7 *The sum of the residues of any elliptic function, f equals zero on every P_a if a is chosen so that there are no poles on ∂P_a.*

Proof: Choose a such that there are no poles of f on the boundary of P_a. By periodicity,

$$\int_{\partial P_a} f(z)\, dz = 0$$

because the integrals over opposite sides of the parallelogram cancel out because the values of f are the same on these sides and the orientations are opposite. It follows from the residue theorem that the sum of the residues in P_a equals 0.

Theorem 30.8 *Let P_a be a period parallelogram for a nonconstant elliptic function, f which has order equal to m. Then f assumes every value in $f(P_a)$ exactly m times.*

Proof: Let $c \in f(P_a)$ and consider $P_{a'}$ such that $f^{-1}(c) \cap P_{a'} = f^{-1}(c) \cap P_a$ and $P_{a'}$ contains the same poles and zeros of $f - c$ as P_a but $P_{a'}$ has no zeros of $f(z) - c$ or poles of f on its boundary. Thus $f'(z) / (f(z) - c)$ is also an elliptic function and so Theorem 30.7 applies. Consider

$$\frac{1}{2\pi i} \int_{\partial P_{a'}} \frac{f'(z)}{f(z) - c} dz.$$

By the argument principle, this equals $N_z - N_p$ where N_z equals the number of zeros of $f(z) - c$ and N_p equals the number of the poles of $f(z)$. From Theorem 30.7 this must equal zero because it is the sum of the residues of $f' / (f - c)$ and so $N_z = N_p$. Now N_p equals the number of poles in P_a counted according to multiplicity.

There is an even better theorem than this one.

Theorem 30.9 *Let f be a non constant elliptic function and suppose it has poles $p_1, \cdots, p_m$ and zeros, $z_1, \cdots, z_m$ in P_α, listed according to multiplicity where ∂P_α contains no poles or zeros of f. Then $\sum_{k=1}^m z_k - \sum_{k=1}^m p_k \in M$, the module of periods.*

Proof: You can assume ∂P_a contains no poles or zeros of f because if it did, then you could consider a slightly shifted period parallelogram, $P_{a'}$ which contains no new zeros and poles but which has all the old ones but no poles or zeros on its boundary. By Theorem 26.8 on Page 710

$$\frac{1}{2\pi i} \int_{\partial P_a} z \frac{f'(z)}{f(z)} dz = \sum_{k=1}^m z_k - \sum_{k=1}^m p_k. \tag{30.2}$$

Denoting by $\gamma(z, w)$ the straight oriented line segment from z to w,

$$\begin{aligned} & \int_{\partial P_a} z \frac{f'(z)}{f(z)} dz \\ = & \int_{\gamma(a, a+w_1)} z \frac{f'(z)}{f(z)} dz + \int_{\gamma(a+w_1+w_2, a+w_2)} z \frac{f'(z)}{f(z)} dz \\ & + \int_{\gamma(a+w_1, a+w_2+w_1)} z \frac{f'(z)}{f(z)} dz + \int_{\gamma(a+w_2, a)} z \frac{f'(z)}{f(z)} dz \end{aligned}$$

$$\begin{aligned} = & \int_{\gamma(a, a+w_1)} (z - (z + w_2)) \frac{f'(z)}{f(z)} dz \\ & + \int_{\gamma(a, a+w_2)} (z - (z + w_1)) \frac{f'(z)}{f(z)} dz \end{aligned}$$

Now near these line segments $\frac{f'(z)}{f(z)}$ is analytic and so there exists a primitive, $g_{w_i}(z)$ on $\gamma(a, a + w_i)$ by Corollary 24.32 on Page 663 which satisfies $e^{g_{w_i}(z)} = f(z)$. Therefore,

$$= -w_2 (g_{w_1}(a + w_1) - g_{w_1}(a)) - w_1 (g_{w_2}(a + w_2) - g_{w_2}(a)).$$

Now by periodicity of f it follows $f(a+w_1) = f(a) = f(a+w_2)$. Hence

$$g_{w_i}(a+w_1) - g_{w_i}(a) = 2m\pi i$$

for some integer, m because

$$e^{g_{w_i}(a+w_i)} - e^{g_{w_i}(a)} = f(a+w_i) - f(a) = 0.$$

Therefore, from 30.2, there exist integers, k, l such that

$$\begin{aligned}
& \frac{1}{2\pi i}\int_{\partial P_a} z\frac{f'(z)}{f(z)}dz \\
= & \frac{1}{2\pi i}\left[-w_2\left(g_{w_1}(a+w_1) - g_{w_1}(a)\right) - w_1\left(g_{w_2}(a+w_2) - g_{w_2}(a)\right)\right] \\
= & \frac{1}{2\pi i}\left[-w_2(2k\pi i) - w_1(2l\pi i)\right] \\
= & -w_2k - w_1l \in M.
\end{aligned}$$

From 30.2 it follows

$$\sum_{k=1}^{m} z_k - \sum_{k=1}^{m} p_k \in M.$$

This proves the theorem.

Hille says this relation is due to Liouville. There is also a simple corollary which follows from the above theorem applied to the elliptic function, $f(z) - c$.

Corollary 30.10 *Let f be a non constant elliptic function and suppose the function, $f(z) - c$ has poles $p_1, \cdots, p_m$ and zeros, $z_1, \cdots, z_m$ on P_α, listed according to multiplicity where ∂P_α contains no poles or zeros of $f(z) - c$. Then $\sum_{k=1}^{m} z_k - \sum_{k=1}^{m} p_k \in M$, the module of periods.*

30.1.1 The Unimodular Transformations

Definition 30.11 *Suppose f is a nonconstant elliptic function and the module of periods is of the form $\{aw_1 + bw_2\}$ where a, b are integers and w_1/w_2 is not real. Then by analogy with linear algebra, $\{w_1, w_2\}$ is referred to as a basis. The unimodular transformations will refer to matrices of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where all entries are integers and

$$ad - bc = \pm 1.$$

These linear transformations are also called the modular group.

The following is an interesting lemma which ties matrices with the fractional linear transformations.

Lemma 30.12 *Define*

$$\phi\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right) \equiv \frac{az+b}{cz+d}.$$

Then

$$\phi(AB) = \phi(A) \circ \phi(B), \tag{30.3}$$

$\phi(A)(z) = z$ *if and only if*

$$A = cI$$

where I is the identity matrix and $c \neq 0$. Also if $f(z) = \frac{az+b}{cz+d}$, then $f^{-1}(z)$ exists if and only if $ad - cb \neq 0$. Furthermore it is easy to find f^{-1}.

Proof: The equation in 30.3 is just a simple computation. Now suppose $\phi(A)(z) = z$. Then letting $A = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$, this requires

$$az + b = z(cz + d)$$

and so $az + b = cz^2 + dz$. Since this is to hold for all z it follows $c = 0 = b$ and $a = d$. The other direction is obvious.

Consider the claim about the existence of an inverse. Let $ad - cb \neq 0$ for $f(z) = \frac{az+b}{cz+d}$. Then

$$f(z) = \phi\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right)$$

It follows $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)^{-1}$ exists and equals $\frac{1}{ad-bc}\left(\begin{array}{cc} d & -b \\ -c & a \end{array}\right)$. Therefore,

$$\begin{aligned} z &= \phi(I)(z) = \phi\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\left(\frac{1}{ad-bc}\left(\begin{array}{cc} d & -b \\ -c & a \end{array}\right)\right)\right)(z) \\ &= \phi\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\right) \circ \phi\left(\left(\frac{1}{ad-bc}\left(\begin{array}{cc} d & -b \\ -c & a \end{array}\right)\right)\right)(z) \\ &= f \circ f^{-1}(z) \end{aligned}$$

which shows f^{-1} exists and it is easy to find.

Next suppose f^{-1} exists. I need to verify the condition $ad - cb \neq 0$. If f^{-1} exists, then from the process used to find it, you see that it must be a fractional linear transformation. Letting $A = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$ so $\phi(A) = f$, it follows there exists a matrix B such that

$$\phi(BA)(z) = \phi(B) \circ \phi(A)(z) = z.$$

However, it was shown that this implies BA is a nonzero multiple of I which requires that A^{-1} must exist. Hence the condition must hold.

Theorem 30.13 *If f is a nonconstant elliptic function with a basis $\{w_1, w_2\}$ for the module of periods, then $\{w_1', w_2'\}$ is another basis, if and only if there exists a unimodular transformation,* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$ *such that*

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{30.4}$$

Proof: Since $\{w_1, w_2\}$ is a basis, there exist integers, a, b, c, d such that 30.4 holds. It remains to show the transformation determined by the matrix is unimodular. Taking conjugates,

$$\begin{pmatrix} \overline{w_1'} \\ \overline{w_2'} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \overline{w_1} \\ \overline{w_2} \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix}$$

Now since $\{w_1', w_2'\}$ is also given to be a basis, there exits another matrix having all integer entries, $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ such that

$$\begin{pmatrix} \overline{w_1} \\ \overline{w_2} \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} \overline{w_1'} \\ \overline{w_2'} \end{pmatrix}$$

and

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} w_1' \\ w_2' \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'} \end{pmatrix}.$$

However, since w_1'/w_2' is not real, it is routine to verify that

$$\det \begin{pmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'} \end{pmatrix} \neq 0.$$

Therefore,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

and so $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det \begin{pmatrix} e & f \\ g & h \end{pmatrix} = 1$. But the two matrices have all integer entries and so both determinants must equal either 1 or -1.

Next suppose

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{30.5}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unimodular. I need to verify that $\{w_1', w_2'\}$ is a basis. If $w \in M$, there exist integers, m, n such that

$$w = mw_1 + nw_2 = \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

From 30.5

$$\pm \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and so

$$w = \pm \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} w_1' \\ w_2' \end{pmatrix}$$

which is an integer linear combination of $\{w_1', w_2'\}$. It only remains to verify that w_1'/w_2' is not real.

Claim: Let w_1 and w_2 be nonzero complex numbers. Then w_2/w_1 is not real if and only if

$$w_1\overline{w_2} - \overline{w_1}w_2 = \det \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix} \neq 0$$

Proof of the claim: Let $\lambda = w_2/w_1$. Then

$$w_1\overline{w_2} - \overline{w_1}w_2 = \overline{\lambda}w_1\overline{w_1} - \overline{w_1}\lambda w_1 = \left(\overline{\lambda} - \lambda\right) |w_1|^2$$

Thus the ratio is not real if and only if $\left(\overline{\lambda} - \lambda\right) \neq 0$ if and only if $w_1\overline{w_2} - \overline{w_1}w_2 \neq 0$.

Now to verify w_2'/w_1' is not real,

$$\begin{aligned} & \det \begin{pmatrix} w_1' & \overline{w_1'} \\ w_2' & \overline{w_2'} \end{pmatrix} \\ = & \det \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix} \right) \\ = & \pm \det \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix} \neq 0 \end{aligned}$$

This proves the theorem.

30.1.2 The Search For An Elliptic Function

By Theorem 30.5 and 30.7 if you want to find a nonconstant elliptic function it must fail to be analytic and also have either no terms in its Laurent expansion which are of the form $b_1 (z-a)^{-1}$ or else these terms must cancel out. It is simplest to look for a function which simply does not have them. Weierstrass looked for a function of the form

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right) \tag{30.6}$$

where w consists of all numbers of the form $aw_1 + bw_2$ for a, b integers. Sometimes people write this as $\wp(z, w_1, w_2)$ to emphasize its dependence on the periods, w_1 and w_2 but I won't do so. It is understood there exist these periods, which are given. This is a reasonable thing to try. Suppose you formally differentiate the right side. Never mind whether this is justified for now. This yields

$$\wp'(z) = \frac{-2}{z^3} - \sum_{w \neq 0} \frac{-2}{(z-w)^3} = \sum_w \frac{-2}{(z-w)^3}$$

which is clearly periodic having both periods w_1 and w_2. Therefore, $\wp(z+w_1) - \wp(z)$ and $\wp(z+w_2) - \wp(z)$ are both constants, c_1 and c_2 respectively. The reason for this is that since $\wp'$ is periodic with periods w_1 and w_2, it follows $\wp'(z+w_i) - \wp'(z) = 0$ as long as z is not a period. From 30.6 you can see right away that

$$\wp(z) = \wp(-z)$$

Indeed

$$\begin{aligned} \wp(-z) &= \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(-z-w)^2} - \frac{1}{w^2} \right) \\ &= \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(-z+w)^2} - \frac{1}{w^2} \right) = \wp(z). \end{aligned}$$

and so

$$\begin{aligned} c_1 &= \wp\left(-\frac{w_1}{2} + w_1\right) - \wp\left(-\frac{w_1}{2}\right) \\ &= \wp\left(\frac{w_1}{2}\right) - \wp\left(-\frac{w_1}{2}\right) = 0 \end{aligned}$$

which shows the constant for $\wp(z+w_1) - \wp(z)$ must equal zero. Similarly the constant for $\wp(z+w_2) - \wp(z)$ also equals zero. Thus $\wp$ is periodic having the two periods w_1, w_2.

Of course to justify this, you need to consider whether the series of 30.6 converges. Consider the terms of the series.

$$\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| = |z| \left| \frac{2w - z}{(z-w)^2 w^2} \right|$$

If $|w| > 2|z|$, this can be estimated more. For such w,

$$\begin{aligned} &\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| \\ = \ & |z| \left| \frac{2w - z}{(z-w)^2 w^2} \right| \leq |z| \frac{(5/2)|w|}{|w|^2 (|w| - |z|)^2} \\ \leq \ & |z| \frac{(5/2)|w|}{|w|^2 ((1/2)|w|)^2} = |z| \frac{10}{|w|^3}. \end{aligned}$$

It follows the series in 30.6 converges uniformly and absolutely on every compact set, K provided $\sum_{w\neq 0}\frac{1}{|w|^3}$ converges. This question is considered next.

Claim: There exists a positive number, k such that for all pairs of integers, m,n, not both equal to zero,

$$\frac{|mw_1+nw_2|}{|m|+|n|}\geq k>0.$$

Proof of claim: If not, there exists m_k and n_k such that

$$\lim_{k\to\infty}\frac{m_k}{|m_k|+|n_k|}w_1+\frac{n_k}{|m_k|+|n_k|}w_2=0$$

However, $\left(\frac{m_k}{|m_k|+|n_k|},\frac{n_k}{|m_k|+|n_k|}\right)$ is a bounded sequence in $\mathbb{R}^2$ and so, taking a subsequence, still denoted by k, you can have

$$\left(\frac{m_k}{|m_k|+|n_k|},\frac{n_k}{|m_k|+|n_k|}\right)\to(x,y)\in\mathbb{R}^2$$

and so there are real numbers, x,y such that $xw_1+yw_2=0$ contrary to the assumption that w_2/w_1 is not equal to a real number. This proves the claim.

Now from the claim,

$$\begin{aligned}&\sum_{w\neq 0}\frac{1}{|w|^3}\\ =&\sum_{(m,n)\neq(0,0)}\frac{1}{|mw_1+nw_2|^3}\leq\sum_{(m,n)\neq(0,0)}\frac{1}{k^3\left(|m|+|n|\right)^3}\\ =&\frac{1}{k^3}\sum_{j=1}^{\infty}\sum_{|m|+|n|=j}\frac{1}{\left(|m|+|n|\right)^3}=\frac{1}{k^3}\sum_{j=1}^{\infty}\frac{4j}{j^3}<\infty.\end{aligned}$$

Now consider the series in 30.6. Letting $z\in B(0,R)$,

$$\begin{aligned}\wp(z)\equiv&\frac{1}{z^2}+\sum_{w\neq 0,|w|\leq R}\left(\frac{1}{(z-w)^2}-\frac{1}{w^2}\right)\\ &+\sum_{w\neq 0,|w|>R}\left(\frac{1}{(z-w)^2}-\frac{1}{w^2}\right)\end{aligned}$$

and the last series converges uniformly on $B(0,R)$ to an analytic function. Thus $\wp$ is a meromorphic function and also the argument given above involving differentiation of the series termwise is valid. Thus $\wp$ is an elliptic function as claimed. This is called the Weierstrass $\wp$ function. This has proved the following theorem.

Theorem 30.14 *The function $\wp$ defined above is an example of an elliptic function. On any compact set, $\wp$ equals a rational function added to a series which is uniformly and absolutely convergent on the compact set.*

30.1.3 The Differential Equation Satisfied By $\wp$

For z not a pole,

$$\wp'(z) = \frac{-2}{z^3} - \sum_{w \neq 0} \frac{2}{(z-w)^3}$$

Also since there are no poles of order 1 you can obtain a primitive for $\wp$, $-\zeta$.[2] To do so, recall

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

where for $|z| < R$ this is the sum of a rational function with a uniformly convergent series. Therefore, you can take the integral along any path, $\gamma(0, z)$ from 0 to z which misses the poles of $\wp$. By the uniform convergence of the above integral, you can interchange the sum with the integral and obtain

$$\zeta(z) = \frac{1}{z} + \sum_{w \neq 0} \frac{1}{z-w} + \frac{z}{w^2} + \frac{1}{w} \tag{30.7}$$

This function is odd. Here is why.

$$\zeta(-z) = \frac{1}{-z} + \sum_{w \neq 0} \frac{1}{-z-w} - \frac{z}{w^2} + \frac{1}{w}$$

while

$$\begin{aligned} -\zeta(z) &= \frac{1}{-z} + \sum_{w \neq 0} \frac{-1}{z-w} - \frac{z}{w^2} - \frac{1}{w} \\ &= \frac{1}{-z} + \sum_{w \neq 0} \frac{-1}{z+w} - \frac{z}{w^2} + \frac{1}{w}. \end{aligned}$$

Now consider 30.7. It will be used to find the Laurent expansion about the origin for ζ which will then be differentiated to obtain the Laurent expansion for $\wp$ at the origin. Since $w \neq 0$ and the interest is for z near 0 so $|z| < |w|$,

$$\begin{aligned} \frac{1}{z-w} + \frac{z}{w^2} + \frac{1}{w} &= \frac{z}{w^2} + \frac{1}{w} - \frac{1}{w} \frac{1}{1 - \frac{z}{w}} \\ &= \frac{z}{w^2} + \frac{1}{w} - \frac{1}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w} \right)^k \\ &= -\frac{1}{w} \sum_{k=2}^{\infty} \left(\frac{z}{w} \right)^k \end{aligned}$$

[2] I don't know why it is traditional to refer to this antiderivative as $-\zeta$ rather than ζ but I am following the convention. I think it is to minimize the number of minus signs in the next expression.

From 30.7

$$\begin{aligned}\zeta(z) &= \frac{1}{z}+\sum_{w\neq 0}\left(-\sum_{k=2}^{\infty}\frac{z^k}{w^{k+1}}\right)\\ &= \frac{1}{z}-\sum_{k=2}^{\infty}\sum_{w\neq 0}\frac{z^k}{w^{k+1}}=\frac{1}{z}-\sum_{k=2}^{\infty}\sum_{w\neq 0}\frac{z^{2k-1}}{w^{2k}}\end{aligned}$$

because the sum over odd powers must be zero because for each $w\neq 0$, there exists $-w\neq 0$ such that the two terms $\frac{z^{2k}}{w^{2k+1}}$ and $\frac{z^{2k}}{(-w)^{2k+1}}$ cancel each other. Hence

$$\zeta(z)=\frac{1}{z}-\sum_{k=2}^{\infty}G_k z^{2k-1}$$

where $G_k=\sum_{w\neq 0}\frac{1}{w^{2k}}$. Now with this,

$$\begin{aligned}-\zeta'(z) &= \wp(z)=\frac{1}{z^2}+\sum_{k=2}^{\infty}G_k(2k-1)z^{2k-2}\\ &= \frac{1}{z^2}+3G_2z^2+5G_3z^4+\cdots\end{aligned}$$

Therefore,

$$\wp'(z)=\frac{-2}{z^3}+6G_2z+20G_3z^3+\cdots$$

$$\begin{aligned}\wp'(z)^2 &= \frac{4}{z^6}-\frac{24G_2}{z^2}-80G_3+\cdots\\ 4\wp(z)^3 &= 4\left(\frac{1}{z^2}+3G_2z^2+5G_3z^4\cdots\right)^3\\ &= \frac{4}{z^6}+\frac{36}{z^2}G_2+60G_3+\cdots\end{aligned}$$

and finally

$$60G_2\wp(z)=\frac{60G_2}{z^2}+0+\cdots$$

where in the above, the positive powers of z are not listed explicitly. Therefore,

$$\wp'(z)^2-4\wp(z)^3+60G_2\wp(z)+140G_3=\sum_{n=1}^{\infty}a_nz^n$$

In deriving the equation it was assumed $|z|<|w|$ for all $w=aw_1+bw_2$ where a,b are integers not both zero. The left side of the above equation is periodic with respect to w_1 and w_2 where w_2/w_1 is not a real number. The only possible poles of the left side are at 0, w_1, w_2, and w_1+w_2, the vertices of the parallelogram determined by w_1 and w_2. This follows from the original formula for $\wp(z)$. However, the above equation shows the left side has no pole at 0. Since the left side is periodic with periods w_1 and w_2, it follows it has no pole at the other vertices of this parallelogram either. Therefore, the left side is periodic and has no poles. Consequently, it equals a constant by Theorem 30.5. But the right side of the above equation shows this constant is 0 because this side equals zero when $z = 0$. Therefore, $\wp$ satisfies the differential equation,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2\wp(z) + 140G_3 = 0.$$

It is traditional to define $60G_2 \equiv g_2$ and $140G_3 \equiv g_3$. Then in terms of these new quantities the differential equation is

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Suppose e_1, e_2 and e_3 are zeros of the polynomial $4w^3 - g_2 w - g_3 = 0$. Then the above equation can be written in the form

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3). \tag{30.8}$$

30.1.4 A Modular Function

The next task is to find the e_i in 30.8. First recall that $\wp$ is an even function. That is $\wp(-z) = \wp(z)$. This follows from 30.6 which is listed here for convenience.

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{w\neq 0}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right) \tag{30.9}$$

Thus

$$\begin{aligned}\wp(-z) &= \frac{1}{z^2} + \sum_{w\neq 0}\left(\frac{1}{(-z-w)^2} - \frac{1}{w^2}\right)\\ &= \frac{1}{z^2} + \sum_{w\neq 0}\left(\frac{1}{(-z+w)^2} - \frac{1}{w^2}\right) = \wp(z).\end{aligned}$$

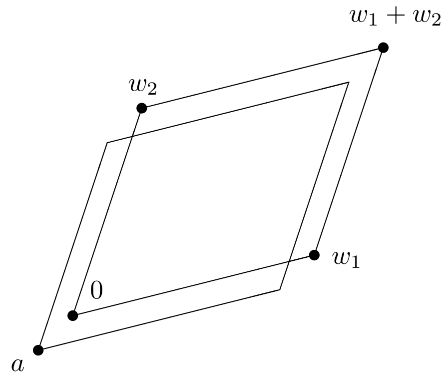
Therefore, $\wp(w_1 - z) = \wp(z - w_1) = \wp(z)$ and so $-\wp'(w_1 - z) = \wp'(z)$. Letting $z = w_1/2$, it follows $\wp'(w_1/2) = 0$. Similarly, $\wp'(w_2/2) = 0$ and $\wp'((w_1 + w_2)/2) = 0$. Therefore, from 30.8

$$0 = 4(\wp(w_1/2) - e_1)(\wp(w_1/2) - e_2)(\wp(w_1/2) - e_3).$$

It follows one of the e_i must equal $\wp(w_1/2)$. Similarly, one of the e_i must equal $\wp(w_2/2)$ and one must equal $\wp((w_1 + w_2)/2)$.

Lemma 30.15 *The numbers, $\wp(w_1/2), \wp(w_2/2)$, and $\wp((w_1 + w_2)/2)$ are distinct.*

Proof: Choose P_a, a period parallelogram which contains the pole 0, and the points $w_1/2$, $w_2/2$, and $(w_1+w_2)/2$ but no other pole of $\wp(z)$. Also ∂P_a^* does not contain any zeros of the elliptic function, $z \to \wp(z) - \wp(w_1/2)$. This can be done by shifting P_0 slightly because the poles are only at the points $aw_1 + bw_2$ for a, b integers and the zeros of $\wp(z) - \wp(w_1/2)$ are discreet.



If $\wp(w_2/2) = \wp(w_1/2)$, then $\wp(z) - \wp(w_1/2)$ has two zeros, $w_2/2$ and $w_1/2$ and since the pole at 0 is of order 2, this is the order of $\wp(z) - \wp(w_1/2)$ on P_a hence by Theorem 30.8 on Page 822 these are the only zeros of this function on P_a. It follows by Corollary 30.10 on Page 824 which says the sum of the zeros minus the sum of the poles is in M, $\frac{w_1}{2} + \frac{w_2}{2} \in M$. Thus there exist integers, a, b such that

$$\frac{w_1 + w_2}{2} = aw_1 + bw_2$$

which implies $(2a-1)w_1 + (2b-1)w_2 = 0$ contradicting w_2/w_1 not being real. Similar reasoning applies to the other pairs of points in $\{w_1/2, w_2/2, (w_1+w_2)/2\}$. For example, consider $(w_1+w_2)/2$ and choose P_a such that its boundary contains no zeros of the elliptic function, $z \to \wp(z) - \wp((w_1+w_2)/2)$ and P_a contains no poles of $\wp$ on its interior other than 0. Then if $\wp(w_2/2) = \wp((w_1+w_2)/2)$, it follows from Theorem 30.8 on Page 822 $w_2/2$ and $(w_1+w_2)/2$ are the only two zeros of $\wp(z) - \wp((w_1+w_2)/2)$ on P_a and by Corollary 30.10 on Page 824

$$\frac{w_1 + w_1 + w_2}{2} = aw_1 + bw_2 \in M$$

for some integers a, b which leads to the same contradiction as before about w_1/w_2 not being real. The other cases are similar. This proves the lemma.

Lemma 30.15 proves the e_i are distinct. Number the e_i such that

$$e_1 = \wp(w_1/2), e_2 = \wp(w_2/2)$$

and

$$e_3 = \wp((w_1+w_2)/2).$$

To summarize, it has been shown that for complex numbers, w_1 and w_2 with w_2/w_1 not real, an elliptic function, $\wp$ has been defined. Denote this function as

$\wp(z) = \wp(z, w_1, w_2)$. This in turn determined numbers, e_i as described above. Thus these numbers depend on w_1 and w_2 and as described above,

$$\begin{aligned} e_1(w_1, w_2) &= \wp\left(\frac{w_1}{2}, w_1, w_2\right), \; e_2(w_1, w_2) = \wp\left(\frac{w_2}{2}, w_1, w_2\right) \\ e_3(w_1, w_2) &= \wp\left(\frac{w_1 + w_2}{2}, w_1, w_2\right). \end{aligned}$$

Therefore, using the formula for $\wp$, 30.9,

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

you see that if the two periods w_1 and w_2 are replaced with tw_1 and tw_2 respectively, then

$$e_i(tw_1, tw_2) = t^{-2} e_i(w_1, w_2).$$

Let τ denote the complex number which equals the ratio, w_2/w_1 which was assumed in all this to not be real. Then

$$e_i(w_1, w_2) = w_1^{-2} e_i(1, \tau)$$

Now define the function, $\lambda(\tau)$

$$\lambda(\tau) \equiv \frac{e_3(1, \tau) - e_2(1, \tau)}{e_1(1, \tau) - e_2(1, \tau)} \left(= \frac{e_3(w_1, w_2) - e_2(w_1, w_2)}{e_1(w_1, w_2) - e_2(w_1, w_2)} \right). \tag{30.10}$$

This function is meromorphic for $\operatorname{Im} \tau > 0$ or for $\operatorname{Im} \tau < 0$. However, since the denominator is never equal to zero the function must actually be analytic on both the upper half plane and the lower half plane. It never is equal to 0 because $e_3 \neq e_2$ and it never equals 1 because $e_3 \neq e_1$. This is stated as an observation.

Observation 30.16 *The function, $\lambda(\tau)$ is analytic for τ in the upper half plane and never assumes the values 0 and 1.*

This is a very interesting function. Consider what happens when

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and the matrix is unimodular. By Theorem 30.13 on Page 826 $\{w_1', w_2'\}$ is just another basis for the same module of periods. Therefore, $\wp(z, w_1, w_2) = \wp(z, w_1', w_2')$ because both are defined as sums over the same values of w, just in different order which does not matter because of the absolute convergence of the sums on compact subsets of $\mathbb{C}$. Since $\wp$ is unchanged, it follows $\wp'(z)$ is also unchanged and so the numbers, e_i are also the same. However, they might be permuted in which case the function $\lambda(\tau)$ defined above would change. What would it take for $\lambda(\tau)$ to not change? In other words, for which unimodular transformations will λ be left unchanged? This happens if and only if no permuting takes place for the e_i. This occurs if $\wp\left(\frac{w_1}{2}\right) = \wp\left(\frac{w_1'}{2}\right)$ and $\wp\left(\frac{w_2}{2}\right) = \wp\left(\frac{w_2'}{2}\right)$. If

$$\frac{w_1'}{2} - \frac{w_1}{2} \in M,\ \frac{w_2'}{2} - \frac{w_2}{2} \in M$$

then $\wp\left(\frac{w_1}{2}\right) = \wp\left(\frac{w_1'}{2}\right)$ and so e_1 will be unchanged and similarly for e_2 and e_3. This occurs exactly when

$$\frac{1}{2}\left(\left(a-1\right)w_1 + bw_2\right) \in M,\ \frac{1}{2}\left(cw_1 + \left(d-1\right)w_2\right) \in M.$$

This happens if a and d are odd and if b and c are even. Of course the stylish way to say this is

$$a \equiv 1 \bmod 2,\ d \equiv 1 \bmod 2,\ b \equiv 0 \bmod 2,\ c \equiv 0 \bmod 2. \tag{30.11}$$

This has shown that for unimodular transformations satisfying 30.11 λ is unchanged. Letting τ be defined as above,

$$\tau' = \frac{w_2'}{w_1'} \equiv \frac{cw_1 + dw_2}{aw_1 + bw_2} = \frac{c + d\tau}{a + b\tau}.$$

Thus for unimodular transformations, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying 30.11, or more succinctly,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2 \tag{30.12}$$

it follows that

$$\lambda\left(\frac{c + d\tau}{a + b\tau}\right) = \lambda\left(\tau\right). \tag{30.13}$$

Furthermore, this is the only way this can happen.

Lemma 30.17 *$\lambda(\tau) = \lambda(\tau')$ if and only if*

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

where 30.12 holds.

Proof: It only remains to verify that if $\wp\left(w_1'/2\right) = \wp\left(w_1/2\right)$ then it is necessary that

$$\frac{w_1'}{2} - \frac{w_1}{2} \in M$$

with a similar requirement for w_2 and w_2'. If $\frac{w_1'}{2} - \frac{w_1}{2} \notin M$, then there exist integers, m, n such that

$$\frac{-w_1'}{2} + mw_1 + nw_2$$

is in the interior of P_0, the period parallelogram whose vertices are $0, w_1, w_1 + w_2$, and w_2. Therefore, it is possible to choose small a such that P_a contains the pole, 0, $\frac{w_1}{2}$, and $\frac{-w_1'}{2} + mw_1 + nw_2$ but no other poles of $\wp$ and in addition, ∂P_a^* contains no zeros of $z \to \wp(z) - \wp\left(\frac{w_1}{2}\right)$. Then the order of this elliptic function is 2. By assumption, and the fact that $\wp$ is even,

$$\wp\left(\frac{-w_1'}{2} + mw_1 + nw_2\right) = \wp\left(\frac{-w_1'}{2}\right) = \wp\left(\frac{w_1'}{2}\right) = \wp\left(\frac{w_1}{2}\right).$$

It follows both $\frac{-w_1'}{2} + mw_1 + nw_2$ and $\frac{w_1}{2}$ are zeros of $\wp(z) - \wp\left(\frac{w_1}{2}\right)$ and so by Theorem 30.8 on Page 822 these are the only two zeros of this function in P_a. Therefore, from Corollary 30.10 on Page 824

$$\frac{w_1}{2} - \frac{w_1'}{2} + mw_1 + nw_2 \in M$$

which shows $\frac{w_1}{2} - \frac{w_1'}{2} \in M$. This completes the proof of the lemma.

Note the condition in the lemma is equivalent to the condition 30.13 because you can relabel the coefficients. The message of either version is that the coefficient of τ in the numerator and denominator is odd while the constant in the numerator and denominator is even.

Next, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$ and therefore,

$$\lambda\left(\frac{2+\tau}{1}\right) = \lambda(\tau + 2) = \lambda(\tau). \tag{30.14}$$

Thus λ is periodic of period 2.

Thus λ leaves invariant a certain subgroup of the unimodular group. According to the next definition, λ is an example of something called a modular function.

Definition 30.18 *When an analytic or meromorphic function is invariant under a group of linear transformations, it is called an automorphic function. A function which is automorphic with respect to a subgroup of the modular group is called a modular function or an elliptic modular function.*

Now consider what happens for some other unimodular matrices which are not congruent to the identity mod 2. This will yield other functional equations for λ in addition to the fact that λ is periodic of period 2. As before, these functional equations come about because $\wp$ is unchanged when you change the basis for M, the module of periods. In particular, consider the unimodular matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{30.15}$$

Consider the first of these. Thus

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} w_1 \\ w_1 + w_2 \end{pmatrix}$$

Hence $\tau' = w_2'/w_1' = (w_1 + w_2)/w_1 = 1 + \tau$. Then from the definition of λ,

$$\begin{aligned}
\lambda(\tau') &= \lambda(1+\tau) \\
&= \frac{\wp\left(\frac{w_1'+w_2'}{2}\right) - \wp\left(\frac{w_2'}{2}\right)}{\wp\left(\frac{w_1'}{2}\right) - \wp\left(\frac{w_2'}{2}\right)} \\
&= \frac{\wp\left(\frac{w_1+w_2+w_1}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)} \\
&= \frac{\wp\left(\frac{w_2}{2}+w_1\right) - \wp\left(\frac{w_1+w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)} \\
&= \frac{\wp\left(\frac{w_2}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)} \\
&= -\frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)} \\
&= -\frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right) + \wp\left(\frac{w_2}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)} \\
&= -\frac{\left(\frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right)}\right)}{1 + \left(\frac{\wp\left(\frac{w_2}{2}\right) - \wp\left(\frac{w_1+w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right)}\right)} \\
&= \frac{\left(\frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right)}\right)}{\left(\frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right)}\right) - 1} \\
&= \frac{\lambda(\tau)}{\lambda(\tau) - 1}. \qquad (30.16)
\end{aligned}$$

Summarizing the important feature of the above,

$$\lambda(1+\tau) = \frac{\lambda(\tau)}{\lambda(\tau)-1}. \qquad (30.17)$$

Next consider the other unimodular matrix in 30.15. In this case $w_1' = w_2$ and $w_2' = w_1$. Therefore, $\tau' = w_2'/w_1' = w_1/w_2 = 1/\tau$. Then

$$\begin{aligned}
\lambda(\tau') &= \lambda(1/\tau) \\
&= \frac{\wp\left(\frac{w_1'+w_2'}{2}\right) - \wp\left(\frac{w_2'}{2}\right)}{\wp\left(\frac{w_1'}{2}\right) - \wp\left(\frac{w_2'}{2}\right)} \\
&= \frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_1}{2}\right)}{\wp\left(\frac{w_2}{2}\right) - \wp\left(\frac{w_1}{2}\right)} \\
&= \frac{e_3 - e_1}{e_2 - e_1} = -\frac{e_3 - e_2 + e_2 - e_1}{e_1 - e_2} \\
&= -(\lambda(\tau) - 1) = -\lambda(\tau) + 1. \qquad (30.18)
\end{aligned}$$

You could try other unimodular matrices and attempt to find other functional equations if you like but this much will suffice here.

30.1.5 A Formula For λ

Recall the formula of Mittag-Leffler for $\cot(\pi\alpha)$ given in 29.15. For convenience, here it is.

$$\frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2} = \pi \cot \pi\alpha.$$

As explained in the derivation of this formula it can also be written as

$$\sum_{n=-\infty}^{\infty} \frac{\alpha}{\alpha^2 - n^2} = \pi \cot \pi\alpha.$$

Differentiating both sides yields

$$\begin{aligned}
\pi^2 \csc^2(\pi\alpha) &= \sum_{n=-\infty}^{\infty} \frac{\alpha^2 + n^2}{(\alpha^2 - n^2)^2} \\
&= \sum_{n=-\infty}^{\infty} \frac{(\alpha+n)^2 - 2\alpha n}{(\alpha+n)^2 (\alpha-n)^2} \\
&= \sum_{n=-\infty}^{\infty} \frac{(\alpha+n)^2}{(\alpha+n)^2 (\alpha-n)^2} - \overbrace{\sum_{n=-\infty}^{\infty} \frac{2\alpha n}{(\alpha^2 - n^2)^2}}^{=0} \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{(\alpha-n)^2}. \qquad (30.19)
\end{aligned}$$

Now this formula can be used to obtain a formula for $\lambda(\tau)$. As pointed out above, λ depends only on the ratio w_2/w_1 and so it suffices to take $w_1 = 1$ and

$w_2 = \tau$. Thus

$$\lambda(\tau) = \frac{\wp\left(\frac{1+\tau}{2}\right) - \wp\left(\frac{\tau}{2}\right)}{\wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau}{2}\right)}. \tag{30.20}$$

From the original formula for $\wp$,

$$\begin{aligned}
&\wp\left(\frac{1+\tau}{2}\right) - \wp\left(\frac{\tau}{2}\right) \\
=\ & \frac{1}{\left(\frac{1+\tau}{2}\right)^2} - \frac{1}{\left(\frac{\tau}{2}\right)^2} + \sum_{(k,m)\neq(0,0)} \frac{1}{\left(k - \frac{1}{2} + \left(m - \frac{1}{2}\right)\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\
=\ & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} + \left(m - \frac{1}{2}\right)\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\
=\ & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} + \left(m - \frac{1}{2}\right)\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\
=\ & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} + \left(-m - \frac{1}{2}\right)\tau\right)^2} - \frac{1}{\left(k + \left(-m - \frac{1}{2}\right)\tau\right)^2} \\
=\ & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(\frac{1}{2} + \left(m + \frac{1}{2}\right)\tau - k\right)^2} - \frac{1}{\left(\left(m + \frac{1}{2}\right)\tau - k\right)^2}.
\end{aligned} \tag{30.21}$$

Similarly,

$$\begin{aligned}
&\wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau}{2}\right) \\
=\ & \frac{1}{\left(\frac{1}{2}\right)^2} - \frac{1}{\left(\frac{\tau}{2}\right)^2} + \sum_{(k,m)\neq(0,0)} \frac{1}{\left(k - \frac{1}{2} + m\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\
=\ & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} + m\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\
=\ & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} - m\tau\right)^2} - \frac{1}{\left(k + \left(-m - \frac{1}{2}\right)\tau\right)^2} \\
=\ & \sum_{(k,m)\in\mathbb{Z}^2} \frac{1}{\left(\frac{1}{2} + m\tau - k\right)^2} - \frac{1}{\left(\left(m + \frac{1}{2}\right)\tau - k\right)^2}.
\end{aligned} \tag{30.22}$$

Now use 30.19 to sum these over k. This yields,

$$\begin{aligned}
&\wp\left(\frac{1+\tau}{2}\right) - \wp\left(\frac{\tau}{2}\right) \\
=\ & \sum_m \frac{\pi^2}{\sin^2\left(\pi\left(\frac{1}{2} + \left(m + \frac{1}{2}\right)\tau\right)\right)} - \frac{\pi^2}{\sin^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)} \\
=\ & \sum_m \frac{\pi^2}{\cos^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)} - \frac{\pi^2}{\sin^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)}
\end{aligned}$$

and

$$\wp\left(\frac{1}{2}\right)-\wp\left(\frac{\tau}{2}\right) = \sum_m \frac{\pi^2}{\sin^2\left(\pi\left(\frac{1}{2}+m\tau\right)\right)}-\frac{\pi^2}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}$$
$$= \sum_m \frac{\pi^2}{\cos^2\left(\pi m\tau\right)}-\frac{\pi^2}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}.$$

The following interesting formula for λ results.

$$\lambda(\tau)=\frac{\sum_m \frac{1}{\cos^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}-\frac{1}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}}{\sum_m \frac{1}{\cos^2(\pi m\tau)}-\frac{1}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}}. \tag{30.23}$$

From this it is obvious $\lambda(-\tau)=\lambda(\tau)$. Therefore, from 30.18,

$$-\lambda(\tau)+1=\lambda\left(\frac{1}{\tau}\right)=\lambda\left(\frac{-1}{\tau}\right) \tag{30.24}$$

(It is good to recall that λ has been defined for $\tau\notin\mathbb{R}$.)

30.1.6 Mapping Properties Of λ

The two functional equations, 30.24 and 30.17 along with some other properties presented above are of fundamental importance. For convenience, they are summarized here in the following lemma.

Lemma 30.19 *The following functional equations hold for λ.*

$$\lambda(1+\tau)=\frac{\lambda(\tau)}{\lambda(\tau)-1}, 1=\lambda(\tau)+\lambda\left(\frac{-1}{\tau}\right) \tag{30.25}$$

$$\lambda(\tau+2)=\lambda(\tau), \tag{30.26}$$

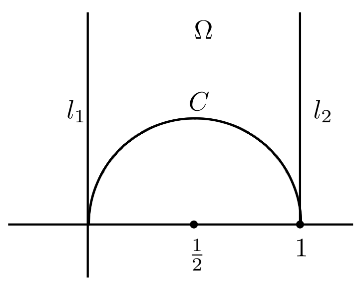
$\lambda(z)=\lambda(w)$ *if and only if there exists a unimodular matrix,*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\sim\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\bmod 2$$

such that

$$w=\frac{az+b}{cz+d} \tag{30.27}$$

Consider the following picture.



In this picture, l_1 is the y axis and l_2 is the line, $x = 1$ while C is the top half of the circle centered at $\left(\frac{1}{2}, 0\right)$ which has radius $1/2$. Note the above formula implies λ has real values on l_1 which are between 0 and 1. This is because 30.23 implies

$$\begin{aligned}\lambda(ib) &= \frac{\sum_m \frac{1}{\cos^2\left(\pi\left(m+\frac{1}{2}\right)ib\right)} - \frac{1}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)ib\right)}}{\sum_m \frac{1}{\cos^2(\pi m ib)} - \frac{1}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)ib\right)}} \\ &= \frac{\sum_m \frac{1}{\cosh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)} + \frac{1}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)}}{\sum_m \frac{1}{\cosh^2(\pi m b)} + \frac{1}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)}} \in (0,1). \qquad (30.28)\end{aligned}$$

This follows from the observation that

$$\cos(ix) = \cosh(x), \ \sin(ix) = i\sinh(x).$$

Thus it is clear from 30.28 that $\lim_{b\to 0+}\lambda(ib) = 1$.

Next I need to consider the behavior of $\lambda(\tau)$ as $\operatorname{Im}(\tau) \to \infty$. From 30.23 listed here for convenience,

$$\lambda(\tau) = \frac{\sum_m \frac{1}{\cos^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)} - \frac{1}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}}{\sum_m \frac{1}{\cos^2(\pi m\tau)} - \frac{1}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}}, \qquad (30.29)$$

it follows

$$\begin{aligned}\lambda(\tau) &= \frac{\frac{1}{\cos^2\left(\pi\left(-\frac{1}{2}\right)\tau\right)} - \frac{1}{\sin^2\left(\pi\left(-\frac{1}{2}\right)\tau\right)} + \frac{1}{\cos^2\left(\pi\frac{1}{2}\tau\right)} - \frac{1}{\sin^2\left(\pi\frac{1}{2}\tau\right)} + A(\tau)}{1+B(\tau)} \\ &= \frac{\frac{2}{\cos^2\left(\pi\left(\frac{1}{2}\right)\tau\right)} - \frac{2}{\sin^2\left(\pi\left(\frac{1}{2}\right)\tau\right)} + A(\tau)}{1+B(\tau)} \qquad (30.30)\end{aligned}$$

Where $A(\tau), B(\tau) \to 0$ as $\operatorname{Im}(\tau) \to \infty$. I took out the $m = 0$ term involving $1/\cos^2(\pi m\tau)$ in the denominator and the $m = -1$ and $m = 0$ terms in the numerator of 30.29. In fact, $e^{-i\pi(a+ib)}A(a+ib), e^{-i\pi(a+ib)}B(a+ib)$ converge to zero uniformly in a as $b \to \infty$.

Lemma 30.20 *For A, B defined in 30.30, $e^{-i\pi(a+ib)}C(a+ib) \to 0$ uniformly in a for $C = A, B$.*

Proof: From 30.23,

$$e^{-i\pi\tau}A(\tau) = \sum_{\substack{m\neq 0\\ m\neq -1}} \frac{e^{-i\pi\tau}}{\cos^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)} - \frac{e^{-i\pi\tau}}{\sin^2\left(\pi\left(m+\frac{1}{2}\right)\tau\right)}$$

Now let $\tau = a + ib$. Then letting $\alpha_m = \pi\left(m + \frac{1}{2}\right)$,

$$\begin{aligned}\cos(\alpha_m a + i\alpha_m b) &= \cos(\alpha_m a)\cosh(\alpha_m b) - i\sinh(\alpha_m b)\sin(\alpha_m a) \\ \sin(\alpha_m a + i\alpha_m b) &= \sin(\alpha_m a)\cosh(\alpha_m b) + i\cos(\alpha_m a)\sinh(\alpha_m b)\end{aligned}$$

Therefore,

$$\begin{aligned}\left|\cos^2(\alpha_m a + i\alpha_m b)\right| &= \cos^2(\alpha_m a)\cosh^2(\alpha_m b) + \sinh^2(\alpha_m b)\sin^2(\alpha_m a) \\ &\geq \sinh^2(\alpha_m b).\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\sin^2(\alpha_m a + i\alpha_m b)\right| &= \sin^2(\alpha_m a)\cosh^2(\alpha_m b) + \cos^2(\alpha_m a)\sinh^2(\alpha_m b) \\ &\geq \sinh^2(\alpha_m b).\end{aligned}$$

It follows that for $\tau = a + ib$ and b large

$$\begin{aligned}&\left|e^{-i\pi\tau}A(\tau)\right| \\ \leq\ & \sum_{\substack{m\neq 0\\ m\neq -1}} \frac{2e^{\pi b}}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)} \\ \leq\ & \sum_{m=1}^{\infty} \frac{2e^{\pi b}}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)} + \sum_{m=-\infty}^{-2} \frac{2e^{\pi b}}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)} \\ =\ & 2\sum_{m=1}^{\infty} \frac{2e^{\pi b}}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)} = 4\sum_{m=1}^{\infty} \frac{e^{\pi b}}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)}\end{aligned}$$

Now a short computation shows

$$\frac{\frac{e^{\pi b}}{\sinh^2\left(\pi\left(m+1+\frac{1}{2}\right)b\right)}}{\frac{e^{\pi b}}{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)}} = \frac{\sinh^2\left(\pi\left(m+\frac{1}{2}\right)b\right)}{\sinh^2\left(\pi\left(m+\frac{3}{2}\right)b\right)} \leq \frac{1}{e^{3\pi b}}.$$

Therefore, for $\tau = a + ib$,

$$\begin{aligned}\left|e^{-i\pi\tau}A(\tau)\right| &\leq 4\frac{e^{\pi b}}{\sinh\left(\frac{3\pi b}{2}\right)}\sum_{m=1}^{\infty}\left(\frac{1}{e^{3\pi b}}\right)^m \\ &\leq 4\frac{e^{\pi b}}{\sinh\left(\frac{3\pi b}{2}\right)}\frac{1/e^{3\pi b}}{1-(1/e^{3\pi b})}\end{aligned}$$

which converges to zero as $b \to \infty$. Similar reasoning will establish the claim about $B(\tau)$. This proves the lemma.

Lemma 30.21 $\lim_{b\to\infty} \lambda(a+ib)\, e^{-i\pi(a+ib)} = 16$ *uniformly in* $a \in \mathbb{R}$.

Proof: From 30.30 and Lemma 30.20, this lemma will be proved if it is shown

$$\lim_{b\to\infty}\left(\frac{2}{\cos^2\left(\pi\left(\frac{1}{2}\right)(a+ib)\right)} - \frac{2}{\sin^2\left(\pi\left(\frac{1}{2}\right)(a+ib)\right)}\right)e^{-i\pi(a+ib)} = 16$$

uniformly in $a \in \mathbb{R}$. Let $\tau = a + ib$ to simplify the notation. Then the above expression equals

$$\begin{aligned}
&\left(\frac{8}{\left(e^{i\frac{\pi}{2}\tau}+e^{-i\frac{\pi}{2}\tau}\right)^2}+\frac{8}{\left(e^{i\frac{\pi}{2}\tau}-e^{-i\frac{\pi}{2}\tau}\right)^2}\right)e^{-i\pi\tau} \\
=&\left(\frac{8e^{i\pi\tau}}{\left(e^{i\pi\tau}+1\right)^2}+\frac{8e^{i\pi\tau}}{\left(e^{i\pi\tau}-1\right)^2}\right)e^{-i\pi\tau} \\
=&\ \frac{8}{\left(e^{i\pi\tau}+1\right)^2}+\frac{8}{\left(e^{i\pi\tau}-1\right)^2} \\
=&\ 16\frac{1+e^{2\pi i\tau}}{\left(1-e^{2\pi i\tau}\right)^2}.
\end{aligned}$$

Now

$$\begin{aligned}
\left|\frac{1+e^{2\pi i\tau}}{\left(1-e^{2\pi i\tau}\right)^2}-1\right| &= \left|\frac{1+e^{2\pi i\tau}}{\left(1-e^{2\pi i\tau}\right)^2}-\frac{\left(1-e^{2\pi i\tau}\right)^2}{\left(1-e^{2\pi i\tau}\right)^2}\right| \\
&\leq \frac{\left|3e^{2\pi i\tau}-e^{4\pi i\tau}\right|}{\left(1-e^{-2\pi b}\right)^2} \leq \frac{3e^{-2\pi b}+e^{-4\pi b}}{\left(1-e^{-2\pi b}\right)^2}
\end{aligned}$$

and this estimate proves the lemma.

Corollary 30.22 $\lim_{b\to\infty}\lambda(a+ib)=0$ *uniformly in* $a \in \mathbb{R}$*. Also* $\lambda(ib)$ *for* $b>0$ *is real and is between* 0 *and 1,* λ *is real on the line,* l_2 *and on the curve,* C *and* $\lim_{b\to 0+}\lambda(1+ib)=-\infty$.

Proof: From Lemma 30.21,

$$\left|\lambda(a+ib)\,e^{-i\pi(a+ib)}-16\right|<1$$

for all a provided b is large enough. Therefore, for such b,

$$|\lambda(a+ib)| \leq 17e^{-\pi b}.$$

30.28 proves the assertion about $\lambda(-bi)$ real.

By the first part, $\lim_{b\to\infty}|\lambda(ib)|=0$. Now from 30.24

$$\lim_{b\to 0+}\lambda(ib)=\lim_{b\to 0+}\left(1-\lambda\left(\frac{-1}{ib}\right)\right)=\lim_{b\to 0+}\left(1-\lambda\left(\frac{i}{b}\right)\right)=1. \tag{30.31}$$

by Corollary 30.22.

Next consider the behavior of λ on line l_2 in the above picture. From 30.17 and 30.28,

$$\lambda(1+ib)=\frac{\lambda(ib)}{\lambda(ib)-1}<0$$

and so as $b \to 0+$ in the above, $\lambda(1+ib) \to -\infty$.

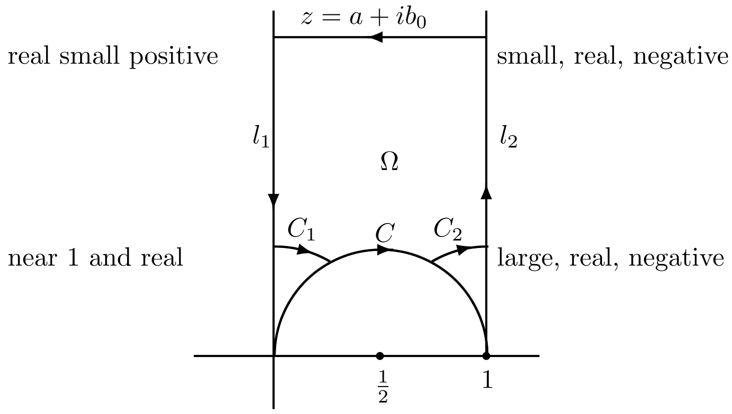
It is left as an exercise to show that the map $\tau \to 1 - \frac{1}{\tau}$ maps l_2 onto the curve, C. Therefore, by 30.25, for $\tau \in l_2$,

$$\lambda\left(1 - \frac{1}{\tau}\right) = \frac{\lambda\left(\frac{-1}{\tau}\right)}{\lambda\left(\frac{-1}{\tau}\right) - 1} \tag{30.32}$$

$$= \frac{1 - \lambda(\tau)}{(1 - \lambda(\tau)) - 1} = \frac{\lambda(\tau) - 1}{\lambda(\tau)} \in \mathbb{R} \tag{30.33}$$

It follows λ is real on the boundary of Ω in the above picture. This proves the corollary.

Now, following Alfors [2], cut off Ω by considering the horizontal line segment, $z = a + ib_0$ where b_0 is very large and positive and $a \in [0,1]$. Also cut Ω off by the images of this horizontal line, under the transformations $z = \frac{1}{\tau}$ and $z = 1 - \frac{1}{\tau}$. These are arcs of circles because the two transformations are fractional linear transformations. It is left as an exercise for you to verify these arcs are situated as shown in the following picture. The important thing to notice is that for b_0 large the points of these circles are close to the origin and $(1,0)$ respectively. The following picture is a summary of what has been obtained so far on the mapping by λ.



In the picture, the descriptions are of λ acting on points of the indicated boundary of Ω. Consider the oriented contour which results from $\lambda(z)$ as z moves first up l_2 as indicated, then along the line $z = a + ib$ and then down l_1 and then along C_1 to C and along C till C_2 and then along C_2 to l_2. As indicated in the picture, this involves going from a large negative real number to a small negative real number and then over a smooth curve which stays small to a real positive number and from there to a real number near 1. $\lambda(z)$ stays fairly near 1 on C_1 provided b_0 is large so that the circle, C_1 has very small radius. Then along C, $\lambda(z)$ is real until it hits C_2. What about the behavior of λ on C_2? For $z \in C_2$, it follows from the definition of C_2 that $z = 1 - \frac{1}{\tau}$ where τ is on the line, $a + ib_0$. Therefore, by Lemma 30.21, 30.17, and 30.24

$$\begin{aligned}\lambda(z) &= \lambda\left(1-\frac{1}{\tau}\right)=\frac{\lambda\left(\frac{-1}{\tau}\right)}{\lambda\left(\frac{-1}{\tau}\right)-1}=\frac{\lambda\left(\frac{1}{\tau}\right)}{\lambda\left(\frac{1}{\tau}\right)-1}\\ &= \frac{1-\lambda(\tau)}{(1-\lambda(\tau))-1}=\frac{\lambda(\tau)-1}{\lambda(\tau)}=1-\frac{1}{\lambda(\tau)}\end{aligned}$$

which is approximately equal to

$$1-\frac{1}{16e^{i\pi(a+ib_0)}}=1-\frac{e^{\pi b_0}e^{-ia\pi}}{16}.$$

These points are essentially on a large half circle in the upper half plane which has radius approximately $\frac{e^{\pi b_0}}{16}$.

Now let $w\in\mathbb{C}$ with $\operatorname{Im}(w)\neq 0$. Then for b_0 large enough, the motion over the boundary of the truncated region indicated in the above picture results in λ tracing out a large simple closed curve oriented in the counter clockwise direction which includes w on its interior if $\operatorname{Im}(w)>0$ but which excludes w if $\operatorname{Im}(w)<0$.

Theorem 30.23 *Let Ω be the domain described above. Then λ maps Ω one to one and onto the upper half plane of $\mathbb{C}$, $\{z\in\mathbb{C}$ such that $\operatorname{Im}(z)>0\}$. Also, the line $\lambda(l_1)=(0,1),\lambda(l_2)=(-\infty,0)$, and $\lambda(C)=(1,\infty)$.*

Proof: Let $\operatorname{Im}(w)>0$ and denote by γ the oriented contour described above and illustrated in the above picture. Then the winding number of $\lambda\circ\gamma$ about w equals 1. Thus

$$\frac{1}{2\pi i}\int_{\lambda\circ\gamma}\frac{1}{z-w}dz=1.$$

But, splitting the contour integrals into l_2,the top line, l_1,C_1,C, and C_2 and changing variables on each of these, yields

$$1=\frac{1}{2\pi i}\int_\gamma\frac{\lambda'(z)}{\lambda(z)-w}dz$$

and by the theorem on counting zeros, Theorem 25.20 on Page 694, the function, $z\to\lambda(z)-w$ has exactly one zero inside the truncated Ω. However, this shows this function has exactly one zero inside Ω because b_0 was arbitrary as long as it is sufficiently large. Since w was an arbitrary element of the upper half plane, this verifies the first assertion of the theorem. The remaining claims follow from the above description of λ, in particular the estimate for λ on C_2. This proves the theorem.

Note also that the argument in the above proof shows that if $\operatorname{Im}(w)<0$, then w is not in $\lambda(\Omega)$. However, if you consider the reflection of Ω about the y axis, then it will follow that λ maps this set one to one onto the lower half plane. The argument will make significant use of Theorem 25.22 on Page 696 which is stated here for convenience.

Theorem 30.24 *Let $f : B(a,R) \to \mathbb{C}$ be analytic and let*

$$f(z) - \alpha = (z-a)^m g(z), \ \infty > m \geq 1$$

where $g(z) \neq 0$ in $B(a,R)$. ($f(z) - \alpha$ has a zero of order m at $z = a$.) Then there exist $\varepsilon, \delta > 0$ with the property that for each z satisfying $0 < |z - \alpha| < \delta$, there exist points,

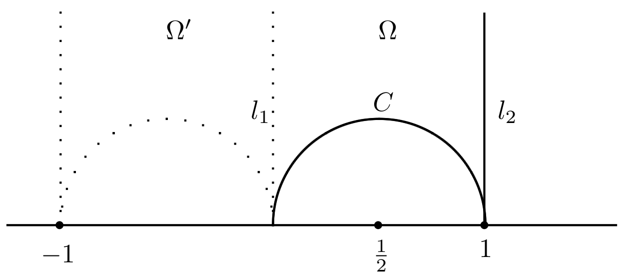
$$\{a_1, \cdots, a_m\} \subseteq B(a,\varepsilon),$$

such that

$$f^{-1}(z) \cap B(a,\varepsilon) = \{a_1, \cdots, a_m\}$$

and each a_k is a zero of order 1 for the function $f(\cdot) - z$.

Corollary 30.25 *Let Ω be the region above. Consider the set of points, $Q = \overline{\Omega} \cup \Omega' \setminus \{0,1\}$ described by the following picture.*



Then $\lambda(Q) = \mathbb{C} \setminus \{0,1\}$. Also $\lambda'(z) \neq 0$ for every z in $\cup_{k=-\infty}^{\infty} (Q + 2k) \equiv H$.

Proof: By Theorem 30.23, this will be proved if it can be shown that $\lambda(\Omega') = \{z \in \mathbb{C} : \text{Im}(z) < 0\}$. Consider λ_1 defined on Ω' by

$$\lambda_1(x+iy) \equiv \overline{\lambda(-x+iy)}.$$

Claim: λ_1 is analytic.

Proof of the claim: You just verify the Cauchy Riemann equations. Letting $\lambda(x+iy) = u(x,y) + iv(x,y)$,

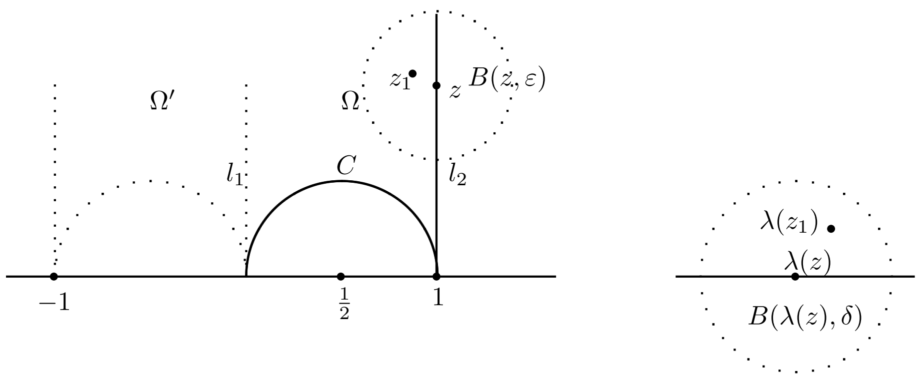
$$\begin{aligned} \lambda_1(x+iy) &= u(-x,y) - iv(-x,y) \\ &\equiv u_1(x,y) + iv(x,y). \end{aligned}$$

Then $u_{1x}(x,y) = -u_x(-x,y)$ and $v_{1y}(x,y) = -v_y(-x,y) = -u_x(-x,y)$ since λ is analytic. Thus $u_{1x} = v_{1y}$. Next, $u_{1y}(x,y) = u_y(-x,y)$ and $v_{1x}(x,y) = v_x(-x,y) = -u_y(-x,y)$ and so $u_{1y} = -v_x$.

Now recall that on l_1, λ takes real values. Therefore, $\lambda_1 = \lambda$ on l_1, a set with a limit point. It follows $\lambda = \lambda_1$ on $\Omega' \cup \Omega$. By Theorem 30.23 λ maps Ω one to one onto the upper half plane. Therefore, from the definition of $\lambda_1 = \lambda$, it follows λ maps Ω' one to one onto the lower half plane as claimed. This has shown that λ

is one to one on $\Omega \cup \Omega'$. This also verifies from Theorem 25.22 on Page 696 that $\lambda' \neq 0$ on $\Omega \cup \Omega'$.

Now consider the lines l_2 and C. If $\lambda'(z) = 0$ for $z \in l_2$, a contradiction can be obtained. Pick such a point. If $\lambda'(z) = 0$, then z is a zero of order $m \geq 2$ of the function, $\lambda - \lambda(z)$. Then by Theorem 25.22 there exist $\delta, \varepsilon > 0$ such that if $w \in B(\lambda(z), \delta)$, then $\lambda^{-1}(w) \cap B(z, \varepsilon)$ contains at least m points.



In particular, for $z_1 \in \Omega \cap B(z, \varepsilon)$ sufficiently close to $z, \lambda(z_1) \in B(\lambda(z), \delta)$ and so the function $\lambda - \lambda(z_1)$ has at least two distinct zeros. These zeros must be in $B(z, \varepsilon) \cap \Omega$ because $\lambda(z_1)$ has positive imaginary part and the points on l_2 are mapped by λ to a real number while the points of $B(z, \varepsilon) \setminus \overline{\Omega}$ are mapped by λ to the lower half plane thanks to the relation, $\lambda(z+2) = \lambda(z)$. This contradicts λ one to one on Ω. Therefore, $\lambda' \neq 0$ on l_2. Consider C. Points on C are of the form $1 - \frac{1}{\tau}$ where $\tau \in l_2$. Therefore, using 30.33,

$$\lambda\left(1 - \frac{1}{\tau}\right) = \frac{\lambda(\tau) - 1}{\lambda(\tau)}.$$

Taking the derivative of both sides,

$$\lambda'\left(1 - \frac{1}{\tau}\right)\left(\frac{1}{\tau^2}\right) = \frac{\lambda'(\tau)}{\lambda(\tau)^2} \neq 0.$$

Since λ is periodic of period 2 it follows $\lambda'(z) \neq 0$ for all $z \in \cup_{k=-\infty}^{\infty}(Q + 2k)$.

Lemma 30.26 *If* $\operatorname{Im}(\tau) > 0$ *then there exists a unimodular* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *such that*

$$\frac{c + d\tau}{a + b\tau}$$

is contained in the interior of Q. In fact, $\left|\frac{c+d\tau}{a+b\tau}\right| \geq 1$ *and*

$$-1/2 \leq \operatorname{Re}\left(\frac{c+d\tau}{a+b\tau}\right) \leq 1/2.$$

Proof: Letting a basis for the module of periods of $\wp$ be $\{1,\tau\}$, it follows from Theorem 30.3 on Page 820 that there exists a basis for the same module of periods, $\{w_1', w_2'\}$ with the property that for $\tau' = w_2'/w_1'$

$$|\tau'| \geq 1, \ \frac{-1}{2} \leq \operatorname{Re}\tau' \leq \frac{1}{2}.$$

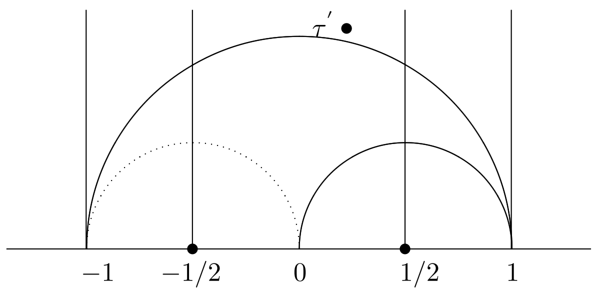
Since this is a basis for the same module of periods, there exists a unimodular matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix}.$$

Hence,

$$\tau' = \frac{w_2'}{w_1'} = \frac{c+d\tau}{a+b\tau}.$$

Thus τ' is in the interior of H. In fact, it is on the interior of $\Omega' \cup \Omega \equiv Q$.



30.1.7 A Short Review And Summary

With this lemma, it is easy to extend Corollary 30.25. First, a simple observation and review is a good idea. Recall that when you change the basis for the module of periods, the Weierstrass $\wp$ function does not change and so the set of e_i used in defining λ also do not change. Letting the new basis be $\{w_1', w_2'\}$, it was shown that

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

for some unimodular transformation, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Letting $\tau = w_2/w_1$ and $\tau' = w_2'/w_1'$

$$\tau' = \frac{c + d\tau}{a + b\tau} \equiv \phi(\tau)$$

Now as discussed earlier

$$\begin{aligned} \lambda(\tau') &= \lambda(\phi(\tau)) \equiv \frac{\wp\left(\frac{w_1'+w_2'}{2}\right) - \wp\left(\frac{w_2'}{2}\right)}{\wp\left(\frac{w_1'}{2}\right) - \wp\left(\frac{w_2'}{2}\right)} \\ &= \frac{\wp\left(\frac{1+\tau'}{2}\right) - \wp\left(\frac{\tau'}{2}\right)}{\wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau'}{2}\right)} \end{aligned}$$

These numbers in the above fraction must be the same as $\wp\left(\frac{1+\tau}{2}\right), \wp\left(\frac{\tau}{2}\right)$, and $\wp\left(\frac{1}{2}\right)$ but they might occur differently. This is because $\wp$ does not change and these numbers are the zeros of a polynomial having coefficients involving only numbers and $\wp(z)$. It could happen for example that $\wp\left(\frac{1+\tau'}{2}\right) = \wp\left(\frac{\tau}{2}\right)$ in which case this would change the value of λ. In effect, you can keep track of all possibilities by simply permuting the e_i in the formula for $\lambda(\tau)$ given by $\frac{e_3 - e_2}{e_1 - e_2}$. Thus consider the following permutation table.

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{array}.$$

Corresponding to this list of 6 permutations, all possible formulas for $\lambda(\phi(\tau))$ can be obtained as follows. Letting $\tau' = \phi(\tau)$ where ϕ is a unimodular matrix corresponding to a change of basis,

$$\lambda(\tau') = \frac{e_3 - e_2}{e_1 - e_2} = \lambda(\tau) \tag{30.34}$$

$$\lambda(\tau') = \frac{e_1 - e_3}{e_2 - e_3} = \frac{e_3 - e_2 + e_2 - e_1}{e_3 - e_2} = 1 - \frac{1}{\lambda(\tau)} = \frac{\lambda(\tau) - 1}{\lambda(\tau)} \tag{30.35}$$

$$\begin{aligned} \lambda(\tau') &= \frac{e_2 - e_1}{e_3 - e_1} = -\left[\frac{e_3 - e_2 - (e_1 - e_2)}{e_1 - e_2}\right]^{-1} \\ &= -[\lambda(\tau) - 1]^{-1} = \frac{1}{1 - \lambda(\tau)} \end{aligned} \tag{30.36}$$

$$\begin{aligned} \lambda(\tau') &= \frac{e_3 - e_1}{e_2 - e_1} = -\left[\frac{e_3 - e_2 - (e_1 - e_2)}{e_1 - e_2}\right] \\ &= -[\lambda(\tau) - 1] = 1 - \lambda(\tau) \end{aligned} \tag{30.37}$$

$$\lambda(\tau') = \frac{e_2 - e_3}{e_1 - e_3} = \frac{e_3 - e_2}{e_3 - e_2 - (e_1 - e_2)} = \frac{1}{1 - \frac{1}{\lambda(\tau)}} = \frac{\lambda(\tau)}{\lambda(\tau) - 1} \tag{30.38}$$

$$\lambda(\tau') = \frac{e_1 - e_3}{e_3 - e_2} = \frac{1}{\lambda(\tau)} \tag{30.39}$$

Corollary 30.27 *$\lambda'(\tau) \neq 0$ for all τ in the upper half plane, denoted by P_+.*

Proof: Let $\tau \in P_+$. By Lemma 30.26 there exists ϕ a unimodular transformation and τ' in the interior of Q such that $\tau' = \phi(\tau)$. Now from the definition of λ in terms of the e_i, there is at worst a permutation of the e_i and so it might be the case that $\lambda(\phi(\tau)) \neq \lambda(\tau)$ but it is the case that $\lambda(\phi(\tau)) = \xi(\lambda(\tau))$ where $\xi'(z) \neq 0$. Here ξ is one of the functions determined by 30.34 - 30.39. (Since $\lambda(\tau) \notin \{0,1\}$, $\xi'(\lambda(z)) \neq 0$. This follows from the above possibilities for ξ listed above in 30.34 - 30.39.) All the possibilities are $\xi(z) =$

$$z, \frac{z-1}{z}, \frac{1}{1-z}, 1-z, \frac{z}{z-1}, \frac{1}{z}$$

and these are the same as the possibilities for ξ^{-1}. Therefore, $\lambda'(\phi(\tau))\phi'(\tau) = \xi'(\lambda(\tau))\lambda'(\tau)$ and so $\lambda'(\tau) \neq 0$ as claimed.

Now I will present a lemma which is of major significance. It depends on the remarkable mapping properties of the modular function and the monodromy theorem from analytic continuation. A review of the monodromy theorem will be listed here for convenience. First recall the definition of the concept of function elements and analytic continuation.

Definition 30.28 *A function element is an ordered pair, (f, D) where D is an open ball and f is analytic on D. (f_0, D_0) and (f_1, D_1) are direct continuations of each other if $D_1 \cap D_0 \neq \emptyset$ and $f_0 = f_1$ on $D_1 \cap D_0$. In this case I will write $(f_0, D_0) \sim (f_1, D_1)$. A chain is a finite sequence, of disks, $\{D_0, \cdots, D_n\}$ such that $D_{i-1} \cap D_i \neq \emptyset$. If (f_0, D_0) is a given function element and there exist function elements, (f_i, D_i) such that $\{D_0, \cdots, D_n\}$ is a chain and $(f_{j-1}, D_{j-1}) \sim (f_j, D_j)$ then (f_n, D_n) is called the analytic continuation of (f_0, D_0) along the chain $\{D_0, \cdots, D_n\}$. Now suppose γ is an oriented curve with parameter interval $[a, b]$ and there exists a chain, $\{D_0, \cdots, D_n\}$ such that $\gamma^* \subseteq \cup_{k=1}^n D_k, \gamma(a)$ is the center of $D_0, \gamma(b)$ is the center of D_n, and there is an increasing list of numbers in $[a, b], a = s_0 < s_1 \cdots < s_n = b$ such that $\gamma([s_i, s_{i+1}]) \subseteq D_i$ and (f_n, D_n) is an analytic continuation of (f_0, D_0) along the chain. Then (f_n, D_n) is called an analytic continuation of (f_0, D_0) along the curve γ. (γ will always be a continuous curve. Nothing more is needed.)*

Then the main theorem is the monodromy theorem listed next, Theorem 27.19 and its corollary on Page 749.

Theorem 30.29 *Let Ω be a simply connected subset of $\mathbb{C}$ and suppose $(f, B(a, r))$ is a function element with $B(a, r) \subseteq \Omega$. Suppose also that this function element can be analytically continued along every curve through a. Then there exists G analytic on Ω such that G agrees with f on $B(a, r)$.*

Here is the lemma.

Lemma 30.30 *Let λ be the modular function defined on P_+ the upper half plane. Let V be a simply connected region in $\mathbb{C}$ and let $f : V \to \mathbb{C}\setminus\{0,1\}$ be analytic and nonconstant. Then there exists an analytic function, $g : V \to P_+$ such that $\lambda \circ g = f$.*

Proof: Let $a \in V$ and choose r_0 small enough that $f(B(a,r_0))$ contains neither 0 nor 1. You need only let $B(a,r_0) \subseteq V$. Now there exists a unique point in Q, τ_0 such that $\lambda(\tau_0) = f(a)$. By Corollary 30.25, $\lambda'(\tau_0) \neq 0$ and so by the open mapping theorem, Theorem 25.22 on Page 696, There exists $B(\tau_0, R_0) \subseteq P_+$ such that λ is one to one on $B(\tau_0, R_0)$ and has a continuous inverse. Then picking r_0 still smaller, it can be assumed $f(B(a,r_0)) \subseteq \lambda(B(\tau_0,R_0))$. Thus there exists a local inverse for λ, λ_0^{-1} defined on $f(B(a,r_0))$ having values in $B(\tau_0,R_0) \cap \lambda^{-1}(f(B(a,r_0)))$. Then defining $g_0 \equiv \lambda_0^{-1} \circ f$, $(g_0, B(a,r_0))$ is a function element. I need to show this can be continued along every curve starting at a in such a way that each function in each function element has values in P_+.

Let $\gamma : [\alpha,\beta] \to V$ be a continuous curve starting at $a, (\gamma(\alpha) = a)$ and suppose that if $t < T$ there exists a nonnegative integer m and a function element $(g_m, B(\gamma(t), r_m))$ which is an analytic continuation of $(g_0, B(a,r_0))$ along γ where $g_m(\gamma(t)) \in P_+$ and each function in every function element for $j \leq m$ has values in P_+. Thus for some small $T > 0$ this has been achieved.

Then consider $f(\gamma(T)) \in \mathbb{C}\setminus\{0,1\}$. As in the first part of the argument, there exists a unique $\tau_T \in Q$ such that $\lambda(\tau_T) = f(\gamma(T))$ and for r small enough there is an analytic local inverse, λ_T^{-1} between $f(B(\gamma(T),r))$ and $\lambda^{-1}(f(B(\gamma(T),r))) \cap B(\tau_T, R_T) \subseteq P_+$ for some $R_T > 0$. By the assumption that the analytic continuation can be carried out for $t < T$, there exists $\{t_0, \cdots, t_m = t\}$ and function elements $(g_j, B(\gamma(t_j), r_j)), j = 0, \cdots, m$ as just described with $g_j(\gamma(t_j)) \in P_+, \lambda \circ g_j = f$ on $B(\gamma(t_j), r_j)$ such that for $t \in [t_m, T], \gamma(t) \in B(\gamma(T), r)$. Let

$$I = B(\gamma(t_m), r_m) \cap B(\gamma(T), r).$$

Then since λ_T^{-1} is a local inverse, it follows for all $z \in I$

$$\lambda(g_m(z)) = f(z) = \lambda\left(\lambda_T^{-1} \circ f(z)\right)$$

Pick $z_0 \in I$. Then by Lemma 30.19 on Page 840 there exists a unimodular mapping of the form

$$\phi(z) = \frac{az+b}{cz+d}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$$

such that

$$g_m(z_0) = \phi\left(\lambda_T^{-1} \circ f(z_0)\right).$$

Since both $g_m(z_0)$ and $\phi\left(\lambda_T^{-1} \circ f(z_0)\right)$ are in the upper half plane, it follows $ad - cb = 1$ and ϕ maps the upper half plane to the upper half plane. Note the pole of ϕ is real and all the sets being considered are contained in the upper half plane so ϕ is analytic where it needs to be.

Claim: For all $z \in I$,

$$g_m(z) = \phi \circ \lambda_T^{-1} \circ f(z). \tag{30.40}$$

Proof: For $z = z_0$ the equation holds. Let

$$A = \left\{ z \in I : g_m(z) = \phi\left(\lambda_T^{-1} \circ f(z)\right) \right\}.$$

Thus $z_0 \in I$. If $z \in I$ and if w is close enough to z, then $w \in I$ also and so both sides of 30.40 with w in place of z are in $\lambda_m^{-1}(f(I))$. But by construction, λ is one to one on this set and since λ is invariant with respect to ϕ,

$$\lambda(g_m(w)) = \lambda\left(\lambda_T^{-1} \circ f(w)\right) = \lambda\left(\phi \circ \lambda_T^{-1} \circ f(w)\right)$$

and consequently, $w \in A$. This shows A is open. But A is also closed in I because the functions are continuous. Therefore, $A = I$ and so 30.40 is obtained.

Letting $f(z) \in f(B(\gamma(T)), r)$,

$$\lambda\left(\phi\left(\lambda_T^{-1}(f(z))\right)\right) = \lambda\left(\lambda_T^{-1}(f(z))\right) = f(z)$$

and so $\phi \circ \lambda_T^{-1}$ is a local inverse for λ on $f(B(\gamma(T)), r)$. Let the new function element be $\left(\overbrace{\phi \circ \lambda_T^{-1} \circ f}^{g_{m+1}}, B(\gamma(T), r)\right)$. This has shown the initial function element can be continued along every curve through a.

By the monodromy theorem, there exists g analytic on V such that g has values in P_+ and $g = g_0$ on $B(a, r_0)$. By the construction, it also follows $\lambda \circ g = f$. This last claim is easy to see because $\lambda \circ g = f$ on $B(a, r_0)$, a set with a limit point so the equation holds for all $z \in V$. This proves the lemma.

30.2 The Picard Theorem Again

Having done all this work on the modular function which is important for its own sake, there is an easy proof of the Picard theorem. In fact, this is the way Picard did it in 1879. I will state it slightly differently since it is no trouble to do so, [27].

Theorem 30.31 *Let f be meromorphic on $\mathbb{C}$ and suppose f misses three distinct points, a, b, c. Then f is a constant function.*

Proof: Let $\phi(z) \equiv \frac{z-a}{z-c}\frac{b-c}{b-a}$. Then $\phi(c) = \infty, \phi(a) = 0$, and $\phi(b) = 1$. Now consider the function, $h = \phi \circ f$. Then h misses the three points $\infty, 0$, and 1. Since h is meromorphic and does not have ∞ in its values, it must actually be analytic.

Thus h is an entire function which misses the two values 0 and 1. If h is not constant, then by Lemma 30.30 there exists a function, g analytic on $\mathbb{C}$ which has values in the upper half plane, P_+ such that $\lambda \circ g = h$. However, g must be a constant because there exists ψ an analytic map on the upper half plane which maps the upper half plane to $B(0,1)$. You can use the Riemann mapping theorem or more simply, $\psi(z) = \frac{z-i}{z+i}$. Thus $\psi \circ g$ equals a constant by Liouville's theorem. Hence g is a constant and so h must also be a constant because $\lambda(g(z)) = h(z)$. This proves f is a constant also. This proves the theorem.

30.3 Exercises

1. Show the set of modular transformations is a group. Also show those modular transformations which are congruent mod 2 to the identity as described above is a subgroup.

2. Suppose f is an elliptic function with period module M. If $\{w_1, w_2\}$ and $\{w_1', w_2'\}$ are two bases, show that the resulting period parallelograms resulting from the two bases have the same area.

3. Given a module of periods with basis $\{w_1, w_2\}$ and letting a typical element of this module be denoted by w as described above, consider the product

$$\sigma(z) \equiv z \prod_{w \neq 0} \left(1 - \frac{z}{w}\right) e^{(z/w) + \frac{1}{2}(z/w)^2}.$$

Show this product converges uniformly on compact sets, is an entire function, and satisfies

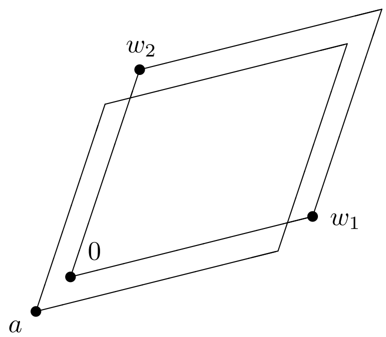
$$\sigma'(z) / \sigma(z) = \zeta(z)$$

where $\zeta(z)$ was defined above as a primitive of $\wp(z)$ and is given by

$$\zeta(z) = \frac{1}{z} + \sum_{w \neq 0} \frac{1}{z-w} + \frac{z}{w^2} + \frac{1}{w}.$$

4. Show $\zeta(z + w_i) = \zeta(z) + \eta_i$ where η_i is a constant.

5. Let P_a be the parallelogram shown in the following picture.



Show that $\frac{1}{2\pi i}\int_{\partial P_a}\zeta(z)\,dz = 1$ where the contour is taken once around the parallelogram in the counter clockwise direction. Next evaluate this contour integral directly to obtain Legendre's relation,

$$\eta_1 w_2 - \eta_2 w_1 = 2\pi i.$$

6. For σ defined in Problem 3, 4 explain the following steps. For $j = 1, 2$

$$\frac{\sigma'(z+w_j)}{\sigma(z+w_j)} = \zeta(z+w_j) = \zeta(z) + \eta_j = \frac{\sigma'(z)}{\sigma(z)} + \eta_j$$

Therefore, there exists a constant, C_j such that

$$\sigma(z+w_j) = C_j \sigma(z) e^{\eta_j z}.$$

Next show σ is an odd function, $(\sigma(-z) = -\sigma(z))$ and then let $z = -w_j/2$ to find $C_j = -e^{\frac{\eta_j w_j}{2}}$ and so

$$\sigma(z+w_j) = -\sigma(z) e^{\eta_j\left(z+\frac{w_j}{2}\right)}. \tag{30.41}$$

7. Show any **even** elliptic function, f with periods w_1 and w_2 for which 0 is neither a pole nor a zero can be expressed in the form

$$f(0)\prod_{k=1}^{n}\frac{\wp(z)-\wp(a_k)}{\wp(z)-\wp(b_k)}$$

where C is some constant. Here $\wp$ is the Weierstrass function which comes from the two periods, w_1 and w_2. **Hint:** You might consider the above function in terms of the poles and zeros on a period parallelogram and recall that an entire function which is elliptic is a constant.

8. Suppose f is any elliptic function with $\{w_1, w_2\}$ a basis for the module of periods. Using Theorem 30.9 and 30.41 show that there exists constants $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ such that for some constant C,

$$f(z) = C\prod_{k=1}^{n}\frac{\sigma(z-a_k)}{\sigma(z-b_k)}.$$

Hint: You might try something like this: By Theorem 30.9, it follows that if $\{\alpha_k\}$ are the zeros and $\{b_k\}$ the poles in an appropriate period parallelogram, $\sum\alpha_k - \sum b_k$ equals a period. Replace α_k with a_k such that $\sum a_k - \sum b_k = 0$. Then use 30.41 to show that the given formula for f is bi periodic. Anyway, you try to arrange things such that the given formula has the same poles as f. Remember an entire elliptic function equals a constant.

9. Show that the map $\tau \to 1 - \frac{1}{\tau}$ maps l_2 onto the curve, C in the above picture on the mapping properties of λ.

10. Modify the proof of Theorem 30.23 to show that $\lambda(\Omega) \cap \{z \in \mathbb{C} : \text{Im}(z) < 0\} = \emptyset$.

Part IV

Stochastic Processes, An Introduction

Random Variables And Basic Probability

Caution: This material on probability and stochastic processes may be half baked in places. I have not yet rewritten it several times. This is not to say that nothing else is half baked. However, the probability is higher here.

Recall Lemma 11.3 on Page 305 which is stated here for convenience.

Lemma 31.1 *Let M be a metric space with the closed balls compact and suppose λ is a measure defined on the Borel sets of M which is finite on compact sets. Then there exists a unique Radon measure, $\overline{\lambda}$ which equals λ on the Borel sets. In particular λ must be both inner and outer regular on all Borel sets.*

Also recall from earlier the following fundamental result which is called the Borel Cantelli lemma.

Lemma 31.2 *Let $(\Omega, \mathcal{F}, \lambda)$ be a measure space and let $\{A_i\}$ be a sequence of measurable sets satisfying*

$$\sum_{i=1}^{\infty} \lambda(A_i) < \infty.$$

Then letting S denote the set of $\omega \in \Omega$ which are in infinitely many A_i, it follows S is a measurable set and $\lambda(S) = 0$.

Proof: $S = \cap_{k=1}^{\infty} \cup_{m=k}^{\infty} A_m$. Therefore, S is measurable and also

$$\lambda(S) \leq \lambda(\cup_{m=k}^{\infty} A_m) \leq \sum_{m=k}^{\infty} \lambda(A_k)$$

and this converges to 0 as $k \to \infty$ because of the convergence of the series. This proves the lemma.

Definition 31.3 *A probability space is a measure space, $(\Omega, \mathcal{F}, P)$ where P is a measure satisfying $P(\Omega) = 1$. A random vector is a measurable function, $\mathbf{X} : \Omega \to \mathbb{R}^p$. This might also be called a random variable. Define the following σ algebra.*

$$\mathcal{H}_{\mathbf{X}} \equiv \left\{\mathbf{X}^{-1}(E) : E \text{ is Borel in } \mathbb{R}^p\right\}$$

Thus $\mathcal{H}_{\mathbf{X}} \subseteq \mathcal{F}$. This is also often written as $\sigma(\mathbf{X})$. For E a Borel set in $\mathbb{R}^p$ define

$$\lambda_{\mathbf{X}}(E) \equiv P\left(\mathbf{X}^{-1}(E)\right).$$

This is called the distribution of the random variable, $\mathbf{X}$. If

$$\int_\Omega |\mathbf{X}(\omega)|\, dP < \infty$$

then define

$$E(\mathbf{X}) \equiv \int_\Omega \mathbf{X} dP$$

where the integral is defined in the obvious way componentwise.

Lemma 31.4 *For $\mathbf{X}$ a random vector defined above, $\lambda_{\mathbf{X}}$ is inner and outer regular, Borel, and its completion is a Radon measure and $\lambda_{\mathbf{X}}(\mathbb{R}^p) = 1$. Furthermore, if h is any bounded Borel measurable function,*

$$\int_\Omega h(\mathbf{X}(\omega))\, dP = \int_{\mathbb{R}^p} h(\mathbf{x})\, d\lambda_{\mathbf{X}}.$$

Proof: The assertions about $\lambda_{\mathbf{X}}$ follow from Lemma 11.3 on Page 305 listed above. It remains to verify the formula involving the integrals. Suppose first

$$h(\mathbf{x}) = c\mathcal{X}_E(\mathbf{x})$$

where E is a Borel set in $\mathbb{R}^p$. Then the left side equals

$$\int_\Omega c\mathcal{X}_E(\mathbf{X}(\omega))\, dP = \int_{[\mathbf{X}\in E]} c dP = c\lambda_{\mathbf{X}}(E)$$

The right side equals

$$\int_{\mathbb{R}^p} c\mathcal{X}_E(\mathbf{x})\, d\lambda_{\mathbf{X}} = c\lambda_{\mathbf{X}}(E).$$

Similarly, if h is any Borel simple function, the same result will hold. For an arbitrary bounded Borel function, h, there exists a sequence of Borel simple functions, $\{s_n\}$ converging to h. Hence, by the dominated convergence theorem,

$$\int_\Omega h(\mathbf{X}(\omega))\, dP = \lim_{n\to\infty} \int_\Omega s_n(\mathbf{X}(\omega))\, dP = \lim_{n\to\infty} \int s_n(\mathbf{x})\, d\lambda_{\mathbf{X}} = \int_{\mathbb{R}^p} h(\mathbf{x})\, d\lambda_{\mathbf{X}}.$$

This proves the lemma.

Obviously, h could also be vector valued and Borel measurable and the same argument would work or else you could simply consider the component functions of $\mathbf{h}$.

Definition 31.5 *A finite set of random vectors, $\{\mathbf{X}_k\}_{k=1}^n$ is independent if whenever $F_k \in \mathcal{H}_{\mathbf{X}_k}$ $(\sigma(\mathbf{X}_k))$,*

$$P\left(\cap_{k=1}^n F_k\right) = \prod_{k=1}^n P(F_k).$$

More generally, if $\{\mathcal{F}_i\}_{i\in I}$ is any set of σ algebras, they are said to be independent if whenever $A_{i_k} \in \mathcal{F}_{i_k}$ for $k = 1, 2, \cdots, m$, then

$$P\left(\cap_{k=1}^m A_{i_k}\right) = \prod_{k=1}^m P(A_{i_k}).$$

Lemma 31.6 *If $\{\mathbf{X}_k\}_{k=1}^r$ are independent and if g_k is a Borel measurable function, then $\{g_k(\mathbf{X}_k)\}_{k=1}^n$ is also independent. Furthermore, if the random variables have values in $\mathbb{R}$ and they are all bounded, then*

$$E\left(\prod_{i=1}^r X_i\right) = \prod_{i=1}^r E(X_i).$$

Proof: First consider the claim about $\{g_k(\mathbf{X}_k)\}_{k=1}^r$. Letting O be an open set in $\mathbb{R}$,

$$(g_k \circ \mathbf{X}_k)^{-1}(O) = \mathbf{X}_k^{-1}\left(g_k^{-1}(O)\right) = \mathbf{X}_k^{-1}(\text{Borel set}) \in \mathcal{H}_{\mathbf{X}_k}.$$

It follows $(g_k \circ \mathbf{X}_k)^{-1}(E)$ is in $\mathcal{H}_{\mathbf{X}_k}$ whenever E is Borel. Thus $\mathcal{H}_{g_k\circ\mathbf{X}_k} \subseteq \mathcal{H}_{\mathbf{X}_k}$ and this proves the first part of the lemma.

Now let $\left\{s_n^i\right\}_{n=1}^\infty$ be a bounded sequence of simple functions measurable in $\mathcal{H}_{X_i}$ which converges to X_i uniformly. (Since X_i is bounded, such a sequence exists by breaking X_i into positive and negative parts and using Theorem 8.27 on Page 190.) Say

$$s_n^i(\omega) = \sum_{k=1}^{m_n} c_k^{n,i} \mathcal{X}_{E_k^{n,i}}(\omega)$$

where the E_k are disjoint elements of $\mathcal{H}_{X_i}$ and some might be empty. This is for convenience in keeping the same index on the top of the sum. Then since all the random variables are bounded, there is no problem about existence of any of the above. Then from the assumption that the X_i are independent,

$$\begin{aligned}
E\left(\prod_{i=1}^{r} X_i\right) &= \int_\Omega \prod_{i=1}^{r} X_i(\omega)\,dP = \lim_{n\to\infty}\int_\Omega \prod_{i=1}^{r} s_n^i(\omega)\,dP \\
&= \lim_{n\to\infty}\int_\Omega \prod_{i=1}^{r}\sum_{k=1}^{m_n} c_k^{n,i}\mathcal{X}_{E_k^{n,i}}(\omega)\,dP \\
&= \lim_{n\to\infty}\int_\Omega \sum_{k_1,k_2,\cdots,k_r} c_{k_1}^{n,1}c_{k_2}^{n,2}\cdots c_{k_r}^{n,r}\mathcal{X}_{E_{k_1}^{n,1}}\mathcal{X}_{E_{k_2}^{n,2}}\cdots\mathcal{X}_{E_{k_r}^{n,r}}dP \\
&= \lim_{n\to\infty}\sum_{k_1,k_2,\cdots,k_r}\int_\Omega c_{k_1}^{n,1}c_{k_2}^{n,2}\cdots c_{k_r}^{n,r}\mathcal{X}_{E_{k_1}^{n,1}}\mathcal{X}_{E_{k_2}^{n,2}}\cdots\mathcal{X}_{E_{k_r}^{n,r}}dP \\
&= \lim_{n\to\infty}\sum_{k_1,k_2,\cdots,k_r} c_{k_1}^{n,1}c_{k_2}^{n,2}\cdots c_{k_r}^{n,r}\prod_{i=1}^{r} P\left(E_{k_i}^{n,i}\right) \\
&= \lim_{n\to\infty}\prod_{i=1}^{r}\int_\Omega s_n^i(\omega)\,dP = \prod_{i=1}^{r} E(X_i).
\end{aligned}$$

This proves the lemma.

31.1 The Characteristic Function

Definition 31.7 *Let $\mathbf{X}$ be a random variable as above. The characteristic function is*

$$\phi_{\mathbf{X}}(\mathbf{t}) \equiv E\left(e^{i\mathbf{t}\cdot\mathbf{X}}\right) \equiv \int_\Omega e^{i\mathbf{t}\cdot\mathbf{X}(\omega)}dP = \int_{\mathbb{R}^p} e^{i\mathbf{t}\cdot\mathbf{x}}d\lambda_{\mathbf{X}}$$

the last equation holding by Lemma 31.4.

Recall the following fundamental lemma and definition, Lemma 19.12 on Page 522.

Definition 31.8 *For $T \in \mathcal{G}^*$, define $FT, F^{-1}T \in \mathcal{G}^*$ by*

$$FT(\phi) \equiv T(F\phi),\ F^{-1}T(\phi) \equiv T\left(F^{-1}\phi\right)$$

Lemma 31.9 *F and F^{-1} are both one to one, onto, and are inverses of each other.*

The main result on characteristic functions is the following.

Theorem 31.10 *Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors with values in $\mathbb{R}^p$ and suppose $E\left(e^{i\mathbf{t}\cdot\mathbf{X}}\right) = E\left(e^{i\mathbf{t}\cdot\mathbf{Y}}\right)$ for all $\mathbf{t} \in \mathbb{R}^p$. Then $\lambda_{\mathbf{X}} = \lambda_{\mathbf{Y}}$.*

Proof: For $\psi \in \mathcal{G}$, let $\lambda_{\mathbf{X}}(\psi) \equiv \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}}$ and $\lambda_{\mathbf{Y}}(\psi) \equiv \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{Y}}$. Thus both $\lambda_{\mathbf{X}}$ and $\lambda_{\mathbf{Y}}$ are in $\mathcal{G}^*$. Then letting $\psi \in \mathcal{G}$ and using Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i\mathbf{t}\cdot\mathbf{y}}\psi(\mathbf{t})\,dtd\lambda_{\mathbf{Y}} &= \int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i\mathbf{t}\cdot\mathbf{y}}d\lambda_{\mathbf{Y}}\psi(\mathbf{t})\,dt \\
&= \int_{\mathbb{R}^p} E\left(e^{i\mathbf{t}\cdot\mathbf{Y}}\right)\psi(\mathbf{t})\,dt \\
&= \int_{\mathbb{R}^p} E\left(e^{i\mathbf{t}\cdot\mathbf{X}}\right)\psi(\mathbf{t})\,dt \\
&= \int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i\mathbf{t}\cdot\mathbf{x}}d\lambda_{\mathbf{X}}\psi(\mathbf{t})\,dt \\
&= \int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i\mathbf{t}\cdot\mathbf{x}}\psi(\mathbf{t})\,dtd\lambda_{\mathbf{X}}.
\end{aligned}$$

Thus $\lambda_{\mathbf{Y}}\left(F^{-1}\psi\right) = \lambda_{\mathbf{X}}\left(F^{-1}\psi\right)$. Since $\psi \in \mathcal{G}$ is arbitrary and F^{-1} is onto, this implies $\lambda_{\mathbf{X}} = \lambda_{\mathbf{Y}}$ in $\mathcal{G}^*$. But $\mathcal{G}$ is dense in $C_0(\mathbb{R}^p)$ and so $\lambda_{\mathbf{X}} = \lambda_{\mathbf{Y}}$ as measures. This proves the theorem.

31.2 Conditional Probability

Here I will consider the concept of conditional probability depending on the theory of differentiation of general Radon measures and leading to the Doob Dynkin lemma.

If $\mathbf{X}, \mathbf{Y}$ are two random vectors defined on a probability space having values in $\mathbb{R}^{p_1}$ and $\mathbb{R}^{p_2}$ respectively, and if E is a Borel set in the appropriate space, then $(\mathbf{X}, \mathbf{Y})$ is a random vector with values in $\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ and $\lambda_{(\mathbf{X},\mathbf{Y})}(E \times \mathbb{R}^{p_2}) = \lambda_{\mathbf{X}}(E)$, $\lambda_{(\mathbf{X},\mathbf{Y})}(\mathbb{R}^{p_1} \times E) = \lambda_{\mathbf{Y}}(E)$. Thus, by Theorem 18.12 on Page 505, there exist probability measures, denoted here by $\lambda_{\mathbf{X}|\mathbf{y}}$ and $\lambda_{\mathbf{Y}|\mathbf{x}}$, such that whenever E is a Borel set in $\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$,

$$\int_{\mathbb{R}^{p_1}\times\mathbb{R}^{p_2}} \mathcal{X}_E d\lambda_{(\mathbf{X},\mathbf{Y})} = \int_{\mathbb{R}^{p_1}}\int_{\mathbb{R}^{p_2}} \mathcal{X}_E d\lambda_{\mathbf{Y}|\mathbf{x}}d\lambda_{\mathbf{X}},$$

and

$$\int_{\mathbb{R}^{p_1}\times\mathbb{R}^{p_2}} \mathcal{X}_E d\lambda_{(\mathbf{X},\mathbf{Y})} = \int_{\mathbb{R}^{p_2}}\int_{\mathbb{R}^{p_1}} \mathcal{X}_E d\lambda_{\mathbf{X}|\mathbf{y}}d\lambda_{\mathbf{Y}}.$$

Definition 31.11 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be two random vectors defined on a probability space. The conditional probability measure of* $\mathbf{Y}$ *given* $\mathbf{X}$ *is the measure* $\lambda_{\mathbf{Y}|\mathbf{x}}$ *in the above. Similarly the conditional probability measure of* $\mathbf{X}$ *given* $\mathbf{Y}$ *is the measure* $\lambda_{\mathbf{X}|\mathbf{y}}$.

More generally, one can use the theory of slicing measures to consider any finite list of random vectors, $\{\mathbf{X}_i\}$, defined on a probability space with $\mathbf{X}_i \in \mathbb{R}^{p_i}$, and write the following for E a Borel set in $\prod_{i=1}^n \mathbb{R}^{p_i}$.

$$\int_{\mathbb{R}^{p_1}\times\cdots\times\mathbb{R}^{p_n}} \mathcal{X}_E d\lambda_{(\mathbf{X}_1,\cdots,\mathbf{X}_n)} = \int_{\mathbb{R}^{p_1}}\int_{\mathbb{R}^{p_2}\times\cdots\times\mathbb{R}^{p_n}} \mathcal{X}_E d\lambda_{(\mathbf{X}_2,\cdots,\mathbf{X}_n)|\mathbf{x}_1}d\lambda_{\mathbf{X}_1}$$

$$= \int_{\mathbb{R}^{p_1}} \int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_3} \times \cdots \times \mathbb{R}^{p_n}} \mathcal{X}_E d\lambda_{(\mathbf{X}_3, \cdots, \mathbf{X}_n)|\mathbf{x}_1 \mathbf{x}_2} d\lambda_{\mathbf{X}_2|\mathbf{x}_1} d\lambda_{\mathbf{X}_1}$$

$$\vdots$$

$$= \int_{\mathbb{R}^{p_1}} \cdots \int_{\mathbb{R}^{p_n}} \mathcal{X}_E d\lambda_{\mathbf{X}_n|\mathbf{x}_1 \mathbf{x}_2 \cdots \mathbf{x}_{n-1}} d\lambda_{\mathbf{X}_{n-1}|\mathbf{x}_1 \cdots \mathbf{x}_{n-2}} \cdots d\lambda_{\mathbf{X}_2|\mathbf{x}_1} d\lambda_{\mathbf{X}_1}. \tag{31.1}$$

Obviously, this could have been done in any order in the iterated integrals by simply modifying the "given" variables, those occurring after the symbol $|$, to be those which have been integrated in an outer level of the iterated integral.

Definition 31.12 *Let $\{\mathbf{X}_1, \cdots, \mathbf{X}_n\}$ be random vectors defined on a probability space having values in $\mathbb{F}^{p_1}, \cdots, \mathbb{F}^{p_n}$ respectively. The random vectors are independent if for every E a Borel set in $\mathbb{F}^{p_1} \times \cdots \times \mathbb{F}^{p_n}$,*

$$\int_{\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n}} \mathcal{X}_E d\lambda_{(\mathbf{X}_1, \cdots, \mathbf{X}_n)}$$

$$= \int_{\mathbb{R}^{p_1}} \cdots \int_{\mathbb{R}^{p_n}} \mathcal{X}_E d\lambda_{\mathbf{X}_n} d\lambda_{\mathbf{X}_{n-1}} \cdots d\lambda_{\mathbf{X}_2} d\lambda_{\mathbf{X}_1} \tag{31.2}$$

and the iterated integration may be taken in any order. If $\mathcal{A}$ is any set of random vectors defined on a probability space, $\mathcal{A}$ is independent if any finite set of random vectors from $\mathcal{A}$ is independent.

Thus, the random vectors are independent exactly when the dependence on the givens in 31.1 can be dropped.

Does this amount to the same thing as discussed earlier? These two random vectors, $\mathbf{X}, \mathbf{Y}$ were independent if whenever $A \in \mathcal{H}_{\mathbf{X}}$ $(\sigma(\mathbf{X}))$ and $B \in \mathcal{H}_{\mathbf{Y}}$, $P(A \cap B) = P(A) P(B)$. Suppose the above definition and A and B as described. Let $A = \mathbf{X}^{-1}(E)$ and $B = \mathbf{Y}^{-1}(F)$. Then

$$\begin{aligned} P(A \cap B) &= P((\mathbf{X}, \mathbf{Y}) \in E \times F) \\ &= \int_{\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}} \mathcal{X}_E(\mathbf{x}) \mathcal{X}_F(\mathbf{y}) \, d\lambda_{(\mathbf{X}, \mathbf{Y})} \\ &= \int_{\mathbb{R}^{p_1}} \int_{\mathbb{R}^{p_2}} \mathcal{X}_E(\mathbf{x}) \mathcal{X}_F(\mathbf{y}) \, d\lambda_{\mathbf{Y}|\mathbf{x}} d\lambda_{\mathbf{X}} \\ &= \int_{\mathbb{R}^{p_1}} \int_{\mathbb{R}^{p_2}} \mathcal{X}_E(\mathbf{x}) \mathcal{X}_F(\mathbf{y}) \, d\lambda_{\mathbf{Y}} d\lambda_{\mathbf{X}} \\ &= \lambda_{\mathbf{X}}(E) \lambda_Y(F) = P(A) P(B) \end{aligned}$$

Next suppose $P(A \cap B) = P(A) P(B)$ where $A \in \mathcal{H}_{\mathbf{X}}$ and $B \in \mathcal{H}_{\mathbf{Y}}$, $A = \mathbf{X}^{-1}(E)$ and $B = \mathbf{Y}^{-1}(F)$. Can it be asserted $\lambda_{\mathbf{X}|\mathbf{y}} = \lambda_{\mathbf{X}}$? In this case, for all E Borel in

$\mathbb{R}^{p_1}$ and F Borel in $\mathbb{R}^{p_2}$,

$$\begin{aligned}&\int_{\mathbb{R}^{p_1}}\int_{\mathbb{R}^{p_2}}\mathcal{X}_E(\mathbf{x})\,\mathcal{X}_F(\mathbf{y})\,d\lambda_{\mathbf{Y}}d\lambda_{\mathbf{X}}\\ =\;& P(A)\,P(B)=P(A\cap B)\\ =\;& \int_{\mathbb{R}^{p_1}\times\mathbb{R}^{p_2}}\mathcal{X}_E(\mathbf{x})\,\mathcal{X}_F(\mathbf{y})\,d\lambda_{(\mathbf{X},\mathbf{Y})}\\ =\;& \int_{\mathbb{R}^{p_1}}\int_{\mathbb{R}^{p_2}}\mathcal{X}_E(\mathbf{x})\,\mathcal{X}_F(\mathbf{y})\,d\lambda_{\mathbf{Y}|\mathbf{x}}d\lambda_{\mathbf{X}}\end{aligned}$$

and so, by uniqueness of the slicing measures, $d\lambda_{\mathbf{Y}|\mathbf{x}}=d\lambda_{\mathbf{Y}}$. A similar argument shows $d\lambda_{\mathbf{X}|\mathbf{y}}=d\lambda_{\mathbf{X}}$. Thus this amounts to the same thing discussed earlier.

Proposition 31.13 *Equations 31.2 and 31.1 hold with $\mathcal{X}_E$ replaced by any nonnegative Borel measurable function and for any bounded continuous function.*

Proof: The two equations hold for simple functions in place of $\mathcal{X}_E$ and so an application of the monotone convergence theorem applied to an increasing sequence of simple functions converging pointwise to a given nonnegative Borel measurable function yields the conclusion of the proposition in the case of the nonnegative Borel function. For a bounded continuous function, one can apply the result just established to the positive and negative parts of the real and imaginary parts of the function.

Lemma 31.14 *Let $\mathbf{X}_1,\cdots,\mathbf{X}_n$ be random vectors with values in $\mathbb{R}^{p_1},\cdots,\mathbb{R}^{p_n}$ respectively and let $\mathbf{g}:\mathbb{R}^{p_1}\times\cdots\times\mathbb{R}^{p_n}\to\mathbb{R}^k$ be Borel measurable. Then $\mathbf{g}(\mathbf{X}_1,\cdots,\mathbf{X}_n)$ is a random vector with values in $\mathbb{R}^k$ and if $h:\mathbb{R}^k\to[0,\infty)$, then*

$$\int_{\mathbb{R}^k}h(\mathbf{y})\,d\lambda_{\mathbf{g}(\mathbf{X}_1,\cdots,\mathbf{X}_n)}(y)=$$

$$\int_{\mathbb{R}^{p_1}\times\cdots\times\mathbb{R}^{p_n}}h(\mathbf{g}(\mathbf{x}_1,\cdots,\mathbf{x}_n))\,d\lambda_{(\mathbf{X}_1,\cdots,\mathbf{X}_n)}. \tag{31.3}$$

If $\mathbf{X}_i$ is a random vector with values in $\mathbb{R}^{p_i}, i=1,2,\cdots$ and if $\mathbf{g}_i:\mathbb{R}^{p_i}\to\mathbb{R}^{k_i}$, where $\mathbf{g}_i$ is Borel measurable, then the random vectors $\mathbf{g}_i(\mathbf{X}_i)$ are also independent whenever the $\mathbf{X}_i$ are independent.

Proof: First let E be a Borel set in $\mathbb{R}^k$. From the definition,

$$\begin{aligned}\int_{\mathbb{R}^k}\mathcal{X}_E d\lambda_{\mathbf{g}(\mathbf{X}_1,\cdots,\mathbf{X}_n)} &= \int_{\mathbb{R}^{p_1}\times\cdots\times\mathbb{R}^{p_n}}\mathcal{X}_{\mathbf{g}^{-1}(E)}d\lambda_{(\mathbf{X}_1,\cdots,\mathbf{X}_n)}\\ &= \int_{\mathbb{R}^{p_1}\times\cdots\times\mathbb{R}^{p_n}}\mathcal{X}_E(\mathbf{g}(\mathbf{x}_1,\cdots,\mathbf{x}_n))\,d\lambda_{(\mathbf{X}_1,\cdots,\mathbf{X}_n)}.\end{aligned}$$

This proves 31.3 in the case when h is $\mathcal{X}_E$. To prove it in the general case, approximate the nonnegative Borel measurable function with simple functions for which the formula is true, and use the monotone convergence theorem.

It remains to prove the last assertion that functions of independent random vectors are also independent random vectors. Let E be a Borel set in $\mathbb{R}^{k_1}\times\cdots\times\mathbb{R}^{k_n}$. Then for

$$\pi_i(\mathbf{x}_1,\cdots,\mathbf{x}_n)\equiv\mathbf{x}_i,$$

$$\int_{\mathbb{R}^{k_1}\times\cdots\times\mathbb{R}^{k_n}}\mathcal{X}_E d\lambda_{(\mathbf{g}_1(\mathbf{X}_1),\cdots,\mathbf{g}_n(\mathbf{X}_n))}$$

$$\equiv\int_{\mathbb{R}^{p_1}\times\cdots\times\mathbb{R}^{p_n}}\mathcal{X}_E\circ(\mathbf{g}_1\circ\pi_1,\cdots,\mathbf{g}_n\circ\pi_n)\,d\lambda_{(\mathbf{X}_1,\cdots,\mathbf{X}_n)}$$

$$=\int_{\mathbb{R}^{p_1}}\cdots\int_{\mathbb{R}^{p_n}}\mathcal{X}_E\circ(\mathbf{g}_1\circ\pi_1,\cdots,\mathbf{g}_n\circ\pi_n)\,d\lambda_{\mathbf{X}_n}\cdots d\lambda_{\mathbf{X}_1}$$

$$=\int_{\mathbb{R}^{k_1}}\cdots\int_{\mathbb{R}^{k_n}}\mathcal{X}_E d\lambda_{\mathbf{g}_n(\mathbf{X}_n)}\cdots d\lambda_{\mathbf{g}_1(\mathbf{X}_1)}$$

and this proves the last assertion.

Proposition 31.15 *Let $\nu_1,\cdots,\nu_n$ be Radon probability measures defined on $\mathbb{R}^p$. Then there exists a probability space and independent random vectors $\{\mathbf{X}_1,\cdots,\mathbf{X}_n\}$ defined on this probability space such that $\lambda_{\mathbf{X}_i}=\nu_i$.*

Proof: Let $(\Omega,\mathcal{S},P)\equiv((\mathbb{R}^p)^n,\mathcal{S}_1\times\cdots\times\mathcal{S}_n,\nu_1\times\cdots\times\nu_n)$ where this is just the product σ algebra and product measure which satisfies the following for measurable rectangles.

$$(\nu_1\times\cdots\times\nu_n)\left(\prod_{i=1}^n E_i\right)=\prod_{i=1}^n\nu_i(E_i).$$

Now let $\mathbf{X}_i(\mathbf{x}_1,\cdots,\mathbf{x}_i,\cdots,\mathbf{x}_n)=\mathbf{x}_i$. Then from the definition, if E is a Borel set in $\mathbb{R}^p$,

$$\lambda_{\mathbf{X}_i}(E)\equiv P\{\mathbf{X}_i\in E\}$$

$$=(\nu_1\times\cdots\times\nu_n)(\mathbb{R}^p\times\cdots\times E\times\cdots\times\mathbb{R}^p)=\nu_i(E).$$

Let $\mathcal{M}$ consist of all Borel sets of $(\mathbb{R}^p)^n$ such that

$$\int_{\mathbb{R}^p}\cdots\int_{\mathbb{R}^p}\mathcal{X}_E(\mathbf{x}_1,\cdots,\mathbf{x}_n)\,d\lambda_{\mathbf{X}_1}\cdots d\lambda_{\mathbf{X}_n}=\int_{(\mathbb{R}^p)^n}\mathcal{X}_E d\lambda_{(\mathbf{X}_1,\cdots,\mathbf{X}_n)}.$$

From what was just shown and the definition of $(\nu_1\times\cdots\times\nu_n)$ that $\mathcal{M}$ contains all sets of the form $\prod_{i=1}^n E_i$ where each $E_i\in$ Borel sets of $\mathbb{R}^p$. Therefore, $\mathcal{M}$ contains the algebra of all finite disjoint unions of such sets. It is also clear that $\mathcal{M}$ is a monotone class and so by the theorem on monotone classes, $\mathcal{M}$ equals the Borel sets. Therefore, the given random vectors are independent and this proves the proposition.

The following Lemma was proved earlier in a different way.

Lemma 31.16 *If $\{X_i\}_{i=1}^n$ are independent random variables having values in $\mathbb{R}$,*

$$E\left(\prod_{i=1}^n X_i\right) = \prod_{i=1}^n E(X_i).$$

Proof: By Lemma 31.14 and denoting by P the product, $\prod_{i=1}^n X_i$,

$$\begin{aligned} E\left(\prod_{i=1}^n X_i\right) &= \int_{\mathbb{R}} z d\lambda_P(z) = \int_{\mathbb{R}\times\mathbb{R}} \prod_{i=1}^n x_i d\lambda_{(X_1,\cdots,X_n)} \\ &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{i=1}^n x_i d\lambda_{X_1} \cdots d\lambda_{X_n} = \prod_{i=1}^n E(X_i). \end{aligned}$$

There is a way to tell if random vectors are independent by using their characteristic functions.

Proposition 31.17 *If $\mathbf{X}_1$ and $\mathbf{X}_2$ are random vectors having values in $\mathbb{R}^{p_1}$ and $\mathbb{R}^{p_2}$ respectively, then the random vectors are independent if and only if*

$$E\left(e^{iP}\right) = \prod_{j=1}^2 E\left(e^{i\mathbf{t}_j\cdot\mathbf{X}_j}\right)$$

where $P \equiv \sum_{j=1}^2 \mathbf{t}_j \cdot \mathbf{X}_j$ for $\mathbf{t}_j \in \mathbb{R}^{p_j}$. More generally, if $\mathbf{X}_i$ is a random vector having values in $\mathbb{R}^{p_i}$ for $i = 1, 2, \cdots, n$, and if $P = \sum_{j=1}^n \mathbf{t}_j \cdot \mathbf{X}_j$, then the random vectors are independent if and only if

$$E\left(e^{iP}\right) = \prod_{j=1}^n E\left(e^{i\mathbf{t}_j\cdot\mathbf{X}_j}\right).$$

The proof of this proposition will depend on the following lemma.

Lemma 31.18 *Let $\mathbf{Y}$ be a random vector with values in $\mathbb{R}^p$ and let f be bounded and measurable with respect to the Radon measure, $\lambda_{\mathbf{Y}}$, and satisfy*

$$\int f(\mathbf{y}) e^{i\mathbf{t}\cdot\mathbf{y}} d\lambda_{\mathbf{Y}} = 0$$

for all $\mathbf{t} \in \mathbb{R}^p$. Then $f(\mathbf{y}) = 0$ for $\lambda_{\mathbf{Y}}$ a.e. $\mathbf{y}$.

Proof: The proof is just like the proof of Theorem 31.10 on Page 860 applied to the measure, $f(\mathbf{y}) d\lambda_{\mathbf{Y}}$. Thus $\int_E f(\mathbf{y}) d\lambda_{\mathbf{Y}} = 0$ for all E Borel. Hence $f(\mathbf{y}) = 0$ a.e.

Proof of the proposition: If the $\mathbf{X}_j$ are independent, the formula follows from Lemma 31.16 and Lemma 31.14.

Now suppose the formula holds. Then

$$\int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_1}} e^{i\mathbf{t}_1\cdot\mathbf{x}_1} e^{i\mathbf{t}_2\cdot\mathbf{x}_2} d\lambda_{\mathbf{X}_1} d\lambda_{\mathbf{X}_2} = E\left(e^{iP}\right)$$

$$= \int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_1}} e^{i\mathbf{t}_1\cdot\mathbf{x}_1} e^{i\mathbf{t}_2\cdot\mathbf{x}_2} d\lambda_{\mathbf{X}_1|\mathbf{x}_2} d\lambda_{\mathbf{X}_2}.$$

Now apply Lemma 31.18 to conclude that

$$\int_{\mathbb{R}^{p_1}} e^{i\mathbf{t}_1\cdot\mathbf{x}_1} d\lambda_{\mathbf{X}_1} = \int_{\mathbb{R}^{p_1}} e^{i\mathbf{t}_1\cdot\mathbf{x}_1} d\lambda_{\mathbf{X}_1|\mathbf{x}_2} \tag{31.4}$$

for $\lambda_{\mathbf{X}_2}$ a.e. $\mathbf{x}_2$, the exceptional set depending on $\mathbf{t}_1$. Therefore, taking the union of all exceptional sets corresponding to $\mathbf{t}_1 \in \mathbb{Q}^{p_1}$, it follows by continuity and the dominated convergence theorem that 31.4 holds for all $\mathbf{t}_1$ whenever $\mathbf{x}_2$ is not an element of this exceptional set of measure zero. Therefore, for such $\mathbf{x}_2$, Theorem 31.10 applies and it follows $\lambda_{\mathbf{X}_1|\mathbf{x}_2} = \lambda_{\mathbf{X}_1}$ for $\lambda_{\mathbf{X}_2}$ a.e. $\mathbf{x}_2$. Hence, if E is a Borel set in $\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, $\int_{\mathbb{R}^{p_1+p_2}} \mathcal{X}_E d\lambda_{(\mathbf{X}_1,\mathbf{X}_2)} = \int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_1}} \mathcal{X}_E d\lambda_{\mathbf{X}_1|\mathbf{x}_2} d\lambda_{\mathbf{X}_2} = \int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_1}} \mathcal{X}_E d\lambda_{\mathbf{X}_1} d\lambda_{\mathbf{X}_2}$. A repeat of the above argument will give the iterated integral in the reverse order or else one could apply Fubini's theorem to obtain this. The proposition also holds if 2 is replaced with n and the argument is a longer version of what was just presented. This proves the proposition.

With this preparation, it is time to present the Doob Dynkin lemma.I am not entirely sure what the Doob Dynkin lemma says actually. What follows is a generalization of what is identified as a special case of this lemma in [42]. I am not sure I have the right generalization. However, it is a very interesting lemma regardless of its name.

Lemma 31.19 *Suppose* $\mathbf{X}, \mathbf{Y}_1, \mathbf{Y}_2, \cdots, \mathbf{Y}_k$ *are random vectors,* $\mathbf{X}$ *having values in* $\mathbb{R}^n$ *and* $\mathbf{Y}_j$ *having values in* $\mathbb{R}^{p_j}$ *and*

$$\mathbf{X}, \mathbf{Y}_j \in L^1(\Omega).$$

Suppose $\mathbf{X}$ *is* $\mathcal{H}_{(\mathbf{Y}_1,\cdots,Y_k)}$ *measurable. Thus*

$$\left\{\mathbf{X}^{-1}(E) : E \text{ Borel}\right\} \subseteq \left\{(\mathbf{Y}_1, \cdots, \mathbf{Y}_k)^{-1}(F) : F \text{ is Borel in } \prod_{j=1}^{k} \mathbb{R}^{p_j}\right\}$$

Then there exists a Borel function, $\mathbf{g} : \prod_{j=1}^{k} \mathbb{R}^{p_j} \to \mathbb{R}^n$ *such that*

$$\mathbf{X} = \mathbf{g}(\mathbf{Y}_1, \mathbf{Y}_2, \cdots, \mathbf{Y}_k).$$

Proof: For the sake of brevity, denote by $\mathbf{Y}$ the vector $(\mathbf{Y}_1, \cdots, \mathbf{Y}_k)$ and by $\mathbf{y}$ the vector $(\mathbf{y}_1, \cdots, \mathbf{y}_k)$ and let $\prod_{j=1}^{k} \mathbb{R}^{p_j} \equiv \mathbb{R}^P$. For E a Borel set of $\mathbb{R}^n$,

$$\begin{aligned}\int_{\mathbf{Y}^{-1}(E)} \mathbf{X} dP &= \int_{\mathbb{R}^n \times \mathbb{R}^P} \mathcal{X}_{\mathbb{R}^n \times E}(\mathbf{x}, \mathbf{y}) \, \mathbf{x} d\lambda_{(\mathbf{X},\mathbf{Y})} \\ &= \int_E \int_{\mathbb{R}^n} \mathbf{x} d\lambda_{\mathbf{X}|\mathbf{y}} d\lambda_{\mathbf{Y}}. \end{aligned} \tag{31.5}$$

Consider the function

$$\mathbf{y} \to \int_{\mathbb{R}^n} \mathbf{x} d\lambda_{\mathbf{X}|\mathbf{y}}.$$

Since $d\lambda_{\mathbf{Y}}$ is a Radon measure having inner and outer regularity, it follows the above function is equal to a Borel function for $\lambda_{\mathbf{Y}}$ a.e. $\mathbf{y}$. This function will be denoted by $\mathbf{g}$. Then from 31.5

$$\begin{aligned}\int_{\mathbf{Y}^{-1}(E)} \mathbf{X} dP &= \int_E \mathbf{g}(\mathbf{y})\, d\lambda_{\mathbf{Y}} = \int_{\mathbb{R}^p} \mathcal{X}_E(\mathbf{y})\, \mathbf{g}(\mathbf{y})\, d\lambda_{\mathbf{Y}} \\ &= \int_\Omega \mathcal{X}_E(\mathbf{Y}(\omega))\, \mathbf{g}(\mathbf{Y}(\omega))\, dP \\ &= \int_{\mathbf{Y}^{-1}(E)} \mathbf{g}(\mathbf{Y}(\omega))\, dP\end{aligned}$$

and since $\mathbf{Y}^{-1}(E)$ is an arbitrary element of $\mathcal{H}_{\mathbf{Y}}$, this shows that since $\mathbf{X}$ is $\mathcal{H}_{\mathbf{Y}}$ measurable,

$$\mathbf{X} = E(\mathbf{X}|\mathcal{H}_{\mathbf{Y}}) = \mathbf{g}(\mathbf{Y}).$$

This proves the lemma.

Note also that as part of this argument, it is shown that if the symbol,

$$E(\mathbf{X}|\mathbf{y}_1, \cdots, \mathbf{y}_k)$$

is defined as the function $\mathbf{g}$ in the above, then for a.e. ω,

$$E\left(\mathbf{X}|\mathcal{H}_{(\mathbf{Y}_1,\mathbf{Y}_2,\cdots,\mathbf{Y}_k)}\right)(\omega) = E(\mathbf{X}|\mathbf{Y}_1(\omega), \mathbf{Y}_2(\omega), \cdots, \mathbf{Y}_k(\omega))$$

which is a fairly attractive formula.

31.3 The Multivariate Normal Distribution

Definition 31.20 *A random vector,* $\mathbf{X}$, *with values in* $\mathbb{R}^p$ *has a multivariate normal distribution written as* $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ *if for all Borel* $E \subseteq \mathbb{R}^p$,

$$\lambda_{\mathbf{X}}(E) = \int_{\mathbb{R}^p} \mathcal{X}_E(\mathbf{x}) \frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^* \Sigma^{-1} (\mathbf{x}-\mathbf{m})} dx$$

for μ *a given vector and* Σ *a given positive definite symmetric matrix.*

Theorem 31.21 *For* $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma), \mathbf{m} = E(\mathbf{X})$ *and*

$$\Sigma = E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^*\right).$$

Proof: Let R be an orthogonal transformation such that

$$R\Sigma R^* = D = diag\left(\sigma_1^2, \cdots, \sigma_p^2\right).$$

Changing the variable by $\mathbf{x}-\mathbf{m}=R^{*}\mathbf{y}$,

$$\begin{aligned}
E(\mathbf{X}) &\equiv \int_{\mathbb{R}^p} \mathbf{x} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^{*}\Sigma^{-1}(\mathbf{x}-\mathbf{m})} dx \left(\frac{1}{(2\pi)^{p/2}\det(\Sigma)^{1/2}}\right) \\
&= \int_{\mathbb{R}^p} (R^{*}\mathbf{y}+\mathbf{m}) e^{-\frac{1}{2}\mathbf{y}^{*}D^{-1}\mathbf{y}} dy \left(\frac{1}{(2\pi)^{p/2}\prod_{i=1}^{p}\sigma_i}\right) \\
&= \mathbf{m}\int_{\mathbb{R}^p} e^{-\frac{1}{2}\mathbf{y}^{*}D^{-1}\mathbf{y}} dy \left(\frac{1}{(2\pi)^{p/2}\prod_{i=1}^{p}\sigma_i}\right) = \mathbf{m}
\end{aligned}$$

by Fubini's theorem and the easy to establish formula

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} e^{-\frac{y^2}{2\sigma^2}} dy = 1.$$

Next let $M \equiv E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^{*}\right)$. Thus, changing the variable as above by $\mathbf{x}-\mathbf{m}=R^{*}\mathbf{y}$

$$\begin{aligned}
M &= \int_{\mathbb{R}^p} (\mathbf{x}-\mathbf{m})(\mathbf{x}-\mathbf{m})^{*} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^{*}\Sigma^{-1}(\mathbf{x}-\mathbf{m})} dx \left(\frac{1}{(2\pi)^{p/2}\det(\Sigma)^{1/2}}\right) \\
&= R^{*}\int_{\mathbb{R}^p} \mathbf{y}\mathbf{y}^{*} e^{-\frac{1}{2}\mathbf{y}^{*}D^{-1}\mathbf{y}} dy \left(\frac{1}{(2\pi)^{p/2}\prod_{i=1}^{p}\sigma_i}\right) R
\end{aligned}$$

Therefore,

$$(RMR^{*})_{ij} = \int_{\mathbb{R}^p} y_i y_j e^{-\frac{1}{2}\mathbf{y}^{*}D^{-1}\mathbf{y}} dy \left(\frac{1}{(2\pi)^{p/2}\prod_{i=1}^{p}\sigma_i}\right) = 0,$$

so; RMR^{*} is a diagonal matrix.

$$(RMR^{*})_{ii} = \int_{\mathbb{R}^p} y_i^2 e^{-\frac{1}{2}\mathbf{y}^{*}D^{-1}\mathbf{y}} dy \left(\frac{1}{(2\pi)^{p/2}\prod_{i=1}^{p}\sigma_i}\right).$$

Using Fubini's theorem and the easy to establish equations,

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} e^{-\frac{y^2}{2\sigma^2}} dy = 1,\ \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} y^2 e^{-\frac{y^2}{2\sigma^2}} dy = \sigma^2,$$

it follows $(RMR^{*})_{ii} = \sigma_i^2$. Hence $RMR^{*} = D$ and so $M = R^{*}DR = \Sigma$. This proves the theorem.

Theorem 31.22 *Suppose* $\mathbf{X}_1 \sim N_p(\mathbf{m}_1, \Sigma_1)$, $\mathbf{X}_2 \sim N_p(\mathbf{m}_2, \Sigma_2)$ *and the two random vectors are independent. Then*

$$\mathbf{X}_1 + \mathbf{X}_2 \sim N_p(\mathbf{m}_1 + \mathbf{m}_2, \Sigma_1 + \Sigma_2). \tag{31.6}$$

Also, if $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ *then* $-\mathbf{X} \sim N_p(-\mathbf{m}, \Sigma)$. *Furthermore, if* $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ *then*

$$E\left(e^{i\mathbf{t}\cdot\mathbf{X}}\right) = e^{i\mathbf{t}\cdot\mathbf{m}} e^{-\frac{1}{2}\mathbf{t}^*\Sigma\mathbf{t}} \tag{31.7}$$

Also if a is a constant and $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ *then* $a\mathbf{X} \sim N_p\left(a\mathbf{m}, a^2\Sigma\right)$.

Proof: Consider $E\left(e^{i\mathbf{t}\cdot\mathbf{X}}\right)$ for $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$.

$$E\left(e^{i\mathbf{t}\cdot\mathbf{X}}\right) \equiv \frac{1}{(2\pi)^{p/2}(\det \Sigma)^{1/2}} \int_{\mathbb{R}^p} e^{i\mathbf{t}\cdot\mathbf{x}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^*\Sigma^{-1}(\mathbf{x}-\mathbf{m})} dx.$$

Let R be an orthogonal transformation such that

$$R\Sigma R^* = D = diag\left(\sigma_1^2, \cdots, \sigma_p^2\right).$$

Then let $R(\mathbf{x}-\mathbf{m}) = \mathbf{y}$. Then

$$E\left(e^{i\mathbf{t}\cdot\mathbf{X}}\right) = \frac{1}{(2\pi)^{p/2}\prod_{i=1}^p \sigma_i} \int_{\mathbb{R}^p} e^{i\mathbf{t}\cdot(R^*\mathbf{y}+\mathbf{m})} e^{-\frac{1}{2}\mathbf{y}^* D^{-1}\mathbf{y}} dx.$$

Therefore

$$E\left(e^{i\mathbf{t}\cdot\mathbf{X}}\right) = \frac{1}{(2\pi)^{p/2}\prod_{i=1}^p \sigma_i} \int_{\mathbb{R}^p} e^{i\mathbf{s}\cdot(\mathbf{y}+R\mathbf{m})} e^{-\frac{1}{2}\mathbf{y}^* D^{-1}\mathbf{y}} dx$$

where $\mathbf{s} = R\mathbf{t}$. This equals

$$e^{i\mathbf{t}\cdot\mathbf{m}} \prod_{i=1}^p \left(\int_{\mathbb{R}} e^{is_i y_i} e^{-\frac{1}{2\sigma_i^2} y_i^2} dy_i\right) \frac{1}{\sqrt{2\pi}\sigma_i}$$

$$= e^{i\mathbf{t}\cdot\mathbf{m}} \prod_{i=1}^p \left(\int_{\mathbb{R}} e^{is_i\sigma_i u} e^{-\frac{1}{2}u^2} du\right) \frac{1}{\sqrt{2\pi}}$$

$$= e^{i\mathbf{t}\cdot\mathbf{m}} \prod_{i=1}^p e^{-\frac{1}{2}s_i^2\sigma_i^2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(u-is_i\sigma_i)^2} du$$

$$= e^{i\mathbf{t}\cdot\mathbf{m}} e^{-\frac{1}{2}\sum_{i=1}^p s_i^2\sigma_i^2} = e^{i\mathbf{t}\cdot\mathbf{m}} e^{-\frac{1}{2}\mathbf{t}^*\Sigma\mathbf{t}}$$

This proves 31.7.

Since $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent, $e^{i\mathbf{t}\cdot\mathbf{X}_1}$ and $e^{i\mathbf{t}\cdot\mathbf{X}_2}$ are also independent. Hence

$$E\left(e^{i\mathbf{t}\cdot\mathbf{X}_1+\mathbf{X}_2}\right) = E\left(e^{i\mathbf{t}\cdot\mathbf{X}_1}\right) E\left(e^{i\mathbf{t}\cdot\mathbf{X}_2}\right).$$

Thus,

$$\begin{aligned} E\left(e^{i\mathbf{t}\cdot\mathbf{X}_1+\mathbf{X}_2}\right) &= E\left(e^{i\mathbf{t}\cdot\mathbf{X}_1}\right) E\left(e^{i\mathbf{t}\cdot\mathbf{X}_2}\right) \\ &= e^{i\mathbf{t}\cdot\mathbf{m}_1} e^{-\frac{1}{2}\mathbf{t}^*\Sigma_1\mathbf{t}} e^{i\mathbf{t}\cdot\mathbf{m}_2} e^{-\frac{1}{2}\mathbf{t}^*\Sigma_2\mathbf{t}} \\ &= e^{i\mathbf{t}\cdot(\mathbf{m}_1+\mathbf{m}_2)} e^{-\frac{1}{2}\mathbf{t}^*(\Sigma_1+\Sigma_2)\mathbf{t}} \end{aligned}$$

which is the characteristic function of a random vector distributed as

$$N_p\left(\mathbf{m}_1+\mathbf{m}_2, \Sigma_1+\Sigma_2\right).$$

Now it follows that $\mathbf{X}_1+\mathbf{X}_2 \sim N_p\left(\mathbf{m}_1+\mathbf{m}_2, \Sigma_1+\Sigma_2\right)$ by Theorem 31.10. This proves 31.6.

The assertion about $-\mathbf{X}$ is also easy to see because

$$\begin{aligned}
E\left(e^{i\mathbf{t}\cdot(-\mathbf{X})}\right) &= E\left(e^{i(-\mathbf{t})\cdot\mathbf{X}}\right) \\
&= \frac{1}{(2\pi)^{p/2}\left(\det\Sigma\right)^{1/2}}\int_{\mathbb{R}^p} e^{i(-\mathbf{t})\cdot\mathbf{x}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^*\Sigma^{-1}(\mathbf{x}-\mathbf{m})}dx \\
&= \frac{1}{(2\pi)^{p/2}\left(\det\Sigma\right)^{1/2}}\int_{\mathbb{R}^p} e^{i\mathbf{t}\cdot\mathbf{x}} e^{-\frac{1}{2}(\mathbf{x}+\mathbf{m})^*\Sigma^{-1}(\mathbf{x}+\mathbf{m})}dx
\end{aligned}$$

which is the characteristic function of a random variable which is $N(-\mathbf{m},\Sigma)$. Theorem 31.10 again implies $-\mathbf{X} \sim N(-\mathbf{m},\Sigma)$. Finally consider the last claim. You apply what is known about $\mathbf{X}$ with $\mathbf{t}$ replaced with $a\mathbf{t}$ and then massage things. This gives the characteristic function for $a\mathbf{X}$ is given by

$$E\left(\exp\left(i\mathbf{t}\cdot a\mathbf{X}\right)\right) = \exp\left(i\mathbf{t}\cdot a\mathbf{m}\right)\exp\left(-\frac{1}{2}\mathbf{t}^*\Sigma a^2\mathbf{t}\right)$$

which is the characteristic function of a normal random vector having mean $a\mathbf{m}$ and covariance $a^2\Sigma$. This proves the theorem.

Following [42] a random vector has a generalized normal distribution if its characteristic function is given as

$$e^{i\mathbf{t}\cdot\mathbf{m}}e^{-\frac{1}{2}\mathbf{t}^*\Sigma\mathbf{t}} \tag{31.8}$$

where Σ is symmetric and has nonnegative eigenvalues. For a random real valued variable, $\mathbf{m}$ is scalar and so is Σ so the characteristic function of such a generalized normally distributed random variable is

$$e^{it\mu}e^{-\frac{1}{2}t^2\sigma^2} \tag{31.9}$$

These generalized normal distributions do not require Σ to be invertible, only that the eigenvalues by nonnegative. In one dimension this would correspond the characteristic function of a dirac measure having point mass 1 at μ. In higher dimensions, it could be a mixture of such things with more familiar things. I won't try very hard to distinguish between generalized normal distributions and normal distributions in which the covariance matrix has all positive eigenvalues.

Here are some other interesting results about normal distributions found in [42]. The next theorem has to do with the question whether a random vector is normally distributed in the above generalized sense.

Theorem 31.23 *Let* $\mathbf{X} = (X_1, \cdots, X_p)$ *where each* X_i *is a real valued random variable. Then* $\mathbf{X}$ *is normally distributed in the above generalized sense if and only* *if every linear combination,* $\sum_{j=1}^{p} a_i X_i$ *is normally distributed. In this case the mean of* $\mathbf{X}$ *is*

$$\mathbf{m} = (E(X_1), \cdots, E(X_p))$$

and the covariance matrix for $\mathbf{X}$ *is*

$$\Sigma_{jk} = E\left((X_j - m_j)(X_k - m_k)^*\right).$$

Proof: Suppose first $\mathbf{X}$ is normally distributed. Then its characteristic function is of the form

$$\phi_{\mathbf{X}}(\mathbf{t}) = E\left(e^{i\mathbf{t}\cdot\mathbf{X}}\right) = e^{i\mathbf{t}\cdot\mathbf{m}} e^{-\frac{1}{2}\mathbf{t}^*\Sigma\mathbf{t}}.$$

Then letting $\mathbf{a} = (a_1, \cdots, a_p)$

$$E\left(e^{it\sum_{j=1}^{p} a_i X_i}\right) = E\left(e^{it\mathbf{a}\cdot\mathbf{X}}\right) = e^{it\mathbf{a}\cdot\mathbf{m}} e^{-\frac{1}{2}\mathbf{a}^*\Sigma\mathbf{a}t^2}$$

which is the characteristic function of a normally distributed random variable with mean $\mathbf{a}\cdot\mathbf{m}$ and variance $\sigma^2 = \mathbf{a}^*\Sigma\mathbf{a}$. This proves half of the theorem.

Next suppose $\sum_{j=1}^{p} a_j X_j = \mathbf{a}\cdot\mathbf{X}$ is normally distributed with mean μ and variance σ^2 so that its characteristic function is given in 31.9. I will now relate μ and σ^2 to various quantities involving the X_j. Letting $m_j = E(X_j), \mathbf{m} = (m_1, \cdots, m_p)^*$

$$\begin{aligned} \mu &= \sum_{j=1}^{p} a_j E(X_j) = \sum_{j=1}^{p} a_j m_j,\ \sigma^2 = E\left(\left(\sum_{j=1}^{p} a_j X_j - \sum_{j=1}^{p} a_j m_j\right)^2\right) \\ &= E\left(\left(\sum_{j=1}^{p} a_j (X_j - m_j)\right)^2\right) = \sum_{j,k} a_j a_k E((X_j - m_j)(X_k - m_k)) \end{aligned}$$

It follows the mean of the normally distributed random variable, $\mathbf{a}\cdot\mathbf{X}$ is

$$\mu = \sum_j a_j m_j = \mathbf{a}\cdot\mathbf{m}$$

and its variance is

$$\sigma^2 = \mathbf{a}^* E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^*\right)\mathbf{a}$$

Therefore,

$$E\left(e^{it\mathbf{a}\cdot\mathbf{X}}\right) = e^{it\mu} e^{-\frac{1}{2}t^2\sigma^2}$$

$$= e^{it\mathbf{a}\cdot\mathbf{m}} e^{-\frac{1}{2}t^2\mathbf{a}^* E((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^*)\mathbf{a}}.$$

Then letting $\mathbf{s} = t\mathbf{a}$ this shows

$$\begin{aligned} E\left(e^{i\mathbf{s}\cdot\mathbf{X}}\right) &= e^{i\mathbf{s}\cdot\mathbf{m}} e^{-\frac{1}{2}\mathbf{s}^* E((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^*)\mathbf{s}} \\ &= e^{i\mathbf{s}\cdot\mathbf{m}} e^{-\frac{1}{2}\mathbf{s}^*\Sigma\mathbf{s}} \end{aligned}$$

which is the characteristic function of a normally distributed random variable with $\mathbf{m}$ given above and Σ given by

$$\Sigma_{jk} = E\left(\left(X_j - m_j\right)\left(X_k - m_k\right)\right).$$

This proves the theorem.

Corollary 31.24 *Let $\mathbf{X} = (X_1, \cdots, X_p), \mathbf{Y} = (Y_1, \cdots, Y_p)$ where each X_i, Y_i is a real valued random variable. Suppose also that for every $\mathbf{a} \in \mathbb{R}^p$, $\mathbf{a} \cdot \mathbf{X}$ and $\mathbf{a} \cdot \mathbf{Y}$ are both normally distributed with the same mean and variance. Then $\mathbf{X}$ and $\mathbf{Y}$ are both multivariate normal random vectors with the same mean and variance.*

Proof: In the Proof of Theorem 31.23 the proof implies that the characteristic functions of $\mathbf{a} \cdot \mathbf{X}$ and $\mathbf{a} \cdot \mathbf{Y}$ are both of the form

$$e^{itm} e^{-\frac{1}{2}\sigma^2 t^2}.$$

Then as in the proof of that theorem, it must be the case that

$$m = \sum_{j=1}^{p} a_j m_j$$

where $E(X_i) = m_i = E(Y_i)$ and

$$\begin{aligned} \sigma^2 &= \mathbf{a}^* E\left(\left(\mathbf{X} - \mathbf{m}\right)\left(\mathbf{X} - \mathbf{m}\right)^*\right)\mathbf{a} \\ &= \mathbf{a}^* E\left(\left(\mathbf{Y} - \mathbf{m}\right)\left(\mathbf{Y} - \mathbf{m}\right)^*\right)\mathbf{a} \end{aligned}$$

and this last equation must hold for every $\mathbf{a}$. Therefore,

$$E\left(\left(\mathbf{X} - \mathbf{m}\right)\left(\mathbf{X} - \mathbf{m}\right)^*\right) = E\left(\left(\mathbf{Y} - \mathbf{m}\right)\left(\mathbf{Y} - \mathbf{m}\right)^*\right) \equiv \Sigma$$

and so the characteristic function of both $\mathbf{X}$ and $\mathbf{Y}$ is $e^{i\mathbf{s}\cdot\mathbf{m}} e^{-\frac{1}{2}\mathbf{s}^*\Sigma\mathbf{s}}$ as in the proof of Theorem 31.23. This proves the corollary.

Theorem 31.25 *Suppose $\mathbf{X} = (X_1, \cdots, X_p)$ is normally distributed with mean $\mathbf{m}$ and covariance Σ. Then if X_1 is uncorrelated with any of the X_i, meaning*

$$E\left(\left(X_1 - m_1\right)\left(X_j - m_j\right)\right) = 0 \text{ for } j > 1,$$

then X_1 and $(X_2, \cdots, X_p)$ are both normally distributed and the two random vectors are independent. Here $m_j \equiv E(X_j)$. More generally, if the covariance matrix is a diagonal matrix, the random variables, $\{X_1, \cdots, X_p\}$ are linearly independent.

Proof: From Theorem 31.21

$$\Sigma = E\left(\left(\mathbf{X} - \mathbf{m}\right)\left(\mathbf{X} - \mathbf{m}\right)^*\right).$$

Then by assumption,

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \mathbf{0} \\ \mathbf{0} & \Sigma_{p-1} \end{pmatrix}. \tag{31.10}$$

I need to verify that if $E \in \mathcal{H}_{X_1}$ $(\sigma(X_1))$ and $F \in \mathcal{H}_{(X_2,\cdots,X_p)}$ $(\sigma(X_2,\cdots,X_p))$, then

$$P(E \cap F) = P(E) P(F).$$

Let $E = X_1^{-1}(A)$ and

$$F = (X_2,\cdots,X_p)^{-1}(B)$$

where A and B are Borel sets in $\mathbb{R}$ and $\mathbb{R}^{p-1}$ respectively. Thus I need to verify that

$$P([(X_1,(X_2,\cdots,X_p)) \in (A,B)]) =$$
$$\mu_{(X_1,(X_2,\cdots,X_p))}(A \times B) = \mu_{X_1}(A)\, \mu_{(X_2,\cdots,X_p)}(B). \tag{31.11}$$

Using 31.10, Fubini's theorem, and definitions,

$$\mu_{(X_1,(X_2,\cdots,X_p))}(A \times B) =$$

$$\int_{\mathbb{R}^p} \mathcal{X}_{A\times B}(\mathbf{x}) \frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^* \Sigma^{-1}(\mathbf{x}-\mathbf{m})} dx$$

$$= \int_{\mathbb{R}} \mathcal{X}_A(x_1) \int_{\mathbb{R}^{p-1}} \mathcal{X}_B(X_2,\cdots,X_p) \cdot \frac{1}{(2\pi)^{(p-1)/2} \sqrt{2\pi}\,(\sigma_1^2)^{1/2} \det(\Sigma_{p-1})^{1/2}} e^{\frac{-(x_1-m_1)^2}{2\sigma_1^2}} \cdot e^{\frac{-1}{2}(\mathbf{x}'-\mathbf{m}')^* \Sigma_{p-1}^{-1}(\mathbf{x}'-\mathbf{m}')} dx' dx_1$$

where $\mathbf{x}' = (x_2,\cdots,x_p)$ and $\mathbf{m}' = (m_2,\cdots,m_p)$. Now this equals

$$\int_{\mathbb{R}} \mathcal{X}_A(x_1) \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{\frac{-(x_1-m_1)^2}{2\sigma_1^2}} \int_B \frac{1}{(2\pi)^{(p-1)/2} \det(\Sigma_{p-1})^{1/2}} \cdot \tag{31.12}$$
$$e^{\frac{-1}{2}(\mathbf{x}'-\mathbf{m}')^* \Sigma_{p-1}^{-1}(\mathbf{x}'-\mathbf{m}')} dx' dx. \tag{31.13}$$

In case $B = \mathbb{R}^{p-1}$, the inside integral equals 1 and

$$\begin{aligned} \lambda_{X_1}(A) &= \lambda_{(X_1,(X_2,\cdots,X_p))}\left(A \times \mathbb{R}^{p-1}\right) \\ &= \int_{\mathbb{R}} \mathcal{X}_A(x_1) \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{\frac{-(x_1-m_1)^2}{2\sigma_1^2}} dx_1 \end{aligned}$$

which shows X_1 is normally distributed as claimed. Similarly, letting $A = \mathbb{R}$,

$$\begin{aligned} &\lambda_{(X_2,\cdots,X_p)}(B) \\ =\ &\lambda_{(X_1,(X_2,\cdots,X_p))}(\mathbb{R} \times B) \\ =\ &\int_B \frac{1}{(2\pi)^{(p-1)/2} \det(\Sigma_{p-1})^{1/2}} e^{\frac{-1}{2}(\mathbf{x}'-\mathbf{m}')^* \Sigma_{p-1}^{-1}(\mathbf{x}'-\mathbf{m}')} dx' \end{aligned}$$

and $(X_2, \cdots, X_p)$ is also normally distributed with mean $\mathbf{m}'$ and covariance Σ_{p-1}. Now from 31.12, 31.11 follows. In case the covariance matrix is diagonal, the above reasoning extends in an obvious way to prove the random variables, $\{X_1, \cdots, X_p\}$ are independent.

However, another way to prove this is to use Proposition 31.17 on Page 865 and consider the characteristic function. Let $E(X_j) = m_j$ and

$$P = \sum_{j=1}^{p} t_j X_j.$$

Then since $\mathbf{X}$ is normally distributed and the covariance is a diagonal,

$$D \equiv \begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_p^2 \end{pmatrix}$$

,

$$\begin{aligned} E\left(e^{iP}\right) &= E\left(e^{i\mathbf{t}\cdot(\mathbf{X}-\mathbf{m})}\right) = e^{i\mathbf{t}\cdot\mathbf{m}} e^{-\frac{1}{2}\mathbf{t}^*\Sigma\mathbf{t}} \\ &= \exp\left(\sum_{j=1}^{p} i t_j m_j - \frac{1}{2} t_j^2 \sigma_j^2\right) \\ &= \prod_{j=1}^{p} \exp\left(i t_j m_j - \frac{1}{2} t_j^2 \sigma_j^2\right) \end{aligned} \tag{31.14}$$

Also,

$$\begin{aligned} E\left(e^{it_j X_j}\right) &= E\left(\exp\left(i t_j X_j + \sum_{k \neq j} i0X_k\right)\right) \\ &= \exp\left(i t_j m_j - \frac{1}{2} t_j^2 \sigma_j^2\right) \end{aligned}$$

With 31.14, this shows

$$E\left(e^{iP}\right) = \prod_{j=1}^{p} E\left(e^{it_j X_j}\right)$$

which shows by Proposition 31.17 that the random variables,

$$\{X_1, \cdots, X_p\}$$

are independent. This proves the theorem.

31.4 The Central Limit Theorem

The central limit theorem is one of the most marvelous theorems in mathematics. It can be proved through the use of characteristic functions. Recall for $\mathbf{x} \in \mathbb{R}^p$,

$$||\mathbf{x}||_\infty \equiv \max\{|x_j|, j = 1, \cdots, p\}.$$

Also recall the definition of the distribution function for a random vector, $\mathbf{X}$.

$$F_{\mathbf{X}}(\mathbf{x}) \equiv P(X_j \leq x_j, j = 1, \cdots, p).$$

Definition 31.26 *Let $\{\mathbf{X}_n\}$ be random vectors with values in $\mathbb{R}^p$. Then $\{\lambda_{\mathbf{X}_n}\}_{n=1}^\infty$ is called "tight" if for all $\varepsilon > 0$ there exists a compact set, K_ε such that*

$$\lambda_{\mathbf{X}_n}([\mathbf{x} \notin K_\varepsilon]) < \varepsilon$$

for all $\lambda_{\mathbf{X}_n}$.

Lemma 31.27 *If $\mathbf{X}_n, \mathbf{X}$ are random vectors with values in $\mathbb{R}^p$ such that*

$$\lim_{n\to\infty} \phi_{\mathbf{X}_n}(\mathbf{t}) = \phi_{\mathbf{X}}(\mathbf{t})$$

for all $\mathbf{t}$, then $\{\lambda_{\mathbf{X}_n}\}_{n=1}^\infty$ is tight.

Proof: Let $\mathbf{e}_j$ be the j^{th} standard unit basis vector.

$$\begin{aligned}
& \left|\frac{1}{u}\int_{-u}^{u}\left(1-\phi_{\mathbf{X}_n}(t\mathbf{e}_j)\right)dt\right| \\
= & \left|\frac{1}{u}\int_{-u}^{u}\left(1-\int_{\mathbb{R}^p}e^{itx_j}d\lambda_{\mathbf{X}_n}\right)dt\right| \\
= & \left|\int_{\mathbb{R}^p}\frac{1}{u}\int_{-u}^{u}\left(1-e^{itx_j}\right)dtd\lambda_{\mathbf{X}_n}(x)\right| \\
= & \left|2\int_{\mathbb{R}^p}\left(1-\frac{\sin(ux_j)}{ux_j}\right)d\lambda_{\mathbf{X}_n}(x)\right| \\
\geq & \ 2\int_{\left[|x_j|\geq\frac{2}{u}\right]}\left(1-\frac{1}{|ux_j|}\right)d\lambda_{\mathbf{X}_n}(x) \\
\geq & \ 2\int_{\left[|x_j|\geq\frac{2}{u}\right]}\left(1-\frac{1}{|u|(2/u)}\right)d\lambda_{\mathbf{X}_n}(x) \\
= & \int_{\left[|x_j|\geq\frac{2}{u}\right]}1d\lambda_{\mathbf{X}_n}(x) \\
= & \ \lambda_{\mathbf{X}_n}\left(\left[\mathbf{x}:|x_j|\geq\frac{2}{u}\right]\right).
\end{aligned}$$

If $\varepsilon > 0$ is given, there exists $r > 0$ such that if $u \leq r$,

$$\frac{1}{u}\int_{-u}^{u}(1-\phi_{\mathbf{X}}(t\mathbf{e}_j))\,dt < \varepsilon/p$$

for all $j = 1, \cdots, p$ and so, by the dominated convergence theorem, the same is true with $\phi_{\mathbf{X}_n}$ in place of $\phi_{\mathbf{X}}$ provided n is large enough, say $n \geq N(u)$. Thus, if $u \leq r$, and $n \geq N(u)$,

$$\lambda_{\mathbf{X}_n}\left(\left[\mathbf{x} : |x_j| \geq \frac{2}{u}\right]\right) < \varepsilon/p$$

for all $j \in \{1, \cdots, p\}$. It follows that for $u \leq r$ and $n \geq N(u)$,

$$\lambda_{\mathbf{X}_n}\left(\left[\mathbf{x} : ||\mathbf{x}||_{\infty} \geq \frac{2}{u}\right]\right) < \varepsilon.$$

This proves the lemma because there are only finitely many measures, $\lambda_{\mathbf{X}_n}$ for $n < N(u)$ and the compact set can be enlarged finitely many times to obtain a single compact set, K_ε such that for all $n, \lambda_{\mathbf{X}_n}([\mathbf{x} \notin K_\varepsilon]) < \varepsilon$. This proves the lemma.

Lemma 31.28 *If $\phi_{\mathbf{X}_n}(\mathbf{t}) \to \phi_{\mathbf{X}}(\mathbf{t})$ for all $\mathbf{t}$, then whenever $\psi \in \mathfrak{S}$,*

$$\lambda_{\mathbf{X}_n}(\psi) \equiv \int_{\mathbb{R}^p} \psi(\mathbf{y})\, d\lambda_{\mathbf{X}_n}(y) \to \int_{\mathbb{R}^p} \psi(\mathbf{y})\, d\lambda_{\mathbf{X}}(y) \equiv \lambda_{\mathbf{X}}(\psi)$$

as $n \to \infty$.

Proof: Recall that if $\mathbf{X}$ is any random vector, its characteristic function is given by

$$\phi_{\mathbf{X}}(\mathbf{y}) \equiv \int_{\mathbb{R}^p} e^{i\mathbf{y}\cdot\mathbf{x}} d\lambda_{\mathbf{X}}(x).$$

Also remember the inverse Fourier transform. Letting $\psi \in \mathfrak{S}$, the Schwartz class,

$$\begin{aligned} F^{-1}(\lambda_{\mathbf{X}})(\psi) &\equiv \lambda_{\mathbf{X}}\left(F^{-1}\psi\right) \equiv \int_{\mathbb{R}^p} F^{-1}\psi d\lambda_{\mathbf{X}} \\ &= \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p}\int_{\mathbb{R}^p} e^{i\mathbf{y}\cdot\mathbf{x}}\psi(\mathbf{x})\, dx d\lambda_{\mathbf{X}}(y) \\ &= \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \psi(\mathbf{x}) \int_{\mathbb{R}^p} e^{i\mathbf{y}\cdot\mathbf{x}} d\lambda_{\mathbf{X}}(y)\, dx \\ &= \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \psi(\mathbf{x})\, \phi_{\mathbf{X}}(\mathbf{x})\, dx \end{aligned}$$

and so, considered as elements of $\mathfrak{S}^*$,

$$F^{-1}(\lambda_{\mathbf{X}}) = \phi_{\mathbf{X}}(\cdot)(2\pi)^{-(p/2)} \in L^{\infty}.$$

By the dominated convergence theorem

$$\begin{aligned} (2\pi)^{p/2} F^{-1}(\lambda_{\mathbf{X}_n})(\psi) &\equiv \int_{\mathbb{R}^p} \phi_{\mathbf{X}_n}(\mathbf{t})\, \psi(\mathbf{t})\, dt \\ &\to \int_{\mathbb{R}^p} \phi_{\mathbf{X}}(\mathbf{t})\, \psi(\mathbf{t})\, dt \\ &= (2\pi)^{p/2} F^{-1}(\lambda_{\mathbf{X}})(\psi) \end{aligned}$$

whenever $\psi \in \mathfrak{S}$. Thus

$$\begin{aligned}\lambda_{\mathbf{X}_n}(\psi) &= FF^{-1}\lambda_{\mathbf{X}_n}(\psi) \equiv F^{-1}\lambda_{\mathbf{X}_n}(F\psi) \to F^{-1}\lambda_{\mathbf{X}}(F\psi) \\ &\equiv FF^{-1}\lambda_{\mathbf{X}}(\psi) = \lambda_{\mathbf{X}}(\psi).\end{aligned}$$

This proves the lemma.

Lemma 31.29 *If* $\phi_{\mathbf{X}_n}(\mathbf{t}) \to \phi_{\mathbf{X}}(\mathbf{t})$, *then if* ψ *is any bounded uniformly continuous function,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}_n} = \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}}.$$

Proof: Let $\varepsilon > 0$ be given, let ψ be a bounded function in $C^\infty(\mathbb{R}^p)$. Now let $\eta \in C_c^\infty(Q_r)$ where $Q_r \equiv [-r,r]^p$ satisfy the additional requirement that $\eta = 1$ on $Q_{r/2}$ and $\eta(\mathbf{x}) \in [0,1]$ for all $\mathbf{x}$. By Lemma 31.27 the set, $\{\lambda_{\mathbf{X}_n}\}_{n=1}^\infty$, is tight and so if $\varepsilon > 0$ is given, there exists r sufficiently large such that for all n,

$$\int_{[\mathbf{x}\notin Q_{r/2}]} |1-\eta| |\psi| d\lambda_{\mathbf{X}_n} < \frac{\varepsilon}{3},$$

and

$$\int_{[\mathbf{x}\notin Q_{r/2}]} |1-\eta| |\psi| d\lambda_{\mathbf{X}} < \frac{\varepsilon}{3}.$$

Thus,

$$\begin{aligned}\left|\int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}_n} - \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}}\right| &\le \left|\int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}_n} - \int_{\mathbb{R}^p} \psi\eta d\lambda_{\mathbf{X}_n}\right| + \\ \left|\int_{\mathbb{R}^p} \psi\eta d\lambda_{\mathbf{X}_n} - \int_{\mathbb{R}^p} \psi\eta d\lambda_{\mathbf{X}}\right| &+ \left|\int_{\mathbb{R}^p} \psi\eta d\lambda_{\mathbf{X}} - \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}}\right| \\ &\le \frac{2\varepsilon}{3} + \left|\int_{\mathbb{R}^p} \psi\eta d\lambda_{\mathbf{X}_n} - \int_{\mathbb{R}^p} \psi\eta d\lambda_{\mathbf{X}}\right| < \varepsilon\end{aligned}$$

whenever n is large enough by Lemma 31.28 because $\psi\eta \in \mathfrak{S}$. This establishes the conclusion of the lemma in the case where ψ is also infinitely differentiable. To consider the general case, let ψ only be uniformly continuous and let $\psi_k = \psi * \phi_k$ where ϕ_k is a mollifier whose support is in $(-(1/k),(1/k))^p$. Then ψ_k converges uniformly to ψ and so the desired conclusion follows for ψ after a routine estimate.

Definition 31.30 *Let* μ *be a Radon measure on* $\mathbb{R}^p$. *A Borel set,* A, *is a* μ *continuity set if* $\mu(\partial A) = 0$ *where* $\partial A \equiv \overline{A} \setminus$ *interior* (A).

The main result is the following continuity theorem. More can be said about the equivalence of various criteria [9].

Theorem 31.31 *If* $\phi_{\mathbf{X}_n}(\mathbf{t}) \to \phi_{\mathbf{X}}(\mathbf{t})$ *then* $\lambda_{\mathbf{X}_n}(A) \to \lambda_{\mathbf{X}}(A)$ *whenever* A *is a* $\lambda_{\mathbf{X}}$ *continuity set.*

Proof: First suppose K is a closed set and let

$$\psi_k(\mathbf{x}) \equiv (1 - k \operatorname{dist}(\mathbf{x},K))^{+}.$$

Thus, since K is closed $\lim_{k\to\infty}\psi_k(\mathbf{x}) = \mathcal{X}_K(\mathbf{x})$. Choose k large enough that

$$\int_{\mathbb{R}^p} \psi_k d\lambda_{\mathbf{X}} \leq \lambda_{\mathbf{X}}(K) + \varepsilon.$$

Then by Lemma 31.29, applied to the bounded uniformly continuous function ψ_k,

$$\limsup_{n\to\infty} \lambda_{\mathbf{X}_n}(K) \leq \limsup_{n\to\infty} \int \psi_k d\lambda_{\mathbf{X}_n} = \int \psi_k d\lambda_{\mathbf{X}} \leq \lambda_{\mathbf{X}}(K) + \varepsilon.$$

Since ε is arbitrary, this shows

$$\limsup_{n\to\infty} \lambda_{\mathbf{X}_n}(K) \leq \lambda_{\mathbf{X}}(K)$$

for all K closed.

Next suppose V is open and let

$$\psi_k(\mathbf{x}) = 1 - \left(1 - k \operatorname{dist}\left(\mathbf{x}, \mathbf{V}^C\right)\right)^{+}.$$

Thus $\psi_k(\mathbf{x}) \in [0,1]$, $\psi_k = 1$ if $\operatorname{dist}\left(\mathbf{x},V^C\right) \geq 1/k$, and $\psi_k = 0$ on V^C. Since V is open, it follows

$$\lim_{k\to\infty} \psi_k(\mathbf{x}) = \mathcal{X}_V(\mathbf{x}).$$

Choose k large enough that

$$\int \psi_k d\lambda_{\mathbf{X}} \geq \lambda_{\mathbf{X}}(V) - \varepsilon.$$

Then by Lemma 31.29,

$$\liminf_{n\to\infty} \lambda_{\mathbf{X}_n}(V) \geq \liminf_{n\to\infty} \int \psi_k(\mathbf{x}) d\lambda_{\mathbf{X}_n} =$$

$$= \int \psi_k(\mathbf{x}) d\lambda_{\mathbf{X}} \geq \lambda_{\mathbf{X}}(V) - \varepsilon$$

and since ε is arbitrary,

$$\liminf_{n\to\infty} \lambda_{\mathbf{X}_n}(V) \geq \lambda_{\mathbf{X}}(V).$$

Now let $\lambda_{\mathbf{X}}(\partial A) = 0$ for A a Borel set.

$$\begin{aligned} \lambda_{\mathbf{X}}(\mathit{interior}(A)) &\leq \liminf_{n\to\infty} \lambda_{\mathbf{X}_n}(\mathit{interior}(A)) \leq \liminf_{n\to\infty} \lambda_{\mathbf{X}_n}(A) \leq \\ \limsup_{n\to\infty} \lambda_{\mathbf{X}_n}(A) &\leq \limsup_{n\to\infty} \lambda_{\mathbf{X}_n}\left(\overline{A}\right) \leq \lambda_{\mathbf{X}}\left(\overline{A}\right). \end{aligned}$$

But $\lambda_{\mathbf{X}}(\textit{interior}(A)) = \lambda_{\mathbf{X}}(\overline{A})$ by assumption and so $\lim_{n\to\infty}\lambda_{\mathbf{X}_n}(A) = \lambda_{\mathbf{X}}(A)$ as claimed. This proves the theorem.

As an application of this theorem the following is a version of the central limit theorem in the situation in which the limit distribution is multivariate normal. It concerns a sequence of random vectors, $\{\mathbf{X}_k\}_{k=1}^{\infty}$, which are identically distributed, have finite mean $\mathbf{m}$, and satisfy

$$\sup_k E\left(|\mathbf{X}_k|^2\right) < \infty. \tag{31.15}$$

Theorem 31.32 *Let $\{\mathbf{X}_k\}_{k=1}^{\infty}$ be random vectors satisfying 31.15, which are independent and identically distributed with mean $\mathbf{m}$ and positive definite covariance $\mathbf{\Sigma} \equiv E\left((\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^*\right)$. Let*

$$\mathbf{Z}_n \equiv \sum_{j=1}^{n} \frac{\mathbf{X}_j - \mathbf{m}}{\sqrt{n}}. \tag{31.16}$$

Then for $\mathbf{Z} \sim N_p(\mathbf{0}, \mathbf{\Sigma})$,

$$\lim_{n\to\infty} F_{\mathbf{Z}_n}(\mathbf{x}) = F_{\mathbf{Z}}(\mathbf{x}) \tag{31.17}$$

for all $\mathbf{x}$.

Proof: The characteristic function of $\mathbf{Z}_n$ is given by

$$\phi_{\mathbf{Z}_n}(\mathbf{t}) = E\left(e^{i\mathbf{t}\cdot\sum_{j=1}^{n}\frac{\mathbf{X}_j-\mu}{\sqrt{n}}}\right) = \prod_{j=1}^{n} E\left(e^{i\mathbf{t}\cdot\left(\frac{\mathbf{X}_j-\mu}{\sqrt{n}}\right)}\right).$$

By Taylor's theorem,

$$e^{ix} = 1 + ix - \frac{e^{i\theta x}x^2}{2}$$

for some $\theta \in [0,1]$ which depends on x. Denoting $\mathbf{X}_j$ as $\mathbf{X}$, this implies

$$e^{i\mathbf{t}\cdot\left(\frac{\mathbf{X}-\mu}{\sqrt{n}}\right)} = 1 + i\mathbf{t}\cdot\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}} - e^{i\theta\mathbf{t}\cdot\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}}\frac{(\mathbf{t}\cdot(\mathbf{X}-\mathbf{m}))^2}{2n}$$

where θ depends on $\mathbf{X}$ and $\mathbf{t}$ and is in $[0,1]$. The above equals

$$1 + i\mathbf{t}\cdot\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}} - \frac{(\mathbf{t}\cdot(\mathbf{X}-\mathbf{m}))^2}{2n}$$

$$+\left(1 - e^{i\theta\mathbf{t}\cdot\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}}\right)\frac{(\mathbf{t}\cdot(\mathbf{X}-\mathbf{m}))^2}{2n}.$$

Thus

$$\phi_{\mathbf{Z}_n}(\mathbf{t}) = \prod_{j=1}^{n}\left[1 - E\left(\frac{(\mathbf{t}\cdot(\mathbf{X}-\mathbf{m}))^2}{2n}\right)\right.$$

$$+E\left(\left(1-e^{i\theta\mathbf{t}\cdot\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}}\right)\frac{(\mathbf{t}\cdot(\mathbf{X}-\mathbf{m}))^2}{2n}\right)\Bigg]$$

$$=\prod_{j=1}^{n}\left[1-\frac{1}{2n}\mathbf{t}^*\Sigma\mathbf{t}+\frac{1}{2n}E\left(\left(1-e^{i\theta\mathbf{t}\cdot\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}}\right)(\mathbf{t}\cdot(\mathbf{X}-\mathbf{m}))^2\right)\right]. \tag{31.18}$$

(Note $(\mathbf{t}\cdot(\mathbf{X}-\mathbf{m}))^2=\mathbf{t}^*(\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^*\mathbf{t}$.) Now here is a simple inequality for complex numbers whose moduli are no larger than one. I will give a proof of this at the end. It follows easily by induction.

$$|z_1\cdots z_n-w_1\cdots w_n|\leq\sum_{k=1}^{n}|z_k-w_k|. \tag{31.19}$$

Also for each $\mathbf{t}$, and all n large enough,

$$\left|\frac{1}{2n}E\left(\left(1-e^{i\theta\mathbf{t}\cdot\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}}\right)(\mathbf{t}\cdot(\mathbf{X}-\mathbf{m}))^2\right)\right|<1.$$

Applying 31.19 to 31.18,

$$\phi_{\mathbf{Z}_n}(\mathbf{t})=\prod_{j=1}^{n}\left(1-\frac{1}{2n}\mathbf{t}^*\Sigma\mathbf{t}\right)+e_n$$

where

$$\begin{aligned}|e_n| &\leq \sum_{j=1}^{n}\left|\frac{1}{2n}E\left(\left(1-e^{i\theta\mathbf{t}\cdot\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}}\right)(\mathbf{t}\cdot(\mathbf{X}-\mathbf{m}))^2\right)\right| \\ &= \frac{1}{2}\left|E\left(\left(1-e^{i\theta\mathbf{t}\cdot\frac{\mathbf{X}-\mathbf{m}}{\sqrt{n}}}\right)(\mathbf{t}\cdot(\mathbf{X}-\mathbf{m}))^2\right)\right|\end{aligned}$$

which converges to 0 as $n\to\infty$ by the Dominated Convergence theorem. Therefore,

$$\lim_{n\to\infty}\left|\phi_{\mathbf{Z}_n}(\mathbf{t})-\left(1-\frac{\mathbf{t}^*\Sigma\mathbf{t}}{2n}\right)^n\right|=0$$

and so

$$\lim_{n\to\infty}\phi_{\mathbf{Z}_n}(\mathbf{t})=e^{-\frac{1}{2}\mathbf{t}^*\Sigma\mathbf{t}}=\phi_{\mathbf{Z}}(\mathbf{t})$$

where $\mathbf{Z}\sim N_p(\mathbf{0},\mathbf{\Sigma})$. Therefore, $F_{\mathbf{Z}_n}(\mathbf{x})\to F_{\mathbf{Z}}(\mathbf{x})$ for all $\mathbf{x}$ because $R_{\mathbf{x}}\equiv\prod_{k=1}^{p}(-\infty,x_k]$ is a set of $\lambda_{\mathbf{Z}}$ continuity due to the assumption that $\lambda_{\mathbf{Z}}\ll m_p$ which is implied by $\mathbf{Z}\sim N_p(\mathbf{0},\mathbf{\Sigma})$. This proves the theorem.

Here is the proof of the little inequality used above. The inequality is obviously true if $n=1$. Assume it is true for n. Then since all the numbers have absolute value no larger than one,

$$\begin{aligned}\left|\prod_{i=1}^{n+1}z_i-\prod_{i=1}^{n+1}w_i\right| &\leq \left|\prod_{i=1}^{n+1}z_i-z_{n+1}\prod_{i=1}^{n}w_i\right| \\ &\quad+\left|z_{n+1}\prod_{i=1}^{n}w_i-\prod_{i=1}^{n+1}w_i\right|\end{aligned}$$

$$\leq \left|\prod_{i=1}^{n} z_i - \prod_{i=1}^{n} w_i\right| + |z_{n+1} - w_{n+1}|$$
$$\leq \sum_{k=1}^{n+1} |z_k - w_k|$$

by induction.

Suppose $\mathbf{X}$ is a random vector with covariance Σ and mean $\mathbf{m}$, and suppose also that Σ^{-1} exists. Consider $\Sigma^{-(1/2)}(\mathbf{X}-\mathbf{m}) \equiv \mathbf{Y}$. Then $E(\mathbf{Y}) = 0$ and

$$\begin{aligned} E(\mathbf{YY}^*) &= E\left(\Sigma^{-(1/2)}(\mathbf{X}-\mathbf{m})(\mathbf{X}^*-\mu)\Sigma^{-(1/2)}\right) \\ &= \Sigma^{-(1/2)}E((\mathbf{X}-\mathbf{m})(\mathbf{X}^*-\mu))\Sigma^{-(1/2)} = I. \end{aligned}$$

Thus $\mathbf{Y}$ has zero mean and covariance I. This implies the following corollary to Theorem 31.32.

Corollary 31.33 *Let independent identically distributed random variables,*

$$\{\mathbf{X}_j\}_{j=1}^{\infty}$$

have mean $\mathbf{m}$ *and positive definite covariance* $\mathbf{\Sigma}$ *where* $\mathbf{\Sigma}^{-1}$ *exists. Then if*

$$\mathbf{Z}_n \equiv \sum_{j=1}^{n} \mathbf{\Sigma}^{-(1/2)} \frac{(\mathbf{X}_j - \mu)}{\sqrt{n}},$$

it follows that for $\mathbf{Z} \sim N_p(\mathbf{0}, I)$,

$$F_{\mathbf{Z}_n}(\mathbf{x}) \to F_{\mathbf{Z}}(\mathbf{x})$$

for all $\mathbf{x}$.

31.5 Brownian Motion

Definition 31.34 *A stochastic process is a set of random vectors,* $\{\mathbf{X}_t\}_{t\geq 0}$.

Brownian motion is a special kind of stochastic process. I will construct it and then summarize its properties.

First recall the Kolmogorov extension theorem listed next for convenience. In this theorem, M_t was a metric space having closed balls compact and I was a totally ordered index set. From now on M_t will equal $\mathbb{R}^n$ and the index set, I will be $[0, \infty)$.

Theorem 31.35 *(Kolmogorov extension theorem) For each finite set*

$$J = (t_1, \cdots, t_n) \subseteq I,$$

suppose there exists a Borel probability measure, $\nu_J = \nu_{t_1 \cdots t_n}$ *defined on the Borel sets of* $\prod_{t \in J} M_t$ *such that if*

$$(t_1, \cdots, t_n) \subseteq (s_1, \cdots, s_p),$$

then

$$\nu_{t_1 \cdots t_n}(F_{t_1} \times \cdots \times F_{t_n}) = \nu_{s_1 \cdots s_p}(G_{s_1} \times \cdots \times G_{s_p}) \tag{31.20}$$

where if $s_i = t_j$, *then* $G_{s_i} = F_{t_j}$ *and if* s_i *is not equal to any of the indices,* t_k, *then* $G_{s_i} = M'_{s_i}$. *Then there exists a probability space,* $(\Omega, P, \mathcal{F})$ *and measurable functions,* $X_t : \Omega \to M_t$ *for each* $t \in I$ *such that for each* $(t_1 \cdots t_n) \subseteq I$,

$$\nu_{t_1 \cdots t_n}(F_{t_1} \times \cdots \times F_{t_n}) = P([X_{t_1} \in F_{t_1}] \cap \cdots \cap [X_{t_n} \in F_{t_n}]). \tag{31.21}$$

Definition 31.36 *For* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $t > 0$ *define*

$$p(t, \mathbf{x}, \mathbf{y}) \equiv \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|\mathbf{y} - \mathbf{x}|^2}{2t}\right).$$

Then considered as a function of $\mathbf{y}$ *this is a normal distribution with mean* $\mathbf{x}$ *and covariance* tI. *In case* $t = 0$ *this is defined to be the measure* $\delta_{\mathbf{x}}$ *which is defined by*

$$\delta_{\mathbf{x}}(E) = 1 \text{ if } \mathbf{x} \in E \text{ and } 0 \text{ if } \mathbf{x} \notin E.$$

Now define for each increasing list $(t_1, t_2, \cdots, t_k)$, a measure defined as follows. For F a Borel set in $\mathbb{R}^{nk}$,

$$\begin{aligned} \nu_{t_1 t_2 \cdots t_k}(F) &\equiv \int_F p(t_1, \mathbf{x}, \mathbf{y}_1) p(t_2 - t_1, \mathbf{y}_1, \mathbf{y}_2) \\ &\quad \cdots p(t_k - t_{k-1}, \mathbf{y}_{k-1}, \mathbf{y}_k) \, dy_1 dy_2 \cdots dy_k. \end{aligned} \tag{31.22}$$

Since $\int_{\mathbb{R}^n} p(s, \mathbf{x}, \mathbf{y}) \, dy = 1$ whenever $s \geq 0$, this shows the conditions of the Kolmogorov extension theorem are satisfied for these measures and therefore there exists a probability space, $(\Omega, \mathcal{F}, P)$ and measurable functions, $\mathbf{B}_t$ for each $t \geq 0$ such that whenever the F_j are Borel sets,

$$\nu_{t_1 t_2 \cdots t_k}(F_{t_1} \times \cdots \times F_{t_k}) = P((\mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_k}) \in F_{t_1} \times \cdots \times F_{t_k})$$

Lemma 31.37 *Letting* $\mathbf{Z} = (\mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_k}) \in \mathbb{R}^{nk}$ *it follows* $\mathbf{Z}$ *is normally distributed. Its mean is*

$$\begin{pmatrix} \mathbf{x} & \cdots & \mathbf{x} \end{pmatrix} \in \mathbb{R}^{nk}$$

and its covariance is the $nk \times nk$ *matrix*

$$\begin{pmatrix} t_1 I_n & t_1 I_n & t_1 I_n & \cdots & t_1 I_n \\ t_1 I_n & t_2 I_n & t_2 I_n & \cdots & t_2 I_n \\ t_1 I_n & t_2 I_n & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ t_1 I_n & t_2 I_n & \cdots & \cdots & t_k I_n \end{pmatrix}.$$

Proof: To show this use Theorem 31.23. The components of $\mathbf{B}_{t_j}$ are independent and normally distributed because $\mathbf{B}_{t_j}$ is distributed as $\mathbf{y} \to p(t_j, \mathbf{x}, \mathbf{y})$ which is defined above. The off diagonal terms of the correlation matrix are zero and so by Theorem 31.25 the components are independent and all normally distributed. Denote by $B_{t_j r}$ the r^{th} component of $\mathbf{B}_{t_j}$. Thus the mean of $B_{t_j r}$ is x_r and the variance of $B_{t_j r}$ is t_j. Also $\mathbf{a} \cdot \mathbf{B}_{t_j}$ is normally distributed with mean $\sum a_r x_r$. To verify $\mathbf{Z}$ is normally distributed, it suffices to show that $\mathbf{a} \cdot \mathbf{Z}$ is normally distributed for $\mathbf{a} = (\mathbf{a}_1, \cdots, \mathbf{a}_k)$. Consider the case where $k = 2$. Then $\mathbf{Z}$ has values in $\mathbb{R}^{2n}$. I will directly calculate the characteristic function for $\mathbf{Z}$ in this case and then note that a similar pattern will hold for larger k.

$$
\begin{aligned}
& E(\exp(i\mathbf{u} \cdot \mathbf{Z})) \\
= & \int\int p(t_1, \mathbf{x}, \mathbf{y}_1)\, p(t_2 - t_1, \mathbf{y}_1, \mathbf{y}_2)\, e^{i\mathbf{u}_1 \cdot \mathbf{y}_1} e^{i\mathbf{u}_2 \cdot \mathbf{y}_2} dy_2 dy_1 \\
= & \int p(t_1, \mathbf{x}, \mathbf{y}_1)\, e^{i\mathbf{u}_1 \cdot \mathbf{y}_1} \int p(t_2 - t_1, \mathbf{y}_1, \mathbf{y}_2)\, e^{i\mathbf{u}_2 \cdot \mathbf{y}_2} dy_2 dy_1 \\
= & \int p(t_1, \mathbf{x}, \mathbf{y}_1)\, e^{i\mathbf{u}_1 \cdot \mathbf{y}_1} \left(\exp\left(i\mathbf{u}_2 \cdot \mathbf{y}_1 + \left(-\frac{1}{2} \left(\mathbf{u}_2^* (t_2 - t_1) I \mathbf{u}_2 \right) \right) \right) \right) dy_1 \\
= & \exp\left(-\frac{1}{2} \left(\mathbf{u}_2^* (t_2 - t_1) I \mathbf{u}_2 \right) \right) \int p(t_1, \mathbf{x}, \mathbf{y}_1)\, e^{i\mathbf{u}_1 \cdot \mathbf{y}_1} \left(\exp\left(i\mathbf{u}_2 \cdot \mathbf{y}_1 \right) \right) dy_1 \\
= & \exp\left(-\frac{1}{2} \left(\mathbf{u}_2^* (t_2 - t_1) I \mathbf{u}_2 \right) \right) \int p(t_1, \mathbf{x}, \mathbf{y}_1)\, e^{i(\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{y}_1} dy_1 \\
= & \exp\left(-\frac{1}{2} \left(\mathbf{u}_2^* (t_2 - t_1) I \mathbf{u}_2 \right) \right) \exp\left(-\frac{1}{2} (\mathbf{u}_1 + \mathbf{u}_2)^* t_1 I (\mathbf{u}_1 + \mathbf{u}_2) \right) \\
& \cdot \exp\left(i (\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{x} \right) \\
= & \exp\left(-\frac{1}{2} \left[\left(\mathbf{u}_2^* (t_2 - t_1) I \mathbf{u}_2 \right) + (\mathbf{u}_1 + \mathbf{u}_2)^* t_1 I (\mathbf{u}_1 + \mathbf{u}_2) \right] \right) \\
& \cdot \exp\left(i (\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{x} \right).
\end{aligned}
$$

The expression $\left(\mathbf{u}_2^* (t_2 - t_1) I \mathbf{u}_2 \right) + (\mathbf{u}_1 + \mathbf{u}_2)^* t_1 I (\mathbf{u}_1 + \mathbf{u}_2)$ equals

$$
\begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \begin{pmatrix} t_1 I & t_1 I \\ t_1 I & t_2 I \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}
$$

and the expression $i (\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{x}$ equals

$$
i \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x} & \mathbf{x} \end{pmatrix}
$$

and so in the case that $k = 2$, this shows $\mathbf{Z}$ is normally distributed with mean $\begin{pmatrix} \mathbf{x} & \mathbf{x} \end{pmatrix}$ and covariance

$$
\begin{pmatrix} t_1 I & t_1 I \\ t_1 I & t_2 I \end{pmatrix}.
$$

The pattern continues in this way. In general the mean is

$$
\begin{pmatrix} \mathbf{x} & \cdots & \mathbf{x} \end{pmatrix}
$$

and the covariance is of the form

$$\begin{pmatrix} t_1 I_k & t_1 I_k & t_1 I_k & \cdots & t_1 I_k \\ t_1 I_k & t_2 I_k & t_2 I_k & \cdots & t_2 I_k \\ t_1 I_k & t_2 I_k & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ t_1 I_k & t_2 I_k & \cdots & \cdots & t_k I_k \end{pmatrix}$$

This proves the lemma.

Continuing to follow [42],

$$E\left(\left|\mathbf{B}_t - \mathbf{x}\right|^2\right) = \sum_{r=1}^{n} E\left(\left(B_{tr} - x_r\right)^2\right) = \sum_{r=1}^{n} t = nt.$$

Then let $s < t$.

$$\begin{aligned} & E\left(\left(\mathbf{B}_t - \mathbf{x}\right)^* \left(\mathbf{B}_s - \mathbf{x}\right)\right) \\ = & \int \int p\left(s, \mathbf{x}, \mathbf{y}_1\right) p\left(t - s, \mathbf{y}_1, \mathbf{y}_2\right) \left(\mathbf{y}_1 - \mathbf{x}\right) \cdot \left(\mathbf{y}_2 - \mathbf{x}\right) dy_1 dy_2 \\ = & \int p\left(s, \mathbf{x}, \mathbf{y}_1\right) \left(\mathbf{y}_1 - \mathbf{x}\right) \cdot \int p\left(t - s, \mathbf{y}_1, \mathbf{y}_2\right) \left(\mathbf{y}_2 - \mathbf{x}\right) dy_2 dy_1 \end{aligned}$$

Consider the inner integral.

$$\begin{aligned} & \int p\left(t - s, \mathbf{y}_1, \mathbf{y}_2\right) \left(\mathbf{y}_2 - \mathbf{x}\right) dy_2 \\ = & \int p\left(t - s, \mathbf{y}_1, \mathbf{y}_2\right) \left(\left(\mathbf{y}_2 - \mathbf{y}_1\right) + \left(\mathbf{y}_1 - \mathbf{x}\right)\right) dy_2 \\ = & \int p\left(t - s, \mathbf{y}_1, \mathbf{y}_2\right) \left(\mathbf{y}_1 - \mathbf{x}\right) dy_2 = \left(\mathbf{y}_1 - \mathbf{x}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} E\left(\left(\mathbf{B}_t - \mathbf{x}\right)^* \left(\mathbf{B}_s - \mathbf{x}\right)\right) &= \int p\left(s, \mathbf{x}, \mathbf{y}_1\right) \left|\mathbf{y}_1 - \mathbf{x}\right|^2 dy_1 \\ &= E\left(\left|\mathbf{B}_s - \mathbf{x}\right|^2\right) = ns \end{aligned}$$

Now for $t \geq s$,

$$\begin{aligned} E\left(\left|\mathbf{B}_t - \mathbf{B}_s\right|^2\right) &= E\left(\left|\mathbf{B}_t - \mathbf{x}\right|^2 + \left|\mathbf{B}_s - \mathbf{x}\right|^2 - 2\left(\mathbf{B}_t - \mathbf{x}\right) \cdot \left(\mathbf{B}_s - \mathbf{x}\right)\right) \\ &= nt + ns - 2ns = n\left(t - s\right). \end{aligned}$$

Lemma 31.38 $\mathbf{B}_t$ *has independent increments. This means if* $t_1 < t_2 < \cdots < t_k$, *the random variables,*

$$\mathbf{B}_{t_1}, \mathbf{B}_{t_2} - \mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_k} - \mathbf{B}_{t_{k-1}}$$

are independent. In addition, these random variables are normally distributed.

Proof: $\mathbf{B}_{t_1}$ is normal and so is each of the $\mathbf{B}_{t_j}$. Also I claim that $\mathbf{B}_{t_j} - \mathbf{B}_{t_{j-1}}$ is normal with mean $\mathbf{0}$. I will show this next.

$$E\left(\exp\left(i\mathbf{u}\cdot\left(\mathbf{B}_{t_j} - \mathbf{B}_{t_{j-1}}\right)\right)\right) = E\left(\exp\left(i\mathbf{u}\cdot\mathbf{B}_{t_j}\right)\exp\left(-i\mathbf{u}\cdot\mathbf{B}_{t_{j-1}}\right)\right)$$

$$\begin{aligned}
&= \int\int p\left(t_{j-1},\mathbf{x},\mathbf{y}_1\right)p\left(t_j - t_{j-1},\mathbf{y}_1,\mathbf{y}_2\right)\exp\left(-i\mathbf{u}\cdot\mathbf{y}_1\right)\exp\left(i\mathbf{u}\cdot\mathbf{y}_2\right)dy_2dy_1 \\
&= \int p\left(t_{j-1},\mathbf{x},\mathbf{y}_1\right)\exp\left(-i\mathbf{u}\cdot\mathbf{y}_1\right)\int p\left(t_j - t_{j-1},\mathbf{y}_1,\mathbf{y}_2\right)\exp\left(i\mathbf{u}\cdot\mathbf{y}_2\right)dy_2dy_1 \\
&= \int p\left(t_{j-1},\mathbf{x},\mathbf{y}_1\right)\exp\left(-i\mathbf{u}\cdot\mathbf{y}_1\right)\exp\left(i\mathbf{u}\cdot\mathbf{y}_1\right)\exp\left(-\frac{1}{2}\left(t_j - t_{j-1}\right)|\mathbf{u}|^2\right)dy_1 \\
&= \exp\left(-\frac{1}{2}\left(t_j - t_{j-1}\right)|\mathbf{u}|^2\right). \qquad (31.23)
\end{aligned}$$

Therefore, $\mathbf{B}_{t_j} - \mathbf{B}_{t_{j-1}}$ is normal with covariance $\left(t_j - t_{j-1}\right)I$ and mean $\mathbf{0}$.

Next let $\mathbf{Z} = \left(\mathbf{B}_{t_1}, \mathbf{B}_{t_2} - \mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_k} - \mathbf{B}_{t_{k-1}}\right)$. I need to verify $\mathbf{Z}$ is normally distributed. Let $\mathbf{u} = \left(\mathbf{u}_1, \cdots, \mathbf{u}_k\right)$.

$$E\left(\exp\left(i\mathbf{u}\cdot\mathbf{Z}\right)\right) = E\left(\exp\left(i\mathbf{u}_1\cdot\mathbf{B}_{t_1}\right)\prod_{r=2}^{k}\exp\left(i\mathbf{u}_r\cdot\left(\mathbf{B}_{t_r} - \mathbf{B}_{t_{r-1}}\right)\right)\right)$$

$$\begin{aligned}
= &\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} p\left(t_1,\mathbf{x},\mathbf{y}_1\right)p\left(t_2 - t_1,\mathbf{y}_1,\mathbf{y}_2\right)\cdots p\left(t_k - t_{k-1},\mathbf{y}_{k-1},\mathbf{y}_k\right)\cdot \\
&\exp\left(i\mathbf{u}_1\cdot\mathbf{y}_1\right)\prod_{r=2}^{k}\exp\left(i\mathbf{u}_r\cdot\left(\mathbf{y}_r - \mathbf{y}_{r-1}\right)\right)dy_kdy_{k-1}\cdots dy_1.
\end{aligned}$$

The inside integral is

$$\int p\left(t_k - t_{k-1},\mathbf{y}_{k-1},\mathbf{y}_k\right)\exp\left(i\mathbf{u}_k\cdot\left(\mathbf{y}_k - \mathbf{y}_{k-1}\right)\right) = \exp\left(-\frac{1}{2}\left(t_k - t_{k-1}\right)|\mathbf{u}_k|^2\right)$$

which has no y variables left so I can factor it out and then work on the next inside integral which gives

$$\exp\left(-\frac{1}{2}\left(t_{k-1} - t_{k-2}\right)|\mathbf{u}_{k-1}|^2\right)$$

which also can be factored out. Continuing this way eventually obtains

$$\prod_{j=1}^{k}\exp\left(-\frac{1}{2}\left(t_j - t_{j-1}\right)|\mathbf{u}_k|^2\right)\int p\left(t_1,\mathbf{x},\mathbf{y}_1\right)\exp\left(i\mathbf{u}_1\cdot\mathbf{y}_1\right)dy_1$$

$$= \prod_{j=1}^{k}\exp\left(-\frac{1}{2}\left(t_j - t_{j-1}\right)|\mathbf{u}_k|^2\right)\exp\left(i\mathbf{u}_1\cdot\mathbf{x}\right)\exp\left(-\frac{1}{2}t_1|\mathbf{u}_1|^2\right).$$

Now let $\mathbf{m} = (\mathbf{x}, \mathbf{0}, \cdots, \mathbf{0})$ and let Σ be the matrix,

$$= \begin{pmatrix} t_1 I_n & & & 0 \\ & (t_2 - t_1) I_n & & \\ & & \ddots & \\ 0 & & & (t_k - t_{k-1}) I_n \end{pmatrix}$$

Then the above reduces to

$$\exp(i\mathbf{u} \cdot \mathbf{m}) \exp\left(-\frac{1}{2}\mathbf{u}^* \Sigma \mathbf{u}\right)$$

which shows that $\mathbf{Z}$ is normally distributed with covariance Σ and mean $\mathbf{m}$. It also shows that since the covariance matrix is diagonal, the component functions of $\mathbf{Z}$ are independent. This proves the lemma.

Next I will consider an estimate for the Brownian motion. By 31.23 the characteristic function for $\mathbf{B}_t - \mathbf{B}_s$ and $t > s$ is

$$E(\exp(i\mathbf{u} \cdot (\mathbf{B}_t - \mathbf{B}_s))) = \exp\left(-\frac{1}{2}(t-s)|\mathbf{u}|^2\right)$$

It follows upon taking a partial derivative with respect to u_j

$$\begin{aligned} & E(i(B_{tj} - B_{sj}) \exp(i\mathbf{u} \cdot (\mathbf{B}_t - \mathbf{B}_s))) \\ = & \ (-u_j(t-s)) \exp\left(-\frac{1}{2}(t-s)|\mathbf{u}|^2\right) \end{aligned}$$

and then taking another partial derivative with respect to u_j yields

$$\begin{aligned} & E\left(-(B_{tj} - B_{sj})^2 \exp(i\mathbf{u} \cdot (\mathbf{B}_t - \mathbf{B}_s))\right) \\ = & \ (-(t-s) - u_j(t-s)) \exp\left(-\frac{1}{2}(t-s)|\mathbf{u}|^2\right) \\ & + (-u_j(t-s))^2 \exp\left(-\frac{1}{2}(t-s)|\mathbf{u}|^2\right) \end{aligned}$$

This looks pretty good. Lets take another derivative with respect to u_j

$$E\left(-i(B_{tj} - B_{sj})^3 \exp(i\mathbf{u} \cdot (\mathbf{B}_t - \mathbf{B}_s))\right)$$

$$\begin{aligned} = & \ (-(t-s)) \exp\left(-\frac{1}{2}(t-s)|\mathbf{u}|^2\right) + (-(t-s) - u_j(t-s))(-u_j(t-s)) \cdot \\ & \exp\left(-\frac{1}{2}(t-s)|\mathbf{u}|^2\right) \\ & 2u_j(t-s)^2 \exp\left(-\frac{1}{2}(t-s)|\mathbf{u}|^2\right) + (-u_j(t-s))^3 \exp\left(-\frac{1}{2}(t-s)|\mathbf{u}|^2\right) \end{aligned}$$

Finally take yet another derivative with respect to u_j and then let $\mathbf{u} = \mathbf{0}$.

$$E\left((B_{tj} - B_{sj})^4\right) = (t-s)^2 + 2(t-s)^2 = 3(t-s)^2 .$$

This shows

$$E\left(\sum_{j=1}^{n} (B_{tj} - B_{sj})^4\right) = 3n(t-s)^2 .$$

But also $\sum_{j=1}^{n} (B_{tj} - B_{sj})^4 \geq (1/n) |\mathbf{B}_t - \mathbf{B}_s|^2$ and so

$$E\left(|\mathbf{B}_t - \mathbf{B}_s|^4\right) \leq 3n^2 (t-s)^2 . \tag{31.24}$$

With more work, you can show the $3n^2$ can be replaced with $n(n+2)$ but it is the inequality which is of interest here.

Before going further here is an interesting elementary lemma.

Lemma 31.39 *Let D be a dense subset of an interval, $I = [0,T]$ and suppose $\mathbf{X} : D \to \mathbb{R}^n$ satisfies*

$$|\mathbf{X}(d) - \mathbf{X}(d')| \leq C |d - d'|^\gamma$$

for all $d', d \in D$. Then $\mathbf{X}$ extends uniquely to $\mathbf{Y}$ defined on $[0,T]$ such that

$$|\mathbf{Y}(t) - \mathbf{Y}(t')| \leq C |t - t'|^\gamma .$$

Proof: Let $t \in I$ and let $d_k \to t$ where $d_k \in D$. Then $\{\mathbf{X}(d_k)\}$ is a Cauchy sequence because $|\mathbf{X}(d_k) - \mathbf{X}(d_m)| \leq C |d_k - d_m|^\gamma$. Therefore, $\mathbf{X}(d_k)$ converges. The thing it converges to will be called $\mathbf{Y}(t)$. Note this is well defined, giving $\mathbf{X}(t)$ if $t \in D$. Also, if $d_k \to t$ and $d_k' \to t$, then $|\mathbf{X}(d_k) - \mathbf{X}(d_k')| \leq C |d_k - d_k'|^\gamma$ and so $\mathbf{X}(d_k)$ and $\mathbf{X}(d_k')$ converge to the same thing. Therefore, it makes sense to define $\mathbf{Y}(t) \equiv \lim_{d \to t} \mathbf{X}(d)$. It only remains to verify the estimate. But letting $|d - t|$ and $|d' - t'|$ be small enough,

$$\begin{aligned} |\mathbf{Y}(t) - \mathbf{Y}(t')| &= |\mathbf{X}(d) - \mathbf{X}(d')| \\ &\leq C |d' - d| + \varepsilon \leq C |t - t'| + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves the existence part of the lemma. Uniqueness follows from observing that $\mathbf{Y}(t)$ must equal $\lim_{d \to t} \mathbf{X}(d)$. This proves the lemma.

The following is a very interesting theorem called the Kolmogorov Čentsov continuity theorem[33].

Theorem 31.40 *Suppose $\mathbf{X}_t$ is a random vector for each $t \in [0, \infty)$. Suppose also that for all $T > 0$ there exists a constant, C and positive numbers, α, β such that*

$$E\left(|\mathbf{X}_t - \mathbf{X}_s|^\alpha\right) \leq C |t - s|^{1+\beta} \tag{31.25}$$

Then there exist random vectors, $\mathbf{Y}_t$ such that for a.e. $\omega, t \to \mathbf{Y}_t(\omega)$ is continuous and $P([|\mathbf{X}_t - \mathbf{Y}_t| > 0]) = 0$.

Proof: Let r_j^m denote $j\left(\frac{T}{2^m}\right)$ where $j \in \{0,1,\cdots,2^m\}$. Also let $D_m = \left\{r_j^m\right\}_{j=1}^{2^m}$ and $D = \cup_{m=1}^{\infty} D_m$. Consider the set,

$$[|\mathbf{X}_t - \mathbf{X}_s| > \delta]$$

for $k = 1,2,\cdots$. By 31.25,

$$\begin{aligned} P([|\mathbf{X}_t - \mathbf{X}_s| > \delta])\,\delta^{\alpha} &\leq \int_{[|\mathbf{X}_t - \mathbf{X}_s| > \delta]} |\mathbf{X}_t - \mathbf{X}_s|^{\alpha}\, dP \\ &\leq C\,|t-s|^{1+\beta}. \end{aligned} \tag{31.26}$$

Letting $t = r_{j+1}^k$, $s = r_j^k$,and $\delta = 2^{-\gamma k}$ where

$$\gamma \in \left(0, \frac{\beta}{\alpha}\right),$$

this yields

$$P\left(\left[\left|\mathbf{X}_{r_{j+1}^k} - \mathbf{X}_{r_j^k}\right| > 2^{-\gamma k}\right]\right) \leq C 2^{\alpha\gamma k}\left(T 2^{-k}\right)^{1+\beta}.$$

There are 2^k of these differences and so letting

$$E_k = \left[\left|\mathbf{X}_{r_{j+1}^k} - \mathbf{X}_{r_j^k}\right|^{\alpha} > 2^{-\gamma k} \text{ for all } j\right]$$

it follows

$$P(E_k) \leq C 2^{\alpha\gamma k}\left(T 2^{-k}\right)^{1+\beta} 2^k = C 2^{k(\alpha\gamma - \beta)} T^{1+\beta}.$$

Since $\gamma < \beta/\alpha$,

$$\sum_{k=1}^{\infty} P(E_k) \leq C T^{1+\beta} \sum_{k=1}^{\infty} 2^{k(\alpha\gamma-\beta)} < \infty$$

and so by the Borel Cantelli lemma, Lemma 31.2, there exists a set of measure zero, E, such that if $\omega \notin E$, then ω is in only finitely many E_k. In other words, for $\omega \notin E$, there exists $N(\omega)$ such that if $k > N(\omega)$, then for each j,

$$\left|\mathbf{X}_{r_{j+1}^k}(\omega) - \mathbf{X}_{r_j^k}(\omega)\right| \leq 2^{-\gamma k}. \tag{31.27}$$

Claim: If $n \geq N(\omega)$ for $\omega \notin E$ and if $d, d' \in D_m$ for $m > n$ such that $|d - d'| < T 2^{-n}$, then

$$|\mathbf{X}_{d'}(\omega) - \mathbf{X}_d(\omega)| \leq 2\sum_{j=n+1}^{m} 2^{-\gamma j}.$$

Proof of the claim: Suppose $d' < d$. Suppose first $m = n+1$. Then $d = (k+1)\,T 2^{-(n+1)}$ and $d' = kT 2^{-(n+1)}$. Then from 31.27

$$|\mathbf{X}_{d'}(\omega) - \mathbf{X}_d(\omega)| \leq 2^{-\gamma(n+1)} \leq 2\sum_{j=n+1}^{n+1} 2^{-\gamma j}.$$

Suppose the claim is true for some $m > n$. Then let $d, d' \in D_{m+1}$ with $|d - d'| < T2^{-n}$. Let $d' \leq d_1' \leq d_1 \leq d$ where d_1, d_1' are in D_m and d_1' is the smallest element of D_m which is at least as large as d' and d_1 is the largest element of D_m which is no larger than d. Then $|d' - d_1'| \leq T2^{-(m+1)}$ and $|d_1 - d| \leq T2^{-(m+1)}$ while all of these are still in D_{m+1} which contains D_m. Therefore, from 31.27 and induction,

$$\begin{aligned} & |\mathbf{X}_{d'}(\omega) - \mathbf{X}_d(\omega)| \\ \leq & \left|\mathbf{X}_{d'}(\omega) - \mathbf{X}_{d_1'}(\omega)\right| + \left|\mathbf{X}_{d_1'}(\omega) - \mathbf{X}_{d_1}(\omega)\right| + |\mathbf{X}_{d_1}(\omega) - \mathbf{X}_d(\omega)| \\ \leq & \ 2 \times 2^{-\gamma(m+1)} + 2\sum_{j=n+1}^{m} 2^{-\gamma j} = 2\sum_{j=n+1}^{m+1} 2^{-\gamma j} \end{aligned}$$

which proves the claim.

From this estimate, it follows that if $d, d' \in D$ and $|d - d'| < T2^{-n}$ where $n \geq N(\omega)$, then d, d' are both in some D_m where $m > n$ and so

$$\begin{aligned} |\mathbf{X}_{d'}(\omega) - \mathbf{X}_d(\omega)| & \leq 2\sum_{j=n+1}^{m} 2^{-\gamma j} \leq 2\sum_{j=n+1}^{\infty} 2^{-\gamma j} \\ & = \frac{2}{1 - 2^{-\gamma}}\left(2^{-\gamma}\right)^{n+1}. \end{aligned} \tag{31.28}$$

Now let $d, d' \in D$ and suppose $|d - d'| < T2^{-N(\omega)}$. Then let $n \geq N(\omega)$ such that

$$T2^{-(n+1)} \leq |d - d'| < T2^{-n}$$

Then from 31.28,

$$\begin{aligned} |\mathbf{X}_{d'}(\omega) - \mathbf{X}_d(\omega)| & \leq \frac{2}{T^{\gamma}(1 - 2^{-\gamma})}\left(T2^{-n+1}\right)^{\gamma} \\ & \leq \frac{2}{T^{\gamma}(1 - 2^{-\gamma})}(|d - d'|)^{\gamma} \end{aligned}$$

which shows $t \to \mathbf{X}_t(\omega)$ is Holder continuous on D.

By Lemma 31.39, one can define $\mathbf{Y}_t(\omega)$ to be the unique function which extends $d \to \mathbf{X}_d(\omega)$ off D for $\omega \notin E$ and let $\mathbf{Y}_t(\omega) = 0$ if $\omega \in E$. Then $\omega \to \mathbf{Y}_t(\omega)$ is measurable because it is the pointwise limit of measurable functions. It remains to verify the claim that $\mathbf{Y}_t(\omega) = \mathbf{X}_t(\omega)$ a.e.

$$\mathcal{X}_{[|\mathbf{Y}_t - \mathbf{X}_t| > \varepsilon]}(\omega) \leq \liminf_{d \to t} \mathcal{X}_{[|\mathbf{X}_d - \mathbf{X}_t| > \varepsilon]}(\omega)$$

and so by Fatou's theorem

$$\begin{aligned} P([|\mathbf{Y}_t - \mathbf{X}_t| > \varepsilon]) & = \int \mathcal{X}_{[|\mathbf{Y}_t - \mathbf{X}_t| > \varepsilon]}(\omega)\, dP \\ & \leq \int \liminf_{d \to t} \mathcal{X}_{[|\mathbf{X}_d - \mathbf{X}_t| > \varepsilon]}(\omega)\, dP \\ & \leq \liminf_{d \to t} \int \mathcal{X}_{[|\mathbf{X}_d - \mathbf{X}_t| > \varepsilon]}(\omega)\, dP \\ & \leq \liminf_{d \to t} \frac{C}{\varepsilon^{\alpha}}|d - t|^{1+\beta} = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} P\left(\left[\left|\mathbf{Y}_t-\mathbf{X}_t\right|>0\right]\right) &= P\left(\cup_{k=1}^{\infty}\left[\left|\mathbf{Y}_t-\mathbf{X}_t\right|>\frac{1}{k}\right]\right) \\ &\leq \sum_{k=1}^{\infty} P\left(\left[\left|\mathbf{Y}_t-\mathbf{X}_t\right|>\frac{1}{k}\right]\right)=0. \end{aligned}$$

This proves the theorem.

Definition 31.41 *Let $\mathbf{X}_t$ and $\mathbf{Y}_t$ be random vectors for each $t \in [0,\infty)$. Then $\mathbf{Y}_t$ is said to be a version of $\mathbf{X}_t$ if there exists a set of measure zero, E such that for $\omega \notin E$, $\mathbf{X}_t(\omega) = \mathbf{Y}_t(\omega)$ a.e. ω for all $t \in [0,T)$.*

In terms of this definition, the following corollary follows.

Corollary 31.42 *Letting $\mathbf{B}_t$ be Brownian motion defined above, $\mathbf{B}_t$ has a Holder continuous version.*

Proof: This follows from Theorem 31.40 and 31.24.

An important observation related to this corollary is the product measurability of Brownian motion.

Corollary 31.43 *Let $\mathcal{B}$ be the Borel sets on $[0,T]$ and let $\mathcal{F}$ be the σ algebra for the underlying probability space. Then there exists a set of measure zero, $N \in \mathcal{F}$ such that $(t,\omega) \to \mathcal{X}_N(\omega)\mathbf{B}_t(\omega)$ is $\mathcal{B}\times\mathcal{F}$ measurable.*

Proof: Let N be the set of measure zero off which $t \to \mathbf{B}_t(\omega)$ is continuous. Letting $t_k^m = 2^{-m}Tk$ consider for $\omega \notin N$

$$\mathbf{B}_t^m(\omega) \equiv \sum_{k=1}^{2^m} \mathbf{B}_{t_k^m}(\omega)\,\mathcal{X}_{[t_{k-1}^m, t_k^m)}(t).$$

Then this is a finite sum of $\mathcal{B}\times\mathcal{F}$ measurable functions and so $\mathbf{B}^m$ is itself $\mathcal{B}\times\mathcal{F}$ measurable. Also, by continuity of $t \to \mathbf{B}_t(\omega)$, it follows that $\lim_{m\to\infty}\mathcal{X}_N(\omega)\mathbf{B}_t^m(\omega) = \mathcal{X}_N(\omega)\mathbf{B}_t(\omega)$ and this shows the result.

From now on, when $\mathbf{B}_t$ is referred to, it will mean $\mathcal{X}_N\mathbf{B}_t$ so that $(t,\omega) \to \mathbf{B}_t(\omega)$ is product measurable and $t \to \mathbf{B}_t(\omega)$ will also be continuous.

Summary of properties of Brownian motion

The above development has proved the following theorem on Brownian motion.

Theorem 31.44 *There exists a probability space, $(\Omega,\mathcal{F},P)$ and random vectors, B_t for $t \in [0,\infty)$ which satisfy the following properties.*

1. *For $\mathbf{Z} = (\mathbf{B}_{t_1},\cdots,\mathbf{B}_{t_k}) \in \mathbb{R}^{nk}$ it follows $\mathbf{Z}$ is normally distributed. Its mean is*

$$\left(\begin{array}{ccc} \mathbf{x} & \cdots & \mathbf{x} \end{array}\right) \in \mathbb{R}^{nk}$$

2. $\mathbf{B}_t$ *has independent increments. This means if* $t_1 < t_2 < \cdots < t_k$*, the random variables,*

$$\mathbf{B}_{t_1}, \mathbf{B}_{t_2} - \mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_k} - \mathbf{B}_{t_{k-1}}$$

are independent and normally distributed. Note this implies the k^{th} *components must also be independent. Also* $\mathbf{B}_{t_j} - \mathbf{B}_{t_{j-1}}$ *is normal with covariance* $(t_j - t_{j-1}) I$ *and mean* $\mathbf{0}$*. In addition to this, the* k^{th} *component of* $\mathbf{B}_t$ *is normally distributed with density function*

$$p(t, x_k, y) \equiv \frac{1}{(2\pi t)^{1/2}} \exp\left(-\frac{|y - x_k|^2}{2t}\right)$$

This follows from the distribution of $\mathbf{B}_t$ *which has a density function*

$$p(t, \mathbf{x}, \mathbf{y}) \equiv \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|\mathbf{y} - \mathbf{x}|^2}{2t}\right)$$

3. $E\left(|\mathbf{B}_t - \mathbf{B}_s|^4\right) \leq 3n^2 (t-s)^2$, *For* $t > s$,

$$\begin{aligned} E\left(|\mathbf{B}_t - \mathbf{B}_s|^2\right) &= n(t-s), E\left((\mathbf{B}_t - \mathbf{x})^* (\mathbf{B}_s - \mathbf{x})\right) = ns \\ E(\mathbf{B}_t - \mathbf{B}_s) &= \mathbf{0}, \end{aligned}$$

4. $t \to \mathbf{B}_t(\omega)$ *is Holder continuous and* $(t, \omega) \to \mathbf{B}_t(\omega)$ *is* $\mathcal{B} \times \mathcal{F}$ *measurable.*

Conditional Expectation And Martingales

32.1 Conditional Expectation

Definition 32.1 *Let $(\Omega, \mathcal{M}, P)$ be a probability space and let $\mathcal{S} \subseteq \mathcal{F}$ be two σ algebras contained in $\mathcal{M}$. Let f be $\mathcal{F}$ measurable and in $L^1(\Omega)$. Then $E(f|\mathcal{S})$, called the conditional expectation of f with respect to $\mathcal{S}$ is defined as follows:*

$$E(f|\mathcal{S}) \text{ is } \mathcal{S} \text{ measurable}$$

For all $E \in \mathcal{S}$,

$$\int_E E(f|\mathcal{S})\, dP = \int_E f dP$$

Lemma 32.2 *The above is well defined. Also, if $\mathcal{S} \subseteq \mathcal{F}$ then*

$$E(X|\mathcal{S}) = E(E(X|\mathcal{F})|\mathcal{S}). \tag{32.1}$$

If Z is bounded and measurable in $\mathcal{F}$ then

$$ZE(X|\mathcal{F}) = E(ZX|\mathcal{F}). \tag{32.2}$$

Proof: Let a finite measure on $\mathcal{S}$, μ be given by

$$\mu(E) \equiv \int_E f dP.$$

Then $\mu \ll P$ and so by the Radon Nikodym theorem, there exists a unique $\mathcal{S}$ measurable function, $E(f|\mathcal{S})$ such that

$$\int_E f dP \equiv \mu(E) = \int_E E(f|\mathcal{S})\, dP$$

for all $E \in \mathcal{S}$.

Let $F \in \mathcal{S}$. Then

$$\begin{aligned}\int_F E(E(X|\mathcal{F})|\mathcal{S})\,dP &\equiv \int_F E(X|\mathcal{F})\,dP \\ &\equiv \int_F X dP \equiv \int_F E(X|\mathcal{S})\,dP\end{aligned}$$

and so, by uniqueness, $E(E(X|\mathcal{F})|\mathcal{S}) = E(X|\mathcal{S})$. This shows 32.1.

To establish 32.2, note that if $Z = \mathcal{X}_F$ where $F \in \mathcal{F}$,

$$\int \mathcal{X}_F E(X|\mathcal{F})\,dP = \int \mathcal{X}_F X dP = \int E(\mathcal{X}_F X|\mathcal{F})\,dP$$

which shows 32.2 in the case where Z is the characteristic function of a set in $\mathcal{F}$. It follows this also holds for simple functions. Let $\{s_n\}$ be a sequence of simple functions which converges uniformly to Z and let $F \in \mathcal{F}$. Then by what was just shown,

$$\int_F s_n E(X|\mathcal{F})\,dP = \int_F s_n X dP.$$

Then passing to the limit using the dominated convergence theorem, yields

$$\int_F Z E(X|\mathcal{F})\,dP = \int_F Z X dP \equiv \int_F E(ZX|\mathcal{F})\,dP.$$

Since this holds for every $F \in \mathcal{F}$, this shows 32.2.

The next major result is a generalization of Jensen's inequality whose proof depends on the following lemma about convex functions.

Lemma 32.3 *Let I be an open interval on $\mathbb{R}$ and let ϕ be a convex function defined on I. Then there exists a sequence $\{(a_n, b_n)\}$ such that*

$$\phi(x) = \sup\{a_n x + b_n, n = 1, \cdots\}.$$

Proof: Let $x \in I$ and let $t > x$. Then by convexity of ϕ,

$$\begin{aligned}\frac{\phi(x + \lambda(t-x)) - \phi(x)}{\lambda(t-x)} &\leq \frac{\phi(x)(1-\lambda) + \lambda\phi(t) - \phi(x)}{\lambda(t-x)} \\ &= \frac{\phi(t) - \phi(x)}{t-x}.\end{aligned}$$

Therefore $t \to \frac{\phi(t)-\phi(x)}{t-x}$ is increasing if $t > x$. If $t < x$

$$\begin{aligned}\frac{\phi(x + \lambda(t-x)) - \phi(x)}{\lambda(t-x)} &\geq \frac{\phi(x)(1-\lambda) + \lambda\phi(t) - \phi(x)}{\lambda(t-x)} \\ &= \frac{\phi(t) - \phi(x)}{t-x}\end{aligned}$$

and so $t \to \frac{\phi(t)-\phi(x)}{t-x}$ is increasing for $t \neq x$. Let

$$a_x \equiv \inf \left\{ \frac{\phi(t) - \phi(x)}{t - x} : t > x \right\}.$$

Then if $t_1 < x$, and $t > x$,

$$\frac{\phi(t_1) - \phi(x)}{t_1 - x} \leq a_x \leq \frac{\phi(t) - \phi(x)}{t - x}.$$

Thus for all $t \in I$,

$$\phi(t) \geq a_x (t - x) + \phi(x). \tag{32.3}$$

Pick $t_2 > x$. Then for all $t \in (x, t_2)$

$$a_x \leq \frac{\phi(t) - \phi(x)}{t - x} \leq \frac{\phi(t_2) - \phi(x)}{t_2 - x}$$

and so

$$a_x (t - x) + \phi(x) \leq \phi(t) \leq \left(\frac{\phi(t_2) - \phi(x)}{t_2 - x} \right) (t - x) + \phi(x). \tag{32.4}$$

Pick $t_3 < x$. Then for $t_3 < t < x$

$$a_x \geq \frac{\phi(t) - \phi(x)}{t - x} \geq \frac{\phi(t_3) - \phi(x)}{t_3 - x}$$

and so

$$a_x (t - x) + \phi(x) \leq \phi(t) \leq \left(\frac{\phi(t_3) - \phi(x)}{t_3 - x} \right) (t - x) + \phi(x). \tag{32.5}$$

32.4 and 32.5 imply ϕ is continuous. Let

$$\psi(x) \equiv \sup \{ a_r (x - r) + \phi(r) : r \in \mathbb{Q} \cap I \}.$$

Then ψ is convex on I so ψ is continuous. Also $\psi(r) \geq \phi(r)$ so by 32.3,

$$\psi(x) \geq \phi(x) \geq \sup \{ a_r (x - r) + \phi(r) \} \equiv \psi(x).$$

Thus $\psi(x) = \phi(x)$ and letting $\mathbb{Q} \cap I = \{r_n\}$, $a_n = a_{r_n}$ and $b_n = a_{r_n} r_n + \phi(r_n)$. This proves the lemma.

Lemma 32.4 *If $X \leq Y$, then $E(X|\mathcal{S}) \leq E(Y|\mathcal{S})$ a.e. Also*

$$X \to E(X|\mathcal{S})$$

is linear.

Proof: Let $A \in \mathcal{S}$.

$$\int_A E(X|\mathcal{S})\, dP \equiv \int_A X dP$$
$$\leq \int_A Y dP \equiv \int_A E(Y|\mathcal{S})\, dP.$$

Hence $E(X|\mathcal{S}) \leq E(Y|\mathcal{S})$ a.e. as claimed. It is obvious $X \to E(X|\mathcal{S})$ is linear.

Theorem 32.5 *(Jensen's inequality)Let $X(\omega) \in I$ and let $\phi : I \to \mathbb{R}$ be convex. Suppose*

$$E(|X|), E(|\phi(X)|) < \infty.$$

Then

$$\phi(E(X|\mathcal{S})) \leq E(\phi(X)|\mathcal{S}).$$

Proof: Let $\phi(x) = \sup\{a_n x + b_n\}$. Letting $A \in \mathcal{S}$,

$$\frac{1}{P(A)} \int_A E(X|\mathcal{S})\, dP = \frac{1}{P(A)} \int_A X dP \in I \text{ a.e.}$$

whenever $P(A) \neq 0$. Hence $E(X|\mathcal{S})(\omega) \in I$ a.e. and so it makes sense to consider $\phi(E(X|\mathcal{S}))$. Now

$$a_n E(X|\mathcal{S}) + b_n = E(a_n X + b_n|\mathcal{S}) \leq E(\phi(X)|\mathcal{S}).$$

Thus

$$\sup\{a_n E(X|\mathcal{S}) + b_n\}$$
$$= \phi(E(X|\mathcal{S})) \leq E(\phi(X)|\mathcal{S}) \text{ a.e.}$$

which proves the theorem.

32.2 Discrete Martingales

Definition 32.6 *Let $\mathcal{S}_k$ be an increasing sequence of σ algebras which are subsets of $\mathcal{S}$ and X_k be a sequence of real-valued random variables with $E(|X_k|) < \infty$ such that X_k is $\mathcal{S}_k$ measurable. Then this sequence is called a martingale if*

$$E(X_{k+1}|\mathcal{S}_k) = X_k,$$

a submartingale if

$$E(X_{k+1}|\mathcal{S}_k) \geq X_k,$$

and a supermartingale if

$$E(X_{k+1}|\mathcal{S}_k) \leq X_k.$$

An upcrossing occurs when a sequence goes from a up to b. Thus it crosses the interval, $[a, b]$ in the up direction, hence upcrossing. More precisely,

Definition 32.7 *Let $\{x_i\}_{i=1}^{I}$ be any sequence of real numbers, $I \leq \infty$. Define an increasing sequence of integers $\{m_k\}$ as follows. m_1 is the first integer ≥ 1 such that $x_{m_1} \leq a$, m_2 is the first integer larger than m_1 such that $x_{m_2} \geq b$, m_3 is the first integer larger than m_2 such that $x_{m_3} \leq a$, etc. Then each sequence, $\left\{x_{m_{2k-1}}, \cdots, x_{m_{2k}}\right\}$, is called an upcrossing of $[a,b]$.*

Proposition 32.8 *Let $\{X_i\}_{i=1}^{n}$ be a finite sequence of real random variables defined on Ω where $(\Omega, \mathcal{S}, P)$ is a probability space. Let $U_{[a,b]}(\omega)$ denote the number of upcrossings of $X_i(\omega)$ of the interval $[a,b]$. Then $U_{[a,b]}$ is a random variable.*

Proof: Let $X_0(\omega) \equiv a+1$, let $Y_0(\omega) \equiv 0$, and let $Y_k(\omega)$ remain 0 for $k = 0, \cdots, l$ until $X_l(\omega) \leq a$. When this happens (if ever), $Y_{l+1}(\omega) \equiv 1$. Then let $Y_i(\omega)$ remain 1 for $i = l+1, \cdots, r$ until $X_r(\omega) \geq b$ when $Y_{r+1}(\omega) \equiv 0$. Let $Y_k(\omega)$ remain 0 for $k \geq r+1$ until $X_k(\omega) \leq a$ when $Y_k(\omega) \equiv 1$ and continue in this way. Thus the upcrossings of $X_i(\omega)$ are identified as unbroken strings of ones for Y_k with a zero at each end, with the possible exception of the last string of ones which may be missing the zero at the upper end and may or may not be an upcrossing.

Note also that Y_0 is measurable because it is identically equal to 0 and that if Y_k is measurable, then Y_{k+1} is measurable because the only change in going from k to $k+1$ is a change from 0 to 1 or from 1 to 0 on a measurable set determined by X_k. Now let

$$Z_k(\omega) = \begin{cases} 1 \text{ if } Y_k(\omega) = 1 \text{ and } Y_{k+1}(\omega) = 0, \\ 0 \text{ otherwise,} \end{cases}$$

if $k < n$ and

$$Z_n(\omega) = \begin{cases} 1 \text{ if } Y_n(\omega) = 1 \text{ and } X_n(\omega) \geq b, \\ 0 \text{ otherwise.} \end{cases}$$

Thus $Z_k(\omega) = 1$ exactly when an upcrossing has been completed and each Z_i is a random variable.

$$U_{[a,b]}(\omega) = \sum_{k=1}^{n} Z_k(\omega)$$

so $U_{[a,b]}$ is a random variable as claimed.

The following corollary collects some key observations found in the above construction.

Corollary 32.9 *$U_{[a,b]}(\omega) \leq$ the number of unbroken strings of ones in the sequence, $\{Y_k(\omega)\}$ there being at most one unbroken string of ones which produces no upcrossing. Also*

$$Y_i(\omega) = \psi_i\left(\{X_j(\omega)\}_{j=1}^{i-1}\right), \tag{32.6}$$

where ψ_i is some function of the past values of $X_j(\omega)$.

Lemma 32.10 *Let ϕ be a convex and increasing function and suppose*

$$\{(X_n, \mathcal{S}_n)\}$$

is a submartingale. Then if $E(|\phi(X_n)|) < \infty$, *it follows*

$$\{(\phi(X_n), \mathcal{S}_n)\}$$

is also a submartingale.

Proof: It is given that $E(X_{n+1}, \mathcal{S}_n) \geq X_n$ and so

$$\phi(X_n) \leq \phi(E(X_{n+1}, \mathcal{S}_n)) \leq E(\phi(X_{n+1}) | \mathcal{S}_n)$$

by Jensen's inequality.

The following is called the upcrossing lemma.

Lemma 32.11 *(upcrossing lemma) Let* $\{(X_i, \mathcal{S}_i)\}_{i=1}^n$ *be a submartingale and let* $U_{[a,b]}(\omega)$ *be the number of upcrossings of* $[a, b]$. *Then*

$$E\left(U_{[a,b]}\right) \leq \frac{E(|X_n|) + |a|}{b - a}.$$

Proof: Let $\phi(x) \equiv a + (x - a)^+$ so that ϕ is an increasing convex function always at least as large as a. By Lemma 32.10 it follows that $\{(\phi(X_k), \mathcal{S}_k)\}$ is also a submartingale.

$$\phi(X_{k+r}) - \phi(X_k) = \sum_{i=k+1}^{k+r} \phi(X_i) - \phi(X_{i-1})$$

$$= \sum_{i=k+1}^{k+r} (\phi(X_i) - \phi(X_{i-1})) Y_i + \sum_{i=k+1}^{k+r} (\phi(X_i) - \phi(X_{i-1}))(1 - Y_i).$$

Observe that Y_i is $\mathcal{S}_{i-1}$ measurable from its construction in Proposition 32.8, Y_i depending only on X_j for $j < i$. Therefore, letting

$$A_i \equiv \{\omega : Y_i(\omega) = 0\},$$

$$E\left(\sum_{i=k+1}^{k+r} (\phi(X_i) - \phi(X_{i-1}))(1 - Y_i)\right)$$

$$\begin{aligned} &= \sum_{i=k+1}^{k+r} \int_\Omega (\phi(X_i) - \phi(X_{i-1}))(1 - Y_i)\, dP \\ &= \sum_{i=k+1}^{k+r} \int_{A_i} (\phi(X_i) - \phi(X_{i-1}))\, dP \end{aligned}$$

because if $Y_i = 1$, $(\phi(X_i) - \phi(X_{i-1}))(1 - Y_i) = 0$. Continuing this chain of formulas, and using the fact that A_i is $\mathcal{S}_{i-1}$ measurable, it follows from the definition of conditional expectation and the definition of a submartingale,

$$\begin{aligned} &= \sum_{i=k+1}^{k+r} \int_{A_i} E(\phi(X_i), \mathcal{S}_{i-1}) \, dP - \int_{A_i} \phi(X_{i-1}) \, dP \\ &\geq \sum_{i=k+1}^{k+r} \int_{A_i} \phi(X_{i-1}) \, dP - \int_{A_i} \phi(X_{i-1}) \, dP = 0. \end{aligned} \tag{32.7}$$

Now let the unbroken strings of ones for $\{Y_i(\omega)\}$ be

$$\{k_1, \cdots, k_1 + r_1\}, \{k_2, \cdots, k_2 + r_2\}, \cdots, \{k_m, \cdots, k_m + r_m\} \tag{32.8}$$

where $m = V(\omega) \equiv$ the number of unbroken strings of ones in the sequence $\{Y_i(\omega)\}$. By Corollary 32.9 $V(\omega) \geq U_{[a,b]}(\omega)$.

$$\begin{aligned} &\phi(X_n(\omega)) - \phi(X_1(\omega)) \\ &= \sum_{k=1}^{n} (\phi(X_k(\omega)) - \phi(X_{k-1}(\omega))) Y_k(\omega) \\ &\quad + \sum_{k=1}^{n} (\phi(X_k(\omega)) - \phi(X_{k-1}(\omega)))(1 - Y_k(\omega)). \end{aligned}$$

The first sum in the above reduces to summing over the unbroken strings of ones because the terms in which $Y_i(\omega) = 0$ contribute nothing. implies

$$\begin{aligned} &\phi(X_n(\omega)) - \phi(X_1(\omega)) \\ &\geq U_{[a,b]}(\omega)(b - a) + 0 + \\ &\sum_{k=1}^{n} (\phi(X_k(\omega)) - \phi(X_{k-1}(\omega)))(1 - Y_k(\omega)) \end{aligned} \tag{32.9}$$

where the zero on the right side results from a string of ones which does not produce an upcrossing. It is here that it is important that $\phi(x) \geq a$. Such a string begins with $\phi(X_k(\omega)) = a$ and results in an expression of the form $\phi(X_{k+m}(\omega)) - \phi(X_k(\omega)) \geq 0$ since $\phi(X_{k+m}(\omega)) \geq a$. If X_k had not been replaced with $\phi(X_k)$, it would have been possible for $\phi(X_{k+m}(\omega))$ to be less than a and the zero in the above could have been a negative number This would have been inconvenient.

Next take the expected value of both sides in 32.9. Using 32.7, this results in

$$\begin{aligned} E(\phi(X_n) - \phi(X_1)) &\geq (b - a) E(U_{[a,b]}) \\ &\quad + E\left(\sum_{k=1}^{n} (\phi(X_k) - \phi(X_{k-1}))(1 - Y_k)\right) \\ &\geq (b - a) E(U_{[a,b]}) \end{aligned}$$

and this proves the lemma.

The reason for this lemma is to prove the amazing submartingale convergence theorem.

Theorem 32.12 *(submartingale convergence theorem) Let*

$$\{(X_i, \mathcal{S}_i)\}_{i=1}^{\infty}$$

be a submartingale with $K \equiv \sup E(|X_n|) < \infty$. *Then there exists a random variable,* X, *such that* $E(|X|) \leq K$ *and*

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \ \ a.e.$$

Proof: Let $a, b \in \mathbb{Q}$ and let $a < b$. Let $U_{[a,b]}^n(\omega)$ be the number of upcrossings of $\{X_i(\omega)\}_{i=1}^n$. Then let

$$U_{[a,b]}(\omega) \equiv \lim_{n\to\infty} U_{[a,b]}^n(\omega) = \text{ number of upcrossings of } \{X_i\}.$$

By the upcrossing lemma,

$$E\left(U_{[a,b]}^n\right) \leq \frac{E(|X_n|) + |a|}{b-a} \leq \frac{K + |a|}{b-a}$$

and so by the monotone convergence theorem,

$$E\left(U_{[a,b]}\right) \leq \frac{K + |a|}{b-a} < \infty$$

which shows $U_{[a,b]}(\omega)$ is finite a.e., for all $\omega \notin S_{[a,b]}$ where $P\left(S_{[a,b]}\right) = 0$. Define

$$S \equiv \cup\left\{S_{[a,b]} : a, b \in \mathbb{Q},\, a < b\right\}.$$

Then $P(S) = 0$ and if $\omega \notin S$, $\{X_k\}_{k=1}^{\infty}$ has only finitely many upcrossings of every interval having rational endpoints. Thus, for $\omega \notin S$,

$$\begin{aligned} \limsup_{k\to\infty} X_k(\omega) &= \liminf_{k\to\infty} X_k(\omega) \\ &= \lim_{k\to\infty} X_k(\omega) \equiv X_\infty(\omega). \end{aligned}$$

Letting $X_\infty(\omega) = 0$ for $\omega \in S$, Fatou's lemma implies

$$\int_\Omega |X_\infty|\, dP = \int_\Omega \liminf_{n\to\infty} |X_n|\, dP \leq \liminf_{n\to\infty} \int_\Omega |X_n|\, dP \leq K$$

and so this proves the theorem.

Another very interesting result about submartingales is the Doob submartingale estimate.

Theorem 32.13 *Let $\{(X_i, \mathcal{S}_i)\}_{i=1}^{\infty}$ be a submartingale. Then*

$$P\left(\max_{1\leq k\leq n} X_k \geq \lambda\right) \leq \frac{1}{\lambda}\int_{\Omega} X_n^+ dP$$

Proof: Let

$$\begin{aligned} A_1 &\equiv [X_1 \geq \lambda], A_2 \equiv [X_2 \geq \lambda] \setminus A_1, \\ \cdots, A_k &\equiv [X_k \geq \lambda] \setminus \left(\cup_{i=1}^{k-1} A_i\right) \cdots \end{aligned}$$

Thus each A_k is $\mathcal{S}_k$ measurable, the A_k are disjoint, and their union equals $[\max_{1\leq k\leq n} X_k \geq \lambda]$. Therefore from the definition of a submartingale and Jensen's inequality,

$$\begin{aligned} P\left(\max_{1\leq k\leq n} X_k \geq \lambda\right) &= \sum_{k=1}^{n} P(A_k) \leq \frac{1}{\lambda}\sum_{k=1}^{n}\int_{A_k} X_k dP \\ &\leq \frac{1}{\lambda}\sum_{k=1}^{n}\int_{A_k} E(X_n|\mathcal{S}_k)\, dP \\ &\leq \frac{1}{\lambda}\sum_{k=1}^{n}\int_{A_k} E(X_n|\mathcal{S}_k)^+\, dP \\ &\leq \frac{1}{\lambda}\sum_{k=1}^{n}\int_{A_k} E\left(X_n^+|\mathcal{S}_k\right) dP \\ &= \frac{1}{\lambda}\sum_{k=1}^{n}\int_{A_k} X_n^+ dP = \frac{1}{\lambda}\int_{\Omega} X_n^+ dP. \end{aligned}$$

This proves the theorem.

Filtrations And Martingales

Definition 33.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration is an* ***increasing*** *collection of σ algebras contained in $\mathcal{F}$, one for each $t \in \mathbb{R}$, $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$. Let $f : [0,\infty) \times \Omega \to \mathbb{R}$ be $\mathcal{B} \times \mathcal{F}$ measurable where $\mathcal{B}$ is the σ algebra of Borel sets. Then f is said to be $\mathcal{F}_t$ adapted if for each t, $f(t,\cdot)$ is $\mathcal{F}_t$ measurable. A function is called an adapted step function if it is of the form*

$$f(t,\omega) = \sum_{j=0}^{\infty} e_j(\omega) \mathcal{X}_{[t_j,t_{j+1})}(t)$$

where e_j is $\mathcal{F}_{t_j}$ measurable and $\cup_{j=0}^{\infty}[t_j, t_{j+1}) = \mathbb{R}$. Of course you can replace $[0,\infty)$ in the above with $[0,T]$ and this is the case of most interest. Another convention followed will be to assume that $(\Omega, \mathcal{F}_t, P)$ is a complete measure space. Thus all sets of measure zero from $\mathcal{F}$ are in $\mathcal{F}_t$. If it is not complete, you simply replace it with its completion. This goes for $\mathcal{F}$ as well.

The act of replacing the measure spaces with their completions is completely harmless. The only important idea which needs consideration is that of independence. Suppose the σ algebras, $\mathcal{G}$ and $\mathcal{H}$ are independent and you then consider their completions. Will the new σ algebras also be independent?

Lemma 33.2 *Suppose the σ algebras, $\mathcal{G}$ and $\mathcal{H}$ are independent and let $\mathcal{G}'$ and $\mathcal{H}'$ be the σ algebras of the completions. Then $\mathcal{G}'$ and $\mathcal{H}'$ are also independent.*

Proof: Let $E \in \mathcal{G}'$ and $F \in \mathcal{H}'$. Then there exists $A \supseteq E, B \supseteq F$ such that $A \in \mathcal{G}$ and $B \in \mathcal{H}$ and $P(A) = P(E), P(F) = P(B)$. It follows that

$$P(E \cap F) = P(A \cap B) = P(A) P(B) = P(E) P(F).$$

This proves the lemma.

This rather simple lemma shows there is no difficulty in assuming that any filtration contains the sets of measure zero. Next is an important lemma about approximation with simple adapted functions. It is based on the proof outlined in [33] which they say is from [38].

Lemma 33.3 *Let $\mathcal{F}_t$ be a filtration as described above and suppose f is adapted, $\mathcal{B} \times \mathcal{F}$ measurable, and uniformly bounded. Then there exists a sequence of uniformly bounded adapted step functions, f_n such that*

$$\lim_{n\to\infty} P\left(\int_0^T (f(t,\omega) - f_n(t,\omega))^2 \, dt > \varepsilon\right) = 0. \tag{33.1}$$

Also in this case, there is a subsequence, f_{n_k}

$$\lim_{k\to\infty} \int_\Omega \int_0^T (f(t,\omega) - f_{n_k}(t,\omega))^2 \, dt dP = 0.$$

Proof: Extend f to equal 0 for $t \notin [0,T]$. Let $t_j = j2^{-n}$ and let $\phi_n(t)$ denote the step function which equals $j2^{-n}$ on the interval $[j2^{-n}, (j+1)2^{-n})$. It follows easily that if $s \geq 0$, then $\phi_n(t-s) + s \in [t - 2^{-n}, t)$. Now let

$$f_h(t) \equiv f * \psi_h(t)$$

where $\psi_h(t)$ equals $1/h$ on $[0,h]$ and zero elsewhere. Thus

$$f_h(t) = \int f(t-s) \psi_h(s) \, ds = \frac{1}{h}\int_{t-h}^t f(s) \, ds.$$

From now on, $\omega \notin E$ where E is the exceptional set of measure zero on which $\int_{-\infty}^\infty f(t,\omega)^2 \, dt = \infty$. Consider

$$\int_0^T \int_0^1 |f(\phi_n(t-s)+s,\omega) - f(t,\omega)|^2 \, dsdt.$$

$$\int_0^T \int_0^1 |f(\phi_n(t-s)+s,\omega) - f(t,\omega)|^2 \, dsdt \leq$$

$$3\left(\int_0^T \int_0^1 |f(\phi_n(t-s)+s,\omega) - f_h(\phi_n(t-s)+s,\omega)|^2 \, dsdt \right. \tag{33.2}$$

$$+ \int_0^T \int_0^1 |f_h(\phi_n(t-s)+s,\omega) - f_h(t,\omega)|^2 \, dsdt \tag{33.3}$$

$$\left. + \int_0^T \int_0^1 |f_h(t,\omega) - f(t,\omega)|^2 \, dsdt\right) \tag{33.4}$$

Consider the term in 33.2. There are disjoint intervals such that $\phi_n(t-s)$ is constant on these intervals. Therefore, the inside integral of this term must be of the form

$$\sum_{k=1}^{m_n} \int_{I_k} |f(c_k+s,\omega) - f_h(c_k+s,\omega)|^2 \, ds$$

where the intervals, I_k are disjoint, c_k is of the form $j2^{-n}$, and the union of these intervals equals $[0,1]$. Therefore, there are other disjoint intervals, J_k such that this term equals

$$\sum_{k=1}^{m_n} \int_{J_k} |f(s,\omega) - f_h(s,\omega)|^2 \, ds \leq \int_{\mathbb{R}} |f(s,\omega) - f_h(s,\omega)|^2 \, ds$$

and this last converges to 0 by standard considerations as $h \to 0$. Of course the necessary smallness of h will depend on ω. To see this is so, use Jensen's inequality to obtain the term of 33.2 is dominated by

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}} |f(s,\omega) - f_h(s,\omega)|^2 \, dsdt \\
= \; & T \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (f(s,\omega) - f(s-r,\omega)) \psi_h(r) \, dr \right|^2 ds \\
\leq \; & T \int_{\mathbb{R}} \int_{\mathbb{R}} |f(s,\omega) - f(s-r,\omega)|^2 \psi_h(r) \, drds \\
\leq \; & T \int_0^h \psi_h(r) \int_{\mathbb{R}} |f(s,\omega) - f(s-r,\omega)|^2 \, dsdr \\
\leq \; & T \int_0^h \psi_h(r) \, \varepsilon dr
\end{aligned}$$

by continuity of translation in L^2 provided h is small enough. Therefore, this converges to 0 as $h \to 0$.

Next consider the term of 33.4

$$\begin{aligned}
\int_0^T \int_0^1 |f_h(t,\omega) - f(t,\omega)|^2 \, dsdt &= \int_0^1 \int_0^T |f_h(t,\omega) - f(t,\omega)|^2 \, dtds \\
&= \int_0^T |f_h(t,\omega) - f(t,\omega)|^2 \, dt \\
&= \int_0^T \left| \int (f(t-s,\omega) - f(t,\omega)) \psi_h(s) \, ds \right|^2 dt \\
&\leq \int_0^T \int (f(t-s,\omega) - f(t,\omega))^2 \psi_h(s) \, dsdt \\
&\leq \int_0^h \psi_h(s) \int (f(t-s,\omega) - f(t,\omega))^2 \, dtds \\
&\leq \int_0^h \psi_h(s) \, \varepsilon ds = \varepsilon
\end{aligned}$$

whenever h is small enough due to continuity of translation in L^2. Thus the terms in 33.2 and 33.4 both converge to 0 as $h \to 0$.

Therefore, there exists small positive h such that

$$\begin{aligned} & \int_0^T \int_0^1 \left|f\left(\phi_n\left(t-s\right)+s,\omega\right)-f\left(t,\omega\right)\right|^2 dsdt \\ \leq\ & \frac{\varepsilon}{2}+3\int_0^T \int_0^1 \left|f_h\left(\phi_n\left(t-s\right)+s,\omega\right)-f_h\left(t,\omega\right)\right|^2 dsdt \end{aligned}$$

Letting M be the uniform bound on f, and using the fact that $\left|\phi_n\left(t-s\right)+s-t\right|<2^{-n}$, it follows the above expression is dominated by

$$\frac{\varepsilon}{2}+\frac{M}{h}2^{-n}<\varepsilon$$

provided n is chosen large enough. Therefore, this has shown

$$\begin{aligned} & \lim_{n\to\infty}\int_0^T \int_0^1 \left|f\left(\phi_n\left(t-s\right)+s,\omega\right)-f\left(t,\omega\right)\right|^2 dsdt \\ =\ & \lim_{n\to\infty}\int_0^1 \int_0^T \left|f\left(\phi_n\left(t-s\right)+s,\omega\right)-f\left(t,\omega\right)\right|^2 dtds=0. \end{aligned}$$

Therefore, if $\varepsilon>0$ is given, the above expression is less than ε provided n is large enough, depending on ω. However, this requires that for some $s_n\in[0,1]$

$$\int_0^T \left|f\left(\phi_n\left(t-s_n\right)+s_n,\omega\right)-f\left(t,\omega\right)\right|^2 dt<\varepsilon.$$

I have now shown that for every $\omega\notin E$, the above expression holds for all n sufficiently large. Therefore,

$$\lim_{n\to\infty}P\left(\int_0^T \left|f\left(\phi_n\left(t-s_n\right)+s_n,\omega\right)-f\left(t,\omega\right)\right|^2 dt\geq\varepsilon\right)=0.$$

Let $f_n\left(t,\omega\right)=f\left(\phi_n\left(t-s_n\right)+s_n,\omega\right)$. Then f_n is clearly a bounded adapted step function.

It only remains to verify the last claim. From 33.1 there exists a subsequence, n_k such that

$$P\left(\int_0^T \left(f_{n_k}\left(t,\omega\right)-f\left(t,\omega\right)\right)^2 dt\geq 2^{-k}\right)\leq 2^{-k}$$

Then letting $A_k=\left[\omega:\int_0^T\left(f_{n_k}\left(t,\omega\right)-f\left(t,\omega\right)\right)^2 dt\geq 2^{-k}\right]$ it follows from the Borel Cantelli lemma that a.e. ω is in only finitely many of these A_k. Therefore, for each ω off a set of measure zero E,

$$\int_0^T \left(f_{n_k}\left(t,\omega\right)-f\left(t,\omega\right)\right)^2 dt<2^{-k}$$

for all k large enough. From the definition of these f_{n_k} given above, they are uniformly bounded. Therefore, by the dominated convergence theorem,

$$\lim_{k\to\infty}\int_\Omega\int_0^T \left(f_{n_k}(t,\omega)-f(t,\omega)\right)^2 dtdP=0.$$

This proves the theorem.

The following corollary is what is really desired. It removes the assumption that f is uniformly bounded.

Corollary 33.4 *Let $\mathcal{F}_t$ be a filtration as described above and suppose f is adapted and $\mathcal{B}\times\mathcal{F}$ measurable such that for a.e. ω,*

$$\int_0^T f(t,\omega)^2\,dt<\infty. \tag{33.5}$$

Then there exists a sequence of uniformly bounded adapted step functions, ϕ_n such that

$$\lim_{n\to\infty}P\left(\int_0^T \left(f(t,\omega)-\phi_n(t,\omega)\right)^2 dt>\varepsilon\right)=0. \tag{33.6}$$

Thus

$$\phi_n(t,\omega)=\sum_{j=0}^{m_n-1}e_j^n(\omega)\,\mathcal{X}_{[t_j^n,t_{j+1}^n)}(t)$$

where $t_0^n=0$ and e_j^n is $\mathcal{F}_{t_j^n}$ measurable. Furthermore, if f is in $L^2([0,T]\times\Omega)$, there exists a subsequence $\{\phi_{n_k}\}$ such that

$$\lim_{k\to\infty}\int_\Omega\int_0^T \left(f(t,\omega)-\phi_{n_k}(t,\omega)\right)^2 dtdP=0$$

Proof: Let f_M be given by the following for $M\in\mathbb{N}$

$$f_M(t,\omega)=\begin{cases} M \text{ if } f(t,\omega)>M \\ f(t,\omega) \text{ if } f(t,\omega)\in[-M,M] \\ -M \text{ if } f(t,\omega)<-M\end{cases}$$

Then f_M satisfies all the conditions of Lemma 33.3. Letting $\varepsilon>0$ be given, it follows there exists ϕ_M a uniformly bounded adapted step function such that

$$P\left(\int_0^T \left(f_M(t,\omega)-\phi_M(t,\omega)\right)^2 dt>\frac{\delta}{4}\right)<\varepsilon.$$

Also for large enough M,

$$P\left(\int_0^T \left(f(t,\omega)-f_M(t,\omega)\right)^2 dt>\frac{\delta}{4}\right)<\varepsilon \tag{33.7}$$

This is because for E the set of measure zero such that 33.5 does not hold,

$$\Omega \setminus E = \cup_{M=1}^{\infty} \left[\omega : \int_0^T (f(t,\omega) - f_M(t,\omega))^2 \, dt \leq \frac{\delta}{2} \right]$$

Therefore, picking M large enough that 33.7 holds,

$$\begin{aligned} & P\left(\int_0^T (f(t,\omega) - \phi_M(t,\omega))^2 \, dt > \delta \right) \\ \leq \; & P\left(\int_0^T 2(f(t,\omega) - f_M(t,\omega))^2 \, dt > \frac{\delta}{2} \right) \\ & + P\left(\int_0^T 2(f_M(t,\omega) - \phi_M(t,\omega))^2 \, dt > \frac{\delta}{2} \right) \\ < \; & \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves the first part of the corollary.

Next suppose $f \in L^2([0,T] \times \Omega)$. By 33.7 there exists a subsequence, $\{f_{M_k}\}$ such that

$$P\left(\int_0^T (f(t,\omega) - f_{M_k}(t,\omega))^2 \, dt > 2^{-k} \right) < 2^{-k}$$

and so by the Borel Canelli lemma there exists a set of measure zero E such that for $\omega \notin E$ and all k large enough,

$$\int_0^T (f(t,\omega) - f_{M_k}(t,\omega))^2 \, dt \leq 2^{-k}.$$

Now by construction, $|f_{M_k}| \leq |f|$ and so the dominated convergence theorem implies

$$\lim_{k \to \infty} \left(\int_\Omega \int_0^T (f(t,\omega) - f_{M_k}(t,\omega))^2 \, dt dP \right)^{1/2} = 0.$$

Therefore, there exists a subsequence, $\{M_k\}$ such that

$$\left(\int_\Omega \int_0^T (f(t,\omega) - f_{M_k}(t,\omega))^2 \, dt dP \right)^{1/2} < 2^{-(\bar{k}+1)}.$$

Then by Lemma 33.3 there exists an adapted bounded step function, ϕ_{M_k} such that

$$\left(\int_\Omega \int_0^T (f_{M_k}(t,\omega) - \phi_{M_k}(t,\omega))^2 \, dt dP \right)^{1/2} < 2^{-(\bar{k}+1)}.$$

It follows

$$\left(\int_\Omega \int_0^T (f(t,\omega) - \phi_{M_k}(t,\omega))^2 \, dt dP \right)^{1/2} < 2^{-k}.$$

This proves the corollary.

The following corollary states things a little differently.

Corollary 33.5 *Let $\mathcal{F}_t$ be a filtration as described above and suppose f is adapted and $\mathcal{B} \times \mathcal{F}$ measurable such that for a.e. ω,*

$$\int_0^T f(t,\omega)^2 \, dt < \infty.$$

Then there exists a sequence of uniformly bounded adapted step functions, ϕ_n such that

$$\lim_{n\to\infty} P\left(\int_0^T (f(t,\omega) - \phi_n(t,\omega))^2 \, dt \le 2^{-n}\right) = 1.$$

Proof: By Corollary 33.4, there exists a sequence of bounded adapted step functions, $\{\phi_n\}$ such that

$$\lim_{n\to\infty} P\left(\int_0^T (f(t,\omega) - \phi_n(t,\omega))^2 \, dt > \varepsilon\right) = 0.$$

Therefore, selecting a sequence of ε as $\varepsilon = 2^{-k}$, there exists a subsequence, ϕ_{n_k} such that

$$P\left(\int_0^T \left(f(t,\omega) - \phi_{n_k}(t,\omega)\right)^2 dt > 2^{-k}\right) < 2^{-k}.$$

From the Borel Cantelli lemma ω which is in infinitely many of the sets

$$\left[\omega : \int_0^T \left(f(t,\omega) - \phi_{n_k}(t,\omega)\right)^2 dt > 2^{-k}\right]$$

has measure zero. Therefore, for ω not in this set,

$$\int_0^T \left(f(t,\omega) - \phi_{n_k}(t,\omega)\right)^2 dt \le 2^{-k}$$

for all k sufficiently large and so

$$\lim_{k\to\infty} P\left(\int_0^T \left(f(t,\omega) - \phi_{n_k}(t,\omega)\right)^2 dt \le 2^{-k}\right) = 1.$$

This proves the corollary.

Note that virtually no change in the argument would yield the above results with the exponent $p \ge 1$ replacing 2. I will use this fact whenever convenient.

Definition 33.6 *Let $\{B_t\}$ be one dimensional Brownian motion as described in Theorem 31.44. Let $\mathcal{F}_t$ be the smallest σ algebra which is complete and contains all sets of the form*

$$(B_{t_1}, \cdots, B_{t_k})^{-1} (F_1 \times \cdots \times F_k)$$

for all finite increasing sequences $t_1, \cdots, t_k$ such that $0 \leq t_1 < t_2 \cdots < t_k \leq t$ and F_j Borel. Thus $\mathcal{F}_t$ is a filtration. Another way to say it is that $\mathcal{F}_t$ is the smallest σ algebra contained in $\mathcal{F}$ which is complete and such that for every increasing sequence, $t_1, \cdots, t_k$ such that $0 \leq t_1 < t_2 \cdots < t_k \leq t$, it follows that $(B_{t_1}, \cdots, B_{t_k}) : \Omega \to \mathbb{R}^n$ is measurable with respect to $\mathcal{F}_t$.

Recall B_t has independent increments. This is why the following lemma is so significant.

Lemma 33.7 *$\mathcal{F}_t$ also equals the smallest σ algebra which is complete and contains all sets of the form*

$$\left(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_k} - B_{t_{k-1}}\right)^{-1} (F_1 \times \cdots \times F_k).$$

In other words, one can consider instead the independent increments when defining $\mathcal{F}_t$.

Proof: $\mathcal{F}_t'$ is the smallest σ algebra such that $(B_{t_1}, \cdots, B_{t_k})$ is measurable for every increasing sequence, $t_1, \cdots, t_k$ such that $0 \leq t_1 < t_2 \cdots < t_k \leq t$. The 0 function is clearly measurable because $0^{-1}(E) = \Omega$ if $0 \in E$ and if $0 \notin E$, $0^{-1}(E) = \emptyset$. Therefore, for every increasing sequence, as just described, $\left(0, -B_{t_1}, \cdots, -B_{t_{k-1}}\right)$ is $\mathcal{F}_t$ measurable. Therefore, $\left(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_k} - B_{t_{k-1}}\right)$ is $\mathcal{F}_t'$ measurable. Now suppose for all such increasing sequences, $\left(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_k} - B_{t_{k-1}}\right)$ is $\mathcal{F}_t'$ measurable. Then in particular,

$$(0, \cdots 0, B_{t_i}, 0 \cdots, 0)$$

must be $\mathcal{F}_t'$ measurable because B_{t_i} is. Therefore, adding in k of these, it follows $(B_{t_1}, \cdots, B_{t_k})$ is $\mathcal{F}_t'$ measurable. In other words, a σ algebra is measurable for all

$$(B_{t_1}, B_{t_2}, \cdots, B_{t_k})$$

if and only if it is measurable for all sequences,

$$\left(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_k} - B_{t_{k-1}}\right)$$

Taking the completion, this proves the lemma.

33.1 Continuous Martingales

Definition 33.8 *Let $(\Omega, \mathcal{M}, P)$ be a probability space and let $\mathcal{M}_t$ be a filtration. Then a stochastic process, $M_t, t \geq 0$ is called a martingale if*

$$M_t \text{ is } \mathcal{M}_t \text{ measurable for all } t \geq 0$$

$$M_t \in L^1(\Omega) \text{ for all } t \geq 0$$

$$E(M_s | \mathcal{M}_t) = M_t \text{ whenever } s \geq t.$$

Theorem 33.9 *Brownian motion for $S \leq t$ with respect to the filtration described above is a martingale.*

Proof: First, $\mathbf{B}_t$ is $\mathcal{F}_t$ measurable. It is necessary to verify $\mathbf{B}_t$ is in $L^1(\Omega)$. But by Theorem 31.44

$$\begin{aligned}
\int_\Omega |\mathbf{B}_t(\omega)|\, dP &\leq \int_\Omega |\mathbf{B}_t(\omega)|^2\, dP \\
&\leq 2\int_\Omega |\mathbf{B}_t - \mathbf{B}_0|^2\, dP + 2\int_\Omega |\mathbf{B}_0|^2 \\
&\leq 2n(t-s) + 2\int_\Omega |\mathbf{x}|^2\, dP < \infty.
\end{aligned}$$

Next it must be shown that $E(\mathbf{B}_s|\mathcal{F}_t) = \mathbf{B}_t$ whenever $s \geq t$. Let $F \in \mathcal{F}_t$ and let $s \geq t$. Then

$$\begin{aligned}
\int_F E(\mathbf{B}_s - \mathbf{B}_t|\mathcal{F}_t)\, dP &= \int (\mathbf{B}_s - \mathbf{B}_t)\, \mathcal{X}_F dP \\
&= \int_\Omega (\mathbf{B}_s - \mathbf{B}_t)\, dP \int_\Omega \mathcal{X}_F dP = \mathbf{0}.
\end{aligned}$$

Hence

$$\begin{aligned}
E(\mathbf{B}_s|\mathcal{F}_t) &= E(\mathbf{B}_s - \mathbf{B}_t|\mathcal{F}_t) + E(\mathbf{B}_t|\mathcal{F}_t) \\
&= \mathbf{0} + \mathbf{B}_t
\end{aligned}$$

and this proves the theorem.

Lemma 33.10 *Let B_t be real valued Brownian motion. Then $B_t^2 - t$ is a martingale.*

Proof: The idea is to exploit the fact the increments $(B_s - B_t)$ for $s > t$ are independent of $\mathcal{F}_t$. Thus you write things in terms of $(B_s - B_t)$. It is easy to see that $(B_s - B_t)^2 - 2B_t^2 + 2B_sB_t = B_s^2 - B_t^2$. Therefore, using the fact that Brownian motion is a martingale,

$$\begin{aligned}
E\left(B_s^2 - B_t^2|\mathcal{F}_t\right) &= E\left((B_s - B_t)^2 - 2B_t^2 + 2B_sB_t|\mathcal{F}_t\right) \\
&= E\left((B_s - B_t)^2|\mathcal{F}_t\right) + E\left(-2B_t^2 + 2B_sB_t|\mathcal{F}_t\right) \\
&= E\left((B_s - B_t)^2|\mathcal{F}_t\right) - 2B_t^2 + 2B_tE(B_s|\mathcal{F}_t) \\
&= E\left((B_s - B_t)^2|\mathcal{F}_t\right) - 2B_t^2 + 2B_t^2 \\
&= E\left((B_s - B_t)^2|\mathcal{F}_t\right)
\end{aligned}$$

Now for $A \in \mathcal{F}_t$,

$$\int_A E\left((B_s - B_t)^2|\mathcal{F}_t\right) dP \equiv \int_A (B_s - B_t)^2\, dP$$

$$= \int \mathcal{X}_A dP \int (B_s - B_t)^2 \, dP = \int_A (s-t) \, dP.$$

Since this holds for all $A \in \mathcal{F}_t$, it follows $E\left((B_s - B_t)^2 |\mathcal{F}_t\right) = (s-t)$ and so

$$\begin{aligned} E\left(B_s^2 - s|\mathcal{F}_t\right) &= E\left(B_s^2 - B_t^2|\mathcal{F}_t\right) + B_t^2 - s \\ &= (s-t) + B_t^2 - s = B_t^2 - t. \end{aligned}$$

Suppose B_t is one dimensional Brownian motion. Then the increments, $B_s - B_t$ for $s > t$ are independent of $\mathcal{F}_t$ where $\mathcal{F}_t$ is the smallest complete σ algebra such that $(B_{t_1}, \cdots, B_{t_k})$ is measurable for every sequence $0 \leq t_1 < t_2 < \cdots < t_k \leq t$. Consider the problem of finding $E\left((B_s - B_t)^m\right)$ for various values of m. It was shown earlier that for $m = 1$, the answer is 0 and for $m = 2$, the answer is $s - t$.

By independence,

$$E\left(e^{iB_s u}\right) = E\left(e^{i(B_s - B_t)u}\right) E\left(e^{iB_t u}\right).$$

Therefore, from the earlier observations about the characteristic function of normaly distributed random variables and using the fact the mean of B_t is x and the variance is t,

$$\frac{e^{iux} e^{-\frac{1}{2}u^2 s}}{e^{iux} e^{-\frac{1}{2}u^2 t}} = E\left(e^{i(B_s - B_t)u}\right)$$

and so

$$E\left(e^{i(B_s - B_t)u}\right) = e^{-\frac{1}{2}u^2 (s-t)}. \tag{33.8}$$

Therefore,

$$E\left(i(B_s - B_t) e^{i(B_s - B_t)u}\right) = u(t-s) e^{\frac{1}{2}u^2(t-s)}$$

$$\begin{aligned} E\left(-(B_s - B_t)^2 e^{i(B_s - B_t)u}\right) &= e^{\frac{1}{2}u^2(t-s)} t - e^{\frac{1}{2}u^2(t-s)} s \\ &\quad + u^2 e^{\frac{1}{2}u^2(t-s)} t^2 - 2u^2 e^{\frac{1}{2}u^2(t-s)} ts \\ &\quad + u^2 e^{\frac{1}{2}u^2(t-s)} s^2 \end{aligned} \tag{33.9}$$

$$\begin{aligned} E\left(-i(B_s - B_t)^3 e^{i(B_s - B_t)u}\right) &= 3ue^{\frac{1}{2}u^2(t-s)} t^2 - 6ue^{\frac{1}{2}u^2(t-s)} ts \\ &\quad + 3ue^{\frac{1}{2}u^2(t-s)} s^2 + u^3 e^{\frac{1}{2}u^2(t-s)} t^3 \\ &\quad - 3u^3 e^{\frac{1}{2}u^2(t-s)} t^2 s + 3u^3 e^{\frac{1}{2}u^2(t-s)} ts^2 \\ &\quad - u^3 e^{\frac{1}{2}u^2(t-s)} s^3 \end{aligned} \tag{33.10}$$

$$\begin{aligned}
E\left((B_s-B_t)^4 e^{i(B_s-B_t)u}\right) &= 3e^{\frac{1}{2}u^2(t-s)}t^2+6u^2e^{\frac{1}{2}u^2(t-s)}t^3 \\
&\quad -18u^2e^{\frac{1}{2}u^2(t-s)}t^2s-6e^{\frac{1}{2}u^2(t-s)}ts \\
&\quad +18u^2e^{\frac{1}{2}u^2(t-s)}ts^2+3e^{\frac{1}{2}u^2(t-s)}s^2 \\
&\quad -6u^2e^{\frac{1}{2}u^2(t-s)}s^3+u^4e^{\frac{1}{2}u^2(t-s)}t^4 \\
&\quad -4u^4e^{\frac{1}{2}u^2(t-s)}t^3s+6u^4e^{\frac{1}{2}u^2(t-s)}t^2s^2 \\
&\quad -4u^4e^{\frac{1}{2}u^2(t-s)}ts^3+u^4e^{\frac{1}{2}u^2(t-s)}s^4 \qquad (33.11)
\end{aligned}$$

Now plug in $u=0$ to get these expected values. Considering 33.9,

$$-E\left((B_s-B_t)^2\right)=t-s \qquad (33.12)$$

which was already noted. Next let $u=0$ in 33.10 to obtain

$$-iE\left((B_s-B_t)^3\right)=0. \qquad (33.13)$$

Finally, consider 33.11. This gives

$$\begin{aligned}
E\left((B_s-B_t)^4\right) &= 3t^2-6ts+3s^2 \\
&= 3(s-t)^2. \qquad (33.14)
\end{aligned}$$

Clearly one could go on like this but this is enough for now. You might conjecture that $E\left((B_s-B_t)^m\right)=(m-1)(s-t)^{m/2}$ for m even and 0 for m odd. Lets simply call it $g_m(s-t)$ for now.

Example 33.11 *Find a martingale which involves B_t^3.*

It is the exploitation of the independence of the increments which is of significance. The idea is to write B_s^3 in terms of B_t and powers of increments, (B_s-B_t). To aid in doing this, just write the Taylor series expansion of the function s^3. Thus

$$s^3=t^3+\left(3t^2\right)(s-t)+3t(s-t)^2+(s-t)^3.$$

It follows

$$B_s^3=B_t^3+3B_t^2(B_s-B_t)+3B_t(B_s-B_t)^2+(B_s-B_t)^3.$$

Then for $s>t$,

$$E\left(B_s^3|\mathcal{F}_t\right)=E\left(B_t^3+3B_t^2(B_s-B_t)+3B_t(B_s-B_t)^2+(B_s-B_t)^3|\mathcal{F}_t\right)$$

$$\begin{aligned}
&= B_t^3+B_t^2E(B_s-B_t|\mathcal{F}_t)+3B_tE\left((B_s-B_t)^2|\mathcal{F}_t\right)+E\left((B_s-B_t)^3|\mathcal{F}_t\right) \\
&= B_t^3+3B_tE\left((B_s-B_t)^2|\mathcal{F}_t\right)+E\left((B_s-B_t)^3|\mathcal{F}_t\right) \qquad (33.15)
\end{aligned}$$

Consider $E\left((B_s - B_t)^m |\mathcal{F}_t\right)$. If $A \in \mathcal{F}_t$,then by independence,

$$\begin{aligned}
\int_A E\left((B_s - B_t)^m |\mathcal{F}_t\right) dP &= \int_A (B_s - B_t)^m dP \\
&= \int \mathcal{X}_A dP \int (B_s - B_t)^m dP \\
&= \int_A g_m (s-t) \, dP
\end{aligned}$$

which shows

$$g_m (s-t) = E\left((B_s - B_t)^m |\mathcal{F}_t\right). \tag{33.16}$$

Then considering 33.15 in light of 33.12 and 33.13, this leads to

$$E\left(B_s^3 | \mathcal{F}_t\right) = B_t^3 + 3(s-t) B_t$$

It follows

$$\begin{aligned}
E\left(B_s^3 - 3B_s s | \mathcal{F}_t\right) &= B_t^3 + 3(s-t) B_t - 3sE\left(B_s|\mathcal{F}_t\right) \\
&= B_t^3 + 3(s-t) B_t - 3sB_t \\
&= B_t^3 - 3tB_t
\end{aligned}$$

This shows $B_t^3 - 3tB_t$ is a martingale.

Of course you can keep going this way. Here is yet another example.

Lemma 33.12 *$B_t^4 - 6tB_t^2 + 3t^2$ is a martingale.*

Proof: The Taylor series for $y^4 - 6ty^2 + 3t^2$ considered a function of y expanded about x is

$$\begin{aligned}
&\left(x^4 - 6tx^2 + 3t^2\right) + \left(-12tx + 4x^3\right)(y-x) \\
&+ \left(-6t + 6x^2\right)(y-x)^2 + 4x(y-x)^3 + (y-x)^4
\end{aligned}$$

and so $B_s^4 - 6sB_s^2 + 3s^2$ equals

$$\begin{aligned}
&\left(B_t^4 - 6sB_t^2 + 3s^2\right) + \left(-12sB_t + 4B_t^3\right)(B_s - B_t) \\
&+ \left(-6s + 6B_t^2\right)(B_s - B_t)^2 + 4B_t (B_s - B_t)^3 + (B_s - B_t)^4
\end{aligned}$$

Using 33.16 and taking conditional expectations using the formulas, 33.12 - 33.14, $E\left(B_s^4 - 6sB_s^2 + 3s^2|\mathcal{F}_t\right)$ equals

$$\left(B_t^4 - 6sB_t^2 + 3s^2\right) + \left(-6s + 6B_t^2\right)(s-t) + 3(s-t)^2$$

and this simplifies to

$$B_t^4 - 6B_t^2 t + 3t^2$$

This proves the lemma.

33.2 Doob's Martingale Estimate

The next big result is an important inequality involving martingales which is due to Doob. First here is a simple lemma.

Lemma 33.13 *Let $\{\mathcal{M}_t\}$ be a filtration and let $\{M_t\}$ be a real valued martingale for $t \in [S,T]$. Then for $\lambda > 0$ and any $p \geq 1$, if A_t is a $\mathcal{M}_t$ measurable subset of $[|M_t| \geq \lambda]$, then*

$$P(A_t) \leq \frac{1}{\lambda^p} \int_{A_t} |M_T|^p \, dP.$$

Proof: From Jensen's inequality,

$$\begin{aligned} \lambda^p P(A_t) &\leq \int_{A_t} |M_t|^p \, dP = \int_{A_t} |E(M_T|\mathcal{M}_t)|^p \, dP \\ &\leq \int_{A_t} E(|M_T|^p |\mathcal{M}_t) \, dP = \int_{A_t} |M_T|^p \, dP \end{aligned}$$

and this proves the lemma.

The next theorem is the main result.

Theorem 33.14 *Let $\{\mathcal{M}_t\}$ be a filtration and let $\{M_t\}$ be a real valued continuous*[1] *martingale for $t \in [S,T]$. Then for all $\lambda > 0$ and $p \geq 1$,*

$$P\left(\left[\sup_{t\in[S,T]} |M_t| \geq \lambda\right]\right) \leq \frac{1}{\lambda^p} \int_\Omega |M_T|^p \, dP$$

Proof: Let $S \leq t_0^m < t_1^m < \cdots < t_{N_m}^m = T$ where $t_{j+1}^m - t_j^m = (T-S)\,2^{-m}$. First consider $m = 1$.

$$A_{t_0^1} \equiv \left\{\omega \in \Omega : \left|M_{t_0^1}(\omega)\right| \geq \lambda\right\},\ A_{t_1^1} \equiv \left\{\omega \in \Omega : \left|M_{t_1^1}(\omega)\right| \geq \lambda\right\} \setminus A_{t_0^1}$$

$$A_{t_2^1} \equiv \left\{\omega \in \Omega : \left|M_{t_2^1}(\omega)\right| \geq \lambda\right\} \setminus \left(A_{t_0^1} \cup A_{t_0^1}\right).$$

Do this type of construction for $m = 2, 3, 4, \cdots$ yielding disjoint sets, $\left\{A_{t_j^m}\right\}_{j=0}^{2^m}$ whose union equals

$$\cup_{t\in D_m} [|M_t| \geq \lambda]$$

[1] $t \to M_t(\omega)$ is continuous for a.e. ω.

where $D_m = \left\{t_j^m\right\}_{j=0}^{2^m}$. Thus $D_m \subseteq D_{m+1}$. Then also, $D \equiv \cup_{m=1}^{\infty} D_m$ is dense and countable. From Lemma 33.13,

$$\begin{aligned} P\left(\cup_{t\in D_m}\left[|M_t| \geq \lambda\right]\right) &= \sum_{j=0}^{2^m} P\left(A_{t_j^m}\right) \\ &\leq \frac{1}{\lambda^p}\sum_{j=0}^{2^m}\int_{A_{t_j^m}} |M_T|^p \, dP \\ &\leq \frac{1}{\lambda^p}\int_{\Omega} |M_T|^p \, dP. \end{aligned}$$

Let $m \to \infty$ in the above to obtain

$$P\left(\cup_{t\in D}\left[|M_t| \geq \lambda\right]\right) \leq \frac{1}{\lambda^p}\int_{\Omega} |M_T|^p \, dP. \tag{33.17}$$

From now on, assume that for a.e. $\omega \in \Omega,\ t \to M_t(\omega)$ is continuous. Then with this assumption, the following claim holds.

Claim: For $\lambda > \varepsilon > 0$,

$$\cup_{t\in D}\left[|M_t| \geq \lambda - \varepsilon\right] \supseteq \left[\sup_{t\in[S,T]} |M_t| \geq \lambda\right]$$

Proof of the claim: Suppose $\omega \in \left[\sup_{t\in[S,T]} |M_t| \geq \lambda\right]$. Then there exists s such that $|M_s(\omega)| > \lambda - \varepsilon$. By continuity, this situation persists for all t near to s. In particular, it is true for some $t \in D$. This proves the claim.

Letting P' denote the outer measure determined by P it follows from the claim and 33.17 that

$$\begin{aligned} P'\left(\left[\sup_{t\in[S,T]} |M_t| \geq \lambda\right]\right) &\leq P\left(\cup_{t\in D}\left[|M_t| \geq \lambda - \varepsilon\right]\right) \\ &\leq \frac{1}{(\lambda-\varepsilon)^p}\int_{\Omega} |M_T|^p \, dP. \end{aligned}$$

Since ε is arbitrary, this shows

$$P'\left(\left[\sup_{t\in[S,T]} |M_t| \geq \lambda\right]\right) \leq \frac{1}{\lambda^p}\int_{\Omega} |M_T|^p \, dP.$$

It would be interesting to consider whether $\left[\sup_{t\in[S,T]} |M_t| \geq \lambda\right]$ is measurable. However, this follows from the continuity of $t \to M_t(\omega)$ which implies

$$\left[\sup_{t\in[S,T]} |M_t| \geq \lambda\right] = \left[\sup_{t\in D} |M_t| \geq \lambda\right],$$

a measurable set due to countability of D. This proves the theorem.

The Itô Integral

In all this, B_t will be a martingale for the filtration, $\mathcal{H}_t$ and the increments, $B_s - B_t$ will be independent of $\mathcal{H}_t$ for $s > t$. I will define the Itô integral on $[0, T]$ where T is arbitrary. In doing so, I will also define it on $[0, t]$. First the integral is defined on uniformly bounded adapted step functions. Let ϕ be such a function. Thus

$$\phi(t, \omega) = \sum_{j=0}^{n-1} \phi_j(\omega) \mathcal{X}_{[t_j, t_{j+1})}(t). \tag{34.1}$$

Then for $t \in [t_k, t_{k+1})$,

$$\int_0^t \phi dB(\omega) \equiv \sum_{j=0}^{k-1} \phi_j(\omega) \left(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)\right) + \phi_k(\omega)(B_t(\omega) - B_{t_k}(\omega)). \tag{34.2}$$

The verification that this is well defined and linear is essentially the same as it is in the context of the Riemann integral from calculus. To show linearity on such step functions, you simply take a common refinement and if s is one of the new partition points in $[t_i, t_{i+1})$, you replace the term $\phi_i \mathcal{X}_{[t_i, t_{i+1})}(t)$ with the sum of the two terms, $\phi_i \mathcal{X}_{[t_i, s)}(t) + \phi_i \mathcal{X}_{[s,, t_{i+1})}(t)$.

Lemma 34.1 *Let $s > t$ and let ϕ be bounded and $\mathcal{H}_t$ measurable. Then*

$$\exp(\phi(B_s - B_t)) \exp\left(-\frac{1}{2}\phi^2(s - t)\right) \tag{34.3}$$

is a function in $L^1(\Omega)$.

Proof: The given function is dominated by

$$h(\omega) \equiv \exp(M |B_s - B_t|)$$

where $|\phi(\omega)| < M$. Then using the technique of the distribution function, Theorem 9.39 on Page 232,

$$\begin{aligned}
\int_\Omega h dP &= \int_0^\infty P(h > \lambda)\, d\lambda \\
&= \int_0^\infty P(\exp(M|B_s - B_t|) > \lambda)\, d\lambda \\
&\leq \int_0^\infty P\left(|B_s - B_t| > \frac{\ln(\lambda)}{M}\right) d\lambda \\
&= \int_1^\infty P\left(|B_s - B_t| > \frac{\ln(\lambda)}{M}\right) d\lambda \\
&\quad + \int_0^1 P(|B_s - B_t| > \text{nonpositive})\, d\lambda \\
&= 1 + \frac{2}{\sqrt{2\pi(s-t)}} \int_1^\infty \int_{\ln(\lambda)/M}^\infty e^{-\frac{x^2}{2(s-t)}} dx d\lambda \\
&= 1 + C \int_0^\infty \int_1^{e^{Mx}} e^{-\frac{x^2}{2(s-t)}} d\lambda dx \leq 1 + C \int_0^\infty e^{-\frac{x^2}{2(s-t)}} e^{Mx} dx < \infty.
\end{aligned}$$

This proves the lemma.

Lemma 34.2 *Let ϕ be a uniformly bounded adapted step function on $[0, T]$. Let*

$$\xi(t) \equiv \exp\left(\int_0^t \phi dB - \frac{1}{2}\int_0^t \phi^2 dr\right)$$

Then $\xi(t)$ is a continuous $\mathcal{H}_t$ martingale.

Proof: That $\xi(t)$ is continuous follows from the fact ϕ is bounded and the continuity of B_t. It remains to verify it is a martingale. Let ϕ be given by 34.1 and suppose $t_j \leq t < s < t_{j+1}$. Then

$$E(\xi(s)|\mathcal{H}_t) = E\left(\frac{\xi(s)}{\xi(t)}\xi(t)|\mathcal{H}_t\right) = \xi(t) E\left(\frac{\xi(s)}{\xi(t)}|\mathcal{H}_t\right). \tag{34.4}$$

From the definition of the integral on step functions given above and using the assumption on t and s just mentioned, it follows this equals

$$\begin{aligned}
&\xi(t) E\left(\frac{\exp\left(\int_0^s \phi dB - \frac{1}{2}\int_0^s \phi^2 dr\right)}{\exp\left(\int_0^t \phi dB - \frac{1}{2}\int_0^t \phi^2 dr\right)}|\mathcal{H}_t\right) \\
= &\ \xi(t) E\left(\exp(\phi_j(B_s - B_t)) \exp\left(-\frac{1}{2}\phi_j^2(s-t)\right)|\mathcal{H}_t\right).
\end{aligned}$$

Letting A be $\mathcal{H}_t$ measurable, it follows from independence of the increment, $B_s - B_t$ and $\mathcal{H}_t$ that

$$\begin{aligned}
&\int_A E\left(\exp\left(\phi_j\left(B_s - B_t\right)\right)\exp\left(-\frac{1}{2}\phi_j^2\left(s-t\right)\right)|\mathcal{H}_t\right)dP \\
= &\int_\Omega \mathcal{X}_A \exp\left(\phi_j\left(B_s - B_t\right)\right)\exp\left(-\frac{1}{2}\phi_j^2\left(s-t\right)\right)dP \\
= &\int_\Omega \mathcal{X}_A dP \int_\Omega \exp\left(\phi_j\left(B_s - B_t\right)\right)\exp\left(-\frac{1}{2}\phi_j^2\left(s-t\right)\right)dP. \qquad (34.5)
\end{aligned}$$

Since ϕ_j is bounded and measurable in $\mathcal{H}_{t_j}$, there exists a sequence of simple functions, $\{\alpha_n\}$ which converges to ϕ_j uniformly. Say

$$\alpha_n(\omega) = \sum_{i=1}^{m_n} c_i \mathcal{X}_{E_i}(\omega)$$

where each $E_i \in \mathcal{H}_t$ (In fact, $E_i \in \mathcal{H}_{t_j}$.). Then by the dominated convergence theorem, or simply the boundedness of ϕ_j and the uniform convergence of α_n to ϕ_j,

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \exp\left(\alpha_n\left(B_s - B_t\right)\right)\exp\left(-\frac{1}{2}\alpha_n^2\left(s-t\right)\right)dP \\
= &\int_\Omega \exp\left(\phi_j\left(B_s - B_t\right)\right)\exp\left(-\frac{1}{2}\phi_j^2\left(s-t\right)\right)dP.
\end{aligned}$$

However, by independence of the increments again

$$\begin{aligned}
&\int_\Omega \exp\left(\alpha_n\left(B_s - B_t\right)\right)\exp\left(-\frac{1}{2}\alpha_n^2\left(s-t\right)\right)dP \\
= &\sum_{i=1}^{m_n}\int_{E_i} \exp\left(c_i\left(B_s - B_t\right)\right)\exp\left(-\frac{1}{2}c_i^2\left(s-t\right)\right)dP \\
= &\sum_{i=1}^{m_n}\int_{E_i} dP\int_\Omega \exp\left(c_i\left(B_s - B_t\right)\right)\exp\left(-\frac{1}{2}c_i^2\left(s-t\right)\right)dP.
\end{aligned}$$

However,

$$\begin{aligned}
&\int_\Omega \exp\left(c_i\left(B_s - B_t\right)\right)\exp\left(-\frac{1}{2}c_i^2\left(s-t\right)\right)dP \\
= &\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\left(s-t\right)}}e^{-\frac{x^2}{2(s-t)}}e^{c_i x}e^{-\frac{1}{2}c_i^2(s-t)}dx = 1
\end{aligned}$$

which follows from completing the square in the exponents and recognizing the integrand as a normal distribution. Therefore, 34.5 reduces to $\int_A dP$ and so $E\left(\frac{\xi(s)}{\xi(t)}|\mathcal{H}_t\right) = 1$ which shows from 34.4 that

$$E\left(\xi\left(s\right)|\mathcal{H}_t\right) = \xi\left(t\right).$$

The next case is when $s = t_{j+1}$ and $t \in (t_j, t_{j+1})$. The argument goes the same way. In this case,

$$E\left(\frac{\xi(s)}{\xi(t)}|\mathcal{H}_t\right)$$

$$\begin{aligned} = \quad & E\left(\exp\left(\phi_j\left(B_{t_{j+1}} - B_{t_j}\right) - \phi_j\left(B_t - B_{t_j}\right)\right)\cdot\right. \\ & \left.\exp\left(-\frac{1}{2}\left(\phi_j^2\left(t_{j+1} - t_j\right) - \phi_j^2\left(t - t_j\right)\right)\right)|\mathcal{H}_t\right) \end{aligned}$$

$$= E\left(\exp\left(\phi_j\left(B_{t_{j+1}} - B_t\right)\right)\exp\left(-\frac{1}{2}\phi_j^2\left(t_{j+1} - t\right)\right)|\mathcal{H}_t\right).$$

Now it is just a repeat of the above argument to show this is 1.

All other cases follow easily from this. For example, suppose $t_{j-1} \leq t < t_j < t_{j+1} \leq s < t_{j+2}$. Then from the two cases considered above,

$$\begin{aligned} E(\xi(s)|\mathcal{H}_t) &= E\left(E\left(E\left(\xi(s)|\mathcal{H}_{t_{j+1}}\right)|\mathcal{H}_{t_j}\right)|\mathcal{H}_t\right) \\ &= E\left(E\left(\xi(t_{j+1})|\mathcal{H}_{t_j}\right)|\mathcal{H}_t\right) \\ &= E(\xi(t_j)|\mathcal{H}_t) = \xi(t). \end{aligned}$$

Continuing in this way shows $\xi(t)$ is a martingale. This proves the lemma.

Note also that this shows

$$E(\xi(T)) = E(\xi(T)|\mathcal{H}_0) = \xi(0) = 1.$$

Now from Doob's martingale estimate, Theorem 33.14,

$$P\left(\max_{t\in[0,T]}\xi(t) \geq \lambda\right) \leq \frac{1}{\lambda}E(\xi(T)) = \frac{1}{\lambda}.$$

If ϕ were replaced by $\alpha\phi$ and ξ_α were obtained by replacing ϕ with $\alpha\phi$, the same estimate would follow. Thus

$$\begin{aligned} & P\left(\max_t\left(\int_0^t \phi dB - \frac{\alpha}{2}\int_0^t \phi^2 ds\right) > \lambda\right) \\ = \quad & P\left(\max_t\left(\int_0^t \alpha\phi dB - \frac{1}{2}\int_0^t (\alpha\phi)^2 ds\right) > \alpha\lambda\right) \\ = \quad & P\left(\max_t\left(\exp\left(\int_0^t \alpha\phi dB - \frac{1}{2}\int_0^t (\alpha\phi)^2 ds\right)\right) > e^{\alpha\lambda}\right) \\ = \quad & P\left(\max_t(\xi_\alpha(t)) > e^{\alpha\lambda}\right) \leq \frac{1}{e^{\alpha\lambda}}E(\xi_\alpha(T)) = e^{-\alpha\lambda} \qquad (34.6) \end{aligned}$$

Summarizing this gives the following very significant inequality in which α, λ are two arbitrary positive constants independent of ϕ.

$$P\left(\max_t\left(\int_0^t \phi dB - \frac{\alpha}{2}\int_0^t \phi^2 ds\right) > \lambda\right) \leq e^{-\alpha\lambda}. \qquad (34.7)$$

Now recall how adapted functions can be approximated by adapted step functions. This was proved in Corollary 33.4 which is stated here for convenience.

Corollary 34.3 *Let $\mathcal{F}_t$ be a filtration and suppose f is adapted and $\mathcal{B} \times \mathcal{F}$ measurable such that for a.e. ω,*

$$\int_0^T f(t,\omega)^2\, dt < \infty. \tag{34.8}$$

Then there exists a sequence of uniformly bounded adapted step functions, ϕ_n such that

$$\lim_{n\to\infty} P\left(\int_0^T (f(t,\omega) - \phi_n(t,\omega))^2\, dt > \varepsilon\right) = 0. \tag{34.9}$$

Thus

$$\phi_n(t,\omega) = \sum_{j=0}^{m_n-1} e_j^n(\omega)\, \mathcal{X}_{[t_j^n, t_{j+1}^n)}(t)$$

where $t_0^n = 0$ and e_j^n is $\mathcal{F}_{t_j^n}$ measurable. Furthermore, if f is in $L^2([0,T] \times \Omega)$, there exists a subsequence $\{\phi_{n_k}\}$ such that

$$\lim_{k\to\infty} \int_\Omega \int_0^T \left(f(t,\omega) - \phi_{n_k}(t,\omega)\right)^2 dtdP = 0$$

From this corollary, the following fundamental lemma will make possible the definition of the Itô integral. It pertains to the filtration, $\mathcal{H}_t$ with respect to which B_t is a martingale and such that for $s > t, B_s - B_t$ is independent of $\mathcal{H}_t$.

Lemma 34.4 *Suppose f is $\mathcal{H}_t$ adapted and $\mathcal{B} \times \mathcal{F}$ measurable such that for a.e. ω,*

$$\int_0^T f(t,\omega)^2\, dt < \infty. \tag{34.10}$$

Then there exists a sequence of bounded adapted step functions, $\{\phi_k\}$ and a set of measure zero, E, such that for $\omega \notin E$,

$$\int_0^T (f(t,\omega) - \phi_k(t,\omega))^2\, dt \leq 2^{-k}$$

for all k sufficiently large (depending on $\omega \notin E$). Also, for $\omega \notin E$, there exists $K(\omega)$ such that if $k > l \geq K(\omega)$, then

$$\int_0^T (\phi_k(s,\omega) - \phi_l(s,\omega))^2\, ds < 2^{-(k-2)}.$$

Proof: By Corollary 33.4 stated above, there exists a subsequence, of the $\{\phi_n\}$ mentioned in this corollary, $\{\phi_{n_k}\}$ such that

$$P\left(\int_0^T \left(f(t,\omega) - \phi_{n_k}(t,\omega)\right)^2 dt > 2^{-k}\right) < 2^{-k}.$$

Now let $A_k \equiv \left[\omega : \int_0^T \left(f(t,\omega) - \phi_{n_k}(t,\omega)\right)^2 dt > 2^{-k}\right]$. Then from the above,

$$\sum_{k=1}^{\infty} P(A_k) < \infty$$

and so by the Borel Cantelli lemma, the set, E of points ω contained in infinitely many of the A_k has measure zero. Therefore, for $\omega \notin E$, there exists $K(\omega)$ such that for $k > K(\omega), \omega \notin A_k$ and so

$$\int_0^T \left(f(t,\omega) - \phi_{n_k}(t,\omega)\right)^2 dt \leq 2^{-k}$$

Denote $\phi_k = \phi_{n_k}$.

For $\omega \notin E$ and $l > k > K(\omega)$ described above, and $t \leq T$,

$$\begin{aligned} \int_0^t (\phi_k(s,\omega) - \phi_l(s,\omega))^2 ds &\leq 2\int_0^t (\phi_k(s,\omega) - f(s,\omega))^2 ds \\ +2\int_0^t (f(s,\omega) - \phi_l(s,\omega))^2 ds &\leq 2^{k-1} + 2^{l-1} \leq 2^{k-2}. \end{aligned}$$

This proves the lemma.

Now recall the fundamental estimate, 34.7,

$$P\left(\max_t \left(\int_0^t \phi dB - \frac{\alpha}{2}\int_0^t \phi^2 ds\right) > \lambda\right) \leq e^{-\alpha\lambda}. \tag{34.11}$$

which was valid for bounded adapted step functions, ϕ. This lemma makes possible the following theorem which is the basis for the definition of the Itô integral.

Theorem 34.5 *Suppose f is $\mathcal{H}_t$ adapted and $\mathcal{B} \times \mathcal{F}$ measurable such that for a.e. ω,*

$$\int_0^T f(t,\omega)^2 dt < \infty,$$

and let $\{\phi_n\}$ be a sequence of bounded adapted step functions such that off a set of measure zero E,

$$\int_0^T (f(t,\omega) - \phi_n(t,\omega))^2 dt \leq 2^{-n} \tag{34.12}$$

for all n sufficiently large, the existence of such a sequence being provided by Lemma 34.4; then there exists a set of measure zero E, such that if $\omega \notin E$, then

$$\left\{\int_0^t \phi_n dB(\omega)\right\}$$

is uniformly Cauchy for $t \in [0,T]$. Furthermore, if $\{\psi_n\}$ is another sequence of bounded adapted step functions satisfying 34.12, then for ω off a set of measure zero,

$$\lim_{n\to\infty} \left(\max_t \left|\int_0^t \phi_n dB(\omega) - \int_0^t \psi_n dB(\omega)\right|\right) = 0.$$

Proof: By Lemma 34.4, for $\omega \notin E$ the exceptional set of measure zero, there exists $N(\omega)$ such that if $n > m \geq N(\omega)$,

$$\int_0^T (\phi_n(s,\omega) - \phi_m(s,\omega))^2 ds < 2^{-(m-2)}$$

In the estimate 34.11 let

$$e^{-\alpha\lambda} = \theta^{-m},\ \alpha = \left(\frac{3}{2}\right)^{m-2}$$

where $\theta > 1$. Hence $\lambda = \left(\frac{2}{3}\right)^{m-2} m \ln\theta$. Thus for $n > m \geq N(\omega)$,

$$P\left(\max_t \left(\int_0^t (\phi_n - \phi_m) dB - \frac{1}{2}\left(\frac{3}{2}\right)^{m-2} \int_0^t (\phi_n - \phi_m)^2 ds\right) > \left(\frac{2}{3}\right)^{m-2} m \ln\theta\right) < \theta^{-m}$$

By the Borel Cantelli lemma again, there exists a set of measure zero E containng the earlier exceptional set such that for $\omega \notin E$ there exists $N(\omega)$ large enough that for $n > m \geq N(\omega)$,

$$\max_t \left(\int_0^t (\phi_n - \phi_m) dB(\omega) - \frac{1}{2}\left(\frac{3}{2}\right)^{m-2} \int_0^t (\phi_n - \phi_m)^2 ds\right) \leq \left(\frac{2}{3}\right)^{m-2} m \ln\theta$$

$$\int_0^T (\phi_n - \phi_m)^2 ds < 2^{-(m-2)}.$$

Therefore for such ω,

$$\max_t \left(\int_0^t (\phi_n - \phi_m) dB(\omega) - \frac{1}{2}\left(\frac{3}{2}\right)^{m-2} 2^{-(m-2)}\right) \leq \left(\frac{2}{3}\right)^{m-2} m \ln\theta$$

and switching ϕ_n and ϕ_m,

$$\max_t \left(\int_0^t (\phi_m - \phi_n) dB(\omega) - \frac{1}{2}\left(\frac{3}{2}\right)^{m-2} 2^{-(m-2)}\right) \leq \left(\frac{2}{3}\right)^{m-2} m \ln\theta.$$

Therefore,

$$\max_t \left(\left|\int_0^t (\phi_m - \phi_n) dB(\omega)\right| \leq \left(\frac{2}{3}\right)^{m-2} m \ln\theta + \frac{1}{2}\left(\frac{3}{4}\right)^{m-2}\right).$$

It follows that for ω off a set of measure zero, $\left\{\int_0^t \phi_n dB(\omega)\right\}$ is a Cauchy sequence. Adjusting the above constants, there exists a constant, $r < 1$ and a positive constant, C such that

$$\max_t \left(\left|\int_0^t (\phi_m - \phi_n) dB(\omega)\right| \leq Cr^m\right)$$

whenever $n > m$ and m large enough.

If $\{\psi_n\}$ is another such sequence satisfying 34.12 then for n large enough,

$$\int_0^T (\phi_n(s,\omega) - \psi_n(s,\omega))^2 \, ds < 2^{-(n-2)}$$

and so the same estimate yields, for all ω off a set of measure zero,

$$\max_t \left(\left| \int_0^t (\psi_n - \phi_n) \, dB(\omega) \right| \leq Cr^n \right)$$

which shows that for ω off a set of measure zero,

$$\lim_{n\to\infty} \left(\max_t \left| \int_0^t \phi_n dB(\omega) - \int_0^t \psi_n dB(\omega) \right| \right) = 0.$$

This proves the theorem.

Note there is no loss of generality in starting the integral at 0. All the above works with no change for integrals on the interval, $[S,T]$ with no significant change.

With this theorem the following definition and conclusion is well defined.

Definition 34.6 *Suppose f is $\mathcal{H}_t$ adapted and $\mathcal{B}\times\mathcal{F}$ measurable such that for $\omega \notin E$ a set of measure zero,*

$$\int_S^T f(t,\omega)^2 \, dt < \infty.$$

Then there exists a sequence of adapted bounded step functions, $\{\phi_n\}$ satisfying

$$\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2 \, dt \leq 2^{-n}$$

for $\omega \notin E$, a set of measure zero. Then for $t \in [S,T]$, the Itô integral is defined by

$$\int_S^t f dB(\omega) = \lim_{n\to\infty} \int_S^t \phi_n dB(\omega).$$

Furthermore, for these ω, $t \to \int_S^t f dB(\omega)$ is continuous because by Theorem 34.5 the convergence of $\int_S^t \phi_n dB(\omega)$ is uniform on $[0,T]$.

Definition 34.7 *Suppose f is $\mathcal{H}_t$ adapted and $\mathcal{B}\times\mathcal{F}$ measurable such that for a.e. ω,*

$$\int_S^T f(t,\omega)^2 \, dt < \infty.$$

I will denote such functions by saying they are in $W(\mathcal{H})$.

34.1 Properties Of The Itô Integral

Theorem 34.8 *Let $f, g \in W(\mathcal{H})$ and let $0 \leq S \leq U \leq T$. Then the following hold.*

$$\int_S^T f dB = \int_S^U f dB + \int_U^T f dB \tag{34.13}$$

$$\int_S^T (af + bg)\, dB = a \int_S^T f dB + b \int_S^T f dB \tag{34.14}$$

$$\int_S^T f dB \text{ is } \mathcal{H}_T \text{ measurable.} \tag{34.15}$$

$$E\left(\int_S^T f dB\right) = 0 \tag{34.16}$$

if for some sequence of adapted step functions, $\{\phi_n\}$,

$$\lim_{n\to\infty} E\left(\int_S^T \phi_n dB\right) = E\left(\int_S^T f dB\right).$$

Proof: This holds essentially because it holds for any ϕ a step function. For example, if

$$\phi(t,\omega) = \sum_{j=1}^n e_j(\omega) \mathcal{X}_{[t_j, t_{j+1})}(t),$$

Then without loss of generality it can be assumed U is one of the t_j say t_k. Therefore,

$$\begin{aligned} \int_S^T \phi dB &= \sum_{j=1}^{k-1} e_j(\omega) \left(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)\right) \\ &\quad + \sum_{j=k}^n e_j(\omega) \left(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)\right) \\ &= \int_S^U \phi dB + \int_U^T \phi dB \end{aligned}$$

It follows 34.13 must hold in the limit. 34.14 is somewhat more obvious. Consider 34.16. Let ϕ be as above. Then recall that e_j is $\mathcal{H}_{t_j}$ measurable and so

$$\begin{aligned} E\left(\sum_{j=1}^n e_j(\omega) \left(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)\right)\right) &= \sum_{j=1}^n E(e_j) E\left(B_{t_{j+1}} - B_{t_j}\right) \\ &= \sum_{j=1}^n E(e_j)\, 0 = 0 \end{aligned}$$

What is used here is the independence of the increments, $B_s - B_t$ to $\mathcal{H}_t$.

34.15 must also hold because if ϕ is as above, $\int_S^T \phi dB$ is $\mathcal{H}_T$ measurable and $\int_S^T f dB$ is a pointwise a.e. limit of these.

The next theorem is called the Itô isometry. It pertains to the case where $f \in L^2([S,T] \times \Omega)$. Thus

$$\int_\Omega \int_0^T f(t,\omega)^2 \, dt dP < \infty$$

which says more than just

$$P\left(\int_0^T f(t,\omega)^2 \, dt < \infty\right) = 1.$$

Recall Corollary 33.4 again.

Corollary 34.9 *Let $\mathcal{F}_t$ be a filtration and suppose f is adapted and $\mathcal{B} \times \mathcal{F}$ measurable such that for a.e. ω,*

$$\int_0^T f(t,\omega)^2 \, dt < \infty. \tag{34.17}$$

Then there exists a sequence of uniformly bounded adapted step functions, ϕ_n such that

$$\lim_{n\to\infty} P\left(\int_0^T (f(t,\omega) - \phi_n(t,\omega))^2 \, dt > \varepsilon\right) = 0. \tag{34.18}$$

Thus

$$\phi_n(t,\omega) = \sum_{j=0}^{m_n-1} e_j^n(\omega) \mathcal{X}_{[t_j^n, t_{j+1}^n)}(t)$$

where $t_0^n = 0$ and e_j^n is $\mathcal{F}_{t_j^n}$ measurable. Furthermore, if f is in $L^2([0,T] \times \Omega)$, there exists a subsequence $\{\phi_{n_k}\}$ such that

$$\lim_{k\to\infty} \int_\Omega \int_0^T \left(f(t,\omega) - \phi_{n_k}(t,\omega)\right)^2 dt dP = 0$$

The following theorem is the Itô isometry.All of this is still in the context of the filtration, $\mathcal{H}_t$ with respect to which B_t is a martingale and the increments, $B_s - B_t$ are independent of $\mathcal{H}_t$ whenever $s > t$.

Theorem 34.10 *Let f be $\mathcal{H}_t$ adapted and in $L^2([S,T] \times \Omega)$. Then*

$$\left|\left|\int_S^T f dB\right|\right|_{L^2(\Omega)} = ||f||_{L^2([0,T]\times\Omega)}.$$

Proof: First let $\phi(t,\omega) = \sum_{j=0}^{n-1} e_j(\omega)\mathcal{X}_{[t_j,t_{j+1})}(t)$ be a uniformly bounded adapted step function. Then

$$\int_S^T \phi dB = \sum_{j=0}^{n-1} e_j\left(B_{t_{j+1}} - B_{t_j}\right).$$

Then

$$\left(\int_S^T \phi dB\right)^2 = \sum_{i,j} e_j\left(B_{t_{j+1}} - B_{t_j}\right) e_i\left(B_{t_{i+1}} - B_{t_i}\right).$$

Consider these terms. First consider one in which $i \neq j$, say

$$e_i e_j\left(B_{t_{i+1}} - B_{t_i}\right)\left(B_{t_{j+1}} - B_{t_j}\right).$$

By independence of the increments, it follows

$$\begin{aligned} & \int_\Omega e_i e_j\left(B_{t_{i+1}} - B_{t_i}\right)\left(B_{t_{j+1}} - B_{t_j}\right) dP \\ = & \int_\Omega e_i e_j\left(B_{t_{i+1}} - B_{t_i}\right) dP \int_\Omega \left(B_{t_{j+1}} - B_{t_j}\right) dP = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_\Omega \left(\int_S^T \phi dB\right)^2 dP &= \int_\Omega \sum_{j=0}^{n-1} e_j^2\left(B_{t_{j+1}} - B_{t_j}\right)^2 dP \\ &= \sum_{j=0}^{n-1} \int_\Omega e_j^2\left(B_{t_{j+1}} - B_{t_j}\right)^2 dP \\ &= \sum_{j=0}^{n-1} \int_\Omega e_j^2 dP \int_\Omega \left(B_{t_{j+1}} - B_{t_j}\right)^2 dP \\ &= \sum_{j=0}^{n-1} \int_\Omega e_j^2 dP\left(t_{j+1} - t_j\right) dP = \int_\Omega \int_0^T \phi^2 dt dP. \end{aligned}$$

This proves the Itô isometry on bounded adapted step functions.

By Corollary 33.4, there exists a sequence of adapted bounded step functions, $\{\phi_n\}$ which converges to f in $L^2([S,T]\times\Omega)$ such that also

$$\int_S^T f dB(\omega) = \lim_{n\to\infty} \int_S^T \phi_n dB(\omega) \text{ a.e.}$$

Therefore, from what was just shown

$$\left\{\int_S^T \phi_n dB\right\}_{n=1}^\infty$$

is a Cauchy sequence in $L^2(\Omega)$. Therefore, a subsequence of it converges a.e. However, this requires the thing to which it converges in $L^2(\Omega)$ must be $\int_S^T f dB(\omega)$. Therefore,

$$\left|\left|\int_S^T f dB\right|\right|_{L^2(\Omega)} = \lim_{n\to\infty}\left|\left|\int_S^T \phi_n dB\right|\right| = \lim_{n\to\infty}||\phi_n||_{L^2([S,T]\times\Omega)} = ||f||_{L^2([S,T]\times\Omega)}.$$

This proves the theorem.

This theorem also gives another way to define the Itô integral when

$$f \in L^2([S,T]\times\Omega)$$

in addition to being in $W(\mathcal{H})$.

Lemma 34.11 *If $f \in W(\mathcal{H})$ and is in $L^2([S,T]\times\Omega)$, let $\{\phi_n\}$ be any sequence of bounded adapted step functions converging to f in $L^2([S,T]\times\Omega)$. Then*

$$\int_S^T f dB = \lim_{n\to\infty}\int_S^T \phi_n dB$$

in $L^2(\Omega)$.

Proof: This is immediate from the following.

$$\left|\left|\int_S^T f dB - \int_S^T \phi_n dB\right|\right|_{L^2(\Omega)} = \left|\left|\int_S^T (f-\phi_n) dB\right|\right|_{L^2(\Omega)} = ||f-\phi_n||_{L^2([S,T]\times\Omega)}$$

which converges to 0.

Letting $f \in W(\mathcal{H})$, one can consider the stochastic process $\int_S^t f(s,\omega) dB(\omega)$. From the construction of the Itô integral above, this is a continuous function of t for a.e. ω. It turns out that if f is also in $L^2([S,T]\times\Omega)$, then this stochastic process is also an $\mathcal{H}_t$ martingale.

Theorem 34.12 *$I(t,\omega) \equiv \int_S^t f(s,\omega) dB(\omega)$ is an $\mathcal{H}_t$ martingale if*

$$f \in L^2([S,T]\times\Omega).$$

Proof: Let

$$I_n(t,\omega) = \int_S^t \phi_n(s,\omega) dB(\omega).$$

where ϕ_n is a bounded adapted step function such that

$$\int_S^t f dB = \lim_{n\to\infty}\int_S^t \phi_n dB$$

in $L^2(\Omega)$ for each $t \in [S,T]$ and for ω not in a suitable set of measure zero,

$$I(t,\omega) \equiv \int_S^t f dB(\omega) = \lim_{n\to\infty} \int_S^t \phi_n dB(\omega) \tag{34.19}$$

uniformly for $t \in [S,T]$.

In fact, $I_n(t,\omega)$ is a martingale. Let $s > t$. Then

$$E(I_n(s,\omega)|\mathcal{F}_t) = E\left(\int_S^t \phi_n dB + \int_t^s \phi_n dB|\mathcal{F}_t\right).$$

Now $\int_S^t \phi_n dB$ is measurable in $\mathcal{F}_t$ and so this reduces to

$$\int_S^t \phi_n dB + E\left(\sum_{t\leq t_j^n < t_{j+1}^n \leq s} e_j^n \left(B_{t_{j+1}} - B_{t_j}\right)|\mathcal{F}_t\right).$$

By Lemma 32.2,

$$\begin{aligned}
&= \int_S^t \phi_n dB + E\left(E\left(\sum_{t\leq t_j^n < t_{j+1}^n \leq s} e_j^n \left(B_{t_{j+1}} - B_{t_j}\right)|\mathcal{F}_{t_j}\right)|\mathcal{F}_t\right) \\
&= \int_S^t \phi_n dB + E\left(\sum_{t\leq t_j^n < t_{j+1}^n \leq s} E\left(e_j^n \left(B_{t_{j+1}} - B_{t_j}\right)|\mathcal{F}_{t_j}\right)|\mathcal{F}_t\right) \\
&= \int_S^t \phi_n dB + E\left(\sum_{t\leq t_j^n < t_{j+1}^n \leq s} e_j^n E\left(\left(B_{t_{j+1}} - B_{t_j}\right)|\mathcal{F}_{t_j}\right)|\mathcal{F}_t\right) \\
&= \int_S^t \phi_n dB + E\left(\sum_{t\leq t_j^n < t_{j+1}^n \leq s} e_j^n \cdot 0|\mathcal{F}_t\right) = \int_S^t \phi_n dB = I_n(t,\omega).
\end{aligned}$$

Thus $I_n(t,\omega)$ is a martingale. Let $s > t$. Since $I_n(r,\cdot) \to I(r,\cdot)$ in $L^2(\Omega)$ for each r, Jensen's inequality implies

$$\begin{aligned}
& \int_\Omega |E(I_n(s,\omega)|\mathcal{F}_t) - E(I(s,\omega)|\mathcal{F}_t)| \, dP \\
\leq & \int_\Omega E(|I_n(s,\omega) - I(s,\omega)|\,|\mathcal{F}_t) \, dP \\
= & \int_\Omega |I_n(s,\omega) - I(s,\omega)| \, dP
\end{aligned}$$

which converges to 0. It follows that for $F \in \mathcal{F}_t$, and $s > t$,

$$\begin{aligned}
\int_F I(t,\omega)\,dP &= \lim_{k\to\infty}\int_F I_n(t,\omega)\,dP \\
&= \lim_{k\to\infty}\int_F E\left(I_n(s,\omega)\,|\mathcal{F}_t\right)dP \\
&= \int_F E\left(I(s,\omega)\,|\mathcal{F}_t\right)dP.
\end{aligned}$$

What about the measurability of I? This follows from the pointwise convergence described in 34.19 the measurability of I_n and completness of the measure. This proves the theorem.

Example 34.13 *Find* $\int_0^t B_s(\omega)\,dB(\omega)$ *assuming* $B_0(0) = 0$.

Let $\phi_n(s,\omega) = \sum_{j=0}^n B_{t_j}(\omega)\,\mathcal{X}_{[t_j,t_{j+1})}(s)$ where $|t_{j+1} - t_j|$ is constant in j and equals t/n. Then $\phi_n \to B_s$ in $L^2([0,t]\times\Omega)$ and it is clear that ϕ_n is adapted. Therefore,

$$\int_S^T \phi_n(t,\omega)\,dB(\omega) \to \int_S^T B_t(\omega)\,dB(\omega)$$

in $L^2(\Omega)$. But by definition,

$$\int_S^T \phi_n(t,\omega)\,dB(\omega) = \sum_{j=0}^n B_{t_j}(\omega)\left(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)\right)$$

and a little algebra shows this equals

$$\begin{aligned}
&\sum_{j=0}^n \frac{1}{2}\left(B_{t_{j+1}}^2 - B_{t_j}^2\right) - \sum_{j=0}^n \frac{1}{2}\left(B_{t_{j+1}} - B_{t_j}\right)^2 \\
= \;& \frac{1}{2}B_t(\omega)^2 - \frac{1}{2}\sum_{j=0}^n \left(B_{t_{j+1}} - B_{t_j}\right)^2.
\end{aligned}$$

Now

$$\int_\Omega \left|\sum_{j=0}^n \left(B_{t_{j+1}} - B_{t_j}\right)^2 - t\right|^2 dP$$

$$= \int_\Omega \sum_{i,j}\left(B_{t_{j+1}} - B_{t_j}\right)^2\left(B_{t_{i+1}} - B_{t_i}\right)^2 - 2t\sum_j\left(B_{t_{j+1}} - B_{t_j}\right)^2 + t^2 dP$$

$$= \sum_{i,j}(t_{j+1} - t_j)(t_{i+1} - t_i) - 2t\sum_j (t_{j+1} - t_j) + t^2 = t^2 - 2t^2 + t^2 = 0$$

and so

$$\int_\Omega \left|\frac{1}{2}B_t(\omega)^2 - \frac{1}{2}\sum_{j=0}^n\left(B_{t_{j+1}} - B_{t_j}\right)^2 - \left(\frac{1}{2}B_t(\omega)^2 - \frac{1}{2}t\right)\right|^2 dP = 0$$

which shows

$$\int_S^T B_t(\omega)\, dB(\omega) = \frac{1}{2} B_t(\omega)^2 - \frac{1}{2} t.$$

This is contrary to what any student would know; that from the symbols involved,

$$\int_S^T B_t(\omega)\, dB_t(\omega) = \frac{1}{2} B_t(\omega)^2 - \frac{1}{2} B_0(\omega)^2 = \frac{1}{2} B_t(\omega)^2.$$

Here you get the extra term, $-\frac{1}{2}t$.

Stochastic Processes

35.1 An Important Filtration

Recall the theorem about Brownian motion which is listed here for convenience.

Theorem 35.1 *There exists a probability space, $(\Omega, \mathcal{F}, P)$ and random vectors, B_t for $t \in [0, \infty)$ which satisfy the following properties.*

1. *For $\mathbf{Z} = (\mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_k}) \in \mathbb{R}^{nk}$ it follows $\mathbf{Z}$ is normally distributed. Its mean is*
$$\begin{pmatrix} \mathbf{x} & \cdots & \mathbf{x} \end{pmatrix} \in \mathbb{R}^{nk}$$

2. *$\mathbf{B}_t$ has independent increments. This means if $t_1 < t_2 < \cdots < t_k$, the random variables,*
$$\mathbf{B}_{t_1}, \mathbf{B}_{t_2} - \mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_k} - \mathbf{B}_{t_{k-1}}$$
are independent and normally distributed. Note this implies the k^{th} components must also be independent. Also $\mathbf{B}_{t_j} - \mathbf{B}_{t_{j-1}}$ is normal with covariance $(t_j - t_{j-1}) I$ and mean $\mathbf{0}$. In addition to this, the k^{th} component of $\mathbf{B}_t$ is normally distributed with density function
$$p(t, x_k, y) \equiv \frac{1}{(2\pi t)^{1/2}} \exp\left(-\frac{|y - x_k|^2}{2t} \right)$$
This follows from the distribution of $\mathbf{B}_t$ which has a density function
$$p(t, \mathbf{x}, \mathbf{y}) \equiv \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|\mathbf{y} - \mathbf{x}|^2}{2t} \right)$$

3. *$E\left(|\mathbf{B}_t - \mathbf{B}_s|^4 \right) \leq 3n^2 (t - s)^2$, For $t > s$,*
$$\begin{aligned} E\left(|\mathbf{B}_t - \mathbf{B}_s|^2 \right) &= n(t - s), \\ E\left((\mathbf{B}_t - \mathbf{x})^* (\mathbf{B}_s - \mathbf{x}) \right) &= ns, \\ E(\mathbf{B}_t - \mathbf{B}_s) &= \mathbf{0}, \end{aligned}$$

4. $t \to \mathbf{B}_t(\omega)$ *is Holder continuous.*

Observation 35.2 *Let $\mathbf{B}_t$ be n dimensional Brownian motion as discussed above. Then considering the k^{th} component of $\mathbf{B}_t, B_{kt}$, it follows B_{kt} is one dimensional Brownian motion. Letting $\mathcal{H}_t$ denote the completion of the smallest σ algebra containing*

$$\left(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k}\right)^{-1}(B)$$

for all B a Borel set in $\mathbb{R}^{nk}$ for all sequences, $0 \leq s_1 < s_2 \cdots < s_k \leq t$, it follows that for $s > t$, $B_{ks} - B_{kt}$ is independent of $\mathcal{H}_t$. Also, from the fact discussed above in Theorem 33.9 that $\mathbf{B}_t$ is a martingale, it follows the same is true of B_{kt}. Thus all the above theory can be applied for this $\mathcal{H}_t$ and integrating with respect to one of the components of n dimensional Brownian motion.

One other thing should be pointed out although it was mentioned above and that is the distribution of $\mathbf{B}_s - \mathbf{B}_t$ for $s > t$. The density for $\mathbf{B}_t$ as described above is

$$\frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|\mathbf{y}-\mathbf{x}|^2}{2t}\right)$$

and so by Theorem 31.22 on Page 868

$$E\left(e^{i\mathbf{u}\cdot\mathbf{B}_t}\right) = e^{i\mathbf{u}\cdot\mathbf{x}} e^{-\frac{1}{2}\mathbf{u}^* tI\mathbf{u}}.$$

Recall also from the above that $\mathbf{B}_s - \mathbf{B}_t$ is independent to $\mathbf{B}_t$. Therefore,

$$\begin{aligned} E\left(e^{i\mathbf{u}\cdot\mathbf{B}_s}\right) &= E\left(e^{i\mathbf{u}\cdot\mathbf{B}_t} e^{i\mathbf{u}\cdot(\mathbf{B}_s-\mathbf{B}_t)}\right) \\ &= E\left(e^{i\mathbf{u}\cdot\mathbf{B}_t}\right) E\left(e^{i\mathbf{u}\cdot(\mathbf{B}_s-\mathbf{B}_t)}\right) \end{aligned}$$

and so

$$\begin{aligned} E\left(e^{i\mathbf{u}\cdot(\mathbf{B}_s-\mathbf{B}_t)}\right) &= \frac{e^{i\mathbf{u}\cdot\mathbf{x}} e^{-\frac{1}{2}\mathbf{u}^* sI\mathbf{u}}}{e^{i\mathbf{u}\cdot\mathbf{x}} e^{-\frac{1}{2}\mathbf{u}^* tI\mathbf{u}}} \\ &= e^{-\frac{1}{2}\mathbf{u}^*(s-t)I\mathbf{u}}. \end{aligned}$$

This implies the following lemma.

Lemma 35.3 *Let $\mathbf{B}_t = (B_{1t}, \cdots, B_{nt})$ be n dimensional Brownian motion. Then for $s > t, B_{is} - B_{it}$ and $B_{js} - B_{jt}$ are linearly independent if $j \neq i$.*

Proof: The covariance matrix of the distribution of $\mathbf{B}_s - \mathbf{B}_t$ is $(s-t)$ times the identity and so the components of this vector are linearly independent.

Lemma 35.4 *Letting $\mathcal{H}_t$ denote the completion of the smallest σ algebra containing*

$$\left(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k}\right)^{-1}(B)$$

for all B a Borel set in $\mathbb{R}^{nk}$ for all sequences, $0 \leq s_1 < s_2 \cdots < s_k \leq t$ as defined above, $\mathcal{H}_t$ is also equal to the completion of the smallest σ algebra containing

$$(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(B)$$

for all B an open set in $\mathbb{R}^{nk}$ for all sequences, $0 \leq s_1 < s_2 \cdots < s_k \leq t$. In addition to this, $\mathcal{H}_t$ is equal to the completion of the smallest σ algebra containing

$$(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(B)$$

for all B an open set in $\mathbb{R}^{nk}$ for all sequences, $0 \leq s_1 < s_2 \cdots < s_k \leq t$ such that the s_j are rational numbers.

Proof: The first claim reducing to inverse images of open sets is not hard. Define $\mathcal{G}_t$ to be the smallest σ algebra such that $(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(U) \in \mathcal{G}_t$ for U open and $0 \leq s_1 < s_2 \cdots < s_k \leq t$. Now let

$$\mathcal{S}_{(s_1,\cdots,s_k)} \equiv \left\{E \text{ Borel such that } (\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(E) \in \mathcal{G}_t\right\}$$

Then $\mathcal{S}_{(s_1,\cdots,s_k)}$ contains the open sets and so it also contains the Borel sets because it is a σ algebra. Hence $\mathcal{G}_t$ contains all sets of the form $(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(E)$ for all E Borel and $0 \leq s_1 < s_2 \cdots < s_k \leq t$. It follows $\mathcal{G}_t$ is the smallest σ algebra containing the sets of the form $(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(B)$ for B Borel and so its completion equals $\mathcal{H}_t$.

The second claim is more interesting. In this claim, it suffices to consider finite increasing sequences of rational numbers, rather than just finite increasing sequences. Let $0 \leq s_1 < s_2 \cdots < s_k \leq t$ and let $0 \leq t_1^n < t_2^n \cdots < t_k^n \leq t$ be an increasing sequence of rational numbers such that $\lim_{n\to\infty} t_k^n = s_k$. It has been proven that off a set of measure zero, $t \to \mathbf{B}_t(\omega)$ is continuous. To simplify the presentation, I will assume without loss of generality that this set of measure zero is empty. If not, you could simply delete it and consider a slightly modified Ω. Another way to see this is not a loss of generality is that $\mathcal{H}_t$ is complete and so contains all subsets of sets of measure zero. Let O be an open set and let

$$O = \cup_{m=1}^{\infty} O_m, \ \cdots O_m \subseteq \overline{O_m} \subseteq O_{m+1} \cdots$$

Then by continuity of $t \to \mathbf{B}_t(\omega)$,

$$\begin{aligned}
(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(\overline{O_m}) &\supseteq \cup_{l=1}^{\infty} \cap_{p \geq l} \left(\mathbf{B}_{t_1^p}, \cdots, \mathbf{B}_{t_k^p}\right)^{-1}(O_m) \\
&\supseteq (\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(O_m)
\end{aligned}$$

It follows upon taking the union over all m,

$$\begin{aligned}
(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(O) &= \cup_m (\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(\overline{O_m}) \\
&\supseteq \cup_m \cup_l \cap_{p \geq l} \left(\mathbf{B}_{t_1^p}, \cdots, \mathbf{B}_{t_k^p}\right)^{-1}(O_m) \\
&\supseteq \cup_m (\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(O_m) \\
&= (\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(O)
\end{aligned}$$

Thus

$$\cup_m \cup_l \cap_{p\geq l} \left(\mathbf{B}_{t_1^p}, \cdots, \mathbf{B}_{t_k^p}\right)^{-1} (O_m) = (\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1} (O)$$

and so the smallest σ algebra containing $(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1} (B)$ for B open and $0 \leq s_1 < s_2 \cdots < s_k \leq t$ an arbitrary sequence of numbers is the same as the smallest σ algebra containing $(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1} (B)$ for B open and $0 \leq s_1 < s_2 \cdots < s_k \leq t$ an increasing sequence of rational numbers.

35.2 Itô Processes

Let $\mathbf{B} = (B_1, \cdots, B_n)$ be n dimensional Brownian motion and let $\mathcal{H}_t$ be the filtration defined above. Thus B_k is a martingale with respect to $\mathcal{H}_t$ and the increments, $B_{ks} - B_{kt}$ are independent of $\mathcal{H}_t$ whenever $s > t$. Let X_k for $k = 1, 2, \cdots, m$ be a stochastic process satisfying for $t \in [0, T]$,

$$X_{kt} - X_{k0} = \int_0^t u_k (s, \omega) \, ds + \sum_{l=1}^{n} \int_0^t v_{kl} (s, \omega) \, dB_l. \tag{35.1}$$

Written in simpler form,

$$\mathbf{X}_t - \mathbf{X}_0 = \int_0^t \mathbf{u} (s) \, ds + \int_0^t V (s) \, d\mathbf{B} \tag{35.2}$$

where $V (s)$ is an $m \times n$ matrix. For now, assume u_k and v_{kl} are all $\mathcal{H}_t$ adapted uniformly bounded step functions.

Also assume g is a C^2 function defined on $\mathbb{R} \times \mathbb{R}^m$ for which all partial derivatives are uniformly bounded and let $\left\{t_j^r\right\}_{j=0}^{n_r}$ be partitions of $[0, T]$ such that for $\Delta (r) \equiv \sup_j \left\{t_{j+1}^r - t_j^r\right\}$, $\lim_{r\to\infty} \Delta (r) = 0$ and also all discontinuities of all the step functions, v_{kl} and u_k are contained in $\left\{t_j^r\right\}_{j=0}^{n_r}$. Then suppressing the superscript on t_j^r for the sake of simpler notation,

$$g (T, \mathbf{X}_T) - g (0, \mathbf{X}_0) = \sum_{j=0}^{n_r - 1} g \left(t_{j+1}, \mathbf{X}_{t_{j+1}}\right) - g \left(t_j, \mathbf{X}_{t_j}\right)$$

$$= \sum_{j=0}^{n_r - 1} \frac{\partial g}{\partial t} \left(t_j, \mathbf{X}_{t_j}\right) \Delta t_j + D_2 g \left(t_j, \mathbf{X}_{t_j}\right) \Delta \mathbf{X}_{t_j}$$

$$+ \frac{1}{2} \left(\frac{\partial^2 g}{\partial t^2} \left(t_j + \theta \Delta t_j, \mathbf{X}_{t_j} + \theta \Delta \mathbf{X}_{t_j}\right) \Delta t_j^2 \right. \tag{35.3}$$

$$+ D_2 \left(D_2 g \left(t_j + \theta \Delta t_j, \mathbf{X}_{t_j} + \theta \Delta \mathbf{X}_{t_j}\right) \left(\Delta \mathbf{X}_{t_j}\right)\right) \Delta \mathbf{X}_{t_j}$$

$$\left. + \, 2 D_2 \left(\frac{\partial g}{\partial t}\right) \left(t_j + \theta \Delta t_j, \mathbf{X}_{t_j} + \theta \Delta \mathbf{X}_{t_j}\right) \Delta t_j \Delta \mathbf{X}_{t_j} \right). \tag{35.4}$$

Now from 35.2 and the assumptions that all discontinuities of all step functions are in the partition, it follows the matrix, V and the vector, $\mathbf{u}$ must be of the form

$$V(s,\omega) = \sum_{j=0}^{n_r-1} V^j(\omega)\,\mathcal{X}_{[t_j,t_{j+1})}(s)\,,\ \mathbf{u}(s,\omega) = \sum_{j=0}^{n_r-1} \mathbf{u}^j(\omega)\,\mathcal{V}_{[t_j,t_{j+1})}(s)$$

It follows from 35.2 that

$$\Delta \mathbf{X}_{t_j} = \mathbf{u}^j \Delta t_j + V^j \Delta \mathbf{B}_{t_j} \tag{35.5}$$

It follows from this that 35.3 - 35.4 can be written as

$$= \sum_{j=0}^{n_r-1} \frac{\partial g}{\partial t}\left(t_j, \mathbf{X}_{t_j}\right)\Delta t_j + D_2 g\left(t_j, \mathbf{X}_{t_j}\right)\left(\mathbf{u}^j \Delta t_j + V^j \Delta \mathbf{B}_{t_j}\right) \tag{35.6}$$

$$+\frac{1}{2}\left(\frac{\partial^2 g}{\partial t^2}\left(t_j + \theta\Delta t_j, \mathbf{X}_{t_j} + \theta \Delta \mathbf{X}_{t_j}\right)\Delta t_j^2\right. \tag{35.7}$$

$$+2D_2\left(\frac{\partial g}{\partial t}\right)\left(t_j + \theta\Delta t_j, \mathbf{X}_{t_j} + \theta \Delta \mathbf{X}_{t_j}\right)\Delta t_j\left(\mathbf{u}^j \Delta t_j + V^j \Delta \mathbf{B}_{t_j}\right) \tag{35.8}$$

$$+D_2\left(D_2 g\left(t_j + \theta\Delta t_j, \mathbf{X}_{t_j} + \theta \Delta \mathbf{X}_{t_j}\right)\left(\mathbf{u}^j \Delta t_j + V^j \Delta \mathbf{B}_{t_j}\right)\right)\cdot \tag{35.9}$$

$$\left.\left(\mathbf{u}^j \Delta t_j + V^j \Delta \mathbf{B}_{t_j}\right)\right). \tag{35.10}$$

Since $\Delta(r) \to 0$ as $r \to \infty$, this simplifies to an expression of the form

$$= e(r) + \sum_{j=0}^{n_r-1} \frac{\partial g}{\partial t}\left(t_j, \mathbf{X}_{t_j}\right)\Delta t_j + D_2 g\left(t_j, \mathbf{X}_{t_j}\right)\left(\mathbf{u}^j \Delta t_j + V^j \Delta \mathbf{B}_{t_j}\right)$$

$$\begin{aligned}&+\frac{1}{2}\left(2D_2\left(\frac{\partial g}{\partial t}\right)\left(t_j + \theta\Delta t_j, \mathbf{X}_{t_j} + \theta \Delta \mathbf{X}_{t_j}\right)V^j \Delta t_j \Delta \mathbf{B}_{t_j}\right.\\ &+\left(\mathbf{u}^j \Delta t_j + V^j \Delta \mathbf{B}_{t_j}\right)^T\\ &H\left(t_j + \theta\Delta t_j, \mathbf{X}_{t_j} + \theta \Delta \mathbf{X}_{t_j}\right)\left(\mathbf{u}^j \Delta t_j + V^j \Delta \mathbf{B}_{t_j}\right)\end{aligned} \tag{35.11}$$

where $e(r) \to 0$ as $r \to \infty$ and H is the Hessian matrix of second partial derivatives of g taken with respect to the x variables. The $e(r)$ in the above is obtained by the inclusion of all the terms which have a Δt_j^2 in them. There are lots of other terms which are of the form

$$\sum_{j=0}^{n_r-1} a^j \Delta t_j \Delta B_{kt_j}$$

where a^j is bounded independent of r. In fact, all such terms can be included in

$e(r)$ off a set of measure zero. I will show this now.

$$\begin{aligned}\int_\Omega \left|\sum_{j=0}^{n_r-1} a^j \Delta t_j \Delta B_{kt_j}\right| dP &\leq C\int_\Omega \sum_{j=0}^{n_r-1} \Delta t_j \left|\Delta B_{kt_j}\right| dP \\ &\leq C\sum_{j=0}^{n_r-1} \Delta t_j \int_\Omega \left|\Delta B_{kt_j}\right| dP \\ &= C\sum_{j=0}^{n_r-1} \Delta t_j \left(\int_\Omega \left|\Delta B_{kt_j}\right|^2 dP\right)^{1/2} \\ &\leq C\Delta(r)^{1/2} T\end{aligned}$$

which converges to 0. Therefore, there exists a set of measure zero off which terms of this form converge to 0 as $r \to \infty$ upon taking a further subsequence if necessary. Therefore, the above expression simplifies further and yields

$$g(T, \mathbf{X}_T) - g(0, \mathbf{X}_0) =$$

$$e(r) + \sum_{j=0}^{n_r-1} \frac{\partial g}{\partial t}\left(t_j, \mathbf{X}_{t_j}\right)\Delta t_j + D_2 g\left(t_j, \mathbf{X}_{t_j}\right)\left(\mathbf{u}^j \Delta t_j + V^j \Delta \mathbf{B}_{t_j}\right)$$

$$+\sum_{j=0}^{n_r-1} \frac{1}{2}\left(\left(V^j \Delta \mathbf{B}_{t_j}\right)^T H\left(t_j + \theta\Delta t_j, \mathbf{X}_{t_j} + \theta\Delta\mathbf{X}_{t_j}\right)\left(V^j \Delta \mathbf{B}_{t_j}\right)\right) \tag{35.12}$$

where $e(r) \to 0$ off a set of measure zero. Consider the last term. This term is of the form

$$\sum_{j=0}^{n_r-1} \frac{1}{2}\left(\left(V^j \Delta \mathbf{B}_{t_j}\right)^T H\left(t_j, \mathbf{X}_{t_j}\right)\left(V^j \Delta \mathbf{B}_{t_j}\right)\right) + \sum_{j=0}^{n_r-1} \frac{1}{2}\cdot \tag{35.13}$$

$$\left(\left(V^j \Delta \mathbf{B}_{t_j}\right)^T \left(H\left(t_j + \theta\Delta t_j, \mathbf{X}_{t_j} + \theta\Delta\mathbf{X}_{t_j}\right) - H\left(t_j, \mathbf{X}_{t_j}\right)\right)\left(V^j \Delta \mathbf{B}_{t_j}\right)\right). \tag{35.14}$$

Now the term in 35.14 is of the form

$$\sum_{j=0}^{n_r-1} \Delta\mathbf{B}_{t_j}^T M_r \Delta\mathbf{B}_{t_j}$$

where $M_r \to 0$ as $r \to \infty$ and is uniformly bounded. It follows that for a.e. ω, the above expression converges to 0 and also

$$\left|\sum_{j=0}^{n_r-1} \Delta\mathbf{B}_{t_j}^T M_r \Delta\mathbf{B}_{t_j}\right| \leq C\sum_{j=0}^{n_r-1} \left|\Delta\mathbf{B}_{t_j}\right|^2. \tag{35.15}$$

Now by independence of the increments,

$$\int_\Omega \left(\sum_{j=0}^{n_r-1} \left|\Delta \mathbf{B}_{t_j}\right|^2 \right)^2 dP = \int_\Omega \sum_{i,j} \left|\Delta \mathbf{B}_{t_j}\right|^2 \left|\Delta \mathbf{B}_{t_i}\right|^2 dP$$

$$= \sum_{i \neq j} \int_\Omega \left|\Delta \mathbf{B}_{t_j}\right|^2 dP \int_\Omega \left|\Delta \mathbf{B}_{t_i}\right|^2 dP + \sum_{i=1}^{n_r-1} \int_\Omega \left|\Delta \mathbf{B}_{t_j}\right|^4 dP$$

$$\begin{aligned} &\leq \sum_{i,j} (t_{j+1} - t_j)(t_{i+1} - t_i) + \sum_{i=1}^{n_r-1} 3n^2 (t_{j+1} - t_j)^2 \\ &\leq T^2 + 3n^2 \Delta(r) T. \end{aligned}$$

Thus

$$\left\{ \left| \sum_{j=0}^{n_r-1} \Delta \mathbf{B}_{t_j}^T M_r \Delta \mathbf{B}_{t_j} \right| \right\}_r$$

is uniformly integrable and so by the Vitali convergence theorem,

$$\lim_{r \to \infty} \int_\Omega \left| \sum_{j=0}^{n_r-1} \Delta \mathbf{B}_{t_j}^T M_r \Delta \mathbf{B}_{t_j} \right| dP = 0.$$

Passing to a further subsequence, 35.12 is of the form

$$g(T, \mathbf{X}_T) - g(0, \mathbf{X}_0) =$$

$$e(r) + \sum_{j=0}^{n_r-1} \frac{\partial g}{\partial t} \left(t_j, \mathbf{X}_{t_j}\right) \Delta t_j + D_2 g\left(t_j, \mathbf{X}_{t_j}\right) \left(\mathbf{u}^j \Delta t_j + V^j \Delta \mathbf{B}_{t_j}\right)$$

$$+ \sum_{j=0}^{n_r-1} \frac{1}{2} \left(\Delta \mathbf{B}_{t_j}^T \left(V^j\right)^T H\left(t_j, \mathbf{X}_{t_j}\right) V^j \Delta \mathbf{B}_{t_j} \right) \tag{35.16}$$

where for a.e. ω, $e(r) \to 0$. It remains to consider the last term in the above as $r \to \infty$. Denote by A^j the symmetric matrix

$$\left(V^j\right)^T H\left(t_j, \mathbf{X}_{t_j}\right) V^j$$

in the above. Note this is measurable in $\mathcal{H}_{t_j}$.

Claim: Letting $A^j = \left(V^j\right)^T H\left(t_j, \mathbf{X}_{t_j}\right) V^j$, there exists a subsequence, $r \to \infty$ and a set of measure zero such that for ω not in this set of measure zero,

$$\sum_{j=0}^{n_r-1} \left(\Delta \mathbf{B}_{t_j}^T A^j \Delta \mathbf{B}_{t_j} \right) - \sum_{j=0}^{n_r-1} tr\left(A^j\right) \Delta t_j \to 0.$$

Proof of the claim:

$$\int_\Omega \left(\sum_{j=0}^{n_r-1} \left(\Delta \mathbf{B}_{t_j}^T A^j \Delta \mathbf{B}_{t_j} \right) - \sum_{j=0}^{n-1} tr\left(A^j\right) \Delta t_j \right)^2 dP$$

$$= \sum_{i,j} \int_\Omega \left(\Delta \mathbf{B}_{t_j}^T A^j \Delta \mathbf{B}_{t_j} - tr\left(A^j\right) \Delta t_j \right) \left(\Delta \mathbf{B}_{t_i}^T A^i \Delta \mathbf{B}_{t_i} - tr\left(A^i\right) \Delta t_i \right) dP \quad (35.17)$$

Consider a term in which $j > i$. The integrand is of the form

$$\Delta \mathbf{B}_{t_j}^T A^j \Delta \mathbf{B}_{t_j} \Delta \mathbf{B}_{t_i}^T A^i \Delta \mathbf{B}_{t_i} - tr\left(A^i\right) \Delta t_i \Delta \mathbf{B}_{t_j}^T A^j \Delta \mathbf{B}_{t_j}$$
$$- tr\left(A^j\right) \Delta t_j \Delta \mathbf{B}_{t_i}^T A^i \Delta \mathbf{B}_{t_i} + tr\left(A^j\right) \Delta t_j tr\left(A^i\right) \Delta t_i \quad (35.18)$$

Consider the first term.

$$\int_\Omega \Delta \mathbf{B}_{t_j}^T A^j \Delta \mathbf{B}_{t_j} \Delta \mathbf{B}_{t_i}^T A^i \Delta \mathbf{B}_{t_i} dP$$
$$= \int_\Omega \left(\sum_{\alpha,\beta} \Delta B_{\alpha t_j} A^j_{\alpha\beta} \Delta B_{\beta t_j} \right) \left(\sum_{\sigma,\tau} \Delta B_{\sigma t_i} A^i_{\sigma\tau} \Delta B_{\tau t_i} \right) dP$$
$$= \sum_{\alpha,\beta,\sigma,\tau} \int_\Omega \Delta B_{\alpha t_j} A^j_{\alpha\beta} \Delta B_{\beta t_j} \Delta B_{\sigma t_i} A^i_{\sigma\tau} \Delta B_{\tau t_i} dP$$
$$= \sum_{\alpha,\beta,\sigma,\tau} \int_\Omega \Delta B_{\alpha t_j} \Delta B_{\beta t_j} dP \int_\Omega \Delta B_{\sigma t_i} A^j_{\alpha\beta} A^i_{\sigma\tau} \Delta B_{\tau t_i} dP$$

Therefore, using independence of the components of n dimensional Brownian motion which independence results from the covariance matrix for the joint distribution being diagonal along with the independence of increments in time, this reduces to

$$= \sum_{\alpha,\sigma,\tau} \int_\Omega \Delta B^2_{\alpha t_j} dP \int_\Omega A^j_{\alpha\alpha} \Delta B_{\sigma t_i} A^i_{\sigma\tau} \Delta B_{\tau t_i} dP$$
$$= \sum_{\sigma,\tau} (t_{j+1} - t_j) \int_\Omega \sum_\alpha A^j_{\alpha\alpha} \Delta B_{\sigma t_i} A^i_{\sigma\tau} \Delta B_{\tau t_i} dP$$
$$= (t_{j+1} - t_j) \int_\Omega tr\left(A^j\right) \sum_{\sigma,\tau} \Delta B_{\sigma t_i} A^i_{\sigma\tau} \Delta B_{\tau t_i} dP$$
$$= \int_\Omega \Delta t_j tr\left(A^j\right) \Delta \mathbf{B}_{t_i}^T A^i \Delta \mathbf{B}_{t_i} dP$$

which shows that this term in the case where $j > i$ cancels with the third term of 35.18. Now consider the second term of 35.18 again in the case where $j > i$. This yields

$$
\begin{aligned}
& -\int_{\Omega} tr\left(A^{i}\right)\Delta t_{i}\Delta\mathbf{B}_{t_{j}}^{T}A^{j}\Delta\mathbf{B}_{t_{j}}dP \\
= & -\int_{\Omega} tr\left(A^{i}\right)\Delta t_{i}\sum_{\alpha,\beta}\Delta B_{\alpha t_{j}}A_{\alpha\beta}^{j}\Delta B_{\beta t_{j}}dP
\end{aligned}
$$

$$
=-\sum_{\alpha,\beta}\int_{\Omega}\Delta B_{\alpha t_{j}}\Delta B_{\beta t_{j}}dP\int_{\Omega} tr\left(A^{i}\right)\Delta t_{i}A_{\alpha\beta}^{j}dP
$$

$$
\begin{aligned}
= & -\sum_{\alpha}\int_{\Omega}\Delta B_{\alpha t_{j}}^{2}dP\int_{\Omega} tr\left(A^{i}\right)\Delta t_{i}A_{\alpha\alpha}^{j}dP \\
= & -\int_{\Omega}\left(\Delta t_{j}\right)tr\left(A^{i}\right)\Delta t_{i}\sum_{\alpha}A_{\alpha\alpha}^{j}dP \\
= & -\int_{\Omega}\left(\Delta t_{j}\right)tr\left(A^{i}\right)\Delta t_{i}tr\left(A^{j}\right)dP
\end{aligned}
$$

and so this second term cancels with the last term of 35.18. It follows the only terms to consider in 35.17 are those for which $j=i$. Thus 35.17 is of the form

$$
\sum_{i}\int_{\Omega}\left(\left(\Delta\mathbf{B}_{t_{i}}^{T}A^{i}\Delta\mathbf{B}_{t_{i}}\right)^{2}-2\Delta\mathbf{B}_{t_{i}}^{T}A^{i}\Delta\mathbf{B}_{t_{i}}tr\left(A^{i}\right)\Delta t_{i}+\left(tr\left(A^{i}\right)\Delta t_{i}\right)^{2}\right)dP. \tag{35.19}
$$

First consider the second term.

$$
\begin{aligned}
& \int_{\Omega}\Delta\mathbf{B}_{t_{i}}^{T}A^{i}\Delta\mathbf{B}_{t_{i}}tr\left(A^{i}\right)\Delta t_{i}dP \\
= & \sum_{\alpha,\beta}\int_{\Omega}\Delta B_{\alpha t_{i}}A_{\alpha\beta}^{i}\Delta B_{\beta t_{i}}tr\left(A^{i}\right)\Delta t_{i}dP \\
= & \sum_{\alpha,\beta}\int_{\Omega}\Delta B_{\alpha t_{i}}\Delta B_{\beta t_{i}}dP\int_{\Omega}A_{\alpha\beta}^{i}tr\left(A^{i}\right)\Delta t_{i}dP \\
= & \sum_{\alpha}\int_{\Omega}\Delta B_{\alpha t_{i}}^{2}dP\int_{\Omega}A_{\alpha\alpha}^{i}tr\left(A^{i}\right)\Delta t_{i}dP \\
= & \Delta t_{i}\int_{\Omega}\left(\sum_{\alpha}A_{\alpha\alpha}^{i}\right)tr\left(A^{i}\right)\Delta t_{i}dP \\
= & \int_{\Omega} tr\left(A^{i}\right)^{2}\Delta t_{i}^{2}dP\geq 0
\end{aligned}
$$

It follows 35.17 is dominated above by

$$
\sum_{i}\int_{\Omega}\left(\left(\Delta\mathbf{B}_{t_{i}}^{T}A^{i}\Delta\mathbf{B}_{t_{i}}\right)^{2}+\left(tr\left(A^{i}\right)\Delta t_{i}\right)^{2}\right)dP
$$

Consider the first term.

$$\begin{aligned}
&\int_\Omega \left(\Delta \mathbf{B}_{t_i}^T A^i \Delta \mathbf{B}_{t_i}\right)^2 dP \\
= &\int_\Omega \left(\sum_{\alpha,\beta} \Delta B_{\alpha t_i} A^i_{\alpha\beta} \Delta B_{\beta t_i}\right)\left(\sum_{\sigma,\tau} \Delta B_{\sigma t_i} A^i_{\sigma\tau} \Delta B_{\tau t_i}\right) dP \\
&= \sum_{\alpha,\beta,\sigma,\tau} \int_\Omega \Delta B_{\alpha t_i} A^i_{\alpha\beta} \Delta B_{\beta t_i} \Delta B_{\sigma t_i} A^i_{\sigma\tau} \Delta B_{\tau t_i} dP
\end{aligned}$$

$$= \sum_{\alpha,\beta,\sigma,\tau} \int_\Omega A^i_{\alpha\beta} A^i_{\sigma\tau} dP \int_\Omega \Delta B_{\alpha t_i} \Delta B_{\beta t_i} \Delta B_{\sigma t_i} \Delta B_{\tau t_i} dP. \tag{35.20}$$

There are two ways in which the term of this sum will not equal zero. One way is for $\alpha = \beta$ and $\sigma = \tau$. In this situation, the above is dominated by

$$\begin{aligned}
&\sum_{\alpha,\sigma} \int_\Omega A^i_{\alpha\alpha} A^i_{\sigma\sigma} dP \int_\Omega \Delta B^2_{\alpha t_i} \Delta B^2_{\sigma t_i} dP \\
= &\sum_{\alpha\neq\sigma} \int_\Omega A^i_{\alpha\alpha} A^i_{\sigma\sigma} dP \Delta t_i^2 + \sum_\alpha \int_\Omega \left(A^i_{\alpha\alpha}\right)^2 dP 3 \left(\Delta t_i\right)^2 \\
\leq &\sum_{\alpha,\sigma} \int_\Omega A^i_{\alpha\alpha} A^i_{\sigma\sigma} dP \Delta t_i^2 + \sum_\alpha \int_\Omega \left(A^i_{\alpha\alpha}\right)^2 dP 3 \left(\Delta t_i\right)^2 \\
= &\int_\Omega tr\left(A^i\right)^2 dP \Delta t_i^2 + 3 \int_\Omega \sum_\alpha \left(A^i_{\alpha\alpha}\right)^2 dP \left(\Delta t_i\right)^2
\end{aligned}$$

The other way in which the expression in 35.20 is not zero is for $\alpha = \sigma$ and $\beta = \tau$. If this happens, the expression is of the form

$$\begin{aligned}
&\sum_{\alpha,\beta} \int_\Omega A^i_{\alpha\beta} A^i_{\alpha\beta} dP \int_\Omega \Delta B_{\alpha t_i} \Delta B_{\beta t_i} \Delta B_{\alpha t_i} \Delta B_{\beta t_i} dP \\
= &\sum_{\alpha,\beta} \int_\Omega A^i_{\alpha\beta} A^i_{\alpha\beta} dP \int_\Omega \Delta B^2_{\alpha t_i} \Delta B^2_{\beta t_i} dP \\
= &\sum_{\alpha\pm\beta} \int_\Omega A^i_{\alpha\beta} A^i_{\alpha\beta} dP \int_\Omega \Delta B^2_{\alpha t_i} dP \int_\Omega \Delta B^2_{\beta t_i} dP \\
&+ \sum_\alpha \int_\Omega \left(A^i_{\alpha\alpha}\right)^2 dP 3 \Delta t_i^2 \\
= &\sum_{\alpha\pm\beta} \int_\Omega A^i_{\alpha\beta} A^i_{\alpha\beta} dP \Delta t_i^2 + \sum_\alpha \int_\Omega \left(A^i_{\alpha\alpha}\right)^2 dP 3 \Delta t_i^2 .
\end{aligned}$$

Thus the expression in 35.20 is dominated by

$$C \sum_{i=1}^{n_r - 1} \Delta t_i^2$$

which converges to 0 as $r \to \infty$. This proves the claim.

Now returning to 35.16 first note that

$$\lim_{r\to\infty} \sum_{j=0}^{n_r-1} tr\left(A^j\right) \Delta t_j = \int_0^T tr\left(V^T H\left(t, \mathbf{X}_t\right) V\right) dt.$$

Therefore, there exists a subsequence, still denoted by r and a set of measure zero off of which the last term of 35.16 converges to

$$\frac{1}{2}\int_0^T tr\left(V^T H\left(t, \mathbf{X}_t\right) V\right) dt.$$

It follows that off a set of measure zero, you can pass to the limit in 35.16 and conclude

$$g\left(T, \mathbf{X}_T\right) - g\left(0, \mathbf{X}_0\right) =$$

$$\int_0^T \left(\frac{\partial g}{\partial t}\left(t, \mathbf{X}_t\right) + D_2 g\left(t, \mathbf{X}_t\right) \mathbf{u} + \frac{1}{2} tr\left(V^T H\left(t, \mathbf{X}_t\right) V\right)\right) dt$$
$$+ \int_0^T D_2 g\left(t, \mathbf{X}_t\right) V d\mathbf{B}$$

This lengthy computation has mostly proved the following lemma.

Lemma 35.5 *Let $\mathcal{H}_t$ be the filtration defined above and let $\mathbf{B}$ be n dimensional Brownian motion. Suppose $\mathbf{X}_t$ is a vector valued stochastic process for $t \in [0, T]$ defined by the following for a.e. ω*

$$\mathbf{X}_t - \mathbf{X}_0 = \int_0^t \mathbf{u}\left(s, \cdot\right) ds + \int_0^t V\left(s, \cdot\right) d\mathbf{B}$$

where all entries of $\mathbf{u}$ and V are $\mathcal{H}_t$ adapted uniformly bounded step functions. Then if g is a C^2 function such that all partial derivatives are uniformly bounded, then for all $t \in [0, T]$,

$$g\left(t, \mathbf{X}_t\right) - g\left(0, \mathbf{X}_0\right) =$$

$$\int_0^t \left(\frac{\partial g}{\partial t}\left(s, \mathbf{X}_s\right) + D_2 g\left(s, \mathbf{X}_s\right) \mathbf{u} + \frac{1}{2} tr\left(V^T H\left(s, \mathbf{X}_s\right) V\right)\right) ds$$
$$+ \int_0^t D_2 g\left(s, \mathbf{X}_s\right) V d\mathbf{B}$$

Proof: Let $\{t_k\}$ be the rational numbers in $[0, T]$. The above computation shows that for each t_k, there exists a set of measure zero, E_k such that if $\omega \notin E_k$, then

$$g\left(t_k, \mathbf{X}_{t_k}\right) - g\left(0, \mathbf{X}_0\right) =$$

$$\int_0^{t_k} \left(\frac{\partial g}{\partial t} (s, \mathbf{X}_s) + D_2 g (s, \mathbf{X}_s) \mathbf{u} + tr \left(V^T H (s, \mathbf{X}_s) V \right) \right) ds$$
$$+ \int_0^{t_k} D_2 g (s, \mathbf{X}_s) V d\mathbf{B}$$

Letting $E = \cup_{k=1}^{\infty} E_k$, it follows E has measure zero and the above formula holds for all t_k. By continuity the above must hold for all $t \in [0, T]$. This proves the lemma.

Now let $V(t, \omega)$ and $\mathbf{u}(t, \omega)$ will be $\mathcal{B} \times \mathcal{F}$ measurable, both $\mathbf{u}$ and V are $\mathcal{H}_t$ adapted, and the components of V and $\mathbf{u}$ satisfy

$$P \left(\int_0^T v_{ij}^2 ds < \infty \right) = 1, \ P \left(\int_0^T |u_k| \, ds < \infty \right) = 1. \tag{35.21}$$

Definition 35.6 $\mathbf{X}_t$ *is called an Itô process if for,* $\mathbf{u}, V$ *described above and a measurable function,* $\mathbf{X}_0$ *such that*

$$\mathbf{X}_t - \mathbf{X}_0 = \int_0^t \mathbf{u}(s, \omega) \, ds + \int_0^t V(s, \omega) \, d\mathbf{B}.$$

Lemma 35.5 shows that if g is a C^2 function defined on $\mathbb{R} \times \mathbb{R}^m$ which has all the second partial derivatives uniformly bounded, then if $\mathbf{u}$ and V have components which are uniformly bounded adapted step functions, then $g(t, \mathbf{X})$ is also an Itô process as described in that lemma. The next step is to remove the assumption that $\mathbf{u}$ and V are step functions.

Lemma 35.7 *Let* $\mathcal{H}_t$ *be the filtration defined above and let* $\mathbf{B}$ *be* n *dimensional Brownian motion. Suppose* $\mathbf{X}_t$ *is a vector valued stochastic process for* $t \in [0, T]$ *defined by the following for a.e.* ω

$$\mathbf{X}_t - \mathbf{X}_0 = \int_0^t \mathbf{u}(s, \cdot) \, ds + \int_0^t V(s, \cdot) \, d\mathbf{B} \tag{35.22}$$

where all entries of $\mathbf{u}$ *and* V *are* $\mathcal{H}_t$ *adapted and satisfy 35.21. Then if* g *is a* C^2 *function such that all second order partial derivatives are uniformly bounded, then for all* $t \in [0, T]$,

$$g(t, \mathbf{X}_t) - g(0, \mathbf{X}_0) =$$

$$\int_0^t \left(\frac{\partial g}{\partial t} (s, \mathbf{X}_s) + D_2 g (s, \mathbf{X}_s) \mathbf{u} + \frac{1}{2} tr \left(V^T H (s, \mathbf{X}_s) V \right) \right) ds$$
$$+ \int_0^t D_2 g (s, \mathbf{X}_s) V d\mathbf{B} \tag{35.23}$$

where H *is the Hessian matrix of* g *whose* ij^{th} *entry is*

$$\frac{\partial^2 g}{\partial x_i \partial x_j} (s, \mathbf{X}_s(\omega)).$$

Proof: From 35.21 there exist sequences, $\left\{\mathbf{u}^l\right\}_{l=1}^{\infty}$ and $\left\{V^l\right\}_{l=1}^{\infty}$ such that the entries of $\mathbf{u}^l$ and V^l are adapted step functions and in addition there is a set of measure zero, E, such that if $\omega \notin E$, then the components of $\mathbf{u}^l$ and V^l satisfy

$$\int_0^T \left|u_k^l - u_k\right| dt < 2^{-l}, \int_0^T \left|v_{kj}^l - v_{kj}\right|^2 dt < 2^{-l} \tag{35.24}$$

for all l large enough, depending on ω of course. Then from 35.22 define

$$\mathbf{X}_t^l - \mathbf{X}_0^l = \int_0^t \mathbf{u}^l(s,\cdot)\,ds + \int_0^t V^l(s,\cdot)\,d\mathbf{B}$$

and it follows that for $\omega \notin E$, and all $t \in [0,T]$,

$$\lim_{l\to\infty} \mathbf{X}_t^l(\omega) = \mathbf{X}_t(\omega). \tag{35.25}$$

In addition to this, it follows from 35.24 it follows there is a subsequence such that for $\omega \notin E$,

$$V^l(t,\omega) \to V(t,\omega), \mathbf{u}^l(t,\omega) \to \mathbf{u}(t,\omega) \text{ a.e. } t.$$

By Lemma 35.5 for a.e. ω,

$$g\left(t,\mathbf{X}_t^l\right) - g\left(0,\mathbf{X}_0^l\right) =$$

$$\int_0^t \left(\frac{\partial g}{\partial t}\left(s,\mathbf{X}_s^l\right) + D_2 g\left(s,\mathbf{X}_s^l\right)\mathbf{u}^l + \frac{1}{2} tr\left(V^{lT} H\left(s,\mathbf{X}_s^l\right)V^l\right)\right) ds$$
$$+ \int_0^t D_2 g\left(s,\mathbf{X}_s^l\right) V^l d\mathbf{B} \tag{35.26}$$

for all $t \in [0,T]$. Now 35.24 implies for each $\omega \notin E, \int_0^T \left|\left|V^l\right|\right|^2 dt$ is uniformly bounded independent of l. Consider the third term.

$$\int_0^T \left|tr\left(V^{lT} H\left(s,\mathbf{X}_s^l\right)V^l\right) - tr\left(V^T H\left(s,\mathbf{X}_s\right)V\right)\right| dt \tag{35.27}$$

$$\leq \int_0^T \left|tr\left(V^{lT} H\left(s,\mathbf{X}_s^l\right)V^l - V^T H\left(s,\mathbf{X}_s^l\right)V\right)\right| dt$$
$$+ \int_0^T \left|tr\left(V^T\left(H\left(s,\mathbf{X}_s^l\right) - H\left(s,\mathbf{X}_s\right)\right)V\right)\right| dt$$

The second term in the above expression converges to 0 by the dominated convergence theorem. It can be dominated by $C\left|tr\left(V^T V\right)\right|$, a function in L^1 where here C does not depend on l but on the uniform bound of the second derivatives of g. Consider the first term. The integrand is dominated by

$$\left|tr\left(\left(V^{lT} - V^T\right)HV^l\right)\right| + \left|tr\left(V^T H\left(V^l - V\right)\right)\right|$$

The integral of both of these converges to 0. Consider the first one.

$$\begin{aligned}\int_0^T \left|tr\left(\left(V^{lT}-V^T\right)HV^l\right)\right| dt &\leq C\int_0^T \left|\left|V^l-V\right|\right| \left|\left|V^l\right|\right| dt \\ &\leq C'\int_0^T \left|\left|V^l-V\right|\right|^2 dt\end{aligned}$$

which converges to 0 as $l\to\infty$. This shows that for each $t\in[0,T]$, one can pass to the limit in the third term of 35.26 and eliminate the superscript, l. Passing to the limit in the second term of 35.26 follows from 35.24 and the boundedness of $D_2g\left(s,\mathbf{X}_s^l\right)$ which results from the assumption that all the partial derivatives of g are uniformly bounded. Passing to the limit in the first term of 35.26 follows from 35.25. The last term involving $d\mathbf{B}$ is of the form

$$\int_0^t D_2g\left(s,\mathbf{X}^l\right)V^l d\mathbf{B}=\sum_j\int_0^t \left(D_2g\left(s,\mathbf{X}^l\right)V^l\right)_{ij} dB_j.$$

Then from the definition of the Itô integral given above, there is a subsequence still denoted by l such that for a.e. ω, the above converges uniformly in t to

$$\sum_j\int_0^t \left(D_2g\left(s,\mathbf{X}\right)V\right)_{ij} dB_j$$

Thus for a dense subset of $[0,T]$, D, there exists an exceptional set of measure zero such that 35.23 holds for all $t\in D$. By continuity of the Ito integral, this continues to hold for all $t\in[0,T]$. This proves the lemma.

It remains to remove the assumption that the partial derivatives of g are bounded. This results in the following theorem which is the main result.

Theorem 35.8 *Let $\mathcal{H}_t$ be the filtration defined above and let $\mathbf{B}$ be n dimensional Brownian motion. Suppose $\mathbf{X}_t$ is a vector valued stochastic process for $t\in[0,T]$ defined by the following for a.e. ω*

$$\mathbf{X}_t-\mathbf{X}_0=\int_0^t \mathbf{u}(s,\cdot)\,ds+\int_0^t V(s,\cdot)\,d\mathbf{B} \tag{35.28}$$

where all entries of $\mathbf{u}$ and V are $\mathcal{H}_t$ adapted and satisfy 35.21. Then if $\mathbf{g}$ is a C^2 function with values in $\mathbb{R}^p$, it follows that for a.e. ω and for all $t\in[0,T]$,

$$g_k(t,\mathbf{X}_t)-g_k(0,\mathbf{X}_0)=$$

$$\int_0^t \left(\frac{\partial g_k}{\partial t}(s,\mathbf{X}_s)+D_2g_k(s,\mathbf{X}_s)\,\mathbf{u}+\frac{1}{2}tr\left(V^TH_k(s,\mathbf{X}_s)V\right)\right)ds$$
$$+\int_0^t D_2g_k(s,\mathbf{X}_s)\,Vd\mathbf{B} \tag{35.29}$$

where H_k is the Hessian matrix of g_k whose ij^{th} entry is

$$\frac{\partial^2 g_k}{\partial x_i \partial x_j}(s, \mathbf{X}_s(\omega)).$$

Proof: There is no loss of generality in proving it only for the case where g has values in $\mathbb{R}$ because you obtain the above formula by simply considering the k^{th} component of $\mathbf{g}$. Assume then that g has values in $\mathbb{R}$. Let $\psi_N \in C_c^\infty(B(\mathbf{0}, 2N))$ such that ψ_N equals 1 on $B(\mathbf{0}, N)$ and ψ_N has values in $[0,1]$. Then let $g_N \equiv g\psi_N$. Thus g_N has uniformly bounded derivatives. By Lemma 35.7, there exists a set of measure zero, E_N such that for $\omega \notin E_N$, and all $t \in [0,T]$,

$$g_N(t, \mathbf{X}_t) - g_N(0, \mathbf{X}_0) =$$

$$\int_0^t \left(\frac{\partial g_N}{\partial t}(s, \mathbf{X}_s) + D_2 g_N(s, \mathbf{X}_s)\mathbf{u} + \frac{1}{2} tr\left(V^T H_N(s, \mathbf{X}_s) V\right)\right) ds + \int_0^t D_2 g_N(s, \mathbf{X}_s) V d\mathbf{B} \tag{35.30}$$

Let $E = \cup_{N=1}^\infty E_N$. Then for $\omega \notin E$, the above formula holds for all N. By continuity of $\mathbf{X}$, it follows that for all N large enough, the values of $\mathbf{X}(t, \omega)$ are in $B(\mathbf{0}, N)$ and so you can delete the subscript of N in the above. This proves the theorem.

How do people remember this? Letting $\mathbf{Y}(t, \omega) \equiv \mathbf{g}(t, \mathbf{X}(t, \omega))$, 35.29 can be considered formally as

$$\begin{aligned} dY_k &= \left(\frac{\partial g_k}{\partial t}(t, \mathbf{X}_t) + D_2 g_k(t, \mathbf{X}_t)\mathbf{u} + \frac{1}{2} tr\left(V^T H_k(t, \mathbf{X}_t) V\right)\right) dt \\ &\quad + D_2 g_k(t, \mathbf{X}_t) d\mathbf{B} \end{aligned}$$

and 35.28 can be written as

$$d\mathbf{X}_t = \mathbf{u}dt + V d\mathbf{B}.$$

I think this is not too bad but one can write an easier to remember formula which reduces to this one,

$$dY_k = \frac{\partial g_k}{\partial t}(t, \mathbf{X}_t) dt + D_2 g_k(t, \mathbf{X}_t) d\mathbf{X}_t + \frac{1}{2} d\mathbf{X}_t^T H_k(t, \mathbf{X}_t) d\mathbf{X}_t$$

under the convention that $dtdB_k = 0, dt^2 = 0$, and, $dB_i dB_j = \delta_{ij} dt$.

Example 35.9 *Let $g(t, x) = x^2$ and let $X_t = B_t$ where $B_0 = 0$.*

In this case,

$$Y_t = B_t^2$$

and so

$$dY_t = 2B_t dB_t + \frac{1}{2} \cdot 2dB_t^2$$

and so

$$\begin{aligned} B_t^2 - 0 &= 2\int_0^t BdB + \int_0^t dt \\ &= 2\int_0^t BdB + t. \end{aligned}$$

This yields

$$\int_0^t BdB = \frac{1}{2}\left(B_t^2 - t\right)$$

which was encountered earlier.

Example 35.10 *Let $g(t,x) = x^3$ and $X_t = B_t, Y_t = X_t^3$ where $B_0 = 0$.*

Then

$$\begin{aligned} dY_t &= 3X_t^2 dX_t + \frac{1}{2} 6X_t dX_t^2 \\ &= 3X_t^2 dB_t + 3B_t\left(dB_t^2\right) \\ &= 3B_t^2 dB_t + 3B_t dt \end{aligned}$$

and so

$$B_t^3 = 3\left(\int_0^t B^2 dB + \int_0^t Bdt\right)$$

Example 35.11 *Let $Y_t = tB_t$ where $X_t = B_t$ and where $B_0 = 0$. Then*

$$\begin{aligned} dY_t &= B_t dt + tdB_t + \frac{1}{2} 0 dB_t^2 \\ &= B_t dt + tdB_t \end{aligned}$$

and so

$$Y_t = tB_t = \int_0^t Bdt + \int_0^t tdB$$

35.3 Some Representation Theorems

First recall the following important lemma, Lemma 35.4 on Page 934 which has to do with the filtrations described above involving n dimensional Brownian motion. The most important part of this lemma is that one can consider only the increasing lists of rational numbers rather than all increasing lists of real numbers in defining the filtration. Here is the lemma.

Lemma 35.12 *Letting $\mathcal{H}_t$ denote the completion of the smallest σ algebra containing*

$$(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(B)$$

for all B a Borel set in $\mathbb{R}^{nk}$ for all sequences, $0 \leq s_1 < s_2 \cdots < s_k \leq t$ as defined above, $\mathcal{H}_t$ is also equal to the completion of the smallest σ algebra containing

$$\left(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k}\right)^{-1}(B)$$

for all B an open set in $\mathbb{R}^{nk}$ for all sequences, $0 \leq s_1 < s_2 \cdots < s_k \leq t$. In addition to this, $\mathcal{H}_t$ is equal to the completion of the smallest σ algebra containing

$$\left(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k}\right)^{-1}(B)$$

for all B an open set in $\mathbb{R}^{nk}$ for all sequences, $0 \leq s_1 < s_2 \cdots < s_k \leq t$ such that the s_j are rational numbers.

Also recall the Doob Dynkin theorem, Theorem 31.19 on Page 866 which is listed here.

Lemma 35.13 *Suppose $\mathbf{X}, \mathbf{Y}_1, \mathbf{Y}_2, \cdots, \mathbf{Y}_k$ are random vectors, $\mathbf{X}$ having values in $\mathbb{R}^n$ and $\mathbf{Y}_j$ having values in $\mathbb{R}^{p_j}$ and*

$$\mathbf{X}, \mathbf{Y}_j \in L^1(\Omega).$$

Suppose $\mathbf{X}$ is $\mathcal{H}_{(\mathbf{Y}_1, \cdots, \mathbf{Y}_k)}$ measurable. Thus

$$\left\{\mathbf{X}^{-1}(E) : E \text{ Borel}\right\} \subseteq \left\{(\mathbf{Y}_1, \cdots, \mathbf{Y}_k)^{-1}(F) : F \text{ is Borel in } \prod_{j=1}^{k} \mathbb{R}^{p_j}\right\}$$

Then there exists a Borel function, $\mathbf{g} : \prod_{j=1}^{k} \mathbb{R}^{p_j} \to \mathbb{R}^n$ such that

$$\mathbf{X} = \mathbf{g}(\mathbf{Y}).$$

Also recall the submartingale convergence theorem, Theorem 32.12 on Page 900 reviewed below.

Theorem 35.14 *(submartingale convergence theorem) Let*

$$\{(X_i, \mathcal{S}_i)\}_{i=1}^{\infty}$$

be a submartingale with $K \equiv \sup E(|X_n|) < \infty$. Then there exists a random variable, X, such that $E(|X|) \leq K$ and

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \ \ a.e.$$

Also here is a generalization of the Itô isometry presented earlier.

Lemma 35.15 *Let $\mathbf{f}$ be $\mathcal{H}_t$ adapted in the sense that every component is $\mathcal{H}_t$ adapted and $\mathbf{f} \in L^2(\Omega; \mathbb{R}^n)$. Here $\mathcal{H}_t$ is the filtration defined in Lemma 35.12. Then*

$$\left|\left|\int_0^T \mathbf{f}(s)^T d\mathbf{B}\right|\right|_{L^2(\Omega)} = ||\mathbf{f}||_{L^2(\Omega\times[0,T];\mathbb{R}^n)}.$$

Proof: Let $\mathbf{f}$ an adapted bounded step function. Say

$$\mathbf{f}(t)=\sum_{i=0}^{m-1}\mathbf{a}_i\mathcal{X}_{[t_i,t_{i+1})}(t).$$

Then

$$\begin{aligned}&\left|\left|\int_0^T \mathbf{f}(s)^T d\mathbf{B}\right|\right|^2_{L^2(\Omega)}\\ =\;&\int_\Omega \sum_{i,j}\mathbf{a}_i^T\left(\mathbf{B}_{t_{i+1}}-\mathbf{B}_{t_i}\right)\mathbf{a}_j^T\left(\mathbf{B}_{t_{j+1}}-\mathbf{B}_{t_j}\right)dP\end{aligned}$$

For a mixed term in which $i\neq j$, this integrates to 0 by independence of the increments. Thus this reduces to

$$\int_\Omega \sum_i \mathbf{a}_i^T\left(\mathbf{B}_{t_{i+1}}-\mathbf{B}_{t_i}\right)\mathbf{a}_i^T\left(\mathbf{B}_{t_{i+1}}-\mathbf{B}_{t_i}\right)dP.$$

Next one uses the independence of the increments of different components of n dimensional Brownian motion. Thus the above equals

$$\begin{aligned}&\sum_i\int_\Omega\sum_{r,s}a_{ir}\left(\Delta B_{t_ir}\right)a_{is}\left(\Delta B_{t_is}\right)dP\\ =\;&\sum_i\int_\Omega\sum_r a_{ir}^2\left(\Delta B_{t_ir}\right)^2 dP\\ =\;&\sum_i\int_\Omega\sum_r a_{ir}^2 dP\Delta t_i=\sum_i\int_\Omega\left|\mathbf{a}_i\right|^2 dP\Delta t_i\\ =\;&\int_0^T\int_\Omega|\mathbf{f}|^2\,dPdt.\end{aligned}$$

This proves the lemma for step functions. Now for $\mathbf{f}$ as given, let $\{\mathbf{f}_k\}_{k=1}^\infty$ be a sequence of step functions for which

$$\int_0^T \mathbf{f}^T d\mathbf{B}=\lim_{k\to\infty}\int_0^T\mathbf{f}_k^T d\mathbf{B}\text{ a.e., }||\mathbf{f}_k-\mathbf{f}||_{L^2(\Omega\times[0,T];\mathbb{R}^n)}\to 0.$$

Then by what was just shown, $\left\{\int_0^T\mathbf{f}_k^T d\mathbf{B}\right\}$ is Cauchy in $L^2(\Omega)$. Therefore, it converges in $L^2(\Omega)$ to some $g\in L^2(\Omega)$. A subsequence converges to g pointwise which shows $g=\int_0^T\mathbf{f}^T d\mathbf{B}$. Therefore,

$$\begin{aligned}\left|\left|\int_0^T\mathbf{f}^T d\mathbf{B}\right|\right|_{L^2(\Omega)}&=\lim_{k\to\infty}\left|\left|\int_0^T\mathbf{f}_k^T d\mathbf{B}\right|\right|_{L^2(\Omega)}\\ &=\lim_{k\to\infty}||\mathbf{f}_k||_{L^2(\Omega\times[0,T];\mathbb{R}^n)}=||\mathbf{f}||_{L^2(\Omega\times[0,T];\mathbb{R}^n)}\end{aligned}$$

and this proves this higher dimensional version of Itô's isometry.

Now with these lemmas and theorem, it is possible to prove a really interesting lemma about density of certain kinds of functions in $L^2(\Omega, \mathcal{H}_T, P)$.

Lemma 35.16 *Let $\mathcal{H}_t$ be the filtration alluded to in Lemma 35.12 defined in terms of the n dimensional Brownian motion. Then random variables of the form*

$$\phi(\mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_k})$$

where $t_1 < t_2 \cdots < t_k$ is a finite increasing sequence of rational points in $[0, T]$ and $\phi \in C_c^\infty(\mathbb{R}^k)$ are dense in $L^2(\Omega, \mathcal{H}_T, P)$. Here the set of random variables includes all such finite increasing lists of rational points of $[0, T]$.

Proof: Let $g \in L^2(\Omega, \mathcal{H}_T, P)$. Also let $\{t_j\}_{j=1}^\infty$ be the rational points of $[0, T]$. Now letting $\{s_1, \cdots, s_n\} = \{t_1, \cdots, t_n\}$ such that $s_1 < s_2 < \cdots < s_n$, let $\mathcal{H}_n$ denote the smallest σ algebra which contains

$$(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_n})^{-1}(U)$$

for all U an open subset of $\mathbb{R}^n$. By Lemma 35.12 $\mathcal{H}_T$ must be the completion of the σ algebra, $\cup_{n=1}^\infty \mathcal{H}_n$ and so g has a representative which is $\cup_{n=1}^\infty \mathcal{H}_n$ measurable. Therefore, without loss of generality, one can assume g is $\cup_{n=1}^\infty \mathcal{H}_n$ measurable.

Now consider the martingale,

$$\{E(g_M | \mathcal{H}_n)\}_{n=1}^\infty$$

where

$$g_M(\omega) \equiv \begin{cases} g(\omega) \text{ if } g(\omega) \in [-M, M] \\ M \text{ if } g(\omega) > M \\ -M \text{ if } g(\omega) < -M \end{cases}$$

and M is chosen large enough that

$$||g - g_M||_{L^2(\Omega)} < \varepsilon/4. \tag{35.31}$$

Now the terms of this martingale are uniformly bounded by M because

$$|E(g_M | \mathcal{H}_n)| \leq E(|g_M| | \mathcal{H}_n) \leq E(M | \mathcal{H}_n) = M.$$

It follows the martingale is certainly bounded in L^1 and so the martingale convergence theorem stated above can be applied, and so there exists f measurable in $\cup_{n=1}^\infty \mathcal{H}_n$ such that $E(g_M | \mathcal{H}_n)(\omega) \to f(\omega)$ a.e. Also $|f(\omega)| \leq M$. Now letting $A \in \cup_{n=1}^\infty \mathcal{H}_n$, it follows from the dominated convergence theorem that

$$\int_A f dP = \lim_{n \to \infty} \int_A E(g_M | \mathcal{H}_n) dP = \int_A g_M dP$$

Since A is an arbitrary set in $\cup_{n=1}^\infty \mathcal{H}_n$, this shows $f = g_M$. Also, the functions, $\{E(g_M | \mathcal{H}_n)\}_{n=1}^\infty$ are uniformly bounded and so they are bounded in $L^p(\Omega, \cup_{n=1}^\infty \mathcal{H}_n, P)$

for $p > 2$. Therefore, these functions are uniformly integrable and so by the Vitali convergence theorem,

$$\left(\int_\Omega (g_M - E(g_M|\mathcal{H}_n))^2 \, dP\right)^{1/2} \to 0$$

as $n \to \infty$. It follows m can be chosen large enough that

$$||E(g_M|\mathcal{H}_m) - g_M||_{L^2(\Omega)} < \varepsilon/4. \tag{35.32}$$

Now by the Doob Dynkin lemma listed above, there exists a Borel measurable, $h : \mathbb{R}^{nm} \to \mathbb{R}$ such that

$$E(g_M|\mathcal{H}_m) = h(\mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_m}) \text{ a.e.}$$

Of course h is not in $C_c^\infty(\mathbb{R}^{nm})$. Let $\lambda_{(\mathbf{B}_{t_1},\cdots,\mathbf{B}_{t_m})}$ be the distribution of the random vector $(\mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_m})$. Thus $\lambda_{(\mathbf{B}_{t_1},\cdots,\mathbf{B}_{t_m})}$ is a Radon measure and so there exists $\phi \in C_c(\mathbb{R}^{nm})$ such that

$$\begin{aligned}
&\left(\int_\Omega |E(g_M|\mathcal{H}_m) - \phi(\mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_m})|^2 \, dP\right)^{1/2} \\
= &\left(\int_\Omega |h(\mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_m}) - \phi(\mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_m})|^2 \, dP\right)^{1/2}
\end{aligned}$$

$$= \left(\int_{\mathbb{R}^{nm}} |h(\mathbf{x}_1, \cdots, \mathbf{x}_m) - \phi(\mathbf{x}_1, \cdots, \mathbf{x}_m)|^2 \, d\lambda_{(\mathbf{B}_{t_1},\cdots,\mathbf{B}_{t_m})}\right)^{1/2} < \varepsilon/4.$$

By convolving with a mollifier, one can assume that $\phi \in C_c^\infty(\mathbb{R}^{nm})$ also. It follows from 35.31 and 35.32 that

$$\begin{aligned}
&||g - \phi(\mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_m})||_{L^2} \\
\leq\ & ||g - g_M||_{L^2} + ||g_M - E(g_M|\mathcal{H}_m)||_{L^2} \\
& + ||E(g_M|\mathcal{H}_m) - \phi(\mathbf{B}_{t_1}, \cdots, \mathbf{B}_{t_m})||_{L^2} \\
\leq\ & 3\left(\frac{\varepsilon}{4}\right) < \varepsilon
\end{aligned}$$

This proves the lemma.

In this lemma and the following theorems, $\mathcal{H}_T$ continues to be the completion of the smallest σ algebra which contains

$$(\mathbf{B}_{s_1}, \cdots, \mathbf{B}_{s_k})^{-1}(B)$$

for all increasing sequences, $s_1 < \cdots < s_k$ and B a Borel set of $\mathbb{R}^{nk}$.

Lemma 35.17 *Let $\mathbf{h}(t)$ be a deterministic step function of the form*

$$\mathbf{h}(t) = \sum_{i=0}^{m-1} \mathbf{a}_i \mathcal{X}_{[t_i,t_{i+1})}.$$

Then for $\mathbf{h}$ *of this form, linear combinations of functions of the form*

$$\exp\left(\int_0^T \mathbf{h}^T d\mathbf{B} - \frac{1}{2}\int_0^T \mathbf{h}\cdot\mathbf{h}dt\right) \tag{35.33}$$

are dense in $L^2(\Omega, \mathcal{H}_T, P)$.

Proof: I will show in the process of the proof that functions of the form 35.33 are in $L^2(\Omega, P)$. If the conclusion of the lemma is not true, there exists nonzero $g \in L^2(\Omega, \mathcal{H}_T, P)$ such that

$$\begin{aligned} & \int_\Omega g(\omega)\exp\left(\int_0^T \mathbf{h}^T d\mathbf{B} - \frac{1}{2}\int_0^T \mathbf{h}\cdot\mathbf{h}dt\right) dP \\ = \quad & \exp\left(-\frac{1}{2}\int_0^T \mathbf{h}\cdot\mathbf{h}dt\right)\int_\Omega g(\omega)\exp\left(\int_0^T \mathbf{h}^T d\mathbf{B}\right) dP = 0 \end{aligned}$$

for all such $\mathbf{h}$. Letting $\mathbf{h}$ be given as above,

$$\begin{aligned} \int_0^T \mathbf{h}^T d\mathbf{B} &= \sum_{i=0}^{m-1} \mathbf{a}_i^T\left(\mathbf{B}_{t_{i+1}} - \mathbf{B}_{t_i}\right) && (35.34) \\ &= \sum_{i=1}^{m} \mathbf{a}_{i-1}^T \mathbf{B}_{t_i} - \sum_{i=0}^{m-1} \mathbf{a}_i^T \mathbf{B}_{t_i} \\ &= \sum_{i=1}^{m-1}\left(\mathbf{a}_{i-1}^T - \mathbf{a}_i^T\right)\mathbf{B}_{t_i} + \mathbf{a}_0^T \mathbf{B}_{t_0} + \mathbf{a}_{n-1}^T \mathbf{B}_{t_n}. && (35.35) \end{aligned}$$

Also 35.34 shows $\exp\left(\int_0^T \mathbf{h}^T d\mathbf{B}\right)$ is in $L^2(\Omega, P)$. To see this recall the $\mathbf{B}_{t_{i+1}} - \mathbf{B}_{t_i}$ are independent and the density of $\mathbf{B}_{t_{i+1}} - \mathbf{B}_{t_i}$ is

$$C(n, \Delta t_i)\exp\left(-\frac{1}{2}\frac{|\mathbf{x}|^2}{(t_{i+1} - t_i)}\right)$$

so

$$\int_\Omega \left(\exp\left(\int_0^T \mathbf{h}^T d\mathbf{B}\right)\right)^2 dP = \int_\Omega \exp\left(2\int_0^T \mathbf{h}^T d\mathbf{B}\right) dP$$

$$
\begin{aligned}
&= \int_\Omega \exp\left(\sum_{i=0}^{m-1} 2\mathbf{a}_i^T\left(\mathbf{B}_{t_{i+1}}-\mathbf{B}_{t_i}\right)\right)dP \\
&= \int_\Omega \prod_{i=0}^{m-1} \exp\left(2\mathbf{a}_i^T\left(\mathbf{B}_{t_{i+1}}-\mathbf{B}_{t_i}\right)\right)dP \\
&= \prod_{i=0}^{m-1} \int_\Omega \exp\left(2\mathbf{a}_i^T\left(\mathbf{B}_{t_{i+1}}-\mathbf{B}_{t_i}\right)\right)dP \\
&= \prod_{i=0}^{m-1} \int_{\mathbb{R}^n} C\left(n,\Delta t_i\right)\exp\left(2\mathbf{a}_i^T\mathbf{x}\right)\exp\left(-\frac{1}{2}\frac{|\mathbf{x}|^2}{\Delta t_i}\right)dx < \infty
\end{aligned}
$$

Choosing the $\mathbf{a}_i$ appropriately in 35.35, the formula in 35.35 is of the form

$$
\sum_{i=0}^{m-1} \mathbf{y}_i^T\mathbf{B}_{t_i}
$$

where $\mathbf{y}_i$ is an arbitrary vector in $\mathbb{R}^n$. It follows that for all choices of $\mathbf{y}_j \in \mathbb{R}^n$,

$$
\int_\Omega g(\omega)\exp\left(\sum_{j=0}^{m-1}\mathbf{y}_j^T\mathbf{B}_{t_j}(\omega)\right)dP = 0.
$$

Now the mapping

$$
\mathbf{y} = (\mathbf{y}_1,\cdots,\mathbf{y}_m) \to \int_\Omega g(\omega)\exp\left(\sum_{j=0}^{m-1}\mathbf{y}_j^T\mathbf{B}_{t_j}(\omega)\right)dP
$$

is analytic on $\mathbb{C}^{mn}$ and equals zero on $\mathbb{R}^{nm}$ so from standard complex variable theory, this analytic function must equal zero on $\mathbb{C}^{nm}$, not just on $\mathbb{R}^{nm}$. In particular, for all $\mathbf{y} = (\mathbf{y}_1,\cdots,\mathbf{y}_m) \in \mathbb{R}^{nm}$,

$$
\int_\Omega g(\omega)\exp\left(\sum_{j=0}^{m-1} i\mathbf{y}_j^T\mathbf{B}_{t_j}(\omega)\right)dP = 0. \tag{35.36}
$$

Now pick $\phi \in C_c^\infty(\mathbb{R}^n)$. Thus ϕ is in the Schwartz class and from the theory of Fourier transforms,

$$
\phi(\mathbf{x}) = \frac{1}{(2\pi)^{mn/2}}\int_{\mathbb{R}^{mn}} e^{i\mathbf{y}\cdot\mathbf{x}}F\phi(\mathbf{y})\,dy.
$$

In particular,

$$
\begin{aligned}
&\phi\left(\mathbf{B}_{t_0}(\omega),\cdots,\mathbf{B}_{t_{m-1}}(\omega)\right) \\
&= \frac{1}{(2\pi)^{mn/2}}\int_{\mathbb{R}^{mn}}\exp\left(\sum_{j=0}^{m-1} i\mathbf{y}_j^T\mathbf{B}_{t_j}(\omega)\right)F\phi(\mathbf{y})\,dy.
\end{aligned}
$$

Therefore,

$$\int_\Omega g(\omega)\,\phi\left(\mathbf{B}_{t_0}(\omega),\cdots,\mathbf{B}_{t_{m-1}}(\omega)\right)dP$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^{mn/2}}\int_\Omega g(\omega)\int_{\mathbb{R}^{mn}}\exp\left(\sum_{j=0}^{m-1} i\mathbf{y}_j^T\mathbf{B}_{t_j}(\omega)\right)F\phi(\mathbf{y})\,dydP \\
&= \frac{1}{(2\pi)^{mn/2}}\int_{\mathbb{R}^{mn}}\int_\Omega g(\omega)\exp\left(\sum_{j=0}^{m-1} i\mathbf{y}_j^T\mathbf{B}_{t_j}(\omega)\right)dPF\phi(\mathbf{y})\,dy \\
&= \frac{1}{(2\pi)^{mn/2}}\int_{\mathbb{R}^{mn}} 0F\phi(\mathbf{y})\,dy = 0
\end{aligned}$$

which shows by Lemma 35.16 that $g = 0$ after all, contrary to assumption. This proves the lemma.

Why such a funny lemma? It is because of the following computation which depends on Itô's formula. First lets review Itô's formula. For

$$d\mathbf{X}_t = \mathbf{u}dt + Vd\mathbf{B}$$

where $\mathbf{u}$ is a vector and V a matrix, and $\mathbf{Y} = \mathbf{g}(t,\mathbf{X})$,

$$dY_k = \frac{\partial g_k}{\partial t}(t,\mathbf{X}_t)\,dt + D_2 g_k(t,\mathbf{X}_t)\,d\mathbf{X}_t + \frac{1}{2}d\mathbf{X}_t^T H_k(t,\mathbf{X}_t)\,d\mathbf{X}_t$$

where $dtdB_k = 0, dt^2 = 0$, and, $dB_i dB_j = \delta_{ij}dt$. In the above, H_k is the hessian matrix of g_k. Thus this reduces to

$$\begin{aligned}
dY_k &= \frac{\partial g_k}{\partial t}(t,\mathbf{X}_t)\,dt + D_2 g_k(t,\mathbf{X}_t)(\mathbf{u}dt + Vd\mathbf{B}) \\
&\quad + \frac{1}{2}(\mathbf{u}dt + Vd\mathbf{B})^T H_k(t,\mathbf{X}_t)(\mathbf{u}dt + Vd\mathbf{B}) \\
&= \left(\frac{\partial g_k}{\partial t}(t,\mathbf{X}_t) + D_2 g_k(t,\mathbf{X}_t)\,\mathbf{u} + \frac{1}{2}tr\left(V^T H_k V\right)\right)dt + D_2 g_k(t,\mathbf{X}_t)\,Vd\mathbf{B}
\end{aligned}$$

In the above case, referring to 35.33, let

$$X = \int_0^t \mathbf{h}^T d\mathbf{B} - \frac{1}{2}\int_0^t \mathbf{h}\cdot\mathbf{h}dt$$

and $g(x) = e^x$ so $Y = e^X$. Then in this case, g is a scalar valued function of one variable and so the above formula reduces to

$$\begin{aligned}
dY &= \left(-\frac{1}{2}e^X|\mathbf{h}|^2 + \frac{1}{2}e^X tr\left(\mathbf{h}\mathbf{h}^T\right)\right)dt + e^X\mathbf{h}^T d\mathbf{B} \\
&= \left(-\frac{1}{2}e^X|\mathbf{h}|^2 + \frac{1}{2}e^X|\mathbf{h}|^2\right)dt + e^X\mathbf{h}^T d\mathbf{B} \\
&= Y\mathbf{h}^T d\mathbf{B}
\end{aligned}$$

Hence

$$\begin{aligned} Y &= Y_0 + \int_0^t Y \mathbf{h}^T d\mathbf{B} \\ &= 1 + \int_0^t Y \mathbf{h}^T d\mathbf{B}. \end{aligned}$$

Now here is the interesting part of this formula.

$$E\left(\int_0^t Y \mathbf{h}^T d\mathbf{B}\right) = 0$$

because the integrand is an adapted step function and $E\left(\mathbf{B}_{t_{j+1}} - \mathbf{B}_{t_j}\right) = 0$. Therefore, letting $F = Y$,

$$F = E(F) + \int_0^T \mathbf{f}(t, \omega)^T d\mathbf{B} \tag{35.37}$$

It follows that for $F \in L^2(\Omega, \mathcal{H}_T, P)$ of the special form described in Lemma 35.17, there exists an adapted function in $L^2(\Omega; \mathbb{R}^n)$, $\mathbf{f}$ such that 35.37 holds. Does such a function $\mathbf{f}$ exist for all $F \in L^2(\Omega, \mathcal{H}_T, P)$? The answer is yes and this is the content of the next theorem which is called the Itô representation theorem.

Theorem 35.18 *Let $F \in L^2(\Omega, \mathcal{H}_T, P)$. Then there exists a unique $\mathcal{H}_t$ adapted $\mathbf{f} \in L^2(\Omega \times [0, T]; \mathbb{R}^n)$ such that*

$$F = E(F) + \int_0^T \mathbf{f}(s, \omega)^T d\mathbf{B}.$$

Proof: By Lemma 35.17, functions of the form

$$\exp\left(\int_0^T \mathbf{h}^T d\mathbf{B} - \frac{1}{2}\int_0^T \mathbf{h} \cdot \mathbf{h} dt\right)$$

where $\mathbf{h}$ is a vector valued deterministic step function of the sort described in this lemma, are dense in $L^2(\Omega, \mathcal{H}_T, P)$. Given $F \in L^2(\Omega, \mathcal{H}_T, P)$, $\{G_k\}_{k=1}^{\infty}$ be functions in the subspace of linear combinations of the above functions which converge to F in $L^2(\Omega, \mathcal{H}_T, P)$. For each of these functions there exists $\mathbf{f}_k$ an adapted step function such that

$$G_k = E(G_k) + \int_0^T \mathbf{f}_k(s, \omega)^T d\mathbf{B}.$$

Then from the Itô isometry, and the observation that $E(G_k - G_l)^2 \to 0$ as $k, l \to \infty$ by the above definition of G_k in which the G_k converge to F in $L^2(\Omega)$,

$$\begin{aligned}
0 &= \lim_{k,l\to\infty} E\left((G_k - G_l)^2\right) \\
&= \lim_{k,l\to\infty} E\left(\left(E(G_k) + \int_0^T \mathbf{f}_k(s,\omega)^T d\mathbf{B} - \left(E(G_l) + \int_0^T \mathbf{f}_l(s,\omega)^T d\mathbf{B}\right)\right)^2\right) \\
&= \lim_{k,l\to\infty} \left\{E(G_k - G_l)^2 + 2E(G_k - G_l)\int_\Omega \int_0^T (\mathbf{f}_k - \mathbf{f}_l)^T d\mathbf{B}dP \right.\\
&\quad \left. + \int_\Omega \left(\int_0^T (\mathbf{f}_k - \mathbf{f}_l)^T d\mathbf{B}\right)^2 dP\right\} \\
&= \lim_{k,l\to\infty} \left\{E(G_k - G_l)^2 + \int_\Omega \left(\int_0^T (\mathbf{f}_k - \mathbf{f}_l)^T d\mathbf{B}\right)^2 dP\right\} \\
&= \lim_{k,l\to\infty} \int_\Omega \left(\int_0^T (\mathbf{f}_k - \mathbf{f}_l)^T d\mathbf{B}\right)^2 dP = \lim_{k,l\to\infty} ||\mathbf{f}_k - \mathbf{f}_l||_{L^2(\Omega\times[0,T];\mathbb{R}^n)} \qquad (35.38)
\end{aligned}$$

Going from the third to the fourth equations, is justified because $\int_\Omega \int_0^T (\mathbf{f}_k - \mathbf{f}_l)^T d\mathbf{B}dP = 0$ thanks to the independence of the integrals and the fact $\mathbf{f}_k - \mathbf{f}_l$ is an adapted step function.

This shows $\{\mathbf{f}_k\}_{k=1}^\infty$ is a Cauchy sequence in $L^2(\Omega \times [0,T]; \mathbb{R}^n)$. It follows there exists a subsequence and $\mathbf{f} \in L^2(\Omega \times [0,T]; \mathbb{R}^n)$ such that $\mathbf{f}_k$ converges to $\mathbf{f}$ pointwise and in $L^2(\Omega \times [0,T]; \mathbb{R}^n)$ with $\mathbf{f}$ $\mathcal{B}\times\mathcal{H}_T$ measurable. Then by the Itô isometry and the equation

$$G_k = E(G_k) + \int_0^T \mathbf{f}_k(s,\omega)^T d\mathbf{B}$$

you can pass to the limit as $k \to \infty$ and obtain

$$F = E(F) + \int_0^T \mathbf{f}(s,\omega)^T d\mathbf{B}$$

where $\mathbf{f}$ is adapted because of the pointwise convergence. Thus if N is the $\mathcal{B}\times\mathcal{H}_T$ measurable set of $m \times P$ measure zero, where convergence does not take place,

$$\int_0^T \int_\Omega \mathcal{X}_N(t,\omega)\, dPdt = 0$$

and so for a.e. t, $\int_\Omega \mathcal{X}_N dP = 0$ and so $\mathbf{f}_k(t,\omega) \to \mathbf{f}(t,\omega)$ for a.e. ω except for t in a set of measure zero. Now each $\mathbf{f}_k(t,\cdot)$ is $\mathcal{H}_t$ measurable and since $\mathcal{H}_t$ is complete, it follows $\mathbf{f}(t,\cdot)$ is also $\mathcal{H}_t$ measurable. Now redefine $\mathbf{f}(t,\omega) \equiv \mathbf{0}$ for all t in this exceptional set of measure zero. Then the resulting $\mathbf{f}$ must be $\mathcal{H}_t$ adapted. This proves the existence part of this theorem.

It remains to consider the uniqueness. Suppose then that

$$F = E(F) + \int_0^T \mathbf{f}(t,\omega)^T d\mathbf{B} = E(F) + \int_0^T \mathbf{f}_1(t,\omega)^T d\mathbf{B}.$$

Then

$$\int_0^T \mathbf{f}(t,\omega)^T d\mathbf{B} = \int_0^T \mathbf{f}_1(t,\omega)^T d\mathbf{B}$$

and so

$$\int_0^T \left(\mathbf{f}(t,\omega)^T - \mathbf{f}_1(t,\omega)^T\right) d\mathbf{B} = 0$$

and by the Itô isometry,

$$0 = \left|\left|\int_0^T \left(\mathbf{f}(t,\omega)^T - \mathbf{f}_1(t,\omega)^T\right) d\mathbf{B}\right|\right|_{L^2(\Omega)} = ||\mathbf{f} - \mathbf{f}_1||_{L^2(\Omega\times[0,T];\mathbb{R}^n)}$$

which proves uniqueness. This proves the theorem.

With the above major result, here is another interesting representation theorem. Recall that if you have an $\mathcal{H}_t$ adapted function, f and $\mathbf{f} \in L^2(\Omega \times [0,T])$, then $\int_0^t \mathbf{f}^T d\mathbf{B}$ is a martingale. This was proved in Theorem 34.12 for the case of one dimensional Brownian motion but it will end up being true for the general case also. The next theorem is sort of a converse. It starts with an $\mathcal{H}_t$ martingale and represents it as an Itô integral. In this theorem, $\mathcal{H}_t$ continues to be the filtration determined by n dimensional Brownian motion.

Theorem 35.19 *Let M_t be an $\mathcal{H}_t$ martingale and suppose $M_t \in L^2(\Omega)$ for all $t \geq 0$. Then there exists a unique stochastic process, $\mathbf{g}(s,\omega)$ such that $\mathbf{g}$ is $\mathcal{H}_t$ adapted and in $L^2(\Omega \times [0,t])$ for each $t > 0$, and for all $t \geq 0$,*

$$M_t = E(M_0) + \int_0^t \mathbf{g}^T d\mathbf{B}$$

Proof: First suppose $\mathbf{f}$ is an adapted function of the sort that $\mathbf{g}$ is. Then the following claim is the first step in the proof.

Claim: Let $t_1 < t_2$. Then

$$E\left(\int_{t_1}^{t_2} \mathbf{f}^T d\mathbf{B} | \mathcal{H}_{t_1}\right) = 0$$

Proof of claim: First consider the claim in the case that $\mathbf{f}$ is an adapted step function of the form

$$\mathbf{f}(t) = \sum_{i=0}^{n-1} \mathbf{a}_i \mathcal{X}_{[t_i,t_{i+1})}(t)$$

Then

$$\int_{t_1}^{t_2} \mathbf{f}^T d\mathbf{B}(\omega) = \sum_{i=0}^{n-1} \mathbf{a}_i^T(\omega)\left(\mathbf{B}_{t_{i+1}}(\omega) - \mathbf{B}_{t_i}(\omega)\right).$$

Now letting $A \in \mathcal{H}_{t_1}$

$$\begin{aligned}
& \int_A E\left(\int_{t_1}^{t_2} \mathbf{f}^T d\mathbf{B}|\mathcal{H}_{t_1}\right) dP \\
\equiv & \int_A \int_{t_1}^{t_2} \mathbf{f}^T d\mathbf{B} dP = \int_A \sum_{i=0}^{n-1} \mathbf{a}_i^T(\omega)\left(\mathbf{B}_{t_{i+1}}(\omega) - \mathbf{B}_{t_i}(\omega)\right) dP \\
= & \sum_{i=0}^{n-1} \int_\Omega \mathcal{X}_A(\omega) \mathbf{a}_i^T(\omega)\left(\mathbf{B}_{t_{i+1}}(\omega) - \mathbf{B}_{t_i}(\omega)\right) dP \\
= & \sum_{i=0}^{n-1} \int_\Omega \mathcal{X}_A(\omega) \mathbf{a}_i^T(\omega) dP \int_\Omega \left(\mathbf{B}_{t_{i+1}}(\omega) - \mathbf{B}_{t_i}(\omega)\right) dP = 0.
\end{aligned}$$

Since A is arbitrary, $E\left(\int_{t_1}^{t_2} \mathbf{f}^T d\mathbf{B}|\mathcal{H}_{t_1}\right) = 0$. This proves the claim in the case that $\mathbf{f}$ is an adapted step function. The general case follows from this in the usual way. If $\mathbf{f}$ is not a step function, there is a sequence of adapted step functions, $\{\mathbf{f}_k\}$ such that

$$\int_{t_1}^{t_2} \mathbf{f}_k^T d\mathbf{B} \rightarrow \int_{t_1}^{t_2} \mathbf{f}^T d\mathbf{B}$$

in $L^2(\Omega, P)$ and $||\mathbf{f}_k - \mathbf{f}||_{L^2(\Omega \times [t_1,t_2];\mathbb{R}^n)} \rightarrow 0$. Then using the Itô isometry,

$$\begin{aligned}
& \int_\Omega \left(E\left(\int_{t_1}^{t_2} \mathbf{f}_k^T d\mathbf{B}|\mathcal{H}_{t_1}\right) - E\left(\int_{t_1}^{t_2} \mathbf{f}^T d\mathbf{B}|\mathcal{H}_{t_1}\right)\right)^2 dP \\
= & \int_\Omega E\left(\int_{t_1}^{t_2} \mathbf{f}_k^T d\mathbf{B} - \int_{\mathbf{t_1}}^{\mathbf{t_2}} \mathbf{f}^{\mathbf{T}} d\mathbf{B}|\mathcal{H}_{t_1}\right)^2 dP \\
= & \int_\Omega \left(\int_{t_1}^{t_2} \mathbf{f}_k^T d\mathbf{B} - \int_{\mathbf{t_1}}^{\mathbf{t_2}} \mathbf{f}^{\mathbf{T}} d\mathbf{B}\right)^2 dP = \int_\Omega \left(\int_{t_1}^{t_2} \left(\mathbf{f}_k^T - \mathbf{f}^{\mathbf{T}}\right) d\mathbf{B}\right)^2 dP \\
= & ||\mathbf{f}_k - \mathbf{f}||_{L^2(\Omega \times [t_1,t_2];\mathbb{R}^n)} \rightarrow 0
\end{aligned}$$

Thus the desired conclusion holds in the general case as well.

Now to prove the theorem, it follows from Theorem 35.18 and the assumption that M_t is a martingale that for $t > 0$ there exists $\mathbf{f}^t \in L^2(\Omega \times [0,T];\mathbb{R}^n)$ such that

$$\begin{aligned}
M_t &= E(M_t) + \int_0^t \mathbf{f}^t(s,\cdot)^T d\mathbf{B} \\
&= E(M_0) + \int_0^t \mathbf{f}^t(s,\cdot)^T d\mathbf{B}.
\end{aligned}$$

Now let $t_1 < t_2$. Then since M_t is a martingale,

$$M_{t_1} = E(M_{t_2}|\mathcal{H}_{t_1}) = E\left(E(M_0) + \int_0^{t_2} \mathbf{f}^{t_2}(s,\cdot)^T d\mathbf{B}|\mathcal{H}_{t_1}\right)$$

$$= E(M_0) + E\left(\int_0^{t_1} \mathbf{f}^{t_2}(s,\cdot)^T d\mathbf{B} + \int_{t_1}^{t_2} \mathbf{f}^{t_2}(s,\cdot)^T d\mathbf{B} | \mathcal{H}_{t_1}\right)$$

$$\begin{aligned} &= E(M_0) + E\left(\int_0^{t_1} \mathbf{f}^{t_2}(s,\cdot)^T d\mathbf{B} | \mathcal{H}_{t_1}\right) \\ &= E(M_0) + \int_0^{t_1} \mathbf{f}^{t_2}(s,\cdot)^T d\mathbf{B} \end{aligned}$$

because $\int_0^{t_1} \mathbf{f}^{t_2}(s,\cdot)^T d\mathbf{B}$ is $\mathcal{H}_{t_1}$ measurable. Thus

$$M_{t_1} = E(M_0) + \int_0^{t_1} \mathbf{f}^{t_2}(s,\cdot)^T d\mathbf{B} = E(M_0) + \int_0^{t_1} \mathbf{f}^{t_1}(s,\cdot)^T d\mathbf{B}$$

and so

$$0 = \int_0^{t_1} \mathbf{f}^{t_1}(s,\cdot)^T d\mathbf{B} - \int_0^{t_1} \mathbf{f}^{t_2}(s,\cdot)^T d\mathbf{B}$$

and so by the Itô isometry,

$$\left|\left|\mathbf{f}^{t_1} - \mathbf{f}^{t_2}\right|\right|_{L^2(\Omega\times[0,t_1];\mathbb{R}^n)} = 0.$$

Letting $N \in \mathbb{N}$, it follows that

$$M_t = E(M_0) + \int_0^t \mathbf{f}^N(s,\cdot)^T d\mathbf{B}$$

for all $t \leq N$. Let $\mathbf{g} = \mathbf{f}^N$ for $t \in [0, N]$. Then asside from a set of measure zero, this is well defined and for all $t \geq 0$

$$M_t = E(M_0) + \int_0^t \mathbf{g}(s,\cdot)^T d\mathbf{B}$$

This proves the theorem.

Surely this is an incredible theorem. Note it implies all the martingales which are in L^2 for each t must be continuous a.e. Also, any such martingale satisfies $M_0 = E(M_0)$. Isn't that amazing?

35.4 Stochastic Differential Equations

35.4.1 Gronwall's Inequality

The fundamental tool in estimating differential equations is Gronwall's inequality. It is a very elementary result but of enormous significance. I will first give a proof of this important theorem. Also, I will write $X(t)$ rather than X_t.

Lemma 35.20 *Let* $k \geq 0$ *and suppose* $u(t)$ *is a Lebesgue measurable function in* $L^1([0,T])$ *which satisfies*

$$u(t) \leq u_0 + \int_0^t ku(s)\,ds.$$

Then

$$u(t) \leq u_0 e^{kt}.$$

Proof: Let

$$f(t) = u_0 e^{kt} - \left(u_0 + \int_0^t ku(s)\,ds\right).$$

Then $f(0) = 0$ and

$$\begin{aligned} f'(t) &= ku_0 e^{kt} - ku(t) \\ &\geq ku_0 e^{kt} - k\left(u_0 + \int_0^t ku(s)\,ds\right) \\ &= kf(t) \end{aligned}$$

and so $f'(t) - kf(t) \geq 0$ and so

$$\frac{d}{dt}\left(e^{-kt} f(t)\right) \geq 0$$

which implies $f(t) \geq 0$. Hence

$$u(t) \leq u_0 + \int_0^t ku(s)\,ds \leq u_0 e^{kt}.$$

This proves Gronwall's inequality.

35.4.2 Review Of Itô Integrals And A Filtration

Next recall the definition of the Itô integral. The context is that $\mathcal{H}_t$ is a filtration, B_t is a martingale for $\mathcal{H}_t$, and if $s > t$, $B_s - B_t$ is independent of $\mathcal{H}_t$.

Definition 35.21 *Suppose* f *is* $\mathcal{H}_t$ *adapted and* $\mathcal{B} \times \mathcal{F}$ *measurable such that for* $\omega \notin E$ *a set of measure zero,*

$$\int_S^T f(t,\omega)^2\,dt < \infty.$$

Then there exists a sequence of adapted bounded step functions, $\{\phi_n\}$ *satisfying*

$$\int_S^T (f(t,\omega) - \phi_n(t,\omega))^2\,dt \leq 2^{-n}$$

for $\omega \notin E$, a set of measure zero. Then for $t \in [S,T]$, the Itô integral is defined by

$$\int_S^t f dB(\omega) = \lim_{n\to\infty} \int_S^t \phi_n dB(\omega).$$

Furthermore, for these ω, $t \to \int_S^t f dB(\omega)$ is continuous because by Theorem 34.5 the convergence of $\int_S^t \phi_n dB(\omega)$ is uniform on $[0,T]$.

In what follows $\mathbf{B}_t$ will be m dimensional Brownian motion and the filtration will be denoted by $\mathcal{H}_t$ where $\mathcal{H}_t$ is the completion of the smallest σ algebra which contains

$$(\mathbf{B}_{t_0}, \cdots, \mathbf{B}_{t_k})^{-1}(U)$$

whenever $0 \leq t_0 < \cdots < t_k \leq t$ and U is a Borel set. Also, $\mathbf{Z}$, a measurable $\mathbb{R}^n$ valued function will be independent of $\mathcal{H}_t$ for all $t > 0$. Then $\mathcal{H}_t^Z$ will denote the completion of the smallest σ algebra which contains

$$\left(\mathbf{Z}, \mathbf{B}_{t_0}, \cdots, \mathbf{B}_{t_k}\right)^{-1}(U)$$

whenever $0 \leq t_0 < \cdots < t_k \leq t$ and U is a Borel set. Then the following lemma is what is needed to consider certain Itô integrals.

Lemma 35.22 *$\mathbf{B}_t$ is an $\mathcal{H}_t^Z$ martingale and if $s > t$, the increments $\mathbf{B}_s - \mathbf{B}_t$ are independent of $\mathcal{H}_t^Z$.*

Proof: Let A, U_k, V for $k = 0, \cdots, p$ be open sets and let $s > t$ and

$$D = \left(\mathbf{Z}, \mathbf{B}_{t_0}, \cdots, \mathbf{B}_{t_p}\right)^{-1}(A \times U_0 \times \cdots \times U_p), E = (\mathbf{B}_s - \mathbf{B}_t)^{-1}(V).$$

I need to verify that $P(D \cap E) = P(D) P(E)$.

$$D \cap E = \mathbf{Z}^{-1}(A) \cap \cap_{i=0}^p \mathbf{B}_{t_i}^{-1}(U_i) \cap E$$

From independence of $\mathbf{Z}$ to $\mathcal{H}_t$ for all $t > 0$, and independence of the increments, $\mathbf{B}_s - \mathbf{B}_t$ to $\mathcal{H}_t$,

$$\begin{aligned} P(D \cap E) &= P\left(\mathbf{Z}^{-1}(A)\right) P\left(\cap_{i=0}^p \mathbf{B}_{t_i}^{-1}(U_i) \cap E\right) \\ &= P\left(\mathbf{Z}^{-1}(A)\right) P\left(\cap_{i=0}^p \mathbf{B}_{t_i}^{-1}(U_i)\right) P(E) \\ &= P\left(\mathbf{Z}^{-1}(A) \cap \cap_{i=0}^p \mathbf{B}_{t_i}^{-1}(U_i)\right) P(E) \\ &= P(D) P(E). \end{aligned}$$

It follows that for all D an inverse image of an open set and E of the above form where V is open, $P(D \cap E) = P(D) P(E)$. It follows easily this holds for all D and E inverse images of Borel sets. If $D \in \mathcal{H}_t$ and $E = (\mathbf{B}_s - \mathbf{B}_t)^{-1}(V)$ then there exists D_1 an inverse image of a Borel set such that $D_1 \supseteq D$ and $P(D_1 \setminus D) = 0$ so

$$\begin{aligned} P(D \cap E) &= P(D_1 \cap E) \\ &= P(D_1) P(E) = P(D) P(E). \end{aligned}$$

This verifies the independence.

Now let $A \in \mathcal{H}_t^Z$. Then from the above independence result,

$$\begin{aligned} \int_A E\left(\mathbf{B}_s - \mathbf{B}_t | \mathcal{H}_t^Z\right) dP &= \int_A \mathbf{B}_s - \mathbf{B}_t dP \\ &= \int_A dP \int_\Omega \mathbf{B}_s - \mathbf{B}_t dP = 0 \end{aligned}$$

and so

$$\begin{aligned} E\left(\mathbf{B}_s | \mathcal{H}_t^Z\right) &= E\left(\mathbf{B}_s - \mathbf{B}_t + \mathbf{B}_t | \mathcal{H}_t^Z\right) \\ &= \mathbf{0} + E\left(\mathbf{B}_t | \mathcal{H}_t^Z\right) = \mathbf{B}_t. \end{aligned}$$

This proves the lemma.

35.4.3 A Function Space

Lemma 35.23 *Let $C\left([0,T]; L^2(\Omega)^p\right)$ denote the space of continuous functions with values in $L^2(\Omega)^p$ and for each $\lambda \geq 0$, denote for $\mathbf{X} \in C\left([0,T]; L^2(\Omega)^p\right)$*

$$||\mathbf{X}||_\lambda \equiv \max\left\{e^{-\lambda t} ||\mathbf{X}(t)||_{L^2(\Omega)^p} : t \in [0,T]\right\}$$

Then all these norms are equivalent and $\left(C\left([0,T]; L^2(\Omega)^p\right), ||\cdot||_\lambda\right)$ is a Banach space. Suppose also that $\mathcal{G}_t$ is a filtration and that $\mathbf{X}(t)$ is $\mathcal{G}_t$ measurable. Then by changing $\mathbf{X}(t)$ There exists a $\mathcal{B} \times \mathcal{F}$ measurable function, $\mathbf{Y}$ such that for each t, $\mathbf{Y}(t)$ is $\mathcal{G}_t$ measurable and for all t not in a Borel measurable set of measure zero, $\mathbf{X}(t) = \mathbf{Y}(t)$ in $L^2(\Omega)^p$.Furthermore, if $V_\mathcal{G}$ denotes those functions in $C\left([0,T]; L^2(\Omega)^p\right)$ which are $\mathcal{G}_t$ adapted, then $V_\mathcal{G}$ is a closed subspace of

$$C\left([0,T]; L^2(\Omega)^p\right).$$

Proof: The assertion about the norms and the Banach space are all obvious. The main message is about the measurability assertions. Let $P \equiv \{t_0, \cdots, t_n\}$ be a partition of $[0,T]$ of the usual sort where $0 = t_0 < t_1 < \cdots < t_n = T$. Then for $\mathbf{X} \in C\left([0,T]; L^2(\Omega)^p\right)$ and $\mathcal{G}_t$ adapted, consider

$$\mathbf{X}_P(t) \equiv \sum_{k=1}^n \mathbf{X}(t_{k-1}) \mathcal{X}_{[t_{k-1}, t_k)}(t).$$

By the assumption of continuity of $\mathbf{X}$, it follows $||\mathbf{X} - \mathbf{X}_P||_\lambda \to 0$ as $||P|| \to 0$ where $||P||$ is the norm of the partition given by $\max\{|t_{i+1} - t_i|\}$. Picking $\lambda = 0$ for convenience, it follows there exists a sequence of partitions, $\{P_n\}$ such that $||\mathbf{X} - \mathbf{X}_{P_n}||_0 < 2^{-n}$. Therefore, for each $t \in [0,T], \mathbf{X}_{P_n}(t)(\omega) \to \mathbf{X}(t)(\omega)$ a.e. ω. It follows $\mathbf{X}(t)$ is $\mathcal{G}_t$ measurable.

Now each $\mathbf{X}_{P_n}$ is $\mathcal{B} \times \mathcal{F}$ measurable because it is a finite sum of such functions. It follows $\{\mathbf{X}_{P_n}\}$ is a Cauchy sequence in $L^2([0,T] \times \Omega)^p$ and so there exists $\mathbf{Y}$ which is $\mathcal{B} \times \mathcal{F}$ measurable and $\mathbf{X}_{P_n}$ converges to $\mathbf{Y}$ in $L^2([0,T] \times \Omega)^p$. Now

$$t \to \int_\Omega |\mathbf{X}_{P_n} - \mathbf{Y}|^2 \, dP$$

is measurable and so taking the limit as $n \to \infty$, it follows

$$t \to \int_\Omega |\mathbf{X} - \mathbf{Y}|^2 \, dP$$

is also measurable. Also

$$\begin{aligned} & \int_0^T \int_\Omega |\mathbf{X} - \mathbf{Y}|^2 \, dP dt \\ \leq \; & 2 \left(\int_0^T \int_\Omega |\mathbf{X} - \mathbf{X}_{P_n}|^2 \, dP dt + \int_0^T \int_\Omega |\mathbf{X}_{P_n} - \mathbf{Y}|^2 \, dP dt \right) \end{aligned}$$

and both these integrals converge to 0 as $n \to \infty$. Therefore, there is a Borel set of measure zero, $N \subseteq [0,T]$ such that for $t \notin N$,

$$\int_\Omega |\mathbf{X}(t) - \mathbf{Y}(t)|^2 \, dP = 0$$

and it follows that for such t,

$$\mathbf{X}(t) = \mathbf{Y}(t) \text{ in } L^2(\Omega)^p.$$

For t in the exceptional set, modify $\mathbf{Y}(t)$ by setting it equal to 0. Then $\mathbf{Y}$ is $\mathcal{G}_t$ adapted.

It only remains to consider the last claim about $V_\mathcal{G}$. Suppose $\{\mathbf{X}_n\}_{n=1}^\infty \subseteq V_\mathcal{G}$ and $||\mathbf{X}_n - \mathbf{Y}||_\lambda \to 0$. I need to verify $\mathbf{Y}$ is $\mathcal{G}_t$ adapted. However, taking a subsequence, still called n and letting $\lambda = 0$ for convenience, it can be assumed

$$||\mathbf{X}_n - \mathbf{Y}||_0 < 2^{-n}$$

and so it follows for each $t \in [0,T]$, $\mathbf{X}_n(t)(\omega) \to \mathbf{Y}(t)(\omega)$ a.e. ω. It follows since $(\Omega, \mathcal{G}_t, P)$ is complete, $\mathbf{Y}(t)$ is $\mathcal{G}_t$ measurable for each t. This proves the lemma.

35.4.4 An Extension Of The Itô Integral

Now I will give a definition of the Itô integral on $V_\mathcal{G}$. It is assumed here that $\mathcal{G}_t$ is a filtration with the property that m dimensional Brownian motion, $\mathbf{B}_t$ is a martingale for it and if $s > t$, $\mathbf{B}_s - \mathbf{B}_t$ is independent of $\mathcal{G}_t$ so it makes sense to speak of an Itô integral, $\int_0^t \mathbf{X}^T d\mathbf{B}$ given $\mathbf{X}$ is $\mathcal{G}_t$ adapted, $\mathcal{B} \times \mathcal{F}$ measurable and in $L^2([0,T] \times \Omega)^m$. Here is the definition.

Definition 35.24 *Let* $\mathbf{X} \in V_{\mathcal{G}}$. *Then*

$$\int_0^t \mathbf{X}^T d\mathbf{B} \equiv \int_0^t \mathbf{Y}^T d\mathbf{B} \text{ in } L^2(\Omega)^p$$

where $\mathbf{Y}$ *is a* $\mathcal{G}_t$ *adapted function which is* $\mathcal{B} \times \mathcal{F}$ *measurable and* $\mathbf{Y}(t) = \mathbf{X}(t)$ *in* $L^2(\Omega)^p$ *for a.e.* $t \in [0,T]$.

Lemma 35.25 *The above definition is well defined.*

Proof: Suppose both $\mathbf{Y}$ and $\mathbf{Y}_1$ work as described in the definition. Then

$$||\mathbf{Y} - \mathbf{Y}_1||_{L^2([0,T]\times\Omega)^p} = 0$$

and so by the Itô isometry,

$$\left|\left|\int_0^t \left(\mathbf{Y}^T - \mathbf{Y}_1^T\right) d\mathbf{B}\right|\right|_{L^2(\Omega)^p} = 0$$

which shows $\int_0^t \mathbf{Y}^T d\mathbf{B} = \int_0^t \mathbf{Y}_1^T d\mathbf{B}$ in $L^2(\Omega)^p$. This proves the lemma.

For $\mathbf{x} \in \mathbb{R}^p$, $\mathbf{b}(t,\mathbf{x}) \in \mathbb{R}^m$ and $\sigma(t,\mathbf{x})$ will be an $p \times m$ matrix. It is assumed that for given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^p$ the following measurability and Lipschitz conditions hold.

$$t \to \mathbf{b}(t,\mathbf{x}), t \to \sigma(t,\mathbf{x}) \text{ are Lebesgue measurable,} \tag{35.39}$$

$$|\mathbf{b}(t,\mathbf{x}) - \mathbf{b}(t,\mathbf{y})| + |\sigma(t,\mathbf{x}) - \sigma(t,\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|, \tag{35.40}$$

$$|\mathbf{b}(t,\mathbf{x})| + |\sigma(t,\mathbf{x})| \leq C(1 + |\mathbf{x}|). \tag{35.41}$$

In the above, it suffices to have the components of $\mathbf{b}$ and σ measurable. Also,the norm refers to any convenient norm. This does not matter because all the norms on a finite dimensional vector space are equivalent.

Definition 35.26 *Let* $\mathcal{G}_t$ *be a filtration for which* $\mathbf{B}_t$ *is a martingale and such that for* $s > t, \mathbf{B}_s - \mathbf{B}_t$ *is independent of* $\mathcal{G}_t$. *For* $\mathbf{X}$ *product measurable in* $\mathcal{B} \times \mathcal{F}$ *and* $\mathcal{G}_t$ *adapted define*

$$\left(\int_0^t \sigma(s,\mathbf{X}) d\mathbf{B}\right)_k \equiv \int_0^t (\sigma(s,\mathbf{X}))_k d\mathbf{B}$$

where $(\sigma(s,\mathbf{X}))_k$ *refers to the* k^{th} *row of the matrix,* σ.

35.4.5 A Vector Valued Deterministic Integral

Lemma 35.27 *Let* $\mathbf{X} \in C\left([0,T]; L^2(\Omega)^p\right)$ *and let* $\mathbf{b}$ *be given as above. Then*

$$t \to \int_\Omega \mathbf{b}(t,\mathbf{X}(t)) dP$$

is Lebesgue measurable. It is also possible to define an integral, having values in $L^2(\Omega)^p$ *according to the following formula in which* $[a,b] \subseteq [0,T]$.

$$\left(\int_a^b \mathbf{b}(t,\mathbf{X}(t))\,dt, \mathbf{h}\right)_{L^2(\Omega)^p} \equiv \int_a^b (\mathbf{b}(t,\mathbf{X}(t)),\mathbf{h})_{L^2(\Omega)^p}\,dt$$

The integral defined in this way satisfies all the usual algebraic properties for integrals and

$$t \to \int_a^t \mathbf{b}(s,\mathbf{X}(s))\,ds$$

is continuous as a function with values in $L^2(\Omega)^p$. *Also if* $\mathbf{X}$ *is* $\mathcal{G}_t$ *adapted for* $\mathcal{G}_t$ *a filtration, then*

$$\int_a^b \mathbf{b}(s,\mathbf{X}(s))\,ds$$

is $\mathcal{G}_b$ *measurable.*

Proof: Let $\mathbf{X}_{P_n}(t)$ be the sequence of step functions converging to $\mathbf{X}(t)$ which is described in Lemma 35.23. Then on $[t_j,t_{j+1})$,

$$\int_\Omega \mathbf{b}(t,\mathbf{X}_{P_n}(t))\,dP = \int_\Omega \mathbf{b}(t,\mathbf{X}(t_j))\,dP$$

where $\mathbf{X}(t_j) \in L^2(\Omega)^p$. It follows $\mathbf{X}(t_j)$ is the pointwise limit of simple functions of the form $\sum_{k=1}^m c_k \mathcal{X}_{E_k}(\omega)$ which also converge to $\mathbf{X}(t_j)$ in $L^2(\Omega)^p$. Thus for $t \in [t_j,t_{j+1})$,

$$\int_\Omega \mathbf{b}\left(t,\sum_{k=1}^m \mathbf{c}_k \mathcal{X}_{E_k}(\omega)\right) dP = \sum_{k=1}^m \int_{E_k} \mathbf{b}(t,\mathbf{c}_k)\,dP,$$

a measurable function of t. Now let $\{\mathbf{S}_n\}$ be a sequence of these simple functions converging to $\mathbf{X}(t_j)$ in $L^2(\Omega)^p$. Then

$$\begin{aligned}\left|\int_\Omega \mathbf{b}(t,\mathbf{X}(t_j))\,dP - \int_\Omega \mathbf{b}(t,\mathbf{S}_n)\,dP\right| &\leq \int_\Omega K\,|\mathbf{S}_n - \mathbf{X}(t_j)|\,dP \\ &\leq K\,||\mathbf{S}_n - \mathbf{X}(t_j)||_{L^2(\Omega)^p}\end{aligned}$$

and so $\int_\Omega \mathbf{b}(t,\mathbf{X}(t_j))\,dP$ is a pointwise limit of Lebesgue measurable functions on $[t_j,t_{j+1})$. It follows $\int_\Omega \mathbf{b}(t,\mathbf{X}_{P_n}(t))\,dP$ is Lebesgue measurable. Using a similar argument to what was just done, this converges to $\int_\Omega \mathbf{b}(t,\mathbf{X}(t))\,dP$, and so this last integral is also Lebesgue measurable in t. By a repeat of the above arguments or by simple specializing to $\mathbf{b}$ having real values, it follows

$$t \to \int_\Omega |\mathbf{b}(t,\mathbf{X}(t))|\,dP$$

is Lebesgue measurable. Also, using the properties of $\mathbf{b}$ and Holder's inequality, if $\mathbf{X}_n \to \mathbf{X}$ in $C\left([0,T]; L^2(\Omega)^p\right)$,

$$\left| \int_\Omega |\mathbf{b}(t, \mathbf{X}(t))|^2 \, dt - \int_\Omega |\mathbf{b}(t, \mathbf{X}_n(t))|^2 \, dt \right| \leq C \, ||\mathbf{X}(t) - \mathbf{X}_n(t)||^2_{L^2(\Omega)^p} .$$

Therefore, exploiting the same sorts of arguments involving first approximating by step functions and then by simple functions, it follows $t \to \int_\Omega |\mathbf{b}(t, \mathbf{X}(t))|^2 \, dt$ is Lebesgue measurable.

Next consider the definition of the integral. From the first part,

$$t \to (\mathbf{b}(t, \mathbf{X}(t)), \mathbf{h})_{L^2(\Omega)^p}$$

is Lebesgue measurable and also

$$\left| \int_a^b (\mathbf{b}(t, \mathbf{X}(t)), \mathbf{h})_{L^2(\Omega)^p} \, dt \right| \leq \int_a^b ||\mathbf{b}(t, \mathbf{X}(t))||^2_{L^2(\Omega)^p} \, dt \, ||\mathbf{h}||_{L^2(\Omega)^p} .$$

Letting $\Lambda(\mathbf{h}) \equiv \int_a^b (\mathbf{b}(t, \mathbf{X}(t)), \mathbf{h})_{L^2(\Omega)^p} \, dt$, it follows Λ is a continuous linear functional on $L^2(\Omega)^p$ and so by the Riesz representation theorem, there exists a unique element of $L^2(\Omega)^p$ denoted by

$$\int_a^b \mathbf{b}(t, \mathbf{X}(t)) \, dt$$

such that

$$\left(\int_a^b \mathbf{b}(t, \mathbf{X}(t)) \, dt, \mathbf{h} \right)_{L^2(\Omega)^p} = \int_a^b (\mathbf{b}(t, \mathbf{X}(t)), \mathbf{h})_{L^2(\Omega)^p} \, dt.$$

This completes the definition of the integral.

It is obvious the integral satisfies all the usual algebraic properties. Consider the claim about continuity. Let $s < t$. Then

$$\left| \left(\int_a^s \mathbf{b}(r, \mathbf{X}(r)) \, dr - \int_a^t \mathbf{b}(r, \mathbf{X}(r)) \, dr, \mathbf{h} \right)_{L^2(\Omega)^p} \right|$$

$$\leq \left| \left(\int_s^t \mathbf{b}(r, \mathbf{X}(r)), \mathbf{h} \right)_{L^2(\Omega)^p} \right| \leq \int_s^t \left| (\mathbf{b}(r, \mathbf{X}(r)), \mathbf{h})_{L^2(\Omega)^p} \right| dr$$

$$\leq C \int_s^t ||\mathbf{b}(r, \mathbf{X}(r))||_{L^2(\Omega)^p} \, dr \, ||\mathbf{h}||_{L^2(\Omega)^p}$$

$$\leq C' (t-s)^{1/2} \, ||\mathbf{h}||_{L^2(\Omega)^p} .$$

It follows

$$\left|\left| \int_a^s \mathbf{b}(r, \mathbf{X}(r)) \, dr - \int_a^t \mathbf{b}(r, \mathbf{X}(r)) \, dr \right|\right|_{L^2(\Omega)^p} \leq C' |t-s|^{1/2} .$$

The last assertion follows from approximating by step functions as in Lemma 35.23, noting the step functions are $\mathcal{G}_b$ measurable and then passing to a limit to obtain the desired conclusion. This proves the lemma.

35.4.6 The Existence And Uniqueness Theorem

In the next lemma, the two integrals are as defined above.

Lemma 35.28 *Let* $\mathbf{b}$ *and* σ *satisfy 35.39 - 35.41. Let* $\mathbf{Z}$ *be a random vector which is either independent of* $\mathcal{H}_t$ *for all* $t > 0$ *or else is measurable with respect to* $\mathcal{H}_t$ *for all* t *and suppose*

$$\int_\Omega |\mathbf{Z}|^2 \, dP < \infty$$

Let $\mathcal{G}_t = \mathcal{H}_t^Z$ *in the first case and let* $\mathcal{G}_t = \mathcal{H}_t$ *in the second. Then there exists a unique solution,* $\mathbf{X} \in V_{\mathcal{G}}$ *to the integral equation,*

$$\mathbf{X}(t) = \mathbf{Z} + \int_0^t \mathbf{b}(s, \mathbf{X}(s)) \, ds + \int_0^t \sigma(s, \mathbf{X}(s)) \, d\mathbf{B}.$$

Proof: For $\mathbf{X} \in V_{\mathcal{G}}$, supplied with the norm $||\cdot||_\lambda$ described above, let

$$\Phi\mathbf{X}(t) \equiv \mathbf{Z} + \int_0^t \mathbf{b}(s, \mathbf{X}(s)) \, ds + \int_0^t \sigma(s, \mathbf{X}(s)) \, d\mathbf{B}$$

It follows from Corollary 31.43 on Page 890 and Lemmas 34.8 on Page 925 and Lemma 35.27 that $\Phi\mathbf{X}$ is $\mathcal{G}_t$ adapted and $\mathcal{B} \times \mathcal{F}$ measurable. The deterministic integral is a continuous function of t with values in $L^2(\Omega)^p$ by Lemma 35.27. The same is true of the Itô integral. To see this, let $\mathbf{Y}$ be adapted and product measurable and equal to $\mathbf{X}$ for a.e. t. Then by the Itô isometry and the above definition of this integral of functions in $V_{\mathcal{G}}$,

$$\begin{aligned}
& \left|\left| \int_0^t \sigma(r, \mathbf{X}(r)) \, d\mathbf{B} - \int_0^s \sigma(r, \mathbf{X}(r)) \, d\mathbf{B} \right|\right|_{L^2(\Omega)^p}^2 \\
= & \left|\left| \int_0^t \sigma(r, \mathbf{Y}(r)) \, d\mathbf{B} - \int_0^s \sigma(r, \mathbf{Y}(r)) \, d\mathbf{B} \right|\right|_{L^2(\Omega)^p}^2 \\
= & \left|\left| \int_s^t \sigma(r, \mathbf{Y}(r)) \, d\mathbf{B} \right|\right|_{L^2(\Omega)^p}^2 = \int_s^t \int_\Omega |\sigma(r, \mathbf{Y}(r))|^2 \, dP dr \\
\leq & \; 2C \int_s^t \int_\Omega \left(1 + |\mathbf{Y}(r)|^2\right) dP dr = 2C \int_s^t \int_\Omega \left(1 + |\mathbf{X}(r)|^2\right) dP dr
\end{aligned}$$

which converges to 0 as $s \to t$. Thus $\Phi : V_{\mathcal{G}} \to V_{\mathcal{G}}$.

In fact, if λ is large enough, Φ is a contraction mapping. I will show this next. Let $\mathbf{X}, \mathbf{Y} \in V_{\mathcal{G}}$ and let $\mathbf{X}_1, \mathbf{Y}_1$ be corresponding measurable representatives as above. By the Itô isometry,

$$\left|\left|e^{-\lambda t}\left(\int_0^t \sigma\left(s,\mathbf{X}\left(s\right)\right)d\mathbf{B}-\int_0^t \sigma\left(s,\mathbf{Y}\left(s\right)\right)d\mathbf{B}\right)\right|\right|_{L^2(\Omega)^p}^2$$

$$\begin{aligned}
&\leq e^{-\lambda t}\int_0^t\int_\Omega\left|\sigma\left(s,\mathbf{X}_1\left(s\right)\right)-\sigma\left(s,\mathbf{Y}_1\left(s\right)\right)\right|^2 dPds \\
&\leq Ke^{-\lambda t}\int_0^t\int_\Omega\left|\mathbf{X}_1\left(s\right)-\mathbf{Y}_1\left(s\right)\right|^2 dPds \\
&= Ke^{-\lambda t}\int_0^t e^{\lambda s}\left(e^{-\lambda s}\int_\Omega\left|\mathbf{X}\left(s\right)-\mathbf{Y}\left(s\right)\right|^2 dP\right)ds \\
&\leq K\int_0^t e^{\lambda(s-t)}ds\left|\left|\mathbf{X}-\mathbf{Y}\right|\right|_\lambda \leq K\frac{1}{\lambda}\left|\left|\mathbf{X}-\mathbf{Y}\right|\right|_\lambda .
\end{aligned}$$

Thus if λ is large enough, this term is a contraction. Similar but easier reasoning applies to the deterministic integral in the definition of Φ. Therefore, by the usual contraction mapping theorem, Φ has a unique fixed point in $V_{\mathcal{G}}$. This proves the lemma.

Theorem 35.29 *Let* $\mathbf{b}$ *and* σ *satisfy 35.39 - 35.41. Let* $\mathbf{Z}$ *be a random vector which is either independent of* $\mathcal{H}_t$ *for all* $t>0$ *or else is measurable with respect to* $\mathcal{H}_t$ *for all* t *and suppose*

$$\int_\Omega\left|\mathbf{Z}\right|^2 dP<\infty$$

Let $\mathcal{G}_t=\mathcal{H}_t^Z$ *in the first case and let* $\mathcal{G}_t=\mathcal{H}_t$ *in the second. Then there exists a* $\mathcal{B}\times\mathcal{F}$ *measurable function,* $\mathbf{Y}\in L^2\left(\left[0,T\right]\times\Omega\right)^p$ *and a set of measure zero,* N *such that for* $\omega\notin N$*,*

$$\mathbf{Y}\left(t\right)\left(\omega\right)=\mathbf{Z}+\int_0^t\mathbf{b}\left(s,\mathbf{Y}\left(s\right)\left(\omega\right)\right)ds+\int_0^t\sigma\left(s,\mathbf{Y}\left(s\right)\right)d\mathbf{B}\left(\omega\right)$$

for all $t\in\left[0,T\right]$.

Proof: Let $\mathbf{X}$ be the solution of Lemma 35.28. Thus

$$\mathbf{X}\left(t\right)=\mathbf{Z}+\int_0^t\mathbf{b}\left(s,\mathbf{X}\left(s\right)\right)ds+\int_0^t\sigma\left(s,\mathbf{X}\left(s\right)\right)d\mathbf{B}.$$

Now let $\widetilde{\mathbf{Y}}$ be $\mathcal{B}\times\mathcal{F}$ measurable, adapted, in $L^2\left(\left[0,T\right]\times\Omega\right)^p$, and $\widetilde{\mathbf{Y}}\left(t\right)=\mathbf{X}\left(t\right)$ a.e. t. Letting $\mathbf{h}\in L^2\left(\Omega\right)^p$, the definition of the first integral in the above implies

$$\left|\left(\int_0^t\mathbf{b}\left(s,\mathbf{X}\left(s\right)\right)ds-\int_0^t\mathbf{b}\left(s,\widetilde{\mathbf{Y}}\left(s\right)\left(\cdot\right)\right)ds,\mathbf{h}\right)_{L^2(\Omega)^p}\right|$$

$$\leq \int_0^t \left(\mathbf{b}(s, \mathbf{X}(s)) - \mathbf{b}\left(s, \widetilde{\mathbf{Y}}(s)\right), \mathbf{h}\right)_{L^2(\Omega)^p} ds$$
$$\leq K \int_0^t \left|\left|\mathbf{X}(s) - \widetilde{\mathbf{Y}}(s)\right|\right|_{L^2(\Omega)^p} ds \, ||\mathbf{h}||_{L^2(\Omega)^p} = 0$$

and so

$$\int_0^t \mathbf{b}(s, \mathbf{X}(s)) \, ds = \int_0^t \mathbf{b}\left(s, \widetilde{\mathbf{Y}}(s)\right) ds \text{ in } L^2(\Omega)^p.$$

It follows that in $L^2(\Omega)^p$,

$$\mathbf{X}(t) = \mathbf{Z} + \int_0^t \mathbf{b}\left(s, \widetilde{\mathbf{Y}}(s)\right) ds + \int_0^t \sigma\left(s, \widetilde{\mathbf{Y}}(s)\right) d\mathbf{B} \tag{35.42}$$

where now the first integral on the right is the usual thing given by

$$\int_0^t \mathbf{b}\left(s, \widetilde{\mathbf{Y}}(s)\right) ds(\omega) = \int_0^t \mathbf{b}\left(s, \widetilde{\mathbf{Y}}(s)(\omega)\right) ds$$

Also, for a.e. $\omega, t \to \widetilde{\mathbf{Y}}(t)(\omega)$ is a function in $L^2(0,T)$ and from the theory of the Itô integral, $\int_0^t \sigma\left(s, \widetilde{\mathbf{Y}}(s)\right) d\mathbf{B}$ is a continuous function of t for ω not in a set of measure zero. Therefore, for ω off a set of measure zero, the right side of 35.42 is continuous in t. It also delivers an adapted product measurable function, $\mathbf{Y}$. Thus for a.e. $\omega, \mathbf{Y}(t)(\omega)$ is a continuous function of t and

$$\mathbf{Y}(t) = \mathbf{Z} + \int_0^t \mathbf{b}\left(s, \widetilde{\mathbf{Y}}(s)\right) ds + \int_0^t \sigma\left(s, \widetilde{\mathbf{Y}}(s)\right) d\mathbf{B}$$

and so $\mathbf{Y}(t) = \mathbf{X}(t)$ in $L^2(\Omega)^p$ for all t. Now this also shows $\mathbf{Y}(t) = \widetilde{\mathbf{Y}}(t)$ for a.e. t. Hence by the Itô isometry, the right side of the above is unchanged in $L^2(\Omega)^p$ if $\widetilde{\mathbf{Y}}$ is replaced by $\mathbf{Y}$. By the argument just given, the resulting right side is continuous in t for a.e. ω and so there exists a set of measure zero such that for ω not in this set,

$$\mathbf{Y}(t)(\omega) = \mathbf{Z} + \int_0^t \mathbf{b}(s, \mathbf{Y}(s)(\omega)) \, ds + \int_0^t \sigma(s, \mathbf{Y}(s)) \, d\mathbf{B}(\omega)$$

and both sides are continuous functions of t. This proves the theorem.

Note there were two cases given for the initial condition in the above theorem. The second is not very interesting. If $\mathbf{Z}$ is $\mathcal{H}_0$ measurable, then since $\mathbf{B}_0 = \mathbf{x}$, a constant, it follows $\mathcal{H}_0 = \{\emptyset, \Omega\}$ so $\mathbf{Z}$ is a constant. However, if $\mathbf{Z}$ is a constant, then it satisfies the first condition.

Not surprisingly, the solution to the above theorem is unique. This is stated as the following corollary which is the main result.

Corollary 35.30 *Let* $\mathbf{b}$ *and* σ *satisfy 35.39 - 35.41. Let* $\mathbf{Z}$ *be a random vector which is independent of* $\mathcal{H}_t$ *for all* $t > 0$ *and suppose*

$$\int_\Omega |\mathbf{Z}|^2 \, dP < \infty$$

Then there exists a unique $\mathcal{H}_t^Z$ adapted solution, $\mathbf{X} \in L^2([0,T]\times\Omega)^n$ to the integral equation,

$$\mathbf{X}(t) = \mathbf{Z} + \int_0^t \mathbf{b}(s,\mathbf{X}(s))\,ds + \int_0^t \sigma(s,\mathbf{X}(s))\,d\mathbf{B} \text{ a.e. } \omega \tag{35.43}$$

in the sense that if $\widetilde{\mathbf{X}}$ is another solution, then there exists a set of measure zero, N such that for $\omega \notin N, \widetilde{\mathbf{X}}(t) = \mathbf{X}(t)$ for all $t \in [0,T]$.

Proof: The existence part of this proof is already done. Let N denote the union of the exceptional sets corresponding to $\mathbf{X}$ and $\widetilde{\mathbf{X}}$. Then from 35.43 and the various assumptions on $\mathbf{b}$ and σ,it follows that for $\omega \notin N$,

$$\begin{aligned}\left|\mathbf{X}(t) - \widetilde{\mathbf{X}}(t)\right|^2 &\leq 2K^2T\int_0^t \left|\mathbf{X}(s) - \widetilde{\mathbf{X}}(s)\right|^2 ds \\ &\quad +2\left|\int_0^t \left(\sigma(s,\mathbf{X}(s)) - \sigma\left(s,\widetilde{\mathbf{X}}(s)\right)\right) d\mathbf{B}\right|^2.\end{aligned}$$

Then by the Itô isometry, this implies

$$\begin{aligned}&\left|\left|\mathbf{X}(t) - \widetilde{\mathbf{X}}(t)\right|\right|^2_{L^2(\Omega)^n} \\ \leq\ & 2K^2T\int_0^t \left|\left|\mathbf{X}(s) - \widetilde{\mathbf{X}}(s)\right|\right|^2_{L^2(\Omega)^n} ds \\ &+2\int_0^t \left|\left|\sigma(s,\mathbf{X}(s)) - \sigma\left(s,\widetilde{\mathbf{X}}(s)\right)\right|\right|^2 ds \\ \leq\ & C_T\int_0^t \left|\left|\mathbf{X}(s) - \widetilde{\mathbf{X}}(s)\right|\right|^2_{L^2(\Omega)^n} ds \qquad (35.44)\end{aligned}$$

and by assumption both $\mathbf{X}$ and $\widetilde{\mathbf{X}}$ are in $L^2([0,T]\times\Omega)^n$ so $t \to \left|\left|\mathbf{X}(t) - \widetilde{\mathbf{X}}(t)\right|\right|^2_{L^2(\Omega)^n}$ is in $L^1([0,T])$. By Gronwall's inequality, $\widetilde{\mathbf{X}}(t) = \mathbf{X}(t)$ in $L^2(\Omega)^n$ for all t. It follows there exists a set of measure zero, N_1 such that for $\omega \notin N_1$,

$$\int_0^T \left|\widetilde{\mathbf{X}}(t) - \mathbf{X}(t)\right|^2 dt = 0$$

But for $\omega \notin N$, the functions, $\widetilde{\mathbf{X}}$ and $\mathbf{X}$ are continuous and so if $\omega \notin N_1 \cup N, \widetilde{\mathbf{X}}(t) = \mathbf{X}(t)$ for all t. This proves the corollary.

Note that if different initial conditions had been given, say $\mathbf{Z}$ and $\mathbf{Z}_1$, the above argument for uniqueness also gives a continuous dependence result with no effort. In fact, 35.44 then would take the form

$$\left|\left|\mathbf{X}(t) - \widetilde{\mathbf{X}}(t)\right|\right|^2_{L^2(\Omega)^n} \leq 3\left|\left|\mathbf{Z} - \mathbf{Z}_1\right|\right|^2_{L^2(\Omega)^n} + C_T\int_0^t \left|\left|\mathbf{X}(s) - \widetilde{\mathbf{X}}(s)\right|\right|^2_{L^2(\Omega)^n} ds$$

and Gronwall's inequality would then imply

$$\left|\left|\mathbf{X}(t)-\widetilde{\mathbf{X}}(t)\right|\right|^2_{L^2(\Omega)^n} \leq C\,||\mathbf{Z}-\mathbf{Z}_1||^2_{L^2(\Omega)^n}$$

for some constant, C.

The equivalent form of the above integral equation,

$$\mathbf{X}(t)=\mathbf{Z}+\int_0^t \mathbf{b}(s,\mathbf{X}(s))\,ds+\int_0^t \sigma(s,\mathbf{X}(s))\,d\mathbf{B}$$

is

$$d\mathbf{X}=\mathbf{b}(t,\mathbf{X})\,dt+\sigma(t,\mathbf{X}(t))\,d\mathbf{B},\ \mathbf{X}(0)=\mathbf{Z}.$$

35.4.7 Some Simple Examples

Here are some examples of simple stochastic differential equations which are solved using the Itô formula.

Example 35.31 *In this example, $m=n=1$ and B is one dimensional Brownian motion. The differential equation is*

$$dX=h(t)\,XdB,\ X(0)=1$$

Obviously, one would want to do something like $\frac{dX}{X}=h(t)\,dB$. However, you have to follow the rules. Let $g(x)=\ln(x)$ and $Y=g(X)$. Then by the Itô formula,

$$\begin{aligned} dY &= \frac{1}{X}dX+\frac{1}{2}\left(\frac{-1}{X^2}\right)dX^2 \\ &= \frac{1}{X}h(t)\,XdB-\frac{1}{2}\frac{1}{X^2}h(t)^2X^2dB^2 \\ &= h(t)\,dB-\frac{1}{2}h(t)^2\,dt \end{aligned}$$

and also $Y(0)=0$. Therefore, $Y(t)=\ln(X(t))=\int_0^t h(s)\,dB-\frac{1}{2}\int_0^t h(t)^2\,dt$ and so

$$X(t)=\exp\left(\int_0^t h(s)\,dB-\frac{1}{2}\int_0^t h(s)^2\,ds\right)$$

Note the extra term, $-\frac{1}{2}\int_0^t h(s)^2\,ds$.

Example 35.32 *Let $m=n=1$.*

$$dX=f(t)\,Xdt+h(t)\,XdB,\ X(0)=1.$$

In this case it is a lot like the above example but it has an extra $f(t)Xdt$. This suggests something useful might be obtained by letting $Y = \ln(X)$ as was done earlier. Thus

$$\begin{aligned} dY &= \frac{1}{X}(Xf(t)dt + h(t)XdB) + \frac{1}{2}\left(\frac{-1}{X^2}\right)(f(t)dt + h(t)XdB)^2 \\ &= \frac{1}{X}(Xf(t)dt + h(t)XdB) + \frac{1}{2}\left(\frac{-1}{X^2}\right)h(t)^2 X^2 dB^2 \\ &= \frac{1}{X}\left(Xf(t)dt - \frac{1}{2}Xh(t)^2 dt + h(t)XdB\right) \\ &= \left(f(t) - \frac{1}{2}h(t)^2\right)dt + h(t)dB \end{aligned}$$

and so $\ln(X) = \int_0^t \left(f(s) - \frac{1}{2}h(s)^2\right)ds + \int_0^t h(s)dB$ and so

$$X(t) = \exp\left(\int_0^t \left(f(s) - \frac{1}{2}h(s)^2\right)ds + \int_0^t h(s)dB\right)$$

The next example is a model for stock prices. Learn this model and get rich.

Example 35.33 *For $P(t)$ the price of stock,*

$$dP = \mu P dt + \sigma P dB$$

In this model, μ is called the drift and σ is called the volatility.

It is just a special case of the above model in which $f(t) = \mu$ and $h(t) = \sigma$. Then from the above,

$$P(t) = \exp\left(t\mu - \frac{1}{2}t\sigma^2 + \sigma B_t\right)$$

Example 35.34 *This example is called the Brownian bridge.*

$$dX = \frac{-X}{1-t} + dB,\ X(0) = 0.$$

This is also a special case in which $f(t) = 1/(t-1)$ and $h(t) = 1$. Thus the solution is

$$\begin{aligned} X(t) &= \exp\left(\int_0^t \left(\frac{1}{t-1} - \frac{1}{2}\right)ds + B_t\right) \\ &= \exp\left(\frac{1}{2}\int_0^t \left(\frac{t-3}{t-1}\right)ds + B_t\right) \\ &= \exp\left(\frac{1}{2}t\left(\frac{t-3}{t-1}\right) + B_t\right) \end{aligned}$$

Before doing another example I will give a simple lemma on integration by parts. In this lemma $\mathbf{B}$ will denote m dimensional Brownian motion.

Lemma 35.35 *Let $(t,\omega) \to \mathbf{g}(t,\omega)$ be an $\mathcal{G}_t$ adapted measurable function such that*

$$P\left(\int_0^t |\mathbf{g}(s,\omega)|^2 \, ds < \infty\right) = 1$$

where $\mathbf{B}_t$ is a martingale with respect to the filtration $\mathcal{G}_t$ and the increments, $\mathbf{B}_s - \mathbf{B}_t$ for $s > t$ are independent of $\mathcal{G}_t$ so that the Itô integral, $\int_0^t \mathbf{g}^T d\mathbf{B}$ is defined. Suppose also that $t \to \mathbf{g}(t,\omega)$ is C^1 and $\mathbf{B}_0 = \mathbf{0}$. Then

$$\int_0^t \mathbf{g}^T(s,\omega)\, d\mathbf{B} = \mathbf{g}^T(t,\omega)\, \mathbf{B}_t(\omega) - \int_0^t \frac{\partial \mathbf{g}^T}{\partial t}(s,\omega)\, \mathbf{B}(s)\, ds \ \text{a.e.}$$

Proof: Let $\mathbf{g}_n(t) \equiv \sum_{k=0}^{n-1} \mathbf{g}(t_k)\, \mathcal{X}_{[t_k,t_{k+1})}(t)$ where $t_k = k(t/n)$. Then by the definition of the Itô integral,

$$\begin{aligned}
\int_0^t \mathbf{g}^T d\mathbf{B} &= \lim_{n\to\infty} \int_0^t \mathbf{g}_n^T d\mathbf{B} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \mathbf{g}(t_k)^T \left(\mathbf{B}_{t_{k+1}} - \mathbf{B}_{t_k}\right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \mathbf{g}(t_{k-1})^T \mathbf{B}_{t_k} - \sum_{k=0}^{n-1} \mathbf{g}(t_k)^T \mathbf{B}_{t_k} \right) \\
&= \lim_{n\to\infty} \left[\left(\mathbf{g}(t_{n-1})^T \mathbf{B}_t \right) - \sum_{k=1}^{n-1} \left(\mathbf{g}(t_k)^T - \mathbf{g}(t_{k-1})^T \right) \mathbf{B}_{t_k} \right] \\
&= \mathbf{g}(t,\omega)\, \mathbf{B}_t(\omega) - \int_0^t \frac{\partial \mathbf{g}^T}{\partial t}(s,\omega)\, \mathbf{B}(s,\omega)\, ds \ \text{a.e. } \omega.
\end{aligned}$$

Example 35.36 *Linear systems of equations. Here $\mathbf{B}$ is m dimensional Brownian motion. The equation of interest is*

$$d\mathbf{X} = (A\mathbf{X} + \mathbf{h}(t))\, dt + K d\mathbf{B},\ \mathbf{X}(0) = \mathbf{X}_0$$

where $\mathbf{X}_0$ is a random vector in $\mathbb{R}^m$ which is independent of $\mathcal{H}_t$ for all $t \geq 0$ and A, K are constant $m \times m$ matrices. Then I will show

$$\mathbf{X}(t) = e^{At}\left(\mathbf{X}_0 + \int_0^t \left(e^{-As}\mathbf{h}(s) + e^{-As} AK\mathbf{B}(s) \right) ds + e^{-At} K\mathbf{B}(t) \right)$$

Let $\mathbf{Y}(t) = e^{-At}\mathbf{X}(t)$. Then from the above,

$$\begin{aligned}
d\mathbf{Y} &= -Ae^{-At}\mathbf{X} + e^{-At} I d\mathbf{X} + \frac{1}{2} e^{-At} d\mathbf{X}^T 0 d\mathbf{X} \\
&= -Ae^{-At}\mathbf{X} + e^{-At} I \left((A\mathbf{X} + \mathbf{h}(t))\, dt + K d\mathbf{B} \right) \\
&= e^{-At}\mathbf{h}(t)\, dt + e^{-At} K d\mathbf{B}.
\end{aligned}$$

Hence using integration by parts,

$$\begin{aligned}
e^{-At}\mathbf{X}(t) - \mathbf{X}_0 &= \int_0^t e^{-As}\mathbf{h}(s)\, ds + \int_0^t e^{-As} K d\mathbf{B} \\
&= \int_0^t e^{-As}\mathbf{h}(s)\, ds + e^{-At} K\mathbf{B}(t) + A \int_0^t e^{-As} K\mathbf{B}(s)\, ds
\end{aligned}$$

and so

$$\begin{aligned}\mathbf{X}(t) &= e^{At}\left(\mathbf{X}_0 + \int_0^t e^{-As}\mathbf{h}(s)\,ds + e^{-At}K\mathbf{B}(t) + A\int_0^t e^{-As}K\mathbf{B}(s)\,ds\right)\\ &= e^{At}\left(\mathbf{X}_0 + \int_0^t \left(e^{-As}\mathbf{h}(s) + e^{-As}AK\mathbf{B}(s)\right)ds + e^{-At}K\mathbf{B}(t)\right) \quad (35.45)\end{aligned}$$

In this formula e^{-As} is the matrix, $M(t)$ which solves $M' = AM, M(0) = I$. Note that formally differentiating the above equation gives

$$\begin{aligned}\mathbf{X}' &= Ae^{At}\left(\mathbf{X}_0 + \int_0^t \left(e^{-As}\mathbf{h}(s) + e^{-As}AK\mathbf{B}(s)\right)ds + e^{-At}K\mathbf{B}(t)\right)\\ &\quad + e^{At}\left(\left(e^{-At}\mathbf{h}(t) + e^{-At}AK\mathbf{B}(t)\right) - Ae^{-At}K\mathbf{B}(t) + e^{-At}K\frac{d\mathbf{B}}{dt}\right)\end{aligned}$$

and so

$$\begin{aligned}\mathbf{X}' &= A\mathbf{X} + \mathbf{h}(t) + AK\mathbf{B}(t) - AK\mathbf{B}(t) + K\frac{d\mathbf{B}}{dt}\\ &= A\mathbf{X} + \mathbf{h}(t) + K\frac{d\mathbf{B}}{dt}.\end{aligned}$$

Of course this is total nonsense because $\mathbf{B}$ is known to not be differentiable. However, multiplying by dt gives

$$d\mathbf{X} = (A\mathbf{X} + \mathbf{h}(t))\,dt + Kd\mathbf{B}$$

and the formula 35.45 shows $\mathbf{X}(0) = \mathbf{X}_0$. This was the original differential equation. Note that it was not necessary to assume very much about $\mathbf{X}_0$ to write 35.45.

35.5 A Different Proof Of Existence And Uniqueness

The proof given here is much longer and uses Picard Iteration directly. However, it requires much less background material so the over all presentation is shorter.

35.5.1 Gronwall's Inequality

The fundamental tool in estimating differential equations is Gronwall's inequality. It is a very elementary result but of enormous significance. I will first give a proof of this important theorem. Also, I will write $X(t)$ rather than X_t.

Lemma 35.37 *Let $k \geq 0$ and suppose $u(t)$ is a Lebesgue measurable function in $L^1([0,T])$ which satisfies*

$$u(t) \leq u_0 + \int_0^t ku(s)\,ds.$$

Then

$$u(t) \leq u_0 e^{kt}.$$

Proof: Let

$$f(t) = u_0 e^{kt} - \left(u_0 + \int_0^t ku(s)\,ds\right).$$

Then $f(0) = 0$ and

$$\begin{aligned} f'(t) &= ku_0 e^{kt} - ku(t) \\ &\geq ku_0 e^{kt} - k\left(u_0 + \int_0^t ku(s)\,ds\right) \\ &= kf(t) \end{aligned}$$

and so $f'(t) - kf(t) \geq 0$ and so

$$\frac{d}{dt}\left(e^{-kt} f(t)\right) \geq 0$$

which implies $f(t) \geq 0$. Hence

$$u(t) \leq u_0 + \int_0^t ku(s)\,ds \leq u_0 e^{kt}.$$

This proves Gronwall's inequality.

35.5.2 Review Of Itô Integrals

Next recall the definition of the Itô integral. The context is that $\mathcal{H}_t$ is a filtration, B_t is a martingale for $\mathcal{H}_t$, and if $s > t$, $B_s - B_t$ is independent of $\mathcal{H}_t$.

Definition 35.38 *Suppose f is $\mathcal{H}_t$ adapted and $\mathcal{B} \times \mathcal{F}$ measurable such that for $\omega \notin E$ a set of measure zero,*

$$\int_S^T f(t,\omega)^2\,dt < \infty.$$

Then there exists a sequence of adapted bounded step functions, $\{\phi_n\}$ satisfying

$$\int_S^T \left(f(t,\omega) - \phi_n(t,\omega)\right)^2 dt \leq 2^{-n}$$

for $\omega \notin E$, a set of measure zero. Then for $t \in [S,T]$, the Itô integral is defined by

$$\int_S^t f dB(\omega) = \lim_{n\to\infty} \int_S^t \phi_n dB(\omega).$$

Furthermore, for these ω, $t \to \int_S^t f dB(\omega)$ is continuous because by Theorem 34.5 the convergence of $\int_S^t \phi_n dB(\omega)$ is uniform on $[0,T]$.

In what follows $\mathbf{B}_t$ will be m dimensional Brownian motion and the filtration will be denoted by $\mathcal{H}_t$ where $\mathcal{H}_t$ is the completion of the smallest σ algebra which contains

$$(\mathbf{B}_{t_0}, \cdot\cdot\cdot, \mathbf{B}_{t_k})^{-1}(U)$$

whenever $0 \leq t_0 < \cdot\cdot\cdot < t_k \leq t$ and U is a Borel set. Also, $\mathbf{Z}$, a measurable $\mathbb{R}^n$ valued function will be independent of $\mathcal{H}_t$ for all $t > 0$. Then $\mathcal{H}_t^Z$ will denote the completion of the smallest σ algebra which contains

$$\left(\mathbf{Z}, \mathbf{B}_{t_0}, \cdot\cdot\cdot, \mathbf{B}_{t_k}\right)^{-1}(U)$$

whenever $0 \leq t_0 < \cdot\cdot\cdot < t_k \leq t$ and U is a Borel set. Then the following lemma is what is needed to consider certain Itô integrals.

Lemma 35.39 *$\mathbf{B}_t$ is an $\mathcal{H}_t^Z$ martingale and if $s > t$, the increments $\mathbf{B}_s - \mathbf{B}_t$ are independent of $\mathcal{H}_t^Z$.*

Proof: Let A, U_k, V for $k = 0, \cdot\cdot\cdot, p$ be open sets and let $s > t$ and

$$D = \left(\mathbf{Z}, \mathbf{B}_{t_0}, \cdot\cdot\cdot, \mathbf{B}_{t_p}\right)^{-1}(A \times U_0 \times \cdot\cdot\cdot \times U_p), E = (\mathbf{B}_s - \mathbf{B}_t)^{-1}(V).$$

I need to verify that $P(D \cap E) = P(D) P(E)$.

$$D \cap E = \mathbf{Z}^{-1}(A) \cap \cap_{i=0}^p \mathbf{B}_{t_i}^{-1}(U_i) \cap E$$

From independence of $\mathbf{Z}$ to $\mathcal{H}_t$ for all $t > 0$, and independence of the increments, $\mathbf{B}_s - \mathbf{B}_t$ to $\mathcal{H}_t$,

$$\begin{aligned} P(D \cap E) &= P\left(\mathbf{Z}^{-1}(A)\right) P\left(\cap_{i=0}^p \mathbf{B}_{t_i}^{-1}(U_i) \cap E\right) \\ &= P\left(\mathbf{Z}^{-1}(A)\right) P\left(\cap_{i=0}^p \mathbf{B}_{t_i}^{-1}(U_i)\right) P(E) \\ &= P\left(\mathbf{Z}^{-1}(A) \cap \cap_{i=0}^p \mathbf{B}_{t_i}^{-1}(U_i)\right) P(E) \\ &= P(D) P(E). \end{aligned}$$

It follows that for all D an inverse image of an open set and E of the above form where V is open, $P(D \cap E) = P(D) P(E)$. It follows easily this holds for all D and E inverse images of Borel sets. If $D \in \mathcal{H}_t$ and $E = (\mathbf{B}_s - \mathbf{B}_t)^{-1}(V)$ then there exists D_1 an inverse image of a Borel set such that $D_1 \supseteq D$ and $P(D_1 \setminus D) = 0$ so

$$\begin{aligned} P(D \cap E) &= P(D_1 \cap E) \\ &= P(D_1) P(E) = P(D) P(E). \end{aligned}$$

This verifies the independence.

Now let $A \in \mathcal{H}_t^Z$. Then from the above independence result,

$$\begin{aligned} \int_A E\left(\mathbf{B}_s - \mathbf{B}_t | \mathcal{H}_t^Z\right) dP &= \int_A \mathbf{B}_s - \mathbf{B}_t dP \\ &= \int_A dP \int_\Omega \mathbf{B}_s - \mathbf{B}_t dP = 0 \end{aligned}$$

and so

$$\begin{aligned} E\left(\mathbf{B}_s|\mathcal{H}_t^Z\right) &= E\left(\mathbf{B}_s - \mathbf{B}_t + \mathbf{B}_t|\mathcal{H}_t^Z\right) \\ &= \mathbf{0} + E\left(\mathbf{B}_t|\mathcal{H}_t^Z\right) = \mathbf{B}_t. \end{aligned}$$

This proves the lemma.

For $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{b}(t, \mathbf{x}) \in \mathbb{R}^m$ and $\sigma(t, \mathbf{x})$ will be an $n \times m$ matrix. It is assumed that for given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ the following measurability and Lipschitz conditions hold.

$$t \to \mathbf{b}(t, \mathbf{x}), t \to \sigma(t, \mathbf{x}) \text{ are Lebesgue measurable,} \tag{35.46}$$

$$|\mathbf{b}(t, \mathbf{x}) - \mathbf{b}(t, \mathbf{y})| + |\sigma(t, \mathbf{x}) - \sigma(t, \mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|, \tag{35.47}$$

$$|\mathbf{b}(t, \mathbf{x})| + |\sigma(t, \mathbf{x})| \leq C(1 + |\mathbf{x}|). \tag{35.48}$$

In the above, it suffices to have the components of $\mathbf{b}$ and σ measurable. Also,the norm refers to any convenient norm. This does not matter because all the norms on a finite dimensional vector space are equivalent.

Definition 35.40 *Let $\mathcal{G}_t$ be a filtration for which $\mathbf{B}_t$ is a martingale and such that for $s > t, \mathbf{B}_s - \mathbf{B}_t$ is independent of $\mathcal{G}_t$. For $\mathbf{X}$ product measurable in $\mathcal{B} \times \mathcal{F}$ and $\mathcal{G}_t$ adapted define*

$$\left(\int_0^t \sigma(s, \mathbf{X})\, d\mathbf{B}\right)_k \equiv \int_0^t (\sigma(s, \mathbf{X}))_k\, d\mathbf{B}$$

where $(\sigma(s, \mathbf{X}))_k$ denotes the k^{th} row of the matrix, σ.

35.5.3 The Existence And Uniqueness Theorem

Theorem 35.41 *Let $\mathbf{b}$ and σ satisfy 35.46 - 35.48. Let $\mathbf{Z}$ be a random vector which is either independent of $\mathcal{H}_t$ for all $t > 0$ or else is measurable with respect to $\mathcal{H}_t$ for all t and suppose*

$$\int_\Omega |\mathbf{Z}|^2\, dP < \infty$$

Let $\mathcal{G}_t = \mathcal{H}_t^Z$ in the first case and let $\mathcal{G}_t = \mathcal{H}_t$ in the second. Then there exists a solution, $\mathbf{X}$ to the integral equation,

$$\mathbf{X}(t) = \mathbf{Z} + \int_0^t \mathbf{b}(s, \mathbf{X}(s))\, ds + \int_0^t \sigma(s, \mathbf{X}(s))\, d\mathbf{B} \text{ a.e. } \omega$$

This solution satisfies $\mathbf{X} \in L^2([0, T] \times \Omega)^n$.

Proof: Let iterates be defined as follows.

$$\mathbf{X}^1(t) \equiv \mathbf{Z} + \int_0^t \mathbf{b}(s, \mathbf{Z})\, ds + \int_0^t \sigma(s, \mathbf{Z})\, d\mathbf{B}$$

The Itô integral on the right is well defined for all ω not in some set of measure zero because $\mathbf{Z}$ is $\mathcal{G}_t$ adapted. Now also $\mathbf{X}^1$ is $\mathcal{G}_t$ adapted because both integrals in the above yield $\mathcal{G}_t$ adapted functions of t by Theorem 34.8 on Page 925 and the Itô integral yields $\mathcal{B}\times\mathcal{F}$ measurable function by Corollary 31.43 on Page 890 and the convention mentioned after this corollary. Then

$$\mathbf{X}^2(t)\equiv\mathbf{Z}+\int_0^t\mathbf{b}\left(s,\mathbf{X}^1(s)\right)ds+\int_0^t\sigma\left(s,\mathbf{X}^1(s)\right)d\mathbf{B}.$$

Continue this way. Each iteration involves a set of measure zero. Take the union of all these sets, N. Then for $\omega\notin N$

$$\mathbf{X}^{k+1}(t)\equiv\mathbf{Z}+\int_0^t\mathbf{b}\left(s,\mathbf{X}^k(s)\right)ds+\int_0^t\sigma\left(s,\mathbf{X}^k(s)\right)d\mathbf{B}\tag{35.49}$$

and each $\mathbf{X}^k$ is $\mathcal{G}_t$ adapted and $\mathcal{B}\times\mathcal{F}$ measurable. Now

$$\begin{aligned}&\int_\Omega\left|\mathbf{X}^{k+1}(t)-\mathbf{X}^k(t)\right|^2dP\\ \leq\;& TK\int_\Omega\int_0^t\left|\mathbf{X}^k(s)-\mathbf{X}^{k-1}(s)\right|^2ds\\ &+\int_\Omega\left|\int_0^t\sigma\left(s,\mathbf{X}^k(s)\right)-\sigma\left(s,\mathbf{X}^{k-1}(s)\right)d\mathbf{B}\right|^2dP.\end{aligned}$$

Using the Itô isometry,

$$\begin{aligned}\leq\;& TK\int_0^t\int_\Omega\left|\mathbf{X}^k(s)-\mathbf{X}^{k-1}(s)\right|^2dPds\\ &+\int_\Omega\int_0^t\left|\sigma\left(s,\mathbf{X}^k(s)\right)-\sigma\left(s,\mathbf{X}^{k-1}(s)\right)\right|^2dsdP\\ \leq\;& TK\int_0^t\int_\Omega\left|\mathbf{X}^k(s)-\mathbf{X}^{k-1}(s)\right|^2dPds\\ &+K\int_0^t\int_\Omega\left|\mathbf{X}^k(s)-\mathbf{X}^{k-1}(s)\right|^2dPds\\ \leq\;& C_T\int_0^t\int_\Omega\left|\mathbf{X}^k(s)-\mathbf{X}^{k-1}(s)\right|^2dPds\end{aligned}$$

where C_T is a constant depending on T and K. Then iterating this inequality yields

$$\begin{aligned}&\int_\Omega\left|\mathbf{X}^{k+1}(t)-\mathbf{X}^k(t)\right|^2dP\\ \leq\;& C_T^k\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}\int_\Omega\left|\mathbf{X}^1(t_k)-\mathbf{Z}\right|^2dPdt_k\cdots dt_1.\end{aligned}$$

Now

$$\begin{aligned}&\int_\Omega \left|\mathbf{X}^1(t_k)-\mathbf{Z}\right|^2 dP\\ \leq\;& C_T\left(\int_0^T\int_\Omega\left(1+|\mathbf{Z}|^2\right)dPdt\right)\equiv C_{\mathbf{Z}}<\infty.\end{aligned}$$

Then it follows

$$\begin{aligned}\left(\int_\Omega\left|\mathbf{X}^{k+1}(t)-\mathbf{X}^k(t)\right|^2dP\right)^{1/2} &\leq \left(C_T^kC_{\mathbf{Z}}\int_0^t\int_0^{t_1}\cdots\int_0^{t_{k-1}}dt_k\cdots dt_1\right)^{1/2}\\ &\leq \left(C_T^kC_{\mathbf{Z}}\frac{t^k}{k!}\right)^{1/2}\leq\left(C_T^kC_{\mathbf{Z}}\frac{T^k}{k!}\right)^{1/2}.\end{aligned}$$

Since $\sum_{k=0}^\infty\left(C_T^kC_{\mathbf{Z}}\frac{T^k}{k!}\right)^{1/2}<\infty$, it follows $\left\{\mathbf{X}^k\right\}$ converges in $C\left([0,T];L^2(\Omega)^n\right)$ to a function, $\widetilde{\mathbf{X}}$. It follows $\left\{\mathbf{X}^k\right\}$ is also Cauchy in $L^2([0,T]\times\Omega)^n$. Therefore, there exists $\mathbf{X}$, $\mathcal{B}\times\mathcal{F}$ measurable and in $L^2([0,T]\times\Omega)$ such that upon taking a suitable subsequence still denoted by k, $\left\{\mathbf{X}^k\right\}$ converges to $\mathbf{X}$ pointwise and in $L^2([0,T]\times\Omega)^n$. The function, $t\to\int_\Omega\left|\mathbf{X}-\mathbf{X}^k\right|^2dP$ is Lebesgue measurable and $t\to\int_\Omega\left|\mathbf{X}-\widetilde{\mathbf{X}}\right|^2dP$ is the limit so it is also Lebesgue measurable. Also,

$$\int_0^T\int_\Omega\left|\mathbf{X}-\widetilde{\mathbf{X}}\right|^2dPdt\leq 2\left(\int_0^T\int_\Omega\left|\mathbf{X}-\mathbf{X}^k\right|^2dPdt+\int_0^T\int_\Omega\left|\mathbf{X}^k-\widetilde{\mathbf{X}}\right|^2dPdt\right)$$

and both of these integrals converge to 0 as $k\to\infty$, the first because of convergence in $L^2([0,T]\times\Omega)^n$ and the second because of the uniform convergence of $\mathbf{X}^k$ to $\widetilde{\mathbf{X}}$. Therefore, $\mathbf{X}(t)=\widetilde{\mathbf{X}}(t)$ in $L^2(\Omega)^n$ for a.e. t. Changing $\mathbf{X}(t)$ on this exceptional set by setting it equal to 0, it follows $\mathbf{X}$ is product measurable and adapted because it either is identically 0 on a set of measure zero or $\mathbf{X}(t)=\widetilde{\mathbf{X}}(t)$ and it is clear $\widetilde{\mathbf{X}}$ is adapted.

Now recall 35.49. Consider the Itô integral. By the Itô isometry and the Lipschitz property of σ, $\left\{\int_0^t\sigma\left(s,\mathbf{X}^k(s)\right)d\mathbf{B}\right\}$ converges to $\int_0^t\sigma(s,\mathbf{X}(s))\,d\mathbf{B}$ in $L^2(\Omega)^n$ for each t. Using the Lipschitz property of $\mathbf{b}$, and passing to the limit in 35.49 the following equation must hold in $L^2(\Omega)^n$.

$$\widetilde{\mathbf{X}}(t)\equiv\mathbf{Z}+\int_0^t\mathbf{b}(s,\mathbf{X}(s))\,ds+\int_0^t\sigma(s,\mathbf{X}(s))\,d\mathbf{B}\tag{35.50}$$

For a.e. ω, the right side of the above is a continuous function of t. This is true of the Itô integral and it also follows for the deterministic integral because of the observation that for a.e. $\omega, s\to\mathbf{X}(s)(\omega)$ is in $L^2(0,T)$. For ω not in this exceptional set of measure zero, define

$$\mathbf{Y}(t)(\omega)\equiv\mathbf{Z}+\int_0^t\mathbf{b}(s,\mathbf{X}(s)(\omega))\,ds+\int_0^t\sigma(s,\mathbf{X}(s))\,d\mathbf{B}(\omega)\tag{35.51}$$

Thus $\mathbf{Y}(t)$ is adapted by Theorem 34.8 on Page 925 and $\mathbf{Y}$ is product measurable. Also it follows from 35.50 that $\mathbf{Y}(t) = \widetilde{\mathbf{X}}(t)$ in $L^2(\Omega)^n$ and $\mathbf{X}(t) = \widetilde{\mathbf{X}}(t)$ for a.e. t so $\mathbf{Y}(t) = \mathbf{X}(t)$ a.e. t. It follows that

$$\mathbf{Y}(t) \equiv \mathbf{Z} + \int_0^t \mathbf{b}(s, \mathbf{Y}(s))\, ds + \int_0^t \sigma(s, \mathbf{Y}(s))\, d\mathbf{B} \tag{35.52}$$

holds in $L^2(\Omega)^n$ for each t and so equality is also true in $L^2([0,T] \times \Omega)^n$. As before, the right side is a continuous function of t for ω off a set of measure zero. Off a set of measure zero, $t \to \mathbf{Y}(t)(\omega)$ is also continuous. This follows from the definition of $\mathbf{Y}(t)$ in 35.51. Since both sides are product measurable, there exists a set of measure zero, N such that for $\omega \notin N$,

$$\int_0^T \left| \mathbf{Y}(t) - \left(\mathbf{Z} + \int_0^t \mathbf{b}(s, \mathbf{Y}(s))\, ds + \int_0^t \sigma(s, \mathbf{Y}(s))\, d\mathbf{B} \right) \right|^2 dt = 0$$

and the integrand is a continuous function. Therefore, 35.52 holds a.e. and both sides are continuous for ω not in a suitable set of measure zero. This proves the theorem.

Note there were two cases given for the initial condition in the above theorem. The second is not very interesting. If $\mathbf{Z}$ is $\mathcal{H}_0$ measurable, then since $\mathbf{B}_0 = \mathbf{x}$, a constant, it follows $\mathcal{H}_0 = \{\emptyset, \Omega\}$ so $\mathbf{Z}$ is a constant. However, if $\mathbf{Z}$ is a constant, then it satisfies the first condition.

Not surprisingly, the solution to the above theorem is unique. This is stated as the following corollary which is the main result.

Corollary 35.42 *Let* $\mathbf{b}$ *and* σ *satisfy 35.46 - 35.48. Let* $\mathbf{Z}$ *be a random vector which is independent of* $\mathcal{H}_t$ *for all* $t > 0$ *and suppose*

$$\int_\Omega |\mathbf{Z}|^2\, dP < \infty$$

Then there exists a unique $\mathcal{H}_t^Z$ *adapted solution,* $\mathbf{X} \in L^2([0,T] \times \Omega)^n$ *to the integral equation,*

$$\mathbf{X}(t) = \mathbf{Z} + \int_0^t \mathbf{b}(s, \mathbf{X}(s))\, ds + \int_0^t \sigma(s, \mathbf{X}(s))\, d\mathbf{B} \text{ a.e. } \omega \tag{35.53}$$

in the sense that if $\widetilde{\mathbf{X}}$ *is another solution, then there exists a set of measure zero,* N *such that for* $\omega \notin N$, $\widetilde{\mathbf{X}}(t) = \mathbf{X}(t)$ *for all* $t \in [0,T]$.

Proof: The existence part of this proof is already done. Let N denote the union of the exceptional sets corresponding to $\mathbf{X}$ and $\widetilde{\mathbf{X}}$. Then from 35.53 and the various assumptions on $\mathbf{b}$ and σ,it follows that for $\omega \notin N$,

$$\begin{aligned} \left| \mathbf{X}(t) - \widetilde{\mathbf{X}}(t) \right|^2 &\leq 2K^2 T \int_0^t \left| \mathbf{X}(s) - \widetilde{\mathbf{X}}(s) \right|^2 ds \\ &\quad + 2 \left| \int_0^t \left(\sigma(s, \mathbf{X}(s)) - \sigma\left(s, \widetilde{\mathbf{X}}(s)\right) \right) d\mathbf{B} \right|^2. \end{aligned}$$

Then by the Itô isometry, this implies

$$\begin{aligned}&\left|\left|\mathbf{X}(t)-\widetilde{\mathbf{X}}(t)\right|\right|^2_{L^2(\Omega)^n}\\ \leq\ & 2K^2T\int_0^t\left|\left|\mathbf{X}(s)-\widetilde{\mathbf{X}}(s)\right|\right|^2_{L^2(\Omega)^n}ds\\ &+2\int_0^t\left|\left|\sigma(s,\mathbf{X}(s))-\sigma\left(s,\widetilde{\mathbf{X}}(s)\right)\right|\right|^2 ds\\ \leq\ & C_T\int_0^t\left|\left|\mathbf{X}(s)-\widetilde{\mathbf{X}}(s)\right|\right|^2_{L^2(\Omega)^n}ds \qquad (35.54)\end{aligned}$$

and by assumption both $\mathbf{X}$ and $\widetilde{\mathbf{X}}$ are in $L^2([0,T]\times\Omega)^n$ so $t\to\left|\left|\mathbf{X}(t)-\widetilde{\mathbf{X}}(t)\right|\right|^2_{L^2(\Omega)^n}$ is in $L^1([0,T])$. By Gronwall's inequality, $\widetilde{\mathbf{X}}(t)=\mathbf{X}(t)$ in $L^2(\Omega)^n$ for all t. It follows there exists a set of measure zero, N_1 such that for $\omega\notin N_1$,

$$\int_0^T\left|\widetilde{\mathbf{X}}(t)-\mathbf{X}(t)\right|^2 dt=0$$

But for $\omega\notin N$, the functions, $\widetilde{\mathbf{X}}$ and $\mathbf{X}$ are continuous and so if $\omega\notin N_1\cup N$, $\widetilde{\mathbf{X}}(t)=\mathbf{X}(t)$ for all t. This proves the corollary.

Note that if different initial conditions had been given, say $\mathbf{Z}$ and $\mathbf{Z}_1$, the above argument for uniqueness also gives a continuous dependence result with no effort. In fact, 35.54 then would take the form

$$\left|\left|\mathbf{X}(t)-\widetilde{\mathbf{X}}(t)\right|\right|^2_{L^2(\Omega)^n}\leq 3\left|\left|\mathbf{Z}-\mathbf{Z}_1\right|\right|^2_{L^2(\Omega)^n}+C_T\int_0^t\left|\left|\mathbf{X}(s)-\widetilde{\mathbf{X}}(s)\right|\right|^2_{L^2(\Omega)^n}ds$$

and Gronwall's inequality would then imply

$$\left|\left|\mathbf{X}(t)-\widetilde{\mathbf{X}}(t)\right|\right|^2_{L^2(\Omega)^n}\leq C\left|\left|\mathbf{Z}-\mathbf{Z}_1\right|\right|^2_{L^2(\Omega)^n}$$

for some constant, C.

The equivalent form of the above integral equation,

$$\mathbf{X}(t)=\mathbf{Z}+\int_0^t\mathbf{b}(s,\mathbf{X}(s))\,ds+\int_0^t\sigma(s,\mathbf{X}(s))\,d\mathbf{B}$$

is

$$d\mathbf{X}=\mathbf{b}(t,\mathbf{X})\,dt+\sigma(t,\mathbf{X}(t))\,d\mathbf{B},\ \mathbf{X}(0)=\mathbf{Z}.$$

35.5.4 Some Simple Examples

Here are some examples of simple stochastic differential equations which are solved using the Itô formula.

Example 35.43 *In this example, $m = n = 1$ and B is one dimensional Brownian motion. The differential equation is*

$$dX = h(t) X dB, \ X(0) = 1$$

Obviously, one would want to do something like $\frac{dX}{X} = h(t) dB$. However, you have to follow the rules. Let $g(x) = \ln(x)$ and $Y = g(X)$. Then by the Itô formula,

$$\begin{aligned} dY &= \frac{1}{X} dX + \frac{1}{2}\left(\frac{-1}{X^2}\right) dX^2 \\ &= \frac{1}{X} h(t) X dB - \frac{1}{2}\frac{1}{X^2} h(t)^2 X^2 dB^2 \\ &= h(t) dB - \frac{1}{2} h(t)^2 dt \end{aligned}$$

and also $Y(0) = 0$. Therefore, $Y(t) = \ln(X(t)) = \int_0^t h(s) dB - \frac{1}{2}\int_0^t h(t)^2 dt$ and so

$$X(t) = \exp\left(\int_0^t h(s) dB - \frac{1}{2}\int_0^t h(s)^2 ds\right)$$

Note the extra term, $-\frac{1}{2}\int_0^t h(s)^2 ds$.

Example 35.44 *Let $m = n = 1$.*

$$dX = f(t) X dt + h(t) X dB, \ X(0) = 1.$$

In this case it is a lot like the above example but it has an extra $f(t) X dt$. This suggests something useful might be obtained by letting $Y = \ln(X)$ as was done earlier. Thus

$$\begin{aligned} dY &= \frac{1}{X}(X f(t) dt + h(t) X dB) + \frac{1}{2}\left(\frac{-1}{X^2}\right)(f(t) dt + h(t) X dB)^2 \\ &= \frac{1}{X}(X f(t) dt + h(t) X dB) + \frac{1}{2}\left(\frac{-1}{X^2}\right) h(t)^2 X^2 dB^2 \\ &= \frac{1}{X}\left(X f(t) dt - \frac{1}{2} X h(t)^2 dt + h(t) X dB\right) \\ &= \left(f(t) - \frac{1}{2} h(t)^2\right) dt + h(t) dB \end{aligned}$$

and so $\ln(X) = \int_0^t \left(f(s) - \frac{1}{2} h(s)^2\right) ds + \int_0^t h(s) dB$ and so

$$X(t) = \exp\left(\int_0^t \left(f(s) - \frac{1}{2} h(s)^2\right) ds + \int_0^t h(s) dB\right)$$

The next example is a model for stock prices. Learn this model and get rich.

Example 35.45 *For $P(t)$ the price of stock,*

$$dP = \mu P dt + \sigma P dB$$

In this model, μ is called the drift and σ is called the volatility.

It is just a special case of the above model in which $f(t) = \mu$ and $h(t) = \sigma$. Then from the above,

$$P(t) = \exp\left(t\mu - \frac{1}{2}t\sigma^2 + \sigma B_t\right)$$

Example 35.46 *This example is called the Brownian bridge.*

$$dX = \frac{-X}{1-t} + dB,\ X(0) = 0.$$

This is also a special case in which $f(t) = 1/(t-1)$ and $h(t) = 1$. Thus the solution is

$$\begin{aligned} X(t) &= \exp\left(\int_0^t \left(\frac{1}{t-1} - \frac{1}{2}\right) ds + B_t\right) \\ &= \exp\left(\frac{1}{2}\int_0^t \left(\frac{t-3}{t-1}\right) ds + B_t\right) \\ &= \exp\left(\frac{1}{2}t\left(\frac{t-3}{t-1}\right) + B_t\right) \end{aligned}$$

Before doing another example I will give a simple lemma on integration by parts. In this lemma $\mathbf{B}$ will denote m dimensional Brownian motion.

Lemma 35.47 *Let $(t,\omega) \to \mathbf{g}(t,\omega)$ be an $\mathcal{G}_t$ adapted measurable function such that*

$$P\left(\int_0^t |\mathbf{g}(s,\omega)|^2 ds < \infty\right) = 1$$

where $\mathbf{B}_t$ is a martingale with respect to the filtration $\mathcal{G}_t$ and the increments, $\mathbf{B}_s - \mathbf{B}_t$ for $s > t$ are independent of $\mathcal{G}_t$ so that the Itô integral, $\int_0^t \mathbf{g}^T d\mathbf{B}$ is defined. Suppose also that $t \to \mathbf{g}(t,\omega)$ is C^1 and $\mathbf{B}_0 = \mathbf{0}$. Then

$$\int_0^t \mathbf{g}^T(s,\omega)\, d\mathbf{B} = \mathbf{g}^T(t,\omega)\,\mathbf{B}_t(\omega) - \int_0^t \frac{\partial \mathbf{g}^T}{\partial t}(s,\omega)\,\mathbf{B}(s)\, ds \ \ a.e.$$

Proof: Let $\mathbf{g}_n(t) \equiv \sum_{k=0}^{n-1} \mathbf{g}(t_k) \mathcal{X}_{[t_k,t_{k+1})}(t)$ where $t_k = k(t/n)$. Then by the definition of the Itô integral,

$$\begin{aligned}
\int_0^t \mathbf{g}^T d\mathbf{B} &= \lim_{n\to\infty} \int_0^t \mathbf{g}_n^T d\mathbf{B} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \mathbf{g}(t_k)^T \left(\mathbf{B}_{t_{k+1}} - \mathbf{B}_{t_k}\right) \\
&= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \mathbf{g}(t_{k-1})^T \mathbf{B}_{t_k} - \sum_{k=0}^{n-1} \mathbf{g}(t_k)^T \mathbf{B}_{t_k} \right) \\
&= \lim_{n\to\infty} \left[\left(\mathbf{g}(t_{n-1})^T \mathbf{B}_t \right) - \sum_{k=1}^{n-1} \left(\mathbf{g}(t_k)^T - \mathbf{g}(t_{k-1})^T \right) \mathbf{B}_{t_k} \right] \\
&= \mathbf{g}(t,\omega) \mathbf{B}_t(\omega) - \int_0^t \frac{\partial \mathbf{g}^T}{\partial t}(s,\omega) \mathbf{B}(s,\omega)\, ds \text{ a.e. } \omega.
\end{aligned}$$

Example 35.48 *Linear systems of equations. Here $\mathbf{B}$ is m dimensional Brownian motion. The equation of interest is*

$$d\mathbf{X} = (A\mathbf{X} + \mathbf{h}(t))\, dt + K d\mathbf{B},\ \mathbf{X}(0) = \mathbf{X}_0$$

where $\mathbf{X}_0$ is a random vector in $\mathbb{R}^m$ which is independent of $\mathcal{H}_t$ for all $t \geq 0$ and A, K are constant $m \times m$ matrices. Then I will show

$$\mathbf{X}(t) = e^{At} \left(\mathbf{X}_0 + \int_0^t \left(e^{-As}\mathbf{h}(s) + e^{-As} AK\mathbf{B}(s) \right) ds + e^{-At} K\mathbf{B}(t) \right)$$

Let $\mathbf{Y}(t) = e^{-At}\mathbf{X}(t)$. Then from the above,

$$\begin{aligned}
d\mathbf{Y} &= -Ae^{-At}\mathbf{X} + e^{-At} I d\mathbf{X} + \frac{1}{2} e^{-At} d\mathbf{X}^T 0 d\mathbf{X} \\
&= -Ae^{-At}\mathbf{X} + e^{-At} I \left((A\mathbf{X} + \mathbf{h}(t))\, dt + K d\mathbf{B} \right) \\
&= e^{-At}\mathbf{h}(t)\, dt + e^{-At} K d\mathbf{B}.
\end{aligned}$$

Hence using integration by parts,

$$\begin{aligned}
e^{-At}\mathbf{X}(t) - \mathbf{X}_0 &= \int_0^t e^{-As}\mathbf{h}(s)\, ds + \int_0^t e^{-As} K d\mathbf{B} \\
&= \int_0^t e^{-As}\mathbf{h}(s)\, ds + e^{-At} K\mathbf{B}(t) + A \int_0^t e^{-As} K\mathbf{B}(s)\, ds
\end{aligned}$$

and so

$$\begin{aligned}
\mathbf{X}(t) &= e^{At} \left(\mathbf{X}_0 + \int_0^t e^{-As}\mathbf{h}(s)\, ds + e^{-At} K\mathbf{B}(t) + A \int_0^t e^{-As} K\mathbf{B}(s)\, ds \right) \\
&= e^{At} \left(\mathbf{X}_0 + \int_0^t \left(e^{-As}\mathbf{h}(s) + e^{-As} AK\mathbf{B}(s) \right) ds + e^{-At} K\mathbf{B}(t) \right) \quad (35.55)
\end{aligned}$$

In this formula e^{-As} is the matrix, $M(t)$ which solves $M' = AM, M(0) = I$. Note that formally differentiating the above equation gives

$$\begin{aligned}\mathbf{X}' &= Ae^{At}\left(\mathbf{X}_0 + \int_0^t \left(e^{-As}\mathbf{h}(s) + e^{-As}AK\mathbf{B}(s)\right) ds + e^{-At}K\mathbf{B}(t)\right) \\ &\quad + e^{At}\left(\left(e^{-At}\mathbf{h}(t) + e^{-At}AK\mathbf{B}(t)\right) - Ae^{-At}K\mathbf{B}(t) + e^{-At}K\frac{d\mathbf{B}}{dt}\right)\end{aligned}$$

and so

$$\begin{aligned}\mathbf{X}' &= A\mathbf{X} + \mathbf{h}(t) + AK\mathbf{B}(t) - AK\mathbf{B}(t) + K\frac{d\mathbf{B}}{dt} \\ &= A\mathbf{X} + \mathbf{h}(t) + K\frac{d\mathbf{B}}{dt}.\end{aligned}$$

Of course this is total nonsense because $\mathbf{B}$ is known to not be differentiable. However, multiplying by dt gives

$$d\mathbf{X} = (A\mathbf{X} + \mathbf{h}(t))\, dt + Kd\mathbf{B}$$

and the formula 35.55 shows $\mathbf{X}(0) = \mathbf{X}_0$. This was the original differential equation. Note that it was not necessary to assume very much about $\mathbf{X}_0$ to write 35.55.

Probability In Infinite Dimensions

I am following the book by Da Prato and Zabczyk for much of this material. [15].

36.1 Expected Value Covariance And Correlation

Let $(\Omega, \mathcal{F}, P)$ be a probability space. First recall the notion of expected value for a scalar valued random variable, X denoted by

$$E(X) \equiv \int_{\Omega} X(\omega)\, dP = \int_{\mathbb{R}} x d\lambda_X$$

where λ_X is a Radon measure which satisfies

$$\lambda_X(G) \equiv P(X(\omega) \in G).$$

To speak of the expected value, it is necessary that $X \in L^1(\Omega; \mathbb{R})$. Now the variance is defined as

$$E\left((X - E(X))^2\right) \equiv \int_{\Omega} (X(\omega) - E(X))^2\, dP$$

and it is necessary that $X \in L^2(\Omega; \mathbb{R})$.

What about random vectors where $\mathbf{X}$ has values in $\mathbb{R}^p$? In this case, the expected value would be a vector in $\mathbb{R}^p$ given by

$$E(\mathbf{X}) \equiv \int_{\Omega} \mathbf{X}(\omega)\, dP$$

and the thing which takes the place of the variance is the covariance. This is a linear transformation mapping $\mathbb{R}^p$ to $\mathbb{R}^p$ just as the variance could be considered a linear transformation mapping $\mathbb{R}$ to $\mathbb{R}$. The covariance is defined as

$$E\left((\mathbf{X}-E(\mathbf{X}))(\mathbf{X}-E(\mathbf{X}))^*\right)$$

Written in terms of the tensor product, this is

$$E\left(\left(\mathbf{X}-E\left(\mathbf{X}\right)\right)\otimes\left(\mathbf{X}-E\left(\mathbf{X}\right)\right)\right).$$

Recall the way this works.

$$\mathbf{u}\otimes\mathbf{v}\left(\mathbf{w}\right)\equiv\left(\mathbf{w},\mathbf{v}\right)\mathbf{u}.$$

If there are two random vectors, $\mathbf{X}$ and $\mathbf{Y},\mathbf{X}$ having values in $\mathbb{R}^p$ and $\mathbf{Y}$ having values in $\mathbb{R}^q$, the correlation is the linear transformation defined by

$$E\left(\left(\mathbf{X}-E\left(\mathbf{X}\right)\right)\otimes\left(\mathbf{Y}-E\left(\mathbf{Y}\right)\right)\right)$$

or in terms of matrices,

$$E\left(\left(\mathbf{X}-E\left(\mathbf{X}\right)\right)\left(\mathbf{Y}-E\left(\mathbf{Y}\right)\right)^*\right)$$

This all makes sense provided $\mathbf{X}$ and $\mathbf{Y}$ are in $L^2\left(\Omega;\mathbb{R}^r\right)$ where $r=p$ or q because you can simply integrate the entries of the matrix which results when you write $\left(\mathbf{X}-E\left(\mathbf{X}\right)\right)\left(\mathbf{Y}-E\left(\mathbf{Y}\right)\right)^*$.

What does it all mean in the case where $X\in L^2\left(\Omega;H\right)$ and $Y\in L^2\left(\Omega;G\right)$ for H,G separable Hilbert spaces? In this case there is no "matrix". This involves the notion of a Hilbert Schmidt operator.

Definition 36.1 *Let H and G be two separable Hilbert spaces and let T map H to G be continuous and linear. Then T is called a Hilbert Schmidt operator if there exists some orthonormal basis for H, $\{e_j\}$ such that*

$$\sum_j\left|\left|Te_j\right|\right|^2<\infty.$$

The collection of all such linear maps will be denoted by $\mathcal{L}_2\left(H,G\right)$.

For convenience I have restated Theorem 14.40 on Page 14.40 here.

Theorem 36.2 *$\mathcal{L}_2\left(H,G\right)\subseteq\mathcal{L}\left(H,G\right)$ and $\mathcal{L}_2\left(H,G\right)$ is a separable Hilbert space with norm given by*

$$\left|\left|T\right|\right|_{\mathcal{L}_2}\equiv\left(\sum_k\left|\left|Te_k\right|\right|^2\right)^{1/2}$$

where $\{e_k\}$ is some orthonormal basis for H. Also

$$\left|\left|T\right|\right|\leq\left|\left|T\right|\right|_{\mathcal{L}_2}. \tag{36.1}$$

All Hilbert Schmidt opearators are compact. Also if $X\in H$ and $Y\in G$, then

$$Y\otimes X\in\mathcal{L}_2\left(H,G\right),$$

and

$$\left|\left|Y\otimes X\right|\right|_{\mathcal{L}_2}=\left|\left|X\right|\right|_H\left|\left|Y\right|\right|_G. \tag{36.2}$$

Now if $X \in L^1(\Omega, H)$,

$$E(X) \equiv \int_\Omega X dP.$$

There is no problem here. Next, consider the correlation.

Theorem 36.3 *Suppose $X \in L^2(\Omega; H)$ and $Y \in L^2(\Omega; G)$ where H and G are separable Hilbert spaces. Then $(Y - E(Y)) \otimes (X - E(X)) \in L^1(\Omega; \mathcal{L}_2(H, G))$ and the correlation is defined by*

$$\begin{aligned} cor(Y, X) &\equiv E((Y - E(Y)) \otimes (X - E(X))) \\ &\in \mathcal{L}_2(H, G) \end{aligned}$$

while the covariance is defined by

$$\begin{aligned} cov(X, X) &\equiv E((X - E(X)) \otimes (X - E(X))) \\ &\in \mathcal{L}_2(H, H). \end{aligned}$$

Proof: It suffices to verify the claim about $cor(Y, X)$. First consider the issue of measurability. I need to verify that $\omega \to (Y(\omega) - E(Y)) \otimes (X(\omega) - E(X))$ is measurable. Since $\mathcal{L}_2(H, G)$ is a separable Hilbert space, it suffices to use the Pettis theorem and verify

$$\omega \to ((Y(\omega) - E(Y)) \otimes (X(\omega) - E(X)), A)_{\mathcal{L}_2}$$

is measurable for every $A \in \mathcal{L}_2(H, G)$. Therefore, letting $\{e_k\}$ be an orthonormal basis in H ,

$$\begin{aligned} & ((Y(\omega) - E(Y)) \otimes (X(\omega) - E(X)), A)_{\mathcal{L}_2} \\ \equiv & \sum_k ((Y(\omega) - E(Y)) \otimes (X(\omega) - E(X))(e_k), A(e_k)) \\ = & \sum_k (e_k, X(\omega) - E(X))((Y(\omega) - E(Y)), A(e_k)). \end{aligned}$$

Each term in this sum is measurable and so this shows measurability.

Finally, from Theorem 14.40,

$$\begin{aligned} & \int_\Omega ||(Y(\omega) - E(Y)) \otimes (X(\omega) - E(X))||_{\mathcal{L}_2} dP \\ \leq & \int_\Omega ||Y(\omega) - E(Y)|| \, ||X(\omega) - E(X)|| \, dP \\ \leq & ||X - E(X)||_{L^2(\Omega;H)} ||Y - E(Y)||_{L^2(\Omega;G)}. \end{aligned}$$

This proves the theorem.

36.2 Independence

Recall that for X a random variable, $\sigma(X)$ is the smallest σ algebra containing all the sets of the form $X^{-1}(F)$ where F is Borel. Since such sets, $X^{-1}(F)$ for F Borel form a σ algebra it follows $\sigma(X) = \left\{X^{-1}(F) : F \text{ is Borel}\right\}$.

Definition 36.4 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A finite set of random vectors, $\{X_k\}_{k=1}^n$ is independent if whenever $F_k \in \sigma(X_k)$,*

$$P\left(\cap_{k=1}^n F_k\right) = \prod_{k=1}^n P(F_k).$$

More generally, if $\{\mathcal{F}_j\}_{j\in J}$ are σ algebras, they are said to be independent if whenever $I \subseteq J$ is a finite set of indices and $A_i \in \mathcal{F}_i$,

$$P\left(\cap_{i\in I} A_i\right) = \prod_{i\in I} P(A_i).$$

Recall the following lemma.

Lemma 36.5 *If $\{X_k\}_{k=1}^r$ are independent and if g_k is a Borel measurable function, then $\{g_k(X_k)\}_{k=1}^n$ is also independent. Furthermore, if the random variables have values in $\mathbb{R}$ and they are all bounded, then*

$$E\left(\prod_{i=1}^r X_i\right) = \prod_{i=1}^r E(X_i).$$

Proof: First consider the claim about $\{g_k(X_k)\}_{k=1}^r$. Letting O be an open set in $\mathbb{R}$,

$$(g_k \circ X_k)^{-1}(O) = X_k^{-1}\left(g_k^{-1}(O)\right) = X_k^{-1}(\text{Borel set}) \in \sigma(X_k).$$

It follows $(g_k \circ X_k)^{-1}(E)$ is in $\sigma(X_k)$ whenever E is Borel. Thus $\sigma(g_k \circ X_k) \subseteq \sigma(X_k)$ and this proves the first part of the lemma.

Now let $\left\{s_n^i\right\}_{n=1}^\infty$ be a bounded sequence of simple functions measurable in $\sigma(X_i)$ which converges to X_i uniformly. (Since X_i is bounded, such a sequence exists by breaking X_i into positive and negative parts and using Theorem 8.27 on Page 190.) Say

$$s_n^i(\omega) = \sum_{k=1}^{m_n} c_k^{n,i} \mathcal{X}_{E_k^{n,i}}(\omega)$$

where the E_k are disjoint elements of $\sigma(X_i)$ and some might be empty. This is for convenience in keeping the same index on the top of the sum. Then since all the random variables are bounded, there is no problem about existence of any of the above. Then from the assumption that the X_i are independent,

$$\begin{aligned}
E\left(\prod_{i=1}^{r} X_i\right) &= \int_\Omega \prod_{i=1}^{r} X_i(\omega)\, dP = \lim_{n\to\infty}\int_\Omega \prod_{i=1}^{r} s_n^i(\omega)\, dP \\
&= \lim_{n\to\infty}\int_\Omega \prod_{i=1}^{r}\sum_{k=1}^{m_n} c_k^{n,i}\mathcal{X}_{E_k^{n,i}}(\omega)\, dP \\
&= \lim_{n\to\infty}\int_\Omega \sum_{k_1,k_2,\cdots,k_r} c_{k_1}^{n,1}c_{k_2}^{n,2}\cdots c_{k_r}^{n,r}\mathcal{X}_{E_{k_1}^{n,1}}\mathcal{X}_{E_{k_2}^{n,2}}\cdots\mathcal{X}_{E_{k_r}^{n,r}}dP \\
&= \lim_{n\to\infty}\sum_{k_1,k_2,\cdots,k_r}\int_\Omega c_{k_1}^{n,1}c_{k_2}^{n,2}\cdots c_{k_r}^{n,r}\mathcal{X}_{E_{k_1}^{n,1}}\mathcal{X}_{E_{k_2}^{n,2}}\cdots\mathcal{X}_{E_{k_r}^{n,r}}dP \\
&= \lim_{n\to\infty}\sum_{k_1,k_2,\cdots,k_r} c_{k_1}^{n,1}c_{k_2}^{n,2}\cdots c_{k_r}^{n,r}\prod_{i=1}^{r} P\left(E_{k_i}^{n,i}\right) \\
&= \lim_{n\to\infty}\prod_{i=1}^{r}\int_\Omega s_n^i(\omega)\, dP = \prod_{i=1}^{r} E(X_i)\,.
\end{aligned}$$

This proves the lemma.

Next consider the case where you have an independent set of σ algebras. First here are some preliminary results.

Definition 36.6 *Let Ω be a set and let $\mathcal{K}$ be a collection of subsets of Ω. Then $\mathcal{K}$ is called a π system if $\emptyset \in \mathcal{K}$ and whenever $A, B \in \mathcal{K}$, it follows $A \cap B \in \mathcal{K}$.*

Obviously an example of a π system is the set of measurable rectangles. Note how simple this definition is. It does not involve an algebra. Now recall the fundamental lemma on π systems, Lemma 9.72 on Page 257.

The following lemma is helpful when you try to verify such a set of σ algebras is independent. It says you only need to check things on π systems contained in the σ algebras.

Lemma 36.7 *Suppose $\{\mathcal{F}_i\}_{i\in I}$ is a set of σ algebras contained in $\mathcal{F}$ where $\mathcal{F}$ is a σ algebra of sets of Ω. Suppose that $\mathcal{K}_i \subseteq \mathcal{F}_i$ is a π system and $\mathcal{F}_i = \sigma(\mathcal{K}_i)$. Suppose also that whenever J is a finite subset of I and $A_j \in \mathcal{K}_j$ for $j \in J$, it follows*

$$P\left(\cap_{j\in J} A_j\right) = \prod_{j\in J} P(A_j)\,.$$

Then $\{\mathcal{F}_i\}_{i\in I}$ is independent.

Proof: I need to verify that for all $n \in \mathbb{N}$, if $\{j_1, j_2, \cdots, j_n\} \subseteq I$ and $A_{j_k} \subseteq \mathcal{F}_{j_k}$, then

$$P\left(\cap_{k=1}^{n} A_{j_k}\right) = \prod_{k=1}^{n} P\left(A_{j_k}\right).$$

Pick $\left(A_{j_1}\cdot\cdot\cdot,A_{j_{n-1}}\right)\in\mathcal{K}_{j_1}\times\cdot\cdot\cdot,\mathcal{K}_{j_{n-1}}$ and let

$$\mathcal{G}_{\left(A_{j_1}\cdots,A_{j_{n-1}}\right)}\equiv\left\{A_{j_n}\in\mathcal{F}_{j_n}:P\left(\cap_{k=1}^{n}A_{j_k}\right)=\prod_{k=1}^{n}P\left(A_{j_k}\right)\right\}$$

Then by hypothesis, $\mathcal{K}_{j_n}\subseteq\mathcal{G}_{\left(A_{j_1}\cdots,A_{j_{n-1}}\right)}$. If $A_{j_n}\in\mathcal{G}_{\left(A_{j_1}\cdots,A_{j_{n-1}}\right)}$,

$$\begin{aligned}\prod_{k=1}^{n-1}P\left(A_{j_k}\right) &= P\left(\cap_{k=1}^{n-1}A_{j_k}\right)\\ &= P\left(\left(\cap_{k=1}^{n-1}A_{j_k}\cap A_{j_n}^{C}\right)\cup\left(\cap_{k=1}^{n-1}A_{j_k}\cap A_{j_n}\right)\right)\\ &= P\left(\cap_{k=1}^{n-1}A_{j_k}\cap A_{j_n}^{C}\right)+P\left(\cap_{k=1}^{n-1}A_{j_k}\cap A_{j_n}\right)\\ &= P\left(\cap_{k=1}^{n-1}A_{j_k}\cap A_{j_n}^{C}\right)+\prod_{k=1}^{n}P\left(A_{j_k}\right)\end{aligned}$$

and so

$$\begin{aligned}P\left(\cap_{k=1}^{n-1}A_{j_k}\cap A_{j_n}^{C}\right) &= \prod_{k=1}^{n-1}P\left(A_{j_k}\right)\left(1-P\left(A_{j_n}\right)\right)\\ &= \prod_{k=1}^{n-1}P\left(A_{j_k}\right)P\left(A_{j_n}^{C}\right)\end{aligned}$$

showing if $A_{j_n}\in\mathcal{G}_{\left(A_{j_1}\cdots,A_{j_{n-1}}\right)}$, then so is $A_{j_n}^{C}$. It is clear that $\mathcal{G}_{\left(A_{j_1}\cdots,A_{j_{n-1}}\right)}$ is closed with respect to disjoint unions also. Therefore, by Lemma 9.72 $\mathcal{G}_{\left(A_{j_1}\cdots,A_{j_{n-1}}\right)}=\mathcal{F}_{j_n}$.

Next fix $A_{j_n}\in\mathcal{F}_{j_n}$ and $\left(A_{j_1}\cdot\cdot\cdot,A_{j_{n-2}}\right)\in\mathcal{K}_{j_1}\times\cdot\cdot\cdot,\mathcal{K}_{j_{n-2}}$. Let

$$\mathcal{G}_{\left(A_{j_1}\cdots,A_{j_{n-2}}\right)}\equiv\left\{A_{j_{n-1}}\in\mathcal{F}_{j_{n-1}}:P\left(\cap_{k=1}^{n}A_{j_k}\right)=\prod_{k=1}^{n}P\left(A_{j_k}\right)\right\}$$

It was just shown $\mathcal{G}_{\left(A_{j_1}\cdots,A_{j_{n-2}}\right)}\supseteq\mathcal{K}_{j_{n-1}}$. Also by similar reasoning to the above, it follows $\mathcal{G}_{\left(A_{j_1}\cdots,A_{j_{n-2}}\right)}$ satisfies the conditions needed to apply Lemma 9.72 on Page 257 and so whenever $A_{j_n},A_{j_{n-1}}$ are in $\mathcal{F}_{j_n}$ and $\mathcal{F}_{j_{n-1}}$ respectively and

$$\left(A_{j_1}\cdot\cdot\cdot,A_{j_{n-2}}\right)\in\mathcal{K}_{j_1}\times\cdot\cdot\cdot,\mathcal{K}_{j_{n-2}},$$

it follows $P\left(\cap_{k=1}^{n}A_{j_k}\right)=\prod_{k=1}^{n}P\left(A_{j_k}\right)$. Continue this way to obtain the desired result. This proves the lemma.

What is a useful π system for $\mathcal{B}(E)$ where E is a Banach space?

Recall the fundamental lemma used to prove the Pettis theorem. It was proved on Page 579 but here I want to show that in addition, the set D' can be taken as a subset of a given dense subspace, M of E'. Thus I will present next a generalization of that important lemma. You might consider whether the following lemma can be generalized even more.

Lemma 36.8 *If E is a separable Banach space with B' the closed unit ball in E', and if M is a dense subspace of E', then there exists a sequence $\{f_n\}_{n=1}^{\infty} \equiv D' \subseteq B' \cap M$ with the property that for every $x \in E$,*

$$||x|| = \sup_{f \in D'} |f(x)|$$

The set, D' also has the property that if $f \in D'$ then $-f \in D'$.

Proof: Let $\{a_n\} \equiv D$ be a dense subset of E. Define a subset of $\mathbb{C}^n$, C_n by

$$\{(f(a_1), \cdots, f(a_n)) : f \in B' \cap M\}$$

Then C_n is a bounded subset of $\mathbb{C}^n$ and so it is separable. Note each f delivers a point in $\mathbb{C}^n$. Now let $\{f_k^n\}_{k=1}^{\infty} \subseteq B' \cap M$ be such that the points

$$\{(f_k^n(a_1), \cdots, f_k^n(a_n))\}_{k=1}^{\infty}$$

are dense in C_n. Then $D' \equiv \cup_{n=1}^{\infty} \{f_k^n\}_{k=1}^{\infty}$.

It remains to verify D' works. Pick $x \in E$. I need to show there exists f_k^n such that $||x|| < |f_k^n(x)| + \varepsilon$. By a standard exercise in the Hahn Banach theorem, there exists $f \in B'$ such that $f(x) = ||x||$. Next choose $a_n \in D$ such that $||x - a_n||_E < \varepsilon/4$. Since M is dense, there exists $g \in M \cap B'$ such that $|g(a_n) - f(a_n)| < \varepsilon/4$. Finally, there exists $f_k^n \in D'$ such that $|f_k^n(a_n) - g(a_n)| < \varepsilon/4$. Then

$$\begin{aligned} ||\,||x|| - |f_k^n(x)|\,|| &= ||f(x)| - |f_k^n(x)|| \\ &\leq ||f(x)| - |f(a_n)|| + ||f(a_n)| - |g(a_n)|| \\ &\quad + ||g(a_n)| - |f_k^n(a_n)|| + ||f_k^n(a_n)| - |f_k^n(x)|| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

It follows $||x|| < |f_k^n(x)| + \varepsilon$ and this proves the lemma because for every $f \in D'$ you can simply include $-f$.

Lemma 36.9 *Let E be a separable real Banach space. Sets of the form*

$$\{x \in E : x_i^*(x) \leq \alpha_i, i = 1, 2, \cdots, m\}$$

where $x_i^* \in M$, a dense subspace of E' and $\alpha_i \in [-\infty, \infty)$ are a π system, and denoting this π system by $\mathcal{K}$, it follows $\sigma(\mathcal{K}) = \mathcal{B}(E)$.

Proof: The sets described are obviously a π system. I want to show $\sigma(\mathcal{K})$ contains the closed balls because then $\sigma(\mathcal{K})$ contains the open balls and hence the open sets and the result will follow. Let $D' \subseteq B' \cap M$ be described in Lemma 36.8.

Then

$$\begin{aligned}
& \{x \in E : ||x-a|| \leq r\} \\
= & \left\{x \in E : \sup_{f \in D'} |f(x-a)| \leq r\right\} \\
= & \left\{x \in E : \sup_{f \in D'} |f(x) - f(a)| \leq r\right\} \\
= & \cap_{f \in D'} \{x \in E : f(a) - r \leq f(x) \leq f(a) + r\} \\
= & \cap_{f \in D'} \{x \in E : f(x) \leq f(a) + r \text{ and } (-f)(x) \leq r - f(a)\}
\end{aligned}$$

which equals a countable intersection of sets of the given π system. Therefore, every closed ball is contained in $\sigma(\mathcal{K})$. It follows easily that every open ball is also contained in $\sigma(\mathcal{K})$ because

$$B(a,r) = \cup_{n=1}^{\infty} \overline{B\left(a, r - \frac{1}{n}\right)}.$$

Since the Banach space is separable, it is completely separable and so every open set is the countable union of balls. This shows the open sets are in $\sigma(\mathcal{K})$ and so $\sigma(\mathcal{K}) \supseteq \mathcal{B}(E)$. However, all the sets in the π system are closed hence Borel because they are inverse images of closed sets. Therefore, $\sigma(\mathcal{K}) \subseteq \mathcal{B}(E)$ and so $\sigma(\mathcal{K}) = \mathcal{B}(E)$. This proves the lemma.

Next suppose you have some random variables having values in a separable Banach space, E, $\{X_i\}_{i \in I}$. How can you tell if they are independent? To show they are independent, you need to verify that

$$P\left(\cap_{k=1}^{n} X_{i_k}^{-1}(F_{i_k})\right) = \prod_{k=1}^{n} P\left(X_{i_k}^{-1}(F_{i_k})\right)$$

whenever the F_{i_k} are Borel sets in E. It is desirable to find a way to do this easily.

Lemma 36.10 *Let $\mathcal{K}$ be a π system of sets of E, a separable real Banach space and let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to E$ be a random variable. Then*

$$X^{-1}(\sigma(\mathcal{K})) = \sigma\left(X^{-1}(\mathcal{K})\right)$$

Proof: First note that $X^{-1}(\sigma(\mathcal{K}))$ is a σ algebra which contains $X^{-1}(\mathcal{K})$ and so it contains $\sigma\left(X^{-1}(\mathcal{K})\right)$. Now let

$$\mathcal{G} \equiv \left\{A \in \sigma(\mathcal{K}) : X^{-1}(A) \in \sigma\left(X^{-1}(\mathcal{K})\right)\right\}$$

Then $\mathcal{G} \supseteq \mathcal{K}$. If $A \in \mathcal{G}$, then

$$X^{-1}(A) \in \sigma\left(X^{-1}(\mathcal{K})\right)$$

and so

$$X^{-1}(A)^C = X^{-1}\left(A^C\right) \in \sigma\left(X^{-1}(\mathcal{K})\right)$$

because $\sigma\left(X^{-1}(\mathcal{K})\right)$ is a σ algebra. Hence $A^C \in \mathcal{G}$. Finally suppose $\{A_i\}$ is a sequence of disjoint sets of $\mathcal{G}$. Then

$$X^{-1}\left(\cup_{i=1}^{\infty} A_i\right) = \cup_{i=1}^{\infty} X^{-1}(A_i) \in \sigma\left(X^{-1}(\mathcal{K})\right)$$

again because $\sigma\left(X^{-1}(\mathcal{K})\right)$ is a σ algebra. It follows from Lemma 9.72 on Page 257 that $\mathcal{G} \supseteq \sigma(\mathcal{K})$ and this shows that whenever $A \in \sigma(\mathcal{K}), X^{-1}(A) \in \sigma\left(X^{-1}(\mathcal{K})\right)$. Thus $X^{-1}(\sigma(\mathcal{K})) \subseteq \sigma\left(X^{-1}(\mathcal{K})\right)$ and this proves the lemma.

With this lemma, here is the desired result about independent random variables. Essentially, you can reduce to the case of random vectors having values in $\mathbb{R}^n$.

Theorem 36.11 *The random variables, $\{X_i\}_{i\in I}$ are independent if whenever*

$$\{i_1, \cdots, i_n\} \subseteq I,$$

$m_{i_1}, \cdots, m_{i_n}$ are positive integers, and $\mathbf{g}_{m_{i_1}}, \cdots, \mathbf{g}_{m_{i_n}}$ are respectively in $(M)^{m_{i_1}}, \cdots, (M)^{m_n}$ for M a dense subspace of E', $\left\{\mathbf{g}_{m_{i_j}} \circ X_{i_j}\right\}_{j=1}^{n}$ are independent random vectors having values in $\mathbb{R}^{m_{i_1}}, \cdots, \mathbb{R}^{m_{i_n}}$ respectively.

Proof: Let $\mathcal{K}$ denote sets of the form

$$\{x \in E : x_i^*(x) \leq \alpha_i, i = 1, 2, \cdots m\}$$

as described in Lemma 36.9. Then as proved in this lemma, $\sigma(\mathcal{K}) = \mathcal{B}(E)$. Then the random vectors are independent if whenever

$$\{i_1, \cdots, i_n\} \subseteq I$$

and $A_{i_1}, \cdots, A_{i_n}$ sets of $\sigma(X_{i_1}), \cdots, \sigma(X_{i_n})$ respectively,

$$P\left(\cap_{j=1}^{n} A_{i_j}\right) = \prod_{j=1}^{n} P\left(A_{i_j}\right).$$

By Lemma 9.72 on Page 257 if $\mathcal{K}_{i_j}$ is a π system contained in $\sigma\left(X_{i_j}\right)$ such that $\sigma\left(\mathcal{K}_{i_j}\right) = \sigma\left(X_{i_j}\right)$, then it suffices to check only the case where the A_{i_j} is in $\mathcal{K}_{i_j}$. So what will serve for such a collection of π systems? Let

$$\mathcal{K}_{i_j} \equiv X_{i_j}^{-1}(\mathcal{K}) \equiv \left\{X_{i_j}^{-1}(A) : A \in \mathcal{K}\right\}.$$

This is clearly a π system contained in $\sigma\left(X_{i_j}\right)$ and by Lemma 36.10

$$\sigma\left(\mathcal{K}_{i_j}\right) = \sigma\left(X_{i_j}^{-1}(\mathcal{K})\right) = X_{i_j}^{-1}(\sigma(\mathcal{K})) \equiv \sigma\left(X_{i_j}\right).$$

Thus it suffices to show that whenever $B_{i_1}, \cdots, B_{i_n}$ are sets of $\mathcal{K}$,

$$P\left(\cap_{j=1}^{n} X_{i_j}^{-1}\left(B_{i_j}\right)\right) = \prod_{j=1}^{n} P\left(X_{i_j}^{-1}\left(B_{i_j}\right)\right)$$

Let $B_{i_j} = \left\{x \in E : \mathbf{g}_{m_{i_j}}(x) \in A_{i_j}\right\}$ where $A_{i_j} = \prod_{j=1}^{m_j}(-\infty, \alpha_i]$ and $\mathbf{g}_{m_j} \in M^{m_j}$. It follows

$$X_{i_j}^{-1}\left(B_{i_j}\right) = \left(\mathbf{g}_{m_{i_j}} \circ X_{i_j}\right)^{-1}\left(A_{i_j}\right).$$

Then by the assumption the random vectors $\mathbf{g}_{m_{i_j}} \circ X_{i_j}$ are independent,

$$\begin{aligned} P\left(\cap_{j=1}^{n} X_{i_j}^{-1}\left(B_{i_j}\right)\right) &= P\left(\cap_{j=1}^{n}\left(\mathbf{g}_{m_{i_j}} \circ X_{i_j}\right)^{-1}\left(A_{i_j}\right)\right) \\ &= \prod_{j=1}^{n} P\left(\left(\mathbf{g}_{m_{i_j}} \circ X_{i_j}\right)^{-1}\left(A_{i_j}\right)\right) \\ &= \prod_{j=1}^{n} P\left(X_{i_j}^{-1}\left(B_{i_j}\right)\right) \end{aligned}$$

and this proves the theorem.

Procedure 36.12 *Suppose you have random vectors, $\{X_i\}_{i\in I}$ having values in a real separable Banach space, E. Then they are independent if the $\mathbb{R}^k$ valued random vectors, $\{\mathbf{g}\circ X_i\}_{i\in I, \mathbf{g}\in M^k}$ are independent for M a dense subspace of E'. You check whether these are independent to determine whether $\{X_i\}_{i\in I}$ are independent.*

The above assertion also goes the other way as you may want to show.

So how can you determine whether random vectors having values in $\mathbb{R}^n$ are independent? Recall an earlier proposition which relates independence of random vectors with characteristic functions. It is proved starting on Page 865.

Proposition 36.13 *Let $\{\mathbf{X}_k\}_{k=1}^{n}$ be random vectors such that $\mathbf{X}_k$ has values in $\mathbb{R}^{p_k}$. Then the random vectors are independent if and only if*

$$E\left(e^{iP}\right) = \prod_{j=1}^{n} E\left(e^{i\mathbf{t}_j\cdot\mathbf{X}_j}\right)$$

where $P \equiv \sum_{j=1}^{n} \mathbf{t}_j \cdot \mathbf{X}_j$ for $\mathbf{t}_j \in \mathbb{R}^{p_j}$.

36.3 Conditional Expectation

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X \in L^1(\Omega; \mathbb{R})$. Also let $\mathcal{G} \subseteq \mathcal{F}$ where $\mathcal{G}$ is also a σ algebra. Then the usual conditional expectation is defined by

$$\int_A X dP = \int_A E(X|\mathcal{G})\, dP$$

where $E(X|\mathcal{G})$ is $\mathcal{G}$ measurable and $A \in \mathcal{G}$ is arbitrary. Recall this is an application of the Radon Nikodym theorem. Also recall $E(X|\mathcal{G})$ is unique up to a set of measure zero.

I want to do something like this here. Denote by $L^1(\Omega; E, \mathcal{G})$ those functions in $L^1(\Omega; E)$ which are measurable with respect to $\mathcal{G}$.

Theorem 36.14 *Let E be a separable Banach space and let $X \in L^1(\Omega; E, \mathcal{F})$ where X is measurable with respect to $\mathcal{F}$. Then there exists a unique $Z \in L^1(\Omega; E, \mathcal{G})$ such that for all $A \in \mathcal{G}$,*

$$\int_A X dP = \int_A Z dP$$

Denoting this Z as $E(X|\mathcal{G})$, it follows

$$||E(X|\mathcal{G})|| \leq E(||X|| \ |\mathcal{G}).$$

Proof: First consider uniqueness. Suppose Z' is another in $L^1(\Omega; E, \mathcal{G})$ which works. Then let $A \equiv \{\omega : ||Z(\omega) - Z'(\omega)|| > \delta\}$ where $\delta > 0$. Now let $D = \{a_k\}$ be a countable dense subset of E and consider the balls $B\left(a_k, \frac{1}{4}||a_k||\right) \equiv B_k$ for which $||a_k|| > \delta$. Since each of these balls has radius larger than $\delta/4$ and the $\{a_k\}$ are dense, it follows the union of these balls includes $\{x \in E : ||x|| > \delta\}$.

Now let

$$\begin{aligned} A_k &\equiv \{\omega : ||Z(\omega) - Z'(\omega)|| > \delta\} \\ &\cap \left\{\omega : ||Z(\omega) - Z'(\omega) - a_k|| < \frac{1}{2}||a_k||\right\}. \end{aligned}$$

If $P(A) > 0$, then for some $k, P(A_k) > 0$ because the balls B_k cover the set $\{x : ||x|| > \delta\}$. Then

$$\begin{aligned} \frac{1}{2}||a_k|| P(A_k) &\geq \int_{A_k} ||Z' - Z + a_k|| dP \\ &\geq \left|\left|\int_{A_k} (Z' - Z + a_k) dP\right|\right| \\ &= \left|\left|\int_{A_k} a_k dP\right|\right| = ||a_k|| P(A_k) \end{aligned}$$

which is a contradiction. Hence $P(A) = 0$ after all. It follows $Z' = Z$ a.e. because $\delta > 0$ is arbitrary. This establishes the uniqueness part of the theorem.

Next I will show Z exists. To do this recall Theorem 21.19 on Page 588 which is stated below for convenience.

Theorem 36.15 *An E valued function, X, is Bochner integrable if and only if X is strongly measurable and*

$$\int_\Omega ||X(\omega)|| dP < \infty. \tag{36.3}$$

In this case there exists a sequence of simple functions $\{X_n\}$ *satisfying*

$$\int_\Omega ||X_n(\omega) - X_m(\omega)||\, dP \to 0 \text{ as } m,n \to \infty. \tag{36.4}$$

$X_n(\omega)$ *converging pointwise to* $X(\omega)$,

$$||X_n(\omega)|| \le 2\,||X(\omega)|| \tag{36.5}$$

and

$$\lim_{n\to\infty} \int_\Omega ||X(\omega) - X_n(\omega)||\, dP = 0. \tag{36.6}$$

Now let $\{X_n\}$ be the simple functions just defined and let

$$X_n(\omega) = \sum_{k=1}^m x_k \mathcal{X}_{F_k}(\omega)$$

where $F_k \in \mathcal{F}$, the F_k being disjoint. Then define

$$Z_n \equiv \sum_{k=1}^m x_k E(\mathcal{X}_{F_k}|\mathcal{G}).$$

Thus, if $A \in \mathcal{G}$,

$$\begin{aligned}\int_A Z_n dP &= \sum_{k=1}^m x_k \int_A E(\mathcal{X}_{F_k}|\mathcal{G})\, dP \\ &= \sum_{k=1}^m x_k \int_A \mathcal{X}_{F_k} dP \\ &= \sum_{k=1}^m x_k P(F_k) = \int_A X_n dP \end{aligned} \tag{36.7}$$

Then since $E(\mathcal{X}_{F_k}|\mathcal{G}) \ge 0$,

$$\begin{aligned} E(||Z_n||) &\le \sum_{k=1}^m ||x_k|| \int E(\mathcal{X}_{F_k}|\mathcal{G})\, dP \\ &= \sum_{k=1}^m ||x_k|| \int \mathcal{X}_{F_k} dP = E(||X_n||). \end{aligned}$$

Similarly,

$$E(||Z_n - Z_m||) \le E(||X_n - X_m||)$$

and this last term converges to 0 as $n, m \to \infty$ by the properties of the X_n. Therefore, $\{Z_n\}$ is a Cauchy sequence in $L^1(\Omega; E; \mathcal{G})$. It follows it converges to Z in

$L^1(\Omega; E, \mathcal{G})$. Then letting $A \in \mathcal{G}$, and using 36.7,

$$\begin{aligned}\int_A Z dP &= \int \mathcal{X}_A Z dP \\ &= \lim_{n\to\infty} \int \mathcal{X}_A Z_n dP \\ &= \lim_{n\to\infty} \int_A Z_n dP \\ &= \lim_{n\to\infty} \int_A X_n dP \\ &= \int_A X dP.\end{aligned}$$

It remains to verify $||E(X|\mathcal{G})|| \leq E(||X|| \ |\mathcal{G})$. This follows similar to the above. Letting Z_n and Z have the same meaning as above and $A \in \mathcal{G}$,

$$\begin{aligned}E(\mathcal{X}_A ||Z_n||) &= \int_A \left|\left|\sum_{k=1}^m x_k E(\mathcal{X}_{F_k}|\mathcal{G})\right|\right| dP \\ &\leq \sum_{k=1}^m ||x_k|| \int_A E(\mathcal{X}_{F_k}|\mathcal{G}) dP \\ &= \sum_{k=1}^m ||x_k|| \int_A \mathcal{X}_{F_k} dP \\ &= \int_A \sum_{k=1}^m ||x_k|| \mathcal{X}_{F_k} dP \\ &= E(\mathcal{X}_A ||X_n||).\end{aligned}$$

Therefore,

$$\begin{aligned}\int_A ||Z|| dP &= \int \mathcal{X}_A ||Z|| dP \\ &= \lim_{n\to\infty} \int \mathcal{X}_A ||Z_n|| dP \\ &\leq \lim_{n\to\infty} E(\mathcal{X}_A ||X_n||) \\ &= \int_A ||X|| dP\end{aligned}$$

Thus for all $A \in \mathcal{G}$,

$$\int_A ||E(X|\mathcal{G})|| dP \leq \int_A ||X|| dP = \int_A E(||X|| \ |\mathcal{G}) dP$$

which shows

$$||E(X|\mathcal{G})|| \leq E(||X|| \ |\mathcal{G})$$

as claimed. This proves the theorem.

In the case where E is reflexive, one could also use Corollary 21.48 on Page 617 to get the above result. You would define a vector measure on $\mathcal{G}$,

$$\nu(F) \equiv \int_F X dP$$

and then you would use the fact that reflexive separable Banach spaces have the Radon Nikodym property to obtain $Z \in L^1(\Omega; E, \mathcal{G})$ such that

$$\nu(F) = \int_F X dP = \int_F Z dP.$$

The function, Z whose existence and uniqueness is guaranteed by Theorem 36.15 is called $E(X|\mathcal{G})$.

36.4 Probability Measures And Tightness

Here and in what remains, $\mathcal{B}(E)$ will denote the Borel sets of E where E is a topological space, usually at least a Banach space. Because of the fact that probability measures are finite, you can use a simpler definition of what it means for a measure to be regular. Recall that there were two ingredients, inner regularity which said that the measure of a set is the supremum of the measures of compact subsets and outer regularity which says that the measure of a set is the infimum of the measures of the open sets which contain the given set. Here the definition will be similar but instead of using compact sets, closed sets are substituted. Thus the following definition is a little different than the earlier one. I will show, however, that in many interesting cases, this definition of regularity is actually the same as the earlier one.

Definition 36.16 *A measure, μ defined on $\mathcal{B}(E)$ will be called inner regular if for all $F \in \mathcal{B}(E)$,*

$$\mu(F) = \sup\{\mu(K) : K \subseteq F \text{ and } K \text{ is closed}\}$$

A measure, μ defined on $\mathcal{B}(E)$ will be called outer regular if for all $F \in \mathcal{B}(E)$,

$$\mu(F) = \inf\{\mu(V) : V \supseteq F \text{ and } V \text{ is open}\}$$

When a measure is both inner and outer regular, it is called regular.

For probability measures, regularity tends to come free.

Lemma 36.17 *Let μ be a finite measure defined on $\mathcal{B}(E)$ where E is a metric space. Then μ is regular.*

Proof: First note every open set is the countable union of closed sets and every closed set is the countable intersection of open sets. Here is why. Let V be an open set and let

$$K_k \equiv \left\{x \in V : \text{dist}\left(x, V^C\right) \geq 1/k\right\}.$$

Then clearly the union of the K_k equals V. Next, for K closed let

$$V_k \equiv \{x \in E : \text{dist}(x, K) < 1/k\}.$$

Clearly the intersection of the V_k equals K. Therefore, letting V denote an open set and K a closed set,

$$\begin{aligned} \mu(V) &= \sup\{\mu(K) : K \subseteq V \text{ and } K \text{ is closed}\} \\ \mu(K) &= \inf\{\mu(V) : V \supseteq K \text{ and } V \text{ is open}\}. \end{aligned}$$

Also since V is open and K is closed,

$$\begin{aligned} \mu(V) &= \inf\{\mu(U) : U \supseteq V \text{ and } V \text{ is open}\} \\ \mu(K) &= \sup\{\mu(L) : L \subseteq K \text{ and } L \text{ is closed}\} \end{aligned}$$

In words, μ is regular on open and closed sets. Let

$$\mathcal{F} \equiv \{F \in \mathcal{B}(E) \text{ such that } \mu \text{ is regular on } F\}.$$

Then $\mathcal{F}$ contains the open sets. I want to show $\mathcal{F}$ is a σ algebra and then it will follow $\mathcal{F} = \mathcal{B}(E)$.

First I will show $\mathcal{F}$ is closed with respect to complements. Let $F \in \mathcal{F}$. Then since μ is finite and F is inner regular, there exists $K \subseteq F$ such that $\mu(F \setminus K) < \varepsilon$. But $K^C \setminus F^C = F \setminus K$ and so $\mu\left(K^C \setminus F^C\right) < \varepsilon$ showning that F^C is outer regular. I have just approximated the measure of F^C with the measure of K^C, an open set containing F^C. A similar argument works to show F^C is inner regular. You start with $V \supseteq F$ such that $\mu(V \setminus F) < \varepsilon$, note $F^C \setminus V^C = V \setminus F$, and then conclude $\mu\left(F^C \setminus V^C\right) < \varepsilon$, thus approximating F^C with the closed subset, V^C.

Next I will show $\mathcal{F}$ is closed with respect to taking countable unions. Let $\{F_k\}$ be a sequence of sets in $\mathcal{F}$. Then μ is inner regular on each of these so there exist $\{K_k\}$ such that $K_k \subseteq F_k$ and $\mu(F_k \setminus K_k) < \varepsilon/2^{k+1}$. First choose m large enough that

$$\mu\left(\left(\cup_{k=1}^{\infty} F_k\right) \setminus \left(\cup_{k=1}^{m} F_k\right)\right) < \frac{\varepsilon}{2}.$$

Then

$$\mu\left(\left(\cup_{k=1}^{m} F_k\right) \setminus \left(\cup_{k=1}^{m} K_k\right)\right) \leq \sum_{k=1}^{m} \frac{\varepsilon}{2^{k+1}} < \frac{\varepsilon}{2}$$

and so

$$\begin{aligned} \mu\left(\left(\cup_{k=1}^{\infty} F_k\right) \setminus \left(\cup_{k=1}^{m} K_k\right)\right) &\leq \mu\left(\left(\cup_{k=1}^{\infty} F_k\right) \setminus \left(\cup_{k=1}^{m} F_k\right)\right) \\ &\quad + \mu\left(\left(\cup_{k=1}^{m} F_k\right) \setminus \left(\cup_{k=1}^{m} K_k\right)\right) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

showing μ is inner regular on $\cup_{k=1}^{\infty} F_k$. Since μ is outer regular on F_k, there exists V_k such that $\mu(V_k \setminus F_k) < \varepsilon/2^k$. Then

$$\begin{aligned}\mu\left(\left(\cup_{k=1}^{\infty} V_k\right) \setminus \left(\cup_{k=1}^{\infty} F_k\right)\right) &\leq \sum_{k=1}^{\infty} \mu(V_k \setminus F_k) \\ &< \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon\end{aligned}$$

and this shows μ is outer regular on $\cup_{k=1}^{\infty} F_k$ and this proves the lemma.

Lemma 36.18 *Let μ be a finite measure on $\mathcal{B}(E)$, the Borel sets of E, a separable complete metric space. Then if C is a closed set,*

$$\mu(C) = \sup\{\mu(K) : K \subseteq C \text{ and } K \text{ is compact.}\}$$

Proof: Let $\{a_k\}$ be a countable dense subset of C. Thus $\cup_{k=1}^{\infty} B\left(a_k, \frac{1}{n}\right) \supseteq C$. Therefore, there exists m_n such that

$$\mu\left(C \setminus \overline{\cup_{k=1}^{m_n} B\left(a_k, \frac{1}{n}\right)}\right) \equiv \mu(C \setminus C_n) < \frac{\varepsilon}{2^n}.$$

Now let $K = C \cap \left(\cap_{n=1}^{\infty} C_n\right)$. Then K is a subset of C_n for each n and so for each $\varepsilon > 0$ there exists an ε net for K since C_n has a $1/n$ net, namely $a_1, \cdots, a_{m_n}$. Since K is closed, it is complete and so it is also compact. Now

$$\mu(C \setminus K) = \mu\left(\cup_{n=1}^{\infty}(C \setminus C_n)\right) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Thus $\mu(C)$ can be approximated by $\mu(K)$ for K a compact subset of C. This proves the lemma.

This shows that for a finite measure on the Borel sets of a separable metric space, the above definition of regular coincides with the earlier one.

Now here is a definition of what it means for a set of measures to be tight.

Definition 36.19 *Let Λ be a set of probability measures defined on the Borel sets of a topological space. Then Λ is "tight" if for all $\varepsilon > 0$ there exists a compact set, K_ε such that*

$$\mu([x \notin K_\varepsilon]) < \varepsilon$$

for all $\mu \in \Lambda$.

Lemma 36.18 implies a single probability measure on the Borel sets of a separable metric space is tight. The proof of that lemma generalizes slightly to give a simple criterion for a set of measures to be tight.

Lemma 36.20 *Let E be a separable complete metric space and let Λ be a set of Borel probability measures. Then Λ is tight if and only if for every $\varepsilon > 0$ and $r > 0$ there exists a finite collection of balls, $\{B(a_i, r)\}_{i=1}^m$ such that*

$$\mu\left(\cup_{i=1}^m \overline{B(a_i, r)}\right) > 1 - \varepsilon$$

for every $\mu \in \Lambda$.

Proof: If Λ is tight, then there exists a compact set, K_ε such that

$$\mu(K_\varepsilon) > 1 - \varepsilon$$

for all $\mu \in \Lambda$. Then consider the open cover, $\{B(x, r) : x \in K_\varepsilon\}$. Finitely many of these cover K_ε and this yields the above condition.

Now suppose the above condition and let

$$C_n \equiv \cup_{i=1}^{m_n} \overline{B(a_i^n, 1/n)}$$

satisfy $\mu(C_n) > 1 - \varepsilon/2^n$ for all $\mu \in \Lambda$. Then let $K_\varepsilon \equiv \cap_{n=1}^\infty C_n$. Then as in Lemma 36.18 $\mu(K_\varepsilon) > 1 - \varepsilon$ for all $\mu \in \Lambda$.

Prokhorov's theorem is an important result which also involves tightness. In order to give a proof of this important theorem, it is necessary to consider some simple results from topology which are interesting for their own sake.

Theorem 36.21 *Let H be a compact metric space. Then there exists a compact subset of $[0, 1]$, K and a continuous function, θ which maps K onto H.*

Proof: Without loss of generality, it can be assumed H is an infinite set since otherwise the conclusion is trivial. You could pick finitely many points of $[0, 1]$ for K.

Since H is compact, it is totally bounded. Therefore, there exists a 1 net for H $\{h_i\}_{i=1}^{m_1}$. Letting $H_i^1 \equiv \overline{B(h_i, 1)}$, it follows H_i^1 is also a compact metric space and so there exists a 1/2 net for each H_i^1, $\{h_j^i\}_{j=1}^{m_i}$. Then taking the intersection of $\overline{B\left(h_j^i, \frac{1}{2}\right)}$ with H_i^1 to obtain sets denoted by H_j^2 and continuing this way, one can obtain compact subsets of H, $\{H_k^i\}$ which satisfies: each H_j^i is contained in some H_k^{i-1}, each H_j^i is compact with diameter less than i^{-1}, each H_j^i is the union of sets of the form H_k^{i+1} which are contained in it. Denoting by $\{H_j^i\}_{j=1}^{m_i}$ those sets corresponding to a superscript of i, it can also be assumed $m_i < m_{i+1}$. If this is not so, simply add in another point to the i^{-1} net. Now let $\{I_j^i\}_{j=1}^{m_i}$ be disjoint closed intervals in $[0, 1]$ each of length no longer than 2^{-m_i} which have the property that I_j^i is contained in I_k^{i-1} for some k. Letting $K_i \equiv \cup_{j=1}^{m_i} I_j^i$, it follows K_i is a sequence of nested compact sets. Let $K = \cap_{i=1}^\infty K_i$. Then each $x \in K$ is the intersection of a unique sequence of these closed intervals, $\{I_{j_k}^k\}_{k=1}^\infty$. Define $\theta x \equiv \cap_{k=1}^\infty H_{j_k}^k$. Since the diameters of the H_j^i converge to 0 as $i \to \infty$, this function is well defined. It is continuous because if $x_n \to x$, then ultimately x_n and x are both in $I_{j_k}^k$, the k^{th} closed interval in the sequence whose intersection is x. Hence, $d(\theta x_n, \theta x) \leq diameter(H_{j_k}^k) \leq 1/k$. To see the map is onto, let $h \in H$. Then from the construction, there exists a sequence $\left\{H_{j_k}^k\right\}_{k=1}^\infty$ of the above sets whose intersection equals h. Then $\theta\left(\cap_{i=1}^\infty I_{j_k}^k\right) = h$. This proves the theorem.

Note θ is maybe not one to one.

As an important corollary, it follows that the continuous functions defined on any compact metric space is separable.

Corollary 36.22 *Let H be a compact metric space and let $C(H)$ denote the continuous functions defined on H with the usual norm,*

$$||f||_\infty \equiv \max\{|f(x)| : x \in H\}$$

Then $C(H)$ is separable.

Proof: The proof is by contradiction. Suppose $C(H)$ is not separable. Let $\mathcal{H}_k$ denote a maximal collection of functions of $C(H)$ with the property that if $f, g \in \mathcal{H}_k$, then $||f - g||_\infty \geq 1/k$. The existence of such a maximal collection of functions is a consequence of a simple use of the Hausdorff maximallity theorem. Then $\cup_{k=1}^\infty \mathcal{H}_k$ is dense. Therefore, it cannot be countable by the assumption that $C(H)$ is not separable. It follows that for some $k, \mathcal{H}_k$ is uncountable. Now by Theorem 36.21 there exists a continuous function, θ defined on a compact subset, K of $[0,1]$ which maps K onto H. Now consider the functions defined on K

$$\mathcal{G}_k \equiv \{f \circ \theta : f \in \mathcal{H}_k\}.$$

Then $\mathcal{G}_k$ is an uncountable set of continuous functions defined on K with the property that the distance between any two of them is at least as large as $1/k$. This contradicts separability of $C(K)$ which follows from the Weierstrass approximation theorem in which the separable countable set of functions is the restrictions of polynomials that involve only rational coefficients. This proves the corollary. Now here is Prokhorov's theorem.

Theorem 36.23 *Let $\Lambda = \{\mu_n\}_{n=1}^\infty$ be a sequence of probability measures defined on $\mathcal{B}(E)$ where E is a separable Banach space. If Λ is tight then there exists a probability measure, λ and a subsequence of $\{\mu_n\}_{n=1}^\infty$, still denoted by $\{\mu_n\}_{n=1}^\infty$ such that whenever ϕ is a continuous bounded complex valued function defined on E,*

$$\lim_{n\to\infty} \int \phi d\mu_n = \int \phi d\lambda.$$

Proof: By tightness, there exists an increasing sequence of compact sets, $\{K_n\}$ such that

$$\mu(K_n) > 1 - \frac{1}{n}$$

for all $\mu \in \Lambda$. Now letting $\mu \in \Lambda$ and $\phi \in C(K_n)$ such that $||\phi||_\infty \leq 1$, it follows

$$\left|\int_{K_n} \phi d\mu\right| \leq \mu(K_n) \leq 1$$

and so the restrictions of the measures of Λ to K_n are contained in the unit ball of $C(K_n)'$. Recall from the Riesz representation theorem, the dual space of $C(K_n)$ is a space of complex Borel measures. Theorem 13.37 on Page 356 implies the unit ball of $C(K_n)'$ is weak $*$ sequentially compact. This follows from the observation that $C(K_n)$ is separable which is proved in Corollary 36.22 and leads to the fact that the unit ball in $C(K_n)'$ is actually metrizable by Theorem 13.37 on Page 356. Therefore, there exists a subsequence of Λ, $\{\mu_{1k}\}$ such that their restrictions to K_1 converge weak $*$ to a measure, $\lambda_1 \in C(K_1)'$. That is, for every $\phi \in C(K_1)$,

$$\lim_{k\to\infty}\int_{K_1}\phi d\mu_{1k} = \int_{K_1}\phi d\lambda_1$$

By the same reasoning, there exists a further subsequence $\{\mu_{2k}\}$ such that the restrictions of these measures to K_2 converge weak $*$ to a measure $\lambda_2 \in C(K_2)'$ etc. Continuing this way,

$$\begin{array}{l} \mu_{11},\mu_{12},\mu_{13},\cdots \to \text{ Weak}* \text{ in } C(K_1)' \\ \mu_{21},\mu_{22},\mu_{23},\cdots \to \text{ Weak}* \text{ in } C(K_2)' \\ \mu_{31},\mu_{32},\mu_{33},\cdots \to \text{ Weak}* \text{ in } C(K_3)' \\ \qquad\qquad\vdots \end{array}$$

Here the j^{th} sequence is a subsequence of the $(j-1)^{th}$. Let λ_n denote the measure in $C(K_n)'$ to which the sequence $\{\mu_{nk}\}_{k=1}^{\infty}$ converges weak$*$. Let $\{\mu_n\} \equiv \{\mu_{nn}\}$, the diagonal sequence. Thus this sequence is ultimately a subsequence of every one of the above sequences and so μ_n converges weak$*$ in $C(K_m)'$ to λ_m for each m.

Claim: For $p > n$, the restriction of λ_p to the Borel sets of K_n equals λ_n.

Proof of claim: Let H be a compact subset of K_n. Then there are sets, V_l open in K_n which are decreasing and whose intersection equals H. This follows because this is a metric space. Then let $H \prec \phi_l \prec V_l$. It follows

$$\begin{aligned} \lambda_n(V_l) &\geq \int_{K_n}\phi_l d\lambda_n = \lim_{k\to\infty}\int_{K_n}\phi_l d\mu_k \\ &= \lim_{k\to\infty}\int_{K_p}\phi_l d\mu_k = \int_{K_p}\phi_l d\lambda_p \geq \lambda_p(H). \end{aligned}$$

Now considering the ends of this inequality, let $l \to \infty$ and pass to the limit to conclude

$$\lambda_n(H) \geq \lambda_p(H).$$

Similarly,

$$\begin{aligned} \lambda_n(H) &\leq \int_{K_n}\phi_l d\lambda_n = \lim_{k\to\infty}\int_{K_n}\phi_l d\mu_k \\ &= \lim_{k\to\infty}\int_{K_p}\phi_l d\mu_k = \int_{K_p}\phi_l d\lambda_p \leq \lambda_p(V_l). \end{aligned}$$

Then passing to the limit as $l \to \infty$, it follows

$$\lambda_n (H) \leq \lambda_p (H) .$$

Thus the restriction of $\lambda_p, \lambda_p|_{K_n}$ to the compact sets of K_n equals λ_n. Then by inner regularity it follows the two measures, $\lambda_p|_{K_n}$, and λ_n are equal on all Borel sets of K_n. Recall that for finite measures on separable metric spaces, regularity is obtained for free.

It is fairly routine to exploit regularity of the measures to verify that $\lambda_m (F) \geq 0$ for all F a Borel subset of K_m. (Whenever $\phi \geq 0, \int_{K_m} \phi d\lambda_m \geq 0$ because $\int_{K_m} \phi d\mu_k \geq 0$. Now you can approximate $\mathcal{X}_F$ with a suitable nonnegative ϕ using regularity of the measure.) Also, letting $\phi \equiv 1$,

$$1 \geq \lambda_m (K_m) \geq 1 - \frac{1}{m}. \tag{36.8}$$

Define for F a Borel set,

$$\lambda (F) \equiv \lim_{n\to\infty} \lambda_n (F \cap K_n) .$$

The limit exists because the sequence on the right is increasing due to the above observation that $\lambda_n = \lambda_m$ on the Borel subsets of K_m whenever $n > m$. Thus for $n > m$

$$\lambda_n (F \cap K_n) \geq \lambda_n (F \cap K_m) = \lambda_m (F \cap K_m) .$$

Now let $\{F_k\}$ be a sequence of disjoint Borel sets. Then

$$\begin{aligned} \lambda \left(\cup_{k=1}^{\infty} F_k\right) &\equiv \lim_{n\to\infty} \lambda_n \left(\cup_{k=1}^{\infty} F_k \cap K_n\right) = \lim_{n\to\infty} \lambda_n \left(\cup_{k=1}^{\infty} (F_k \cap K_n)\right) \\ &= \lim_{n\to\infty} \sum_{k=1}^{\infty} \lambda_n (F_k \cap K_n) = \sum_{k=1}^{\infty} \lambda (F_k) \end{aligned}$$

the last equation holding by the monotone convergence theorem.

It remains to verify

$$\lim_{k\to\infty} \int \phi d\mu_k = \int \phi d\lambda$$

for every ϕ bounded and continuous. This is where tightness is used again. Suppose $||\phi||_\infty < M$. Then as noted above,

$$\lambda_n (K_n) = \lambda (K_n)$$

because for $p > n, \lambda_p (K_n) = \lambda_n (K_n)$ and so letting $p \to \infty$, the above is obtained. Also, from 36.8,

$$\begin{aligned} \lambda \left(K_n^C\right) &= \lim_{p\to\infty} \lambda_p \left(K_n^C \cap K_p\right) \\ &\leq \limsup_{p\to\infty} \left(\lambda_p (K_p) - \lambda_p (K_n)\right) \\ &\leq \limsup_{p\to\infty} \left(\lambda_p (K_p) - \lambda_n (K_n)\right) \\ &\leq \limsup_{p\to\infty} \left(1 - \left(1 - \frac{1}{n}\right)\right) = \frac{1}{n} \end{aligned}$$

Consequently,

$$\left|\int \phi d\mu_k - \int \phi d\lambda\right| \leq \left|\int_{K_n^C} \phi d\mu_k + \int_{K_n} \phi d\mu_k - \left(\int_{K_n} \phi d\lambda + \int_{K_n^C} \phi d\lambda\right)\right|$$

$$\leq \left|\int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\lambda_n\right| + \left|\int_{K_n^C} \phi d\mu_k - \int_{K_n^C} \phi d\lambda\right|$$

$$\leq \left|\int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\lambda_n\right| + \left|\int_{K_n^C} \phi d\mu_k\right| + \left|\int_{K_n^C} \phi d\lambda\right|$$

$$\leq \left|\int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\lambda_n\right| + \frac{M}{n} + \frac{M}{n}$$

First let n be so large that $2M/n < \varepsilon/2$ and then pick k large enough that the above expression is less than ε. This proves the theorem.

Definition 36.24 *Let E be a Banach space and let μ and the sequence of probability measures, $\{\mu_n\}$ defined on $\mathcal{B}(E)$ satisfy*

$$\lim_{n\to\infty} \int \phi d\mu_n = \int \phi d\mu.$$

for every ϕ a bounded continuous function. Then μ_n is said to converge weakly to μ.

36.5 A Major Existence And Convergence Theorem

Here is an interesting lemma about weak convergence.

Lemma 36.25 *Let μ_n converge weakly to μ and let U be an open set with $\mu(\partial U) = 0$. Then*

$$\lim_{n\to\infty} \mu_n(U) = \mu(U).$$

Proof: Let $\{\psi_k\}$ be a sequence of bounded continuous functions which decrease to $\mathcal{X}_{\overline{U}}$. Also let $\{\phi_k\}$ be a sequence of bounded continuous functions which increase to $\mathcal{X}_U$. For example, you could let

$$\psi_k(x) \equiv (1 - k\operatorname{dist}(x, U))^+,$$
$$\phi_k(x) \equiv 1 - \left(1 - k\operatorname{dist}\left(x, U^C\right)\right)^+.$$

Let $\varepsilon > 0$ be given. Then since $\mu(\partial U) = 0$, the dominated convergence theorem implies there exists $\psi = \psi_k$ and $\phi = \phi_k$ such that

$$\varepsilon > \int \psi d\mu - \int \phi d\mu$$

Next use the weak convergence to pick N large enough that if $n \geq N$,

$$\int \psi d\mu_n \leq \int \psi d\mu + \varepsilon, \int \phi d\mu_n \geq \int \phi d\mu - \varepsilon.$$

Therefore, for n this large,

$$\mu(U), \mu_n(U) \in \left[\int \phi d\mu - \varepsilon, \int \psi d\mu + \varepsilon\right]$$

and so

$$|\mu(U) - \mu_n(U)| < 3\varepsilon.$$

since ε is arbitrary, this proves the lemma.

Definition 36.26 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and let* $X : \Omega \to E$ *be a random variable where here* E *is some topological space. Then one can define a probability measure,* λ_X *on* $\mathcal{B}(E)$ *as follows:*

$$\lambda_X(F) \equiv P([X \in F])$$

More generally, if μ *is a probability measure on* $\mathcal{B}(E)$*, and* X *is a random variable defined on a probability space,* $\mathcal{L}(X) = \mu$ *means*

$$\mu(F) \equiv P([X \in F]).$$

The following amazing theorem is due to Skorokhod. It starts with a measure, μ on $\mathcal{B}(E)$ and produces a random variable, X for which $\mathcal{L}(X) = \mu$. It also has something to say about the convergence of a sequence of such random variables.

Theorem 36.27 *Let* E *be a separable Banach space and let* $\{\mu_n\}$ *be a sequence of Borel probability measures defined on* $\mathcal{B}(E)$ *such that* μ_n *converges weakly to* μ *another probability measure on* $\mathcal{B}(E)$*. Then there exist random variables,* X_n, X *defined on the probability space,* $([0,1), \mathcal{B}([0,1)), m)$ *where* m *is one dimensional Lebesgue measure such that*

$$\mathcal{L}(X) = \mu,\ \mathcal{L}(X_n) = \mu_n, \tag{36.9}$$

each random variable, X, X_n *is continuous off a set of measure zero, and*

$$X_n(\omega) \to X(\omega)\ \ m \text{ a.e.}$$

Proof: Let $\{a_k\}$ be a countable dense subset of E.

Construction of sets in E

First I will describe a construction. Letting $C \in \mathcal{B}(E)$ and $r > 0$,

$$\begin{aligned} C_1^r &\equiv C \cap B(a_1, r), C_2^r \equiv B(a_2, r) \cap C \setminus C_1^r, \cdots, \\ C_n^r &\equiv B(a_n, r) \cap C \setminus \left(\cup_{k=1}^{n-1} C_k^r\right). \end{aligned}$$

Thus the sets, C_k^r for $k = 1, 2, \cdots$ are disjoint Borel sets whose union is all of C. Now let $C = E$, the whole Banach space. Also let $\{r_k\}$ be a decreasing sequence of positive numbers which converges to 0. Let

$$A_k \equiv E_k^{r_1},\ k = 1, 2, \cdots$$

Thus $\{A_k\}$ is a sequence of Borel sets, $A_k \subseteq B(a_k, r_1)$, and the union of the A_k equals E. For $(i_1, \cdots, i_m) \in \mathbb{N}^m$, suppose $A_{i_1,\cdots,i_m}$ has been defined. Then for $k \in \mathbb{N}$,

$$A_{i_1,\cdots,i_m k} \equiv (A_{i_1,\cdots,i_m})_k^{r_{m+1}}$$

Thus $A_{i_1,\cdots,i_m k} \subseteq B(a_k, r_{m+1})$, is a Borel set, and

$$\cup_{k=1}^{\infty} A_{i_1,\cdots,i_m k} = A_{i_1,\cdots,i_m}. \tag{36.10}$$

Also note that $A_{i_1,\cdots,i_m}$ could be empty. This is because $A_{i_1,\cdots,i_m k} \subseteq B(a_k, r_{m+1})$ but $A_{i_1,\cdots,i_m} \subseteq B(a_{i_m}, r_m)$ which might have empty intersection with $B(a_k, r_{m+1})$. However, applying 36.10 repeatedly,

$$E = \cup_{i_1} \cdots \cup_{i_m} A_{i_1,\cdots,i_m}$$

and also, the construction shows the Borel sets, $A_{i_1,\cdots,i_m}$ are disjoint.

Construction of intervals depending on the measure

Next I will construct intervals, $I_{i_1,\cdots,i_n}^{\nu}$ in $[0,1)$ corresponding to these $A_{i_1,\cdots,i_n}$. In what follows, $\nu = \mu_n$ or μ. These intervals will depend on the measure chosen as indicated in the notation.

$$I_1^{\nu} \equiv [0, \nu(A_1)), \cdots, I_j^{\nu} \equiv \left[\sum_{k=1}^{j-1} \nu(A_k), \sum_{k=1}^{j} \nu(A_k)\right)$$

for $j = 1, 2, \cdots$. Note these are disjoint intervals whose union is $[0,1)$. Also note

$$m\left(I_j^{\nu}\right) = \nu(A_j).$$

The endpoints of these intervals as well as their lengths depend on the measures of the sets A_k. Now supposing $I_{i_1,\cdots,i_m}^{\nu} = [\alpha, \beta)$ where $\beta - \alpha = \nu(A_{i_1\cdots,i_m})$, define

$$I_{i_1\cdots,i_m,j}^{\nu} \equiv \left[\alpha + \sum_{k=1}^{j-1} \nu(A_{i_1\cdots,i_m,k}), \alpha + \sum_{k=1}^{j} \nu(A_{i_1\cdots,i_m,k})\right)$$

Thus $m\left(I_{i_1\cdots,i_m,j}^{\nu}\right) = \nu(A_{i_1\cdots,i_m,j})$ and

$$\nu(A_{i_1\cdots,i_m}) = \sum_{k=1}^{\infty} \nu(A_{i_1\cdots,i_m,k}) = \sum_{k=1}^{\infty} m\left(I_{i_1\cdots,i_m,k}^{\nu}\right) = \beta - \alpha,$$

the intervals, $I_{i_1\cdots,i_m,j}^{\nu}$ being disjoint and

$$I_{i_1\cdots,i_m}^{\nu} = \cup_{j=1}^{\infty} I_{i_1\cdots,i_m,j}^{\nu}.$$

Choosing the sequence $\{r_k\}$ in an auspicious manner

There are at most countably many positive numbers, r such that for $\nu = \mu_n$ or $\mu, \nu(\partial B(a_i, r)) > 0$. This is because ν is a finite measure. Taking the countable union of these countable sets, there are only countably many r such that $\nu(\partial B(a_i, r)) > 0$ for some a_i. Let the sequence avoid all these bad values of r. Thus for

$$F \equiv \cup_{m=1}^{\infty} \cup_{k=1}^{\infty} \partial B(a_k, r_m)$$

and $\nu = \mu$ or $\mu_n, \nu(F) = 0$.

Claim 1: $\partial A_{i_1, \cdots, i_k} \subseteq F$.

Proof of claim: Suppose C is a Borel set for which $\partial C \subseteq F$. I need to show $\partial C_k^{r_i} \in F$. First consider $k = 1$. Then $C_1^{r_i} \equiv B(a_1, r_i) \cap C$. If $x \in \partial C_1^{r_i}$, then $B(x, \delta)$ contains points of $B(a_1, r_i) \cap C$ and points of $B(a_1, r_i)^C \cup C^C$ for every $\delta > 0$. First suppose $x \in B(a_1, r_i)$. Then a small enough neighborhood of x has no points of $B(a_1, r_i)^C$ and so every $B(x, \delta)$ has points of C and points of C^C so that $x \in \partial C \subseteq F$ by assumption. If $x \in \partial C_1^{r_i}$, then it can't happen that $||x - a_1|| > r_i$ because then there would be a neighborhood of x having no points of $C_1^{r_i}$. The only other case to consider is that $||x - a_i|| = r_i$ but this says $x \in F$. Now assume $\partial C_j^{r_i} \subseteq F$ for $j \leq k - 1$ and consider $\partial C_k^{r_i}$.

$$\begin{aligned} C_k^{r_i} &\equiv B(a_k, r_i) \cap C \setminus \cup_{j=1}^{k-1} C_j^{r_i} \\ &= B(a_k, r_i) \cap C \cap \left(\cap_{j=1}^{k-1} \left(C_j^{r_i} \right)^C \right) \end{aligned} \tag{36.11}$$

Consider $x \in \partial C_k^{r_i}$. If $x \in \text{int}(B(a_k, r_i) \cap C)$ (int $\equiv$ *interior*) then a small enough ball about x contains no points of $(B(a_k, r_i) \cap C)^C$ and so every ball about x must contain points of

$$\left(\cap_{j=1}^{k-1} \left(C_j^{r_i} \right)^C \right)^C = \cup_{j=1}^{k-1} C_j^{r_i}$$

Since there are only finitely many sets in the union, there exists $s \leq k - 1$ such that every ball about x contains points of $C_s^{r_i}$ but from 36.11, every ball about x contains points of $(C_s^{r_i})^C$ which implies $x \in \partial C_s^{r_i} \subseteq F$ by induction. It is not possible that $||x - a_k|| > r_i$ and yet have x in $\partial C_k^{r_i}$. This follows from the description in 36.11. If $||x - a_k|| = r_i$ then by definition, $x \in F$. The only other case to consider is that $x \notin \text{int}(B(a_k, r_i) \cap C)$ but $x \in B(a_k, r_i)$. From 36.11, every ball about x contains points of C. However, since $x \in B(a_k, r_i)$, a small enough ball is contained in $B(a_k, r_i)$. Therefore, every ball about x must also contain points of C^C since otherwise, $x \in \text{int}(B(a_k, r_i) \cap C)$. Thus $x \in \partial C \subseteq F$ by assumption. Now apply what was just shown to the case where $C = E$, the whole space. In this case, $\partial E \subseteq F$ because $\partial E = \emptyset$. Then keep applying what was just shown to the $A_{i_1, \cdots, i_n}$. This proves the claim.

From the claim, $\nu(\text{int}(A_{i_1, \cdots, i_n})) = \nu(A_{i_1, \cdots, i_n})$ whenever $\nu = \mu$ or μ_n.

Some functions on $[0, 1)$

By the axiom of choice, there exists $x_{i_1,\cdots,i_m} \in \text{int}(A_{i_1,\cdots,i_m})$ whenever $\text{int}(A_{i_1,\cdots,i_m}) \neq \emptyset$. For $\nu = \mu_n$ or μ, define the following functions. For $\omega \in I^{\nu}_{i_1,\cdots,i_m}$

$$Z^{\nu}_m(\omega) \equiv x_{i_1,\cdots,i_m}.$$

This defines the functions, $Z^{\mu_n}_m$ and Z^{μ}_m. Note these functions have the same values but on slightly different intervals. Here is an important claim.

Claim 2: For a.e. $\omega \in [0,1), \lim_{n\to\infty} Z^{\mu_n}_m(\omega) = Z^{\mu}_m(\omega)$.

Proof of the claim: This follows from the weak convergence of μ_n to μ and Lemma 36.25. This lemma implies $\mu_n(\text{int}(A_{i_1,\cdots,i_m})) \to \mu(\text{int}(A_{i_1,\cdots,i_m}))$. Thus by the construction described above, $\mu_n(A_{i_1,\cdots,i_m}) \to \mu(A_{i_1,\cdots,i_m})$ because of claim 1 and the construction of F in which it is always a set of measure zero. It follows that if $\omega \in \text{int}\left(I^{\mu}_{i_1,\cdots,i_m}\right)$, then for all n large enough, $\omega \in \text{int}\left(I^{\mu_n}_{i_1,\cdots,i_m}\right)$ and so $Z^{\mu_n}_m(\omega) = Z^{\mu}_m(\omega)$. Note this convergence is very far from being uniform.

Claim 3: For $\nu = \mu_n$ or $\mu, \{Z^{\nu}_m\}_{m=1}^{\infty}$ is uniformly Cauchy independent of n.

Proof of the claim: For $\omega \in I^{\nu}_{i_1,\cdots,i_m}$, then by the construction, $\omega \in I^{\nu}_{i_1,\cdots,i_m,i_{m+1}\cdots,i_n}$ for some $i_{m+1}\cdot\cdot,i_n$. Therefore, $Z^{\nu}_m(\omega)$ and $Z^{\nu}_n(\omega)$ are both contained in $A_{i_1,\cdots,i_m}$ which is contained in $B(a_{i_m}, r_m)$. Since $\omega \in [0,1)$ was arbitrary, and $r_m \to 0$, it follows these functions are uniformly Cauchy as claimed.

Let $X^{\nu}(\omega) = \lim_{m\to\infty} Z^{\nu}_m(\omega)$. Since each Z^{ν}_m is continuous off a set of measure zero, it follows from the uniform convergence that X^{ν} is also continuous off a set of measure zero.

Claim 4: For a.e. ω,

$$\lim_{n\to\infty} X^{\mu_n}(\omega) = X^{\mu}(\omega).$$

Proof of the claim: From Claim 3 and letting $\varepsilon > 0$ be given, there exists m large enough that for all n,

$$||Z^{\mu_n}_m - X^{\mu_n}||_{\infty} < \varepsilon/3, \ ||Z^{\mu}_m - X^{\mu}||_{\infty} < \varepsilon/3.$$

Now pick $\omega \in [0,1)$ such that ω is not equal to any of the end points of any of the intervals, $\left\{I^{\nu}_{i_1,\cdots,i_m}\right\}$, a set of measure zero. Then by Claim 2, there exists N such that if $n \geq N$, then $\left|\left|Z^{\mu_n}_m(\omega) - Z^{\mu}_m(\omega)\right|\right|_E < \varepsilon/3$. Therefore, for such n and this ω,

$$\begin{aligned} ||X^{\mu_n}(\omega) - X^{\mu}(\omega)||_E &\leq ||X^{\mu_n}(\omega) - Z^{\mu_n}_m(\omega)||_E + ||Z^{\mu_n}_m(\omega) - Z^{\mu}_m(\omega)||_E \\ &\quad + ||Z^{\mu}_m(\omega) - X^{\mu}(\omega)|| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

This proves the claim.

Showing $\mathcal{L}(X^{\nu}) = \nu$.

This has mostly proved the theorem except for the claim that $\mathcal{L}(X^{\nu}) = \nu$ for $\nu = \mu_n$ and μ. To do this, I will first show $m\left((X^{\nu})^{-1}(\partial A_{i_1,\cdots,i_m})\right) = 0$. By the construction, $\nu(\partial A_{i_1,\cdots,i_m}) = 0$. Let $\varepsilon > 0$ be given and let $\delta > 0$ be small enough that

$$H_\delta \equiv \{x \in E : \text{dist}(x, \partial A_{i_1,\cdots,i_m}) \leq \delta\}$$

is a set of measure less than $\varepsilon/2$. Denote by $\mathcal{G}_k$ the sets of the form $A_{i_1,\cdots,i_k}$ where $(i_1, \cdots, i_k) \in \mathbb{N}^k$. Recall also that corresponding to $A_{i_1,\cdots,i_k}$ is an interval, $I^\nu_{i_1,\cdots,i_k}$ having length equal to $\nu(A_{i_1,\cdots,i_k})$. Denote by $\mathcal{B}_k$ those sets of $\mathcal{G}_k$ which have nonempty intersection with H_δ and let the corresponding intervals be denoted by $\mathcal{I}^\nu_k$. If $\omega \notin \cup\mathcal{I}^\nu_k$, then from the construction, $Z^\nu_p(\omega)$ is at a distance of at least δ from $\partial A_{i_1,\cdots,i_m}$ for all $p \geq k$ and so, passing to the limit as $p \to \infty$, it follows $X^\nu(\omega) \notin \partial A_{i_1,\cdots,i_m}$. Therefore,

$$(X^\nu)^{-1}(\partial A_{i_1,\cdots,i_m}) \subseteq \cup\mathcal{I}^\nu_k$$

Recall that $A_{i_1,\cdots,i_k} \subseteq B(a_{i_k}, r_k)$ and the $r_k \to 0$. Therefore, if k is large enough,

$$\nu(\cup\mathcal{B}_k) < \varepsilon$$

because $\cup\mathcal{B}_k$ approximates H_δ closely (In fact, $\cap_{k=1}^\infty(\cup\mathcal{B}_k) = H_\delta$.). Therefore,

$$\begin{aligned} m\left((X^\nu)^{-1}(\partial A_{i_1,\cdots,i_m})\right) &\leq m(\cup\mathcal{I}^\nu_k) \\ &= \sum_{I^\nu_{i_1,\cdots,i_k} \in \mathcal{I}^\nu_k} m\left(I^\nu_{i_1,\cdots,i_k}\right) \\ &= \sum_{A_{i_1,\cdots,i_k} \in \mathcal{B}_k} \nu(A_{i_1,\cdots,i_k}) \\ &= \nu(\cup\mathcal{B}_k) < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows $m\left((X^\nu)^{-1}(\partial A_{i_1,\cdots,i_m})\right) = 0$.

If $\omega \in I^\nu_{i_1,\cdots,i_m}$, then from the construction, $Z^\nu_p(\omega) \in \text{int}(A_{i_1,\cdots,i_m})$ for all $p \geq k$. Therefore, taking a limit, as $p \to \infty$,

$$X^\nu(\omega) \in \text{int}(A_{i_1,\cdots,i_m}) \cup \partial A_{i_1,\cdots,i_m}$$

and so

$$I^\nu_{i_1,\cdots,i_m} \subseteq (X^\nu)^{-1}(\text{int}(A_{i_1,\cdots,i_m}) \cup \partial A_{i_1,\cdots,i_m})$$

but also, if $X^\nu(\omega) \in \text{int}(A_{i_1,\cdots,i_m})$, then $Z^\nu_p(\omega) \in \text{int}(A_{i_1,\cdots,i_m})$ for all p large enough and so

$$\begin{aligned} & (X^\nu)^{-1}(\text{int}(A_{i_1,\cdots,i_m})) \\ \subseteq\ & I^\nu_{i_1,\cdots,i_m} \\ \subseteq\ & (X^\nu)^{-1}(\text{int}(A_{i_1,\cdots,i_m}) \cup \partial A_{i_1,\cdots,i_m}) \end{aligned}$$

Therefore,

$$\begin{aligned} & m\left((X^{\nu})^{-1}(\operatorname{int}(A_{i_1,\cdots,i_m}))\right) \\ \leq\ & m\left(I^{\nu}_{i_1,\cdots,i_m}\right) \\ \leq\ & m\left((X^{\nu})^{-1}(\operatorname{int}(A_{i_1,\cdots,i_m}))\right) + m\left((X^{\nu})^{-1}(\partial A_{i_1,\cdots,i_m})\right) \\ =\ & m\left((X^{\nu})^{-1}(\operatorname{int}(A_{i_1,\cdots,i_m}))\right) \end{aligned}$$

which shows

$$m\left((X^{\nu})^{-1}(\operatorname{int}(A_{i_1,\cdots,i_m}))\right) = m\left(I^{\nu}_{i_1,\cdots,i_m}\right) = \nu\left(A_{i_1,\cdots,i_m}\right). \tag{36.12}$$

Also

$$\begin{aligned} & m\left((X^{\nu})^{-1}(\operatorname{int}(A_{i_1,\cdots,i_m}))\right) \\ \leq\ & m\left((X^{\nu})^{-1}(A_{i_1,\cdots,i_m})\right) \\ \leq\ & m\left((X^{\nu})^{-1}(\operatorname{int}(A_{i_1,\cdots,i_m}) \cup \partial A_{i_1,\cdots,i_m})\right) \\ =\ & m\left((X^{\nu})^{-1}(\operatorname{int}(A_{i_1,\cdots,i_m}))\right) \end{aligned}$$

Hence from 36.12,

$$\begin{aligned} \nu\left(A_{i_1,\cdots,i_m}\right) &= m\left((X^{\nu})^{-1}(\operatorname{int}(A_{i_1,\cdots,i_m}))\right) \\ &= m\left((X^{\nu})^{-1}(A_{i_1,\cdots,i_m})\right) \end{aligned} \tag{36.13}$$

Now let U be an open set in E. Then letting

$$H_k = \left\{x \in U : \operatorname{dist}\left(x, U^C\right) \geq r_k\right\}$$

it follows

$$\cup_k H_k = U.$$

Next consider the sets of $\mathcal{G}_k$ which have nonempty intersection with $H_k, \mathcal{H}_k$. Then H_k is covered by $\mathcal{H}_k$ and every set of $\mathcal{H}_k$ is contained in U, the sets of $\mathcal{H}_k$ also being disjoint. Then from 36.13,

$$\begin{aligned} m\left((X^{\nu})^{-1}(\cup\mathcal{H}_k)\right) &= \sum_{A\in\mathcal{H}_k} m\left((X^{\nu})^{-1}(A)\right) \\ &= \sum_{A\in\mathcal{H}_k} \nu(A) = \nu(\cup\mathcal{H}_k). \end{aligned}$$

Therefore, letting $k \to \infty$ and passing to the limit in the above,

$$m\left((X^{\nu})^{-1}(U)\right) = \nu(U).$$

Since this holds for every open set, it is routine to verify using regularity that it holds for every Borel set and so $\mathcal{L}(X^{\nu}) = \nu$ as claimed. This proves the theorem.

36.6 Characteristic Functions

Recall the characteristic function for a random variable having values in $\mathbb{R}^n$. I will give a review of this to begin with. Then the concept will be generalized to random variables (vectors) which have values in a real separable Banach space.

Definition 36.28 *Let* $\mathbf{X}$ *be a random variable. The characteristic function is*

$$\phi_{\mathbf{X}}(\mathbf{t}) \equiv E\left(e^{i\mathbf{t}\cdot\mathbf{X}}\right) \equiv \int_\Omega e^{i\mathbf{t}\cdot\mathbf{X}(\omega)}dP = \int_{\mathbb{R}^p} e^{i\mathbf{t}\cdot\mathbf{x}}d\lambda_{\mathbf{X}}$$

the last equation holding by Lemma 31.4 on Page 858.

Recall the following fundamental lemma and definition, Lemma 19.12 on Page 522.

Definition 36.29 *For* $T \in \mathcal{G}^*$, *define* $FT, F^{-1}T \in \mathcal{G}^*$ *by*

$$FT(\phi) \equiv T(F\phi),\ F^{-1}T(\phi) \equiv T\left(F^{-1}\phi\right)$$

Lemma 36.30 F *and* F^{-1} *are both one to one, onto, and are inverses of each other.*

The main result on characteristic functions is the following is in Theorem 31.10 on Page 860 which is stated here for convenience.

Theorem 36.31 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be random vectors with values in* $\mathbb{R}^p$ *and suppose* $E\left(e^{i\mathbf{t}\cdot\mathbf{X}}\right) = E\left(e^{i\mathbf{t}\cdot\mathbf{Y}}\right)$ *for all* $\mathbf{t} \in \mathbb{R}^p$. *Then* $\lambda_{\mathbf{X}} = \lambda_{\mathbf{Y}}$.

I want to do something similar for random variables which have values in a separable real Banach space, E instead of $\mathbb{R}^p$.

Corollary 36.32 *Let* $\mathcal{K}$ *be a* π *system of subsets of* Ω *and suppose two probability measures,* μ *and* ν *defined on* $\sigma(\mathcal{K})$ *are equal on* $\mathcal{K}$. *Then* $\mu = \nu$.

Proof: This follows from the Lemma 9.72 on Page 257. Let

$$\mathcal{G} \equiv \{E \in \sigma(\mathcal{K}) : \mu(E) = \nu(E)\}$$

Then $\mathcal{K} \subseteq \mathcal{G}$, since μ and ν are both probability measures, it follows that if $E \in \mathcal{G}$, then so is E^C. Since these are measures, if $\{A_i\}$ is a sequence of disjoint sets from $\mathcal{G}$ then

$$\mu\left(\cup_{i=1}^\infty A_i\right) = \sum_i \mu(A_i) = \sum_i \nu(A_i) = \nu\left(\cup_{i=1}^\infty A\right)$$

and so from Lemma 9.72, $\mathcal{G} = \sigma(\mathcal{K})$. This proves the corollary.

Next recall the following fundamental lemma used to prove Pettis' theorem. It is proved on Page 579 but is stated here for convenience.

Lemma 36.33 *If E is a separable Banach space with B' the closed unit ball in E', then there exists a sequence $\{f_n\}_{n=1}^{\infty} \equiv D' \subseteq B'$ with the property that for every $x \in E$,*

$$||x|| = \sup_{f \in D'} |f(x)|$$

Definition 36.34 *Let E be a separable real Banach space. A cylindrical set is one which is of the form*

$$\{x \in E : x_i^*(x) \in \Gamma_i, i = 1, 2, \cdots, m\}$$

where here $x_i^* \in E'$ and Γ_i is a Borel set in $\mathbb{R}$.

It is obvious that $\emptyset$ is a cylindrical set and that the intersection of two cylindrical sets is another cylindrical set. Thus the cylindrical sets form a π system. What is the smallest σ algebra containing the cylindrical sets? Letting $\{f_n\}_{n=1}^{\infty} = D'$ be the sequence of Lemma 36.33 it follows that

$$\begin{aligned}
& \{x \in E : ||x - a|| \leq \delta\} \\
= & \left\{x \in E : \sup_{f \in D'} |f(x - a)| \leq \delta\right\} \\
= & \left\{x \in E : \sup_{f \in D'} |f(x) - f(a)| \leq \delta\right\} \\
= & \cap_{n=1}^{\infty} \left\{x \in E : f_n(x) \in \overline{B(f_n(a), \delta)}\right\}
\end{aligned}$$

which yields a countable intersection of cylindrical sets. It follows the smallest σ algebra containing the cylindrical sets contains the closed balls and hence the open balls and consequently the open sets and so it contains the Borel sets. However, each cylindrical set is a Borel set and so in fact this σ algebra equals $\mathcal{B}(E)$.

From Corollary 36.32 it follows that two probability measures which are equal on the cylindrical sets are equal on the Borel sets, $\mathcal{B}(E)$.

Definition 36.35 *Let μ be a probability measure on a real separable Banach space, E. Then for $x^* \in E'$,*

$$\phi_\mu(x^*) \equiv \int_E e^{ix^*(x)} d\mu(x).$$

ϕ_μ is called the characteristic function for the measure μ.

Note this is a little different than earlier when the symbol $\phi_X(\mathbf{t})$ was used and X was a random variable. Here the focus is more on the measure than a random variable, X such that $\mathcal{L}(X) = \mu$ but it does not matter much because of Skorokhod's theorem presented above. The fundamental result is the following theorem.

Theorem 36.36 *Let μ and ν be two probability measures on $\mathcal{B}(E)$ where E is a separable real Banach space. Suppose*

$$\phi_\mu(x^*) = \phi_\nu(x^*)$$

for all $x^* \in E'$. Then $\mu = \nu$.

Proof: Let $x_1^*, \cdots, x_n^*$ be in E' and define for A a Borel set of $\mathbb{R}^n$,

$$\begin{aligned} \widetilde{\mu}(A) &\equiv \mu(\{x \in E : (x_1^*(x), \cdots, x_n^*(x)) \in A\}), \\ \widetilde{\nu}(A) &\equiv \nu(\{x \in E : (x_1^*(x), \cdots, x_n^*(x)) \in A\}). \end{aligned} \tag{36.14}$$

Note these sets in the parentheses are cylindrical sets. Letting $\boldsymbol{\lambda} \in \mathbb{R}^n$, consider in the definition of the characteristic function, $\lambda_1 x_1^* + \cdots + \lambda_n x_n^* \in E'$. Thus

$$\int_E e^{i(\lambda_1 x_1^*(x) + \cdots + \lambda_n x_n^*(x))} d\mu = \int_E e^{i(\lambda_1 x_1^*(x) + \cdots + \lambda_n x_n^*(x))} d\nu$$

Now if F is a Borel measurable subset of $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \mathcal{X}_F(\mathbf{y})\, d\widetilde{\mu}(y) = \widetilde{\mu}(F)$$

$$\begin{aligned} &\equiv \mu(\{x \in E : (x_1^*(x), \cdots, x_n^*(x)) \in F\}) \\ &= \int_E \mathcal{X}_F(x_1^*(x), \cdots, x_n^*(x))\, d\mu \end{aligned}$$

and using the usual approximations involving simple functions, it follows that for any f bounded and Borel measurable,

$$\int_{\mathbb{R}^n} f(\mathbf{y})\, d\widetilde{\mu}(y) = \int_E f((x_1^*(x), \cdots, x_n^*(x)))\, d\mu(x).$$

Similarly,

$$\int_{\mathbb{R}^n} f(\mathbf{y})\, d\widetilde{\nu}(y) = \int_E f((x_1^*(x), \cdots, x_n^*(x)))\, d\nu(x),$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^n} e^{i\boldsymbol{\lambda}\cdot\mathbf{y}} d\widetilde{\mu}(y) &= \int_E e^{i(\lambda_1 x_1^*(x) + \cdots + \lambda_n x_n^*(x))} d\mu \\ &= \int_E e^{i(\lambda_1 x_1^*(x) + \cdots + \lambda_n x_n^*(x))} d\nu \\ &= \int_{\mathbb{R}^n} e^{i\boldsymbol{\lambda}\cdot\mathbf{y}} d\widetilde{\nu}(y) \end{aligned}$$

which shows from Theorem 36.31 that $\widetilde{\nu} = \widetilde{\mu}$ on the Borel sets of $\mathbb{R}^n$. However, from the definition of these measures in 36.14 this says nothing more than $\mu = \nu$

on any cylindrical set. Hence by Corollary 36.32 this shows $\mu = \nu$ on $\mathcal{B}(E)$. This proves the theorem.

Finally, I will consider the relation between the characteristic function and independence of random variables. Recall an earlier proposition which relates independence of random vectors with characteristic functions. It is proved starting on Page 865 in the case of two random variables and concludes with the observation that the general case is entirely similar but more tedious to write down.

Proposition 36.37 *Let $\{\mathbf{X}_k\}_{k=1}^n$ be random vectors such that $\mathbf{X}_k$ has values in $\mathbb{R}^{p_k}$. Then the random vectors are independent if and only if*

$$E\left(e^{iP}\right) = \prod_{j=1}^{n} E\left(e^{i\mathbf{t}_j\cdot\mathbf{X}_j}\right)$$

where $P \equiv \sum_{j=1}^n \mathbf{t}_j \cdot \mathbf{X}_j$ for $\mathbf{t}_j \in \mathbb{R}^{p_j}$.

It turns out there is a generalization of the above proposition to the case where the random variables have values in a real separable Banach space. Before proving this recall an earlier theorem which had to do with reducing to the case where the random variables had values in $\mathbb{R}^n$. It is restated here for convenience.

Theorem 36.38 *The random variables, $\{X_i\}_{i\in I}$ are independent if whenever*

$$\{i_1, \cdots, i_n\} \subseteq I,$$

$m_{i_1}, \cdots, m_{i_n}$ are positive integers, and $\mathbf{g}_{m_{i_1}}, \cdots, \mathbf{g}_{m_{i_n}}$ are in $(E')^{m_{i_1}}, \cdots, (E')^{m_n}$ respectively, $\left\{\mathbf{g}_{m_{i_j}} \circ X_{i_j}\right\}_{j=1}^n$ are independent random vectors having values in $\mathbb{R}^{m_{i_1}}, \cdots, \mathbb{R}^{m_{i_n}}$ respectively.

Now here is the theorem about independence and the characteristic functions.

Theorem 36.39 *Let $\{X_k\}_{k=1}^n$ be random variables having values in E, a real separable Banach space. Then the random variables are independent if and only if*

$$E\left(e^{iP}\right) = \prod_{j=1}^{n} E\left(e^{it_j^*(X_j)}\right)$$

where $P \equiv \sum_{j=1}^n t_j^*(X_j)$ for $t_j^* \in E'$.

Proof: If the random variables are independent, then so are the random variables, $t_j^*(X_j)$ and so the equation follows.

The interesting case is when the equation holds. Can you draw the conclusion the random variables are independent? By Theorem 36.38, it suffices to show the random variables $\{\mathbf{g}_{m_k} \circ X_k\}_{k=1}^n$ are independent. This happens if whenever $\mathbf{t}_{m_k} \in \mathbb{R}^{m_k}$ and

$$P = \sum_{k=1}^{n} \mathbf{t}_{m_k} \cdot (\mathbf{g}_{m_k} \circ X_k),$$

it follows

$$E\left(e^{iP}\right)=\prod_{j=1}^{n}E\left(e^{i\mathbf{t}_{m_k}\cdot\left(\mathbf{g}_{m_k}\circ X_k\right)}\right). \tag{36.15}$$

Now consider one of these terms in the exponent on the right.

$$\begin{aligned}\mathbf{t}_{m_k}\cdot\left(\mathbf{g}_{m_k}\circ X_k\right)(\omega) &= \sum_{j=1}^{m_k}t_jx_j^*\left(X_k(\omega)\right)\\ &= y_k^*\left(X_k(\omega)\right)\end{aligned}$$

where $y^*\equiv\sum_{j=1}^{m_k}t_jx_j^*$. Therefore, 36.15 reduces to

$$E\left(e^{i\sum_{k=1}^{n}y_k^*(X_k)}\right)=\prod_{k=1}^{n}E\left(e^{iy_k^*(X_k)}\right)$$

which is assumed to hold. Therefore, the random variables are independent. This proves the theorem.

There is an obvious corollary which is useful.

Corollary 36.40 *Let $\{X_k\}_{k=1}^n$ be random variables having values in E, a real separable Banach space. Then the random variables are independent if and only if*

$$E\left(e^{iP}\right)=\prod_{j=1}^{n}E\left(e^{it_j^*(X_j)}\right)$$

where $P\equiv\sum_{j=1}^{n}t_j^*(X_j)$ for $t_j^*\in M$ where M is a dense subset of E'.

Proof: The easy direction follows from Theorem 36.39. Suppose then the above equation holds for all $t_j^*\in M$. Then let $t_j^*\in E'$ and let $\left\{t_{nj}^*\right\}$ be a sequence in M such that

$$\lim_{n\to\infty}t_{nj}^*=t_j^* \text{ in } E'$$

Then define

$$P\equiv\sum_{j=1}^{n}t_j^*X_j,\ P_n\equiv\sum_{j=1}^{n}t_{nj}^*X_j.$$

It follows

$$\begin{aligned}E\left(e^{iP}\right) &= \lim_{n\to\infty}E\left(e^{iP_n}\right)\\ &= \lim_{n\to\infty}\prod_{j=1}^{n}E\left(e^{it_{nj}^*(X_j)}\right)\\ &= \prod_{j=1}^{n}E\left(e^{it_j^*(X_j)}\right)\end{aligned}$$

and this proves the corollary.

36.7 Convolution

Lemma 36.18 on Page 1002 makes possible a definition of convolution of two probability measures defined on $\mathcal{B}(E)$ where E is a separable Banach space as well as some other interesting theorems which held earlier in the context of locally compact spaces. I will first show a little theorem about density of continuous functions in $L^p(E)$ and then define the convolution of two finite measures. First here is a simple technical lemma.

Lemma 36.41 *Suppose K is a compact subset of U an open set in E a metric space. Then there exists $\delta > 0$ such that*

$$\operatorname{dist}(x, K) + \operatorname{dist}\left(x, U^C\right) \geq \delta \text{ for all } x \in E.$$

Proof: For each $x \in K$, there exists a ball, $B(x, \delta_x)$ such that $B(x, 3\delta_x) \subseteq U$. Finitely many of these balls cover K because K is compact, say $\{B(x_i, \delta_{x_i})\}_{i=1}^m$. Let

$$0 < \delta < \min\left(\delta_{x_i} : i = 1, 2, \cdots, m\right).$$

Now pick any $x \in K$. Then $x \in B(x_i, \delta_{x_i})$ for some x_i and so $B(x, \delta) \subseteq B(x_i, 2\delta_{x_i}) \subseteq U$. Therefore, for any $x \in K, \operatorname{dist}\left(x, U^C\right) \geq \delta$. If $x \in B(x_i, 2\delta_{x_i})$ for some x_i, it follows $\operatorname{dist}\left(x, U^C\right) \geq \delta$ because then $B(x, \delta) \subseteq B(x_i, 3\delta_{x_i}) \subseteq U$. If $x \notin B(x_i, 2\delta_{x_i})$ for any of the x_i, then $x \notin B(y, \delta)$ for any $y \in K$ because all these sets are contained in some $B(x_i, 2\delta_{x_i})$. Consequently $\operatorname{dist}(x, K) \geq \delta$. This proves the lemma.

From this lemma, there is an easy corollary.

Corollary 36.42 *Suppose K is a compact subset of U, an open set in E a metric space. Then there exists a uniformly continuous function f defined on all of E, having values in $[0, 1]$ such that $f(x) = 0$ if $x \notin U$ and $f(x) = 1$ if $x \in K$.*

Proof: Consider

$$f(x) \equiv \frac{\operatorname{dist}\left(x, U^C\right)}{\operatorname{dist}\left(x, U^C\right) + \operatorname{dist}(x, K)}.$$

Then some algebra yields

$$|f(x) - f(x')| \leq$$

$$\frac{1}{\delta}\left(\left|\operatorname{dist}\left(x, U^C\right) - \operatorname{dist}\left(x', U^C\right)\right| + \left|\operatorname{dist}(x, K) - \operatorname{dist}(x', K)\right|\right)$$

where δ is the constant of Lemma 36.41. Now it is a general fact that

$$\left|\operatorname{dist}(x, S) - \operatorname{dist}(x', S)\right| \leq d(x, x').$$

Therefore,

$$|f(x) - f(x')| \leq \frac{2}{\delta} d(x, x')$$

and this proves the corollary.

Now suppose μ is a finite measure defined on the Borel sets of a separable Banach space, E. It was shown above that μ is inner and outer regular. Lemma 36.18 on Page 1002 shows that μ is inner regular in the usual sense with respect to compact sets. This makes possible the following theorem.

Theorem 36.43 *Let μ be a finite measure on $\mathcal{B}(E)$ where E is a separable Banach space and let $f \in L^p(E;\mu)$. Then for any $\varepsilon > 0$, there exists a uniformly continuous, bounded g defined on E such that*

$$||f-g||_{L^p(E)} < \varepsilon.$$

Proof: As usual in such situations, it suffices to consider only $f \geq 0$. Then by Theorem 8.27 on Page 190 and an application of the monotone convergence theorem, there exists a simple measurable function,

$$s(x) \equiv \sum_{k=1}^{m} c_k \mathcal{X}_{A_k}(x)$$

such that $||f-s||_{L^p(E)} < \varepsilon/2$. Now by regularity of μ there exist compact sets, K_k and open sets, V_k such that $2\sum_{k=1}^m |c_k| \mu(V_k \setminus K)^{1/p} < \varepsilon/2$ and by Corollary 36.42 there exist uniformly continuous functions g_k having values in $[0,1]$ such that $g_k = 1$ on K_k and 0 on V_k^C. Then consider

$$g(x) = \sum_{k=1}^{m} c_k g_k(x).$$

This function is bounded and uniformly continuous. Furthermore,

$$\begin{aligned}
||s-g||_{L^p(E)} &\leq \left(\int_E \left|\sum_{k=1}^m c_k \mathcal{X}_{A_k}(x) - \sum_{k=1}^m c_k g_k(x)\right|^p d\mu\right)^{1/p} \\
&\leq \left(\int_E \left(\sum_{k=1}^m |c_k| \left|\mathcal{X}_{A_k}(x) - g_k(x)\right|\right)^p\right)^{1/p} \\
&\leq \sum_{k=1}^m |c_k| \left(\int_E \left|\mathcal{X}_{A_k}(x) - g_k(x)\right|^p d\mu\right)^{1/p} \\
&\leq \sum_{k=1}^m |c_k| \left(\int_{V_k \setminus K_k} 2^p d\mu\right)^{1/p} \\
&= 2\sum_{k=1}^m |c_k| \mu(V_k \setminus K)^{1/p} < \varepsilon/2.
\end{aligned}$$

Therefore,

$$||f-g||_{L^p} \leq ||f-s||_{L^p} + ||s-g||_{L^p} < \varepsilon/2 + \varepsilon/2.$$

This proves the theorem.

Lemma 36.44 *Let $A \in \mathcal{B}(E)$ where μ is a finite measure on $\mathcal{B}(E)$ for E a separable Banach space. Also let $x_i \in E$ for $i = 1, 2, \cdots, m$. Then for $\mathbf{x} \in E^m$,*

$$\mathbf{x} \to \mu\left(A + \sum_{i=1}^m x_i\right), \ \mathbf{x} \to \mu\left(A - \sum_{i=1}^m x_i\right)$$

are Borel measurable functions. Furthermore, the above functions are

$$\mathcal{B}(E) \times \cdots \times \mathcal{B}(E)$$

measurable where the above denotes the product measurable sets as described in Theorem 9.75 on Page 260.

Proof: First consider the case where $A = U$, an open set. Let

$$\mathbf{y} \in \left\{\mathbf{x} \in E^m : \mu\left(U + \sum_{i=1}^m x_i\right) > \alpha\right\} \tag{36.16}$$

Then from Lemma 36.18 on Page 1002 there exists a compact set, $K \subseteq U + \sum_{i=1}^m y_i$ such that $\mu(K) > \alpha$. Then if $\mathbf{y}'$ is close enough to $\mathbf{y}$, it follows $K \subseteq U + \sum_{i=1}^m y_i'$ also. Therefore, for all $\mathbf{y}'$ close enough to $\mathbf{y}$,

$$\mu\left(U + \sum_{i=1}^m y_i'\right) \geq \mu(K) > \alpha.$$

In other words the set described in 36.16 is an open set and so $\mathbf{y} \to \mu\left(U + \sum_{i=1}^m y_i\right)$ is Borel measurable whenever U is an open set in E.

Define a π system, $\mathcal{K}$ to consist of all open sets in E. Then define $\mathcal{G}$ as

$$\left\{A \in \sigma(\mathcal{K}) = \mathcal{B}(E) : \mathbf{y} \to \mu\left(A + \sum_{i=1}^m y_i\right) \text{ is Borel measurable}\right\}$$

I just showed $\mathcal{G} \supseteq \mathcal{K}$. Now suppose $A \in \mathcal{G}$. Then

$$\mu\left(A^C + \sum_{i=1}^m y_i\right) = \mu(E) - \mu\left(A + \sum_{i=1}^m y_i\right)$$

and so $A^C \in \mathcal{G}$ whenever $A \in \mathcal{G}$. Next suppose $\{A_i\}$ is a sequence of disjoint sets of $\mathcal{G}$. Then

$$\begin{aligned} \mu\left((\cup_{i=1}^\infty A_i) + \sum_{j=1}^m y_j\right) &= \mu\left(\cup_{i=1}^\infty\left(A_i + \sum_{j=1}^m y_j\right)\right) \\ &= \sum_{i=1}^\infty \mu\left(A_i + \sum_{j=1}^m y_j\right) \end{aligned}$$

and so $\cup_{i=1}^{\infty} A_i \in \mathcal{G}$ because it is the sum of Borel measurable functions. By the lemma on π systems, Lemma 9.72 on Page 257, it follows $\mathcal{G} = \sigma(\mathcal{K}) = \mathcal{B}(E)$. Similarly, $\mathbf{x} \to \mu\left(A - \sum_{j=1}^{m} x_j\right)$ is also Borel measurable whenever $A \in \mathcal{B}(E)$. Finally note that

$$\mathcal{B}(E) \times \cdots \times \mathcal{B}(E)$$

contains the open sets of E^m because the separability of E implies the existence of a countable basis for the topology of E^m consisting of sets of the form

$$\prod_{i=1}^{m} U_i$$

where the U_i come from a countable basis for E. Since every open set is the countable union of sets like the above, each being a measurable box, the open sets are contained in

$$\mathcal{B}(E) \times \cdots \times \mathcal{B}(E)$$

which implies $\mathcal{B}(E^m) \subseteq \mathcal{B}(E) \times \cdots \times \mathcal{B}(E)$ also. This proves the lemma.

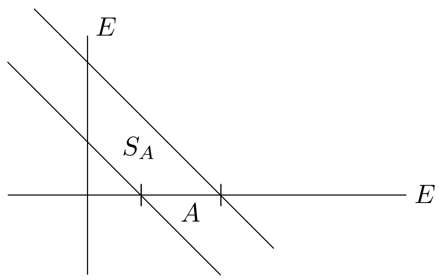
With this lemma, it is possible to define the convolution of two finite measures.

Definition 36.45 *Let μ and ν be two finite measures on $\mathcal{B}(E)$, for E a separable Banach space. Then define a new measure, $\mu * \nu$ on $\mathcal{B}(E)$ as follows*

$$\mu * \nu(A) \equiv \int_E \nu(A - x)\, d\mu(x).$$

This is well defined because of Lemma 36.44 which says that $x \to \nu(A - x)$ is Borel measurable.

Here is an interesting theorem about convolutions. However, first here is a little lemma. The following picture is descriptive of the set described in the following lemma.



Lemma 36.46 *For A a Borel set in E, a separable Banach space, define*

$$S_A \equiv \{(x, y) \in E \times E : x + y \in A\}$$

Then $S_A \in \mathcal{B}(E) \times \mathcal{B}(E)$, the σ algebra of product measurable sets, the smallest σ algebra which contains all the sets of the form $A \times B$ where A and B are Borel.

Proof: Let $\mathcal{K}$ denote the open sets in E. Then $\mathcal{K}$ is a π system. Let

$$\mathcal{G} \equiv \{A \in \sigma(\mathcal{K}) = \mathcal{B}(E) : S_A \in \mathcal{B}(E) \times \mathcal{B}(E)\}.$$

Then $\mathcal{K} \subseteq \mathcal{G}$ because if $U \in \mathcal{K}$ then S_U is an open set in $E \times E$ and all open sets are in $\mathcal{B}(E) \times \mathcal{B}(E)$ because a countable basis for the topology of $E \times E$ are sets of the form $B \times C$ where B and C come from a countable basis for E. Therefore, $\mathcal{K} \subseteq \mathcal{G}$. Now let $A \in \mathcal{G}$. For $(x, y) \in E \times E$, either $x + y \in A$ or $x + y \notin A$. Hence $E \times E = S_A \cup S_{A^C}$ which shows that if $A \in \mathcal{G}$ then so is A^C. Finally if $\{A_i\}$ is a sequence of disjoint sets of $\mathcal{G}$

$$S_{\cup_{i=1}^{\infty} A_i} = \cup_{i=1}^{\infty} S_{A_i}$$

and this shows that $\mathcal{G}$ is also closed with respect to countable unions of disjoint sets. Therefore, by the lemma on π systems, Lemma 9.72 on Page 257 it follows $\mathcal{G} = \sigma(\mathcal{K}) = \mathcal{B}(E)$. This proves the lemma.

Theorem 36.47 *Let μ, ν, and λ be finite measures on $\mathcal{B}(E)$ for E a separable Banach space. Then*

$$\mu * \nu = \nu * \mu \tag{36.17}$$

$$(\mu * \nu) * \lambda = \mu * (\nu * \lambda) \tag{36.18}$$

If μ is the distribution for an E valued random variable, X and if ν is the distribution for an E valued random variable, Y, and X and Y are independent, then $\mu * \nu$ is the distribution for the random variable, $X + Y$.

Proof: First consider 36.17. Letting $A \in \mathcal{B}(E)$, the following computation holds from Fubini's theorem and Lemma 36.46

$$\begin{aligned}
\mu * \nu(A) &\equiv \int_E \nu(A - x)\, d\mu(x) = \int_E \int_E \mathcal{X}_{S_A}(x, y)\, d\nu(y)\, d\mu(x) \\
&= \int_E \int_E \mathcal{X}_{S_A}(x, y)\, d\mu(x)\, d\nu(y) = \nu * \mu(A).
\end{aligned}$$

Next consider 36.18. Using 36.17 whenever convenient,

$$\begin{aligned}
(\mu * \nu) * \lambda(A) &\equiv \int_E (\mu * \nu)(A - x)\, d\lambda(x) \\
&= \int_E \int_E \nu(A - x - y)\, d\mu(y)\, d\lambda(x)
\end{aligned}$$

while

$$\begin{aligned}
\mu * (\nu * \lambda)(A) &\equiv \int_E (\nu * \lambda)(A - y)\, d\mu(y) \\
&= \int_E \int_E \nu(A - y - x)\, d\lambda(x)\, d\mu(y) \\
&= \int_E \int_E \nu(A - y - x)\, d\mu(y)\, d\lambda(x).
\end{aligned}$$

The necessary product measurability comes from Lemma 36.44.

Recall

$$(\mu * \nu)(A) \equiv \int_E \nu(A - x)\, d\mu(x).$$

Therefore, if s is a simple function, $s(x) = \sum_{k=1}^n c_k \mathcal{X}_{A_k}(x)$,

$$\begin{aligned}
\int_E s d(\mu * \nu) &= \sum_{k=1}^n c_k \int_E \nu(A_k - x)\, d\mu(x) \\
&= \int_E \sum_{k=1}^n c_k \nu(A_k - x)\, d\mu(x) \\
&= \int_E \int_E s(x+y)\, d\nu(x)\, d\mu(y)
\end{aligned}$$

Approximating with simple functions it follows that whenever f is bounded and measurable or nonnegative and measurable,

$$\int_E f d(\mu * \nu) = \int_E \int_E f(x+y)\, d\nu(y)\, d\mu(x) \tag{36.19}$$

Therefore, letting $Z = X + Y$, and λ the distribution of Z, it follows from independence of X and Y that for $t^* \in E'$,

$$\phi_\lambda(t^*) \equiv E\left(e^{it^*(Z)}\right) = E\left(e^{it^*(X+Y)}\right) = E\left(e^{it^*(X)}\right) E\left(e^{it^*(Y)}\right)$$

But also, it follows from 36.19

$$\begin{aligned}
\phi_{(\mu*\nu)}(t^*) &= \int_E e^{it^*(z)} d(\mu * \nu)(z) \\
&= \int_E \int_E e^{it^*(x+y)} d\nu(y)\, d\mu(x) \\
&= \int_E \int_E e^{it^*(x)} e^{it^*(y)} d\nu(y)\, d\mu(x) \\
&= \left(\int_E e^{it^*(y)} d\nu(y)\right)\left(\int_E e^{it^*(x)} d\mu(x)\right) \\
&= E\left(e^{it^*(X)}\right) E\left(e^{it^*(Y)}\right)
\end{aligned}$$

Since $\phi_\lambda(t^*) = \phi_{(\mu*\nu)}(t^*)$, it follows $\lambda = \mu * \nu$. This proves the theorem.

Note the last part of this argument shows the characteristic function of a convolution equals the product of the characteristic functions.

36.8 The Multivariate Normal Distribution

Here I give a review of the main theorems and definitions about multivariate normal random variables. Recall that for a random vector (variable), $\mathbf{X}$ having values in

$\mathbb{R}^p$, $\lambda_{\mathbf{X}}$ is the law of $\mathbf{X}$ defined by

$$P([\mathbf{X} \in E]) = \lambda_{\mathbf{X}}(E)$$

for all E a Borel set in $\mathbb{R}^p$. In different notaion, $\mathcal{L}(\mathbf{X}) = \lambda_{\mathbf{X}}$. Then the following definitions and theorems are proved and presented starting on Page 867

Definition 36.48 *A random vector, $\mathbf{X}$, with values in $\mathbb{R}^p$ has a multivariate normal distribution written as $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ if for all Borel $E \subseteq \mathbb{R}^p$,*

$$\lambda_{\mathbf{X}}(E) = \int_{\mathbb{R}^p} \mathcal{X}_E(\mathbf{x}) \frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^* \Sigma^{-1}(\mathbf{x}-\mathbf{m})} dx$$

for μ a given vector and Σ a given positive definite symmetric matrix.

Theorem 36.49 *For $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma), \mathbf{m} = E(\mathbf{X})$ and*

$$\Sigma = E\left((\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^*\right).$$

Theorem 36.50 *Suppose $\mathbf{X}_1 \sim N_p(\mathbf{m}_1, \Sigma_1)$, $\mathbf{X}_2 \sim N_p(\mathbf{m}_2, \Sigma_2)$ and the two random vectors are independent. Then*

$$\mathbf{X}_1 + \mathbf{X}_2 \sim N_p(\mathbf{m}_1 + \mathbf{m}_2, \Sigma_1 + \Sigma_2). \tag{36.20}$$

Also, if $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ then $-\mathbf{X} \sim N_p(-\mathbf{m}, \Sigma)$. Furthermore, if $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ then

$$E\left(e^{i\mathbf{t}\cdot\mathbf{X}}\right) = e^{i\mathbf{t}\cdot\mathbf{m}} e^{-\frac{1}{2}\mathbf{t}^*\Sigma\mathbf{t}} \tag{36.21}$$

Also if a is a constant and $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ then $a\mathbf{X} \sim N_p\left(a\mathbf{m}, a^2\Sigma\right)$.

Following [42] a random vector has a generalized normal distribution if its characteristic function is given as

$$e^{i\mathbf{t}\cdot\mathbf{m}} e^{-\frac{1}{2}\mathbf{t}^*\Sigma\mathbf{t}} \tag{36.22}$$

where Σ is symmetric and has nonnegative eigenvalues. For a random real valued variable, $\mathbf{m}$ is scalar and so is Σ so the characteristic function of such a generalized normally distributed random variable is

$$e^{itm} e^{-\frac{1}{2}t^2\sigma^2} \tag{36.23}$$

These generalized normal distributions do not require Σ to be invertible, only that the eigenvalues by nonnegative. In one dimension this would correspond the characteristic function of a dirac measure having point mass 1 at m. In higher dimensions, it could be a mixture of such things with more familiar things. I will often not bother to distinguish between generalized normal and normal distributions.

Here are some other interesting results about normal distributions found in [42]. The next theorem has to do with the question whether a random vector is normally distributed in the above generalized sense. It is proved on Page 870.

Theorem 36.51 *Let* $\mathbf{X} = (X_1, \cdot\cdot\cdot, X_p)$ *where each* X_i *is a real valued random variable. Then* $\mathbf{X}$ *is normally distributed in the above generalized sense if and only if every linear combination,* $\sum_{j=1}^{p} a_i X_i$ *is normally distributed. In this case the mean of* $\mathbf{X}$ *is*

$$\mathbf{m} = (E(X_1), \cdot\cdot\cdot, E(X_p))$$

and the covariance matrix for $\mathbf{X}$ *is*

$$\Sigma_{jk} = E((X_j - m_j)(X_k - m_k))$$

where $m_j = E(X_j)$.

Also proved there is the interesting corollary listed next.

Corollary 36.52 *Let* $\mathbf{X} = (X_1, \cdot\cdot\cdot, X_p), \mathbf{Y} = (Y_1, \cdot\cdot\cdot, Y_p)$ *where each* X_i, Y_i *is a real valued random variable. Suppose also that for every* $\mathbf{a} \in \mathbb{R}^p$, $\mathbf{a} \cdot \mathbf{X}$ *and* $\mathbf{a} \cdot \mathbf{Y}$ *are both normally distributed with the same mean and variance. Then* $\mathbf{X}$ *and* $\mathbf{Y}$ *are both multivariate normal random vectors with the same mean and variance.*

Theorem 36.53 *Suppose* $\mathbf{X} = (X_1, \cdot\cdot\cdot, X_p)$ *is normally distributed with mean* $\mathbf{m}$ *and covariance* Σ. *Then if* X_1 *is uncorrelated with any of the* X_i,

$$E((X_1 - m_1)(X_j - m_j)) = 0 \text{ for } j > 1$$

then X_1 *and* $(X_2, \cdot\cdot\cdot, X_p)$ *are both normally distributed and the two random vectors are independent. Here* $m_j \equiv E(X_j)$.

Next I will consider the question of existence of independent random variables having a given law.

Lemma 36.54 *Let* μ *be a probability measure on* $\mathcal{B}(E)$, *the Borel subsets of a separable real Banach space. Then there exists a probability space* $(\Omega, \mathcal{F}, P)$ *and two independent random variables,* X, Y *mapping* Ω *to* E *such that* $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$.

Proof: First note that if A, B are Borel sets of E then $A \times B$ is a Borel set in $E \times E$ where the norm on $E \times E$ is given by

$$||(x, y)|| \equiv \max(||x||, ||y||).$$

This can be proved by letting A be open and considering

$$\mathcal{G} \equiv \{B \in \mathcal{B}(E) : A \times B \in \mathcal{B}(A \times B)\}.$$

Show $\mathcal{G}$ is a σ algebra and it contains the open sets. Therefore, this will show $A \times B$ is in $\mathcal{B}(A \times B)$ whenever A is open and B is Borel. Next repeat a similar argument to show that this is true whenever either set is Borel. Since E is separable, it is completely separable and so is $E \times E$. Thus every open set in $E \times E$ is the union of balls from a countable set. However, these balls are of the form $B_1 \times B_2$ where B_i is a ball in E. Now let

$$\mathcal{K} \equiv \{A \times B : A, B \text{ are Borel}\}$$

Then $\mathcal{K} \subseteq \mathcal{B}(E \times E)$ as was just shown and also every open set from $E \times E$ is in $\sigma(\mathcal{K})$. It follows $\sigma(\mathcal{K})$ equals the σ algebra of product measurable sets, $\mathcal{B}(E) \times \mathcal{B}(E)$ and you can consider the product measure, $\mu \times \mu$. By Skorokhod's theorem, Theorem 36.27, there exists (X, Y) a random variable with values in $E \times E$ and a probability space, $(\Omega, \mathcal{F}, P)$ such that $\mathcal{L}((X, Y)) = \mu \times \mu$. Then for A, B Borel sets in E

$$P(X \in A, Y \in B) = (\mu \times \mu)(A \times B) = \mu(A)\mu(B).$$

Also, $P(X \in A) = P(X \in A, Y \in E) = \mu(A)$ and similarly, $P(Y \in B) = \mu(B)$ showing $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$ and X, Y are independent.

Now here is an interesting theorem in [15].

Theorem 36.55 *Suppose ν is a probability measure on the Borel sets of $\mathbb{R}$ and suppose that ξ and ζ are independent random variables such that $\mathcal{L}(\xi) = \mathcal{L}(\zeta) = \nu$ and whenever $\alpha^2 + \beta^2 = 1$ it follows $\mathcal{L}(\alpha\xi + \beta\zeta) = \nu$. Then*

$$\mathcal{L}(\xi) = N\left(0, \sigma^2\right)$$

for some $\sigma \geq 0$. Also if $\mathcal{L}(\xi) = \mathcal{L}(\zeta) = N\left(0, \sigma^2\right)$ where ξ, ζ are independent, then if $\alpha^2 + \beta^2 = 1$, it follows $\mathcal{L}(\alpha\xi + \beta\zeta) = N\left(0, \sigma^2\right)$.

Proof: Let ξ, ζ be independent random variables with $\mathcal{L}(\xi) = \mathcal{L}(\zeta) = \nu$ and whenever $\alpha^2 + \beta^2 = 1$ it follows $\mathcal{L}(\alpha\xi + \beta\zeta) = \nu$.

By independence of ξ and ζ,

$$\begin{aligned}
\phi_\nu(t) &\equiv \phi_{\alpha\xi+\alpha\zeta}(t) \\
&= E\left(e^{it(\alpha\xi+\beta\zeta)}\right) \\
&= E\left(e^{it\alpha\xi}\right) E\left(e^{it\beta\zeta}\right) \\
&= \phi_\xi(\alpha t)\,\phi_\zeta(\beta t) \\
&\equiv \phi_\nu(\alpha t)\,\phi_\nu(\beta t)
\end{aligned}$$

In simpler terms and suppressing the subscript,

$$\phi(t) = \phi(\cos(\theta)\,t)\,\phi(\sin(\theta)\,t). \tag{36.24}$$

Since ν is a probability measure, $\phi(0) = 1$. Also, letting $\theta = \pi/4$, this yields

$$\phi(t) = \phi\left(\frac{\sqrt{2}}{2}t\right)^2 \tag{36.25}$$

and so if ϕ has real values, then $\phi(t) \geq 0$.

Next I will show ϕ is real. To do this, it follows from the definition of ϕ_ν,

$$\phi_\nu(-t) \equiv \int_{\mathbb{R}} e^{-itx} d\nu = \overline{\int_{\mathbb{R}} e^{itx} d\nu} = \overline{\phi_\nu(t)}.$$

Then letting $\theta = \pi$,

$$\phi(t) = \phi(-t) \cdot \phi(0) = \phi(-t) = \overline{\phi(t)}$$

showing ϕ has real values. It is positive near 0 because $\phi(0) = 1$ and ϕ is a continuous function of t thanks to the dominated convergence theorem. However, this and 36.25 implies it is positive everywhere. Here is why. If not, let t_m be the smallest positive value of t where $\phi(t) = 0$. Then $t_m > 0$ by continuity. Now from 36.25, an immediate contradiction results. Therefore, $\phi(t) > 0$ for all $t > 0$. Similar reasoning yields the same conclusion for $t < 0$.

Next note that $\phi(t) = \phi(-t)$ also implies ϕ depends only on $|t|$ because it takes the same value for t as for $-t$. More simply, ϕ depends only on t^2. Thus one can define a new function of the form $\phi(t) = f(t^2)$ and 36.24 implies the following for $\alpha \in [0,1]$.

$$f(t^2) = f(\alpha^2 t^2) f((1-\alpha^2) t^2).$$

Taking ln of both sides, one obtains the following for $g(t^2) \equiv \ln f(t^2)$.

$$\ln f(t^2) = \ln f(\alpha^2 t^2) + \ln f((1-\alpha^2) t^2).$$

Now letting $x = \alpha^2 t^2$ and $y = (1-\alpha^2) t^2$, it follows that for all $x \geq 0$

$$\ln f(x+y) = \ln f(x) + \ln f(y).$$

Hence $\ln f(x) = kx$ and so $\ln f(t^2) = kt^2$ and so $\phi(t) = f(t^2) = e^{kt^2}$for all t. The constant, k must be nonpositive because $\phi(t)$ is bounded due to its definition. Therefore, the characteristic function of ν is

$$\phi_\nu(t) = e^{-\frac{1}{2}t^2\sigma^2}$$

for some $\sigma \geq 0$. That is, ν is the law of a generalized normal random variable.

Note the other direction of the implication is obvious. If $\xi, \zeta \sim N(0,\sigma)$ and they are independent, then if $\alpha^2 + \beta^2 = 1$, it follows

$$\alpha\xi + \beta\zeta \sim N(0, \sigma^2)$$

because

$$\begin{aligned} E\left(e^{it(\alpha\xi+\beta\zeta)}\right) &= E\left(e^{it\alpha\xi}\right) E\left(e^{it\beta\zeta}\right) \\ &= e^{-\frac{1}{2}(\alpha t)^2\sigma^2} e^{-\frac{1}{2}(\beta t)^2\sigma^2} \\ &= e^{-\frac{1}{2}t^2\sigma^2}, \end{aligned}$$

the characteristic function for a random variable which is $N(0,\sigma)$. This proves the theorem.

The next theorem is a useful gimick for showing certain random variables are independent in the context of normal distributions.

Theorem 36.56 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be random vectors having values in* $\mathbb{R}^p$ *and* $\mathbb{R}^q$ *respectively. Suppose also that* $(\mathbf{X},\mathbf{Y})$ *is multivariate normally distributed and*

$$E\left((\mathbf{X}-E(\mathbf{X}))(\mathbf{Y}-E(\mathbf{Y}))^*\right)=\mathbf{0}.$$

Then $\mathbf{X}$ *and* $\mathbf{Y}$ *are independent random vectors.*

Proof: Let $\mathbf{Z}=(\mathbf{X},\mathbf{Y}), m=p+q$. Then by hypothesis, the characteristic function of $\mathbf{Z}$ is of the form

$$E\left(e^{i\mathbf{t}\cdot\mathbf{Z}}\right)=e^{i\mathbf{t}\cdot\mathbf{m}}e^{-\frac{1}{2}i\mathbf{t}^*\Sigma\mathbf{t}}$$

where $\mathbf{m}=(\mathbf{m_X},\mathbf{m_Y})=E(\mathbf{Z})=E(\mathbf{X},\mathbf{Y})$ and

$$\begin{aligned}\Sigma &= \begin{pmatrix} E\left((\mathbf{X}-E(\mathbf{X}))(\mathbf{X}-E(\mathbf{X}))^*\right) & \mathbf{0} \\ \mathbf{0} & E\left((\mathbf{Y}-E(\mathbf{Y}))(\mathbf{Y}-E(\mathbf{Y}))^*\right)\end{pmatrix} \\ &\equiv \begin{pmatrix} \Sigma_{\mathbf{X}} & \mathbf{0} \\ \mathbf{0} & \Sigma_{\mathbf{Y}}\end{pmatrix}.\end{aligned}$$

Therefore, letting $\mathbf{t}=(\mathbf{u},\mathbf{v})$ where $\mathbf{u}\in\mathbb{R}^p$ and $\mathbf{v}\in\mathbb{R}^q$

$$\begin{aligned}E\left(e^{i\mathbf{t}\cdot\mathbf{Z}}\right) &= E\left(e^{i(\mathbf{u},\mathbf{v})\cdot(\mathbf{X},\mathbf{Y})}\right)=E\left(e^{i(\mathbf{u}\cdot\mathbf{X}+\mathbf{v}\cdot\mathbf{Y})}\right) \\ &= e^{i\mathbf{u}\cdot\mathbf{m_X}}e^{-\frac{1}{2}\mathbf{u}^*\Sigma_{\mathbf{X}}\mathbf{u}}e^{i\mathbf{v}\cdot\mathbf{m_Y}}e^{-\frac{1}{2}\mathbf{v}^*\Sigma_{\mathbf{Y}}\mathbf{v}} \\ &= E\left(e^{i\mathbf{u}\cdot\mathbf{X}}\right)E\left(e^{i\mathbf{v}\cdot\mathbf{Y}}\right). \qquad (36.26)\end{aligned}$$

Where the last equality needs to be justified. When this is done it will follow from Proposition 36.37 on Page 1017 which is proved on Page 996 that $\mathbf{X}$ and $\mathbf{Y}$ are independent. Thus all that remains is to verify

$$E\left(e^{i\mathbf{u}\cdot\mathbf{X}}\right)=e^{i\mathbf{u}\cdot\mathbf{m_X}}e^{-\frac{1}{2}\mathbf{u}^*\Sigma_{\mathbf{X}}\mathbf{u}},\ E\left(e^{i\mathbf{v}\cdot\mathbf{Y}}\right)=e^{i\mathbf{v}\cdot\mathbf{m_Y}}e^{-\frac{1}{2}\mathbf{v}^*\Sigma_{\mathbf{Y}}\mathbf{v}}.$$

However, this follows from 36.26. To get the first formula, let $\mathbf{v}=\mathbf{0}$. To get the second, let $\mathbf{u}=\mathbf{0}$. This proves the Theorem.

Note that to verify the conclusion of this theorem, it suffices to show

$$E(X_i-E(X_i)(Y_j-E(Y_j)))=0.$$

Next are some technical lemmas. The first is like an earlier result but will require more work because it will not be assumed a certain function is bounded.

Lemma 36.57 *Let* $(\Omega,\mathcal{F},P)$ *be a probability space and let* $X:\Omega\to E$ *be a random variable, where* E *is a real separable Banach space. Also let* $\mathcal{L}(X)=\mu$*, a probability measure defined on* $\mathcal{B}(E)$*, the Borel sets of* E*. Suppose* $h:E\to\mathbb{R}$ *is continuous and also suppose* $h\circ X$ *is in* $L^1(\Omega)$*. Then*

$$\int_\Omega (h\circ X)\,dP=\int_E h(x)\,d\mu.$$

Proof: Let $\{a_i\}_{i=1}^{\infty}$ be a countable dense subset of $\mathbb{R}$. Let $B_i^n \equiv B\left(a_i, \frac{1}{n}\right) \subseteq \mathbb{R}$ and define Borel sets, $A_i^n \subseteq E$ as follows:

$$A_1^n = h^{-1}\left(B_1^n\right), \ A_{k+1}^n \equiv h^{-1}\left(B_{k+1}^n\right) \setminus \left(\cup_{i=1}^k A_i^n\right).$$

Thus $\{A_i^n\}_{i=1}^{\infty}$ are disjoint Borel sets, with $h\left(A_i^n\right) \subseteq B_i^n$. Also let b_i^n denote the endpoint of $B\left(a_i, \frac{1}{n}\right)$ which is closer to 0.

$$h_i^n \equiv \begin{cases} b_i^n \text{ if } h^{-1}\left(B_i^n\right) \neq \emptyset \\ 0 \text{ if } h^{-1}\left(B_i^n\right) = \emptyset \end{cases}$$

Then define

$$h^n(x) \equiv \sum_{i=1}^{\infty} h_i^n \mathcal{X}_{A_i^n}(x)$$

Thus $|h^n(x)| \leq |h(x)|$ for all $x \in E$ and $|h^n(x) - h(x)| \leq 1/n$ for all $x \in E$. Then $h^n \circ X$ is in $L^1(\Omega)$ and for all n,

$$|h^n \circ X(\omega)| \leq |h \circ X(\omega)|.$$

Let

$$h_k^n(x) \equiv \sum_{i=1}^{k} h_i^n \mathcal{X}_{A_i^n}(x).$$

Then from the construction in which the $\{A_i^n\}_{i=1}^{\infty}$ are disjoint,

$$|h_k^n(X(\omega))| \leq |h(X(\omega))|, \ |h_k^n(x)| \leq |h(x)|$$

and $|h_k^n(x)|$ is increasing in k.

$$\begin{aligned} \int_{\Omega} |h_k^n(X(\omega))| \, dP &= \int_{\Omega} \sum_{i=1}^{k} |h_i^n| \, \mathcal{X}_{A_i^n}(X(\omega)) \, dP \\ &= \int_{\Omega} \sum_{i=1}^{k} |h_i^n| \, \mathcal{X}_{X^{-1}\left(A_i^n\right)}(\omega) \, dP \\ &= \sum_{i=1}^{k} |h_i^n| \, P\left(X^{-1}\left(A_i^n\right)\right) \\ &= \sum_{i=1}^{k} |h_i^n| \, \mu\left(A_i^n\right) \\ &= \int_E |h_k^n(x)| \, d\mu. \end{aligned}$$

By the monotone convergence theorem, and letting $k \to \infty$,

$$\int_{\Omega} |h^n(X(\omega))| \, dP = \int_E |h^n(x)| \, d\mu.$$

Now by the uniform convergence in the construction, you can let $n \to \infty$ and obtain

$$\int_{\Omega} |h(X(\omega))| \, dP = \int_E |h(x)| \, d\mu.$$

Thus $h \in L^1(E, \mu)$. It is obviously Borel measurable, being the limit of a sequence of Borel measurable functions. Now similar reasoning to the above and using the dominated convergence theorem when necessary yields

$$\int_{\Omega} h^n(X(\omega)) \, dP = \int_E h^n(x) \, d\mu$$

Now another application of the dominated convergence theorem yields

$$\int_{\Omega} h(X(\omega)) \, dP = \int_E h(x) \, d\mu.$$

This proves the lemma.

36.9 Gaussian Measures

36.9.1 Definitions And Basic Properties

First suppose $\mathbf{X}$ is a random vector having values in $\mathbb{R}^n$ and its distribution function is $N(\mathbf{m},\Sigma)$ where $\mathbf{m}$ is the mean and Σ is the covariance. Then the characteristic function of $\mathbf{X}$ or equivalently, the characteristic function of its distribution is

$$e^{i\mathbf{t}\cdot\mathbf{m}} e^{-\frac{1}{2}\mathbf{t}^*\Sigma\mathbf{t}}$$

What is the distribution of $\mathbf{a}\cdot\mathbf{X}$ where $\mathbf{a} \in \mathbb{R}^n$? In other words, if you take a linear functional and do it to $\mathbf{X}$ to get a scalar valued random variable, what is the distribution of this scalar valued random variable? Let $Y = \mathbf{a}\cdot\mathbf{X}$. Then

$$E\left(e^{itY}\right) = E\left(e^{it\mathbf{a}\cdot\mathbf{X}}\right)$$

which from the above formula is

$$e^{i\mathbf{a}\cdot\mathbf{m}t} e^{-\frac{1}{2}\mathbf{a}^*\Sigma\mathbf{a}t^2}$$

which is the characteristic function of a random variable whose distribution is $N(\mathbf{a}\cdot\mathbf{m}, \mathbf{a}^*\Sigma\mathbf{a})$. In other words, it is normally distributed having mean equal to $\mathbf{a}\cdot\mathbf{m}$ and variance equal to $\mathbf{a}^*\Sigma\mathbf{a}$. Obviously such a concept generalizes to a Banach space in place of $\mathbb{R}^n$ and this motivates the following definition.

Definition 36.58 *Let E be a real separable Banach space. A probability measure, μ defined on $\mathcal{B}(E)$ is called a Gaussian measure if for every $h \in E'$, the law of h considered as a random variable defined on the probability space, $(E, \mathcal{B}(E), \mu)$ is normal. That is, for $A \subseteq \mathbb{R}$ a Borel set,*

$$\lambda_h(A) \equiv \mu\left(h^{-1}(A)\right)$$

is given by

$$\int_A \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-m)^2} dx$$

for some σ and m. A Gaussian measure is called symmetric if m is always equal to 0.

Lemma 36.59 *Let $\mu = \mathcal{L}(X)$ where X is a random variable defined on a probability space, $(\Omega, \mathcal{F}, P)$ which has values in E, a Banach space. Suppose also that for all $\phi \in E', \phi \circ X$ is normally distributed. Then μ is a Gaussian measure. Conversely, suppose μ is a Gaussian measure on $\mathcal{B}(E)$ and X is a random variable having values in E such that $\mathcal{L}(X) = \mu$. Then for every $h \in E'$, $h \circ X$ is normally distributed.*

Proof: First suppose μ is a Gaussian measure and X is a random variable such that $\mathcal{L}(X) = \mu$. Then if F is a Borel set in $\mathbb{R}$, and $h \in E'$

$$\begin{aligned} P\left((h \circ X)^{-1}(F)\right) &= P\left(X^{-1}\left(h^{-1}(F)\right)\right) \\ &= \mu\left(h^{-1}(F)\right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_F e^{-\frac{|x-m|^2}{2\sigma^2}} dx \end{aligned}$$

for some m and σ^2 showing that $h \circ X$ is normally distributed.

Next suppose $h \circ X$ is normally distributed whenever $h \in E'$ and $\mathcal{L}(X) = \mu$. Then letting F be a Borel set in $\mathbb{R}$, I need to verify

$$\mu\left(h^{-1}(F)\right) = \frac{1}{\sqrt{2\pi}\sigma} \int_F e^{-\frac{|x-m|^2}{2\sigma^2}} dx.$$

However, this is easy because

$$\begin{aligned} \mu\left(h^{-1}(F)\right) &= P\left(X^{-1}\left(h^{-1}(F)\right)\right) \\ &= P\left((h \circ X)^{-1}(F)\right) \end{aligned}$$

which is given to equal

$$\frac{1}{\sqrt{2\pi}\sigma} \int_F e^{-\frac{|x-m|^2}{2\sigma^2}} dx$$

for some m and σ^2. This proves the lemma.

Here is another important observation. Suppose X is as just described, a random variable having values in E such that $\mathcal{L}(X) = \mu$ and suppose $h_1, \cdots, h_n$ are each in E'. Then for scalars, $t_1, \cdots, t_n$,

$$\begin{aligned} & t_1 h_1 \circ X + \cdots + t_n h_n \circ X \\ = \ & (t_1 h_1 + \cdots + t_n h_n) \circ X \end{aligned}$$

and this last is assumed to be normally distributed because $(t_1 h_1 + \cdots + t_n h_n) \in E'$. Therefore, by Theorem 36.51

$$(h_1 \circ X, \cdot \cdot \cdot, h_n \circ X)$$

is distributed as a multivariate normal.

Obviously there exist examples of Gaussian measures defined on E, a Banach space. Here is why. Let ξ be a random variable defined on a probability space, $(\Omega, \mathcal{F}, P)$ which is normally distributed with mean 0 and variance σ^2. Then let $X(\omega) \equiv \xi(\omega) e$ where $e \in E$. Then let $\mu \equiv \mathcal{L}(X)$. For A a Borel set of $\mathbb{R}$ and $h \in E'$,

$$\begin{aligned}\mu([h(x) \in A]) &\equiv P([X(\omega) \in [x : h(x) \in A]]) \\ &= P([h \circ X \in A]) = P([\xi(\omega) h(e) \in A]) \\ &= \frac{1}{|h(e)| \sigma \sqrt{2\pi}} \int_A e^{-\frac{1}{2|h(e)|^2 \sigma^2} x^2} dx\end{aligned}$$

because $h(e) \xi$ is a random variable which has variance $|h(e)|^2 \sigma^2$ and mean 0. Thus μ is indeed a Gaussian measure. Similarly, one can consider finite sums of the form

$$\sum_{i=1}^{n} \xi_i(\omega) e_i$$

where the ξ_i are independent normal random variables having mean 0 for convenience. However, this is a rather trivial case. It is much more interesting to consider the case of infinite sums of random variables.

36.10 Gaussian Measures For A Separable Hilbert Space

First recall the Kolmogorov extension theorem, Theorem 11.11 on Page 310 which is stated here for convenience. In this theorem, I is an ordered index set, possibly infinite, even uncountable.

Theorem 36.60 *(Kolmogorov extension theorem) For each finite set*

$$J = (t_1, \cdot \cdot \cdot, t_n) \subseteq I,$$

suppose there exists a Borel probability measure, $\nu_J = \nu_{t_1 \cdots t_n}$ defined on the Borel sets of $\prod_{t \in J} M_t$ where M_t is a locally compact metric space such that if

$$(t_1, \cdot \cdot \cdot, t_n) \subseteq (s_1, \cdot \cdot \cdot, s_p),$$

then

$$\nu_{t_1 \cdots t_n}(F_{t_1} \times \cdots \times F_{t_n}) = \nu_{s_1 \cdots s_p}(G_{s_1} \times \cdots \times G_{s_p}) \tag{36.27}$$

where if $s_i = t_j$, then $G_{s_i} = F_{t_j}$ and if s_i is not equal to any of the indices, t_k, then $G_{s_i} = M_{s_i}$. Then there exists a probability space, $(\Omega, P, \mathcal{F})$ and measurable functions, $X_t : \Omega \to M_t$ for each $t \in I$ such that for each $(t_1 \cdots t_n) \subseteq I$,

$$\nu_{t_1 \cdots t_n} (F_{t_1} \times \cdots \times F_{t_n}) = P([X_{t_1} \in F_{t_1}] \cap \cdots \cap [X_{t_n} \in F_{t_n}]) . \tag{36.28}$$

Lemma 36.61 *There exists a sequence, $\{\xi_k\}_{k=1}^{\infty}$ of random variables such that*

$$\mathcal{L}(\xi_k) = N(0,1)$$

and $\{\xi_k\}_{k=1}^{\infty}$ is independent.

Proof: Let $i_1 < i_2 \cdots < i_n$ be positive integers and define

$$\mu_{i_1 \cdots i_n} (F_1 \times \cdots \times F_n) \equiv \frac{1}{(\sqrt{2\pi})^n} \int_{F_1 \times \cdots \times F_n} e^{-|\mathbf{x}|^2/2} dx.$$

Then for the index set equal to $\mathbb{N}$ the measures satisfy the necessary consistency condition for the Kolmogorov theorem above. Therefore, there exists a probability space, $(\Omega, P, \mathcal{F})$ and measurable functions, $\xi_k : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} & P([\xi_{i_1} \in F_{i_1}] \cap [\xi_{i_2} \in F_{i_2}] \cdots \cap [\xi_{i_n} \in F_{i_n}]) \\ = \ & \mu_{i_1 \cdots i_n} (F_1 \times \cdots \times F_n) \\ = \ & P([\xi_{i_1} \in F_{i_1}]) \cdots P([\xi_{i_n} \in F_{i_n}]) \end{aligned}$$

which shows the random variables are independent as well as normal with mean 0 and variance 1. This proves the Lemma.

Now let H be a separable Hilbert space. Consider

$$\sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k$$

where $\{e_k\}$ is a complete orthonormal set of vectors and

$$\sum_{k=1}^{\infty} \lambda_k < \infty, \ \lambda_k \geq 0.$$

Thus if $A = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k$, it follows A is a nuclear operator and $Ae_k = \lambda_k e_k$.

Now define for $a \in H$,

$$X(\omega) \equiv a + \sum_{k=1}^{\infty} \sqrt{\lambda_k} \xi_k (\omega) e_k \tag{36.29}$$

Claim: The series converges a.e. and in $L^2(\Omega)$.

Proof of claim: Let $X_n(\omega) \equiv a + \sum_{k=1}^{n} \sqrt{\lambda_k} \xi_k(\omega) e_k$. Then

$$X_n(\omega) = a + \sum_{k=1}^{n} \sqrt{\lambda_k} \xi_k (\omega) e_k$$

$$\equiv a + Y_n(\omega). \tag{36.30}$$

For $n > m$,

$$|Y_n(\omega) - Y_m(\omega)|_H^2 = \sum_{k=m}^{n} \lambda_k \xi_k(\omega)^2 \tag{36.31}$$

and

$$\int_\Omega \sum_{k=1}^{\infty} \lambda_k \xi_k(\omega)^2 \, dP = \sum_{k=1}^{\infty} \lambda_k < \infty$$

and so for a.e. ω, 36.31 shows $\{Y_n(\omega)\}$ is a Cauchy sequence in H and therefore, converges.

The series also converges in $L^2(\Omega; H)$ because

$$\int_\Omega \left(\sum_{k=m}^{n} \sqrt{\lambda_k} \xi_k(\omega) e_k, \sum_{j=m}^{n} \sqrt{\lambda_j} \xi_j(\omega) e_j \right) dP$$

$$= \int_\Omega \sum_{k=m}^{n} \lambda_k |\xi_k(\omega)|^2 \, dP = \sum_{k=m}^{n} \lambda_k \leq \sum_{k=m}^{\infty} \lambda_k$$

which by assumption converges to 0 as $m \to \infty$. Thus the partial sums form a Cauchy sequence in $L^2(\Omega; H)$. This proves the claim.

Letting

$$(h, X(\omega))_n \equiv (h, a) + \sum_{k=1}^{n} \sqrt{\lambda_k} \xi_k(\omega) (h, e_k)$$

it follows

$$(h, X(\omega))_n \to (h, X(\omega)) \ a.e.$$

and an application of the dominated convergence theorem, shows that since the ξ_k are normal and independent, it follows from Theorem 36.50 and the description of the characteristic function of a normally distributed random variable,

$$\begin{aligned} E\left(e^{it(h,X)}\right) &= \lim_{n\to\infty} E\left(e^{it(h,X)_n}\right) \\ &= \lim_{n\to\infty} e^{it(h,a)} e^{-t^2 \sum_{k=1}^{n} \lambda_k (h,e_k)^2} \\ &= e^{it(h,a)} e^{-t^2 \sum_{k=1}^{\infty} \lambda_k (h,e_k)^2} \end{aligned}$$

and this is the characteristic function for a random variable with mean (h, a) and variance $\sum_{k=1}^{\infty} \lambda_k (h, e_k)^2$, this last series converging because

$$\sum_{k=1}^{M} \lambda_k (h, e_k)^2 \leq C \sum_{k=1}^{\infty} (h, e_k)^2 = C |h|^2.$$

Thus for every $h \in H', h \circ X$ is normally distributed. Therefore, Lemma 36.59 implies the following theorem.

Theorem 36.62 *Let $X(\omega)$ be given by 36.29 as described above. Then letting $\mu \equiv \mathcal{L}(X)$, it follows μ is a Gaussian measure on the separable Hilbert space, H.*

36.11 Abstract Wiener Spaces

This material follows [11], [30] and [25]. More can be found on this subject in these references. Here H will be a separable Hilbert space.

Definition 36.63 *Cylinder sets in H are of the form*

$$\{x \in H : ((x, e_1), \cdots, (x, e_n)) \in F\}$$

where $F \in \mathcal{B}(\mathbb{R}^n)$, the Borel sets of $\mathbb{R}^n$ and $\{e_k\}$ are orthonormal. Denote this collection of cylinder sets as $\mathcal{C}$.

Lemma 36.64 *$\sigma(\mathcal{C})$, the smallest σ algebra containing $\mathcal{C}$, contains the Borel sets of $H, \mathcal{B}(H)$.*

Proof: It follows from the definition of these cylinder sets that if $f_i(x) \equiv (x, e_i)$, so that $f_i \in H'$, then with respect to $\sigma(\mathcal{C})$, each f_i is measurable. It follows that every linear combination of the f_i is also measurable with respect to $\sigma(\mathcal{C})$. However, this set of linear combinations is dense in H' and so the conclusion of the lemma follows from Lemma 36.9 on Page 993. This proves the lemma.

Definition 36.65 *Define ν on the cylinder sets, $\mathcal{C}$ by the following rule. For $\{e_k\}$ a complete orthonormal set in H,*

$$\begin{aligned} & \nu(\{x \in H : ((x, e_1), \cdots, (x, e_n)) \in F\}) \\ \equiv \; & \frac{1}{(2\pi)^{n/2}} \int_F e^{-|\mathbf{x}|^2/2} dx. \end{aligned}$$

Lemma 36.66 *The above definition is well defined.*

Proof: Let $\{f_k\}$ be another orthonormal set such that

$$\begin{aligned} & \{x \in H : ((x, e_1), \cdots, (x, e_n)) \in F\} \\ = \; & \{x \in H : ((x, f_1), \cdots, (x, f_n)) \in G\} \end{aligned}$$

Then it needs to be the case that ν gives the same result for the two equal cylinder sets. Let

$$L = \sum_i e_i \otimes f_i.$$

Thus $Lf_i = e_i$ and L maps H one to one and onto and preserves norms and

$$L^* = \sum_i f_i \otimes e_i$$

and maps e_i to f_i and has the same properties of being one to one and onto H and preserving norms.

Let

$$x \in \{x \in H : ((x, e_1), \cdots, (x, e_n)) \in F\} \equiv A.$$

Then by definition,

$$((x,e_1),\cdots,(x,e_n))\in F$$

and so

$$((x,Lf_1),\cdots,(x,Lf_n))\in F$$

which implies

$$((L^*x,f_1),\cdots,(L^*x,f_n))\in F$$

Thus, since L,L^* are one to one and onto,

$$\begin{aligned} A &= \{x\in H:((x,e_1),\cdots,(x,e_n))\in F\} \\ &= \{x\in H:((L^*x,f_1),\cdots,(L^*x,f_n))\in F\} \\ &= \{x\in LH:((L^*x,f_1),\cdots,(L^*x,f_n))\in F\} \\ &= \{L^*x\in H:((L^*x,f_1),\cdots,(L^*x,f_n))\in F\} \\ &= \{y\in H:((y,f_1),\cdots,(y,f_n))\in F\} \\ &= \{x\in H:((x,f_1),\cdots,(x,f_n))\in G\} \end{aligned}$$

If $\boldsymbol{\alpha}\in F$, then there exists $y\in H$ such that

$$((y,f_1),\cdots,(y,f_n))\in F$$

and this y must be in the set,

$$\{x\in H:((x,f_1),\cdots,(x,f_n))\in G\}$$

which shows

$$((y,f_1),\cdots,(y,f_n))\in G.$$

Hence $F\subseteq G$. Similarly, $G\subseteq F$ and ν is well defined. This proves the lemma.

It would be natural to try to extend ν to the σ algebra determined by $\mathcal{C}$ and obtain a measure defined on this σ algebra. However, this is always impossible if the Hilbert space, H is infinite dimensional.

Proposition 36.67 *ν cannot be extended to a measure defined on $\sigma(\mathcal{C})$ whenever H is infinite dimensional.*

Proof: Let $\{e_n\}$ be a complete orthonormal set of vectors in H. Then first note that H is a cylinder set.

$$H=\{x\in H:(x,e_1)\in\mathbb{R}\}$$

and so

$$\nu(H)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-x^2/2}dx=1.$$

However, H is also equal to the countable union of the sets,

$$A_n\equiv\{x\in H:((x,e_1)_H,\cdots,(x,e_{a_n})_H)\in B(\mathbf{0},n)\}$$

where $a_n \to \infty$.

$$\begin{aligned}\nu(A_n) &\equiv \frac{1}{\left(\sqrt{2\pi}\right)^{a_n}} \int_{B(\mathbf{0},n)} e^{-|\mathbf{x}|^2/2} dx \\ &\leq \frac{1}{\left(\sqrt{2\pi}\right)^{a_n}} \int_{-n}^{n} \cdots \int_{-n}^{n} e^{-|\mathbf{x}|^2/2} dx_1 \cdots dx_{a_n} \\ &= \left(\frac{\int_{-n}^{n} e^{-x^2/2} dx}{\sqrt{2\pi}}\right)^{a_n}\end{aligned}$$

Now pick a_n so large that the above is smaller than $1/2^{n+1}$. This can be done because for no matter what choice of n,

$$\frac{\int_{-n}^{n} e^{-x^2/2} dx}{\sqrt{2\pi}} < 1.$$

Then

$$\sum_{n=1}^{\infty} \nu(A_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.$$

This proves the proposition and shows something else must be done to get a measure from ν.

Definition 36.68 *Let H be a separable Hilbert space and let $||\cdot||$ be a norm defined on H which has the following property. Whenever $\{e_n\}$ is an orthonormal sequence of vectors in H and $\mathcal{F}(\{e_n\})$ consists of the set of all orthogonal projections onto the span of finitely many of the e_k the following condition holds. For every $\varepsilon > 0$ there exists $P_\varepsilon \in \mathcal{F}(\{e_n\})$ such that if $P \in \mathcal{F}(\{e_n\})$ and $PP_\varepsilon = 0$, then*

$$\nu(\{x \in H : ||Px|| > \varepsilon\}) < \varepsilon.$$

Then $||\cdot||$ is called Gross measurable.

The following lemma is a fundamental result about Gross measurable norms. It is about the continuity of $||\cdot||$. It is obvious that with respect to the topology determined by $||\cdot||$ that $x \to ||x||$ is continuous. However, it would be interesting if this were the case with respect to the topology determined by the norm on $H, |\cdot|$. This lemma shows this is the case and so the funny condition above implies $x \to ||x||$ is a continuous, hence Borel measurable function.

Lemma 36.69 *Let $||\cdot||$ be Gross measurable. Then there exists $c > 0$ such that*

$$||x|| \leq c\,|x|$$

for all $x \in H$. Furthermore, the above definition is well defined.

Proof: First it is important to consider the question whether the above definition is well defined. To do this note that on PH, the two norms are equivalent because PH is a finite dimensional space. Let $G = \{y \in PH : ||y|| > \varepsilon\}$ so G is an open set in PH. Then

$$\{x \in H : ||Px|| > \varepsilon\}$$

equals

$$\{x \in H : Px \in G\}$$

which equals a set of the form

$$\{x \in H : ((x, e_{i_1})_H, \cdots, (x, e_{i_m})_H) \in G'\}$$

for G' an open set in $\mathbb{R}^m$ and so everything makes sense in the above definition.

Now it is necessary to verify $||\cdot|| \leq c\,|\cdot|$. If it is not so, there exists e_1 such that

$$||e_1|| \geq 1,\ |e_1| = 1.$$

Suppose $\{e_k\}_{k=1}^n$ have been chosen such that each is a unit vector in H and $||e_k|| \geq k$. Then considering $\text{span}\,(e_1, \cdots, e_n)^{\perp}$ if for every $x \in \text{span}\,(e_1, \cdots, e_n)^{\perp}$, $||x|| \leq c\,|x|$, then if $z \in H$ is arbitrary, $z = x + y$ where $y \in \text{span}\,(e_1, \cdots, e_n)$ and so since the two norms are equivalent on a finite dimensional subspace, there exists c' corresponding to $\text{span}\,(e_1, \cdots, e_n)$ such that

$$\begin{aligned} ||z||^2 &\leq (||x|| + ||y||)^2 \leq 2\,||x||^2 + 2\,||y||^2 \\ &\leq 2c^2\,|x|^2 + 2c'\,|y|^2 \\ &\leq \left(2c^2 + 2c'^2\right)\left(|x|^2 + |y|^2\right) \\ &= \left(2c^2 + 2c'^2\right)|z|^2 \end{aligned}$$

and the lemma is proved. Therefore it can be assumed, there exists

$$e_{n+1} \in \text{span}\,(e_1, \cdots, e_n)^{\perp}$$

such that $|e_{n+1}| = 1$ and $||e_{n+1}|| \geq n + 1$.

This constructs an orthonormal set of vectors, $\{e_k\}$. Letting $0 < \varepsilon < \frac{1}{2}$, it follows since $||\cdot||$ is measurable, there exists $P_\varepsilon \in \mathcal{F}(\{e_n\})$ such that if $PP_\varepsilon = 0$ where $P \in \mathcal{F}(\{e_n\})$, then

$$\nu(\{x \in H : ||Px|| > \varepsilon\}) < \varepsilon.$$

Say P_ε is the projection onto the span of finitely many of the e_k, the last one being e_N. Then for $n > N$ and P_n the projection onto e_n, it follows $P_\varepsilon P_n = 0$ and from the definition of ν,

$$\begin{aligned} \varepsilon &> \nu(\{x \in H : ||P_n x|| > \varepsilon\}) \\ &= \nu(\{x \in H : |(x, e_n)|\,||e_{n+1}|| > \varepsilon\}) \\ &= \nu(\{x \in H : |(x, e_n)| > \varepsilon/\,||e_{n+1}||\}) \\ &\geq \nu(\{x \in H : |(x, e_n)| > \varepsilon/\,(n+1)\}) \\ &> \frac{1}{\sqrt{2\pi}} \int_{\varepsilon/(n+1)}^{\infty} e^{-x^2/2} dx \end{aligned}$$

which yields a contradiction for all n large enough. This proves the lemma.

What are examples of Gross measurable norms defined on a separable Hilbert space, H? The following lemma gives an important example.

Lemma 36.70 *Let H be a separable Hilbert space and let $A \in \mathcal{L}_2(H,H)$, a Hilbert Schmidt operator. Thus A is a continuous linear operator with the property that for any orthonormal set, $\{e_k\}$,*

$$\sum_{k=1}^{\infty} |Ae_k|^2 < \infty.$$

Then define $||\cdot||$ by

$$||x|| \equiv |Ax|_H .$$

Then if $||\cdot||$ is a norm, it is measurable[1].

Proof: Let $\{e_k\}$ be an orthonormal sequence. Let P_n denote the orthogonal projection onto $\text{span}(e_1, \cdots, e_n)$. Let $\varepsilon > 0$ be given. Since A is a Hilbert Schmidt operator, there exists N such that

$$\sum_{k=N}^{\infty} |Ae_k|^2 < \alpha$$

where α is chosen very small. In fact, α is chosen such that $\alpha < \varepsilon^2/r^2$ where r is sufficiently large that

$$\frac{2}{\sqrt{2\pi}} \int_r^{\infty} e^{-t^2/2} dt < \varepsilon. \tag{36.32}$$

Let P denote an orthogonal projection in $\mathcal{F}(\{e_k\})$ such that $PP_N = 0$. Thus P is the projection on to $\text{span}(e_{i_1}, \cdots, e_{i_m})$ where each $i_k > N$. Then

$$\begin{aligned} & \nu(\{x \in H : ||Px|| > \varepsilon\}) \\ = \; & \nu(\{x \in H : |APx| > \varepsilon\}) \end{aligned}$$

Now $Px = \sum_{j=1}^m (x, e_{i_j}) e_{i_j}$ and the above reduces to

$$\nu\left(\left\{x \in H : \left|\sum_{j=1}^m (x, e_{i_j}) Ae_{i_j}\right| > \varepsilon\right\}\right) \leq$$

[1] If it is only a seminorm, it satisfies the same conditions.

$$
\begin{aligned}
& \nu\left(\left\{x \in H : \left(\sum_{j=1}^{m}\left|\left(x, e_{i_j}\right)\right|^2\right)^{1/2}\left(\sum_{j=1}^{m}\left|Ae_{i_j}\right|^2\right)^{1/2} > \varepsilon\right\}\right) \\
\leq\ & \nu\left(\left\{x \in H : \left(\sum_{j=1}^{m}\left|\left(x, e_{i_j}\right)\right|^2\right)^{1/2}\alpha^{1/2} > \varepsilon\right\}\right) \\
=\ & \nu\left(\left\{x \in H : \left(\sum_{j=1}^{m}\left|\left(x, e_{i_j}\right)\right|^2\right)^{1/2} > \frac{\varepsilon}{\alpha^{1/2}}\right\}\right) \\
=\ & \nu\left(\left\{x \in H : \left(\left(x, e_{i_1}\right), \cdots, \left(x, e_{i_m}\right)\right) \in B\left(\mathbf{0}, \frac{\varepsilon}{\alpha^{1/2}}\right)^C\right\}\right) \\
\leq\ & \nu\left(\left\{x \in H : \max\left\{\left|\left(x, e_{i_j}\right)\right|\right\} > \frac{\varepsilon}{\sqrt{m}\alpha^{1/2}}\right\}\right)
\end{aligned}
$$

This is no larger than

$$
\begin{aligned}
& \frac{1}{\left(\sqrt{2\pi}\right)^m}\int_{|t_1|>\frac{\varepsilon}{\sqrt{m}\sqrt{\alpha}}}\int_{|t_2|>\frac{\varepsilon}{\sqrt{m}\sqrt{\alpha}}}\cdots\int_{|t_m|>\frac{\varepsilon}{\sqrt{m}\sqrt{\alpha}}} e^{-|\mathbf{t}|^2/2}dt_m\cdots dt_1 \\
& = \left(\frac{2\int_{\varepsilon/\left(\sqrt{m}\alpha^{1/2}\right)}^{\infty} e^{-t^2/2}dt}{\sqrt{2\pi}}\right)^m
\end{aligned}
$$

which by Jensen's inequality is no larger than

$$
\begin{aligned}
\frac{2\int_{\varepsilon/\left(\sqrt{m}\alpha^{1/2}\right)}^{\infty} e^{-mt^2/2}dt}{\sqrt{2\pi}} &= \frac{2\frac{1}{\sqrt{m}}\int_{\varepsilon/\left(\alpha^{1/2}\right)}^{\infty} e^{-t^2/2}dt}{\sqrt{2\pi}} \\
&\leq \frac{2\int_{\varepsilon/(\varepsilon/r)}^{\infty} e^{-t^2/2}dt}{\sqrt{2\pi}} \\
&= \frac{2\int_{r}^{\infty} e^{-t^2/2}dt}{\sqrt{2\pi}} < \varepsilon
\end{aligned}
$$

By 36.32. This proves the lemma.

Definition 36.71 *A triple, (i, H, B) is called an abstract Wiener space if B is a separable Banach space and H is a separable Hilbert space such that H is dense and continuously embedded in B and the norm $||\cdot||$ on B is Gross measurable.*

Next consider a weaker norm for H which comes from the inner product

$$
(x, y)_E \equiv \sum_{k=1}^{\infty}\frac{1}{k^2}(x, e_k)_H (y, e_k)_H .
$$

Then let E be the completion of H with respect to this new norm. Thus $\{ke_k\}$ is a complete orthonormal basis for E. This follows from the density of H in E along with the obvious observation that in the above inner product, $\{ke_k\}$ is an orthonormal set of vectors.

Lemma 36.72 *There exists a countably additive Gaussian measure, λ defined on $\mathcal{B}(E)$. This measure is the law of the random variable,*

$$X(\omega) \equiv \sum_{k=1}^{\infty} \xi_k(\omega) e_k,$$

where $\{\xi_k\}$ denotes a sequence of independent normally distributed random variables having mean 0 and variance 1, the series converging pointwise a.e. in E. Also

$$k^2 (X(\omega), e_k)_E = \xi_k(\omega) \text{ a.e.}$$

Proof: Observe that $\sum_{k=1}^{\infty} \frac{1}{k^2} (ke_k) \otimes (ke_k)$ is a nuclear operator on the Hilbert space, E. Letting $\{\xi_k\}$ be a sequence of independent random variables each normally distributed with mean 0 and variance 1, it follows as in Theorem 36.62 that

$$X(\omega) \equiv \sum_{k=1}^{\infty} \frac{1}{k} \xi_k(\omega) ke_k = \sum_{k=1}^{\infty} \xi_k(\omega) e_k \tag{36.33}$$

is a random variable with values in E and $\mathcal{L}(X)$ is a Gaussian measure on $\mathcal{B}(E)$, the series converging pointwise a.e. in E. Let λ be the name of this Gaussian measure and denote the probability space on which the ξ_k are defined as $(\Omega, \mathcal{F}, P)$. Thus for $F \in \mathcal{B}(E)$,

$$\lambda(F) \equiv P(\{\omega \in \Omega : X(\omega) \in F\})$$

Finally, denoting by X_N, the partial sum,

$$X_N(\omega) \equiv \sum_{k=1}^{N} \xi_k(\omega) e_k,$$

the definition of $(\cdot,\cdot)_E$ on H and a simple computation yields

$$\begin{aligned} \xi_k(\omega) &= \lim_{N\to\infty} k^2 (X_N(\omega), e_k)_E \\ &= k^2 (X(\omega), e_k)_E . \end{aligned} \tag{36.34}$$

One can pass to the limit because $X_N(\omega)$ converges to $X(\omega)$ in E. This proves the lemma.

Theorem 36.73 *Let (i, H, B) be an abstract Wiener space. Then there exists a Gaussian measure on the Borel sets of B.*

Proof: Let E be defined above as the completion of H with respect to that weaker norm. Then from Lemma 36.72 and $X(\omega)$ given above in 36.33,

$$k^2 (X(\omega), e_k)_E = \xi_k(\omega) \text{ a.e. } \omega.$$

Let $\{e_n\}$ be a complete orthonormal set for H. There exists an increasing sequence of projections, $\{Q_n\} \subseteq \mathcal{F}(\{e_n\})$ such that $Q_n x \to x$ in H for each $x \in H$. Say Q_n is the orthogonal projection onto $\text{span}(e_1, \cdots, e_{p_n})$. Then since $||\cdot||$ is measurable, these can be chosen such that if Q is the orthogonal projection onto $\text{span}(e_1, \cdots, e_k)$ for some $k > p_n$ then

$$\nu\left(\left\{x : ||Qx - Q_n x|| > 2^{-n}\right\}\right) < 2^{-n}.$$

In particular,

$$\nu\left(\left\{x : ||Q_n x - Q_m x|| > 2^{-m}\right\}\right) < 2^{-m}$$

whenever $n \geq m$.

I would like to consider the infinite series,

$$S(\omega) \equiv \sum_{k=1}^{\infty} k^2 (X(\omega), e_k)_E e_k \in B.$$

converging in B but of course this might make no sense because the series might not converge. It was shown above that the series converges in E but it has not been shown to converge in B.

Suppose the series did converge a.e. Then let $f \in B'$ and consider the random variable $f \circ S$ which maps Ω to $\mathbb{R}$. I would like to verify this is normally distributed. First note that the following finite sum is weakly measurable and separably valued so it is strongly measurable with values in B.

$$S_{p_n}(\omega) \equiv \sum_{k=1}^{p_n} k^2 (X(\omega), e_k)_E e_k,$$

Since $f \in B'$ which is a subset of H' due to the assumption that H is dense in B, there exists a unique $v \in H$ such that $f(x) = (x, v)$ for all $x \in H$. Then from the above sum,

$$f(S_{p_n}(\omega)) = (S_{p_n}(\omega), v) = \sum_{k=1}^{p_n} k^2 (X(\omega), e_k)_E (e_k, v)$$

which by Lemma 36.72 equals

$$\sum_{k=1}^{p_n} (e_k, v)_H \xi_k(\omega)$$

a finite linear combination of the independent $N(0,1)$ random variables, $\xi_k(\omega)$. Then it follows

$$\omega \to f(S_{p_n}(\omega))$$

is also normally distributed and has mean 0 and variance equal to

$$\sum_{k=1}^{p_n} (e_k, v)_H^2 .$$

Then it seems reasonable to suppose

$$\begin{aligned} E\left(e^{itf\circ S}\right) &= \lim_{n\to\infty} E\left(e^{itf\circ S_{p_n}}\right) \\ &= \lim_{n\to\infty} e^{-t^2\sum_{k=1}^{p_n}(e_k,v)_H^2} \\ &= e^{-t^2\sum_{k=1}^{\infty}(e_k,v)_H^2} \\ &= e^{-t^2|v|_H^2} \end{aligned} \tag{36.35}$$

the characteristic function of a random variable which is $N\left(0, |v|_H^2\right)$. Thus at least formally, this would imply for all $f \in B', f \circ S$ is normally distributed and so if $\mu = \mathcal{L}(S)$, then by Lemma 36.59 it follows μ is a Gaussian measure.

What is missing to make the above a proof? First of all, there is the issue of the sum. Next there is the problem of passing to the limit in the little argument above in which the characteristic function is used.

First consider the the sum. Note that $Q_n X(\omega) \in H$. Then for any $n > p_m$,

$$P\left(\left\{\omega \in \Omega : ||S_n(\omega) - S_{p_m}(\omega)|| > 2^{-m}\right\}\right)$$

$$\begin{aligned} &= P\left(\left\{\omega \in \Omega : \left|\left|\sum_{k=p_m+1}^{n} k^2 (X(\omega), e_k)_E e_k\right|\right| > 2^{-m}\right\}\right) \\ &= P\left(\left\{\omega \in \Omega : \left|\left|\sum_{k=p_m+1}^{n} \xi_k(\omega) e_k\right|\right| > 2^{-m}\right\}\right) \end{aligned} \tag{36.36}$$

Let Q be the orthogonal projection onto $\operatorname{span}(e_1, \cdots, e_n)$. Define

$$F \equiv \left\{x \in (Q - Q_m) H : ||x|| > 2^{-m}\right\}$$

Then continuing the chain of equalities ending with 36.36,

$$= P\left(\left\{\omega \in \Omega : \sum_{k=p_m+1}^{n} \xi_k(\omega) e_k \in F\right\}\right)$$

$$\begin{aligned} &= P\left(\left\{\omega \in \Omega : \left(\xi_n(\omega), \cdots, \xi_{p_m+1}(\omega)\right) \in F'\right\}\right) \\ &= \nu\left(\left\{x \in H : \left((x, e_n)_H, \cdots, (x, e_{p_m+1})_H\right) \in F'\right\}\right) \\ &= \nu\left(\left\{x \in H : Q(x) - Q_m(x) \in F\right\}\right) \\ &= \nu\left(\left\{x \in H : ||Q(x) - Q_m(x)|| > 2^{-m}\right\}\right) < 2^{-m}. \end{aligned}$$

This has shown that

$$P\left(\left\{\omega \in \Omega : ||S_n(\omega) - S_{p_m}(\omega)|| > 2^{-m}\right\}\right) < 2^{-m} \tag{36.37}$$

for all $n > p_m$. In particular, the above is true if $n = p_n$ for $n > m$.

If $\{S_{p_n}(\omega)\}$ fails to converge, then ω must be contained in the set,

$$A \equiv \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} \left\{\omega \in \Omega : ||S_{p_n}(\omega) - S_{p_m}(\omega)|| > 2^{-m}\right\} \tag{36.38}$$

because if ω is in the complement of this set,

$$\cup_{m=1}^{\infty} \cap_{n=m}^{\infty} \left\{\omega \in \Omega : ||S_{p_n}(\omega) - S_{p_m}(\omega)|| \leq 2^{-m}\right\},$$

it follows $\{S_{p_n}(\omega)\}_{n=1}^{\infty}$ is a Cauchy sequence and so it must converge. However, the set in 36.38 is a set of measure 0 because of 36.37 and the observation that for all m,

$$\begin{aligned} P(A) &\leq \sum_{n=m}^{\infty} P\left(\left\{\omega \in \Omega : ||S_{p_n}(\omega) - S_{p_m}(\omega)|| > 2^{-m}\right\}\right) \\ &\leq \sum_{n=m}^{\infty} \frac{1}{2^m} \end{aligned}$$

Thus the subsequence $\{S_{p_n}\}$ of the sequence of partial sums of the above series does converge pointwise in B and so the dominated convergence theorem also verifies that the computations involving the characteristic function in 36.35 are correct.

The random variable obtained as the limit of the partial sums, $\{S_{p_n}(\omega)\}$ described above is strongly measurable because each $S_{p_n}(\omega)$ is strongly measurable due to each of these being weakly measurable and separably valued. Thus the measure given as the law of S defined as

$$S(\omega) \equiv \lim_{n\to\infty} S_{p_n}(\omega)$$

is defined on the Borel sets of B.This proves the theorem.

Also, there is an important observation from the proof which I will state as the following corollary.

Corollary 36.74 *Let (i, H, B) be an abstract Wiener space. Then there exists a Gaussian measure on the Borel sets of B. This Gaussian measure equals $\mathcal{L}(S)$ where $S(\omega)$ is the a.e. limit of a subsequence of the sequence of partial sums,*

$$S_{p_n}(\omega) \equiv \sum_{k=1}^{p_n} \xi_k(\omega) e_k$$

for $\{\xi_k\}$ a sequence of independent random variables which are normal with mean 0 and variance 1 which are defined on a probability space, $(\Omega, \mathcal{F}, P)$. Furthermore, for any $k > p_n$,

$$P\left(\left\{\omega \in \Omega : ||S_k(\omega) - S_{p_n}(\omega)|| > 2^{-n}\right\}\right) < 2^{-n}.$$

36.12 White Noise

In an abstract Wiener space as discussed above there is a Gaussian measure, μ defined on the Borel sets of B. This measure is the law of a random variable having values in B which is the limit of a subsequence of a sequence of partial sums. I will show here that the sequence of partial sums also converges pointwise a.e.

First is a simple definition and lemma about random variables whose distribution is symmetric.

Definition 36.75 *Let X be a random variable defined on a probability space, $(\Omega, \mathcal{F}, P)$ having values in a Banach space, E. Then it has a symmetric distribution if whenever A is a Borel set,*

$$P([X \in A]) = P([X \in -A])$$

In terms of the distribution,

$$\lambda_X = \lambda_{-X}.$$

It is good to observe that if X, Y are independent random variables defined on a probability space, $(\Omega, \mathcal{F}, P)$ such that each has symmetric distribution, then $X + Y$ also has symmetric distribution. Here is why. Let A be a Borel set in E. Then by Theorem 36.47 on Page 1023,

$$\begin{aligned} \lambda_{X+Y}(A) &= \int_E \lambda_X (A - z)\, d\lambda_Y (z) \\ &= \int_E \lambda_{-X} (A - z)\, d\lambda_{-Y} (z) \\ &= \lambda_{-(X+Y)} (A) \end{aligned}$$

By induction, it follows that if you have n independent random variables each having symmetric distribution, then their sum has symmetric distribution.

Here is a simple lemma about random variables having symmetric distributions. It will depend on Lemma 36.57 on Page 1029.

Lemma 36.76 *Let $\mathbf{X} \equiv (X_1, \cdots, X_n)$ and Y be random variables defined on a probability space, $(\Omega, \mathcal{F}, P)$ such that $X_i, i = 1, 2, \cdots, n$ and Y have values in E a separable Banach space. Thus $\mathbf{X}$ has values in E^n. Suppose also that $\{X_1, \cdots, X_n, Y\}$ are independent and that Y has symmetric distribution. Then if $A \in \mathcal{B}(E^n)$, it follows*

$$\begin{aligned} & P\left([\mathbf{X} \in A] \cap \left[\left\|\sum_{i=1}^n X_i + Y\right\| < r\right]\right) \\ = \; & P\left([\mathbf{X} \in A] \cap \left[\left\|\sum_{i=1}^n X_i - Y\right\| < r\right]\right) \end{aligned}$$

You can also change the inequalities in the obvious way, $<$ to $\leq$, $>$ to $\geq$.

Proof: Denote by $\lambda_{\mathbf{X}}$ and λ_Y the distribution measures for $\mathbf{X}$ and Y respectively. Since the random variables are independent, the distribution for the random variable, $(\mathbf{X}, Y)$ mapping into E^{n+1} is $\lambda_{\mathbf{X}} \times \lambda_Y$ where this denotes product measure. Since the Banach space is separable, the Borel sets are contained in the product measurable sets. Then by symmetry of the distribution of Y

$$\begin{aligned}
& P\left([\mathbf{X} \in A] \cap \left[\left|\left|\sum_{i=1}^n X_i + Y\right|\right| < r\right]\right) \\
= & \int_{E^n \times E} \mathcal{X}_A(\mathbf{x}) \mathcal{X}_{B(0,r)}\left(\sum_{i=1}^n x_i + y\right) d(\lambda_{\mathbf{X}} \times \lambda_Y)(\mathbf{x},y) \\
= & \int_E \int_{E^n} \mathcal{X}_A(\mathbf{x}) \mathcal{X}_{B(0,r)}\left(\sum_{i=1}^n x_i + y\right) d\lambda_{\mathbf{X}} d\lambda_Y \\
= & \int_E \int_{E^n} \mathcal{X}_A(\mathbf{x}) \mathcal{X}_{B(0,r)}\left(\sum_{i=1}^n x_i + y\right) d\lambda_{\mathbf{X}} d\lambda_{-Y} \\
= & \int_{E^n \times E} \mathcal{X}_A(\mathbf{x}) \mathcal{X}_{B(0,r)}\left(\sum_{i=1}^n x_i + y\right) d(\lambda_{\mathbf{X}} \times \lambda_{-Y})(\mathbf{x},y) \\
= & P\left([\mathbf{X} \in A] \cap \left[\left|\left|\sum_{i=1}^n X_i + (-Y)\right|\right| < r\right]\right)
\end{aligned}$$

This proves the lemma. Other cases are similar.

Now here is a really interesting lemma.

Lemma 36.77 *Let E be a real separable Banach space. Assume $\xi_1, \cdots, \xi_N$ are independent random variables having values in E, a separable Banach space which have symmetric distributions. Also let $S_k = \sum_{i=1}^k \xi_i$. Then for any $r > 0$,*

$$P\left(\left[\sup_{k \leq N} ||S_k|| > r\right]\right) \leq 2P([||S_N|| > r]).$$

Proof: First of all,

$$\begin{aligned}
& P\left(\left[\sup_{k \leq N} ||S_k|| > r\right]\right) \\
= & P\left(\left[\sup_{k \leq N} ||S_k|| > r \text{ and } ||S_N|| > r\right]\right) \\
& + P\left(\left[\sup_{k \leq N-1} ||S_k|| > r \text{ and } ||S_N|| \leq r\right]\right) \\
\leq & P([||S_N|| > r]) + P\left(\left[\sup_{k \leq N-1} ||S_k|| > r \text{ and } ||S_N|| \leq r\right]\right). \quad (36.39)
\end{aligned}$$

I need to estimate the second of these terms. Let

$$A_1 \equiv [||S_1|| > r], \cdots, A_k \equiv [||S_k|| > r, \ ||S_j|| \leq r \text{ for } j < k].$$

Thus A_k consists of those ω where $||S_k(\omega)|| > r$ for the first time at k. Thus

$$\left[\sup_{k\leq N-1} ||S_k|| > r \text{ and } ||S_N|| \leq r\right] = \cup_{j=1}^{N-1} A_j \cap [||S_N|| \leq r]$$

and the sets in the above union are disjoint. Consider $A_j \cap [||S_N|| \leq r]$. For ω in this set,

$$||S_j(\omega)|| > r,\ ||S_i(\omega)|| \leq r \text{ if } i < j.$$

Since $||S_N(\omega)|| \leq r$ in this set, it follows

$$\left|\left|S_j(\omega) + \sum_{i=j+1}^{N} \xi_i(\omega)\right|\right| \leq r$$

and so from the symmetry of the distributions and Lemma 36.76 the following computation is valid.

$$P(A_j \cap [||S_N|| \leq r]) \tag{36.40}$$

$$= P\left(\cap_{i=1}^{j-1} [||S_i|| \leq r] \cap [||S_j|| > r] \cap \left[\left|\left|S_j + \sum_{i=j+1}^{N} \xi_i\right|\right| \leq r\right]\right) \tag{36.41}$$

Now $\cap_{i=1}^{j-1} [||S_i|| \leq r] \cap [||S_j|| > r]$ is of the form

$$\left[(\xi_1, \cdots, \xi_j) \in A\right]$$

for some Borel set, A. Then letting $Y = \sum_{i=j+1}^{N} \xi_i$ in Lemma 36.76 and $X_i = \xi_i$, 36.41 equals

$$\begin{aligned} & P\left(\cap_{i=1}^{j-1} [||S_i|| \leq r] \cap [||S_j|| > r] \cap \left[\left|\left|S_j - \sum_{i=j+1}^{N} \xi_i\right|\right| \leq r\right]\right) \\ = \ & P\left(\cap_{i=1}^{j-1} [||S_i|| \leq r] \cap [||S_j|| > r] \cap [||S_j - (S_N - S_j)|| \leq r]\right) \\ = \ & P\left(\cap_{i=1}^{j-1} [||S_i|| \leq r] \cap [||S_j|| > r] \cap [||2S_j - S_N|| \leq r]\right) \end{aligned}$$

Now since $||S_j(\omega)|| > r$,

$$\begin{aligned} [||2S_j - S_N|| \leq r] & \subseteq [2||S_j|| - ||S_N|| \leq r] \\ & \subseteq [2r - ||S_N|| < r] \\ & = [||S_N|| > r] \end{aligned}$$

and so, referring to 36.40, this has shown

$$P(A_j \cap [||S_N|| \leq r])$$

$$\begin{aligned}
&= P\left(\cap_{i=1}^{j-1}[||S_i|| \leq r] \cap [||S_j|| > r] \cap [||2S_j(\omega) - S_N(\omega)|| \leq r]\right) \\
&\leq P\left(\cap_{i=1}^{j-1}[||S_i|| \leq r] \cap [||S_j|| > r] \cap [||S_N(\omega)|| > r]\right) \\
&= P(A_j \cap [||S_N(\omega)|| > r]).
\end{aligned}$$

It follows that

$$\begin{aligned}
P\left(\left[\sup_{k \leq N-1} ||S_k|| > r \text{ and } ||S_N|| \leq r\right]\right) &= \sum_{i=1}^{N-1} P(A_j \cap [||S_N|| \leq r]) \\
&\leq \sum_{i=1}^{N-1} P(A_j \cap [||S_N|| > r]) \\
&\leq P([||S_N|| > r])
\end{aligned}$$

and using 36.39, this proves the lemma.

This interesting lemma will now be used to prove the following which concludes a sequence of partial sums converges given a subsequence of the sequence of partial sums converges.

Lemma 36.78 *Let $\{\zeta_k\}$ be a sequence of random variables having values in a separable real Banach space, E whose distributions are symmetric. Letting $S_k \equiv \sum_{i=1}^k \zeta_i$, suppose $\{S_{n_k}\}$ converges a.e. Also suppose that for every $m > n_k$,*

$$P\left(\left[||S_m - S_{n_k}||_E > 2^{-k}\right]\right) < 2^{-k}. \tag{36.42}$$

Then in fact,

$$S_k(\omega) \to S(\omega) \ \ a.e.\omega \tag{36.43}$$

Proof: Let $n_k \leq l \leq m$. Then by Lemma 36.77

$$P\left(\left[\sup_{n_k < l \leq m} ||S_l - S_{n_k}|| > 2^{-k}\right]\right) \leq 2P\left(\left[||S_m - S_{n_k}|| > 2^{-k}\right]\right)$$

In using this lemma, you could renumber the ζ_i so that the sum

$$\sum_{j=n_k+1}^{l} \zeta_j$$

corresponds to

$$\sum_{j=1}^{l-n_k} \xi_j$$

where $\xi_j = \zeta_{j+n_k}$. Then using 36.42,

$$P\left(\left[\sup_{n_k < l \leq m} ||S_l - S_{n_k}|| > 2^{-k}\right]\right) \leq 2P\left(\left[||S_m - S_{n_k}|| > 2^{-k}\right]\right) < 2^{-(k-1)}$$

If $S_l(\omega)$ fails to converge then ω must be in infinitely many of the sets,

$$\left[\sup_{n_k<l}||S_l - S_{n_k}|| > 2^{-k}\right]$$

each of which has measure no more than $2^{-(k-1)}$. Thus ω must be in a set of measure zero. This proves the lemma.

Now with this preparation, here is the theorem about white noise.

Theorem 36.79 *Let (i, H, B) be an abstract Wiener space. Then there exists a Gaussian measure on the Borel sets of B. This Gaussian measure equals $\mathcal{L}(S)$ where $S(\omega)$ is the a.e. limit of the sequence of partial sums,*

$$S_n(\omega) \equiv \sum_{k=1}^{n} \xi_k(\omega) e_k$$

for $\{\xi_k\}$ a sequence of independent random variables which are normal with mean 0 and variance 1 which are defined on a probability space, $(\Omega, \mathcal{F}, P)$ and $\{e_k\}$ is a complete orthonormal sequence in H.

Proof: By Corollary 36.74 there is a subsequence, $\{S_{p_n}\}$ of these partial sums which converge pointwise a.e. to $S(\omega)$. However, this corollary also states that

$$P\left(\left\{\omega \in \Omega : ||S_k(\omega) - S_{p_n}(\omega)|| > 2^{-n}\right\}\right) < 2^{-n}$$

whenever $k > p_n$ and so by Lemma 36.78 the original sequence of partial sums also converges a.e. The reason this lemma applies is that $\xi_k(\omega) e_k$ has symmetric distribution. This proves the corollary.

36.13 Existence Of Abstract Wiener Spaces

It turns out that if E is a separable Banach space, then it is the top third of an abstract Wiener space. This is what will be shown in this section. Therefore, it follows from the above that there exists a Gaussian measure on E which is the law of an a.e. convergent series as discussed above. First here is a little lemma which is interesting for its own sake.

Lemma 36.80 *Let E be a separable Banach space. Then there exists an increasing sequence of subspaces, $\{F_n\}$ such that $\dim(F_{n+1}) - \dim(F_n) \leq 1$ and equals 1 for all n if the dimension of E is infinite. Also $\cup_{n=1}^{\infty} F_n$ is dense in E.*

Proof: Since E is separable, so is $\partial B(0,1)$, the boundary of the unit ball. Let $\{w_k\}_{k=1}^{\infty}$ be a countable dense subset of $\partial B(0,1)$.

Let $z_1 = w_1$. Let $F_1 = \mathbb{F} z_1$. Suppose F_n has been obtained and equals $\text{span}(z_1, \cdots, z_n)$ where $\{z_1, \cdots, z_n\}$ is independent, $||z_k|| = 1$, and if $n \neq m$,

$$||z_m - z_n|| \geq \frac{1}{2}.$$

Claim: F_n is closed. Let

$$y_k \equiv \sum_{j=1}^{n} c_j^k z_j \tag{36.44}$$

be such that $y_k \to y$. I need to verify $y \in F_n$. Let $\mathbf{c}_k = \left(c_1^k, \cdots, c_n^k\right)$. Suppose first $\{\mathbf{c}_k\}$ is unbounded in $\mathbb{F}^n$. Then taking a subsequence, still denoted by $\mathbf{c}_k$, it can be assumed $|\mathbf{c}_k| \to \infty$. Therefore,

$$0 = \lim_{k\to\infty} \frac{y_k}{|\mathbf{c}_k|} = \lim_{k\to\infty} \sum_{j=1}^{n} \frac{c_j^k}{|\mathbf{c}_k|} z_j. \tag{36.45}$$

Then taking another subsequence it can also be assumed that

$$\frac{\mathbf{c}_k}{|\mathbf{c}_k|} \to \mathbf{c},\ |\mathbf{c}| = 1.$$

but then 36.45 implies

$$0 = \sum_{j=1}^{n} c_j z_j$$

and this is a contradiction since the z_j are independent.

Thus it must be the case that $\{\mathbf{c}_k\}$ is bounded in $\mathbb{F}^n$. But now, taking a suitable subsequence such that $\mathbf{c}_k \to \mathbf{c}$, it follows from 36.44 that

$$y \equiv \sum_{j=1}^{n} c_j z_j$$

so $y \in F_n$. This shows F_n is closed and this proves the claim.

If F_n contains $\{w_k\}$, let $F_m = F_n$ for all $m > n$. Otherwise, pick $w \in \{w_k\}$ to be the point of $\{w_k\}$ having the smallest subscript which is not contained in F_n. Then w is at a positive distance, λ from F_n because F_n is closed. Therefore, there exists $y \in F_n$ such that $\lambda \leq ||y - w|| \leq 2\lambda$. Let $z_{n+1} = \frac{w-y}{||w-y||}$. It follows

$$w = ||w - y||\, z_{n+1} + y \in \text{span}\left(z_1, \cdots, z_{n+1}\right) \equiv F_{n+1}$$

Then if $m < n + 1$,

$$\begin{aligned} ||z_{n+1} - z_m|| &= \left|\left|\frac{w-y}{||w-y||} - z_m\right|\right| \\ &= \left|\left|\frac{w-y}{||w-y||} - \frac{||w-y||\, z_m}{||w-y||}\right|\right| \\ &\geq \frac{1}{2\lambda} ||w - y - ||w-y||\, z_m|| \\ &\geq \frac{\lambda}{2\lambda} = \frac{1}{2}. \end{aligned}$$

This has shown the existence of an increasing sequence of subspaces, $\{F_n\}$ as described above. It remains to show the union of these subspaces is dense. First note that the union of these subspaces must contain the $\{w_k\}$ because if w_m is missing, then it would contradict the construction at the m^{th} step. That one should have been chosen. However, $\{w_k\}$ is dense in $\partial B(0,1)$. If $x \in E$ and $x \neq 0$, then $\frac{x}{||x||} \in \partial B(0,1)$ then there exists

$$w_m \in \{w_k\} \subseteq \cup_{n=1}^{\infty} F_n$$

such that $\left|\left|w_m - \frac{x}{||x||}\right|\right| < \frac{\varepsilon}{||x||}$. But then

$$||||x||\, w_m - x|| < \varepsilon$$

and so $||x||\, w_m$ is a point of $\cup_{n=1}^{\infty} F_n$ which is within ε of x. This proves $\cup_{n=1}^{\infty} F_n$ is dense as desired. This proves the lemma.

Lemma 36.81 *Let E be a separable Banach space. Then there exists a sequence $\{e_n\}$ of points of E such that whenever $|\boldsymbol{\beta}| \leq 1$ for $\boldsymbol{\beta} \in \mathbb{F}^n$,*

$$\sum_{k=1}^{n} \beta_k e_k \in B(0,1)$$

the unit ball in E.

Proof: By Lemma 36.80, let $\{z_1, \cdot\cdot\cdot, z_n\}$ be a basis for F_n where $\cup_{n=1}^{\infty} F_n$ is dense in E. Then let α_1 be such that $e_1 \equiv \alpha_1 z_1 \in B(0,1)$. Thus $|\beta_1 e_1| \in B(0,1)$ whenever $|\beta_1| \leq 1$. Suppose α_i has been chosen for $i = 1, 2, \cdot\cdot\cdot, n$ such that for all $\boldsymbol{\beta} \in D_n \equiv \{\boldsymbol{\alpha} \in \mathbb{F}^n : |\boldsymbol{\alpha}| \leq 1\}$, it follows

$$\sum_{k=1}^{n} \beta_k \alpha_k z_k \in B(0,1).$$

Then

$$C_n \equiv \left\{ \sum_{k=1}^{n} \beta_k \alpha_k z_k : \boldsymbol{\beta} \in D_n \right\}$$

is a compact subset of $B(0,1)$ and so it is at a positive distance from the complement of $B(0,1)$, δ. Now let $0 < \alpha_{n+1} < \delta / ||z_{n+1}||$. Then for $\boldsymbol{\beta} \in D_{n+1}$,

$$\sum_{k=1}^{n} \beta_k \alpha_k z_k \in C_n$$

and so

$$\begin{aligned} \left|\left| \sum_{k=1}^{n+1} \beta_k \alpha_k z_k - \sum_{k=1}^{n} \beta_k \alpha_k z_k \right|\right| &= ||\beta_{n+1} \alpha_{n+1} z_{n+1}|| \\ &< ||\alpha_{n+1} z_{n+1}|| < \delta \end{aligned}$$

which shows

$$\sum_{k=1}^{n+1} \beta_k \alpha_k z_k \in B(0,1).$$

This proves the lemma. Let $e_k \equiv \alpha_k z_k$.

Now the main result is the following. It says that any separable Banach space is the upper third of some abstract Wiener space.

Theorem 36.82 *Let E be a real separable Banach space with norm $||\cdot||$. Then there exists a separable Hilbert space, H such that H is dense in E and the inclusion map is continuous. Furthermore, if ν is the Gaussian measure defined earlier on the cylinder sets of $H, ||\cdot||$ is Gross measurable.*

Proof: Let $\{e_k\}$ be the points of E described in Lemma 36.81. Then let H_0 denote the subspace of all finite linear combinations of the $\{e_k\}$. It follows H_0 is dense in E. Next decree that $\{e_k\}$ is an orthonormal basis for H_0. Thus for

$$\sum_{k=1}^{n} c_k e_k, \sum_{j=1}^{n} d_j e_k \in H_0,$$

$$\left(\sum_{k=1}^{n} c_k e_k, \sum_{j=1}^{n} d_j e_j\right)_{H_0} \equiv \sum_{k=1}^{n} c_k d_k$$

this being well defined because the $\{e_k\}$ are linearly independent. Let the norm on H_0 be denoted by $|\cdot|_{H_0}$. Let H_1 be the completion of H_0 with respect to this norm.

I want to show that $|\cdot|_{H_0}$ is stronger than $||\cdot||$. Suppose then that

$$\left|\sum_{k=1}^{n} \beta_k e_k\right|_{H_0} \leq 1.$$

It follows then from the definition of $|\cdot|_{H_0}$ that

$$\left|\sum_{k=1}^{n} \beta_k e_k\right|_{H_0}^2 = \sum_{k=1}^{n} \beta_k^2 \leq 1$$

and so from the construction of the e_k, it follows that

$$\left|\left|\sum_{k=1}^{n} \beta_k e_k\right|\right| < 1$$

Stated more simply, this has just shown that if $h \in H_0$ then since $\left|h/|h|_{H_0}\right|_{H_0} \leq 1$, it follows that

$$||h|| / |h|_{H_0} < 1$$

and so

$$||h|| < |h|_{H_0} .$$

It follows that the completion of H_0 must lie in E because this shows that every Cauchy sequence in H_0 is a Cauchy sequence in E. Thus H_1 embedds continuously into E and is dense in E. Denote its norm by $|\cdot|_{H_1}$.

Now consider the Hilbert Schmidt operator,

$$A = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k$$

where each $\lambda_k > 0$ and $\sum_k \lambda_k^2 < \infty$. This operator is clearly one to one. Let

$$H \equiv AH_1.$$

and for $x \in H$, define

$$|x|_H \equiv \left|A^{-1}x\right|_{H_1} .$$

Since each e_k is in H it follows that H is dense in E. Note also that $H \subseteq H_1$ because A maps H_1 to H_1.

$$Ax \equiv \sum_{k=1}^{\infty} \lambda_k (x, e_k) e_k$$

and the series converges in H_1 because

$$\sum_{k=1}^{\infty} \lambda_k |(x, e_k)| \leq \left(\sum_{k=1}^{\infty} \lambda_k^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |(x, e_k)|^2\right)^{1/2} < \infty.$$

Also H is a Hilbert space with inner product given by

$$(x, y)_H \equiv \left(A^{-1}x, A^{-1}y\right)_{H_1} .$$

H is complete because if $\{x_n\}$ is a Cauchy sequence in H, this is the same as $\left\{A^{-1}x_n\right\}$ being a Cauchy sequence in H_1 which implies $A^{-1}x_n \to y$ for some $y \in H_1$. Then it follows $x_n = A\left(A^{-1}x_n\right) \to Ay$ in H.

For $x \in H \subseteq H_1$,

$$||x|| \leq |x|_{H_1} = \left|AA^{-1}x\right|_{H_1} \leq ||A|| \left|A^{-1}x\right|_{H_1} \equiv ||A|| \, |x|_H$$

and so the embedding of H into E is continuous. Why is $||\cdot||$ a measurable norm on H? Note first that for $x \in H \subseteq H_1$,

$$|Ax|_H \equiv \left|A^{-1}Ax\right|_{H_1} = |x|_{H_1} \geq ||x||_E . \tag{36.46}$$

Therefore, if it can be shown A is a Hilbert Schmidt operator on H, the desired measurability will follow from Lemma 36.70 on Page 1040.

Claim: A is a Hilbert Schmidt operator on H.

Proof of the claim: From the definition of the inner product in H, it follows an orthonormal basis for H is $\{\lambda_k e_k\}$. This is because

$$(\lambda_k e_k, \lambda_j e_j)_H \equiv \left(\lambda_k A^{-1} e_k, \lambda_j A^{-1} e_j\right)_{H_1} = (e_k, e_j)_{H_1} = \delta_{jk}.$$

To show that A is Hilbert Schmidt, it suffices to show that

$$\sum_k |A(\lambda_k e_k)|_H^2 < \infty$$

because this is the definition of an operator being Hilbert Schmidt. However, the above equals

$$\sum_k \left|A^{-1} A(\lambda_k e_k)\right|_{H_1}^2 = \sum_k \lambda_k^2 < \infty.$$

This proves the claim.

Now consider 36.46. By Lemma 36.70, it follows the norm $||x||' \equiv |Ax|_H$ is Gross measurable on H. Therefore, $||\cdot||_E$ is also Gross measurable because it is smaller. This proves the theorem.

Using Theorem 36.73 and Theorem 36.82 this proves the following important corollary.

Corollary 36.83 *Let E be any real separable Banach space and let $\{\xi_k\}$ be any sequence of independent random variables such that $\mathcal{L}(\xi_k) = N(0,1)$. Then there exists a sequence, $\{e_k\} \subseteq E$ such that*

$$X(\omega) \equiv \sum_{k=1}^{\infty} \xi_k(\omega) e_k$$

converges a.e. and its law is a Gaussian measure defined on $\mathcal{B}(E)$.

36.14 Fernique's Theorem

The following is an interesting lemma.

Lemma 36.84 *Suppose μ is a symmetric Gaussian measure on the real separable Banach space, E. Then there exists a probability space, $(\Omega, \mathcal{F}, P)$ and independent random variables, X and Y mapping Ω to E such that $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$. Also, the two random variables,*

$$\frac{1}{\sqrt{2}}(X - Y), \ \frac{1}{\sqrt{2}}(X + Y)$$

are independent and

$$\mathcal{L}\left(\frac{1}{\sqrt{2}}(X - Y)\right) = \mathcal{L}\left(\frac{1}{\sqrt{2}}(X + Y)\right) = \mu.$$

Proof: Letting $X' \equiv \frac{1}{\sqrt{2}}(X+Y)$ and $Y' \equiv \frac{1}{\sqrt{2}}(X-Y)$, it follows from Theorem 36.38 on Page 1017, that X' and Y' are independent if whenever $h_1, \cdots, h_m \in E'$ and $g_1, \cdots, g_k \in E'$, the two random vectors,

$$(h_1 \circ X', \cdots, h_m \circ X') \text{ and } (g_1 \circ Y', \cdots, g_k \circ Y')$$

are linearly independent. Now consider linear combinations

$$\sum_{j=1}^{m} t_j h_j \circ X' + \sum_{i=1}^{k} s_i g_i \circ Y'.$$

This equals

$$\begin{aligned}
&\frac{1}{\sqrt{2}} \sum_{j=1}^{m} t_j h_j (X) + \frac{1}{\sqrt{2}} \sum_{j=1}^{m} t_j h_j (Y) \\
&+ \frac{1}{\sqrt{2}} \sum_{i=1}^{k} s_i g_i (X) - \frac{1}{\sqrt{2}} \sum_{i=1}^{k} s_i g_i (Y) \\
= &\ \frac{1}{\sqrt{2}} \left(\sum_{j=1}^{m} t_j h_j + \sum_{i=1}^{k} s_i g_i \right) (X) \\
&+ \frac{1}{\sqrt{2}} \left(\sum_{j=1}^{m} t_j h_j - \sum_{i=1}^{k} s_i g_i \right) (Y)
\end{aligned}$$

and this is the sum of two independent normally distributed random variables so it is also normally distributed. Therefore, by Theorem 36.51

$$(h_1 \circ X', \cdots, h_m \circ X', g_1 \circ Y', \cdots, g_k \circ Y')$$

is a random variable with multivariate normal distribution and by Theorem 36.56 the two random vectors

$$(h_1 \circ X', \cdots, h_m \circ X') \text{ and } (g_1 \circ Y', \cdots, g_k \circ Y')$$

are linearly independent if

$$E((h_i \circ X')(g_j \circ Y')) = 0$$

for all i, j. This is what I will show next.

$$\begin{aligned}
&E((h_i \circ X')(g_j \circ Y')) \\
= &\ \frac{1}{4} E((h_i(X) + h_i(Y))(g_j(X) - g_j(Y))) \\
= &\ \frac{1}{4} E(h_i(X) g_j(X)) - \frac{1}{4} E(h_i(X) g_j(Y)) \\
&+ \frac{1}{4} E(h_i(Y) g_j(X)) - \frac{1}{4} E(h_i(Y) g_j(Y))
\end{aligned} \tag{36.47}$$

Now from the above observation after the definition of Gaussian measure $h_i(X) g_j(X)$ and $h_i(Y) g_j(Y)$ are both in L^1 because each term in each product is normally distributed. Therefore, by Lemma 36.57,

$$\begin{aligned} E(h_i(X) g_j(X)) &= \int_\Omega h_i(Y) g_j(Y) \, dP \\ &= \int_E h_i(y) g_j(y) \, d\mu \\ &= \int_\Omega h_i(X) g_j(X) \, dP \\ &= E(h_i(Y) g_j(Y)) \end{aligned}$$

and so 36.47 reduces to

$$\frac{1}{4}(E(h_i(Y) g_j(X) - h_i(X) g_j(Y))) = 0$$

because $h_i(X)$ and $g_j(Y)$ are independent due to the assumption that X and Y are independent. Thus

$$E(h_i(X) g_j(Y)) = E(h_i(X)) E(g_j(Y)) = 0$$

due to the assumption that μ is symmetric which implies the mean of these random variables equals 0. The other term works out similarly. This has proved the independence of the random variables, X' and Y'.

Next consider the claim they have the same law and it equals μ. To do this, I will use Theorem 36.36 on Page 1016. Thus I need to show

$$E\left(e^{ih(X')}\right) = E\left(e^{ih(Y')}\right) = E\left(e^{ih(X)}\right) \tag{36.48}$$

for all $h \in E'$. Pick such an h. Then $h \circ X$ is normally distributed and has mean 0. Therefore, for some σ,

$$E\left(e^{ith \circ X}\right) = e^{-\frac{1}{2}t^2\sigma^2}.$$

Now since X and Y are independent,

$$\begin{aligned} E\left(e^{ith \circ X'}\right) &= E\left(e^{ith\left(\frac{1}{\sqrt{2}}\right)(X+Y)}\right) \\ &= E\left(e^{ith\left(\frac{1}{\sqrt{2}}\right)X}\right) E\left(e^{ith\left(\frac{1}{\sqrt{2}}\right)Y}\right) \end{aligned}$$

the product of two characteristic functions of two random variables, $\frac{1}{\sqrt{2}}X$ and $\frac{1}{\sqrt{2}}Y$. The variance of these two random variables which are normally distributed with zero mean is $\frac{1}{2}\sigma^2$ and so

$$E\left(e^{ith \circ X'}\right) = e^{-\frac{1}{2}\left(\frac{1}{2}\sigma^2\right)} e^{-\frac{1}{2}\left(\frac{1}{2}\sigma^2\right)} = e^{-\frac{1}{2}\sigma^2} = E\left(e^{ith \circ X}\right).$$

Similar reasoning shows $E\left(e^{ith\circ Y'}\right) = E\left(e^{ith\circ Y}\right) = E\left(e^{ith\circ X}\right)$. Letting $t = 1$, this yields 36.48. This proves the lemma.

With this preparation, here is an incredible theorem due to Fernique.

Theorem 36.85 *Let μ be a symmetric Gaussian measure on $\mathcal{B}(E)$ where E is a real separable Banach space. Then for λ sufficiently small and positive,*

$$\int_R e^{\lambda ||x||^2} d\mu < \infty.$$

More specifically, if λ and r are chosen such that

$$\ln\left(\frac{\mu([x : ||x|| > r])}{\mu\left(\overline{B(0,r)}\right)}\right) + 25\lambda r^2 < -1,$$

then

$$\int_R e^{\lambda ||x||^2} d\mu \leq \exp\left(\lambda r^2\right) + \frac{e^2}{e^2 - 1}.$$

Proof: Let X, Y be independent random variables having values in E such that $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$. Then by Lemma 36.84

$$\frac{1}{\sqrt{2}}(X - Y), \frac{1}{\sqrt{2}}(X + Y)$$

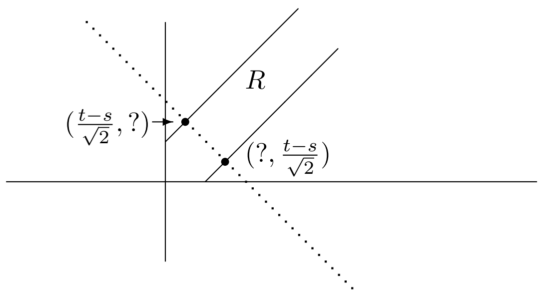
are also independent and have the same law. Now let $0 \leq s \leq t$ and use independence of the above random variables along with the fact they have the same law as X and Y to obtain

$$P(||X|| \leq s, ||Y|| > t) = P(||X|| \leq s) P(||Y|| > t)$$

$$\begin{aligned}
&= P\left(\left|\left|\frac{1}{\sqrt{2}}(X - Y)\right|\right| \leq s\right) P\left(\left|\left|\frac{1}{\sqrt{2}}(X + Y)\right|\right| > t\right) \\
&= P\left(\left|\left|\frac{1}{\sqrt{2}}(X - Y)\right|\right| \leq s, \left|\left|\frac{1}{\sqrt{2}}(X + Y)\right|\right| > t\right) \\
&\leq P\left(\frac{1}{\sqrt{2}}\left|||X|| - ||Y||\right| \leq s, \ \frac{1}{\sqrt{2}}(||X|| + ||Y||) > t\right).
\end{aligned}$$

Now consider the following picture in which the region, R represents the points, $(||X||, ||Y||)$ such that

$$\frac{1}{\sqrt{2}}\left|||X|| - ||Y||\right| \leq s \text{ and } \frac{1}{\sqrt{2}}(||X|| + ||Y||) > t.$$



Therefore, continuing with the chain of inequalities above,

$$\begin{aligned} & P(||X|| \leq s) P(||Y|| > t) \\ \leq & P\left(||X|| > \frac{t-s}{\sqrt{2}}, ||Y|| > \frac{t-s}{\sqrt{2}}\right) \\ = & P\left(||X|| > \frac{t-s}{\sqrt{2}}\right)^2. \end{aligned}$$

Since X, Y have the same law, this can be written as

$$P(||X|| > t) \leq \frac{P\left(||X|| > \frac{t-s}{\sqrt{2}}\right)^2}{P(||X|| \leq s)}.$$

Now define a sequence as follows. $t_0 \equiv r > 0$ and $t_{n+1} \equiv r + \sqrt{2} t_n$. Also, in the above inequality, let $s \equiv r$ and then it follows

$$\begin{aligned} P(||X|| > t_{n+1}) & \leq \frac{P\left(||X|| > \frac{t_{n+1}-r}{\sqrt{2}}\right)^2}{P(||X|| \leq r)} \\ & = \frac{P(||X|| > t_n)^2}{P(||X|| \leq r)}. \end{aligned}$$

Let

$$\alpha_n(r) \equiv \frac{P(||X|| > t_n)}{P(||X|| \leq r)}.$$

Then it follows

$$\alpha_{n+1}(r) \leq \alpha_n(r)^2, \ \alpha_0(r) = \frac{P(||X|| > r)}{P(||X|| \leq r)}.$$

Consequently, $\alpha_n(r) \leq \alpha_0(r)^{2^n}$ and also

$$\begin{aligned} P(||X|| > t_n) & = \alpha_n(r) P(||X|| \leq r) \\ & \leq P(||X|| \leq r) \alpha_0(r)^{2^n} \\ & = P(||X|| \leq r) e^{\ln(\alpha_0(r)) 2^n}. \end{aligned} \tag{36.49}$$

Now using the distribution function technique and letting $\lambda > 0$,

$$\begin{aligned}\int_E e^{\lambda||x||^2} d\mu &= \int_0^\infty \mu\left(\left[e^{\lambda||x||^2} > t\right]\right) dt \\ &= 1 + \int_1^\infty \mu\left(\left[e^{\lambda||x||^2} > t\right]\right) dt \\ &= 1 + \int_1^\infty P\left(\left[e^{\lambda||X||^2} > t\right]\right) dt. \end{aligned} \tag{36.50}$$

From 36.49,

$$P\left(\left[\exp\left(\lambda\,||X||^2\right) > \exp\left(\lambda t_n^2\right)\right]\right) \leq P\left(\left[||X|| \leq r\right]\right) e^{\ln(\alpha_0(r))2^n}.$$

Now split the above improper integral into intervals, $\left(\exp\left(\lambda t_n^2\right), \exp\left(\lambda t_{n+1}^2\right)\right)$ for $n = 0, 1, \cdots$ and note that $P\left(\left[e^{\lambda||X||^2} > t\right]\right)$ is decreasing in t. Then from 36.50,

$$\int_E e^{\lambda||x||^2} d\mu \leq \exp\left(\lambda r^2\right) + \sum_{n=0}^\infty \int_{\exp\left(\lambda t_n^2\right)}^{\exp\left(\lambda t_{n+1}^2\right)} P\left(\left[e^{\lambda||X||^2} > t\right]\right) dt$$

$$\begin{aligned} &\leq \exp\left(\lambda r^2\right) + \sum_{n=0}^\infty P\left(\left[e^{\lambda||X||^2} > \exp\left(\lambda t_n^2\right)\right]\right)\left(\exp\left(\lambda t_{n+1}^2\right) - \exp\left(\lambda t_n^2\right)\right) \\ &\leq \exp\left(\lambda r^2\right) + \sum_{n=0}^\infty P\left(\left[||X|| \leq r\right]\right) e^{\ln(\alpha_0(r))2^n} \exp\left(\lambda t_{n+1}^2\right) \\ &\leq \exp\left(\lambda r^2\right) + \sum_{n=0}^\infty e^{\ln(\alpha_0(r))2^n} \exp\left(\lambda t_{n+1}^2\right). \end{aligned}$$

It remains to estimate t_{n+1}. From the description of the t_n,

$$t_n = \left(\sum_{k=0}^n \left(\sqrt{2}\right)^k\right) r = r\frac{\left(\sqrt{2}\right)^{n+1} - 1}{\sqrt{2} - 1} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} r \left(\sqrt{2}\right)^n$$

and so

$$t_{n+1} \leq 5r\left(\sqrt{2}\right)^n$$

Therefore,

$$\int_E e^{\lambda||x||^2} d\mu \leq \exp\left(\lambda r^2\right) + \sum_{n=0}^\infty e^{\ln(\alpha_0(r))2^n + \lambda 25 r^2 2^n}.$$

Now first pick r large enough that $\ln\left(\alpha_0\left(r\right)\right) < -2$ and then let λ be small enough that $25\lambda r^2 < 1$ or some such scheme and you obtain $\ln\left(\alpha_0\left(r\right)\right) + \lambda 25 r^2 < -1$. Then for this choice of r and λ, or for any other choice which makes $\ln(\alpha_0(r)) + \lambda 25 r^2 < -1$,

$$\begin{aligned}\int_E e^{\lambda ||x||^2} d\mu &\leq \exp\left(\lambda r^2\right) + \sum_{n=0}^{\infty} e^{-2^n} \\ &\leq \exp\left(\lambda r^2\right) + \sum_{n=0}^{\infty} e^{-2n} \\ &= \exp\left(\lambda r^2\right) + \frac{e^2}{e^2 - 1}.\end{aligned}$$

This proves the theorem.

Note this theorem implies all moments exist for Gaussian measures.

36.15 Reproducing Kernels

Suppose μ is a symmetric Gaussian measure on a real separable Banach space, E. Recall this means that for $\phi \in E'$, $\mathcal{L}(\phi) = N\left(0, \sigma^2\right)$ for some σ^2. So what is σ^2 in terms of ϕ? By definition

$$\sigma^2 = E\left(\phi^2\right) = \int_E \phi(x)^2 \, d\mu \tag{36.51}$$

and so $\phi \in L^2(E)$. Thus you can consider $E' \subseteq L^2(E)$. Let $\overline{E'}$ denote the closure of E' in $L^2(E)$. Then $\overline{E'}$ is a Hilbert space with inner product given by

$$(\phi, \psi) \equiv \int_E \phi(x) \psi(x) \, d\mu.$$

For $\phi \in L^2(E)$, denote by $R^{-1}\phi$ the element of E given by the Bochner integral,

$$R^{-1}\phi \equiv \int_E x \phi(x) \, d\mu. \tag{36.52}$$

It is necessary to verify this integral makes sense. By Fernique's theorem, Theorem 36.85,

$$\int_E ||x||^2 \, d\mu \leq C \int_E e^{\lambda ||x||^2} d\mu < \infty$$

and so by the Cauchy Schwarz inequality,

$$\int_E ||x\phi(x)|| \, d\mu \leq \left(\int_E ||x||^2 \, d\mu\right)^{1/2} \left(\int_E |\phi(x)|^2 \, d\mu\right)^{1/2} < \infty.$$

Also in 36.52 the integrand is weakly measurable and is separably valued so the Bochner integral makes sense as claimed and the integrand is in $L^1(E; E)$.

The map, R^{-1} is clearly linear and it is also one to one on $\overline{E'}$ because if $R^{-1}\phi = 0$, then there exists a sequence $\{\phi_n\} \subseteq E'$ converging to ϕ in $L^2(E)$. Therefore,

$$0 = \phi_n\left(\int_E x\phi(x)\,d\mu\right) = \int_E \phi_n(x)\,\phi(x)\,d\mu \to \int_E \phi(x)^2\,d\mu$$

and so $\phi(x) = 0$ a.e. x.

Now define

$$H \equiv \left\{R^{-1}\phi : \phi \in \overline{E'}\right\}$$

and let an inner product be given by

$$\left(R^{-1}\phi, R^{-1}\psi\right)_H \equiv \int_E \phi(x)\,\psi(x)\,d\mu.$$

Since R^{-1} is one to one, the inner product is well defined and the map, $R^{-1} : \overline{E'} \to H$ is one to one, onto, and preserves norms. Therefore, H is also a Hilbert space.

Now before making the next observation, note that by Fernique's theorem, Theorem 36.85, there exists $\lambda > 0$ such that

$$\int_E \|x\|^2\,d\mu = \frac{1}{\lambda}\int_E \lambda\,\|x\|^2\,d\mu \leq \frac{1}{\lambda}\int_E e^{\lambda\|x\|^2}d\mu \equiv C_\mu < \infty.$$

This implies H embedds continuously into E because

$$\begin{aligned}
\left\|R^{-1}\phi\right\|_E &= \left\|\int_E x\phi(x)\,d\mu\right\|_E \leq \int_E \|x\|_E\,|\phi(x)|\,d\mu \\
&\leq \left(\int_E \|x\|^2\,d\mu\right)^{1/2}\left(\int_E \phi(x)^2\,d\mu\right)^{1/2} \\
&= C_\mu\left(\int_E \phi(x)^2\,d\mu\right)^{1/2} \equiv C_\mu\left\|R^{-1}\phi\right\|_H
\end{aligned}$$

Now it follows from all this that H is a Hilbert space which embedds continuously into E and for $\phi \in E'$,

$$\begin{aligned}
\|i^*\phi\|_{H'} &\equiv \sup_{|h|_H \leq 1, h\in H} \phi(h) \\
&= \sup_{|R^{-1}\psi|_H \leq 1, \psi\in E'} \phi\left(R^{-1}(\psi)\right) \\
&= \sup_{|R^{-1}\psi|_H \leq 1, \psi\in E'} \phi\left(\int_E x\psi(x)\,d\mu\right) \\
&= \sup_{\|\psi\|_{L^2(E)} \leq 1, \psi\in E'} \int_E \phi(x)\,\psi(x)\,d\mu \\
&= \|\phi\|_{L^2(E)} = \sigma.
\end{aligned}$$

by 36.51.

Finally, I claim that H must be dense in E. To see this, suppose it is not the case. Then by a standard use of the Hahn Banch theorem, there would exist $\phi \in E'$ such that $\phi(H) = 0$ but $\phi \neq 0$. But then

$$0 = \phi\left(R^{-1}\phi\right) \equiv \phi\left(\int_E x\phi(x)\,d\mu\right) = \int_E \phi(x)^2\,d\mu$$

which is a contradiction. In fact, this shows even more than H is dense in E. It also shows that $R^{-1}(E) \equiv H^0 \subseteq H$ is dense in E.

Definition 36.86 *Let μ be a symmetric Gaussian measure on a real separable Banach space, E. Then a Hilbert space, H is said to be a reproducing kernel space for μ if $H \subseteq E$ with the inclusion map continuous, and for every $\phi \in E'$,*

$$\mathcal{L}(\phi) = N\left(0, |\phi|_H\right)$$

where

$$||i^*\phi||_{H'} \equiv \sup_{||h||_H \leq 1, h \in H} \phi(h) = ||\phi||_{L^2(E)}$$

Implicit in this definition are the inclusions,

$$H \subseteq E,\ E' \subseteq H' \subseteq L^2(E,\mu)$$

where i is the inclusion map of H into E.

Now I need a technical lemma before proving the next theorem.

Lemma 36.87 *Suppose A is a reflexive Banach space and $A \subseteq B$ with the inclusion map, i continuous. Also suppose M is a subset of B' which separates the points of A. Then i^*M is dense in A'.*

Proof: Suppose that i^*M is not dense in A'. Then there exists $a^* \in A' \setminus \left(\overline{i^*M}\right)$. It follows from a standard construction using the Hahn Banach theorem there exists $a^{**} \in A''$ such that $a^{**}(a^*) \neq 0$ but $a^{**}(i^*b^*) = 0$ for all $b^* \in M$. Since A is reflexive, there exists $a \neq 0$ such that $a \in A$ and $Ja = a^{**}$ where J is the James map from A to A^{**} given by $Ja(a^*) \equiv a^*(a)$. Then for all $b^* \in M$,

$$0 = a^{**}(i^*b^*) = Ja(i^*b^*) = i^*b^*(a) \equiv b^*(a).$$

Since M separates the points, this implies that $a = 0$ contrary to $a \neq 0$. This proves the lemma.

The above discussion proves the existence part of the following theorem.

Theorem 36.88 *If μ is a symmetric Gaussian measure on E, a real separable Banach space, then there exists a unique reproducing kernel space, H satisfying the above properties of Definition 36.86. Also it must be the case that H is dense in E and that the inverse Riesz map, R^{-1} from H' to H is given by*

$$R^{-1}\phi \equiv \int_E x\phi(x)\,d\mu$$

Proof: For $\phi \in L^2(E,\mu)$ define

$$S^{-1}\phi \equiv \int_E x\phi(x)\,d\mu.$$

The integrand is weakly measurable and separably valued and Fernique's theorem implies

$$\int_E ||x||_E |\phi(x)|\,d\mu \leq \left(\int_E ||x||^2\,d\mu\right)^{1/2}\left(\int_E |\phi(x)|^2\,d\mu\right)^{1/2} < \infty$$

so S^{-1} is defined on all of $L^2(E,\mu)$.

$S^{-1}(E')$ is dense in E because if not, there would exist $\phi \in E'$ such that $\phi \neq 0$ but $\phi\left(S^{-1}(E')\right) = 0$ but then

$$0 = \phi\left(S^{-1}(\phi)\right) = \phi\left(\int_E x\phi(x)\,d\mu\right) = \int_E \phi(x)^2\,d\mu,$$

a contradiction to $\phi \neq 0$.

Now suppose H_1 and H are two reproducing kernel spaces for μ. Let $\phi \in H_1' \subseteq L^2(E,\mu)$. Since the norm on H' equals the $L^2(E)$ norm, it follows that for $\phi \in E', \psi \in H'$ and R the Riesz map from H to H',

$$\begin{aligned}
\phi\left(R^{-1}(\psi)\right) &= R\left(R^{-1}(\phi)\right)\left(R^{-1}\psi\right) \\
&= \left(R^{-1}\phi, R^{-1}\psi\right)_H = (\phi,\psi)_{H'} \\
&= \int_E \phi(x)\,\psi(x)\,d\mu = \phi\left(S^{-1}(\psi)\right).
\end{aligned}$$

Therefore, $R^{-1} = S^{-1}$ on H'. A similar conclusion holds for H_1.

By Lemma 36.87, E' must be dense in H_1' and also dense in H'. Thus $S^{-1}(E')$ is dense in both H and H_1. Letting $\phi \in H_1'$ there exists $\{\phi_n\}$ a sequence in E' which converges to ϕ in H_1'. Which implies $\phi_n \to \phi$ in $L^2(E,\mu)$ because the norms are the same. But then ϕ_n is also a Cauchy sequence in H' which shows $\phi \in H'$. Thus $H_1' \subseteq H'$ and similarly, $H' \subseteq H_1'$. It follows $H = S^{-1}(H') = S^{-1}(H_1') = H_1$. This proves the theorem.

Definition 36.89 *For μ a symmetric Gaussian measure on E a real separable Banach space, denote by H_μ the reproducing kernel Hilbert space described in Definition 36.86.*

Here is an interesting formula. Letting E be a real separable Banach space and μ a symmetric Gaussian measure on $\mathcal{B}(E)$, with H_μ the reproducing kernel space, consider $h, g \in H_\mu$. Then letting $\phi_h, \phi_g \in H_\mu'$ such that $R^{-1}\phi_h = h$ and $R^{-1}\phi_g = g$,

$$(h,g)_{H_\mu} \equiv \int_E \phi_h(x)\,\phi_g(x)\,d\mu. \tag{36.53}$$

Also, for $x \in H$,

$$(h,x)_{H_\mu} = \left(R^{-1}\phi_h, x\right)_{H_\mu} \equiv \phi_h(x),$$

a similar formula holding for g in place of h. Now using this in 36.53 yields the following interesting formula.

$$(h,g)_{H_\mu} \equiv \int_E (h,x)_{H_\mu} (g,x)_{H_\mu} \, d\mu.$$

Next consider the question of how to identify reproducing kernels and how to tell whether a given probability measure is a Gaussian measure.

Before the next theorem is proved, recall the following two theorems proved on Pages 1029 and 1027 respectively.

Theorem 36.90 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be random vectors having values in* $\mathbb{R}^p$ *and* $\mathbb{R}^q$ *respectively. Suppose also that* $(\mathbf{X},\mathbf{Y})$ *is normally distributed and*

$$E\left((\mathbf{X}-\mathbf{E}(\mathbf{X}))(\mathbf{Y}-\mathbf{E}(\mathbf{Y}))^*\right) = \mathbf{0}.$$

Then $\mathbf{X}$ *and* $\mathbf{Y}$ *are independent random vectors.*

Theorem 36.91 *Suppose* ν *is a probability measure on the Borel sets of* $\mathbb{R}$ *and suppose that* ξ *and* ζ *are independent random variables such that* $\mathcal{L}(\xi) = \mathcal{L}(\zeta) = \nu$ *and whenever* $\alpha^2+\beta^2 = 1$ *it follows* $\mathcal{L}(\alpha\xi+\beta\zeta) = \nu$*. Then*

$$\mathcal{L}(\xi) = N\left(0,\sigma^2\right)$$

for some $\sigma \geq 0$*. Also if* $\mathcal{L}(\xi) = \mathcal{L}(\zeta) = N\left(0,\sigma^2\right)$ *where* ξ,ζ *are independent, then if* $\alpha^2+\beta^2=1$*, it follows* $\mathcal{L}(\alpha\xi+\beta\zeta) = N\left(0,\sigma^2\right)$.

Also recall the following theorem and corollary proved on Page 870.

Theorem 36.92 *Let* $\mathbf{X} = (X_1,\cdots,X_p)$ *where each* X_i *is a real valued random variable. Then* $\mathbf{X}$ *is normally distributed in the above generalized sense if and only if every linear combination,* $\sum_{j=1}^p a_i X_i$ *is normally distributed. In this case the mean of* $\mathbf{X}$ *is*

$$\mathbf{m} = (E(X_1),\cdots,E(X_p))$$

and the covariance matrix for $\mathbf{X}$ *is*

$$\Sigma_{jk} = E\left((X_j - m_j)(X_k - m_k)\right)$$

where $m_j = E(X_j)$.

Corollary 36.93 *Let* $\mathbf{X} = (X_1,\cdots,X_p), \mathbf{Y} = (Y_1,\cdots,Y_p)$ *where each* X_i, Y_i *is a real valued random variable. Suppose also that for every* $\mathbf{a} \in \mathbb{R}^p$*,* $\mathbf{a}\cdot\mathbf{X}$ *and* $\mathbf{a}\cdot\mathbf{Y}$ *are both normally distributed with the same mean and variance. Then* $\mathbf{X}$ *and* $\mathbf{Y}$ *are both multivariate normal random vectors with the same mean and covariance.*

Lemma 36.94 *Let $M \subseteq E'$, where E is a real separable Banach space, be such that $\sigma(M) = \mathcal{B}(E)$. Also suppose X, Y are two E valued random variables such that for all $n \in \mathbb{N}$, and $\vec{\phi} \in M^n, \mathcal{L}\left(\vec{\phi} \circ X\right) = \mathcal{L}\left(\vec{\phi} \circ Y\right)$. That is, for all $F \in \mathcal{B}(\mathbb{R}^n)$,*

$$P\left(\left[\vec{\phi} \circ X \in F\right]\right) = P\left(\left[\vec{\phi} \circ Y \in F\right]\right)$$

Then $\mathcal{L}(X) = \mathcal{L}(Y)$.

Proof: Define $\mathcal{F}$ as the π system which consists of cylindrical sets of the form

$$\left\{x \in E : \vec{\phi}(x) \in \prod_{i=1}^{m} G_i, G_i \in \mathcal{B}(\mathbb{R})\right\}$$

where $\vec{\phi} \in M^m$for some $m \in \mathbb{N}$. Thus $\sigma(\mathcal{F}) = \sigma(M) = \mathcal{B}(E)$ and $\mathcal{F}$ is clearly a π system.

Now define

$$\mathcal{G} \equiv \{F \in \sigma(\mathcal{F}) = \mathcal{B}(E) : P([X \in F]) = P([Y \in F])\}.$$

First suppose $F \in \mathcal{F}$. Then

$$F = \vec{\phi}^{-1}\left(\prod_{i=1}^{m} G_i\right)$$

where each G_i is in $\mathcal{B}(\mathbb{R})$. Thus

$$\begin{aligned} P([X \in F]) &= P\left(\left[X \in \vec{\phi}^{-1}\left(\prod_{i=1}^{m} G_i\right)\right]\right) \\ &= P\left(\left[\vec{\phi}(X) \in \left(\prod_{i=1}^{m} G_i\right)\right]\right) \\ &= P\left(\left[\vec{\phi}(Y) \in \left(\prod_{i=1}^{m} G_i\right)\right]\right) \\ &= P([Y \in F]) \end{aligned}$$

and so $\mathcal{F} \subseteq \mathcal{G}$. If $A \in \mathcal{G}$ then

$$\begin{aligned} P\left(\left[X \in A^C\right]\right) &= 1 - P([X \in A]) \\ &= 1 - P([Y \in A]) = P\left(\left[Y \in A^C\right]\right) \end{aligned}$$

and so $\mathcal{G}$ is closed under complements. Next, if $\{A_i\}$ is a sequence of disjoint sets of $\mathcal{G}$,

$$\begin{aligned} P([X \in \cup_{i=1}^{\infty} A_i]) &= P(\cup_{i=1}^{\infty} [X \in A_i]) \\ &= \sum_{i=1}^{\infty} P([X \in A_i]) \\ &= \sum_{i=1}^{\infty} P([Y \in A_i]) \\ &= P([Y \in \cup_{i=1}^{\infty} A_i]). \end{aligned}$$

It follows from the lemma about π systems, Lemma 9.72 on Page 257 that $\mathcal{G} = \sigma(\mathcal{F}) = \mathcal{B}(E)$ and this says $\mathcal{L}(X) = \mathcal{L}(Y)$. This proves the lemma.

So when do the conditions of this lemma hold? It seems a fairly strong assumption to have $\mathcal{L}\left(\vec{\phi} \circ X\right) = \mathcal{L}\left(\vec{\phi} \circ Y\right)$ for all $\vec{\phi} \in M^n$ for any $n \in \mathbb{N}$. In the next corollary, this condition will hold. This corollary says that if $\sigma(M) = \mathcal{B}(E)$, then in verifying a probability measure is Gaussian, you only need to consider $\phi \in M$ rather than all $\phi \in E'$.

Corollary 36.95 *Suppose E is a real separable Banach space and μ is a probability measure on $\mathcal{B}(E)$. Suppose M is a subspace of E' with the property that $\sigma(M)$[2]$=$ $\mathcal{B}(E)$ such that each $\phi \in M$ considered as a random variable on the probability space, $(E, \mathcal{B}(E), \mu)$ is normally distributed with mean 0 and variance $||\phi||_{L^2(E)}^2$. Then μ is a symmetric Gaussian measure.*

Proof: Let X, Y be independent random variables having values in E such that $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$. As indicated earlier, such a thing exists. You apply Skorokhod's theorem to the product measure $\mu \times \mu$ to find (X, Y) having values in $E \times E$ such that $\mathcal{L}((X, Y)) = \mu \times \mu$.

I want to show that for all $\phi \in E', \mathcal{L}(\phi) = N(0, \sigma^2)$ for some σ. I know this is true for $\phi \in M$. Therefore, by Theorem 36.91, if $\phi \in M$ and $\alpha^2 + \beta^2 = 1$, then

$$\mathcal{L}(\phi(\alpha X + \beta Y)) = \mathcal{L}(\phi(X)),$$

and both random variables are normally distributed with 0 mean. Now take $\vec{\phi} \in M^n$ and consider $\mathbf{a} \cdot \vec{\phi}$ which is also in M because M is a subspace. Then from the above,

$$\mathcal{L}\left(\mathbf{a} \cdot \vec{\phi}(\alpha X + \beta Y)\right) = \mathcal{L}\left(\mathbf{a} \cdot \left(\alpha \vec{\phi}(X) + \beta \vec{\phi}(Y)\right)\right) = \mathcal{L}\left(\mathbf{a} \cdot \vec{\phi}(X)\right).$$

and the random variables, $\mathbf{a} \cdot \vec{\phi}(X)$ and $\mathbf{a} \cdot \left(\alpha \vec{\phi}(X) + \beta \vec{\phi}(Y)\right)$ are both normally distributed with 0 mean and have the same distribution. Then by Corollary 36.93

$$\begin{aligned} \mathcal{L}\left(\vec{\phi}(X)\right) &= \mathcal{L}\left(\alpha \vec{\phi}(X) + \beta \vec{\phi}(Y)\right) \\ &= \mathcal{L}\left(\vec{\phi}(\alpha X + \beta Y)\right) \end{aligned}$$

[2]Recall this means the smallest σ algebra such that each function in M is measurable.

and both are equal to a multvariate normal distribution. Now applying Lemma 36.94, it follows

$$\mathcal{L}(X) = \mathcal{L}(\alpha X + \beta Y) = \mu \tag{36.54}$$

whenever $\alpha^2 + \beta^2 = 1$.

I want to verify $\mathcal{L}(\phi) = N\left(0, ||\phi||^2_{L^2(E)}\right)$ for all $\phi \in E'$. I have just shown that whenever X, Y are independent with $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$, then if $\alpha^2 + \beta^2 = 1$, it follows 36.54 holds. Now take an arbitrary $\phi \in E'$. It follows

$$\mathcal{L}(\phi(X)) = \mathcal{L}(\phi(\alpha X + \beta Y)) = \mathcal{L}(\alpha\phi(X) + \beta\phi(Y)) \tag{36.55}$$

whenever X, Y are independent having $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$ and $\alpha^2 + \beta^2 = 1$. This is a good time to state Theorem 36.91 again. Here it is:

Theorem 36.96 *Suppose ν is a probability measure on the Borel sets of $\mathbb{R}$ and suppose that ξ and ζ are independent random variables such that $\mathcal{L}(\xi) = \mathcal{L}(\zeta) = \nu$ and whenever $\alpha^2 + \beta^2 = 1$ it follows $\mathcal{L}(\alpha\xi + \beta\zeta) = \nu$. Then*

$$\mathcal{L}(\xi) = N\left(0, \sigma^2\right)$$

for some $\sigma \geq 0$. Also if $\mathcal{L}(\xi) = \mathcal{L}(\zeta) = N\left(0, \sigma^2\right)$ where ξ, ζ are independent, then if $\alpha^2 + \beta^2 = 1$, it follows $\mathcal{L}(\alpha\xi + \beta\zeta) = N\left(0, \sigma^2\right)$.

Fixing $\phi \in E'$, define $\nu(F) \equiv \mu([\phi(X) \in F]) = \mu([\phi(Y) \in F])$ so ν is a probability measure on the Borel sets of $\mathbb{R}$ and $\mathcal{L}(\phi \circ X) = \mathcal{L}(\phi \circ X) = \nu$. Furthermore, from 36.55 $\nu = \mathcal{L}(\alpha\phi(X) + \beta\phi(Y))$ whenever $\alpha^2 + \beta^2 = 1$ and $\phi \circ X$ and $\phi \circ Y$ are independent. Therefore by Theorem 36.91 $\mathcal{L}(\phi \circ X) = N\left(0, \sigma^2\right)$. In other words, letting $(\Omega, \mathcal{F}, P)$ be the probability space on which X is defined,

$$\int_E e^{it\phi(x)} d\mu = \int_\Omega e^{it\phi(X(\omega))} dP = e^{-\frac{1}{2}\sigma^2 t^2}$$

which shows $\mathcal{L}(\phi) = N\left(0, \sigma^2\right)$ because the characteristic function of the random variable, ϕ equals the characteristic function of one which has law equal to $N\left(0, \sigma^2\right)$. This proves the corollary.

In the above corollary, you could say μ was Gaussian by verifying it worked as it should with regard to $\phi \in M$ where M was a suitable subset of E'. In this next theorem, this result will be combined with one which also gives a way to identify the reproducing kernel space.

Theorem 36.97 *Suppose E is a real separable Banach space and μ is a probability measure on $\mathcal{B}(E)$. Suppose M is a subspace of E' with the property that $\sigma(M) = \mathcal{B}(E)$ such that each $\phi \in M$ considered as a random variable on the probability space, $(E, \mathcal{B}(E), \mu)$ is normally distributed with mean 0 and variance $||\phi||^2_{L^2(E)}$. Then μ is a symmetric Gaussian measure. Next also assume M separates the points of E and that H is a Hilbert space continuously embedded in E with the property that for every $\phi \in M$,*

$$||i^*\phi||_{H'} = ||\phi||_{L^2(E)}.$$

Then it follows

$$||i^*\phi||_{H'} = ||\phi||_{L^2(E)}$$

for all $\phi \in E'$ *and* H *is the reproducing kernel space for* μ.

Proof: By Corollary 36.95 it follows μ is a Gaussian measure. It remains to verify the assertion that H is the reproducing kernel space. To do this I will first show $\overline{M} = \overline{E'}$ where the closure is taken in $L^2(E;\mu)$. If this is not so, then there exists $\phi \in E' \setminus \overline{M}$. Then there exists $\phi_0 \in \overline{M}$ such that $\phi - \phi_0 \neq 0$ is orthogonal to $\overline{M}$, the orthogonality being in $L^2(E)$. Furthermore, there exists a sequence, $\{\phi_{0n}\} \subseteq M$ such that $\phi_{0n} \to \phi_0$ in $L^2(E)$.

From the first part, whenever $\psi \in E', \mathcal{L}(\psi) = N\left(0, ||\phi||^2_{L^2(E)}\right)$.(This is what it means for μ to be Gaussian.) Let $\{\psi_1, \cdots, \psi_m\} \subseteq M$. Then an arbitrary linear combination of these ψ_j and $\phi - \phi_{0n}$ is another function in E' and so its law is normal with zero mean. It follows from Theorem 36.92 that

$$(\phi - \phi_{0n}, \psi_1, \cdots, \psi_m) \tag{36.56}$$

is a multivariate normal random vector having values in $\mathbb{R}^{m+1}$. Now the random vector,

$$(\phi - \phi_0, \psi_1, \cdots, \psi_m) \tag{36.57}$$

is the limit in $L^2(E)$ as $n \to \infty$ of the random vectors in 36.56 and so the means and covariances of the vectors in 36.56 converge. Thus the vector in 36.57 is also a multivariate normal random vector. By Theorem 36.90 it follows the two random vectors, $\phi - \phi_0$ and $(\psi_1, \cdots, \psi_m)$ are independent. Now it is easy to see that

$$\sigma(M) = \cup_{F \subseteq M, F \text{ finite}} \sigma(F).$$

Therefore, $\sigma(\phi - \phi_0)$ and $\sigma(M) = \mathcal{B}(E)$ are independent. This implies $\phi - \phi_0 = 0$. Here is why. Since $\phi - \phi_0$ is the limit in $L^2(E)$ of continuous functions, and since μ is regular by Lemma 36.17 on Page 1000 it can be assumed by changing $\phi - \phi_0$ on a set of measure zero that $\phi - \phi_0$ is Borel measurable. Now here is an interesting lemma.

Lemma 36.98 *Suppose* $\mathcal{G}$ *and* $\mathcal{F}$ *are two* σ *algebras on a probability space,* $(\Omega, \mathcal{S}, P)$ *and suppose they are independent and that* $\mathcal{G} \subseteq \mathcal{F}$. *Then if* $A \in \mathcal{G}$ *it follows either* $P(A) = 0$ *or* $P(A) = 1$.

Proof of the lemma: Let $A \in \mathcal{G}$. Then

$$\begin{aligned} \int_\Omega \mathcal{X}_A(\omega)\, dP &= \int_\Omega \mathcal{X}_A(\omega)^2\, dP \\ &= \left(\int_\Omega \mathcal{X}_A dP\right)\left(\int_\Omega \mathcal{X}_A dP\right) \end{aligned}$$

and so $P(A) = P(A)^2$. This proves the lemma.

Now continuing with the proof of Theorem 36.97, $\sigma(\phi - \phi_0) \subseteq \mathcal{B}(E)$ and the two are independent so every set in $\sigma(\phi - \phi_0)$ has measure either 0 or 1. Thus

$$\mu([|\phi - \phi_0| > \alpha])$$

always equals either 0 or 1. However, $t \to \mu\left(\left[|\phi - \phi_0|^2 > t\right]\right)$ is a decreasing function and so if t is large enough this function equals 0 since otherwise it always equals 1 and

$$\int_0^\infty \mu\left(\left[|\phi - \phi_0|^2 > t\right]\right) dt = \infty$$

contrary to $\phi - \phi_0$ being in L^2. Therefore, there exists M such that

$$\mu([x \in E : |\phi(x) - \phi_0(x)| \leq M]) = 1$$

This is impossible if the distribution of $\phi - \phi_0$ is normal having positive variance which is the case if $\phi - \phi_0 \neq 0$ in $L^2(E)$. Therefore, $\phi = \phi_0$ which shows that $\overline{M} = \overline{E'}$ in $L^2(E)$ as claimed.

With this, it is time to show $||i^*\phi||_{H'} = ||\phi||_{L^2(E)}$ whenever $\phi \in E'$ given this holds for $\phi \in M$. It is here that the assumption that M separates the points of E is used for the first time. Thus, assume

$$||i^*\phi||_{H'} = ||\phi||_{L^2(E)} \tag{36.58}$$

for all $\phi \in M$. Let $f \in E'$. Then $i^*f \in H'$. By Lemma 36.87 there exists a sequence, $\{\phi_n\} \subseteq M$ such that $i^*\phi_n \to i^*f$ in H'. Hence by 36.58 it is also the case that $\phi_n \to f$ in $L^2(E)$. Let $\phi, \psi \in M$. Then by 36.58 it is routine to verify

$$(i^*\phi, i^*\psi)_{H'} = \int_E \phi(x)\psi(x)\,d\mu.$$

Also recall that by Fernique's theorem and $f \in E'$

$$\int_E ||x||\,|f(x)|\,d\mu \leq ||f|| \int_E ||x||^2\,d\mu < \infty$$

Therefore, letting $\phi \in M$,

$$\begin{aligned}
\phi\left(\int_E xf(x)\,d\mu\right) &= \int_E \phi(x) f(x)\,d\mu \\
&= \lim_{n\to\infty} \int \phi(x)\phi_n(x)\,d\mu \\
&= \lim_{n\to\infty} (i^*\phi, i^*\phi_n)_{H'} = (i^*\phi, i^*f)_{H'} \\
&= \phi\left(R^{-1}(i^*f)\right)
\end{aligned}$$

where R^{-1} is the inverse of the Riesz map, R from H to H' which is defined by

$$Rx(y) = (x, y)_H.$$

Now here is where M separates the points is used. The above equation shows since ϕ is arbitrary that

$$\int_E xf(x)\,d\mu = R^{-1}(i^*f)$$

for any $f \in E'$. This also shows that for all $f, g \in E'$

$$(i^*f, i^*g)_{H'} = \int_E f(x)\,g(x)\,d\mu$$

because

$$\begin{aligned}(i^*f, i^*g)_{H'} &= \left(R^{-1}(i^*f), R^{-1}(i^*g)\right)_H \\ &= i^*f\left(R^{-1}(i^*g)\right) = f\left(\int_E xg(x)\,d\mu\right) \\ &= \int_E f(x)\,g(x)\,d\mu.\end{aligned}$$

Therefore, letting $f \in E'$, and noting that by Lemma 36.87 i^*E' is dense in H'

$$\begin{aligned}||i^*f||_{H'} &\equiv \sup_{||h||_H \leq 1} f(h) = \sup_{||\psi||_{H'} \leq 1} f\left(R^{-1}\psi\right) \\ &= \sup_{||i^*\psi||_{H'} \leq 1, \psi \in E'} f\left(R^{-1}i^*\psi\right) \\ &= \sup_{||\psi||_{L^2(E)} \leq 1, \psi \in E'} \int_E f(x)\,\psi(x)\,d\mu = ||f||_{L^2(E)}\end{aligned}$$

and this proves the theorem. Since H has the properties of the reproducing kernel space, it equals H_μ.

36.16 Reproducing Kernels And White Noise

Consider the case where E is a real separable Banach space and μ is a symmetric Gaussian measure on the Borel sets of E. Then as discussed in Theorem 36.88 there exists a unique reproducing kernel space, H_μ such that H_μ is a dense subset of E, E' is a dense subset of H'_μ, and the norm on H'_μ is the same as the norm in $L^2(E,\mu)$, H_μ being a closed subspace of $L^2(E,\mu)$. Thus

$$H_\mu \subseteq E, \quad E' \subseteq H'_\mu \subseteq L^2(E,\mu).$$

Also recall $H^0_\mu \equiv R^{-1}(E')$ where R is the Riesz map from H_μ to H'_μ satisfying

$$Rx(y) \equiv (x,y)_{H_\mu}.$$

Then the theorem about white noise to be proved here is the following.

Theorem 36.99 *Let E be a real separable Banach space and let μ be a Gaussian measure defined on the Borel sets of E and let H_μ be the reproducing kernel space for E. Suppose also that there exists an orthonormal complete basis for H_μ, $\{e_n\} \subseteq H_\mu^0$ such that for $\phi_n \in E'$ defined by $e_n = R^{-1}\phi_n$, $\text{span}(\{\phi_n\})$ is also dense in E'. Then if $\{\xi_j\}$ is a sequence of independent random variables having mean 0 and variance 1, which are defined on a probability space, $(\Omega, \mathcal{F}, P)$ it follows*

$$X(\omega) \equiv \sum_{i=1}^{\infty} \xi_i(\omega) e_i \tag{36.59}$$

converges in E a.e. and $\mathcal{L}(X) = \mu$.

Proof: Let $\phi_i = R(e_i)$ where R is the Riesz map from H_μ to H_μ'. Then it follows ϕ_i is normally distributed with mean 0 and variance σ^2. What is σ^2? By definition, and the properties of the reproducing kernel space,

$$\begin{aligned}\sigma^2 &= \int_E \phi_i^2(x)\, d\mu \equiv (\phi_i, \phi_i)_{H_\mu'} \\ &= \left(R^{-1}\phi_i, R^{-1}\phi_i\right)_{H_\mu} = (e_i, e_i)_{H_\mu} = 1.\end{aligned}$$

I claim that it is also the case that $\{\phi_i\}$ are independent. First note that if $\alpha_i \in \mathbb{R}$, then $\sum_{i=1}^{p} \alpha_i \phi_{n_i} \in E'$ and so is also normally distributed. Hence by Theorem 31.23 on Page 870, $\left(\phi_{n_1}, \cdots, \phi_{n_p}\right)$ is multivariate normal. Now

$$\begin{aligned}E\left(\phi_{n_j}\phi_{n_k}\right) &\equiv \int_E \phi_{n_j}(x)\, \phi_{n_k}(x)\, d\mu \\ &= \left(\phi_{n_j}, \phi_{n_k}\right)_{H_\mu'} = \left(e_{n_j}, e_{n_k}\right)_{H_\mu} = \delta_{jk}\end{aligned}$$

and so the covariance matrix is a diagonal. It follows from Theorem 31.25 on Page 872 that $\left\{\phi_{n_j}\right\}_{j=1}^{p}$ is independent. This establishes the claim and shows that a special case of the theorem involves the consideration of

$$\sum_{k=1}^{\infty} \phi_k(x)\, e_k. \tag{36.60}$$

Here the probability space is E and the measure is μ. Now this special case is easier to work with and the plan is to consider this special case first, showing that the above sum in 36.60 converges to x for a.e. $x \in E$ and then extending to the general case. The advantage of considering this special case first is that you have a candidate for the function to which the series converges which has known distribution.

Let $S(x) \equiv x$ and let $S_N(x) \equiv \sum_{n=1}^{N} \phi_n(x)\, e_n$. First of all, observe that $(E, \mathcal{B}(E), \mu)$ is a probability space and S maps E to E and $\mathcal{L}(S) = \mu$. Thus S has known distribution and it is reasonable to try and get $S_N(x)$ to converge to $S(x)$.

Let

$$\mathcal{L}(S_N) \equiv \mu_N,\ \mathcal{L}(S - S_N) \equiv \mu_N^{\perp}.$$

First note that S_N and $S_M - S_N$ for $M > N$ are independent random variables by the first part of this argument. Letting $\phi \in \operatorname{span}(\{\phi_n\})$

$$\phi\left(\sum_{k=1}^{N} \phi_n(x) e_n\right) = \sum_{k=1}^{N} \phi_n(x) \phi(e_n)$$

and this series converges for each $x \in E$ as $N \to \infty$ because $\phi_k(e_n) = (e_k, e_n)_{H_\mu} = \delta_{kn}$ which implies that if n is sufficiently large $\phi(e_n) = 0$ so the above sequence of partial sums is eventually constant.

Therefore, letting $\phi \in \operatorname{span}(\{\phi_n\})$,

$$\begin{aligned}
& E(\exp(i\phi(S - S_N)))\, E(\exp(i\phi(S_N))) \\
= & \lim_{M\to\infty} E(\exp(i\phi(S_M - S_N)))\, E(\exp(i\phi(S_N))) \\
= & \lim_{M\to\infty} E(\exp(i\phi(S_M - S_N)) \exp(i\phi(S_N))) \\
= & E(\exp(i\phi(S - S_N)) \exp(i\phi(S_N)))
\end{aligned}$$

Now $\operatorname{span}(\{\phi_n\})$ is dense in E' by assumption and so it follows from Corollary 36.40 on Page 1018 that $S - S_N$ and S_N are independent random variables.

It follows from Theorem 36.47 on Page 1023 that

$$\mu = \mu_N * \mu_N^{\perp} \tag{36.61}$$

because $S = S_N + (S - S_N)$ and the two random variables, S_N and $S - S_N$ were just shown to be independent.

Next I will show the measures $\{\mu_N^{\perp}\}$ are tight. This will set things up for an application of Prokhorov's theorem. Let $\varepsilon > 0$ be given. Since μ is a finite measure and E is separable, it follows from Lemmas 36.17 and 36.18 there exists a compact set, $K \subseteq E$ such that

$$\mu(K) \geq 1 - \varepsilon.$$

By 36.61,

$$1 - \varepsilon \leq \mu(K) = \int_E \mu_N^{\perp}(K - x)\, d\mu_N(x)$$

and so there exists $x_N \in E$ such that $\mu_N^{\perp}(K - x_N) \geq 1 - \varepsilon$. However, each ϕ_k has a symmetric distribution since they are each normally distributed with mean 0. Also, S has a symmetric distribution and so it follows so does $S - S_N$. Thus

$$\mu_N^{\perp}(-K + x_N) = \mu_N^{\perp}(K - x_N) \geq 1 - \varepsilon.$$

Now note

$$(-K + x_N) \cap (K - x_N) \subseteq \frac{K - K}{2}$$

because if $x \in (-K + x_N) \cap (K - x_N)$, then $x = -k_1 + x_N = k_2 - x_N$ and so

$$2x = k_2 - k_1 \in K - K.$$

Therefore,

$$\mu_N^{\perp}\left(\frac{K-K}{2}\right) \geq \mu_N^{\perp}\left((-K + x_N) \cap (K - x_N)\right) \geq 1 - 2\varepsilon.$$

This follows easily because

$$\begin{aligned} & \mu_N^{\perp}\left((-K + x_N)^C \cup (K - x_N)^C\right) \\ \leq \; & \mu_N^{\perp}\left((-K + x_N)^C\right) + \mu_N^{\perp}\left((K - x_N)^C\right) \leq 2\varepsilon. \end{aligned}$$

Now note that $\frac{K-K}{2}$ is a compact set because if $f : E \times E \to E$ is given by $f(x, y) = \frac{x-y}{2}$ then f is continuous and so it maps the compact subset of $E \times E$, $K \times K$ to the compact set $\frac{K-K}{2}$. Since this set is not dependent on N this shows $\left\{\mu_N^{\perp}\right\}$ is tight.

By Prokhorov's theorem, Theorem 36.23 on Page 1004 every subsequence of $\left\{\mu_N^{\perp}\right\}$ has a subsequence which converges weakly. I will show they all converge weakly to the measure δ_0 defined by $\delta_0(F) = 1$ if $0 \in F$ and $\delta_0(F) = 0$ if $0 \notin F$. From this it will follow that $\mu_N^{\perp}$ must converge weakly to δ_0. When this is established, it will lead to the desired conclusion.

Let $\phi \in \operatorname{span}(\{\phi_n\})$ and let $\left\{\mu_k^{\perp}\right\}$ denote a weakly convergent subsequence of $\left\{\mu_N^{\perp}\right\}$. Say this converges weakly to ν. Then recall that since $\phi \in \operatorname{span}(\{\phi_n\})$,

$$\lim_{k\to\infty} \phi(S(x) - S_k(x)) = 0$$

and so letting $Y : E \to E$ be given by $Y(x) \equiv 0$,

$$\begin{aligned} 1 &= E(\exp(i\phi(Y))) = \lim_{k\to\infty} E(\exp(i\phi(S - S_k))) \\ &\equiv \lim_{k\to\infty} \int_E \exp(i\phi(x))\, d\mu_k^{\perp}(x) = \int_E \exp(i\phi(x))\, d\nu. \end{aligned}$$

By density of $\operatorname{span}(\{\phi_n\})$ in E', it follows that

$$1 = E(\exp(i\phi(Y))) = \int_E \exp(i\phi(x))\, d\nu$$

for all $\phi \in E'$. If $\lambda = \mathcal{L}(Y)$, then for all $\phi \in E'$,

$$\int_E \exp(i\phi(x))\, d\lambda = E(\exp(i\phi(Y))) = \int_E \exp(i\phi(x))\, d\nu$$

By Theorem 36.36 it follows $\lambda = \nu$ and so $\nu = \mathcal{L}(Y)$ where $Y(x) \equiv 0$. As discussed above, this has shown that $\left\{\mu_N^{\perp}\right\}$ converges weakly to ν where $\nu = \mathcal{L}(Y)$. Also,

$\nu = \delta_0$ because by Theorem 36.36 the two measures, ν and δ_0, have the same characteristic functions.

Now consider $B(0,\varepsilon)^C \subseteq E$. $\delta_0\left(\partial\left(B(0,\varepsilon)^C\right)\right) = 0$ and so it follows by Lemma 36.25 on Page 1007 that

$$\lim_{N\to\infty} \mu_N^{\perp}\left(B(0,\varepsilon)^C\right) = \delta_0\left(B(0,\varepsilon)^C\right) = 0.$$

Therefore,

$$\begin{aligned} 0 &= \lim_{N\to\infty} \mu_N^{\perp}\left(B(0,\varepsilon)^C\right) \\ &= \lim_{N\to\infty} \mu\left(\left[S - S_N \in B(0,\varepsilon)^C\right]\right) \\ &= \lim_{N\to\infty} \mu\left(\left[||S - S_N||_E \geq \varepsilon\right]\right). \end{aligned}$$

Which shows S_N converges in probability to S. In particular, there exists a subsequence n_k such that for all $m > n_k$,

$$\mu\left(\left[||S - S_{n_k}|| > 2^{-k}\right]\right) < 2^{-k}$$

$$\mu\left(\left[||S_m - S_{n_k}|| > 2^{-k}\right]\right) < 2^{-k} \tag{36.62}$$

Then $\{S_{n_k}\}$ converges pointwise a.e. to S. Now it follows from Lemma 36.78 on Page 1049 it follows that since the distributions of the ϕ_k are symmetric that $\{S_n\}$ converges pointwise a.e. to S. Letting $\phi \in E'$,

$$\phi(S_n(x)) = \sum_{k=1}^{n} \phi_k(x)\,\phi(e_k)$$

which is a normally distributed random variable having mean 0 and variance $\sum_{k=1}^{n} \phi(e_k)^2$. Therefore,

$$E(\exp(it\phi(S_n))) = e^{-\frac{1}{2}t^2 \sum_{k=1}^{n} \phi(e_k)^2}$$

and so, passing to the limit, yields

$$E(\exp(it\phi(S))) = e^{-\frac{1}{2}t^2 \sum_{k=1}^{\infty} \phi(e_k)^2}$$

the series in the last term converging because it equals

$$\sum_{k=1}^{\infty} \left(R^{-1}\phi, e_k\right)^2 = \left|R^{-1}\phi\right|_{H_\mu}^2 = |\phi|_{H'_\mu}^2 = ||\phi||_{L^2(E,\mu)}^2.$$

Thus $\phi(S)$ is normally distributed with mean 0 and variance $||\phi||_{L^2(E)}^2$. Hence $\mathcal{L}(S) = \mu$ because if $\nu = \mathcal{L}(S)$, the above has just shown, for $\psi \in E'$,

$$\phi_\nu(\psi) = e^{-\frac{1}{2}||\psi||_{L^2(E)}^2}$$

while

$$\phi_\mu(\psi) \equiv \int_E e^{i\psi(x)} d\mu = e^{-\frac{1}{2}||\psi||^2_{L^2(E)}}$$

due to the observation that since μ is Gaussian, each $\psi \in E'$ is normal with mean 0 and variance equal to $||\psi||^2_{L^2(E)}$. Since the two measures have the same characteristic functions, they are equal by Theorem 36.36.

It only remains to consider the general case described in 36.59. Consider the sum,

$$X_n(\omega) \equiv \sum_{i=1}^{n} \xi_i(\omega) e_i.$$

Letting $\nu_{nm} = \mathcal{L}(X_n - X_m)$ for $m < n$, I would like to show that $\nu_{nm} = \mathcal{L}(S_n - S_m)$ defined above in terms of the sums involving the ϕ_k. The reason for this is that if these are the same, then

$$\mu\left(\left[||S_m - S_{n_k}|| > 2^{-k}\right]\right) = P\left(||X_m - X_{n_k}|| > 2^{-k}\right) < 2^{-k} \tag{36.63}$$

and the pointwise a.e. convergence of $\{X_n\}$ will follow as above using Lemma 36.78 on Page 1049 and then the same characteristic function argument will show $X(\omega)$ defined in 36.59 has $\mathcal{L}(X) = \mu$. But the ξ_i are given to be independent and normally distributed with mean 0 and variance 1 so

$$E(\exp(i\phi(X_n - X_m))) = e^{-\frac{1}{2}\sum_{k=m+1}^{n} \phi(e_k)^2}$$

which is the same as the result of

$$E(\exp(i\phi(S_n - S_m)))$$

and this implies the desired result. Thus 36.63 implies $\{X_{n_k}\}_{k=1}^{\infty}$ converges pointwise a.e. and then Lemma 36.78 on Page 1049 applies again to conclude $\{X_n\}$ converges pointwise a.e. That $\mathcal{L}(X) = \mu$ follows as before from a characteristic function argument. This proves the theorem.

Part V

Sobolev Spaces

Weak Derivatives

37.1 Weak $*$ Convergence

A very important sort of convergence in applications of functional analysis is the concept of weak or weak $*$ convergence. It is important because it allows you to assert the existence of a convergent subsequence of a given bounded sequence. The only problem is the convergence is very weak so it does not tell you as much as you would like. Nevertheless, it is a very useful concept. The big theorems in the subject are the Eberlein Smulian theorem and the Banach Alaoglu theorem about the weak or weak $*$ compactness of the closed unit balls in either a Banach space or its dual space. These theorems are proved in Yosida [52]. Here I will present a special case which turns out to be by far the most important in applications and it is not hard to get from the Riesz representation theorem for L^p. First I define weak and weak $*$ convergence.

Definition 37.1 *Let X' be the dual of a Banach space X and let $\{x_n^*\}$ be a sequence of elements of X'. Then x_n^* converges weak $*$ to x^* if and only if for all $x \in X$,*

$$\lim_{n\to\infty} x_n^* (x) = x^* (x) .$$

A sequence in $X, \{x_n\}$ converges weakly to $x \in X$ if and only if for all $x^* \in X'$

$$\lim_{n\to\infty} x^* (x_n) = x^* (x) .$$

The main result is contained in the following lemma.

Lemma 37.2 *Let X' be the dual of a Banach space, X and suppose X is separable. Then if $\{x_n^*\}$ is a bounded sequence in X', there exists a weak $*$ convergent subsequence.*

Proof: Let D be a dense countable set in X. Then the sequence, $\{x_n^* (x)\}$ is bounded for all x and in particular for all $x \in D$. Use the Cantor diagonal process to obtain a subsequence, still denoted by n such that $x_n^* (d)$ converges for each $d \in D$.

Now let $x \in X$ be completely arbitrary. In fact $\{x_n^*(x)\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given and pick $d \in D$ such that for all n

$$|x_n^*(x) - x_n^*(d)| < \frac{\varepsilon}{3}.$$

This is possible because D is dense. By the first part of the proof, there exists N_ε such that for all $m, n > N_\varepsilon$,

$$|x_n^*(d) - x_m^*(d)| < \frac{\varepsilon}{3}.$$

Then for such m, n,

$$\begin{aligned} |x_n^*(x) - x_m^*(x)| &\leq |x_n^*(x) - x_n^*(d)| + |x_n^*(d) - x_m^*(d)| \\ + |x_m^*(d) - x_m^*(x)| &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since ε is arbitrary, this shows $\{x_n^*(x)\}$ is a Cauchy sequence for all $x \in X$.

Now define $f(x) \equiv \lim_{n\to\infty} x_n^*(x)$. Since each x_n^* is linear, it follows f is also linear. In addition to this,

$$|f(x)| = \lim_{n\to\infty} |x_n^*(x)| \leq K\,||x||$$

where K is some constant which is larger than all the norms of the x_n^*. Such a constant exists because the sequence, $\{x_n^*\}$ was bounded. This proves the lemma.

The lemma implies the following important theorem.

Theorem 37.3 *Let Ω be a measurable subset of $\mathbb{R}^n$ and let $\{f_k\}$ be a bounded sequence in $L^p(\Omega)$ where $1 < p \leq \infty$. Then there exists a weak $*$ convergent subsequence.*

Proof: Since $L^{p'}(\Omega)$ is separable, this follows from the Riesz representation theorem.

Note that from the Riesz representation theorem, it follows that if $p < \infty$, then the sequence converges weakly.

37.2 Test Functions And Weak Derivatives

In elementary courses in mathematics, functions are often thought of as things which have a formula associated with them and it is the formula which receives the most attention. For example, in beginning calculus courses the derivative of a function is defined as the limit of a difference quotient. You start with one function which tends to be identified with a formula and, by taking a limit, you get another formula for the derivative. A jump in abstraction occurs as soon as you encounter the derivative of a function of n variables where the derivative is defined as a certain linear transformation which is determined not by a formula but by what it does to vectors. When this is understood, it reduces to the usual idea in one dimension. The idea of weak partial derivatives goes further in the direction of defining something in terms of what it does rather than by a formula, and extra generality is obtained when it is used. In particular, it is possible to differentiate almost anything if the notion of what is meant by the derivative is sufficiently weak. This has the advantage of allowing the consideration of the weak partial derivative of a function without having to agonize over the important question of existence but it has the disadvantage of not being able to say much about the derivative. Nevertheless, it is the idea of weak partial derivatives which makes it possible to use functional analytic techniques in the study of partial differential equations and it is shown in this chapter that the concept of weak derivative is useful for unifying the discussion of some very important theorems. Certain things which shold be true are.

Let $\Omega \subseteq \mathbb{R}^n$. A distribution on Ω is defined to be a linear functional on $C_c^\infty(\Omega)$, called the space of test functions. The space of all such linear functionals will be denoted by $\mathcal{D}^*(\Omega)$. Actually, more is sometimes done here. One imposes a topology on $C_c^\infty(\Omega)$ making it into a topological vector space, and when this has been done, $\mathcal{D}'(\Omega)$ is defined as the dual space of this topological vector space. To see this, consult the book by Yosida [52] or the book by Rudin [46].

Example: The space $L^1_{loc}(\Omega)$ may be considered as a subset of $\mathcal{D}^*(\Omega)$ as follows.

$$f(\phi) \equiv \int_\Omega f(\mathbf{x})\,\phi(\mathbf{x})\,dx$$

for all $\phi \in C_c^\infty(\Omega)$. Recall that $f \in L^1_{loc}(\Omega)$ if $f\mathcal{X}_K \in L^1(\Omega)$ whenever K is compact.

Example: $\delta_x \in \mathcal{D}^*(\Omega)$ where $\delta_{\mathbf{x}}(\phi) \equiv \phi(\mathbf{x})$.

It will be observed from the above two examples and a little thought that $\mathcal{D}^*(\Omega)$ is truly enormous. The derivative of a distribution will be defined in such a way that it agrees with the usual notion of a derivative on those distributions which are also continuously differentiable functions. With this in mind, let f be the restriction to Ω of a smooth function defined on $\mathbb{R}^n$. Then $D_{x_i}f$ makes sense and for $\phi \in C_c^\infty(\Omega)$

$$D_{x_i}f(\phi) \equiv \int_\Omega D_{x_i}f(\mathbf{x})\,\phi(\mathbf{x})\,dx = -\int_\Omega fD_{x_i}\phi dx = -f(D_{x_i}\phi).$$

This motivates the following definition.

Definition 37.4 *For* $T \in \mathcal{D}^*(\Omega)$

$$D_{x_i}T(\phi) \equiv -T(D_{x_i}\phi).$$

Of course one can continue taking derivatives indefinitely. Thus,

$$D_{x_ix_j}T \equiv D_{x_i}\left(D_{x_j}T\right)$$

and it is clear that all mixed partial derivatives are equal because this holds for the functions in $C_c^\infty(\Omega)$. In this weak sense, the derivative of almost anything exists, even functions that may be discontinuous everywhere. However the notion of “derivative” is very weak, hence the name, “weak derivatives”.

Example: Let $\Omega = \mathbb{R}$ and let

$$H(x) \equiv \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ if } x < 0. \end{cases}$$

Then

$$DH(\phi) = -\int H(x)\,\phi'(x)\,dx = \phi(0) = \delta_0(\phi).$$

Note that in this example, DH is not a function.

What happens when Df is a function?

Theorem 37.5 *Let $\Omega = (a,b)$ and suppose that f and Df are both in $L^1(a,b)$. Then f is equal to a continuous function a.e., still denoted by f and*

$$f(x) = f(a) + \int_a^x Df(t)\,dt.$$

In proving Theorem 37.5 the following lemma is useful.

Lemma 37.6 *Let $T \in \mathcal{D}^*(a,b)$ and suppose $DT = 0$. Then there exists a constant C such that*

$$T(\phi) = \int_a^b C\phi dx.$$

Proof: $T(D\phi) = 0$ for all $\phi \in C_c^\infty(a,b)$ from the definition of $DT = 0$. Let

$$\phi_0 \in C_c^\infty(a,b),\ \int_a^b \phi_0(x)\,dx = 1,$$

and let

$$\psi_\phi(x) = \int_a^x [\phi(t) - \left(\int_a^b \phi(y)\,dy\right)\phi_0(t)]dt$$

for $\phi \in C_c^\infty(a,b)$. Thus $\psi_\phi \in C_c^\infty(a,b)$ and

$$D\psi_\phi = \phi - \left(\int_a^b \phi(y)\,dy\right)\phi_0.$$

Therefore,

$$\phi = D\psi_\phi + \left(\int_a^b \phi(y)\,dy\right)\phi_0$$

and so

$$T(\phi) = T(D\psi_\phi) + \left(\int_a^b \phi(y)\,dy\right)T(\phi_0) = \int_a^b T(\phi_0)\,\phi(y)\,dy.$$

Let $C = T\phi_0$. This proves the lemma.

Proof of Theorem 37.5 Since f and Df are both in $L^1(a,b)$,

$$Df(\phi) - \int_a^b Df(x)\phi(x)\,dx = 0.$$

Consider

$$f(\cdot) - \int_a^{(\cdot)} Df(t)\,dt$$

and let $\phi \in C_c^\infty(a,b)$.

$$D\left(f(\cdot) - \int_a^{(\cdot)} Df(t)\,dt\right)(\phi)$$

$$\equiv -\int_a^b f(x)\phi'(x)\,dx + \int_a^b \left(\int_a^x Df(t)\,dt\right)\phi'(x)\,dx$$

$$\begin{aligned} &= Df(\phi) + \int_a^b \int_t^b Df(t)\phi'(x)\,dxdt \\ &= Df(\phi) - \int_a^b Df(t)\phi(t)\,dt = 0. \end{aligned}$$

By Lemma 37.6, there exists a constant, C, such that

$$\left(f(\cdot) - \int_a^{(\cdot)} Df(t)\,dt\right)(\phi) = \int_a^b C\phi(x)\,dx$$

for all $\phi \in C_c^\infty(a,b)$. Thus

$$\int_a^b \{\left(f(x) - \int_a^x Df(t)\,dt\right) - C\}\phi(x)\,dx = 0$$

for all $\phi \in C_c^\infty(a,b)$. It follows from Lemma 37.9 in the next section that

$$f(x) - \int_a^x Df(t)\,dt - C = 0 \text{ a.e. } x.$$

Thus let $f(a) = C$ and write

$$f(x) = f(a) + \int_a^x Df(t)\,dt.$$

This proves Theorem 37.5.

Theorem 37.5 says that

$$f(x) = f(a) + \int_a^x Df(t)\,dt$$

whenever it makes sense to write $\int_a^x Df(t)\,dt$, if Df is interpreted as a weak derivative. Somehow, this is the way it ought to be. It follows from the fundamental theorem of calculus that $f'(x)$ exists for a.e. x where the derivative is taken in the sense of a limit of difference quotients and $f'(x) = Df(x)$. This raises an interesting question. Suppose f is continuous on $[a,b]$ and $f'(x)$ exists in the classical sense for a.e. x. Does it follow that

$$f(x) = f(a) + \int_a^x f'(t)\,dt?$$

The answer is no. To see an example, consider Problem 4 on Page 445 which gives an example of a function which is continuous on $[0,1]$, has a zero derivative for a.e. x but climbs from 0 to 1 on $[0,1]$. Thus this function is not recovered from integrating its classical derivative.

In summary, if the notion of weak derivative is used, one can at least give meaning to the derivative of almost anything, the mixed partial derivatives are always equal, and, in one dimension, one can recover the function from integrating its derivative. None of these claims are true for the classical derivative. Thus weak derivatives are convenient and rule out pathologies.

37.3 Weak Derivatives In L^p_{loc}

Definition 37.7 *Let U be an open set in $\mathbb{R}^n$.$f \in L^p_{loc}(U)$ if $f\mathcal{X}_K \in L^p$ whenever K is a compact subset of U.*

Definition 37.8 *For $\alpha = (k_1, \cdots, k_n)$ where the k_i are nonnegative integers, define*

$$|\alpha| \equiv \sum_{i=1}^n |k_{x_i}|,\ D^\alpha f(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} f(\mathbf{x})}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}.$$

Also define ϕ_k to be a mollifier if $\operatorname{spt}(\phi_k) \subseteq B\left(\mathbf{0}, \frac{1}{k}\right), \phi_k \geq 0, \int \phi_k dx = 1$, and $\phi_k \in C_c^\infty\left(B\left(\mathbf{0}, \frac{1}{k}\right)\right)$. In the case a Greek letter like δ or ε is used as a subscript, it will mean $\operatorname{spt}(\phi_\delta) \subseteq B(\mathbf{0}, \delta), \phi_\delta \geq 0, \int \phi_\delta dx = 1$, and $\phi_\delta \in C_c^\infty(B(\mathbf{0}, \delta))$. You can always get a mollifier by letting $\phi \geq 0, \phi \in C_c^\infty(B(\mathbf{0}, 1)), \int \phi dx = 1$,and then defining $\phi_k(\mathbf{x}) \equiv k^n \phi(k\mathbf{x})$ or in the case of a Greek subscript, $\phi_\delta(\mathbf{x}) = \frac{1}{\delta^n}\phi\left(\frac{\mathbf{x}}{\delta}\right)$.

Consider the case where u and $D^\alpha u$ for $|\alpha| = 1$ are each in $L^p_{loc}(\mathbb{R}^n)$. The next lemma is the one alluded to in the proof of Theorem 37.5.

Lemma 37.9 *Suppose $f \in L^1_{loc}(U)$ and suppose*

$$\int f\phi dx = 0$$

for all $\phi \in C_c^\infty(U)$. Then $f(\mathbf{x}) = 0$ a.e. $\mathbf{x}$.

Proof: Without loss of generality f is real valued. Let

$$E \equiv \{\mathbf{x} : f(\mathbf{x}) > \varepsilon\}$$

and let

$$E_m \equiv E \cap B(0,m).$$

Is $m(E_m) = 0$? If not, there exists an open set, V, and a compact set K satisfying

$$K \subseteq E_m \subseteq V \subseteq B(0,m),\ m(V \setminus K) < 4^{-1} m(E_m),$$

$$\int_{V \setminus K} |f|\, dx < \varepsilon 4^{-1} m(E_m).$$

Let H and W be open sets satisfying

$$K \subseteq H \subseteq \overline{H} \subseteq W \subseteq \overline{W} \subseteq V$$

and let

$$\overline{H} \prec g \prec W$$

where the symbol, $\prec$, in the above implies $\operatorname{spt}(g) \subseteq W$, g has all values in $[0,1]$, and $g(\mathbf{x}) = 1$ on $\overline{H}$. Then let ϕ_δ be a mollifier and let $h \equiv g * \phi_\delta$ for δ small enough that

$$K \prec h \prec V.$$

Thus

$$\begin{aligned} 0 &= \int fh dx = \int_K f dx + \int_{V \setminus K} fh dx \\ &\geq \varepsilon m(K) - \varepsilon 4^{-1} m(E_m) \\ &\geq \varepsilon \left(m(E_m) - 4^{-1} m(E_m) \right) - \varepsilon 4^{-1} m(E_m) \\ &\geq 2^{-1} \varepsilon m(E_m). \end{aligned}$$

Therefore, $m(E_m) = 0$, a contradiction. Thus

$$m(E) \leq \sum_{m=1}^{\infty} m(E_m) = 0$$

and so, since $\varepsilon > 0$ is arbitrary,

$$m(\{\mathbf{x} : f(\mathbf{x}) > 0\}) = 0.$$

Similarly $m(\{\mathbf{x} : f(\mathbf{x}) < 0\}) = 0$. This proves the lemma.

This lemma allows the following definition.

Definition 37.10 *Let U be an open subset of $\mathbb{R}^n$ and let $u \in L^1_{loc}(U)$. Then $D^\alpha u \in L^1_{loc}(U)$ if there exists a function $g \in L^1_{loc}(U)$, necessarily unique by Lemma 37.9, such that for all $\phi \in C_c^\infty(U)$,*

$$\int_U g\phi dx = D^\alpha u(\phi) \equiv \int_U (-1)^{|\alpha|} u(D^\alpha \phi)\, dx.$$

Then $D^\alpha u$ is defined to equal g when this occurs.

Lemma 37.11 *Let $u \in L^1_{loc}(\mathbb{R}^n)$ and suppose $u_{,i} \in L^1_{loc}(\mathbb{R}^n)$, where the subscript on the u following the comma denotes the i^{th} weak partial derivative. Then if ϕ_ε is a mollifier and $u_\varepsilon \equiv u * \phi_\varepsilon$, it follows $u_{\varepsilon,i} \equiv u_{,i} * \phi_\varepsilon$.*

Proof: If $\psi \in C_c^\infty(\mathbb{R}^n)$, then

$$\begin{aligned} \int u(\mathbf{x}-\mathbf{y})\,\psi_{,i}(\mathbf{x})\,dx &= \int u(\mathbf{z})\,\psi_{,i}(\mathbf{z}+\mathbf{y})\,dz \\ &= -\int u_{,i}(\mathbf{z})\,\psi(\mathbf{z}+\mathbf{y})\,dz \\ &= -\int u_{,i}(\mathbf{x}-\mathbf{y})\,\psi(\mathbf{x})\,dx. \end{aligned}$$

Therefore,

$$\begin{aligned} u_{\varepsilon,i}(\psi) &= -\int u_\varepsilon \psi_{,i} = -\int\int u(\mathbf{x}-\mathbf{y})\,\phi_\varepsilon(\mathbf{y})\,\psi_{,i}(\mathbf{x})\,d\;ydx \\ &= -\int\int u(\mathbf{x}-\mathbf{y})\,\psi_{,i}(\mathbf{x})\,\phi_\varepsilon(\mathbf{y})\,dxdy \\ &= \int\int u_{,i}(\mathbf{x}-\mathbf{y})\,\psi(\mathbf{x})\,\phi_\varepsilon(\mathbf{y})\,dxdy \\ &= \int u_{,i} * \phi_\varepsilon(\mathbf{x})\,\psi(\mathbf{x})\,dx. \end{aligned}$$

The technical questions about product measurability in the use of Fubini's theorem may be resolved by picking a Borel measurable representative for u. This proves the lemma.

What about the product rule? Does it have some form in the context of weak derivatives?

Lemma 37.12 *Let U be an open set, $\psi \in C^\infty(U)$ and suppose $u, u_{,i} \in L^p_{loc}(U)$. Then $(u\psi)_{,i}$ and $u\psi$ are in $L^p_{loc}(U)$ and*

$$(u\psi)_{,i} = u_{,i}\psi + u\psi_{,i}.$$

Proof: Let $\phi \in C_c^\infty(U)$ then

$$\begin{aligned} (u\psi)_{,i}(\phi) &\equiv -\int_U u\psi\phi_{,i}dx \\ &= -\int_U u[(\psi\phi)_{,i} - \phi\psi_{,i}]dx \\ &= \int_U \left(u_{,i}\psi\phi + u\psi_{,i}\phi\right)dx \\ &= \int_U \left(u_{,i}\psi + u\psi_{,i}\right)\phi dx \end{aligned}$$

This proves the lemma.

Recall the notation for the gradient of a function.

$$\nabla u(\mathbf{x}) \equiv (u_{,1}(\mathbf{x}) \cdots \ u_{,n}(\mathbf{x}))^T$$

thus

$$Du(\mathbf{x})\,\mathbf{v} = \nabla u(\mathbf{x}) \cdot \mathbf{v}.$$

37.4 Morrey's Inequality

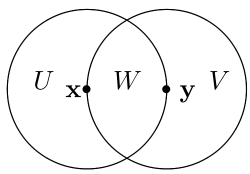
The following inequality will be called Morrey's inequality. It relates an expression which is given pointwise to an integral of the p^{th} power of the derivative.

Lemma 37.13 *Let $u \in C^1(\mathbb{R}^n)$ and $p > n$. Then there exists a constant, C, depending only on n such that for any $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$,*

$$|u(\mathbf{x}) - u(\mathbf{y})|$$

$$\leq C \left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)} |\nabla u(\mathbf{z})|^p dz \right)^{1/p} \left(|\mathbf{x} - \mathbf{y}|^{(1-n/p)} \right). \tag{37.1}$$

Proof: In the argument C will be a generic constant which depends on n. Consider the following picture.



This is a picture of two balls of radius r in $\mathbb{R}^n$, U and V having centers at $\mathbf{x}$ and $\mathbf{y}$ respectively, which intersect in the set, W. The center of U is on the boundary of V and the center of V is on the boundary of U as shown in the picture. There exists a constant, C, independent of r depending only on n such that

$$\frac{m(W)}{m(U)} = \frac{m(W)}{m(V)} = C.$$

You could compute this constant if you desired but it is not important here.

Define the average of a function over a set, $E \subseteq \mathbb{R}^n$ as follows.

$$\fint_E f dx \equiv \frac{1}{m(E)} \int_E f dx.$$

Then

$$\begin{aligned}
|u(\mathbf{x}) - u(\mathbf{y})| &= \fint_W |u(\mathbf{x}) - u(\mathbf{y})|\, dz \\
&\leq \fint_W |u(\mathbf{x}) - u(\mathbf{z})|\, dz + \fint_W |u(\mathbf{z}) - u(\mathbf{y})|\, dz \\
&= \frac{C}{m(U)} \left[\int_W |u(\mathbf{x}) - u(\mathbf{z})|\, dz + \int_W |u(\mathbf{z}) - u(\mathbf{y})|\, dz \right] \\
&\leq C \left[\fint_U |u(\mathbf{x}) - u(\mathbf{z})|\, dz + \fint_V |u(\mathbf{y}) - u(\mathbf{z})|\, dz \right]
\end{aligned}$$

Now consider these two terms. Using spherical coordinates and letting U_0 denote the ball of the same radius as U but with center at $\mathbf{0}$,

$$\begin{aligned}
&\fint_U |u(\mathbf{x}) - u(\mathbf{z})|\, dz \\
&= \frac{1}{m(U_0)} \int_{U_0} |u(\mathbf{x}) - u(\mathbf{z} + \mathbf{x})|\, dz \\
&= \frac{1}{m(U_0)} \int_0^r \rho^{n-1} \int_{S^{n-1}} |u(\mathbf{x}) - u(\rho \mathbf{w} + \mathbf{x})|\, d\sigma(w)\, d\rho \\
&\leq \frac{1}{m(U_0)} \int_0^r \rho^{n-1} \int_{S^{n-1}} \int_0^\rho |\nabla u(\mathbf{x} + t\mathbf{w}) \cdot \mathbf{w}|\, dt d\sigma d\rho \\
&\leq \frac{1}{m(U_0)} \int_0^r \rho^{n-1} \int_{S^{n-1}} \int_0^\rho |\nabla u(\mathbf{x} + t\mathbf{w})|\, dt d\sigma d\rho \\
&\leq C \frac{1}{r} \int_0^r \int_{S^{n-1}} \int_0^r |\nabla u(\mathbf{x} + t\mathbf{w})|\, dt d\sigma d\rho \\
&= C \frac{1}{r} \int_0^r \int_{S^{n-1}} \int_0^r \frac{|\nabla u(\mathbf{x} + t\mathbf{w})|}{t^{n-1}} t^{n-1} dt d\sigma d\rho
\end{aligned}$$

$$\begin{aligned}
&= C \int_{S^{n-1}} \int_0^r \frac{|\nabla u(\mathbf{x} + t\mathbf{w})|}{t^{n-1}} t^{n-1} dt d\sigma \\
&= C \int_{U_0} \frac{|\nabla u(\mathbf{x} + \mathbf{z})|}{|\mathbf{z}|^{n-1}} dz \\
&\leq C \left(\int_{U_0} |\nabla u(\mathbf{x} + \mathbf{z})|^p\, dz \right)^{1/p} \left(\int_U |\mathbf{z}|^{p' - np'} \right)^{1/p'} \\
&= C \left(\int_U |\nabla u(\mathbf{z})|^p\, dz \right)^{1/p} \left(\int_{S^{n-1}} \int_0^r \rho^{p' - np'} \rho^{n-1} d\rho d\sigma \right)^{(p-1)/p} \\
&= C \left(\int_U |\nabla u(\mathbf{z})|^p\, dz \right)^{1/p} \left(\int_{S^{n-1}} \int_0^r \frac{1}{\rho^{\frac{n-1}{p-1}}} d\rho d\sigma \right)^{(p-1)/p}
\end{aligned}$$

$$
\begin{aligned}
&= C\left(\frac{p-1}{p-n}\right)^{(p-1)/p}\left(\int_U |\nabla u(\mathbf{z})|^p\,dz\right)^{1/p} r^{1-\frac{n}{p}} \\
&= C\left(\frac{p-1}{p-n}\right)^{(p-1)/p}\left(\int_U |\nabla u(\mathbf{z})|^p\,dz\right)^{1/p} |\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}}
\end{aligned}
$$

Similarly,

$$
⨍_V |u(\mathbf{y}) - u(\mathbf{z})|\,dz \le C\left(\frac{p-1}{p-n}\right)^{(p-1)/p}\left(\int_V |\nabla u(\mathbf{z})|^p\,dz\right)^{1/p} |\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}}
$$

Therefore,

$$
|u(\mathbf{x}) - u(\mathbf{y})| \le C\left(\frac{p-1}{p-n}\right)^{(p-1)/p}\left(\int_{B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|)} |\nabla u(\mathbf{z})|^p\,dz\right)^{1/p} |\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}}
$$

because $B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|) \supseteq V \cup U$. This proves the lemma.

The following corollary is also interesting

Corollary 37.14 *Suppose* $u \in C^1(\mathbb{R}^n)$. *Then*

$$
|u(\mathbf{y}) - u(\mathbf{x}) - \nabla u(\mathbf{x}) \cdot (\mathbf{y}-\mathbf{x})|
$$

$$
\le C\left(\frac{1}{m(B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|))}\int_{B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|)} |\nabla u(\mathbf{z}) - \nabla u(\mathbf{x})|^p dz\right)^{1/p} |\mathbf{x}-\mathbf{y}|. \tag{37.2}
$$

Proof: This follows easily from letting $g(\mathbf{y}) \equiv u(\mathbf{y}) - u(\mathbf{x}) - \nabla u(\mathbf{x}) \cdot (\mathbf{y}-\mathbf{x})$. Then $g \in C^1(\mathbb{R}^n)$, $g(\mathbf{x}) = 0$, and $\nabla g(\mathbf{z}) = \nabla u(\mathbf{z}) - \nabla u(\mathbf{x})$. From Lemma 37.13,

$$
\begin{aligned}
& |u(\mathbf{y}) - u(\mathbf{x}) - \nabla u(\mathbf{x}) \cdot (\mathbf{y}-\mathbf{x})| \\
=\ & |g(\mathbf{y})| = |g(\mathbf{y}) - g(\mathbf{x})| \\
\le\ & C\left(\int_{B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|)} |\nabla u(\mathbf{z}) - \nabla u(\mathbf{x})|^p dz\right)^{1/p} |\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}} \\
=\ & C\left(\frac{1}{m(B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|))}\int_{B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|)} |\nabla u(\mathbf{z}) - \nabla u(\mathbf{x})|^p dz\right)^{1/p} |\mathbf{x}-\mathbf{y}|.
\end{aligned}
$$

This proves the corollary.

It may be interesting at this point to recall the definition of differentiability on Page 115. If you knew the above inequality held for ∇u having components in $L^1_{loc}(\mathbb{R}^n)$, then at Lebesgue points of ∇u, the above would imply $Du(\mathbf{x})$ exists. This is exactly the approach taken below.

37.5 Rademacher's Theorem

The inequality of Corollary 37.14 can be extended to the case where u and $u_{,i}$ are in $L^p_{loc}(\mathbb{R}^n)$ for $p > n$. This leads to an elegant proof of the differentiability a.e. of a Lipschitz continuous function as well as a more general theorem.

Theorem 37.15 *Suppose u and all its weak partial derivatives, $u_{,i}$ are in $L^p_{loc}(\mathbb{R}^n)$. Then there exists a set of measure zero, E such that if $\mathbf{x}, \mathbf{y} \notin E$ then inequalities 37.2 and 37.1 are both valid. Furthermore, u equals a continuous function a.e.*

Proof: Let $u \in L^p_{loc}(\mathbb{R}^n)$ and $\psi_k \in C_c^\infty(\mathbb{R}^n), \psi_k \geq 0$, and $\psi_k(\mathbf{z}) = 1$ for all $\mathbf{z} \in B(\mathbf{0}, k)$. Then it is routine to verify that

$$u\psi_k,\ (u\psi_k)_{,i} \in L^p(\mathbb{R}^n).$$

Here is why:

$$\begin{aligned}
(u\psi_k)_{,i}(\phi) &\equiv -\int_{\mathbb{R}^n} u\psi_k \phi_{,i} dx \\
&= -\int_{\mathbb{R}^n} u\psi_k \phi_{,i} dx - \int_{\mathbb{R}^n} u\psi_{k,i}\phi dx + \int_{\mathbb{R}^n} u\psi_{k,i}\phi dx \\
&= -\int_{\mathbb{R}^n} u(\psi_k\phi)_{,i} dx + \int_{\mathbb{R}^n} u\psi_{k,i}\phi dx \\
&= \int_{\mathbb{R}^n} \left(u_{,i}\psi_k + u\psi_{k,i}\right)\phi dx
\end{aligned}$$

which shows

$$(u\psi_k)_{,i} = u_{,i}\psi_k + u\psi_{k,i}$$

as expected.

Let ϕ_ε be a mollifier and consider

$$(u\psi_k)_\varepsilon \equiv u\psi_k * \phi_\varepsilon.$$

By Lemma 37.11 on Page 1086,

$$(u\psi_k)_{\varepsilon,i} = (u\psi_k)_{,i} * \phi_\varepsilon.$$

Therefore

$$(u\psi_k)_{\varepsilon,i} \to (u\psi_k)_{,i} \text{ in } L^p(\mathbb{R}^n) \tag{37.3}$$

and

$$(u\psi_k)_\varepsilon \to u\psi_k \text{ in } L^p(\mathbb{R}^n) \tag{37.4}$$

as $\varepsilon \to 0$. By 37.4, there exists a subsequence $\varepsilon \to 0$ such that for $|\mathbf{z}| < k$ and for each $i = 1, 2, \cdots, n$

$$(u\psi_k)_{\varepsilon,i}(\mathbf{z}) \to (u\psi_k)_{,i}(\mathbf{z}) = u_{,i}(\mathbf{z}) \text{ a.e.}$$

$$(u\psi_k)_\varepsilon(\mathbf{z}) \to u\psi_k(\mathbf{z}) = u(\mathbf{z}) \text{ a.e.} \tag{37.5}$$

Denoting the exceptional set by E_k, let

$$\mathbf{x}, \mathbf{y} \notin \cup_{k=1}^{\infty} E_k \equiv E$$

and let k be so large that

$$B(\mathbf{0},k) \supseteq B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|).$$

Then by 37.1 and for $\mathbf{x}, \mathbf{y} \notin E$,

$$|(u\psi_k)_\varepsilon(\mathbf{x}) - (u\psi_k)_\varepsilon(\mathbf{y})|$$

$$\leq C\left(\int_{B(\mathbf{x},2|\mathbf{y}-\mathbf{x}|)} |\nabla(u\psi_k)_\varepsilon|^p dz\right)^{1/p} |\mathbf{x}-\mathbf{y}|^{(1-n/p)}$$

where C depends only on n. Similarly, by 37.2,

$$|(u\psi_k)_\varepsilon(\mathbf{x}) - (u\psi_k)_\varepsilon(\mathbf{y}) - \nabla(u\psi_k)_\varepsilon(\mathbf{x})\cdot(\mathbf{y}-\mathbf{x})| \leq$$

$$C\left(\frac{1}{m(B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|))}\int_{B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|)} |\nabla(u\psi_k)_\varepsilon(\mathbf{z}) - \nabla(u\psi_k)_\varepsilon(\mathbf{x})|^p dz\right)^{1/p} |\mathbf{x}-\mathbf{y}|.$$

Now by 37.5 and 37.3 passing to the limit as $\varepsilon \to 0$ yields

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq C\left(\int_{B(\mathbf{x},2|\mathbf{y}-\mathbf{x}|)} |\nabla u|^p dz\right)^{1/p} |\mathbf{x}-\mathbf{y}|^{(1-n/p)} \tag{37.6}$$

and

$$|u(\mathbf{y}) - u(\mathbf{x}) - \nabla u(\mathbf{x})\cdot(\mathbf{y}-\mathbf{x})|$$

$$\leq C\left(\frac{1}{m(B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|))}\int_{B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|)} |\nabla u(\mathbf{z}) - \nabla u(\mathbf{x})|^p dz\right)^{1/p} |\mathbf{x}-\mathbf{y}|. \tag{37.7}$$

Redefining u on the set of mesure zero, E yields 37.6 for all $\mathbf{x}, \mathbf{y}$. This proves the theorem.

Corollary 37.16 *Let $u, u_{,i} \in L^p_{loc}(\mathbb{R}^n)$ for $i = 1, \cdots, n$ and $p > n$. Then the representative of u described in Theorem 37.15 is differentiable a.e.*

Proof: From Theorem 37.15

$$|u(\mathbf{y}) - u(\mathbf{x}) - \nabla u(\mathbf{x})\cdot(\mathbf{y}-\mathbf{x})|$$

$$\leq C\left(\frac{1}{m(B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|))}\int_{B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|)} |\nabla u(\mathbf{z}) - \nabla u(\mathbf{x})|^p dz\right)^{1/p} |\mathbf{x}-\mathbf{y}|. \tag{37.8}$$

and at every Lebesgue point, $\mathbf{x}$ of ∇u

$$\lim_{\mathbf{y}\to\mathbf{x}}\left(\frac{1}{m\left(B\left(\mathbf{x},2\left|\mathbf{x}-\mathbf{y}\right|\right)\right)}\int_{B(\mathbf{x},2|\mathbf{x}-\mathbf{y}|)}\left|\nabla u\left(\mathbf{z}\right)-\nabla u\left(\mathbf{x}\right)\right|^{p}dz\right)^{1/p}=0$$

and so at each of these points,

$$\lim_{\mathbf{y}\to\mathbf{x}}\frac{\left|u\left(\mathbf{y}\right)-u\left(\mathbf{x}\right)-\nabla u\left(\mathbf{x}\right)\cdot\left(\mathbf{y}-\mathbf{x}\right)\right|}{\left|\mathbf{x}-\mathbf{y}\right|}=0$$

which says that u is differentiable at $\mathbf{x}$ and $Du\left(\mathbf{x}\right)\left(\mathbf{v}\right)=\nabla u\left(\mathbf{x}\right)\cdot\left(\mathbf{v}\right)$. See Page 115. This proves the corollary.

Definition 37.17 *Now suppose u is Lipschitz on $\mathbb{R}^n$,*

$$\left|u\left(\mathbf{x}\right)-u\left(\mathbf{y}\right)\right|\leq K\left|\mathbf{x}-\mathbf{y}\right|$$

for some constant K. Define $\text{Lip}\left(u\right)$ *as the smallest value of K that works in this inequality.*

The following corollary is known as Rademacher's theorem. It states that every Lipschitz function is differentiable a.e.

Corollary 37.18 *If u is Lipschitz continuous then u is differentiable a.e. and* $\left|\left|u_{,i}\right|\right|_{\infty}\leq\text{Lip}\left(u\right)$.

Proof: This is done by showing that Lipschitz continuous functions have weak derivatives in $L^{\infty}\left(\mathbb{R}^n\right)$ and then using the previous results. Let

$$D_{\mathbf{e}_i}^{h}u\left(\mathbf{x}\right)\equiv h^{-1}\left[u\left(\mathbf{x}+h\mathbf{e}_i\right)-u\left(\mathbf{x}\right)\right].$$

Then $D_{\mathbf{e}_i}^{h}u$ is bounded in $L^{\infty}\left(\mathbb{R}^n\right)$ and

$$||D_{\mathbf{e}_i}^{h}u||_{\infty}\leq\text{Lip}\left(u\right).$$

It follows that $D_{\mathbf{e}_i}^{h}u$ is contained in a ball in $L^{\infty}\left(\mathbb{R}^n\right)$, the dual space of $L^{1}\left(\mathbb{R}^n\right)$. By Theorem 37.3 on Page 1080, there is a subsequence $h\to0$ such that

$$D_{\mathbf{e}_i}^{h}u\rightharpoonup w,\ ||w||_{\infty}\leq\text{Lip}\left(u\right)$$

where the convergence takes place in the weak $*$ topology of $L^{\infty}(\mathbb{R}^n)$. Let $\phi\in C_c^{\infty}\left(\mathbb{R}^n\right)$. Then

$$\int w\phi dx=\lim_{h\to0}\int D_{\mathbf{e}_i}^{h}u\phi dx$$

$$=\lim_{h\to0}\int u\left(\mathbf{x}\right)\frac{\left(\phi\left(\mathbf{x}-h\mathbf{e}_i\right)-\phi\left(\mathbf{x}\right)\right)}{h}dx$$

$$=-\int u\left(\mathbf{x}\right)\phi_{,i}\left(\mathbf{x}\right)dx.$$

Thus $w=u_{,i}$ and $u_{,i}\in L^{\infty}\left(\mathbb{R}^n\right)$ for each i. Hence $u,u_{,i}\in L_{loc}^{p}\left(\mathbb{R}^n\right)$ for all $p>n$ and so u is differentiable a.e. by Corollary 37.16. This proves the corollary.

37.6 Change Of Variables Formula Lipschitz Maps

With Rademacher's theorem, one can give a general change of variables formula involving Lipschitz maps.

Definition 37.19 *Let E be a Lebesgue measurable set.$\mathbf{x} \in E$ is a point of density if*

$$\lim_{r\to 0} \frac{m(E\cap B(\mathbf{x},r))}{m(B(\mathbf{x},r))} = 1.$$

You see that if $\mathbf{x}$ were an interior point of E, then this limit will equal 1. However, it is sometimes the case that the limit equals 1 even when $\mathbf{x}$ is not an interior point. In fact, these points of density make sense even for sets that have empty interior.

Lemma 37.20 *Let E be a Lebesgue measurable set. Then there exists a set of measure zero, N, such that if $\mathbf{x} \in E \setminus N$, then $\mathbf{x}$ is a point of density of E.*

Proof: Consider the function, $f(\mathbf{x}) = \mathcal{X}_E(\mathbf{x})$. This function is in $L^1_{loc}(\mathbb{R}^n)$. Let N^C denote the Lebesgue points of f. Then for $\mathbf{x} \in E \setminus N$,

$$\begin{aligned} 1 &= \mathcal{X}_E(\mathbf{x}) = \lim_{r\to 0} \frac{1}{m_n(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} \mathcal{X}_E(\mathbf{y})\, dm_n \\ &= \lim_{r\to 0} \frac{m_n(B(\mathbf{x},r)\cap E)}{m_n(B(\mathbf{x},r))}. \end{aligned}$$

In this section, Ω will be a Lebesgue measurable set in $\mathbb{R}^n$ and $\mathbf{h} : \Omega \to \mathbb{R}^n$ will be Lipschitz. Recall the following definition and theorems. See Page 10.17 for the proofs and more discussion.

Definition 37.21 *Let $\mathcal{F}$ be a collection of balls that cover a set, E, which have the property that if $\mathbf{x} \in E$ and $\varepsilon > 0$, then there exists $B \in \mathcal{F}$, diameter of $B < \varepsilon$ and $\mathbf{x} \in B$. Such a collection covers E in the sense of Vitali.*

Theorem 37.22 *Let $E \subseteq \mathbb{R}^n$ and suppose $\overline{m_n}(E) < \infty$ where $\overline{m_n}$ is the outer measure determined by m_n, n dimensional Lebesgue measure, and let $\mathcal{F}$, be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m_n}(E \setminus \cup_{j=1}^\infty B_j) = 0$.*

Now this theorem implies a simple lemma which is what will be used.

Lemma 37.23 *Let V be an open set in $\mathbb{R}^r, m_r(V) < \infty$. Then there exists a sequence of disjoint open balls $\{B_i\}$ having radii less than δ and a set of measure 0, T, such that*

$$V = (\cup_{i=1}^\infty B_i) \cup T.$$

As in the proof of the change of variables theorem given earlier, the first step is to show that $\mathbf{h}$ maps Lebesgue measurable sets to Lebesgue measurable sets. In showing this the key result is the next lemma which states that $\mathbf{h}$ maps sets of measure zero to sets of measure zero.

Lemma 37.24 *If $m_n(T) = 0$ then $m_n(\mathbf{h}(T)) = 0$.*

Proof: Let V be an open set containing T whose measure is less than ε. Now using the Vitali covering theorem, there exists a sequence of disjoint balls $\{B_i\}$, $B_i = B(\mathbf{x}_i, r_i)$ which are contained in V such that the sequence of enlarged balls, $\left\{\widehat{B}_i\right\}$, having the same center but 5 times the radius, covers T. Then

$$m_n(\mathbf{h}(T)) \leq m_n\left(\mathbf{h}\left(\cup_{i=1}^{\infty}\widehat{B}_i\right)\right)$$

$$\leq \sum_{i=1}^{\infty} m_n\left(\mathbf{h}\left(\widehat{B}_i\right)\right)$$

$$\begin{aligned}
&\leq \sum_{i=1}^{\infty}\alpha(n)(\mathrm{Lip}(\mathbf{h}))^n 5^n r_i^n = 5^n(\mathrm{Lip}(\mathbf{h}))^n\sum_{i=1}^{\infty}m_n(B_i) \\
&\leq (\mathrm{Lip}(\mathbf{h}))^n 5^n m_n(V) \leq \varepsilon(\mathrm{Lip}(\mathbf{h}))^n 5^n.
\end{aligned}$$

Since ε is arbitrary, this proves the lemma.

With the conclusion of this lemma, the next lemma is fairly easy to obtain.

Lemma 37.25 *If A is Lebesgue measurable, then $\mathbf{h}(A)$ is m_n measurable. Furthermore,*

$$m_n(\mathbf{h}(A)) \leq (\mathrm{Lip}(\mathbf{h}))^n m_n(A). \tag{37.9}$$

Proof: Let $A_k = A \cap B(\mathbf{0}, k), k \in \mathbb{N}$. Let $V \supseteq A_k$ and let $m_n(V) < \infty$. By Lemma 37.23, there is a sequence of disjoint balls $\{B_i\}$ and a set of measure 0, T, such that

$$V = \cup_{i=1}^{\infty} B_i \cup T,\ B_i = B(x_i, r_i).$$

By Lemma 37.24,

$$\overline{m_n}(\mathbf{h}(A_k)) \leq \overline{m_n}(\mathbf{h}(V))$$

$$\leq \overline{m_n}(\mathbf{h}(\cup_{i=1}^{\infty}B_i)) + \overline{m_n}(\mathbf{h}(T)) = \overline{m_n}(\mathbf{h}(\cup_{i=1}^{\infty}B_i))$$

$$\leq \sum_{i=1}^{\infty}\overline{m_n}(\mathbf{h}(B_i)) \leq \sum_{i=1}^{\infty}\overline{m_n}(B(\mathbf{h}(x_i), \mathrm{Lip}(\mathbf{h})\,r_i))$$

$$\leq \sum_{i=1}^{\infty}\alpha(n)(\mathrm{Lip}(\mathbf{h})\,r_i)^n = \mathrm{Lip}(\mathbf{h})^n\sum_{i=1}^{\infty}m_n(B_i) = \mathrm{Lip}(\mathbf{h})^n m_n(V).$$

Therefore,

$$\overline{m_n}(\mathbf{h}(A_k)) \leq \mathrm{Lip}(\mathbf{h})^n m_n(V).$$

Since V is an arbitrary open set containing A_k, it follows from regularity of Lebesgue measure that

$$\overline{m_n}\left(\mathbf{h}\left(A_k\right)\right) \leq \operatorname{Lip}\left(\mathbf{h}\right)^n m_n\left(A_k\right). \tag{37.10}$$

Now let $k \to \infty$ to obtain 37.9. This proves the formula. It remains to show $\mathbf{h}(A)$ is measurable.

By inner regularity of Lebesgue measure, there exists a set, F, which is the countable union of compact sets and a set T with $m_n(T) = 0$ such that

$$F \cup T = A_k.$$

Then $\mathbf{h}(F) \subseteq \mathbf{h}(A_k) \subseteq \mathbf{h}(F) \cup \mathbf{h}(T)$. By continuity of $\mathbf{h}$, $\mathbf{h}(F)$ is a countable union of compact sets and so it is Borel. By 37.10 with T in place of A_k,

$$\overline{m_n}\left(\mathbf{h}\left(T\right)\right) = 0$$

and so $\mathbf{h}(T)$ is m_n measurable. Therefore, $\mathbf{h}(A_k)$ is m_n measurable because m_n is a complete measure and this exhibits $\mathbf{h}(A_k)$ between two m_n measurable sets whose difference has measure 0. Now

$$\mathbf{h}\left(A\right) = \cup_{k=1}^{\infty} \mathbf{h}\left(A_k\right)$$

so $\mathbf{h}(A)$ is also m_n measurable and this proves the lemma.

The following lemma, depending on the Brouwer fixed point theorem and found in Rudin [45], will be important for the following arguments. A slightly more precise version was presented earlier on Page 462 but this version given below will suffice in this context. The idea is that if a continuous function mapping a ball in $\mathbb{R}^k$ to $\mathbb{R}^k$ doesn't move any point very much, then the image of the ball must contain a slightly smaller ball.

Lemma 37.26 *Let $B = B(\mathbf{0}, r)$, a ball in $\mathbb{R}^k$ and let $\mathbf{F} : \overline{B} \to \mathbb{R}^k$ be continuous and suppose for some $\varepsilon < 1$,*

$$\left|\mathbf{F}\left(\mathbf{v}\right) - \mathbf{v}\right| < \varepsilon r$$

for all $\mathbf{v} \in \overline{B}$. Then

$$\mathbf{F}\left(\overline{B}\right) \supseteq \overline{B\left(\mathbf{0}, r\left(1-\varepsilon\right)\right)}.$$

Proof: Suppose $\mathbf{a} \in \overline{B\left(\mathbf{0}, r\left(1-\varepsilon\right)\right)} \setminus \mathbf{F}\left(\overline{B}\right)$ and let

$$\mathbf{G}\left(\mathbf{v}\right) \equiv \frac{r\left(\mathbf{a} - \mathbf{F}\left(\mathbf{v}\right)\right)}{\left|\mathbf{a} - \mathbf{F}\left(\mathbf{v}\right)\right|}.$$

Then by the Brouwer fixed point theorem, $\mathbf{G}(\mathbf{v}) = \mathbf{v}$ for some $\mathbf{v} \in \overline{B}$. Using the formula for $\mathbf{G}$, it follows $|\mathbf{v}| = r$. Taking the inner product with $\mathbf{v}$,

$$\begin{aligned}
(\mathbf{G}(\mathbf{v}), \mathbf{v}) &= |\mathbf{v}|^2 = r^2 = \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} (\mathbf{a} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \\
&= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} (\mathbf{a} - \mathbf{v} + \mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \\
&= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} [(\mathbf{a} - \mathbf{v}, \mathbf{v}) + (\mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v})] \\
&= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} \left[(\mathbf{a}, \mathbf{v}) - |\mathbf{v}|^2 + (\mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v})\right] \\
&\leq \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} \left[r^2 (1 - \varepsilon) - r^2 + r^2 \varepsilon\right] = 0,
\end{aligned}$$

a contradiction. Therefore, $\overline{B(\mathbf{0}, r(1-\varepsilon))} \setminus \mathbf{F}(\overline{B}) = \emptyset$ and this proves the lemma.

Now let Ω be a Lebesgue measurable set and suppose $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous and one to one on Ω. Let

$$N \equiv \{\mathbf{x} \in \Omega : D\mathbf{h}(\mathbf{x}) \text{ does not exist}\} \tag{37.11}$$

$$S \equiv \left\{\mathbf{x} \in \Omega \setminus N : D\mathbf{h}(\mathbf{x})^{-1} \text{ does not exist}\right\} \tag{37.12}$$

Lemma 37.27 *Let $\mathbf{x} \in \Omega \setminus (S \cup N)$. Then if $\varepsilon \in (0,1)$ the following hold for all r small enough.*

$$m_n\left(\mathbf{h}\left(\overline{B(\mathbf{x},r)}\right)\right) \geq m_n(D\mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1-\varepsilon))), \tag{37.13}$$

$$\mathbf{h}(B(\mathbf{x}, r)) \subseteq \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1+\varepsilon)), \tag{37.14}$$

$$m_n\left(\mathbf{h}\left(\overline{B(\mathbf{x},r)}\right)\right) \leq m_n(D\mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1+\varepsilon))) \tag{37.15}$$

If $\mathbf{x} \in \Omega \setminus (S \cup N)$ is also a point of density of Ω, then

$$\lim_{r \to 0} \frac{m_n(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_n(\mathbf{h}(B(\mathbf{x}, r)))} = 1. \tag{37.16}$$

If $\mathbf{x} \in \Omega \setminus N$, then

$$|\det D\mathbf{h}(\mathbf{x})| = \lim_{r \to 0} \frac{m_n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x},r))} \quad a.e. \tag{37.17}$$

Proof: Since $D\mathbf{h}(\mathbf{x})^{-1}$ exists,

$$\begin{aligned}
\mathbf{h}(\mathbf{x} + \mathbf{v}) &= \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x}) \mathbf{v} + o(|\mathbf{v}|) &(37.18) \\
&= \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x}) \left(\mathbf{v} + \overbrace{D\mathbf{h}(\mathbf{x})^{-1} o(|\mathbf{v}|)}^{=o(|\mathbf{v}|)}\right) &(37.19)
\end{aligned}$$

Consequently, when r is small enough, 37.14 holds. Therefore, 37.15 holds. From 37.19, and the assumption that $D\mathbf{h}(\mathbf{x})^{-1}$ exists,

$$D\mathbf{h}(\mathbf{x})^{-1}\mathbf{h}(\mathbf{x}+\mathbf{v})-D\mathbf{h}(\mathbf{x})^{-1}\mathbf{h}(\mathbf{x})-\mathbf{v}=o(|\mathbf{v}|). \tag{37.20}$$

Letting

$$\mathbf{F}(\mathbf{v})=D\mathbf{h}(\mathbf{x})^{-1}\mathbf{h}(\mathbf{x}+\mathbf{v})-D\mathbf{h}(\mathbf{x})^{-1}\mathbf{h}(\mathbf{x}),$$

apply Lemma 37.26 in 37.20 to conclude that for r small enough, whenever $|\mathbf{v}|<r$,

$$D\mathbf{h}(\mathbf{x})^{-1}\mathbf{h}(\mathbf{x}+\mathbf{v})-D\mathbf{h}(\mathbf{x})^{-1}\mathbf{h}(\mathbf{x})\supseteq B(\mathbf{0},(1-\varepsilon)r).$$

Therefore,

$$\mathbf{h}\left(\overline{B(\mathbf{x},r)}\right)\supseteq\mathbf{h}(\mathbf{x})+D\mathbf{h}(\mathbf{x})B(\mathbf{0},(1-\varepsilon)r)$$

which implies

$$m_n\left(\mathbf{h}\left(\overline{B(\mathbf{x},r)}\right)\right)\geq m_n(D\mathbf{h}(\mathbf{x})B(\mathbf{0},r(1-\varepsilon)))$$

which shows 37.13.

Now suppose that $\mathbf{x}$ is a point of density of Ω as well as being a point where $D\mathbf{h}(\mathbf{x})^{-1}$ and $D\mathbf{h}(\mathbf{x})$ exist. Then whenever r is small enough,

$$1-\varepsilon<\frac{m_n(\mathbf{h}(B(\mathbf{x},r)\cap\Omega))}{m_n(\mathbf{h}(B(\mathbf{x},r)))}\leq 1$$

and so

$$\begin{aligned}1-\varepsilon &< \frac{m_n\left(\mathbf{h}\left(B(\mathbf{x},r)\cap\Omega^C\right)\right)}{m_n(\mathbf{h}(B(\mathbf{x},r)))}+\frac{m_n(\mathbf{h}(B(\mathbf{x},r)\cap\Omega))}{m_n(\mathbf{h}(B(\mathbf{x},r)))}\\ &\leq \frac{m_n\left(\mathbf{h}\left(B(\mathbf{x},r)\cap\Omega^C\right)\right)}{m_n(\mathbf{h}(B(\mathbf{x},r)))}+1.\end{aligned}$$

which implies

$$m_n(B(\mathbf{x},r)\setminus\Omega)<\varepsilon\alpha(n)r^n. \tag{37.21}$$

Then for such r,

$$1\geq\frac{m_n(\mathbf{h}(B(\mathbf{x},r)\cap\Omega))}{m_n(\mathbf{h}(B(\mathbf{x},r)))}$$

$$\geq\frac{m_n(\mathbf{h}(B(\mathbf{x},r)))-m_n(\mathbf{h}(B(\mathbf{x},r)\setminus\Omega))}{m_n(\mathbf{h}(B(\mathbf{x},r)))}.$$

From Lemma 37.25, 37.21, and 37.13, this is no larger than

$$1-\frac{\operatorname{Lip}(\mathbf{h})^n\varepsilon\alpha(n)r^n}{m_n(D\mathbf{h}(\mathbf{x})B(\mathbf{0},r(1-\varepsilon)))}.$$

By the theorem on the change of variables for a linear map, this expression equals

$$1-\frac{\mathrm{Lip}\left(\mathbf{h}\right)^{n}\varepsilon\alpha\left(n\right)r^{n}}{\left|\det\left(D\mathbf{h}\left(\mathbf{x}\right)\right)\right|r^{n}\alpha\left(n\right)\left(1-\varepsilon\right)^{n}}\equiv 1-g\left(\varepsilon\right)$$

where $\lim_{\varepsilon\rightarrow 0}g\left(\varepsilon\right)=0$. Then for all r small enough,

$$1\geq\frac{m_{n}\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\cap\Omega\right)\right)}{m_{n}\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\right)\right)}\geq 1-g\left(\varepsilon\right)$$

which shows 37.16 since ε is arbitrary. It remains to verify 37.17.

In case $\mathbf{x}\in S$, for small $\left|\mathbf{v}\right|$,

$$\mathbf{h}\left(\mathbf{x}+\mathbf{v}\right)=\mathbf{h}\left(\mathbf{x}\right)+D\mathbf{h}\left(\mathbf{x}\right)\mathbf{v}+o\left(\left|\mathbf{v}\right|\right)$$

where $\left|o\left(\left|\mathbf{v}\right|\right)\right|<\varepsilon\left|\mathbf{v}\right|$. Therefore, for small enough r,

$$\mathbf{h}\left(B\left(\mathbf{x},r\right)\right)-\mathbf{h}\left(\mathbf{x}\right)\subseteq K+B\left(\mathbf{0},r\varepsilon\right)$$

where K is a compact subset of an $n-1$ dimensional subspace contained in $D\mathbf{h}\left(\mathbf{x}\right)\left(\mathbb{R}^{n}\right)$ which has diameter no more than $2\left|\left|D\mathbf{h}\left(\mathbf{x}\right)\right|\right|r$. By Lemma 10.33 on Page 285,

$$\begin{aligned}m_{n}\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\right)\right) &= m_{n}\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\right)-\mathbf{h}\left(\mathbf{x}\right)\right)\\ &\leq 2^{n}\varepsilon r\left(2\left|\left|D\mathbf{h}\left(\mathbf{x}\right)\right|\right|r+r\varepsilon\right)^{n-1}\end{aligned}$$

and so, in this case, letting r be small enough,

$$\frac{m_{n}\left(\mathbf{h}\left(B\left(\mathbf{x},r\right)\right)\right)}{m_{n}\left(B\left(\mathbf{x},r\right)\right)}\leq\frac{2^{n}\varepsilon r\left(2\left|\left|D\mathbf{h}\left(\mathbf{x}\right)\right|\right|r+r\varepsilon\right)^{n-1}}{\alpha\left(n\right)r^{n}}\leq C\varepsilon.$$

Since ε is arbitrary, the limit as $r\rightarrow 0$ of this quotient equals 0.

If $\mathbf{x}\notin S$, use 37.13 - 37.15 along with the change of variables formula for linear maps. This proves the Lemma.

Since $\mathbf{h}$ is one to one, there exists a measure, μ, defined by

$$\mu\left(E\right)\equiv m_{n}\left(\mathbf{h}\left(E\right)\right)$$

on the Lebesgue measurable subsets of Ω. By Lemma 37.25 $\mu\ll m_{n}$ and so by the Radon Nikodym theorem, there exists a nonnegative function, $J\left(\mathbf{x}\right)$ in $L_{loc}^{1}\left(\mathbb{R}^{n}\right)$ such that whenever E is Lebesgue measurable,

$$\mu\left(E\right)=m_{n}\left(\mathbf{h}\left(E\cap\Omega\right)\right)=\int_{E\cap\Omega}J\left(\mathbf{x}\right)dm_{n}. \tag{37.22}$$

Extend J to equal zero off Ω.

Lemma 37.28 *The function,* $J\left(\mathbf{x}\right)$ *equals* $\left|\det D\mathbf{h}\left(\mathbf{x}\right)\right|$ *a.e.*

Proof: Define

$$Q \equiv \{\mathbf{x} \in \Omega : \mathbf{x} \text{ is not a point of density of } \Omega\} \cup N \cup$$

$$\{\mathbf{x} \in \Omega : \mathbf{x} \text{ is not a Lebesgue point of } J\}.$$

Then Q is a set of measure zero and if $\mathbf{x} \notin Q$, then by 37.17, and 37.16,

$$\begin{aligned}
& |\det D\mathbf{h}(\mathbf{x})| \\
= & \lim_{r\to 0} \frac{m_n(\mathbf{h}(B(\mathbf{x},r)))}{m_n(B(\mathbf{x},r))} \\
= & \lim_{r\to 0} \frac{m_n(\mathbf{h}(B(\mathbf{x},r)))}{m_n(\mathbf{h}(B(\mathbf{x},r)\cap\Omega))} \frac{m_n(\mathbf{h}(B(\mathbf{x},r)\cap\Omega))}{m_n(B(\mathbf{x},r))} \\
= & \lim_{r\to 0} \frac{1}{m_n(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)\cap\Omega} J(\mathbf{y})\, dm_n \\
= & \lim_{r\to 0} \frac{1}{m_n(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} J(\mathbf{y})\, dm_n = J(\mathbf{x}).
\end{aligned}$$

the last equality because J was extended to be zero off Ω. This proves the lemma.

Here is the change of variables formula for Lipschitz mappings. It is a special case of the area formula.

Theorem 37.29 *Let Ω be a Lebesgue measurable set, let $f \geq 0$ be Lebesgue measurable. Then for $\mathbf{h}$ a Lipschitz mapping defined on $\mathbb{R}^n$ which is one to one on Ω,*

$$\int_{\mathbf{h}(\Omega)} f(\mathbf{y})\, dm_n = \int_\Omega f(\mathbf{h}(\mathbf{x}))\, |\det D\mathbf{h}(\mathbf{x})|\, dm_n. \tag{37.23}$$

Proof: Let F be a Borel set. It follows that $\mathbf{h}^{-1}(F)$ is a Lebesgue measurable set. Therefore, by 37.22,

$$\begin{aligned}
& m_n\left(\mathbf{h}\left(\mathbf{h}^{-1}(F)\cap\Omega\right)\right) \qquad (37.24)\\
= & \int_{\mathbf{h}(\Omega)} \mathcal{X}_F(\mathbf{y})\, dm_n = \int_\Omega \mathcal{X}_{\mathbf{h}^{-1}(F)}(\mathbf{x})\, J(\mathbf{x})\, dm_n \\
= & \int_\Omega \mathcal{X}_F(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm_n.
\end{aligned}$$

What if F is only Lebesgue measurable? Note there are no measurability problems with the above expression because $\mathbf{x} \to \mathcal{X}_F(\mathbf{h}(\mathbf{x}))$ is Borel measurable due to the assumption that $\mathbf{h}$ is continuous while J is given to be Lebesgue measurable. However, if F is Lebesgue measurable, not necessarily Borel measurable, then it is no longer clear that $\mathbf{x} \to \mathcal{X}_F(\mathbf{h}(\mathbf{x}))$ is measurable. In fact this is not always even true. However, $\mathbf{x} \to \mathcal{X}_F(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})$ is measurable and 37.24 holds.

Let F be Lebesgue measurable. Then by inner regularity, $F = H \cup N$ where N has measure zero, H is the countable union of compact sets so it is a Borel set, and

$H \cap N = \emptyset$. Therefore, letting N' denote a Borel set of measure zero which contains N,

$$b(\mathbf{x}) \equiv \mathcal{X}_H(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x}) \leq \mathcal{X}_F(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})$$

$$\begin{aligned} &= \mathcal{X}_H(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x}) + \mathcal{X}_N(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x}) \\ &\leq \mathcal{X}_H(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x}) + \mathcal{X}_{N'}(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x}) \equiv u(\mathbf{x}) \end{aligned}$$

Now since N' is Borel,

$$\int_\Omega (u(\mathbf{x}) - b(\mathbf{x}))\, dm_n = \int_\Omega \mathcal{X}_{N'}(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm_n$$

$$= m_n\left(\mathbf{h}\left(\mathbf{h}^{-1}(N') \cap \Omega\right)\right) = m_n(N' \cap \mathbf{h}(\Omega)) = 0$$

and this shows $\mathcal{X}_H(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x}) = \mathcal{X}_F(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})$ except on a set of measure zero. By completeness of Lebesgue measure, it follows $\mathbf{x} \to \mathcal{X}_F(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})$ is Lebesgue measurable and also since $\mathbf{h}$ maps sets of measure zero to sets of measure zero,

$$\begin{aligned} \int_\Omega \mathcal{X}_F(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm_n &= \int_\Omega \mathcal{X}_H(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm_n \\ &= \int_{\mathbf{h}(\Omega)} \mathcal{X}_H(\mathbf{y})\, dm_n \\ &= \int_{\mathbf{h}(\Omega)} \mathcal{X}_F(\mathbf{y})\, dm_n. \end{aligned}$$

It follows that if s is any nonnegative Lebesgue measurable simple function,

$$\int_\Omega s(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm_n = \int_{\mathbf{h}(\Omega)} s(\mathbf{y})\, dm_n \tag{37.25}$$

and now, if $f \geq 0$ is Lebesgue measurable, let s_k be an increasing sequence of Lebesgue measurable simple functions converging pointwise to f. Then since 37.25 holds for s_k, the monotone convergence theorem applies and yields 37.23. This proves the theorem.

It turns out that a Lipschitz function defined on some subset of $\mathbb{R}^n$ always has a Lipschitz extension to all of $\mathbb{R}^n$. The next theorem gives a proof of this. For more on this sort of theorem see Federer [22]. He gives a better but harder theorem than what follows.

Theorem 37.30 *If* $\mathbf{h} : \Omega \to \mathbb{R}^m$ *is Lipschitz, then there exists* $\overline{\mathbf{h}} : \mathbb{R}^n \to \mathbb{R}^m$ *which extends* $\mathbf{h}$ *and is also Lipschitz.*

Proof: It suffices to assume $m = 1$ because if this is shown, it may be applied to the components of $\mathbf{h}$ to get the desired result. Suppose

$$|h(\mathbf{x}) - h(\mathbf{y})| \leq K\,|\mathbf{x} - \mathbf{y}|. \tag{37.26}$$

Define
$$\overline{h}(\mathbf{x}) \equiv \inf\{h(\mathbf{w}) + K|\mathbf{x}-\mathbf{w}| : \mathbf{w}\in\Omega\}. \tag{37.27}$$
If $\mathbf{x}\in\Omega$, then for all $\mathbf{w}\in\Omega$,
$$h(\mathbf{w}) + K|\mathbf{x}-\mathbf{w}| \geq h(\mathbf{x})$$
by 37.26. This shows $h(\mathbf{x}) \leq \overline{h}(\mathbf{x})$. But also you could take $\mathbf{w}=\mathbf{x}$ in 37.27 which yields $\overline{h}(\mathbf{x}) \leq h(\mathbf{x})$. Therefore $\overline{h}(\mathbf{x}) = h(\mathbf{x})$ if $\mathbf{x}\in\Omega$.

Now suppose $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$ and consider $\left|\overline{h}(\mathbf{x}) - \overline{h}(\mathbf{y})\right|$. Without loss of generality assume $\overline{h}(\mathbf{x}) \geq \overline{h}(\mathbf{y})$. (If not, repeat the following argument with $\mathbf{x}$ and $\mathbf{y}$ interchanged.) Pick $\mathbf{w}\in\Omega$ such that
$$h(\mathbf{w}) + K|\mathbf{y}-\mathbf{w}| - \varepsilon < \overline{h}(\mathbf{y}).$$
Then
$$\left|\overline{h}(\mathbf{x}) - \overline{h}(\mathbf{y})\right| = \overline{h}(\mathbf{x}) - \overline{h}(\mathbf{y}) \leq h(\mathbf{w}) + K|\mathbf{x}-\mathbf{w}| - \\ [h(\mathbf{w}) + K|\mathbf{y}-\mathbf{w}| - \varepsilon] \leq K|\mathbf{x}-\mathbf{y}| + \varepsilon.$$
Since ε is arbitrary,
$$\left|\overline{h}(\mathbf{x}) - \overline{h}(\mathbf{y})\right| \leq K|\mathbf{x}-\mathbf{y}|$$
and this proves the theorem.

This yields a simple corollary to Theorem 37.29.

Corollary 37.31 *Let $\mathbf{h}:\Omega\to\mathbb{R}^n$ be Lipschitz continuous and one to one where Ω is a Lebesgue measurable set. Then if $f\geq 0$ is Lebesgue measurable,*
$$\int_{\mathbf{h}(\Omega)} f(\mathbf{y})\, dm_n = \int_\Omega f(\mathbf{h}(\mathbf{x}))\left|\det D\overline{\mathbf{h}}(\mathbf{x})\right| dm_n. \tag{37.28}$$
where $\overline{\mathbf{h}}$ denotes a Lipschitz extension of $\mathbf{h}$.

The Area And Coarea Formulas

38.1 The Area Formula Again

Recall the area formula presented earlier. For convenience, here it is.

Theorem 38.1 *Let $g : \mathbf{h}(A) \to [0, \infty]$ be $\mathcal{H}^n$ measurable where $\mathbf{h}$ is a continuous function and A is a Lebesgue measurable set which satisfies 17.8 - 17.10. That is, U is an open set in $\mathbb{R}^n$ on which $\mathbf{h}$ is defined and $A \subseteq U$ is a Lebesgue measurable set, $m \geq n$, and*

$$\mathbf{h} : U \to \mathbb{R}^m \text{ is continuous,} \tag{38.1}$$

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in A, \tag{38.2}$$

Also assume that for every $\mathbf{x} \in A$, there exists $r_{\mathbf{x}}$ and $L_{\mathbf{x}}$ such that for all $\mathbf{y}, \mathbf{z} \in B(\mathbf{x}, r_{\mathbf{x}})$,

$$|\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y})| \leq L_{\mathbf{x}} |\mathbf{x} - \mathbf{y}| \tag{38.3}$$

Then

$$\mathbf{x} \to (g \circ \mathbf{h})(\mathbf{x}) J(\mathbf{x})$$

is Lebesgue measurable and

$$\int_{\mathbf{h}(A)} g(\mathbf{y}) \, d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) \, dm$$

where $J(\mathbf{x}) = \det(U(\mathbf{x})) = \det\left(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})\right)^{1/2}$.

Obviously, one can obtain improved versions of this important theorem by using Rademacher's theorem and condition 38.3. As mentioned earlier, a function which satisfies 38.3 is called locally Lipschitz at $\mathbf{x}$. Here is a simple lemma which is in the spirit of similar lemmas presented in the chapter on Hausdorff measures.

Lemma 38.2 *Let U be an open set in $\mathbb{R}^n$ and let $\mathbf{h} : U \to \mathbb{R}^m$ where $m \geq n$. Let $A \subseteq U$ and let $\mathbf{h}$ be locally Lipschitz at every point of A. Then if $N \subseteq A$ has Lebesgue measure zero, it follows that $\mathcal{H}^n(\mathbf{h}(N)) = 0$.*

Proof: Let N_k be defined as

$$N_k \equiv \{\mathbf{x} \in N : \text{for some } R_{\mathbf{x}} > 0, |\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y})| \leq k|\mathbf{z} - \mathbf{y}| \text{ for all } \mathbf{y}, \mathbf{z} \in B(\mathbf{x}, R_{\mathbf{x}})\}$$

Thus $N_k \uparrow N$. Let $\varepsilon > 0$ be given and let $U \supseteq V_k \supseteq N$ be open and $m_n(V_k) < \frac{\varepsilon}{5^n k^n}$. Now fix $\delta > 0$. For $\mathbf{x} \in N_k$ let $B(\mathbf{x}, 5r_{\mathbf{x}}) \subseteq V_k$ such that $r_{\mathbf{x}} < \min\left(\frac{\delta}{5k}, R_{\mathbf{x}}\right)$. By the Vitali covering theorem, there exists a disjoint sequence of these balls, $\{B_i\}_{i=1}^{\infty}$ such that $\left\{\widehat{B_i}\right\}_{i=1}^{\infty}$, the corresponding sequence of balls having the same centers but five times the radius covers N_k. Then $\operatorname{diam}\left(\widehat{B_i}\right) < 2\delta/k$. Hence $\left\{\mathbf{h}\left(\widehat{B_i}\right)\right\}_{i=1}^{\infty}$ covers $\mathbf{h}(N_k)$ and $\operatorname{diam}\left(\mathbf{h}\left(\widehat{B_i}\right)\right) < 2\delta$. It follows

$$\begin{aligned}
\mathcal{H}_{2\delta}^n(\mathbf{h}(N_k)) &\leq \sum_{i=1}^{\infty} \alpha(n)\, r\left(\mathbf{h}\left(\widehat{B_i}\right)\right)^n \\
&\leq \sum_{i=1}^{\infty} \alpha(n)\, k^n 5^n r(B_i)^n \\
&= 5^n k^n \sum_{i=1}^{\infty} m_n(B_i) \leq 5^n k^n m_n(V_k) < \varepsilon
\end{aligned}$$

Since δ was arbitrary, this shows $\mathcal{H}^n(\mathbf{h}(N_k)) \leq \varepsilon$. Since k was arbitrary, this shows $\mathcal{H}^n(\mathbf{h}(N)) = \lim_{k\to\infty} \mathcal{H}^n(\mathbf{h}(N_k)) \leq \varepsilon$. Since ε is arbitrary, this shows $\mathcal{H}^n(\mathbf{h}(N)) = 0$. This proves the lemma.

Now with this lemma, here is one of many possible generalizations of the area formula.

Theorem 38.3 *Let U be an open set in $\mathbb{R}^n$ and $\mathbf{h} : U \to \mathbb{R}^m$. Let $\mathbf{h}$ be locally Lipschitz and one to one on A, a Lebesgue measurable subset of U and let $g : \mathbf{h}(A) \to \mathbb{R}$ be a nonnegative $\mathcal{H}^n$ measurable function. Then*

$$\mathbf{x} \to (g \circ \mathbf{h})(\mathbf{x})\, J(\mathbf{x})$$

is Lebesgue measurable and

$$\int_{\mathbf{h}(A)} g(\mathbf{y})\, d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm_n$$

where $J(\mathbf{x}) = \det(U(\mathbf{x})) = \det\left(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})\right)^{1/2}$.

Proof: For $\mathbf{x} \in A$, there exists a ball, $B_{\mathbf{x}}$ on which $\mathbf{h}$ is Lipschitz. By Rademacher's theorem, $\mathbf{h}$ is differentiable a.e. on $B_{\mathbf{x}}$. There is a countable cover of A consisting of such balls on which $\mathbf{h}$ is Lipschitz. Therefore, $\mathbf{h}$ is differentiable on $A_0 \subseteq A$ where $m_n(A \setminus A_0) = 0$. Then by the earlier area formula,

$$\int_{\mathbf{h}(A_0)} g(\mathbf{y})\, d\mathcal{H}^n = \int_{A_0} g(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm_n$$

By Lemma 38.2

$$\begin{aligned}\int_{\mathbf{h}(A)} g(\mathbf{y})\, d\mathcal{H}^n &= \int_{\mathbf{h}(A_0)} g(\mathbf{y})\, d\mathcal{H}^n \\ &= \int_{A_0} g(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm_n = \int_A g(\mathbf{h}(\mathbf{x}))\, J(\mathbf{x})\, dm_n\end{aligned}$$

This proves the theorem.

Note how a special case of this occurs when $\mathbf{h}$ is one to one and C^1. Of course this yields the earlier change of variables formula as a still more special case.

In addition to this, recall the divergence theorem, Theorem 17.50 on Page 492. This theorem was stated for bounded open sets which have a Lipschitz boundary. This definition of Lipschitz boundary involved an assumption that certain Lipschitz mappings had a derivative a.e. Rademacher's theorem makes this assumption redundant. Therefore, the statement of Theorem 17.50 remains valid with the following definition of a Lipschitz boundary.

Definition 38.4 *A bounded open set, $U \subseteq \mathbb{R}^n$ is said to have a Lipschitz boundary and to lie on one side of its boundary if the following conditions hold. There exist open boxes, $Q_1, \cdots, Q_N$,*

$$Q_i = \prod_{j=1}^{n} \left(a_j^i, b_j^i\right)$$

such that $\partial U \equiv \overline{U} \setminus U$ is contained in their union. Also, for each Q_i, there exists k and a Lipschitz function, g_i such that $U \cap Q_i$ is of the form

$$\left\{ \mathbf{x} : (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \prod_{j=1}^{k-1} \left(a_j^i, b_j^i\right) \times \prod_{j=k+1}^{n} \left(a_j^i, b_j^i\right) \text{ and } a_k^i < x_k < g_i(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \right\} \quad (38.4)$$

or else of the form

$$\left\{ \mathbf{x} : (x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) \in \prod_{j=1}^{k-1} \left(a_j^i, b_j^i\right) \times \prod_{j=k+1}^{n} \left(a_j^i, b_j^i\right) \text{ and } g_i(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n) < x_k < b_j^i \right\}. \quad (38.5)$$

Also, there exists an open set, Q_0 such that $Q_0 \subseteq \overline{Q_0} \subseteq U$ and $\overline{U} \subseteq Q_0 \cup Q_1 \cup \cdots \cup Q_N$.

38.2 Mappings That Are Not One To One

Next I will consider the case where $\mathbf{h}$ is not necessarily one to one. Recall the major theorem presented earlier on which the proof of the area formula depended, Theorem 17.25 on Page 466. Here it is.

Theorem 38.5 *Let $\mathbf{h} : U \to \mathbb{R}^m$ where U is an open set in $\mathbb{R}^n$ for $n \leq m$ and suppose $\mathbf{h}$ is locally Lipschitz at every point of a Lebesgue measurable subset, A of U. Also suppose that for every $\mathbf{x} \in A$, $D\mathbf{h}(\mathbf{x})$ exists. Then for $\mathbf{x} \in A$,*

$$J(\mathbf{x}) = \lim_{r\to 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x},r))}, \tag{38.6}$$

where $J(\mathbf{x}) \equiv \det(U(\mathbf{x})) = \det\left(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})\right)^{1/2}$.

The next lemma is a version of Sard's lemma.

Lemma 38.6 *Let $\mathbf{h} : U \to \mathbb{R}^m$ where U is an open set in $\mathbb{R}^n$ for $n \leq m$ and suppose $\mathbf{h}$ is locally Lipschitz at every point of a Lebesgue measurable subset, A of U. Let*

$$N \equiv \{\mathbf{x} \in A : D\mathbf{h}(\mathbf{x}) \text{ does not exist}\} \tag{38.7}$$

and let

$$S \equiv \{\mathbf{x} \in A_0 \equiv A \setminus N : J(\mathbf{x}) = 0\} \tag{38.8}$$

Then $\mathcal{H}^n(\mathbf{h}(S \cup N)) = 0$.

Proof: By Rademacher's theorem, N has measure 0. Therefore, $\mathcal{H}^n(\mathbf{h}(N)) = 0$ by Lemma 38.2.

It remains to show $\mathcal{H}^n(\mathbf{h}(S)) = 0$. Let $S_k = B(\mathbf{0}, k) \cap S$ for k a positive integer large enough that $S_k \neq \emptyset$. By Theorem 17.25 on Page 466 stated above, if $\mathbf{x} \in S_k$, there exists $r_{\mathbf{x}}$ such that $5r_{\mathbf{x}} < \min(R_{\mathbf{x}}, 1)$ and if $r \leq 5r_{\mathbf{x}}$,

$$\frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x},r))} < \frac{\varepsilon}{5^n k^n},\ B(\mathbf{x}, r) \subseteq B(\mathbf{0},k) \cap U \tag{38.9}$$

Then by the Vitali covering theorem, there exists a sequence of disjoint balls of this sort, $\{B_i\}_{i=1}^\infty$ such that the balls having 5 times the radius but the same center, $\left\{\widehat{B_i}\right\}_{i=1}^\infty$ cover S_k. Then $\left\{\mathbf{h}\left(\widehat{B_i}\right)\right\}_{i=1}^\infty$ covers $\mathbf{h}(S_k)$. Then from 38.9

$$\begin{aligned} \mathcal{H}^n(\mathbf{h}(S_k)) &\leq \sum_{i=1}^\infty \mathcal{H}^n\left(\mathbf{h}\left(\widehat{B_i}\right)\right) \leq \sum_{i=1}^\infty 5^n \mathcal{H}^n(\mathbf{h}(B_i)) \\ &\leq \sum_{i=1}^\infty 5^n \frac{\varepsilon}{5^n k^n} m_n(B_i) \leq \frac{\varepsilon}{k^n} m_n(B(\mathbf{0}, k)) = \varepsilon\alpha(n) \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows $\mathcal{H}^n(\mathbf{h}(S_k)) = 0$ and now letting $k \to 0$, it follows $\mathcal{H}^n(\mathbf{h}(S)) = 0$. This proves the lemma.

The following very technical lemma provides the necessary theory to generalize to functions which are not one to one.

Lemma 38.7 *Let* $\mathbf{h} : U \to \mathbb{R}^m$ *where* U *is an open set in* $\mathbb{R}^n$ *for* $n \leq m$ *and suppose* $\mathbf{h}$ *is locally Lipschitz at every point of a Lebesgue measurable subset,* A *of* U. *Let*

$$N \equiv \{\mathbf{x} \in A : D\mathbf{h}(\mathbf{x}) \text{ does not exist}\}$$

and let

$$S \equiv \{\mathbf{x} \in A_0 \equiv A \setminus N : J(\mathbf{x}) = 0\}$$

Let $B = A \setminus (S \cup N)$. *Then there exist measurable disjoint sets,* $\{E_i\}_{i=1}^{\infty}$ *such that* $A = \cup_{i=1}^{\infty} E_i$ *and* $\mathbf{h}$ *is one to one on* E_i. *Furthermore,* $\mathbf{h}^{-1}$ *is Lipschitz on* $\mathbf{h}(E_i)$.

Proof: Let $\mathcal{C}$ be a dense countable subset of B and let $\mathcal{F}$ be a countable dense subset of the invertible elements of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. For i a positive integer and $T \in \mathcal{F}, \mathbf{c} \in \mathcal{C}$

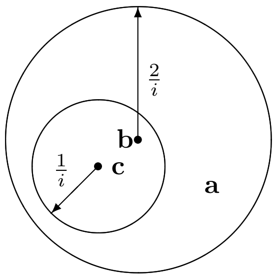
$$E(\mathbf{c}, T, i) \equiv \left\{\mathbf{b} \in B\left(\mathbf{c}, \frac{1}{i}\right) \cap B : (a), (b) \text{ both hold}\right\}$$

where $(a), (b)$ are given by

$$\frac{2}{3}|T\mathbf{v}| \leq |U(\mathbf{b})\mathbf{v}| \text{ for all } \mathbf{v} \tag{a}$$

$$|\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{b}) - D\mathbf{h}(\mathbf{b})(\mathbf{a} - \mathbf{b})| \leq \frac{1}{2}|T(\mathbf{a} - \mathbf{b})|. \tag{b}$$

for all $\mathbf{a} \in B\left(\mathbf{b}, \frac{2}{i}\right)$.



First I will show these sets, $E(\mathbf{c}, T, i)$ cover B and that they are measurable sets. To begin with consider the measurability question. Inequality (a) is the same as saying

$$\frac{2}{3}|T\mathbf{v}| \leq |D\mathbf{h}(\mathbf{b})\mathbf{v}| \text{ for all } \mathbf{v}$$

which is the same as saying

$$\frac{2}{3}|\mathbf{v}| \leq \left|D\mathbf{h}(\mathbf{b})T^{-1}\mathbf{v}\right| \text{ for all } \mathbf{v}.$$

Let $\{\mathbf{v}_i\}$ denote a dense countable subset of $\mathbb{R}^n$. Letting

$$S_i \equiv \left\{\mathbf{b} : \frac{2}{3}|\mathbf{v}_i| \leq \left|D\mathbf{h}(\mathbf{b})T^{-1}\mathbf{v}_i\right|\right\}$$

it follows easily that S_i is measurable because the component functions of the matrix of $D\mathbf{h}(\mathbf{b})$ are limits of difference quotients of continuous functions so they are Borel measurable. (Note that if B were Borel, then S_i would also be Borel.) Now by continuity,

$$\cup_{i=1}^{\infty} S_i = \left\{\mathbf{b} : \frac{2}{3}|\mathbf{v}| \leq \left|D\mathbf{h}(\mathbf{b})T^{-1}\mathbf{v}\right| \text{ for all } \mathbf{v}\right\}$$

and so this set is measurable also. Inequality (b) also determines a measurable set by similar reasoning. It is the same as saying that for all $|\mathbf{v}| < 2/i$,

$$|\mathbf{h}(\mathbf{b}+\mathbf{v}) - \mathbf{h}(\mathbf{b}) - D\mathbf{h}(\mathbf{b})(\mathbf{v})| \leq \frac{1}{2}|T(\mathbf{v})|$$

Use $\{\mathbf{v}_i\}$ a countable dense subset of $B(\mathbf{0},2/i)$ in a similar fashion to (a).

Next I need to show these sets cover B. Let $\mathbf{x} \in B$. Then pick $\mathbf{c}_i \in B\left(\mathbf{x}, \frac{1}{i}\right)$ and $T_i \in B\left(U(\mathbf{x}), \frac{1}{i}\right)$. I need to show that $\mathbf{x} \in E(\mathbf{c}_i, T_i, i)$ for i large enough. For i large enough, $\left|\left|T_i U(\mathbf{x})^{-1}\right|\right| < \frac{3}{2}$. Therefore, for such i

$$\left|T_i U(\mathbf{x})^{-1}(\mathbf{v})\right| < \frac{3}{2}|\mathbf{v}|$$

for all $\mathbf{v}$ and so

$$|T_i\mathbf{w}| < \frac{3}{2}|U(\mathbf{x})\mathbf{w}|$$

for all $\mathbf{w}$. Next consider (b). An equivalent norm is $\mathbf{v} \to |U(\mathbf{x})\mathbf{v}|$ and so, for i large enough,

$$|\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})(\mathbf{a}-\mathbf{x})| \leq \frac{1}{8}|U(\mathbf{x})(\mathbf{a}-\mathbf{x})| \tag{38.10}$$

whenever $|\mathbf{a}-\mathbf{x}| < 2/i$. Now also, for i large enough, $\left|\left|U(\mathbf{x})T_i^{-1}\right|\right| < 4$ and so for all $\mathbf{w}$,

$$\left|U(\mathbf{x})T_i^{-1}\mathbf{w}\right| < 4|\mathbf{w}|$$

which implies

$$|U(\mathbf{x})\mathbf{v}| < 4|T_i\mathbf{v}|.$$

Applying this in 38.10 yields

$$|\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})(\mathbf{a}-\mathbf{x})| \leq \frac{1}{2}|T_i(\mathbf{a}-\mathbf{x})|$$

with implies $\mathbf{x} \in E(\mathbf{c}_i, T_i, i)$.

Next I need to show $\mathbf{h}$ is one to one on $E(\mathbf{c}, T, i)$. Suppose $\mathbf{b}_1, \mathbf{b}_2 \in E(\mathbf{c}, T, i)$. From (b) and (a),

$$\begin{aligned} |T(\mathbf{b}_2 - \mathbf{b}_1)| &\leq \frac{3}{2}|U(\mathbf{b}_1)(\mathbf{b}_2 - \mathbf{b}_1)| = \frac{3}{2}|D\mathbf{h}(\mathbf{b}_1)(\mathbf{b}_2 - \mathbf{b}_1)| \\ &\leq \frac{3}{2}\frac{1}{2}|T(\mathbf{b}_2 - \mathbf{b}_1)| \end{aligned}$$

which is a contradiction unless $\mathbf{b}_2 = \mathbf{b}_1$.

There are clearly countably many $E(\mathbf{c}, T, i)$. Denote them as $\{F_i\}_{i=1}^{\infty}$. Then let $E_1 = F_1$ and if $E_1, \cdots, E_m$ have been chosen, let

$$E_{m+1} = F_{m+1} \setminus \cup_{i=1}^{m} E_i.$$

Thus the E_i are disjoint measurable sets whose union is B and $\mathbf{h}$ is one to one on each E_i.

Now consider one of the E_i. This is a subset of some $E(\mathbf{c}, T, i)$. Let $\mathbf{a}, \mathbf{b} \in E_i$. Then using (a) and (b),

$$\begin{aligned} |T(\mathbf{a} - \mathbf{b})| &\leq \frac{3}{2}|U(\mathbf{b})(\mathbf{a} - \mathbf{b})| \\ &= \frac{3}{2}|D\mathbf{h}(\mathbf{b})(\mathbf{a} - \mathbf{b})| \\ &\leq \frac{3}{2}|\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{b})| + \frac{3}{4}|T(\mathbf{a} - \mathbf{b})|. \end{aligned}$$

Hence

$$\frac{1}{4}|T(\mathbf{a} - \mathbf{b})| \leq \frac{3}{2}|\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{b})|$$

Since $\mathbf{v} \to |T\mathbf{v}|$ is an equivalent norm, there exists some $r > 0$ such that $|T\mathbf{v}| \geq r|\mathbf{v}|$ for all $\mathbf{v}$. Therefore,

$$|\mathbf{a} - \mathbf{b}| \leq \frac{6}{r}|\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{b})|.$$

In other words,

$$\left|\mathbf{h}^{-1}(\mathbf{h}(\mathbf{a})) - \mathbf{h}^{-1}(\mathbf{h}(\mathbf{b}))\right| = |\mathbf{a} - \mathbf{b}| \leq \frac{6}{r}|\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{b})|.$$

which completes the proof.

With these lemmas, here is the main theorem which is a generalization of Theorem 38.3. First remember that from Lemma 17.18 on Page 462 a locally Lipschitz function maps Lebesgue measurable sets to Hausdorff measurable sets.

Theorem 38.8 *Let U be an open set in $\mathbb{R}^n$ and $\mathbf{h} : U \to \mathbb{R}^m$. Let $\mathbf{h}$ be locally Lipschitz on A, a Lebesgue measurable subset of U and let $g : \mathbf{h}(A) \to \mathbb{R}$ be a nonnegative $\mathcal{H}^n$ measurable function. Also let*

$$\#(\mathbf{y}) \equiv \textit{ Number of elements of } \mathbf{h}^{-1}(\mathbf{y})$$

Then $\#$ is $\mathcal{H}^n$ measurable,

$$\mathbf{x} \to (g \circ \mathbf{h})(\mathbf{x}) J(\mathbf{x})$$

is Lebesgue measurable, and

$$\int_{\mathbf{h}(A)} \#(\mathbf{y}) g(\mathbf{y}) d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n$$

where $J(\mathbf{x}) = \det(U(\mathbf{x})) = \det\left(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})\right)^{1/2}$.

Proof: Let $B = A \setminus (S \cup N)$ where S is the set of points where $J(\mathbf{x}) = 0$ and N is the set of points, $\mathbf{x}$ of A where $D\mathbf{h}(\mathbf{x})$ does not exist. Also from Lemma 38.7 there exists $\{E_i\}_{i=1}^{\infty}$, a sequence of disjoint measurable sets whose union equals B such that $\mathbf{h}$ is one to one on each E_i. Then from Theorem 38.3

$$\begin{aligned} & \int_A g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n \\ = & \int_B g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n = \sum_{i=1}^{\infty} \int_{E_i} g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n \\ = & \sum_{i=1}^{\infty} \int_{\mathbf{h}(E_i)} g(\mathbf{y}) d\mathcal{H}^n = \int_{\mathbf{h}(B)} \left(\sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) \right) g(\mathbf{y}) d\mathcal{H}^n. \end{aligned} \tag{38.11}$$

Now $\#(\mathbf{y}) = \left(\sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y})\right)$ on $\mathbf{h}(B)$ and $\#$ differs from this $\mathcal{H}^n$ measurable function only on $\mathbf{h}(S \cup N)$, which by Lemma 38.6 is a set of $\mathcal{H}^n$ measure zero. Therefore, $\#$ is $\mathcal{H}^n$ measurable and the last term of 38.11 equals

$$\int_{\mathbf{h}(A)} \left(\sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) \right) g(\mathbf{y}) d\mathcal{H}^n = \int_{\mathbf{h}(A)} \#(\mathbf{y}) g(\mathbf{y}) d\mathcal{H}^n.$$

This proves the theorem.

38.3 The Coarea Formula

The coarea formula involves a function, $\mathbf{h}$ which maps a subset of $\mathbb{R}^n$ to $\mathbb{R}^m$ where $m \leq n$ instead of $m \geq n$ as in the area formula. The symbol, $\text{Lip}(\mathbf{h})$ will denote the Lipschitz constant for $\mathbf{h}$.

It is possible to obtain the coarea formula as a computation involving the area formula and some simple linear algebra and this is the approach taken here. To begin with, here is the necessary linear algebra.

Theorem 38.9 *Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix for $m \leq n$. Then*

$$p_{BA}(t) = t^{n-m} p_{AB}(t),$$

so the eigenvalues of BA and AB are the same including multiplicities except that BA has $n - m$ extra zero eigenvalues.

Proof: Use block multiplication to write

$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}$$

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}^{-1}\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$$

It follows that $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$ and $\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}$ have the same characteristic polynomials because the two matrices are simlar. Thus

$$\det\begin{pmatrix} tI - AB & 0 \\ -B & tI \end{pmatrix} = \det\begin{pmatrix} tI & 0 \\ -B & tI - BA \end{pmatrix}$$

and so noting that BA is an $n \times n$ matrix and AB is an $m \times m$ matrix,

$$t^m \det(tI - BA) = t^n \det(tI - AB)$$

and so $\det(tI - BA) = p_{BA}(t) = t^{n-m}\det(tI - AB) = t^{n-m}p_{AB}(t)$. This proves the theorem.

The following corollary is what will be used to prove the coarea formula.

Corollary 38.10 *Let A be an $m \times n$ matrix. Then*

$$\det(I + AA^*) = \det(I + A^*A).$$

Proof: Assume $m \leq n$. From Theorem 38.9 AA^* and A^*A have the eigenvalues, $\lambda_1, \cdots .\lambda_m$, necessarily nonnegative, with the same multiplicities and some zero eigenvalues which have differing multiplicities. The eigenvalues, $\lambda_1, \cdots .\lambda_m$ are the zeros of $p_{AA^*}(t)$. Thus there is an orthogonal transformation, P such that

$$A^*A = P\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_m & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{pmatrix}P^*.$$

Therefore,

$$I + A^*A = P\begin{pmatrix} \lambda_1 + 1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_m + 1 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{pmatrix}P^*$$

and so

$$\det(I + A^*A) = \det\begin{pmatrix} \lambda_1 + 1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m + 1 \end{pmatrix} = \det(I + AA^*).$$

This proves the corollary.

The other main ingredient is the following version of the chain rule.

Theorem 38.11 *Let* $\mathbf{h}$ *and* $\mathbf{g}$ *be locally Lipschitz mappings from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *with* $\mathbf{h}(\mathbf{g}(\mathbf{x})) = \mathbf{x}$ *on* A*, a Lebesgue measurable set. Then for a.e.* $\mathbf{x} \in A$*,* $D\mathbf{g}(\mathbf{h}(\mathbf{x}))$*,* $D\mathbf{h}(\mathbf{x})$*, and* $D(\mathbf{h} \circ \mathbf{g})(\mathbf{x})$ *all exist and*

$$I = D(\mathbf{g} \circ \mathbf{h})(\mathbf{x}) = D\mathbf{g}(\mathbf{h}(\mathbf{x}))\, D\mathbf{h}(\mathbf{x}).$$

The proof of this theorem is based on the following lemma.

Lemma 38.12 *If* $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^n$ *is locally Lipschitz, then if* $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ *for all* $\mathbf{x} \in A$*, then* $\det(D\mathbf{h}(\mathbf{x})) = 0$ *a.e.*

Proof: By the case of the Area formula which involves mappings which are not one to one, $0 = \int_{\{\mathbf{0}\}} \#(\mathbf{y})\, dy = \int_A |\det(D\mathbf{h}(\mathbf{x}))|\, dx$ and so $\det(D\mathbf{h}(\mathbf{x})) = 0$ a.e.

Proof of the theorem: On A, $\mathbf{g}(\mathbf{h}(\mathbf{x})) - \mathbf{x} = \mathbf{0}$ and so by the lemma, there exists a set of measure zero, N_1 such that if $\mathbf{x} \notin N_1$, $D(\mathbf{g} \circ \mathbf{h})(\mathbf{x}) - I = \mathbf{0}$. Let M be the set of measure zero of points in $\mathbf{h}(\mathbb{R}^n)$ where $\mathbf{g}$ fails to be differentiable and let $N_2 \equiv \mathbf{g}(M) \cap A$, also a set of measure zero because locally Lipschitz maps take sets of measure zero to sets of measure zero. Finally let N_3 be the set of points where $\mathbf{h}$ fails to be differentiable. Then if $\mathbf{x} \notin N_1 \cup N_2 \cup N_3$, the chain rule implies $I = D(\mathbf{g} \circ \mathbf{h})(\mathbf{x}) = D\mathbf{g}(\mathbf{h}(\mathbf{x}))\, D\mathbf{h}(\mathbf{x})$. This proves the theorem.

Lemma 38.13 *Let* $\mathbf{h} : \mathbb{R}^p \to \mathbb{R}^m$ *be Lipschitz continuous and* $\delta > 0$*. Then if* $A \subseteq \mathbb{R}^p$ *is either open or compact,*

$$\mathbf{y} \to \mathcal{H}^s_\delta\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$$

is Borel measurable.

Proof: Suppose first that A is compact and suppose for $\delta > 0$,

$$\mathcal{H}^s_\delta\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) < t$$

Then there exist sets S_i, satisfying

$$\operatorname{diam}(S_i) < \delta,\ A \cap \mathbf{h}^{-1}(\mathbf{y}) \subseteq \cup_{i=1}^\infty S_i,$$

and

$$\sum_{i=1}^\infty \alpha(s)\,(r(S_i))^s < t.$$

I claim these sets can be taken to be open sets. Choose $\lambda > 1$ but close enough to 1 that

$$\sum_{i=1}^{\infty} \alpha(s)(\lambda r(S_i))^s < t$$

Replace S_i with $S_i + B(0, \eta_i)$ where η_i is small enough that

$$\operatorname{diam}(S_i) + 2\eta_i < \lambda \operatorname{diam}(S_i).$$

Then

$$\operatorname{diam}(S_i + B(0, \eta_i)) \leq \lambda \operatorname{diam}(S_i)$$

and so $r(S_i + B(0, \eta_i)) \leq \lambda r(S_i)$. Thus

$$\sum_{i=1}^{\infty} \alpha(s) r(S_i + B(0, \eta_i))^s < t.$$

Hence you could replace S_i with $S_i + B(0, \eta_i)$ and so one can assume the sets S_i are open.

Claim: If $\mathbf{z}$ is close enough to $\mathbf{y}$, then $A \cap \mathbf{h}^{-1}(\mathbf{z}) \subseteq \cup_{i=1}^{\infty} S_i$.

Proof: If not, then there exists a sequence $\{\mathbf{z}_k\}$ such that

$$\mathbf{z}_k \to \mathbf{y},$$

and

$$\mathbf{x}_k \in (A \cap \mathbf{h}^{-1}(\mathbf{z}_k)) \setminus \cup_{i=1}^{\infty} S_i.$$

By compactness of A, there exists a subsequence still denoted by k such that

$$\mathbf{z}_k \to \mathbf{y},\ \mathbf{x}_k \to \mathbf{x} \in A \setminus \cup_{i=1}^{\infty} S_i.$$

Hence

$$\mathbf{h}(\mathbf{x}) = \lim_{k\to\infty} \mathbf{h}(\mathbf{x}_k) = \lim_{k\to\infty} \mathbf{z}_k = \mathbf{y}.$$

But $\mathbf{x} \notin \cup_{i=1}^{\infty} S_i$ contrary to the assumption that $A \cap \mathbf{h}^{-1}(\mathbf{y}) \subseteq \cup_{i=1}^{\infty} S_i$.

It follows from this claim that whenever $\mathbf{z}$ is close enough to $\mathbf{y}$,

$$\mathcal{H}_\delta^s\left(A \cap \mathbf{h}^{-1}(\mathbf{z})\right) < t.$$

This shows

$$\left\{\mathbf{z} \in \mathbb{R}^p : \mathcal{H}_\delta^s\left(A \cap \mathbf{h}^{-1}(\mathbf{z})\right) < t\right\}$$

is an open set and so $\mathbf{y} \to \mathcal{H}_\delta^s\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$ is Borel measurable whenever A is compact. Now let V be an open set and let

$$A_k \uparrow V,\ A_k \text{ compact.}$$

Then

$$\mathcal{H}_\delta^s\left(V \cap \mathbf{h}^{-1}(\mathbf{y})\right) = \lim_{k\to\infty} \mathcal{H}_\delta^s\left(A_k \cap \mathbf{h}^{-1}(\mathbf{y})\right)$$

so $\mathbf{y} \to \mathcal{H}_\delta^s\left(V \cap \mathbf{h}^{-1}(\mathbf{y})\right)$ is Borel measurable for all V open. This proves the lemma.

Lemma 38.14 *Let $\mathbf{h} : \mathbb{R}^p \to \mathbb{R}^m$ be Lipschitz continuous. Suppose A is either open or compact in $\mathbb{R}^p$. Then $\mathbf{y} \to \mathcal{H}^s\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$ is also Borel measurable and*

$$\int_{\mathbb{R}^m} \mathcal{H}^s\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \leq 2^m (\mathrm{Lip}(\mathbf{h}))^m \frac{\alpha(s)\alpha(m)}{\alpha(s+m)} \mathcal{H}^{s+m}(A)$$

In particular, if $s = n - m$ and $p = n$

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \leq 2^m (\mathrm{Lip}(\mathbf{h}))^m \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} m_n(A)$$

Proof: From Lemma 38.13 $\mathbf{y} \to \mathcal{H}^s_\delta\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$ is Borel measurable for each $\delta > 0$. Without loss of generality, $\mathcal{H}^{s+m}(A) < \infty$. Now let B_i be closed sets with $\mathrm{diam}(B_i) < \delta, A \subseteq \cup_{i=1}^\infty B_i$, and

$$\mathcal{H}^{s+m}_\delta(A) + \varepsilon > \sum_{i=1}^\infty \alpha(s+m) r(B_i)^{s+m}.$$

Note each B_i is compact so $\mathbf{y} \to \mathcal{H}^s_\delta\left(B_i \cap \mathbf{h}^{-1}(\mathbf{y})\right)$ is Borel measurable. Thus

$$\begin{aligned}
& \int_{\mathbb{R}^m} \mathcal{H}^s_\delta\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \\
\leq & \int_{\mathbb{R}^m} \sum_i \mathcal{H}^s_\delta\left(B_i \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \\
= & \sum_i \int_{\mathbb{R}^m} \mathcal{H}^s_\delta\left(B_i \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \\
\leq & \sum_i \int_{\mathbf{h}(B_i)} \mathcal{H}^s_\delta(B_i) dy \\
= & \sum_i m_m(\mathbf{h}(B_i)) \mathcal{H}^s_\delta(B_i) \\
\leq & \sum_i (\mathrm{Lip}(\mathbf{h}))^m 2^m \alpha(m) r(B_i)^m \alpha(s) r(B_i)^s \\
= & (\mathrm{Lip}(\mathbf{h}))^m \frac{\alpha(m)\alpha(s)}{\alpha(m+s)} 2^m \sum_i \alpha(s+m) r(B_i)^{m+s} \\
\leq & (\mathrm{Lip}(\mathbf{h}))^m \frac{\alpha(m)\alpha(s)}{\alpha(m+s)} 2^m \mathcal{H}^{s+m}(A)
\end{aligned}$$

Taking a limit as $\delta \to 0$ this proves the lemma.

Next I will show that whenever A is Lebesgue measurable,

$$\mathbf{y} \to \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$$

is m_m measurable and the above estimate holds.

Lemma 38.15 *Let A be a Lebesgue measurable subset of $\mathbb{R}^n$ and let $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. Then*

$$\mathbf{y} \to \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$$

is Lebesgue measurable. Furthermore, for all A Lebesgue measurable,

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \leq 2^m \left(\text{Lip}(\mathbf{h})\right)^m \frac{\alpha(n-m)\,\alpha(m)}{\alpha(n)} m_n(A)$$

Proof: Let A be a bounded Lebesgue measurable set in $\mathbb{R}^n$. Then by inner and outer regularity of Lebesgue measure there exists an increasing sequence of compact sets, $\{K_k\}$ contained in A and a decreasing sequence of open sets, $\{V_k\}$ containing A such that $m_n(V_k \setminus K_k) < 2^{-k}$. Thus $m_n(V_1) \leq m_n(A) + 1$. By Lemma 38.14

$$\int_{\mathbb{R}^m} \mathcal{H}_\delta^{n-m}\left(V_1 \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy < 2^m \left(\text{Lip}(\mathbf{h})\right)^m \frac{\alpha(n-m)\,\alpha(m)}{\alpha(n)} \left(m_n(A) + 1\right).$$

Then

$$\mathcal{H}_\delta^{n-m}\left(K_k \cap \mathbf{h}^{-1}(\mathbf{y})\right) \leq \mathcal{H}_\delta^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) \leq \mathcal{H}_\delta^{n-m}\left(V_k \cap \mathbf{h}^{-1}(\mathbf{y})\right) \tag{38.12}$$

By Lemma 38.14

$$\begin{aligned}
&= \int_{\mathbb{R}^m} \left(\mathcal{H}_\delta^{n-m}\left(V_k \cap \mathbf{h}^{-1}(\mathbf{y})\right) - \mathcal{H}_\delta^{n-m}\left(K_k \cap \mathbf{h}^{-1}(\mathbf{y})\right)\right) dy \\
&= \int_{\mathbb{R}^m} \mathcal{H}_\delta^{n-m}\left((V_k - K_k) \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \\
&\leq 2^m \left(\text{Lip}(\mathbf{h})\right)^m \frac{\alpha(n-m)\,\alpha(m)}{\alpha(n)} m_n(V_k \setminus K_k) \\
&< 2^m \left(\text{Lip}(\mathbf{h})\right)^m \frac{\alpha(n-m)\,\alpha(m)}{\alpha(n)} 2^{-k}
\end{aligned}$$

Let the Borel measurable functions, g and f be defined by

$$g(\mathbf{y}) \equiv \lim_{k\to\infty} \mathcal{H}_\delta^{n-m}\left(V_k \cap \mathbf{h}^{-1}(\mathbf{y})\right),\ f(\mathbf{y}) \equiv \lim_{k\to\infty} \mathcal{H}_\delta^{n-m}\left(K_k \cap \mathbf{h}^{-1}(\mathbf{y})\right)$$

It follows from the dominated convergence theorem and 38.12 that

$$f(\mathbf{y}) \leq \mathcal{H}_\delta^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) \leq g(\mathbf{y})$$

and

$$\int_{\mathbb{R}^m} \left(g(\mathbf{y}) - f(\mathbf{y})\right) dy = 0.$$

By completeness of m_m, this establishes $\mathbf{y} \to \mathcal{H}_\delta^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$ is Lebesgue measurable. Then by Lemma 38.14 again,

$$\int_{\mathbb{R}^m} \mathcal{H}_\delta^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \leq 2^m \left(\text{Lip}(\mathbf{h})\right)^m \frac{\alpha(n-m)\,\alpha(m)}{\alpha(n)} m_n(A).$$

Letting $\delta \to 0$ and using the monotone convergence theorem yields the desired inequality for $\mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right)$.

The case where A is not bounded can be handled by considering $A_r = A \cap B(\mathbf{0}, r)$ and letting $r \to \infty$. This proves the lemma.

By fussing with the isodiametric inequality one can remove the factor of 2^m in the above inequalities obtaining much more attractive formulas. This is done in [20]. See also [36] which follows [20] and [22]. This last reference probably has the most complete treatment of these topics.

With these lemmas, it is now possible to give a proof of the coarea formula.

Define $\Lambda(n, m)$ as all possible ordered lists of m numbers taken from $\{1, 2, \cdots, n\}$.

Lemma 38.16 *Let A be a measurable set in $\mathbb{R}^n$ and let $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map where $m \leq n$ which is differentiable at every point of A and for which*

$$J\mathbf{h}(\mathbf{x}) \equiv \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right)^{1/2} \neq 0.$$

Then the following formula holds along with all measurability assertions needed for it to make sense.

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy = \int_A J\mathbf{h}(\mathbf{x})\, dx \tag{38.13}$$

Proof: For $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{i} \in \Lambda(n, m)$, with $\mathbf{i} = (i_1, \cdots, i_m)$, define $\mathbf{x_i} \equiv (x_{i_1}, \cdots, x_{i_m})$, and $\pi_{\mathbf{i}}\mathbf{x} \equiv \mathbf{x_i}$. Also for $\mathbf{i} \in \Lambda(n, m)$, let $\mathbf{i}_c \in \Lambda(n, n-m)$ consist of the remaining indices taken in order. For $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m$ where $m \leq n$, define $J\mathbf{h}(\mathbf{x}) \equiv \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right)^{1/2}$. For each $\mathbf{i} \in \Lambda(n, m)$, define

$$\mathbf{h}^{\mathbf{i}}(\mathbf{x}) \equiv \begin{pmatrix} \mathbf{h}(\mathbf{x}) \\ \mathbf{x}_{\mathbf{i}_c} \end{pmatrix}.$$

By Lemma 38.7, there exist disjoint measurable sets $\left\{F_j^{\mathbf{i}}\right\}_{j=1}^{\infty}$ such that $\mathbf{h}^{\mathbf{i}}$ is one to one on $F_j^{\mathbf{i}}$, $\left(\mathbf{h}^{\mathbf{i}}\right)^{-1}$ is Lipschitz on $\mathbf{h}^{\mathbf{i}}\left(F_j^{\mathbf{i}}\right)$, and

$$\cup_{j=1}^{\infty} F_j^{\mathbf{i}} = \left\{\mathbf{x} \in A : \det\left(D\mathbf{h}^{\mathbf{i}}(\mathbf{x})\right) \neq 0\right\}.$$

For $\mathbf{x} \in A$, $\det\left(D_{\mathbf{x_i}}\mathbf{h}(\mathbf{x})\right) \neq 0$ for some $\mathbf{i} \in \Lambda(n, m)$. But $\det\left(D_{\mathbf{x_i}}\mathbf{h}(\mathbf{x})\right) = \det\left(D\mathbf{h}^{\mathbf{i}}(\mathbf{x})\right)$ and so $\mathbf{x} \in F_j^{\mathbf{i}}$ for some $\mathbf{i}$ and j. Hence

$$\cup_{\mathbf{i},j} F_j^{\mathbf{i}} = A.$$

Now let $\left\{E_j^{\mathbf{i}}\right\}$ be measurable sets such that $E_j^{\mathbf{i}} \subseteq F_k^{\mathbf{i}}$ for some k, the sets are disjoint, and their union coincides with $\cup_{\mathbf{i},j} F_j^{\mathbf{i}}$. Then

$$\int_A J\mathbf{h}(\mathbf{x})\, dx = \sum_{\mathbf{i} \in \Lambda(n,m)} \sum_{j=1}^{\infty} \int_{E_j^{\mathbf{i}} \cap A} \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right)^{1/2} dx. \tag{38.14}$$

Let $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz extension of $\left(\mathbf{h}^{\mathbf{i}}\right)^{-1}$ so $\mathbf{g} \circ \mathbf{h}^{\mathbf{i}}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in E_j^{\mathbf{i}}$. First, using Theorem 38.11, and the fact that Lipschitz mappings take sets of measure zero to sets of measure zero, replace $E_j^{\mathbf{i}}$ with a measurable set, $\widetilde{E}_j^{\mathbf{i}} \subseteq E_j^{\mathbf{i}}$ such that $E_j^{\mathbf{i}} \setminus \widetilde{E}_j^{\mathbf{i}}$ has measure zero and

$$D\mathbf{h}^{\mathbf{i}}(\mathbf{g}(\mathbf{y}))\, D\mathbf{g}(\mathbf{y}) = I$$

on $\mathbf{h}^{\mathbf{i}}\left(\widetilde{E}_j^{\mathbf{i}}\right)$. Changing the variables using the area formula, the expression in 38.14 equals

$$\int_A J\mathbf{h}(\mathbf{x})\, dx =$$

$$\sum_{\mathbf{i} \in \Lambda(n,m)} \sum_{j=1}^{\infty} \int_{\mathbf{h}^{\mathbf{i}}\left(\widetilde{E}_j^{\mathbf{i}} \cap A\right)} \det\left(D\mathbf{h}(\mathbf{g}(\mathbf{y}))\, D\mathbf{h}(\mathbf{g}(\mathbf{y}))^*\right)^{1/2} \left|\det D\mathbf{h}^{\mathbf{i}}(\mathbf{g}(\mathbf{y}))\right|^{-1} dy. \tag{38.15}$$

Note the integrands are all Borel measurable functions because they are continuous functions of the entries of matrices which entries come from taking limits of difference quotients of continuous functions. Thus,

$$\int_{\widetilde{E}_j^{\mathbf{i}} \cap A} \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right)^{1/2} dx =$$

$$\int_{\mathbb{R}^n} \mathcal{X}_{\mathbf{h}^{\mathbf{i}}\left(\widetilde{E}_j^{\mathbf{i}} \cap A\right)}(\mathbf{y}) \det\left(D\mathbf{h}(\mathbf{g}(\mathbf{y}))\, D\mathbf{h}(\mathbf{g}(\mathbf{y}))^*\right)^{1/2} \left|\det D\mathbf{h}^{\mathbf{i}}(\mathbf{g}(\mathbf{y}))\right|^{-1} dy$$

$$= \int_{\mathbb{R}^m} \int_{\pi_{\mathbf{i}_c}\left(\mathbf{h}^{-1}(\mathbf{y}_1) \cap \widetilde{E}_j^{\mathbf{i}} \cap A\right)} \det\left(D\mathbf{h}(\mathbf{g}(\mathbf{y}))\, D\mathbf{h}(\mathbf{g}(\mathbf{y}))^*\right)^{1/2} \left|\det D\mathbf{h}_{\mathbf{x}_{\mathbf{i}}}(\mathbf{g}(\mathbf{y}))\right|^{-1} dy_2 dy_1 \tag{38.16}$$

where $\mathbf{y}_1 = \mathbf{h}(\mathbf{x})$ and $\mathbf{y}_2 = \mathbf{x}_{\mathbf{i}_c}$. Thus

$$\mathbf{y}_2 = \pi_{\mathbf{i}_c}\mathbf{g}(\mathbf{y}) = \pi_{\mathbf{i}_c}\mathbf{g}\left(\mathbf{h}^{\mathbf{i}}(\mathbf{x})\right) = \mathbf{x}_{\mathbf{i}_c}. \tag{38.17}$$

Now consider the inner integral in 38.16 in which $\mathbf{y}_1$ is fixed. The integrand equals

$$\det\left[\begin{pmatrix} D_{\mathbf{x}_{\mathbf{i}}}\mathbf{h}(\mathbf{g}(\mathbf{y})) & D_{\mathbf{x}_{\mathbf{i}_c}}\mathbf{h}(\mathbf{g}(\mathbf{y})) \end{pmatrix} \begin{pmatrix} D_{\mathbf{x}_{\mathbf{i}}}\mathbf{h}(\mathbf{g}(\mathbf{y}))^* \\ D_{\mathbf{x}_{\mathbf{i}_c}}\mathbf{h}(\mathbf{g}(\mathbf{y}))^* \end{pmatrix}\right]^{1/2} \left|\det D\mathbf{h}_{\mathbf{x}_{\mathbf{i}}}(\mathbf{g}(\mathbf{y}))\right|^{-1}. \tag{38.18}$$

I want to massage the above expression slightly. Since $\mathbf{y}_1$ is fixed, and $\mathbf{y}_1 = \mathbf{h}(\pi_{\mathbf{i}}\mathbf{g}(\mathbf{y}), \pi_{\mathbf{i}_c}\mathbf{g}(\mathbf{y})) = \mathbf{h}(\mathbf{g}(\mathbf{y}))$, it follows from 38.17 that

$$\begin{aligned} \mathbf{0} &= D_{\mathbf{x}_{\mathbf{i}}}\mathbf{h}(\mathbf{g}(\mathbf{y}))\, D_{\mathbf{y}_2}\pi_{\mathbf{i}}\mathbf{g}(\mathbf{y}) + D_{\mathbf{x}_{\mathbf{i}_c}}\mathbf{h}(\mathbf{g}(\mathbf{y}))\, D_{\mathbf{y}_2}\pi_{\mathbf{i}_c}\mathbf{g}(\mathbf{y}) \\ &= D_{\mathbf{x}_{\mathbf{i}}}\mathbf{h}(\mathbf{g}(\mathbf{y}))\, D_{\mathbf{y}_2}\pi_{\mathbf{i}}\mathbf{g}(\mathbf{y}) + D_{\mathbf{x}_{\mathbf{i}_c}}\mathbf{h}(\mathbf{g}(\mathbf{y})). \end{aligned}$$

Letting $A \equiv D_{\mathbf{x_i}}\mathbf{h}(\mathbf{g}(\mathbf{y}))$ and $B \equiv D_{\mathbf{y}_2}\pi_{\mathbf{i}}\mathbf{g}(\mathbf{y})$ and using the above formula, 38.18 is of the form

$$\begin{aligned} & \det\left[\begin{pmatrix} A & -AB \end{pmatrix}\begin{pmatrix} A^* \\ -B^*A^* \end{pmatrix}\right]^{1/2} \left|\det A\right|^{-1} \\ = & \det\left[A^*A + ABB^*A^*\right]^{1/2}\left|\det A\right|^{-1} \\ = & \det\left[A^*\left(I + BB^*\right)A\right]^{1/2}\left|\det A\right|^{-1} \\ = & \det\left(I + BB^*\right)^{1/2}, \end{aligned}$$

which, by Corollary 38.10, equals $\det\left(I + B^*B\right)^{1/2}$. (Note the size of the identity changes in these two expressions, the first being an $m \times m$ matrix and the second being a $n - m \times n - m$ matrix.)

By 38.17 $\pi_{\mathbf{i}_c}\mathbf{g}(\mathbf{y}) = \mathbf{y}_2$ and so,

$$\det\left(I + B^*B\right)^{1/2} = \det\left[\begin{pmatrix} B^* & I \end{pmatrix}\begin{pmatrix} B \\ I \end{pmatrix}\right]^{1/2}$$

$$= \det\left[\begin{pmatrix} D_{\mathbf{y}_2}\pi_{\mathbf{i}}\mathbf{g}(\mathbf{y})^* & D_{\mathbf{y}_2}\pi_{\mathbf{i}_c}\mathbf{g}(\mathbf{y})^* \end{pmatrix}\begin{pmatrix} D_{\mathbf{y}_2}\pi_{\mathbf{i}}\mathbf{g}(\mathbf{y}) \\ D_{\mathbf{y}_2}\pi_{\mathbf{i}_c}\mathbf{g}(\mathbf{y}) \end{pmatrix}\right]^{1/2}$$

$$= \det\left(D_{\mathbf{y}_2}\mathbf{g}(\mathbf{y})^* D_{\mathbf{y}_2}\mathbf{g}(\mathbf{y})\right)^{1/2}.$$

Therefore, 38.16 reduces to

$$\int_{\widetilde{E}_j^{\mathbf{i}} \cap A} \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right)^{1/2} dx =$$

$$\int_{\mathbb{R}^m} \int_{\pi_{\mathbf{i}_c}\left(\mathbf{h}^{-1}(\mathbf{y}_1) \cap \widetilde{E}_j^{\mathbf{i}} \cap A\right)} \det\left(D_{\mathbf{y}_2}\mathbf{g}(\mathbf{y})^* D_{\mathbf{y}_2}\mathbf{g}(\mathbf{y})\right)^{1/2} dy_2 dy_1. \tag{38.19}$$

By the area formula applied to the inside integral, this integral equals

$$\mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y}_1) \cap \widetilde{E}_j^{\mathbf{i}} \cap A\right)$$

and so

$$\begin{aligned} & \int_{\widetilde{E}_j^{\mathbf{i}} \cap A} \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right)^{1/2} dx \\ = & \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y}_1) \cap \widetilde{E}_j^{\mathbf{i}} \cap A\right) dy_1. \end{aligned}$$

Using Lemma 38.15, along with the inner regularity of Lebesgue measure, $\widetilde{E}_j^{\mathbf{i}}$ can be replaced with $E_j^{\mathbf{i}}$. Therefore, summing the terms over all $\mathbf{i}$ and j,

$$\int_A \det\left(D\mathbf{h}(\mathbf{x})\, D\mathbf{h}(\mathbf{x})^*\right)^{1/2} dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y}) \cap A\right) dy.$$

This proves the lemma.

Now the following is the Coarea formula.

Corollary 38.17 *Let A be a measurable set in $\mathbb{R}^n$ and let $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map where $m \leq n$. Then the following formula holds along with all measurability assertions needed for it to make sense.*

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m} \left(A \cap \mathbf{h}^{-1} (\mathbf{y}) \right) dy = \int_A J\mathbf{h} (\mathbf{x}) \, dx \tag{38.20}$$

where $J\mathbf{h} (\mathbf{x}) \equiv \det \left(D\mathbf{h} (\mathbf{x}) \, D\mathbf{h} (\mathbf{x})^* \right)^{1/2}$.

Proof: By Lemma 38.15 again, this formula is true for all measurable $A \subseteq \mathbb{R}^n \setminus S$. It remains to verify the formula for all measurable sets, A, whether or not they intersect S.

Consider the case where

$$A \subseteq S \equiv \{\mathbf{x} : J(D\mathbf{h}(\mathbf{x})) = 0\}.$$

Let A be compact so that by Lemma 38.14, $\mathbf{y} \to \mathcal{H}^{n-m} \left(A \cap \mathbf{h}^{-1} (\mathbf{y}) \right)$ is Borel. For $\varepsilon > 0$, define $\mathbf{k}$, $\mathbf{p} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ by

$$\mathbf{k} (\mathbf{x}, \mathbf{y}) \equiv \mathbf{h} (\mathbf{x}) + \varepsilon \mathbf{y}, \;\; \mathbf{p} (\mathbf{x}, \mathbf{y}) \equiv \mathbf{y}.$$

Then

$$D\mathbf{k} (\mathbf{x}, \mathbf{y}) = (D\mathbf{h} (\mathbf{x}) \, , \varepsilon I) = (UR, \varepsilon I)$$

where the dependence of U and R on $\mathbf{x}$ has been suppressed. Thus

$$J\mathbf{k}^2 = \det (UR, \varepsilon I) \begin{pmatrix} R^* U \\ \varepsilon I \end{pmatrix} = \det \left(U^2 + \varepsilon^2 I \right)$$

$$= \det \left(Q^* D Q Q^* D Q + \varepsilon^2 I \right) = \det \left(D^2 + \varepsilon^2 I \right)$$

$$= \prod_{i=1}^{m} \left(\lambda_i^2 + \varepsilon^2 \right) \in [\varepsilon^{2m}, C^2 \varepsilon^2] \tag{38.21}$$

since one of the λ_i equals 0. All the eigenvalues must be bounded independent of $\mathbf{x}$, since $||D\mathbf{h} (\mathbf{x})||$ is bounded independent of $\mathbf{x}$ due to the assumption that $\mathbf{h}$ is Lipschitz. Since $J\mathbf{k} \neq 0$, the first part of the argument implies

$$\varepsilon C m_{n+m} \left(A \times \overline{B(\mathbf{0},1)} \right) \geq \int_{A \times \overline{B(\mathbf{0},1)}} |J\mathbf{k}| \, dm_{n+m}$$

$$= \int_{\mathbb{R}^m} \mathcal{H}^n \left(\mathbf{k}^{-1} (\mathbf{y}) \cap A \times \overline{B(\mathbf{0},1)} \right) dy$$

Which by Lemma 38.14,

$$\geq C_{nm} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m} \left(\mathbf{k}^{-1} (\mathbf{y}) \cap \mathbf{p}^{-1} (\mathbf{w}) \cap A \times \overline{B(\mathbf{0},1)} \right) dw dy \tag{38.22}$$

where $C_{nm} = \frac{\alpha(n)}{\alpha(n-m)\alpha(m)}$.

Claim:

$$\mathcal{H}^{n-m}\left(\mathbf{k}^{-1}(\mathbf{y})\cap\mathbf{p}^{-1}(\mathbf{w})\cap A\times\overline{B(\mathbf{0},1)}\right)$$
$$\geq\mathcal{X}_{\overline{B(\mathbf{0},1)}}(\mathbf{w})\,\mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y}-\varepsilon\mathbf{w})\cap A\right).$$

Proof of the claim: If $\mathbf{w}\notin\overline{B(\mathbf{0},1)}$, there is nothing to prove so assume $\mathbf{w}\in\overline{B(\mathbf{0},1)}$. For such $\mathbf{w}$,

$$(\mathbf{x},\mathbf{w}_1)\in\mathbf{k}^{-1}(\mathbf{y})\cap\mathbf{p}^{-1}(\mathbf{w})\cap A\times\overline{B(\mathbf{0},1)}$$

if and only if $\mathbf{h}(\mathbf{x})+\varepsilon\mathbf{w}_1=\mathbf{y}$, $\mathbf{w}_1=\mathbf{w}$, and $\mathbf{x}\in A$, if and only if

$$(\mathbf{x},\mathbf{w}_1)\in\mathbf{h}^{-1}(\mathbf{y}-\varepsilon\mathbf{w})\cap A\times\{\mathbf{w}\}.$$

Therefore for $\mathbf{w}\in\overline{B(\mathbf{0},1)}$,

$$\mathcal{H}^{n-m}\left(\mathbf{k}^{-1}(\mathbf{y})\cap\mathbf{p}^{-1}(\mathbf{w})\cap A\times\overline{B(\mathbf{0},1)}\right)$$
$$\geq\mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y}-\varepsilon\mathbf{w})\cap A\times\{\mathbf{w}\}\right)=\mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y}-\varepsilon\mathbf{w})\cap A\right).$$

(Actually equality holds in the claim.) From the claim, 38.22 is at least as large as

$$C_{nm}\int_{\mathbb{R}^m}\int_{\overline{B(\mathbf{0},1)}}\mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y}-\varepsilon\mathbf{w})\cap A\right)dwdy \tag{38.23}$$
$$=C_{nm}\int_{\overline{B(\mathbf{0},1)}}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y}-\varepsilon\mathbf{w})\cap A\right)dydw$$
$$=\frac{\alpha(n)}{\alpha(n-m)}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}\left(\mathbf{h}^{-1}(\mathbf{y})\cap A\right)dy. \tag{38.24}$$

The use of Fubini's theorem is justified because the integrand is Borel measurable.

Now by 38.24, it follows since $\varepsilon>0$ is arbitrary,

$$\int_{\mathbb{R}^m}\mathcal{H}^{n-m}\left(A\cap\mathbf{h}^{-1}(\mathbf{y})\right)dy=0=\int_A J\mathbf{h}(\mathbf{x})\,dx.$$

Since this holds for arbitrary compact sets in S, it follows from Lemma 38.15 and inner regularity of Lebesgue measure that the equation holds for all measurable subsets of S. This completes the proof of the coarea formula.There is a simple corollary to this theorem in the case of locally Lipschitz maps.

Corollary 38.18 *Let* $\mathbf{h}:\mathbb{R}^n\to\mathbb{R}^m$ *where* $m\leq n$ *and* $\mathbf{h}$ *is locally Lipschitz. Then the Coarea formula, 38.13, holds for* $\mathbf{h}$.

Proof: The assumption that $\mathbf{h}$ is locally Lipschitz implies that for each $r>0$ it follows $\mathbf{h}$ is Lipschitz on $B(\mathbf{0},r)$. To see this, cover the compact set, $\overline{B(\mathbf{0},r)}$ with finitely many balls on which $\mathbf{h}$ is Lipschitz.

Let $A \subseteq B(\mathbf{0},r)$ and let $\mathbf{h}_r$ be Lipschitz with

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}_r(\mathbf{x})$$

for $\mathbf{x} \in B(\mathbf{0},r+1)$. Then

$$\int_A J(D\mathbf{h}(\mathbf{x}))\,dx = \int_A J(D\mathbf{h}_r(\mathbf{x}))dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}_r^{-1}(\mathbf{y})\right) dy$$

$$= \int_{\mathbf{h}_r(A)} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}_r^{-1}(\mathbf{y})\right) dy = \int_{\mathbf{h}(A)} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy$$

$$= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy$$

Now for arbitrary measurable A the above shows for $k = 1, 2, \cdots$

$$\int_{A \cap B(\mathbf{0},k)} J(D\mathbf{h}(\mathbf{x}))\,dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap B(\mathbf{0},k) \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy.$$

Use the monotone convergence theorem to obtain 38.13.

From the definition of Hausdorff measure it follows $\mathcal{H}^0(E)$ equals the number of elements in E. Thus, if $n = m$, the Coarea formula implies

$$\int_A J(D\mathbf{h}(\mathbf{x}))\,dx = \int_{\mathbf{h}(A)} \mathcal{H}^0\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy = \int_{\mathbf{h}(A)} \#(y)\,dy$$

This gives a version of Sard's theorem by letting $S = A$.

38.4 A Nonlinear Fubini's Theorem

Coarea formula holds for $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m, n \geq m$ if whenever A is a Lebesgue measurable subset of $\mathbb{R}^n$,the following formula is valid.

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy = \int_A J\mathbf{h}(\mathbf{x})\,dx \tag{38.25}$$

Note this is the same as

$$\int_A J(D\mathbf{h}(\mathbf{x}))dx = \int_{\mathbf{h}(A)} \mathcal{H}^{n-m}\left(A \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy$$

because if $\mathbf{y} \notin \mathbf{h}(A)$, then $\mathbf{h}^{-1}(\mathbf{y}) = \emptyset$. Now let

$$s(\mathbf{x}) = \sum_{i=1}^{p} c_i \mathcal{X}_{E_i}(\mathbf{x})$$

where E_i is measurable and $c_i \geq 0$. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} s(\mathbf{x}) J((D\mathbf{h}(\mathbf{x}))) \, dx &= \sum_{i=1}^{p} c_i \int_{E_i} J(D\mathbf{h}(\mathbf{x})) \, dx \\
&= \sum_{i=1}^{p} c_i \int_{\mathbf{h}(E_i)} \mathcal{H}^{n-m} \left(E_i \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \\
&= \int_{\mathbf{h}(\mathbb{R}^n)} \sum_{i=1}^{p} c_i \mathcal{H}^{n-m} \left(E_i \cap \mathbf{h}^{-1}(\mathbf{y})\right) dy \\
&= \int_{\mathbf{h}(\mathbb{R}^n)} \left[\int_{\mathbf{h}^{-1}(\mathbf{y})} s \, d\mathcal{H}^{n-m} \right] dy
\end{aligned}$$

$$= \int_{\mathbf{h}(\mathbb{R}^n)} \left[\int_{\mathbf{h}^{-1}(\mathbf{y})} s \, d\mathcal{H}^{n-m} \right] dy. \tag{38.26}$$

Theorem 38.19 *Let $g \geq 0$ be Lebesgue measurable and let*

$$\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m, \ n \geq m$$

satisfy the Coarea formula. For example, it could be locally Lipschitz. Then

$$\int_{\mathbb{R}^n} g(\mathbf{x}) J((D\mathbf{h}(\mathbf{x}))) \, dx = \int_{\mathbf{h}(\mathbb{R}^n)} \left[\int_{\mathbf{h}^{-1}(\mathbf{y})} g \, d\mathcal{H}^{n-m} \right] dy.$$

Proof: Let $s_i \uparrow g$ where s_i is a simple function satisfying 38.26. Then let $i \to \infty$ and use the monotone convergence theorem to replace s_i with g. This proves the change of variables formula.

Note that this formula is a nonlinear version of Fubini's theorem. The "$n - m$ dimensional surface", $\mathbf{h}^{-1}(\mathbf{y})$, plays the role of $\mathbb{R}^{n-m}$ and $\mathcal{H}^{n-m}$ is like $n - m$ dimensional Lebesgue measure. The term, $J((D\mathbf{h}(\mathbf{x})))$, corrects for the error occurring because of the lack of flatness of $\mathbf{h}^{-1}(\mathbf{y})$.

Integration On Manifolds

You can do integration on various manifolds by using the Hausdorff measure of an appropriate dimension. However, it is possible to discuss this through the use of the Riesz representation theorem and some of the machinery for accomplishing this is interesting for its own sake so I will present this alternate point of view.

39.1 Partitions Of Unity

This material has already been mostly discussed starting on Page 481. However, that was a long time ago and it seems like it might be good to go over it again and so, for the sake of convenience, here it is again.

Definition 39.1 *Let $\mathfrak{C}$ be a set whose elements are subsets of $\mathbb{R}^n$.*[1] *Then $\mathfrak{C}$ is said to be locally finite if for every $\mathbf{x} \in \mathbb{R}^n$, there exists an open set, $U_{\mathbf{x}}$ containing $\mathbf{x}$ such that $U_{\mathbf{x}}$ has nonempty intersection with only finitely many sets of $\mathfrak{C}$.*

Lemma 39.2 *Let $\mathfrak{C}$ be a set whose elements are open subsets of $\mathbb{R}^n$ and suppose $\cup\mathfrak{C} \supseteq H$, a closed set. Then there exists a countable list of open sets, $\{U_i\}_{i=1}^{\infty}$ such that each U_i is bounded, each U_i is a subset of some set of $\mathfrak{C}$, and $\cup_{i=1}^{\infty} U_i \supseteq H$.*

Proof: Let $W_k \equiv B(\mathbf{0}, k), W_0 = W_{-1} = \emptyset$. For each $\mathbf{x} \in H \cap \overline{W_k}$ there exists an open set, $U_{\mathbf{x}}$ such that $U_{\mathbf{x}}$ is a subset of some set of $\mathfrak{C}$ and $U_{\mathbf{x}} \subseteq W_{k+1} \setminus \overline{W_{k-1}}$. Then since $H \cap \overline{W_k}$ is compact, there exist finitely many of these sets, $\left\{U_i^k\right\}_{i=1}^{m(k)}$ whose union contains $H \cap \overline{W_k}$. If $H \cap \overline{W_k} = \emptyset$, let $m(k) = 0$ and there are no such sets obtained.The desired countable list of open sets is $\cup_{k=1}^{\infty} \left\{U_i^k\right\}_{i=1}^{m(k)}$. Each open set in this list is bounded. Furthermore, if $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{x} \in W_k$ where k is the first positive integer with $\mathbf{x} \in W_k$. Then $W_k \setminus \overline{W_{k-1}}$ is an open set containing $\mathbf{x}$ and this open set can have nonempty intersection only with with a set of $\left\{U_i^k\right\}_{i=1}^{m(k)} \cup \left\{U_i^{k-1}\right\}_{i=1}^{m(k-1)}$, a finite list of sets. Therefore, $\cup_{k=1}^{\infty} \left\{U_i^k\right\}_{i=1}^{m(k)}$ is locally finite.

The set, $\{U_i\}_{i=1}^{\infty}$ is said to be a locally finite cover of H. The following lemma gives some important reasons why a locally finite list of sets is so significant. First

[1]The definition applies with no change to a general topological space in place of $\mathbb{R}^n$.

of all consider the rational numbers, $\{r_i\}_{i=1}^{\infty}$ each rational number is a closed set.

$$\mathbb{Q} = \{r_i\}_{i=1}^{\infty} = \cup_{i=1}^{\infty}\overline{\{r_i\}} \neq \overline{\cup_{i=1}^{\infty}\{r_i\}} = \mathbb{R}$$

The set of rational numbers is definitely not locally finite.

Lemma 39.3 *Let $\mathfrak{C}$ be locally finite. Then*

$$\overline{\cup\mathfrak{C}} = \cup\left\{\overline{H} : H \in \mathfrak{C}\right\}.$$

Next suppose the elements of $\mathfrak{C}$ are open sets and that for each $U \in \mathfrak{C}$, there exists a differentiable function, ψ_U having $\operatorname{spt}(\psi_U) \subseteq U$. Then you can define the following finite sum for each $\mathbf{x} \in \mathbb{R}^n$

$$f(\mathbf{x}) \equiv \sum \{\psi_U(\mathbf{x}) : \mathbf{x} \in U \in \mathfrak{C}\}.$$

Furthermore, f is also a differentiable function[2] *and*

$$Df(\mathbf{x}) = \sum \{D\psi_U(\mathbf{x}) : \mathbf{x} \in U \in \mathfrak{C}\}.$$

Proof: Let $\mathbf{p}$ be a limit point of $\cup\mathfrak{C}$ and let W be an open set which intersects only finitely many sets of $\mathfrak{C}$. Then $\mathbf{p}$ must be a limit point of one of these sets. It follows $\mathbf{p} \in \cup\left\{\overline{H} : H \in \mathfrak{C}\right\}$ and so $\overline{\cup\mathfrak{C}} \subseteq \cup\left\{\overline{H} : H \in \mathfrak{C}\right\}$. The inclusion in the other direction is obvious.

Now consider the second assertion. Letting $\mathbf{x} \in \mathbb{R}^n$, there exists an open set, W intersecting only finitely many open sets of $\mathfrak{C}$, $U_1, U_2, \cdots, U_m$. Then for all $\mathbf{y} \in W$,

$$f(\mathbf{y}) = \sum_{i=1}^{m} \psi_{U_i}(\mathbf{y})$$

and so the desired result is obvious. It merely says that a finite sum of differentiable functions is differentiable. Recall the following definition.

Definition 39.4 *Let K be a closed subset of an open set, U. $K \prec f \prec U$ if f is continuous, has values in $[0,1]$, equals 1 on K, and has compact support contained in U.*

Lemma 39.5 *Let U be a bounded open set and let K be a closed subset of U. Then there exist an open set, W, such that $W \subseteq \overline{W} \subseteq U$ and a function, $f \in C_c^{\infty}(U)$ such that $K \prec f \prec U$.*

Proof: The set, K is compact so is at a positive distance from U^C. Let

$$W \equiv \left\{\mathbf{x} : \operatorname{dist}(\mathbf{x}, K) < 3^{-1}\operatorname{dist}\left(K, U^C\right)\right\}.$$

[2] If each ψ_U were only continuous, one could conclude f is continuous. Here the main interest is differentiable.

Also let

$$W_1 \equiv \left\{ \mathbf{x} : \text{dist}\left(\mathbf{x}, K\right) < 2^{-1} \text{dist}\left(K, U^C\right) \right\}$$

Then it is clear

$$K \subseteq W \subseteq \overline{W} \subseteq W_1 \subseteq \overline{W_1} \subseteq U$$

Now consider the function,

$$h\left(\mathbf{x}\right) \equiv \frac{\text{dist}\left(\mathbf{x}, W_1^C\right)}{\text{dist}\left(\mathbf{x}, W_1^C\right) + \text{dist}\left(\mathbf{x}, \overline{W}\right)}$$

Since $\overline{W}$ is compact it is at a positive distance from W_1^C and so h is a well defined continuous function which has compact support contained in $\overline{W}_1$, equals 1 on W, and has values in $[0,1]$. Now let ϕ_k be a mollifier. Letting

$$k^{-1} < \min\left(\text{dist}\left(K, W^C\right), 2^{-1} \text{dist}\left(\overline{W}_1, U^C\right)\right),$$

it follows that for such k,the function, $h * \phi_k \in C_c^\infty\left(U\right)$, has values in $[0,1]$, and equals 1 on K. Let $f = h * \phi_k$.

The above lemma is used repeatedly in the following.

Lemma 39.6 *Let K be a closed set and let $\{V_i\}_{i=1}^\infty$ be a locally finite list of bounded open sets whose union contains K. Then there exist functions, $\psi_i \in C_c^\infty\left(V_i\right)$ such that for all $\mathbf{x} \in K$,*

$$1 = \sum_{i=1}^\infty \psi_i\left(\mathbf{x}\right)$$

and the function $f\left(\mathbf{x}\right)$ given by

$$f\left(\mathbf{x}\right) = \sum_{i=1}^\infty \psi_i\left(\mathbf{x}\right)$$

is in $C^\infty\left(\mathbb{R}^n\right)$.

Proof: Let $K_1 = K \setminus \cup_{i=2}^\infty V_i$. Thus K_1 is compact because $K_1 \subseteq V_1$. Let

$$K_1 \subseteq W_1 \subseteq \overline{W}_1 \subseteq V_1$$

Thus $W_1, V_2, \cdots, V_n$ covers K and $\overline{W}_1 \subseteq V_1$. Suppose $W_1, \cdots, W_r$ have been defined such that $\overline{W_i} \subseteq V_i$ for each i, and $W_1, \cdots, W_r, V_{r+1}, \cdots, V_n$ covers K. Then let

$$K_{r+1} \equiv K \setminus \left(\left(\cup_{i=r+2}^\infty V_i\right) \cup \left(\cup_{j=1}^r W_j\right)\right).$$

It follows K_{r+1} is compact because $K_{r+1} \subseteq V_{r+1}$. Let W_{r+1} satisfy

$$K_{r+1} \subseteq W_{r+1} \subseteq \overline{W}_{r+1} \subseteq V_{r+1}$$

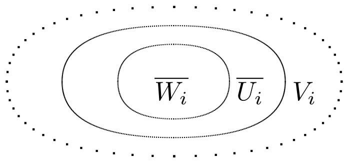
Continuing this way defines a sequence of open sets, $\{W_i\}_{i=1}^\infty$ with the property

$$\overline{W_i} \subseteq V_i, \ K \subseteq \cup_{i=1}^\infty W_i.$$

Note $\{W_i\}_{i=1}^{\infty}$ is locally finite because the original list, $\{V_i\}_{i=1}^{\infty}$ was locally finite. Now let U_i be open sets which satisfy

$$\overline{W}_i \subseteq U_i \subseteq \overline{U}_i \subseteq V_i.$$

Similarly, $\{U_i\}_{i=1}^{\infty}$ is locally finite.



Since the set, $\{W_i\}_{i=1}^{\infty}$ is locally finite, it follows $\overline{\cup_{i=1}^{\infty} W_i} = \cup_{i=1}^{\infty} \overline{W_i}$ and so it is possible to define ϕ_i and γ, infinitely differentiable functions having compact support such that

$$\overline{U}_i \prec \phi_i \prec V_i, \ \cup_{i=1}^{\infty} \overline{W}_i \prec \gamma \prec \cup_{i=1}^{\infty} U_i.$$

Now define

$$\psi_i(\mathbf{x}) = \begin{cases} \gamma(\mathbf{x})\phi_i(\mathbf{x}) / \sum_{j=1}^{\infty} \phi_j(\mathbf{x}) \text{ if } \sum_{j=1}^{\infty} \phi_j(\mathbf{x}) \neq 0, \\ 0 \text{ if } \sum_{j=1}^{\infty} \phi_j(\mathbf{x}) = 0. \end{cases}$$

If $\mathbf{x}$ is such that $\sum_{j=1}^{\infty} \phi_j(\mathbf{x}) = 0$, then $\mathbf{x} \notin \cup_{i=1}^{\infty} \overline{U_i}$ because ϕ_i equals one on $\overline{U_i}$. Consequently $\gamma(\mathbf{y}) = 0$ for all $\mathbf{y}$ near $\mathbf{x}$ thanks to the fact that $\cup_{i=1}^{\infty} \overline{U_i}$ is closed and so $\psi_i(\mathbf{y}) = 0$ for all $\mathbf{y}$ near $\mathbf{x}$. Hence ψ_i is infinitely differentiable at such $\mathbf{x}$. If $\sum_{j=1}^{\infty} \phi_j(\mathbf{x}) \neq 0$, this situation persists near $\mathbf{x}$ because each ϕ_j is continuous and so ψ_i is infinitely differentiable at such points also thanks to Lemma 39.3. Therefore ψ_i is infinitely differentiable. If $\mathbf{x} \in K$, then $\gamma(\mathbf{x}) = 1$ and so $\sum_{j=1}^{\infty} \psi_j(\mathbf{x}) = 1$. Clearly $0 \leq \psi_i(\mathbf{x}) \leq 1$ and $\text{spt}(\psi_j) \subseteq V_j$. This proves the theorem.

The method of proof of this lemma easily implies the following useful corollary.

Corollary 39.7 *If H is a compact subset of V_i for some V_i there exists a partition of unity such that $\psi_i(x) = 1$ for all $x \in H$ in addition to the conclusion of Lemma 39.6.*

Proof: Keep V_i the same but replace V_j with $\widetilde{V_j} \equiv V_j \setminus H$. Now in the proof above, applied to this modified collection of open sets, if $j \neq i$, $\phi_j(x) = 0$ whenever $x \in H$. Therefore, $\psi_i(x) = 1$ on H.

Theorem 39.8 *Let H be any closed set and let $\mathfrak{C}$ be any open cover of H. Then there exist functions $\{\psi_i\}_{i=1}^{\infty}$ such that $\text{spt}(\psi_i)$ is contained in some set of $\mathfrak{C}$ and ψ_i is infinitely differentiable having values in $[0,1]$ such that on H, $\sum_{i=1}^{\infty} \psi_i(\mathbf{x}) = 1$. Furthermore, the function, $f(\mathbf{x}) \equiv \sum_{i=1}^{\infty} \psi_i(\mathbf{x})$ is infinitely differentiable on $\mathbb{R}^n$. Also, $\text{spt}(\psi_i) \subseteq U_i$ where U_i is a bounded open set with the property that $\{U_i\}_{i=1}^{\infty}$ is locally finite and each U_i is contained in some set of $\mathfrak{C}$.*

Proof: By Lemma 39.2 there exists an open cover of H composed of bounded open sets, U_i such that each U_i is a subset of some set of $\mathfrak{C}$ and the collection, $\{U_i\}_{i=1}^{\infty}$ is locally finite. Then the result follows from Lemma 39.6 and Lemma 39.3.

Corollary 39.9 *Let H be any closed set and let $\{V_i\}_{i=1}^{m}$ be a finite open cover of H. Then there exist functions $\{\phi_i\}_{i=1}^{m}$ such that $\operatorname{spt}(\phi_i) \subseteq V_i$ and ϕ_i is infinitely differentiable having values in $[0,1]$ such that on H, $\sum_{i=1}^{m} \phi_i(\mathbf{x}) = 1$.*

Proof: By Theorem 39.8 there exists a set of functions, $\{\psi_i\}_{i=1}^{\infty}$ having the properties listed in this theorem relative to the open covering, $\{V_i\}_{i=1}^{m}$. Let $\phi_1(\mathbf{x})$ equal the sum of all $\psi_j(\mathbf{x})$ such that $\operatorname{spt}(\psi_j) \subseteq V_1$. Next let $\phi_2(\mathbf{x})$ equal the sum of all $\psi_j(\mathbf{x})$ which have not already been included and for which $\operatorname{spt}(\psi_j) \subseteq V_2$. Continue in this manner. Since the open sets, $\{U_i\}_{i=1}^{\infty}$ mentioned in Theorem 39.8 are locally finite, it follows from Lemma 39.3 that each ϕ_i is infinitely differentiable having support in V_i. This proves the corollary.

39.2 Integration On Manifolds

Manifolds are things which locally appear to be $\mathbb{R}^n$ for some n. The extent to which they have such a local appearance varies according to various analytical characteristics which the manifold possesses.

Definition 39.10 *Let $U \subseteq \mathbb{R}^n$ be an open set and let $\mathbf{h} : U \to \mathbb{R}^m$. Then for $r \in [0,1)$, $\mathbf{h} \in C^{k,r}(U)$ for k a nonnegative integer means that $D^{\alpha}\mathbf{h}$ exists for all $|\alpha| \leq k$ and each $D^{\alpha}\mathbf{h}$ is Holder continuous with exponent r. That is*

$$|D^{\alpha}\mathbf{h}(\mathbf{x}) - D^{\alpha}\mathbf{h}(\mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|^{r}.$$

Also $\mathbf{h} \in C^{k,r}(\overline{U})$ if it is the restriction of a function of $C^{k,r}(\mathbb{R}^n)$ to U.

Definition 39.11 *Let Γ be a closed subset of $\mathbb{R}^p$ where $p \geq n$. Suppose $\Gamma = \cup_{i=1}^{\infty}\Gamma_i$ where $\Gamma_i = \Gamma \cap W_i$ for W_i a bounded open set. Suppose also $\{W_i\}_{i=1}^{\infty}$ is locally finite. This means every bounded open set intersects only finitely many. Also suppose there are open bounded sets, U_i having Lipschitz boundaries and functions $\mathbf{h}_i : U_i \to \Gamma_i$ which are one to one, onto, and in $C^{m,1}(U_i)$. Suppose also there exist functions, $\mathbf{g}_i : W_i \to U_i$ such that $\mathbf{g}_i$ is $C^{m,1}(W_i)$, and $\mathbf{g}_i \circ \mathbf{h}_i = \mathrm{id}$ on U_i while $\mathbf{h}_i \circ \mathbf{g}_i = \mathrm{id}$ on Γ_i. The collection of sets, Γ_j and mappings, $\mathbf{g}_j$, $\{(\Gamma_j, \mathbf{g}_j)\}$ is called an atlas and an individual entry in the atlas is called a chart. Thus $(\Gamma_j, \mathbf{g}_j)$ is a chart. Then Γ as just described is called a $C^{m,1}$ manifold. The number, m is just a nonnegative integer. When $m = 0$ this would be called a Lipschitz manifold, the least smooth of the manifolds discussed here.*

For example, take $p = n + 1$ and let

$$\mathbf{h}_i(\mathbf{u}) = (u_1, \cdots, u_i, \phi_i(\mathbf{u}), u_{i+1}, \cdots, u_n)^T$$

for $\mathbf{u} = (u_1, \cdots, u_i, u_{i+1}, \cdots, u_n)^T \in U_i$ for $\phi_i \in C^{m,1}(U_i)$ and $\mathbf{g}_i : U_i \times \mathbb{R} \to U_i$ given by

$$\mathbf{g}_i(u_1, \cdots, u_i, y, u_{i+1}, \cdots, u_n) \equiv \mathbf{u}$$

for $i = 1, 2, \cdots, p$. Then for $\mathbf{u} \in U_i$, the definition gives

$$\mathbf{g}_i \circ \mathbf{h}_i(\mathbf{u}) = \mathbf{g}_i(u_1, \cdots, u_i, \phi_i(\mathbf{u}), u_{i+1}, \cdots, u_n) = \mathbf{u}$$

and for $\Gamma_i \equiv \mathbf{h}_i(U_i)$ and $(u_1, \cdots, u_i, \phi_i(\mathbf{u}), u_{i+1}, \cdots, u_n)^T \in \Gamma_i$,

$$\begin{aligned} & \mathbf{h}_i \circ \mathbf{g}_i(u_1, \cdots, u_i, \phi_i(\mathbf{u}), u_{i+1}, \cdots, u_n) \\ = \; & \mathbf{h}_i(\mathbf{u}) = (u_1, \cdots, u_i, \phi_i(\mathbf{u}), u_{i+1}, \cdots, u_n)^T. \end{aligned}$$

This example can be used to describe the boundary of a bounded open set and since $\phi_i \in C^{m,1}(U_i)$, such an open set is said to have a $C^{m,1}$ boundary. Note also that in this example, U_i could be taken to be $\mathbb{R}^n$ or if U_i is given, both $\mathbf{h}_i$ and and $\mathbf{g}_i$ can be taken as restrictions of functions defined on all of $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively.

The symbol, I will refer to an increasing list of n indices taken from $\{1, \cdots, p\}$. Denote by $\Lambda(p, n)$ the set of all such increasing lists of n indices.

Let

$$J_i(\mathbf{u}) \equiv \left[\sum_{I \in \Lambda(p,n)} \left(\frac{\partial(x^{i_1} \cdots x^{i_n})}{\partial(u^1 \cdots u^n)} \right)^2 \right]^{1/2}$$

where here the sum is taken over all possible increasing lists of n indices, I, from $\{1, \cdots, p\}$ and $\mathbf{x} = \mathbf{h}_i \mathbf{u}$. Thus there are $\binom{p}{n}$ terms in the sum. In this formula, $\frac{\partial(x^{i_1} \cdots x^{i_n})}{\partial(u^1 \cdots u^n)}$ is defined to be the determinant of the following matrix.

$$\begin{pmatrix} \frac{\partial x^{i_1}}{\partial u_1} & \cdots & \frac{\partial x^{i_1}}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial x^{i_n}}{\partial u_1} & \cdots & \frac{\partial x^{i_n}}{\partial u_n} \end{pmatrix}.$$

Note that if $p = n$ there is only one term in the sum, the absolute value of the determinant of $D\mathbf{x}(\mathbf{u})$. Define a positive linear functional, Λ on $C_c(\Gamma)$ as follows: First let $\{\psi_i\}$ be a C^∞ partition of unity subordinate to the open sets, $\{W_i\}$. Thus $\psi_i \in C_c^\infty(W_i)$ and $\sum_i \psi_i(\mathbf{x}) = 1$ for all $\mathbf{x} \in \Gamma$. Then

$$\Lambda f \equiv \sum_{i=1}^{\infty} \int_{\mathbf{g}_i \Gamma_i} f \psi_i(\mathbf{h}_i(\mathbf{u})) J_i(\mathbf{u}) \, du. \tag{39.1}$$

Is this well defined?

Lemma 39.12 *The functional defined in 39.1 does not depend on the choice of atlas or the partition of unity.*

Proof: In 39.1, let $\{\psi_i\}$ be a C^∞ partition of unity which is associated with the atlas $(\Gamma_i, \mathbf{g}_i)$ and let $\{\eta_i\}$ be a C^∞ partition of unity associated in the same manner with the atlas $(\Gamma_i', \mathbf{g}_i')$. In the following argument, the local finiteness of the Γ_i implies that all sums are finite. Using the change of variables formula with $\mathbf{u} = \left(\mathbf{g}_i \circ \mathbf{h}_j'\right) \mathbf{v}$

$$\sum_{i=1}^\infty \int_{\mathbf{g}_i \Gamma_i} \psi_i f\left(\mathbf{h}_i\left(\mathbf{u}\right)\right) J_i\left(\mathbf{u}\right) du = \tag{39.2}$$

$$\sum_{i=1}^\infty \sum_{j=1}^\infty \int_{\mathbf{g}_i \Gamma_i} \eta_j \psi_i f\left(\mathbf{h}_i\left(\mathbf{u}\right)\right) J_i\left(\mathbf{u}\right) du = \sum_{i=1}^\infty \sum_{j=1}^\infty \int_{\mathbf{g}_j'\left(\Gamma_i \cap \Gamma_j'\right)} \cdot$$

$$\eta_j\left(\mathbf{h}_j'\left(\mathbf{v}\right)\right) \psi_i\left(\mathbf{h}_j'\left(\mathbf{v}\right)\right) f\left(\mathbf{h}_j'\left(\mathbf{v}\right)\right) J_i\left(\mathbf{u}\right) \left|\frac{\partial\left(u^1 \cdots u^n\right)}{\partial\left(v^1 \cdots v^n\right)}\right| dv$$

$$= \sum_{i=1}^\infty \sum_{j=1}^\infty \int_{\mathbf{g}_j'\left(\Gamma_i \cap \Gamma_j'\right)} \eta_j\left(\mathbf{h}_j'\left(\mathbf{v}\right)\right) \psi_i\left(\mathbf{h}_j'\left(\mathbf{v}\right)\right) f\left(\mathbf{h}_j'\left(\mathbf{v}\right)\right) J_j\left(\mathbf{v}\right) dv. \tag{39.3}$$

Thus

$$\text{the definition of } \Lambda f \text{ using } \left(\Gamma_i, \mathbf{g}_i\right) \equiv$$

$$\sum_{i=1}^\infty \int_{\mathbf{g}_i \Gamma_i} \psi_i f\left(\mathbf{h}_i\left(\mathbf{u}\right)\right) J_i\left(\mathbf{u}\right) du =$$

$$\sum_{i=1}^\infty \sum_{j=1}^\infty \int_{\mathbf{g}_j'\left(\Gamma_i \cap \Gamma_j'\right)} \eta_j\left(\mathbf{h}_j'\left(\mathbf{v}\right)\right) \psi_i\left(\mathbf{h}_j'\left(\mathbf{v}\right)\right) f\left(\mathbf{h}_j'\left(\mathbf{v}\right)\right) J_j\left(\mathbf{v}\right) dv$$

$$= \sum_{j=1}^\infty \int_{\mathbf{g}_j'\left(\Gamma_j'\right)} \eta_j\left(\mathbf{h}_j'\left(\mathbf{v}\right)\right) f\left(\mathbf{h}_j'\left(\mathbf{v}\right)\right) J_j\left(\mathbf{v}\right) dv$$

$$\text{the definition of } \Lambda f \text{ using } \left(V_i, \mathbf{g}_i'\right).$$

This proves the lemma.

This lemma and the Riesz representation theorem for positive linear functionals implies the part of the following theorem which says the functional is well defined.

Theorem 39.13 *Let Γ be a $C^{m,1}$ manifold. Then there exists a unique Radon measure, μ, defined on Γ such that whenever f is a continuous function having compact support which is defined on Γ and $(\Gamma_i, \mathbf{g}_i)$ denotes an atlas and $\{\psi_i\}$ a partition of unity subordinate to this atlas,*

$$\Lambda f = \int_\Gamma f d\mu = \sum_{i=1}^\infty \int_{\mathbf{g}_i \Gamma_i} \psi_i f\left(\mathbf{h}_i\left(\mathbf{u}\right)\right) J_i\left(\mathbf{u}\right) du. \tag{39.4}$$

Also, a subset, A, of Γ is μ measurable if and only if for all $r, \mathbf{g}_r\left(\Gamma_r \cap A\right)$ is ν_r measurable where ν_r is the measure defined by

$$\nu_r\left(\mathbf{g}_r\left(\Gamma_r \cap A\right)\right) \equiv \int_{\mathbf{g}_r\left(\Gamma_r \cap A\right)} J_r\left(\mathbf{u}\right) du$$

Proof: To begin, here is a claim.

Claim : A set, $S \subseteq \Gamma_i$, has μ measure zero if and only if $\mathbf{g}_i S$ has measure zero in $\mathbf{g}_i \Gamma_i$ with respect to the measure, ν_i.

Proof of the claim: Let $\varepsilon > 0$ be given. By outer regularity, there exists a set, $V \subseteq \Gamma_i$, open[3] in Γ such that $\mu(V) < \varepsilon$ and $S \subseteq V \subseteq \Gamma_i$. Then $\mathbf{g}_i V$ is open in $\mathbb{R}^n$ and contains $\mathbf{g}_i S$. Letting $h \prec \mathbf{g}_i V$ and $h_1(\mathbf{x}) \equiv h(\mathbf{g}_i(\mathbf{x}))$ for $\mathbf{x} \in \Gamma_i$ it follows $h_1 \prec V$. By Corollary 39.7 on Page 1126 there exists a partition of unity such that $\text{spt}(h_1) \subseteq \{\mathbf{x} \in \mathbb{R}^p : \psi_i(\mathbf{x}) = 1\}$. Thus $\psi_j h_1(\mathbf{h}_j(u)) = 0$ unless $j = i$ when this reduces to $h_1(\mathbf{h}_i(u))$. It follows

$$\begin{aligned}
\varepsilon &\geq \mu(V) \geq \int_V h_1 d\mu = \int_\Gamma h_1 d\mu \\
&= \sum_{j=1}^{\infty} \int_{\mathbf{g}_j \Gamma_j} \psi_j h_1(\mathbf{h}_j(\mathbf{u})) J_j(\mathbf{u}) \, du \\
&= \int_{\mathbf{g}_i \Gamma_i} h_1(\mathbf{h}_i(\mathbf{u})) J_i(\mathbf{u}) \, du = \int_{\mathbf{g}_i \Gamma_i} h(\mathbf{u}) J_i(\mathbf{u}) \, du \\
&= \int_{\mathbf{g}_i V} h(\mathbf{u}) J_i(\mathbf{u}) \, du
\end{aligned}$$

Now this holds for all $h \prec \mathbf{g}_i V$ and so

$$\int_{\mathbf{g}_i V} J_i(\mathbf{u}) \, du \leq \varepsilon.$$

Since ε is arbitrary, this shows $\mathbf{g}_i V$ has measure no more than ε with respect to the measure, ν_i. Since ε is arbitrary, $\mathbf{g}_i S$ has measure zero.

Consider the converse. Suppose $\mathbf{g}_i S$ has ν_i measure zero. Then there exists an open set, $O \subseteq \mathbf{g}_i \Gamma_i$ such that $O \supseteq \mathbf{g}_i S$ and

$$\int_O J_i(\mathbf{u}) \, du < \varepsilon.$$

Thus $\mathbf{h}_i(O)$ is open in Γ and contains S. Let $h \prec \mathbf{h}_i(O)$ be such that

$$\int_\Gamma h d\mu + \varepsilon > \mu(\mathbf{h}_i(O)) \geq \mu(S) \tag{39.5}$$

As in the first part, Corollary 39.7 on Page 1126 implies there exists a partition of unity such that $h(\mathbf{x}) = 0$ off the set,

$$\{\mathbf{x} \in \mathbb{R}^p : \psi_i(\mathbf{x}) = 1\}$$

[3]This means V is the intersection of an open set with Γ. Equivalently, it means that V is an open set in the traditional way regarding Γ as a metric space with the metric it inherits from $\mathbb{R}^m$.

and so as in this part of the argument,

$$\begin{aligned}
\int_{\Gamma} h d\mu &\equiv \sum_{j=1}^{\infty} \int_{\mathbf{g}_j U_j} \psi_j h\left(\mathbf{h}_j\left(\mathbf{u}\right)\right) J_j\left(\mathbf{u}\right) du \\
&= \int_{\mathbf{g}_i \Gamma_i} h\left(\mathbf{h}_i\left(\mathbf{u}\right)\right) J_i\left(\mathbf{u}\right) du \\
&= \int_{O \cap \mathbf{g}_i \Gamma_i} h\left(\mathbf{h}_i\left(\mathbf{u}\right)\right) J_i\left(\mathbf{u}\right) du \\
&\leq \int_{O} J_i\left(\mathbf{u}\right) du < \varepsilon \qquad (39.6)
\end{aligned}$$

and so from 39.5 and 39.6 $\mu(S) \leq 2\varepsilon$. Since ε is arbitrary, this proves the claim.

For the last part of the theorem, it suffices to let $A \subseteq \Gamma_r$ because otherwise, the above argument would apply to $A \cap \Gamma_r$. Thus let $A \subseteq \Gamma_r$ be μ measurable. By the regularity of the measure, there exists an F_σ set, F and a G_δ set, G such that $\Gamma_r \supseteq G \supseteq A \supseteq F$ and $\mu(G \setminus F) = 0$.(Recall a G_δ set is a countable intersection of open sets and an F_σ set is a countable union of closed sets.) Then since $\overline{\Gamma_r}$ is compact, it follows each of the closed sets whose union equals F is a compact set. Thus if $F = \cup_{k=1}^{\infty} F_k$, $\mathbf{g}_r(F_k)$ is also a compact set and so $\mathbf{g}_r(F) = \cup_{k=1}^{\infty} \mathbf{g}_r(F_k)$ is a Borel set. Similarly, $\mathbf{g}_r(G)$ is also a Borel set. Now by the claim,

$$\int_{\mathbf{g}_r(G \setminus F)} J_r\left(\mathbf{u}\right) du = 0.$$

Since $\mathbf{g}_r$ is one to one,

$$\mathbf{g}_r G \setminus \mathbf{g}_r F = \mathbf{g}_r\left(G \setminus F\right)$$

and so

$$\mathbf{g}_r(F) \subseteq \mathbf{g}_r(A) \subseteq \mathbf{g}_r(G)$$

where $\mathbf{g}_r(G) \setminus \mathbf{g}_r(F)$ has measure zero. By completeness of the measure, ν_r, $\mathbf{g}_r(A)$ is measurable. It follows that if $A \subseteq \Gamma$ is μ measurable, then $\mathbf{g}_r(\Gamma_r \cap A)$ is ν_r measurable for all r. The converse is entirely similar. This proves the theorem.

Corollary 39.14 *Let $f \in L^1(\Gamma; \mu)$ and suppose $f(\mathbf{x}) = 0$ for all $\mathbf{x} \notin \Gamma_r$ where $(\Gamma_r, \mathbf{g}_r)$ is a chart. Then*

$$\int_{\Gamma} f d\mu = \int_{\Gamma_r} f d\mu = \int_{\mathbf{g}_r \Gamma_r} f\left(\mathbf{h}_r\left(\mathbf{u}\right)\right) J_r\left(\mathbf{u}\right) du. \qquad (39.7)$$

Furthermore, if $\{(\Gamma_i, \mathbf{g}_i)\}$ is an atlas and $\{\psi_i\}$ is a partition of unity as described earlier, then for any $f \in L^1(\Gamma, \mu)$,

$$\int_{\Gamma} f d\mu = \sum_{r=1}^{\infty} \int_{\mathbf{g}_r \Gamma_r} \psi_r f\left(\mathbf{h}_r\left(\mathbf{u}\right)\right) J_r\left(\mathbf{u}\right) du. \qquad (39.8)$$

Proof: Let $f \in L^1(\Gamma,\mu)$ with $f=0$ off Γ_r. Without loss of generality assume $f \geq 0$ because if the formulas can be established for this case, the same formulas are obtained for an arbitrary complex valued function by splitting it up into positive and negative parts of the real and imaginary parts in the usual way. Also, let $K \subseteq \Gamma_r$ a compact set. Since μ is a Radon measure there exists a sequence of continuous functions, $\{f_k\}$, $f_k \in C_c(\Gamma_r)$, which converges to f in $L^1(\Gamma,\mu)$ and for μ a.e. $\mathbf{x}$. Take the partition of unity, $\{\psi_i\}$ to be such that

$$K \subseteq \{\mathbf{x} : \psi_r(\mathbf{x}) = 1\}.$$

Therefore, the sequence $\{f_k(\mathbf{h}_r(\cdot))\}$ is a Cauchy sequence in the sense that

$$\lim_{k,l\to\infty} \int_{\mathbf{g}_r(K)} |f_k(\mathbf{h}_r(\mathbf{u})) - f_l(\mathbf{h}_r(\mathbf{u}))| J_r(\mathbf{u})\, du = 0$$

It follows there exists g such that

$$\int_{\mathbf{g}_r(K)} |f_k(\mathbf{h}_r(\mathbf{u})) - g(\mathbf{u})| J_r(\mathbf{u})\, du \to 0,$$

and

$$g \in L^1(\mathbf{g}_r K; \nu_r).$$

By the pointwise convergence and the claim used in the proof of Theorem 39.13,

$$g(\mathbf{u}) = f(\mathbf{h}_r(\mathbf{u}))$$

for μ a.e. $\mathbf{h}_r(\mathbf{u}) \in K$. Therefore,

$$\begin{aligned}\int_K f d\mu &= \lim_{k\to\infty}\int_K f_k d\mu = \lim_{k\to\infty}\int_{\mathbf{g}_r(K)} f_k(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u})\, du \\ &= \int_{\mathbf{g}_r(K)} g(\mathbf{u}) J_r(\mathbf{u})\, du = \int_{\mathbf{g}_r(K)} f(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u})\, du. \qquad (39.9)\end{aligned}$$

Now let $\cdots K_j \subseteq K_{j+1} \cdots$ and $\cup_{j=1}^{\infty} K_j = \Gamma_r$ where K_j is compact for all j. Replace K in 39.9 with K_j and take a limit as $j \to \infty$. By the monotone convergence theorem,

$$\int_{\Gamma_r} f d\mu = \int_{\mathbf{g}_r(\Gamma_r)} f(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u})\, du.$$

This establishes 39.7.

To establish 39.8, let $f \in L^1(\Gamma,\mu)$ and let $\{(\Gamma_i, \mathbf{g}_i)\}$ be an atlas and $\{\psi_i\}$ be a partition of unity. Then $f\psi_i \in L^1(\Gamma,\mu)$ and is zero off Γ_i. Therefore, from what was just shown,

$$\begin{aligned}\int_\Gamma f d\mu &= \sum_{i=1}^{\infty} \int_{\Gamma_i} f\psi_i d\mu \\ &= \sum_{r=1}^{\infty} \int_{\mathbf{g}_r(\Gamma_r)} \psi_r f(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u})\, du\end{aligned}$$

39.3 Comparison With $\mathcal{H}^n$

The above gives a measure on a manifold, Γ. I will now show that the measure obtained is nothing more than $\mathcal{H}^n$, the n dimensional Hausdorff measure. Recall $\Lambda(p,n)$ was the set of all increasing lists of n indices taken from $\{1,2,\cdots,p\}$

Recall

$$J_i(\mathbf{u}) \equiv \left[\sum_{I\in\Lambda(p,n)} \left(\frac{\partial\left(x^{i_1}\cdots x^{i_n}\right)}{\partial\left(u^1\cdots u^n\right)}\right)^2\right]^{1/2}$$

where here the sum is taken over all possible increasing lists of n indices, I, from $\{1,\cdots,p\}$ and $\mathbf{x}=\mathbf{h}_i\mathbf{u}$ and the functional was given as

$$\Lambda f \equiv \sum_{i=1}^{\infty}\int_{\mathbf{g}_i\Gamma_i} f\psi_i(\mathbf{h}_i(\mathbf{u}))\,J_i(\mathbf{u})\,du \tag{39.10}$$

where the $\{\psi_i\}_{i=1}^{\infty}$ was a partition of unity subordinate to the open sets, $\{W_i\}_{i=1}^{\infty}$ as described above. I will show

$$J_i(\mathbf{u}) = \det\left(D\mathbf{h}(\mathbf{u})^* D\mathbf{h}(\mathbf{u})\right)^{1/2}$$

and then use the area formula. The key result is really a special case of the Binet Cauchy theorem and this special case is presented in the next lemma.

Lemma 39.15 *Let $A=(a_{ij})$ be a real $p\times n$ matrix in which $p\geq n$. For $I\in\Lambda(p,n)$ denote by A_I the $n\times n$ matrix obtained by deleting from A all rows except for those corresponding to an element of I. Then*

$$\sum_{I\in\Lambda(p,n)} \det(A_I)^2 = \det(A^*A)$$

Proof: For $(j_1,\cdots,j_n)\in\Lambda(p,n)$, define $\theta(j_k)\equiv k$. Then let for $\{k_1,\cdots,k_n\}=\{j_1,\cdots,j_n\}$ define

$$\operatorname{sgn}(k_1,\cdots,k_n)\equiv\operatorname{sgn}(\theta(k_1),\cdots,\theta(k_n)).$$

Then from the definition of the determinant and matrix multiplication,

$$\begin{aligned}\det(A^*A) &= \sum_{i_1,\cdots,i_n}\operatorname{sgn}(i_1,\cdots,i_n)\sum_{k_1=1}^{p}a_{k_1i_1}a_{k_11}\sum_{k_2=1}^{p}a_{k_2i_2}a_{k_22}\\ &\quad\cdots\sum_{k_n=1}^{p}a_{k_ni_n}a_{k_nn}\end{aligned}$$

$$=\sum_{J\in\Lambda(p,n)}\sum_{\{k_1,\cdots,k_n\}=J}\sum_{i_1,\cdots,i_n}\operatorname{sgn}(i_1,\cdots,i_n)\,a_{k_1i_1}a_{k_11}a_{k_2i_2}a_{k_22}\cdots a_{k_ni_n}a_{k_nn}$$

$$= \sum_{J\in\Lambda(p,n)} \sum_{\{k_1,\cdots,k_n\}=J} \sum_{i_1,\cdots,i_n} \text{sgn}(i_1,\cdot\cdot\cdot,i_n)\, a_{k_1 i_1} a_{k_2 i_2} \cdots a_{k_n i_n} \cdot a_{k_1 1} a_{k_2 2} \cdots a_{k_n n}$$

$$= \sum_{J\in\Lambda(p,n)} \sum_{\{k_1,\cdots,k_n\}=J} \text{sgn}(k_1,\cdot\cdot\cdot,k_n) \det(A_J)\, a_{k_1 1} a_{k_2 2} \cdots a_{k_n n}$$

$$= \sum_{J\in\Lambda(p,n)} \det(A_J) \det(A_J)$$

and this proves the lemma.

It follows from this lemma that

$$J_i(\mathbf{u}) = \det\left(D\mathbf{h}(\mathbf{u})^* D\mathbf{h}(\mathbf{u})\right)^{1/2}.$$

From 39.10 and the area formula, the functional equals

$$\begin{aligned}\Lambda f &\equiv \sum_{i=1}^{\infty} \int_{\mathbf{g}_i \Gamma_i} f\psi_i(\mathbf{h}_i(\mathbf{u}))\, J_i(\mathbf{u})\, du \\ &= \sum_{i=1}^{\infty} \int_{\Gamma_i} f\psi_i(\mathbf{y})\, d\mathcal{H}^n = \int_{\Gamma} f(\mathbf{y})\, d\mathcal{H}^n.\end{aligned}$$

Now $\mathcal{H}^n$ is a Borel measure defined on Γ which is finite on all compact subsets of Γ. This finiteness follows from the above formula. If K is a compact subset of Γ, then there exists an open set, W whose closure is compact and a continuous function with compact support, f such that $K \prec f \prec W$. Then $\mathcal{H}^n(K) \leq \int_\Gamma f(\mathbf{y})\, d\mathcal{H}^n < \infty$ because of the above formula.

Lemma 39.16 *$\mu = \mathcal{H}^n$ on every μ measurable set.*

Proof: The Riesz representation theorem shows that

$$\int_\Gamma f d\mu = \int_\Gamma f d\mathcal{H}^n$$

for every continuous function having compact support. Therefore, since every open set is the countable union of compact sets, it follows $\mu = \mathcal{H}^n$ on all open sets. Since compact sets can be obtained as the countable intersection of open sets, these two measures are also equal on all compact sets. It follows they are also equal on all countable unions of compact sets. Suppose now that E is a μ measurable set of finite measure. Then there exist sets, F, G such that G is the countable intersection of open sets each of which has finite measure and F is the countable union of compact sets such that $\mu(G \setminus F) = 0$ and $F \subseteq E \subseteq G$. Thus $\mathcal{H}^n(G \setminus F) = 0$,

$$\mathcal{H}^n(G) = \mu(G) = \mu(F) = \mathcal{H}^n(F)$$

By completeness of $\mathcal{H}^n$ it follows E is $\mathcal{H}^n$ measurable and $\mathcal{H}^n(E) = \mu(E)$. If E is not of finite measure, consider $E_r \equiv E \cap B(\mathbf{0}, r)$. This is contained in the compact set $\Gamma \cap \overline{B(\mathbf{0}, r)}$ and so $\mu(E_r)$ if finite. Thus from what was just shown, $\mathcal{H}^n(E_r) = \mu(E_r)$ and so, taking $r \to \infty$ $\mathcal{H}^n(E) = \mu(E)$.

This shows you can simply use $\mathcal{H}^n$ for the measure on Γ.

Basic Theory Of Sobolev Spaces

Definition 40.1 *Let U be an open set of $\mathbb{R}^n$. Define $X^{m,p}(U)$ as the set of all functions in $L^p(U)$ whose weak partial derivatives up to order m are also in $L^p(U)$ where $1 \leq p$. The norm[1] in this space is given by*

$$||u||_{m,p} \equiv \left(\int_U \sum_{|\alpha| \leq m} |D^\alpha u|^p \, dx \right)^{1/p}.$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| \equiv \sum \alpha_i$. Here $D^{\mathbf{0}} u \equiv u$. $C^\infty\left(\overline{U}\right)$ is defined to be the set of functions which are restrictions to U of a function in $C_c^\infty(\mathbb{R}^n)$. Thus $C^\infty\left(\overline{U}\right) \subseteq W^{m,p}(U)$. The Sobolev space, $W^{m,p}(U)$ is defined to be the closure of $C^\infty\left(\overline{U}\right)$ in $X^{m,p}(U)$ with respect to the above norm. Denote this norm by $||u||_{W^{m,p}(U)}$, $||u||_{X^{m,p}(U)}$, or $||u||_{m,p,U}$ when it is important to identify the open set, U.

Also the following notation will be used pretty consistently.

Definition 40.2 *Let u be a function defined on U. Define*

$$\widetilde{u}(\mathbf{x}) \equiv \begin{cases} u(\mathbf{x}) & \text{if } \mathbf{x} \in U \\ 0 & \text{if } \mathbf{x} \notin U \end{cases}.$$

Theorem 40.3 *Both $X^{m,p}(U)$ and $W^{m,p}(U)$ are separable reflexive Banach spaces provided $p > 1$.*

Proof: Define $\Lambda : X^{m,p}(U) \to L^p(U)^w$ where w equals the number of multi indices, α, such that $|\alpha| \leq m$ as follows. Letting $\{\alpha_i\}_{i=1}^w$ be the set of all multi indices with $\alpha_1 = \mathbf{0}$,

$$\Lambda(u) \equiv (D^{\alpha_1} u, D^{\alpha_2} u, \cdots, D^{\alpha_w} u) = (u, D^{\alpha_2} u, \cdots, D^{\alpha_w} u).$$

[1] You could also let the norm be given by $||u||_{m,p} \equiv \sum_{|\alpha| \leq m} ||D^\alpha u||_p$ or $||u||_{m,p} \equiv \max\left\{ ||D^\alpha u||_p : |\alpha| \leq m \right\}$ because all norms are equivalent on $\mathbb{R}^p$ where p is the number of multi indices no larger than m. This is used whenever convenient.

Then Λ is one to one because one of the multi indices is $\mathbf{0}$. Also $\Lambda\left(X^{m,p}\left(U\right)\right)$ is a closed subspace of $L^{p}\left(U\right)^{w}$. To see this, suppose $\left(u_{k}, D^{\alpha_{2}}u_{k}, \cdot\cdot\cdot, D^{\alpha_{w}}u_{k}\right) \rightarrow \left(f_{1}, f_{2}, \cdot\cdot\cdot, f_{w}\right)$ in $L^{p}\left(U\right)^{w}$. Then $u_{k} \rightarrow f_{1}$ in $L^{p}\left(U\right)$ and $D^{\alpha_{j}}u_{k} \rightarrow f_{j}$ in $L^{p}\left(U\right)$. Therefore, letting $\phi \in C_{c}^{\infty}\left(U\right)$ and letting $k \rightarrow \infty$,

$$\begin{array}{llll}\int_{U}\left(D^{\alpha_{j}}u_{k}\right)\phi dx & = (-1)^{|\alpha|}\int_{U}u_{k}D^{\alpha_{j}}\phi dx & \rightarrow (-1)^{|\alpha|}\int_{U}f_{1}D^{\alpha_{j}}\phi dx & \equiv D^{\alpha_{j}}\left(f_{1}\right)\left(\phi\right) \\ \quad\downarrow & & & \\ \int_{U}f_{j}\phi dx & & & \end{array}$$

It follows $D^{\alpha_{j}}\left(f_{1}\right) = f_{j}$ and so $\Lambda\left(X^{m,p}\left(U\right)\right)$ is closed as claimed. This is clearly also a subspace of $L^{p}\left(U\right)^{w}$ and so it follows that $\Lambda\left(X^{m,p}\left(U\right)\right)$ is a reflexive Banach space. This is because $L^{p}\left(U\right)^{w}$, being the product of reflexive Banach spaces, is reflexive and any closed subspace of a reflexive Banach space is reflexive. Now Λ is an isometry of $X^{m,p}\left(U\right)$ and $\Lambda\left(X^{m,p}\left(U\right)\right)$ which shows that $X^{m,p}\left(U\right)$ is a reflexive Banach space. Finally, $W^{m,p}\left(U\right)$ is a closed subspace of the reflexive Banach space, $X^{m,p}\left(U\right)$ and so it is also reflexive. To see $X^{m,p}\left(U\right)$ is separable, note that $L^{p}\left(U\right)^{w}$ is separable because it is the finite product of the separable hence completely separable metric space, $L^{p}\left(U\right)$ and $\Lambda\left(X^{m,p}\left(U\right)\right)$ is a subset of $L^{p}\left(U\right)^{w}$. Therefore, $\Lambda\left(X^{m,p}\left(U\right)\right)$ is separable and since Λ is an isometry, it follows $X^{m,p}\left(U\right)$ is separable also. Now $W^{m,p}\left(U\right)$ must also be separable because it is a subset of $X^{m,p}\left(U\right)$.

The following theorem is obvious but is worth noting because it says that if a function has a weak derivative in $L^{p}\left(U\right)$ on a large open set, U then the restriction of this weak derivative is also the weak derivative for any smaller open set.

Theorem 40.4 *Suppose U is an open set and $U_{0} \subseteq U$ is another open set. Suppose also $D^{\alpha}u \in L^{p}\left(U\right)$. Then for all $\psi \in C_{c}^{\infty}\left(U_{0}\right)$,*

$$\int_{U_{0}}\left(D^{\alpha}u\right)\psi dx = (-1)^{|\alpha|}\int_{U_{0}}u\left(D^{\alpha}\psi\right).$$

The following theorem is a fundamental approximation result for functions in $X^{m,p}\left(U\right)$.

Theorem 40.5 *Let U be an open set and let U_{0} be an open subset of U with the property that* dist $\left(\overline{U_{0}}, U^{C}\right) > 0$. *Then if $u \in X^{m,p}\left(U\right)$ and $\widetilde{u}$ denotes the zero extention of u off U,*

$$\lim_{l\rightarrow\infty}||\widetilde{u}*\phi_{l}-u||_{X^{m,p}\left(U_{0}\right)} = 0.$$

Proof: Always assume l is large enough that $1/l <$ dist $\left(\overline{U_{0}}, U^{C}\right)$. Thus for $\mathbf{x} \in U_{0}$,

$$\widetilde{u}*\phi_{l}\left(\mathbf{x}\right) = \int_{B\left(\mathbf{0},\frac{1}{l}\right)}u\left(\mathbf{x}-\mathbf{y}\right)\phi_{l}\left(\mathbf{y}\right)dy. \tag{40.1}$$

The theorem is proved if it can be shown that $D^{\alpha}(\widetilde{u}*\phi_l)\to D^{\alpha}u$ in $L^p(U_0)$. Let $\psi\in C_c^{\infty}(U_0)$

$$\begin{aligned} D^{\alpha}(\widetilde{u}*\phi_l)(\psi) &\equiv (-1)^{|\alpha|}\int_{U_0}(\widetilde{u}*\phi_l)(D^{\alpha}\psi)\,dx \\ &= (-1)^{|\alpha|}\int_{U_0}\int \widetilde{u}(\mathbf{y})\,\phi_l(\mathbf{x}-\mathbf{y})(D^{\alpha}\psi)(\mathbf{x})\,dydx \\ &= (-1)^{|\alpha|}\int_{U}u(\mathbf{y})\int_{U_0}\phi_l(\mathbf{x}-\mathbf{y})(D^{\alpha}\psi)(\mathbf{x})\,dxdy. \end{aligned}$$

Also,

$$\begin{aligned} \left(\widetilde{D^{\alpha}u}*\phi_l\right)(\psi) &\equiv \int_{U_0}\left(\int \widetilde{D^{\alpha}u}(\mathbf{y})\,\phi_l(\mathbf{x}-\mathbf{y})\,dy\right)\psi(\mathbf{x})\,dx \\ &= \int_{U_0}\left(\int_U D^{\alpha}u(\mathbf{y})\,\phi_l(\mathbf{x}-\mathbf{y})\,dy\right)\psi(\mathbf{x})\,dx \\ &= \int_{U_0}\left(\int_U u(\mathbf{y})(D^{\alpha}\phi_l)(\mathbf{x}-\mathbf{y})\,dy\right)\psi(\mathbf{x})\,dx \\ &= \int_U u(\mathbf{y})\int_{U_0}(D^{\alpha}\phi_l)(\mathbf{x}-\mathbf{y})\,\psi(\mathbf{x})\,dxdy \\ &= (-1)^{|\alpha|}\int_U u(\mathbf{y})\int_{U_0}\phi_l(\mathbf{x}-\mathbf{y})(D^{\alpha}\psi)(\mathbf{x})\,dxdy. \end{aligned}$$

It follows that $D^{\alpha}(\widetilde{u}*\phi_l)=\left(\widetilde{D^{\alpha}u}*\phi_l\right)$ as weak derivatives defined on $C_c^{\infty}(U_0)$. Therefore,

$$\begin{aligned} \left|\left|D^{\alpha}(\widetilde{u}*\phi_l)-D^{\alpha}u\right|\right|_{L^p(U_0)} &= \left|\left|\widetilde{D^{\alpha}u}*\phi_l-D^{\alpha}u\right|\right|_{L^p(U_0)} \\ &\leq \left|\left|\widetilde{D^{\alpha}u}*\phi_l-\widetilde{D^{\alpha}u}\right|\right|_{L^p(\mathbb{R}^n)}\to 0. \end{aligned}$$

This proves the theorem.

As part of the proof of the theorem, the following corollary was established.

Corollary 40.6 *Let U_0 and U be as in the above theorem. Then for all l large enough and ϕ_l a mollifier,*

$$D^{\alpha}(\widetilde{u}*\phi_l)=\left(\widetilde{D^{\alpha}u}*\phi_l\right) \tag{40.2}$$

as distributions on $C_c^{\infty}(U_0)$.

Definition 40.7 *Let U be an open set. $C^{\infty}(U)$ denotes the set of functions which are defined and infinitely differentiable on U.*

Note that $f(x) = \frac{1}{x}$ is a function in $C^{\infty}(0,1)$. However, it is not equal to the restriction to $(0,1)$ of some function which is in $C_c^{\infty}(\mathbb{R})$. This illustrates the distinction between $C^{\infty}(U)$ and $C^{\infty}(\overline{U})$. The set, $C^{\infty}(\overline{U})$ is a subset of $C^{\infty}(U)$. The following theorem is known as the Meyer Serrin theorem.

Theorem 40.8 *(Meyer Serrin) Let U be an open subset of $\mathbb{R}^n$. Then if $\delta > 0$ and $u \in X^{m,p}(U)$, there exists $J \in C^{\infty}(U)$ such that $||J - u||_{m,p,U} < \delta$.*

Proof: Let $\cdots U_k \subseteq \overline{U_k} \subseteq U_{k+1} \cdots$ be a sequence of open subsets of U whose union equals U such that $\overline{U_k}$ is compact for all k. Also let $U_{-3} = U_{-2} = U_{-1} = U_0 = \emptyset$. Now define $V_k \equiv U_{k+1} \setminus \overline{U_{k-1}}$. Thus $\{V_k\}_{k=1}^{\infty}$ is an open cover of U. Note the open cover is locally finite and therefore, there exists a partition of unity subordinate to this open cover, $\{\eta_k\}_{k=1}^{\infty}$ such that each $\text{spt}(\eta_k) \in C_c(V_k)$. Let ψ_m denote the sum of all the η_k which are non zero at some point of V_m. Thus

$$\text{spt}(\psi_m) \subseteq U_{m+2} \setminus \overline{U_{m-2}}, \psi_m \in C_c^{\infty}(U), \sum_{m=1}^{\infty} \psi_m(\mathbf{x}) = 1 \tag{40.3}$$

for all $\mathbf{x} \in U$, and $\psi_m u \in W^{m,p}(U_{m+2})$.

Now let ϕ_l be a mollifier and consider

$$J \equiv \sum_{m=0}^{\infty} u\psi_m * \phi_{l_m} \tag{40.4}$$

where l_m is chosen large enough that the following two conditions hold:

$$\text{spt}\left(u\psi_m * \phi_{l_m}\right) \subseteq U_{m+3} \setminus \overline{U_{m-3}}, \tag{40.5}$$

$$\left|\left|(u\psi_m) * \phi_{l_m} - u\psi_m\right|\right|_{m,p,U_{m+3}} = \left|\left|(u\psi_m) * \phi_{l_m} - u\psi_m\right|\right|_{m,p,U} < \frac{\delta}{2^{m+5}}, \tag{40.6}$$

where 40.6 is obtained from Theorem 40.5. Because of 40.3 only finitely many terms of the series in 40.4 are nonzero and therefore, $J \in C^{\infty}(U)$. Now let $N > 10$, some large value.

$$\begin{aligned} ||J - u||_{m,p,U_{N-3}} &= \left|\left|\sum_{k=0}^{N} \left(u\psi_k * \phi_{l_k} - u\psi_k\right)\right|\right|_{m,p,U_{N-3}} \\ &\leq \sum_{k=0}^{N} \left|\left|u\psi_k * \phi_{l_k} - u\psi_k\right|\right|_{m,p,U_{N-3}} \\ &\leq \sum_{k=0}^{N} \frac{\delta}{2^{m+5}} < \delta. \end{aligned}$$

Now apply the monotone convergence theorem to conclude that $||J - u||_{m,p,U} \leq \delta$. This proves the theorem.

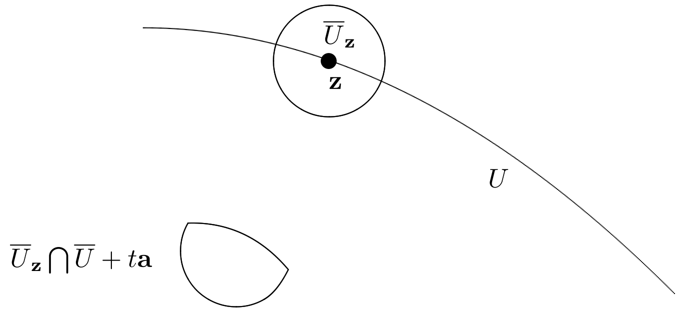
Note that $J = 0$ on ∂U. Later on, you will see that this is pathological.

In the study of partial differential equations it is the space $W^{m,p}(U)$ which is of the most use, not the space $X^{m,p}(U)$. This is because of the density of $C^{\infty}\left(\overline{U}\right)$. Nevertheless, for reasonable open sets, U, the two spaces coincide.

Definition 40.9 *An open set, $U \subseteq \mathbb{R}^n$ is said to satisfy the segment condition if for all $\mathbf{z} \in \overline{U}$, there exists an open set $U_{\mathbf{z}}$ containing $\mathbf{z}$ and a vector $\mathbf{a}$ such that*

$$\overline{U} \cap \overline{U}_{\mathbf{z}} + t\mathbf{a} \subseteq U$$

for all $t \in (0,1)$.



You can imagine open sets which do not satisfy the segment condition. For example, a pair of circles which are tangent at their boundaries. The condition in the above definition breaks down at their point of tangency.

Here is a simple lemma which will be used in the proof of the following theorem.

Lemma 40.10 *If $u \in W^{m,p}(U)$ and $\psi \in C_c^{\infty}(\mathbb{R}^n)$, then $u\psi \in W^{m,p}(U)$.*

Proof: Let $|\alpha| \leq m$ and let $\phi \in C_c^{\infty}(U)$. Then

$$\begin{aligned}
(D_{x_i}(u\psi))(\phi) &\equiv -\int_U u\psi\phi_{,x_i}dx \\
&= -\int_U u\left((\psi\phi)_{,x_i} - \phi\psi_{,x_i}\right)dx \\
&= (D_{x_i}u)(\psi\phi) + \int_U u\psi_{,x_i}\phi dx \\
&= \int_U \left(\psi D_{x_i}u + u\psi_{,x_i}\right)\phi dx
\end{aligned}$$

Therefore, $D_{x_i}(u\psi) = \psi D_{x_i}u + u\psi_{,x_i} \in L^p(U)$. In other words, the product rule holds. Now considering the terms in the last expression, you can do the same argument with each of these as long as they all have derivatives in $L^p(U)$. Therefore, continuing this process the lemma is proved.

Theorem 40.11 *Let U be an open set and suppose there exists a locally finite covering*[2] *of $\overline{U}$ which is of the form $\{U_i\}_{i=1}^{\infty}$ such that each U_i is a bounded open set which satisfies the conditions of Definition 40.9. Thus there exist vectors, $\mathbf{a}_i$ such that for all $t \in (0,1)$,*

$$\overline{U_i} \cap U + t\mathbf{a}_i \subseteq U.$$

Then $C^{\infty}\left(\overline{U}\right)$ is dense in $X^{m,p}(U)$ and so $W^{m,p}(U) = X^{m,p}(U)$.

Proof: Let $\{\psi_i\}_{i=1}^{\infty}$ be a partition of unity subordinate to the given open cover with $\psi_i \in C_c^{\infty}(U_i)$ and let $u \in X^{m,p}(U)$. Thus

$$u = \sum_{k=1}^{\infty} \psi_k u.$$

Consider U_k for some k. Let $\mathbf{a}_k$ be the special vector associated with U_k such that

$$t\mathbf{a}_k + \overline{U} \cap \overline{U_k} \subseteq U \tag{40.7}$$

for all $t \in (0,1)$ and consider only t small enough that

$$\operatorname{spt}(\psi_k) - t\mathbf{a}_k \subseteq U_k \tag{40.8}$$

Pick $l(t) > 1/t$ which is also large enough that

$$t\mathbf{a}_k + \overline{U} \cap \overline{U_k} + B\left(\mathbf{0}, \frac{1}{l(t)}\right) \subseteq U, \ \overline{\operatorname{spt}(\psi_k) + B\left(\mathbf{0}, \frac{1}{l(t_k)}\right)} - t\mathbf{a}_k \subseteq U_k. \tag{40.9}$$

This can be done because $t\mathbf{a}_k + \overline{U} \cap \overline{U_k}$ is a compact subset of U and so has positive distance to U^C and $\operatorname{spt}(\psi_k) - t\mathbf{a}_k$ is a compact subset of U_k having positive distance to U_k^C. Let t_k be such a value for t and for ϕ_l a mollifier, define

$$v_{t_k}(\mathbf{x}) \equiv \int_{\mathbb{R}^n} \widetilde{u}(\mathbf{x} + t_k\mathbf{a}_k - \mathbf{y})\, \psi_k(\mathbf{x} + t_k\mathbf{a}_k - \mathbf{y})\, \phi_{l(t_k)}(\mathbf{y})\, dy \tag{40.10}$$

where as usual, $\widetilde{u}$ is the zero extention of u off U. For $v_{t_k}(\mathbf{x}) \neq 0$, it is necessary that $\mathbf{x} + t_k\mathbf{a}_k - \mathbf{y} \in \operatorname{spt}(\psi_k)$ for some $\mathbf{y} \in B\left(\mathbf{0}, \frac{1}{l(t_k)}\right)$. Therefore, using 40.9, for $v_{t_k}(\mathbf{x}) \neq 0$, it is necessary that

$$\mathbf{x} \in \mathbf{y} - t_k\mathbf{a}_k + U \cap \operatorname{spt}(\psi_k) \subseteq B\left(\mathbf{0}, \frac{1}{l(t_k)}\right) + \operatorname{spt}(\psi_k) - t_k\mathbf{a}_k$$

$$\subseteq \overline{B\left(\mathbf{0}, \frac{1}{l(t_k)}\right) + \operatorname{spt}(\psi_k) - t_k\mathbf{a}_k} \subseteq U_k$$

[2] This is never a problem in $\mathbb{R}^n$. In fact, every open covering has a locally finite subcovering in $\mathbb{R}^n$ or more generally in any metric space due to Stone's theorem. These are issues best left to you in case you are interested. I am usually interested in bounded sets, U, and for these, there is a finite covering.

showing that v_{t_k} has compact support in U_k. Now change variables in 40.10 to obtain

$$v_{t_k}(\mathbf{x}) \equiv \int_{\mathbb{R}^n} \widetilde{u}(\mathbf{y})\,\psi_k(\mathbf{y})\,\phi_{l(t_k)}(\mathbf{x}+t_k\mathbf{a}_k-\mathbf{y})\,dy. \tag{40.11}$$

For $\mathbf{x} \in U \cap U_k$, the above equals zero unless

$$\mathbf{y} - t_k\mathbf{a}_k - \mathbf{x} \in B\left(\mathbf{0}, \frac{1}{l(t_k)}\right)$$

which implies by 40.9 that

$$\mathbf{y} \in t_k\mathbf{a}_k + U \cap U_k + B\left(\mathbf{0}, \frac{1}{l(t_k)}\right) \subseteq U$$

Therefore, for such $\mathbf{x} \in U \cap U_k$,40.11 reduces to

$$\begin{aligned} v_{t_k}(\mathbf{x}) &= \int_{\mathbb{R}^n} u(\mathbf{y})\,\psi_k(\mathbf{y})\,\phi_{l(t_k)}(\mathbf{x}+t_k\mathbf{a}_k-\mathbf{y})\,dy \\ &= \int_U u(\mathbf{y})\,\psi_k(\mathbf{y})\,\phi_{l(t_k)}(\mathbf{x}+t_k\mathbf{a}_k-\mathbf{y})\,dy. \end{aligned}$$

It follows that for $|\alpha| \leq m$, and $\mathbf{x} \in U \cap U_k$

$$\begin{aligned} D^\alpha v_{t_k}(\mathbf{x}) &= \int_U u(\mathbf{y})\,\psi_k(\mathbf{y})\,D^\alpha\phi_{l(t_k)}(\mathbf{x}+t_k\mathbf{a}_k-\mathbf{y})\,dy \\ &= \int_U D^\alpha(u\psi_k)(\mathbf{y})\,\phi_{l(t_k)}(\mathbf{x}+t_k\mathbf{a}_k-\mathbf{y})\,dy \\ &= \int_{\mathbb{R}^n} \widetilde{D^\alpha(u\psi_k)}(\mathbf{y})\,\phi_{l(t_k)}(\mathbf{x}+t_k\mathbf{a}_k-\mathbf{y})\,dy \\ &= \int_{\mathbb{R}^n} \widetilde{D^\alpha(u\psi_k)}(\mathbf{x}+t_k\mathbf{a}_k-\mathbf{y})\,\phi_{l(t_k)}(\mathbf{y})\,dy. \end{aligned} \tag{40.12}$$

Actually, this formula holds for all $\mathbf{x} \in U$. If $\mathbf{x} \in U$ but $\mathbf{x} \notin U_k$, then the left side of the above formula equals zero because, as noted above, $\text{spt}(v_{t_k}) \subseteq U_k$. The integrand of the right side equals zero unless

$$\mathbf{x} \in B\left(\mathbf{0},\frac{1}{l(t_k)}\right) + \text{spt}(\psi_k) - t_k\mathbf{a}_k \subseteq U_k$$

by 40.9 and here $\mathbf{x} \notin U_k$.

Next an estimate is obtained for $||D^\alpha v_{t_k} - D^\alpha(u\psi_k)||_{L^p(U)}$. By 40.12,

$$\begin{aligned} &||D^\alpha v_{t_k} - D^\alpha(u\psi_k)||_{L^p(U)} \\ \leq\ & \left(\int_U \left(\int_{\mathbb{R}^n} \left|\widetilde{D^\alpha(u\psi_k)}(\mathbf{x}+t_k\mathbf{a}_k-\mathbf{y}) - \widetilde{D^\alpha(u\psi_k)}(\mathbf{x})\right| \phi_{l(t_k)}(\mathbf{y})\,dy\right)^p dx\right)^{1/p} \\ \leq\ & \int_{\mathbb{R}^n} \phi_{l(t_k)}(\mathbf{y}) \left(\int_U \left|\widetilde{D^\alpha(u\psi_k)}(\mathbf{x}+t_k\mathbf{a}_k-\mathbf{y}) - \widetilde{D^\alpha(u\psi_k)}(\mathbf{x})\right|^p dx\right)^{1/p} dy \\ \leq\ & \frac{\varepsilon}{2^k} \end{aligned}$$

whenever t_k is taken small enough. Pick t_k this small and let $w_k \equiv v_{t_k}$. Thus

$$||D^\alpha w_k - D^\alpha (u\psi_k)||_{L^p(U)} \leq \frac{\varepsilon}{2^k}$$

and $w_k \in C_c^\infty(\mathbb{R}^n)$. Now let

$$J(\mathbf{x}) \equiv \sum_{k=1}^{\infty} w_k.$$

Since the U_k are locally finite and $\operatorname{spt}(w_k) \subseteq U_k$ for each k, it follows $D^\alpha J = \sum_{k=0}^{\infty} D^\alpha w_k$ and the sum is always finite. Similarly,

$$D^\alpha \sum_{k=1}^{\infty} (\psi_k u) = \sum_{k=1}^{\infty} D^\alpha (\psi_k u)$$

and the sum is always finite. Therefore,

$$\begin{aligned} ||D^\alpha J - D^\alpha u||_{L^p(U)} &= \left|\left|\sum_{k=1}^{\infty} D^\alpha w_k - D^\alpha (\psi_k u)\right|\right|_{L^p(U)} \\ &\leq \sum_{k=1}^{\infty} ||D^\alpha w_k - D^\alpha (\psi_k u)||_{L^p(U)} \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon. \end{aligned}$$

By choosing t_k small enough, such an inequality can be obtained for

$$\left|\left|D^\beta J - D^\beta u\right|\right|_{L^p(U)}$$

for each multi index, β such that $|\beta| \leq m$. Therefore, there exists $J \in C_c^\infty(\mathbb{R}^n)$ such that

$$||J - u||_{W^{m,p}(U)} \leq \varepsilon K$$

where K equals the number of multi indices no larger than m. Since ε is arbitrary, this proves the theorem.

Corollary 40.12 *Let U be an open set which has the segment property. Then $W^{m,p}(U) = X^{m,p}(U)$.*

Proof: Start with an open covering of $\overline{U}$ whose sets satisfy the segment condition and obtain a locally finite refinement consisting of bounded sets which are of the sort in the above theorem.

Now consider a situation where $\mathbf{h} : U \to V$ where U and V are two open sets in $\mathbb{R}^n$ and $D^\alpha \mathbf{h}$ exists and is continuous and bounded if $|\alpha| < m - 1$ and $D^\alpha \mathbf{h}$ is Lipschitz if $|\alpha| = m - 1$.

Definition 40.13 *Whenever $\mathbf{h} : U \to V$, define $\mathbf{h}^*$ mapping the functions which are defined on V to the functions which are defined on U as follows.*

$$\mathbf{h}^* f(\mathbf{x}) \equiv f(\mathbf{h}(\mathbf{x})).$$

$\mathbf{h} : U \to V$ is bilipschitz if $\mathbf{h}$ is one to one, onto and Lipschitz and $\mathbf{h}^{-1}$ is also one to one, onto and Lipschitz.

Theorem 40.14 *Let* $\mathbf{h} : U \to V$ *be one to one and onto where* U *and* V *are two open sets. Also suppose that* $D^{\alpha}\mathbf{h}$ *and* $D^{\alpha}\left(\mathbf{h}^{-1}\right)$ *exist and are Lipschitz continuous if* $|\alpha| \leq m-1$ *for* m *a positive integer. Then*

$$\mathbf{h}^* : W^{m,p}(V) \to W^{m,p}(U)$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $\left(\mathbf{h}^{-1}\right)^*$.

Proof: It is clear that $\mathbf{h}^*$ is linear. It is required to show it is one to one and continuous. First suppose $\mathbf{h}^* f = 0$. Then

$$0 = \int_V |f(\mathbf{h}(\mathbf{x}))|^p \, dx$$

and so $f(\mathbf{h}(\mathbf{x})) = 0$ for a.e. $\mathbf{x} \in U$. Since $\mathbf{h}$ is Lipschitz, it takes sets of measure zero to sets of measure zero. Therefore, $f(\mathbf{y}) = 0$ a.e. This shows $\mathbf{h}^*$ is one to one.

By the Meyer Serrin theorem, Theorem 40.8, it suffices to verify that $\mathbf{h}^*$ is continuous on functions in $C^{\infty}(V)$. Let f be such a function. Then using the chain rule and product rule, $(\mathbf{h}^* f)_{,i}(\mathbf{x}) = f_{,k}(\mathbf{h}(\mathbf{x})) h_{k,i}(\mathbf{x})$,

$$\begin{aligned}
(\mathbf{h}^* f)_{,ij}(\mathbf{x}) &= \left(f_{,k}(\mathbf{h}(\mathbf{x})) h_{k,i}(\mathbf{x})\right)_{,j} \\
&= f_{,kl}(\mathbf{h}(\mathbf{x})) h_{l,j}(\mathbf{x}) h_{k,i}(\mathbf{x}) + f_{,k}(\mathbf{h}(\mathbf{x})) h_{k,ij}(\mathbf{x})
\end{aligned}$$

etc. In general, for $|\alpha| \leq m-1$, succsessive applications of the product rule and chain rule yield that $D^{\alpha}(\mathbf{h}^* f)(\mathbf{x})$ has the form

$$D^{\alpha}(\mathbf{h}^* f)(\mathbf{x}) = \sum_{|\beta| \leq |\alpha|} \mathbf{h}^* \left(D^{\beta} f\right)(\mathbf{x}) \, g_{\beta}(\mathbf{x})$$

where g_{β} is a bounded Lipschitz function with Lipschitz constant dependent on $\mathbf{h}$ and its derivatives. It only remains to take one more derivative of the functions, $D^{\alpha} f$ for $|\alpha| = m-1$. This can be done again but this time you have to use Rademacher's theorem which assures you that the derivative of a Lipschitz function exists a.e. in order to take the partial derivative of the $g_{\beta}(\mathbf{x})$. When this is done, the above formula remains valid for all $|\alpha| \leq m$. Therefore, using the change of variables formula for multiple integrals, Corollary 37.31 on Page 1101,

$$\begin{aligned}
\int_U |D^{\alpha}(\mathbf{h}^* f)(\mathbf{x})|^p \, dx &\leq C_{m,p,\mathbf{h}} \sum_{|\beta| \leq m} \int_U \left|\mathbf{h}^* \left(D^{\beta} f\right)(\mathbf{x})\right|^p dx \\
&= C_{m,p,\mathbf{h}} \sum_{|\beta| \leq m} \int_U \left|\left(D^{\beta} f\right)(\mathbf{h}(\mathbf{x}))\right|^p dx \\
&= C_{m,p,\mathbf{h}} \sum_{|\beta| \leq m} \int_V \left|\left(D^{\beta} f\right)(\mathbf{y})\right|^p \left|\det D\mathbf{h}^{-1}(\mathbf{y})\right| dy \\
&\leq C_{m,p,\mathbf{h},\mathbf{h}^{-1}} \, ||f||_{m,p,V}
\end{aligned}$$

This shows $\mathbf{h}^*$ is continuous on $C^{\infty}(V) \cap W^{m,p}(U)$ and since this set is dense, this proves $\mathbf{h}^*$ is continuous. The same argument applies to $\left(\mathbf{h}^{-1}\right)^*$ and now the definitions of $\mathbf{h}^*$ and $\left(\mathbf{h}^{-1}\right)^*$ show these are inverses.

40.1 Embedding Theorems For $W^{m,p}(\mathbb{R}^n)$

Recall Theorem 37.15 which is listed here for convenience.

Theorem 40.15 *Suppose $u, u_{,i} \in L^p_{loc}(\mathbb{R}^n)$ for $i = 1, \cdots, n$ and $p > n$. Then u has a representative, still denoted by u, such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,*

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq C \left(\int_{B(\mathbf{x}, 2|\mathbf{y}-\mathbf{x}|)} |\nabla u|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|^{(1-n/p)}. \tag{40.13}$$

This amazing result shows that every $u \in W^{m,p}(\mathbb{R}^n)$ has a representative which is continuous provided $p > n$.

Using the above inequality, one can give an important embedding theorem.

Definition 40.16 *Let X, Y be two Banach spaces and let $f : X \to Y$ be a function. Then f is a compact map if whenever S is a bounded set in X, it follows that $f(S)$ is precompact in Y.*

Theorem 40.17 *Let U be a bounded open set and for u a function defined on $\mathbb{R}^n$, let $r_U u(\mathbf{x}) \equiv u(\mathbf{x})$ for $\mathbf{x} \in \overline{U}$. Then if $p > n$, $r_U : W^{1,p}(\mathbb{R}^n) \to C(\overline{U})$ is continuous and compact.*

Proof: First suppose $u_k \to 0$ in $W^{1,p}(\mathbb{R}^n)$. Then if $r_U u_k$ does not converge to 0, it follows there exists a sequence, still denoted by k and $\varepsilon > 0$ such that $u_k \to 0$ in $W^{1,p}(\mathbb{R}^n)$ but $||r_U u_k||_\infty \geq \varepsilon$. Selecting a further subsequence which is still denoted by k, you can also assume $u_k(\mathbf{x}) \to 0$ a.e. Pick such an $\mathbf{x}_0 \in U$ where this convergence takes place. Then from 40.13, for all $\mathbf{x} \in \overline{U}$,

$$|u_k(\mathbf{x})| \leq |u_k(\mathbf{x}_0)| + C\,||u_k||_{1,p,\mathbb{R}^n} \operatorname{diam}(U)$$

showing that u_k converges uniformly to 0 on $\overline{U}$ contrary to $||r_U u_k||_\infty \geq \varepsilon$. Therefore, r_U is continuous as claimed.

Next let S be a bounded subset of $W^{1,p}(\mathbb{R}^n)$ with $||u||_{1,p} < M$ for all $u \in S$. Then for $u \in S$

$$r^p m_n([|u| > r] \cap U) \leq \int_{[|u|>r]\cap U} |u|^p \, dm_n \leq M^p$$

and so

$$m_n([|u| > r] \cap U) \leq \frac{M^p}{r^p}.$$

Now choosing r large enough, $M^p/r^p < m_n(U)$ and so, for such r, there exists $\mathbf{x}_u \in U$ such that $|u(\mathbf{x}_u)| \leq r$. Therefore from 40.13, whenever $\mathbf{x} \in U$,

$$\begin{aligned} |u(\mathbf{x})| &\leq |u(\mathbf{x}_u)| + CM \operatorname{diam}(U)^{1-n/p} \\ &\leq r + CM \operatorname{diam}(U)^{1-n/p} \end{aligned}$$

showing that $\{r_U u : u \in S\}$ is uniformly bounded. But also, for $\mathbf{x},\mathbf{y} \in \overline{U}$,40.13 implies

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq CM|\mathbf{x}-\mathbf{y}|^{1-\frac{n}{p}}$$

showing that $\{r_U u : u \in S\}$ is equicontinuous. By the Ascoli Arzela theorem, it follows $r_U(S)$ is precompact and so r_U is compact.

Definition 40.18 *Let $\alpha \in (0,1]$ and K a compact subset of $\mathbb{R}^n$*

$$C^\alpha(K) \equiv \{f \in C(K) : \rho_\alpha(f) + ||f|| \equiv ||f||_\alpha < \infty\}$$

where

$$||f|| \equiv ||f||_\infty \equiv \sup\{|f(\mathbf{x})| : \mathbf{x} \in K\}$$

and

$$\rho_\alpha(f) \equiv \sup\left\{\frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\alpha} : \mathbf{x},\mathbf{y} \in K,\ \mathbf{x} \neq \mathbf{y}\right\}.$$

Then $(C^\alpha(K), ||\cdot||_\alpha)$ is a complete normed linear space called a Holder space.

The verification that this is a complete normed linear space is routine and is left for you. More generally, one considers the following class of Holder spaces.

Definition 40.19 *Let K be a compact subset of $\mathbb{R}^n$ and let $\lambda \in (0,1]$. $C^{m,\lambda}(K)$ denotes the set of functions, u which are restrictions of functions defined on $\mathbb{R}^n$ to Ksuch that for $|\alpha| \leq m$,*

$$D^\alpha u \in C(K)$$

and if $|\alpha| = m$,

$$D^\alpha u \in C^\lambda(K).$$

Thus $C^{0,\lambda}(K) = C^\lambda(K)$. The norm of a function in $C^{m,\lambda}(K)$ is given by

$$||u||_{m,\lambda} \equiv \sup_{|\alpha|=m} \rho_\lambda(D^\alpha u) + \sum_{|\alpha|\leq m} ||D^\alpha u||_\infty.$$

Lemma 40.20 *Let m be a positive integer, K a compact subset of $\mathbb{R}^n$, and let $0 < \beta < \lambda \leq 1$. Then the identity map from $C^{m,\lambda}(K)$ into $C^{m,\beta}(K)$ is compact.*

Proof: First note that the containment is obvious because for any function, f, if

$$\rho_\lambda(f) \equiv \sup\left\{\frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\lambda} : \mathbf{x},\mathbf{y} \in K,\ \mathbf{x} \neq \mathbf{y}\right\} < \infty,$$

Then

$$\begin{aligned}
\rho_\beta(f) &\equiv \sup\left\{\frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\beta} : \mathbf{x},\mathbf{y} \in K,\ \mathbf{x} \neq \mathbf{y}\right\} \\
&= \sup\left\{\frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\lambda}|\mathbf{x}-\mathbf{y}|^{\lambda-\beta} : \mathbf{x},\mathbf{y} \in K,\ \mathbf{x} \neq \mathbf{y}\right\} \\
&\leq \sup\left\{\frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\lambda}\operatorname{diam}(K)^{\lambda-\beta} : \mathbf{x},\mathbf{y} \in K,\ \mathbf{x} \neq \mathbf{y}\right\} < \infty.
\end{aligned}$$

Suppose the identity map, id, is not compact. Then there exists $\varepsilon > 0$ and a sequence, $\{f_k\}_{k=1}^{\infty} \subseteq C^{m,\lambda}(K)$ such that $||f_k||_{m,\lambda} < M$ for all k but $||f_k - f_l||_{\beta} \geq \varepsilon$ whenever $k \neq l$. By the Ascoli Arzela theorem, there exists a subsequence of this, still denoted by f_k such that $\sum_{|\alpha|\leq m} ||D^{\alpha}(f_l - f_k)||_{\infty} < \delta$ where δ satisfies

$$0 < \delta < \min\left(\frac{\varepsilon}{2}, \left(\frac{\varepsilon}{8}\right)\left(\frac{\varepsilon}{8M}\right)^{\beta/(\lambda-\beta)}\right). \tag{40.14}$$

Therefore, $\sup_{|\alpha|=m} \rho_{\beta}(D^{\alpha}(f_k - f_l)) \geq \varepsilon - \delta$ for all $k \neq l$. It follows that there exist pairs of points and a multi index, α with $|\alpha| = m$, $\{\mathbf{x}_{kl}, \mathbf{y}_{kl}, \alpha\}$ such that

$$\frac{\varepsilon - \delta}{2} < \frac{|(D^{\alpha}f_k - D^{\alpha}f_l)(\mathbf{x}_{kl}) - ((D^{\alpha}f_k - D^{\alpha}f_l)(\mathbf{y}_{kl}))|}{|\mathbf{x}_{kl} - \mathbf{y}_{kl}|^{\beta}} \leq 2M |\mathbf{x}_{kl} - \mathbf{y}_{kl}|^{\lambda-\beta} \tag{40.15}$$

and so considering the ends of the above inequality,

$$\left(\frac{\varepsilon - \delta}{4M}\right)^{1/(\lambda-\beta)} < |\mathbf{x}_{kl} - \mathbf{y}_{kl}|.$$

Now also, since $\sum_{|\alpha|\leq m} ||D^{\alpha}(f_l - f_k)||_{\infty} < \delta$, it follows from the first inequality in 40.15 that

$$\frac{\varepsilon - \delta}{2} < \frac{2\delta}{\left(\frac{\varepsilon-\delta}{4M}\right)^{\beta/(\lambda-\beta)}}.$$

Since $\delta < \varepsilon/2$, this implies

$$\frac{\varepsilon}{4} < \frac{2\delta}{\left(\frac{\varepsilon}{8M}\right)^{\beta/(\lambda-\beta)}}$$

and so

$$\left(\frac{\varepsilon}{8}\right)\left(\frac{\varepsilon}{8M}\right)^{\beta/(\lambda-\beta)} < \delta$$

contrary to 40.14. This proves the lemma.

Corollary 40.21 *Let $p > n, U$ and r_U be as in Theorem 40.17 and let m be a nonnegative integer. Then $r_U : W^{m+1,p}(\mathbb{R}^n) \to C^{m,\lambda}(\overline{U})$ is continuous as a map into $C^{m,\lambda}(\overline{U})$ for all $\lambda \in [0, 1 - \frac{n}{p}]$ and r_U is compact if $\lambda < 1 - \frac{n}{p}$.*

Proof: Suppose $u_k \to 0$ in $W^{m+1,p}(\mathbb{R}^n)$. Then from 40.13, if $\lambda \leq 1 - \frac{n}{p}$ and $|\alpha| = m$

$$\rho_{\lambda}(D^{\alpha}u_k) \leq C ||D^{\alpha}u_k||_{1,p} \operatorname{diam}(U)^{1-\frac{n}{p}-\lambda}.$$

Therefore, $\rho_{\lambda}(D^{\alpha}u_k) \to 0$. From Theorem 40.17 it follows that for $|\alpha| \leq m$,

$$||D^{\alpha}u_k||_{\infty} \to 0$$

and so $||u_k||_{m,\lambda} \to 0$. This proves the claim about continuity. The claim about compactness for $\lambda < 1 - \frac{n}{p}$ follows from Lemma 40.20 and this.

(Bounded in $W^{m,p}(\mathbb{R}^n) \xrightarrow{r_U}$ Bounded in $C^{m,1-\frac{n}{p}}(\overline{U}) \xrightarrow{\text{id}}$ Compact in $C^{m,\lambda}(\overline{U})$.)

It is just as important to consider the case where $p < n$. To do this case the following lemma due to Gagliardo [23] will be of interest. See also [1].

Lemma 40.22 *Suppose $n \geq 2$ and w_j does not depend on the j^{th} component of $\mathbf{x}$, x_j. Then*

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} |w_j(\mathbf{x})|\, dm_n \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^{n-1}} |w_j(\mathbf{x})|^{n-1} dm_{n-1} \right)^{1/(n-1)}.$$

In this inequality, assume all the functions are continuous so there can be no measurability questions.

Proof: First note that for $n = 2$ the inequality reduces to the statement

$$\int \int |w_1(x_2)| |w_2(x_1)|\, dx_1 dx_2 \leq \int |w_1(x_2)|\, dx_2 \int |w_2(x_1)|\, dx_1$$

which is obviously true. Suppose then that the inequality is valid for some n. Using Fubini's theorem, Holder's inequality, and the induction hypothesis,

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}} \prod_{j=1}^{n+1} |w_j(\mathbf{x})|\, dm_{n+1} \\
= &\int_{\mathbb{R}} \int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})| \prod_{j=1}^{n} |w_j(\mathbf{x})|\, dm_n dx_{n+1} \\
= &\int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})| \int_{\mathbb{R}} \prod_{j=1}^{n} |w_j(\mathbf{x})|\, dx_{n+1} dm_n \\
= &\int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})| \left(\prod_{j=1}^{n} \int_{\mathbb{R}} |w_j(\mathbf{x})|^n dx_{n+1} \right)^{1/n} dm_n \\
= &\int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})| \prod_{j=1}^{n} \left(\int_{\mathbb{R}} |w_j(\mathbf{x})|^n dx_{n+1} \right)^{1/n} dm_n
\end{aligned}$$

$$\begin{aligned}
\leq &\left(\int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})|^n dm_n \right)^{1/n} \cdot \\
&\left(\int_{\mathbb{R}^n} \left(\prod_{j=1}^{n} \left(\int_{\mathbb{R}} |w_j(\mathbf{x})|^n dx_{n+1} \right)^{1/n} \right)^{n/(n-1)} dm_n \right)^{(n-1)/n} \\
= &\left(\int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})|^n dm_n \right)^{1/n} \cdot \\
&\left(\int_{\mathbb{R}^n} \prod_{j=1}^{n} \left(\int_{\mathbb{R}} |w_j(\mathbf{x})|^n dx_{n+1} \right)^{1/(n-1)} dm_n \right)^{(n-1)/n}
\end{aligned}$$

$$\leq \left(\int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})|^n \, dm_n\right)^{1/n} \cdot$$
$$\left(\prod_{j=1}^{n} \left(\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} |w_j(\mathbf{x})|^n \, dx_{n+1}\right) dm_{n-1}\right)^{1/(n-1)}\right)^{(n-1)/n}$$
$$= \left(\int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})|^n \, dm_n\right)^{1/n} \prod_{j=1}^{n} \left(\int_{\mathbb{R}^n} |w_j(\mathbf{x})|^n \, dm_n\right)^{1/n}$$
$$= \prod_{j=1}^{n+1} \left(\int_{\mathbb{R}^n} |w_j(\mathbf{x})|^n \, dm_n\right)^{1/n}$$

This proves the lemma.

Lemma 40.23 *If $\phi \in C_c^\infty(\mathbb{R}^n)$ and $n \geq 1$, then*

$$||\phi||_{n/(n-1)} \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^{n} \left|\left|\frac{\partial \phi}{\partial x_j}\right|\right|_1.$$

Proof: The case where $n = 1$ is obvious if $n/(n-1)$ is interpreted as ∞. Assume then that $n > 1$ and note that for $a_i \geq 0$,

$$n \prod_{i=1}^{n} a_i \leq \left(\sum_{j=1}^{n} a_i\right)^n$$

In fact, the term on the left is one of many terms of the expression on the right. Therefore, taking n^{th} roots

$$\prod_{i=1}^{n} a_i^{1/n} \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^{n} a_i.$$

Then observe that for each $j = 1, 2, \cdots, n$,

$$|\phi(\mathbf{x})| \leq \int_{-\infty}^{\infty} |\phi_{,j}(\mathbf{x})| \, dx_j$$

so

$$||\phi||_{n/(n-1)}^{n/(n-1)} \equiv \int_{\mathbb{R}^n} |\phi(\mathbf{x})|^{n/(n-1)} \, dm_n$$
$$\leq \int_{\mathbb{R}^n} \prod_{j=1}^{n} \left(\int_{-\infty}^{\infty} |\phi_{,j}(\mathbf{x})| \, dx_j\right)^{1/(n-1)} dm_n$$

and from Lemma 40.22 this is dominated by

$$\leq \prod_{j=1}^{n} \left(\int_{\mathbb{R}^n} |\phi_{,j}(\mathbf{x})| \, dm_n\right)^{1/(n-1)}.$$

Hence $\prod_{i=1}^n a_i^{1/n} \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^n a_i$

$$\begin{aligned}
||\phi||_{n/(n-1)} &\leq \prod_{j=1}^n \left(\int_{\mathbb{R}^n} \left|\phi_{,j}(\mathbf{x})\right| dm_n\right)^{1/n} \\
&\leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^n \int_{\mathbb{R}^n} \left|\phi_{,j}(\mathbf{x})\right| dm_n \\
&= \frac{1}{\sqrt[n]{n}} \sum_{j=1}^n \left|\left|\phi_{,i}\right|\right|_1
\end{aligned}$$

and this proves the lemma.

The above lemma is due to Gagliardo and Nirenberg.

With this lemma, it is possible to prove a major embedding theorem which follows.

Theorem 40.24 *Let $1 \leq p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. Then if $f \in W^{1,p}(\mathbb{R}^n)$,*

$$||f||_q \leq \frac{1}{\sqrt[n]{n}} \frac{(n-1)p}{n-p} ||f||_{1,p,\mathbb{R}^n}.$$

Proof: From the definition of $W^{1,p}(\mathbb{R}^n)$, $C_c^1(\mathbb{R}^n)$ is dense in $W^{1,p}$. Here $C_c^1(\mathbb{R}^n)$ is the space of continuous functions having continuous derivatives which have compact support. The desired inequality will be established for such ϕ and then the density of this set in $W^{1,p}(\mathbb{R}^n)$ will be exploited to obtain the inequality for all $f \in W^{1,p}(\mathbb{R}^n)$. First note that the case where $p = 1$ follows immediately from the above lemma and so it is only necessary to consider the case where $p > 1$.

Let $\phi \in C_c^1(\mathbb{R}^n)$ and consider $|\phi|^r$ where $r > 1$. Then a short computation shows $|\phi|^r \in C_c^1(\mathbb{R}^n)$ and

$$\left||\phi|^r_{,i}\right| = r|\phi|^{r-1}\left|\phi_{,i}\right|.$$

Therefore, from Lemma 40.23,

$$\begin{aligned}
&\left(\int |\phi|^{\frac{rn}{n-1}} dm_n\right)^{(n-1)/n} \\
\leq\ & \frac{r}{\sqrt[n]{n}} \sum_{i=1}^n \int |\phi|^{r-1}\left|\phi_{,i}\right| dm_n \\
\leq\ & \frac{r}{\sqrt[n]{n}} \sum_{i=1}^n \left(\int \left|\phi_{,i}\right|^p\right)^{1/p} \left(\int \left(|\phi|^{r-1}\right)^{p/(p-1)} dm_n\right)^{(p-1)/p}.
\end{aligned}$$

Now choose r such that

$$\frac{(r-1)p}{p-1} = \frac{rn}{n-1}.$$

That is, let $r = \frac{p(n-1)}{n-p} > 1$ and so $\frac{rn}{n-1} = \frac{np}{n-p}$. Then this reduces to

$$\left(\int |\phi|^{\frac{np}{n-p}} dm_n \right)^{(n-1)/n} \leq \frac{r}{\sqrt[n]{n}} \sum_{i=1}^{n} \left(\int |\phi_{,i}|^p \right)^{1/p} \left(\int |\phi|^{\frac{np}{n-p}} dm_n \right)^{(p-1)/p}.$$

Also, $\frac{n-1}{n} - \frac{p-1}{p} = \frac{n-p}{np}$ and so, dividing both sides by the last term yields

$$\left(\int |\phi|^{\frac{np}{n-p}} dm_n \right)^{\frac{n-p}{np}} \leq \frac{r}{\sqrt[n]{n}} \sum_{i=1}^{n} \left(\int |\phi_{,i}|^p \right)^{1/p} \leq \frac{r}{\sqrt[n]{n}} ||\phi||_{1,p,\mathbb{R}^n}.$$

Letting $q = \frac{np}{n-p}$, it follows $\frac{1}{q} = \frac{n-p}{np} = \frac{1}{p} - \frac{1}{n}$ and

$$||\phi||_q \leq \frac{r}{\sqrt[n]{n}} ||\phi||_{1,p,\mathbb{R}^n}.$$

Now let $f \in W^{m,p}(\mathbb{R}^n)$ and let $||\phi_k - f||_{1,p,\mathbb{R}^n} \to 0$ as $k \to \infty$. Taking another subsequence, if necessary, you can also assume $\phi_k(\mathbf{x}) \to f(\mathbf{x})$ a.e. Therefore, by Fatou's lemma,

$$\begin{aligned} ||f||_q &\leq \liminf_{k\to\infty} \left(\int_{\mathbb{R}^n} |\phi_k(\mathbf{x})|^q dm_n \right)^{1/q} \\ &\leq \liminf_{k\to\infty} \frac{r}{\sqrt[n]{n}} ||\phi_k||_{1,p,\mathbb{R}^n} = ||f||_{1,p,\mathbb{R}^n}. \end{aligned}$$

This proves the theorem.

Corollary 40.25 *Suppose $mp < n$. Then $W^{m,p}(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n)$ where $q = \frac{np}{n-mp}$ and the identity map,* $\text{id} : W^{m,p}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ *is continuous.*

Proof: This is true if $m = 1$ according to Theorem 40.24. Suppose it is true for $m-1$ where $m > 1$. If $u \in W^{m,p}(\mathbb{R}^n)$ and $|\alpha| \leq 1$, then $D^\alpha u \in W^{m-1,p}(\mathbb{R}^n)$ so by induction, for all such α,

$$D^\alpha u \in L^{\frac{np}{n-(m-1)p}}(\mathbb{R}^n).$$

Thus $u \in W^{1,q_1}(\mathbb{R}^n)$ where

$$q_1 = \frac{np}{n-(m-1)p}$$

By Theorem 40.24, it follows that $u \in L^q(\mathbb{R}^n)$ where

$$\frac{1}{q} = \frac{n-(m-1)p}{np} - \frac{1}{n} = \frac{n-mp}{np}.$$

This proves the corollary.

There is another similar corollary of the same sort which is interesting and useful.

Corollary 40.26 *Suppose $m \geq 1$ and j is a nonnegative integer satisfying $jp < n$. Then*

$$W^{m+j,p}(\mathbb{R}^n) \subseteq W^{m,q}(\mathbb{R}^n)$$

for

$$q \equiv \frac{np}{n-jp} \tag{40.16}$$

and the identity map is continuous.

Proof: If $|\alpha| \leq m$, then $D^\alpha u \in W^{j,p}(\mathbb{R}^n)$ and so by Corollary 40.25, $D^\alpha u \in L^q(\mathbb{R}^n)$ where q is given above. This means $u \in W^{m,q}(\mathbb{R}^n)$.

The above corollaries imply yet another interesting corollary which involves embeddings in the Holder spaces.

Corollary 40.27 *Suppose $jp < n < (j+1)p$ and let m be a positive integer. Let U be any bounded open set in $\mathbb{R}^n$. Then letting r_U denote the restriction to $\overline{U}$, $r_U : W^{m+j,p}(\mathbb{R}^n) \to C^{m-1,\lambda}(\overline{U})$ is continuous for every $\lambda \leq \lambda_0 \equiv (j+1) - \frac{n}{p}$ and if $\lambda < (j+1) - \frac{n}{p}$, then r_U is compact.*

Proof: From Corollary 40.26 $W^{m+j,p}(\mathbb{R}^n) \subseteq W^{m,q}(\mathbb{R}^n)$ where q is given by 40.16. Therefore,

$$\frac{np}{n-jp} > n$$

and so by Corollary 40.21, $W^{m,q}(\mathbb{R}^n) \subseteq C^{m-1,\lambda}(\overline{U})$ for all λ satisfying

$$0 < \lambda < 1 - \frac{(n-jp)\,n}{np} = \frac{p(j+1)-n}{p} = (j+1) - \frac{n}{p}.$$

The assertion about compactness follows from the compactness of the embedding of $C^{m-1,\lambda_0}(\overline{U})$ into $C^{m-1,\lambda}(\overline{U})$ for $\lambda < \lambda_0$. See Lemma 40.20.

There are other embeddings of this sort available. You should see Adams [1] for a more complete listing of these. Next are some theorems about compact embeddings. This requires some consideration of which subsets of $L^p(U)$ are compact. The main theorem is the following. See [1].

Theorem 40.28 *Let K be a bounded subset of $L^p(U)$ and suppose that for all $\varepsilon > 0$, there exist a $\delta > 0$ such that if $|\mathbf{h}| < \delta$, then*

$$\int_{\mathbb{R}^n} |\widetilde{u}(\mathbf{x}+\mathbf{h}) - \widetilde{u}(\mathbf{x})|^p \, dx < \varepsilon^p \tag{40.17}$$

Suppose also that for each $\varepsilon > 0$ there exists an open set, $G \subseteq U$ such that $\overline{G}$ is compact and for all $u \in K$,

$$\int_{U \setminus \overline{G}} |u(\mathbf{x})|^p \, dx < \varepsilon^p \tag{40.18}$$

Then K is precompact in $L^p(\mathbb{R}^n)$.

Proof: To save fussing first consider the case where $U = \mathbb{R}^n$ so that $\widetilde{u} = u$. Suppose the two conditions hold and let ϕ_k be a mollifier of the form $\phi_k(\mathbf{x}) = k^n \phi(k\mathbf{x})$ where $\operatorname{spt}(\phi) \subseteq B(\mathbf{0}, 1)$. Consider

$$K_k \equiv \{u * \phi_k : u \in K\}.$$

and verify the conditions for the Ascoli Arzela theorem for these functions defined on $\overline{G}$. Say $||u||_p \leq M$ for all $u \in K$.

First of all, for $u \in K$ and $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned} |u * \phi_k(\mathbf{x})|^p &\leq \left(\int |u(\mathbf{x}-\mathbf{y})\,\phi_k(\mathbf{y})|\,dy\right)^p \\ &= \left(\int |u(\mathbf{y})\,\phi_k(\mathbf{x}-\mathbf{y})|\,dy\right)^p \\ &\leq \int |u(\mathbf{y})|^p\,\phi_k(\mathbf{x}-\mathbf{y})\,dy \\ &\leq \left(\sup_{\mathbf{z}\in\mathbb{R}^n} \phi_k(\mathbf{z})\right)\int |u(\mathbf{y})|\,dy \leq M\left(\sup_{\mathbf{z}\in\mathbb{R}^n}\phi_k(\mathbf{z})\right) \end{aligned}$$

showing the functions in K_k are uniformly bounded.

Next suppose $\mathbf{x}, \mathbf{x}_1 \in K_k$ and consider

$$\begin{aligned} & |u * \phi_k(\mathbf{x}) - u * \phi_k(\mathbf{x}_1)| \\ \leq & \int |u(\mathbf{x}-\mathbf{y}) - u(\mathbf{x}_1-\mathbf{y})|\,\phi_k(\mathbf{y})\,dy \\ \leq & \left(\int |u(\mathbf{x}-\mathbf{y}) - u(\mathbf{x}_1-\mathbf{y})|^p\,dy\right)^{1/p}\left(\int \phi_k(\mathbf{y})^q\,dy\right)^q \end{aligned}$$

which by assumption 40.17 is small independent of the choice of u whenever $|\mathbf{x}-\mathbf{x}_1|$ is small enough. Note that k is fixed in the above. Therefore, the set, K_k is precompact in $C(\overline{G})$ thanks to the Ascoli Arzela theorem. Next consider how well $u \in K$ is approximated by $u * \phi_k$ in $L^p(\mathbb{R}^n)$. By Minkowski's inequality,

$$\begin{aligned} & \left(\int |u(\mathbf{x}) - u * \phi_k(\mathbf{x})|^p\,dx\right)^{1/p} \\ \leq & \left(\int\left(\int |u(\mathbf{x}) - u(\mathbf{x}-\mathbf{y})|\,\phi_k(\mathbf{y})\,dy\right)^p dx\right)^{1/p} \\ \leq & \int_{B\left(\mathbf{0},\frac{1}{k}\right)} \phi_k(\mathbf{y})\left(\int |u(\mathbf{x}) - u(\mathbf{x}-\mathbf{y})|^p\,dx\right)^{1/p} dy. \end{aligned}$$

Now let $\eta > 0$ be given. From 40.17 there exists k large enough that for all $u \in K$,

$$\int_{B\left(\mathbf{0},\frac{1}{k}\right)} \phi_k(\mathbf{y})\left(\int |u(\mathbf{x}) - u(\mathbf{x}-\mathbf{y})|^p\,dx\right)^{1/p} dy \leq \int_{B\left(\mathbf{0},\frac{1}{k}\right)} \phi_k(\mathbf{y})\,\eta\,dy = \eta.$$

Now let $\varepsilon > 0$ be given and let δ and G correspond to ε as given in the hypotheses and let $1/k < \delta$ and also k is large enough that for all $u \in K$,

$$||u - u * \phi_k||_p < \varepsilon$$

as in the above inequality. By the Ascoli Arzela theorem there exists an

$$\left(\frac{\varepsilon}{m\left(\overline{G} + B(\mathbf{0}, 1)\right)}\right)^{1/p}$$

net for K_k in $C\left(\overline{G}\right)$. That is, there exist $\{u_i\}_{i=1}^m \subseteq K$ such that for any $u \in K$,

$$||u * \phi_k - u_j * \phi_k||_\infty < \left(\frac{\varepsilon}{m\left(\overline{G} + B(\mathbf{0}, 1)\right)}\right)^{1/p}$$

for some j. Letting $u \in K$ be given, let $u_j \in \{u_i\}_{i=1}^m \subseteq K$ be such that the above inequality holds. Then

$$\begin{aligned} ||u - u_j||_p &\leq ||u - u * \phi_k||_p + ||u * \phi_k - u_j * \phi_k||_p + ||u_j * \phi_k - u_j||_p \\ &< 2\varepsilon + ||u * \phi_k - u_j * \phi_k||_p \end{aligned}$$

$$\begin{aligned} &\leq 2\varepsilon + \left(\int_{\overline{G}+B(\mathbf{0},1)} |u * \phi_k - u_j * \phi_k|^p \, dx\right)^{1/p} \\ &\quad + \left(\int_{\mathbb{R}^n \setminus \left(\overline{G}+B(\mathbf{0},1)\right)} |u * \phi_k - u_j * \phi_k|^p \, dx\right)^{1/p} \\ &\leq 2\varepsilon + \varepsilon^{1/p} \\ &\quad + \left(\int_{\mathbb{R}^n \setminus \left(\overline{G}+B(\mathbf{0},1)\right)} \left(\int |u(\mathbf{x}-\mathbf{y}) - u_j(\mathbf{x}-\mathbf{y})| \, \phi_k(\mathbf{y}) \, dy\right)^p dx\right)^{1/p} \\ &\leq 2\varepsilon + \varepsilon^{1/p} \\ &\quad + \int \phi_k(\mathbf{y}) \left(\int_{\mathbb{R}^n \setminus \left(\overline{G}+B(\mathbf{0},1)\right)} \left(|u(\mathbf{x}-\mathbf{y})| + |u_j(\mathbf{x}-\mathbf{y})|\right)^p dx\right)^{1/p} dy \\ &\leq 2\varepsilon + \varepsilon^{1/p} + \int \phi_k(\mathbf{y}) \left(\int_{\mathbb{R}^n \setminus \overline{G}} \left(|u(\mathbf{x})| + |u_j(\mathbf{x})|\right)^p dx\right)^{1/p} dy \\ &\leq 2\varepsilon + \varepsilon^{1/p} + 2^{p-1} \int \phi_k(\mathbf{y}) \left(\int_{\mathbb{R}^n \setminus \overline{G}} \left(|u(\mathbf{x})|^p + |u_j(\mathbf{x})|^p\right) dx\right)^{1/p} dy \\ &\leq 2\varepsilon + \varepsilon^{1/p} + 2^{p-1} 2^{1/p} \varepsilon \end{aligned}$$

and since $\varepsilon > 0$ is arbitrary, this shows that K is totally bounded and is therefore precompact.

Now for an arbitrary open set, U and K given in the hypotheses of the theorem, let $\widetilde{K} \equiv \{\widetilde{u} : u \in K\}$ and observe that $\widetilde{K}$ is precompact in $L^p(\mathbb{R}^n)$. But this is the same as saying that K is precompact in $L^p(U)$. This proves the theorem.

Actually the converse of the above theorem is also true [1] but this will not be needed so I have left it as an exercise for anyone interested.

Lemma 40.29 *Let $u \in W^{1,1}(U)$ for U an open set and let $\phi \in C_c^\infty(U)$. Then there exists a constant,*

$$C\left(\phi, ||u||_{1,1,U}\right),$$

depending only on the indicated quantities such that whenever $\mathbf{v} \in \mathbb{R}^n$ with $|\mathbf{v}| <$ $\text{dist}\left(\text{spt}(\phi), U^C\right)$, *it follows that*

$$\int_{\mathbb{R}^n} \left|\widetilde{\phi u}(\mathbf{x}+\mathbf{v}) - \widetilde{\phi u}(\mathbf{x})\right| dx \leq C\left(\phi, ||u||_{1,1,U}\right)|\mathbf{v}|.$$

Proof: First suppose $u \in C^\infty\left(\overline{U}\right)$. Then for any $\mathbf{x} \in \text{spt}(\phi) \cup (\text{spt}(\phi) - \mathbf{v}) \equiv G_{\mathbf{v}}$, the chain rule implies

$$\begin{aligned} |\phi u(\mathbf{x}+\mathbf{v}) - \phi u(\mathbf{x})| &\leq \int_0^1 \sum_{i=1}^n \left|(\phi u)_{,i}(\mathbf{x}+t\mathbf{v}) v_i\right| dt \\ &\leq \int_0^1 \sum_{i=1}^n \left|\left(\phi_{,i} u + u_{,i}\phi\right)(\mathbf{x}+t\mathbf{v})\right| dt\,|\mathbf{v}|. \end{aligned}$$

Therefore, for such u,

$$\begin{aligned} &\int_{\mathbb{R}^n} \left|\widetilde{\phi u}(\mathbf{x}+\mathbf{v}) - \widetilde{\phi u}(\mathbf{x})\right| dx \\ = &\int_{G_{\mathbf{v}}} |\phi u(\mathbf{x}+\mathbf{v}) - \phi u(\mathbf{x})| dx \\ \leq &\int_{G_{\mathbf{v}}} \int_0^1 \sum_{i=1}^n \left|\left(\phi_{,i} u + u_{,i}\phi\right)(\mathbf{x}+t\mathbf{v})\right| dt dx\,|\mathbf{v}| \\ \leq &\int_0^1 \int_{G_{\mathbf{v}}} \sum_{i=1}^n \left|\left(\phi_{,i} u + u_{,i}\phi\right)(\mathbf{x}+t\mathbf{v})\right| dx dt\,|\mathbf{v}| \\ \leq &\ C\left(\phi, ||u||_{1,1,U}\right)|\mathbf{v}| \end{aligned}$$

where C is a continuous function of $||u||_{1,1,U}$. Now for general $u \in W^{1,1}(U)$, let $u_k \to u$ in $W^{1,1}(U)$ where $u_k \in C^\infty(\overline{U})$. Then for $|\mathbf{v}| < \text{dist}\left(\text{spt}(\phi), U^C\right)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left|\widetilde{\phi u}(\mathbf{x}+\mathbf{v}) - \widetilde{\phi u}(\mathbf{x})\right| dx \\
= &\int_{G_{\mathbf{v}}} |\phi u(\mathbf{x}+\mathbf{v}) - \phi u(\mathbf{x})| \, dx \\
= &\lim_{k\to\infty} \int_{G_{\mathbf{v}}} |\phi u_k(\mathbf{x}+\mathbf{v}) - \phi u_k(\mathbf{x})| \, dx \\
\leq &\lim_{k\to\infty} C\left(\phi, ||u_k||_{1,1,U}\right) |\mathbf{v}| \\
= &C\left(\phi, ||u||_{1,1,U}\right) |\mathbf{v}|.
\end{aligned}$$

This proves the lemma.

Lemma 40.30 *Let U be a bounded open set and define for $p > 1$*

$$S \equiv \left\{u \in W^{1,1}(U) \cap L^p(U) : ||u||_{1,1,U} + ||u||_{L^p(U)} \leq M\right\} \tag{40.19}$$

and let $\phi \in C_c^\infty(U)$ and

$$S_1 \equiv \{u\phi : u \in S\}. \tag{40.20}$$

Then S_1 is precompact in $L^q(U)$ where $1 \leq q < p$.

Proof: This depends on Theorem 40.28. The second condition is satisfied by taking $G \equiv \text{spt}(\phi)$. Thus, for $w \in S_1$,

$$\int_{U\setminus\overline{G}} |w(\mathbf{x})|^q \, dx = 0 < \varepsilon^p.$$

It remains to satisfy the first condition. It is necessary to verify there exists $\delta > 0$ such that if $|\mathbf{v}| < \delta$, then

$$\int_{\mathbb{R}^n} \left|\widetilde{\phi u}(\mathbf{x}+\mathbf{v}) - \widetilde{\phi u}(\mathbf{x})\right|^q dx < \varepsilon^p. \tag{40.21}$$

Let $\text{spt}(\phi) \cup (\text{spt}(\phi) - \mathbf{v}) \equiv G_{\mathbf{v}}$. Now if h is any measurable function, and if $\theta \in (0,1)$ is chosen small enough that $\theta q < 1$,

$$\begin{aligned}
\int_{G_{\mathbf{v}}} |h|^q \, dx &= \int_{G_{\mathbf{v}}} |h|^{\theta q} |h|^{(1-\theta)q} \, dx \\
&\leq \left(\int_{G_{\mathbf{v}}} |h| \, dx\right)^{\theta q} \left(\int_{G_{\mathbf{v}}} \left(|h|^{(1-\theta)q}\right)^{\frac{1}{1-\theta q}}\right)^{1-\theta q} \\
&= \left(\int_{G_{\mathbf{v}}} |h| \, dx\right)^{\theta q} \left(\int_{G_{\mathbf{v}}} |h|^{\frac{(1-\theta)q}{1-\theta q}}\right)^{1-\theta q}.
\end{aligned} \tag{40.22}$$

Now let θ also be small enough that there exists $r > 1$ such that

$$r \frac{(1-\theta) q}{1-\theta q} = p$$

and use Holder's inequality in the last factor of the right side of 40.22. Then 40.22 is dominated by

$$\begin{aligned} & \left(\int_{G_{\mathbf{v}}} |h| \, dx \right)^{\theta q} \left(\int_{G_{\mathbf{v}}} |h|^p \right)^{\frac{1-\theta q}{r}} \left(\int_{G_{\mathbf{v}}} 1 dx \right)^{1/r'} \\ = \; & C \left(||h||_{L^p(G_{\mathbf{v}})}, m_n(G_{\mathbf{v}}) \right) \left(\int_{G_{\mathbf{v}}} |h| \, dx \right)^{\theta q}. \end{aligned}$$

Therefore, for $u \in S$,

$$\int_{\mathbb{R}^n} \left| \widetilde{\phi u}(\mathbf{x}+\mathbf{v}) - \widetilde{\phi u}(\mathbf{x}) \right|^q dx = \int_{G_{\mathbf{v}}} |\phi u(\mathbf{x}+\mathbf{v}) - \phi u(\mathbf{x})|^q \, dx$$

$$\leq C \left(||\phi u(\cdot + \mathbf{v}) - \phi u(\cdot)||_{L^p(G_{\mathbf{v}})}, m_n(G_{\mathbf{v}}) \right) \left(\int_{G_{\mathbf{v}}} |\phi u(\mathbf{x}+\mathbf{v}) - \phi u(\mathbf{x})| \, dx \right)^{\theta q}$$

$$\leq C \left(2 \, ||\phi u(\cdot)||_{L^p(U)}, m_n(U) \right) \left(\int_{G_{\mathbf{v}}} |\phi u(\mathbf{x}+\mathbf{v}) - \phi u(\mathbf{x})| \, dx \right)^{\theta q}$$

$$\begin{aligned} \leq \; & C(\phi, M, m_n(U)) \left(\int_{G_{\mathbf{v}}} |\phi u(\mathbf{x}+\mathbf{v}) - \phi u(\mathbf{x})| \, dx \right)^{\theta q} \\ = \; & C(\phi, M, m_n(U)) \left(\int_{\mathbb{R}^n} \left| \widetilde{\phi u}(\mathbf{x}+\mathbf{v}) - \widetilde{\phi u}(\mathbf{x}) \right| dx \right)^{\theta q}. \end{aligned} \tag{40.23}$$

Now by Lemma 40.29,

$$\int_{\mathbb{R}^n} \left| \widetilde{\phi u}(\mathbf{x}+\mathbf{v}) - \widetilde{\phi u}(\mathbf{x}) \right| dx \leq C \left(\phi, ||u||_{1,1,U} \right) |\mathbf{v}| \tag{40.24}$$

and so from 40.23 and 40.24, and adjusting the constants

$$\begin{aligned} \int_{\mathbb{R}^n} \left| \widetilde{\phi u}(\mathbf{x}+\mathbf{v}) - \widetilde{\phi u}(\mathbf{x}) \right|^q dx \; & \leq \; C(\phi, M, m_n(U)) \left(C \left(\phi, ||u||_{1,1,U} \right) |\mathbf{v}| \right)^{\theta q} \\ & = \; C(\phi, M, m_n(U)) \, |\mathbf{v}|^{\theta q} \end{aligned}$$

which verifies 40.21 whenever $|\mathbf{v}|$ is sufficiently small. This proves the lemma because the conditions of Theorem 40.28 are satisfied.

Theorem 40.31 *Let U be a bounded open set and define for $p > 1$*

$$S \equiv \left\{ u \in W^{1,1}(U) \cap L^p(U) : ||u||_{1,1,U} + ||u||_{L^p(U)} \leq M \right\} \tag{40.25}$$

Then S is precompact in $L^q(U)$ where $1 \leq q < p$.

Proof: If suffices to show that every sequence, $\{u_k\}_{k=1}^{\infty} \subseteq S$ has a subsequence which converges in $L^q(U)$. Let $\{K_m\}_{m=1}^{\infty}$ denote a sequence of compact subsets of U with the property that $K_m \subseteq K_{m+1}$ for all m and $\cup_{m=1}^{\infty} K_m = U$. Now let $\phi_m \in C_c^{\infty}(U)$ such that $\phi_m(\mathbf{x}) \in [0,1]$ and $\phi_m(\mathbf{x}) = 1$ for all $\mathbf{x} \in K_m$. Let $S_m \equiv \{\phi_m u : u \in S\}$. By Lemma 40.30 there exists a subsequence of $\{u_k\}_{k=1}^{\infty}$, denoted here by $\{u_{1,k}\}_{k=1}^{\infty}$ such that $\{\phi_1 u_{1,k}\}_{k=1}^{\infty}$ converges in $L^q(U)$. Now S_2 is also precompact in $L^q(U)$ and so there exists a subsequence of $\{u_{1,k}\}_{k=1}^{\infty}$, denoted by $\{u_{2,k}\}_{k=1}^{\infty}$ such that $\{\phi_2 u_{2,k}\}_{k=1}^{\infty}$ converges in $L^2(U)$. Thus it is also the case that $\{\phi_1 u_{2,k}\}_{k=1}^{\infty}$ converges in $L^q(U)$. Continue taking subsequences in this manner such that for all $l \leq m$, $\{\phi_l u_{m,k}\}_{k=1}^{\infty}$ converges in $L^q(U)$. Let $\{w_m\}_{m=1}^{\infty} = \{u_{m,m}\}_{m=1}^{\infty}$ so that $\{w_k\}_{k=m}^{\infty}$ is a subsequence of $\{u_{m,k}\}_{k=1}^{\infty}$. Then it follows for all k, $\{\phi_k w_m\}_{m=1}^{\infty}$ must converge in $L^q(U)$. For $u \in S$,

$$\begin{aligned} ||u - \phi_k u||_{L^q(U)}^q &= \int_U |u|^q (1-\phi_k)^q \, dx \\ &\leq \left(\int_U |u|^p \, dx \right)^{q/p} \left(\int_U (1-\phi_k)^{qr} \, dx \right)^{1/r} \\ &\leq M \left(\int_U (1-\phi_k)^{qr} \, dx \right)^{1/r} \end{aligned}$$

where $q/p + 1/r = 1$. Now $\phi_l(\mathbf{x}) \to \mathcal{X}_U(\mathbf{x})$ and so the integrand in the last integral converges to 0 by the dominated convergence theorem. Therefore, k may be chosen large enough that for all $u \in S$,

$$||u - \phi_k u||_{L^q(U)}^q \leq \left(\frac{\varepsilon}{3} \right)^q .$$

Fix such a value of k. Then

$$\begin{aligned} &||w_q - w_p||_{L^q(U)} \leq \\ &||w_q - \phi_k w_q||_{L^q(U)} + ||\phi_k w_q - \phi_k w_p||_{L^q(U)} + ||w_p - \phi_k w_p||_{L^q(U)} \\ &\leq \frac{2\varepsilon}{3} + ||\phi_k w_q - \phi_k w_p||_{L^q(U)} . \end{aligned}$$

But $\{\phi_k w_m\}_{m=1}^{\infty}$ converges in $L^q(U)$ and so the last term in the above is less than $\varepsilon/3$ whenever p, q are large enough. Thus $\{w_m\}_{m=1}^{\infty}$ is a Cauchy sequence and must therefore converge in $L^q(U)$. This proves the theorem.

40.2 An Extension Theorem

Definition 40.32 *An open subset, U, of $\mathbb{R}^n$ has a Lipschitz boundary if it satisfies the following conditions. For each $p \in \partial U \equiv \overline{U} \setminus U$, there exists an open set, Q, containing p, an open interval (a,b), a bounded open box $B \subseteq \mathbb{R}^{n-1}$, and an orthogonal transformation R such that*

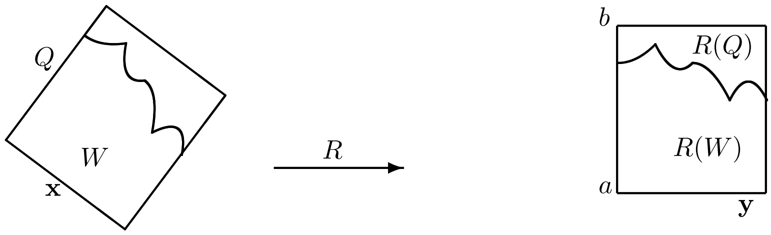
$$RQ = B \times (a,b), \tag{40.26}$$

$$R(Q \cap U) = \{\mathbf{y} \in \mathbb{R}^n : \widehat{\mathbf{y}} \in B,\ a < y_n < g(\widehat{\mathbf{y}})\} \tag{40.27}$$

where g is Lipschitz continuous on $\overline{B}, a < \min\{g(\mathbf{x}) : \mathbf{x} \in \overline{B}\}$, and

$$\widehat{\mathbf{y}} \equiv (y_1, \cdots, y_{n-1}).$$

Letting $W = Q \cap U$ the following picture describes the situation.



The following lemma is important.

Lemma 40.33 *If U is an open subset of $\mathbb{R}^n$ which has a Lipschitz boundary, then it satisfies the segment condition and so $X^{m,p}(U) = W^{m,p}(U)$.*

Proof: For $\mathbf{x} \in \partial U$, simply look at a single open set, $Q_{\mathbf{x}}$ described in the above which contains $\mathbf{x}$. Then consider an open set whose intersection with U is of the form $R^T(\{\mathbf{y} : \widehat{\mathbf{y}} \in B, g(\widehat{\mathbf{y}}) - \varepsilon < y_n < g(\widehat{\mathbf{y}})\})$ and a vector of the form $\varepsilon R^T(-\mathbf{e}_n)$ where ε is chosen smaller than $\min\{g(\mathbf{x}) : \mathbf{x} \in \overline{B}\} - a$. There is nothing to prove for points of U.

One way to extend many of the above theorems to more general open sets than $\mathbb{R}^n$ is through the use of an appropriate extension theorem. In this section, a fairly general one will be presented.

Lemma 40.34 *Let $B \times (a, b)$ be as described in Definition 40.32 and let*

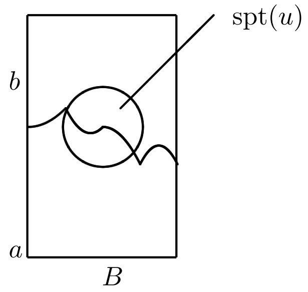
$$V^- \equiv \{(\widehat{\mathbf{y}}, y_n) : y_n < g(\widehat{\mathbf{y}})\},\ V^+ \equiv \{(\widehat{\mathbf{y}}, y_n) : y_n > g(\widehat{\mathbf{y}})\},$$

for g a Lipschitz function of the sort described in this definition. Suppose u^+ and u^- are Lipschitz functions defined on $\overline{V^+}$ and $\overline{V^-}$ respectively and suppose that $u^+(\widehat{\mathbf{y}}, g(\widehat{\mathbf{y}})) = u^-(\widehat{\mathbf{y}}, g(\widehat{\mathbf{y}}))$ for all $\widehat{\mathbf{y}} \in B$. Let

$$u(\widehat{\mathbf{y}}, y_n) \equiv \begin{cases} u^+(\widehat{\mathbf{y}}, y_n) & \text{if } (\widehat{\mathbf{y}}, y_n) \in V^+ \\ u^-(\widehat{\mathbf{y}}, y_n) & \text{if } (\widehat{\mathbf{y}}, y_n) \in V^- \end{cases}$$

and suppose $\operatorname{spt}(u) \subseteq B \times (a, b)$. Then extending u to be 0 off of $B \times (a, b)$, u is continuous and the weak partial derivatives, $u_{,i}$, are all in $L^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $p > 1$ and $u_{,i} = (u^+)_{,i}$ on V^+ and $u_{,i} = (u^-)_{,i}$ on V^-.

Proof: Consider the following picture which is descriptive of the situation.



Note first that u is Lipschitz continuous. To see this, consider $|u(\mathbf{y}_1) - u(\mathbf{y}_2)|$ where $\left(\widehat{\mathbf{y}}_i, y_n^i\right) = \mathbf{y}_i$. There are various cases to consider depending on whether y_n^i is above $g(\widehat{\mathbf{y}}_i)$. Suppose $y_n^1 < g(\widehat{\mathbf{y}}_1)$ and $y_n^2 > g(\widehat{\mathbf{y}}_2)$. Then letting $K \geq \max\left(\operatorname{Lip}(u^+), \operatorname{Lip}(u^-)\right)$,

$$\begin{aligned}
\left|u\left(\widehat{\mathbf{y}}_1, y_n^1\right) - u\left(\widehat{\mathbf{y}}_2, y_n^2\right)\right| &\leq \left|u\left(\widehat{\mathbf{y}}_1, y_n^1\right) - u\left(\widehat{\mathbf{y}}_2, y_n^1\right)\right| + \left|u\left(\widehat{\mathbf{y}}_2, y_n^1\right) - u\left(\widehat{\mathbf{y}}_2, g(\widehat{\mathbf{y}}_2)\right)\right| \\
&\quad + \left|u\left(\widehat{\mathbf{y}}_2, g(\widehat{\mathbf{y}}_2)\right) - u\left(\widehat{\mathbf{y}}_2, y_n^2\right)\right| \\
&\leq K\left|\widehat{\mathbf{y}}_1 - \widehat{\mathbf{y}}_2\right| + K\left[g(\widehat{\mathbf{y}}_2) - y_n^1 + y_n^2 - g(\widehat{\mathbf{y}}_2)\right] \\
&= K\left(\left|\widehat{\mathbf{y}}_1 - \widehat{\mathbf{y}}_2\right| + \left|y_n^1 - y_n^2\right|\right) \leq K\sqrt{n}\left|\mathbf{y}_1 - \mathbf{y}_2\right|
\end{aligned}$$

The other cases are similar. Thus u is a Lipschitz continuous function which has compact support. By Corollary 37.18 on Page 1092 it follows that $u_{,i} \in L^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $p > 1$. It remains to verify $u_{,i} = (u^+)_{,i}$ on V^+ and $u_{,i} = (u^-)_{,i}$ on V^-. The last claim is obvious from the definition of weak derivatives.

Lemma 40.35 *In the situation of Lemma 40.34 let* $u \in C^1\left(\overline{V^-}\right) \cap C_c^1(B \times (a,b))$[3] *and define*

$$w(\widehat{\mathbf{y}}, y_n) \equiv \begin{cases} u(\widehat{\mathbf{y}}, y_n) & \text{if } \widehat{\mathbf{y}} \in B \text{ and } y_n \leq g(\widehat{\mathbf{y}}), \\ u(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n), & \text{if } \widehat{\mathbf{y}} \in B \text{ and } y_n > g(\widehat{\mathbf{y}}) \\ 0 & \text{if } \widehat{\mathbf{y}} \notin B. \end{cases}$$

Then $w \in W^{1,p}(\mathbb{R}^n)$ *and there exists a constant,* C *depending only on* $\operatorname{Lip}(g)$ *and dimension such that*

$$||w||_{W^{1,p}(\mathbb{R}^n)} \leq C\,||u||_{W^{1,p}(V^-)}.$$

Denote w *by* $E_0 u$*. Thus* $E_0(u)(\mathbf{y}) = u(\mathbf{y})$ *for all* $\mathbf{y} \in V^-$ *but* $E_0 u = w$ *is defined on all of* $\mathbb{R}^n$*. Also,* E_0 *is a linear mapping.*

Proof: As in the previous lemma, w is Lipschitz continuous and has compact support so it is clear $w \in W^{1,p}(\mathbb{R}^n)$. The main task is to fine $w_{,i}$ for $\widehat{\mathbf{y}} \in B$ and

[3]This means that $\operatorname{spt}(u) \subseteq B \times (a,b)$ and $u \in C^1\left(\overline{V^-}\right)$.

$y_n > g(\widehat{\mathbf{y}})$ and then to extract an estimate of the right sort. Denote by U the set of points of $\mathbb{R}^n$ with the property that $(\widehat{\mathbf{y}}, y_n) \in U$ if and only if $\widehat{\mathbf{y}} \notin B$ or $\widehat{\mathbf{y}} \in B$ and $y_n > g(\widehat{\mathbf{y}})$. Then letting $\phi \in C_c^\infty(U)$, suppose first that $i < n$. Then

$$\int_U w(\widehat{\mathbf{y}}, y_n)\,\phi_{,i}(\mathbf{y})\,dy$$

$$\equiv \lim_{h\to 0} \int_U \phi(\mathbf{y}) \frac{u\left(\widehat{\mathbf{y}} - h\mathbf{e}_i^{n-1}, 2g\left(\widehat{\mathbf{y}} - h\mathbf{e}_i^{n-1}\right) - y_n\right) - u\left(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n\right)}{h} dy \tag{40.28}$$

$$\begin{aligned} = \; & \lim_{h\to 0} \left\{ \frac{-1}{h} \int_U \phi(\mathbf{y}) \left[D_1 u(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n) \left(h\mathbf{e}_i^{n-1}\right) \right.\right. \\ & \left. +2D_2 u(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n) \left(g\left(\widehat{\mathbf{y}} - h\mathbf{e}_i^{n-1}\right) - g(\widehat{\mathbf{y}})\right)\right] dy \\ & \left. + \frac{-1}{h} \int_U \phi(\mathbf{y}) \left[o\left(g\left(\widehat{\mathbf{y}} - h\mathbf{e}_i^{n-1}\right) - g(\widehat{\mathbf{y}})\right) + o(h)\right] dy \right\} \end{aligned}$$

where $\mathbf{e}_i^{n-1}$ is the unit vector in $\mathbb{R}^{n-1}$ having all zeros except for a 1 in the i^{th} position. Now by Rademacher's theorem, $Dg(\widehat{\mathbf{y}})$ exists for a.e. $\widehat{\mathbf{y}}$ and so except for a set of measure zero, the expression, $g\left(\widehat{\mathbf{y}} - h\mathbf{e}_i^{n-1}\right) - g(\widehat{\mathbf{y}})$ is $o(h)$ and also for $\widehat{\mathbf{y}}$ not in the exceptional set,

$$g\left(\widehat{\mathbf{y}} - h\mathbf{e}_i^{n-1}\right) - g(\widehat{\mathbf{y}}) = -hDg(\widehat{\mathbf{y}})\,\mathbf{e}_i^{n-1} + o(h).$$

Therefore, since the integrand in 40.28 has compact support and because of the Lipschitz continuity of all the functions, the dominated convergence theorem may be applied to obtain

$$\int_U w(\widehat{\mathbf{y}}, y_n)\,\phi_{,i}(\mathbf{y})\,dy =$$

$$\int_U \phi(\mathbf{y}) \left[-D_1 u(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n)\left(\mathbf{e}_i^{n-1}\right) + 2D_2 u(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n)\left(Dg(\widehat{\mathbf{y}})\,\mathbf{e}_i^{n-1}\right)\right] dy$$

$$= \int_U \phi(\mathbf{y}) \left[-\frac{\partial u}{\partial y_i}(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n) + 2\frac{\partial u}{\partial y_n}(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n)\frac{\partial g(\widehat{\mathbf{y}})}{\partial y_i} \right] dy$$

and so

$$w_{,i}(\mathbf{y}) = \frac{\partial u}{\partial y_i}(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n) - 2\frac{\partial u}{\partial y_n}(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n)\frac{\partial g(\widehat{\mathbf{y}})}{\partial y_i} \tag{40.29}$$

whenever $i < n$ which is what you would expect from a formal application of the chain rule. Next suppose $i = n$.

$$\begin{aligned}
&\int_U w(\widehat{\mathbf{y}}, y_n)\,\phi_{,n}(\mathbf{y})\,dy \\
=\;& \lim_{h\to 0}\int_U \frac{u(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - (y_n - h)) - u(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n)}{h}\phi(\mathbf{y})\,dy \\
=\;& \lim_{h\to 0}\int_U \frac{D_2 u(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n)\,h + o(h)}{h}\phi(\mathbf{y})\,dy \\
=\;& \int_U \frac{\partial u}{\partial y_n}(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n)\,\phi(\mathbf{y})\,dy
\end{aligned}$$

showing that

$$w_{,n}(\mathbf{y}) = \frac{-\partial u}{\partial y_n}(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n) \tag{40.30}$$

which is also expected.

From the definnition, for $\mathbf{y} \in \mathbb{R}^n \setminus U \equiv \{(\widehat{\mathbf{y}}, y_n) : y_n \leq g(\widehat{\mathbf{y}})\}$ it follows $w_{,i} = u_{,i}$ and on $U, w_{,i}$ is given by 40.29 and 40.30. Consider $||w_{,i}||_{L^p(U)}^p$ for $i < n$. From 40.29

$$||w_{,i}||_{L^p(U)}^p = \int_U \left| \frac{\partial u}{\partial y_i}(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n) - 2\frac{\partial u}{\partial y_n}(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n)\frac{\partial g(\widehat{\mathbf{y}})}{\partial y_i} \right|^p dy$$

$$
\begin{aligned}
&\leq 2^{p-1}\int_U \left|\frac{\partial u}{\partial y_i}\left(\widehat{\mathbf{y}},2g\left(\widehat{\mathbf{y}}\right)-y_n\right)\right|^p \\
&\quad +2^p\left|\frac{\partial u}{\partial y_n}\left(\widehat{\mathbf{y}},2g\left(\widehat{\mathbf{y}}\right)-y_n\right)\right|^p \operatorname{Lip}(g)^p\,dy \\
&\leq 4^p\left(1+\operatorname{Lip}(g)^p\right)\int_U \left|\frac{\partial u}{\partial y_i}\left(\widehat{\mathbf{y}},2g\left(\widehat{\mathbf{y}}\right)-y_n\right)\right|^p \\
&\quad +\left|\frac{\partial u}{\partial y_n}\left(\widehat{\mathbf{y}},2g\left(\widehat{\mathbf{y}}\right)-y_n\right)\right|^p dy \\
&= 4^p\left(1+\operatorname{Lip}(g)^p\right)\int_B\int_{g(\widehat{\mathbf{y}})}^{\infty} \left|\frac{\partial u}{\partial y_i}\left(\widehat{\mathbf{y}},2g\left(\widehat{\mathbf{y}}\right)-y_n\right)\right|^p \\
&\quad +\left|\frac{\partial u}{\partial y_n}\left(\widehat{\mathbf{y}},2g\left(\widehat{\mathbf{y}}\right)-y_n\right)\right|^p dy_n d\widehat{y} \\
&= 4^p\left(1+\operatorname{Lip}(g)^p\right)\int_B\int_{-\infty}^{g(\widehat{\mathbf{y}})} \left|\frac{\partial u}{\partial y_i}\left(\widehat{\mathbf{y}},z_n\right)\right|^p \\
&\quad +\left|\frac{\partial u}{\partial y_n}\left(\widehat{\mathbf{y}},z_n\right)\right|^p dz_n d\widehat{y} \\
&= 4^p\left(1+\operatorname{Lip}(g)^p\right)\int_B\int_{a}^{g(\widehat{\mathbf{y}})} \left|\frac{\partial u}{\partial y_i}\left(\widehat{\mathbf{y}},z_n\right)\right|^p \\
&\quad +\left|\frac{\partial u}{\partial y_n}\left(\widehat{\mathbf{y}},z_n\right)\right|^p dz_n d\widehat{y} \\
&\leq 4^p\left(1+\operatorname{Lip}(g)^p\right)||u||_{1,p,V^-}^p
\end{aligned}
$$

Now by similar reasoning,

$$
\begin{aligned}
||w_{,n}||_{L^p(U)}^p &= \int_U \left|\frac{-\partial u}{\partial y_n}\left(\widehat{\mathbf{y}},2g\left(\widehat{\mathbf{y}}\right)-y_n\right)\right|^p dy \\
&= \int_B\int_{g(\widehat{\mathbf{y}})}^{\infty} \left|\frac{-\partial u}{\partial y_n}\left(\widehat{\mathbf{y}},2g\left(\widehat{\mathbf{y}}\right)-y_n\right)\right|^p dy_n d\widehat{y} \\
&= \int_B\int_{a}^{g(\widehat{\mathbf{y}})} \left|\frac{-\partial u}{\partial y_n}\left(\widehat{\mathbf{y}},z_n\right)\right|^p dz_n d\widehat{y} = ||u_{,n}||_{1,p,V^-}^p .
\end{aligned}
$$

It follows

$$
\begin{aligned}
||w||_{1,p,\mathbb{R}^n}^p &= ||w||_{1,p,U}^p + ||u||_{1,p,V^-}^p \\
&\leq 4^p n\left(1+\operatorname{Lip}(g)^p\right)||u||_{1,p,V^-}^p + ||u||_{1,p,V^-}^p
\end{aligned}
$$

and so

$$||w||_{1,p,\mathbb{R}^n}^p \leq 4^p n\left(2+\operatorname{Lip}(g)^p\right)||u||_{1,p,V^-}^p$$

which implies

$$||w||_{1,p,\mathbb{R}^n} \leq 4n^{1/p}\left(2+\operatorname{Lip}(g)^p\right)^{1/p}||u||_{1,p,V^-}$$

It is obvious that E_0 is a linear mapping. This proves the theorem.

Now recall Definition 40.32, listed here for convenience.

Definition 40.36 *An open subset, U, of $\mathbb{R}^n$ has a Lipschitz boundary if it satisfies the following conditions. For each $p \in \partial U \equiv \overline{U} \setminus U$, there exists an open set, Q, containing p, an open interval (a, b), a bounded open box $B \subseteq \mathbb{R}^{n-1}$, and an orthogonal transformation R such that*

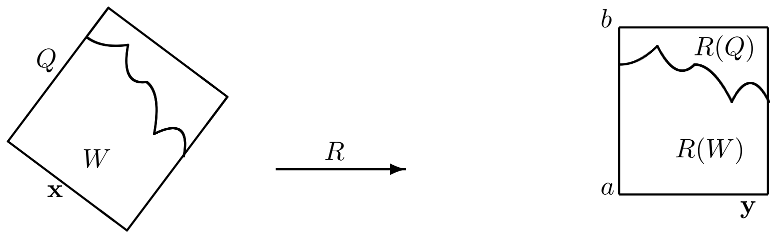
$$RQ = B \times (a, b), \tag{40.31}$$

$$R(Q \cap U) = \{\mathbf{y} \in \mathbb{R}^n : \widehat{\mathbf{y}} \in B,\ a < y_n < g(\widehat{\mathbf{y}})\} \tag{40.32}$$

where g is Lipschitz continuous on $\overline{B}, a < \min\left\{g(\mathbf{x}) : \mathbf{x} \in \overline{B}\right\}$, and

$$\widehat{\mathbf{y}} \equiv (y_1, \cdots, y_{n-1}).$$

Letting $W = Q \cap U$ the following picture describes the situation.



Lemma 40.37 *In the situation of Definition 40.32 let $u \in C^1\left(\overline{U}\right) \cap C_c^1(Q)$ and define*

$$Eu \equiv R^* E_0 \left(R^T\right)^* u.$$

where $\left(R^T\right)^*$ maps $W^{1,p}(U \cap Q)$ to $W^{1,p}(R(W))$. Then E is linear and satisfies

$$||Eu||_{W^{1,p}(\mathbb{R}^n)} \leq C\, ||u||_{W^{1,p}(Q \cap U)},\ Eu(\mathbf{x}) = u(\mathbf{x}) \text{ for } \mathbf{x} \in Q \cap U.$$

where C depends only on the dimension and $\text{Lip}(g)$.

Proof: This follows from Theorem 40.14 and Lemma 40.35.

The following theorem is a general extension theorem for Sobolev spaces.

Theorem 40.38 *Let U be a bounded open set which has Lipschitz boundary. Then for each $p \geq 1$, there exists $E \in \mathcal{L}\left(W^{1,p}(U), W^{1,p}(\mathbb{R}^n)\right)$ such that $Eu(\mathbf{x}) = u(\mathbf{x})$ a.e. $\mathbf{x} \in U$.*

Proof: Let $\partial U \subseteq \cup_{i=1}^p Q_i$ Where the Q_i are as described in Definition 40.36. Also let R_i be the orthogonal trasformation and g_i the Lipschitz functions associated with Q_i as in this definition. Now let $Q_0 \subseteq \overline{Q_0} \subseteq U$ be such that $\overline{U} \subseteq \cup_{i=0}^p Q_i$, and let $\psi_i \in C_c^\infty(Q_i)$ with $\psi_i(\mathbf{x}) \in [0,1]$ and $\sum_{i=0}^p \psi_i(\mathbf{x}) = 1$ on $\overline{U}$. For $u \in C^\infty\left(\overline{U}\right)$, let $E^0(\psi_0 u) \equiv \psi_0 u$ on Q_0 and 0 off Q_0. Thus

$$\left|\left|E^0(\psi_0 u)\right|\right|_{1,p,\mathbb{R}^n} = \left|\left|\psi_0 u\right|\right|_{1,p,U}.$$

For $i \geq 1$, let

$$E^i(\psi_i u) \equiv R_i^* E_0 \left(R^T\right)^* (\psi_i u).$$

Thus, by Lemma 40.37

$$\left|\left|E^1(\psi_i u)\right|\right|_{1,p,\mathbb{R}^n} \leq C \left|\left|\psi_i u\right|\right|_{1,p,Q_i \cap U}$$

where the constant depends on $\operatorname{Lip}(g_i)$ but is independent of $u \in C^\infty\left(\overline{U}\right)$. Now define E as follows.

$$Eu \equiv \sum_{i=0}^p E^i(\psi_i u).$$

Thus for $u \in C^\infty\left(\overline{U}\right)$, it follows $Eu(\mathbf{x}) = u(\mathbf{x})$ for all $\mathbf{x} \in U$. Also,

$$\begin{aligned} \left|\left|Eu\right|\right|_{1,p,\mathbb{R}^n} &\leq \sum_{i=0}^p \left|\left|E^i(\psi_i u)\right|\right| \leq \sum_{i=0}^p C_i \left|\left|\psi_i u\right|\right|_{1,p,Q_i \cap U} \\ &= \sum_{i=0}^p C_i \left|\left|\psi_i u\right|\right|_{1,p,U} \leq \sum_{i=0}^p C_i \left|\left|u\right|\right|_{1,p,U} \\ &\leq (p+1) \sum_{i=0}^p C_i \left|\left|u\right|\right|_{1,p,U} \equiv C \left|\left|u\right|\right|_{1,p,U}. \end{aligned} \tag{40.33}$$

where C depends on the ψ_i and the g_i but is independent of $u \in C^\infty\left(\overline{U}\right)$. Therefore, by density of $C^\infty\left(\overline{U}\right)$ in $W^{1,p}(U)$, E has a unique continuous extension to $W^{1,p}(U)$ still denoted by E satisfying the inequality determined by the ends of 40.33. It remains to verify that $Eu(\mathbf{x}) = u(\mathbf{x})$ a.e. for $\mathbf{x} \in U$.

Let $u_k \to u$ in $W^{1,p}(U)$ where $u_k \in C^\infty\left(\overline{U}\right)$. Therefore, by 40.33, $Eu_k \to Eu$ in $W^{1,p}(\mathbb{R}^n)$. Since $Eu_k(\mathbf{x}) = u_k(\mathbf{x})$ for each k,

$$\begin{aligned} \left|\left|u - Eu\right|\right|_{L^p(U)} &= \lim_{k\to\infty} \left|\left|u_k - Eu_k\right|\right|_{L^p(U)} \\ &= \lim_{k\to\infty} \left|\left|Eu_k - Eu_k\right|\right|_{L^p(U)} = 0 \end{aligned}$$

which shows $u(\mathbf{x}) = Eu(\mathbf{x})$ for a.e. $\mathbf{x} \in U$ as claimed. This proves the theorem.

Definition 40.39 *Let U be an open set. Then $W_0^{m,p}(U)$ is the closure of the set, $C_c^\infty(U)$ in $W^{m,p}(U)$.*

Corollary 40.40 *Let U be a bounded open set which has Lipschitz boundary and let W be an open set containing $\overline{U}$. Then for each $p \geq 1$, there exists $E_W \in \mathcal{L}\left(W^{1,p}(U), W_0^{1,p}(W)\right)$ such that $E_W u(\mathbf{x}) = u(\mathbf{x})$ a.e. $\mathbf{x} \in U$.*

Proof: Let $\psi \in C_c^\infty(W)$ and $\psi = 1$ on U. Then let $E_W u \equiv \psi E u$ where E is the extension operator of Theorem 40.38.

Extension operators of the above sort exist for many open sets, U, not just for bounded ones. In particular, the above discussion would apply to an open set, U, not necessarily bounded, if you relax the condition that the Q_i must be bounded but require the existence of a finite partition of unity $\{\psi_i\}_{i=1}^p$ having the property that ψ_i and $\psi_{i,j}$ are uniformly bounded for all i, j. The proof would be identical to the above. My main interest is in bounded open sets so the above theorem will suffice. Such an extension operator will be referred to as a $(1,p)$ extension operator.

40.3 General Embedding Theorems

With the extension theorem it is possible to give a useful theory of embeddings.

Theorem 40.41 *Let $1 \leq p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and let U be any open set for which there exists a $(1,p)$ extension operator. Then if $u \in W^{1,p}(U)$, there exists a constant independent of u such that*

$$||u||_{L^q(U)} \leq C\,||u||_{1,p,U}\,.$$

If U is bounded and $r < q$, then $\text{id} : W^{1,p}(U) \to L^r(U)$ *is also compact.*

Proof: Let E be the $(1,p)$ extension operator. Then by Theorem 40.24 on Page 1149

$$\begin{aligned} ||u||_{L^q(U)} &\leq ||Eu||_{L^q(\mathbb{R}^n)} \leq \frac{1}{\sqrt[n]{n}} \frac{(n-1)\,p}{(n-p)} ||Eu||_{1,p,\mathbb{R}^n} \\ &\leq C\,||u||_{1,p,U}\,. \end{aligned}$$

It remains to prove the assertion about compactness. If $S \subseteq W^{1,p}(U)$ is bounded then

$$\sup_{u \in S}\left\{||u||_{1,1,U} + ||u||_{L^q(U)}\right\} < \infty$$

and so by Theorem 40.31 on Page 1156, it follows S is precompact in $L^r(U)$.This proves the theorem.

Corollary 40.42 *Suppose $mp < n$ and U is an open set satisfying the segment condition which has a $(1,p)$ extension operator for all p. Then* $\text{id} \in \mathcal{L}(W^{m,p}(U), L^q(U))$ *where $q = \frac{np}{n-mp}$.*

Proof: This is true if $m = 1$ according to Theorem 40.41. Suppose it is true for $m-1$ where $m > 1$. If $u \in W^{m,p}(U)$ and $|\alpha| \leq 1$, then $D^\alpha u \in W^{m-1,p}(U)$ so by induction, for all such α,

$$D^\alpha u \in L^{\frac{np}{n-(m-1)p}}(U)\,.$$

Thus, since U has the segment condition, $u \in W^{1,q_1}(U)$ where

$$q_1 = \frac{np}{n-(m-1)p}$$

By Theorem 40.41, it follows $u \in L^q(\mathbb{R}^n)$ where

$$\frac{1}{q} = \frac{n-(m-1)p}{np} - \frac{1}{n} = \frac{n-mp}{np}.$$

This proves the corollary.

There is another similar corollary of the same sort which is interesting and useful.

Corollary 40.43 *Suppose $m \geq 1$ and j is a nonnegative integer satisfying $jp < n$. Also suppose U has a $(1,p)$ extension operator for all $p \geq 1$ and satisfies the segment condition. Then*

$$\text{id} \in \mathcal{L}\left(W^{m+j,p}(U), W^{m,q}(U)\right)$$

where

$$q \equiv \frac{np}{n-jp}. \tag{40.34}$$

If, in addition to the above, U is bounded and $1 \leq r < q$, then

$$\text{id} \in \mathcal{L}\left(W^{m+j,p}(U), W^{m,r}(U)\right)$$

and is compact.

Proof: If $|\alpha| \leq m$, then $D^\alpha u \in W^{j,p}(U)$ and so by Corollary 40.42, $D^\alpha u \in L^q(U)$ where q is given above. Since U has the segment property, this means $u \in W^{m,q}(U)$. It remains to verify the assertion about compactness of id.

Let S be bounded in $W^{m+j,p}(U)$. Then S is bounded in $W^{m,q}(U)$ by the first part. Now let $\{u_k\}_{k=1}^\infty$ be any sequence in S. The corollary will be proved if it is shown that any such sequence has a convergent subsequence in $W^{m,r}(U)$. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_h\}$ denote the indices satisfying $|\alpha| \leq m$. Then for each of these indices, α,

$$\sup_{u \in S}\left\{||D^\alpha u||_{1,1,U} + ||D^\alpha u||_{L^q(U)}\right\} < \infty$$

and so for each such α, satisfying $|\alpha| \leq m$, it follows from Lemma 40.30 on Page 1155 that $\{D^\alpha u : u \in S\}$ is precompact in $L^r(U)$. Therefore, there exists a subsequence, still denoted by u_k such that $D^{\alpha_1} u_k$ converges in $L^r(U)$. Applying the same lemma, there exists a subsequence of this subsequence such that both $D^{\alpha_1} u_k$ and $D^{\alpha_2} u_k$ converge in $L^r(U)$. Continue taking subsequences until you obtain a subsequence, $\{u_k\}_{k=1}^\infty$ for which $\{D^\alpha u_k\}_{k=1}^\infty$ converges in $L^r(U)$ for all $|\alpha| \leq m$. But this must be a convergent subsequence in $W^{m,r}(U)$ and this proves the corollary.

Theorem 40.44 *Let U be a bounded open set having a $(1,p)$ extension operator and let $p > n$. Then $\text{id} : W^{1,p}(U) \to C\left(\overline{U}\right)$ is continuous and compact.*

Proof: Theorem 40.17 on Page 40.17 implies $r_U : W^{1,p}(\mathbb{R}^n) \to C(\overline{U})$ is continuous and compact. Thus

$$||u||_{\infty,U} = ||Eu||_{\infty,U} \leq C\,||Eu||_{1,p,\mathbb{R}^n} \leq C\,||u||_{1,p,U}\,.$$

This proves continuity. If S is a bounded set in $W^{1,p}(U)$, then define $S_1 \equiv \{Eu : u \in S\}$. Then S_1 is a bounded set in $W^{1,p}(\mathbb{R}^n)$ and so by Theorem 40.17 the set of restrictions to U, is precompact. However, the restrictions to U are just the functions of S. Therefore, id is compact as well as continuous.

Corollary 40.45 *Let $p > n$, let U be a bounded open set having a $(1,p)$ extension operator which also satisfies the segment condition, and let m be a nonnegative integer. Then* id $: W^{m+1,p}(U) \to C^{m,\lambda}(\overline{U})$ *is continuous for all $\lambda \in [0, 1 - \frac{n}{p}]$ and* id *is compact if $\lambda < 1 - \frac{n}{p}$.*

Proof: Let $u_k \to 0$ in $W^{m+1,p}(U)$. Then it follows that for each $|\alpha| \leq m$, $D^\alpha u_k \to 0$ in $W^{1,p}(U)$. Therefore,

$$E(D^\alpha u_k) \to 0 \text{ in } W^{1,p}(\mathbb{R}^n)\,.$$

Then from Morrey's inequality, 40.13 on Page 1144, if $\lambda \leq 1 - \frac{n}{p}$ and $|\alpha| = m$

$$\rho_\lambda(E(D^\alpha u_k)) \leq C\,||E(D^\alpha u_k)||_{1,p,\mathbb{R}^n} \operatorname{diam}(U)^{1-\frac{n}{p}-\lambda}\,.$$

Therefore, $\rho_\lambda(E(D^\alpha u_k)) = \rho_\lambda(D^\alpha u_k) \to 0$. From Theorem 40.44 it follows that for $|\alpha| \leq m, ||D^\alpha u_k||_\infty \to 0$ and so $||u_k||_{m,\lambda} \to 0$. This proves the claim about continuity. The claim about compactness for $\lambda < 1 - \frac{n}{p}$ follows from Lemma 40.20 on Page 1145 and this. (Bounded in $W^{m,p}(U) \overset{\text{id}}{\to}$ Bounded in $C^{m,1-\frac{n}{p}}(\overline{U}) \overset{\text{id}}{\to}$ Compact in $C^{m,\lambda}(\overline{U})$.)

Theorem 40.46 *Suppose $jp < n < (j+1)p$ and let m be a positive integer. Let U be any bounded open set in $\mathbb{R}^n$ which has a $(1,p)$ extension operator for each $p \geq 1$ and the segment property. Then* id $\in \mathcal{L}(W^{m+j,p}(U), C^{m-1,\lambda}(\overline{U}))$ *for every $\lambda \leq \lambda_0 \equiv (j+1) - \frac{n}{p}$ and if $\lambda < (j+1) - \frac{n}{p}$,* id *is compact.*

Proof: From Corollary 40.43 $W^{m+j,p}(U) \subseteq W^{m,q}(U)$ where q is given by 40.34. Therefore,

$$\frac{np}{n - jp} > n$$

and so by Corollary 40.45, $W^{m,q}(U) \subseteq C^{m-1,\lambda}(\overline{U})$ for all λ satisfying

$$0 < \lambda < 1 - \frac{(n - jp)\,n}{np} = \frac{p(j+1) - n}{p} = (j+1) - \frac{n}{p}.$$

The assertion about compactness follows from the compactness of the embedding of $C^{m-1,\lambda_0}(\overline{U})$ into $C^{m-1,\lambda}(\overline{U})$ for $\lambda < \lambda_0$, Lemma 40.20 on Page 1145.

40.4 More Extension Theorems

The theorem about the existence of a $(1, p)$ extension is all that is needed to obtain general embedding theorems for Sobolev spaces. However, a more general theory is needed in order to tie the theory of Sobolev spaces presented thus far to a very appealing description using Fourier transforms. First the problem of extending $W^{k,p}(H)$ to $W^{k,p}(\mathbb{R}^n)$ is considered for H^- a half space

$$H^- \equiv \{\mathbf{y} \in \mathbb{R}^n : y_n < 0\}. \tag{40.35}$$

I am following Adams [1].

Lemma 40.47 *Let H^- be a half space as in 40.35. Let H^+ be the half space in which $y_n < 0$ is replaced with $y_n > 0$. Also let $(\mathbf{y}', y_n) = \mathbf{y}$*

$$u(\mathbf{y}', y_n) \equiv \begin{cases} u^+(\mathbf{y}', y_n) & \text{if } \mathbf{y} \in H^+ \\ u^-(\mathbf{y}', y_n) & \text{if } \mathbf{y} \in H^- \end{cases},$$

suppose $u^+ \in C^\infty\left(\overline{H^+}\right)$ and $u^- \in C^\infty\left(\overline{H^-}\right)$, and that for $l \leq k-1$,

$$D^{l\mathbf{e}_n}\mathbf{u}^+(\mathbf{y}', 0) = D^{l\mathbf{e}_n}\mathbf{u}^-(\mathbf{y}', 0).$$

Then $u \in W^{k,p}(\mathbb{R}^n)$. Furthermore,

$$D^\alpha u(\mathbf{y}', y_n) \equiv \begin{cases} D^\alpha u^+(\mathbf{y}', y_n) & \text{if } \mathbf{y} \in H^+ \\ D^\alpha u^-(\mathbf{y}', y_n) & \text{if } \mathbf{y} \in H^- \end{cases}$$

Proof: Consider the following for $\phi \in C_c^\infty(\mathbb{R}^n)$ and $|\alpha| \leq k$.

$$(-1)^{|\alpha|}\left(\int_{\mathbb{R}^{n-1}}\int_0^\infty u^+ D^\alpha\phi dy_n dy' + \int_{\mathbb{R}^{n-1}}\int_{-\infty}^0 u^- D^\alpha\phi dy_n dy'\right).$$

Integrating by parts, this yields

$$(-1)^{|\alpha|}(-1)^{|\beta|}\left(\int_{\mathbb{R}^{n-1}}\int_0^\infty D^\beta u^+ D^{\alpha_n\mathbf{e}_n}\phi dy_n dy' + \int_{\mathbb{R}^{n-1}}\int_{-\infty}^0 D^\beta u^- D^{\alpha_n\mathbf{e}_n}\phi dy_n dy'\right)$$

where $\beta \equiv (\alpha_1, \alpha_2, \cdots \alpha_{n-1}, 0)$. Do integration by parts on the inside integral and by assumption, the boundary terms will cancel and the whole thing reduces to

$$(-1)^{|\alpha|}(-1)^{|\beta|}(-1)^{\alpha_n}\left(\int_{\mathbb{R}^{n-1}}\int_0^\infty D^\alpha u^+\phi dy_n dy' + \int_{\mathbb{R}^{n-1}}\int_{-\infty}^0 D^\alpha u^-\phi dy_n dy'\right)$$
$$= \left(\int_{\mathbb{R}^{n-1}}\int_0^\infty D^\alpha u^+\phi dy_n dy' + \int_{\mathbb{R}^{n-1}}\int_{-\infty}^0 D^\alpha u^-\phi dy_n dy'\right)$$

which proves the lemma.

Lemma 40.48 *Let H^- be the half space in 40.35 and let $u \in C^\infty\left(\overline{H^-}\right)$. Then there exists a mapping,*

$$E : C^\infty\left(\overline{H^-}\right) \to W^{k,p}\left(\mathbb{R}^n\right)$$

and a constant, C which is independent of $u \in C^\infty\left(\overline{H^-}\right)$ such that E is linear and for all $l \leq k$,

$$\left|\left|Eu\right|\right|_{l,p,\mathbb{R}^n} \leq C \left|\left|u\right|\right|_{l,p,H^-}. \tag{40.36}$$

Proof: Define

$$Eu\left(\mathbf{x}', x_n\right) \equiv \begin{cases} u\left(\mathbf{x}', x_n\right) \text{ if } x_n < 0 \\ \sum_{j=1}^k \lambda_j u\left(\mathbf{x}', -jx_n\right) \text{ if } x_n \geq 0 \end{cases}$$

where the λ_j are chosen in such a way that for $l \leq k-1$,

$$D^{l\mathbf{e}_n} u\left(\mathbf{x}', 0\right) - D^{l\mathbf{e}_n}\left(\sum_{j=1}^k \lambda_j u\right)\left(\mathbf{x}', 0\right) = 0$$

so that Lemma 40.47 may be applied. Do there exist such λ_j? It is necessary to have the following hold for each $r = 0, 1, \cdot \cdot \cdot, k-1$.

$$\sum_{j=1}^k (-j)^r \lambda_j D^{r\mathbf{e}_n} u\left(\mathbf{x}', 0\right) = D^{r\mathbf{e}_n} u\left(\mathbf{x}', 0\right).$$

This is satisfied if

$$\sum_{j=1}^k (-j)^r \lambda_j = 1$$

for $r = 0, 1, \cdot \cdot \cdot, k-1$. This is a system of k equations for the k variables, the λ_j. The matrix of coefficients is of the form

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & -2 & -3 & \cdots & -k \\ 1 & 4 & 9 & \cdots & k^2 \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^k & (-2)^k & (-3)^k & \cdots & (-k)^k \end{pmatrix}$$

This matrix has an inverse because its determinant is nonzero.

Now from Lemma 40.47, it follows from the above description of E that for $|\alpha| \leq k$,

$$D^\alpha (Eu)\left(\mathbf{x}', x_n\right) \equiv \begin{cases} D^\alpha u\left(\mathbf{x}', x_n\right) \text{ if } x_n < 0 \\ \sum_{j=1}^k \lambda_j (-j)^{\alpha_n} \left(D^\alpha u\right)\left(\mathbf{x}', -jx_n\right) \text{ if } x_n \geq 0 \end{cases}$$

It follows that E is linear and there exists a constant, C independent of u such that 40.36 holds. This proves the lemma.

Corollary 40.49 *Let H^- be the half space of 40.35. There exists E with the property that $E : W^{l,p}(H^-) \to W^{l,p}(\mathbb{R}^n)$ and is linear and continuous for each $l \leq k$.*

Proof: This immediate from the density of $C_c^\infty\left(\overline{H^-}\right)$ in $W^{k,p}\left(\overline{H^-}\right)$ and Lemma 40.48.

There is nothing sacred about a half space or this particular half space. It is clear that everything works as well for a half space of the form

$$H_k^- \equiv \{\mathbf{x} : x_k < 0\}.$$

Thus the half space featured in the above discussion is H_n^-.

Corollary 40.50 *Let $\{k_1, \cdots, k_r\} \subseteq \{1, \cdots, n\}$ where the k_i are distinct and let*

$$H_{k_1\cdots k_r}^- \equiv H_{k_1}^- \cap H_{k_2}^- \cap \cdots \cap H_{k_r}^-. \tag{40.37}$$

Then there exists $E : W^{k,p}\left(H_{k_1\cdots k_r}^-\right) \to W^{k,p}(\mathbb{R}^n)$ such that E is linear and continuous.

Proof: Follow the above argument with minor modifications to first extend from $H_{k_1\cdots k_r}^-$ to $H_{k_1\cdots k_{r-1}}^-$ and then from from $H_{k_1\cdots k_{r-1}}^-$ to $H_{k_1\cdots k_{r-2}}^-$ etc.

This easily implies the ability to extend off bounded open sets which near their boundaries look locally like an intersection of half spaces.

Theorem 40.51 *Let U be a bounded open set and suppose $U_0, U_1, \cdots, U_m$ are open sets with the property that $\overline{U} \subseteq \cup_{k=0}^m U_k, \overline{U_0} \subseteq U$, and $\partial U \subseteq \cup_{k=1}^m U_k$. Suppose also there exist one to one and onto functions, $\mathbf{h}_k : \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{h}_k(U_k \cap U) = W_k$ where W_k equals the intersection of a bounded open set with a finite intersection of half spaces, $H_{k_1\cdots k_r}^-$, as in 40.37 such that $\mathbf{h}_k(\partial U \cap U_k) \subseteq \partial H_{k_1\cdots k_r}^-$. Suppose also that for all $|\alpha| \leq k-1$,*

$$D^\alpha \mathbf{h}_k \text{ and } D^\alpha \mathbf{h}_k^{-1}$$

exist and are Lipschitz continuous. Then there letting W be an open set which contains $\overline{U}$, there exists $E : W^{k,p}(U) \to W^{k,p}(W)$ such that E is a linear continuous map from $W^{l,p}(U)$ to $W^{l,p}(W)$ for each $l \leq k$.

Proof: Let $\psi_j \in C_c^\infty(U_j)$, $\psi_j(\mathbf{x}) \in [0,1]$ for all $\mathbf{x} \in \mathbb{R}^n$, and $\sum_{j=0}^m \psi_j(\mathbf{x}) = 1$ on $\overline{U}$. This is a C^∞ partition of unity on $\overline{U}$. By Theorem 40.14 $\left(\mathbf{h}_j^{-1}\right)^* u\psi_j \in W^{k,p}(W_j)$. By the assumption that $\mathbf{h}_j(\partial U \cap U_j) \subseteq \partial H_{k_1\cdots k_r}^-$, the zero extension of $\left(\mathbf{h}_j^{-1}\right)^* u\psi_j$ to the rest of $H_{k_1\cdots k_r}^-$ results in an element of $W^{k,p}\left(H_{k_1\cdots k_r}^-\right)$. Apply Corollary 40.50 to conclude there exists $E_j : W^{k,p}\left(H_{k_1\cdots k_r}^-\right) \to W^{k,p}(\mathbb{R}^n)$ which is continuous and linear. Abusing notation slightly, by using $\left(\mathbf{h}_j^{-1}\right)^* u\psi_j$ as the above zero extension, it follows $E_j\left(\left(\mathbf{h}_j^{-1}\right)^* u\psi_j\right) \in W^{k,p}(\mathbb{R}^n)$. Now let η be a function in $C_c^\infty(\mathbf{h}(W))$ such that $\eta(\mathbf{y}) = 1$ on $\mathbf{h}\left(\overline{U}\right)$. Then Define

$$Eu \equiv \sum_{j=0}^m \mathbf{h}_j^* \eta E_j\left(\left(\mathbf{h}_j^{-1}\right)^*(u\psi_j)\right).$$

Clearly $Eu(\mathbf{x}) = u(\mathbf{x})$ if $\mathbf{x} \in U$. It is also clear that E is linear. It only remains to verify E is continuous. In what follows, C_j will denote a constant which is independent of u which may change from line to line. By Theorem 40.14,

$$\begin{aligned}
||Eu||_{k,p,W} &\leq \sum_{j=0}^{m} \left|\left|\mathbf{h}_j^* \eta E_j \left(\left(\mathbf{h}_j^{-1}\right)^* (u\psi_j)\right)\right|\right|_{k,p,W} \\
&\leq \sum_{j=0}^{m} C_j \left|\left|\eta E_j \left(\left(\mathbf{h}_j^{-1}\right)^* (u\psi_j)\right)\right|\right|_{k,p,\mathbf{h}(W)} \\
&= \sum_{j=0}^{m} C_j \left|\left|\eta E_j \left(\left(\mathbf{h}_j^{-1}\right)^* (u\psi_j)\right)\right|\right|_{k,p,\mathbb{R}^n} \\
&\leq \sum_{j=0}^{m} C_j \left|\left|E_j \left(\left(\mathbf{h}_j^{-1}\right)^* (u\psi_j)\right)\right|\right|_{k,p,\mathbb{R}^n} \\
&\leq \sum_{j=0}^{m} C_j \left|\left|\left(\mathbf{h}_j^{-1}\right)^* (u\psi_j)\right|\right|_{k,p,\mathbf{h}_j(U\cap U_j)} \\
&\leq \sum_{j=0}^{m} C_j \left|\left|u\psi_j\right|\right|_{k,p,U\cap U_k} \\
&\leq \sum_{j=0}^{m} C_j \left|\left|u\right|\right|_{k,p,U\cap U_k} \leq \left(\sum_{j=0}^{m} C_j\right) ||u||_{k,p,U}.
\end{aligned}$$

Similarly $E : W^{l,p}(U) \to W^{l,p}(U)$ for $l \leq k$. This proves the theorem.

Definition 40.52 *When E is a linear continuous map from $W^{l,p}(U)$ to $W^{l,p}(\mathbb{R}^n)$ for each $l \leq k$. it is called a strong (k,p) extension map.*

There is also a very easy sort of extension theorem for the space, $W_0^{m,p}(U)$ which does not require any assumptions on the boundary of U other than $m_n(\partial U) = 0$. First here is the definition of $W_0^{m,p}(U)$.

Definition 40.53 *Denote by $W_0^{m,p}(U)$ the closure of $C_c^\infty(U)$ in $W^{m,p}(U)$.*

Theorem 40.54 *For $u \in W_0^{m,p}(U)$, define*

$$Eu(\mathbf{x}) \equiv \begin{cases} u(\mathbf{x}) \text{ if } \mathbf{x} \in U \\ 0 \text{ if } \mathbf{x} \notin U \end{cases}$$

Then E is a strong (k,p) extension map.

Proof: Letting $l \leq m$, it is clear that for $|\alpha| \leq l$,

$$D^\alpha Eu = \begin{cases} D^\alpha u \text{ for } \mathbf{x} \in U \\ 0 \text{ for } \mathbf{x} \notin U \end{cases}.$$

This follows because, since $m_n(\partial U) = 0$ it suffices to consider $\phi \in C_c^\infty(U)$ and $\phi \in C_c^\infty\left(\overline{U}^C\right)$. Therefore, $||Eu||_{l,p,\mathbb{R}^n} = ||u||_{l,p,U}$.

There are many other extension theorems and if you are interested in pursuing this further, consult Adams [1]. One of the most famous which is discussed in this reference is due to Calderon and depends on the theory of singular integrals.

Sobolev Spaces Based On L^2

41.1 Fourier Transform Techniques

Much insight can be obtained easily through the use of Fourier transform methods. This technique will be developed in this chapter. When this is done, it is necessary to use Sobolev spaces of the form $W^{k,2}(U)$, those Sobolev spaces which are based on $L^2(U)$. It is true there are generalizations which use Fourier transform methods in the context of L^p but the spaces so considered are called Bessel potential spaces. They are not really Sobolev spaces. Furthermore, it is Mihlin's theorem rather than the Plancherel theorem which is the main tool of the analysis. This is a hard theorem.

It is convenient to consider the Schwartz class of functions,$\mathfrak{S}$. These are functions which have infinitely many derivatives and vanish quickly together with their derivatives as $|\mathbf{x}| \to \infty$. In particular, $C_c^\infty(\mathbb{R}^n)$ is contained in $\mathfrak{S}$ which is not true of the functions, $\mathcal{G}$ used earlier in defining the Fourier transforms which are a suspace of $\mathfrak{S}$. Recall the following definition.

Definition 41.1 *$f \in \mathfrak{S}$, the Schwartz class, if $f \in C^\infty(\mathbb{R}^n)$ and for all positive integers N,*

$$\rho_N(f) < \infty$$

where

$$\rho_N(f) = \sup\{(1 + |\mathbf{x}|^2)^N |D^\alpha f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n , \ |\alpha| \leq N\}.$$

Thus $f \in \mathfrak{S}$ if and only if $f \in C^\infty(\mathbb{R}^n)$ and

$$\sup\{|\mathbf{x}^\beta D^\alpha f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\} < \infty \tag{41.1}$$

for all multi indices α and β.

Thus all partial derivatives of a function in $\mathfrak{S}$ are in $L^p(\mathbb{R}^n)$ for all $p \geq 1$. Therefore, for $f \in \mathfrak{S}$, the Fourier and inverse Fourier transforms are given in the usual way,

$$Ff(\mathbf{t}) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{t}\cdot\mathbf{x}} dx, \ F^{-1}f(\mathbf{t}) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{i\mathbf{t}\cdot\mathbf{x}} dx.$$

Also recall that the Fourier transform and its inverse are one to one and onto maps from $\mathfrak{S}$ to $\mathfrak{S}$.

To tie the Fourier transform technique in with what has been done so far, it is necessary to make the following assumption on the set, U. This assumption is made so that it is possible to consider elements of $W^{k,2}(U)$ as restrictions of elements of $W^{k,2}(\mathbb{R}^n)$.

Assumption 41.2 *Assume U satisfies the segment condition and that for any m of interest, there exists $E \in \mathcal{L}(W^{m,p}(U), W^{m,p}(\mathbb{R}^n))$ such that for each $k \leq m$, $E \in \mathcal{L}\left(W^{k,p}(U), W^{k,p}(\mathbb{R}^n)\right)$. That is, there exists a stong (m,p) extension operator.*

Lemma 41.3 *The Schwartz class, $\mathfrak{S}$, is dense in $W^{m,p}(\mathbb{R}^n)$.*

Proof: The set, $\mathbb{R}^n$ satisfies the segment condition and so $C_c^{\infty}(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$. However, $C_c^{\infty}(\mathbb{R}^n) \subseteq \mathfrak{S}$. This proves the lemma.

Recall now Plancherel's theorem which states that $||f||_{0,2,\mathbb{R}^n} = ||Ff||_{0,2,\mathbb{R}^n}$ whenever $f \in L^2(\mathbb{R}^n)$. Also it is routine to verify from the definition of the Fourier transform that for $u \in \mathfrak{S}$,

$$F\partial_k u = ix_k Fu.$$

From this it follows that

$$||D^{\alpha}u||_{0,2,\mathbb{R}^n} = ||\mathbf{x}^{\alpha}Fu||_{0,2,\mathbb{R}^n}.$$

Here $\mathbf{x}^{\alpha}$ denotes the function $\mathbf{x} \to \mathbf{x}^{\alpha}$. Therefore,

$$||u||_{m,2,\mathbb{R}^n} = \left(\int_{\mathbb{R}^n} \sum_{|\alpha| \leq m} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} |Fu(\mathbf{x})|^2 \, dx\right)^{1/2}.$$

Also, it is not hard to verify that

$$\sum_{|\alpha| \leq m} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \leq \left(1 + \sum_{j=1}^{n} x_j^2\right)^m \leq C(n,m) \sum_{|\alpha| \leq m} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n}$$

where $C(n,m)$ is the largest of the multinomial coefficients obtained in the expansion,

$$\left(1 + \sum_{j=1}^{n} x_j^2\right)^m.$$

Therefore, for all $u \in \mathfrak{S}$,

$$||u||_{m,2,\mathbb{R}^n} \leq \left(\int_{\mathbb{R}^n} \left(1 + |\mathbf{x}|^2\right)^m |Fu(\mathbf{x})|^2 \, dx\right)^{1/2} \leq C(n,m) \, ||u||_{m,2,\mathbb{R}^n}. \quad (41.2)$$

This motivates the following definition.

Definition 41.4 *Let* $H^m(\mathbb{R}^n) \equiv$

$$\left\{ u \in L^2(\mathbb{R}^n) : ||u||_{H^m(\mathbb{R}^n)} \equiv \left(\int_{\mathbb{R}^n} \left(1 + |\mathbf{x}|^2\right)^m |Fu(\mathbf{x})|^2 \, dx \right)^{1/2} < \infty \right\}. \quad (41.3)$$

Lemma 41.5 $\mathfrak{S}$ *is dense in* $H^m(\mathbb{R}^n)$ *and* $H^m(\mathbb{R}^n) = W^{2,m}(\mathbb{R}^n)$. *Furthermore, the norms are equivalent.*

Proof: First it is shown that $\mathfrak{S}$ is dense in $H^m(\mathbb{R}^n)$. Let $u \in H^m(\mathbb{R}^n)$. Let $\mu(E) \equiv \int_E \left(1 + |\mathbf{x}|^2\right)^m dx$. Thus μ is a regular measure and $u \in H^m(\mathbb{R}^n)$ just means that $Fu \in L^2(\mu)$, the space of functions which are in $L^2(\mathbb{R}^n)$ with respect to this measure, μ. Therefore, from the regularity of the measure, μ, there exists $u_k \in C_c(\mathbb{R}^n)$ such that

$$||u_k - Fu||_{L^2(\mu)} \to 0.$$

Now let ψ_ε be a mollifier and pick ε_k small enough that

$$\left|\left|u_k * \psi_{\varepsilon_k} - u_k\right|\right|_{L^2(\mu)} < \frac{1}{2^k}.$$

Then $u_k * \psi_{\varepsilon_k} \in C_c^\infty(\mathbb{R}^n) \subseteq \mathfrak{S}$. Therefore, there exists $w_k \in \mathcal{G}$ such that $Fw_k = u_k * \psi_{\varepsilon_k}$. It follows

$$||Fw_k - Fu||_{L^2(\mu)} \leq ||Fw_k - u_k||_{L^2(\mu)} + ||u_k - Fu||_{L^2(\mu)}$$

and these last two terms converge to 0 as $k \to \infty$. Therefore, $w_k \to u$ in $H^m(\mathbb{R}^n)$ and this proves the first part of this lemma.

Now let $u \in H^m(\mathbb{R}^n)$. By what was just shown, there exists a sequence, $u_k \to u$ in $H^m(\mathbb{R}^n)$ where $u_k \in \mathfrak{S}$. It follows from 41.2 that

$$||u_k - u_l||_{H^m} \geq ||u_k - u_l||_{m,2,\mathbb{R}^n}$$

and so $\{u_k\}$ is a Cauchy sequence in $W^{m,2}(\mathbb{R}^n)$. Therefore, there exists $w \in W^{m,2}(\mathbb{R}^n)$ such that

$$||u_k - w||_{m,2,\mathbb{R}^n} \to 0.$$

But this implies

$$0 = \lim_{k\to\infty} ||u_k - w||_{0,2,\mathbb{R}^n} = \lim_{k\to\infty} ||u_k - u||_{0,2,\mathbb{R}^n}$$

showing $u = w$ which verifies $H^m(\mathbb{R}^n) \subseteq W^{2,m}(\mathbb{R}^n)$. The opposite inclusion is proved the same way, using density of $\mathfrak{S}$ and the fact that the norms in both spaces are larger than the norms in $L^2(\mathbb{R}^n)$. The equivalence of the norms follows from the density of $\mathfrak{S}$ and the equivalence of the norms on $\mathfrak{S}$. This proves the lemma.

The conclusion of this lemma with the density of $\mathfrak{S}$ and 41.2 implies you can use either norm, $||u||_{H^m(\mathbb{R}^n)}$ or $||u||_{m,2,\mathbb{R}^n}$ when working with these Sobolev spaces.

What of open sets satisfying Assumption 41.2? How does $W^{m,2}(U)$ relate to the Fourier transform?

Definition 41.6 *Let U be an open set in $\mathbb{R}^n$. Then*

$$H^m(U) \equiv \{u : u = v|_U \text{ for some } v \in H^m(\mathbb{R}^n)\} \tag{41.4}$$

Here the notation, $v|_U$ means v restricted to U. Define the norm in this space by

$$||u||_{H^m(U)} \equiv \inf\left\{||v||_{H^m(\mathbb{R}^n)} : v|_U = u\right\}. \tag{41.5}$$

Lemma 41.7 *$H^m(U)$ is a Banach space.*

Proof: First it is necessary to verify that the given norm really is a norm. Suppose then that $u = 0$. Is $||u||_{H^m(U)} = 0$? Of course it is. Just take $v \equiv 0$. Then $v|_U = u$ and $||v||_{H^m} = 0$. Next suppose $||u||_{H^m(U)} = 0$. Does it follow that $u = 0$? Letting $\varepsilon > 0$ be given, there exists $v \in H^m(\mathbb{R}^n)$ such that $v|_U = u$ and $||v||_{H^m(\mathbb{R}^n)} < \varepsilon$. Therefore,

$$||u||_{0,U} \leq ||v||_{0,\mathbb{R}^n} \leq ||v||_{H^m(U)} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows $u = 0$ a.e. Next suppose $u_i \in H^m(U)$ for $i = 1, 2$. There exists $v_i \in H^m(\mathbb{R}^n)$ such that

$$||v_i||_{H^m(\mathbb{R}^n)} < ||u_i||_{H^m(U)} + \varepsilon.$$

Therefore,

$$\begin{aligned} ||u_1 + u_2||_{H^m(U)} &\leq ||v_1 + v_2||_{H^m(\mathbb{R}^n)} \leq ||v_1||_{H^m(\mathbb{R}^n)} + ||v_2||_{H^m(\mathbb{R}^n)} \\ &\leq ||u_1||_{H^m(U)} + ||u_2||_{H^m(U)} + 2\varepsilon \end{aligned}$$

and since $\varepsilon > 0$ is arbitrary, this shows the triangle inequality.

The interesting question is the one about completeness. Suppose then $\{u_k\}$ is a Cauchy sequence in $H^m(U)$. There exists N_k such that if $k, l \geq N_k$, it follows $||u_k - u_l||_{H^m(U)} < \frac{1}{2^k}$ and the numbers, N_k can be taken to be strictly increasing in k. Thus for $l \geq N_k$, $||u_l - u_{N_k}||_{H^m(U)} < 1/2^l$. Therefore, there exists $w_l \in H^m(\mathbb{R}^n)$ such that

$$w_l|_U = u_l - u_{N_k},\ ||w_l||_{H^m(\mathbb{R}^n)} < \frac{1}{2^l}.$$

Also let $v_{N_k}|_U = u_{N_k}$ with $v_{N_k} \in H^m(\mathbb{R}^n)$ and

$$||v_{N_k}||_{H^m(\mathbb{R}^n)} < ||u_{N_k}||_{H^m(U)} + \frac{1}{2^k}.$$

Now for $l > N_k$, define v_l by $v_l - v_{N_k} = w_{N_k}$ so that $||v_l - v_{N_k}||_{H^m(\mathbb{R}^n)} < 1/2^k$. In particular,

$$\left|\left|v_{N_{k+1}} - v_{N_k}\right|\right|_{H^m(\mathbb{R}^n)} < 1/2^k$$

which shows that $\{v_{N_k}\}_{k=1}^{\infty}$ is a Cauchy sequence. Consequently it must converge to $v \in H^m(\mathbb{R}^n)$. Let $u = v|_U$. Then

$$||u - u_{N_k}||_{H^m(U)} \leq ||v - v_{N_k}||_{H^m(\mathbb{R}^n)}$$

which shows the subsequence, $\{u_{N_k}\}_k$ converges to u. Since $\{u_k\}$ is a Cauchy sequence, it follows it too must converge to u. This proves the lemma.

The main result is next.

Theorem 41.8 *Suppose U satisfies Assumption 41.2. Then for m a nonnegative integer, $H^m(U) = W^{m,2}(U)$ and the two norms are equivalent.*

Proof: Let $u \in H^m(U)$. Then there exists $v \in H^m(\mathbb{R}^n)$ such that $v|_U = u$. Hence $v \in W^{k,2}(\mathbb{R}^n)$ and so all its weak derivatives up to order m are in $L^2(\mathbb{R}^n)$. Therefore, the restrictions of these weak derivitves are in $L^2(U)$. Since U satisfies the segment condition, it follows $u \in W^{m,2}(U)$ which shows $H^m(U) \subseteq W^{m,2}(U)$.

Next take $u \in W^{m,2}(U)$. Then $Eu \in W^{m,2}(\mathbb{R}^n) = H^m(\mathbb{R}^n)$ and this shows $u \in H^m(U)$. This has shown the two spaces are the same. It remains to verify their norms are equivalent. Let $u \in H^m(U)$ and let $v|_U = u$ where $v \in H^m(\mathbb{R}^n)$ and

$$||u||_{H^m(U)} + \varepsilon > ||v||_{H^m(\mathbb{R}^n)}.$$

Then recalling that $||\cdot||_{H^m(\mathbb{R}^n)}$ and $||\cdot||_{m,2,\mathbb{R}^n}$ are equivalent norms for $H^m(\mathbb{R}^n)$, there exists a constant, C such that

$$||u||_{H^m(U)} + \varepsilon > ||v||_{H^m(\mathbb{R}^n)} \geq C\,||v||_{m,2,\mathbb{R}^n} \geq C\,||u||_{m,2,U}$$

Now consider the two Banach spaces,

$$\left(H^m(U), ||\cdot||_{H^m(U)}\right), \left(W^{m,2}(U), ||\cdot||_{m,2,U}\right).$$

The above inequality shows since $\varepsilon > 0$ is arbitrary that $\text{id} : \left(H^m(U), ||\cdot||_{H^m(U)}\right) \to \left(W^{m,2}(U), ||\cdot||_{m,2,U}\right)$ is continuous. By the open mapping theorem, it follows id is continuous in the other direction. Thus there exists a constant, K such that $||u||_{H^m(U)} \leq K\,||u||_{k,2,U}$. Hence the two norms are equivalent as claimed.

Specializing Corollary 40.43 and Theorem 40.46 starting on Page 1166 to the case of $p = 2$ while also assuming more on U yields the following embedding theorems.

Theorem 41.9 *Suppose $m \geq 0$ and j is a nonnegative integer satisfying $2j < n$. Also suppose U is an open set which satisfies Assumption 41.2. Then $\text{id} \in \mathcal{L}\left(H^{m+j}(U), W^{m,q}(U)\right)$ where*

$$q \equiv \frac{2n}{n - 2j}. \tag{41.6}$$

If, in addition to the above, U is bounded and $1 \leq r < q$, then

$$\text{id} \in \mathcal{L}\left(H^{m+j}(U), W^{m,r}(U)\right)$$

and is compact.

Theorem 41.10 *Suppose for j a nonnegative integer, $2j < n < 2(j+1)$ and let m be a positive integer. Let U be any bounded open set in $\mathbb{R}^n$ which satisfies Assumption 41.2. Then $\text{id} \in \mathcal{L}\left(H^{m+j}(U), C^{m-1,\lambda}\left(\overline{U}\right)\right)$ for every $\lambda \leq \lambda_0 \equiv (j+1) - \frac{n}{2}$ and if $\lambda < (j+1) - \frac{n}{2}$, id is compact.*

41.2 Fractional Order Spaces

What has been gained by all this? The main thing is that $H^{m+s}(U)$ makes sense for any $s \in (0,1)$ and m an integer. You simply replace m with $m+s$ in the above for $s \in (0,1)$. This gives what is meant by $H^{m+s}(\mathbb{R}^n)$

Definition 41.11 *For m an integer and $s \in (0,1)$, let $H^{m+s}(\mathbb{R}^n) \equiv$*

$$\left\{ u \in L^2(\mathbb{R}^n) : ||u||_{H^{m+s}(\mathbb{R}^n)} \equiv \left(\int_{\mathbb{R}^n} \left(1 + |\mathbf{x}|^2\right)^{m+s} |Fu(\mathbf{x})|^2 \, dx \right)^{1/2} < \infty \right\}. \tag{41.7}$$

You could also simply refer to $H^t(\mathbb{R}^n)$ where t is a real number replacing the $m+s$ in the above formula with t but I want to emphasize the notion that $t = m+s$ where m is a nonnegative integer. Therefore, I will often write $m+s$. Let U be an open set in $\mathbb{R}^n$. Then

$$H^{m+s}(U) \equiv \left\{ u : u = v|_U \text{ for some } v \in H^{m+s}(\mathbb{R}^n) \right\}. \tag{41.8}$$

Define the norm in this space by

$$||u||_{H^{m+s}(U)} \equiv \inf \left\{ ||v||_{H^{m+s}(\mathbb{R}^n)} : v|_U = u \right\}. \tag{41.9}$$

Lemma 41.12 *$H^{m+s}(U)$ is a Banach space.*

Proof: Just repeat the proof of Lemma 41.7.

The theorem about density of $\mathfrak{S}$ also remains true in $H^{m+s}(\mathbb{R}^n)$. Just repeat the proof of that part of Lemma 41.5 replacing the integer, m, with the symbol, $m+s$.

Lemma 41.13 *$\mathfrak{S}$ is dense in $H^{m+s}(\mathbb{R}^n)$.*

In fact, more can be said.

Corollary 41.14 *Let U be an open set and let $\mathfrak{S}|_U$ denote the restrictions of functions of $\mathfrak{S}$ to U. Then $\mathfrak{S}|_U$ is dense in $H^t(U)$.*

Proof: Let $u \in H^t(U)$ and let $v \in H^t(\mathbb{R}^n)$ such that $v|_U = u$ a.e. Then since $\mathfrak{S}$ is dense in $H^t(\mathbb{R}^n)$, there exists $w \in \mathfrak{S}$ such that

$$||w - v||_{H^t(\mathbb{R}^n)} < \varepsilon.$$

It follows that

$$\begin{aligned} ||u - w||_{H^t(U)} &\leq ||u - v||_{H^t(U)} + ||v - w||_{H^t(U)} \\ &\leq 0 + ||v - w||_{H^t(\mathbb{R}^n)} < \varepsilon. \end{aligned}$$

These fractional order spaces are important when trying to understand the trace on the boundary. The Fourier transform description also makes it very easy to establish interesting inequalities such as interpolation inequalities.

Lemma 41.15 *Let $0 \leq r < s < t$. Then if $u \in H^t(\mathbb{R}^n)$,*

$$||u||_{H^s(\mathbb{R}^n)} \leq ||u||_{H^r(\mathbb{R}^n)}^{\theta} ||u||_{H^t(\mathbb{R}^n)}^{1-\theta}$$

where θ is a positve number such that $\theta r + (1-\theta)t = s$.

Proof: This follows from Holder's inequality applied to the measure μ given by

$$\mu(E) = \int_E |Fu|^2 \, dx$$

Thus

$$\begin{aligned} & \int \left(1+|\mathbf{x}|^2\right)^s |Fu|^2 \, dx \\ = & \int \left(1+|\mathbf{x}|^2\right)^{r\theta} \left(1+|\mathbf{x}|^2\right)^{(1-\theta)t} |Fu|^2 \, dx \\ \leq & \left(\int \left(1+|\mathbf{x}|^2\right)^r |Fu|^2 \, dx\right)^{\theta} \left(\int \left(1+|\mathbf{x}|^2\right)^{(1-\theta)t} |Fu|^2 \, dx\right)^{1-\theta} \\ = & ||u||_{H^r(\mathbb{R}^n)}^{2\theta} ||u||_{H^t(\mathbb{R}^n)}^{2(1-\theta)}. \end{aligned}$$

Taking square roots yields the desired inequality.

Corollary 41.16 *Let U be an open set satisfying Assumption 41.2 and let $p < q$ where p, q are two nonnegative integers. Also let $t \in (p, q)$. Then exists a constant, C independent of $u \in H^q(U)$ such that for all $u \in H^q(U)$,*

$$||u||_{H^t(U)} \leq C ||u||_{H^p(U)}^{\theta} ||u||_{H^q(U)}^{1-\theta}$$

where θ is such that $t = \theta p + (1-\theta) q$.

Proof: Let $E \in \mathcal{L}(H^q(U), H^q(\mathbb{R}^n))$ such that for all positive integers, l less than or equal to q, $E \in \mathcal{L}\left(H^l(U), H^l(\mathbb{R}^n)\right)$. Then $Eu|_U = u$ and $Eu \in H^t(\mathbb{R}^n)$. Therefore, by Lemma 41.15,

$$\begin{aligned} ||u||_{H^t(U)} & \leq ||Eu||_{H^t(\mathbb{R}^n)} \leq ||Eu||_{H^p(\mathbb{R}^n)}^{\theta} ||Eu||_{H^q(\mathbb{R}^n)}^{1-\theta} \\ & \leq C ||u||_{H^p(U)}^{\theta} ||u||_{H^q(U)}^{1-\theta}. \end{aligned}$$

Now recall the very important Theorem 40.14 on Page 1143 which is listed here for convenience.

Theorem 41.17 *Let $\mathbf{h} : U \to V$ be one to one and onto where U and V are two open sets. Also suppose that $D^{\alpha}\mathbf{h}$ and $D^{\alpha}\left(\mathbf{h}^{-1}\right)$ exist and are Lipschitz continuous if $|\alpha| \leq m-1$ for m a positive integer. Then*

$$\mathbf{h}^* : W^{m,p}(V) \to W^{m,p}(U)$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $\left(\mathbf{h}^{-1}\right)^*$.

Is there something like this for the fractional order spaces? Yes there is. However, in order to prove it, it is convenient to use an equivalent norm for $H^{m+s}(\mathbb{R}^n)$ which does not depend explicitly on the Fourier transform. The following theorem is similar to one in [28]. It describes the norm in $H^{m+s}(\mathbb{R}^n)$ in terms which are free of the Fourier transform. This is also called an intrinsic norm [1].

Theorem 41.18 *Let $s \in (0,1)$ and let m be a nonnegative integer. Then an equivalent norm for $H^{m+s}(\mathbb{R}^n)$ is*

$$|||u|||_{m+s}^2 \equiv ||u||_{m,2,\mathbb{R}^n}^2 + \sum_{|\alpha|=m} \int\int |D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2 |\mathbf{x}-\mathbf{y}|^{-n-2s} \, dxdy.$$

Also if $|\beta| \leq m$, there are constants, $m(s)$ and $M(s)$ such that

$$m(s) \int |Fu(\mathbf{z})|^2 \left|\mathbf{z}^\beta\right|^2 |\mathbf{z}|^{2s} \, dz \leq \int\int \left|D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})\right|^2 |\mathbf{x}-\mathbf{y}|^{-n-2s} \, dxdy$$
$$\leq M(s) \int |Fu(\mathbf{z})|^2 \left|\mathbf{z}^\beta\right|^2 |\mathbf{z}|^{2s} \, dz \tag{41.10}$$

Proof: Let $u \in \mathfrak{S}$ which is dense in $H^{m+s}(\mathbb{R}^n)$. The Fourier transform of the function, $\mathbf{y} \to D^\alpha u(\mathbf{x}+\mathbf{y}) - D^\alpha u(\mathbf{y})$ equals

$$\left(e^{i\mathbf{x}\cdot\mathbf{z}} - 1\right) FD^\alpha u(\mathbf{z}).$$

Now by Fubini's theorem and Plancherel's theorem along with the above, taking $|\alpha| = m$,

$$\int\int |D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2 |\mathbf{x}-\mathbf{y}|^{-n-2s} \, dxdy$$
$$= \int\int |D^\alpha u(\mathbf{y}+\mathbf{t}) - D^\alpha u(\mathbf{y})|^2 |\mathbf{t}|^{-n-2s} \, dtdy$$
$$= \int |\mathbf{t}|^{-n-2s} \int |D^\alpha u(\mathbf{y}+\mathbf{t}) - D^\alpha u(\mathbf{y})|^2 \, dydt$$
$$= \int |\mathbf{t}|^{-n-2s} \int \left|\left(e^{i\mathbf{t}\cdot\mathbf{z}} - 1\right) FD^\alpha u(\mathbf{z})\right|^2 dzdt$$
$$= \int |FD^\alpha u(\mathbf{z})|^2 \left(\int |\mathbf{t}|^{-n-2s} \left|\left(e^{i\mathbf{t}\cdot\mathbf{z}} - 1\right)\right|^2 dt\right) dz. \tag{41.11}$$

Consider the inside integral, the one taken with respect to $\mathbf{t}$.

$$G(\mathbf{z}) \equiv \left(\int |\mathbf{t}|^{-n-2s} \left|\left(e^{i\mathbf{t}\cdot\mathbf{z}} - 1\right)\right|^2 dt\right).$$

The essential thing to notice about this function of $\mathbf{z}$ is that it is a positive real number whenever $\mathbf{z} \neq \mathbf{0}$. This is because for small $|\mathbf{t}|$, the integrand is dominated by $C|\mathbf{t}|^{-n+2(1-s)}$. Changing to polar coordinates, you see that

$$\int_{[|\mathbf{t}|\leq 1]} |\mathbf{t}|^{-n-2s} \left|\left(e^{i\mathbf{t}\cdot\mathbf{z}} - 1\right)\right|^2 dt < \infty$$

Next, for $|\mathbf{t}| > 1$, the integrand is bounded by $4|\mathbf{t}|^{-n-2s}$, and changing to polar coordinates shows

$$\int_{[|\mathbf{t}|>1]} |\mathbf{t}|^{-n-2s} \left|\left(e^{i\mathbf{t}\cdot\mathbf{z}}-1\right)\right|^2 dt \leq 4\int_{[|\mathbf{t}|>1]} |\mathbf{t}|^{-n-2s}\, dt < \infty.$$

Now for $\alpha > 0$,

$$\begin{aligned} G(\alpha\mathbf{z}) &= \int |\mathbf{t}|^{-n-2s}\left|\left(e^{i\mathbf{t}\cdot\alpha\mathbf{z}}-1\right)\right|^2 dt \\ &= \int |\mathbf{t}|^{-n-2s}\left|\left(e^{i\alpha\mathbf{t}\cdot\mathbf{z}}-1\right)\right|^2 dt \\ &= \int \left|\frac{\mathbf{r}}{\alpha}\right|^{-n-2s}\left|\left(e^{i\mathbf{r}\cdot\mathbf{z}}-1\right)\right|^2 \frac{1}{\alpha^n} dr \\ &= \alpha^{2s}\int |\mathbf{r}|^{-n-2s}\left|\left(e^{i\mathbf{r}\cdot\mathbf{z}}-1\right)\right|^2 dr = \alpha^{2s}G(\mathbf{z}). \end{aligned}$$

Also G is continuous and strictly positive. Letting

$$0 < m(s) = \min\{G(\mathbf{w}) : |\mathbf{w}| = 1\}$$

and

$$M(s) = \max\{G(\mathbf{w}) : |\mathbf{w}| = 1\},$$

it follows from this, and letting $\alpha = |\mathbf{z}|, \mathbf{w} \equiv \mathbf{z}/|\mathbf{z}|$, that

$$G(\mathbf{z}) \in \left(m(s)|\mathbf{z}|^{2s}, M(s)|\mathbf{z}|^{2s}\right).$$

More can be said but this will suffice. Also observe that for $s \in (0,1)$ and $b > 0$,

$$(1+b)^s \leq 1+b^s,\ 2^{1-s}(1+b)^s \geq 1+b^s.$$

In what follows, $C(s)$ will denote a constant which depends on the indicated quantities which may be different on different lines of the argument. Then from 41.11,

$$\begin{aligned} &\int\int |D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2 |\mathbf{x}-\mathbf{y}|^{-n-2s}\, dxdy \\ &\leq M(s)\int |FD^\alpha u(\mathbf{z})|^2 |\mathbf{z}|^{2s}\, dz \\ &= M(s)\int |Fu(\mathbf{z})|^2 |\mathbf{z}^\alpha|^2 |\mathbf{z}|^{2s}\, dz. \end{aligned}$$

No referrence was made to $|\alpha| = m$ and so this establishes the top half of 41.10. Therefore,

$$\begin{aligned} |||u|||_{m+s}^2 &\equiv ||u||_{m,2,\mathbb{R}^n}^2 + \sum_{|\alpha|=m}\int\int |D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2 |\mathbf{x}-\mathbf{y}|^{-n-2s}\, dxdy \\ &\leq C\int\left(1+|\mathbf{z}|^2\right)^m |Fu(\mathbf{z})|^2\, dz + M(s)\int |Fu(\mathbf{z})|^2 \sum_{|\alpha|=m} |\mathbf{z}^\alpha|^2 |\mathbf{z}|^{2s}\, dz \end{aligned}$$

Recall that

$$\sum_{|\alpha|\leq m} z_1^{2\alpha_1}\cdots z_n^{2\alpha_n} \leq \left(1+\sum_{j=1}^n z_j^2\right)^m \leq C(n,m)\sum_{|\alpha|\leq m} z_1^{2\alpha_1}\cdots z_n^{2\alpha_n}. \quad (41.12)$$

Therefore, where $C(n,m)$ is the largest of the multinomial coefficients obtained in the expansion,

$$\left(1+\sum_{j=1}^n z_j^2\right)^m .$$

Therefore,

$$\begin{aligned} & |||u|||_{m+s}^2 \\ \leq\ & C\int\left(1+|\mathbf{z}|^2\right)^m |Fu(\mathbf{z})|^2\,dz + M(s)\int |Fu(\mathbf{z})|^2 \sum_{|\alpha|=m}|\mathbf{z}^\alpha|^2 |\mathbf{z}|^{2s}\,dz \\ \leq\ & C\int\left(1+|\mathbf{z}|^2\right)^{m+s} |Fu(\mathbf{z})|^2\,dz + M(s)\int |Fu(\mathbf{z})|^2 \left(1+|\mathbf{z}|^2\right)^m |\mathbf{z}|^{2s}\,dz \\ \leq\ & C\int\left(1+|\mathbf{z}|^2\right)^{m+s} |Fu(\mathbf{z})|^2\,dz = C\,||u||_{H^{m+s}(\mathbb{R}^n)}. \end{aligned}$$

It remains to show the other inequality. From 41.11,

$$\int\int |D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2 |\mathbf{x}-\mathbf{y}|^{-n-2s}\,dxdy$$

$$\begin{aligned} \geq\ & m(s)\int |FD^\alpha u(\mathbf{z})|^2 |\mathbf{z}|^{2s}\,dz \\ =\ & m(s)\int |Fu(\mathbf{z})|^2 |\mathbf{z}^\alpha|^2 |\mathbf{z}|^{2s}\,dz. \end{aligned}$$

No reference was made to $|\alpha| = m$ and so this establishes the bottom half of 41.10. Therefore, from 41.12,

$$\begin{aligned} & |||u|||_{m+s}^2 \\ \geq\ & C\int\left(1+|\mathbf{z}|^2\right)^m |Fu(\mathbf{z})|^2\,dz + m(s)\int |Fu(\mathbf{z})|^2 \sum_{|\alpha|=m}|\mathbf{z}^\alpha|^2 |\mathbf{z}|^{2s}\,dz \\ \geq\ & C\int\left(1+|\mathbf{z}|^2\right)^m |Fu(\mathbf{z})|^2\,dz + C\int |Fu(\mathbf{z})|^2 \left(1+|\mathbf{z}|^2\right)^m |\mathbf{z}|^{2s}\,dz \\ =\ & C\int\left(1+|\mathbf{z}|^2\right)^m \left(1+|\mathbf{z}|^{2s}\right)|Fu(\mathbf{z})|^2\,dz \\ \geq\ & C\int\left(1+|\mathbf{z}|^2\right)^m \left(1+|\mathbf{z}|^2\right)^s |Fu(\mathbf{z})|^2\,dz \\ =\ & C\int\left(1+|\mathbf{z}|^2\right)^{m+s} |Fu(\mathbf{z})|^2\,dz = ||u||_{H^{m+s}(\mathbb{R}^n)}. \end{aligned}$$

This proves the theorem.

With the above intrinsic norm, it becomes possible to prove the following version of Theorem 41.17.

Lemma 41.19 *Let $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^n$ be one to one and onto. Also suppose that $D^\alpha \mathbf{h}$ and $D^\alpha \left(\mathbf{h}^{-1}\right)$ exist and are Lipschitz continuous if $|\alpha| \leq m$ for m a positive integer. Then*

$$\mathbf{h}^* : H^{m+s}\left(\mathbb{R}^n\right) \to H^{m+s}\left(\mathbb{R}^n\right)$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $\left(\mathbf{h}^{-1}\right)^*$.

Proof: Let $u \in \mathfrak{S}$. From Theorem 41.17 and the equivalence of the norms in $W^{m,2}\left(\mathbb{R}^n\right)$ and $H^m\left(\mathbb{R}^n\right)$,

$$\begin{aligned}
&||\mathbf{h}^* u||^2_{H^m(\mathbb{R}^n)} + \int\int \textstyle\sum_{|\alpha|=m} \left|D^\alpha \mathbf{h}^* u\left(\mathbf{x}\right) - D^\alpha \mathbf{h}^* u\left(\mathbf{y}\right)\right|^2 \left|\mathbf{x}-\mathbf{y}\right|^{-n-2s} dxdy \\
&\leq C\,||u||^2_{H^m(\mathbb{R}^n)} + \int\int \textstyle\sum_{|\alpha|=m} \left|D^\alpha \mathbf{h}^* u\left(\mathbf{x}\right) - D^\alpha \mathbf{h}^* u\left(\mathbf{y}\right)\right|^2 \left|\mathbf{x}-\mathbf{y}\right|^{-n-2s} dxdy \\
&= C\,||u||^2_{H^m(\mathbb{R}^n)} + \int\int \textstyle\sum_{|\alpha|=m} \Big|\sum_{|\beta(\alpha)|\leq m} \mathbf{h}^*\left(D^{\beta(\alpha)}u\right) g_{\beta(\alpha)}\left(\mathbf{x}\right) \\
&\qquad -\mathbf{h}^*\left(D^{\beta(\alpha)}u\right) g_{\beta(\alpha)}\left(\mathbf{y}\right)\Big|^2 \left|\mathbf{x}-\mathbf{y}\right|^{-n-2s} dxdy \\
&\leq C\,||u||^2_{H^m(\mathbb{R}^n)} + C\int\int \textstyle\sum_{|\alpha|=m}\sum_{|\beta(\alpha)|\leq m} \Big|\mathbf{h}^*\left(D^{\beta(\alpha)}u\right) g_{\beta(\alpha)}\left(\mathbf{x}\right) \\
&\qquad -\mathbf{h}^*\left(D^{\beta(\alpha)}u\right) g_{\beta(\alpha)}\left(\mathbf{y}\right)\Big|^2 \left|\mathbf{x}-\mathbf{y}\right|^{-n-2s} dxdy
\end{aligned} \tag{41.13}$$

A single term in the last sum corresponding to a given α is then of the form,

$$\int\int \left|\mathbf{h}^*\left(D^\beta u\right) g_\beta\left(\mathbf{x}\right) - \mathbf{h}^*\left(D^\beta u\right) g_\beta\left(\mathbf{y}\right)\right|^2 \left|\mathbf{x}-\mathbf{y}\right|^{-n-2s} dxdy \tag{41.14}$$

$$\begin{aligned}
&\leq \Big[\int\int \left|\mathbf{h}^*\left(D^\beta u\right)\left(\mathbf{x}\right) g_\beta\left(\mathbf{x}\right) - \mathbf{h}^*\left(D^\beta u\right)\left(\mathbf{y}\right) g_\beta\left(\mathbf{x}\right)\right|^2 \left|\mathbf{x}-\mathbf{y}\right|^{-n-2s} dxdy + \\
&\int\int \left|\mathbf{h}^*\left(D^\beta u\right)\left(\mathbf{y}\right) g_\beta\left(\mathbf{x}\right) - \mathbf{h}^*\left(D^\beta u\right)\left(\mathbf{y}\right) g_\beta\left(\mathbf{y}\right)\right|^2 \left|\mathbf{x}-\mathbf{y}\right|^{-n-2s} dxdy\Big] \\
&\leq \Big[C\left(\mathbf{h}\right)\int\int \left|\mathbf{h}^*\left(D^\beta u\right)\left(\mathbf{x}\right) - \mathbf{h}^*\left(D^\beta u\right)\left(\mathbf{y}\right)\right|^2 \left|\mathbf{x}-\mathbf{y}\right|^{-n-2s} dxdy + \\
&\int\int \left|\mathbf{h}^*\left(D^\beta u\right)\left(\mathbf{y}\right)\right|^2 \left|g_\beta\left(\mathbf{x}\right) - g_\beta\left(\mathbf{y}\right)\right|^2 \left|\mathbf{x}-\mathbf{y}\right|^{-n-2s} dxdy\Big].
\end{aligned}$$

Changing variables, and then using the names of the old variables to simplify the notation,

$$\leq \Big[C\left(\mathbf{h},\mathbf{h}^{-1}\right)\int\int \left|\left(D^\beta u\right)\left(\mathbf{x}\right) - \left(D^\beta u\right)\left(\mathbf{y}\right)\right|^2 \left|\mathbf{x}-\mathbf{y}\right|^{-n-2s} dxdy +$$

$$\left.\int\int \left|\mathbf{h}^*\left(D^\beta u\right)(\mathbf{y})\right|^2 \left|g_\beta(\mathbf{x})-g_\beta(\mathbf{y})\right|^2 |\mathbf{x}-\mathbf{y}|^{-n-2s}\,dxdy\right].$$

By 41.10,

$$\begin{aligned}\leq\;& C(\mathbf{h})\int |F(u)(\mathbf{z})|^2 \left|\mathbf{z}^\beta\right|^2 |\mathbf{z}|^{2s}\,dz \\ &+\int\int \left|\mathbf{h}^*\left(D^\beta u\right)(\mathbf{y})\right|^2 \left|g_\beta(\mathbf{x})-g_\beta(\mathbf{y})\right|^2 |\mathbf{x}-\mathbf{y}|^{-n-2s}\,dxdy.\end{aligned}$$

In the second term, let $\mathbf{t}=\mathbf{x}-\mathbf{y}$. Then this term is of the form

$$\int \left|\mathbf{h}^*\left(D^\beta u\right)(\mathbf{y})\right|^2 \int \left|g_\beta(\mathbf{y}+\mathbf{t})-g_\beta(\mathbf{y})\right|^2 |\mathbf{t}|^{-n-2s}\,dtdy \tag{41.15}$$

$$\leq\; C\int \left|\mathbf{h}^*\left(D^\beta u\right)(\mathbf{y})\right|^2 dy \leq C\,||u||^2_{H^m(\mathbb{R}^n)}\,. \tag{41.16}$$

because the inside integral equals a constant which depends on the Lipschitz constants and bounds of the function, g_β and these things depend only on $\mathbf{h}$. The reason this integral is finite is that for $|\mathbf{t}|\leq 1$,

$$\left|g_\beta(\mathbf{y}+\mathbf{t})-g_\beta(\mathbf{y})\right|^2 |\mathbf{t}|^{-n-2s} \leq K\,|\mathbf{t}|^2\,|\mathbf{t}|^{-n-2s}$$

and using polar coordinates, you see

$$\int_{[|\mathbf{t}|\leq 1]} \left|g_\beta(\mathbf{y}+\mathbf{t})-g_\beta(\mathbf{y})\right|^2 |\mathbf{t}|^{-n-2s}\,dt<\infty.$$

Now for $|\mathbf{t}|>1$, the integrand in 41.15 is dominated by $4\,|\mathbf{t}|^{-n-2s}$ and using polar coordinates, this yields

$$\int_{[|\mathbf{t}|>1]} \left|g_\beta(\mathbf{y}+\mathbf{t})-g_\beta(\mathbf{y})\right|^2 |\mathbf{t}|^{-n-2s}\,dt \leq 4\int_{[|\mathbf{t}|>1]} |\mathbf{t}|^{-n-2s}\,dt<\infty.$$

It follows 41.14 is dominated by an expression of the form

$$C(\mathbf{h})\int |F(u)(\mathbf{z})|^2 \left|\mathbf{z}^\beta\right|^2 |\mathbf{z}|^{2s}\,dz + C\,||u||^2_{H^m(\mathbb{R}^n)}$$

and so the sum in 41.13 is dominated by

$$\begin{aligned}& C(m,\mathbf{h})\int |F(u)(\mathbf{z})|^2 |\mathbf{z}|^{2s} \sum_{|\beta|\leq m} \left|\mathbf{z}^\beta\right|^2 dz + C\,||u||^2_{H^m(\mathbb{R}^n)} \\ \leq\;& C(m,\mathbf{h})\int |F(u)(\mathbf{z})|^2 \left(1+|\mathbf{z}|^2\right)^s \left(1+|\mathbf{z}|^2\right)^m dz + C\,||u||^2_{H^m(\mathbb{R}^n)} \\ \leq\;& C\,||u||^2_{H^{m+s}(\mathbb{R}^n)}\,.\end{aligned}$$

This proves the theorem because the assertion about $\mathbf{h}^{-1}$ is obvious. Just replace $\mathbf{h}$ with $\mathbf{h}^{-1}$ in the above argument.

Next consider the case where U is an open set.

Lemma 41.20 *Let* $\mathbf{h}(U) \subseteq V$ *where* U *and* V *are open subsets of* $\mathbb{R}^n$ *and suppose that* $\mathbf{h}, \mathbf{h}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ *are both functions in* $C^{m,1}(\mathbb{R}^n)$. *Recall this means* $D^\alpha \mathbf{h}$ *and* $D^\alpha \mathbf{h}^{-1}$ *exist and are Lipschitz continuous for all* $|\alpha| \leq m$. *Then* $\mathbf{h}^* \in \mathcal{L}(H^{m+s}(V), H^{m+s}(U))$.

Proof: Let $u \in H^{m+s}(V)$ and let $v \in H^{m+s}(\mathbb{R}^n)$ such that $v|_V = u$. Then from the above, $\mathbf{h}^* v \in H^{m+s}(\mathbb{R}^n)$ and so $\mathbf{h}^* u \in H^{m+s}(U)$ because $\mathbf{h}^* u = \mathbf{h}^* v|_U$. Then by Lemma 41.19,

$$||\mathbf{h}^* u||_{H^{m+s}(U)} \leq ||\mathbf{h}^* v||_{H^{m+s}(\mathbb{R}^n)} \leq C\, ||v||_{H^{m+s}(\mathbb{R}^n)}$$

Since this is true for all $v \in H^{m+s}(\mathbb{R}^n)$, it follows that

$$||\mathbf{h}^* u||_{H^{m+s}(U)} \leq C\, ||u||_{H^{m+s}(V)}.$$

With harder work, you don't need to have $\mathbf{h}, \mathbf{h}^{-1}$ defined on all of $\mathbb{R}^n$ but I don't feel like including the details so this lemma will suffice.

Another interesting application of the intrinsic norm is the following.

Lemma 41.21 *Let* $\phi \in C^{m,1}(\mathbb{R}^n)$ *and suppose* $\operatorname{spt}(\phi)$ *is compact. Then there exists a constant,* C_ϕ *such that whenever* $u \in H^{m+s}(\mathbb{R}^n)$,

$$||\phi u||_{H^{m+s}(\mathbb{R}^n)} \leq C_\phi\, ||u||_{H^{m+s}(\mathbb{R}^n)}.$$

Proof: It is a routine exercise in the product rule to verify that $||\phi u||_{H^m(\mathbb{R}^n)} \leq C_\phi\, ||u||_{H^m(\mathbb{R}^n)}$. It only remains to consider the term involving the integral. A typical term is

$$\int \int |D^\alpha \phi u(\mathbf{x}) - D^\alpha \phi u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s}\, dxdy.$$

This is a finite sum of terms of the form

$$\int \int \left|D^\gamma \phi(\mathbf{x}) D^\beta u(\mathbf{x}) - D^\gamma \phi(\mathbf{y}) D^\beta u(\mathbf{y})\right|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s}\, dxdy$$

where $|\gamma|$ and $|\beta| \leq m$.

$$\begin{aligned} \leq\ & 2 \int \int |D^\gamma \phi(\mathbf{x})|^2 \left|D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})\right|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s}\, dxdy \\ & + 2 \int \int \left|D^\beta u(\mathbf{y})\right|^2 |D^\gamma \phi(\mathbf{x}) - D^\gamma \phi(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s}\, dxdy \end{aligned}$$

By 41.10 and the Lipschitz continuity of all the derivatives of ϕ, this is dominated by

$$\begin{aligned}
& CM(s)\int |Fu(\mathbf{z})|^2 \left|\mathbf{z}^\beta\right|^2 |\mathbf{z}|^{2s}\,dz \\
& +K\int\int \left|D^\beta u(\mathbf{y})\right|^2 |\mathbf{x}-\mathbf{y}|^2 |\mathbf{x}-\mathbf{y}|^{-n-2s}\,dxdy \\
= \;& CM(s)\int |Fu(\mathbf{z})|^2 \left|\mathbf{z}^\beta\right|^2 |\mathbf{z}|^{2s}\,dz \\
& +K\int \left|D^\beta u(\mathbf{y})\right|^2 \int |\mathbf{t}|^{-n+2(1-s)}\,dtdy \\
\leq \;& C(s)\left(\int |Fu(\mathbf{z})|^2 \left|\mathbf{z}^\beta\right|^2 |\mathbf{z}|^{2s}\,dz + K\int \left|D^\beta u(\mathbf{y})\right|^2 dy\right) \\
\leq \;& C(s)\int \left(1+|\mathbf{y}|^2\right)^{m+s} |Fu(\mathbf{y})|^2\,dy.
\end{aligned}$$

Since there are only finitely many such terms, this proves the lemma.

Corollary 41.22 *Let $t = m + s$ for $s \in [0,1)$ and let U, V be open sets. Let $\phi \in C_c^{m,1}(V)$. This means* $\operatorname{spt}(\phi) \subseteq V$ *and $\phi \in C^{m,1}(\mathbb{R}^n)$. Then if $u \in H^t(U)$ it follows that $u\phi \in H^t(U \cap V)$ and $||u\phi||_{H^t(U\cap V)} \leq C_\phi ||u||_{H^t(U)}$.*

Proof: Let $v|_U = u$ a.e. where $v \in H^t(\mathbb{R}^n)$. Then by Lemma 41.21, $\phi v \in H^t(\mathbb{R}^n)$ and $\phi v|_{U\cap V} = \phi u$ a.e. Therefore, $\phi u \in H^t(U \cap V)$ and

$$||\phi u||_{H^t(U\cap V)} \leq ||\phi v||_{H^t(\mathbb{R}^n)} \leq C_\phi ||v||_{H^t(\mathbb{R}^n)}.$$

Taking the infimum for all such v whose restrictions equal u, this yields

$$||\phi u||_{H^t(U\cap V)} \leq C_\phi ||u||_{H^t(U)}.$$

This proves the corollary.

41.3 Embedding Theorems

The Fourier transform description of Sobolev spaces makes possible fairly easy proofs of various embedding theorems.

Definition 41.23 *Let $C_b^m(\mathbb{R}^n)$ denote the functions which are m times continuously differentiable and for which*

$$\sup_{|\alpha|\leq m}\sup_{x\in\mathbb{R}^n} |D^\alpha u(\mathbf{x})| \equiv ||u||_{C_b^m(\mathbb{R}^n)} < \infty.$$

For U an open set, $C^m\left(\overline{U}\right)$ denotes the functions which are restrictions of $C_b^m(\mathbb{R}^n)$ to U.

It is clear this is a Banach space, the proof being a simple exercise in the use of the fundamental theorem of calculus along with standard results about uniform convergence.

Lemma 41.24 *Let $u \in \mathfrak{S}$ and let $\frac{n}{2} + m < t$. Then there exists C independent of u such that*

$$||u||_{C_b^m(\mathbb{R}^n)} \leq C\,||u||_{H^t(\mathbb{R}^n)}.$$

Proof: Using the fact that the Fourier transform maps $\mathfrak{S}$ to $\mathfrak{S}$ and the definition of the Fourier transform,

$$\begin{aligned}
|D^\alpha u(\mathbf{x})| &\leq C\,||FD^\alpha u||_{L^1(\mathbb{R}^n)} \\
&= C\int |\mathbf{x}^\alpha|\,|Fu(\mathbf{x})|\,dx \\
&\leq C\int \left(1+|\mathbf{x}|^2\right)^{|\alpha|/2} |Fu(\mathbf{x})|\,dx \\
&\leq C\int \left(1+|\mathbf{x}|^2\right)^{m/2}\left(1+|\mathbf{x}|^2\right)^{-t/2}\left(1+|\mathbf{x}|^2\right)^{t/2}|Fu(\mathbf{x})|\,dx \\
&\leq C\left(\int\left(1+|\mathbf{x}|^2\right)^{m-t}dx\right)^{1/2}\left(\int\left(1+|\mathbf{x}|^2\right)^{t}|Fu(\mathbf{x})|^t\right)^{1/2} \\
&\leq C\,||u||_{H^t(\mathbb{R}^n)}
\end{aligned}$$

because for the given values of t and m the first integral is finite. This follows from a use of polar coordinates. Taking sup over all $\mathbf{x} \in \mathbb{R}^n$ and $|\alpha| \leq m$, this proves the lemma.

Corollary 41.25 *Let $u \in H^t(\mathbb{R}^n)$ where $t > m + \frac{n}{2}$. Then u is a.e. equal to a function of $C_b^m(\mathbb{R}^n)$ still denoted by u. Furthermore, there exists a constant, C independent of u such that*

$$||u||_{C_b^m(\mathbb{R}^n)} \leq C\,||u||_{H^t(\mathbb{R}^n)}.$$

Proof: This follows from the above lemma. Let $\{u_k\}$ be a sequence of functions of $\mathfrak{S}$ which converges to u in H^t and a.e. Then by the inequality of the above lemma, this sequence is also Cauchy in $C_b^m(\mathbb{R}^n)$ and taking the limit,

$$||u||_{C_b^m(\mathbb{R}^n)} = \lim_{k\to\infty}||u_k||_{C_b^m(\mathbb{R}^n)} \leq C\lim_{k\to\infty}||u_k||_{H^t(\mathbb{R}^n)} = C\,||u||_{H^t(\mathbb{R}^n)}.$$

What about open sets, U?

Corollary 41.26 *Let $t > m + \frac{n}{2}$ and let U be an open set with $u \in H^t(U)$. Then u is a.e. equal to a function of $C^m\left(\overline{U}\right)$ still denoted by u. Furthermore, there exists a constant, C independent of u such that*

$$||u||_{C^m(\overline{U})} \leq C\,||u||_{H^t(U)}.$$

Proof: Let $u \in H^t(U)$ and let $v \in H^t(\mathbb{R}^n)$ such that $v|_U = u$. Then

$$||u||_{C^m(\overline{U})} \leq ||v||_{C_b^m(\mathbb{R}^n)} \leq C\,||v||_{H^t(\mathbb{R}^n)}.$$

Now taking the inf for all such v yields

$$||u||_{C^m(\overline{U})} \leq C\,||u||_{H^t(U)}.$$

41.4 The Trace On The Boundary Of A Half Space

It is important to consider the restriction of functions in a Sobolev space onto a smaller dimensional set such as the boundary of an open set.

Definition 41.27 *For $u \in \mathfrak{S}$, define γu a function defined on $\mathbb{R}^{n-1}$ by $\gamma u(\mathbf{x}') \equiv u(\mathbf{x}', 0)$ where $\mathbf{x}' \in \mathbb{R}^{n-1}$ is defined by $\mathbf{x} = (\mathbf{x}', x_n)$.*

The following elementary lemma featuring trig. substitutions is the basis for the proof of some of the arguments which follow.

Lemma 41.28 *Consider the integral,*

$$\int_{\mathbb{R}} \left(a^2 + x^2\right)^{-t} dx.$$

for $a > 0$ and $t > 1/2$. Then this integral is of the form $C_t a^{-2t+1}$ where C_t is some constant which depends on t.

Proof: Letting $x = a\tan\theta$,

$$\int_{\mathbb{R}} \left(a^2 + x^2\right)^{-t} dx = a^{-2t+1} \int_{-\pi/2}^{\pi/2} \cos^{2t-2}(\theta)\, d\theta$$

and since $t > 1/2$ the last integral is finite. This yields the desired conclusion and proves the lemma.

Lemma 41.29 *Let $u \in \mathfrak{S}$. Then there exists a constant, C_n, depending on n but independent of $u \in \mathfrak{S}$ such that*

$$F\gamma u(\mathbf{x}') = C_n \int_{\mathbb{R}} Fu(\mathbf{x}', x_n)\, dx_n.$$

Proof: Using the dominated convergence theorem,

$$\int_{\mathbb{R}} Fu(\mathbf{x}', x_n)\, dx_n \equiv \lim_{\varepsilon \to 0} \int_{\mathbb{R}} e^{-(\varepsilon x_n)^2} Fu(\mathbf{x}', x_n)\, dx_n$$

$$\begin{aligned} &\equiv \lim_{\varepsilon\to 0}\int_{\mathbb{R}} e^{-(\varepsilon x_n)^2}\left(\frac{1}{2\pi}\right)^{n/2}\int_{\mathbb{R}^n} e^{-i\left(\mathbf{x}'\cdot\mathbf{y}'+x_ny_n\right)}u\left(\mathbf{y}',y_n\right)dy'dy_ndx_n \\ &= \lim_{\varepsilon\to 0}\left(\frac{1}{2\pi}\right)^{n/2}\int_{\mathbb{R}^n} u\left(\mathbf{y}',y_n\right)e^{-i\mathbf{x}'\cdot\mathbf{y}'}\int_{\mathbb{R}} e^{-(\varepsilon x_n)^2}e^{-ix_ny_n}dx_ndy'dy_n. \end{aligned}$$

Now $-(\varepsilon x_n)^2 - ix_ny_n = -\varepsilon^2\left(x_n+\frac{iy_n}{2}\right)^2 - \varepsilon^2\frac{y_n^2}{4}$ and so the above reduces to an expression of the form

$$\begin{aligned} \lim_{\varepsilon\to 0}K_n\int_{\mathbb{R}}\frac{1}{\varepsilon}e^{-\varepsilon^2\frac{y_n^2}{4}}\int_{\mathbb{R}^{n-1}}u\left(\mathbf{y}',y_n\right)e^{-i\mathbf{x}'\cdot\mathbf{y}'}dy'dy_n &= K_n\int_{\mathbb{R}^n}u\left(\mathbf{y}',0\right)e^{-i\mathbf{x}'\cdot\mathbf{y}'}dy' \\ &= K_nF\gamma u\left(\mathbf{x}'\right) \end{aligned}$$

and this proves the lemma with $C_n \equiv K_n^{-1}$.

Earlier $H^t(\mathbb{R}^n)$ was defined and then for U an open subset of $\mathbb{R}^n$, $H^t(U)$ was defined to be the space of restrictions of functions of $H^t(\mathbb{R}^n)$ to U and a norm was given which made $H^t(U)$ into a Banach space. The next task is to consider $\mathbb{R}^{n-1}\times\{0\}$, a smaller dimensional subspace of $\mathbb{R}^n$ and examine the functions defined on this set, denoted by $\mathbb{R}^{n-1}$ for short which are restrictions of functions in $H^t(\mathbb{R}^n)$. You note this is somewhat different because heuristically, the dimension of the domain of the function is changing. An open set in $\mathbb{R}^n$ is considered an n dimensional thing but $\mathbb{R}^{n-1}$ is only $n-1$ dimensional. I realize this is vague because the standard definition of dimension requires a vector space and an open set is not a vector space. However, think in terms of fatness. An open set is fat in n directions whereas $\mathbb{R}^{n-1}$ is only fat in $n-1$ directions. Therefore, something interesting is likely to happen.

Let $\mathfrak{S}$ denote the Schwartz class of functions on $\mathbb{R}^n$ and $\mathfrak{S}'$ the Schwartz class of functions on $\mathbb{R}^{n-1}$. Also, $\mathbf{y}'\in\mathbb{R}^{n-1}$ while $\mathbf{y}\in\mathbb{R}^n$. Let $u\in\mathfrak{S}$. Then from Lemma 41.29 and $s>0$,

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}}\left(1+|\mathbf{y}'|^2\right)^s\left|F\gamma u\left(\mathbf{y}'\right)\right|^2dy' \\ &= C_n\int_{\mathbb{R}^{n-1}}\left(1+|\mathbf{y}'|^2\right)^s\left|\int_{\mathbb{R}}Fu\left(\mathbf{y}',y_n\right)dy_n\right|^2dy' \end{aligned}$$

$$= C_n\int_{\mathbb{R}^{n-1}}\left(1+|\mathbf{y}'|^2\right)^s\left|\int_{\mathbb{R}}Fu\left(\mathbf{y}',y_n\right)\left(1+|\mathbf{y}|^2\right)^{t/2}\left(1+|\mathbf{y}|^2\right)^{-t/2}dy_n\right|^2dy'$$

Then by the Cauchy Schwarz inequality,

$$\leq C_n\int_{\mathbb{R}^{n-1}}\left(1+|\mathbf{y}'|^2\right)^s\int_{\mathbb{R}}\left|Fu\left(\mathbf{y}',y_n\right)\right|^2\left(1+|\mathbf{y}|^2\right)^t dy_n\int_{\mathbb{R}}\left(1+|\mathbf{y}|^2\right)^{-t}dy_ndy'. \tag{41.17}$$

Consider

$$\int_{\mathbb{R}}\left(1+|\mathbf{y}|^2\right)^{-t}dy_n = \int_{\mathbb{R}}\left(1+|\mathbf{y}'|^2+y_n^2\right)^{-t}dy_n$$

by Lemma 41.28 and taking $a = \left(1 + |\mathbf{y}'|^2\right)^{1/2}$, this equals

$$C_t \left(\left(1 + |\mathbf{y}'|^2\right)^{1/2}\right)^{-2t+1} = C_t \left(1 + |\mathbf{y}'|^2\right)^{(-2t+1)/2}.$$

Now using this in 41.17,

$$\begin{aligned}
& \int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{y}'|^2\right)^s |F\gamma u(\mathbf{y}')|^2 dy' \\
\leq \quad & C_{n,t} \int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{y}'|^2\right)^s \int_{\mathbb{R}} |Fu(\mathbf{y}', y_n)|^2 \left(1 + |\mathbf{y}|^2\right)^t dy_n \cdot \\
& \left(1 + |\mathbf{y}'|^2\right)^{(-2t+1)/2} dy' \\
= \quad & C_{n,t} \int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{y}'|^2\right)^{s+(-2t+1)/2} \int_{\mathbb{R}} |Fu(\mathbf{y}', y_n)|^2 \left(1 + |\mathbf{y}|^2\right)^t dy_n dy'.
\end{aligned}$$

What is the correct choice of t so that the above reduces to $||u||^2_{H^t(\mathbb{R}^n)}$? It is clearly the one for which

$$s + (-2t + 1)/2 = 0$$

which occurs when $t = s + \frac{1}{2}$. Then for this choice of t, the following inequality is obtained for any $u \in \mathfrak{S}$.

$$||\gamma u||_{H^{t-1/2}(\mathbb{R}^{n-1})} \leq C_{n,t} ||u||_{H^t(\mathbb{R}^n)}. \tag{41.18}$$

This has proved part of the following theorem.

Theorem 41.30 *For each $t > 1/2$ there exists a unique mapping*

$$\gamma \in \mathcal{L}\left(H^t(\mathbb{R}^n), H^{t-1/2}(\mathbb{R}^{n-1})\right)$$

which has the property that for $u \in \mathfrak{S}$, $\gamma u(\mathbf{x}') = u(\mathbf{x}', 0)$. In addition to this, γ is onto. In fact, there exists a continuous map, $\zeta \in \mathcal{L}\left(H^{t-1/2}(\mathbb{R}^{n-1}), H^t(\mathbb{R}^n)\right)$ such that $\gamma \circ \zeta = \mathrm{id}$.

Proof: It only remains to verify that γ is onto and that the continuous map, ζ exists. Now define

$$\phi(\mathbf{y}) \equiv \phi(\mathbf{y}', y_n) \equiv \frac{\left(1 + |\mathbf{y}'|^2\right)^{t-1/2}}{\left(1 + |\mathbf{y}|^2\right)^t}.$$

Then for $u \in \mathfrak{S}'$, let

$$\zeta u(\mathbf{x}) \equiv CF^{-1}(\phi Fu)(\mathbf{x}) =$$

$$C \int_{\mathbb{R}^n} e^{i\mathbf{y}\cdot\mathbf{x}} \frac{\left(1 + |\mathbf{y}'|^2\right)^{t-1/2}}{\left(1 + |\mathbf{y}|^2\right)^t} Fu(\mathbf{y}')\, dy \tag{41.19}$$

Here the inside Fourier transform is taken with respect to $\mathbb{R}^{n-1}$ because u is only defined on $\mathbb{R}^{n-1}$ and C will be chosen in such a way that $\gamma \circ \zeta = \text{id}$. First the existence of C such that $\gamma \circ \zeta = \text{id}$ will be shown. Since $u \in \mathfrak{S}'$ it follows

$$\mathbf{y} \to \frac{\left(1+|\mathbf{y}'|^2\right)^{t-1/2}}{\left(1+|\mathbf{y}|^2\right)^t} Fu(\mathbf{y}')$$

is in $\mathfrak{S}$. Hence the inverse Fourier transform of this function is also in $\mathfrak{S}$ and so for $u \in \mathfrak{S}'$, it follows $\zeta u \in \mathfrak{S}$. Therefore, to check $\gamma \circ \zeta = \text{id}$ it suffices to plug in $x_n = 0$. From Lemma 41.28 this yields

$$\begin{aligned}
&\gamma(\zeta u)(\mathbf{x}', 0) \\
&= C \int_{\mathbb{R}^n} e^{i\mathbf{y}'\cdot\mathbf{x}'} \frac{\left(1+|\mathbf{y}'|^2\right)^{t-1/2}}{\left(1+|\mathbf{y}|^2\right)^t} Fu(\mathbf{y}')\, dy \\
&= C \int_{\mathbb{R}^{n-1}} \left(1+|\mathbf{y}'|^2\right)^{t-1/2} e^{i\mathbf{y}'\cdot\mathbf{x}'} Fu(\mathbf{y}') \int_{\mathbb{R}} \frac{1}{\left(1+|\mathbf{y}|^2\right)^t} dy_n dy' \\
&= CC_t \int_{\mathbb{R}^{n-1}} \left(1+|\mathbf{y}'|^2\right)^{t-1/2} e^{i\mathbf{y}'\cdot\mathbf{x}'} Fu(\mathbf{y}') \left(1+|\mathbf{y}'|^2\right)^{\frac{-2t+1}{2}} dy' \\
&= CC_t \int_{\mathbb{R}^{n-1}} e^{i\mathbf{y}'\cdot\mathbf{x}'} Fu(\mathbf{y}')\, dy' = CC_t (2\pi)^{n/2} F^{-1}(Fu)(\mathbf{x}')
\end{aligned}$$

and so the correct value of C is $\left(C_t (2\pi)^{n/2}\right)^{-1}$ to obtain $\gamma \circ \zeta = \text{id}$. It only remains to verify that ζ is continuous. From 41.19, and Lemma 41.28,

$$\begin{aligned}
&||\zeta u||^2_{H^t(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} \left(1+|\mathbf{x}|^2\right)^t |F\zeta u(\mathbf{x})|^2 dx \\
&= C^2 \int_{\mathbb{R}^n} \left(1+|\mathbf{x}|^2\right)^t \left|F\left(F^{-1}(\phi Fu)(\mathbf{x})\right)\right|^2 dx \\
&= C^2 \int_{\mathbb{R}^n} \left(1+|\mathbf{x}|^2\right)^t |\phi(\mathbf{x}) Fu(\mathbf{x}')|^2 dx \\
&= C^2 \int_{\mathbb{R}^n} \left(1+|\mathbf{x}|^2\right)^t \left| \frac{\left(1+|\mathbf{x}'|^2\right)^{t-1/2}}{\left(1+|\mathbf{x}|^2\right)^t} Fu(\mathbf{x}') \right|^2 dx \\
&= C^2 \int_{\mathbb{R}^n} \left(1+|\mathbf{x}|^2\right)^{-t} \left| \left(1+|\mathbf{x}'|^2\right)^{t-1/2} Fu(\mathbf{x}') \right|^2 dx \\
&= C^2 \int_{\mathbb{R}^{n-1}} \left(1+|\mathbf{x}'|^2\right)^{2t-1} |Fu(\mathbf{x}')|^2 \int_{\mathbb{R}} \left(1+|\mathbf{x}|^2\right)^{-t} dx_n dx'
\end{aligned}$$

$$
\begin{aligned}
&= C^2 C_t \int_{\mathbb{R}^{n-1}} \left(1+|\mathbf{x}'|^2\right)^{2t-1} |Fu(\mathbf{x}')|^2 \left(1+|\mathbf{y}'|^2\right)^{\frac{-2t+1}{2}} dx' \\
&= C^2 C_t \int_{\mathbb{R}^{n-1}} \left(1+|\mathbf{x}'|^2\right)^{t-1/2} |Fu(\mathbf{x}')|^2 dx' = C^2 C_t \|u\|^2_{H^{t-1/2}(\mathbb{R}^{n-1})}.
\end{aligned}
$$

This proves the theorem because $\mathfrak{S}$ is dense in $\mathbb{R}^n$.

Actually, the assertion that $\gamma u(\mathbf{x}') = u(\mathbf{x}', 0)$ holds for more functions, u than just $u \in \mathfrak{S}$. I will make no effort to obtain the most general description of such functions but the following is a useful lemma which will be needed when the trace on the boundary of an open set is considered.

Lemma 41.31 *Suppose u is continuous and $u \in H^1(\mathbb{R}^n)$. Then there exists a set of m_1 measure zero, N such that if $x_n \notin N$, then for every $\phi \in L^2(\mathbb{R}^{n-1})$*

$$
(\gamma u, \phi)_H + \int_0^{x_n} (u_{,n}(\cdot, t), \phi)_H \, dt = (u(\cdot, x_n), \phi)_H
$$

where here

$$
(f, g)_H \equiv \int_{\mathbb{R}^{n-1}} f \overline{g} dx',
$$

just the inner product in $L^2(\mathbb{R}^{n-1})$. Furthermore,

$$
u(\cdot, 0) = \gamma u \text{ a.e. } \mathbf{x}'.
$$

Proof: Let $\{u_k\}$ be a sequence of functions from $\mathfrak{S}$ which converges to u in $H^1(\mathbb{R}^n)$ and let $\{\phi_k\}$ denote a countable dense subset of $L^2(\mathbb{R}^{n-1})$. Then

$$
(\gamma u_k, \phi_j)_H + \int_0^{x_n} (u_{k,n}(\cdot, t), \phi_j)_H \, dt = (u_k(\cdot, x_n), \phi_j)_H . \tag{41.20}
$$

Now

$$
\begin{aligned}
&\left(\int_0^\infty \left|(u_k(\cdot, x_n), \phi_j)_H - (u(\cdot, x_n), \phi_j)_H\right|^2 dx_n\right)^{1/2} \\
&= \left(\int_0^\infty \left|(u_k(\cdot, x_n) - u(\cdot, x_n), \phi_j)_H\right|^2 dx_n\right)^{1/2} \\
&\leq \left(\int_0^\infty |u_k(\cdot, x_n) - u(\cdot, x_n)|_H^2 |\phi_j|_H^2 dx_n\right)^{1/2} \\
&= |\phi_j|_H^2 \left(\int_0^\infty |u_k(\cdot, x_n) - u(\cdot, x_n)|_H^2 dx_n\right)^{1/2} \\
&= |\phi_j|_H^2 \left(\int_0^\infty \int_{\mathbb{R}^{n-1}} |u_k(\mathbf{x}', x_n) - u(\mathbf{x}', x_n)|^2 dx' dx_n\right)^{1/2}
\end{aligned}
$$

which converges to zero. Therefore, there exists a set of measure zero, N_j and a subsequence, still denoted by k such that if $x_n \notin N_j$, then

$$
(u_k(\cdot, x_n), \phi_j)_H \to (u(\cdot, x_n), \phi_j)_H .
$$

Now by Theorem 41.30, $\gamma u_k \to \gamma u$ in $H = L^2\left(\mathbb{R}^{n-1}\right)$. It only remains to consider the term of 41.20 which involves an integral.

$$\begin{aligned}
&\left|\int_0^{x_n} \left(u_{k,n}(\cdot,t), \phi_j\right)_H dt - \int_0^{x_n} \left(u_{,n}(\cdot,t), \phi_j\right)_H dt\right| \\
\leq\ & \int_0^{x_n} \left|\left(u_{k,n}(\cdot,t) - u_{,n}(\cdot,t), \phi_j\right)_H\right| dt \\
\leq\ & \int_0^{x_n} \left|u_{k,n}(\cdot,t) - u_{,n}(\cdot,t)\right|_H \left|\phi_j\right|_H dt \\
\leq\ & \left(\int_0^{x_n} \left|u_{k,n}(\cdot,t) - u_{,n}(\cdot,t)\right|_H^2 dt\right)^{1/2} \left(\int_0^{x_n} \left|\phi_j\right|_H^2 dt\right)^{1/2} \\
=\ & x_n^{1/2} \left|\phi_j\right|_H \left(\int_0^{x_n} \int_{\mathbb{R}^{n-1}} \left|u_{k,n}(\mathbf{x}',t) - u_{,n}(\mathbf{x}',t)\right|^2 dx'\right)^{1/2} dt
\end{aligned}$$

and this converges to zero as $k \to \infty$. Therefore, using a diagonal sequence argument, there exists a subsequence, still denoted by k and a set of measure zero, $N \equiv \cup_{j=1}^{\infty} N_j$ such that for $\mathbf{x}' \notin N$, you can pass to the limit in 41.20 and obtain that for all ϕ_j,

$$\left(\gamma u, \phi_j\right)_H + \int_0^{x_n} \left(u_{,n}(\cdot,t), \phi_j\right)_H dt = \left(u(\cdot, x_n), \phi_j\right)_H.$$

By density of $\{\phi_j\}$, this equality holds for all $\phi \in L^2\left(\mathbb{R}^{n-1}\right)$. In particular, the equality holds for every $\phi \in C_c\left(\mathbb{R}^{n-1}\right)$. Since u is uniformly continuous on the compact set, $\operatorname{spt}(\phi) \times [0,1]$, there exists a sequence, $(x_n)_k \to 0$ such that the above equality holds for x_n replaced with $(x_n)_k$ and ϕ in place of ϕ_j. Now taking $k \to \infty$, this uniform continuity implies

$$(\gamma u, \phi)_H = (u(\cdot, 0), \phi)_H$$

This implies since $C_c\left(\mathbb{R}^{n-1}\right)$ is dense in $L^2\left(\mathbb{R}^{n-1}\right)$ that $\gamma u = u(\cdot, 0)$ a.e. and this proves the lemma.

Lemma 41.32 *Suppose U is an open subset of $\mathbb{R}^n$ of the form*

$$U \equiv \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U' \text{ and } 0 < u_n < \phi(\mathbf{u}')\}$$

where U' is an open subset of $\mathbb{R}^{n-1}$ and $\phi(\mathbf{u}')$ is a positive function such that $\phi(\mathbf{u}') \leq \infty$ and

$$\inf\{\phi(\mathbf{u}') : \mathbf{u}' \in U'\} = \delta > 0$$

Suppose $v \in H^t(\mathbb{R}^n)$ such that $v = 0$ a.e. on U. Then $\gamma v = 0$ m_{n-1} a.e. point of U'. Also, if $v \in H^t(\mathbb{R}^n)$ and $\phi \in C_c^{\infty}(\mathbb{R}^n)$, then $\gamma v \gamma \phi = \gamma(\phi v)$.

Proof: First consider the second claim. Let $v \in H^t(\mathbb{R}^n)$ and let $v_k \to v$ in $H^t(\mathbb{R}^n)$ where $v_k \in \mathfrak{S}$. Then from Lemma 41.21 and Theorem 41.30

$$||\gamma(\phi v) - \gamma\phi\gamma v||_{H^{t-1/2}(\mathbb{R}^{n-1})} = \lim_{k\to\infty} ||\gamma(\phi v_k) - \gamma\phi\gamma v_k||_{H^{t-1/2}(\mathbb{R}^{n-1})} = 0$$

because each term in the sequence equals zero due to the observation that for $v_k \in \mathfrak{S}$ and $\phi \in C_c^\infty(U)$, $\gamma(\phi v_k) = \gamma v_k \gamma \phi$.

Now suppose $v = 0$ a.e. on U. Define for $0 < r < \delta$, $v_r(\mathbf{x}) \equiv v(\mathbf{x}', x_n + r)$.

Claim: If $u \in H^t(\mathbb{R}^n)$, then

$$\lim_{r\to 0} ||v_r - v||_{H^t(\mathbb{R}^n)} = 0.$$

Proof of claim: First of all, let $v \in \mathfrak{S}$. Then $v \in H^m(\mathbb{R}^n)$ for all m and so by Lemma 41.15,

$$||v_r - v||_{H^t(\mathbb{R}^n)} \leq ||v_r - v||_{H^m(\mathbb{R}^n)}^\theta ||v_r - v||_{H^{m+1}(\mathbb{R}^n)}^{1-\theta}$$

where $t \in [m, m+1]$. It follows from continuity of translation in $L^p(\mathbb{R}^n)$ that

$$\lim_{r\to 0} ||v_r - v||_{H^m(\mathbb{R}^n)}^\theta ||v_r - v||_{H^{m+1}(\mathbb{R}^n)}^{1-\theta} = 0$$

and so the claim is proved if $v \in \mathfrak{S}$. Now suppose $u \in H^t(\mathbb{R}^n)$ is arbitrary. By density of $\mathfrak{S}$ in $H^t(\mathbb{R}^n)$, there exists $v \in \mathfrak{S}$ such that

$$||u - v||_{H^t(\mathbb{R}^n)} < \varepsilon/3.$$

Therefore,

$$\begin{aligned} ||u_r - u||_{H^t(\mathbb{R}^n)} &\leq ||u_r - v_r||_{H^t(\mathbb{R}^n)} + ||v_r - v||_{H^t(\mathbb{R}^n)} + ||v - u||_{H^t(\mathbb{R}^n)} \\ &= 2\varepsilon/3 + ||v_r - v||_{H^t(\mathbb{R}^n)}. \end{aligned}$$

Now using what was just shown, it follows that for r small enough, $||u_r - u||_{H^t(\mathbb{R}^n)} < \varepsilon$ and this proves the claim.

Now suppose $v \in H^t(\mathbb{R}^n)$. By the claim,

$$||v_r - v||_{H^t(\mathbb{R}^n)} \to 0$$

and so by continuity of γ,

$$\gamma v_r \to \gamma v \text{ in } H^{t-1/2}\left(\mathbb{R}^{n-1}\right). \tag{41.21}$$

Note $v_r = 0$ a.e. on

$$U_r \equiv \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U' \text{ and } -r < u_n < \phi(\mathbf{u}') - r\}$$

Let $\phi \in C_c^\infty(U_r)$ and consider ϕv_r. Then it follows $\phi v_r = 0$ a.e. on $\mathbb{R}^n$. Let $w \equiv 0$. Then $w \in \mathfrak{S}$ and so $\gamma w = 0 = \gamma(\phi v_r) = \gamma\phi\gamma v_r$ in $H^{t-1/2}\left(\mathbb{R}^{n-1}\right)$. It follows that for m_{n-1} a.e. $\mathbf{x}' \in [\phi \neq 0] \cap \mathbb{R}^{n-1}$, $\gamma v_r(\mathbf{x}') = 0$. Now let $U' = \cup_{k=1}^\infty K_k$ where the K_k are compact sets such that $K_k \subseteq K_{k+1}$ and let $\phi_k \in C_c^\infty(U)$ such that ϕ_k has values in $[0,1]$ and $\phi_k(\mathbf{x}') = 1$ if $\mathbf{x}' \in K_k$. Then from what was just shown, $\gamma v_r = 0$ for a.e. point of K_k. Therefore, $\gamma v_r = 0$ for m_{n-1} a.e. point in U'. Therefore, since each $\gamma v_r = 0$, it follows from 41.21 that $\gamma v = 0$ also. This proves the lemma.

Theorem 41.33 *Let $t > 1/2$ and let U be of the form*

$$\{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U' \text{ and } 0 < u_n < \phi(\mathbf{u}')\}$$

where U' is an open subset of $\mathbb{R}^{n-1}$ and $\phi(\mathbf{u}')$ is a positive function such that $\phi(\mathbf{u}') \leq \infty$ and

$$\inf\{\phi(\mathbf{u}') : \mathbf{u}' \in U'\} = \delta > 0.$$

Then there exists a unique

$$\gamma \in \mathcal{L}\left(H^t(U), H^{t-1/2}(U')\right)$$

which has the property that if $u = v|_U$ where v is continuous and also a function of $H^1(\mathbb{R}^n)$, then $\gamma u(\mathbf{x}') = u(\mathbf{x}', 0)$ for a.e. $\mathbf{x}' \in U'$.

Proof: Let $u \in H^t(U)$. Then $u = v|_U$ for some $v \in H^t(\mathbb{R}^n)$. Define

$$\gamma u \equiv \gamma v|_{U'}$$

Is this well defined? The answer is yes because if $v_i|_U = u$ a.e., then $\gamma(v_1 - v_2) = 0$ a.e. on U' which implies $\gamma v_1 = \gamma v_2$ a.e. and so the two different versions of γu differ only on a set of measure zero.

If $u = v|_U$ where v is continuous and also a function of $H^1(\mathbb{R}^n)$, then for a.e. $\mathbf{x}' \in \mathbb{R}^{n-1}$, it follows from Lemma 41.31 on Page 1192 that $\gamma v(\mathbf{x}') = v(\mathbf{x}', 0)$. Hence, it follows that for a.e. $\mathbf{x}' \in U'$, $\gamma u(\mathbf{x}') \equiv u(\mathbf{x}', 0)$.

In particular, γ is determined by $\gamma u(\mathbf{x}') = u(\mathbf{x}', 0)$ on $\mathfrak{S}|_U$ and the density of $\mathfrak{S}|_U$ and continuity of γ shows γ is unique.

It only remains to show γ is continuous. Let $u \in H^t(U)$. Thus there exists $v \in H^t(\mathbb{R}^n)$ such that $u = v|_U$. Then

$$||\gamma u||_{H^{t-1/2}(U')} \leq ||\gamma v||_{H^{t-1/2}(\mathbb{R}^{n-1})} \leq C\,||v||_{H^t(\mathbb{R}^n)}$$

for C independent of v. Then taking the inf for all such $v \in H^t(\mathbb{R}^n)$ which are equal to u a.e. on U, it follows

$$||\gamma u||_{H^{t-1/2}(U')} \leq C\,||v||_{H^t(\mathbb{R}^n)}$$

and this proves γ is continuous.

41.5 Sobolev Spaces On Manifolds

41.5.1 General Theory

The type of manifold, Γ for which Sobolev spaces will be defined on is:

Definition 41.34 *1. Γ is a closed subset of $\mathbb{R}^p$ where $p \geq n$.*

2. $\Gamma = \cup_{i=1}^{\infty}\Gamma_i$ where $\Gamma_i = \Gamma \cap W_i$ for W_i a bounded open set.

3. *$\{W_i\}_{i=1}^{\infty}$ is locally finite.*

4. *There are open bounded sets, U_i and functions $\mathbf{h}_i : U_i \to \Gamma_i$ which are one to one, onto, and in $C^{m,1}(U_i)$. There exists a constant, C, such that $C \geq \operatorname{Lip} \mathbf{h}_r$ for all r.*

5. *There exist functions, $\mathbf{g}_i : W_i \to U_i$ such that $\mathbf{g}_i$ is $C^{m,1}(W_i)$, and $\mathbf{g}_i \circ \mathbf{h}_i = \mathrm{id}$ on U_i while $\mathbf{h}_i \circ \mathbf{g}_i = \mathrm{id}$ on Γ_i.*

This will be referred to as a $C^{m,1}$ manifold.

Lemma 41.35 *Let $\mathbf{g}_i$, $\mathbf{h}_i, U_i, W_i$, and Γ_i be as defined above. Then*

$$\mathbf{g}_i \circ \mathbf{h}_k : U_k \cap \mathbf{h}_k^{-1}(\Gamma_i) \to U_i \cap \mathbf{h}_i^{-1}(\Gamma_k)$$

is $C^{m,1}$. Furthermore, the inverse of this map is $\mathbf{g}_k \circ \mathbf{h}_i$.

Proof: First it is well to show it does indeed map the given open sets. Let $\mathbf{x} \in U_k \cap \mathbf{h}_k^{-1}(\Gamma_i)$. Then $\mathbf{h}_k(\mathbf{x}) \in \Gamma_k \cap \Gamma_i$ and so $\mathbf{g}_i(\mathbf{h}_k(\mathbf{x})) \in U_i$ because $\mathbf{h}_k(\mathbf{x}) \in \Gamma_i$. Now since $\mathbf{h}_k(\mathbf{x}) \in \Gamma_k$, $\mathbf{g}_i(\mathbf{h}_k(\mathbf{x})) \in \mathbf{h}_i^{-1}(\Gamma_k)$ also and this proves the mappings do what they should in terms of mapping the two open sets. That $\mathbf{g}_i \circ \mathbf{h}_k$ is $C^{m,1}$ follows immediately from the chain rule and the assumptions that the functions $\mathbf{g}_i$ and $\mathbf{h}_k$ are $C^{m,1}$. The claim about the inverse follows immediately from the definitions of the functions.

Let $\{\psi_i\}_{i=1}^{\infty}$ be a partition of unity subordinate to the open cover $\{W_i\}$ satisfying $\psi_i \in C_c^{\infty}(W_i)$. Then the following definition provides a norm for $H^{m+s}(\Gamma)$.

Definition 41.36 *Let $s \in (0,1)$ and m is a nonnegative integer. Also let μ denote the surface measure for Γ defined in the last section. A μ measurable function, u is in $H^{m+s}(\Gamma)$ if whenever $\{W_i, \psi_i, \Gamma_i, U_i, \mathbf{h}_i, \mathbf{g}_i\}_{i=1}^{\infty}$ is described above, $\mathbf{h}_i^*(u\psi_i) \in H^{m+s}(U_i)$ and*

$$||u||_{H^{m+s}(\Gamma)} \equiv \left(\sum_{i=1}^{\infty} ||\mathbf{h}_i^*(u\psi_i)||_{H^{m+s}(U_i)}^2 \right)^{1/2} < \infty.$$

Are there functions which are in $H^{m+s}(\Gamma)$? The answer is yes. Just take the restriction to Γ of any function, $u \in C_c^{\infty}(\mathbb{R}^m)$. Then each $\mathbf{h}_i^*(u\psi_i) \in H^{m+s}(U_i)$ and the sum is finite because $\operatorname{spt} u$ has nonempty intersection with only finitely many W_i.

It is not at all obvious this norm is well defined. What if $\{W_i', \psi_i', \Gamma_i', U_i, \mathbf{h}_i', \mathbf{g}_i'\}_{i=1}^{\infty}$ is as described above? Would the two norms be equivalent? If they aren't, then this is not a good way to define $H^{m+s}(\Gamma)$ because it would depend on the choice of partition of unity and functions, $\mathbf{h}_i$ and choice of the open sets, U_i. To begin with pick a particular choice for $\{W_i, \psi_i, \Gamma_i, U_i, \mathbf{h}_i, \mathbf{g}_i\}_{i=1}^{\infty}$.

Lemma 41.37 *$H^{m+s}(\Gamma)$ as just described, is a Banach space.*

Proof: Let $\{u_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $H^{m+s}(\Gamma)$. Then $\{\mathbf{h}_i^*(u_j\psi_i)\}_{j=1}^{\infty}$ is a Cauchy sequence in $H^{m+s}(U_i)$ for each i. Therefore, for each i, there exists $w_i \in H^{m+s}(U_i)$ such that

$$\lim_{j\to\infty} \mathbf{h}_i^*(u_j\psi_i) = w_i \text{ in } H^{m+s}(U_i). \tag{41.22}$$

It is required to show there exists $u \in H^{m+s}(\Gamma)$ such that $w_i = \mathbf{h}_i^*(u\psi_i)$ for each i.

Now from Corollary 39.14 it follows easily by approximating with simple functions that for ever nonnegative μ measurable function, f,

$$\int_\Gamma f d\mu = \sum_{r=1}^{\infty} \int_{\mathbf{g}_r\Gamma_r} \psi_r f(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u}) du.$$

Therefore,

$$\begin{aligned}
\int_\Gamma |u_j - u_k|^2 d\mu &= \sum_{r=1}^{\infty} \int_{\mathbf{g}_r\Gamma_r} \psi_r |u_j - u_k|^2 (\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u}) du \\
&\leq C\sum_{r=1}^{\infty} \int_{\mathbf{g}_r\Gamma_r} \psi_r |u_j - u_k|^2 (\mathbf{h}_r(\mathbf{u})) du \\
&= C\sum_{r=1}^{\infty} ||\mathbf{h}_r^*(\psi_r |u_j - u_k|)||_{0,2,U_r}^2 \\
&\leq C ||u_j - u_k||_{H^{m+s}(\Gamma)}
\end{aligned}$$

and it follows there exists $u \in L^2(\Gamma)$ such that

$$||u_j - u||_{0,2,\Gamma} \to 0.$$

and a subsequence, still denoted by u_j such that $u_j(\mathbf{x}) \to u(\mathbf{x})$ for μ a.e. $\mathbf{x} \in \Gamma$. It is required to show that $u \in H^{m+s}(\Gamma)$ such that $w_i = \mathbf{h}_i^*(u\psi_i)$ for each i. First of all, u is measurable because it is the limit of measurable functions. The pointwise convergence just established and the fact that sets of measure zero on Γ_i correspond to sets of measure zero on U_i which was discussed in the claim found in the proof of Theorem 39.13 on Page 1129 shows that

$$\mathbf{h}_i^*(u_j\psi_i)(\mathbf{x}) \to \mathbf{h}_i^*(u\psi_i)(\mathbf{x})$$

a.e. $\mathbf{x}$. Therefore,

$$\mathbf{h}_i^*(u\psi_i) = w_i$$

and this shows that $\mathbf{h}_i^*(u\psi_i) \in H^{m+s}(U_i)$. It remains to verify that $u \in H^{m+s}(\Gamma)$. This follows from Fatou's lemma. From 41.22,

$$||\mathbf{h}_i^*(u_j\psi_i)||_{H^{m+s}(U_i)}^2 \to ||\mathbf{h}_i^*(u\psi_i)||_{H^{m+s}(U_i)}^2$$

and so

$$\begin{aligned}\sum_{i=1}^{\infty}||\mathbf{h}_i^*(u\psi_i)||^2_{H^{m+s}(U_i)} &\leq \liminf_{j\to\infty}\sum_{i=1}^{\infty}||\mathbf{h}_i^*(u_j\psi_i)||^2_{H^{m+s}(U_i)} \\ &= \liminf_{j\to\infty}||u_j||^2_{H^{m+s}(\Gamma)}<\infty.\end{aligned}$$

This proves the lemma.

In fact any two such norms are equivalent. This follows from the open mapping theorem. Suppose $||\cdot||_1$ and $||\cdot||_2$ are two such norms and consider the norm $||\cdot||_3 \equiv \max(||\cdot||_1, ||\cdot||_2)$. Then $(H^{m+s}(\Gamma), ||\cdot||_3)$ is also a Banach space and the identity map from this Banach space to $(H^{m+s}(\Gamma), ||\cdot||_i)$ for $i = 1, 2$ is continuous. Therefore, by the open mapping theorem, there exist constants, C, C' such that for all $u \in H^{m+s}(\Gamma)$,

$$||u||_1 \leq ||u||_3 \leq C\,||u||_2 \leq C\,||u||_3 \leq CC'\,||u||_1$$

Therefore,

$$||u||_1 \leq C\,||u||_2\,,\ ||u||_2 \leq C'\,||u||_1\,.$$

This proves the following theorem.

Theorem 41.38 *Let Γ be described above. Defining $H^t(\Gamma)$ as in Definition 41.36, any two norms like those given in this definition are equivalent.*

Suppose $(\Gamma, W_i, U_i, \Gamma_i, \mathbf{h}_i, \mathbf{g}_i)$ are as defined above where $\mathbf{h}_i, \mathbf{g}_i$ are $C^{m,1}$ functions. Take W, an open set in $\mathbb{R}^p$ and define $\Gamma' \equiv W \cap \Gamma$. Then letting

$$W_i' \equiv W \cap W_i, \Gamma_i' \equiv W_i' \cap \Gamma,$$

and

$$U_i' \equiv \mathbf{g}_i(\Gamma_i') = \mathbf{h}_i^{-1}(W_i' \cap \Gamma),$$

it follows that U_i' is an open set because $\mathbf{h}_i$ is continuous and $(\Gamma', W_i', U_i', \Gamma_i', \mathbf{h}_i', \mathbf{g}_i')$ is also a $C^{m,1}$ manifold if you define $\mathbf{h}_i'$ to be the restriction of $\mathbf{h}_i$ to U_i' and $\mathbf{g}_i'$ to be the restriction of $\mathbf{g}_i$ to W_i'.

As a case of this, consider a $C^{m,1}$ manifold, Γ where $(\Gamma, W_i, U_i, \Gamma_i, \mathbf{h}_i, \mathbf{g}_i)$ are as described in Definition 41.34 and the submanifold consisting of Γ_i. The next lemma shows there is a simple way to define a norm on $H^t(\Gamma_i)$ which does not depend on dragging in a partition of unity.

Lemma 41.39 *Suppose Γ is a $C^{m,1}$ manifold and $(\Gamma, W_i, U_i, \Gamma_i, \mathbf{h}_i, \mathbf{g}_i)$ are as described in Definition 41.34. Then for $t \in [m, m+s)$, it follows that if $u \in H^t(\Gamma)$, then $u \in H^t(\Gamma_k)$ and the restriction map is continuous. Also an equivalent norm for $H^t(\Gamma_k)$ is given by*

$$|||u|||_t \equiv ||\mathbf{h}_k^* u||_{H^t(U_k)}\,.$$

Proof: Let $u \in H^t(\Gamma)$ and let $(\Gamma_k, W_i', U_i', \Gamma_i', \mathbf{h}_i', \mathbf{g}_i')$ be the sets and functions which define what is meant by Γ_k being a $C^{m,1}$ manifold as described in Definition 41.34. Also let $(\Gamma, W_i, U_i, \Gamma_i, \mathbf{h}_i, \mathbf{g}_i)$ be pertain to Γ in the same way and let $\{\phi_j\}$ be a C^∞ partition of unity for the $\{W_j\}$. Since the $\{W_i'\}$ are locally finite, only finitely many can intersect Γ_k, say $\{W_1', \cdots, W_s'\}$. Also only finitely many of the W_i can intersect Γ_k, say $\{W_1, \cdots, W_q\}$. Then letting $\{\psi_i'\}$ be a C^∞ partition of unity subordinate to the $\{W_i'\}$.

$$\begin{aligned}
&\sum_{i=1}^{\infty} \left|\left|\mathbf{h}_i'^* \left(u\psi_i'\right)\right|\right|_{H^t(U_i')} \\
= &\sum_{i=1}^{s} \left|\left|\mathbf{h}_i'^* \left(\sum_{j=1}^{q} \phi_j u \psi_i'\right)\right|\right|_{H^t(U_i')} \\
\leq &\sum_{i=1}^{s} \sum_{j=1}^{q} \left|\left|\mathbf{h}_i'^* \phi_j u \psi_i'\right|\right|_{H^t(U_i')} \\
= &\sum_{j=1}^{q} \sum_{i=1}^{s} \left|\left|\mathbf{h}_i'^* \phi_j u \psi_i'\right|\right|_{H^t(U_i')} \\
= &\sum_{j=1}^{q} \sum_{i=1}^{s} \left|\left|\left(\mathbf{g}_j \circ \mathbf{h}_i'\right)^* \mathbf{h}_j^* \phi_j u \psi_i'\right|\right|_{H^t(U_i')}.
\end{aligned}$$

By Lemma 41.20 on page 1185, there exists a single constant, C such that the above is dominated by

$$C \sum_{j=1}^{q} \sum_{i=1}^{s} \left|\left|\mathbf{h}_j^* \phi_j u \psi_i'\right|\right|_{H^t(U_j)}.$$

Now by Corollary 41.22 on Page 1186, this is no larger than

$$\begin{aligned}
C \sum_{j=1}^{q} \sum_{i=1}^{s} C_{\psi_i'} \left|\left|\mathbf{h}_j^* \phi_j u\right|\right|_{H^t(U_j)} &\leq C \sum_{j=1}^{q} \sum_{i=1}^{s} \left|\left|\mathbf{h}_j^* \phi_j u\right|\right|_{H^t(U_j)} \\
&\leq C \sum_{j=1}^{q} \left|\left|\mathbf{h}_j^* \phi_j u\right|\right|_{H^t(U_j)} < \infty.
\end{aligned}$$

This shows that u restricted to Γ_k is in $H^t(\Gamma_k)$. It also shows that the restriction map of $H^t(\Gamma)$ to $H^t(\Gamma_k)$ is continuous.

Now consider the norm $|||\cdot|||_t$. For $u \in H^t(\Gamma_k)$, let $(\Gamma_k, W_i', U_i', \Gamma_i', \mathbf{h}_i', \mathbf{g}_i')$ be sets and functions which define an atlas for Γ_k. Since the $\{W_i'\}$ are locally finite, only finitely many can have nonempty intersection with Γ_k, say $\{W_1, \cdots, W_s\}$. Thus $i \leq s$ for some finite s. The problem is to compare $|||\cdot|||_t$ with $||\cdot||_{H^t(\Gamma_k)}$. As above, let $\{\psi_i'\}$ denote a C^∞ partition of unity subordinate to the $\{W_j'\}$. Then

$$\begin{aligned}
|||u|||_t &\equiv ||\mathbf{h}_k^* u||_{H^t(U_k)} = \left|\left|\mathbf{h}_k^* \sum_{j=1}^s \psi_j' u\right|\right|_{H^t(U_k)} \\
&\leq \sum_{j=1}^s ||\mathbf{h}_k^* (\psi_j' u)||_{H^t(U_k)} \\
&= \sum_{j=1}^s \left|\left|(\mathbf{g}_j' \circ \mathbf{h}_k)^* \mathbf{h}_j'^* (\psi_j' u)\right|\right|_{H^t(U_k)} \\
&\leq C \sum_{j=1}^s ||\mathbf{h}_j'^* (\psi_j' u)||_{H^t(U_j')} . \\
&\leq C \left(\sum_{j=1}^s ||\mathbf{h}_j'^* (\psi_j' u)||_{H^t(U_j')}^2 \right)^{1/2} = ||u||_{H^t(\Gamma_k)} .
\end{aligned}$$

where Lemma 41.20 on page 1185 was used in the last step. Now also, from Lemma 41.20 on page 1185

$$\begin{aligned}
||u||_{H^t(\Gamma_k)} &= \left(\sum_{j=1}^s ||\mathbf{h}_j'^* (\psi_j' u)||_{H^t(U_j')}^2 \right)^{1/2} \\
&= \left(\sum_{j=1}^s \left|\left|(\mathbf{g}_k \circ \mathbf{h}_j')^* \mathbf{h}_k^* (\psi_j' u)\right|\right|_{H^t(U_j')}^2 \right)^{1/2} \\
&\leq C \left(\sum_{j=1}^s ||\mathbf{h}_k^* (\psi_j' u)||_{H^t(U_k)}^2 \right)^{1/2} \\
&\leq C \left(\sum_{j=1}^s ||\mathbf{h}_k^* u||_{H^t(U_k)}^2 \right)^{1/2} = C_s ||\mathbf{h}_k^* u||_{H^t(U_k)} = |||u|||_t .
\end{aligned}$$

This proves the lemma.

41.5.2 The Trace On The Boundary

Definition 41.40 *A bounded open subset, Ω, of $\mathbb{R}^n$ has a $C^{m,1}$boundary if it satisfies the following conditions. For each $p \in \Gamma \equiv \overline{\Omega} \setminus \Omega$, there exists an open set, W, containing p, an open interval $(0, b)$, a bounded open box $U' \subseteq \mathbb{R}^{n-1}$, and an affine orthogonal transformation, R_W consisting of a distance preserving linear transformation followed by a translation such that*

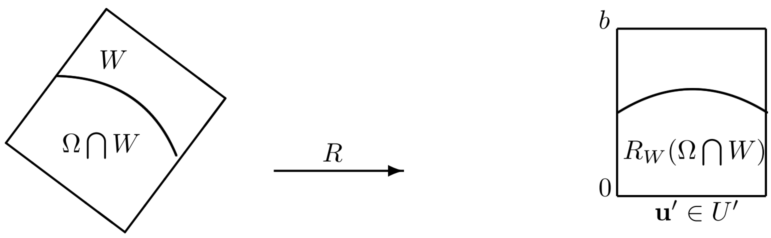
$$R_W W = U' \times (0, b), \tag{41.23}$$

$$R_W(W\cap\Omega) = \{\mathbf{u}\in\mathbb{R}^n : \mathbf{u}'\in U',\ 0<u_n<\phi_W(\mathbf{u}')\}\equiv U_W \tag{41.24}$$

where $\phi_W \in C^{m,1}\left(\overline{U'}\right)$ *meaning* ϕ_W *is the restriction to* U' *of a function, still denoted by* ϕ_W *which is in* $C^{m,1}\left(\mathbb{R}^{n-1}\right)$ *and*

$$\inf\{\phi_W(\mathbf{u}') : \mathbf{u}'\in U'\}>0$$

The following picture depicts the situation.



For the situation described in the above definition, let $\mathbf{h}_W : U'\to\Gamma\cap W$ be defined by

$$\mathbf{h}_W(\mathbf{u}')\equiv R_W^{-1}(\mathbf{u}',\phi_W(\mathbf{u}')),\ \mathbf{g}_W(\mathbf{x})\equiv(R_W\mathbf{x})',\ \mathbf{H}_W(\mathbf{u})\equiv R_W^{-1}(\mathbf{u}',\phi_W(\mathbf{u}')-u_n).$$

where $\mathbf{x}'\equiv(x_1,\cdots,x_{n-1})$ for $\mathbf{x}=(x_1,\cdots,x_n)$. Thus $\mathbf{g}_W\circ\mathbf{h}_W=\text{id}$ on U' and $\mathbf{h}_W\circ\mathbf{g}_W=\text{id}$ on $\Gamma\cap W$. Also note that $\mathbf{H}_W$ is defined on all of $\mathbb{R}^n$ is $C^{m,1}$, and has an inverse with the same properties. To see this, let $\mathbf{G}_W(\mathbf{u})=(\mathbf{u}',\phi_W(\mathbf{u}')-u_n)$. Then $\mathbf{H}_W=R_W^{-1}\circ\mathbf{G}_W$ and $\mathbf{G}_W^{-1}=(\mathbf{u}',\phi_W(\mathbf{u}')-u_n)$ and so $\mathbf{H}_W^{-1}=\mathbf{G}_W^{-1}\circ R_W$. Note also that as indicated in the picture,

$$R_W(W\cap\Omega)=\{\mathbf{u}\in\mathbb{R}^n:\mathbf{u}'\in U' \text{ and } 0<u_n<\phi_W(\mathbf{u}')\}.$$

Since $\Gamma=\partial\Omega$ is compact, there exist finitely many of these open sets, W, denoted by $\{W_i\}_{i=1}^q$ such that $\Gamma\subseteq\cup_{i=1}^q W_i$. Let the corresponding sets, U' be denoted by U_i' and let the functions, ϕ be denoted by ϕ_i. Also let $\mathbf{h}_i=\mathbf{h}_{W_i}$ etc. Now let $\{\psi_i\}_{i=1}^q$ be a C^∞ partition of unity subordinate to the $\{W_i\}_{i=1}^q$. If $u\in H^t(\Omega)$, then by Corollary 41.22 on Page 1186 it follows that $u\psi_i\in H^t(W_i\cap\Omega)$. Now

$$\mathbf{H}_i : U_i\equiv\{\mathbf{u}\in\mathbb{R}^n:\mathbf{u}'\in U_i',\ 0<u_n<\phi_i(\mathbf{u}')\}\to W_i\cap\Omega$$

and $\mathbf{H}_i$ and its inverse are defined on $\mathbb{R}^n$ and are in $C^{m,1}(\mathbb{R}^n)$. Therefore, by Lemma 41.20 on Page 1185,

$$\mathbf{H}_i^*\in\mathcal{L}\left(H^t(W_i\cap\Omega),H^t(U_i)\right).$$

Now it is possible to define the trace on $\Gamma\equiv\partial\Omega$. For $u\in H^t(\Omega)$,

$$\gamma u\equiv\sum_{i=1}^q\mathbf{g}_i^*(\gamma\mathbf{H}_i^*(u\psi_i)). \tag{41.25}$$

I must show it satisfies what it should. Recall the definition of what it means for a function to be in $H^{t-1/2}(\Gamma)$ where $t=m+s$.

Definition 41.41 *Let $s \in (0,1)$ and m is a nonnegative integer. Also let μ denote the surface measure for Γ. A μ measurable function, u is in $H^{m+s}(\Gamma)$ if whenever $\{W_i, \psi_i, \Gamma_i, U_i, \mathbf{h}_i, \mathbf{g}_i\}_{i=1}^{\infty}$ is described above, $\mathbf{h}_i^*(u\psi_i) \in H^{m+s}(U_i)$ and*

$$||u||_{H^{m+s}(\Gamma)} \equiv \left(\sum_{i=1}^{\infty} ||\mathbf{h}_i^*(u\psi_i)||^2_{H^{m+s}(U_i)} \right)^{1/2} < \infty.$$

Recall that all these norms which are obtained from various partitions of unity and functions, $\mathbf{h}_i$ and $\mathbf{g}_i$ are equivalent. Here there are only finitely many W_i so the sum is a finite sum. The theorem is the following.

Theorem 41.42 *Let Ω be a bounded open set having $C^{m,1}$ boundary as discussed above in Definition 41.40. Then for $t \leq m+1$, there exists a unique*

$$\gamma \in \mathcal{L}\left(H^t(\Omega), H^{t-1/2}(\Gamma)\right)$$

which has the property that for μ the measure on the boundary,

$$\gamma u(\mathbf{x}) = u(\mathbf{x}) \text{ for } \mu \text{ a.e. } \mathbf{x} \in \Gamma \text{ whenever } u \in \mathfrak{S}|_{\Omega}. \tag{41.26}$$

Proof: First consider the claim that $\gamma \in \mathcal{L}\left(H^t(\Omega), H^{t-1/2}(\Gamma)\right)$. This involves first showing that for $u \in H^t(\Omega), \gamma u \in H^{t-1/2}(\Gamma)$. To do this, use the above definition.

$$\begin{aligned} \mathbf{h}_j^*(\psi_j(\gamma u)) &= \sum_{i=1}^{q} \mathbf{h}_j^*(\psi_j \mathbf{g}_i^*(\gamma \mathbf{H}_i^*(u\psi_i))) \\ &= \sum_{i=1}^{q} \left(\mathbf{h}_j^*\psi_j\right)\left(\mathbf{h}_j^*(\mathbf{g}_i^*(\gamma \mathbf{H}_i^*(u\psi_i)))\right) \\ &= \sum_{i=1}^{q} \left(\mathbf{h}_j^*\psi_j\right)(\mathbf{g}_i \circ \mathbf{h}_j)^*(\gamma \mathbf{H}_i^*(u\psi_i)) \end{aligned} \tag{41.27}$$

First note that

$$\gamma \mathbf{H}_i^*(u\psi_i) \in H^{t-1/2}(U_i')$$

Now $\mathbf{g}_i \circ \mathbf{h}_j$ and its inverse, $\mathbf{g}_j \circ \mathbf{h}_i$ are both functions in $C^{m,1}\left(\mathbb{R}^{n-1}\right)$ and

$$\mathbf{g}_i \circ \mathbf{h}_j : U_j' \to U_i'.$$

Therefore, by Lemma 41.20 on Page 1185,

$$(\mathbf{g}_i \circ \mathbf{h}_j)^*(\gamma \mathbf{H}_i^*(u\psi_i)) \in H^{t-1/2}\left(U_j'\right)$$

and

$$\left|\left|(\mathbf{g}_i \circ \mathbf{h}_j)^*(\gamma \mathbf{H}_i^*(u\psi_i))\right|\right|_{H^{t-1/2}\left(U_j'\right)} \leq C_{ij} ||\gamma \mathbf{H}_i^*(u\psi_i)||_{H^{t-1/2}(U_i')}.$$

Also $\mathbf{h}_j^* \psi_j \in C^{m,1}\left(U_j'\right)$ and has compact support in U_j' and so by Corollary 41.22 on Page 1186

$$\left(\mathbf{h}_j^* \psi_j\right)\left(\mathbf{g}_i \circ \mathbf{h}_j\right)^* \left(\gamma \mathbf{H}_i^* \left(u\psi_i\right)\right) \in H^{t-1/2}\left(U_j'\right)$$

and

$$\begin{aligned}
&\left|\left|\left(\mathbf{h}_j^* \psi_j\right)\left(\mathbf{g}_i \circ \mathbf{h}_j\right)^* \left(\gamma \mathbf{H}_i^* \left(u\psi_i\right)\right)\right|\right|_{H^{t-1/2}\left(U_j'\right)} \\
\leq\ & C_{ij}\left|\left|\left(\mathbf{g}_i \circ \mathbf{h}_j\right)^* \left(\gamma \mathbf{H}_i^* \left(u\psi_i\right)\right)\right|\right|_{H^{t-1/2}\left(U_j'\right)} \qquad (41.28)\\
\leq\ & C_{ij}\left|\left|\gamma \mathbf{H}_i^* \left(u\psi_i\right)\right|\right|_{H^{t-1/2}\left(U_i'\right)}. \qquad (41.29)
\end{aligned}$$

This shows $\gamma u \in H^{t-1/2}(\Gamma)$ because each $\mathbf{h}_j^*\left(\psi_j(\gamma u)\right) \in H^{t-1/2}\left(U_j'\right)$. Also from 41.29 and 41.27

$$||\gamma u||_{H^{t-1/2}(\Gamma)}^2 \leq \sum_{j=1}^q \left|\left|\mathbf{h}_j^*\left(\psi_j(\gamma u)\right)\right|\right|_{H^{t-1/2}\left(U_j'\right)}^2$$

$$\begin{aligned}
=\ & \sum_{j=1}^q \left|\left|\mathbf{h}_j^*\left(\psi_j(\gamma u)\right)\right|\right|_{H^{t-1/2}\left(U_j'\right)}^2 \\
=\ & \sum_{j=1}^q \left|\left|\sum_{i=1}^q \left(\mathbf{h}_j^* \psi_j\right)\left(\mathbf{g}_i \circ \mathbf{h}_j\right)^* \left(\gamma \mathbf{H}_i^* \left(u\psi_i\right)\right)\right|\right|_{H^{t-1/2}\left(U_j'\right)}^2 \\
\leq\ & C_q \sum_{j=1}^q \sum_{i=1}^q \left|\left|\left(\mathbf{h}_j^* \psi_j\right)\left(\mathbf{g}_i \circ \mathbf{h}_j\right)^* \left(\gamma \mathbf{H}_i^* \left(u\psi_i\right)\right)\right|\right|_{H^{t-1/2}\left(U_j'\right)}^2
\end{aligned}$$

$$\begin{aligned}
\leq\ & C_q \sum_{j=1}^q \sum_{i=1}^q C_{ij}\left|\left|\left(\gamma \mathbf{H}_i^* \left(u\psi_i\right)\right)\right|\right|_{H^{t-1/2}\left(U_i'\right)}^2 \\
\leq\ & C_q \sum_{i=1}^q \left|\left|\left(\gamma \mathbf{H}_i^* \left(u\psi_i\right)\right)\right|\right|_{H^{t-1/2}\left(U_i'\right)}^2
\end{aligned}$$

$$\begin{aligned}
\leq\ & C_q \sum_{i=1}^q \left|\left|\mathbf{H}_i^* \left(u\psi_i\right)\right|\right|_{H^t(R_i(W_i \cap \Omega))}^2 \\
\leq\ & C_q \sum_{i=1}^q \left|\left|u\psi_i\right|\right|_{H^t(W_i \cap \Omega)}^2 \leq C_q\,||u||_{H^t(\Omega)}^2 .
\end{aligned}$$

Does γ satisfy 41.26? Let $\mathbf{x} \in \Gamma$ and $u \in \mathfrak{S}|_\Omega$. Let

$$I_{\mathbf{x}} \equiv \left\{i \in \{1, 2, \cdots, q\} : \mathbf{x} = \mathbf{h}_i\left(\mathbf{u}_i'\right) \text{ for some } \mathbf{u}_i' \in U_i'\right\}.$$

Then

$$\begin{aligned}\gamma u(\mathbf{x}) &= \sum_{i\in I_{\mathbf{x}}} (\gamma \mathbf{H}_i^* (u\psi_i))(\mathbf{g}_i(\mathbf{x})) \\ &= \sum_{i\in I_{\mathbf{x}}} (\gamma \mathbf{H}_i^* (u\psi_i))(\mathbf{g}_i(\mathbf{h}_i(\mathbf{u}_i'))) \\ &= \sum_{i\in I_{\mathbf{x}}} (\gamma \mathbf{H}_i^* (u\psi_i))(\mathbf{u}_i').\end{aligned}$$

Now because $\mathbf{H}_i$ is Lipschitz continuous and $u\psi \in \mathfrak{S}$, it follows that $\mathbf{H}_i^* (u\psi_i) \in H^1(\mathbb{R}^n)$ and is continuous and so by Theorem 41.33 on Page 1195 for a.e. $\mathbf{u}_i'$,

$$\begin{aligned} &= \sum_{i\in I_{\mathbf{x}}} \mathbf{H}_i^* (u\psi_i)(\mathbf{u}_i', 0) \\ &= \sum_{i\in I_{\mathbf{x}}} \mathbf{h}_i^* (u\psi_i)(\mathbf{u}_i') \\ &= \sum_{i\in I_{\mathbf{x}}} (u\psi_i)(\mathbf{h}_i(\mathbf{u}_i')) = u(\mathbf{x}) \text{ for } \mu \text{ a.e.}\mathbf{x}. \end{aligned} \tag{41.30}$$

This verifies 41.26 and completes the proof of the theorem.

Weak Solutions

42.1 The Lax Milgram Theorem

The Lax Milgram theorem is a fundamental result which is useful for obtaining weak solutions to many types of partial differential equations. It is really a general theorem in functional analysis.

Definition 42.1 *Let $A \in \mathcal{L}(V, V')$ where V is a Hilbert space. Then A is said to be coercive if*

$$A(v)(v) \geq \delta ||v||^2$$

for some $\delta > 0$.

Theorem 42.2 *(Lax Milgram) Let $A \in \mathcal{L}(V, V')$ be coercive. Then A maps one to one and onto.*

Proof: The proof that A is onto involves showing $A(V)$ is both dense and closed.

Consider first the claim that $A(V)$ is closed. Let $Ax_n \to y^* \in V'$. Then

$$\delta ||x_n - x_m||_V^2 \leq ||Ax_n - Ax_m||_{V'} ||x_n - x_m||_V .$$

Therefore, $\{x_n\}$ is a Cauchy sequence in V. It follows $x_n \to x \in V$ and since A is continuous, $Ax_n \to Ax$. This shows $A(V)$ is closed.

Now let $R : V \to V'$ denote the Riesz map defined by $Rx(y) = (y, x)$. Recall that the Riesz map is one to one, onto, and preserves norms. Therefore, $R^{-1}(A(V))$ is a closed subspace of V. If there $R^{-1}(A(V)) \neq V$, then $\left(R^{-1}(A(V))\right)^{\perp} \neq \{0\}$. Let $x \in \left(R^{-1}(A(V))\right)^{\perp}$ and $x \neq 0$. Then in particular,

$$0 = \left(x, R^{-1}Ax\right) = R\left(R^{-1}(A(x))\right)(x) = A(x)(x) \geq \delta ||x||_V^2 ,$$

a contradiction to $x \neq 0$. Therefore, $R^{-1}(A(V)) = V$ and so $A(V) = R(V) = V'$.

Since $A(V)$ is both closed and dense, $A(V) = V'$. This shows A is onto.

If $Ax = Ay$, then $0 = A(x-y)(x-y) \geq \delta ||x-y||_V^2$, and this shows A is one to one. This proves the theorem.

Here is a simple example which illustrates the use of the above theorem. In the example the repeated index summation convention is being used. That is, you sum over the repeated indices.

Example 42.3 *Let U be an open subset of $\mathbb{R}^n$ and let V be a closed subspace of $H^1(U)$. Let $\alpha^{ij} \in L^\infty(U)$ for $i,j = 1,2,\cdots,n$. Now define $A : V \to V'$ by*

$$A(u)(v) \equiv \int_U \left(\alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) + u(\mathbf{x}) v(\mathbf{x})\right) dx.$$

Suppose also that

$$\alpha^{ij} v_i v_j \geq \delta |\mathbf{v}|^2$$

whenever $\mathbf{v} \in \mathbb{R}^n$. Then A maps V to V' one to one and onto.

Here is why. It is obvious that A is in $\mathcal{L}(V, V')$. It only remains to verify that it is coercive.

$$\begin{aligned} A(u)(u) &\equiv \int_U \left(\alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) u_{,j}(\mathbf{x}) + u(\mathbf{x}) u(\mathbf{x})\right) dx \\ &\geq \int_U \delta |\nabla u(\mathbf{x})|^2 + |u(\mathbf{x})|^2 dx \\ &\geq \delta ||u||^2_{H^1(U)} \end{aligned}$$

This proves coercivity and verifies the claim.

What has been obtained in the above example? This depends on how you choose V. In Example 42.3 suppose U is a bounded open set with $C^{0,1}$ boundary and $V = H_0^1(U)$ where

$$H_0^1(U) \equiv \left\{u \in H^1(U) : \gamma u = 0\right\}.$$

Also suppose $f \in L^2(U)$. Then you can consider $F \in V'$ by defining

$$F(v) \equiv \int_U f(\mathbf{x}) v(\mathbf{x}) dx.$$

According to the Lax Milgram theorem and the verification of its conditions in Example 42.3, there exists a unique solution to the problem of finding $u \in H_0^1(U)$ such that for all $v \in H_0^1(U)$,

$$\int_U \left(\alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) + u(\mathbf{x}) v(\mathbf{x})\right) dx = \int_U f(\mathbf{x}) v(\mathbf{x}) dx \tag{42.1}$$

In particular, this holds for all $v \in C_c^\infty(U)$. Thus for all such v,

$$\int_U \left(-\left(\alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x})\right)_{,j} + u(\mathbf{x}) - f(\mathbf{x})\right) v(\mathbf{x}) dx = 0.$$

Therefore, in terms of weak derivatives,

$$-\left(\alpha^{ij} u_{,i}\right)_{,j} + u = f$$

and since $u \in H_0^1(U)$, it must be the case that $\gamma u = 0$ on ∂U. This is why the solution to 42.1 is referred to as a weak solution to the boundary value problem

$$-\left(\alpha^{ij}(\mathbf{x})\, u_{,i}(\mathbf{x})\right)_{,j} + u(\mathbf{x}) = f(\mathbf{x}),\ u = 0 \text{ on } \partial U.$$

Of course you then begin to ask the important question whether u really has two derivatives. It is not immediately clear that just because $-\left(\alpha^{ij}(\mathbf{x})\, u_{,i}(\mathbf{x})\right)_{,j} \in L^2(U)$ it follows that the second derivatives of u exist. Actually this will often be true and is discussed somewhat in the next section.

Next suppose you choose $V = H^1(U)$ and let $g \in H^{1/2}(\partial U)$. Define $F \in V'$ by

$$F(v) \equiv \int_U f(\mathbf{x})\, v(\mathbf{x})\, dx + \int_{\partial U} g(\mathbf{x})\, \gamma v(\mathbf{x})\, d\mu.$$

Everything works the same way and you get the existence of a unique $u \in H^1(U)$ such that for all $v \in H^1(U)$,

$$\int_U \left(\alpha^{ij}(\mathbf{x})\, u_{,i}(\mathbf{x})\, v_{,j}(\mathbf{x}) + u(\mathbf{x})\, v(\mathbf{x})\right) dx = \int_U f(\mathbf{x})\, v(\mathbf{x})\, dx + \int_{\partial U} g(\mathbf{x})\, \gamma v(\mathbf{x})\, d\mu \tag{42.2}$$

is satisfied. It you pretend u has all second order derivatives in $L^2(U)$ and apply the divergence theorem, you find that you have obtained a weak solution to

$$-\left(\alpha^{ij} u_{,i}\right)_{,j} + u = f,\ \alpha^{ij} u_{,i} n_j = g \text{ on } \partial U$$

where n_j is the j^{th} component of $\mathbf{n}$, the unit outer normal. Therefore, u is a weak solution to the above boundary value problem.

The conclusion is that the Lax Milgram theorem gives a way to obtain existence and uniqueness of weak solutions to various boundary value problems. The following theorem is often very useful in establishing coercivity. To prove this theorem, here is a definition.

Definition 42.4 *Let U be an open set and $\delta > 0$. Then*

$$U_\delta \equiv \left\{\mathbf{x} \in U : \operatorname{dist}\left(\mathbf{x}, U^C\right) > \delta\right\}.$$

Theorem 42.5 *Let U be a connected bounded open set having $C^{0,1}$ boundary such that for some sequence, $\eta_k \downarrow 0$,*

$$U = \cup_{k=1}^{\infty} U_{\eta_k} \tag{42.3}$$

and U_{η_k} is a connected open set. Suppose $\Gamma \subseteq \partial U$ has positive surface measure and that

$$V \equiv \left\{u \in H^1(U) : \gamma u = 0 \text{ a.e. on } \Gamma\right\}.$$

Then the norm $|||\cdot|||$ given by

$$|||u||| \equiv \left(\int_U |\nabla u|^2\, dx\right)^{1/2}$$

is equivalent to the usual norm on V.

Proof: First it is necessary to verify this is actually a norm. It clearly satisfies all the usual axioms of a norm except for the condition that $|||u||| = 0$ if and only if $u = 0$. Suppose then that $|||u||| = 0$. Let $\delta_0 = \eta_k$ for one of those η_k mentioned above and define

$$u_\delta(\mathbf{x}) \equiv \int_{B(\mathbf{0},\delta)} u(\mathbf{x}-\mathbf{y})\,\phi_\delta(\mathbf{y})\,dy$$

where ϕ_δ is a mollifier having support in $B(\mathbf{0},\delta)$. Then changing the variables, it follows that for $\mathbf{x} \in U_{\delta_0}$

$$u_\delta(\mathbf{x}) = \int_{B(\mathbf{x},\delta)} u(\mathbf{t})\,\phi_\delta(\mathbf{x}-\mathbf{t})\,dt = \int_U u(\mathbf{t})\,\phi_\delta(\mathbf{x}-\mathbf{t})\,dt$$

and so $u_\delta \in C^\infty(U_{\delta_0})$ and

$$\nabla u_\delta(\mathbf{x}) = \int_U u(\mathbf{t})\,\nabla\phi_\delta(\mathbf{x}-\mathbf{t})\,dt = \int_{B(\mathbf{0},\delta)} \nabla u(\mathbf{x}-\mathbf{y})\,\phi_\delta(\mathbf{y})\,dy = 0.$$

Therefore, u_δ equals a constant on U_{δ_0} because U_{δ_0} is a connected open set and u_δ is a smooth function defined on this set which has its gradient equal to $\mathbf{0}$. By Minkowski's inequality,

$$\left(\int_{U_{\delta_0}} |u(\mathbf{x}) - u_\delta(\mathbf{x})|^2\,dx\right)^{1/2} \leq \int_{B(\mathbf{0},\boldsymbol{\delta})} \phi_\delta(\mathbf{y}) \left(\int_{U_{\delta_0}} |u(\mathbf{x}) - u(\mathbf{x}-\mathbf{y})|^2\,dx\right)^{1/2} dy$$

and this converges to 0 as $\delta \to 0$ by continuity of translation in L^2. It follows there exists a sequence of constants, $c_\delta \equiv u_\delta(\mathbf{x})$ such that $\{c_\delta\}$ converges to u in $L^2(U_{\delta_0})$. Consequently, a subsequence, still denoted by u_δ, converges to u a.e. By Eggoroff's theorem there exists a set, N_k having measure no more than $3^{-k}m_n(U_{\delta_0})$ such that u_δ converges to u uniformly on N_k^C. Thus u is constant on N_k^C. Now $\sum_k m_n(N_k) \leq \frac{1}{2}m_n(U_{\delta_0})$ and so there exists $\mathbf{x}_0 \in U_{\delta_0} \setminus \cup_{k=1}^\infty N_k$. Therefore, if $\mathbf{x} \notin N_k$ it follows $u(\mathbf{x}) = u(\mathbf{x}_0)$ and so, if $u(\mathbf{x}) \neq u(\mathbf{x}_0)$ it must be the case that $\mathbf{x} \in \cap_{k=1}^\infty N_k$, a set of measure zero. This shows that u equals a constant a.e. on $U_{\delta_0} = U_{\eta_k}$. Since k is arbitrary, 42.3 shows u is a.e. equal to a constant on U. Therefore, u equals the restriction of a function of $\mathfrak{S}$ to U and so γu equals this constant in $L^2(\partial\Omega)$. Since the surface measure of Γ is positive, the constant must equal zero. Therefore, $|||\cdot|||$ is a norm.

It remains to verify that it is equivalent to the usual norm. It is clear that $|||u||| \leq ||u||_{1,2}$. What about the other direction? Suppose it is not true that for some constant, K, $||u||_{1,2} \leq K\,|||u|||$. Then for every $k \in \mathbb{N}$, there exists $u_k \in V$ such that

$$||u_k||_{1,2} > k\,|||u_k|||.$$

Replacing u_k with $u_k/\,||u_k||_{1,2}$, it can be assumed that $||u_k||_{1,2} = 1$ for all k. Therefore, using the compactness of the embedding of $H^1(U)$ into $L^2(U)$, there exists a subsequence, still denoted by u_k such that

$$
\begin{aligned}
u_k &\rightarrow u \text{ weakly in } V, & (42.4)\\
u_k &\rightarrow u \text{ strongly in } L^2(U), & (42.5)\\
|||u_k||| &\rightarrow 0, & (42.6)\\
u_k &\rightarrow u \text{ weakly in } (V, |||\cdot|||). & (42.7)
\end{aligned}
$$

From 42.6 and 42.7, it follows $u = 0$. Therefore, $|u_k|_{L^2(U)} \rightarrow 0$. This with 42.6 contradicts the fact that $||u_k||_{1,2} = 1$ and this proves the equivalence of the two norms.

The proof of the above theorem yields the following interesting corollary.

Corollary 42.6 *Let U be a connected open set with the property that for some sequence, $\eta_k \downarrow 0$,*

$$U = \cup_{k=1}^{\infty} U_{\eta_k}$$

for U_{η_k} a connected open set and suppose $u \in W^{1,p}(U)$ and $\nabla u = 0$ a.e. Then u equals a constant a.e.

Example 42.7 *Let U be a bounded open connected subset of $\mathbb{R}^n$ and let V be a closed subspace of $H^1(U)$ defined by*

$$V \equiv \left\{u \in H^1(U) : \gamma u = 0 \text{ on } \Gamma\right\}$$

where the surface measure of Γ is positive.

Let $\alpha^{ij} \in L^{\infty}(U)$ for $i, j = 1, 2, \cdots, n$ and define $A : V \rightarrow V'$ by

$$A(u)(v) \equiv \int_U \alpha^{ij}(\mathbf{x})\, u_{,i}(\mathbf{x})\, v_{,j}(\mathbf{x})\, dx.$$

for

$$\alpha^{ij} v_i v_j \geq \delta |\mathbf{v}|^2$$

whenever $\mathbf{v} \in \mathbb{R}^n$. Then A maps V to V' one to one and onto.

This follows from Theorem 42.5 using the equivalent norm defined there. Define $F \in V'$ by

$$\int_U f(\mathbf{x})\, v(\mathbf{x})\, dx + \int_{\partial U \setminus \Gamma} g(\mathbf{x})\, \gamma v(\mathbf{x})\, dx$$

for $f \in L^2(U)$ and $g \in H^{1/2}(\partial U)$. Then the equation,

$$Au = F \text{ in } V'$$

which is equivalent to $u \in V$ and for all $v \in V$,

$$\int_U \alpha^{ij}(\mathbf{x})\, u_{,i}(\mathbf{x})\, v_{,j}(\mathbf{x})\, dx = \int_U f(\mathbf{x})\, v(\mathbf{x})\, dx + \int_{\partial U \setminus \Gamma} g(\mathbf{x})\, \gamma v(\mathbf{x})\, d\mu$$

is a weak solution for the boundary value problem,

$$-\left(\alpha^{ij} u_{,i}\right)_{,j} = f \text{ in } U,\ \alpha^{ij} u_{,i} n_j = g \text{ on } \partial U \setminus \Gamma,\ u = 0 \text{ on } \Gamma$$

as you can verify by using the divergence theorem formally.

Korn's Inequality

A fundamental inequality used in elasticity to obtain coercivity and then apply the Lax Milgram theorem or some other theorem is Korn's inequality. The proof given here of this fundamental result follows [41] and [19].

43.1 A Fundamental Inequality

The proof of Korn's inequality depends on a fundamental inequality involving negative Sobolev space norms. The theorem to be proved is the following.

Theorem 43.1 *Let $f \in L^2(\Omega)$ where Ω is a bounded Lipschitz domain. Then there exist constants, C_1 and C_2 such that*

$$C_1 \left|\left|f\right|\right|_{0,2,\Omega} \leq \left(\left|\left|f\right|\right|_{-1,2,\Omega} + \sum_{i=1}^{n} \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\Omega} \right) \leq C_2 \left|\left|f\right|\right|_{0,2,\Omega},$$

where here $\left|\left|\cdot\right|\right|_{0,2,\Omega}$ represents the L^2 norm and $\left|\left|\cdot\right|\right|_{-1,2,\Omega}$ represents the norm in the dual space of $H_0^1(\Omega)$, denoted by $H^{-1}(\Omega)$.

Similar conventions will apply for any domain in place of Ω. The proof of this theorem will proceed through the use of several lemmas.

Lemma 43.2 *Let U^- denote the set,*

$$\{(\mathbf{x}, x_n) \in \mathbb{R}^n : x_n < g(\mathbf{x})\}$$

where $g : \mathbb{R}^{n-1} \to \mathbb{R}$ is Lipschitz and denote by U^+ the set

$$\{(\mathbf{x}, x_n) \in \mathbb{R}^n : x_n > g(\mathbf{x})\}.$$

Let $f \in L^2(U^-)$ and extend f to all of $\mathbb{R}^n$ in the following way.

$$f(\mathbf{x}, x_n) \equiv -3f(\mathbf{x}, 2g(\mathbf{x}) - x_n) + 4f(\mathbf{x}, 3g(\mathbf{x}) - 2x_n).$$

Then there is a constant, C_g, depending on g such that

$$||f||_{-1,2,\mathbb{R}^n}+\sum_{i=1}^n\left|\left|\frac{\partial f}{\partial x_i}\right|\right|_{-1,2,\mathbb{R}^n}\leq C_g\left(||f||_{-1,2,U^-}+\sum_{i=1}^n\left|\left|\frac{\partial f}{\partial x_i}\right|\right|_{-1,2,U^-}\right).$$

Proof: Let $\phi\in C_c^\infty\left(\mathbb{R}^n\right)$. Then,

$$\int_{\mathbb{R}^n}f\frac{\partial\phi}{\partial x_n}dx=\int_{U^+}\frac{\partial\phi}{\partial x_n}\left[-3f\left(\mathbf{x},2g\left(\mathbf{x}\right)-x_n\right)+4f\left(\mathbf{x},3g\left(\mathbf{x}\right)-2x_n\right)\right]dx$$

$$+\int_{U^-}f\frac{\partial\phi}{\partial x_n}dx. \tag{43.1}$$

Consider the first integral on the right in 43.1. Changing the variables, letting $y_n=2g\left(\mathbf{x}\right)-x_n$ in the first term of the integrand and $3g\left(\mathbf{x}\right)-2x_n$ in the next, it equals

$$-3\int_{U^-}\frac{\partial\phi}{\partial x_n}\left(\mathbf{x},2g\left(\mathbf{x}\right)-y_n\right)f\left(\mathbf{x},y_n\right)dy_ndx$$

$$+2\int_{U^-}\frac{\partial\phi}{\partial x_n}\left(\mathbf{x},\frac{3}{2}g\left(\mathbf{x}\right)-\frac{y_n}{2}\right)f\left(\mathbf{x},y_n\right)dy_ndx.$$

For $\left(\mathbf{x},y_n\right)\in U^-$, and defining

$$\psi\left(\mathbf{x},y_n\right)\equiv\phi\left(\mathbf{x},y_n\right)+3\phi\left(\mathbf{x},2g\left(\mathbf{x}\right)-y_n\right)-4\phi\left(\mathbf{x},\frac{3}{2}g\left(\mathbf{x}\right)-\frac{y_n}{2}\right),$$

it follows $\psi=0$ when $y_n=g\left(\mathbf{x}\right)$ and so

$$\int_{\mathbb{R}^n}f\frac{\partial\phi}{\partial x_n}dx=\int_{U^-}\frac{\partial\psi}{\partial y_n}f\left(\mathbf{x},y_n\right)dxdy_n.$$

Now from the definition of ψ given above,

$$||\psi||_{1,2,U^-}\leq C_g\,||\phi||_{1,2,U^-}\leq C_g\,||\phi||_{1,2,\mathbb{R}^n}$$

and so

$$\left|\left|\frac{\partial f}{\partial x_n}\right|\right|_{-1,2,\mathbb{R}^n}\equiv$$

$$\sup\left\{\int_{\mathbb{R}^n}f\frac{\partial\phi}{\partial x_n}dx:\phi\in C_c^\infty\left(\mathbb{R}^n\right),||\phi||_{1,2,\mathbb{R}^n}\leq 1\right\}\leq$$

$$\sup\left\{\left|\int_{U^-}f\frac{\partial\psi}{\partial x_n}dxdy_n\right|:\psi\in H_0^1\left(U^-\right),||\psi||_{1,2,U^-}\leq C_g\right\}$$

$$=C_g\left|\left|\frac{\partial f}{\partial x_n}\right|\right|_{-1,2,U^-} \tag{43.2}$$

It remains to establish a similar inequality for the case where the derivatives are taken with respect to x_i for $i < n$. Let $\phi \in C_c^{\infty}(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_i} dx = \int_{U^-} f \frac{\partial \phi}{\partial x_i} dx$$

$$\int_{U^+} \frac{\partial \phi}{\partial x_i} \left[-3f(\mathbf{x}, g(\mathbf{x}) - x_n) + 4f(\mathbf{x}, 3g(\mathbf{x}) - 2x_n) \right] dx.$$

Changing the variables as before, this last integral equals

$$-3 \int_{U^-} D_i \phi(\mathbf{x}, 2g(\mathbf{x}) - y_n) f(\mathbf{x}, y_n) \, dy_n dx$$

$$+ 2 \int_{U^-} D_i \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x}) - \frac{y_n}{2}\right) f(\mathbf{x}, y_n) \, dy_n dx. \tag{43.3}$$

Now let

$$\psi_1(\mathbf{x}, y_n) \equiv \phi(\mathbf{x}, 2g(\mathbf{x}) - y_n), \; \psi_2(\mathbf{x}, y_n) \equiv \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x}) - \frac{y_n}{2}\right).$$

Then

$$\frac{\partial \psi_1}{\partial x_i} = D_i \phi(\mathbf{x}, 2g(\mathbf{x}) - y_n) + D_n \phi(\mathbf{x}, 2g(\mathbf{x}) - y_n) \, 2 D_i g(\mathbf{x}),$$

$$\frac{\partial \psi_2}{\partial x_i} = D_i \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x}) - \frac{y_n}{2}\right) + D_n \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x}) - \frac{y_n}{2}\right) \frac{3}{2} D_i g(\mathbf{x}).$$

Also

$$\frac{\partial \psi_1}{\partial y_n}(\mathbf{x}, y_n) = -D_n \phi(\mathbf{x}, 2g(\mathbf{x}) - y_n),$$

$$\frac{\partial \psi_2}{\partial y_n}(\mathbf{x}, y_n) = \left(\frac{-1}{2}\right) D_n \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x}) - \frac{y_n}{2}\right).$$

Therefore,

$$\frac{\partial \psi_1}{\partial x_i}(\mathbf{x}, y_n) = D_i \phi(\mathbf{x}, 2g(\mathbf{x}) - y_n) - 2 \frac{\partial \psi_1}{\partial y_n}(\mathbf{x}, y_n) D_i g(\mathbf{x}),$$

$$\frac{\partial \psi_2}{\partial x_i}(\mathbf{x}, y_n) = D_i \phi\left(\mathbf{x}, \frac{3}{2} g(\mathbf{x}) - \frac{y_n}{2}\right) - 3 \frac{\partial \psi_2}{\partial y_n}(\mathbf{x}, y_n) D_i g(\mathbf{x}).$$

Using this in 43.3, the integrals in this expression equal

$$-3 \int_{U^-} \left[\frac{\partial \psi_1}{\partial x_i}(\mathbf{x}, y_n) + 2 \frac{\partial \psi_1}{\partial y_n}(\mathbf{x}, y_n) D_i g(\mathbf{x}) \right] f(\mathbf{x}, y_n) \, dy_n dx +$$

$$2 \int_{U^-} \left[\frac{\partial \psi_2}{\partial x_i}(\mathbf{x}, y_n) + 3 \frac{\partial \psi_2}{\partial y_n}(\mathbf{x}, y_n) D_i g(\mathbf{x}) \right] f(\mathbf{x}, y_n) \, dy_n dx$$

$$= \int_{U^-} \left[-3 \frac{\partial \psi_1 (\mathbf{x}, y)}{\partial x_i} + 2 \frac{\partial \psi_2 (\mathbf{x}, y_n)}{\partial x_i} \right] f (\mathbf{x}, y_n) \, dy_n dx.$$

Therefore,

$$\int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i} f dx = \int_{U^-} \left[\frac{\partial \phi}{\partial x_i} - 3 \frac{\partial \psi_1}{\partial x_i} + 2 \frac{\partial \psi_2}{\partial x_i} \right] f dx dy_n$$

and also

$$\phi (\mathbf{x}, g (\mathbf{x})) - 3\psi_1 (\mathbf{x}, g (\mathbf{x})) + 2\psi_2 (\mathbf{x}, g (\mathbf{x})) =$$

$$\phi (\mathbf{x}, g (\mathbf{x})) - 3\phi (\mathbf{x}, g (\mathbf{x})) + 2\phi (\mathbf{x}, g (\mathbf{x})) = 0$$

and so $\phi - 3\psi_1 + 2\psi_2 \in H_0^1 (U^-)$. It also follows from the definition of the functions, ψ_i and the assumption that g is Lipschitz, that

$$||\psi_i||_{1,2,U^-} \leq C_g \, ||\phi||_{1,2,U^-} \leq C_g \, ||\phi||_{1,2,\mathbb{R}^n} \, . \tag{43.4}$$

Therefore,

$$\left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\mathbb{R}^n} \equiv \sup \left\{ \left| \int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_i} dx \right| : ||\phi||_{1,2,\mathbb{R}^n} \leq 1 \right\}$$

$$= \sup \left\{ \left| \int_{U^-} f \left[\frac{\partial \phi}{\partial x_i} - 3 \frac{\partial \psi_1}{\partial x_i} + 2 \frac{\partial \psi_2}{\partial x_i} \right] dx \right| : ||\phi||_{1,2,\mathbb{R}^n} \leq 1 \right\}$$

$$\leq C_g \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,U^-}$$

where C_g is a constant which depends on g. This inequality along with 43.2 yields

$$\sum_{i=1}^{n} \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\mathbb{R}^n} \leq C_g \left(\sum_{i=1}^{n} \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,U^-} \right).$$

The inequality,

$$||f||_{-1,2,\mathbb{R}^n} \leq C_g \, ||f||_{-1,2,U^-}$$

follows from 43.4 and the equation,

$$\int_{\mathbb{R}^n} f \phi dx = \int_{U^-} f \phi dx - 3 \int_{U^-} f (\mathbf{x}, y_n) \, \psi_1 (\mathbf{x}, y_n) \, dx dy_n$$

$$+2 \int_{U^-} f (\mathbf{x}, y_n) \, \psi_2 (\mathbf{x}, y_n) \, dx dy_n$$

which results in the same way as before by changing variables using the definition of f off U^-. This proves the lemma.

The next lemma is a simple application of Fourier transforms.

Lemma 43.3 *If $f \in L^2 (\mathbb{R}^n)$, then the following formula holds.*

$$C_n \, ||f||_{0,2,\mathbb{R}^n} = \sum_{i=1}^{n} \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\mathbb{R}^n} + ||f||_{-1,2,\mathbb{R}^n}$$

Proof: For $\phi \in C_c^\infty(\mathbb{R}^n)$

$$||\phi||_{1,2,\mathbb{R}^n} \equiv \left(\int_{\mathbb{R}^n} \left(1 + |\mathbf{t}|^2\right) |F\phi|^2 \, dt \right)^{1/2}$$

is an equivalent norm to the usual Sobolev space norm for $H_0^1(\mathbb{R}^n)$ and is used in the following argument which depends on Plancherel's theorem and the fact that $F\left(\frac{\partial \phi}{\partial x_i}\right) = t_i F(\phi)$.

$$\left|\left|\frac{\partial f}{\partial x_i}\right|\right|_{-1,2,\mathbb{R}^n} \equiv \sup\left\{ \left| \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i} \overline{f} dx \right| : ||\phi||_{1,2} \leq 1 \right\}$$

$$= C_n \sup\left\{ \left| \int_{\mathbb{R}^n} t_i (F\phi) \overline{(Ff)} dt \right| : ||\phi||_{1,2} \leq 1 \right\}$$

$$= C_n \sup\left\{ \left| \int_{\mathbb{R}^n} \frac{t_i (F\phi) \left(1 + |\mathbf{t}|^2\right)^{1/2}}{\left(1 + |\mathbf{t}|^2\right)^{1/2}} \overline{(Ff)} dt \right| : ||\phi||_{1,2} \leq 1 \right\}$$

$$= C_n \left(\int \frac{|Ff|^2 t_i^2}{\left(1 + |\mathbf{t}|^2\right)} dt \right)^{1/2} \tag{43.5}$$

Also,

$$||f||_{-1,2} \equiv \sup\left\{ \left| \int_{\mathbb{R}^n} \phi \overline{f} dx \right| : ||\phi||_{1,2} \leq 1 \right\}$$

$$= C_n \sup\left\{ \left| \int_{\mathbb{R}^n} (F\phi) \left(\overline{Ff}\right) dx \right| : ||\phi||_{1,2} \leq 1 \right\}$$

$$= C_n \sup\left\{ \left| \int_{\mathbb{R}^n} \frac{F\phi \left(1 + |\mathbf{t}|^2\right)^{1/2}}{\left(1 + |\mathbf{t}|^2\right)^{1/2}} \overline{(Ff)} dt \right| : ||\phi||_{1,2} \leq 1 \right\}$$

$$= C_n \left(\int_{\mathbb{R}^n} \frac{|Ff|^2}{\left(1 + |\mathbf{t}|^2\right)} dt \right)^{1/2}$$

This along with 43.5 yields the conclusion of the lemma because

$$\sum_{i=1}^n \left|\left|\frac{\partial f}{\partial x_i}\right|\right|_{-1,2}^2 + ||f||_{-1,2}^2 = C_n \int_{\mathbb{R}^n} |Ff|^2 \, dx = C_n \, ||f||_{0,2}^2 .$$

Now consider Theorem 43.1. First note that by Lemma 43.2 and U^- defined there, Lemma 43.3 implies that for f extended as in Lemma 43.2,

$$||f||_{0,2,U^-} \leq ||f||_{0,2,\mathbb{R}^n} = C_n \left(||f||_{-1,2,\mathbb{R}^n} + \sum_{i=1}^n \left|\left|\frac{\partial f}{\partial x_i}\right|\right|_{-1,2,\mathbb{R}^n} \right)$$

$$\leq C_{gn}\left(||f||_{-1,2,U^-} + \sum_{i=1}^n \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,U^-} \right). \tag{43.6}$$

Let Ω be a bounded open set having Lipschitz boundary which lies locally on one side of its boundary. Let $\{Q_i\}_{i=0}^p$ be cubes of the sort used in the proof of the divergence theorem such that $\overline{Q_0} \subseteq \Omega$ and the other cubes cover the boundary of Ω. Let $\{\psi_i\}$ be a C^∞ partition of unity with $\operatorname{spt}(\psi_i) \subseteq Q_i$ and let $f \in L^2(\Omega)$. Then for $\phi \in C_c^\infty(\Omega)$ and ψ one of these functions in the partition of unity,

$$\left|\left| \frac{\partial (f\psi)}{\partial x_i} \right|\right|_{-1,2,\Omega} \leq \sup_{||\phi||_{1,2}\leq 1} \left| \int_\Omega f \frac{\partial}{\partial x_i}(\psi\phi)\, dx \right| + \sup_{||\phi||_{1,2}\leq 1} \left| \int_\Omega f\phi \frac{\partial \psi}{\partial x_i} dx \right|$$

Now if $||\phi||_{1,2} \leq 1$, then for a suitable constant, C_ψ,

$$||\psi\phi||_{1,2} \leq C_\psi ||\phi||_{1,2} \leq C_\psi, \ \left|\left| \phi \frac{\partial \psi}{\partial x_i} \right|\right|_{1,2} \leq C_\psi.$$

Therefore,

$$\left|\left| \frac{\partial (f\psi)}{\partial x_i} \right|\right|_{-1,2,\Omega} \leq \sup_{||\eta||_{1,2}\leq C_\psi} \left| \int_\Omega f \frac{\partial \eta}{\partial x_i} dx \right| + \sup_{||\eta||_{1,2}\leq C_\psi} \left| \int_\Omega f\eta dx \right|$$

$$\leq C_\psi \left(\left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\Omega} + ||f||_{-1,2,\Omega} \right). \tag{43.7}$$

Now using 43.7 and 43.6

$$||f\psi_j||_{0,2,\Omega} \leq C_g \left(||f\psi_j||_{-1,2,\Omega} + \sum_{i=1}^n \left|\left| \frac{\partial (f\psi_j)}{\partial x_i} \right|\right|_{-1,2,\Omega} \right)$$

$$\leq C_{\psi_j} C_g \left(||f||_{-1,2,\Omega} + \sum_{i=1}^n \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\Omega} \right).$$

Therefore, letting $C = \sum_{j=1}^p C_{\psi_j} C_g$,

$$||f||_{0,2,\Omega} \leq \sum_{j=1}^p ||f\psi_j||_{0,2,\Omega} \leq C \left(||f||_{-1,2,\Omega} + \sum_{i=1}^n \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\Omega} \right). \tag{43.8}$$

This proves the hard half of the inequality of Theorem 43.1.

To complete the proof, let $\overline{f}$ denote the zero extension of f off Ω. Then

$$||f||_{-1,2,\Omega} + \sum_{i=1}^n \left|\left| \frac{\partial f}{\partial x_i} \right|\right|_{-1,2,\Omega} \leq ||\overline{f}||_{-1,2,\mathbb{R}^n} + \sum_{i=1}^n \left|\left| \frac{\partial \overline{f}}{\partial x_i} \right|\right|_{-1,2,\mathbb{R}^n}$$

$$\leq C_n ||\overline{f}||_{0,2,\mathbb{R}^n} = C_n ||f||_{0,2,\Omega}.$$

This along with 43.8 proves Theorem 43.1.

43.2 Korn's Inequality

The inequality in this section is known as Korn's second inequality. It is also known as coercivity of strains. For $\mathbf{u}$ a vector valued function in $\mathbb{R}^n$, define

$$\boldsymbol{\varepsilon}_{ij}(\mathbf{u}) \equiv \frac{1}{2}(u_{i,j} + u_{j,i})$$

This is known as the strain or small strain. Korn's inequality says that the norm given by,

$$|||\mathbf{u}||| \equiv \left(\sum_{i=1}^{n} ||u_i||_{0,2,\Omega}^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} ||\boldsymbol{\varepsilon}_{ij}(\mathbf{u})||_{0,2,\Omega}^2 \right)^{1/2} \tag{43.9}$$

is equivalent to the norm,

$$||\mathbf{u}|| \equiv \left(\sum_{i=1}^{n} ||u_i||_{0,2,\Omega}^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \left|\left| \frac{\partial u_i}{\partial x_j} \right|\right|_{0,2,\Omega}^2 \right)^{1/2} \tag{43.10}$$

It is very significant because it is the strain as just defined which occurs in many of the physical models proposed in continuum mechanics. The inequality is far from obvious because the strains only involve certain combinations of partial derivatives.

Theorem 43.4 *(Korn's second inequality) Let Ω be any domain for which the conclusion of Theorem 43.1 holds. Then the two norms in 43.9 and 43.10 are equivalent.*

Proof: Let $\mathbf{u}$ be such that $u_i \in H^1(\Omega)$ for each $i = 1, \cdots, n$. Note that

$$\frac{\partial^2 u_i}{\partial x_j, \partial x_k} = \frac{\partial}{\partial x_j}(\boldsymbol{\varepsilon}_{ik}(\mathbf{u})) + \frac{\partial}{\partial x_k}(\boldsymbol{\varepsilon}_{ij}(\mathbf{u})) - \frac{\partial}{\partial x_i}(\boldsymbol{\varepsilon}_{jk}(\mathbf{u})).$$

Therefore, by Theorem 43.1,

$$\left|\left| \frac{\partial u_i}{\partial x_j} \right|\right|_{0,2,\Omega} \leq C \left[\left|\left| \frac{\partial u_i}{\partial x_j} \right|\right|_{-1,2,\Omega} + \sum_{k=1}^{n} \left|\left| \frac{\partial^2 u_i}{\partial x_j, \partial x_k} \right|\right|_{-1,2,\Omega} \right]$$

$$\leq C \left[\left|\left| \frac{\partial u_i}{\partial x_j} \right|\right|_{-1,2,\Omega} + \sum_{r,s,p} \left|\left| \frac{\partial \boldsymbol{\varepsilon}_{rs}(\mathbf{u})}{\partial x_p} \right|\right|_{-1,2,\Omega} \right]$$

$$\leq C \left[\left|\left| \frac{\partial u_i}{\partial x_j} \right|\right|_{-1,2,\Omega} + \sum_{r,s} ||\boldsymbol{\varepsilon}_{rs}(\mathbf{u})||_{0,2,\Omega} \right].$$

But also by this theorem,

$$||u_i||_{-1,2,\Omega} + \sum_{p} \left|\left| \frac{\partial u_i}{\partial x_p} \right|\right|_{-1,2,\Omega} \leq C\, ||u_i||_{0,2,\Omega}$$

and so

$$\left|\left|\frac{\partial u_i}{\partial x_j}\right|\right|_{0,2,\Omega} \leq C\left[||u_i||_{0,2,\Omega} + \sum_{r,s} ||\varepsilon_{rs}(\mathbf{u})||_{0,2,\Omega}\right]$$

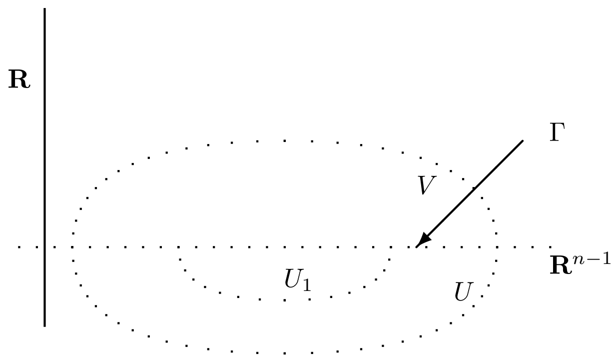
This proves the theorem.

Note that Ω did not need to be bounded. It suffices to be able to conclude the result of Theorem 43.1 which would hold whenever the boundary of Ω can be covered with finitely many boxes of the sort to which Lemma 43.2 can be applied.

Elliptic Regularity And Nirenberg Differences

44.1 The Case Of A Half Space

Regularity theorems are concerned with obtaining more regularity given a weak solution. This extra regularity is essential in order to obtain error estimates for various problems. In this section a regularity is given for weak solutions to various elliptic boundary value problems. To save on notation, I will use the repeated index summation convention. Thus you sum over repeated indices. Consider the following picture.



Here V is an open set,

$$U \equiv \{\mathbf{y} \in V : y_n < 0\}, \Gamma \equiv \{\mathbf{y} \in V : y_n = 0\}$$

and U_1 is an open set as shown for which $U_1 \subseteq V \cap U$. Assume also that V is bounded. Suppose

$$\begin{aligned} f &\in L^2(U), \\ \alpha^{rs} &\in C^{0,1}\left(\overline{U}\right), \end{aligned} \tag{44.1}$$

$$\alpha^{rs}(\mathbf{y})\, v_r v_s \geq \delta \left|\mathbf{v}\right|^2, \ \delta > 0. \tag{44.2}$$

The following technical lemma gives the essential ideas.

Lemma 44.1 *Suppose*

$$\begin{aligned} w &\in H^1(U), &(44.3)\\ \alpha^{rs} &\in C^{0,1}\left(\overline{U}\right), &(44.4)\\ h_s &\in H^1(U), &(44.5)\\ f &\in L^2(U). &(44.6)\end{aligned}$$

and

$$\int_U \alpha^{rs}(\mathbf{y}) \frac{\partial w}{\partial y^r}\frac{\partial z}{\partial y^s} dy + \int_U h_s(\mathbf{y}) \frac{\partial z}{\partial y^s} dy = \int_U fz dy \tag{44.7}$$

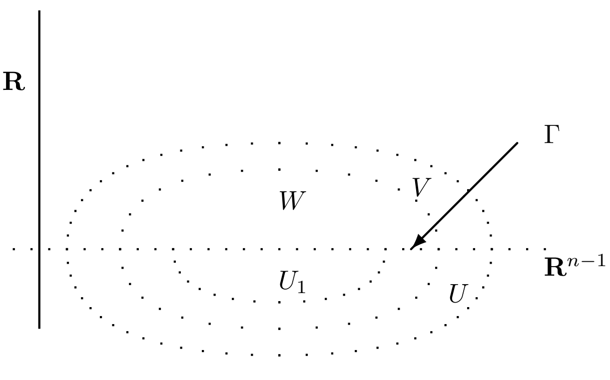
for all $z \in H^1(U)$ *having the property that* $\operatorname{spt}(z) \subseteq V$. *Then* $w \in H^2(U_1)$ *and for some constant* C, *independent of* f, w, *and* g, *the following estimate holds.*

$$||w||^2_{H^2(U_1)} \leq C\left(||w||^2_{H^1(U)} + ||f||^2_{L^2(U)} + \sum_s ||h_s||^2_{H^1(U)}\right). \tag{44.8}$$

Proof: Define for small real h,

$$D_k^h l(\mathbf{y}) \equiv \frac{1}{h}\left(l(\mathbf{y} + h\mathbf{e}_k) - l(\mathbf{y})\right).$$

Let $U_1 \subseteq \overline{U_1} \subseteq W \subseteq \overline{W} \subseteq V$ and let $\eta \in C_c^\infty(W)$ with $\eta(\mathbf{y}) \in [0,1]$, and $\eta = 1$ on $\overline{U_1}$ as shown in the following picture.



For h small $(3h < \operatorname{dist}\left(\overline{W}, V^C\right))$, let

$$z(\mathbf{y}) \equiv \frac{1}{h}\left\{\eta^2(\mathbf{y} - h\mathbf{e}_k)\left[\frac{w(\mathbf{y}) - w(\mathbf{y} - h\mathbf{e}_k)}{h}\right]\right.$$

$$-\eta^2(\mathbf{y})\left[\frac{w(\mathbf{y}+h\mathbf{e}_k)-w(\mathbf{y})}{h}\right]\Bigg\} \tag{44.9}$$

$$\equiv \quad -D_k^{-h}\left(\eta^2 D_k^h w\right), \tag{44.10}$$

where here $k<n$. Thus z can be used in equation 44.7. Begin by estimating the left side of 44.7.

$$\int_U \alpha^{rs}(\mathbf{y})\frac{\partial w}{\partial y^r}\frac{\partial z}{\partial y^s}dy$$

$$= \frac{1}{h}\int_U \alpha^{rs}(\mathbf{y}+h\mathbf{e}_k)\frac{\partial w}{\partial y^r}(\mathbf{y}+h\mathbf{e}_k)\frac{\partial\left(\eta^2 D_k^h w\right)}{\partial y^s}dy$$

$$-\frac{1}{h}\int_U \alpha^{rs}(\mathbf{y})\frac{\partial w}{\partial y^r}\frac{\partial\left(\eta^2 D_k^h w\right)}{\partial y^s}dy$$

$$=\int_U \alpha^{rs}(\mathbf{y}+h\mathbf{e}_k)\frac{\partial\left(D_k^h w\right)}{\partial y^r}\frac{\partial\left(\eta^2 D_k^h w\right)}{\partial y^s}dy+$$

$$\frac{1}{h}\int_U\left(\alpha^{rs}(\mathbf{y}+h\mathbf{e}_k)-\alpha^{rs}(\mathbf{y})\right)\frac{\partial w}{\partial y^r}\frac{\partial\left(\eta^2 D_k^h w\right)}{\partial y^s}dy \tag{44.11}$$

Now

$$\frac{\partial\left(\eta^2 D_k^h w\right)}{\partial y^s}=2\eta\frac{\partial\eta}{\partial y^s}D_k^h w+\eta^2\frac{\partial\left(D_k^h w\right)}{\partial y^s}. \tag{44.12}$$

therefore,

$$= \int_U \eta^2\alpha^{rs}(\mathbf{y}+h\mathbf{e}_k)\frac{\partial\left(D_k^h w\right)}{\partial y^r}\frac{\partial\left(D_k^h w\right)}{\partial y^s}dy$$

$$+\left\{\int_{W\cap U}\alpha^{rs}(\mathbf{y}+h\mathbf{e}_k)\frac{\partial\left(D_k^h w\right)}{\partial y^r}2\eta\frac{\partial\eta}{\partial y^s}D_k^h w dy\right.$$

$$\left.+\frac{1}{h}\int_{W\cap U}\left(\alpha^{rs}(\mathbf{y}+h\mathbf{e}_k)-\alpha^{rs}(\mathbf{y})\right)\frac{\partial w}{\partial y^r}\frac{\partial\left(\eta^2 D_k^h w\right)}{\partial y^s}dy\right\}\equiv A.+\{B.\}. \tag{44.13}$$

Now consider these two terms. From 44.2,

$$A.\geq\delta\int_U\eta^2\left|\nabla D_k^h w\right|^2 dy. \tag{44.14}$$

Using the Lipschitz continuity of α^{rs} and 44.12,

$$B.\leq C(\eta,\mathrm{Lip}(\alpha),\alpha)\Big\{\left|\left|D_k^h w\right|\right|_{L^2(W\cap U)}\left|\left|\eta\nabla D_k^h w\right|\right|_{L^2(W\cap U;\mathbb{R}^n)}+$$

$$\left|\left|\eta\nabla w\right|\right|_{L^2(W\cap U;\mathbb{R}^n)}\left|\left|\eta\nabla D_k^h w\right|\right|_{L^2(W\cap U;\mathbb{R}^n)}$$

$$+\left|\left|\eta\nabla w\right|\right|_{L^2(W\cap U;\mathbb{R}^n)}\left|\left|D_k^h w\right|\right|_{L^2(W\cap U)}\Big\}. \tag{44.15}$$

$$\leq C(\eta, \text{Lip}(\alpha), \alpha) C_\varepsilon \left(\left|\left|D_k^h w\right|\right|^2_{L^2(W\cap U)} + ||\eta \nabla w||^2_{L^2(W\cap U;\mathbb{R}^n)} \right) +$$

$$\varepsilon C(\eta, \text{Lip}(\alpha), \alpha) \left(\left|\left|\eta \nabla D_k^h w\right|\right|^2_{L^2(W\cap U;\mathbb{R}^n)} + \left|\left|D_k^h w\right|\right|^2_{L^2(W\cap U)} \right). \tag{44.16}$$

Now

$$\left|\left|D_k^h w\right|\right|_{L^2(W)} \leq ||\nabla w||^2_{L^2(U;\mathbb{R}^n)}. \tag{44.17}$$

To see this, observe that if w is smooth, then

$$\left(\int_W \left| \frac{w(\mathbf{y} + h\mathbf{e}_k) - w(\mathbf{y})}{h} \right|^2 dy \right)^{1/2}$$

$$\leq \left(\int_W \left| \frac{1}{h} \int_0^h \nabla w(\mathbf{y} + t\mathbf{e}_k) \cdot \mathbf{e}_k dt \right|^2 dy \right)^{1/2}$$

$$\leq \left(\int_0^h \left(\int_W |\nabla w(\mathbf{y} + t\mathbf{e}_k) \cdot \mathbf{e}_k|^2 dy \right)^{1/2} \frac{dt}{h} \right) \leq ||\nabla w||_{L^2(U;\mathbb{R}^n)}$$

so by density of such functions in $H^1(U)$, 44.17 holds. Therefore, changing ε, yields

$$B. \leq C_\varepsilon (\eta, \text{Lip}(\alpha), \alpha) ||\nabla w||^2_{L^2(U;\mathbb{R}^n)} + \varepsilon \left|\left|\eta \nabla D_k^h w\right|\right|^2_{L^2(W\cap U;\mathbb{R}^n)}. \tag{44.18}$$

With 44.14 and 44.18 established, consider the other terms of 44.7.

$$\begin{aligned}
\left| \int_U fz dy \right| &\leq \left| \int_U f\left(-D_k^{-h} \eta^2 D_k^h w\right) dy \right| \\
&\leq \left(\int_U |f|^2 dy \right)^{1/2} \left(\int_U \left| D_k^{-h} \left(\eta^2 D_k^h w\right) \right|^2 dy \right)^{1/2} \\
&\leq ||f||_{L^2(U)} \left|\left|\nabla\left(\eta^2 D_k^h w\right)\right|\right|_{L^2(U;\mathbb{R}^n)} \\
&\leq ||f||_{L^2(U)} \left(\left|\left|2\eta \nabla \eta D_k^h w\right|\right|_{L^2(U;\mathbb{R}^n)} + \left|\left|\eta^2 \nabla D_k^h w\right|\right|_{L^2(U;\mathbb{R}^n)} \right) \\
&\leq C ||f||_{L^2(U)} ||\nabla w||_{L^2(U;\mathbb{R}^n)} + ||f||_{L^2(U)} \left|\left|\eta \nabla D_k^h w\right|\right|_{L^2(U;\mathbb{R}^n)} \\
&\leq C_\varepsilon \left(||f||^2_{L^2(U)} + ||\nabla w||^2_{L^2(U;\mathbb{R}^n)} \right) + \varepsilon \left|\left|\eta \nabla D_k^h w\right|\right|^2_{L^2(U;\mathbb{R}^n)} \quad (44.19)
\end{aligned}$$

$$\begin{aligned}
&\left|\int_U h_s(\mathbf{y})\frac{\partial z}{\partial y^s}dy\right| \\
\leq\ & \left|\int_U h_s(\mathbf{y})\frac{\partial\left(-D_k^{-h}\left(\eta^2 D_k^h w\right)\right)}{\partial y^s}dy\right| \\
\leq\ & \left|\int_U D_k^h h_s(\mathbf{y})\frac{\partial\left(\left(\eta^2 D_k^h w\right)\right)}{\partial y^s}\right| \\
\leq\ & \int_U\left|D_k^h h_s 2\eta\frac{\partial\eta}{\partial y^s}D_k^h w\right|dy+\int_U\left|\left(\eta D_k^h h_s\right)\left(\eta\frac{\partial\left(D_k^h w\right)}{\partial y^s}\right)\right|dy \\
\leq\ & C\sum_s ||h_s||_{H^1(U)}\left(||w||_{H^1(U)}+\left|\left|\eta\nabla D_k^h w\right|\right|_{L^2(U;\mathbb{R}^n)}\right) \\
\leq\ & C_\varepsilon\sum_s ||h_s||_{H^1(U)}^2+||w||_{H^1(U)}^2+\varepsilon\left|\left|\eta\nabla D_k^h w\right|\right|_{L^2(U;\mathbb{R}^n)}^2. \qquad (44.20)
\end{aligned}$$

The following inequalities in 44.14,44.18, 44.19and 44.20 are summarized here.

$$A. \geq \delta\int_U \eta^2\left|\nabla D_k^h w\right|^2 dy,$$

$$B. \leq C_\varepsilon(\eta,\mathrm{Lip}(\alpha),\alpha)\,||\nabla w||_{L^2(U;\mathbb{R}^n)}^2+\varepsilon\left|\left|\eta\nabla D_k^h w\right|\right|_{L^2(W\cap U;\mathbb{R}^n)}^2,$$

$$\left|\int_U fz dy\right| \leq C_\varepsilon\left(||f||_{L^2(U)}^2+||\nabla w||_{L^2(U;\mathbb{R}^n)}^2\right)+\varepsilon\left|\left|\eta\nabla D_k^h w\right|\right|_{L^2(U;\mathbb{R}^n)}^2$$

$$\begin{aligned}
\left|\int_U h_s(\mathbf{y})\frac{\partial z}{\partial y^s}dy\right| \leq\ & C_\varepsilon\sum_s ||h_s||_{H^1(U)}^2 \\
& +||w||_{H^1(U)}^2+\varepsilon\left|\left|\eta\nabla D_k^h w\right|\right|_{L^2(U;\mathbb{R}^n)}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\delta\left|\left|\eta\nabla D_k^h w\right|\right|_{L^2(U;\mathbb{R}^n)}^2 \\
\leq\ & C_\varepsilon(\eta,\mathrm{Lip}(\alpha),\alpha)\,||\nabla w||_{L^2(U;\mathbb{R}^n)}^2+\varepsilon\left|\left|\eta\nabla D_k^h w\right|\right|_{L^2(U;\mathbb{R}^n)}^2 \\
&+C_\varepsilon\sum_s ||h_s||_{H^1(U)}^2+||w||_{H^1(U)}^2+\varepsilon\left|\left|\eta\nabla D_k^h w\right|\right|_{L^2(U;\mathbb{R}^n)}^2 \\
&+C_\varepsilon\left(||f||_{L^2(U)}^2+||\nabla w||_{L^2(U;\mathbb{R}^n)}^2\right)+\varepsilon\left|\left|\eta\nabla D_k^h w\right|\right|_{L^2(U;\mathbb{R}^n)}^2.
\end{aligned}$$

Letting ε be small enough and adjusting constants yields

$$\left|\left|\nabla D_k^h w\right|\right|_{L^2(U_1;\mathbb{R}^n)}^2 \leq \left|\left|\eta\nabla D_k^h w\right|\right|_{L^2(U;\mathbb{R}^n)}^2 \leq$$

$$C\left(||w||_{H^1(U)}^2+||f||_{L^2(U)}^2+C_\varepsilon\sum_s||h_s||_{H^1(U)}^2\right)$$

where the constant, C, depends on $\eta, \mathrm{Lip}(\alpha), \alpha, \delta$. Since this holds for all h small enough, it follows $\frac{\partial w}{\partial y^k}\in H^1(U_1)$ and

$$\left|\left|\nabla\frac{\partial w}{\partial y^k}\right|\right|_{L^2(U_1;\mathbb{R}^n)}^2\leq$$

$$C\left(||w||_{H^1(U)}^2+||f||_{L^2(U)}^2+C_\varepsilon\sum_s||h_s||_{H^1(U)}^2\right) \tag{44.21}$$

for each $k<n$. It remains to estimate $\left|\left|\frac{\partial^2 w}{\partial y_n^2}\right|\right|_{L^2(U_1)}^2$. To do this return to 44.7 which must hold for all $z\in C_c^\infty(U_1)$. Therefore, using 44.7 it follows that for all $z\in C_c^\infty(U_1)$,

$$\int_U\alpha^{rs}(\mathbf{y})\frac{\partial w}{\partial y^r}\frac{\partial z}{\partial y^s}dy=-\int_U\frac{\partial h_s}{\partial y^s}zdy+\int_U fzdy.$$

Now from the Lipschitz assumption on α^{rs}, it follows

$$\begin{aligned} F &\equiv \sum_{r,s\leq n-1}\frac{\partial}{\partial y^s}\left(\alpha^{rs}\frac{\partial w}{\partial y^r}\right) \\ &\quad+\sum_{s\leq n-1}\frac{\partial}{\partial y^s}\left(\alpha^{ns}\frac{\partial w}{\partial y^n}\right)-\sum_s\frac{\partial h_s}{\partial y^s}+f \\ &\in L^2(U_1)\end{aligned}$$

and

$$||F||_{L^2(U_1)}\leq C\left(||w||_{H^1(U)}^2+||f||_{L^2(U)}^2+C_\varepsilon\sum_s||h_s||_{H^1(U)}^2\right). \tag{44.22}$$

Therefore, from density of $C_c^\infty(U_1)$ in $L^2(U_1)$,

$$-\frac{\partial}{\partial y^n}\left(\alpha^{nn}(\mathbf{y})\frac{\partial w}{\partial y^n}\right)=F,\text{ no sum on } n$$

and so

$$-\frac{\partial\alpha^{nn}}{\partial y^n}\frac{\partial w}{\partial y^n}-\alpha^{nn}\frac{\partial^2 w}{\partial(y^n)^2}=F$$

By 44.2 $\alpha^{nn}(\mathbf{y})\geq\delta$ and so it follows from 44.22 that there exists a constant,C depending on δ such that

$$\left|\frac{\partial^2 w}{\partial(y^n)^2}\right|_{L^2(U_1)}\leq C\left(|F|_{L^2(U_1)}+||w||_{H^1(U)}\right)$$

which with 44.21 and 44.22 implies the existence of a constant, C depending on δ such that

$$||w||^2_{H^2(U_1)} \leq C\left(||w||^2_{H^1(U)} + ||f||^2_{L^2(U)} + C_\varepsilon \sum_s ||h_s||^2_{H^1(U)}\right),$$

proving the lemma.

What if more regularity is known for f , h_s, α^{rs} and w? Could more be said about the regularity of the solution? The answer is yes and is the content of the next corollary.

First here is some notation. For α a multi-index with $|\alpha| = k-1$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ define

$$D^h_\alpha l(\mathbf{y}) \equiv \prod_{k=1}^n \left(D^h_k\right)^{\alpha_k} l(\mathbf{y}).$$

Also, for α and τ multi indices, $\tau < \alpha$ means $\tau_i < \alpha_i$ for each i.

Corollary 44.2 *Suppose in the context of Lemma 44.1 the following for* $k \geq 1$.

$$\begin{aligned} w &\in H^k(U), \\ \alpha^{rs} &\in C^{k-1,1}\left(\overline{U}\right), \\ h_s &\in H^k(U), \\ f &\in H^{k-1}(U), \end{aligned}$$

and

$$\int_U \alpha^{rs}(\mathbf{y}) \frac{\partial w}{\partial y^r}\frac{\partial z}{\partial y^s} dy + \int_U h_s(\mathbf{y}) \frac{\partial z}{\partial y^s} dy = \int_U fz dy \tag{44.23}$$

for all $z \in H^1(U)$ *or* $H^1_0(U)$ *such that* $\operatorname{spt}(z) \subseteq V$. *Then there exists* C *independent of* w *such that*

$$||w||_{H^{k+1}(U_1)} \leq C\left(||f||_{H^{k-1}(U)} + \sum_s ||h_s||_{H^k(U)} + ||w||_{H^k(U)}\right). \tag{44.24}$$

Proof: The proof involves the following claim which is proved using the conclusion of Lemma 44.1 on Page 1220.

Claim : If $\alpha = (\alpha', 0)$ where $|\alpha'| \leq k-1$, then there exists a constant independent of w such that

$$||D^\alpha w||_{H^2(U_1)} \leq C\left(||f||_{H^{k-1}(U)} + \sum_s ||h_s||_{H^k(U)} + ||w||_{H^k(U)}\right). \tag{44.25}$$

Proof of claim: First note that if $|\alpha| = 0$, then 44.25 follows from Lemma 44.1 on Page 1220. Now suppose the conclusion of the claim holds for all $|\alpha| \leq j-1$ where $j < k$. Let $|\alpha| = j$ and $\alpha = (\alpha', 0)$. Then for $z \in H^1(U)$ having compact support in V, it follows that for h small enough,

$$D^{-h}_\alpha z \in H^1(U),\ \operatorname{spt}\left(D^h_\alpha z\right) \subseteq V.$$

Therefore, you can replace z in 44.23 with $D_\alpha^{-h} z$. Now note that you can apply the following manipulation.

$$\int_U p(\mathbf{y}) D_\alpha^{-h} z(\mathbf{y})\, dy = \int_U D_\alpha^h p(\mathbf{y}) z(\mathbf{y})\, dy$$

and obtain

$$\int_U \left(D_\alpha^h \left(\alpha^{rs} \frac{\partial w}{\partial y^r} \right) \frac{\partial z}{\partial y^s} + D_\alpha^h (h_s) \frac{\partial z}{\partial y^s} \right) dy = \int_U \left(\left(D_\alpha^h f \right) z \right) dy. \tag{44.26}$$

Letting $h \to 0$, this gives

$$\int_U \left(D^\alpha \left(\alpha^{rs} \frac{\partial w}{\partial y^r} \right) \frac{\partial z}{\partial y^s} + D^\alpha (h_s) \frac{\partial z}{\partial y^s} \right) dy = \int_U ((D^\alpha f) z)\, dy.$$

Now

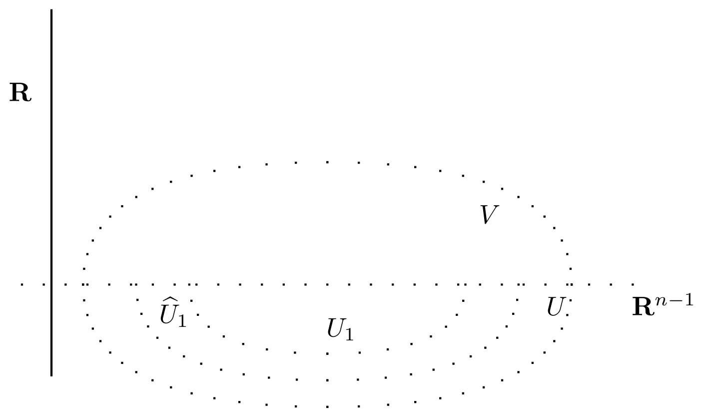
$$D^\alpha \left(\alpha^{rs} \frac{\partial w}{\partial y^r} \right) = \alpha^{rs} \frac{\partial (D^\alpha w)}{\partial y^r} + \sum_{\tau < \alpha} C(\tau) D^{\alpha - \tau} (\alpha^{rs}) \frac{\partial (D^\tau w)}{\partial y^r}$$

where $C(\tau)$ is some coefficient. Therefore, from 44.26,

$$\int_U \alpha^{rs} \frac{\partial (D^\alpha w)}{\partial y^r} \frac{\partial z}{\partial y^s} dy + \int_U \left(\sum_{\tau < \alpha} C(\tau) D^{\alpha - \tau} (\alpha^{rs}) \frac{\partial (D^\tau w)}{\partial y^r} + D^\alpha (h_s) \right) \frac{\partial z}{\partial y^s} dy$$

$$= \int_U (D^\alpha f) z dy. \tag{44.27}$$

Let $\widehat{U}_1$ be as indicated in the following picture



Now apply the induction hypothesis to $\widehat{U}_1$ in order to write

$$\left\| \frac{\partial (D^\tau w)}{\partial y^r} \right\|_{H^1(\widehat{U}_1)} \leq ||D^\tau w||_{H^2(\widehat{U}_1)}$$

$$\leq C\left(||f||_{H^{k-1}(U)}+\sum_s ||h_s||_{H^k(U)}+||w||_{H^k(U)}\right).$$

Since $\alpha^{rs} \in C^{k-1,1}\left(\overline{U}\right)$, it follows that each term from the sum in 44.27 satisfies an inequality of the form

$$\left|\left|C(\tau) D^{\alpha-\tau}(\alpha^{rs})\frac{\partial(D^\tau w)}{\partial y^r}\right|\right|_{H^1\left(\widehat{U}_1\right)} \leq$$

$$C\left(||f||_{H^{k-1}(U)}+\sum_s ||h_s||_{H^k(U)}+||w||_{H^k(U)}\right)$$

and consequently,

$$\left|\left|\sum_{\tau<\alpha} C(\tau) D^{\alpha-\tau}(\alpha^{rs})\frac{\partial(D^\tau w)}{\partial y^r}+D^\alpha(h_s)\right|\right|_{H^1\left(\widehat{U}_1\right)} \leq$$

$$C\left(||f||_{H^{k-1}(U)}+\sum_s ||h_s||_{H^k(U)}+||w||_{H^k(U)}\right). \tag{44.28}$$

Now consider 44.27. The equation remains true if you replace U with $\widehat{U}_1$ and require that $\text{spt}(z) \subseteq \widehat{U}_1$. Therefore, by Lemma 44.1 on Page 1220 there exists a constant, C independent of w such that

$$||D^\alpha w||_{H^2(U_1)} \leq C\left(||D^\alpha f||_{L^2\left(\widehat{U}_1\right)}+||D^\alpha w||_{H^1\left(\widehat{U}_1\right)}+\right.$$

$$\left.+\sum_s\left|\left|\sum_{\tau<\alpha} C(\tau) D^{\alpha-\tau}(\alpha^{rs})\frac{\partial(D^\tau w)}{\partial y^r}+D^\alpha(h_s)\right|\right|_{H^1\left(\widehat{U}_1\right)}\right)$$

and by 44.28, this implies

$$||D^\alpha w||_{H^2(U_1)} \leq C\left(||f||_{H^{k-1}(U)}+||w||_{H^k(U)}+\sum_s ||h_s||_{H^k(U)}\right)$$

which proves the Claim.

To establish 44.24 it only remains to verify that if $|\alpha| \leq k+1$, then

$$||D^\alpha w||_{L^2(U_1)} \leq C\left(||f||_{H^{k-1}(U)}+||w||_{H^k(U)}+\sum_s ||h_s||_{H^k(U)}\right). \tag{44.29}$$

If $|\alpha| < k+1$, there is nothing to show because it is given that $w \in H^k(U)$. Therefore, assume $|\alpha| = k+1$. If α_n equals 0 the conclusion follows from the claim because in this case, you can subtract 1 from a pair of positive α_i and obtain a new multi index, β such that $|\beta| = k-1$ and $\beta_n = 0$ and then from the claim,

$$||D^\alpha w||_{L^2(U_1)} \leq \left|\left|D^\beta w\right|\right|_{H^2(U_1)} \leq C\left(||f||_{H^{k-1}(U)} + ||w||_{H^k(U)} + \sum_s ||h_s||_{H^k(U)}\right).$$

If $\alpha_n = 1$, then subtract 1 from some positive α_i and consider

$$\beta = (\alpha_1, \cdot\cdot\cdot, \alpha_i - 1, \alpha_{i+1}, \cdot\cdot\cdot, \alpha_{n-1}, 0)$$

Then from the claim,

$$||D^\alpha w||_{L^2(U_1)} \leq \left|\left|D^\beta w\right|\right|_{H^2(U_1)} \leq C\left(||f||_{H^{k-1}(U)} + ||w||_{H^k(U)} + \sum_s ||h_s||_{H^k(U)}\right).$$

Suppose 44.29 holds for $\alpha_n \leq j-1$ where $j-1 \geq 1$ and consider α for which $|\alpha| = k+1$ and $\alpha_n = j$. Let

$$\beta \equiv (\alpha_1, \cdot\cdot\cdot, \alpha_{n-1}, \alpha_n - 2).$$

Thus $D^\alpha = D^\beta D_n^2$. Restricting 44.23 to $z \in C_c^\infty(U_1)$ and using the density of this set of functions in $L^2(U_1)$, it follows that

$$-\frac{\partial}{\partial y^s}\left(\alpha^{rs}(\mathbf{y})\frac{\partial w}{\partial y^r}\right) - \frac{\partial h_s}{\partial y^s} = f.$$

Therefore, from the product rule,

$$\frac{\partial \alpha^{rs}}{\partial y^s}\frac{\partial w}{\partial y^r} + \alpha^{rs}\frac{\partial^2 w}{\partial y^s \partial y^r} + \frac{\partial h_s}{\partial y^s} = -f$$

and so

$$\begin{aligned}\alpha^{nn} D_n^2 w \quad = \quad & -\left(\frac{\partial \alpha^{rs}}{\partial y^s}\frac{\partial w}{\partial y^r} + \sum_{r\leq n-1}\sum_{s\leq n-1}\alpha^{rs}\frac{\partial^2 w}{\partial y^s \partial y^r} + \right. \\ & \left. \sum_s \alpha^{ns}\frac{\partial^2 w}{\partial y^s \partial y^n} + \sum_r \alpha^{rn}\frac{\partial^2 w}{\partial y^n \partial y^r} + \frac{\partial h_s}{\partial y^s} + f\right).\end{aligned}$$

As noted earlier, the condition, 44.2 implies $\alpha^{nn}(\mathbf{y}) \geq \delta > 0$ and so

$$\begin{aligned}D_n^2 w \quad = \quad & -\frac{1}{\alpha^{nn}}\left(\frac{\partial \alpha^{rs}}{\partial y^s}\frac{\partial w}{\partial y^r} + \sum_{r\leq n-1}\sum_{s\leq n-1}\alpha^{rs}\frac{\partial^2 w}{\partial y^s \partial y^r} + \right. \\ & \left. \sum_s \alpha^{ns}\frac{\partial^2 w}{\partial y^s \partial y^n} + \sum_r \alpha^{rn}\frac{\partial^2 w}{\partial y^n \partial y^r} + \frac{\partial h_s}{\partial y^s} + f\right).\end{aligned}$$

It follows from $D^\alpha = D^\beta D_n^2$ that

$$D^\alpha w = D^\beta \left[-\frac{1}{\alpha^{nn}} \left(\frac{\partial \alpha^{rs}}{\partial y^s} \frac{\partial w}{\partial y^r} + \sum_{r \leq n-1} \sum_{s \leq n-1} \alpha^{rs} \frac{\partial^2 w}{\partial y^s \partial y^r} + \sum_s \alpha^{ns} \frac{\partial^2 w}{\partial y^s \partial y^n} + \sum_r \alpha^{rn} \frac{\partial^2 w}{\partial y^n \partial y^r} + \frac{\partial h_s}{\partial y^s} + f \right) \right].$$

Now you note that terms like $D^\beta \left(\frac{\partial^2 w}{\partial y^s \partial y^n} \right)$ have $\alpha_n = j - 1$ and so, from the induction hypothesis along with the assumptions on the given functions,

$$||D^\alpha w||_{L^2(U_1)} \leq C \left(||f||_{H^{k-1}(U)} + ||w||_{H^k(U)} + \sum_s ||h_s||_{H^k(U)} \right).$$

This proves the corollary.

44.2 The Case Of Bounded Open Sets

The main interest in all this is in the application to bounded open sets. Recall the following definition.

Definition 44.3 *A bounded open subset, Ω, of $\mathbb{R}^n$ has a $C^{m,1}$ boundary if it satisfies the following conditions. For each $p \in \Gamma \equiv \overline{\Omega} \setminus \Omega$, there exists an open set, W, containing p, an open interval $(0, b)$, a bounded open box $U' \subseteq \mathbb{R}^{n-1}$, and an affine orthogonal transformation, R_W consisting of a distance preserving linear transformation followed by a translation such that*

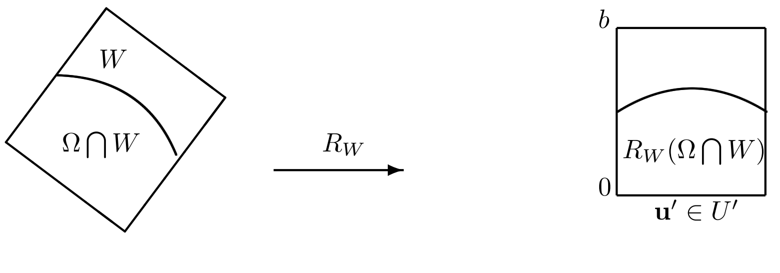
$$R_W W = U' \times (0, b), \tag{44.30}$$

$$R_W (W \cap \Omega) = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U', \ 0 < u_n < \phi_W (\mathbf{u}')\} \tag{44.31}$$

where $\phi_W \in C^{m,1}\left(\overline{U'}\right)$ meaning ϕ_W is the restriction to U' of a function, still denoted by ϕ_W which is in $C^{m,1}\left(\mathbb{R}^{n-1}\right)$ and

$$\inf \{\phi_W (\mathbf{u}') : \mathbf{u}' \in U'\} > 0$$

The following picture depicts the situation.



For the situation described in the above definition, let $\mathbf{h}_W : U' \to \Gamma \cap W$ be defined by

$$\mathbf{h}_W(\mathbf{u}') \equiv R_W^{-1}(\mathbf{u}', \phi_W(\mathbf{u}')),\ \mathbf{g}_W(\mathbf{x}) \equiv (R_W\mathbf{x})',\ \mathbf{H}_W(\mathbf{u}) \equiv R_W^{-1}(\mathbf{u}', \phi_W(\mathbf{u}') - u_n).$$

where $\mathbf{x}' \equiv (x_1, \cdots, x_{n-1})$ for $\mathbf{x} = (x_1, \cdots, x_n)$. Thus $\mathbf{g}_W \circ \mathbf{h}_W = \text{id}$ on U' and $\mathbf{h}_W \circ \mathbf{g}_W = \text{id}$ on $\Gamma \cap W$. Also note that $\mathbf{H}_W$ is defined on all of $\mathbb{R}^n$ is $C^{m,1}$, and has an inverse with the same properties. To see this, let $\mathbf{G}_W(\mathbf{u}) = (\mathbf{u}', \phi_W(\mathbf{u}') - u_n)$. Then $\mathbf{H}_W = R_W^{-1} \circ \mathbf{G}_W$ and $\mathbf{G}_W^{-1} = (\mathbf{u}', \phi_W(\mathbf{u}') - u_n)$ and so $\mathbf{H}_W^{-1} = \mathbf{G}_W^{-1} \circ R_W$. Note also that as indicated in the picture,

$$R_W(W \cap \Omega) = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U' \text{ and } 0 < u_n < \phi_W(\mathbf{u}')\}.$$

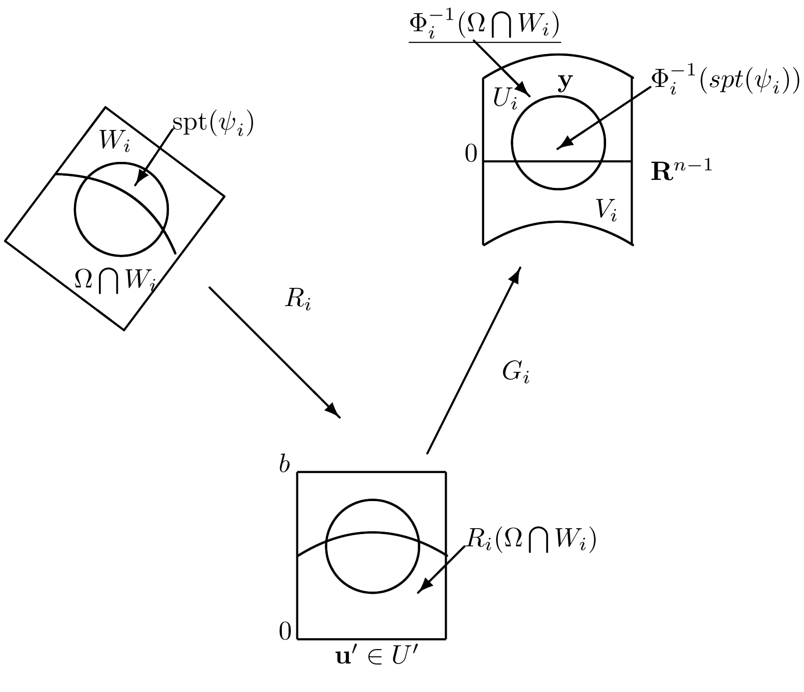
Since $\Gamma = \partial\Omega$ is compact, there exist finitely many of these open sets, W, denoted by $\{W_i\}_{i=1}^q$ such that $\Gamma \subseteq \cup_{i=1}^q W_i$. Let the corresponding sets, U' be denoted by U_i' and let the functions, ϕ be denoted by ϕ_i. Also let $\mathbf{h}_i = \mathbf{h}_{W_i}, G_{W_i} = G_i$ etc. Now let

$$\mathbf{\Phi}_i : \mathbf{G}_i R_i(\Omega \cap W) \equiv V_i \to \Omega \cap W_i$$

be defined by

$$\mathbf{\Phi}_i(\mathbf{y}) \equiv R_i^{-1} \circ \mathbf{G}_i^{-1}(\mathbf{y}).$$

Thus $\mathbf{\Phi}_i, \mathbf{\Phi}_i^{-1} \in C^{m,1}(\mathbb{R}^n)$. The following picture might be helpful.



Therefore, by Lemma 41.20 on Page 1185, it follows that for $t \in [m, m+1)$,

$$\mathbf{\Phi}_i^* \in \mathcal{L}\left(H^t\left(W_i \cap \Omega\right), H^t\left(V_i\right)\right).$$

Assume

$$a^{ij}(\mathbf{x})\, v_i v_j \geq \delta\, |\mathbf{v}|^2 . \tag{44.32}$$

Lemma 44.4 *Let W be one of the sets described in the above definition and let $m \geq 1$. Let $W_1 \subseteq \overline{W_1} \subseteq W$ where W_1 is an open set. Suppose also that*

$$\begin{aligned} u &\in H^1(\Omega), \\ \alpha^{rs} &\in C^{0,1}\left(\overline{\Omega}\right), \\ f &\in L^2(\Omega), \\ h_k &\in H^1(\Omega), \end{aligned}$$

and that for all $v \in H^1(\Omega \cap W)$ such that $\operatorname{spt}(v) \subseteq \Omega \cap W$,

$$\int_\Omega a^{ij}(\mathbf{x})\, u_{,i}(\mathbf{x})\, v_{,j}(\mathbf{x})\, dx + \int_\Omega h_k(\mathbf{x})\, v_{,k}(\mathbf{x})\, dx = \int_\Omega f(\mathbf{x})\, v(\mathbf{x})\, dx. \tag{44.33}$$

Then there exists a constant, C, independent of f, u, and g such that

$$||u||_{H^2(\Omega\cap W_1)}^2 \leq C\left(||f||_{L^2(\Omega)}^2 + ||u||_{H^1(\Omega)}^2 + \sum_k ||h_k||_{H^1(\Omega)}^2\right). \tag{44.34}$$

Proof: Let

$$E \equiv \left\{v \in H^1(\Omega \cap W) : \operatorname{spt}(v) \subseteq W\right\}$$

u restricted to $W \cap \Omega$ is in $H^1(\Omega \cap W)$ and

$$\int_{\Omega\cap W} a^{ij}(\mathbf{x})\, u_{,i} v_{,j} dx + \int_\Omega h_k(\mathbf{x})\, v_{,k}(\mathbf{x})\, dx = \int_\Omega f(\mathbf{x})\, v(\mathbf{x})\, dx \text{ for all } v \in E. \tag{44.35}$$

Now let $\mathbf{\Phi}_i(\mathbf{y}) = \mathbf{x}$. For this particular W, denote Φ_i more simply by Φ, $U_i \equiv \Phi_i(\Omega \cap W_i)$ by U, and V_i by V. Denoting the coordinates of V by $\mathbf{y}$, and letting $u(\mathbf{x}) \equiv w(\mathbf{y})$ and $v(\mathbf{x}) \equiv z(\mathbf{y})$, it follows that in terms of the new coordinates, 44.35 takes the form

$$\begin{aligned} &\int_U a^{ij}(\Phi(\mathbf{y}))\frac{\partial w}{\partial y^r}\frac{\partial y^r}{\partial x^i}\frac{\partial z}{\partial y^s}\frac{\partial y^s}{\partial x^j}\left|\det D\Phi(\mathbf{y})\right| dy \\ &+\int_U h_k(\Phi(\mathbf{y}))\frac{\partial z}{\partial y^l}\frac{\partial y^l}{\partial x^k}\left|\det D\Phi(\mathbf{y})\right| dx \\ = &\int_U f(\Phi(\mathbf{y}))\, z(\mathbf{y})\left|\det D\Phi(\mathbf{y})\right| dy \end{aligned}$$

Let

$$\alpha^{rs}(\mathbf{y}) \equiv a^{ij}(\Phi(\mathbf{y}))\frac{\partial y^r}{\partial x^i}\frac{\partial y^s}{\partial x^j}\left|\det D\Phi(\mathbf{y})\right|, \tag{44.36}$$

$$\widetilde{h}_l(\mathbf{y}) \equiv h_k(\Phi(\mathbf{y})) \frac{\partial y^l}{\partial x^k} |\det D\Phi(\mathbf{y})|, \tag{44.37}$$

and

$$\widetilde{f}(\mathbf{y}) \equiv \Phi^* f |\det D\Phi|(\mathbf{y}) \equiv f(\Phi(\mathbf{y})) |\det D\Phi(\mathbf{y})|. \tag{44.38}$$

Now the function on the right in 44.36 is in $C^{0,1}\left(\overline{U}\right)$. This is because of the assumption that $m \geq 1$ in the statement of the lemma. This function is therefore a finite product of bounded functions in $C^{0,1}\left(\overline{U}\right)$.

The function $\widetilde{h}_l$ defined in 44.37 is in $H^1(U)$ and

$$\left|\left|\widetilde{h}_l\right|\right|_{H^1(U)} \leq C \sum_k ||h_k||_{H^1(\Omega\cap W)}$$

again because $m \geq 1$.

Finally, the right side of 44.38 is a function in $L^2(U)$ by Lemma 41.20 on Page 1185 and the observation that $|\det D\Phi(\cdot)| \in C^{0,1}\left(\overline{U}\right)$ which follows from the assumption of the lemma that $m \geq 1$ so $\Phi \in C^{1,1}(\mathbb{R}^n)$. Also

$$\left|\left|\widetilde{f}\right|\right|_{L^2(U)} \leq C ||f||_{L^2(\Omega\cap W)}.$$

Therefore, 44.35 is of the form

$$\int_U \alpha^{rs}(\mathbf{y}) w_{,r} z_{,s} dy + \int_U \widetilde{h}_l z_{,l} dy = \int_U \widetilde{f} z dy, \tag{44.39}$$

for all z in $H^1(U)$ having support in V.

Claim: There exists $r > 0$ independent of $\mathbf{y} \in \overline{U}$ such that for all $\mathbf{y} \in \overline{U}$,

$$\alpha^{rs}(\mathbf{y}) v_r v_s \geq r |\mathbf{v}|^2.$$

Proof of the claim: If this is not so, there exist vectors, $\mathbf{v}^n, |\mathbf{v}^n| = 1$, and $\mathbf{y}_n \in \overline{U}$ such that $\alpha^{rs}(\mathbf{y}_n) v_r^n v_s^n \leq \frac{1}{n}$. Taking a subsequence, there exists $\mathbf{y} \in \overline{U}$ and $|\mathbf{v}| = 1$ such that $\alpha^{rs}(\mathbf{y}) v_r v_s = 0$ contradicting 44.32.

Therefore, by Lemma 44.1, there exists a constant, C, independent of f, g, and w such that

$$||w||^2_{H^2(\Phi^{-1}(W_1\cap\Omega))} \leq C \left(\left|\left|\widetilde{f}\right|\right|^2_{L^2(U)} + ||w||^2_{H^1(U)} + \sum_l \left|\left|\widetilde{h}_l\right|\right|^2_{H^1(U)} \right).$$

Therefore,

$$\begin{aligned} ||u||^2_{H^2(W_1\cap\Omega)} &\leq C \left(||f||^2_{L^2(W\cap\Omega)} + ||w||^2_{H^1(W\cap\Omega)} + \sum_k ||h_k||^2_{H^1(W\cap\Omega)} \right) \\ &\leq C \left(||f||^2_{L^2(\Omega)} + ||w||^2_{H^1(\Omega)} + \sum_k ||h_k||^2_{H^1(\Omega)} \right). \end{aligned}$$

which proves the lemma.

With this lemma here is the main result.

Theorem 44.5 *Let Ω be a bounded open set with $C^{1,1}$ boundary as in Definition 44.3, let $f \in L^2(\Omega), h_k \in H^1(\Omega)$, and suppose that for all $\mathbf{x} \in \overline{\Omega}$,*

$$a^{ij}(\mathbf{x}) v_i v_j \geq \delta |\mathbf{v}|^2 .$$

Suppose also that $u \in H^1(\Omega)$ and

$$\int_\Omega a^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) dx + \int_\Omega h_k(\mathbf{x}) v_{,k}(\mathbf{x}) dx = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) dx$$

for all $v \in H^1(\Omega)$. Then $u \in H^2(\Omega)$ and for some C independent of f, g, and u,

$$||u||^2_{H^2(\Omega)} \leq C \left(||f||^2_{L^2(\Omega)} + ||u||^2_{H^1(\Omega)} + \sum_k ||h_k||^2_{H^1(\Omega)} \right).$$

Proof: Let the W_i for $i = 1, \cdots, l$ be as described in Definition 44.3. Thus $\partial\Omega \subseteq \cup_{j=1}^l W_j$. Then let $C_1 \equiv \partial\Omega \setminus \cup_{i=2}^l W_i$, a closed subset of W_1. Let D_1 be an open set satisfying

$$C_1 \subseteq D_1 \subseteq \overline{D_1} \subseteq W_1.$$

Then $D_1, W_2, \cdots, W_l$ cover $\partial\Omega$. Let $C_2 = \partial\Omega \setminus \left(D_1 \cup \left(\cup_{i=3}^l W_i\right)\right)$. Then C_2 is a closed subset of W_2. Choose an open set, D_2 such that

$$C_2 \subseteq D_2 \subseteq \overline{D_2} \subseteq W_2.$$

Thus $D_1, D_2, W_3 \cdots, W_l$ covers $\partial\Omega$. Continue in this way to get $\overline{D_i} \subseteq W_i$, and $\partial\Omega \subseteq \cup_{i=1}^l D_i$, and D_i is an open set. Now let

$$D_0 \equiv \Omega \setminus \cup_{i=1}^l \overline{D_i}.$$

Also, let $\overline{D_i} \subseteq V_i \subseteq \overline{V_i} \subseteq W_i$. Therefore, $D_0, V_1, \cdots, V_l$ covers Ω. Then the same estimation process used above yields

$$||u||_{H^2(D_0)} \leq C \left(||f||^2_{L^2(\Omega)} + ||u||^2_{H^1(\Omega)} + \sum_k ||h_k||^2_{H^1(\Omega)} \right).$$

From Lemma 44.4

$$||u||_{H^2(V_i \cap \Omega)} \leq C \left(||f||^2_{L^2(\Omega)} + ||u||^2_{H^1(\Omega)} + \sum_k ||h_k||^2_{H^1(\Omega)} \right)$$

also. This proves the theorem since

$$||u||_{H^2(\Omega)} \leq \sum_{i=1}^l ||u||_{H^2(V_i \cap \Omega)} + ||u||_{H^2(D_0)}.$$

What about the Dirichlet problem? The same differencing procedure as above yields the following.

Theorem 44.6 *Let Ω be a bounded open set with $C^{1,1}$ boundary as in Definition 44.3, let $f \in L^2(\Omega), h_k \in H^1(\Omega)$, and suppose that for all $\mathbf{x} \in \overline{\Omega}$,*

$$a^{ij}(\mathbf{x}) v_i v_j \geq \delta |\mathbf{v}|^2 .$$

Suppose also that $u \in H_0^1(\Omega)$ and

$$\int_\Omega a^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) dx + \int_\Omega h_k(\mathbf{x}) v_{,k}(\mathbf{x}) dx = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) dx$$

for all $v \in H_0^1(\Omega)$. Then $u \in H^2(\Omega)$ and for some C independent of f, g, and u,

$$||u||^2_{H^2(\Omega)} \leq C \left(||f||^2_{L^2(\Omega)} + ||u||^2_{H^1(\Omega)} + \sum_k ||h_k||^2_{H^1(\Omega)} \right).$$

What about higher regularity?

Lemma 44.7 *Let W be one of the sets described in Definition 44.3 and let $m \geq k$. Let $W_1 \subseteq \overline{W_1} \subseteq W$ where W_1 is an open set. Suppose also that*

$$\begin{aligned} u &\in H^k(\Omega), \\ \alpha^{rs} &\in C^{k-1,1}\left(\overline{\Omega}\right), \\ f &\in H^{k-1}(\Omega), \\ h_s &\in H^k(\Omega), \end{aligned}$$

and that for all $v \in H^1(\Omega \cap W)$ such that $\operatorname{spt}(v) \subseteq \Omega \cap W$,

$$\int_\Omega a^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) dx + \int_\Omega h_s(\mathbf{x}) v_{,s}(\mathbf{x}) dx = \int_\Omega f(\mathbf{x}) v(\mathbf{x}) dx. \quad (44.40)$$

Then there exists a constant, C, independent of f, u, and g such that

$$||u||^2_{H^{k+1}(\Omega\cap W_1)} \leq C \left(||f||^2_{H^{k-1}(\Omega)} + ||u||^2_{H^k(\Omega)} + \sum_s ||h_s||^2_{H^k(\Omega)} \right). \quad (44.41)$$

Proof: Let

$$E \equiv \left\{ v \in H^k(\Omega \cap W) : \operatorname{spt}(v) \subseteq W \right\}$$

u restricted to $W \cap \Omega$ is in $H^k(\Omega \cap W)$ and

$$\int_{\Omega\cap W} a^{ij}(\mathbf{x}) u_{,i} v_{,j} dx + \int_\Omega h_s(\mathbf{x}) v_{,s}(\mathbf{x}) dx$$

$$= \int_\Omega f(\mathbf{x}) v(\mathbf{x}) dx \text{ for all } v \in E. \quad (44.42)$$

Now let $\mathbf{\Phi}_i(\mathbf{y}) = \mathbf{x}$. For this particular W, denote Φ_i more simply by Φ, $U_i \equiv \Phi_i(\Omega \cap W_i)$ by U, and V_i by V. Denoting the coordinates of V by $\mathbf{y}$, and letting

$u(\mathbf{x}) \equiv w(\mathbf{y})$ and $v(\mathbf{x}) \equiv z(\mathbf{y})$, it follows that in terms of the new coordinates, 44.35 takes the form

$$\begin{aligned} &\int_U a^{ij}(\Phi(\mathbf{y})) \frac{\partial w}{\partial y^r} \frac{\partial y^r}{\partial x^i} \frac{\partial z}{\partial y^s} \frac{\partial y^s}{\partial x^j} |\det D\Phi(\mathbf{y})| \, dy \\ &+ \int_U h_k(\Phi(\mathbf{y})) \frac{\partial z}{\partial y^l} \frac{\partial y^l}{\partial x^k} |\det D\Phi(\mathbf{y})| \, dx \\ = &\int_U f(\Phi(\mathbf{y})) z(\mathbf{y}) |\det D\Phi(\mathbf{y})| \, dy \end{aligned}$$

Let

$$\alpha^{rs}(\mathbf{y}) \equiv a^{ij}(\Phi(\mathbf{y})) \frac{\partial y^r}{\partial x^i} \frac{\partial y^s}{\partial x^j} |\det D\Phi(\mathbf{y})|, \tag{44.43}$$

$$\widetilde{h_l}(\mathbf{y}) \equiv h_k(\Phi(\mathbf{y})) \frac{\partial y^l}{\partial x^k} |\det D\Phi(\mathbf{y})|, \tag{44.44}$$

and

$$\widetilde{f}(\mathbf{y}) \equiv \Phi^* f |\det D\Phi|(\mathbf{y}) \equiv f(\Phi(\mathbf{y})) |\det D\Phi(\mathbf{y})|. \tag{44.45}$$

Now the function on the right in 44.43 is in $C^{k,1}(\overline{U})$. This is because of the assumption that $m \geq k$ in the statement of the lemma. This function is therefore a finite product of bounded functions in $C^{k,1}(\overline{U})$.

The function $\widetilde{h_l}$ defined in 44.44 is in $H^k(U)$ and

$$\left|\left|\widetilde{h_l}\right|\right|_{H^k(U)} \leq C \sum_s ||h_s||_{H^k(\Omega \cap W)}$$

again because $m \geq k$.

Finally, the right side of 44.45 is a function in $H^{k-1}(U)$ by Lemma 41.20 on Page 1185 and the observation that $|\det D\Phi(\cdot)| \in C^{k-1,1}(\overline{U})$ which follows from the assumption of the lemma that $m \geq k$ so $\Phi \in C^{k-1,1}(\mathbb{R}^n)$. Also

$$\left|\left|\widetilde{f}\right|\right|_{H^{k-1}(U)} \leq C \, ||f||_{H^{k-1}(\Omega \cap W)}.$$

Therefore, 44.42 is of the form

$$\int_U \alpha^{rs}(\mathbf{y}) w_{,r} z_{,s} dy + \int_U \widetilde{h_l} z_{,l} dy = \int_U \widetilde{f} z dy, \tag{44.46}$$

for all z in $H^1(U)$ having support in V.

Claim: There exists $r > 0$ independent of $\mathbf{y} \in \overline{U}$ such that for all $\mathbf{y} \in \overline{U}$,

$$\alpha^{rs}(\mathbf{y}) v_r v_s \geq r |\mathbf{v}|^2.$$

Proof of the claim: If this is not so, there exist vectors, $\mathbf{v}^n, |\mathbf{v}^n| = 1$, and $\mathbf{y}_n \in \overline{U}$ such that $\alpha^{rs}(\mathbf{y}_n) v_r^n v_s^n \leq \frac{1}{n}$. Taking a subsequence, there exists $\mathbf{y} \in \overline{U}$ and $|\mathbf{v}| = 1$ such that $\alpha^{rs}(\mathbf{y}) v_r v_s = 0$ contradicting 44.32.

Therefore, by Corollary 44.2, there exists a constant, C, independent of f, g, and w such that

$$||w||^2_{H^{k+1}(\Phi^{-1}(W_1\cap\Omega))} \leq C\left(\left|\left|\widetilde{f}\right|\right|^2_{H^{k-1}(U)} + ||w||^2_{H^k(U)} + \sum_l \left|\left|\widetilde{h_l}\right|\right|^2_{H^k(U)}\right).$$

Therefore,

$$\begin{aligned} ||u||^2_{H^{k+1}(W_1\cap\Omega)} &\leq C\left(||f||^2_{H^{k-1}(W\cap\Omega)} + ||w||^2_{H^k(W\cap\Omega)} + \sum_s ||h_s||^2_{H^k(W\cap\Omega)}\right) \\ &\leq C\left(||f||^2_{H^{k-1}(\Omega)} + ||w||^2_{H^k(\Omega)} + \sum_s ||h_s||^2_{H^k(\Omega)}\right). \end{aligned}$$

which proves the lemma.

Now here is a theorem which generalizes the one above in the case where more regularity is known.

Theorem 44.8 *Let Ω be a bounded open set with $C^{k,1}$ boundary as in Definition 44.3, let $f \in H^{k-1}(\Omega), h_s \in H^k(\Omega)$, and suppose that for all $\mathbf{x} \in \overline{\Omega}$,*

$$a^{ij}(\mathbf{x}) v_i v_j \geq \delta |\mathbf{v}|^2.$$

Suppose also that $u \in H^k(\Omega)$ and

$$\int_\Omega a^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x})\, dx + \int_\Omega h_k(\mathbf{x}) v_{,k}(\mathbf{x})\, dx = \int_\Omega f(\mathbf{x}) v(\mathbf{x})\, dx$$

for all $v \in H^k(\Omega)$. Then $u \in H^{k+1}(\Omega)$ and for some C independent of f, g, and u,

$$||u||^2_{H^{k+1}(\Omega)} \leq C\left(||f||^2_{H^{k-1}(\Omega)} + ||u||^2_{H^k(\Omega)} + \sum_s ||h_s||^2_{H^k(\Omega)}\right).$$

Proof: Let the W_i for $i = 1, \cdots, l$ be as described in Definition 44.3. Thus $\partial\Omega \subseteq \cup_{j=1}^l W_j$. Then let $C_1 \equiv \partial\Omega \setminus \cup_{i=2}^l W_i$, a closed subset of W_1. Let D_1 be an open set satisfying

$$C_1 \subseteq D_1 \subseteq \overline{D_1} \subseteq W_1.$$

Then $D_1, W_2, \cdots, W_l$ cover $\partial\Omega$. Let $C_2 = \partial\Omega \setminus \left(D_1 \cup \left(\cup_{i=3}^l W_i\right)\right)$. Then C_2 is a closed subset of W_2. Choose an open set, D_2 such that

$$C_2 \subseteq D_2 \subseteq \overline{D_2} \subseteq W_2.$$

Thus $D_1, D_2, W_3 \cdots, W_l$ covers $\partial\Omega$. Continue in this way to get $\overline{D_i} \subseteq W_i$, and $\partial\Omega \subseteq \cup_{i=1}^l D_i$, and D_i is an open set. Now let

$$D_0 \equiv \Omega \setminus \cup_{i=1}^l \overline{D_i}.$$

Also, let $\overline{D_i} \subseteq V_i \subseteq \overline{V_i} \subseteq W_i$. Therefore, $D_0, V_1, \cdot\cdot\cdot, V_l$ covers Ω. Then the same estimation process used above yields

$$||u||_{H^{k+1}(D_0)} \leq C\left(||f||^2_{H^{k-1}(\Omega)} + ||u||^2_{H^k(\Omega)} + \sum_k ||h_k||^2_{H^k(\Omega)}\right).$$

From Lemma 44.7

$$||u||_{H^{k+1}(V_i\cap\Omega)} \leq C\left(||f||^2_{H^{k-1}(\Omega)} + ||u||^2_{H^k(\Omega)} + \sum_k ||h_k||^2_{H^k(\Omega)}\right)$$

also. This proves the theorem since

$$||u||_{H^{k+1}(\Omega)} \leq \sum_{i=1}^{l} ||u||_{H^{k+1}(V_i\cap\Omega)} + ||u||_{H^{k+1}(D_0)}.$$

Interpolation In Banach Space

45.1 An Assortment Of Important Theorems

45.1.1 Weak Vector Valued Derivatives

In this section, several significant theorems are presented. Unless indicated otherwise, the measure will be Lebesgue measure. First here is a lemma.

Lemma 45.1 *Suppose $g \in L^1([a,b];X)$ where X is a Banach space. Then if $\int_a^b g(t)\phi(t)\,dt = 0$ for all $\phi \in C_c^\infty(a,b)$, then $g(t) = 0$ a.e.*

Proof: Let E be a measurable subset of (a,b) and let $K \subseteq E \subseteq V \subseteq (a,b)$ where K is compact, V is open and $m(V \setminus K) < \varepsilon$. Let $K \prec h \prec V$ as in the proof of the Riesz representation theorem for positive linear functionals. Enlarging K slightly and convolving with a mollifier, it can be assumed $h \in C_c^\infty(a,b)$. Then

$$\begin{aligned}\left|\int_a^b \mathcal{X}_E(t)\,g(t)\,dt\right| &= \left|\int_a^b (\mathcal{X}_E(t) - h(t))\,g(t)\,dt\right| \\ &\leq \int_a^b |\mathcal{X}_E(t) - h(t)|\,||g(t)||\,dt \\ &\leq \int_{V\setminus K} ||g(t)||\,dt.\end{aligned}$$

Now let $K_n \subseteq E \subseteq V_n$ with $m(V_n \setminus K_n) < 2^{-n}$. Then from the above,

$$\left|\int_a^b \mathcal{X}_E(t)\,g(t)\,dt\right| \leq \int_a^b \mathcal{X}_{V_n\setminus K_n}(t)\,||g(t)||\,dt$$

and the integrand of the last integral converges to 0 a.e. as $n \to \infty$ because $\sum_n m(V_n \setminus K_n) < \infty$. By the dominated convergence theorem, this last integral converges to 0. Therefore, whenever $E \subseteq (a,b)$,

$$\int_a^b \mathcal{X}_E(t)\, g(t)\, dt = 0.$$

Since the endpoints have measure zero, it also follows that for any measurable E, the above equation holds.

Now $g \in L^1([a,b];X)$ and so it is measurable. Therefore, $g([a,b])$ is separable. Let D be a countable dense subset and let E denote the set of linear combinations of the form $\sum_i a_i d_i$ where a_i is a rational point of $\mathbb{F}$ and $d_i \in D$. Thus E is countable. Denote by Y the closure of E in X. Thus Y is a separable closed subspace of X which contains all the values of g.

Now let $S_n \equiv g^{-1}(B(y_n, ||y_n||/2))$ where $E = \{y_n\}_{n=1}^{\infty}$. Therefore, $\cup_n S_n = g^{-1}(X \setminus \{0\})$. This follows because if $x \in Y$ and $x \neq 0$, then in $B\left(x, \frac{||x||}{4}\right)$ there is a point of E, y_n. Therefore, $||y_n|| > \frac{3}{4}||x||$ and so $\frac{||y_n||}{2} > \frac{3||x||}{8} > \frac{||x||}{4}$ so $x \in B(y_n, ||y_n||/2)$. It follows that if each S_n has measure zero, then $g(t) = 0$ for a.e. t. Suppose then that for some n, the set, S_n has positive mesure. Then from what was shown above,

$$\begin{aligned} ||y_n|| &= \left|\left|\frac{1}{m(S_n)}\int_{S_n} g(t)\, dt - y_n\right|\right| = \left|\left|\frac{1}{m(S_n)}\int_{S_n} g(t) - y_n dt\right|\right| \\ &\leq \frac{1}{m(S_n)}\int_{S_n} ||g(t) - y_n||\, dt \leq \frac{1}{m(S_n)}\int_{S_n} ||y_n||/2 dt = ||y_n||/2 \end{aligned}$$

and so $y_n = 0$ which implies $S_n = \emptyset$, a contradiction to $m(S_n) > 0$. This contradiction shows each S_n has measure zero and so as just explained, $g(t) = 0$ a.e.

Definition 45.2 *For $f \in L^1(a,b;X)$, define an extension, $\overline{f}$ defined on*

$$[2a-b, 2b-a] = [a-(b-a), b+(b-a)]$$

as follows.

$$\overline{f}(t) \equiv \begin{cases} f(t) & \text{if } t \in [a,b] \\ f(2a-t) & \text{if } t \in [2a-b, a] \\ f(2b-t) & \text{if } t \in [b, 2b-a] \end{cases}$$

Definition 45.3 *Also if $f \in L^p(a,b;X)$ and $h > 0$, define for $t \in [a,b]$, $f_h(t) \equiv \overline{f}(t-h)$ for all $h < b-a$. Thus the map $f \to f_h$ is continuous and linear on $L^p(a,b;X)$. It is continuous because*

$$\begin{aligned} \int_a^b ||f_h(t)||^p dt &= \int_a^{a+h} ||f(2a-t+h)||^p dt + \int_a^{b-h} ||f(t)||^p dt \\ &= \int_a^{a+h} ||f(t)||^p dt + \int_a^{b-h} ||f(t)||^p dt \leq 2||f||_p^p. \end{aligned}$$

The following lemma is on continuity of translation in $L^p(a,b;X)$.

Lemma 45.4 *Let $\overline{f}$ be as defined in Definition 45.2. Then for $f \in L^p(a,b;X)$ for $p \in [1,\infty)$,*

$$\lim_{\delta \to 0} \int_a^b \left|\left|\overline{f}(t-\delta) - f(t)\right|\right|_X^p dt = 0.$$

Proof: Regarding the measure space as (a,b) with Lebesgue measure, by Lemma 21.42 there exists $g \in C_c(a,b;X)$ such that $||f-g||_p < \varepsilon$. Here the norm is the norm in $L^p(a,b;X)$. Therefore,

$$\begin{aligned} ||f_h - f||_p &\leq ||f_h - g_h||_p + ||g_h - g||_p + ||g - f||_p \\ &\leq \left(2^{1/p}+1\right)||f-g||_p + ||g_h - g||_p \\ &< \left(2^{1/p}+1\right)\varepsilon + \varepsilon \end{aligned}$$

whenever h is sufficiently small. This is because of the uniform continuity of g. Therefore, since $\varepsilon > 0$ is arbitrary, this proves the lemma.

Definition 45.5 *Let $f \in L^1(a,b;X)$. Then the distributional derivative in the sense of X valued distributions is given by*

$$f'(\phi) \equiv -\int_a^b f(t)\,\phi'(t)\,dt$$

Then $f' \in L^1(a,b;X)$ if there exists $h \in L^1(a,b;X)$ such that for all $\phi \in C_c^\infty(a,b)$,

$$f'(\phi) = \int_a^b h(t)\,\phi(t)\,dt.$$

Then f' is defined to equal h. Here f and f' are considered as vector valued distributions in the same way as was done for scalar valued functions.

Lemma 45.6 *The above definition is well defined.*

Proof: Suppose both h and g work in the definition. Then for all $\phi \in C_c^\infty(a,b)$,

$$\int_a^b (h(t) - g(t))\,\phi(t)\,dt = 0.$$

Therefore, by Lemma 45.1, $h(t) - g(t) = 0$ a.e.

The other thing to notice about this is the following lemma. It follows immediately from the definition.

Lemma 45.7 *Suppose $f, f' \in L^1(a,b;X)$. Then if $[c,d] \subseteq [a,b]$, it follows that $\left(f|_{[c,d]}\right)' = f'|_{[c,d]}$. This notation means the restriction to $[c,d]$.*

Recall that in the case of scalar valued functions, if you had both f and its weak derivative, f' in $L^1(a,b)$, then you were able to conclude that f is almost everywhere equal to a continuous function, still denoted by f and

$$f(t) = f(a) + \int_a^t f'(s)\,ds.$$

In particular, you can define $f(a)$ to be the initial value of this continuous function. It turns out that an identical theorem holds in this case. To begin with here is the same sort of lemma which was used earlier for the case of scalar valued functions. It says that if $f' = 0$ where the derivative is taken in the sense of X valued distributions, then f equals a constant.

Lemma 45.8 *Suppose $f \in L^1(a,b;X)$ and for all $\phi \in C_c^\infty(a,b)$,*

$$\int_a^b f(t)\,\phi'(t)\,dt = 0.$$

Then there exists a constant, $a \in X$ such that $f(t) = a$ a.e.

Proof: Let $\phi_0 \in C_c^\infty(a,b)$, $\int_a^b \phi_0(x)\,dx = 1$ and define for $\phi \in C_c^\infty(a,b)$

$$\psi_\phi(x) \equiv \int_a^x [\phi(t) - \left(\int_a^b \phi(y)\,dy\right)\phi_0(t)]dt$$

Then $\psi_\phi \in C_c^\infty(a,b)$ and $\psi'_\phi = \phi - \left(\int_a^b \phi(y)\,dy\right)\phi_0$. Then

$$\begin{aligned}\int_a^b f(t)(\phi(t))\,dt &= \int_a^b f(t)\left(\psi'_\phi(t) + \left(\int_a^b \phi(y)\,dy\right)\phi_0(t)\right)dt \\ &= \int_a^b f(t)\,\psi'_\phi(t)\,dt + \left(\int_a^b \phi(y)\,dy\right)\int_a^b f(t)\,\phi_0(t)\,dt \\ &= \left(\int_a^b \left(\int_a^b f(t)\,\phi_0(t)\,dt\right)\phi(y)\,dy\right).\end{aligned}$$

It follows that for all $\phi \in C_c^\infty(a,b)$,

$$\int_a^b \left(f(t) - \left(\int_a^b f(t)\,\phi_0(t)\,dt\right)\right)\phi(t)\,dt$$

and so by Lemma 45.1,

$$f(t) - \left(\int_a^b f(t)\,\phi_0(t)\,dt\right) = 0 \text{ a.e.}$$

This proves the lemma.

Theorem 45.9 *Suppose f, f' both are in $L^1(a,b;X)$ where the derivative is taken in the sense of X valued distributions. Then there exists a unique point of X, denoted by $f(a)$ such that the following formula holds a.e. t.*

$$f(t) = f(a) + \int_a^t f'(s)\, ds$$

Proof:

$$\int_a^b \left(f(t) - \int_a^t f'(s)\, ds \right) \phi'(t)\, dt = \int_a^b f(t)\phi'(t)\, dt - \int_a^b \int_a^t f'(s)\phi'(t)\, dsdt.$$

Now consider $\int_a^b \int_a^t f'(s)\phi'(t)\, dsdt$. Let $\Lambda \in X'$. Then it is routine from approximating f' with simple functions to verify

$$\Lambda\left(\int_a^b \int_a^t f'(s)\phi'(t)\, dsdt \right) = \int_a^b \int_a^t \Lambda(f'(s))\phi'(t)\, dsdt.$$

Now the ordinary Fubini theorem can be applied to obtain

$$\begin{aligned} &= \int_a^b \int_s^b \Lambda(f'(s))\phi'(t)\, dtds \\ &= \Lambda\left(\int_a^b \int_s^b f'(s)\phi'(t)\, dtds \right). \end{aligned}$$

Since X' separates the points of X, it follows

$$\int_a^b \int_a^t f'(s)\phi'(t)\, dsdt = \int_a^b \int_s^b f'(s)\phi'(t)\, dtds.$$

Therefore,

$$\begin{aligned} &\int_a^b \left(f(t) - \int_a^t f'(s)\, ds \right) \phi'(t)\, dt \\ &= \int_a^b f(t)\phi'(t)\, dt - \int_a^b \int_s^b f'(s)\phi'(t)\, dtds \\ &= \int_a^b f(t)\phi'(t)\, dt - \int_a^b f'(s) \int_s^b \phi'(t)\, dtds \\ &= \int_a^b f(t)\phi'(t)\, dt + \int_a^b f'(s)\phi(s)\, ds = 0. \end{aligned}$$

Therefore, by Lemma 45.8, there exists a constant, denoted as $f(a)$ such that

$$f(t) - \int_a^t f'(s)\, ds = f(a)$$

and this proves the theorem.

The integration by parts formula is also important.

Corollary 45.10 *Suppose $f, f' \in L^1(a,b;X)$ and suppose $\phi \in C^1([a,b])$. Then the following integration by parts formula holds.*

$$\int_a^b f(t)\phi'(t)\,dt = f(b)\phi(b) - f(a)\phi(a) - \int_a^b f'(t)\phi(t)\,dt.$$

Proof: From Theorem 45.9

$$\begin{aligned}
&\int_a^b f(t)\phi'(t)\,dt \\
= &\int_a^b \left(f(a) + \int_a^t f'(s)\,ds\right)\phi'(t)\,dt \\
= &f(a)(\phi(b)-\phi(a)) + \int_a^b \int_a^t f'(s)\,ds\phi'(t)\,dt \\
= &f(a)(\phi(b)-\phi(a)) + \int_a^b f'(s)\int_s^b \phi'(t)\,dtds \\
= &f(a)(\phi(b)-\phi(a)) + \int_a^b f'(s)(\phi(b)-\phi(s))\,ds \\
= &f(a)(\phi(b)-\phi(a)) - \int_a^b f'(s)\phi(s)\,ds + (f(b)-f(a))\phi(b) \\
= &f(b)\phi(b) - f(a)\phi(a) - \int_a^b f'(s)\phi(s)\,ds.
\end{aligned}$$

The interchange in order of integration is justified as in the proof of Theorem 45.9.

With this integration by parts formula, the following interesting lemma is obtained. This lemma shows why it was appropriate to define $\overline{f}$ as in Definition 45.2.

Lemma 45.11 *Let $\overline{f}$ be given in Definition 45.2 and suppose $f,\ f' \in L^1(a,b;X)$. Then $\overline{f},\ \overline{f}' \in L^1(2a-b, 2b-a; X)$ also and*

$$\overline{f}'(t) \equiv \begin{cases} f'(t) & \text{if } t \in [a,b] \\ -f(2a-t) & \text{if } t \in [2a-b,a] \\ -f(2b-t) & \text{if } t \in [b, 2b-a] \end{cases} \tag{45.1}$$

Proof: It is clear from the definition of $\overline{f}$ that $\overline{f} \in L^1(2a-b, 2b-a; X)$ and that in fact

$$||\overline{f}||_{L^1(2a-b,2b-a;X)} \leq 3\,||f||_{L^1(a,b;X)}. \tag{45.2}$$

Let $\phi \in C_c^\infty(2a-b, 2b-a)$. Then from the integration by parts formula,

$$\begin{aligned}
&\int_{2a-b}^{2b-a} \overline{f}(t)\,\phi'(t)\,dt \\
= & \int_a^b f(t)\,\phi'(t)\,dt + \int_b^{2b-a} f(2b-t)\,\phi'(t)\,dt + \int_{2a-b}^a f(2a-t)\,\phi'(t)\,dt \\
= & \int_a^b f(t)\,\phi'(t)\,dt + \int_a^b f(u)\,\phi'(2b-u)\,du + \int_a^b f(u)\,\phi'(2a-u)\,du \\
= & f(b)\,\phi(b) - f(a)\,\phi(a) - \int_a^b f'(t)\,\phi(t)\,dt - f(b)\,\phi(b) + f(a)\,\phi(2b-a) \\
& + \int_a^b f'(u)\,\phi(2b-u)\,du - f(b)\,\phi(2a-b) \\
& + f(a)\,\phi(a) + \int_a^b f'(u)\,\phi(2a-u)\,du \\
= & -\int_a^b f'(t)\,\phi(t)\,dt + \int_a^b f'(u)\,\phi(2b-u)\,du + \int_a^b f'(u)\,\phi(2a-u)\,du \\
= & -\int_a^b f'(t)\,\phi(t)\,dt - \int_b^{2b-a} -f'(2b-t)\,\phi(t)\,dt - \int_{2a-b}^a -f'(2a-t)\,\phi(t)\,dt \\
= & -\int_{2a-b}^{2b-a} \overline{f}'(t)\,\phi(t)\,dt
\end{aligned}$$

where $\overline{f}'(t)$ is given in 45.1. This proves the lemma.

Definition 45.12 *Let V be a Banach space and let H be a Hilbert space. (Typically $H = L^2(\Omega)$) Suppose $V \subseteq H$ is dense in H meaning that the closure in H of V gives H. Then it is often the case that H is identified with its dual space, and then because of the density of V in H, it is possible to write*

$$V \subseteq H = H' \subseteq V'$$

When this is done, H is called a pivot space. Another notation which is often used is $\langle f, g\rangle$ to denote $f(g)$ for $f \in V'$ and $g \in V$. This may also be written as $\langle f, g\rangle_{V',V}$

The next theorem is an example of a trace theorem. In this theorem, $f \in L^p(0,T;V)$ while $f' \in L^p(0,T;V')$. It makes no sense to consider the initial values of f in V because it is not even continuous with values in V. However, because of the derivative of f it will turn out that f is continuous with values in a larger space and so it makes sense to consider initial values of f in this other space. This other space is called a trace space.

Theorem 45.13 *Let V and H be a Banach space and Hilbert space as described in Definition 45.12. Suppose $f \in L^p(0,T;V)$ and $f' \in L^{p'}(0,T;V')$. Then f is* *a.e. equal to a continuous function mapping* $[0,T]$ *to* H. *Furthermore, there exists* $f(0) \in H$ *such that*

$$\frac{1}{2}|f(t)|_H^2 - \frac{1}{2}|f(0)|_H^2 = \int_0^t \langle f'(s), f(s)\rangle\, ds, \tag{45.3}$$

and for all $t \in [0,T]$,

$$\int_0^t f'(s)\, ds \in H, \tag{45.4}$$

and for a.e. $t \in [0,T]$,

$$f(t) = f(0) + \int_0^t f'(s)\, ds \text{ in } H, \tag{45.5}$$

Here f' *is being taken in the sense of* V' *valued distributions and* $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $p \geq 2$.

Proof: Let $\Psi \in C_c^\infty(-T, 2T)$ satisfy $\Psi(t) = 1$ if $t \in [-T/2, 3T/2]$ and $\Psi(t) \geq 0$. For $t \in \mathbb{R}$, define

$$\widehat{f}(t) \equiv \begin{cases} \overline{f}(t)\Psi(t) & \text{if } t \in [-T, 2T] \\ 0 \text{ if } t \notin [-T, 2T] \end{cases}$$

and

$$f_n(t) \equiv \int_{-1/n}^{1/n} \widehat{f}(t-s)\phi_n(s)\, ds \tag{45.6}$$

where ϕ_n is a mollifier having support in $(-1/n, 1/n)$. Then by Minkowski's inequality

$$\left|\left|f_n - \widehat{f}\right|\right|_{L^p(\mathbb{R};V)} = \left(\int_{\mathbb{R}} \left|\left|\widehat{f}(t) - \int_{-1/n}^{1/n} \widehat{f}(t-s)\phi_n(s)\, ds\right|\right|_V^p dt\right)^{1/p}$$

$$\begin{aligned}
&= \left(\int_{\mathbb{R}} \left|\left|\int_{-1/n}^{1/n} \left(\widehat{f}(t) - \widehat{f}(t-s)\right)\phi_n(s)\, ds\right|\right|_V^p dt\right)^{1/p} \\
&\leq \left(\int_{\mathbb{R}} \left(\int_{-1/n}^{1/n} \left|\left|\widehat{f}(t) - \widehat{f}(t-s)\right|\right|_V \phi_n(s)\, ds\right)^p dt\right)^{1/p} \\
&\leq \int_{-1/n}^{1/n} \phi_n(s) \left(\int_{\mathbb{R}} \left|\left|\widehat{f}(t) - \widehat{f}(t-s)\right|\right|_V^p dt\right)^{1/p} ds \\
&\leq \int_{-1/n}^{1/n} \phi_n(s)\, \varepsilon ds = \varepsilon
\end{aligned}$$

provided n is large enough. This follows from Lemma 45.4 about continuity of translation. Since $\varepsilon > 0$ is arbitrary, it follows $f_n \to \widehat{f}$ in $L^p(\mathbb{R}; V)$. Similarly,

$f_n \to f$ in $L^2(\mathbb{R}; H)$. This follows because $p \geq 2$ and the norm in V and norm in H are related by $|x|_H \leq C\,||x||_V$ for some constant, C. Now

$$\widehat{f}(t) = \begin{cases} \Psi(t) f(t) \text{ if } t \in [0,T], \\ \Psi(t) f(2T-t) \text{ if } t \in [T,2T], \\ \Psi(t) f(-t) \text{ if } t \in [0,T], \\ 0 \text{ if } t \notin [-T,2T]. \end{cases}$$

An easy modification of the argument of Lemma 45.11 yields

$$\widehat{f'}(t) = \begin{cases} \Psi'(t) f(t) + \Psi(y) f'(t) \text{ if } t \in [0,T], \\ \Psi'(t) f(2T-t) - \Psi(t) f'(2T-t) \text{ if } t \in [T,2T], \\ \Psi'(t) f(-t) - \Psi(t) f'(-t) \text{ if } t \in [-T,0], \\ 0 \text{ if } t \notin [-T,2T]. \end{cases}.$$

Recall

$$\begin{aligned} f_n(t) &= \int_{-1/n}^{1/n} \widehat{f}(t-s)\,\phi_n(s)\,ds = \int_{\mathbb{R}} \widehat{f}(t-s)\,\phi_n(s)\,ds \\ &= \int_{\mathbb{R}} \widehat{f}(s)\,\phi_n(t-s)\,ds. \end{aligned}$$

Therefore,

$$\begin{aligned} f_n'(t) &= \int_{\mathbb{R}} \widehat{f}(s)\,\phi_n'(t-s)\,ds = \int_{-T-\frac{1}{n}}^{2T+\frac{1}{n}} \widehat{f}(s)\,\phi_n'(t-s)\,ds \\ &= \int_{-T-\frac{1}{n}}^{2T+\frac{1}{n}} \widehat{f'}(s)\,\phi_n(t-s)\,ds = \int_{\mathbb{R}} \widehat{f'}(s)\,\phi_n(t-s)\,ds \\ &= \int_{\mathbb{R}} \widehat{f'}(t-s)\,\phi_n(s)\,ds = \int_{-1/n}^{1/n} \widehat{f'}(t-s)\,\phi_n(s)\,ds \end{aligned}$$

and it follows from the first line above that f_n' is continuous with values in V for all $t \in \mathbb{R}$. Also note that both f_n' and f_n equal zero if $t \notin [-T,2T]$ whenever n is large enough. Exactly similar reasoning to the above shows that $f_n' \to \widehat{f'}$ in $L^{p'}(\mathbb{R}; V')$.

Now let $\phi \in C_c^\infty(0,T)$.

$$\begin{aligned} \int_{\mathbb{R}} |f_n(t)|_H^2\,\phi'(t)\,dt &= \int_{\mathbb{R}} (f_n(t), f_n(t))_H\,\phi'(t)\,dt \qquad (45.7) \\ = -\int_{\mathbb{R}} 2\,(f_n'(t), f_n(t))\,\phi(t)\,dt &= -\int_{\mathbb{R}} 2\,\langle f_n'(t), f_n(t)\rangle\,\phi(t)\,dt \end{aligned}$$

Now

$$\begin{aligned} &\left|\int_{\mathbb{R}} \langle f_n'(t), f_n(t)\rangle\,\phi(t)\,dt - \int_{\mathbb{R}} \langle f'(t), f(t)\rangle\,\phi(t)\,dt\right| \\ \leq\ &\int_{\mathbb{R}} \left(|\langle f_n'(t) - f'(t), f_n(t)\rangle| + |\langle f'(t), f_n(t) - f(t)\rangle|\right)\phi(t)\,dt. \end{aligned}$$

From the first part of this proof which showed that $f_n \to \widehat{f}$ in $L^p(\mathbb{R};V)$ and $f_n' \to \widehat{f}'$ in $L^{p'}(\mathbb{R};V')$, an application of Holder's inequality shows the above converges to 0 as $n \to \infty$. Therefore, passing to the limit as $n \to \infty$ in the 45.8,

$$\int_{\mathbb{R}} \left|\widehat{f}(t)\right|_H^2 \phi'(t)\,dt = -\int_{\mathbb{R}} 2\left\langle \widehat{f}'(t), \widehat{f}(t)\right\rangle \phi(t)\,dt$$

which shows $t \to \left|\widehat{f}(t)\right|_H^2$ equals a continuous function a.e. and it also has a weak derivative equal to $2\left\langle \widehat{f}', \widehat{f}\right\rangle$.

It remains to verify that $\widehat{f}$ is continuous on $[0,T]$. Of course $\widehat{f} = f$ on this interval. Let N be large enough that $f_n(-T) = 0$ for all $n > N$. Then for $m, n > N$ and $t \in [-T, 2T]$

$$\begin{aligned}
|f_n(t) - f_m(t)|_H^2 &= 2\int_{-T}^t (f_n'(s) - f_m'(s), f_n(s) - f_m(s))\,ds \\
&= 2\int_{-T}^t \langle f_n'(s) - f_m'(s), f_n(s) - f_m(s)\rangle_{V',V}\,ds \\
&\leq 2\int_{\mathbb{R}} ||f_n'(s) - f_m'(s)||_{V'}\,||f_n(s) - f_m(s)||_V\,ds \\
&\leq 2\,||f_n - f_m||_{L^{p'}(\mathbb{R};V')}\,||f_n - f_m||_{L^p(\mathbb{R};V)}
\end{aligned}$$

which shows from the above that $\{f_n\}$ is uniformly Cauchy on $[-T, 2T]$ with values in H. Therefore, there exists g a continuous function defined on $[-T, 2T]$ having values in H such that

$$\lim_{n\to\infty} \max\{|f_n(t) - g(t)|_H ; t \in [-T, 2T]\} = 0.$$

However, $g = \widehat{f}$ a.e. because f_n converges to f in $L^p(0,T;V)$. Therefore, taking a subsequence, the convergence is a.e. It follows from the fact that $V \subseteq H = H' \subseteq V'$ and Theorem 45.9 there exists $f(0) \in V'$ such that for a.e. t,

$$f(t) = f(0) + \int_0^t f'(s)\,ds \text{ in } V'$$

Now $g = f$ a.e. and g is continuous with values in H hence continuous with values in $V'$and so

$$g(t) = f(0) + \int_0^t f'(s)\,ds \text{ in } V'$$

for all t. Since g is continuous with values in H it is continuous with values in V'. Taking the limit as $t \downarrow 0$ in the above, $g(a) = \lim_{t\to 0+} g(t) = f(0)$, showing that $f(0) \in H$. Therefore, for a.e. t,

$$f(t) = f(0) + \int_0^t f'(s)\,ds \text{ in } H, \int_0^t f'(s)\,ds \in H.$$

This proves the theorem.

Note that if $f \in L^p(0,T;V)$ and $f' \in L^{p'}(0,T;V')$, then you can consider the initial value of f and it will be in H. What if you start with something in H? Is it an initial condition for a function $f \in L^p(0,T;V)$ such that $f' \in L^{p'}(0,T;V')$? This is worth thinking about. If it is not so, what is the space of initial values? How can you give this space a norm? What are its properties? It turns out that if V is a closed subspace of the Sobolev space, $W^{1,p}(\Omega)$ which contains $W_0^{1,p}(\Omega)$ for $p \geq 2$ and $H = L^2(\Omega)$ the answer to the above question is yes. Not surprisingly, there are many generalizations of the above ideas.

45.1.2 Some Imbedding Theorems

The next theorem is very useful in getting estimates in partial differential equations. It is called Erling's lemma.

Definition 45.14 *Let E, W be Banach spaces such that $E \subseteq W$ and the injection map from E into W is continuous. The injection map is said to be compact if every bounded set in E has compact closure in W. In other words, if a sequence is bounded in E it has a convergent subsequence converging in W. This is also referred to by saying that bounded sets in E are precompact in W.*

Theorem 45.15 *Let $E \subseteq W \subseteq X$ where the injection map is continuous from W to X and compact from E to W. Then for every $\varepsilon > 0$ there exists a constant, C_ε such that for all $u \in E$,*

$$||u||_W \leq \varepsilon ||u||_E + C_\varepsilon ||u||_X$$

Proof: Suppose not. Then there exists $\varepsilon > 0$ and for each $n \in \mathbb{N}$, u_n such that

$$||u_n||_W > \varepsilon ||u_n||_E + n ||u_n||_X$$

Now let $v_n = u_n / ||u_n||_E$. Therefore, $||v_n||_E = 1$ and

$$||v_n||_W > \varepsilon + n ||v_n||_X$$

It follows there exists a subsequence, still denoted by v_n such that v_n converges to v in W. However, the above inequality shows that $||v_n||_X \to 0$. Therefore, $v = 0$. But then the above inequality would imply that $||v_n|| > \varepsilon$ and passing to the limit yields $0 > \varepsilon$, a contradiction.

Definition 45.16 *Define $C([a,b];X)$ the space of functions continuous at every point of $[a,b]$ having values in X.*

You should verify that this is a Banach space with norm

$$||u||_{\infty,X} = \max\left\{||u_{n_k}(t) - u(t)||_X : t \in [a,b]\right\}.$$

The following theorem is an infinite dimensional version of the Ascoli Arzela theorem.

Theorem 45.17 *Let $q > 1$ and let $E \subseteq W \subseteq X$ where the injection map is continuous from W to X and compact from E to W. Let S be defined by*

$$\left\{u \text{ such that } ||u(t)||_E + ||u'||_{L^q([a,b];X)} \leq R \text{ for all } t \in [a,b]\right\}.$$

Then $S \subseteq C([a,b];W)$ and if $\{u_n\} \subseteq S$, there exists a subsequence, $\{u_{n_k}\}$ which converges to a function $u \in C([a,b];W)$ in the following way.

$$\lim_{k\to\infty} ||u_{n_k} - u||_{\infty,W} = 0.$$

Proof: First consider the issue of S being a subset of $C([a,b];W)$. By Theorem 45.9 on Page 1243 the following holds in X for $u \in S$.

$$u(t) - u(s) = \int_s^t u'(r)\,dr.$$

Thus $S \subseteq C([a,b];X)$. Let $\varepsilon > 0$ be given. Then by Theorem 45.15 there exists a constant, C_ε such that for all $u \in W$

$$||u||_W \leq \frac{\varepsilon}{4R}||u||_E + C_\varepsilon ||u||_X.$$

Therefore, for all $u \in S$,

$$\begin{aligned} ||u(t) - u(s)||_W &\leq \frac{\varepsilon}{6R}||u(t) - u(s)||_E + C_\varepsilon ||u(t) - u(s)||_X \\ &\leq \frac{\varepsilon}{3} + C_\varepsilon \left|\left|\int_s^t u'(r)\,dr\right|\right|_X \\ &\leq \frac{\varepsilon}{3} + C_\varepsilon \int_s^t ||u'(r)||_X\,dr \leq \frac{\varepsilon}{3} + C_\varepsilon R|t-s|^{1/q}. \quad (45.8) \end{aligned}$$

Since ε is arbitrary, it follows $u \in C([a,b];W)$.

Let $D = \mathbb{Q} \cap [a,b]$ so D is a countable dense subset of $[a,b]$. Let $D = \{t_n\}_{n=1}^\infty$. By compactness of the embedding of E into W, there exists a subsequence $u_{(n,1)}$ such that as $n \to \infty$, $u_{(n,1)}(t_1)$ converges to a point in W. Now take a subsequence of this, called $(n,2)$ such that as $n \to \infty, u_{(n,2)}(t_2)$ converges to a point in W. It follows that $u_{(n,2)}(t_1)$ also converges to a point of W. Continue this way. Now consider the diagonal sequence, $u_k \equiv u_{(k,k)}$ This sequence is a subsequence of $u_{(n,l)}$ whenever $k > l$. Therefore, $u_k(t_j)$ converges for all $t_j \in D$.

Claim: Let $\{u_k\}$ be as just defined, converging at every point of $[a,b]$. Then $\{u_k\}$ converges at every point of $[a,b]$.

Proof of claim: Let $\varepsilon > 0$ be given. Let $t \in [a,b]$. Pick $t_m \in D \cap [a,b]$ such that in 45.8 $C_\varepsilon R|t - t_m| < \varepsilon/3$. Then there exists N such that if $l, n > N$, then $||u_l(t_m) - u_n(t_m)||_X < \varepsilon/3$. It follows that for $l, n > N$,

$$\begin{aligned} ||u_l(t) - u_n(t)||_X &\leq ||u_l(t) - u_l(t_m)|| + ||u_l(t_m) - u_n(t_m)|| \\ &\quad + ||u_n(t_m) - u_n(t)|| \\ &\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} < 2\varepsilon \end{aligned}$$

Since ε was arbitrary, this shows $\{u_k(t)\}_{k=1}^{\infty}$ is a Cauchy sequence. Since W is complete, this shows this sequence converges.

Now for $t \in [a,b]$, it was just shown that if $\varepsilon > 0$ there exists N_t such that if $n, m > N_t$, then

$$||u_n(t) - u_m(t)|| < \frac{\varepsilon}{3}.$$

Now let $s \neq t$. Then

$$||u_n(s) - u_m(s)|| \leq ||u_n(s) - u_n(t)|| + ||u_n(t) - u_m(t)|| + ||u_m(t) - u_m(s)||$$

From 45.8

$$||u_n(s) - u_m(s)|| \leq 2\left(\frac{\varepsilon}{3} + C_\varepsilon R\,|t-s|^{1/q}\right) + ||u_n(t) - u_m(t)||$$

and so it follows that if δ is sufficiently small and $s \in B(t,\delta)$, then when $n, m > N_t$

$$||u_n(s) - u_m(s)|| < \varepsilon.$$

Since $[a,b]$ is compact, there are finitely many of these balls, $\{B(t_i,\delta)\}_{i=1}^{p}$, such that for $s \in B(t_i,\delta)$ and $n, m > N_{t_i}$, the above inequality holds. Let $N > \max\left\{N_{t_1}, \cdots, N_{t_p}\right\}$. Then if $m, n > N$ and $s \in [a,b]$ is arbitrary, it follows the above inequality must hold. Therefore, this has shown the following claim.

Claim: Let $\varepsilon > 0$ be given. Then there exists N such that if $m, n > N$, then $||u_n - u_m||_{\infty,W} < \varepsilon$.

Now let $u(t) = \lim_{k\to\infty} u_k(t)$.

$$||u(t) - u(s)||_W \leq ||u(t) - u_n(t)||_W + ||u_n(t) - u_n(s)||_W + ||u_n(s) - u(s)||_W \tag{45.9}$$

Let N be in the above claim and fix $n > N$. Then

$$||u(t) - u_n(t)||_W = \lim_{m\to\infty} ||u_m(t) - u_n(t)||_W \leq \varepsilon$$

and similarly, $||u_n(s) - u(s)||_W \leq \varepsilon$. Then if $|t-s|$ is small enough, 45.8 shows the middle term in 45.9 is also smaller than ε. Therefore, if $|t-s|$ is small enough,

$$||u(t) - u(s)||_W < 3\varepsilon.$$

Thus u is continuous. Finally, let N be as in the above claim. Then letting $m, n > N$, it follows that for all $t \in [a,b]$,

$$||u_m(t) - u_n(t)|| < \varepsilon.$$

Therefore, letting $m \to \infty$, it follows that for all $t \in [a,b]$,

$$||u(t) - u_n(t)|| \leq \varepsilon.$$

and so $||u - u_n||_{\infty,W} \leq \varepsilon$. Since ε is arbitrary, this proves the theorem.

The next theorem is another such imbedding theorem. It is often used in partial differential equations.

Theorem 45.18 *Let $E \subseteq W \subseteq X$ where the injection map is continuous from W to X and compact from E to W. Let $p \geq 1$, let $q > 1$, and define*

$$S \equiv \{u \in L^p([a,b];E) : u' \in L^q([a,b];X)$$
$$\text{and } ||u||_{L^p([a,b];E)} + ||u'||_{L^q([a,b];X)} \leq R\}.$$

Then S is precompact in $L^p([a,b];W)$. This means that if $\{u_n\}_{n=1}^{\infty} \subseteq S$, it has a subsequence $\{u_{n_k}\}$ which converges in $L^p([a,b];W)$.

Proof: By Proposition 6.12 on Page 136 it suffices to show S has an η net in $L^p([a,b];W)$ for each $\eta > 0$.

If not, there exists $\eta > 0$ and a sequence $\{u_n\} \subseteq S$, such that

$$||u_n - u_m|| \geq \eta \tag{45.10}$$

for all $n \neq m$ and the norm refers to $L^p([a,b];W)$. Let

$$a = t_0 < t_1 < \cdots < t_n = b, \; t_k - t_{k-1} = T/k.$$

Now define

$$\overline{u}_n(t) \equiv \sum_{i=1}^{k} \overline{u}_{n_i} \mathcal{X}_{[t_{i-1},t_i)}(t), \; \overline{u}_{n_i} \equiv \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} u_n(s)\, ds.$$

The idea is to show that $\overline{u}_n$ approximates u_n well and then to argue that a subsequence of the $\{\overline{u}_n\}$ is a Cauchy sequence yielding a contradiction to 45.10.

Therefore,

$$u_n(t) - \overline{u}_n(t) = \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n(t) - \overline{u}_n(s))\, ds \mathcal{X}_{[t_{i-1},t_i)}(t).$$

It follows from Jensen's inequality that

$$\begin{aligned} & ||u_n(t) - \overline{u}_n(t)||_W^p \\ = & \sum_{i=1}^{k} \left|\left| \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n(t) - u_n(s))\, ds \right|\right|_W^p \mathcal{X}_{[t_{i-1},t_i)}(t) \\ \leq & \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} ||u_n(t) - u_n(s)||_W^p\, ds \mathcal{X}_{[t_{i-1},t_i)}(t) \end{aligned}$$

and so

$$\begin{aligned} & \int_a^b ||(u_n(t) - \overline{u}_n(s))||_W^p\, ds \\ \leq & \int_a^b \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} ||u_n(t) - u_n(s)||_W^p\, ds \mathcal{X}_{[t_{i-1},t_i)}(t)\, dt \\ = & \sum_{i=1}^{k} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} ||u_n(t) - u_n(s)||_W^p\, ds dt. \end{aligned} \tag{45.11}$$

From Theorems 45.15 and 45.9, if $\varepsilon > 0$, there exists C_ε such that

$$||u_n(t) - u_n(s)||_W^p \leq \varepsilon ||u_n(t) - u_n(s)||_E^p + C_\varepsilon ||u_n(t) - u_n(s)||_X^p$$

$$\begin{aligned}
&\leq 2^{p-1}\varepsilon \left(||u_n(t)||^p + ||u_n(s)||^p\right) + C_\varepsilon \left|\left|\int_s^t u_n'(r)\,dr\right|\right|_X^p \\
&\leq 2^{p-1}\varepsilon \left(||u_n(t)||^p + ||u_n(s)||^p\right) + C_\varepsilon \left(\int_s^t ||u_n'(r)||_X\,dr\right)^p \\
&\leq 2^{p-1}\varepsilon \left(||u_n(t)||^p + ||u_n(s)||^p\right) \\
&\quad + C_\varepsilon \left(\left(\int_s^t ||u_n'(r)||_X^q\,dr\right)^{1/q} |t-s|^{1/q'}\right)^p \\
&= 2^{p-1}\varepsilon \left(||u_n(t)||^p + ||u_n(s)||^p\right) + C_\varepsilon R^{p/q} |t-s|^{p/q'}.
\end{aligned}$$

This is substituted in to 45.11 to obtain

$$\int_a^b ||(u_n(t) - \overline{u}_n(s))||_W^p\,ds \leq$$

$$\begin{aligned}
&\sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i} \left(2^{p-1}\varepsilon\left(||u_n(t)||^p + ||u_n(s)||^p\right)\right. \\
&\left. + C_\varepsilon R^{p/q} |t-s|^{p/q'}\right) dsdt \\
= &\sum_{i=1}^k 2^p \varepsilon \int_{t_{i-1}}^{t_i} ||u_n(t)||_W^p + C_\varepsilon R^{p/q} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i} |t-s|^{p/q'}\,dsdt \\
= \; &2^p\varepsilon \int_a^b ||u_n(t)||^p\,dt + C_\varepsilon R^{p/q} \sum_{i=1}^k \frac{1}{(t_i - t_{i-1})} (t_i - t_{i-1})^{p/q'} \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i} dsdt \\
= \; &2^p\varepsilon \int_a^b ||u_n(t)||^p\,dt + C_\varepsilon R^{p/q} \sum_{i=1}^k \frac{1}{(t_i - t_{i-1})} (t_i - t_{i-1})^{p/q'} (t_i - t_{i-1})^2 \\
\leq \; &2^p\varepsilon R^p + C_\varepsilon R^{p/q} \sum_{i=1}^k (t_i - t_{i-1})^{1+p/q'} = 2^p\varepsilon R^p + C_\varepsilon R^{p/q} k \left(\frac{T}{k}\right)^{1+p/q'}.
\end{aligned}$$

Taking ε so small that $2^p\varepsilon R^p < \eta^p/8^p$ and then choosing k sufficiently large, it follows

$$||u_n - \overline{u}_n||_{L^p([a,b];W)} < \frac{\eta}{4}.$$

Now use compactness of the embedding of E into W to obtain a subsequence such that $\{\overline{u}_n\}$ is Cauchy in $L^p(a,b;W)$ and use this to contradict 45.10. Suppose $\overline{u}_n(t) = \sum_{i=1}^k u_i^n \mathcal{X}_{[t_{i-1},t_i)}(t)$. Thus

$$||\overline{u}_n(t)||_E = \sum_{i=1}^k ||u_i^n||_E \mathcal{X}_{[t_{i-1},t_i)}(t)$$

and so

$$R \geq \int_a^b ||\overline{u}_n(t)||_E^p \, dt = \frac{T}{k} \sum_{i=1}^k ||u_i^n||_E^p$$

Therefore, the $\{u_i^n\}$ are all bounded. It follows that after taking subsequences k times there exists a subsequence $\{u_{n_k}\}$ such that u_{n_k} is a Cauchy sequence in $L^p(a,b;W)$. You simply get a subsequence such that $u_i^{n_k}$ is a Cauchy sequence in W for each i. Then denoting this subsequence by n,

$$\begin{aligned} ||u_n - u_m||_{L^p(a,b;W)} &\leq ||u_n - \overline{u}_n||_{L^p(a,b;W)} \\ &\quad + ||\overline{u}_n - \overline{u}_m||_{L^p(a,b;W)} + ||\overline{u}_m - u_m||_{L^p(a,b;W)} \\ &\leq \frac{\eta}{4} + ||\overline{u}_n - \overline{u}_m||_{L^p(a,b;W)} + \frac{\eta}{4} < \eta \end{aligned}$$

provided m, n are large enough, contradicting 45.10. This proves the theorem.

45.2 The K Method

This considers the problem of interpolating Banach spaces. The idea is to build a Banach space between two others in a systematic way, thus constructing a new Banach space from old ones. The first method of defining intermediate Banach spaces is called the K method. For more on this topic as well as the other topics on interpolation see [8] which is what I am following. See also [50]. There is far more on these subjects in these books than what I am presenting here! My goal is to present only enough to give an introduction to the topic and to use it in presenting more theory of Sobolev spaces.

In what follows a topological vector space is a vector space in which vector addition and scalar multiplication are continuous. That is $\cdot : \mathbb{F} \times X \to X$ is continuous and $+ : X \times X \to X$ is also continuous.

A common example of a topological vector space is the dual space, X' of a Banach space, X with the weak $*$ topology. For $S \subseteq X$ a finite set, define

$$B_S(x^*, r) \equiv \{y^* \in X' : |y^*(x) - x^*(x)| < r \text{ for all } x \in S\}$$

Then the $B_S(x^*, r)$ for S a finite subset of X and $r > 0$ form a basis for the topology on X' called the weak $*$ topology. You can check that the vector space operations are continuous.

Definition 45.19 *Let A_0 and A_1 be two Banach spaces with norms $||\cdot||_0$ and $||\cdot||_1$ respectively, also written as $||\cdot||_{A_0}$ and $||\cdot||_{A_1}$ and let X be a topological vector space such that $A_i \subseteq X$ for $i = 1, 2,$ and the identity map from A_i to X is continuous. For each $t > 0$, define a norm on $A_0 + A_1$ by*

$$K(t, a) \equiv ||a||_t \equiv \inf\{||a_0||_0 + t\,||a_1||_1 : a_0 + a_1 = a\}.$$

This is short for $K(t, a, A_0, A_1)$. Thus $K(t, a, A_1, A_0)$ will mean

$$K(t, a, A_1, A_0) \equiv \inf\left\{||a_1||_{A_1} + t\,||a_0||_{A_0} : a_0 + a_1 = a\right\}$$

but the default is $K(t,a,A_0,A_1)$ *if* $K(t,a)$ *is written.*

The following lemma is an interesting exercise.

Lemma 45.20 $(A_0+A_1,K(t,\cdot))$ *is a Banach space and all the norms,* $K(t,\cdot)$ *are equivalent.*

Proof: First, why is $K(t,\cdot)$ a norm? It is clear that $K(t,a)\geq 0$ and that if $a=0$ then $K(t,a)=0$. Is this the only way this can happen? Suppose $K(t,a)=0$. Then there exist $a_{0n}\in A_0$ and $a_{1n}\in A_1$ such that $||a_{0n}||_0\to 0$, $||a_{1n}||_1\to 0$, and $a=a_{0n}+a_{1n}$. Since the embedding of A_i into X is continuous and since X is a topological vector space[1], it follows

$$a=a_{0n}+a_{1n}\to 0$$

and so $a=0$.

Let α be a nonzero scalar. Then

$$\begin{aligned}
K(t,\alpha a) &= \inf\{||a_0||_0+t\,||a_1||_1 : a_0+a_1=\alpha a\}\\
&= \inf\left\{|\alpha|\left|\left|\frac{a_0}{\alpha}\right|\right|_0+t\,|\alpha|\left|\left|\frac{a_1}{\alpha}\right|\right|_1 : \frac{a_0}{\alpha}+\frac{a_1}{\alpha}=a\right\}\\
&= |\alpha|\inf\left\{\left|\left|\frac{a_0}{\alpha}\right|\right|_0+t\left|\left|\frac{a_1}{\alpha}\right|\right|_1 : \frac{a_0}{\alpha}+\frac{a_1}{\alpha}=a\right\}\\
&= |\alpha|\inf\{||a_0||_0+t\,||a_1||_1 : a_0+a_1=a\}=|\alpha|\,K(t,a).
\end{aligned}$$

It remains to verify the triangle inequality. Let $\varepsilon>0$ be given. Then there exist $a_0,a_1,b_0,$ and b_1 in $A_0,A_1,A_0,$ and A_1 respectively such that $a_0+a_1=a, b_0+b_1=b$ and

$$\begin{aligned}
\varepsilon+K(t,a)+K(t,b) &> ||a_0||_0+t\,||a_1||_1+||b_0||_0+t\,||b_1||_1\\
&\geq ||a_0+b_0||_0+t\,||b_1+a_1||_1\geq K(t,a+b).
\end{aligned}$$

This has shown that $K(t,\cdot)$ is at least a norm. Are all these norms equivalent? If $0<s<t$ then it is clear that $K(t,a)\geq K(s,a)$. To show there exists a constant, C such that $CK(s,a)\geq K(t,a)$ for all a,

$$\begin{aligned}
\frac{t}{s}K(s,a) &\equiv \frac{t}{s}\inf\{||a_0||_0+s\,||a_1||_1 : a_0+a_1=a\}\\
&= \inf\left\{\frac{t}{s}\,||a_0||_0+s\frac{t}{s}\,||a_1||_1 : a_0+a_1=a\right\}\\
&= \inf\left\{\frac{t}{s}\,||a_0||_0+t\,||a_1||_1 : a_0+a_1=a\right\}\\
&\geq \inf\{||a_0||_0+t\,||a_1||_1 : a_0+a_1=a\}=K(t,a).
\end{aligned}$$

Therefore, the two norms are equivalent as hoped.

[1]Vector addition is continuous is the property which is used here.

Finally, it is required to verify that $(A_0 + A_1, K(t, \cdot))$ is a Banach space. Since all these norms are equivalent, it suffices to only consider the norm, $K(1, \cdot)$. Let $\{a_{0n} + a_{1n}\}_{n=1}^{\infty}$ be a Cauchy sequence in $A_0 + A_1$. Then for m, n large enough,

$$K(1, a_{0n} + a_{1n} - (a_{0m} + a_{1m})) < \varepsilon.$$

It follows there exist $x_n \in A_0$ and $y_n \in A_1$ such that $x_n + y_n = 0$ for every n and whenever m, n are large enough,

$$||a_{0n} + x_n - (a_{0m} + x_m)||_0 + ||a_{1n} + y_n - (a_{1m} + y_m)||_1 < \varepsilon$$

Hence $\{a_{1n} + y_n\}$ is a Cauchy sequence in A_1 and $\{a_{0n} + x_n\}$ is a Cauchy sequence in A_0. Let

$$\begin{aligned} a_{0n} + x_n &\to a_0 \in A_0 \\ a_{1n} + y_n &\to a_1 \in A_1. \end{aligned}$$

Then

$$\begin{aligned} K(1, a_{0n} + a_{1n} - (a_0 + a_1)) &= K(1, a_{0n} + x_n + a_{1n} + y_n - (a_0 + a_1)) \\ &\leq ||a_{0n} + x_n - a_0||_0 + ||a_{1n} + y_n - a_1||_1 \end{aligned}$$

which converges to 0. Thus $A_0 + A_1$ is a Banach space as claimed.

With this, there exists a method for constructing a Banach space which lies between $A_0 \cap A_1$ and $A_0 + A_1$.

Definition 45.21 *Let* $1 \leq q < \infty, 0 < \theta < 1$. *Define* $(A_0, A_1)_{\theta,q}$ *to be those elements of* $A_0 + A_1, a$, *such that*

$$||a||_{\theta,q} \equiv \left[\int_0^{\infty} \left(t^{-\theta} K(t, a, A_0, A_1)\right)^q \frac{dt}{t}\right]^{1/q} < \infty.$$

Theorem 45.22 $(A_0, A_1)_{\theta,q}$ *is a normed linear space satisfying*

$$A_0 \cap A_1 \subseteq (A_0, A_1)_{\theta,q} \subseteq A_0 + A_1, \tag{45.12}$$

with the inclusion maps continuous, and

$$\left((A_0, A_1)_{\theta,q}, ||\cdot||_{\theta,q}\right) \text{ is a Banach space.} \tag{45.13}$$

If $a \in A_0 \cap A_1$, *then*

$$||a||_{\theta,q} \leq \left(\frac{1}{q\theta(1-\theta)}\right)^{1/q} ||a||_1^{\theta} ||a||_0^{1-\theta}. \tag{45.14}$$

If $A_0 \subseteq A_1$ *with* $||\cdot||_0 \geq ||\cdot||_1$, *then*

$$A_0 \cap A_1 = A_0 \subseteq (A_0, A_1)_{\theta,q} \subseteq A_1 = A_0 + A_1.$$

Also, if bounded sets in A_0 have compact closures in A_1 then the same is true if A_1 is replaced with $(A_0, A_1)_{\theta,q}$. Finally, if

$$T \in \mathcal{L}(A_0, B_0), T \in \mathcal{L}(A_1, B_1), \tag{45.15}$$

and T is a linear map from $A_0 + A_1$ to $B_0 + B_1$ where the A_i and B_i are Banach spaces with the properties described above, then it follows

$$T \in \mathcal{L}\left((A_0, A_1)_{\theta,q}, (B_0, B_1)_{\theta,q}\right) \tag{45.16}$$

and if M is its norm, and M_0 and M_1 are the norms of T as a map in $\mathcal{L}(A_0, B_0)$ and $\mathcal{L}(A_1, B_1)$ respectively, then

$$M \leq M_0^{1-\theta} M_1^{\theta}. \tag{45.17}$$

Proof: Suppose first $a \in A_0 \cap A_1$. Then

$$\begin{aligned} ||a||_{\theta,q}^q &\equiv \int_0^r \left(t^{-\theta} K(t,a)\right)^q \frac{dt}{t} + \int_r^\infty \left(t^{-\theta} K(t,a)\right)^q \frac{dt}{t} \quad (45.18) \\ &\leq \int_0^r \left(t^{-\theta} ||a||_1 t\right)^q \frac{dt}{t} + \int_r^\infty \left(t^{-\theta} ||a||_0\right)^q \frac{dt}{t} \\ &= ||a||_1^q \int_0^r t^{q(1-\theta)-1} dt + ||a||_0^q \int_r^\infty t^{-1-\theta q} dt \\ &= ||a||_1^q \frac{r^{q-q\theta}}{q - q\theta} + ||a||_0^q \frac{r^{-\theta q}}{\theta q} < \infty \quad (45.19) \end{aligned}$$

Which shows the first inclusion of 45.12. The above holds for all $r > 0$ and in particular for the value of r which minimizes the expression on the right in 45.19, $r = ||a||_0 / ||a||_1$. Therefore, doing some calculus,

$$||a||_{\theta,q}^q \leq \frac{1}{\theta q(1+\theta)} ||a||_0^{q(1-\theta)} ||a||_1^{q\theta}$$

which shows 45.14. This also verifies that the inclusion map is continuous in 45.12.

Now consider the second inclusion in 45.12. The inclusion is obvious because $(A_0, A_1)_{\theta,q}$ is given to be a subset of A_0+A_1. It remains to verify the inclusion map is continuous. Therefore, suppose $a_n \to 0$ in $(A_0, A_1)_{\theta,q}$. Since $a_n \to 0$ in $(A_0, A_1)_{\theta,q}$, it follows the function, $t \to t^{-\theta} K(t, a_n)$ converges to zero in $L^q(0,\infty)$ with respect to the measure, dt/t. Therefore, taking another subsequence, still denoted as a_n, you can assume this function converges to 0 a.e. Pick such a t where this convergence takes place. Then $K(t, a_n) \to 0$ as $n \to \infty$ and so $a_n \to 0$ in $A_0 + A_1$. this shows that if $a_n \to 0$ in $(A_0, A_1)_{\theta,q}$, then there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \to 0$ in $A_0 + A_1$. It follows that if $a_n \to 0$ in $(A_0, A_1)_{\theta,q}$, then $a_n \to 0$ in $A_0 + A_1$. This proves the continuity of the embedding.

What about 45.13? Suppose $\{a_n\}$ is a Cauchy sequence in $(A_0, A_1)_{\theta,q}$. Then there exists $a \in A_0 + A_1$ such that $a_n \to a$ in $A_0 + A_1$ because $A_0 + A_1$ is a Banach space. Thus, $K(t, a_n) \to K(t, a)$ for all $t > 0$. Therefore, by Fatou's lemma,

$$\begin{aligned}\left(\int_0^\infty \left(t^{-\theta} K(t, a)\right)^q \frac{dt}{t}\right)^{1/q} &\leq \lim\inf_{n\to\infty} \left(\int_0^\infty \left(t^{-\theta} K(t, a_n)\right)^q \frac{dt}{t}\right)^{1/q} \\ &\leq \max\left\{||a_n||_{\theta,q} : n \in \mathbb{N}\right\} < \infty\end{aligned}$$

and so $a \in (A_0, A_1)_{\theta,q}$. Now

$$||a - a_n||_{\theta,q} \leq \lim\inf_{m\to\infty} \left(\int_0^\infty \left(t^{-\theta} K(t, a_n - a_m)\right)^q \frac{dt}{t}\right)^{1/q} < \varepsilon$$

whenever n is large enough. Thus $(A_0, A_1)_{\theta,q}$ is complete as claimed.

Next suppose $A_0 \subseteq A_1$ and the inclusion map is compact. In this case, $A_0 \cap A_1 = A_0$ and so it has been shown above that $A_0 \subseteq (A_0, A_1)_{\theta,q}$. It remains to show that every bounded subset, S, contained in A_0 has an η net in $(A_0, A_1)_{\theta,q}$. Recall the inequality, 45.14

$$\begin{aligned}||a||_{\theta,q} &\leq \left(\frac{1}{q\theta(1-\theta)}\right)^{1/q} ||a||_1^\theta\, ||a||_0^{1-\theta} \\ &= \frac{C}{\varepsilon} ||a||_1^\theta\, \varepsilon\, ||a||_0^{1-\theta}.\end{aligned}$$

Now this implies

$$||a||_{\theta,q} \leq \left(\frac{C}{\varepsilon}\right)^{1/\theta} \theta\, ||a||_1 + \varepsilon^{1/(1-\theta)} (1-\theta)\, ||a||_0$$

By compactness of the embedding of A_0 into A_1, it follows there exists an $\varepsilon^{(1+\theta)/\theta}$ net for S in $A_1, \{a_1, \cdots, a_p\}$. Then for $a \in S$, there exists k such that $||a - a_k||_1 < \varepsilon^{(1+\theta)/\theta}$. It follows

$$\begin{aligned}||a - a_k||_{\theta,q} &\leq \left(\frac{C}{\varepsilon}\right)^{1/\theta} \theta\, ||a - a_k||_1 + \varepsilon^{1/(1-\theta)} (1-\theta)\, ||a - a_k||_0 \\ &\leq \left(\frac{C}{\varepsilon}\right)^{1/\theta} \theta \varepsilon^{(1+\theta)/\theta} + \varepsilon^{1/(1-\theta)} (1-\theta)\, 2M \\ &= C^{1/\theta} \theta \varepsilon + \varepsilon^{1/(1-\theta)} (1-\theta)\, 2M\end{aligned}$$

where M is large enough that $||a||_0 \leq M$ for all $a \in S$. Since ε is arbitrary, this shows the existence of a η net and proves the compactness of the embedding into $(A_0, A_1)_{\theta,q}$.

It remains to verify the assertions 45.15-45.17. Let $T \in \mathcal{L}(A_0, B_0), T \in \mathcal{L}(A_1, B_1)$ with T a linear map from $A_0 + A_1$ to $B_0 + B_1$. Let $a \in (A_0, A_1)_{\theta,q} \subseteq A_0 + A_1$ and consider $Ta \in B_0 + B_1$. Denote by $K(t, \cdot)$ the norm described above for both $A_0 + A_1$ and $B_0 + B_1$ since this will cause no confusion. Then

$$||Ta||_{\theta,q} \equiv \left(\int_0^\infty \left(t^{-\theta} K(t, Ta)\right)^q \frac{dt}{t}\right)^{1/q}. \tag{45.20}$$

Now let $a_0 + a_1 = a$ and so $Ta_0 + Ta_1 = Ta$

$$\begin{aligned} K(t, Ta) &\leq ||Ta_0||_0 + t\,||Ta_1||_1 \leq M_0\,||a_0||_0 + M_1 t\,||a_1||_1 \\ &\leq M_0 \left(||a_0||_0 + t\left(\frac{M_1}{M_0}\right) ||a_1||_1 \right) \end{aligned}$$

and so, taking inf for all $a_0 + a_1 = a$, yields

$$K(t, Ta) \leq M_0 K\left(t\left(\frac{M_1}{M_0}\right), a\right)$$

It follows from 45.20 that

$$\begin{aligned} ||Ta||_{\theta,q} &\equiv \left(\int_0^\infty \left(t^{-\theta} K(t, Ta)\right)^q \frac{dt}{t}\right)^{1/q} \\ &\leq \left(\int_0^\infty \left(t^{-\theta} M_0 K\left(t\left(\frac{M_1}{M_0}\right), a\right)\right)^q \frac{dt}{t}\right)^{1/q} \\ &= M_0 \left(\int_0^\infty \left(t^{-\theta} K\left(t\left(\frac{M_1}{M_0}\right), a\right)\right)^q \frac{dt}{t}\right)^{1/q} \\ &= M_0 \left(\int_0^\infty \left(\left(\frac{M_0}{M_1} s\right)^{-\theta} K(s, a)\right)^q \frac{ds}{s}\right)^{1/q} \\ &= M_1^\theta M_0^{(1-\theta)} \left(\int_0^\infty \left(s^{-\theta} K(s, a)\right)^q \frac{ds}{s}\right)^{1/q} = M_1^\theta M_0^{(1-\theta)}\, ||a||_{\theta,q}\,. \end{aligned}$$

This shows $T \in \mathcal{L}\left((A_0, A_1)_{\theta,q}, (B_0, B_1)_{\theta,q}\right)$ and if M is the norm of $T, M \leq M_0^{1-\theta} M_1^\theta$ as claimed. This proves the theorem.

45.3 The J Method

There is another method known as the J method.

Definition 45.23 *For A_0 and A_1 Banach spaces as described above, and $a \in A_0 \cap A_1$,*

$$J(t, a) \equiv \max\left(||a||_{A_0}, t\,||a||_{A_1}\right). \tag{45.21}$$

this is short for $J(t, a, A_0, A_1)$. Thus

$$J(t, a, A_1, A_0) \equiv \max\left(||a||_{A_1}, t\,||a||_{A_0}\right)$$

but unless indicated otherwise, A_0 will come first. Now for $\theta \in (0, 1)$ and $q \geq 1$, define a space, $(A_0, A_1)_{\theta,q,J}$ as follows. The space, $(A_0, A_1)_{\theta,q,J}$ will consist of those elements, a, of $A_0 + A_1$ which can be written in the form

$$a = \int_0^\infty u(t)\frac{dt}{t} \equiv \lim_{\varepsilon \to 0+} \int_\varepsilon^1 u(t)\frac{dt}{t} + \lim_{r \to \infty} \int_1^r u(t)\frac{dt}{t} \tag{45.22}$$

the limits taking place in $A_0 + A_1$ *with the norm*

$$K(1,a) \equiv \inf_{a=a_0+a_1} \left(||a_0||_{A_0} + ||a_1||_{A_1} \right),$$

where $u(t)$ *is strongly measurable with values in* $A_0 \cap A_1$ *and bounded on every compact subset of* $(0,\infty)$ *such that*

$$\left(\int_0^\infty \left(t^{-\theta} J(t, u(t), A_0, A_1) \right)^q \frac{dt}{t} \right)^{1/q} < \infty. \tag{45.23}$$

For such $a \in A_0 + A_1$, *define*

$$||a||_{\theta,q,J} \equiv \inf_u \left\{ \left(\int_0^\infty \left(t^{-\theta} J(t, u(t), A_0, A_1) \right)^q \frac{dt}{t} \right)^{1/q} \right\} \tag{45.24}$$

where the infimum is taken over all u satisfying 45.22 and 45.23.

Note that a norm on $A_0 \times A_1$ would be

$$||(a_0, a_1)|| \equiv \max \left(||a_0||_{A_0}, t\, ||a_1||_{A_1} \right)$$

and so $J(t,\cdot)$ is the restriction of this norm to the subspace of $A_0 \times A_1$ defined by $\{(a,a) : a \in A_0 \cap A_1\}$. Also for each $t > 0$ $J(t,\cdot)$ is a norm on $A_0 \cap A_1$ and furthermore, any two of these norms are equivalent. In fact, it is easy to see that for $0 < t < s$, $\frac{t}{s} J(s,a) \le J(t,a) \le J(s,a)$.

The following lemma is significant and follows immediately from the above definition.

Lemma 45.24 *Suppose* $a \in (A_0, A_1)_{\theta,q,J}$ *and* $a = \int_0^\infty u(t) \frac{dt}{t}$ *where u is described above. Then letting* $r > 1$,

$$u_r(t) \equiv \begin{cases} u(t) & \text{if } t \in \left(\frac{1}{r}, r\right) \\ 0 & \text{otherwise} \end{cases}.$$

it follows that

$$\int_0^\infty u_r(t) \frac{dt}{t} \in A_0 \cap A_1.$$

Proof: The integral equals $\int_{1/r}^r u(t) \frac{dt}{t}$. $\int_{1/r}^r \frac{1}{t} dt = 2 \ln r < \infty$. Now u_r is measurable in $A_0 \cap A_1$ and bounded. Therefore, there exists a sequence of measurable simple functions, $\{s_n\}$ having values in $A_0 \cap A_1$ which converges pointwise and uniformly to u_r. It can also be assumed $J(r, s_n(t)) \le J(r, u_r(t))$ for all $t \in [1/r, r]$. Therefore,

$$\lim_{n,m\to\infty} \int_{1/r}^r J(r, s_m - s_n) \frac{dt}{t} = 0.$$

It follows from the definition of the Bochner integral that

$$\lim_{n\to\infty} \int_{1/r}^r s_n \frac{dt}{t} = \int_{1/r}^r u_r \frac{dt}{t} \in A_0 \cap A_1.$$

This proves the lemma.

The remarkable thing is that the two spaces, $(A_0, A_1)_{\theta,q}$ and $(A_0, A_1)_{\theta,q,J}$ coincide and have equivalent norms. The following important lemma, called the fundamental lemma of interpolation theory in [8] is used to prove this. This lemma is really incredible.

Lemma 45.25 *Suppose for* $a \in A_0+A_1$, $\lim_{t\to 0+} K(t,a) = 0$ *and* $\lim_{t\to\infty} \frac{K(t,a)}{t} = 0$. *Then for any* $\varepsilon > 0$, *there is a representation,*

$$a = \sum_{i=-\infty}^{\infty} u_i = \lim_{n,m\to\infty} \sum_{i=-m}^{n} u_i,\ u_i \in A_0 \cap A_1, \tag{45.25}$$

the convergences taking place in $A_0 + A_1$, *such that*

$$J\left(2^i, u_i\right) \leq 3(1+\varepsilon) K\left(2^i, a\right). \tag{45.26}$$

Proof: For each i, there exist $a_{0,i} \in A_0$ and $a_{1,i} \in A_1$ such that

$$a = a_{0,i} + a_{1,i},$$

and

$$(1+\varepsilon) K\left(2^i, a\right) \geq ||a_{0,i}||_{A_0} + 2^i ||a_{1,i}||_{A_1}. \tag{45.27}$$

This follows directly from the definition of $K(t,a)$. From the assumed limit conditions on $K(t,a)$,

$$\lim_{i\to\infty} ||a_{1,i}||_{A_1} = 0,\ \lim_{i\to-\infty} ||a_{0,i}||_{A_0} = 0. \tag{45.28}$$

Then let $u_i \equiv a_{0,i} - a_{0,i-1} = a_{1,i-1} - a_{1,i}$. The reason these are equal is $a = a_{0,i} + a_{1,i} = a_{0,i-1} + a_{1,i-1}$. Then

$$\sum_{i=-m}^{n} u_i = a_{0,n} - a_{0,-(m+1)} = a_{1,-(m+1)} - a_{1,n}.$$

It follows $a - \sum_{i=-m}^{n} u_i = a - \left(a_{0,n} - a_{0,-(m+1)}\right) = a_{0,-(m+1)} + a_{1,n}$, and both terms converge to zero as m and n converge to ∞ by 45.28. Therefore,

$$K\left(1, a - \sum_{i=-m}^{n} u_i\right) \leq \left|\left|a_{0,-(m+1)}\right|\right| + ||a_{1,n}||$$

and so this shows $a = \sum_{i=-\infty}^{\infty} u_i$ which is one of the claims of the lemma. Also

$$J\left(2^i, u_i\right) \equiv \max\left(||u_i||_{A_0}, 2^i ||u_i||_{A_1}\right) \leq ||u_i||_{A_0} + 2^i ||u_i||_{A_1}$$

$$\leq ||a_{0,i}||_{A_0} + 2^i ||a_{1,i}||_{A_1} + \overbrace{||a_{0,i-1}||_{A_0} + 2^i ||a_{1,i-1}||_{A_1}}^{\leq 2\left(||a_{0,i-1}||_{A_0} + 2^{i-1}||a_{1,i-1}||_{A_1}\right)}$$

$$\leq (1+\varepsilon) K\left(2^i, a\right) + 2(1+\varepsilon) K\left(2^{i-1}, a\right) \leq 3(1+\varepsilon) K\left(2^i, a\right)$$

because $t \to K(t,a)$ is nondecreasing. This proves the lemma.

Lemma 45.26 *If* $a \in A_0 \cap A_1$, *then* $K(t,a) \leq \min\left(1, \frac{t}{s}\right) J(s,a)$.

Proof: If $s \geq t$, then $\min\left(1, \frac{t}{s}\right) = \frac{t}{s}$ and so

$$\begin{aligned}\min\left(1, \frac{t}{s}\right) J(s,a) &= \frac{t}{s} \max\left(||a||_{A_0}, s\,||a||_{A_1}\right) \geq \left(\frac{t}{s}\right) s\,||a||_{A_1} \\ &= t\,||a||_{A_1} \geq K(t,a).\end{aligned}$$

Now in case $s < t$, then $\min\left(1, \frac{t}{s}\right) = 1$ and so

$$\begin{aligned}\min\left(1, \frac{t}{s}\right) J(s,a) &= \max\left(||a||_{A_0}, s\,||a||_{A_1}\right) \geq ||a||_{A_0} \\ &\geq K(t,a).\end{aligned}$$

This proves the lemma.

Theorem 45.27 *Let* A_0, A_1, K *and* J *be as described above. Then for all* $q \geq 1$ *and* $\theta \in (0,1)$,

$$(A_0, A_1)_{\theta,q} = (A_0, A_1)_{\theta,q,J}$$

and furthermore, the norms are equivalent.

Proof: Begin with $a \in (A_0, A_1)_{\theta,q}$. Thus

$$||a||_{\theta,q}^q = \int_0^\infty \left(t^{-\theta} K(t,a)\right)^q \frac{dt}{t} < \infty \tag{45.29}$$

and it is necessary to produce $u(t)$ as described above,

$$a = \int_0^\infty u(t) \frac{dt}{t} \text{ where } \int_0^\infty \left(t^{-\theta} J(t, u(t))\right)^q \frac{dt}{t} < \infty.$$

From 45.29, $\lim_{t\to 0+} K(t,a) = 0$ since $t \to K(t,a)$ is nondecreasing and so if its limit is positive, the integrand would have a non integrable singularity like $t^{-\theta q - 1}$. Next consider what happens to $\frac{K(t,a)}{t}$ as $t \to \infty$.

Claim: $t \to \frac{K(t,a)}{t}$ is decreasing.

Proof of the claim: Choose $a_0 \in A_0$ and $a_1 \in A_1$ such that $a_0 + a_1 = a$ and

$$K(t,a) + \varepsilon t > ||a_0||_{A_0} + t\,||a_1||_{A_1}$$

let $s > t$. Then

$$\frac{K(t,a) + t\varepsilon}{t} \geq \frac{||a_0||_{A_0} + t\,||a_1||_{A_1}}{t} \geq \frac{||a_0||_{A_0} + s\,||a_1||_{A_1}}{s} \geq \frac{K(s,a)}{s}.$$

Since ε is arbitrary, this proves the claim.

Let $r \equiv \lim_{t\to\infty} \frac{K(t,a)}{t}$. Is $r = 0$? Suppose to the contrary that $r > 0$. Then the integrand of 45.29, is at least as large as

$$t^{-\theta q} K(t,a)^{q-1} \frac{K(t,a)}{t} \geq t^{-\theta q} K(t,a)^{q-1} r$$

$$\geq t^{-\theta q} (tr)^{q-1} r \geq r^q t^{q(1-\theta)-1}$$

whose integral is infinite. Therefore, $r = 0$.

Lemma 45.25, implies there exist $u_i \in A_0 \cap A_1$ such that $a = \sum_{i=-\infty}^{\infty} u_i$, the convergence taking place in $A_0 + A_1$with the inequality of that Lemma holding,

$$J\left(2^i, u_i\right) \leq 3(1+\varepsilon) K\left(2^i, a\right).$$

For i an integer and $t \in [2^{i-1}, 2^i)$, let

$$u(t) \equiv u_i / \ln 2.$$

Then

$$a = \sum_{i=-\infty}^{\infty} u_i = \int_0^{\infty} u(t) \frac{dt}{t}. \tag{45.30}$$

Now

$$\begin{aligned}
\|a\|_{\theta,q,J}^q &\leq \int_0^{\infty} \left(t^{-\theta} J(t, u(t))\right)^q \frac{dt}{t} \\
&= \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^i} \left(t^{-\theta} J\left(t, \frac{u_i}{\ln 2}\right)\right)^q \frac{dt}{t} \\
&\leq \left(\frac{1}{\ln 2}\right)^q \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^i} \left(t^{-\theta} J\left(2^i, u_i\right)\right)^q \frac{dt}{t} \\
&\leq \left(\frac{1}{\ln 2}\right)^q \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^i} \left(t^{-\theta} 3(1+\varepsilon) K\left(2^i, a\right)\right)^q \frac{dt}{t}
\end{aligned}$$

Using the above claim, $\frac{K(2^i,a)}{2^i} \leq \frac{K(2^{i-1},a)}{2^{i-1}}$ and so $K\left(2^i, a\right) \leq 2K\left(2^{i-1}, a\right)$. Therefore, the above is no larger than

$$\begin{aligned}
&\leq 2\left(\frac{1}{\ln 2}\right)^q \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^i} \left(t^{-\theta} 3(1+\varepsilon) K\left(2^{i-1}, a\right)\right)^q \frac{dt}{t} \\
&\leq 2\left(\frac{1}{\ln 2}\right)^q \sum_{i=-\infty}^{\infty} \int_{2^{i-1}}^{2^i} \left(t^{-\theta} 3(1+\varepsilon) K(t, a)\right)^q \frac{dt}{t} \\
&= 2\left(\frac{3(1+\varepsilon)}{\ln 2}\right)^q \int_0^{\infty} \left(t^{-\theta} K(t,a)\right)^q \frac{dt}{t} \equiv 2\left(\frac{3(1+\varepsilon)}{\ln 2}\right)^q \|a\|_{\theta,q}^q. \quad (45.31)
\end{aligned}$$

This has shown that if $a \in (A_0, A_1)_{\theta,q}$, then by 45.30 and 45.31, $a \in (A_0, A_1)_{\theta,q,J}$ and

$$||a||^q_{\theta,q,J} \leq 2 \left(\frac{3(1+\varepsilon)}{\ln 2} \right)^q ||a||^q_{\theta,q} . \tag{45.32}$$

It remains to prove the other inclusion and norm inequality, both of which are much easier to obtain. Thus, let $a \in (A_0, A_1)_{\theta,q,J}$ with

$$a = \int_0^\infty u(t) \frac{dt}{t} \tag{45.33}$$

where u is a strongly measurable function having values in $A_0 \cap A_1$ and for which

$$\int_0^\infty \left(t^{-\theta} J(t, u(t)) \right)^q dt < \infty. \tag{45.34}$$

$$K(t, a) = K\left(t, \int_0^\infty u(s) \frac{ds}{s}\right) \leq \int_0^\infty K(t, u(s)) \frac{ds}{s}.$$

Now by 45.26, this is dominated by an expression of the form

$$\leq \int_0^\infty \min\left(1, \frac{t}{s}\right) J(s, u(s)) \frac{ds}{s} = \int_0^\infty \min\left(1, \frac{1}{s}\right) J(ts, u(ts)) \frac{ds}{s}$$

where the equation follows from a change of variable. From Minkowski's inequality,

$$\begin{aligned} ||a||_{\theta,q} &\equiv \left(\int_0^\infty \left(t^{-\theta} K(t, a) \right)^q \frac{dt}{t} \right)^{1/q} \\ &\leq \left(\int_0^\infty \left(t^{-\theta} \int_0^\infty \min\left(1, \frac{1}{s}\right) J(ts, u(ts)) \frac{ds}{s} \right)^q \frac{dt}{t} \right)^{1/q} \end{aligned}$$

$$\leq \int_0^\infty \left(\int_0^\infty \left(t^{-\theta} \min\left(1, \frac{1}{s}\right) J(ts, u(ts)) \right)^q \frac{dt}{t} \right)^{1/q} \frac{ds}{s}.$$

Now change the variable in the inside integral to obtain, letting $t = \tau s$,

$$\leq \int_0^\infty \min\left(1, \frac{1}{s}\right) \left(\int_0^\infty \left(t^{-\theta} J(ts, u(ts)) \right)^q \frac{dt}{t} \right)^{1/q} \frac{ds}{s}$$

$$\begin{aligned} &= \int_0^\infty \min\left(1, \frac{1}{s}\right) s^\theta \frac{ds}{s} \left(\int_0^\infty \left(\tau^{-\theta} J(\tau, u(\tau)) \right)^q \frac{d\tau}{\tau} \right)^{1/q} \\ &= \left(\frac{1}{(1-\theta)\theta} \right) \left(\int_0^\infty \left(\tau^{-\theta} J(\tau, u(\tau)) \right)^q \frac{d\tau}{\tau} \right)^{1/q} . \end{aligned}$$

This has shown that

$$||a||_{\theta,q} \leq \left(\frac{1}{(1-\theta)\theta} \right) \left(\int_0^\infty \left(\tau^{-\theta} J(\tau, u(\tau)) \right)^q \frac{d\tau}{\tau} \right)^{1/q} < \infty$$

for all u satisfying 45.33 and 45.34. Therefore, taking the infimum it follows $a \in (A_0, A_1)_{\theta,q}$ and

$$||a||_{\theta,q} \leq \left(\frac{1}{(1-\theta)\,\theta}\right) ||a||_{\theta,q,J}\,.$$

This proves the theorem.

45.4 Duality And Interpolation

In this section it will be assumed that $A_0 \cap A_1$ is dense in A_i for $i = 0, 1$. This is done so that $A_i' \subseteq (A_0 \cap A_1)'$ and the inclusion map is continuous. Thus it makes sense to add something in A_0' to something in A_1'.

What is the dual space of $(A_0, A_1)_{\theta,q}$? The answer is based on the following

lemma, [8]. Remember that

$$J(t,a) = \max\left(||a||_{A_0}, t\,||a||_{A_1}\right)$$

and this is a norm on $A_0 \cap A_1$ and

$$K(t,a) = \inf\left\{||a_0||_{A_0} + t\,||a_1||_{A_1} : a = a_0 + a_1\right\}.$$

As mentioned above, $A_0' + A_1' \subseteq (A_0 \cap A_1)'$. In fact these two are equal. This is the first part of the following lemma.

Lemma 45.28 *Suppose $A_0 \cap A_1$ is dense in A_i, $i = 0, 1$. Then*

$$A_0' + A_1' = (A_0 \cap A_1)', \tag{45.35}$$

and for $a' \in A_0' + A_1' = (A_0 \cap A_1)'$,

$$K(t,a') = \sup_{a \in A_0 \cap A_1} \frac{|a'(a)|}{J(t^{-1}, a)}. \tag{45.36}$$

Thus $K(t,\cdot)$ is an equivalent norm to the usual operator norm on $(A_0 \cap A_1)'$ taken with respect to $J\left(t^{-1},\cdot\right)$. If, in addition to this, A_i is reflexive, then for $a' \in A_0' \cap A_1'$, and $a \in A_0 \cap A_1$,

$$J(t,a')\,K\left(t^{-1},a\right) \geq |a'(a)|\,. \tag{45.37}$$

Proof: First consider the claim that $A_0' + A_1' = (A_0 \cap A_1)'$. As noted above, $\subseteq$ is clear. Define a norm on $A_0 \times A_1$ as follows.

$$||(a_0, a_1)||_{A_0 \times A_1} \equiv \max\left(||a_0||_{A_0}, t^{-1}\,||a_1||_{A_1}\right). \tag{45.38}$$

Let $a' \in (A_0 \cap A_1)'$. Let

$$E \equiv \{(a,a) : a \in A_0 \cap A_1\}$$

with the norm $J\left(t^{-1},a\right)\equiv\max\left(||a||_{A_0},t^{-1}||a||_{A_1}\right)$. Now define λ on E, the subspace of $A_0\times A_1$ by

$$\lambda\left((a,a)\right)\equiv a'\left(a\right).$$

Thus λ is a continuous linear map on E and in fact,

$$\left|\lambda\left((a,a)\right)\right|=\left|a'\left(a\right)\right|\leq||a'||\,J\left(t^{-1},a\right).$$

By the Hahn Banach theorem there exists an extension of λ to all of $A_0\times A_1$. This extension is of the form $(a_0',a_1')\in A_0'\times A_1'$. Thus

$$\left(a_0',a_1'\right)\left((a,a)\right)=a_0'\left(a\right)+a_1'\left(a\right)=a'\left(a\right)$$

and therefore, $a_0'+a_1'=a'$ provided $a_0'+a_1'$ is continuous. But

$$\begin{aligned}\left|\left(a_0'+a_1'\right)(a)\right| &= \left|a_0'\left(a\right)+a_1'\left(a\right)\right|\leq\left|a_0'\left(a\right)\right|+\left|a_1'\left(a\right)\right|\\ &\leq ||a_0'||\,||a||_{A_0}+||a_1'||\,||a||_{A_1}\\ &\leq ||a_0'||\,||a||_{A_0}+t\,||a_1'||\,t^{-1}\,||a||_{A_1}\\ &\leq \left(||a_0'||+t\,||a_1'||\right)J\left(t^{-1},a\right)\end{aligned}$$

which shows that $a_0'+a_1'$ is continuous and in fact

$$||a_0'+a_1'||_{(A_0\cap A_1)'}\leq\left(||a_0'||+t\,||a_1'||\right).$$

This proves the first part of the lemma.

Claim: With this definition of the norm in 45.38, the operator norm of $(a_0',a_1')\in(A_0\times A_1)'=A_0'\times A_1'$ is

$$||(a_0',a_1')||_{(A_0\times A_1)'}=||a_0'||_{A_0'}+t\,||a_1'||_{A_1'}. \tag{45.39}$$

Proof of the claim: $\left|\left(a_0',a_1'\right)(a_0,a_1)\right|\leq||a_0'||\,||a_0||+||a_1'||\,||a_1||$. Now suppose that $||a_0||=\max\left(||a_0||,t^{-1}||a_1||\right)$. Then this is no larger than

$$\left(||a_0'||+t\,||a_1'||\right)||a_0||=\left(||a_0'||+t\,||a_1'||\right)\max\left(||a_0||,t^{-1}||a_1||\right).$$

The other case is that $t^{-1}||a_1||=\max\left(||a_0||,t^{-1}||a_1||\right)$. In this case,

$$\begin{aligned}\left|\left(a_0',a_1'\right)(a_0,a_1)\right| &\leq ||a_0'||\,||a_0||+||a_1'||\,||a_1||\\ &\leq ||a_0'||\,t^{-1}\,||a_1||+||a_1'||\,||a_1||\\ &= \left(||a_0'||+t\,||a_1'||\right)t^{-1}\,||a_1||\\ &= \left(||a_0'||+t\,||a_1'||\right)\max\left(||a_0||,t^{-1}||a_1||\right).\end{aligned}$$

This shows $||(a_0',a_1')||_{(A_0\times A_1)'}\leq\left(||a_0'||+t\,||a_1'||\right)$. Is equality achieved? Let a_{0n} and a_{1n} be points of A_0 and A_1 respectively such that $||a_{0n}||,||a_{1n}||\leq 1$ and $\lim_{n\to\infty}a_i'\left(a_{in}\right)=||a_i'||$. Then

$$\left(a_0',a_1'\right)\left(a_{0n},ta_{1n}\right)\to||a_0'||+t\,||a_1'||$$

and also, $||(a_{0n}, ta_{1n})||_{A_0\times A_1} = \max\left(||a_{0n}||, t^{-1}t\,||a_{1n}||_{A_1}\right) \leq 1$. Therefore, equality is indeed achieved and this proves the claim.

Consider 45.36. Take $a' \in A_0' + A_1' = (A_0 \cap A_1)'$ and let

$$E \equiv \{(a, a) \in A_0 \times A_1 : a \in A_0 \cap A_1\}.$$

Now define a linear map, λ on E as before.

$$\lambda((a, a)) \equiv a'(a).$$

If $a' = \widetilde{a}_0' + \widetilde{a}_1'$,

$$\begin{aligned} |\lambda((a,a))| &\leq ||\widetilde{a}_0'||_{A_0'}\,||a||_{A_0} + ||\widetilde{a}_1'||_{A_1'}\,||a||_{A_1} \\ &= ||\widetilde{a}_0'||_{A_0'}\,||a||_{A_0} + t\,||\widetilde{a}_1'||_{A_1'}\,t^{-1}\,||a||_{A_1} \\ &\leq (||\widetilde{a}_0'|| + t\,||\widetilde{a}_1'||)\,||(a,a)||_{A_0\times A_1} \end{aligned}$$

so λ is continuous on the subspace, E of $A_0 \times A_1$ and

$$||\lambda||_{E'} \leq ||\widetilde{a}_0'|| + t\,||\widetilde{a}_1'||. \tag{45.40}$$

By the Hahn Banach theorem, there exists an extension of λ defined on all of $A_0 \times A_1$ with the same norm. Thus, from 45.39, there exists $(a_0', a_1') \in (A_0 \times A_1)'$ which is an extension of λ such that

$$||(a_0', a_1')||_{(A_0\times A_1)'} = ||a_0'||_{A_0'} + t\,||a_1'||_{A_1'} = ||\lambda||_{E'}$$

and for all $a \in A_0 \cap A_1$,

$$a_0'(a) + a_1'(a) = \lambda((a,a)) = a'(a).$$

It follows that $a_0' + a_1' = a'$ in $(A_0 \cap A_1)'$. Therefore, from 45.40,

$$\begin{aligned} ||\lambda||_{E'} &\leq \inf\left\{||\widetilde{a}_0'||_{A_0} + t\,||\widetilde{a}_1'||_{A_1} : a' = \widetilde{a}_0' + \widetilde{a}_1'\right\} \equiv K(t, a') && (45.41) \\ &\leq ||a_0'||_{A_0'} + t\,||a_1'||_{A_1'} = ||\lambda||_{E'} \equiv \sup_{a\in A_0\cap A_1} \frac{|a'(a)|}{J(t^{-1}, a)} && (45.42) \end{aligned}$$

because on E, $J\left(t^{-1}, a\right) = ||(a,a)||_{A_0\times A_1}$ which proves 45.36.

To obtain 45.37 in the case that A_i is reflexive, apply 45.36 to the case where A_i'' plays the role of A_i in 45.36. Thus, for $a'' \in A_0'' + A_1''$,

$$K(t, a'') = \sup_{a'\in A_0'\cap A_1'} \frac{|a''(a')|}{J(t^{-1}, a')}.$$

Now $a'' = a_1'' + a_0'' = \eta_1 a_1 + \eta_0 a_0$ where η_i is the map from A_i to A_i'' which is onto and preserves norms, given by $\eta a(a') \equiv a'(a)$. Therefore, letting $a_1 + a_0 = a$

$$\begin{aligned} K(t, a) &= K(t, a'') = \sup_{a'\in A_0'\cap A_1'} \frac{|a''(a')|}{J(t^{-1}, a')} \\ &= \sup_{a'\in A_0'\cap A_1'} \frac{|(\eta_1 a_1 + \eta_0 a_0)(a')|}{J(t^{-1}, a')} = \sup_{a'\in A_0'\cap A_1'} \frac{|(a'(a_1 + a_0))|}{J(t^{-1}, a')} \end{aligned}$$

and so

$$K(t,a) = \sup_{a'\in A_0'\cap A_1'} \frac{|a'(a)|}{J(t^{-1},a')}$$

Changing $t \to t^{-1}$,

$$K\left(t^{-1},a\right) J(t,a') \geq |a'(a)|.$$

which proves the lemma.

Consider $(A_0,A_1)'_{\theta,q}$.

Definition 45.29 *Let $q \geq 1$. Then $\lambda^{\theta,q}$ will denote the sequences, $\{\alpha_i\}_{i=-\infty}^{\infty}$ such that*

$$\sum_{i=-\infty}^{\infty} \left(|\alpha_i| 2^{-i\theta}\right)^q < \infty.$$

For $\alpha \in \lambda^{\theta,q}$,

$$||\alpha||_{\lambda^{\theta,q}} \equiv \left(\sum_{i=-\infty}^{\infty} \left(|\alpha_i| 2^{-i\theta}\right)^q\right)^{1/q}.$$

Thus $\alpha \in \lambda^{\theta,q}$ means $\left\{\alpha_i 2^{-i\theta}\right\} \in l_q$.

Lemma 45.30 *Let $f(t) \geq 0$, and let $f(t) = \alpha_i$ for $t \in [2^i, 2^{i+1})$ where $\alpha \in \lambda^{\theta,q}$. Then there exists a constant, C, such that*

$$\left|\left|t^{-\theta} f\right|\right|_{L^q\left(0,\infty;\frac{dt}{t}\right)} \leq C\,||\alpha||_{\lambda^{\theta,q}}. \tag{45.43}$$

Also, if whenever $\alpha \in \lambda^{\theta,q}$, and $\alpha_i \geq 0$ for all i,

$$\sum_i f\left(2^i\right) 2^{-i}\alpha_i \leq C\,||\alpha||_{\lambda^{\theta,q}}, \tag{45.44}$$

then

$$\left|\left|\left\{f\left(2^i\right)\right\}_{i=-\infty}^{\infty}\right|\right|_{\lambda^{1-\theta,q'}} \leq C. \tag{45.45}$$

Proof: Consider 45.43.

$$\int_0^\infty \left(t^{-\theta} f(t)\right)^q \frac{dt}{t} = \sum_i \int_{2^i}^{2^{i+1}} t^{-\theta q}\alpha_i^q \frac{dt}{t}$$

$$\leq \sum_i \int_{2^i}^{2^{i+1}} \left(2^{-i\theta}\alpha_i\right)^q \frac{dt}{t} = \ln 2 \sum_i \left(2^{-i\theta}\alpha_i\right)^q = \ln 2\,||\alpha||_{\lambda^{\theta,q}}^q.$$

45.45 is next. By 45.44, whenever $\alpha \in \lambda^{\theta,q}$,

$$\left|\sum_i \left(f\left(2^i\right) 2^{-(1-\theta)i}\right) 2^{-\theta i}\alpha_i\right| \leq C\left|\left|\left\{2^{-\theta i}|\alpha_i|\right\}\right|\right|_{l_q}.$$

It follows from the Riesz representation theorem that $\left\{f\left(2^{i}\right)2^{-(1-\theta)i}\right\}$ is in $l_{q'}$ and

$$\left|\left|\left\{f\left(2^{i}\right)2^{-(1-\theta)i}\right\}\right|\right|_{l_{q'}}=\left|\left|\left\{f\left(2^{i}\right)\right\}\right|\right|_{\lambda^{1-\theta,q'}}\leq C.$$

This proves the lemma.

The dual space of $(A_0,A_1)_{\theta,q,J}$ is discussed next.

Lemma 45.31 *Let $\theta\in(0,1)$ and let $q\geq 1$. Then,*

$$(A_0,A_1)'_{\theta,q,J}\subseteq(A'_1,A'_0)_{1-\theta,q'}$$

and the inclusion map is continuous.

Proof: Let $a'\in(A_0,A_1)'_{\theta,q,J}$. Now

$$A_0\cap A_1\subseteq(A_0,A_1)_{\theta,q,J}$$

and if

$$a\in(A_0,A_1)_{\theta,q,J},$$

then a has a representation of the form

$$a=\int_0^\infty u(t)\frac{dt}{t}$$

where

$$\int_0^\infty\left(t^{-\theta}J(t,u(t))\right)^q\frac{dt}{t}<\infty$$

where

$$J(t,u(t))=\max\left(||u(t)||_{A_0},t\,||u(t)||_{A_1}\right)$$

for $u(t)\in A_0\cap A_1$. Now let

$$u_r(t)\equiv\begin{cases}u(t)\text{ if }t\in\left(\frac{1}{r},r\right)\\ 0\text{ otherwise}\end{cases}.$$

Then $\int_0^\infty\left(t^{-\theta}J(t,u_r(t))\right)^q\frac{dt}{t}<\infty$ and

$$a_r\equiv\int_0^\infty u_r(t)\frac{dt}{t}\in A_0\cap A_1$$

by Lemma 45.24. Also

$$||a-a_r||^q_{\theta,q,J}\leq\int_0^{\frac{1}{r}}\left(t^{-\theta}J(t,u(t))\right)^q\frac{dt}{t}+$$

$$\int_r^\infty\left(t^{-\theta}J(t,u(t))\right)^q\frac{dt}{t}$$

which is small whenever r is large enough thanks to the dominated convergence theorem. Therefore, $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta,q,J}$ and so

$$(A_0, A_1)'_{\theta,q,J} \subseteq (A_0 \cap A_1)' = A'_0 + A'_1,$$

the equality following from Lemma 45.28.

It follows $a' \in A'_0 + A'_1$ and so, by Lemma 45.28, there exists $b_i \in A_0 \cap A_1$ such that

$$K\left(2^{-i}, a', A'_0, A'_1\right) - \varepsilon \min\left(1, 2^{-i}\right) \leq \frac{a'(b_i)}{J(2^i, b_i, A_0, A_1)}.$$

Now let $\alpha \in \lambda^{\theta,q}$ with $\alpha_i \geq 0$ for all i and let

$$a_\infty \equiv \sum_i J\left(2^i, b_i, A_0, A_1\right)^{-1} b_i \alpha_i. \tag{45.46}$$

Consider first whether a_∞ makes sense before proceeding further.

$$a_\infty \equiv \sum_i \frac{b_i 2^{i\theta}}{\max\left(||b_i||_{A_0}, 2^i ||b_i||_{A_1}\right)} 2^{-i\theta}\alpha_i.$$

Now

$$\left|\left|\frac{b_i 2^{i\theta}}{\max\left(||b_i||_{A_0}, 2^i ||b_i||_{A_1}\right)}\right|\right|_{A_0+A_1} \leq \begin{cases} 2^{i\theta} \text{ if } i < 0 \\ 2^{-i(1-\theta)} \text{ if } i \geq 0 \end{cases}. \tag{45.47}$$

This is fairly routine to verify. Consider the case where $i \geq 0$. Then

$$\left|\left|\frac{b_i 2^{i\theta}}{\max\left(||b_i||_{A_0}, 2^i ||b_i||_{A_1}\right)}\right|\right|_{A_0+A_1} \leq \left|\left|\frac{b_i 2^{i\theta}}{2^i ||b_i||_{A_1}}\right|\right|_{A_0+A_1} \leq 2^{-i(1-\theta)}$$

because $||b_i||_{A_1} \geq ||b_i||_{A_0+A_1}$. Therefore,

$$\sum_{i=0}^{M} \left|\left|\frac{b_i 2^{i\theta}}{\max\left(||b_i||_{A_0}, 2^i ||b_i||_{A_1}\right)} 2^{-i\theta}\alpha_i\right|\right|_{A_0+A_1} \leq$$

$$\sum_{i=0}^{M} 2^{-i(1-\theta)} 2^{-i\theta}\alpha_i \leq \left(\sum_{i=0}^{\infty} 2^{-i(1-\theta)q'}\right)^{1/q'} \left(\sum_{i=0}^{\infty} 2^{-iq\theta}\alpha_i^q\right)^{1/q} < \infty$$

and similarly,

$$\sum_{i=-\infty}^{0} \left|\left|\frac{b_i 2^{i\theta}}{\max\left(||b_i||_{A_0}, 2^i ||b_i||_{A_1}\right)} 2^{-i\theta}\alpha_i\right|\right|_{A_0+A_1}$$

converges. Therefore, a_∞ makes sense in $A_0 + A_1$ and also from 45.47, we see that

$$\left\{\frac{||b_i||_{A_0+A_1} 2^{i\theta}}{J(2^i, b_i)}\right\} \in \lambda^{(1-\theta)q'}$$

Now let

$$u(t) \equiv \frac{\alpha_i b_i}{J(2^i, b_i)\ln 2} \text{ on } [2^{i-1}, 2^i).$$

Then

$$\begin{aligned}\int_0^\infty u(t)\frac{dt}{t} &= \sum_i \int_{2^{i-1}}^{2^i} \frac{\alpha_i b_i}{J(2^i, b_i)\ln 2}\frac{dt}{t} \\ &= \sum_i \frac{\alpha_i b_i}{J(2^i, b_i)} = a_\infty.\end{aligned}$$

Also

$$\int_0^\infty \left(t^{-\theta} J(t, u(t))\right)^q \frac{dt}{t} \leq \sum_i \int_{2^{i-1}}^{2^i} \left(2^{(1-i)\theta} J\left(2^i, u\left(2^{i-1}\right)\right)\right) \frac{dt}{t}$$

$$\leq \sum_i \left[2^{-(i-1)\theta} J\left(2^i, u\left(2^{i-1}\right)\right)\right]^q \ln 2$$

$$= \sum_i \left[2^{-(i-1)\theta} \frac{J\left(2^i, b_i\right)\alpha_i}{J(2^i, b_i)\ln 2}\right]^q \ln 2$$

$$= C\sum_i \left(2^{-i\theta}|\alpha_i|\right)^q < \infty \tag{45.48}$$

and so $||a_\infty||_{\theta,q,J} < \infty$. Now for a' as above, $a' \in (A_0, A_1)'_{\theta,q,J} \subseteq (A_0 + A_1)'$, and so since the sum for a_∞ converges in $A_0 + A_1$, we have

$$a'(a_\infty) = \sum_i J\left(2^i, b_i\right)^{-1} \alpha_i a'(b_i).$$

Therefore,

$$\begin{aligned}a'(a_\infty) &\geq \sum_i \left[K\left(2^{-i}, a'\right) - \varepsilon \min\left(1, 2^{-i}\right)\right]\alpha_i \\ &= \sum_i K\left(2^{-i}, a'\right)\alpha_i - \sum_i \varepsilon \min\left(1, 2^{-i}\right)\alpha_i \\ &= \sum_i K\left(2^{-i}, a'\right)\alpha_i - O(\varepsilon) \end{aligned} \tag{45.49}$$

The reason for this is that $\alpha \in \lambda^{\theta,q}$ so $\left\{\alpha_i 2^{-i\theta}\right\} \in l_q$. Therefore,

$$\begin{aligned}
&\sum_i \varepsilon \min\left(1, 2^{-i}\right)\alpha_i \\
= \; & \varepsilon\left\{\sum_{i=0}^{\infty} 2^{-i}\alpha_i + \sum_{i=-\infty}^{-1}\alpha_i\right\} \\
= \; & \varepsilon\left\{\sum_{i=0}^{\infty} 2^{-i\theta}2^{(\theta-1)i}\alpha_i + \sum_{i=-\infty}^{-1}\alpha_i 2^{-i\theta}2^{i\theta}\right\} \\
\leq \; & \varepsilon\left\{\left(\sum_i \left|\alpha_i 2^{-i\theta}\right|^q\right)^{1/q}\left(\sum_{i=0}^{\infty}\left(2^{(\theta-1)i}\right)^{q'}\right)^{1/q'}\right. \\
& \left. + \left(\sum_i \left|\alpha_i 2^{-i\theta}\right|^q\right)^{1/q}\left(\sum_{i=0}^{\infty}\left(2^{\theta i}\right)^{q'}\right)^{1/q'}\right\} \\
< \; & C\varepsilon.
\end{aligned}$$

Also

$$\left|a'\left(a_\infty\right)\right| \leq \left|\left|a'\right|\right|_{(A_0,A_1)'_{\theta,q,J}}\left|\left|a_\infty\right|\right|_{(A_0,A_1)_{\theta,q,J}}.$$

Now from the definition of K,

$$K\left(2^{-i}, a', A_0', A_1'\right) = 2^{-i}K\left(2^i, a', A_1', A_0'\right)$$

and so from 45.49

$$\begin{aligned}
\sum_i 2^{-i}K\left(2^i, a', A_1', A_0'\right)\alpha_i - O\left(\varepsilon\right) &\leq a'\left(a_\infty\right) \\
&\leq \left|\left|a'\right|\right|_{(A_0,A_1)'_{\theta,q,J}} C_\theta \left|\left|\alpha\right|\right|_{\lambda^{\theta,q}}.
\end{aligned}$$

Since ε is arbitrary, it follows that whenever, $\alpha \in \lambda^{\theta,q}, \alpha_i \geq 0$,

$$\sum_i 2^{-i}K\left(2^i, a', A_1', A_0'\right)\alpha_i \leq \left|\left|a'\right|\right|_{(A_0,A_1)'_{\theta,q,J}} C_\theta \left|\left|\alpha\right|\right|_{\lambda^{\theta,q}}.$$

By Lemma 45.30, $\left\{K\left(2^i, a', A_1', A_0'\right)\right\} \in \lambda^{1-\theta,q'}$ and

$$\left|\left|\left\{K\left(2^i, a', A_1', A_0'\right)\right\}\right|\right|_{\lambda^{1-\theta,q'}} \leq \left|\left|a'\right|\right|_{(A_0,A_1)'_{\theta,q,J}} C_\theta.$$

Therefore,

$$\left(\frac{1}{\ln 2}\int_0^\infty \left(K\left(t,a',A_1',A_0'\right)t^{-(1-\theta)}\right)^{q'}\frac{dt}{t}\right)^{1/q'}$$
$$= \left(\sum_i \frac{1}{\ln 2}\int_{2^i}^{2^{i+1}} \left(K\left(t,a',A_1',A_0'\right)t^{-(1-\theta)}\right)^{q'}\frac{dt}{t}\right)^{1/q'}$$
$$\leq \left(\sum_i \left(2^{-i(1-\theta)}K\left(2^i,a',A_1',A_0'\right)\right)^{q'}\right)^{1/q'}$$
$$\leq \|a'\|_{(A_0,A_1)'_{\theta,q,J}} C_\theta.$$

Thus

$$\|a'\|_{\left(A_1',A_0'\right)_{1-\theta,q'}} \equiv \left\|t^{-(1-\theta)}K\left(t,a',A_1',A_0'\right)\right\|_{L^{q'}\left(0,\infty,\frac{dt}{t}\right)} \leq C\,\|a'\|_{(A_0,A_1)'_{\theta,q,J}}$$

which shows that $(A_0,A_1)'_{\theta,q,J} \subseteq (A_1',A_0')_{1-\theta,q'}$ with the inclusion map continuous. This proves the lemma.

Lemma 45.32 *If A_i is reflexive for $i=0,1$ and if $A_0\cap A_1$ is dense in A_i, then*

$$(A_1',A_0')_{1-\theta,q',J} \subseteq (A_0,A_1)'_{\theta,q}$$

and the inclusion map is continuous.

Proof: Let $a' \in (A_1',A_0')_{1-\theta,q',J}$. Thus, there exists u^* bounded on compact subsets of $(0,\infty)$ and measurable with values in $A_0\cap A_1$ and

$$a' = \int_0^\infty u^*(t)\,\frac{dt}{t}, \tag{45.50}$$

$$\int_0^\infty \left(t^{-(1-\theta)}J\left(t,u^*(t)\right)\right)^{q'}\frac{dt}{t} < \infty.$$

Then

$$a' = \sum_{i=-\infty}^{\infty}\int_{2^i}^{2^{i+1}} u^*(t)\,\frac{dt}{t} \equiv \sum_{i=-\infty}^{\infty} a_i'$$

where $a_i' \in A_1'\cap A_0'$, the convergence taking place in $A_1'+A_0'$. Now let $a\in A_0\cap A_1$.

From Lemma 45.28

$$\begin{aligned}
|a'(a)| &\leq \sum_{i=-\infty}^{\infty} |a_i'(a)| \\
&\leq \sum_{i=-\infty}^{\infty} J\left(2^{-i}, a_i', A_0', A_1'\right) K\left(2^i, a, A_0, A_1\right) \\
&= \sum_{i=-\infty}^{\infty} 2^{-i} J\left(2^i, a_i', A_1', A_0'\right) K\left(2^i, a, A_0, A_1\right) \\
&\leq \left(\sum_i \left(2^{-(1-\theta)i} J\left(2^i, a_i', A_1', A_0'\right)\right)^{q'}\right)^{1/q'} \cdot \\
&\quad \left(\sum_i \left(2^{-\theta i} K\left(2^i, a, A_0, A_1\right)\right)^q\right)^{1/q} \\
&\leq C\left[\int_0^\infty \left(t^{-(1-\theta)} J\left(t, u^*(t), A_1', A_0'\right)\right)^{q'} \frac{dt}{t}\right]^{1/q'} \cdot \\
&\quad \left[\int_0^\infty \left(t^{-\theta} K\left(t, a, A_0, A_1\right)\right)^q \frac{dt}{t}\right]^{1/q}.
\end{aligned}$$

In going from the sums to the integrals, express the first sum as a sum of integrals on $[2^i, 2^{i+1})$ and the second sum as a sum of integrals on $(2^{i-1}, 2^i]$.

Taking the infimum over all u^* representing a',

$$|a'(a)| \leq C\, ||a'||_{\left(A_1', A_0'\right)_{1-\theta, q', J}} ||a||_{\theta, q}.$$

It follows $a' \in (A_0, A_1)'_{\theta,q}$ and $||a'||_{(A_0,A_1)'_{\theta,q}} \leq C\, ||a'||_{\left(A_1', A_0'\right)_{1-\theta,q',J}}$ which proves the lemma.

With these two lemmas the main result follows.

Theorem 45.33 *Suppose $A_0 \cap A_1$ is dense in A_i and A_i is reflexive. Then*

$$(A_1', A_0')_{1-\theta, q'} = (A_0, A_1)'_{\theta, q}$$

and the norms are equivalent.

Proof: By Theorem 45.27, and the last two lemmas,

$$\begin{aligned}
(A_0, A_1)'_{\theta,q} &= (A_0, A_1)'_{\theta,q,J} \subseteq (A_1', A_0')_{1-\theta,q'} \\
&= (A_1', A_0')_{1-\theta,q',J} \subseteq (A_0, A_1)'_{\theta,q}.
\end{aligned}$$

This proves the theorem.

Trace Spaces

46.1 Definition And Basic Theory Of Trace Spaces

As in the case of interpolation spaces, suppose A_0 and A_1 are two Banach spaces which are continuously embedded in some topological vector space, X.

Definition 46.1 *Define a norm on* $A_0 + A_1$ *as follows.*

$$||a||_{A_0+A_1} \equiv \inf \left\{ ||a_0||_{A_0} + ||a_1||_{A_1} : a_0 + a_1 = a \right\} \tag{46.1}$$

Lemma 46.2 $A_0 + A_1$ *with the norm just described is a Banach space.*

Proof: This was already explained in the treatment of the K method of interpolation. It is just $K(1, a)$.

Definition 46.3 *Take* f' *in the sense of distributions for any*

$$f \in L^1_{loc}(0, \infty; A_0 + A_1)$$

as follows.

$$f'(\phi) \equiv \int_0^\infty -f(t)\,\phi'(t)\,dt$$

whenever $\phi \in C_c^\infty(0, \infty)$. *Define a Banach space,* $W(A_0, A_1, p, \theta) = W$ *where* $p \geq 1, \theta \in (0, 1)$. *Let*

$$||f||_W \equiv \max \left(\left|\left|t^\theta f\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_0\right)}, \left|\left|t^\theta f'\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_1\right)} \right) \tag{46.2}$$

and let W *consist of* $f \in L^1_{loc}(0, \infty; A_0 + A_1)$ *such that* $||f||_W < \infty$.

Note that to be in W, $f(t) \in A_0$ and $f'(t) \in A_1$.

Lemma 46.4 *If* $f \in W$, *then*

$$Trace(f) \equiv f(0) \equiv \lim_{t \to 0} f(t)$$

exists in $A_0 + A_1$. *Also* $Z \equiv \{f \in W : f(0) = 0\}$ *is a closed subspace of* W.

Proof: Let $0 < s < t$. Let $\nu + \frac{1}{p} = \theta$. Then

$$\int_0^\infty ||\tau^\nu g(\tau)||^p \, d\tau = \int_0^\infty \left|\left|\tau^\theta g(\tau)\right|\right|^p \frac{d\tau}{\tau}$$

so that $t^\nu f' \in L^p(0,\infty;A_1)$, the measure in this case being usual Lebesgue measure. Then

$$f(t) - f(s) = \int_s^t f'(\tau)\, d\tau = \int_s^t \tau^\nu f'(\tau)\, \tau^{-\nu} d\tau.$$

For $\frac{1}{p} + \frac{1}{p'} = 1$, $\nu p' = \left(\theta - \frac{1}{p}\right) p' < 1$ because $\theta < 1 = \frac{1}{p'} + \frac{1}{p}$. Therefore,

$$\begin{aligned}
&||f(t) - f(s)||_{A_0+A_1} \\
\leq\;& \int_s^t ||f'(\tau)||_{A_0+A_1} \, d\tau && (46.3) \\
\leq\;& \int_s^t ||f'(\tau)||_{A_1} \, d\tau = \int_s^t ||\tau^\nu f'(\tau)||_{A_1} \tau^{-\nu} d\tau \\
\leq\;& \left(\int_s^t ||\tau^\nu f'(\tau)||_{A_1}^p \, d\tau\right)^{1/p} \left(\int_s^t \tau^{-\nu p'} d\tau\right)^{1/p'} \\
\leq\;& ||f||_W \left(\frac{t^{1-\nu p'}}{1-\nu p'} - \frac{s^{1-\nu p'}}{1-\nu p'}\right) && (46.4) \\
\leq\;& ||f||_W \frac{t^{1-\nu p'}}{1-\nu p'}.
\end{aligned}$$

which converges to 0 as $t \to 0$. This shows that $\lim_{t\to 0+} f(t)$ exists in $A_0 + A_1$.

Clearly Z is a subspace. Let $f_n \to f$ in W and suppose $f_n \in Z$. Then since $f \in W$, 46.4 implies f is continuous. Using 46.4 and replacing f with $f_n - f_m$ and then taking a limit as $s \to 0$,

$$||f_n(t) - f_m(t)||_{A_0+A_1} \leq ||f_n - f_m||_W \, C_\nu t^{1-\nu p'}$$

Taking a subsequence, it can be assumed $f_n(t)$ converges to $f(t)$ a.e. But the above inequality shows that $f_n(t)$ is a Cauchy sequence in $C([0,\beta];A_0+A_1)$ for all $\beta < \infty$. Therefore, $f_n(t) \to f(t)$ for all t. Also,

$$||f_n(t)||_{A_0+A_1} \leq C_\nu \, ||f_n||_W \, t^{1-\nu p'} \leq K t^{1-\nu p'}$$

for some K depending on $\max\{||f_n|| : n \geq 1\}$ and so

$$||f(t)||_{A_0+A_1} \leq K t^{1-\nu p'}$$

which implies $f(0) = 0$. Thus Z is closed.

Definition 46.5 *Let W be a Banach space and let Z be a closed subspace. Then the quotient space, denoted by W/Z consists of the set of equivalence classes $[x]$*

where the equivalence relation is defined by $x \backsim y$ *means* $x - y \in Z$. *Then* W/Z *is a vector space if the operations are defined by* $\alpha[x] \equiv [\alpha x]$ *and* $[x] + [y] \equiv [x + y]$ *and these vector space operations are well defined. The norm on the quotient space is defined as* $||[x]|| \equiv \inf\{||x + z|| : z \in Z\}$.

The verification of the algebraic claims made in the above definition is left to the reader. It is routine. What is not as routine is the following lemma. However, it is similar to some topics in the presentation of the K method of interpolation.

Lemma 46.6 *Let* W *be a Banach space and let* Z *be a closed subspace of* W. *Then* W/Z *with the norm described above is a Banach space.*

Proof: That W/Z is a vector space is left to the reader. Why is $||\cdot||$ a norm? Suppose $\alpha \neq 0$. Then

$$\begin{aligned} ||\alpha[x]|| &= ||[\alpha x]|| \equiv \inf\{||\alpha x + z|| : z \in Z\} \\ &= \inf\{||\alpha x + \alpha z|| : z \in Z\} \\ &= |\alpha| \inf\{||x + z|| : z \in Z\} = |\alpha| \, ||[x]||. \end{aligned}$$

Now let $||[x]|| \geq ||x + z_1|| - \varepsilon$ and let $||[y]|| \geq ||y + z_2|| - \varepsilon$ where $z_i \in Z$. Then

$$\begin{aligned} ||[x] + [y]|| &\equiv ||[x + y]|| \leq ||x + y + z_1 + z_2|| \\ &\leq ||x + z_1|| + ||y + z_2|| \leq ||[x]|| + ||[y]|| + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, this shows the triangle inequality. Clearly, $||[x]|| \geq 0$. It remains to show that the only way $||[x]|| = 0$ is for $x \in Z$. Suppose then that $||[x]|| = 0$. This means there exist $z_n \in Z$ such that $||x + z_n|| \to 0$. Therefore, $-x$ is a limit of a sequence of points of Z and since Z is closed, this requires $-x \in Z$. Hence $x \in Z$ also because Z is a subspace. This shows $||\cdot||$ is a norm on W/Z. It remains to verify that W/Z is a Banach space.

Suppose $\{[x_n]\}$ is a Cauchy sequence in W/Z and suppose $||[x_n] - [x_{n+1}]|| < \frac{1}{2^{n+1}}$. Let $x_1' = x_1$. If x_n' has been chosen let $x_{n+1}' = x_{n+1} + z_{n+1}$ where $z_{n+1} \in Z$ be such that

$$\begin{aligned} \left|\left|x_{n+1}' - x_n'\right|\right| &\leq ||[x_{n+1} - x_n]|| + \frac{1}{2^{(n+1)}} \\ &= ||[x_{n+1}] - [x_n]|| + \frac{1}{2^{(n+1)}} < \frac{1}{2^n}. \end{aligned}$$

It follows $\{x_n'\}$ is a Cauchy sequence in W and so it must converge to some $x \in W$. Now

$$||[x] - [x_n]|| = ||[x - x_n]|| = ||[x - x_n']|| \leq ||x - x_n'||$$

which converges to 0. Now if $\{[x_n]\}$ is just a Cauchy sequence, there exists a subsequence satisfying $\left|\left|[x_{n_k}] - [x_{n_{k+1}}]\right|\right| < \frac{1}{2^{k+1}}$ and so from the first part, the subsequence converges to some $[x] \in W/Z$ and so the original Cauchy sequence also converges. therefore, W/Z is a Banach space as claimed.

Definition 46.7 *Define* $T(A_0, A_1, p, \theta) = T$, *to consist of*

$$\left\{a \in A_0 + A_1 : a = \lim_{t\to 0+} f(t) \text{ for some } f \in W(A_0, A_1, p, \theta)\right\},$$

the limit taking place in A_0+A_1. *Let* γf *be defined for* $f \in W$ *by* $\gamma f \equiv \lim_{t\to 0+} f(t)$. *Thus* $T = \gamma(W)$. *As above* $Z \equiv \{f \in W : \gamma f = 0\} = \ker(\gamma)$.

Lemma 46.8 T *is a Banach space with norm given by*

$$||a||_T \equiv \inf\{||f||_W : f(0) = a\}. \tag{46.5}$$

Proof: Define a mapping, $\psi : W/Z \to T$ by

$$\psi([f]) \equiv \gamma f.$$

Then ψ is one to one and onto. Also

$$||[f]|| \equiv \inf\{||f+g|| : g \in Z\} = \inf\{||h||_W : \gamma h = \gamma f\} = ||\gamma(f)||_T.$$

Therefore, the Banach space, W/Z and T are isometric and so T must be a Banach space since W/Z is.

The following is an important interpolation inequality.

Theorem 46.9 *If* $a \in T$, *then*

$$||a||_T = \inf\left\{||t^\theta f||^{1-\theta}_{L^p\left(0,\infty,\frac{dt}{t};A_0\right)} ||t^\theta f'||^{\theta}_{L^p\left(0,\infty,\frac{dt}{t};A_1\right)}\right\} \tag{46.6}$$

where the infimum is taken over all $f \in W$ *such that* $a = f(0)$. *Also, if* $a \in A_0 \cap A_1$, *then* $a \in T$ *and*

$$||a||_T \leq K ||a||^{1-\theta}_{A_1} ||a||^{\theta}_{A_0} \tag{46.7}$$

for some constant K. *Also*

$$A_0 \cap A_1 \subseteq T(A_0, A_1, p, \theta) \subseteq A_0 + A_1 \tag{46.8}$$

and the inclusion maps are continuous.

Proof: First suppose $f(0) = a$ where $f \in W$. Then letting $f_\lambda(t) \equiv f(\lambda t)$, it follows that $f_\lambda(0) = a$ also and so

$$\begin{aligned}
||a||_T &\leq \max\left(||t^\theta f_\lambda||_{L^p\left(0,\infty,\frac{dt}{t};A_0\right)}, ||t^\theta (f_\lambda)'||_{L^p\left(0,\infty,\frac{dt}{t};A_1\right)}\right) \\
&= \max\left(\lambda^{-\theta} ||t^\theta f||_{L^p\left(0,\infty,\frac{dt}{t};A_0\right)}, \lambda^{1-\theta} ||t^\theta f'||_{L^p\left(0,\infty,\frac{dt}{t};A_1\right)}\right) \\
&\equiv \max\left(\lambda^{-\theta} R, \lambda^{1-\theta} S\right).
\end{aligned}$$

Now choose $\lambda = R/S$ to obtain

$$||a||_T \leq R^{1-\theta} S^\theta = ||t^\theta f||^{1-\theta}_{L^p\left(0,\infty,\frac{dt}{t};A_0\right)} ||t^\theta f'||^{\theta}_{L^p\left(0,\infty,\frac{dt}{t};A_1\right)}.$$

Thus

$$||a||_T \leq \inf \left\{ \left|\left|t^\theta f\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_0\right)}^{1-\theta} \left|\left|t^\theta f'\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_1\right)}^{\theta} \right\}.$$

Next choose $f \in W$ such that $f(0) = a$ and $||f||_W \approx ||a||_T$. More precisely, pick $f \in W$ such that $f(0) = a$ and $||a||_T > -\varepsilon + ||f||_W$. Also let

$$R \equiv \left|\left|t^\theta f\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_0\right)}, S \equiv \left|\left|t^\theta f'\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_1\right)}.$$

Then as before,

$$\left|\left|t^\theta f_\lambda\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_0\right)} = \lambda^{-\theta} R,\ \left|\left|t^\theta (f_\lambda)'\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_1\right)} = \lambda^{1-\theta} S. \tag{46.9}$$

so that $||f||_W = \max(R,S)$. Then, changing the variables, letting $\lambda = R/S$,

$$\left|\left|t^\theta f_\lambda\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_0\right)} = \left|\left|t^\theta (f_\lambda)'\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_1\right)} = R^{1-\theta} S^\theta \tag{46.10}$$

Since $f_\lambda(0) = a$, $f_\lambda \in W$, and it is always the case that for positive R, S, $R^{1-\theta}S^\theta \leq \max(R,S)$, this shows that

$$\begin{aligned} ||a||_T &\leq \max\left(\left|\left|t^\theta f_\lambda\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_0\right)}, \left|\left|t^\theta (f_\lambda)'\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_1\right)} \right) \\ &= R^{1-\theta} S^\theta \leq \max(R,S) = ||f||_W < ||a||_T + \varepsilon, \end{aligned}$$

the first inequality holding because $||a||_T$ is the infimum of such things on the right. This shows 46.6.

It remains to verify 46.7. To do this, let $\psi \in C^\infty([0,\infty))$, with $\psi(0) = 1$ and $\psi(t) = 0$ for all $t > 1$. Then consider the special $f \in W$ which is given by $f(t) \equiv a\psi(t)$ where $a \in A_0 \cap A_1$. Thus $f \in W$ and $f(0) = a$ so $a \in T(A_0, A_1, p, \theta)$. From the first part, there exists a constant, K such that

$$\begin{aligned} ||a||_T &\leq \left|\left|t^\theta f\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_0\right)}^{1-\theta} \left|\left|t^\theta f'\right|\right|_{L^p\left(0,\infty,\frac{dt}{t};A_1\right)}^{\theta} \\ &\leq K ||a||_{A_0}^{1-\theta} ||a||_{A_1}^{\theta} \end{aligned}$$

This shows 46.7 and the first inclusion in 46.8. From the inequality just obtained,

$$\begin{aligned} ||a||_T &\leq K\left((1-\theta)||a||_{A_0} + \theta ||a||_{A_1}\right) \\ &\leq K ||a||_{A_0 \cap A_1}. \end{aligned}$$

This shows the first inclusion map of 46.8 is continuous.

Now take $a \in T$. Let $f \in W$ be such that $a = f(0)$ and

$$||a||_T + \varepsilon > ||f||_W \geq ||a||_T.$$

By 46.4,

$$||a - f(t)||_{A_0 + A_1} \leq C_\nu t^{1-\nu p'} ||f||_W$$

where $\frac{1}{p} + \nu = \theta$, and so

$$||a||_{A_0+A_1} \leq ||f(t)||_{A_0+A_1} + C_\nu t^{1-\nu p'} ||f||_W .$$

Now $||f(t)||_{A_0+A_1} \leq ||f(t)||_{A_0}$.

$$\begin{aligned} ||a||_{A_0+A_1} &\leq t^\nu ||f(t)||_{A_0+A_1} t^{-\nu} + C_\nu t^{1-\nu p'} ||f||_W \\ &\leq t^\nu ||f(t)||_{A_0} t^{-\nu} + C_\nu t^{1-\nu p'} ||f||_W \end{aligned}$$

Therefore, recalling that $\nu p' < 1$, and integrating both sides from 0 to 1,

$$||a||_{A_0+A_1} \leq C_\nu ||f||_W \leq C_\nu (||a||_T + \varepsilon) .$$

To see this,

$$\begin{aligned} \int_0^1 t^\nu ||f(t)||_{A_0} t^{-\nu} dt &\leq \left(\int_0^1 \left(t^\nu ||f(t)||_{A_0} \right)^p dt \right)^{1/p} \left(\int_0^1 t^{-\nu p'} dt \right)^{1/p'} \\ &\leq C ||f||_W . \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this verifies the second inclusion and continuity of the inclusion map completing the proof of the theorem.

The interpolation inequality, 46.7 is very significant. The next result concerns bounded linear transformations.

Theorem 46.10 *Now suppose A_0, A_1 and B_0, B_1 are pairs of Banach spaces such that A_i embeds continuously into a topological vector space, X and B_i embeds continuously into a topological vector space, Y. Suppose also that $L \in \mathcal{L}(A_0, B_0)$ and $L \in \mathcal{L}(A_1, B_1)$ where the operator norm of L in these spaces is $K_i, i = 0, 1$. Then*

$$L \in \mathcal{L}(A_0 + A_1, B_0 + B_1) \tag{46.11}$$

with

$$||La||_{B_0+B_1} \leq \max(K_0, K_1) ||a||_{A_0+A_1} \tag{46.12}$$

and

$$L \in \mathcal{L}(T(A_0, A_1, p, \theta), T(B_0, B_1, p, \theta)) \tag{46.13}$$

and for K the operator norm,

$$K \leq K_0^{1-\theta} K_1^\theta . \tag{46.14}$$

Proof: To verify 46.11, let $a \in A_0 + A_1$ and pick $a_0 \in A_0$ and $a_1 \in A_1$ such that

$$||a||_{A_0+A_1} + \varepsilon > ||a_0||_{A_0} + ||a_1||_{A_1} .$$

Then

$$||L(a)||_{B_0+B_1} = ||La_0 + La_1||_{B_0+B_1} \leq ||La_0||_{B_0} + ||La_1||_{B_1}$$

$$\leq K_0 \|a_0\|_{A_0} + K_1 \|a_1\|_{A_1} \leq \max(K_0, K_1) \left(\|a\|_{A_0+A_1} + \varepsilon\right).$$

This establishes 46.12. Now consider the other assertions.

Let $a \in T(A_0, A_1, p, \theta)$ and pick $f \in W(A_0, A_1, p, \theta)$ such that $\gamma f = a$ and

$$\|a\|_{T(A_0,A_1,p,\theta)} + \varepsilon > \left\|t^\theta f\right\|^{1-\theta}_{L^p\left(0,\infty,\frac{dt}{t},A_0\right)} \left\|t^\theta f'\right\|^{\theta}_{L^p\left(0,\infty,\frac{dt}{t},A_1\right)}.$$

Then consider Lf. Since L is continuous on $A_0 + A_1$,

$$Lf(0) = La$$

and $Lf \in W(B_0, B_1, p, \theta)$. Therefore, by Theorem 46.9,

$$\begin{aligned}
\|La\|_{T(B_0,B_1,p,\theta)} &\leq \left\|t^\theta Lf\right\|^{1-\theta}_{L^p\left(0,\infty,\frac{dt}{t},A_0\right)} \left\|t^\theta Lf'\right\|^{\theta}_{L^p\left(0,\infty,\frac{dt}{t},A_1\right)} \\
&\leq K_0^{1-\theta} K_1^\theta \left\|t^\theta f\right\|^{1-\theta}_{L^p\left(0,\infty,\frac{dt}{t},A_0\right)} \left\|t^\theta f'\right\|^{\theta}_{L^p\left(0,\infty,\frac{dt}{t},A_1\right)} \\
&\leq K_0^{1-\theta} K_1^\theta \left(\|a\|_{T(A_0,A_1,p,\theta)} + \varepsilon\right).
\end{aligned}$$

and since $\varepsilon > 0$ is arbitrary, this proves the theorem.

46.2 Equivalence Of Trace And Interpolation Spaces

Trace spaces are equivalent to interpolation spaces. In showing this, a more general sort of trace space than that presented earlier will be used.

Definition 46.11 *Define for m a positive integer, $V^m = V^m(A_0, A_1, p, \theta)$ to be the set of functions, u such that*

$$t \to t^\theta u(t) \in L^p\left(0, \infty, \frac{dt}{t}; A_0\right) \tag{46.15}$$

and

$$t \to t^{\theta+m-1} u^{(m)}(t) \in L^p\left(0, \infty, \frac{dt}{t}; A_1\right). \tag{46.16}$$

V^m is a Banach space with the norm

$$\|u\|_{V^m} \equiv \max\left(\left\|t^\theta u(t)\right\|_{L^p\left(0,\infty,\frac{dt}{t};A_0\right)}, \left\|t^{\theta+m-1} u^{(m)}(t)\right\|_{L^p\left(0,\infty,\frac{dt}{t};A_1\right)}\right).$$

Thus V^m equals W in the case when $m = 1$. More generally, as in [8] different exponents are used for the two L^p spaces, p_0 in place of p for the space corresponding to A_0 and p_1 in place of p for the space corresponding to A_1.

Definition 46.12 *Denote by $T^m(A_0, A_1, p, \theta)$ the set of all $a \in A_0 + A_1$ such that for some $u \in V^m$,*

$$a = \lim_{t \to 0+} u(t) \equiv trace(u), \tag{46.17}$$

the limit holding in $A_0 + A_1$. For the norm

$$\|a\|_{T^m} \equiv \inf\left\{\|u\|_{V^m} : trace(u) = a\right\}. \tag{46.18}$$

The case when $m = 1$ was discussed in Section 46.1. Note it is not known at this point whether $\lim_{t\to 0+} u(t)$ even exists for every $u \in V^m$. Of course, if $m = 1$ this was shown earlier but it has not been shown for $m > 1$. The following theorem is absolutely amazing. Note the lack of dependence on m of the right side!

Theorem 46.13 *The following hold.*

$$T^m (A_0, A_1, p, \theta) = (A_0, A_1)_{\theta,p,J} = (A_0, A_1)_{\theta,p}\,. \tag{46.19}$$

Proof: It is enough to show the first equality because of Theorem 45.27 which identifies $(A_0, A_1)_{\theta,p,J}$ and $(A_0, A_1)_{\theta,p}$. Let $a \in T^m$. Then there exists $u \in V^m$ such that

$$a = \lim_{t\to 0+} u(t) \text{ in } A_0 + A_1.$$

The first task is to modify this $u(t)$ to get a better one which is more usable in order to show $a \in (A_0, A_1)_{\theta,p,J}$. Remember, it is required to find $w(t) \in A_0 \cap A_1$ for all $t \in (0, \infty)$ and $a = \int_0^\infty w(t)\,\frac{dt}{t}$, a representation which is not known at this time. To get such a thing, let

$$\phi \in C_c^\infty (0, \infty), \operatorname{spt}(\phi) \subseteq [\alpha, \beta] \tag{46.20}$$

with $\phi \geq 0$ and

$$\int_0^\infty \phi(t)\,\frac{dt}{t} = 1. \tag{46.21}$$

Then define

$$\widetilde{u}(t) \equiv \int_0^\infty \phi\left(\frac{t}{\tau}\right) u(\tau)\,\frac{d\tau}{\tau} = \int_0^\infty \phi(s)\, u\left(\frac{t}{s}\right) \frac{ds}{s}. \tag{46.22}$$

Claim: $\lim_{t\to 0+} \widetilde{u}(t) = a$ and $\lim_{t\to\infty} \widetilde{u}^{(k)}(t) = 0$ in $A_0 + A_1$ for all $k \leq m$.

Proof of the claim: From 46.22 and 46.21 it follows that for $||\cdot||$ referring to $||\cdot||_{A_0+A_1}$,

$$\begin{aligned} ||\widetilde{u}(t) - a|| &\leq \int_0^\infty \left|\left|u\left(\frac{t}{s}\right) - a\right|\right| \phi(s)\,\frac{ds}{s} \\ &= \int_0^\infty ||u(\tau) - a||\,\phi\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \\ &= \int_{t/\beta}^{t/\alpha} ||u(\tau) - a||\,\phi\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \\ &\leq \int_{t/\beta}^{t/\alpha} \varepsilon\phi\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} = \varepsilon \int_\alpha^\beta \phi(s)\,\frac{ds}{s} = \varepsilon \end{aligned}$$

whenever t is small enough due to the convergence of $u(t)$ to a in $A_0 + A_1$.

Now consider what occurs when $t \to \infty$. For $||\cdot||$ referring to the norm in A_0,

$$\widetilde{u}^{(k)}(t) = \int_0^\infty \phi^{(k)}\left(\frac{t}{\tau}\right)\frac{1}{\tau^k}\,u(\tau)\,\frac{d\tau}{\tau}$$

and so

$$\left|\left|\widetilde{u}^{(k)}(t)\right|\right|_{A_0} \leq C_k \int_{t/\beta}^{t/\alpha} ||u(\tau)||_{A_0} \frac{d\tau}{\tau}$$

$$\leq C\left(\int_{t/\beta}^{t/\alpha} \frac{d\tau}{\tau}\right)^{1/p'} \left(\int_{t/\beta}^{t/\alpha} ||u(\tau)||_{A_0}^p \frac{d\tau}{\tau}\right)^{1/p}.$$

Now $\left(\frac{\beta}{t}\right)^{\theta} \tau^{\theta} \geq 1$ for $\tau \geq t/\beta$ and so the above expression

$$\leq C\left(\ln\frac{\beta}{\alpha}\right)^{1/p'} \left(\frac{\beta}{t}\right)^{\theta} \left(\int_{t/\beta}^{\infty} \left(\tau^{\theta} ||u(\tau)||_{A_0}\right)^p \frac{d\tau}{\tau}\right)^{1/p}$$

and so $\lim_{t\to\infty} \left|\left|\widetilde{u}^{(k)}(t)\right|\right|_{A_0} = 0$ and therefore, this also holds in $A_0 + A_1$. This proves the claim.

Thus $\widetilde{u}$ has the same properties as u in terms of having a as its trace. $\widetilde{u}$ is used to build the desired w, representing a as an integral. Define

$$v(t) \equiv \frac{(-1)^m t^m}{(m-1)!} \widetilde{u}^{(m)}(t) = \frac{(-1)^m}{(m-1)!} \int_0^{\infty} \frac{t^m}{\tau^m} \phi^{(m)}\left(\frac{t}{\tau}\right) u(\tau) \frac{d\tau}{\tau}$$

$$= \frac{(-1)^m}{(m-1)!} \int_0^{\infty} s^m \phi^{(m)}(s)\, u\left(\frac{t}{s}\right) \frac{ds}{s}. \tag{46.23}$$

Then from the claim, and integration by parts in the last step,

$$\int_0^{\infty} v\left(\frac{1}{t}\right) \frac{dt}{t} = \int_0^{\infty} v(t) \frac{dt}{t} = \frac{(-1)^m}{(m-1)!} \int_0^{\infty} t^{m-1} \widetilde{u}^{(m)}(t)\, dt = a. \tag{46.24}$$

Thus $v\left(\frac{1}{t}\right)$ represents a in the way desired for $(A_0, A_1)_{\theta,p,J}$ if it is also true that $v\left(\frac{1}{t}\right) \in A_0 \cap A_1$ and $t \to t^{-\theta} v\left(\frac{1}{t}\right)$ is in $L^p\left(0, \infty, \frac{dt}{t}; A_0\right)$ and $t \to t^{1-\theta} v\left(\frac{1}{t}\right)$ is in $L^p\left(0, \infty, \frac{dt}{t}; A_1\right)$. First consider whether $v(t) \in A_0 \cap A_1$. $v(t) \in A_0$ for each t from 46.23 and the assumption that $u \in L^p\left(0, \infty, \frac{dt}{t}; A_0\right)$. To verify $v(t) \in A_1$, integrate by parts in 46.23 to obtain

$$v(t) = \frac{(-1)^m}{(m-1)!} \int_0^{\infty} \phi^{(m)}(s) \left(s^{m-1} u\left(\frac{t}{s}\right)\right) ds \tag{46.25}$$

$$= \frac{1}{(m-1)!} \int_0^{\infty} \phi(s) \frac{d^m}{ds^m} \left(s^{m-1} u\left(\frac{t}{s}\right)\right) ds$$

$$= \frac{(-1)^m}{(m-1)!} \int_0^{\infty} \phi(s) \frac{t^m}{s^{m+1}} u^{(m)}\left(\frac{t}{s}\right) ds \in A_1$$

The last step may look very mysterious. If so, consider the case where $m = 2$.

$$\begin{aligned} & \phi(s)\left(su\left(\frac{t}{s}\right)\right)'' \\ = \; & \phi(s)\left(-\frac{t}{s}u'\left(\frac{t}{s}\right)+u\left(\frac{t}{s}\right)\right)' \\ = \; & \phi(s)\left(\left(-\frac{t}{s}\right)u''\left(\frac{t}{s}\right)\left(-\frac{t}{s^2}\right)+\frac{t}{s^2}u'\left(\frac{t}{s}\right)-\frac{t}{s^2}u'\left(\frac{t}{s}\right)\right) \\ = \; & \phi(s)\frac{t^2}{s^3}u''\left(\frac{t}{s}\right). \end{aligned}$$

You can see the same pattern will take place for other values of m.

Now

$$||a||_{\theta,p,J} \leq \left(\int_0^\infty \left(t^{-\theta}J\left(t, v\left(\frac{1}{t}\right)\right)\right)^p \frac{dt}{t}\right)^{1/p}$$

$$\leq C_p\left\{\int_0^\infty \left[\left(t^{-\theta}\left\|v\left(\frac{1}{t}\right)\right\|_{A_0}\right)+\left(t^{1-\theta}\left\|v\left(\frac{1}{t}\right)\right\|_{A_1}\right)\right]^p \frac{dt}{t}\right\}^{1/p}$$

$$\leq C_p\left\{\left(\int_0^\infty \left(t^{-\theta}\left\|v\left(\frac{1}{t}\right)\right\|_{A_0}\right)^p \frac{dt}{t}\right)^{1/p} + \left(\int_0^\infty \left(t^{1-\theta}\left\|v\left(\frac{1}{t}\right)\right\|_{A_1}\right)^p \frac{dt}{t}\right)^{1/p}\right\}. \tag{46.26}$$

The first term equals

$$\begin{aligned} & \left(\int_0^\infty \left(t^{-\theta}\left\|v\left(\frac{1}{t}\right)\right\|_{A_0}\right)^p \frac{dt}{t}\right)^{1/p} \\ = \; & \left(\int_0^\infty \left(t^{\theta}\left\|v(t)\right\|_{A_0}\right)^p \frac{dt}{t}\right)^{1/p} \end{aligned}$$

$$= \left(\int_0^\infty \left(t^{\theta}\left\|\int_0^\infty s^m\phi^{(m)}(s)\,u\left(\frac{t}{s}\right)\frac{ds}{s}\right\|_{A_0}\right)^p \frac{dt}{t}\right)^{1/p}$$

$$\leq \int_0^\infty \left(\int_0^\infty \left(t^{\theta}s^m\left|\phi^{(m)}(s)\right|\left\|u\left(\frac{t}{s}\right)\right\|_{A_0}\right)^p \frac{dt}{t}\right)^{1/p} \frac{ds}{s}$$

$$\leq \int_0^\infty s^m\left|\phi^{(m)}(s)\right|\left(\int_0^\infty \left(t^{\theta}\left\|u\left(\frac{t}{s}\right)\right\|_{A_0}\right)^p \frac{dt}{t}\right)^{1/p} \frac{ds}{s}$$

$$= \int_0^\infty s^{\theta+m} \left|\phi^{(m)}(s)\right| \frac{ds}{s} \left(\int_0^\infty \left(\tau^\theta ||u(\tau)||_{A_0}\right)^p \frac{d\tau}{\tau}\right)^{1/p}$$

$$= C \left(\int_0^\infty \left(\tau^\theta ||u(\tau)||_{A_0}\right)^p \frac{d\tau}{\tau}\right)^{1/p}. \tag{46.27}$$

The second term equals

$$\left(\int_0^\infty \left(t^{1-\theta} \left\|v\left(\frac{1}{t}\right)\right\|_{A_1}\right)^p \frac{dt}{t}\right)^{1/p} = \left(\int_0^\infty \left(t^{\theta-1} ||v(t)||_{A_1}\right)^p \frac{dt}{t}\right)^{1/p}$$

$$= \left(\int_0^\infty \left(t^{\theta-1} \left\|\frac{1}{(m-1)!} \int_0^\infty \phi(s) \frac{t^m}{s^m} u^{(m)}\left(\frac{t}{s}\right) \frac{ds}{s}\right\|_{A_1}\right)^p \frac{dt}{t}\right)^{1/p}$$

$$\leq \int_0^\infty \left(\int_0^\infty \left(\left(\frac{t^{\theta+m-1}}{s^m}\right) |\phi(s)| \left\|u^{(m)}\left(\frac{t}{s}\right)\right\|_{A_1}\right)^p \frac{dt}{t}\right)^{1/p} \frac{ds}{s}$$

$$\leq \int_0^\infty \frac{|\phi(s)|}{s^m} \left(\int_0^\infty \left(t^{\theta+m-1} \left\|u^{(m)}\left(\frac{t}{s}\right)\right\|_{A_1}\right)^p \frac{dt}{t}\right)^{1/p} \frac{ds}{s}$$

$$= \int_0^\infty \frac{|\phi(s)|}{s^m} s^{\theta+m-1} \left(\int_0^\infty \left(\tau^{\theta+m-1} \left\|u^{(m)}(\tau)\right\|_{A_1}\right)^p \frac{d\tau}{\tau}\right)^{1/p} \frac{ds}{s}$$

$$= C \left(\int_0^\infty \left(\tau^{\theta+m-1} \left\|u^{(m)}(\tau)\right\|_{A_1}\right)^p \frac{d\tau}{\tau}\right)^{1/p}. \tag{46.28}$$

Now from the estimates on the two terms in 46.26 found in 46.27 and 46.28, and the simple estimate,

$$2\max(\alpha, \beta) \geq \alpha + \beta,$$

it follows

$$||a||_{\theta,p,J} \tag{46.29}$$

$$\leq C \max\left(\left(\int_0^\infty \left(\tau^\theta ||u(\tau)||_{A_0}\right)^p \frac{d\tau}{\tau}\right)^{1/p}\right. \tag{46.30}$$

$$\left., \left(\int_0^\infty \left(\tau^{\theta+m-1} \left\|u^{(m)}(\tau)\right\|_{A_1}\right)^p \frac{d\tau}{\tau}\right)^{1/p}\right) \tag{46.31}$$

which shows that after taking the infimum over all u whose trace is a, it follows $a \in (A_0, A_1)_{\theta,p,J}$.

$$||a||_{\theta,p,J} \leq C\, ||a||_{T^m} \tag{46.32}$$

Thus $T^m(A_0, A_1, \theta, p) \subseteq (A_0, A_1)_{\theta,p,J}$.

Is $(A_0, A_1)_{\theta,p,J} \subseteq T^m(A_0, A_1, \theta, p)$? Let $a \in (A_0, A_1)_{\theta,p,J}$. There exists u having values in $A_0 \cap A_1$ and such that

$$a = \int_0^\infty u(t)\frac{dt}{t} = \int_0^\infty u\left(\frac{1}{t}\right)\frac{dt}{t},$$

in $A_0 + A_1$ such that

$$\int_0^\infty \left(t^{-\theta} J(t, u(t))\right)^p dt < \infty, \text{ where } J(t,a) = \max\left(||a||_{A_0}, t\,||a||_{A_1}\right).$$

Then let

$$w(t) \equiv \int_t^\infty \left(1 - \frac{t}{\tau}\right)^{m-1} u\left(\frac{1}{\tau}\right)\frac{d\tau}{\tau} = \tag{46.33}$$

$$\int_0^{1/t} (1 - st)^{m-1} u(s)\frac{ds}{s} = \int_0^1 (1-\tau)^{m-1} u\left(\frac{\tau}{t}\right)\frac{d\tau}{\tau}. \tag{46.34}$$

It is routine to verify from 46.33 that

$$w^{(m)}(t) = (m-1)!\,(-1)^m \frac{u\left(\frac{1}{t}\right)}{t^m}. \tag{46.35}$$

For example, consider the case where $m = 2$.

$$\begin{aligned}\left(\int_t^\infty \left(1 - \frac{t}{\tau}\right) u\left(\frac{1}{\tau}\right)\frac{d\tau}{\tau}\right)'' &= \left(0 + \int_t^\infty \left(-\frac{1}{\tau}\right) u\left(\frac{1}{\tau}\right)\frac{d\tau}{\tau}\right)' \\ &= \frac{1}{t^2} u\left(\frac{1}{t}\right).\end{aligned}$$

Also from 46.33, it follows that $trace\,(w) = a$. It remains to verify $w \in V^m$. From 46.35,

$$\left(\int_0^\infty \left(t^{\theta+m-1}\left|\left|w^{(m)}(t)\right|\right|_{A_1}\right)^p \frac{dt}{t}\right)^{1/p} =$$

$$C_m\left(\int_0^\infty \left(t^{\theta-1}\left|\left|u\left(\frac{1}{t}\right)\right|\right|_{A_1}\right)^p \frac{dt}{t}\right)^{1/p} = C_m\left(\int_0^\infty \left(t^{1-\theta}\,||u(t)||_{A_1}\right)^p \frac{dt}{t}\right)^{1/p}$$

$$\leq C_m\left(\int_0^\infty \left(t^{-\theta} J(t, u(t))\right)^p \frac{dt}{t}\right)^{1/p} < \infty. \tag{46.36}$$

It remains to consider $\left(\int_0^\infty \left(t^\theta\,||w(t)||_{A_0}\right)^p \frac{dt}{t}\right)^{1/p}$. From 46.34,

$$\begin{aligned}&\left(\int_0^\infty \left(t^\theta\,||w(t)||_{A_0}\right)^p \frac{dt}{t}\right)^{1/p} \\ =\ &\left(\int_0^\infty \left(t^\theta\left|\left|\int_0^1 (1-\tau)^{m-1} u\left(\frac{\tau}{t}\right)\frac{d\tau}{\tau}\right|\right|_{A_0}\right)^p \frac{dt}{t}\right)^{1/p}\end{aligned}$$

$$= \left(\int_0^\infty \left(t^{-\theta}\left|\left|\int_0^1 (1-\tau)^{m-1} u(\tau t)\frac{d\tau}{\tau}\right|\right|_{A_0}\right)^p \frac{dt}{t}\right)^{1/p}$$

$$\leq \int_0^1 \left(\int_0^\infty \left(t^{-\theta}(1-\tau)^{m-1}||u(\tau t)||_{A_0}\right)^p \frac{dt}{t}\right)^{1/p} \frac{d\tau}{\tau}$$

$$= \int_0^1 \tau^\theta (1-\tau)^{m-1}\left(\int_0^\infty \left(s^{-\theta}||u(s)||_{A_0}\right)^p \frac{ds}{s}\right)^{1/p} \frac{d\tau}{\tau}$$

$$= \left(\int_0^1 \tau^{\theta-1}(1-\tau)^{m-1} d\tau\right)\left(\int_0^\infty \left(s^{-\theta}||u(s)||_{A_0}\right)^p \frac{ds}{s}\right)^{1/p}$$

$$\leq C\left(\int_0^\infty \left(s^{-\theta}||u(s)||_{A_0}\right)^p \frac{ds}{s}\right)^{1/p}$$

$$\leq C\left(\int_0^\infty \left(t^{-\theta}J(t,u(t))\right)^p dt\right)^{1/p} < \infty. \tag{46.37}$$

It follows that

$$||w||_{V^m} \equiv$$

$$\max\left(\left(\int_0^\infty \left(t^\theta ||w(t)||_{A_0}\right)^p \frac{dt}{t}\right)^{1/p}, \left(\int_0^\infty \left(t^{\theta+m-1}\left|\left|w^{(m)}(t)\right|\right|_{A_1}\right)^p \frac{dt}{t}\right)^{1/p}\right)$$

$$\leq C\left(\int_0^\infty \left(t^{-\theta}J(t,u(t))\right)^p dt\right)^{1/p} < \infty$$

which shows that $a \in T^m(A_0, A_1, \theta, p)$. Taking the infimum,

$$||a||_{T^m} \leq C\,||a||_{\theta,p,J}\,.$$

This together with 46.32 proves the theorem.

By Theorem 46.13 and Theorem 45.33, we obtain the following important corollary describing the dual space of a trace space.

Corollary 46.14 *Let $A_0 \cap A_1$ be dense in A_i for $i = 0, 1$ and suppose that A_i is reflexive for $i = 0, 1$. Then for $\infty > p \geq 1$,*

$$T^m(A_0, A_1, \theta, p)' = T^m(A_1', A_0', 1-\theta, p')$$

Traces Of Sobolev Spaces And Fractional Order Spaces

47.1 Traces Of Sobolev Spaces On The Boundary Of A Half Space

In this section we consider the trace of $W^{m,p}\left(\mathbb{R}_{+}^{n}\right)$ onto a Sobolev space of functions defined on $\mathbb{R}^{n-1}$. This latter Sobolev space will be defined in terms of the following theory in such a way that the trace map is continuous. We already know the trace map is continuous as a map from $W^{m,p}\left(\mathbb{R}_{+}^{n}\right)$ to $W^{m-1,p}\left(\mathbb{R}^{n-1}\right)$ but we can do much better than this using the above theory.

Definition 47.1 *Let $\theta \in (0,1)$ and let Ω be an open subset of $\mathbb{R}^{m}$. We define*

$$W^{\theta,p}(\Omega) \equiv T\left(W^{1,p}(\Omega), L^{p}(\Omega), p, 1-\theta\right).$$

Thus, from the above general theory, $W^{1,p}(\Omega) \hookrightarrow W^{\theta,p}(\Omega) \hookrightarrow L^{p}(\Omega) = L^{p}(\Omega) + W^{1,p}(\Omega)$. Now we consider the trace map for Sobolev space.

Lemma 47.2 *Let $\phi \in C^{\infty}\left(\overline{\mathbb{R}_{+}^{n}}\right)$. Then $\gamma\phi(\mathbf{x}') \equiv \phi(\mathbf{x}',0)$. Then $\gamma : C^{\infty}\left(\overline{\mathbb{R}_{+}^{n}}\right) \to L^{p}\left(\mathbb{R}^{n-1}\right)$ is continuous as a map from $W^{1,p}\left(\mathbb{R}_{+}^{n}\right)$ to $L^{p}\left(\mathbb{R}^{n-1}\right)$.*

Proof: We know

$$\phi(\mathbf{x}', x_n) = \gamma\phi(\mathbf{x}') + \int_0^{x_n} \frac{\partial \phi(\mathbf{x}', t)}{\partial t} dt$$

Then by Jensen's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} |\gamma\phi(\mathbf{x}')|^p \, dx' \\
= &\int_0^1 \int_{\mathbb{R}^{n-1}} |\gamma\phi(\mathbf{x}')|^p \, dx' dx_n \\
\leq &\ C \int_0^1 \int_{\mathbb{R}^{n-1}} |\phi(\mathbf{x}', x_n)|^p \, dx' dx_n \\
&+ C \int_0^1 \int_{\mathbb{R}^{n-1}} \left| \int_0^{x_n} \frac{\partial \phi(\mathbf{x}', t)}{\partial t} dt \right|^p dx' dx_n \\
\leq &\ C \|\phi\|_{0,p,\mathbb{R}^{n-1}}^p + C \int_0^1 x_n^{p-1} \int_{\mathbb{R}^{n-1}} \int_0^{x_n} \left| \frac{\partial \phi(\mathbf{x}', t)}{\partial t} \right|^p dt dx' dx_n \\
\leq &\ C \|\phi\|_{0,p,\mathbb{R}^{n-1}}^p + C \int_0^1 x_n^{p-1} \int_{\mathbb{R}^{n-1}} \int_0^{\infty} \left| \frac{\partial \phi(\mathbf{x}', t)}{\partial t} \right|^p dt dx' dx_n \\
\leq &\ C \|\phi\|_{0,p,\mathbb{R}^{n-1}}^p + \frac{C}{p} \int_{\mathbb{R}^{n-1}} \int_0^{\infty} \left| \frac{\partial \phi(\mathbf{x}', t)}{\partial t} \right|^p dt dx' \\
\leq &\ C \|\phi\|_{1,p,\mathbb{R}^n_+}^p
\end{aligned}$$

This proves the lemma.

Definition 47.3 *We define the trace,* $\gamma : W^{1,p}(\mathbb{R}^n_+) \to L^p(\mathbb{R}^{n-1})$ *as follows.* $\gamma\phi(\mathbf{x}') \equiv \phi(\mathbf{x}', 0)$ *whenever* $\phi \in C^\infty\left(\overline{\mathbb{R}^n_+}\right)$. *For* $u \in W^{1,p}\left(\overline{\mathbb{R}^n_+}\right)$, *we define* $\gamma u \equiv \lim_{k\to\infty} \gamma\phi_k$ *in* $L^p(\mathbb{R}^{n-1})$ *where* $\phi_k \to u$ *in* $W^{1,p}\left(\overline{\mathbb{R}^n_+}\right)$. *Then the above lemma shows this is well defined.*

Also from this lemma we obtain a constant, C such that

$$\|\phi\|_{0,p,\mathbb{R}^{n-1}} \leq C \|\phi\|_{1,p,\mathbb{R}^n_+}$$

and we see the same constant holds for all $u \in W^{1,p}(\mathbb{R}^n_+)$. Now we will assert more than this. From the definition of the norm in the trace space, if $f \in C^\infty\left(\overline{\mathbb{R}^n_+}\right)$, and we let $\theta = 1 - \frac{1}{p}$, then

$$\begin{aligned}
&\|\gamma f\|_{1-\frac{1}{p},p,\mathbb{R}^{n-1}} \\
\leq &\ \max\left(\left(\int_0^\infty \left(t^{1/p} \|f(t)\|_{1,p,\mathbb{R}^{n-1}} \right)^p \frac{dt}{t} \right)^{1/p} \right. \\
&\left. , \left(\int_0^\infty \left(t^{1/p} \|f'(t)\|_{0,p,\mathbb{R}^{n-1}} \right)^p \frac{dt}{t} \right)^{1/p} \right) \\
\leq &\ C \|f\|_{1,p,\mathbb{R}^n_+}.
\end{aligned}$$

Thus, if $f \in W^{1,p}(\mathbb{R}^n_+)$, we may define $\gamma f \in W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ according to the rule,

$$\gamma f = \lim_{k\to\infty} \gamma\phi_k,$$

where $\phi_k \to f$ in $W^{1,p}\left(\mathbb{R}_+^n\right)$ and $\phi_k \in C^\infty\left(\overline{\mathbb{R}_+^n}\right)$. This shows the continuity part of the following lemma.

Lemma 47.4 *The trace map, γ, is a continuous map from $W^{1,p}\left(\mathbb{R}_+^n\right)$ onto*

$$W^{1-\frac{1}{p},p}\left(\mathbb{R}^{n-1}\right).$$

Furthermore, for $f \in W^{1,p}\left(\mathbb{R}_+^n\right)$,

$$\gamma f = f(0) = \lim_{t\to 0+} f(t)$$

the limit taking place in $L^p\left(\mathbb{R}^{n-1}\right)$.

Proof: It remains to verify γ is onto along with the displayed equation. But by definition, things in $W^{1-\frac{1}{p},p}\left(\mathbb{R}^{n-1}\right)$ are of the form $\lim_{t\to 0+} f(t)$ where $f \in L^p\left(0,\infty; W^{1,p}\left(\mathbb{R}^{n-1}\right)\right)$, and $f' \in L^p\left(0,\infty; L^p\left(\mathbb{R}^{n-1}\right)\right)$, the limit taking place in

$$W^{1,p}\left(\mathbb{R}^{n-1}\right) + L^p\left(\mathbb{R}^{n-1}\right) = L^p\left(\mathbb{R}^{n-1}\right),$$

and

$$\left(\int_0^\infty ||f(t)||_{1,p,\mathbb{R}^{n-1}}^p dt\right)^{1/p} + \left(\int_0^\infty ||f'(t)||_{0,p}^p dt\right)^{1/p} < \infty.$$

Then taking a measurable representative, we see $f \in W^{1,p}\left(\mathbb{R}_+^n\right)$ and $f_{,x_n} = f'$. Also, as an equation in $L^p\left(\mathbb{R}^{n-1}\right)$, the following holds for all $t > 0$.

$$f(\cdot,t) = f(0) + \int_0^t f_{,x_n}(\cdot,s)\,ds$$

But we also have that for a.e. $\mathbf{x}'$,the following equation holds for a.e. $t > 0$.

$$f(\mathbf{x}',t) = \gamma f(\mathbf{x}') + \int_0^t f_{,x_n}(\mathbf{x}',s)\,ds, \tag{47.1}$$

showing that

$$\gamma f = f(0) \in W^{1-\frac{1}{p},p}\left(\mathbb{R}^{n-1}\right) \equiv T\left(W^{1,p}(\Omega), L^p(\Omega), p, \frac{1}{p}\right).$$

To see that 47.1 holds, we approximate f with a sequence from $C^\infty\left(\overline{\mathbb{R}_+^n}\right)$ and finally obtain an equation of the form

$$\int_{\mathbb{R}^{n-1}}\int_0^\infty \left[f(\mathbf{x}',t) - \gamma f(\mathbf{x}') - \int_0^t f_{,x_n}(\mathbf{x}',s)\,ds\right]\psi(\mathbf{x}',t)\,dt dx' = 0,$$

which holds for all $\psi \in C_c^\infty\left(\mathbb{R}_+^n\right)$. This proves the lemma.

Thus we lose $\frac{1}{p}$ derivatives when we take the trace of a function in $W^{1,p}\left(\mathbb{R}_+^n\right)$.

47.2 A Right Inverse For The Trace For A Half Space

It is also important to show there is a continuous linear function,

$$R : W^{1-\frac{1}{p},p}\left(\mathbb{R}^{n-1}\right) \to W^{1,p}\left(\mathbb{R}^n_+\right)$$

which has the property that $\gamma\left(Rg\right) = g$. We will define this function as follows.

$$Rg\left(\mathbf{x}', x_n\right) \equiv \int_{\mathbb{R}^{n-1}} g\left(\mathbf{y}'\right)\phi\left(\frac{\mathbf{x}'-\mathbf{y}'}{x_n}\right)\frac{1}{x_n^{n-1}}d\mathbf{y}' \tag{47.2}$$

where ϕ is a mollifier having support in $B\left(\mathbf{0},1\right)$. Then we have the following lemma.

Lemma 47.5 *Let R be defined in 47.2. Then $Rg \in W^{1,p}\left(\mathbb{R}^n_+\right)$ and is a continuous linear map from $W^{1-\frac{1}{p},p}\left(\mathbb{R}^{n-1}\right)$ to $W^{1,p}\left(\mathbb{R}^n_+\right)$ with the property that $\gamma Rg = g$.*

Proof: Let $f \in W^{1,p}\left(\mathbb{R}^n_+\right)$ be such that $\gamma f = g$. Let $\psi\left(x_n\right) \equiv \left(1-x_n\right)_+$ and assume f is Borel measurable by taking a Borel measurable representative. Then for a.e. $\mathbf{x}'$ we have the following formula holding for a.e. x_n.

$$\begin{aligned} & Rg\left(\mathbf{x}', x_n\right) \\ = & \int_{\mathbb{R}^{n-1}}\left[\psi\left(x_n\right)f\left(\mathbf{y}',\psi\left(x_n\right)\right) - \int_0^{\psi(x_n)}\left(\psi f\right)_{,n}\left(\mathbf{y}',t\right)dt\right]\phi\left(\frac{\mathbf{x}'-\mathbf{y}'}{x_n}\right)x_n^{1-n}dy'. \end{aligned}$$

Using the repeated index summation convention to save space, we obtain that in terms of weak derivatives,

$$\begin{aligned} & Rg_{,n}\left(\mathbf{x}', x_n\right) \\ = & \int_{\mathbb{R}^{n-1}}\left[\psi\left(x_n\right)f\left(\mathbf{y}',\psi\left(x_n\right)\right) - \int_0^{\psi(x_n)}\left(\psi f\right)_{,n}\left(\mathbf{y}',t\right)dt\right]\cdot \\ & \left[\phi_{,k}\left(\frac{\mathbf{x}'-\mathbf{y}'}{x_n}\right)\left(\frac{y_k-x_k}{x_n^n}\right) + \phi\left(\frac{\mathbf{x}'-\mathbf{y}'}{x_n}\right)\frac{(1-n)}{x_n^n}\right]dy' \\ = & \int_{\mathbb{R}^{n-1}}\left[\psi\left(x_n\right)f\left(\mathbf{x}'-x_n\mathbf{z}',\psi\left(x_n\right)\right) - \int_0^{\psi(x_n)}\left(\psi f\right)_{,n}\left(\mathbf{x}'-x_n\mathbf{z}',t\right)dt\right]\cdot \\ & \left[\phi_{,k}\left(\mathbf{z}'\right)\left(\frac{y_k-x_k}{x_n^n}\right)z_k + \phi\left(\mathbf{z}'\right)\frac{(1-n)}{x_n^n}\right]x_n^n dz' \end{aligned}$$

and so

$$\begin{aligned} \left|Rg_{,n}\left(\mathbf{x}', x_n\right)\right| \leq & C\left(\phi\right)\left|\int_{B(\mathbf{0},1)}\left[\psi\left(x_n\right)f\left(\mathbf{x}'-x_n\mathbf{z}',\psi\left(x_n\right)\right)\right.\right. \\ & \left.\left. - \int_0^{\psi(x_n)}\left(\psi f\right)_{,n}\left(\mathbf{x}'-x_n\mathbf{z}',t\right)dt\right]\right| \end{aligned}$$

$$\leq \frac{C(\phi)}{x_n^{n-1}}\left\{\int_{B(\mathbf{0},x_n)}|\psi(x_n)f(\mathbf{x}'+\mathbf{y}',\psi(x_n))|\,dy' + \int_{B(\mathbf{0},x_n)}\int_0^{\psi(x_n)}\left|(\psi f)_{,n}(\mathbf{x}'+\mathbf{y}',t)\right|dtdy'\right\}$$

Therefore,

$$\left(\int_0^\infty\int_{\mathbb{R}^{n-1}}|Rg_{,n}(\mathbf{x}',x_n)|^p\,dx'dx_n\right)^{1/p}\leq$$
$$C(\phi)\left(\int_0^\infty\int_{\mathbb{R}^{n-1}}\left(\frac{1}{x_n^{n-1}}\int_{B(\mathbf{0},x_n)}|\psi(x_n)f(\mathbf{x}'+\mathbf{y}',\psi(x_n))|\,dy'\right)^p dx'dx_n\right)^{1/p}$$
$$+C(\phi)\left(\int_0^\infty\int_{\mathbb{R}^{n-1}}\left(\frac{1}{x_n^{n-1}}\int_{B(\mathbf{0},x_n)}\int_0^{\psi(x_n)}\left|(\psi f)_{,n}(\mathbf{x}'+\mathbf{y}',t)\right|dtdy'\right)^p dx'dx_n\right)^{1/p} \tag{47.3}$$

Consider the first term on the right. We change variables, letting $\mathbf{y}'=\mathbf{z}'x_n$. Then this term becomes

$$C(\phi)\left(\int_0^1\int_{\mathbb{R}^{n-1}}\left(\int_{B(\mathbf{0},1)}|\psi(x_n)f(\mathbf{x}'+x_n\mathbf{z}',\psi(x_n))|\,dz'\right)^p dx'dx_n\right)^{1/p}$$
$$\leq C(\phi)\int_{B(\mathbf{0},1)}\left(\int_0^1\int_{\mathbb{R}^{n-1}}|\psi(x_n)f(\mathbf{x}'+x_n\mathbf{z}',\psi(x_n))|^p\,dx'dx_n\right)^{1/p}dz'$$

Now we change variables, letting $t=\psi(x_n)$. This yields

$$=C(\phi)\int_{B(\mathbf{0},1)}\left(\int_0^1\int_{\mathbb{R}^{n-1}}|tf(\mathbf{x}'+x_n\mathbf{z}',t)|^p\,dx'dt\right)^{1/p}dz'\leq C(\phi)\,||f||_{0,p,\mathbb{R}^n_+}\,. \tag{47.4}$$

Now we consider the second term on the right in 47.3. Using the same arguments which were used on the first term involving Minkowski's inequality and changing the variables, we obtain the second term

$$\begin{aligned}&\leq C(\phi)\int_{B(\mathbf{0},1)}\int_0^1\left(\int_0^1\int_{\mathbb{R}^{n-1}}\left|(\psi f)_{,n}(\mathbf{x}'+x_n\mathbf{z}',t)\right|^p d\mathbf{x}'dx_n\right)^{1/p}dtdy'\\&\leq C(\phi)\,||f||_{1,p,\mathbb{R}^n_+}\,.\end{aligned} \tag{47.5}$$

It is somewhat easier to verify that

$$||Rg_{,j}||_{0,p,\mathbb{R}^n_+}\leq C(\phi)\,||f||_{1,p,\mathbb{R}^n_+}\,.$$

Therefore, we have shown that whenever $\gamma f=f(0)=g$,

$$||Rg||_{1,p,\mathbb{R}^n_+}\leq C(\phi)\,||f||_{1,p,\mathbb{R}^n_+}\,.$$

Taking the infimum over all such f and using the definition of the norm in

$$W^{1-\frac{1}{p},p}\left(\mathbb{R}^{n-1}\right),$$

it follows

$$||Rg||_{1,p,\mathbb{R}_+^n} \leq C(\phi)\,||g||_{1-\frac{1}{p},p,\mathbb{R}^{n-1}},$$

showing that this map, R, is continuous as claimed. It is obvious that

$$\lim_{x_n \to 0} Rg(x_n) = g,$$

the convergence taking place in $L^p\left(\mathbb{R}^{n-1}\right)$ because of general results about convolution with mollifiers. This proves the lemma.

47.3 Fractional Order Sobolev Spaces

Definition 47.6 *Let m be a nonnegative integer and let $s = m+\sigma$ where $\sigma \in (0,1)$. Then $W^{s,p}(\Omega)$ will consist of those elements of $W^{m,p}(\Omega)$ for which $D^\alpha u \in W^{\sigma,p}(\Omega)$ for all $|\alpha| = m$. The norm is given by the following.*

$$||u||_{s,p,\Omega} \equiv \left(||u||_{m,p,\Omega}^p + \sum_{|\alpha|=m} ||D^\alpha u||_{\sigma,p,\Omega}^p\right)^{1/p}.$$

Corollary 47.7 *The space, $W^{s,p}(\Omega)$ is a reflexive Banach space whenever $p > 1$.*

Proof: We know from the theory of interpolation spaces that $W^{\sigma,p}(\Omega)$ is reflexive. This is because it is an iterpolation space for the two reflexive spaces, $L^p(\Omega)$ and $W^{1,p}(\Omega)$. Now the formula for the norm of an element in $W^{s,p}(\Omega)$ shows this space is isometric to a closed subspace of $W^{m,p}(\Omega) \times W^{\sigma,p}(\Omega)^k$ for suitable k. Therefore, $W^{s,p}(\Omega)$ is also reflexive.

Theorem 47.8 *The trace map, $\gamma : W^{m,p}\left(\mathbb{R}_+^n\right) \to W^{m-\frac{1}{p},p}\left(\mathbb{R}^{n-1}\right)$ is continuous.*

Proof: Let $f \in \mathfrak{S}$. We let $\sigma = 1 - \frac{1}{p}$ so that $m - \left(\frac{1}{p}\right) = m - 1 + \sigma$. Then from the definition,

$$||\gamma f||_{m-\frac{1}{p},p,\mathbb{R}^{n-1}} = \left(||\gamma f||_{m-1,p,\mathbb{R}^{n-1}}^p + \sum_{|\alpha|=m-1} ||D^\alpha \gamma f||_{1-\frac{1}{p},p,\mathbb{R}^{n-1}}^p\right)^{1/p}$$

and from Lemma 47.4, and the fact that the trace is continuous as a map from $W^{m,p}\left(\mathbb{R}_+^n\right)$ to $W^{m-1,p}\left(\mathbb{R}^{n-1}\right)$,

$$||\gamma f||_{m-\frac{1}{p},p,\mathbb{R}^{n-1}} \leq \left(C_1\,||f||_{m,p,\mathbb{R}_+^n}^p + C_2\,||f||_{m,p,\mathbb{R}^n+}\right)^{1/p} \leq C\,||f||_{m,p,\mathbb{R}^n+}.$$

This proves the theorem.

With the definition of $W^{s,p}(\Omega)$ for s not an integer, we can generalize an earlier theorem.

Theorem 47.9 *Let $\mathbf{h} : U \to V$ where U and V are two open sets and suppose $\mathbf{h}$ is bilipschitz and that $D^\alpha \mathbf{h}$ and $D^\alpha \mathbf{h}^{-1}$ exist and are Lipschitz continuous if $|\alpha| \leq m$ where $m = 0, 1, \cdots$.and $s = m + \sigma$ where $\sigma \in (0, 1)$. Then*

$$\mathbf{h}^* : W^{s,p}(V) \to W^{s,p}(U)$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $(\mathbf{h}^{-1})^*$.

Proof: In case $m = 0$, the conclusion of the theorem is immediate from the general theory of trace spaces. Therefore, we can assume $m \geq 1$. We know from the definition that

$$||\mathbf{h}^* u||_{m+\sigma,p,U} \equiv \left[||\mathbf{h}^* u||^p_{m,p,U} + \sum_{|\alpha|=m} ||D^\alpha (\mathbf{h}^* u)||^p_{\sigma,p,U} \right]^{1/p}$$

Now consider the case when $m = 1$. Then it is routine to verify that

$$D_j \mathbf{h}^* u(\mathbf{x}) = u_{,k}(\mathbf{h}(\mathbf{x})) h_{k,j}(\mathbf{x}).$$

Let $L_k : W^{1,p}(V) \to W^{1,p}(U)$ be defined by

$$L_k v = \mathbf{h}^*(v) h_{k,j}.$$

Then L_k is continuous as a map from $W^{1,p}(V)$ to $W^{1,p}(U)$ and as a map from $L^p(V)$ to $L^p(U)$ and therefore, it follows that L_k is continuous as a map from $W^{\sigma,p}(V)$ to $W^{\sigma,p}(U)$. Therefore,

$$||L_k(v)||_{\sigma,p,U} \leq C_k ||v||_{\sigma,p,U}$$

and so

$$\begin{aligned} ||D_j(\mathbf{h}^* u)||_{\sigma,p,U} &\leq \sum_k ||L_k(u_{,k})||_{\sigma,p,U} \\ &\leq \sum_k C_k ||D_k u||_{\sigma,p,V} \\ &\leq C \left(\sum_k ||D_k u||^p_{\sigma,p,V} \right)^{1/p}. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} ||\mathbf{h}^* u||_{1+\sigma,p,U} &\leq \left[||\mathbf{h}^* u||^p_{1,p,U} + \sum_j C^p \sum_k ||D_k u||^p_{\sigma,p,V} \right]^{1/p} \\ &\leq C \left[||u||^p_{1,p,V} + \sum_k ||D_k u||^p_{\sigma,p,V} \right]^{1/p} = C ||u||_{1+\sigma,p,V}. \end{aligned}$$

The general case is similar. We simply have a more complicated continuous linear operator in place of L_k.

Now we prove an important interpolation inequality for Sobolev spaces.

Theorem 47.10 *Let Ω be an open set in $\mathbb{R}^n$ and let $f \in W^{m+1,p}(\Omega)$ and $\sigma \in (0,1)$. Then for some constant, C, independent of f,*

$$||f||_{m+\sigma,p,\Omega} \leq C\, ||f||_{m+1,p,\Omega}^{1-\sigma} ||f||_{m,p,\Omega}^{\sigma}.$$

Also, if $L \in \mathcal{L}(W^{m,p}(\Omega), W^{m,p}(\Omega))$ for all $m = 0, 1, \cdots$, and $L \circ D^\alpha = D^\alpha \circ L$ on $C^\infty(\overline{\Omega})$, then $L \in \mathcal{L}(W^{m+\sigma,p}(\Omega), W^{m+\sigma,p}(\Omega))$ for any $m = 0, 1, \cdots$.

Proof: Recall from above, $W^{1-\theta,p}(\Omega) \equiv T\left(W^{1,p}(\Omega), L^p(\Omega), p, \theta\right)$. Therefore, from Theorem 46.9, if $f \in W^{1,p}(\Omega)$,

$$||f||_{1-\theta,p,\Omega} \leq K\, ||f||_{1,p,\Omega}^{\theta} ||f||_{0,p,\Omega}^{1-\theta}$$

Therefore,

$$\begin{aligned}
||f||_{m+\sigma,p,\Omega} &\leq \left(||f||_{m,p,\Omega}^p + \sum_{|\alpha|=m} K \left(||D^\alpha f||_{1,p,\Omega}^{1-\sigma} ||D^\alpha f||_{0,p,\Omega}^{\sigma} \right)^p \right)^{1/p} \\
&\leq C \left[||f||_{m,p,\Omega}^p + \left(||f||_{m+1,p,\Omega}^{1-\sigma} ||f||_{m,p,\Omega}^{\sigma} \right)^p \right]^{1/p} \\
&\leq C \left[\left(||f||_{m+1,p,\Omega}^{1-\sigma} ||f||_{m,p,\Omega}^{\sigma} \right)^p + \left(||f||_{m+1,p,\Omega}^{1-\sigma} ||f||_{m,p,\Omega}^{\sigma} \right)^p \right]^{1/p} \\
&\leq C\, ||f||_{m+1,p,\Omega}^{1-\sigma} ||f||_{m,p,\Omega}^{\sigma}.
\end{aligned}$$

This proves the first part. Now we consider the second. Let $\phi \in C^\infty(\overline{\Omega})$

$$||L\phi||_{m+\sigma,p,\Omega} = \left(||L\phi||_{m,p,\Omega}^p + \sum_{|\alpha|=m} ||D^\alpha L\phi||_{\sigma,p,\Omega}^p \right)^{1/p}$$

$$= \left(||L\phi||_{m,p,\Omega}^p + \sum_{|\alpha|=m} ||LD^\alpha \phi||_{T(W^{1,p},L^p,p,1-\sigma)}^p \right)^{1/p}$$

$$= \left(||L\phi||_{m,p,\Omega}^p + \sum_{|\alpha|=m} \left[\inf \left(\left|\left| t^{1-\sigma} L f_\alpha \right|\right|_1^{\sigma} \left|\left| t^{1-\sigma} L f_\alpha' \right|\right|_2^{1-\sigma} \right) \right]^p \right)^{1/p} \tag{47.6}$$

where

$$\inf \left(\left|\left| t^{1-\sigma} L f_\alpha \right|\right|_1^{\sigma} \left|\left| t^{1-\sigma} L f_\alpha' \right|\right|_2^{1-\sigma} \right) =$$

$$\inf \left(\left|\left| t^{1-\sigma} L f_\alpha \right|\right|_{L^p\left(0,\infty;\frac{dt}{t};W^{1,p}(\Omega)\right)}^{\sigma} \left|\left| t^{1-\sigma} L f_\alpha' \right|\right|_{L^p\left(0,\infty;\frac{dt}{t};L^p(\Omega)\right)}^{1-\sigma} \right),$$

$f_\alpha(0) \equiv \lim_{t\to 0} f_\alpha(t) = D^\alpha \phi$ in $W^{1,p}(\Omega) + L^p(\Omega)$, and the infimum is taken over all such functions. Therefore, from 47.6, and letting $||L||_1$ denote the operator norm of L in $W^{1,p}(\Omega)$ and $||L||_2$ denote the operator norm of L in $L^p(\Omega)$,

$$\begin{aligned}
&||L\phi||_{m+\sigma,p,\Omega} \\
\leq\ & \left(||L\phi||^p_{m,p,\Omega} + \sum_{|\alpha|=m} \left[\inf \left(||L||_1^\sigma ||L||_2^{1-\sigma} \left|\left|t^{1-\sigma} f_\alpha\right|\right|_1^\sigma \left|\left|t^{1-\sigma} f'_\alpha\right|\right|_2^{1-\sigma} \right) \right]^p \right)^{1/p} \\
\leq\ & \left(||L\phi||^p_{m,p,\Omega} + \left(||L||_1^\sigma ||L||_2^{1-\sigma} \right)^p \sum_{|\alpha|=m} \left[\inf \left(\left|\left|t^{1-\sigma} f_\alpha\right|\right|_1^\sigma \left|\left|t^{1-\sigma} f'_\alpha\right|\right|_2^{1-\sigma} \right) \right]^p \right)^{1/p} \\
\leq\ & C \left(||\phi||^p_{m,p,\Omega} + \sum_{|\alpha|=m} \left[||D^\alpha \phi||_{\sigma,p,\Omega} \right]^p \right)^{1/p} = C\, ||\phi||_{m+\sigma,p,\Omega}.
\end{aligned}$$

Since $C^\infty\left(\overline{\Omega}\right)$ is dense in all the Sobolev spaces, this inequality establishes the desired result.

Definition 47.11 *We define for $s \geq 0$, $W^{-s,p'}(\mathbb{R}^n)$ to be the dual space of*

$$W^{s,p}(\mathbb{R}^n).$$

Here $\frac{1}{p} + \frac{1}{p'} = 1$.

Note that in the case of $m = 0$ this is consistent with the Riesz representation theorem for the L^p spaces.

Sobolev Spaces On Manifolds

48.1 Basic Definitions

We will be considering the following situation. We have a set, $\Gamma \subseteq \mathbb{R}^m$ where $m > n$, mappings, $\mathbf{h}_i : U_i \rightarrow \Gamma_i = \Gamma \cap W_i$ for W_i an open set in $\mathbb{R}^m$with $\Gamma \subseteq \cup_{i=1}^l W_i$ and U_i is an open subset of $\mathbb{R}^n$. We assume $\mathbf{h}_i$ is of the form

$$\mathbf{h}_i(\mathbf{x}) = \mathbf{H}_i(\mathbf{x}, \mathbf{0}) \tag{48.1}$$

where for some open set, O_i, $\mathbf{H}_i : U_i \times O_i \rightarrow W_i$ is bilipschitz having bilipschitz inverse such that for $\mathbf{G} = \mathbf{H}_i$ or $\mathbf{H}_i^{-1}, D^\alpha \mathbf{G}$ is Lipschitz for $|\alpha| \leq k$.

For example, we could let $m = n + 1$ and let

$$\mathbf{H}_i(\mathbf{x}, y) = \begin{pmatrix} \mathbf{x} \\ \phi(\mathbf{x}) + y \end{pmatrix}$$

where ϕ is a Lipschitz function having $D^\alpha \phi$ Lipschitz for all $|\alpha| \leq k$. This is an example of the sort of thing just described, letting $\mathbf{x} \in U_i \subseteq \mathbb{R}^n$ and $O_i = \mathbb{R}$, becuase it is obvious the inverse of $\mathbf{H}_i$ is given by

$$\mathbf{H}_i^{-1}(\mathbf{x}, y) = \begin{pmatrix} \mathbf{x} \\ y - \phi(\mathbf{x}) \end{pmatrix}.$$

We will also let $\{\psi_i\}_{i=1}^l$ be a partition of unity subordinate to the open cover $\{W_i\}$ satisfying $\psi_i \in C_c^\infty(W_i)$. Then we give the following definition.

Definition 48.1 *We say $u \in W^{s,p}(\Gamma)$ if whenever $\{W_i, \psi_i, \Gamma_i, U_i, \mathbf{h}_i, \mathbf{H}_i\}_{i=1}^l$ is described above, $\mathbf{h}_i^*(u\psi_i) \in W^{s,p}(U_i)$. We define the norm as*

$$||u||_{s,p,\Gamma} \equiv \sum_{i=1}^{l} ||\mathbf{h}_i^*(u\psi_i)||_{s,p,U_i}$$

It is not at all obvious this norm is well defined. What if

$$\{W_i', \phi_i, \Gamma_i, V_i, \mathbf{g}_i, \mathbf{G}_i\}_{i=1}^r$$

is as described above. Would the two norms be equivalent? To begin with we show the following lemma which involves a particular choice for $\{W_i, \psi_i, \Gamma_i, U_i, \mathbf{h}_i, \mathbf{H}_i\}_{i=1}^l$.

Lemma 48.2 *$W^{s,p}(\Gamma)$ as just described, is a reflexive Banach space.*

Proof: Let $L : W^{s,p}(\Gamma) \to \prod_{i=1}^{l} W^{s,p}(U_i)$ be defined by $(Lu)_i \equiv \mathbf{h}_i^*(u\psi_i)$. Let $\{u_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $W^{s,p}(\Gamma)$. Then $\{\mathbf{h}_i^*(u_j\psi_i)\}_{j=1}^{\infty}$ is a Cauchy sequence in $W^{s,p}(U_i)$ for each i. Therefore, for each i, there exists $w_i \in W^{s,p}(U_i)$ such that

$$\lim_{j\to\infty} \mathbf{h}_i^*(u_j\psi_i) = w_i \text{ in } W^{s,p}(U_i).$$

But also, we may take a subsequence such that

$$\left\{\sum_{i=1}^{l} \mathbf{h}_i^*(u_j\psi_i)(\mathbf{x})\right\}_{j=1}^{\infty} = \{u_j(\mathbf{h}(\mathbf{x}))\}_{j=1}^{\infty}$$

converges for a.e. $\mathbf{x}$. Since $\mathbf{h}$ maps sets of measure zero to sets of n dimensional Hausdorff measure zero, it follows that for a.e. $\mathbf{y} \in \Gamma$,

$$u_j(\mathbf{y}) \to u(\mathbf{y}) \text{ a.e.}$$

Therefore, $w_i(\mathbf{x}) = \mathbf{h}_i^*(u\psi_i)(\mathbf{x})$ a.e. and this shows $\mathbf{h}_i^*(u\psi_i) \in W^{s,p}(U_i)$. Thus $u \in W^{s,p}(\Gamma)$ and this shows completness. It is clear $||\cdot||_{s,p,\Gamma}$ is a norm. Thus L is an isometry of $W^{s,p}(\Gamma)$ and a closed subspace of $\prod_{i=1}^{l} W^{s,p}(U_i)$ so this proves the lemma since by Corollary 47.7, $W^{s,p}(U_i)$ is reflexive.

We now show that any two such norms are equivalent.

Suppose $\{W_j', \phi_j, \Gamma_j, V_j, \mathbf{g}_j, \mathbf{G}_j\}_{j=1}^{r}$ and $\{W_i, \psi_i, \Gamma_i, U_i, \mathbf{h}_i, \mathbf{H}_i\}_{i=1}^{l}$ both satisfy the conditions described above. Let $||\cdot||_{s,p,\Gamma}^{1}$ denote the norm defined by

$$||u||_{s,p,\Gamma}^{1} \equiv \sum_{j=1}^{r} \left|\left|\mathbf{g}_j^*(u\phi_j)\right|\right|_{s,p,V_j}$$

$$\leq \sum_{j=1}^{r} \left|\left|\mathbf{g}_j^*\left(\sum_{i=1}^{l} u\phi_j\psi_i\right)\right|\right|_{s,p,V_j} \leq \sum_{j,i} \left|\left|\mathbf{g}_j^*(u\phi_j\psi_i)\right|\right|_{s,p,V_j}$$

$$= \sum_{j,i} \left|\left|\mathbf{g}_j^*(u\phi_j\psi_i)\right|\right|_{s,p,\mathbf{g}_j^{-1}(W_i\cap W_j')} \tag{48.2}$$

Now we may define a new norm $||u||_{s,p,\Gamma}^{1,\mathbf{g}}$ by the formula 48.2. This norm is determined by

$$\{W_j' \cap W_i, \psi_i\phi_j, \Gamma_j \cap \Gamma_i, V_j, \mathbf{g}_{i,j}, \mathbf{G}_{i,j}\}$$

where $\mathbf{g}_{i,j} = \mathbf{g}_j$. Thus the identity map is continuous from $\left(W^{s,p}(\Gamma), ||\cdot||_{s,p,\Gamma}^{1,\mathbf{g}}\right)$ to $\left(W^{s,p}(\Gamma), ||\cdot||_{s,p,\Gamma}^{1}\right)$. It follows the two norms, $||\cdot||_{s,p,\Gamma}^{1,\mathbf{g}}$ and $||\cdot||_{s,p,\Gamma}^{1}$, are equivalent by the open mapping theorem. In a similar way, the norms, $||\cdot||_{s,p,\Gamma}^{2,\mathbf{h}}$ and $||\cdot||_{s,p,\Gamma}^{2}$ are equivalent where

$$||u||_{s,p,\Gamma}^{2} \equiv \sum_{j=1}^{l} ||\mathbf{h}_i^*(u\psi_i)||_{s,p,U_i}$$

and

$$||u||_{s,p,\Gamma}^{2,\mathbf{h}} \equiv \sum_{j,i} \left|\left|\mathbf{h}_i^* \left(u\phi_j\psi_i\right)\right|\right|_{s,p,U_i} . = \sum_{j,i} \left|\left|\mathbf{h}_i^* \left(u\phi_j\psi_i\right)\right|\right|_{s,p,\mathbf{h}_i^{-1}\left(W_i\cap W_j'\right)}$$

But from the assumptions on $\mathbf{h}$ and $\mathbf{g}$, in particular the assumption that these are restrictions of functions which are defined on open subsets of $\mathbb{R}^m$ which have Lipschitz derivatives up to order k along with their inverses, we know from Theorem 47.9, there exist constants C_i, independent of u such that

$$\left|\left|\mathbf{h}_i^* \left(u\phi_j\psi_i\right)\right|\right|_{s,p,\mathbf{h}_i^{-1}\left(W_i\cap W_j'\right)} \leq C_1 \left|\left|\mathbf{g}_j^* \left(u\phi_j\psi_i\right)\right|\right|_{s,p,\mathbf{g}_j^{-1}\left(W_i\cap W_j'\right)}$$

and

$$\left|\left|\mathbf{g}_j^* \left(u\phi_j\psi_i\right)\right|\right|_{s,p,\mathbf{g}_j^{-1}\left(W_i\cap W_j'\right)} \leq C_2 \left|\left|\mathbf{h}_i^* \left(u\phi_j\psi_i\right)\right|\right|_{s,p,\mathbf{h}_i^{-1}\left(W_i\cap W_j'\right)} .$$

Therefore, the two norms, $||\cdot||_{s,p,\Gamma}^{1,\mathbf{g}}$ and $||\cdot||_{s,p,\Gamma}^{2,\mathbf{h}}$ are equivalent. It follows that the norms, $||\cdot||_{s,p,\Gamma}^{2}$ and $||\cdot||_{s,p,\Gamma}^{1}$ are equivalent. This proves the following theorem.

Theorem 48.3 *Let Γ be described above. Then we may define $W^{s,p}(\Gamma)$ as in Definition 41.36 and any two norms like those given in this definition are equivalent.*

48.2 The Trace On The Boundary Of An Open Set

Next we generalize earlier theorems about the loss of $\frac{1}{p}$ derivatives on the boundary.

Definition 48.4 *We define*

$$\mathbb{R}_k^{n-1} \equiv \{\mathbf{x} \in \mathbb{R}^n : x_k = 0\}, \widehat{\mathbf{x}}_k \equiv (x_1, \cdot\cdot\cdot, x_{k-1}, 0, x_{k+1}, \cdot\cdot\cdot, x_n).$$

We will say an open set, Ω is $C^{m,1}$ if there exist open sets, $W_i, i = 0, 1, \cdot\cdot\cdot, l$ such that

$$\Omega = \cup_{i=0}^{l} W_i$$

with $\overline{W_0} \subseteq \Omega$, open sets $U_i \subseteq \mathbb{R}_k^{n-1}$ for some k, and open intervals, (a_i, b_i) containing 0 such that for $i \geq 1$,

$$\partial\Omega \cap W_i = \{\widehat{\mathbf{x}}_k + \phi_i(\widehat{\mathbf{x}}_k)\mathbf{e}_k : \widehat{\mathbf{x}}_k \in U_i\},$$

$$\Omega \cap W_i = \{\widehat{\mathbf{x}}_k + (\phi_i(\widehat{\mathbf{x}}_k) + x_k)\mathbf{e}_k : (\widehat{\mathbf{x}}_k, x_k) \in U_i \times (0, b_i)\},$$

where ϕ_i is Lipschitz with partial derivatives up to order m also Lipschitz. Note that it makes no difference whether we use $(0, b_i)$ or $(a_i, 0)$ in the last part of this definition since we can go from one to the other by a simple change if the ϕ_i.

Assume $\Omega \in C^{m-1,1}$. Then, if we define

$$\mathbf{h}_i(\widehat{\mathbf{x}}_k) = \widehat{\mathbf{x}}_k + \phi_i(\widehat{\mathbf{x}}_k)\mathbf{e}_k, \mathbf{H}_i(\mathbf{x}) \equiv \widehat{\mathbf{x}}_k + (\phi_i(\widehat{\mathbf{x}}_k) + x_k)\mathbf{e}_k,$$

and let $\psi_i \in C_c^\infty(W_i)$ with $\sum_{i=0}^l \psi_i(\mathbf{x}) = 1$ on $\overline{\Omega}$, we see that

$$\{W_i, \psi_i, \partial\Omega \cap W_i, U_i, \mathbf{h}_i, \mathbf{H}_i\}_{i=1}^l$$

satisfies all the conditions for defining $W^{s,p}(\partial\Omega)$ for $s \le m$. Let $u \in C^\infty\left(\overline{\Omega}\right)$ and let $\mathbf{h}_i$ be as just described. Using Theorem 47.8, and Theorem 40.14,

$$||\gamma u||_{m-\frac{1}{p},p,\partial\Omega} = \sum_{i=1}^l ||\mathbf{h}_i^*(\psi_i \gamma u)||_{m-\frac{1}{p},p,U_i}$$

$$= \sum_{i=1}^l ||\mathbf{h}_i^*(\psi_i \gamma u)||_{m-\frac{1}{p},p,\mathbb{R}_k^{n-1}} \le C \sum_{i=1}^l ||\mathbf{H}_i^*(\psi_i u)||_{m,p,\mathbb{R}_+^n}$$

$$\le C \sum_{i=1}^l ||\mathbf{H}_i^*(\psi_i u)||_{m,p,U_i\times(0,b_i)} \le C \sum_{i=1}^l ||(\psi_i u)||_{m,p,W_i\cap\Omega}$$

$$\le C \sum_{i=1}^l ||(\psi_i u)||_{m,p,\Omega} \le C \sum_{i=1}^l ||u||_{m,p,\Omega} \le Cl\,||u||_{m,p,\Omega}\,.$$

Now we use the density of $C^\infty\left(\overline{\Omega}\right)$ in $W^{m,p}(\Omega)$ to see that γ extends to a continuous linear map defined on $W^{m,p}(\Omega)$ still called γ such that for all $u \in W^{m,p}(\Omega)$,

$$||\gamma u||_{m-\frac{1}{p},p,\partial\Omega} \le Cl\,||u||_{m,p,\Omega}\,. \tag{48.3}$$

In addition to this, in the case where $m = 1$, we may use Lemma 47.5 to obtain a continuous linear map, R, from $W^{1-\frac{1}{p},p}(\partial\Omega)$ to $W^{1,p}(\Omega)$ which has the property that $\gamma Rg = g$ for every $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$. Letting $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$,

$$g = \sum_{i=1}^l \psi_i g.$$

Then also,

$$\mathbf{h}_i^*(\psi_i g) \in W^{1-\frac{1}{p},p}\left(\mathbb{R}^{n-1}\right)$$

and so from Lemma 47.5, we can extend this to $W^{1,p}\left(\mathbb{R}_+^n\right)$, $R\mathbf{h}_i^*(\psi_i g)$. We may also assume that $R\mathbf{h}_i^*(\psi_i g) \in W^{1,p}(U_i \times (0,b_i))$. We can accomplish this by multiplying by a suitable cut off function in the definition of R or else adjusting the function, ψ occuring in the proof of this lemma so that it vanishes off $(0,b_i)$. Then our extension is

$$Rg = \sum_{i=1}^l \left(\mathbf{H}_i^{-1}\right)^* R\mathbf{h}_i^*(\psi_i g)\,.$$

This works because

$$\begin{aligned}\gamma Rg &\equiv \sum_{i=1}^{l} \gamma \left(\mathbf{H}_i^{-1}\right)^* R\mathbf{h}_i^* \left(\psi_i g\right) \\ &= \sum_{i=1}^{l} \left(\mathbf{H}_i^{-1}\right)^* \gamma R\mathbf{h}_i^* \left(\psi_i g\right) \\ &= \sum_{i=1}^{l} \left(\mathbf{H}_i^{-1}\right)^* \mathbf{h}_i^* \left(\psi_i g\right) = g.\end{aligned}$$

This proves the following theorem about the trace.

Theorem 48.5 *Let $\Omega \in C^{m,1}$. Then there exists a constant, C independent of $u \in W^{m,p}(\Omega)$ and a continuous linear map, $\gamma : W^{m,p}(\Omega) \to W^{m-\frac{1}{p},p}(\partial\Omega)$ such that 48.3 holds. This map satisfies $\gamma u(\mathbf{x}) = u(\mathbf{x})$ for all $u \in C^\infty\left(\overline{\Omega}\right)$. In the case where $m = 1$, we obtain also the existence of a continuous linear map, $R : W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega)$ which has the property that $\gamma Rg = g$ for all $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$.*

The Hausdorff Maximal Theorem

The purpose of this appendix is to prove the equivalence between the axiom of choice, the Hausdorff maximal theorem, and the well-ordering principle. The Hausdorff maximal theorem and the well-ordering principle are very useful but a little hard to believe; so, it may be surprising that they are equivalent to the axiom of choice. First it is shown that the axiom of choice implies the Hausdorff maximal theorem, a remarkable theorem about partially ordered sets.

A nonempty set is partially ordered if there exists a partial order, $\prec$, satisfying

$$x \prec x$$

and

$$\text{if } x \prec y \text{ and } y \prec z \text{ then } x \prec z.$$

An example of a partially ordered set is the set of all subsets of a given set and $\prec \equiv \subseteq$. Note that two elements in a partially ordered sets may not be related. In other words, just because $x,\ y$ are in the partially ordered set, it does not follow that either $x \prec y$ or $y \prec x$. A subset of a partially ordered set, $\mathcal{C}$, is called a chain if $x,\ y \in \mathcal{C}$ implies that either $x \prec y$ or $y \prec x$. If either $x \prec y$ or $y \prec x$ then x and y are described as being comparable. A chain is also called a totally ordered set. $\mathcal{C}$ is a maximal chain if whenever $\widetilde{\mathcal{C}}$ is a chain containing $\mathcal{C}$, it follows the two chains are equal. In other words $\mathcal{C}$ is a maximal chain if there is no strictly larger chain.

Lemma A.1 *Let $\mathcal{F}$ be a nonempty partially ordered set with partial order $\prec$. Then assuming the axiom of choice, there exists a maximal chain in $\mathcal{F}$.*

Proof: Let $\mathcal{X}$ be the set of all chains from $\mathcal{F}$. For $\mathcal{C} \in \mathcal{X}$, let

$$S_{\mathcal{C}} = \{x \in \mathcal{F} \text{ such that } \mathcal{C} \cup \{x\} \text{ is a chain strictly larger than } \mathcal{C}\}.$$

If $S_{\mathcal{C}} = \emptyset$ for any $\mathcal{C}$, then $\mathcal{C}$ is maximal. Thus, assume $S_{\mathcal{C}} \neq \emptyset$ for all $\mathcal{C} \in \mathcal{X}$. Let $f(\mathcal{C}) \in S_{\mathcal{C}}$. (This is where the axiom of choice is being used.) Let

$$g(\mathcal{C}) = \mathcal{C} \cup \{f(\mathcal{C})\}.$$

Thus $g(\mathcal{C}) \supsetneq \mathcal{C}$ and $g(\mathcal{C}) \setminus \mathcal{C} = \{f(\mathcal{C})\} = \{$a single element of $\mathcal{F}\}$. A subset $\mathcal{T}$ of $\mathcal{X}$ is called a tower if

$$\emptyset \in \mathcal{T},$$

$$\mathcal{C} \in \mathcal{T} \text{ implies } g(\mathcal{C}) \in \mathcal{T},$$

and if $\mathcal{S} \subseteq \mathcal{T}$ is totally ordered with respect to set inclusion, then

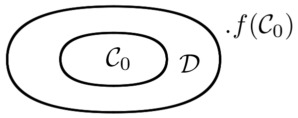
$$\cup \mathcal{S} \in \mathcal{T}.$$

Here $\mathcal{S}$ is a chain with respect to set inclusion whose elements are chains.

Note that $\mathcal{X}$ is a tower. Let $\mathcal{T}_0$ be the intersection of all towers. Thus, $\mathcal{T}_0$ is a tower, the smallest tower. Are any two sets in $\mathcal{T}_0$ comparable in the sense of set inclusion so that $\mathcal{T}_0$ is actually a chain? Let $\mathcal{C}_0$ be a set of $\mathcal{T}_0$ which is comparable to every set of $\mathcal{T}_0$. Such sets exist, $\emptyset$ being an example. Let

$$\mathcal{B} \equiv \{\mathcal{D} \in \mathcal{T}_0 : \mathcal{D} \supsetneq \mathcal{C}_0 \text{ and } f(\mathcal{C}_0) \notin \mathcal{D}\}.$$

The picture represents sets of $\mathcal{B}$. As illustrated in the picture, $\mathcal{D}$ is a set of $\mathcal{B}$ when $\mathcal{D}$ is larger than $\mathcal{C}_0$ but fails to be comparable to $g(\mathcal{C}_0)$. Thus there would be more than one chain ascending from $\mathcal{C}_0$ if $\mathcal{B} \neq \emptyset$, rather like a tree growing upward in more than one direction from a fork in the trunk. It will be shown this can't take place for any such $\mathcal{C}_0$ by showing $\mathcal{B} = \emptyset$.



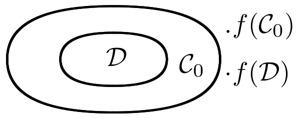
This will be accomplished by showing $\widetilde{\mathcal{T}_0} \equiv \mathcal{T}_0 \setminus \mathcal{B}$ is a tower. Since $\mathcal{T}_0$ is the smallest tower, this will require that $\widetilde{\mathcal{T}_0} = \mathcal{T}_0$ and so $\mathcal{B} = \emptyset$.

Claim: $\widetilde{\mathcal{T}_0}$ is a tower and so $\mathcal{B} = \emptyset$.

Proof of the claim: It is clear that $\emptyset \in \widetilde{\mathcal{T}_0}$ because for $\emptyset$ to be contained in $\mathcal{B}$ it would be required to properly contain $\mathcal{C}_0$ which is not possible. Suppose $\mathcal{D} \in \widetilde{\mathcal{T}_0}$. The plan is to verify $g(\mathcal{D}) \in \widetilde{\mathcal{T}_0}$.

Case 1: $f(\mathcal{D}) \in \mathcal{C}_0$. If $\mathcal{D} \subseteq \mathcal{C}_0$, then since both $\mathcal{D}$ and $\{f(\mathcal{D})\}$ are contained in $\mathcal{C}_0$, it follows $g(\mathcal{D}) \subseteq \mathcal{C}_0$ and so $g(\mathcal{D}) \notin \mathcal{B}$. On the other hand, if $\mathcal{D} \supsetneq \mathcal{C}_0$, then since $\mathcal{D} \in \widetilde{\mathcal{T}_0}$, $f(\mathcal{C}_0) \in \mathcal{D}$ and so $g(\mathcal{D})$ also contains $f(\mathcal{C}_0)$ implying $g(\mathcal{D}) \notin \mathcal{B}$. These are the only two cases to consider because $\mathcal{C}_0$ is comparable to every set of $\mathcal{T}_0$.

Case 2: $f(\mathcal{D}) \notin \mathcal{C}_0$. If $\mathcal{D} \subsetneq \mathcal{C}_0$ it can't be the case that $f(\mathcal{D}) \notin \mathcal{C}_0$ because if this were so, $g(\mathcal{D})$ would not compare to $\mathcal{C}_0$.



Hence if $f(\mathcal{D}) \notin \mathcal{C}_0$, then $\mathcal{D} \supseteq \mathcal{C}_0$. If $\mathcal{D} = \mathcal{C}_0$, then $f(\mathcal{D}) = f(\mathcal{C}_0) \in g(\mathcal{D})$ so

$g(\mathcal{D}) \notin \mathcal{B}$. Therefore, assume $\mathcal{D} \supsetneq \mathcal{C}_0$. Then, since $\mathcal{D}$ is in $\widetilde{\mathcal{T}_0}$, $f(\mathcal{C}_0) \in \mathcal{D}$ and so $f(\mathcal{C}_0) \in g(\mathcal{D})$. Therefore, $g(\mathcal{D}) \in \widetilde{\mathcal{T}_0}$.

Now suppose $\mathcal{S}$ is a totally ordered subset of $\widetilde{\mathcal{T}_0}$ with respect to set inclusion. Then if every element of $\mathcal{S}$ is contained in $\mathcal{C}_0$, so is $\cup\mathcal{S}$ and so $\cup\mathcal{S} \in \widetilde{\mathcal{T}_0}$. If, on the other hand, some chain from $\mathcal{S}$, $\mathcal{C}$, contains $\mathcal{C}_0$ properly, then since $\mathcal{C} \notin \mathcal{B}$, $f(\mathcal{C}_0) \in \mathcal{C} \subseteq \cup\mathcal{S}$ showing that $\cup\mathcal{S} \notin \mathcal{B}$ also. This has proved $\widetilde{\mathcal{T}_0}$ is a tower and since $\mathcal{T}_0$ is the smallest tower, it follows $\widetilde{\mathcal{T}_0} = \mathcal{T}_0$. This has shown roughly that no splitting into more than one ascending chain can occur at any $\mathcal{C}_0$ which is comparable to every set of $\mathcal{T}_0$. Next it is shown that every element of $\mathcal{T}_0$ has the property that it is comparable to all other elements of $\mathcal{T}_0$. This is done by showing that these elements which possess this property form a tower.

Define $\mathcal{T}_1$ to be the set of all elements of $\mathcal{T}_0$ which are comparable to every element of $\mathcal{T}_0$. (Recall the elements of $\mathcal{T}_0$ are chains from the original partial order.)

Claim: $\mathcal{T}_1$ is a tower.

Proof of the claim: It is clear that $\emptyset \in \mathcal{T}_1$ because $\emptyset$ is a subset of every set. Suppose $\mathcal{C}_0 \in \mathcal{T}_1$. It is necessary to verify that $g(\mathcal{C}_0) \in \mathcal{T}_1$. Let $\mathcal{D} \in \mathcal{T}_0$ (Thus $\mathcal{D} \subseteq \mathcal{C}_0$ or else $\mathcal{D} \supsetneq \mathcal{C}_0$.)and consider $g(\mathcal{C}_0) \equiv \mathcal{C}_0 \cup \{f(\mathcal{C}_0)\}$. If $\mathcal{D} \subseteq \mathcal{C}_0$, then $\mathcal{D} \subseteq g(\mathcal{C}_0)$ so $g(\mathcal{C}_0)$ is comparable to $\mathcal{D}$. If $\mathcal{D} \supsetneq \mathcal{C}_0$, then $\mathcal{D} \supseteq g(\mathcal{C}_0)$ by what was just shown ($\mathcal{B} = \emptyset$). Hence $g(\mathcal{C}_0)$ is comparable to $\mathcal{D}$. Since $\mathcal{D}$ was arbitrary, it follows $g(\mathcal{C}_0)$ is comparable to every set of $\mathcal{T}_0$. Now suppose $\mathcal{S}$ is a chain of elements of $\mathcal{T}_1$ and let $\mathcal{D}$ be an element of $\mathcal{T}_0$. If every element in the chain, $\mathcal{S}$ is contained in $\mathcal{D}$, then $\cup\mathcal{S}$ is also contained in $\mathcal{D}$. On the other hand, if some set, $\mathcal{C}$, from $\mathcal{S}$ contains $\mathcal{D}$ properly, then $\cup\mathcal{S}$ also contains $\mathcal{D}$. Thus $\cup\mathcal{S} \in \mathcal{T}_1$ since it is comparable to every $\mathcal{D} \in \mathcal{T}_0$.

This shows $\mathcal{T}_1$ is a tower and proves therefore, that $\mathcal{T}_0 = \mathcal{T}_1$. Thus every set of $\mathcal{T}_0$ compares with every other set of $\mathcal{T}_0$ showing $\mathcal{T}_0$ is a chain in addition to being a tower.

Now $\cup\mathcal{T}_0, g(\cup\mathcal{T}_0) \in \mathcal{T}_0$. Hence, because $g(\cup\mathcal{T}_0)$ is an element of $\mathcal{T}_0$, and $\mathcal{T}_0$ is a chain of these, it follows $g(\cup\mathcal{T}_0) \subseteq \cup\mathcal{T}_0$. Thus

$$\cup\mathcal{T}_0 \supseteq g(\cup\mathcal{T}_0) \supsetneq \cup\mathcal{T}_0,$$

a contradiction. Hence there must exist a maximal chain after all. This proves the lemma.

If X is a nonempty set,$\leq$ is an order on X if

$$x \leq x,$$

and if $x,\ y \in X$, then

$$\text{either } x \leq y \text{ or } y \leq x$$

and

$$\text{if } x \leq y \text{ and } y \leq z \text{ then } x \leq z.$$

$\leq$ is a well order and say that $(X, \leq)$ is a well-ordered set if every nonempty subset of X has a smallest element. More precisely, if $S \neq \emptyset$ and $S \subseteq X$ then there exists an $x \in S$ such that $x \leq y$ for all $y \in S$. A familiar example of a well-ordered set is the natural numbers.

Lemma A.2 *The Hausdorff maximal principle implies every nonempty set can be well-ordered.*

Proof: Let X be a nonempty set and let $a \in X$. Then $\{a\}$ is a well-ordered subset of X. Let

$$\mathcal{F} = \{S \subseteq X : \text{ there exists a well order for } S\}.$$

Thus $\mathcal{F} \neq \emptyset$. For $S_1,\ S_2 \in \mathcal{F}$, define $S_1 \prec S_2$ if $S_1 \subseteq S_2$ and there exists a well order for S_2, $\leq_2$ such that

$$(S_2, \leq_2) \text{ is well-ordered}$$

and if

$$y \in S_2 \setminus S_1 \text{ then } x \leq_2 y \text{ for all } x \in S_1,$$

and if $\leq_1$is the well order of S_1 then the two orders are consistent on S_1. Then observe that $\prec$ is a partial order on $\mathcal{F}$. By the Hausdorff maximal principle, let $\mathcal{C}$ be a maximal chain in $\mathcal{F}$ and let

$$X_\infty \equiv \cup \mathcal{C}.$$

Define an order, $\leq$, on X_∞ as follows. If $x,\ y$ are elements of X_∞, pick $S \in \mathcal{C}$ such that $x,\ y$ are both in S. Then if $\leq_S$ is the order on S, let $x \leq y$ if and only if $x \leq_S y$. This definition is well defined because of the definition of the order, $\prec$. Now let U be any nonempty subset of X_∞. Then $S \cap U \neq \emptyset$ for some $S \in \mathcal{C}$. Because of the definition of $\leq$, if $y \in S_2 \setminus S_1,\ S_i \in \mathcal{C}$, then $x \leq y$ for all $x \in S_1$. Thus, if $y \in X_\infty \setminus S$ then $x \leq y$ for all $x \in S$ and so the smallest element of $S \cap U$ exists and is the smallest element in U. Therefore X_∞ is well-ordered. Now suppose there exists $z \in X \setminus X_\infty$. Define the following order, $\leq_1$, on $X_\infty \cup \{z\}$.

$$x \leq_1 y \text{ if and only if } x \leq y \text{ whenever } x, y \in X_\infty$$

$$x \leq_1 z \text{ whenever } x \in X_\infty.$$

Then let

$$\widetilde{\mathcal{C}} = \{S \in \mathcal{C} \text{ or } X_\infty \cup \{z\}\}.$$

Then $\widetilde{\mathcal{C}}$ is a strictly larger chain than $\mathcal{C}$ contradicting maximality of $\mathcal{C}$. Thus $X \setminus X_\infty = \emptyset$ and this shows X is well-ordered by $\leq$. This proves the lemma.

With these two lemmas the main result follows.

Theorem A.3 *The following are equivalent.*

The axiom of choice

The Hausdorff maximal principle

The well-ordering principle.

Proof: It only remains to prove that the well-ordering principle implies the axiom of choice. Let I be a nonempty set and let X_i be a nonempty set for each $i \in I$. Let $X = \cup\{X_i : i \in I\}$ and well order X. Let $f(i)$ be the smallest element of X_i. Then

$$f \in \prod_{i \in I} X_i.$$

A.1 Exercises

1. Zorn's lemma states that in a nonempty partially ordered set, if every chain has an upper bound, there exists a maximal element, x in the partially ordered set. x is maximal, means that if $x \prec y$, it follows $y = x$. Show Zorn's lemma is equivalent to the Hausdorff maximal theorem.

2. Let X be a vector space. $Y \subseteq X$ is a Hamel basis if every element of X can be written in a unique way as a finite linear combination of elements in Y. Show that every vector space has a Hamel basis and that if Y, Y_1 are two Hamel bases of X, then there exists a one to one and onto map from Y to Y_1.

3. $\uparrow$ Using the Baire category theorem of the chapter on Banach spaces show that any Hamel basis of a Banach space is either finite or uncountable.

4. $\uparrow$ Consider the vector space of all polynomials defined on $[0,1]$. Does there exist a norm, $||\cdot||$ defined on these polynomials such that with this norm, the vector space of polynomials becomes a Banach space (complete normed vector space)?

Bibliography

[1] **Adams R.** *Sobolev Spaces*, Academic Press, New York, San Francisco, London, 1975.

[2] **Alfors, Lars** *Complex Analysis,* McGraw Hill 1966.

[3] **Apostol, T. M.,** *Mathematical Analysis,* Addison Wesley Publishing Co., 1969.

[4] **Apostol, T. M.,** *Calculus second edition,* Wiley, 1967.

[5] **Apostol, T. M.,** *Mathematical Analysis,* Addison Wesley Publishing Co., 1974.

[6] **Ash, Robert,** *Complex Variables*, Academic Press, 1971.

[7] **Baker, Roger**, *Linear Algebra*, Rinton Press 2001.

[8] **Bergh J. and Löfström J**. *Interpolation Spaces,* Springer Verlag 1976.

[9] **Billingsley P.,** *Probability and Measure,* Wiley, 1995.

[10] **Bledsoe W.W.**, *Am. Math. Monthly* vol. 77, PP. 180-182 1970.

[11] **Bogachev Vladimir I.** *Gaussian Measures* American Mathematical Society Mathematical Surveys and Monographs, volume 62 1998.

[12] **Bruckner A. , Bruckner J., and Thomson B.,** *Real Analysis* Prentice Hall 1997.

[13] **Conway J. B.** *Functions of one Complex variable Second edition,* Springer Verlag 1978.

[14] **Cheney, E. W.,** *Introduction To Approximation Theory,* McGraw Hill 1966.

[15] **Da Prato, G.** and **Zabczyk J.,** *Stochastic Equations in Infinite Dimensions*, Cambridge 1992.

[16] **Diestal J.** and **Uhl J.,** *Vector Measures,* American Math. Society, Providence, R.I., 1977.

[17] **Dontchev A.L.** The Graves theorem Revisited, *Journal of Convex Analysis,* Vol. 3, 1996, No.1, 45-53.

[18] **Dunford N.** and **Schwartz J.T.** *Linear Operators,* Interscience Publishers, a division of John Wiley and Sons, New York, part 1 1958, part 2 1963, part 3 1971.

[19] **Duvaut, G.** and **Lions**, **J. L.**, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[20] **Evans L.C.** and **Gariepy,** *Measure Theory and Fine Properties of Functions,* CRC Press, 1992.

[21] **Evans L.C.** *Partial Differential Equations,* Berkeley Mathematics Lecture Notes. 1993.

[22] **Federer H.,** *Geometric Measure Theory,* Springer-Verlag, New York, 1969.

[23] **Gagliardo, E.**, Properieta di alcune classi di funzioni in piu variabili, *Ricerche Mat.* 7 (1958), 102-137.

[24] **Grisvard, P.** *Elliptic problems in nonsmooth domains,* Pittman 1985.

[25] **Gross L.** Abstract Wiener Spaces, Proc. fifth Berkeley Sym. Math. Stat. Prob. 1965.

[26] **Hewitt E.** and **Stromberg K.** *Real and Abstract Analysis,* Springer-Verlag, New York, 1965.

[27] **Hille Einar,** *Analytic Function Theory,* Ginn and Company 1962.

[28] **Hörmander, Lars** *Linear Partial Differrential Operators,* Springer Verlag, *1976.*

[29] **Hörmander L.** Estimates for translation invariant operators in L^p spaces, *Acta Math.* 104 1960, 93-139.

[30] **Hui-Hsiung Kuo** Gaussian Measures in Banach Spaces *Lecture notes in Mathematics* Springer number 463 1975.

[31] **John, Fritz,** *Partial Differential Equations,* Fourth edition, Springer Verlag, 1982.

[32] **Jones F.,** *Lebesgue Integration on Euclidean Space,* Jones and Bartlett 1993.

[33] **Karatzas and Shreve,** *Brownian Motion and Stochastic Calculus, Springer Verlag, 1991.*

[34] **Kuratowski K.** and **Ryll-Nardzewski C.** A general theorem on selectors, *Bull. Acad. Pol. Sc.,* **13,** 397-403.

[35] **Kuttler K.L.** *Basic Analysis.* Rinton Press. November 2001.

[36] **Kuttler K.L.,** *Modern Analysis* CRC Press 1998.

[37] **Levinson, N. and Redheffer, R**. *Complex Variables,* Holden Day, Inc. 1970

[38] **Liptser, R.S. Nd Shiryaev, A. N.** *Statistics of Random Processes.* Vol I General theory. Springer Verlag, New York 1977.

[39] **Markushevich, A.I.,** *Theory of Functions of a Complex Variable,* Prentice Hall, 1965.

[40] **McShane E. J.** *Integration,* Princeton University Press, Princeton, N.J. 1944.

[41] **Nečas J.** and **Hlaváček,** *Mathematical Theory of Elasic and Elasto-Plastic Bodies: An introduction*, Elsevier, 1981.

[42] **Øksendal Bernt** *Stochastic Differential Equations*, Springer 2003.

[43] **Ray W.O.** *Real Analysis,* Prentice-Hall, 1988.

[44] **Rudin, W.,** *Principles of mathematical analysis,* McGraw Hill third edition 1976

[45] **Rudin W.** *Real and Complex Analysis,* third edition, McGraw-Hill, 1987.

[46] **Rudin W.** *Functional Analysis,* second edition, McGraw-Hill, 1991.

[47] **Saks and Zygmund,** *Analytic functions*, 1952. (This book is available on the web. http://www.geocities.com/alex_stef/mylist.html#FuncAn)

[48] **Smart D.R.** *Fixed point theorems* Cambridge University Press, 1974.

[49] **Stein E.** *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, N. J., 1970.

[50] **Triebel H.** *Interpolation Theory, Function Spaces and Differential Operators,* North Holland, Amsterdam, 1978.

[51] **Varga R. S.** *Functional Analysis and Approximation Theory in Numerical Analysis,* SIAM 1984.

[52] **Yosida K.** *Functional Analysis,* Springer-Verlag, New York, 1978.

Index