**REVIEW ARTICLE**

# An Introduction to Conformal Field Theory

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**Abstract.** A comprehensive introduction to two-dimensional conformal field theory is given.

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## 1. Introduction

Conformal field theories have been at the centre of much attention during the last fifteen years since they are relevant for at least three different areas of modern theoretical physics: conformal field theories provide toy models for genuinely interacting quantum field theories, they describe two-dimensional critical phenomena, and they play a central rôle in string theory, at present the most promising candidate for a unifying theory of all forces. Conformal field theories have also had a major impact on various aspects of modern mathematics, in particular the theory of vertex operator algebras and Borcherds algebras, finite groups, number theory and low-dimensional topology.

From an abstract point of view, conformal field theories are Euclidean quantum field theories that are characterised by the property that their symmetry group contains, in addition to the Euclidean symmetries, local conformal transformations, *i.e.* transformations that preserve angles but not lengths. The local conformal symmetry is of special importance in two dimensions since the corresponding symmetry algebra is infinite-dimensional in this case. As a consequence, two-dimensional conformal field theories have an infinite number of conserved quantities, and are completely solvable by symmetry considerations alone.

As a bona fide quantum field theory, the requirement of conformal invariance is very restrictive. In particular, since the theory is scale invariant, all particle-like excitations of the theory are necessarily massless. This might be seen as a strong argument against any possible physical relevance of such theories. However, all particles of any (two-dimensional) quantum field theory are approximately massless in the limit of high energy, and many structural features of quantum field theories are believed to be unchanged in this approximation. Furthermore, it is possible to analyse deformations of conformal field theories that describe integrable massive models [1, 2]. Finally, it might be hoped that a good mathematical understanding of interactions in any model theory should have implications for realistic theories.

The more recent interest in conformal field theories has different origins. In the description of statistical mechanics in terms of Euclidean quantum field theories, conformal field theories describe systems at the critical point, where the correlation length diverges. One simple system where this occurs is the so-called *Ising model*. This model is formulated in terms of a two-dimensional lattice whose lattice sites represent atoms of an (infinite) two-dimensional crystal. Each atom is taken to have a spin variable $\sigma_i$ that can take the values $\pm 1$, and the magnetic energy of the system is the sum over pairs of adjacent atoms

$$E = \sum_{(ij)} \sigma_i \sigma_j \,. \tag{1}$$

If we consider the system at a finite temperature $T$, the thermal average $\langle \cdots \rangle$ behaves as

$$\langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \cdot \langle \sigma_j \rangle \sim \exp\left(-\frac{|i-j|}{\xi}\right) \,, \tag{2}$$

where $|i - j| \gg 1$ and $\xi$ is the so-called *correlation length* that is a function of the temperature $T$. Observable (magnetic) properties can be derived from such correlation functions, and are therefore directly affected by the actual value of $\xi$.

The system possesses a *critical temperature*, at which the correlation length $\xi$ diverges, and the exponential decay in (2) is replaced by a power law. The continuum theory that describes the correlation functions for distances that are large compared to the lattice spacing is then scale invariant. Every scale-invariant two-dimensional local quantum field theory is actually conformally invariant [3], and the critical point of the Ising model is therefore described by a conformal field theory [4]. (The conformal field theory in question will be briefly described at the end of section 4.)

The Ising model is only a rather rough approximation to the actual physical system. However, the continuum theory at the critical point — and in particular the different *critical exponents* that describe the power law behaviour of the correlation functions at the critical point — are believed to be fairly insensitive to the details of the chosen model; this is the idea of *universality*. Thus conformal field theory is a very important method in the study of critical systems.

The second main area in which conformal field theory has played a major rôle is *string theory* [5, 6]. String theory is a generalised quantum field theory in which the basic objects are not point particles (as in ordinary quantum field theory) but one dimensional strings. These strings can either form closed loops (*closed string theory*), or they can have two end-points, in which case the theory is called *open string theory*. Strings interact by joining together and splitting into two; compared to the interaction of point particles where two particles come arbitrarily close together, the interaction of strings is more spread out, and thus many divergencies of ordinary quantum field theory are absent.

Unlike point particles, a string has internal degrees of freedom that describe the different ways in which it can vibrate in the ambient space-time. These different vibrational modes are interpreted as the 'particles' of the theory — in particular, the whole particle spectrum of the theory is determined in terms of one fundamental object. The vibrations of the string are most easily described from the point of view of the so-called *world-sheet*, the two-dimensional surface that the string sweeps out as it propagates through space-time; in fact, as a theory on the world-sheet the vibrations of the string are described by a conformal field theory.

In closed string theory, the oscillations of the string can be decomposed into two waves which move in opposite directions around the loop. These two waves are essentially independent of each other, and the theory therefore factorises into two so-called *chiral conformal field theories*. Many properties of the local theory can be studied separately for the two chiral theories, and we shall therefore mainly analyse the chiral theory in this article. The main advantage of this approach is that the chiral theory can be studied using the powerful tools of complex analysis since its correlation functions are analytic functions. The chiral theories also play a crucial rôle for conformal field theories that are defined on manifolds with boundaries, and that are relevant for the

description of open string theory.

All known consistent string theories can be obtained by compactification from a rather small number of theories. These include the five different supersymmetric string theories in ten dimensions, as well as a number of non-supersymmetric theories that are defined in either ten or twenty-six dimensions. The recent advances in string theory have centered around the idea of *duality*, namely that these theories are further related in the sense that the strong coupling regime of one theory is described by the weak coupling regime of another. A crucial element in these developments has been the realisation that the *solitonic* objects that define the relevant degrees of freedom at strong coupling are *Dirichlet-branes* that have an alternative description in terms of open string theory [7]. In fact, the effect of a Dirichlet brane is completely described by adding certain open string sectors (whose end-points are fixed to lie on the world-volume of the brane) to the theory. The possible Dirichlet branes of a given string theory are then selected by the condition that the resulting theory of open and closed strings must be consistent. These consistency conditions contain (and may be equivalent to) the consistency conditions of conformal field theory on a manifold with a boundary [8–10]. Much of the structure of the theory that we shall explain in this review article is directly relevant for an analysis of these questions, although we shall not discuss the actual consistency conditions (and their solutions) here.

Any review article of a well-developed subject such as conformal field theory will miss out important elements of the theory, and this article is no exception. We have chosen to present one coherent route through some section of the theory and we shall not discuss in any detail alternative view points on the subject. The approach that we have taken is in essence algebraic (although we shall touch upon some questions of analysis), and is inspired by the work of Goddard [11] as well as the mathematical theory of *vertex operator algebras* that was developed by Borcherds [12,13], Frenkel, Lepowsky & Meurman [14], Frenkel, Huang & Lepowsky [15], Zhu [16], Kac [17] and others. This algebraic approach will be fairly familiar to many physicists, but we have tried to give it a somewhat new slant by emphasising the fundamental rôle of the amplitudes. We have also tried to explain some of the more recent developments in the mathematical theory of vertex operator algebras that have so far not been widely appreciated in the physics community, in particular, the work of Zhu.

There exist in essence two other view points on the subject: a functional analytic approach in which techniques from algebraic quantum field theory [18] are employed and which has been pioneered by Wassermann [19] and Gabbiani and Fröhlich [20]; and a geometrical approach that is inspired by string theory (for example the work of Friedan & Shenker [21]) and that has been put on a solid mathematical foundation by Segal [22] (see also Huang [23,24]).

We shall also miss out various recent developments of the theory, in particular the progress in understanding conformal field theories on higher genus Riemann surfaces [25–29], and on surfaces with boundaries [30–35].

Finally, we should mention that a number of treatments of conformal field theory are by now available, in particular the review articles of Ginsparg [36] and Gawedzki [37], and the book by Di Francesco, Mathieu and Sénéchal [38]. We have attempted to be somewhat more general, and have put less emphasis on specific well understood models such as the minimal models or the WZNW models (although they will be explained in due course). We have also been more influenced by the mathematical theory of vertex operator algebras, although we have avoided to phrase the theory in this language.

The paper is organised as follows. In section 2, we outline the general structure of the theory, and explain how the various ingredients that will be subsequently described fit together. Section 3 is devoted to the study of meromorphic conformal field theory; this is the part of the theory that describes in essence what is sometimes called the chiral algebra by physicists, or the vertex operator algebra by mathematicians. We also introduce the most important examples of conformal field theories, and describe standard constructions such as the coset and orbifold construction. In section 4 we introduce the concept of a representation of the meromorphic conformal field theory, and explain the rôle of Zhu's algebra in classifying (a certain class of) such representations. Section 5 deals with higher correlation functions and fusion rules. We explain Verlinde's formula, and give a brief account of the polynomial relations of Moore & Seiberg and their relation to quantum groups. We also describe logarithmic conformal field theories. We conclude in section 6 with a number of general open problems that deserve, in our opinion, more work. Finally, we have included an appendix that contains a brief summary about the different definitions of rationality.

## 2. The General Structure of a Local Conformal Field Theory

Let us begin by describing somewhat sketchily what the general structure of a local conformal field theory is, and how the various structures that will be discussed in detail later fit together.

### 2.1. The Space of States

In essence, a two-dimensional conformal field theory (like any other field theory) is determined by its space of states and the collection of its correlation functions. The space of states is a vector space $\mathcal{H}$ (that may or may not be a Hilbert space), and the correlation functions are defined for collections of vectors in some dense subspace $\mathcal{F}$ of $\mathcal{H}$. These correlation functions are defined on a two-dimensional space-time, which we shall always assume to be of Euclidean signature. We shall mainly be interested in the case where the space-time is a closed compact surface. These surfaces are classified (topologically) by their genus $g$ which counts the number of 'handles'; the simplest such surface is the sphere with $g = 0$, the surface with $g = 1$ is the torus, *etc.* In a first step we shall therefore consider conformal field theories that are defined on the sphere; as we shall explain later, under certain conditions it is possible to associate to such a theory

families of theories that are defined on surfaces of arbitrary genus. This is important in the context of string theory where the perturbative expansion consists of a sum over all such theories (where the genus of the surface plays the rôle of the loop order).

One of the special features of conformal field theory is the fact that the theory is naturally defined on a *Riemann surface* (or *complex curve*), *i.e.* on a surface that possesses suitable complex coordinates. In the case of the sphere, the complex coordinates can be taken to be those of the complex plane that cover the sphere except for the point at infinity; complex coordinates around infinity are defined by means of the coordinate function $\gamma(z) = 1/z$ that maps a neighbourhood of infinity to a neighbourhood of 0. With this choice of complex coordinates, the sphere is usually referred to as the *Riemann sphere*, and this choice of complex coordinates is up to some suitable class of reparametrisations unique. The correlation functions of a conformal field theory that is defined on the sphere are thus of the form

$$\langle V(\psi_1; z_1, \bar{z}_1) \cdots V(\psi_n; z_n, \bar{z}_n) \rangle, \tag{3}$$

where $V(\psi, z)$ is the field that is associated to the state $\psi$, $\psi_i \in \mathcal{F} \subset \mathcal{H}$, and $z_i$ and $\bar{z}_i$ are complex numbers (or infinity). These correlation functions are assumed to be *local*, *i.e.* independent of the order in which the fields appear in (3).

One of the properties that makes two-dimensional conformal field theories exactly solvable is the fact that the theory contains a large (infinite-dimensional) symmetry algebra with respect to which the states in $\mathcal{H}$ fall into representations. This symmetry algebra is directly related (in a way we shall describe below) to a certain preferred subspace $\mathcal{F}_0$ of $\mathcal{F}$ that is characterised by the property that the correlation functions (3) of its states depend only on the complex parameter $z$, but not on its complex conjugate $\bar{z}$. More precisely, a state $\psi \in \mathcal{F}$ is in $\mathcal{F}_0$ if for any collection of $\psi_i \in \mathcal{F} \subset \mathcal{H}$, the correlation functions

$$\langle V(\psi; z, \bar{z}) V(\psi_1; z_1, \bar{z}_1) \cdots V(\psi_n; z_n, \bar{z}_n) \rangle \tag{4}$$

do not depend on $\bar{z}$. The correlation functions that involve only states in $\mathcal{F}_0$ are then analytic functions on the sphere. These correlation functions define the meromorphic (sub)theory [11] that will be the main focus of the next section.§

Similarly, we can consider the subspace of states $\overline{\mathcal{F}}_0$ that consists of those states for which the correlation functions of the form (4) do not depend on $z$. These states define an (anti-)meromorphic conformal field theory which can be analysed by the same methods as a meromorphic conformal field theory. The two meromorphic conformal subtheories encode all the information about the symmetries of the theory, and for the most interesting class of theories, the so-called *finite* or *rational* theories, the whole theory can be reconstructed from them up to some finite ambiguity. In essence, this means that the whole theory is determined by symmetry considerations alone, and this is at the heart of the solvability of the theory.

§ Our use of the term *meromorphic* conformal field theory is different from that employed by, *e.g.*, Schellekens [39].

The correlation functions of the theory determine the *operator product expansion* (OPE) of the conformal fields which expresses the operator product of two fields in terms of a sum of single fields. If $\psi_1$ and $\psi_2$ are two arbitrary states in $\mathcal{F}$ then the OPE of $\psi_1$ and $\psi_2$ is an expansion of the form

$$V(\psi_1; z_1, \bar{z}_1)V(\psi_2; z_2, \bar{z}_2)$$
$$= \sum_i (z_1 - z_2)^{\Delta_i}(\bar{z}_1 - \bar{z}_2)^{\bar{\Delta}_i} \sum_{r,s \geq 0} V(\phi^i_{r,s}; z_2, \bar{z}_2)(z_1 - z_2)^r(\bar{z}_1 - \bar{z}_2)^s, \tag{5}$$

where $\Delta_i$ and $\bar{\Delta}_i$ are real numbers, $r, s \in \mathbb{N}$ and $\phi^i_{r,s} \in \mathcal{F}$. The actual form of this expansion can be read off from the correlation functions of the theory since the identity (5) has to hold in *all* correlation functions, *i.e.*

$$\Big\langle V(\psi_1; z_1, \bar{z}_1)V(\psi_2; z_2, \bar{z}_2)V(\phi_1; w_1, \bar{w}_1) \cdots V(\phi_n; w_n, \bar{w}_n) \Big\rangle$$
$$= \sum_i (z_1 - z_2)^{\Delta_i}(\bar{z}_1 - \bar{z}_2)^{\bar{\Delta}_i} \sum_{r,s \geq 0} (z_1 - z_2)^r(\bar{z}_1 - \bar{z}_2)^s$$
$$\Big\langle V(\phi^i_{r,s}; z_2, \bar{z}_2)V(\phi_1; w_1, \bar{w}_1) \cdots V(\phi_n; w_n, \bar{w}_n) \Big\rangle \tag{6}$$

for all $\phi_j \in \mathcal{F}$. If both states $\psi_1$ and $\psi_2$ belong to the meromorphic subtheory $\mathcal{F}_0$, (6) only depends on $z_i$, and $\phi^i_{r,s}$ also belongs to the meromorphic subtheory $\mathcal{F}_0$. The OPE therefore defines a certain product on the meromorphic fields. Since the product involves the complex parameters $z_i$ in a non-trivial way, it does not directly define an algebra; the resulting structure is usually called a *vertex (operator) algebra* in the mathematical literature [12, 14], and we shall adopt this name here as well.

By virtue of its definition in terms of (6), the operator product expansion is *associative*, *i.e.*

$$\Big( V(\psi_1; z_1, \bar{z}_1)V(\psi_2; z_2, \bar{z}_2) \Big)V(\psi_3; z_3, \bar{z}_3) = V(\psi_1; z_1, \bar{z}_1)\Big( V(\psi_2; z_2, \bar{z}_2)V(\psi_3; z_3, \bar{z}_3) \Big), \tag{7}$$

where the brackets indicate which OPE is evaluated first. If we consider the case where both $\psi_1$ and $\psi_2$ are meromorphic fields (*i.e.* in $\mathcal{F}_0$), then the associativity of the OPE implies that the states in $\mathcal{F}$ form a *representation* of the vertex operator algebra. The same also holds for the vertex operator algebra associated to the anti-meromorphic fields, and we can thus decompose the whole space $\mathcal{F}$ (or $\mathcal{H}$) as

$$\mathcal{H} = \bigoplus_{(j,\bar{j})} \mathcal{H}_{(j,\bar{j})}, \tag{8}$$

where each $\mathcal{H}_{(j,\bar{j})}$ is an (indecomposable) representation of the two vertex operator algebras. Finite theories are characterised by the property that only finitely many indecomposable representations of the two vertex operator algebras occur in (8).

## 2.2. Modular Invariance

The decomposition of the space of states in terms of representations of the two vertex operator algebras throws considerable light on the problem of whether the theory is well-defined on higher Riemann surfaces. One necessary constraint for this (which is believed

also to be sufficient [40]) is that the vacuum correlator on the torus is independent of its parametrisation. Every two-dimensional torus can be described as the quotient space of $\mathbb{R}^2 \simeq \mathbb{C}$ by the relations $z \sim z + w_1$ and $z \sim z + w_2$, where $w_1$ and $w_2$ are not parallel. The complex structure of the torus is invariant under rotations and rescalings of $\mathbb{C}$, and therefore every torus is conformally equivalent to (*i.e.* has the same complex structure as) a torus for which the relations are $z \sim z + 1$ and $z \sim z + \tau$, and $\tau$ is in the upper half plane of $\mathbb{C}$. It is also easy to see that $\tau$, $T(\tau) = \tau + 1$ and $S(\tau) = -1/\tau$ describe conformally equivalent tori; the two maps $T$ and $S$ generate the group $\mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$ that consists of matrices of the form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{where} \quad a, b, c, d \in \mathbb{Z}, \qquad ad - bc = 1, \tag{9}$$

and the matrices $A$ and $-A$ have the same action on $\tau$,

$$\tau \mapsto A\tau = \frac{a\tau + b}{c\tau + d}. \tag{10}$$

The parameter $\tau$ is sometimes called the modular parameter of the torus, and the group $\mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$ is called the modular group (of the torus).

Given a conformal field theory that is defined on the Riemann sphere, the vacuum correlator on the torus can be determined as follows. First, we cut the torus along one of its non-trivial cycles; the resulting surface is a cylinder (or an annulus) whose shape depends on one complex parameter $q$. Since the annulus is a subset of the sphere, the conformal field theory on the annulus is determined in terms of the theory on the sphere. In particular, the states that can propagate in the annulus are precisely the states of the theory as defined on the sphere.

In order to reobtain the torus from the annulus, we have to glue the two ends of the annulus together; in terms of conformal field theory this means that we have to sum over a complete set of states. The vacuum correlator on the torus is therefore described by a trace over the whole space of states, the *partition function* of the theory,

$$\sum_{(j,\bar{j})} \mathrm{Tr}_{\mathcal{H}_{(j,\bar{j})}} \left( \mathcal{O}(q, \bar{q}) \right), \tag{11}$$

where $\mathcal{O}(q, \bar{q})$ is the operator that describes the propagation of the states along the annulus,

$$\mathcal{O}(q, \bar{q}) = q^{L_0 - \frac{c}{24}} \, \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}. \tag{12}$$

Here $L_0$ and $\bar{L}_0$ are the scaling operators of the two vertex operator algebras and $c$ and $\bar{c}$ their central charges; this will be discussed in more detail in the following section. The propagator depends on the actual shape of the annulus that is described in terms of the complex parameter $q$. For a given torus that is described by $\tau$, there is a natural choice for how to cut the torus into an annulus, and the complex parameter $q$ that is associated to this annulus is $q = e^{2\pi i \tau}$. Since the tori that are described by $\tau$ and $A\tau$ (where $A \in \mathrm{SL}(2,\mathbb{Z})$) are equivalent, the vacuum correlator is only well-defined provided that (11) is invariant under this transformation. This provides strong constraints on the spectrum of the theory.

For most conformal field theories (although not for all, see for example [41]) each of the spaces $\boldsymbol{\mathcal{H}}_{(j,\bar{\jmath})}$ is a tensor product of an irreducible representation $\mathcal{H}_j$ of the meromorphic vertex operator algebra and an irreducible representation $\bar{\mathcal{H}}_{\bar{\jmath}}$ of the anti-meromorphic vertex operator algebra. In this case, the vacuum correlator on the torus (11) takes the form

$$\sum_{(j,\bar{\jmath})} \chi_j(q)\,\bar{\chi}_{\bar{\jmath}}(\bar{q})\,, \tag{13}$$

where $\chi_j$ is the *character* of the representation $\mathcal{H}_j$ of the meromorphic vertex operator algebra,

$$\chi_j(\tau) = \mathrm{Tr}_{\mathcal{H}_j}(q^{L_0 - \frac{c}{24}}) \qquad \text{where} \quad q = e^{2\pi i\tau}\,, \tag{14}$$

and likewise for $\bar{\chi}_{\bar{\jmath}}$. One of the remarkable facts about many vertex operator algebras (that has now been proven for a certain class of them [16], see also [42]) is the property that the characters transform into one another under modular transformations,

$$\chi_j(-1/\tau) = \sum_k S_{jk}\,\chi_k(\tau) \qquad \text{and} \qquad \chi_j(\tau+1) = \sum_k T_{jk}\,\chi_k(\tau)\,, \tag{15}$$

where $S$ and $T$ are *constant* matrices, *i.e.* independent of $\tau$. In this case, writing

$$\boldsymbol{\mathcal{H}} = \bigoplus_{i,\bar{\jmath}} M_{i\bar{\jmath}}\,\mathcal{H}_i \otimes \bar{\mathcal{H}}_{\bar{\jmath}}\,, \tag{16}$$

where $M_{i\bar{\jmath}} \in \mathbb{N}$ denotes the multiplicity with which the tensor product $\mathcal{H}_i \otimes \bar{\mathcal{H}}_{\bar{\jmath}}$ appears in $\boldsymbol{\mathcal{H}}$, the torus vacuum correlation function is well defined provided that

$$\sum_{i,\bar{\jmath}} S_{il} M_{i\bar{\jmath}} \bar{S}_{\bar{\jmath}\bar{k}} = \sum_{i,\bar{\jmath}} T_{il} M_{i\bar{\jmath}} \bar{T}_{\bar{\jmath}\bar{k}} = M_{l\bar{k}}\,, \tag{17}$$

and $\bar{S}$ and $\bar{T}$ are the matrices defined as in (15) for the representations of the anti-meromorphic vertex operator algebra. This provides very powerful constraints for the multiplicity matrices $M_{i\bar{\jmath}}$. In particular, in the case of a finite theory (for which each of the two vertex operator algebras has only finitely many irreducible representations) these conditions typically only allow for a finite number of solutions that can be classified; this has been done for the case of the so-called minimal models and the affine theories with group $SU(2)$ by Cappelli, Itzykson and Zuber [43,44] (for a modern proof involving some Galois theory see [45]), and for the affine theories with group $SU(3)$ and the $N=2$ superconformal minimal models by Gannon [46, 47].

This concludes our brief overview over the general structure of a local conformal field theory. For the rest of the paper we shall mainly concentrate on the theory that is defined on the sphere. Let us begin by analysing the meromorphic conformal subtheory in some detail.

## 3. Meromorphic Conformal Field Theory

In this section we shall describe in detail the structure of a meromorphic conformal field theory; our exposition follows closely the work of Goddard [11] and Gaberdiel & Goddard [48], and we refer the reader for some of the mathematical details (that shall be ignored in the following) to these papers.

### 3.1. Amplitudes and Möbius Covariance

As we have explained above, a meromorphic conformal field theory is determined in terms of its space of states $\mathcal{H}_0$, and the amplitudes involving arbitrary elements $\psi_i$ in a dense subspace $\mathcal{F}_0$ of $\mathcal{H}_0$. Indeed, for each state $\psi \in \mathcal{F}_0$, there exists a *vertex operator* $V(\psi, z)$ that creates the state $\psi$ from the vacuum (in a sense that will be described in more detail shortly), and the amplitudes are the vacuum expectation values of the corresponding product of vertex operators,

$$\mathcal{A}(\psi_1, \ldots, \psi_n; z_1, \ldots, z_n) = \langle V(\psi_1, z_1) \cdots V(\psi_n, z_n) \rangle. \tag{18}$$

Each vertex operator $V(\psi, z)$ depends linearly on $\psi$, and the amplitudes are *meromorphic* functions that are defined on the Riemann sphere $\mathbb{P} = \mathbb{C} \cup \{\infty\}$, *i.e.* they are analytic except for possible poles at $z_i = z_j$, $i \neq j$. The operators are furthermore assumed to be *local* in the sense that for $z \neq \zeta$

$$V(\psi, z)V(\phi, \zeta) = \varepsilon \, V(\phi, \zeta)V(\psi, z), \tag{19}$$

where $\varepsilon = -1$ if both $\psi$ and $\phi$ are fermionic, and $\varepsilon = +1$ otherwise. In formulating (19) we have assumed that $\psi$ and $\phi$ are states of definite fermion number; more precisely, this means that $\mathcal{F}_0$ decomposes as

$$\mathcal{F}_0 = \mathcal{F}_0^B \oplus \mathcal{F}_0^F, \tag{20}$$

where $\mathcal{F}_0^B$ and $\mathcal{F}_0^F$ is the subspace of bosonic and fermionic states, respectively, and that both $\psi$ and $\phi$ are either in $\mathcal{F}_0^B$ or in $\mathcal{F}_0^F$. In the following we shall always only consider states of definite fermion number.

In terms of the amplitudes, the locality condition (19) is equivalent to the property that

$$\mathcal{A}(\psi_1, \ldots, \psi_i, \psi_{i+1}, \ldots, \psi_n; z_1, \ldots, z_i, z_{i+1}, \ldots, z_n)$$
$$= \varepsilon_{i,i+1} \mathcal{A}(\psi_1, \ldots, \psi_{i+1}, \psi_i, \ldots, \psi_n; z_1, \ldots, z_{i+1}, z_i, \ldots, z_n), \tag{21}$$

and $\varepsilon_{i,i+1}$ is defined as above. As the amplitudes are essentially independent of the order of the fields, we shall sometimes also write them as

$$\mathcal{A}(\psi_1, \ldots, \psi_n; z_1, \ldots, z_n) = \langle \prod_{i=1}^n V(\psi_i, z_i) \rangle. \tag{22}$$

We may assume that $\mathcal{F}_0$ contains a (bosonic) state $\Omega$ that has the property that its vertex operator $V(\Omega, z)$ is the identity operator; in terms of the amplitudes this means that

$$\langle V(\Omega, z) \prod_{i=1}^{n} V(\psi_i, z_i) \rangle = \langle \prod_{i=1}^{n} V(\psi_i, z_i) \rangle. \tag{23}$$

We call $\Omega$ the *vacuum (state)* of the theory. Given $\Omega$, the state $\psi \in \mathcal{F}_0$ that is associated to the vertex operator $V(\psi, z)$ can then be defined as

$$\psi = V(\psi, 0)\Omega. \tag{24}$$

In conventional quantum field theory, the states of the theory transform in a suitable way under the Poincaré group, and the amplitudes are therefore covariant under Poincaré transformations. In the present context, the rôle of the Poincaré group is played by the group of Möbius transformations $\mathcal{M}$, *i.e.* the group of (complex) automorphisms of the Riemann sphere. These are the transformations of the form

$$z \mapsto \gamma(z) = \frac{az + b}{cz + d}, \qquad \text{where} \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1. \tag{25}$$

We can associate to each element

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}), \tag{26}$$

the Möbius transformation (25), and since $A \in \mathrm{SL}(2,\mathbb{C})$ and $-A \in \mathrm{SL}(2,\mathbb{C})$ define the same Möbius transformation, the group of Möbius transformations $\mathcal{M}$ is isomorphic to $\mathcal{M} \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$. In the following we shall sometimes use elements of $\mathrm{SL}(2,\mathbb{C})$ to denote the corresponding elements of $\mathcal{M}$ where no confusion will result.

It is convenient to introduce a set of generators for $\mathcal{M}$ by

$$e^{\lambda \mathcal{L}_{-1}}(z) = z + \lambda, \qquad e^{\lambda \mathcal{L}_0}(z) = e^{\lambda} z, \qquad e^{\lambda \mathcal{L}_1}(z) = \frac{z}{1 - \lambda z}, \tag{27}$$

where the first transformation is a *translation*, the second is a *scaling*, and the last one is usually referred to as a *special conformal transformation*. Every Möbius transformation can be obtained as a composition of these transformations, and for Möbius transformations with $d \neq 0$, this can be compactly described as

$$\gamma = \exp\left[\frac{b}{d}\mathcal{L}_{-1}\right] d^{-2\mathcal{L}_0} \exp\left[-\frac{c}{d}\mathcal{L}_1\right], \tag{28}$$

where $\gamma$ is given as in (25). In terms of $\mathrm{SL}(2,\mathbb{C})$, the three transformations in (27) are

$$e^{\lambda \mathcal{L}_{-1}} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \qquad e^{\lambda \mathcal{L}_0} = \begin{pmatrix} e^{\frac{1}{2}\lambda} & 0 \\ 0 & e^{-\frac{1}{2}\lambda} \end{pmatrix}, \qquad e^{\lambda \mathcal{L}_1} = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}. \tag{29}$$

The corresponding infinitesimal generators (that are complex $2 \times 2$ matrices with vanishing trace) are then

$$\mathcal{L}_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{L}_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \qquad \mathcal{L}_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{30}$$

They form a basis for the Lie algebra $sl(2,\mathbb{C})$ of $SL(2,\mathbb{C})$, and satisfy the commutation relations

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}, \qquad m, n = 0, \pm 1. \tag{31}$$

As in conventional quantum field theory, the states of the meromorphic theory form a representation of this algebra which can be decomposed into irreducible representations. The (irreducible) representations that are relevant in physics are those that satisfy the condition of *positive energy*. In the present context, since $L_0$ (the operator associated to $\mathcal{L}_0$) can be identified with the energy operator (up to some constant), these are those representations for which the spectrum of $L_0$ is bounded from below. This will follow from the *cluster property* of the amplitudes that will be discussed below. In a given irreducible highest weight representation, let us denote by $\psi$ the state for which $L_0$ takes the minimal value, $h$ say.‖ Using (31) we then have

$$L_0 L_1 \psi = [L_0, L_1]\psi + h L_1 \psi = (h-1)L_1 \psi, \tag{32}$$

where $L_n$ denotes the operator corresponding to $\mathcal{L}_n$. Since $\psi$ is the state with the minimal value for $L_0$, it follows that $L_1\psi = 0$; states with the property

$$L_1\psi = 0 \qquad L_0\psi = h\psi \tag{33}$$

are called *quasiprimary*, and the real number $h$ is called the *conformal weight* of $\psi$. Every quasiprimary state $\psi$ generates a representation of $sl(2,\mathbb{C})$ that consists of the $L_{-1}$-descendants (of $\psi$), *i.e.* the states of the form $L_{-1}^n\psi$ where $n = 0, 1, \ldots$. This infinite-dimensional representation is irreducible unless $h$ is a non-positive half-integer. Indeed,

$$L_1 L_{-1}^n \psi = \sum_{l=0}^{n-1} L_{-1}^l [L_1, L_{-1}] L_{-1}^{n-1-l} \psi \tag{34}$$

$$= 2\sum_{l=0}^{n-1}(h + n - 1 - l)L_{-1}^{n-1}\psi \tag{35}$$

$$= 2n(h + \frac{1}{2}(n-1))L_{-1}^{n-1}\psi, \tag{36}$$

and thus if $h$ is a non-positive half-integer, the state $L_{-1}^n\psi$ with $n = 1 - 2h$ and its $L_{-1}$-descendants define a subrepresentation. In order to obtain an irreducible representation one has to quotient the space of $L_{-1}$-descendants of $\psi$ by this subrepresentation; the resulting irreducible representation is then finite-dimensional.

Since the states of the theory carry a representation of the Möbius group, the amplitudes transform covariantly under Möbius transformations. The transformation rule for general states is quite complicated (we shall give an explicit formula later on), but for quasiprimary states it can be easily described: let $\psi_i$, $i = 1, \ldots, n$ be $n$ quasiprimary states with conformal weights $h_i$, $i = 1, \ldots, n$, then

$$\langle \prod_{i=1}^n V(\psi_i, z_i)\rangle = \prod_{i=1}^n \left(\frac{d\gamma(z_i)}{dz_i}\right)^{h_i} \langle \prod_{i=1}^n V(\psi_i, \gamma(z_i))\rangle, \tag{37}$$

‖ We shall assume here that there is only one such state; this is always true in irreducible representations.

where $\gamma$ is a Möbius transformation as in (25).

Let us denote the operators that implement the Möbius transformations on the space of states by the same symbols as in (27) with $\mathcal{L}_n$ replaced by $L_n$. Then the transformation formulae for the vertex operators are given as

$$e^{\lambda L_{-1}} V(\psi, z) e^{-\lambda L_{-1}} = V(\psi, z + \lambda) \tag{38}$$

$$x^{L_0} V(\psi, z) x^{-L_0} \quad = x^h V(\psi, xz) \tag{39}$$

$$e^{\mu L_1} V(\psi, z) e^{-\mu L_1} \quad = (1 - \mu z)^{-2h} V(\psi, z/(1 - \mu z)), \tag{40}$$

where $\psi$ is quasiprimary with conformal weight $h$. We also write more generally

$$D_\gamma V(\psi, z) D_\gamma^{-1} = \left( \gamma'(z) \right)^h V\left( \psi, \gamma(z) \right), \tag{41}$$

where $D_\gamma$ is given by the same formula as in (28). In this notation we then have

$$D_\gamma \Omega = \Omega \tag{42}$$

for all $\gamma$; this is equivalent to $L_n \Omega = 0$ for $n = 0, \pm 1$. The transformation formula for the vertex operator associated to a quasiprimary field $\psi$ is consistent with the identification of states and fields (24) and the definition of a quasiprimary state (33): indeed, if we apply (39) and (40) to the vacuum, use (42) and set $z = 0$, we obtain

$$x^{L_0} \psi = x^h \psi \qquad \text{and} \qquad e^{\mu L_1} \psi = \psi \tag{43}$$

which implies that $L_1 \psi = 0$ and $L_0 \psi = h \psi$, and is thus in agreement with (33).

The Möbius symmetry constrains the functional form of all amplitudes, but in the case of the one-, two- and three-point functions it actually determines their functional dependence completely. If $\psi$ is a quasiprimary state with conformal weight $h$, then $\langle V(\psi, z) \rangle$ is independent of $z$ because of the translation symmetry, but it follows from (39) that

$$\langle V(\psi, z) \rangle = \lambda^h \langle V(\psi, \lambda z) \rangle. \tag{44}$$

The one-point function can therefore only be non-zero if $h = 0$. Under the assumption of the cluster property to be discussed in the next subsection, the only state with $h = 0$ is the vacuum, $\psi = \Omega$.

If $\psi$ and $\phi$ are two quasiprimary states with conformal weights $h_\psi$ and $h_\phi$, respectively, then the translation symmetry implies that

$$\langle V(\psi, z) V(\phi, \zeta) \rangle = \langle V(\psi, z - \zeta) V(\phi, 0) \rangle = F(z - \zeta), \tag{45}$$

and the scaling symmetry gives

$$\lambda^{h_\psi + h_\phi} F(\lambda x) = F(x), \tag{46}$$

so that

$$F(x) = C x^{-h_\psi - h_\phi}, \tag{47}$$

where $C$ is some constant. On the other hand, the symmetry under special conformal transformations implies that

$$\langle V(\psi, x) V(\phi, 0) \rangle = (1 - \mu x)^{-2h_\psi} \langle V(\psi, x/(1 - \mu x)) V(\phi, 0) \rangle, \tag{48}$$

and therefore, upon comparison with (47), the amplitude can only be non-trivial if $2h_\psi = h_\psi + h_\phi$, *i.e.* $h_\psi = h_\phi$. In this case the amplitude is of the form

$$\langle V(\psi, z)V(\phi, \zeta)\rangle = C(z - \zeta)^{-2h_\psi} . \tag{49}$$

If the amplitude is non-trivial for $\psi = \phi$, the locality condition implies that $h \in \mathbb{Z}$ if $\psi$ is a bosonic field, and $h \in \frac{1}{2} + \mathbb{Z}$ if $\psi$ is fermionic. This is the familiar *Spin-Statistics Theorem*.

Finally, if $\psi_i$ are quasiprimary fields with conformal weights $h_i$, $i = 1, 2, 3$, then

$$\langle V(\psi_1, z_1)V(\psi_2, z_2)V(\psi_3, z_3)\rangle = \prod_{i<j} \left(\frac{a_i - a_j}{z_i - z_j}\right)^{h_{ij}} \langle V(\psi_1, a_1)V(\psi_2, a_2)V(\psi_3, a_3)\rangle , \tag{50}$$

where $h_{12} = h_1 + h_2 - h_3$, *etc.*, and $a_i$ are three distinct arbitrary constants. In deriving (50) we have used the fact that every three points can be mapped to any other three points by means of a Möbius transformation.

### 3.2. The Uniqueness Theorem

It follows directly from (38), (42) and (24) that

$$V(\psi, z)\Omega = e^{zL_{-1}}V(\psi, 0)\Omega = e^{zL_{-1}}\psi . \tag{51}$$

If $V(\psi, z)$ is in addition local, *i.e.* if it satisfies (19) for every $\phi$, $V(\psi, z)$ is uniquely characterised by this property; this is the content of the
**Uniqueness Theorem** [11]: If $U_\psi(z)$ is a local vertex operator that satisfies

$$U_\psi(z)\Omega = e^{zL_{-1}}\psi \tag{52}$$

then

$$U_\psi(z) = V(\psi, z) \tag{53}$$

on a dense subspace of $\mathcal{H}_0$.
**Proof:** Let $\chi \in \mathcal{F}_0$ be arbitrary. Then

$$U_\psi(z)\chi = U_\psi(z)V(\chi, 0)\Omega = \varepsilon_{\chi,\psi}V(\chi, 0)U_\psi(z)\Omega = \varepsilon_{\chi,\psi}V(\chi, 0)e^{zL_{-1}}\psi , \tag{54}$$

where we have used the locality of $U_\psi(z)$ and (52) and $\varepsilon_{\chi,\psi}$ denotes the sign in (19). We can then use (51) and the locality of $V(\psi, z)$ to rewrite this as

$$\varepsilon_{\chi,\psi}V(\chi, 0)e^{zL_{-1}}\psi = \varepsilon_{\chi,\psi}V(\chi, 0)V(\psi, z)\Omega = V(\psi, z)V(\chi, 0)\Omega = V(\psi, z)\chi , \tag{55}$$

and thus the action of $U_\psi(z)$ and $V(\psi, z)$ agrees on the dense subspace $\mathcal{F}_0$.

Given the uniqueness theorem, we can now deduce the transformation property of a general vertex operator under Möbius transformations

$$D_\gamma V(\psi, z)D_\gamma^{-1} = V\left[\left(\frac{d\gamma}{dz}\right)^{L_0} \exp\left(\frac{\gamma''(z)}{2\gamma'(z)}L_1\right)\psi, \gamma(z)\right] . \tag{56}$$

In the special case where $\psi$ is quasiprimary, $\exp(\gamma''(z)/2\gamma'(z)L_1)\psi = \psi$, and (56) reduces to (41). To prove (56), we observe that the uniqueness theorem implies that it is sufficient to evaluate the identity on the vacuum, in which case it becomes

$$D_\gamma e^{zL_{-1}}\psi = e^{\gamma(z)L_{-1}}(cz+d)^{-2L_0}e^{-\frac{c}{cz+d}L_1}\psi\,, \tag{57}$$

where we have written $\gamma$ as in (25). This then follows from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & az+b \\ c & cz+d \end{pmatrix} \tag{58}$$

$$= \begin{pmatrix} 1 & \frac{az+b}{cz+d} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (cz+d)^{-1} & 0 \\ 0 & (cz+d) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{c}{cz+d} & 1 \end{pmatrix} \tag{59}$$

together with the fact that $\mathcal{M} \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

We can now also deduce the behaviour under infinitesimal transformations from (56). For example, if $\gamma$ is an infinitesimal translation, $\gamma(z) = z + \delta$, then to first order in $\delta$, (56) becomes

$$V(\psi,z) + \delta[L_{-1}, V(\psi,z)] = V(\psi,z) + \delta\frac{dV}{dz}(\psi,z)\,, \tag{60}$$

from which we deduce that

$$[L_{-1}, V(\psi,z)] = \frac{dV}{dz}(\psi,z)\,. \tag{61}$$

Similarly, we find that

$$[L_0, V(\psi,z)] = z\frac{d}{dz}V(\psi,z) + V(L_0\psi,z)\,, \tag{62}$$

and

$$[L_1, V(\psi,z)] = z^2\frac{d}{dz}V(\psi,z) + 2zV(L_0\psi,z) + V(L_1\psi,z)\,. \tag{63}$$

If $\psi$ is quasiprimary of conformal weight $h$, the last three equations can be compactly written as

$$[L_n, V(\psi,z)] = z^n\left\{z\frac{d}{dz} + (n+1)h\right\}V(\psi,z) \qquad \text{for } n = 0, \pm 1. \tag{64}$$

Finally, applying (61) to the vacuum we have

$$e^{zL_{-1}}L_{-1}\psi = e^{zL_{-1}}\frac{dV}{dz}(\psi,0)\Omega\,, \tag{65}$$

and this implies, using the uniqueness theorem, that

$$\frac{dV}{dz}(\psi,z) = V(L_{-1}\psi,z)\,. \tag{66}$$

In particular, it follows that the correlation functions of $L_{-1}$-descendants of quasiprimary states can be directly deduced from those that only involve the quasiprimary states themselves.

### 3.3. Factorisation and the Cluster Property

As we have explained above, a meromorphic conformal field theory is determined by its space of states $\mathcal{H}_0$ together with the set of amplitudes that are defined for arbitrary elements in a dense subspace $\mathcal{F}_0$ of $\mathcal{H}_0$. The amplitudes contain all relevant information about the vertex operators; for example the locality and Möbius transformation properties of the vertex operators follow from the corresponding properties of the amplitudes (21), and (37).

In practice, this is however not a good way to define a conformal field theory, since $\mathcal{H}_0$ is always infinite-dimensional (unless the meromorphic conformal field theory consists only of the vacuum), and it is unwieldy to give the correlation functions for arbitrary combinations of elements in an infinite-dimensional (dense) subspace $\mathcal{F}_0$ of $\mathcal{H}_0$. Most (if not all) theories of interest however possess a finite-dimensional subspace $V \subset \mathcal{H}_0$ that is not dense in $\mathcal{H}_0$ but that *generates* $\mathcal{H}_0$ in the sense that $\mathcal{H}_0$ and all its amplitudes can be derived from those only involving states in $V$; this process is called *factorisation.*

The basic idea of factorisation is very simple: given the amplitudes involving states in $V$, we can define the vector space that consists of linear combinations of states of the form

$$\Psi = V(\psi_1, z_1) \cdots V(\psi_n, z_n)\Omega \,, \tag{67}$$

where $\psi_i \in V$, and $z_i \neq z_j$ for $i \neq j$. We identify two such states if their difference vanishes in all amplitudes (involving states in $V$), and denote the resulting vector space by $\widehat{\mathcal{F}}_0$. We then say that $V$ generates $\mathcal{H}_0$ if $\widehat{\mathcal{F}}_0$ is dense in $\mathcal{H}_0$. Finally we can introduce a vertex operator for $\Psi$ by

$$V(\Psi, z) = V(\psi_1, z_1 + z) \cdots V(\psi_n, z_n + z) \,, \tag{68}$$

and the amplitudes involving arbitrary elements in $\widehat{\mathcal{F}}_0$ are thus determined in terms of those that only involve states in $V$. (More details of this construction can be found in [48].) In the following, when we shall give examples of meromorphic conformal field theories, we shall therefore only describe the theory associated to a suitable generating space $V$.

It is easy to check that the locality and Möbius transformation properties of the amplitudes involving only states in $V$ are sufficient to guarantee the corresponding properties for the amplitudes involving arbitrary states in $\widehat{\mathcal{F}}_0$, and therefore for the conformal field theory that is obtained by factorisation from $V$. However, the situation is more complicated with respect to the condition that the states in $\mathcal{H}_0$ are of positive energy, *i.e.* that the spectrum of $L_0$ is bounded from below, since this clearly does not follow from the condition that this is so for the states in $V$. In the case of the meromorphic theory the relevant spectrum condition is actually slightly stronger in that it requires that the spectrum of $L_0$ is non-negative, and that there exists a *unique* state, the vacuum, with $L_0 = 0$. This stronger condition (which we shall always assume from now on) is satisfied for the meromorphic theory obtained by factorisation from $V$

provided the amplitudes in $V$ satisfy the *cluster property*; this states that if we separate the variables of an amplitude into two sets and scale one set towards a fixed point (*e.g.* 0 or $\infty$) the behaviour of the amplitude is dominated by the product of two amplitudes, corresponding to the two sets of variables, multiplied by an appropriate power of the separation, specifically

$$\left\langle \prod_i V(\phi_i, \zeta_i) \prod_j V(\psi_j, \lambda z_j) \right\rangle \sim \left\langle \prod_i V(\phi_i, \zeta_i) \right\rangle \left\langle \prod_j V(\psi_j, z_j) \right\rangle \lambda^{-\Sigma h_j} \quad \text{as } \lambda \to 0\,, (69)$$

where $\phi_i, \psi_j \in V$ have conformal weight $h'_i$ and $h_j$, respectively. (Here $\sim$ means that the two sides of the equation agree up to terms of lower order in $\lambda$.) Because of the Möbius covariance of the amplitudes this is equivalent to

$$\left\langle \prod_i V(\phi_i, \lambda\zeta_i) \prod_j V(\psi_j, z_j) \right\rangle \sim \left\langle \prod_i V(\phi_i, \zeta_i) \right\rangle \left\langle \prod_j V(\psi_j, z_j) \right\rangle \lambda^{-\Sigma h'_i} \quad \text{as } \lambda \to \infty \,.$$

To prove that this implies that the spectrum of $L_0$ is non-negative and that the vacuum is unique, let us introduce the projection operators defined by

$$P_N = \oint_0 u^{L_0 - N - 1} du, \qquad \text{for } N \in \mathbb{Z}/2 \,, \tag{70}$$

where we have absorbed a factor of $1/2\pi i$ into the definition of the symbol $\oint$. In particular, we have

$$P_N \prod_j V(\psi_j, z_j)\Omega = \oint du \, u^{h-N-1} \prod_j V(\psi_j, uz_j) \, \Omega \,, \tag{71}$$

where $h = \sum_j h_j$. It then follows that the $P_N$ are projection operators

$$P_N P_M = 0, \text{ if } N \neq M, \qquad P_N^2 = P_N, \qquad \sum_N P_N = 1 \tag{72}$$

onto the eigenspaces of $L_0$,

$$L_0 P_N = N P_N \,. \tag{73}$$

For $N \leq 0$, we then have

$$\left\langle \prod_i V(\phi_i, \zeta_i) P_N \prod_j V(\psi_j, z_j) \right\rangle = \oint_0 u^{\Sigma h_j - N - 1} \left\langle \prod_i V(\phi_i, \zeta_i) \prod_j V(\psi_j, uz_j) \right\rangle du$$

$$\sim \left\langle \prod_i V(\phi_i, \zeta_i) \right\rangle \left\langle \prod_j V(\psi_j, z_j) \right\rangle \oint_{|u|=\rho} u^{-N-1} du \tag{74}$$

which, by taking $\rho \to 0$, is seen to vanish for $N < 0$ and, for $N = 0$, to give

$$P_0 \prod_j V(\psi_j, z_j)\Omega = \Omega \left\langle \prod_j V(\psi_j, z_j) \right\rangle \,, \tag{75}$$

and so $P_0 \Psi = \Omega \langle \Psi \rangle$. Thus the cluster decomposition property implies that $P_N = 0$ for $N < 0$, *i.e.* that the spectrum of $L_0$ is non-negative, and that $\Omega$ is the unique state with $L_0 = 0$. The cluster property also implies that the space of states can be completely decomposed into irreducible representations of the Lie algebra $sl(2, \mathbb{C})$ that corresponds to the Möbius transformations (see Appendix D of [48]).

### 3.4. The Operator Product Expansion

One of the most important consequences of the uniqueness theorem is that it allows for a direct derivation of the *duality relation* which in turn gives rise to the operator product expansion.

**Duality Theorem** [11]: Let $\psi$ and $\phi$ be states in $\mathcal{F}_0$, then

$$V(\psi, z)V(\phi, \zeta) = V\Big(V(\psi, z - \zeta)\phi, \zeta\Big). \tag{76}$$

**Proof:** By the uniqueness theorem it is sufficient to evaluate both sides on the vacuum, in which case (76) becomes

$$V(\psi, z)V(\phi, \zeta)\Omega = V(\psi, z)e^{\zeta L_{-1}}\phi \tag{77}$$

$$= e^{\zeta L_{-1}}V(\psi, z - \zeta)\phi \tag{78}$$

$$= V\Big(V(\psi, z - \zeta)\phi, \zeta\Big)\Omega, \tag{79}$$

where we have used (38).

For many purposes it is convenient to expand the fields $V(\psi, z)$ in terms of modes

$$V(\psi, z) = \sum_{n \in \mathbb{Z} - h} V_n(\psi) z^{-n-h}, \tag{80}$$

where $\psi$ has conformal weight $h$, *i.e.* $L_0\psi = h\psi$. The modes can be defined in terms of a contour integral as

$$V_n(\psi) = \oint z^{h+n-1} V(\psi, z)dz, \tag{81}$$

where the contour encircles $z = 0$ anticlockwise. In terms of the modes the identity $V(\psi, 0)\Omega = \psi$ implies that

$$V_{-h}(\psi)\Omega = \psi \qquad \text{and} \qquad V_l(\psi)\Omega = 0 \quad \text{for } l > -h. \tag{82}$$

Furthermore, if $\psi$ is quasiprimary, (64) becomes

$$[L_m, V_n(\psi)] = (m(h-1) - n)\, V_{m+n}(\psi) \qquad m = 0, \pm 1. \tag{83}$$

Actually, the equations for $m = 0, -1$ do not require that $\psi$ is quasiprimary as follows from (61) and (62); thus we have that $[L_0, V_n(\psi)] = -nV_n(\psi)$ for all $\psi$, so that $V_n(\psi)$ lowers the eigenvalue of $L_0$ by $n$.

Given the modes of the conformal fields, we can introduce the *Fock space* $\widetilde{\mathcal{F}}_0$ that is spanned by eigenstates of $L_0$ and that forms a dense subspace of the space of states. This space consists of finite linear combinations of vectors of the form

$$\Psi = V_{n_1}(\psi_1)V_{n_2}(\psi_2) \cdots V_{n_N}(\psi_N)\Omega, \tag{84}$$

where $n_i + h_i \in \mathbb{Z}$, $h_i$ is the conformal weight of $\psi_i$, and we may restrict $\psi_i$ to be in the subspace $V$ that generates the whole theory by factorisation. Because of (83) $\Psi$ is an eigenvector of $L_0$ with eigenvalue

$$L_0\Psi = h\psi \qquad \text{where } h = -\sum_{i=1}^{N} n_i. \tag{85}$$

The Fock space $\widetilde{\mathcal{F}}_0$ is a quotient space of the vector space $\mathcal{W}_0$ whose basis is given by the states of the form (84); the subspace by which $\mathcal{W}_0$ has to be divided consists of linear combinations of states of the form (84) that vanish in all amplitudes.

We can also introduce a vertex operator for $\Psi$ by the formula

$$V(\Psi, z) = \oint_{\mathcal{C}_1} z_1^{h_1 + n_1 - 1} V(\psi_1, z + z_1) dz_1 \cdots \oint_{\mathcal{C}_N} z_N^{h_N + n_N - 1} V(\psi_N, z + z_N) dz_N \,, \tag{86}$$

where the $\mathcal{C}_j$ are contours about 0 with $|z_i| > |z_j|$ if $i < j$. The Fock space $\widetilde{\mathcal{F}}_0$ thus satisfies the conditions that we have required of the dense subspace $\mathcal{F}_0$, and we may therefore assume that $\mathcal{F}_0$ *is* actually the Fock space of the theory; from now on we shall always do so.

The duality property of the vertex operators can now be rewritten in terms of modes as

$$\begin{aligned} V(\phi, z) V(\psi, \zeta) &= V(V(\phi, z - \zeta)\psi, \zeta) \\ &= \sum_{n \leq h_\psi} V(V_n(\phi)\psi, \zeta)(z - \zeta)^{-n - h_\phi} \,, \end{aligned} \tag{87}$$

where $L_0\psi = h_\psi \psi$ and $L_0\phi = h_\phi \phi$, and $\psi, \phi \in \mathcal{F}_0$. The sum over $n$ is bounded by $h_\psi$, since $L_0 V_n(\phi)\psi = (h_\psi - n)V_n(\phi)\psi$, and the spectrum condition implies that the theory does not contain any states of negative conformal weight. The equation (87) is known as the *Operator Product Expansion*. The infinite sum converges provided that all other meromorphic fields in a given amplitude are further away from $\zeta$ than $z$.

We can use (87) to derive a formula for the commutation relations of modes as follows.¶ The commutator of two modes $V_m(\phi)$ and $V_n(\psi)$ is given as

$$\begin{aligned} [V_m(\Phi), V_n(\Psi)] &= \oint dz \oint_{|z|>|\zeta|} d\zeta \; z^{m + h_\phi - 1} \zeta^{n + h_\psi - 1} V(\phi, z) V(\psi, \zeta) \\ &\quad - \oint dz \oint_{|\zeta|>|z|} d\zeta \; z^{m + h_\phi - 1} \zeta^{n + h_\psi - 1} V(\phi, z) V(\psi, \zeta) \end{aligned} \tag{88}$$

where the contours on the right-hand side encircle the origin anti-clockwise. We can then deform the two contours so as to rewrite (88) as

$$[V_m(\phi), V_n(\psi)] = \oint_0 \zeta^{n + h_\psi - 1} d\zeta \oint_\zeta z^{m + h_\phi - 1} dz \sum_l V(V_l(\phi)\psi, \zeta)(z - \zeta)^{-l - h_\phi} \,, \tag{89}$$

where the $z$ contour is a small positive circle about $\zeta$ and the $\zeta$ contour is a positive circle about the origin. Only terms with $l \geq 1 - h_\phi$ contribute, and the integral becomes

$$[V_m(\phi), V_n(\psi)] = \sum_{N = -h_\phi + 1}^{h_\psi} \binom{m + h_\phi - 1}{m - N} V_{m+n}(V_N(\phi)\psi) \,. \tag{90}$$

In particular, if $m \geq -h_\phi + 1$, $n \geq -h_\psi + 1$, then $m - N \geq 0$ in the sum, and $m + n \geq N + n \geq N - h_\psi + 1$. This implies that the modes $\{V_m(\psi) : m \geq -h_\psi + 1\}$

¶ To be precise, the following construction *a priori* only defines a Lie bracket for the quotient space of modes where we identify modes whose action on the Fock space of the meromorphic theory coincides.

close as a Lie algebra. The same also holds for the modes $\{V_m(\psi) : m \leq h_\psi - 1\}$, and therefore for their intersection

$$\mathcal{L}^0 = \{V_n(\psi) : -h_\psi + 1 \leq n \leq h_\psi - 1\} . \tag{91}$$

This algebra is sometimes called the *vacuum-preserving algebra* since any element in $\mathcal{L}^0$ annihilates the vacuum. A certain deformation of $\mathcal{L}^0$ defines a finite Lie algebra that can be interpreted as describing the finite $W$-symmetry of the conformal field theory [49]. It is also clear that the subset of all positive, all negative or all zero modes form closed Lie algebras, respectively.

### 3.5. The Inner Product and Null-vectors

We can define an (hermitian) inner product on the Fock space $\mathcal{F}_0$ provided that the amplitudes are hermitian in the following sense: there exists an antilinear involution $\psi \mapsto \overline{\psi}$ for each $\psi \in \mathcal{F}_0$ such that the amplitudes satisfy

$$\left( \langle \prod_{i=1}^{n} V(\psi_i, z_i) \rangle \right)^* = \langle \prod_{i=1}^{n} V(\overline{\psi}_i, \bar{z}_i) \rangle . \tag{92}$$

If this condition is satisfied, we can define an inner product by

$$\langle \psi, \phi \rangle = \lim_{z \to 0} \left\langle V\left( \left( -\frac{1}{\bar{z}^2} \right)^{L_0} \exp\left[ -\frac{1}{\bar{z}} L_1 \right] \overline{\psi}, \frac{1}{\bar{z}} \right) V(\phi, z) \right\rangle . \tag{93}$$

This inner product is hermitian, *i.e.*

$$\langle \psi, \phi \rangle^* = \langle \phi, \psi \rangle \tag{94}$$

since (92) implies that the left-hand-side of (94) is

$$\lim_{z \to 0} \left\langle V\left( \left( -\frac{1}{z^2} \right)^{L_0} \exp\left[ -\frac{1}{z} L_1 \right] \psi, \frac{1}{z} \right) V(\bar{\phi}, \bar{z}) \right\rangle , \tag{95}$$

and the covariance under the Möbius transformation $\gamma(z) = 1/z$ then implies that this equals

$$\lim_{z \to 0} \left\langle V\left( \left( -\frac{1}{\bar{z}^2} \right)^{L_0} \exp\left[ -\frac{1}{\bar{z}} L_1 \right] \bar{\phi}, \frac{1}{\bar{z}} \right) V(\psi, z) \right\rangle . \tag{96}$$

By a similar calculation we find that the adjoint of a vertex operator is given by

$$(V(\psi, \zeta))^\dagger = V\left( \left( \frac{1}{\bar{\zeta}^2} \right)^{L_0} \exp\left[ -\frac{1}{\bar{\zeta}} L_1 \right] \bar{\psi}, \frac{1}{\bar{\zeta}} \right) , \tag{97}$$

where the adjoint is defined to satisfy

$$\langle \chi, V(\psi, \zeta)\phi \rangle = \left\langle (V(\psi, \zeta))^\dagger \chi, \phi \right\rangle . \tag{98}$$

Since $\psi \mapsto \overline{\psi}$ is an involution, we can choose a basis of *real* states, *i.e.* states that satisfy $\overline{\psi} = \psi$. If $\psi$ is a quasiprimary real state, then (97) simplifies to

$$(V(\psi, \zeta))^\dagger = \left( -\frac{1}{\bar{\zeta}^2} \right)^{h} V(\psi, 1/\bar{\zeta}), \tag{99}$$

where $h$ denotes the conformal weight of $\psi$. In this case the adjoint of the mode $V_n(\psi)$ is

$$
\begin{aligned}
(V_n(\psi))^\dagger &= \oint d\bar{z}\,\bar{z}^{h+n-1}\left(-\frac{1}{\bar{z}^2}\right)^h V(\psi, 1/\bar{z}) \\
&= (-1)^h \oint d\bar\zeta\,\bar\zeta^{h-n+1} V(\psi, \bar\zeta) \\
&= (-1)^h V_{-n}(\psi)\,.
\end{aligned}
\tag{100}
$$

By a similar calculation it also follows that the adjoint of the Möbius generators are given as

$$
L_{\pm 1}^\dagger = L_{\mp 1} \qquad\qquad L_0^\dagger = L_0\,.
\tag{101}
$$

All known conformal field theories satisfy (92) and thus possess a hermitian inner product; from now on we shall therefore sometimes assume that the theory has such an inner product.

The inner product can be extended to the vector space $\mathcal{W}_0$ whose basis is given by the states of the form (84). Typically, the inner product is degenerate on $\mathcal{W}_0$, *i.e.* there exist vectors $\mathcal{N} \in \mathcal{W}_0$ for which

$$
\langle \psi, \mathcal{N} \rangle = 0 \qquad \text{for all } \psi \in \mathcal{W}_0.
\tag{102}
$$

Every vector with this property is called a *null-vector*. Because of Möbius covariance, the field corresponding to $\mathcal{N}$ vanishes in all amplitudes, and therefore $\mathcal{N}$ is in the subspace by which $\mathcal{W}_0$ has to be divided in order to obtain the Fock space $\mathcal{F}_0$. Since this is the case for every null-vector of $\mathcal{W}_0$, it follows that the inner product is non-degenerate on $\mathcal{F}_0$.

In general, the inner product may not be positive definite, but there exist many interesting theories for which it is; in this case the theory is called *unitary*. For unitary theories, the spectrum of $L_0$ is always bounded by 0. To see this we observe that if $\psi$ is a quasiprimary state with conformal weight $h$, then

$$
\begin{aligned}
\langle L_{-1}\psi, L_{-1}\psi \rangle &= \langle \psi, L_1 L_{-1}\psi \rangle \\
&= 2h\langle \psi, \psi \rangle\,,
\end{aligned}
\tag{103}
$$

where we have used (101). If the theory is unitary then both sides of (103) have to be non-negative, and thus $h \geq 0$.

### 3.6. Conformal Structure

Up to now we have described what could be called 'meromorphic field theory' rather than 'meromorphic conformal field theory' (and that is, in the mathematical literature, sometimes referred to as a vertex algebra, rather than a vertex operator algebra). Indeed, we have not yet discussed the conformal symmetry of the correlation functions but only its Möbius symmetry. A large part of the structure that we shall discuss in these notes does not actually rely on the presence of a conformal structure, but more advanced

features of the theory do, and therefore the conformal structure is an integral part of the theory.

A meromorphic field theory is called *conformal* if the three Möbius generators $L_0$, $L_{\pm 1}$ are the modes of a field $L$ that is then usually called the *stress-energy tensor* or the *Virasoro field*. Because of (31), (83) and (90), the field in question must be a quasiprimary field of conformal weight 2 that can be expanded as

$$L(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2} . \tag{104}$$

If we write $L(z) = V(\psi_L, z)$, the commutator in (90) becomes

$$
\begin{aligned}
[L_m, L_n] &= \sum_{N=-1}^{2} \binom{m+1}{m_N} V_{m+n} (L_N \psi_L) \\
&= \frac{m(m^2 - 1)}{6} V_{m+n}(L_2 \psi_L) + \frac{m(m+1)}{2} V_{m+n}(L_1 \psi_L) \\
&\quad + (m+1)V_{m+n}(L_0 \psi_L) + V_{m+n}(L_{-1}\psi_L) . 
\end{aligned} \tag{105}
$$

All these expressions can be evaluated further [11][+]: since $L_2 \psi_L$ has conformal weight $h = 0$, the uniqueness of the vacuum implies that it must be proportional to the vacuum vector,

$$L_2 \psi_L = L_2 L_{-2} \Omega = \frac{c}{2} \Omega , \tag{106}$$

where $c$ is some constant. Also, since the vacuum vector acts as the identity operator, $V_n(\Omega) = \delta_{n,0}$. Furthermore, $L_1 \psi_L = 0$ since $L$ is quasiprimary, and $L_0 \psi_L = 2\psi_L$ since $L$ has conformal weight 2. Finally, because of (66),

$$V(L_{-1}\psi, z) = \frac{d}{dz} \sum_n V_n(\psi) z^{-n-h} = -\sum_n (n+h)V_n(\psi) z^{-n-(h+1)} , \tag{107}$$

and since $L_{-1}\psi$ has conformal weight $h + 1$ (if $\psi$ has conformal weight $h$),

$$V_n(L_{-1}\psi) = -(n+h)V_n(\psi) . \tag{108}$$

Putting all of this together we then find that (105) becomes

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} . \tag{109}$$

This algebra is called the *Virasoro algebra* [53], and the parameter $c$ is called the *central charge*. The real algebra defined by (109) is the Lie algebra of the central extension of the group of diffeomorphisms of the circle (see *e.g.* [54]).

If the theory contains a Virasoro field, the states transform in representations of the Virasoro algebra (rather than just the Lie algebra of $sl(2,\mathbb{C})$ that corresponds to the Möbius transformations). Under suitable conditions (for example if the theory is unitary), the space of states can then be completely decomposed into irreducible representations of the Virasoro algebra. Because of the spectrum condition, the relevant

[+] This is also known as the Lüscher-Mack Theorem, see [50–52].

representations are then highest weight representations that are generated from a *primary* state $\psi$, *i.e.* a state satisfying

$$L_0\psi = h\psi \qquad L_n\psi = 0 \quad \text{for } n > 0. \tag{110}$$

If $\psi$ is primary, the commutation relation (83) holds for all $m$, *i.e.*

$$[L_m, V_n(\psi)] = (m(h-1) - n)V_{m+n}(\psi) \qquad \text{for all } m \in \mathbb{Z} \tag{111}$$

as follows from (90) together with (108). In this case the conformal symmetry also leads to an extension of the Möbius transformation formula (41) to arbitrary holomorphic transformations $f$ that are only locally defined,

$$D_f V(\psi, z) D_f^{-1} = (f'(z))^h \, V(\psi, f(z)) \,, \tag{112}$$

where $\psi$ is primary and $D_f$ is a certain product of exponentials of $L_n$ with coefficients that depend on $f$ [55]. The extension of (112) to states that are not primary is also known (but again much more complicated).

### 3.7. Examples

Let us now give a number of examples that exhibit the structures that we have described so far.

*3.7.1. The Free Boson* The simplest conformal field theory is the theory that is associated to a single free boson. In this case $V$ can be taken to be a one-dimensional vector space, spanned by a vector $J$ of weight 1, in which case we write $J(z) \equiv V(J, z)$. The amplitude of an odd number of $J$-fields is defined to vanish, and in the case of an even number it is given by

$$\langle J(z_1) \cdots J(z_{2n}) \rangle = \frac{k^n}{2^n n!} \sum_{\pi \in S_{2n}} \prod_{j=1}^{n} \frac{1}{\left(z_{\pi(j)} - z_{\pi(j+n)}\right)^2} \,, \tag{113}$$

$$= k^n \sum_{\pi \in S'_{2n}} \prod_{j=1}^{n} \frac{1}{\left(z_{\pi(j)} - z_{\pi(j+n)}\right)^2} \,, \tag{114}$$

where $k$ is an arbitrary (real) constant and, in (113), $S_{2n}$ is the permutation group on $2n$ objects, whilst, in (114), the sum is restricted to the subset $S'_{2n}$ of permutations $\pi \in S_{2n}$ such that $\pi(i) < \pi(i+n)$ and $\pi(i) < \pi(j)$ if $1 \le i < j \le n$. It is clear that these amplitudes are meromorphic and local, and it is easy to check that they satisfy the condition of Möbius invariance with the conformal weight of $J$ being 1.

From the amplitudes we can directly read off the operator product expansion of the field $J$ with itself as

$$J(z)J(\zeta) \sim \frac{k}{(z - \zeta)^2} \,, \tag{115}$$

where we use the symbol $\sim$ to indicate equality up to terms that are non-singular at $z = \zeta$. Comparing this with (87), and using (90) we then obtain

$$[J_n, J_m] = nk\delta_{n,-m} \,. \tag{116}$$

This defines (a representation of) the affine algebra $\hat{u}(1)$. $J$ is also sometimes called a $U(1)$-current. The operator product expansion (115) actually contains all the relevant information about the theory since one can reconstruct the amplitudes from it; to this end one defines recursively

$$\langle \, \rangle \quad = 1 \tag{117}$$

$$\langle J(z) \rangle = 0 \tag{118}$$

and

$$\left\langle J(z) \prod_{i=1}^{n} J(\zeta_i) \right\rangle = \sum_{j=1}^{n} \frac{k}{(z - \zeta_i)^2} \left\langle \prod_{\substack{i=1 \\ i \neq j}}^{n} J(\zeta_i) \right\rangle . \tag{119}$$

Indeed, the two sets of amplitudes have the same poles, and their difference describes therefore an entire function; all entire functions on the sphere are constant and it is not difficult to see that the constant is actually zero. The equality between the two sets of amplitudes can also be checked directly.

This theory is actually conformal since the space of states that is obtained by factorisation from these amplitudes contains the state

$$\psi_L = \frac{1}{2k} \, J_{-1} J_{-1} \Omega , \tag{120}$$

which plays the rôle of the stress-energy tensor with central charge $c = 1$. The corresponding field (that is defined by (86)) can actually be given directly as

$$V(\psi_L, z) = L(z) = \frac{1}{2k} \, {}^{\times}_{\times} J(z) J(z) \, {}^{\times}_{\times} , \tag{121}$$

where ${}^{\times}_{\times} \, {}^{\times}_{\times}$ denotes *normal ordering*, which, in this context, means that the singular part of the OPE of $J$ with itself has been subtracted. In fact, it follows from (87) that

$$J(w)J(z) = \frac{1}{(w-z)^2} V(J_1 J_{-1} \Omega, z) + \frac{1}{(w-z)} V(J_0 J_{-1} \Omega, z) \tag{122}$$

$$+ V(J_{-1} J_{-1} \Omega, z) + O(w - z) , \tag{123}$$

and therefore (121) implies (120).

*3.7.2. Affine Theories*  We can generalise this example to the case of an arbitrary finite-dimensional Lie algebra $g$; the corresponding conformal field theory is usually called a *Wess-Zumino-Novikov-Witten model* [56–60], and the following explicit construction of the amplitudes is due to Frenkel & Zhu [61]. Suppose that the matrices $t^a$, $1 \leq a \leq \dim g$, provide a finite-dimensional representation of $g$ so that $[t^a, t^b] = f^{ab}{}_c t^c$, where $f^{ab}{}_c$ are the structure constants of $g$. We introduce a field $J^a(z)$ for each $t^a$, $1 \leq a \leq \dim g$. If $K$ is any matrix which commutes with all the $t^a$, define

$$\kappa^{a_1 a_2 \ldots a_m} = \mathrm{tr}(K t^{a_1} t^{a_2} \cdots t^{a_m}) . \tag{124}$$

The $\kappa^{a_1 a_2 \ldots a_m}$ have the properties that

$$\kappa^{a_1 a_2 a_3 \ldots a_{m-1} a_m} = \kappa^{a_2 a_3 \ldots a_{m-1} a_m a_1} \tag{125}$$

and

$$\kappa^{a_1 a_2 a_3 \ldots a_{m-1} a_m} - \kappa^{a_2 a_1 a_3 \ldots a_{m-1} a_m} = f^{a_1 a_2}{}_b \kappa^{b a_3 \ldots a_{m-1} a_m} \,. \tag{126}$$

With a cycle $\sigma = (i_1, i_2, \ldots, i_m) \equiv (i_2, \ldots, i_m, i_1)$ we associate the function

$$f_\sigma^{a_{i_1} a_{i_2} \ldots a_{i_m}}(z_{i_1}, z_{i_2}, \ldots, z_{i_m}) = \frac{\kappa^{a_{i_1} a_{i_2} \ldots a_{i_m}}}{(z_{i_1} - z_{i_2})(z_{i_2} - z_{i_3}) \cdots (z_{i_{m-1}} - z_{i_m})(z_{i_m} - z_{i_1})} \,. \tag{127}$$

If the permutation $\rho \in S_n$ has no fixed points, it can be written as the product of cycles of length at least 2, $\rho = \sigma_1 \sigma_2 \ldots \sigma_M$. We associate to $\rho$ the product $f_\rho$ of functions $f_{\sigma_1} f_{\sigma_2} \ldots f_{\sigma_M}$ and define $\langle J^{a_1}(z_1) J^{a_2}(z_2) \ldots J^{a_n}(z_n) \rangle$ to be the sum of such functions $f_\rho$ over permutations $\rho \in S_n$ with no fixed point. Graphically, we can construct these amplitudes by summing over all graphs with $n$ vertices where the vertices carry labels $a_j$, $1 \le j \le n$, and each vertex is connected by two directed lines (propagators) to other vertices, one of the lines at each vertex pointing towards it and one away. (In the above notation, the vertex $i$ is connected to $\sigma^{-1}(i)$ and to $\sigma(i)$, and the line from $\sigma^{-1}(i)$ is directed towards $i$, and from $i$ to $\sigma(i)$.) Thus, in a given graph, the vertices are divided into directed loops or cycles, each loop containing at least two vertices. To each loop, we associate a function as in (127) and to each graph we associate the product of functions associated to the loops of which it is composed.

The resulting amplitudes are evidently local and meromorphic, and one can verify that they satisfy the Möbius covariance property with the weight of $J^a$ being 1. They determine the operator product expansion to be of the form*

$$J^a(z) J^b(w) \sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c J^c(w)}{(z-w)} \,, \tag{128}$$

and the algebra therefore becomes

$$[J_m^a, J_n^b] = f^{ab}{}_c J_{m+n}^c + m \kappa^{ab} \delta_{m,-n} \,. \tag{129}$$

This is (a representation of) the affine algebra $\hat{g}$ [62–64]. In the particular case where $g$ is simple, $\kappa^{ab} = \mathrm{tr}(K t^a t^b) = k \delta^{ab}$ in a suitable basis, where $k$ is a real number (that is called the *level*). The algebra then becomes

$$[J_m^a, J_n^b] = f^{ab}{}_c J_{m+n}^c + m k \delta^{ab} \delta_{m,-n} \,. \tag{130}$$

Again this theory is conformal since it has a stress-energy tensor given by

$$\psi_L = \frac{1}{2(k+Q)} \sum_a J_{-1}^a J_{-1}^a \Omega \,, \tag{131}$$

where $Q$ is the dual Coxeter number of $g$ (*i.e.* the value of the quadratic Casimir in the adjoint representation divided by the length squared of the longest root). Here the central charge is

$$c = \frac{2k \dim g}{2k + Q} \,, \tag{132}$$

* The terms singular in $(z-w)$ only arise from cycles where the vertices associated to $z$ and $w$ are adjacent. The first term in (128) comes from the 2-cycle involving $z$ and $w$. For every larger cycle in which $z$ and $w$ are adjacent, there exists another cycle where the order of $z$ and $w$ is reversed; the contributions of these two cycles combine to give the second term in (128).

and the corresponding field can be described as

$$L(z) = \frac{1}{2(k+Q)} \sum_a {}_\times^\times J^a(z) J^a(z) {}_\times^\times . \tag{133}$$

Finally, the modes of $L$ can be expressed in terms of the modes of $J^a$ as

$$L_n = \frac{1}{2(k+Q)} \sum_a \sum_m : J^a_m J^a_{n-m} : , \tag{134}$$

where the colons denote normal ordering

$$: J^a_m J^b_l := \begin{cases} J^a_m J^b_l & \text{if } m < 0 \\ J^b_l J^a_m & \text{if } m \geq 0. \end{cases} \tag{135}$$

The construction of the stress-energy-tensor as a bilinear in the *currents* $J^a$ is called the *Sugawara construction* [65]. The construction can be generalised directly to the case where $g$ is semi-simple: in this case, the Sugawara field is the sum of the Sugawara fields associated to each simple factor, and the central charge is the sum of the corresponding central charges.

The conformal field theory associated to the affine algebra $\hat{g}$ is unitary if $k$ is a positive integer [54, 66]; in this case $c \geq 1$.

*3.7.3. Virasoro Theories* Another very simple example of a meromorphic conformal field theory is the theory where $V$ can be taken to be a one-dimensional vector space that is spanned by the (conformal) vector $L$ [4]. Let us denote the corresponding field by $L(z) = V(L, z)$. Again following Frenkel and Zhu [61], we can construct the amplitudes graphically as follows. We sum over all graphs with $n$ vertices, where the vertices are labelled by the integers $1 \leq j \leq n$, and each vertex is connected by two directed lines (propagators) to other vertices, one of the lines at each vertex pointing towards it and one away. In a given graph, the vertices are now divided into loops, each loop containing at least two vertices. To each loop $\ell = (i_1, i_2, \ldots, i_m)$, we associate a function

$$f_\ell(z_{i_1}, z_{i_2}, \ldots, z_{i_m}) = \frac{c/2}{(z_{i_1} - z_{i_2})^2 (z_{i_2} - z_{i_3})^2 \cdots (z_{i_{m-1}} - z_{i_m})^2 (z_{i_m} - z_{i_1})^2} , \tag{136}$$

where $c$ is a real number, and, to a graph, the product of the functions associated to its loops. [Since it corresponds to a factor of the form $(z_i - z_j)^{-2}$ rather than $(z_i - z_j)^{-1}$, each line or propagator might appropriately be represented by a double line.] The amplitudes $\langle L(z_1) L(z_2) \ldots L(z_n) \rangle$ are then obtained by summing the functions associated with the various graphs with $n$ vertices. [Note that graphs related by reversing the direction of any loop contribute equally to this sum.]

These amplitudes determine the operator product expansion to be

$$L(z) L(\zeta) \sim \frac{c/2}{(z - \zeta)^4} + \frac{2L(\zeta)}{(z - \zeta)^2} + \frac{L'(\zeta)}{z - \zeta} , \tag{137}$$

which thus agrees with the operator product expansion of the field $L$ as defined in subsection 3.6.

These pure Virasoro models are unitary if either $c \geq 1$ or $c$ belongs to the *unitary discrete series* [67]

$$c = 1 - \frac{6}{m(m+1)} \qquad m = 2, 3, 4, \ldots \tag{138}$$

The necessity of this condition was established in [67] using the Kac-determinant formula [68] that was proven by Feigin & Fuchs in [69]. The existence of these unitary representations follows from the coset construction (to be explained below) [70].

*3.7.4. Lattice Theories*  Let us recall that a *lattice* $\Lambda$ is a subset of an $n$-dimensional inner product space which has integral coordinates in some basis, $e_j$, $j = 1, \ldots, n$; thus $\Lambda = \{\sum m_j e_j : m_j \in \mathbb{Z}\}$. The lattice is called *Euclidean* if the inner product is positive definite, *i.e.* if $k^2 \geq 0$ for each $k \in \Lambda$, and *integral* if $k \cdot l \in \mathbb{Z}$ for all $k, l \in \Lambda$. An (integral) lattice is *even* if $k^2$ is an even integer for every $k \in \Lambda$.

Suppose $\Lambda$ is an even Euclidean lattice with basis $e_j$, $j = 1, \ldots, n$. Let us introduce an algebra consisting of matrices $\gamma_j$, $1 \leq j \leq n$, such that $\gamma_j^2 = 1$ and $\gamma_i \gamma_j = (-1)^{e_i \cdot e_j} \gamma_j \gamma_i$. If we define $\gamma_k = \gamma_1^{m_1} \gamma_2^{m_2} \ldots \gamma_n^{m_n}$ for $k = m_1 e_1 + m_2 e_2 + \ldots + m_n e_n$, we can define quantities $\epsilon(k_1, k_2, \ldots, k_N)$, taking the values $\pm 1$, by

$$\gamma_{k_1} \gamma_{k_2} \ldots \gamma_{k_N} = \epsilon(k_1, k_2, \ldots, k_N) \gamma_{k_1 + k_2 + \ldots + k_N} \,. \tag{139}$$

We define the theory associated to the lattice $\Lambda$ by taking a basis for $V$ to consist of the states $k \in \Lambda$. For each $k$, the corresponding field $V(k, z)$ has conformal weight $\frac{1}{2}k^2$, and the amplitudes are given by

$$\langle V(k_1, z_1) V(k_2, z_2) \cdots V(k_N, z_n) \rangle = \epsilon(k_1, k_2, \ldots, k_N) \prod_{1 \leq i < j \leq N} (z_i - z_j)^{k_i \cdot k_j} \tag{140}$$

if $k_1 + k_2 + \ldots + k_N = 0$ and zero otherwise. The $\epsilon(k_1, k_2, \ldots, k_N)$ obey the conditions

$$\epsilon(k_1, k_2, \ldots, k_{j-1}, k_j, k_{j+1}, k_{j+2}, \ldots, k_N) \tag{141}$$

$$= (-1)^{k_j \cdot k_{j+1}} \epsilon(k_1, k_2, \ldots, k_{j-1}, k_{j+1}, k_j, k_{j+2}, \ldots, k_N) \,, \tag{142}$$

which guarantees the locality of the amplitudes, and

$$\epsilon(k_1, k_2, \ldots, k_j) \epsilon(k_{j+1}, \ldots, k_N) \tag{143}$$

$$= \epsilon(k_1 + k_2 + \ldots + k_j, k_{j+1} + \ldots + k_N) \epsilon(k_1, k_2, \ldots, k_j, k_{j+1}, \ldots, k_N) \,, \tag{144}$$

which implies the cluster decomposition property (that guarantees the uniqueness of the vacuum). It is also easy to check that the amplitudes satisfy the Möbius covariance condition.

This theory is also conformal, but the Virasoro field cannot be easily described in terms of the fields in $V$. In fact, the theory that is obtained by factorisation from the above amplitudes contains $n$ fields of conformal weight 1, $H^i(z)$, $i = 1, \ldots, n$, whose operator product expansion is

$$H^i(z) H^j(\zeta) \sim \delta^{ij} \frac{1}{(z - \zeta)^2} \,. \tag{145}$$

This is of the same form as (115), and the corresponding modes therefore satisfy

$$[H_m^i, H_n^j] = m\delta^{ij}\delta_{m,-n} \,. \tag{146}$$

The Virasoro field is then given as

$$L(z) = \frac{1}{2} \sum_i {}_\times^\times H^i(z) H^i(z) {}_\times^\times \,, \tag{147}$$

and the central charge is $c = n$. Details of the construction of this unitary meromorphic conformal field theory can be found in [71, 72].

For the case where $\Lambda$ is the root lattice of a simply-laced finite-dimensional Lie algebra, the lattice theory coincides with the affine theory of the corresponding affine Lie algebra at level $k = 1$; this is known as the Frenkel-Kac-Segal construction [73, 74].

Lattice theories that are associated to even self-dual lattices whose dimension is a multiple of 24 provide examples of *local meromorphic conformal field theories*, *i.e.* local modular invariant theories that are meromorphic. Such theories are characterised by the property that all amplitudes depend only on the chiral coordinates (*i.e.* on $z_i$ but not on $\bar{z}_i$), and that the space of states of the complete local theory coincides with (rather than just contains) that of the meromorphic (sub)theory. Recall that for a given lattice $\Lambda$, the *dual lattice* $\Lambda^*$ is the lattice that contains all vectors $y$ for which $x \cdot y \in \mathbb{Z}$ for all $x \in \Lambda$. A lattice is integral if $\Lambda \subset \Lambda^*$, and it is *self-dual* if $\Lambda^* = \Lambda$. The dimension of an even self-dual lattice has to be a multiple of 8.

It is not difficult to prove that a basis of states for the meromorphic Fock space $\mathcal{F}_\Lambda$ that is associated to an even lattice $\Lambda$ can be taken to consist of the states of the form

$$H_{m_1}^{i_1} H_{m_2}^{i_2} \cdots H_{m_N}^{i_N} |k\rangle \,, \tag{148}$$

where $m_1 \leq m_2 \leq \cdots \leq m_N$, $i_j \in \{1, \ldots, n\}$ and $k \in \Lambda$. Here $H_l^i$ are the modes of the currents (145), and $|k\rangle = V(k, 0)\Omega$. The contribution of the meromorphic subtheory to the partition function (11) is therefore

$$\chi_{\mathcal{F}_\Lambda}(\tau) = q^{-\frac{c}{24}} \text{tr}_{\mathcal{F}_\Lambda}(q^{L_0}) = \eta(\tau)^{-\dim \Lambda} \Theta_\Lambda(\tau) \,, \tag{149}$$

where $\eta(\tau)$ is the famous *Dedekind eta-function*,

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \,, \tag{150}$$

$\Theta_\Lambda$ is the theta function of the lattice,

$$\Theta_\Lambda(\tau) = \sum_{k \in \Lambda} q^{\frac{1}{2}k^2} \,, \tag{151}$$

and we have set $q = e^{2\pi i \tau}$. The theta function of a lattice is related to that of its dual by the Jacobi transformation formula

$$\Theta_\Lambda(-1/\tau) = (-i\tau)^{\frac{1}{2} \dim \Lambda} ||\Lambda^*|| \Theta_{\Lambda^*}(\tau) \,, \tag{152}$$

where $||\Lambda|| = |\det(e_i \cdot e_j)|$. Together with the transformation formula of the Dedekind function

$$\eta(-1/\tau) = (-i\tau)^{\frac{1}{2}} \eta(\tau) \,, \tag{153}$$

this implies that the partition function of the meromorphic theory transforms under modular transformations as

$$\chi_{\mathcal{F}_\Lambda}(-1/\tau) = ||\Lambda^*||\chi_{\mathcal{F}_{\Lambda^*}}(\tau) \,. \tag{154}$$

Furthermore, it follows from (149) together with the fact that the spectrum of $L_0$ in $\mathcal{F}_\Lambda$ is integral that

$$\chi_{\mathcal{F}_\Lambda}(\tau + 1) = e^{2\pi i \frac{c}{24}}\chi_{\mathcal{F}_\Lambda}(\tau) \,. \tag{155}$$

If $\Lambda$ is a self-dual lattice, $||\Lambda^*|| = 1$ and the partition function is invariant under the transformation in (154). If in addition $c = n = 24k$ where $k \in \mathbb{Z}$, the partition function of the meromorphic theory is invariant under the whole modular group, and the meromorphic theory defines a local (modular invariant) conformal field theory. For the case where $n = 24$, lattices with these properties have been classified: there exist precisely 24 such lattices, the 23 Niemeier lattices and the Leech lattice (see [75] for a review). The Leech lattice plays a central rôle in the construction of the Monster conformal field theory [14].

*3.7.5. More General W-Algebras* In the above examples, closed formulae for all amplitudes of the generating fields could be given explicitly. There exist however many meromorphic conformal field theories for which this is not the case. These theories are normally defined in terms of the operator product expansion of a set of generating fields (that span $V$) from which the commutation relations of the corresponding modes can be derived; the resulting algebra is then usually called a *W-algebra*. In general, a $W$-algebra is not a Lie algebra in the modes of the generating fields since the operator product expansion (and therefore the associated commutator) of two generating fields may involve normal ordered products of the generating fields rather than just the generating fields themselves.

In principle all amplitudes can be determined from the knowledge of these commutation relations, but it is often difficult to give closed expressions for them. (It is also, *a priori*, not clear whether the power series expansion that can be obtained from these commutation relations will converge to define meromorphic functions with the appropriate singularity structure, although this is believed to be the case for all presently known examples.) The theories that we have described in detail above are in some sense fundamental in that all presently known meromorphic (bosonic) conformal field theories have an (alternative) description as a *coset* or *orbifold* of one of these theories; these constructions will be described in the next subsection.

The first example of a $W$-algebra that is not a Lie algebra in the modes of its generating fields is the so-called $W_3$ algebra [76, 77]; this algebra is generated by the Virasoro algebra $\{L_n\}$, and the modes $W_m$ of a quasiprimary field of conformal weight $h = 3$, subject to (109) and the relations [76–78]

$$[L_m, W_n] = (2m - n)\,W_{m+n}\,,$$
$$[W_m, W_n] = \frac{1}{48}\,(22 + 5c)\,\frac{c}{3 \cdot 5!}\,(m^2 - 4)\,(m^2 - 1)\,m\,\delta_{m,-n} + \frac{1}{3}(m - n)\,\Lambda_{m+n}$$

$$+\frac{1}{48}\left(22+5c\right)\frac{1}{30}\left(m-n\right)\left(2m^2-mn+2n^2-8\right)L_{m+n}\,,\tag{156}$$

where $\Lambda_k$ are the modes of a quasiprimary field of conformal weight $h_\Lambda = 4$. This field is a normal ordered product of $L$ with itself, and its modes are explicitly given as

$$\Lambda_n=\sum_{k=-1}^{\infty}L_{n-k}\,L_k+\sum_{k=-\infty}^{-2}L_k\,L_{n-k}-\frac{3}{10}\left(n+2\right)\left(n+3\right)L_n\,.\tag{157}$$

One can check that this set of commutators satisfies the Jacobi-identity. (This is believed to be equivalent to the associativity of the operator product expansion of the corresponding fields.) Subsequently, various classes of $W$-algebras have been constructed [79–86]. There have also been attempts to construct systematically classes of $W$-algebras [87, 88] following [89]. For a review of these matters see [90].

*3.7.6. Superconformal Field Theories* All examples mentioned up to now have been bosonic theories, *i.e.* theories all of whose fields are bosonic (and therefore have integral conformal weight). The simplest example of a fermionic conformal field theory is the theory generated by a single free fermion field $b(z)$ of conformal weight $h = \frac{1}{2}$ with operator product expansion

$$b(z)b(\zeta)\sim\frac{1}{(z-\zeta)}\,.\tag{158}$$

The corresponding modes

$$b(z)=\sum_{r\in\mathbb{Z}+\frac{1}{2}}b_r z^{-r-\frac{1}{2}}\tag{159}$$

satisfy an anti-commutation relation of the form

$$\{b_r,b_s\}=\delta_{r,-s}\,.\tag{160}$$

The Virasoro field is given in this case by

$$L(z)=\frac{1}{2}\,{}_\times^\times\frac{db(z)}{dz}\,b(z)\,{}_\times^\times\,,\tag{161}$$

and the corresponding modes are

$$L_n=\frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(r-\frac{1}{2}\right):b_{n-r}b_r:\,.\tag{162}$$

They satisfy the Virasoro algebra with central charge $c = \frac{1}{2}$.

This theory has a *supersymmetric* extension whose generating fields are the free fermion $b(z)$ (158) and the free boson $J(z)$ defined by (115) and (116), where we set for convenience $k = 1$. The operator product expansion of $J$ and $b$ is regular, so that the corresponding commutator vanishes. This theory then exhibits *superconformal invariance*; in the present context this means that the theory has in addition to the

stress-energy-tensor $L$, the superpartner field $G$ of conformal weight $3/2$, whose modes satisfy

$$
\begin{aligned}
{[L_m, G_r]} &= \left(\frac{m}{2} - r\right) G_{m+r} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}\,.
\end{aligned}
\tag{163}
$$

Together with the Virasoro commutation relations for $L$ (109), this is called the (Neveu Schwarz [NS] sector of the) $N = 1$ *superconformal algebra.* In terms of $J$ and $b$, $L$ and $G$ can be written as

$$
\begin{aligned}
L(z) &= \frac{1}{2} {}_\times^\times J(z)J(z) {}_\times^\times + \frac{1}{2} {}_\times^\times \frac{db(z)}{dz}b(z) {}_\times^\times \\
G(z) &= J(z)b(z)\,,
\end{aligned}
\tag{164}
$$

where on the right-hand-side of the second line we have omitted the normal ordering since the operator product expansion is regular. The modes of $L$ and $G$ are then given as

$$
\begin{aligned}
L_n &= \frac{1}{2}\sum_{m\in\mathbb{Z}} : J_{n-m}J_m : + \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\left(r - \frac{1}{2}\right) : b_{n-r}b_r : \\
G_r &= \sum_n J_{-n}b_{r+n}\,,
\end{aligned}
\tag{165}
$$

and it is easy to check that they satisfy (109) and (163) with $c = 3/2$.

The $N = 1$ superconformal field theory that is generated by the fields $L$ and $G$ subject to the commutation relations (163) is unitary if [91]

$$
c = \frac{3}{2} - \frac{12}{m(m+2)} \qquad m = 2, 3, 4, \ldots
\tag{166}
$$

There also exist extended superconformal algebras that contain the above algebra as a subalgebra. The most important of these is the $N = 2$ *superconformal algebra* [92, 93] that is the symmetry algebra of the world-sheet conformal field theory of space-time supersymmetric string theories [94]. In the so-called NS sector, this algebra is generated by the modes of the Virasoro algebra (109), the modes of a free boson $J_n$ (116) (where we set again $k = 1$), and the modes of two supercurrents of conformal dimension $h = \frac{3}{2}$, $\{G_\alpha^\pm\}$, $\alpha \in \mathbb{Z} + \frac{1}{2}$, subject to the relations [95]

$$
\begin{aligned}
{[L_m, G_r^\pm]} &= \left(\frac{1}{2}m - r\right) G_{m+r}^\pm \\
{[L_m, J_n]} &= -n\, J_{m+n} \\
{[J_m, G_r^\pm]} &= \pm G_{m+r}^\pm \\
\{G_r^\pm, G_s^\pm\} &= 0 \\
\{G_r^+, G_s^-\} &= 2\,L_{r+s} + (r - s)\, J_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s}\,.
\end{aligned}
\tag{167}
$$

The representation theory of the $N = 2$ superconformal algebra exhibits many interesting and new phenomena [96, 97].

*3.8. The Coset Construction*

There exists a fairly general construction by means of which a meromorphic conformal field theory can be associated to a pair of a meromorphic conformal field theory and a subtheory. In its simplest formulation [70,98] (see also [99–101] for a related construction in a particular case) the pair of theories are affine theories that are associated to a pair $h \subset g$ of finite-dimensional simple Lie algebras. Let us denote by $L_m^g$ and $L_m^h$ the modes of the Sugawara fields (134) of the affine algebras $\hat{g}$ and $\hat{h}$, respectively. If $J_n^a$ is a generator of $\hat{h} \subset \hat{g}$, then

$$[L_m^g, J_n^a] = -n J_{m+n}^a \qquad \text{and} \qquad [L_m^h, J_n^a] = -n J_{m+n}^a \tag{168}$$

since $J_n^a$ are the modes of a primary field of conformal weight $h = 1$. (This can also be checked directly using (134) and (130).) It then follows that

$$\mathcal{K}_m = L_m^g - L_m^h \tag{169}$$

commutes with every generator $J_n^a$ of $\hat{h}$, and therefore with the modes $L_m^h$ (which are bilinear in $J_n^a$). We can thus write $L_m^g$ as the sum of two commuting terms

$$L_m^g = L_m^h + \mathcal{K}_m, \tag{170}$$

and since both $L_m^g$ and $L_m^h$ satisfy the commutation relations of a Virasoro algebra, it follows that this is also the case for $\mathcal{K}_m$, where the corresponding central charge is

$$c^{\mathcal{K}} = c^g - c^h. \tag{171}$$

It is also straightforward to extend the construction to the case where $g$ (and/or $h$) are semi-simple.

In the unitary case, both $c^g \geq c^h \geq 1$, but $c^{\mathcal{K}}$ need not be greater or equal to one. Indeed, we can consider $\hat{g} = \hat{su}(2)_m \oplus \hat{su}(2)_1$ and $h = \hat{su}(2)_{m+1}$, where the index is the level $k$ of (130), and the central charge $c^{\mathcal{K}}$ is then

$$c^{\mathcal{K}} = \frac{3m}{m+2} + 1 - \frac{3(m+1)}{(m+3)} = 1 - \frac{6}{(m+2)(m+3)}, \tag{172}$$

where $m \in \mathbb{Z}_+$ and we have used (132) together with $\dim su(2) = 3$ and $Q_{su(2)} = 4$. By construction, the subspace that is generated by $\mathcal{K}_m$ from the vacuum forms a unitary representation of the Virasoro algebra with $h = 0$ and $c = c^{\mathcal{K}}$, and thus for each value of $c$ in the unitary discrete series (138), the vacuum representation of the Virasoro algebra is indeed unitary [70]. Similar arguments can also be given for the discrete series of the $N = 1$ superconformal algebra (166) [98].

We can generalise this construction directly to the case where instead of the affine theory associated to $\hat{g}$, we consider an arbitrary conformal field theory $\mathcal{H}$ (with stress-energy tensor $L$) that contains, as a subtheory, the affine theory associated to $\hat{h}$. Then by the same arguments as above

$$\mathcal{K} = L - L^h \tag{173}$$

commutes with $\hat{h}$ (and thus with $L^h$), and therefore satisfies a Virasoro algebra with central charge $c^{\mathcal{K}} = c - c^h$. By construction, the Virasoro algebra $\mathcal{K}_m$ leaves the subspace of states

$$\mathcal{H}^h = \left\{ \psi \in \mathcal{H} : J_n^a \psi = 0 \text{ for every } J_n^a \in \hat{h} \text{ with } n \geq 0 \right\} \tag{174}$$

invariant. Furthermore, the operator product expansion of an $h$-current $J^a(z)$ and a vertex operator $V(\psi, \zeta)$ associated to $\psi \in \mathcal{H}^h$ is regular, and therefore the corresponding modes commute. This implies that the operator product expansion of two states in $\mathcal{H}^h$ only contain states that lie in $\mathcal{H}^h$ [102], and thus that we can define a meromorphic field theory whose space of states is $\mathcal{H}^h$. Since the commutator of $L^h$ with any state in $\mathcal{H}^h$ vanishes, it is clear that the Virasoro field for this theory is $\mathcal{K}$; the resulting meromorphic conformal field theory is called the *coset theory*.

Many $W$-algebras can be constructed as cosets of affine theories [81, 102–104]. It is also possible to construct representations of the coset theory from those of $\mathcal{H}$, and to determine the corresponding modular transformation matrix; details of these constructions for the case of certain coset theories of affine theories have been recently worked out in [105].

### 3.9. Orbifolds

There exists another very important construction that associates to a given local (modular invariant) conformal field theory another such theory [106–110]. This construction is possible whenever the theory carries an action of a finite group $G$. A group $G$ acts on the space of states of a conformal field theory $\mathcal{H}$, if each $g \in G$ defines a linear map $g : \mathcal{H} \to \mathcal{H}$ (that leaves the dense subspace $\mathcal{F}$ invariant, $g : \mathcal{F} \to \mathcal{F}$), the composition of maps respects the group structure of $G$, and the amplitudes satisfy

$$\langle V(g\psi_1; z_1, \bar{z}_1) \cdots V(g\psi_n; z_n, \bar{z}_n) \rangle = \langle V(\psi_1; z_1, \bar{z}_1) \cdots V(\psi_n; z_n, \bar{z}_n) \rangle . \tag{175}$$

The space of states of the orbifold theory consists of those states that are invariant under the action of $G$,

$$\mathcal{H}^G = \{ \psi \in \mathcal{H} : g\psi = \psi \quad \text{for all } g \in G. \} , \tag{176}$$

together with additional *twisted sectors*, one for each conjugacy class in $G$. Generically the meromorphic subtheory of the resulting theory consists of those meromorphic fields of the original theory that are invariant under $G$, but in general it may also happen that some of the twisted sectors contain additional meromorphic fields.

The construction of the twisted sectors is somewhat formal in general, and so is the proof that the resulting local theory is always modular invariant. There exist however some examples where the construction is understood in detail, most notably the local meromorphic lattice theories that were discussed in section 3.7.4. Let us for simplicity consider the case where $G = \mathbb{Z}_2 = \{1, \theta\}$, and the action on $\mathcal{H}_\Lambda$ is determined by

$$\theta H^i(z) \theta = -H^i(z) , \qquad \theta V(x, z) \theta = V(-x, z) , \qquad \theta \Omega = \Omega . \tag{177}$$

In this case there exists only one twisted sector, $\mathcal{H}'_\Lambda$, and it is generated by the operators $c_r^i$, $i = 1, \ldots, n$ with $r \in \mathbb{Z} + \frac{1}{2}$, satisfying the commutation relations

$$[c_r^i, c_s^j] = r \delta^{ij} \delta_{r,-s} \,. \tag{178}$$

These act on an irreducible representation space $U$ of the algebra

$$\gamma^i \gamma^j = (-1)^{e_i \cdot e_j} \gamma^j \gamma^i \,, \tag{179}$$

where $e_i$, $i = 1, \ldots, n$ is a basis of $\Lambda$, and where for $\chi \in U$, $c_r^i \chi = 0$ if $r > 0$. The actual orbifold theory consists then of the states in the untwisted $\mathcal{H}_\Lambda$ and the twisted sector $\mathcal{H}'_\Lambda$ that are left invariant by $\theta$, where the action of $\theta$ on $\mathcal{H}_\Lambda$ is given as in (177), and on $\mathcal{H}'_\Lambda$ we have

$$\theta c_r^i \theta = -c_r^i \qquad \theta|_U = \pm 1 \,. \tag{180}$$

The generators of the Virasoro algebra act in the twisted sector as

$$L_m = \frac{1}{2} \sum_{i=1}^n \sum_{r \in \mathbb{Z} + \frac{1}{2}} : c_r^i c_{m-r}^i : + \frac{n}{16} \delta_{m,0} \,. \tag{181}$$

As in the untwisted sector $L_m$ commutes with $\theta$ and is therefore well defined in the orbifold theory.

Since the local meromorphic conformal field theory is already modular invariant, the dimension $n$ of the lattice is a multiple of 24 and the orbifold theory is again a meromorphic conformal field theory. This theory is again bosonic provided the sign in (180) corresponds to the parity of $\dim \Lambda$ divided by 8. With this choice of (180) the orbifold theory defines another local meromorphic conformal field theory [71, 72, 111].

The most important example of this type is the orbifold theory associated to the Leech lattice for which the orbifold theory does not have any states of conformal weight one. This is the famous Monster conformal field theory whose automorphism group is the Monster group [12–14, 112], the largest simple sporadic group. It has been conjectured that this theory is uniquely characterised by the property to be a local meromorphic conformal field theory with $c = 24$ and without any states of conformal weight one [14], but this has not been proven so far.

One can also apply the construction systematically to the other 23 Niemeier lattices. Together with the 24 local meromorphic conformal field theories that are directly associated to the 24 self-dual lattices, this would naively give 48 conformal field theories. However, nine of these theories coincide, and therefore these constructions only produce 39 different local meromorphic conformal field theories [72, 111]. If the above conjecture about the uniqueness of the Monster theory is true, then every local meromorphic conformal field theory at $c = 24$ (other than the Monster theory) contains states of weight one, and therefore an affine subtheory [11]. The theory can then be analysed in terms of this subtheory, and using arguments of modular invariance, Schellekens has suggested that at most 71 local meromorphic conformal field theories exist for $c = 24$ [39]. However this classification has only been done on the level of the partition functions, and it is not clear whether more than one conformal field theory

may correspond to a given partition function. Also, none of these additional theories has been constructed explicitly, and it is not obvious that all 71 partition functions arise from consistent conformal field theories.

## 4. Representations of a Meromorphic Conformal Field Theory

For most local conformal field theories the meromorphic fields form a proper subspace of the space of states. The additional states of the theory transform then in representations of the meromorphic (and the anti-meromorphic) subtheory. Indeed, as we explained in section 2.1, the space of states is a direct sum of subspaces $\mathcal{H}_{(j,\bar{j})}$, each of which forms an indecomposable representation of the two meromorphic conformal field theories. For most conformal field theories of interest (although not for all, see [41]), each $\mathcal{H}_{(j,\bar{j})}$ is a tensor product of an irreducible representation of the meromorphic and the anti-meromorphic conformal subtheory, respectively

$$\mathcal{H}_{(j,\bar{j})} = \mathcal{H}_j \otimes \bar{\mathcal{H}}_{\bar{j}} \,. \tag{182}$$

The local theory is specified in terms of the space of states and the set of all amplitudes involving arbitrary states in $\mathcal{F} \subset \mathcal{H}$. The meromorphic subtheory that we analysed above describes the amplitudes that only involve states in $\mathcal{F}_0$. Similarly, the anti-meromorphic subtheory describes the amplitudes that only involve states in $\overline{\mathcal{F}}_0$. Since the two meromorphic theories commute, a general amplitude involving states from both $\mathcal{F}_0$ and $\overline{\mathcal{F}}_0$ is simply the product of the corresponding meromorphic and anti-meromorphic amplitude. (Indeed, the product of the meromorphic and the anti-meromorphic amplitude has the same poles as the original amplitude.)

If the theory factorises as in (182), one of the summands in (8) is the completion of $\mathcal{F}_0 \otimes \overline{\mathcal{F}}_0$, and we denote it by $\mathcal{H}_{(0,\bar{0})}$. A general amplitude of the theory contains states from different sectors $\mathcal{H}_{(j,\bar{j})}$. Since each $\mathcal{H}_{(j,\bar{j})}$ is a representation of the two vertex operator algebras, we can use the operator product expansion (5) to rewrite a given amplitude in terms of amplitudes that do not involve states in $\mathcal{H}_{(0,\bar{0})}$. It is therefore useful to call an amplitude an *n-point function* if it involves $n$ states from sectors other than $\mathcal{H}_{(0,\bar{0})}$ and an arbitrary number of states from $\mathcal{H}_{(0,\bar{0})}$. In general, each such amplitude can be expressed as a sum of products of a *chiral* amplitude, *i.e.* an amplitude that only depends on the $z_i$, and an *anti-chiral* amplitude, *i.e.* one that only depends on the $\bar{z}_i$. However, for $n = 0, 1, 2, 3$, the sum contains only one term since the functional form of the relevant chiral (and anti-chiral) amplitudes is uniquely determined by Möbius symmetry.

The zero-point functions are simply products of meromorphic amplitudes, and the one-point functions vanish. The two-point functions are usually non-trivial, and they define, in essence, the different representations of the meromorphic and the anti-meromorphic subtheory that are present in the theory. Since these amplitudes factorise into chiral and anti-chiral amplitudes, one can analyse them separately; these chiral amplitudes define then a representation of the meromorphic subtheory.

## 4.1. Highest Weight Representations

A representation of the meromorphic conformal field theory is defined by the collection of amplitudes

$$\left\langle \bar{\phi}(w) V(\psi_1, z_1) \cdots V(\psi_n, z_n) \phi(u) \right\rangle, \tag{183}$$

where $\phi$ and $\bar{\phi}$ are two fixed fields (that describe the generating field of a representation and its conjugate), and $\psi_i$ are quasiprimary fields in the meromorphic conformal field theory. The amplitudes (183) are analytic functions of the variables and transform covariantly under the Möbius transformations as in (37)

$$
\begin{aligned}
\left\langle \bar{\phi}(w) V(\psi_1, z_1) \cdots V(\psi_n, z_n) \phi(u) \right\rangle & \\
= \left( \frac{d\gamma(w)}{dw} \right)^{\bar{h}} & \left( \frac{d\gamma(u)}{du} \right)^{h} \prod_{i=1}^{n} \left( \frac{d\gamma(z_i)}{dz_i} \right)^{h_i} \\
& \left\langle \bar{\phi}(\gamma(w)) V(\psi_1, \gamma(z_1)) \cdots V(\psi_n, \gamma(z_n)) \phi(\gamma(u)) \right\rangle,
\end{aligned} \tag{184}
$$

where $h_i$ is the conformal weight of $\psi_i$, and we call $h$ and $\bar{h}$ the conformal weights of $\phi$ and $\bar{\phi}$, respectively. Since $\phi$ and $\bar{\phi}$ are not meromorphic fields, $h$ and $\bar{h}$ are in general not half-integer, and the amplitudes are typically branched about $u = w$. Because of the Möbius symmetry we can always map the two points $u$ and $w$ to $0$ and $\infty$, respectively (for example by considering the Möbius transformation $\gamma(z) = \frac{z-u}{z-w}$), and we shall from now always do so. In this case we shall write $\phi(0)\rangle = |\phi\rangle$. For the case of $w = \infty$ the situation is slightly more subtle since (183) behaves as $w^{-2\bar{h}}$ for $w \to \infty$; we therefore define

$$\langle \bar{\phi}| = \lim_{w \to \infty} w^{2\bar{h}} \langle \bar{\phi}(w) \,. \tag{185}$$

We can then think of the amplitudes as being the expectation value of the meromorphic fields in the background described by $\phi$ and $\bar{\phi}$.

The main property that distinguishes the amplitudes as *representations* of the meromorphic conformal field theory is the condition that the operator product relations of the meromorphic conformal field theory are preserved by these amplitudes. This is the requirement that the operator product expansion of the meromorphic fields (87) also holds in the amplitudes (183), *i.e.*

$$
\begin{aligned}
\langle \bar{\phi}| V(\psi_1, z_1) \cdots V(\psi_i, z_i) V(\psi_{i+1}, z_{i+1}) \cdots V(\psi_n, z_n) |\phi\rangle & \\
= \sum_{n < h_{i+1}} (z_i - z_{i+1})^{-n-h_i} & \\
& \langle \bar{\phi}| V(\psi_1, z_1) \cdots V(\psi_{i-1}, z_{i-1}) V(V_n(\psi_i)\psi_{i+1}, z_{i+1}) \cdots V(\psi_n, z_n) |\phi\rangle,
\end{aligned} \tag{186}
$$

where $|z_i - z_{i+1}| < |z_j - z_{i+1}|$ for $j \neq i$ and $|z_i - z_{i+1}| < |z_{i+1}|$. In writing (186) we have also implicitly assumed that if $\mathcal{N}$ is a null-state of the meromorphic conformal field theory (*i.e.* a linear combination of states of the form (84) that vanishes in every meromorphic amplitude) then any amplitude (183) involving $\mathcal{N}$ also vanishes; this is

implicit in the above since the operator product expansion of the meromorphic conformal field theory is only determined up to such null-fields by the meromorphic amplitudes.

We call a representation *untwisted* if the amplitudes (183) are single-valued as $z_i$ encircles the origin or infinity; if this is not the case for at least some of the meromorphic fields the representation is called *twisted*. If the representation is untwisted, we can expand the meromorphic fields in terms of their modes as in (80). In this way we can then define the action of $V_n(\psi)$ on the non-meromorphic state $|\phi\rangle$, and thus on arbitrary states of the form

$$V_{n_1}(\psi_1) \cdots V_{n_N}(\psi_N)|\phi\rangle . \tag{187}$$

As we explained in section 3.4, the commutation relations of these modes (90) can be derived from the operator product expansion of the corresponding fields. Since the representation amplitudes (183) preserve these in the sense of (186), it follows that the action of the modes on the states of the form (187) also respects (90), at least up to null-states that vanish in all amplitudes. If we thus define the Fock space $\mathcal{F}$ to be the quotient space of the space of states generated by (187), where we identify states whose difference vanishes in all amplitudes, then $\mathcal{F}$ carries a representation of the Lie algebra of modes (of the meromorphic fields).♯

For most of the following we shall only consider untwisted representations, but there is one important case of a twisted representations, the so-called *Ramond sector* of a fermionic algebra, that should be mentioned here since it can be analysed by very similar methods. In this case the bosonic fields are single-valued as $z$ encircles the origin, and the fermionic fields pick up a minus sign. It is then again possible to expand the meromorphic fields in modes, where the bosonic fields are treated as before, and for a fermionic field we now have

$$V(\chi, z) = \sum_{r \in \mathbb{Z}} V_r(\chi) z^{-r-h} \qquad \text{R-sector of fermionic } \chi. \tag{188}$$

Using the same methods as before, we can deduce the commutation (and anti-commutation) relations of these modes from the operator product expansion of the fields in the meromorphic conformal field theory, and the Fock space of the R-sector representations forms then a representation of this Lie algebra. Indeed, the actual form of the commutation and anti-commutation relations is the same except that the fermionic fields now have integer mode number. For example, the commutation relations of the R-sector of the $N = 1$ superconformal algebra are

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right)G_{m+r} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s} ,
\end{aligned} \tag{189}$$

♯ Strictly speaking, the underlying vector space of this Lie algebra is the vector space of modes, where we identify two modes if their difference vanishes on the Fock space of all representations.

where now $r, s \in \mathbb{Z}$, and this agrees formally with (163).

The physically relevant representations satisfy again the condition that the spectrum of $L_0$ (the zero mode of the conformal stress-energy-tensor) is bounded from below. This implies that there exists a state $\phi_0$ in $\mathcal{F}$ that is annihilated by all modes $V_n(\psi)$ with $n > 0$ (since $V_n(\psi)$ lowers the $L_0$ eigenvalue by $n$ as follows from (83)); such a state is called a *(Virasoro) highest weight state*. If the representation is irreducible, *i.e.* if it does not contain any proper subrepresentations, then the representation generated from $\phi_0$ by the action of the modes reproduces the whole representation space, and we may therefore assume that $\phi$ (and $\bar{\phi}$) are highest weight states. Using the mode expansion of the meromorphic field $V(\psi, z)$, the highest weight property of $\phi$ can be rewritten as the condition that the pole in $z$ of an amplitude involving $V(\psi, z)|\phi\rangle$ is at most of order $h$, where $h$ is the conformal weight of $\psi$. Because of the Möbius covariance of the amplitudes, this is then also equivalent to the condition that the order of the pole in $(z_i - u)$ in (183) is bounded by the conformal weight of $\psi_i$, $h_i$.

If $\phi_0$ is a Virasoro highest weight state, then so is $V_0(\psi)\phi_0$ for any $\psi$. The space of Virasoro highest weight states therefore forms a representation of the zero modes of the meromorphic fields. Conversely, given a representation $R$ of the zero modes we can consider the space of states that is generated from a state in $R$ by the action of the negative modes; this is called the *Verma module*. More precisely, the Verma module $\mathcal{V}$ is the vector space that is spanned by the states of the form

$$V_{n_1}(\psi_1) \cdots V_{n_N}(\psi_N)\phi \qquad \phi \in R, \tag{190}$$

where $n_i < 0$, modulo the relations

$$V_{l_1}(\psi_1) \cdots V_{l_r}(\psi_r)\Big(V_n(\psi)V_m(\chi) - V_m(\chi)V_n(\psi)\Big)V_{l_{r+1}}(\psi_{r+1}) \cdots V_{l_N}(\psi_N)\phi$$
$$= V_{l_1}(\psi_1) \cdots V_{l_r}(\psi_r)\Big[V_n(\psi), V_m(\chi)\Big]V_{l_{r+1}}(\psi_{r+1}) \cdots V_{l_N}(\psi_N)\phi, \tag{191}$$

where $\phi \in R$, and $[V_n(\psi), V_m(\chi)]$ stands for the right-hand-side of (90). If the meromorphic conformal field theory is generated by a finite set of fields, $W^1(z), \ldots W^s(z)$, one can show [104] that the Verma module is spanned by the so-called *Poincaré-Birkhoff-Witt basis* that consists of vectors of the form

$$W^{i_1}_{-m_1} \cdots W^{i_l}_{-m_l}\phi, \tag{192}$$

where $\phi \in R$, $m_j > 0$, $1 \leq i_{j+1} \leq i_j \leq s$ and $m_j \geq m_{j+1}$ if $i_j = i_{j+1}$. The actual Fock space of the representation $\mathcal{F}$ is again a certain quotient of the Verma module, where we set to zero all states that vanish identically in all amplitudes; these states are again called *null-vectors*.

### 4.2. An Illustrative Example

It should be stressed at this stage that the condition to be a representation of the meromorphic conformal field theory is usually *stronger* than that of being a representation of the Lie algebra (or W-algebra) of modes of the generating fields. For example, in the case of the affine theories introduced in section 3.7.2, the latter condition

means that the representation space has to be a representation of the affine algebra (130), and for any value of $k$, there exist infinitely many (non-integral) representations. On the other hand, if $k$ is a positive integer, the meromorphic theory possesses null-vectors, and only those representations of the affine algebra are representations of the meromorphic conformal field theory for which the null-fields act trivially on the representation space; this selects a finite number of representations. For example, if $g = su(2)$, the affine algebra can be written (in the Cartan-Weyl basis) as [66]

$$[H_m, H_n] = \frac{1}{2}km\delta_{m,-n}$$
$$[H_m, J_n^{\pm}] = \pm J_{m+n}^{\pm} \tag{193}$$
$$[J_m^+, J_n^-] = 2H_{m+n} + km\delta_{m,-n},$$

and if $k$ is a positive integer, the vector

$$\mathcal{N} = \left(J_{-1}^+\right)^{k+1}\Omega \tag{194}$$

is a *singular* vector, *i.e.* it is a descendant of the highest weight vector that is annihilated by all modes $V_n(\psi)$ with $n > 0$. This follows from the fact that the positive modes of the generating fields annihilate $\mathcal{N}$ which is obvious for $H_n\mathcal{N} = J_n^+\mathcal{N} = 0$, and for $J_n^-\mathcal{N}$ is a consequence of $L_n\mathcal{N} = 0$ for $n > 0$ together with

$$J_1^-\mathcal{N} = \sum_{l=0}^{k}(J_{-1}^+)^l[J_1^-, J_{-1}^+](J_{-1}^+)^{k-l}\Omega$$
$$= \left[k(k+1) - 2\sum_{l=0}^{k}(k-l)\right](J_{-1}^+)^k\Omega = 0. \tag{195}$$

Every singular vector is a null-vector as follows from (100). In the above example, $\mathcal{N}$ actually generates the null-space in the sense that every null-vector of the meromorphic conformal field theory can be obtained by the action of the modes from $\mathcal{N}$ [113] (see also [64]).

The zero modes of the affine algebra $\hat{su}(2)$ form the finite Lie algebra of $su(2)$. For every (finite-dimensional) representation $R$ of $su(2)$, *i.e.* for every spin $j \in \mathbb{Z}/2$, we can construct a Verma module for $\hat{su}(2)$ whose Virasoro highest weight space transforms as $R$. This (and any irreducible quotient space thereof) defines a representation of the affine algebra $\hat{su}(2)$. On the other hand, the zero mode of the null-vector $\mathcal{N}$ acts on any Virasoro highest weight state $\phi$ as

$$V_0(\mathcal{N})\phi = \left(J_0^+\right)^{k+1}\phi, \tag{196}$$

and in order for the Verma module to define a representation of the meromorphic conformal field theory, (196) must vanish. This implies that $j$ can only take the values $j = 0, 1/2, \ldots, k/2$. Since $\mathcal{N}$ generates all other null-fields [61], one may suspect that this is the only additional condition, and this is indeed correct.

Incidentally, in the case at hand the meromorphic theory is actually unitary, and the allowed representations are precisely those representations of the affine algebra that are unitary with respect to an inner product for which

$$\left(J_n^{\pm}\right)^{\dagger} = J_{-n}^{\mp} \qquad \left(H_m\right)^{\dagger} = H_{-m}. \tag{197}$$

Indeed, if $|j,j\rangle$ is a Virasoro highest weight state with $J_0^+|j,j\rangle = 0$ and $H_0|j,j\rangle = j|j,j\rangle$, then

$$\left(J_{-1}^+|j,j\rangle, J_{-1}^+|j,j\rangle\right) = \left(|j,j\rangle, J_1^- J_{-1}^+|j,j\rangle\right)$$

$$= (k - 2j)\left(|j,j\rangle, |j,j\rangle\right), \tag{198}$$

and if the representation is unitary, this requires that $(k - 2j) \geq 0$, and thus that $j = 0, 1/2, \ldots, k/2$. As it turns out, this is also sufficient to guarantee unitarity. In general, however, the constraints that select the representations of the meromorphic conformal field theory from those of the Lie algebra of modes cannot be understood in terms of unitarity.

### 4.3. Zhu's Algebra and the Classification of Representations

The above analysis suggests that to each representation of the zero modes of the meromorphic fields for which the zero modes of the null-fields vanish, a highest weight representation of the meromorphic conformal field theory can be associated, and that all highest weight representations of a meromorphic conformal field theory can be obtained in this way [114]. This idea has been made precise by Zhu [16] who constructed an algebra, now commonly referred to as *Zhu's algebra*, that describes the algebra of zero modes modulo zero modes of null-vectors, and whose representations are in one-to-one correspondence with those of the meromorphic conformal field theory. The following explanation of Zhu's work follows closely [48].

In a first step we determine the subspace of states whose zero modes always vanish on Virasoro highest weight states. This subspace certainly contains the states of the form $(L_{-1} + L_0)\psi$, where $\psi \in \mathcal{F}_0$ is arbitrary, since (108) implies that

$$V_0\left((L_{-1} + L_0)\psi\right) = V_0\left(L_{-1}\psi\right) + hV_0\left(\psi\right) = 0. \tag{199}$$

Furthermore, the subspace must also contain every state whose zero mode is of the form $V_0(\chi)V_0((L_{-1}+L_0)\psi)$ or $V_0((L_{-1}+L_0)\psi)V_0(\chi)$. In order to describe states of this form more explicitly, it is useful to observe that if both $\phi$ and $\bar{\phi}$ are Virasoro highest weight states

$$\langle\bar{\phi}|V(\psi,1)|\phi\rangle = \sum_l \langle\bar{\phi}|V_l(\psi)|\phi\rangle$$

$$= \langle\bar{\phi}|V_0(\psi)|\phi\rangle, \tag{200}$$

since the highest weight property implies that $V_l(\psi)|\phi\rangle = 0$ for $l > 0$ and similarly, using (100), $\langle\bar{\phi}|V_l(\psi) = 0$ for $l < 0$. Because of the translation symmetry of the amplitudes, this can then be rewritten as

$$\langle\bar{\phi}|\phi(-1)\ \psi\rangle = \langle\bar{\phi}|V_0(\psi)|\phi\rangle. \tag{201}$$

Let us introduce the operators

$$V^{(N)}(\psi) = \oint_0 \frac{dw}{w^{N+1}} V\left((w+1)^{L_0}\psi, w\right), \tag{202}$$

where $N$ is an arbitrary integer, and the contour is a small circle that encircles $w = 0$ but not $w = -1$. Then if both $\phi$ and $\bar{\phi}$ are arbitrary Virasoro highest weight states and $N > 0$, we have that

$$\langle \bar{\phi} | \phi(-1) \, V^{(N)}(\psi) \chi \rangle = 0 \,, \tag{203}$$

since the integrand in (203) does not have any poles at $w = -1$ or $w = \infty$. Because of (201) this implies that the zero mode of the corresponding state vanishes on an arbitrary highest weight state (since the amplitude with any other highest weight state vanishes). Let us denote by $O(\mathcal{F}_0)$ the subspace of $\mathcal{F}_0$ that is generated by states of the form $V^{(N)}(\psi)\chi$ with $N > 0$, and define the quotient space $\mathcal{A}(\mathcal{F}_0) = \mathcal{F}_0/O(\mathcal{F}_0)$. The above then implies that we can associate a zero mode (acting on a highest weight state) to each state in $\mathcal{A}(\mathcal{F}_0)$. We can write (202) in terms of modes as

$$V^{(N)}(\psi) = \sum_{n=0}^{h} \binom{h}{n} V_{-n-N}(\psi) \,, \tag{204}$$

where $\psi$ has conformal weight $h$, and it therefore follows that

$$V^{(1)}(\psi)\Omega = V_{-h-1}(\psi)\Omega + h V_{-h}(\psi)\Omega = (L_{-1} + L_0)\psi \,. \tag{205}$$

Thus $O(\mathcal{F}_0)$ contains the states in (199). Furthermore,

$$\begin{aligned}
V^{(N)}(L_{-1}\psi) &= \oint_0 \frac{dw}{w^{N+1}} (w+1)^{h_\psi+1} \frac{dV(\psi, w)}{dw} \\
&= -\oint_0 dw \frac{d}{dw} \left( \frac{(w+1)^{h_\psi+1}}{w^{N+1}} \right) V(\psi, w) \\
&= (N - h_\psi) V^{(N)}(\psi) + (N+1) V^{(N+1)}(\psi) \,,
\end{aligned} \tag{206}$$

and this implies, for $N \neq -1$,

$$V^{(N+1)}(\psi) = \frac{1}{N+1} V^{(N)}(L_{-1}\psi) - \frac{N - h_\psi}{N+1} V^{(N)}(\psi) \,. \tag{207}$$

Thus $O(\mathcal{F}_0)$ is actually generated by the states of the form $V^{(1)}(\psi)\chi$, where $\psi$ and $\chi$ are arbitrary states in $\mathcal{F}_0$.

As we shall show momentarily, the vector space $\mathcal{A}(\mathcal{F}_0)$ actually has the structure of an associative algebra, where the product is defined by

$$\psi *_L \chi \equiv V^{(0)}(\psi)\chi \,, \tag{208}$$

and $V^{(0)}(\psi)$ is given as in (202) or (204); this algebra is called Zhu's algebra. The analogue of (203) is then

$$\begin{aligned}
\langle \bar{\phi} | \phi(-1) \, V^{(0)}(\psi)\chi \rangle &= (-1)^{h_\psi} \langle V_0(\psi)\bar{\phi} | \phi(-1) \, \chi \rangle \\
&= (-1)^{h_\psi} \langle V_0(\psi)\bar{\phi} | V_0(\chi) | \phi \rangle \tag{209} \\
&= \langle \bar{\phi} | V_0(\psi) V_0(\chi) | \phi \rangle \,, \tag{210}
\end{aligned}$$

and thus the product in $\mathcal{A}(\mathcal{F}_0)$ corresponds indeed to the action of the zero modes.

In order to exhibit the structure of this algebra it is useful to introduce a second set of modes by

$$V_c^{(N)}(\psi) = (-1)^N \oint \frac{dw}{w} \frac{1}{(w+1)} \left(\frac{w+1}{w}\right)^N V\left((w+1)^{L_0}\psi, w\right). \quad (211)$$

These modes are characterised by the property that

$$\langle \bar{\phi} | \phi(-1) \, V_c^{(N)}(\psi) \, \chi \rangle = 0 \qquad \text{for } N > 0, \quad (212)$$

$$\langle \bar{\phi} | \phi(-1) \, V_c^{(0)}(\psi) \, \chi \rangle = \langle \bar{\phi} | \, (V_0(\psi)\phi) \, (-1) \, \chi \rangle. \quad (213)$$

It is obvious from (204) and (211) that $V^{(1)}(\psi) = V_c^{(1)}(\psi)$, and the analogue of (207) is

$$V_c^{(N+1)}(\psi) = -\frac{1}{N+1} V_c^{(N)}(L_{-1}\psi) - \frac{N+h_\psi}{N+1} V_c^{(N)}(\psi). \quad (214)$$

The space $O(\mathcal{F}_0)$ is therefore also generated by the states of the form $V_c^{(1)}(\psi)\chi$. Let us introduce, following (209) and (213), the notation

$$V_L(\psi) \equiv V^{(0)}(\psi) \qquad V_R(\psi) \equiv V_c^{(0)}(\psi) \qquad N(\psi) = V^{(1)}(\psi) = V_c^{(1)}(\psi).$$

We also denote by $N(\mathcal{F}_0)$ the vector space of operators that are spanned by $N(\psi)$ for $\psi \in \mathcal{F}_0$; then $O(\mathcal{F}_0) = N(\mathcal{F}_0)\mathcal{F}_0$. Finally it follows from (82) that $V_L(\psi)\Omega = V_R(\psi)\Omega = \psi$.

The equations (209) and (213) suggest that the modes $V_L(\psi)$ and $V_R(\chi)$ commute up to an operator in $N(\mathcal{F}_0)$. In order to prove this it is sufficient to consider the case where $\psi$ and $\chi$ are both eigenvectors of $L_0$ with eigenvalues $h_\psi$ and $h_\chi$, respectively. Then we have

$$
\begin{aligned}
[V_L(\psi), V_R(\chi)] &= \oint \oint_{|\zeta|>|w|} \frac{d\zeta}{\zeta}(\zeta+1)^{h_\psi} \frac{dw}{w}(w+1)^{h_\chi-1} V(\psi,\zeta)V(\chi,w) \\
&\quad - \oint \oint_{|w|>|\zeta|} \frac{dw}{w}(w+1)^{h_\chi-1} \frac{d\zeta}{\zeta}(\zeta+1)^{h_\psi} V(\chi,w)V(\psi,\zeta) \\
&= \oint_0 \left\{ \oint_w \frac{d\zeta}{\zeta}(\zeta+1)^{h_\psi} V(\psi,\zeta)V(\chi,w) \right\} \frac{dw}{w}(w+1)^{h_\chi-1} \\
&= \sum_n \oint_0 \left\{ \oint_w \frac{d\zeta}{\zeta}(\zeta+1)^{h_\psi} V(V_n(\psi)\chi,w)(\zeta-w)^{-n-h_\psi} \right\} \frac{dw}{w}(w+1)^{h_\chi-1} \\
&= \sum_{h_\chi \geq n \geq 0} \sum_{l=0}^{n+h_\psi-1} (-1)^l \binom{h_\psi}{l+1-n} \\
&\qquad \oint_0 \frac{dw}{w(w+1)} \left(\frac{w+1}{w}\right)^{l+1} (w+1)^{h_\chi-n} V(V_n(\psi)\chi,w) \\
&\in N(\mathcal{F}_0). \quad (215)
\end{aligned}
$$

Because of (214), every element in $N(\mathcal{F}_0)$ can be written as $V_R(\phi)$ for a suitable $\phi$, and (215) thus implies that $[V_L(\psi), N(\chi)] \in N(\mathcal{F}_0)$; hence $V_L(\psi)$ defines an endomorphism of $\mathcal{A}(\mathcal{F}_0)$.

For two endomorphisms, $\Phi_1, \Phi_2$, of $\mathcal{F}_0$, which leave $O(\mathcal{F}_0)$ invariant (so that they induce endomorphisms of $\mathcal{A}(\mathcal{F}_0)$), we shall write $\Phi_1 \approx \Phi_2$ if they agree as endomorphisms of $\mathcal{A}(\mathcal{F}_0)$, *i.e.* if $(\Phi_1 - \Phi_2)\mathcal{F}_0 \subset O(\mathcal{F}_0)$. Similarly we write $\phi \approx 0$ if $\phi \in O(\mathcal{F}_0)$.

In the same way in which the action of $V(\psi, z)$ is uniquely characterised by locality and (51), we can now prove the following

**Uniqueness Theorem for Zhu modes** [48]: Suppose $\Phi$ is an endomorphism of $\mathcal{F}_0$ that leaves $O(\mathcal{F}_0)$ invariant and satisfies

$$\Phi\Omega = \psi$$

$$[\Phi, V_R(\chi)] \in N(\mathcal{F}_0) \quad \text{for all } \chi \in \mathcal{F}_0.$$

Then $\Phi \approx V_L(\psi)$.
**Proof**: This follows from

$$\Phi\chi = \Phi\, V_R(\chi)\Omega \approx V_R(\chi)\,\Phi\,\Omega = V_R(\chi)\psi = V_R(\chi)\,V_L(\psi)\,\Omega \approx V_L(\psi)\chi\,,$$

where we have used that $V_L(\psi)\Omega = V_R(\psi)\Omega = \psi$.

It is then an immediate consequence that

$$V_L(V_L(\psi)\chi) \approx V_L(\psi)V_L(\chi)\,, \tag{216}$$

and a particular case of this (using again the fact that every element in $N(\mathcal{F}_0)$ can be written as $V_L(\phi)$ for some suitable $\phi$) is that

$$V_L(N(\psi)\chi) \approx N(\psi)V_L(\chi)\,. \tag{217}$$

In particular this implies that the product (208) $\phi *_L \psi$ is well-defined for both $\phi, \psi \in \mathcal{A}(\mathcal{F}_0)$. Furthermore, (216) shows that this product is associative, and thus $\mathcal{A}(\mathcal{F}_0)$ has the structure of an algebra.

We can also define a product by $\phi *_R \psi = V_R(\phi)\psi$. Since

$$\phi *_L \psi = V_L(\phi)\psi = V_L(\phi)V_R(\psi)\Omega \approx V_R(\psi)V_L(\phi)\Omega = V_R(\psi)\phi = \psi *_R \phi$$

this defines the reverse ring (or algebra) structure.

As we have explained before, this algebra plays the rôle of the algebra of zero modes. Since it has been constructed in terms of the space of states of the meromorphic theory, all null-relations have been taken into account, and one may therefore expect that its irreducible representations are in one-to-one correspondence with the irreducible representation of the meromorphic conformal field theory. This is indeed true [16], although the proof is rather non-trivial.

## *4.4. Finite (or Rational) Theories*

Since Zhu's algebra plays a central rôle in characterising the structure of a conformal field theory, one may expect that the theories for which it is finite-dimensional are particularly simple and tractable. In the physics literature these theories are sometimes called *rational* [115], although it may seem more appropriate to call them *finite*, and

we shall from now on do so. The name rational originates from the observation that the conformal weights of all states as well as the central charge are rational numbers in these theories [116, 117]. Unfortunately, there is no uniform definition of rationality, and indeed, the notion is used somewhat differently in mathematics and physics; a survey of the most common definitions is given in the appendix. In this paper we shall adopt the convention that a theory is called *finite* if Zhu's algebra is finite-dimensional, and it is called *rational* if it satisfies the conditions of Zhu's definition together with the $C_2$ criterion (see the appendix).

The determination of Zhu's algebra is usually rather difficult since the modes $N(\psi)$ that generate the space $O(\mathcal{F}_0)$ are not homogeneous with respect to $L_0$. It would therefore be interesting to find an equivalent condition for the finiteness of a conformal field theory that is easier to analyse in practice. One such condition that implies (and may be equivalent to) the finiteness of a meromorphic conformal field theory is the so-called $C_2$ condition of Zhu [16]: this is the condition that the quotient space

$$\mathcal{A}_{(1)}(\mathcal{F}_0) = \mathcal{H}/O_{(1)}(\mathcal{F}_0) \tag{218}$$

is finite-dimensional, where $O_{(1)}(\mathcal{F}_0)$ is spanned by the states of the form $V_{-l}(\psi)\chi$ where $l \geq h$, the conformal weight of $\psi$.†† It is not difficult to show that the dimension of Zhu's algebra is bounded by that of the above quotient space [16], *i.e.* $\dim(\mathcal{A}(\mathcal{F}_0)) \leq \dim(\mathcal{A}_{(1)}(\mathcal{F}_0))$. In many cases the two dimensions are actually the same, but this is not true in general; the simplest counter example is the theory associated to the affine algebra for $g = e_8$ at level $k = 1$. As we have mentioned before, this theory can equivalently be described as the meromorphic conformal field theory that is associated to the self-dual root lattice of $e_8$, and it is well known that its only representation is the meromorphic conformal field theory itself [118]; the highest weight space of the vacuum representation is one-dimensional, and Zhu's algebra is therefore also one-dimensional. On the other hand, it is clear that the dimension of $\mathcal{A}_{(1)}(\mathcal{F}_0)$ is at least 249 since the vacuum state and the 248 vectors of the form $J^a_{-1}\Omega$ (where $a$ runs over a basis of the 248-dimensional adjoint representation of $e_8$) are linearly independent in $\mathcal{A}_{(1)}(\mathcal{F}_0)$.

Many examples of finite conformal field theories are known. Of the examples we mentioned in section 3.7, the theories associated to even Euclidean lattices (3.7.4) are always finite (and unitary) [118, 119], the affine theories (3.7.2) are finite if the level $k$ is a positive integer [59–61] (in which case the theory is also unitary), and the Virasoro models (3.7.3) are finite if they belong to the so-called minimal series [4, 120]. This is the case provided the central charge $c$ is of the form

$$c_{p,q} = 1 - \frac{6(p-q)^2}{pq}, \tag{219}$$

where $p, q \geq 2$ are coprime integers. In this case there exist only finitely many irreducible representations of the meromorphic conformal field theory. Each such representation is

††Incidentally, $\mathcal{A}_{(1)}(\mathcal{F}_0)$ also has the structure of an abelian algebra; the significance of this algebra is however not clear at present.

the irreducible quotient space of a Verma module generated from a highest weight state with conformal weight $h$, and the allowed values for $h$ are

$$h_{(r,s)} = \frac{(rp - qs)^2 - (p - q)^2}{4pq}, \tag{220}$$

where $1 \leq r \leq q-1$ and $1 \leq s \leq p-1$, and $(r, s)$ defines the same value as $(q-r, p-s)$. Each of the corresponding Verma modules has two null-vectors at conformal weights $h+rs$ and $h+(p-r)(q-s)$, respectively, and the actual Fock space is the quotient space of the Verma module by the subspace generated by these two null-vectors [68, 69, 121].

There are therefore $(p - 1)(q - 1)/2$ inequivalent irreducible highest weight representations, and Zhu's algebra has dimension $(p - 1)(q - 1)/2$, and is of the form

$$\mathbb{C}[t]/\prod_{(r,s)}(t - h_{(r,s)}). \tag{221}$$

In this case the dimension of Zhu's algebra actually agrees with the dimension of the homogeneous quotient space (218). Indeed, we can choose a basis for (218) to consist of the states of the form $L_{-2}^l\Omega$, where $l = 0, 1, \ldots$. For $c = c_{p,q}$ the meromorphic Verma module has a null-vector at level $(p - 1)(q - 1)$ (since it corresponds to $r = s = 1$), and since the coefficient of $L_{-2}^{(p-1)(q-1)/2}\Omega$ in the null-vector does not vanish [120–122], this allows us to express $L_{-2}^{(p-1)(q-1)/2}\Omega$ in terms of states in $O_{(1)}(\mathcal{F}_0)$, and thus shows that the dimension of $\mathcal{A}_{(1)}(\mathcal{F}_0)$ is indeed $(p - 1)(q - 1)/2$.

The minimal models include the unitary discrete series (138) for which we choose $p = m$ and $q = m + 1$, but they also include non-unitary finite theories. The theory with $(p, q) = (2, 3)$ is trivial since $c = 0$, and the simplest (non-trivial) unitary theory is the so called *Ising model* for which $(p, q) = (3, 4)$ [4]: this is the theory with $c = \frac{1}{2}$, and its allowed representations have conformal weight

$$h = 0 \quad \text{(vacuum)} \qquad h = \frac{1}{2} \quad \text{(energy)} \qquad h = \frac{1}{16} \quad \text{(spin)}. \tag{222}$$

The simplest non-unitary finite theory is the *Yang-Lee edge* theory with $(p, q) = (2, 5)$ [123] for which $c = -\frac{22}{5}$. This theory has only two allowed representations, the vacuum representation with $h = 0$, and the representation with $h = -1/5$. It has also been observed that the theory with $(p, q) = (4, 5)$ can be identified with the *tricritical Ising model* [124].

## 5. Fusion Rules, Correlation Functions and Verlinde's Formula

Upto now we have analysed in detail the meromorphic subtheory and its representations, *i.e.* the zero- and two-point functions of the theory. In order to understand the structure of the theory further we need to analyse next the amplitudes that involve more than two non-trivial representations of the meromorphic subtheory. In a first step we shall consider the three-point functions that describe the allowed *couplings* between the different subspaces of $\mathcal{H}$. We shall then also consider higher correlation functions; their structure is in essence already determined in terms of the three-point functions.

## 5.1. Fusion Rules and the Comultiplication Formula

As we have explained before, the three-point amplitudes factorise into chiral and anti-chiral functions. We can therefore restrict ourselves to discussing the corresponding chiral amplitudes of the meromorphic theory, say. Their functional form is uniquely determined, and one of the essential pieces of information is therefore whether the corresponding amplitudes can be non-trivial or not; this is encoded in the so-called *fusion rules*.

The definition of the fusion rule is actually slightly more complicated since there can also be non-trivial multiplicities. In fact, the problem is rather analogous to that of decomposing a tensor product representation (of a compact group, say) into irreducibles. Because of the Möbius covariance of the amplitudes, it is sufficient to consider the amplitudes of the form

$$\langle \phi_k(\infty) \, V(\psi_1, z_1) \cdots V(\psi_l, z_l) \, \phi_i(u_1)\phi_j(u_2) \rangle \,, \tag{223}$$

where the three non-meromorphic fields are $\phi_i$, $\phi_j$ and $\phi_k$, and we could set $u_1 = 1$ and $u_2 = 0$, for example. The amplitude defines, in essence, an action of the meromorphic fields on the product $\phi_i(u_1)\phi_j(u_2)$, and the amplitude can only be non-trivial if this product representation contains the representation that is conjugate to $\phi_k$. Furthermore, if this representation is contained a finite number of times in the product representation $\phi_i(u_1)\phi_j(u_2)$, there is a finite ambiguity in defining the amplitude. We therefore define, more precisely, the fusion rule $N_{ij}^k$ to be the multiplicity with which the representation conjugate to $\phi_k$ appears in $\phi_i(u_1)\phi_j(u_2)$.

The action of the meromorphic fields (or rather their modes) on the product of the two fields can actually be described rather explicitly using the comultiplication formula [40, 125, 126]:† let us denote by $\mathcal{A}$ the algebra of modes of the meromorphic fields. A comultiplication is a homomorphism

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \,, \qquad V_n(\psi) \mapsto \sum \Delta^{(1)}(V_n(\psi)) \otimes \Delta^{(2)}(V_n(\psi)) \,, \tag{224}$$

and it defines an action on the product of two fields as

$$V_n(\psi)\Big(\phi_i(u_1)\phi_j(u_2)\Big) = \sum \left(\Delta^{(1)}(V_n(\psi))\phi_i\right)(u_1) \left(\Delta^{(2)}(V_n(\psi))\phi_j\right)(u_2) \,. \tag{225}$$

Here the action of the modes of the meromorphic fields on $\phi_i$ or $\phi_j$ is defined as in (187). The comultiplication depends on $u_1$ and $u_2$, and for the modes of a field $\psi$ of conformal weight $h$, it is explicitly given as

$$\begin{aligned}
\Delta_{u_1,u_2}(V_n(\psi)) \;=\; & \sum_{m=1-h}^{n} \binom{n+h-1}{m+h-1} u_1^{n-m} \left(V_m(\psi) \otimes \mathbb{1}\right) \\
& + \varepsilon_1 \sum_{l=1-h}^{n} \binom{n+h-1}{l+h-1} u_2^{n-l} \left(\mathbb{1} \otimes V_l(\psi)\right)
\end{aligned} \tag{226}$$

† An alternative (more mathematical) definition of this tensor product was developed by Huang & Lepowsky [127–130].

$$\Delta_{u_1,u_2}(V_{-n}(\psi)) = \sum_{m=1-h}^{\infty} \begin{pmatrix} n+m-1 \\ n-h \end{pmatrix} (-1)^{m+h-1} u_1^{-(n+m)} \left(V_m(\psi) \otimes \mathbb{1}\right)$$

$$+ \varepsilon_1 \sum_{l=n}^{\infty} \begin{pmatrix} l-h \\ n-h \end{pmatrix} (-u_2)^{l-n} \left(\mathbb{1} \otimes V_{-l}(\psi)\right), \quad (227)$$

where in (226) $n \geq 1-h$, in (227) $n \geq h$ and $\varepsilon_1$ is $\pm 1$ according to whether the left hand vector in the tensor product and the meromorphic field $\psi$ are both fermionic or not.‡ (In (226, 227) $m$ and $l$ are in $\mathbb{Z} - h$.) This formula holds in every amplitude, *i.e.*

$$\left\langle \prod_j V(\chi_j,\zeta_j) V_n(\psi) \Big(\phi_i(u_1)\phi_j(u_2)\Big) \right\rangle$$

$$= \sum \left\langle \prod_j V(\chi_j,\zeta_j) \left(\Delta^{(1)}(V_n(\psi))\phi_i\right)(u_1) \left(\Delta^{(2)}(V_n(\psi))\phi_j\right)(u_2) \right\rangle, \quad (228)$$

where each $\chi_j$ can be a meromorphic or a non-meromorphic field and we have used the notation of (225). In fact, the comultiplication formula can be derived from (228) using the fact that the amplitude $\langle \prod_j V(\chi_j,\zeta_j) V(\psi,z)\phi_i(u_1)\phi_j(u_2)\rangle$ from which the above expression can be obtained by integration has only poles (as a function of $z$) for $z = \zeta_j$ and $z = u_i$ [125, 126].

The above formula is not symmetric under the exchange of $\phi_i$ and $\phi_j$. In fact, it is manifest from the derivation that (228) must also hold if the comultiplication formulae (226) and (227) are replaced by

$$\widetilde{\Delta}_{u_1,u_2}(V_n(\psi)) = \sum_{m=1-h}^{n} \begin{pmatrix} n+h-1 \\ m+h-1 \end{pmatrix} u_1^{n-m} \left(V_m(\psi) \otimes \mathbb{1}\right)$$

$$+ \varepsilon_1 \sum_{l=1-h}^{n} \begin{pmatrix} n+h-1 \\ l+h-1 \end{pmatrix} u_2^{n-l} \left(\mathbb{1} \otimes V_l(\psi)\right), \quad (229)$$

for $n \geq 1-h$, and

$$\widetilde{\Delta}_{u_1,u_2}(V_{-n}(\psi)) = \sum_{m=n}^{\infty} \begin{pmatrix} m-h \\ n-h \end{pmatrix} (-u_1)^{m-n} \left(V_{-m}(\psi) \otimes \mathbb{1}\right)$$

$$+ \varepsilon_1 \sum_{l=1-h}^{\infty} \begin{pmatrix} n+l-1 \\ n-h \end{pmatrix} (-1)^{l+h-1} u_2^{-(n+l)} \left(\mathbb{1} \otimes V_l(\psi)\right), \quad (230)$$

for $n \geq h$. Since the two formulae agree in every amplitude, the product space is therefore the *ring-like* tensor product, *i.e.* the quotient of the direct product by the relations that guarantee that $\Delta = \widetilde{\Delta}$; this construction is based on an idea of Richard Borcherds, *unpublished* (see [125]).

A priori it is not clear whether the actual product space may not be even smaller. However, the fusion rules for a number of models have been calculated with this

‡ It is *a priori* ambiguous whether a given vector in a representation space is fermionic or not. However, once a convention has been chosen for one element, the fermion number of any element that can be obtained from it by the action of the modes of the meromorphic fields is well defined.

definition [126, 131], and the results coincide with those obtained by other methods. Indeed, fusion rules were first determined, for the case of the minimal models, by considering the implications for the amplitudes of the differential equations that follow from the condition that a null-vector of a representation must vanish in all amplitudes [4]: if the central charge $c$ is given in terms of $(p, q)$ as in (219), the highest weight representations are labelled by $(r, s)$, where $h$ is defined by (220) and $1 \leq r \leq q - 1$ and $1 \leq s \leq p - 1$; the fusion rules are then given as

$$(r_1, s_1) \otimes (r_2, s_2) \quad \bigoplus_{r=|r_1-r_2|+1}^{\min(r_1+r_2-1, 2q-1-r_1-r_2)} \quad \bigoplus_{s=|s_1-s_2|+1}^{\min(s_1+s_2-1, 2p-1-s_1-s_2)} \quad (r, s), \quad (231)$$

where $r$ and $s$ attain only every other value, *i.e.* $r$ $(s)$ is even if $r_1 + r_2 - 1$ $(s_1 + s_2 - 1)$ is even, and odd otherwise.

The analysis of [4] was adapted for the Wess-Zumino-Novikov-Witten models in [60]. For a general affine algebra $\hat{g}$, the fusion rules can be determined from the so-called *depth rule*; in the specific case of $g = su(2)$ at level $k$, this leads to

$$j_1 \otimes j_2 = \bigoplus_{j=|j_1-j_2|}^{\min(j_1+j_2, k-j_1-j_2)} j, \quad (232)$$

where $j$ is integer if $j_1 + j_2$ is integer, and half-integer otherwise, and the highest weight representations are labelled by $j = 0, 1/2, \ldots, k/2$. A closed expression for the fusion rules in the general case is provided by the *Kac-Walton formula* [64, 132–134].

Similarly, the fusion rules have been determined for the $W_3$ algebra in [135], the $N = 1$ superconformal minimal models in [136] and the $N = 2$ superconformal minimal models in [137, 138]. For finite theories the fusion rules can also be obtained by performing the analogue of Zhu's construction in each representation space; this was first done (in a slightly different language) by Feigin & Fuchs for the minimal models [139], and later by Frenkel & Zhu for general vertex operator algebras [61]. (As was pointed out by Li [140], the analysis of Frenkel & Zhu only holds under additional assumptions, for example in the rational case.)

One of the advantages of the approach that we have adopted here is the fact that structural properties of fusion can be derived in this framework [141]. For each representation $\mathcal{H}_j$, let us define the subspace $\mathcal{F}_j^-$ of the Fock space $\mathcal{F}_j$ to be the space that is spanned by the vectors of the form

$$V_{-n}(\psi)\Phi \qquad \text{where } \Phi \in \mathcal{F}_j \text{ and } n \geq h(\psi). \quad (233)$$

We call a representation *quasi-rational* provided that the quotient space

$$\mathcal{F}_j / \mathcal{F}_j^- \quad (234)$$

is finite-dimensional. This quotient space (or rather a realisation of it as a subspace of $\mathcal{F}_j$) is usually called the *special subspace*.

It was shown by Nahm [141] that the fusion product of a quasi-rational representation and a highest weight representation contains only finitely many highest weight

representations. He also showed that the special subspace of the fusion product of two quasi-rational representations is finite-dimensional. In fact, if we denote by $d_j^s$ the dimension of the special subspace, we have

$$\sum_k N_{ij}^k d_k^s \leq d_i^s d_j^s \,. \tag{235}$$

In particular, these results imply that the set of quasi-rational representations of a meromorphic conformal field theory is closed under the operation of taking fusion products. It is believed that every representation of a finite meromorphic conformal field theory is quasi-rational [141], but quasi-rationality is a weaker condition and there also exist quasi-rational representations of meromorphic theories that are not finite. The simplest example is the Virasoro theory for which $c$ is given by (219), but $p$ and $q$ are not (coprime) integers greater than one. This theory is not finite, but every highest weight representation with $h = h_{r,s}$ as in (220) and $r, s$ positive integers is quasi-rational. (This is a consequence of the fact that such a representation has a null-vector with conformal weight $h + rs$, whose coefficient of $L_{-1}^{rs}$ is non-zero.) In fact, the collection of all of these representations is closed under fusion, and forms a 'quasi-rational' chiral conformal field theory.

## 5.2. Indecomposable Fusion Products and Logarithmic Theories

In much of the above discussion we have implicitly assumed that the fusion product of any two irreducible representations of the chiral conformal field theory can be completely decomposed into irreducible representations. Whilst this is indeed correct for most theories of interest, there exist a few models where this is not the case. These theories are usually called *logarithmic* theories since, as we shall explain, some of their correlation functions contain logarithms. In this subsection we shall give a brief account of this class of theories; since the general theory has only been developed for theories for which this problem is absent, the present subsection is something of an interlude and not crucial for the rest of this article.

The simplest example of a logarithmic theory is the (quasi-rational) Virasoro model with $p = 2, q = 1$ whose conformal charge is $c = -2$ [142]. As we have explained before, the quasi-rational (irreducible) representations of this theory have a highest weight vector with conformal weight $h = h_{r,s}$ (220), where $r$ and $s$ are positive integers. Since the formula for $h_{r,s}$ has the symmetry (for $p = 2, q = 1$)

$$h_{r,s} = h_{1-r,2-s} = h_{r-1,s-2} \,, \tag{236}$$

we can restrict ourselves to the values $(r, s)$ with $s = 1, 2$.

The vacuum representation is $(r, s) = (1, 1)$ with conformal weight $h = 0$; the null-vector at level $rs = 1$ is $L_{-1}\Omega$. The simplest non-trivial representation is $(r, s) = (1, 2)$ with $h = -1/8$; it has a null-vector at level $rs = 2$. As we have alluded to before, a null-vector of a representation gives rise to a differential equation for the corresponding

amplitude [4]. In the present case, if we denote by $\mu$ the highest weight state with conformal weight $h = -1/8$, the 4-point function involving four times $\mu$ has the form

$$\langle \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4) \rangle = (z_1 - z_3)^{\frac{1}{4}}(z_2 - z_4)^{\frac{1}{4}}\left(x(1-x)\right)^{\frac{1}{4}} F(x) , \qquad (237)$$

where we have used the Möbius symmetry, and $x$ denotes the *cross-ratio*

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} . \qquad (238)$$

The null-vector for $\mu$ gives then rise to a differential equation for $F$ which, in the present case, is given by

$$x(1-x)F''(x) + (1 - 2x)F'(x) - \frac{1}{4}F(x) = 0 . \qquad (239)$$

We can make an ansatz for $F$ as

$$F(x) = x^s \left(a_0 + a_1 x + a_2 x^2 + \cdots\right) , \qquad (240)$$

where $a_0 \neq 0$. The differential equation (239) then determines the $a_i$ recursively provided we can solve the *indicial* equation, *i.e.* the equation that comes from the coefficient of $x^{s-1}$,

$$s(s-1) + s = s^2 = 0 . \qquad (241)$$

Generically, the indicial equation has two distinct roots (that do not differ by an integer), and for each solution of the indicial equation there is a solution of the original differential equation that is of the form (240). However, if the two roots coincide (as in our case), only one solution of the differential equation is of the form (240), and the general solution to (239) is

$$F(x) = AG(x) + B\left[G(x)\log(x) + H(x)\right] , \qquad (242)$$

where $G$ and $H$ are regular at $x = 0$ (since $s = 0$ solves (241)), and $A$ and $B$ are constants. In fact,

$$G(x) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - x\sin^2\varphi}} , \qquad (243)$$

and

$$G(x)\log(x) + H(x) = G(1 - x) . \qquad (244)$$

This implies that the 4-point function necessarily has a logarithmic branch cut: if $F$ is regular at $x = 0$, *i.e.* if we choose $B = 0$, then because of (244), $F$ has a logarithmic branch cut at $x = 1$.

In terms of the representation theory this logarithmic behaviour is related to the property of the fusion product of $\mu$ with itself not to be completely reducible: by considering a suitable limit of $z_1, z_2 \to \infty$ in the above 4-point function we can obtain a state $\Omega'$ satisfying

$$\left\langle \Omega'(\infty)\,\mu(z)\mu(0) \right\rangle = z^{\frac{1}{4}}\left(A + B\log(z)\right) , \qquad (245)$$

where $A$ and $B$ are constants (that depend now on $\Omega'$). We can therefore write

$$\mu(z)\mu(0) \sim z^{\frac{1}{4}}\Big(\omega(0) + \log(z)\Omega(0)\Big), \tag{246}$$

where $\langle \Omega'(\infty)\omega(0)\rangle = A$ and $\langle \Omega'(\infty)\Omega(0)\rangle = B$. Next we consider the transformation of this amplitude under a rotation by $2\pi$; this is implemented by the Möbius transformation $\exp(2\pi i L_0)$,

$$\Big\langle \Omega'(\infty)\, e^{2\pi i L_0}\Big(\mu(z)\mu(0)\Big)\Big\rangle = e^{-\frac{2\pi i}{4}}\Big\langle \Omega'(\infty)\,\mu(e^{2\pi i}z)\mu(0)\Big\rangle$$
$$= z^{\frac{1}{4}}\Big(A + B\log(z) + 2\pi i B\Big), \tag{247}$$

where we have used that the transformation property of vertex operators (39) also holds for non-meromorphic fields. On the other hand, because of (246) we can rewrite

$$\Big\langle \Omega'(\infty)\, e^{2\pi i L_0}\Big(\mu(z)\mu(0)\Big)\Big\rangle = z^{\frac{1}{4}}\Big\langle \Omega'(\infty)\, e^{2\pi i L_0}\Big(\omega(0) + \log(z)\Omega(0)\Big)\Big\rangle. \tag{248}$$

Comparing (247) with (248) we then find that

$$e^{2\pi i L_0}\Omega = \Omega \tag{249}$$
$$e^{2\pi i L_0}\omega = \omega + 2\pi i\Omega, \tag{250}$$

*i.e.* $L_0\Omega = 0$, $L_0\omega = \Omega$. Thus we find that the scaling operator $L_0$ is not diagonalisable, but that it acts as a *Jordan block*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{251}$$

on the space spanned by $\Omega$ and $\omega$. Since $L_0$ *is* diagonalisable in every irreducible representation, it follows that the fusion product is necessarily not completely decomposable. This conclusion holds actually more generally whenever any correlation function contains a logarithm.

One can analyse the fusion product of $\mu$ with itself using the comultiplication formula, and this allows one to determine the structure of the resulting representation $\mathcal{R}_{1,1}$ in detail [143]: the representation is generated from a highest weight state $\omega$ satisfying

$$L_0\omega = \Omega, \qquad L_0\Omega = 0, \qquad L_n\omega = 0 \quad \text{for } n > 0 \tag{252}$$

by the action of the Virasoro algebra. The state $L_{-1}\Omega$ is a null-state of $\mathcal{R}_{1,1}$, but $L_{-1}\omega$ is not null since $L_1 L_{-1}\omega = [L_1, L_{-1}]\omega = 2L_0\omega = 2\Omega$. Schematically the representation can therefore be described as
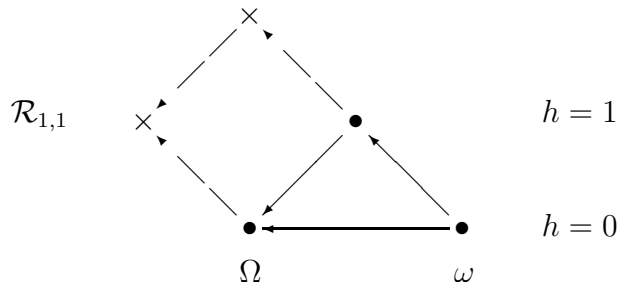
Here each vertex • denotes a state of the representation space, and the vertices × correspond to null-vectors. An arrow $A \longrightarrow B$ indicates that the vertex $B$ is in the image of $A$ under the action of the Virasoro algebra. The representation $\mathcal{R}_{1,1}$ is not irreducible since the states that are obtained by the action of the Virasoro algebra from $\Omega$ form the subrepresentation $\mathcal{H}_0$ of $\mathcal{R}_{1,1}$ (that is actually isomorphic to the vacuum representation). On the other hand, $\mathcal{R}_{1,1}$ is not completely reducible since we cannot find a complementary subspace to $\mathcal{H}_0$ that is a representation by itself; $\mathcal{R}_{1,1}$ is therefore called an *indecomposable* (but reducible) representation.

Actually, $\mathcal{R}_{1,1}$ is the simplest example of a whole class of indecomposable representations that appear in fusion products of the irreducible quasi-rational representations; these indecomposable representations are labelled by $(m, n)$ where now $n = 1$, and their structure is schematically described as
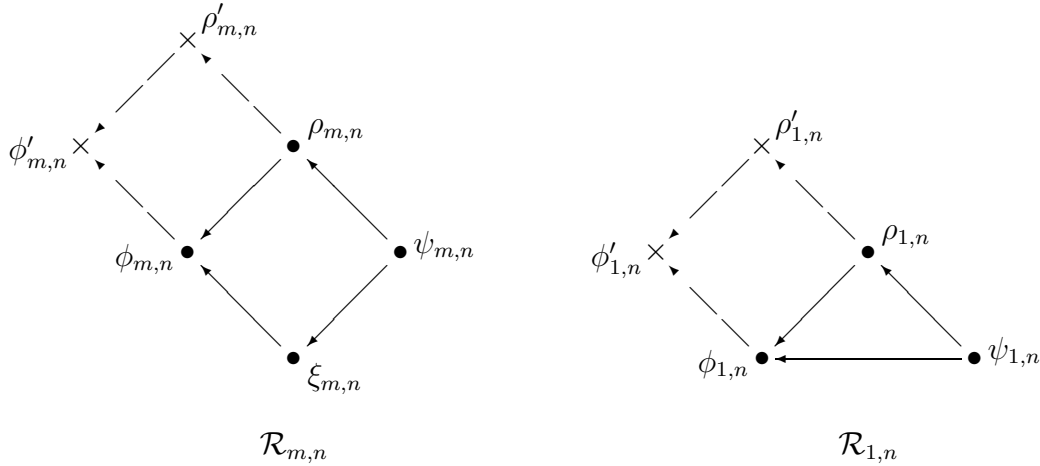


The representation $\mathcal{R}_{m,n}$ is generated from the vector $\psi_{m,n}$ by the action of the Virasoro algebra, where

$$L_0 \psi_{m,n} = h_{(m,n)} \psi_{m,n} + \phi_{m,n} , \tag{253}$$

$$L_0 \phi_{m,n} = h_{(m,n)} \phi_{m,n} , \tag{254}$$

$$L_k \psi_{m,n} = 0 \qquad \text{for } k \geq 2 . \tag{255}$$

If $m = 1$ we have in addition $L_1 \psi_{m,n} = 0$, whereas if $m \geq 2$, $L_1 \psi_{m,n} \neq 0$, and

$$L_1^{(m-1)(2-n)} \psi_{m,n} = \xi_{m,n} , \tag{256}$$

where $\xi_{m,n}$ is a Virasoro highest weight vector of conformal weight $h = h_{(m-1,2-n)}$. The Verma module generated by $\xi_{m,n}$ has a singular vector of conformal weight

$$h_{(m-1,2-n)} + (m-1)(2-n) = \frac{(2(m-1) - (2-n))^2 + 8(m-1)(2-n) - 1}{8} \tag{257}$$

$$= \frac{(2m-n)^2 - 1}{8} = h_{(m,n)} , \tag{258}$$

and this vector is proportional to $\phi_{m,n}$; this singular vector is however not a null-vector in $\mathcal{R}_{m,n}$ since it does not vanish in an amplitude with $\psi_{m,n}$. It was shown in [143] that the

set of representations that consists of all quasi-rational irreducible representations and the above indecomposable representations closes under fusion, *i.e.* any fusion product of two such representations can be decomposed as a direct sum of these representations.

This model is not an isolated example; the same structure is also present for the Virasoro models with $q = 1$ where $p$ is any positive integer [143–145]. Furthermore, the WZNW model on the supergroup $GL(1, 1)$ [146] and gravitationally dressed conformal field theories [147, 148] are also known to define logarithmic theories. It was conjectured by Dong & Mason [149] (using a different language) that logarithms can only occur if the theory is not finite, *i.e.* if the number of irreducible representations is infinite. However, this does not seem to be correct since the triplet algebra [150] has only finitely many irreducible representations, but contains indecomposable representations in their fusion products that lead to logarithmic correlation functions [151]. Logarithmic conformal field theories are not actually pathological; as was shown in [41] a consistent local conformal field theory that satisfies all conditions of a local theory (including modular invariance of the partition function) can be associated to this triplet algebra. The space of states of this local theory is then a certain quotient of the direct sum of tensor products of indecomposable representations of the two chiral algebras.

## 5.3. Verlinde's Formula

If all chiral representations $\mathcal{H}_j$ as well as their fusion products are completely reducible into irreducibles, we can define for each irreducible representation $\mathcal{H}_j$, its *conjugate* representation $\mathcal{H}_{j^\vee}$. The conjugate representation has the property that at least one two-point function involving a state from $\mathcal{H}_j$ and a state from $\mathcal{H}_{j^\vee}$ is non-trivial. Because of Schur's lemma the conjugation map is uniquely defined, and by construction it is clearly an involution, *i.e.* $(j^\vee)^\vee = j$.

If conjugation is defined, the condition that the fusion product of $\mathcal{H}_i$ and $\mathcal{H}_j$ contains the representation $\mathcal{H}_k$ is equivalent to the condition that the three-point function involving a state from $\mathcal{H}_i$, one from $\mathcal{H}_j$ and one from $\mathcal{H}_{k^\vee}$ is non-trivial; thus we can identify $N_{ij}^k$ with the number of different three-point functions of suitable $\phi_i \in \mathcal{H}_i$, $\phi_j \in \mathcal{H}_j$ and $\phi_{k^\vee} \in \mathcal{H}_{k^\vee}$. It is then natural to define

$$N_{ijk} \equiv N_{ij}^{k^\vee} , \tag{259}$$

which is manifestly symmetric under the exchange of $i$, $j$ and $k$.

The fusion product is also associative,

$$\sum_k N_{ij}^k N_{kl}^m = \sum_k N_{ik}^m N_{jl}^k . \tag{260}$$

If we define $\mathbf{N}_i$ to be the matrix with matrix elements

$$(\mathbf{N}_i)_j^k \equiv N_{ij}^k , \tag{261}$$

then (260) can be rewritten as

$$\sum_k (\mathbf{N}_i)_j^k (\mathbf{N}_l)_k^m = \sum_k (\mathbf{N}_l)_j^k (\mathbf{N}_i)_k^m , \tag{262}$$

where we have used (259). Thus the matrices $\mathbf{N}_i$ commute with each other.

Because of (259) the matrices $\mathbf{N}_i$ are normal, *i.e.* they commute with their adjoint (or transpose) since $\mathbf{N}_i^\dagger = \mathbf{N}_{i^\vee}$. This implies that each $\mathbf{N}_i$ can be diagonalised, and since the different $\mathbf{N}_i$ commute, there exists a common matrix $S$ that diagonalises all $\mathbf{N}_i$ simultaneously. If we denote the different eigenvalues of $\mathbf{N}_i$ by $\lambda_i^{(l)}$, we therefore have that

$$N_{ij}^k = \sum_{lm} S_j^l \lambda_i^{(l)} \delta_l^m \left(S^{-1}\right)_m^k = \sum_l S_j^l \lambda_i^{(l)} \left(S^{-1}\right)_l^k . \tag{263}$$

If $j$ is the vacuum representation, *i.e.* $j = 0$, then $N_{i0}^k = \delta_i^k$ if all representations labelled by $i$ are irreducible. In this case, multiplying all three expressions of (263) by $S_k^n$ from the right (and summing over $k$), we find

$$S_i^n = \sum_{lm} S_0^l \lambda_i^{(l)} \delta_l^n = S_0^n \lambda_i^{(n)} . \tag{264}$$

Hence $\lambda_i^{(n)} = S_i^n / S_0^n$, and we can rewrite (263) as

$$N_{ij}^k = \sum_l \frac{S_j^l S_i^l \left(S^{-1}\right)_l^k}{S_0^l} . \tag{265}$$

What we have done so far has been a rather trivial manipulation. However, the deep conjecture is now that the matrix $S$ that diagonalises the fusion rules coincides precisely with the matrix $S$ (15) that describes the modular transformation properties of the characters associated to the irreducible representations [152]. Thus (265) provides an expression for the fusion rules in terms of the modular properties of the corresponding characters; with this interpretation, (265) is called the *Verlinde formula*. This is a remarkable formula, not least because it is not obvious in general why the right-hand-side of (265) should define a non-negative integer. In fact, using techniques from Galois theory, one can show that this property implies severe constraints on the matrix elements of $S$ [153, 154].

The Verlinde formula has been tested for many conformal field theories, and whenever it makes sense (*i.e.* whenever the theory is rational), it is indeed correct. It has also been proven for the case of the WZNW models of the classical groups at integer level [155], and it follows from the polynomial equations of Moore & Seiberg [115] to be discussed below.

## 5.4. Higher Correlation Functions and the Polynomial Relations of Moore & Seiberg

The higher correlation functions of the local theory do not directly factorise into products of chiral and anti-chiral functions, but they can always be written as sums of such products. It is therefore useful to analyse the functional form of these chiral functions. The actual (local) amplitudes (that are linear combinations of products of the chiral and anti-chiral amplitudes) can then be determined from these by the conditions that (i) they have to be local, and (ii) the operator product expansion (7) is indeed associative. These constraints define the so-called *bootstrap equations* [4]; in practice they are rather

difficult to solve, and explicit solutions are only known for a relatively small number of examples [41, 156–164].

The chiral $n$-point functions are largely determined in terms of the three-point functions of the theory. In particular, the number of different solutions for a given set of non-meromorphic fields can be deduced from the fusion rules of the theory.§ Let us consider, as an example, the case of a 4-point function, where the four non-meromorphic fields $\phi_i \in \mathcal{H}_{m_i}$, $i = 1, \ldots, 4$ are inserted at $u_1, \ldots, u_4$. (It will become apparent from the following discussion how this generalises to arbitrary higher correlation functions.) In the limit in which $u_2 \to u_1$ (with $u_3$ and $u_4$ far away), the 4-point function can be thought of as a three-point function whose non-meromorphic field at $u_1 \approx u_2$ is the fusion product of $\phi_1$ and $\phi_2$; we can therefore write every 4-point function involving $\phi_1, \ldots, \phi_4$ as

$$\langle \phi_1(u_1)\phi_2(u_2)\phi_3(u_3)\phi_4(u_4)\rangle = \sum_k \sum_{i=1}^{N_{m_1 m_2}^k} \alpha_{k,i} \left\langle \Phi_{12}^{k,i}(u_1, u_2)\, \phi_3(u_3)\phi_4(u_4) \right\rangle, \tag{266}$$

where $\Phi_{12}^{k,i}(u_1, u_2) \in \mathcal{H}_k$, $\alpha_{k,i}$ are arbitrary constants, and the sum extends over those $k$ for which $N_{m_1 m_2}^k \geq 1$. The number of different three-point functions involving $\Phi_{12}^k(u_1, u_2) \in \mathcal{H}_k$, $\phi_3$ and $\phi_4$, is given by $N_{m_3 m_4}^{k^\vee}$, and the number of different solutions is therefore altogether

$$\sum_k N_{m_1 m_2}^k N_{m_3 m_4}^{k^\vee}. \tag{267}$$

The space of chiral 4-point functions is a vector space (since any linear combination of 4-point functions is again a 4-point function), and in the above we have selected a specific *basis* for this space; in fact the different basis vectors (*i.e.* the solutions in terms of which we have expanded (266)) are characterised by the condition that they can be approximated by a product of three-point functions as $u_2 \to u_1$. In the notation of Moore & Seiberg [40], these solutions are described by

$$\langle \phi_1 | \begin{pmatrix} m_2 \\ m_1 \ k \end{pmatrix}_{u_2;a} (\phi_2) \begin{pmatrix} m_3 \\ k \ m_4 \end{pmatrix}_{u_3;b} (\phi_3) | \phi_4 \rangle, \tag{268}$$

where we have used the Möbius invariance to set, without loss of generality, $u_1 = \infty$ and $u_4 = 0$. Here,

$$\begin{pmatrix} i \\ j \ k \end{pmatrix}_{u;a} (\phi) : \mathcal{H}_k \to \mathcal{H}_j \tag{269}$$

describes the so-called *chiral vertex operator* that is associated to $\phi \in \mathcal{H}_i$; it is the restriction of $\phi(u)$ to $\mathcal{H}_k$, where the image is projected onto $\mathcal{H}_j$ and $a$ labels the different such projections (if $N_{ik}^j \geq 2$). This definition has to be treated with some care since $\phi(u)$

§ Since the functional form of the amplitudes is no longer determined by Möbius symmetry, it is possible (and indeed usually the case) that there are more than one amplitude for a given set of irreducible highest weight representations; see for example the 4-point function that we considered in section 5.2.

is strictly speaking *not* a well-defined operator on the direct sum of the chiral spaces, $\oplus_i \mathcal{H}_i$ — indeed, if it were, there would only be one 4-point function!‖

In the above we have expanded the 4-point functions in terms of a *basis* of functions each of which approximates a product of three-point functions as $u_2 \to u_1$. We could equally consider the basis of functions to consist of those functions that approximate products of three-point functions as $u_2 \to u_3$; in the notation of Moore & Seiberg [40] these are described by

$$\langle \phi_1 | \begin{pmatrix} k \\ m_1\; m_4 \end{pmatrix}_{u_3;c} (\chi) \; | \phi_4 \rangle \cdot \langle \chi | \begin{pmatrix} m_2 \\ k\; m_3 \end{pmatrix}_{u_2 - u_3;d} (\phi_2) | \phi_3 \rangle \,. \tag{270}$$

Since both sets of functions form a basis for the same vector space, their number must be equal; there are (267) elements in the first set of basis vectors, and the number of basis elements of the form (270) is

$$\sum_k N^k_{m_1 m_4} N^{k^\vee}_{m_2 m_3} \,. \tag{271}$$

The two expressions are indeed equal, as follows from (260) upon setting $m_1 = j$, $m_2 = i$, $m_3 = \bar{m}$, $m_4 = l$, and using (259). We can furthermore express the two sets of basis vectors in terms of each other; this is achieved by the so-called *fusing matrix* of Moore & Seiberg [40],

$$\begin{pmatrix} m_2 \\ m_1\; p \end{pmatrix}_{u_2;a} \begin{pmatrix} m_3 \\ p\; m_4 \end{pmatrix}_{u_3;b} = \sum_{q;c,d} F_{pq} \begin{bmatrix} m_2 & m_3 \\ m_1 & m_4 \end{bmatrix}^{cd}_{ab} \begin{pmatrix} q \\ m_1\; m_4 \end{pmatrix}_{u_3;c} \begin{pmatrix} m_2 \\ q\; m_3 \end{pmatrix}_{u_2 - u_3;d} \,. \tag{272}$$

We can also consider the basis of functions that are approximated by products of three-point functions as $u_3 \to u_1$ (rather than $u_2 \to u_1$). These basis functions are described, in the notation of Moore & Seiberg [40], by

$$\langle \phi_1 | \begin{pmatrix} m_3 \\ m_1\; k \end{pmatrix}_{u_3;c} (\phi_3) \begin{pmatrix} m_2 \\ k\; m_4 \end{pmatrix}_{u_2;d} (\phi_2) | \phi_4 \rangle \,. \tag{273}$$

By similar arguments to the above, it is easy to see that the number of such basis vectors is the same as (267) or (271). Furthermore, we can express the basis vectors in (268) in terms of the new basis vectors (273) as

$$\begin{pmatrix} m_2 \\ m_1\; p \end{pmatrix}_{u_2;a} \begin{pmatrix} m_3 \\ p\; m_4 \end{pmatrix}_{u_3;b} = \sum_{q;c,d} B(\pm)_{pq} \begin{bmatrix} m_2 & m_3 \\ m_1 & m_4 \end{bmatrix}^{cd}_{ab} \begin{pmatrix} m_3 \\ m_1\; q \end{pmatrix}_{u_3;c} \begin{pmatrix} m_2 \\ q\; m_4 \end{pmatrix}_{u_2;d} \,, \tag{274}$$

where $B$ is the so-called *braiding matrix*. Since the correlation functions are not single-valued, the braiding matrix depends on the equivalence class of paths along which the configuration $u_2 \approx u_1$ (with $u_4$ far away) is analytically continued to the configuration $u_3 \approx u_1$. In fact, there are two such equivalence classes which differ by a path along which $u_3$ encircles $u_2$ once; we distinguish the corresponding braiding matrices by $B(\pm)$.

‖ It is possible to give an operator description for the chiral theory (at least for the case of the WZNW-models) by considering, instead of $\mathcal{H}_i$, the tensor product of $\mathcal{H}_i$ with a finite-dimensional vector space that is a certain truncation of the corresponding anti-chiral representation $\overline{\mathcal{H}}_{\bar{i}}$ [165, 166]. This also provides a natural interpretation for the quantum group symmetry to be discussed below.

The two matrices (272) and (274) have been derived in the context of 4-point functions, but the notation we have used suggests that the corresponding identities should hold more generally, namely for products of chiral vertex operators in any correlation function. As we can always consider the limit in which the remaining coordinates coalesce (so that the amplitude approximates a 4-point function), this must be true in every consistent conformal field theory. On the other hand, the identities (272) and (274) *can* only be true in general provided that the matrices $F$ and $B$ satisfy a number of consistency conditions; these are usually called the *polynomial* equations [115].

The simplest relation is that which allows to describe the braiding matrix $B$ in terms of the fusing matrix $F$, and the diagonalisable matrix $\Omega$. The latter is defined by

$$\begin{pmatrix} k \\ l\ m \end{pmatrix}_{z;a} = \left(\Omega(\pm)^k_{lm}\right)^b_a \begin{pmatrix} k \\ m\ l \end{pmatrix}_{z;b}, \tag{275}$$

where again the sign $\pm$ distinguishes between clockwise (or anti-clockwise) analytic continuation of the field in $\mathcal{H}_m$ around that in $\mathcal{H}_l$. Since all three representations are irreducible, $\Omega(\pm)$ is just a phase,

$$\left(\Omega(\pm)^k_{lm}\right)^b_a = s_a e^{\pm i\pi(h_k - h_l - h_m)}\delta^b_a, \tag{276}$$

where $s_a = \pm 1$, and $h_i$ is the conformal weight of the highest weight state in $\mathcal{H}_i$. In order to describe now $B$ in terms of $F$, we apply the fusing matrix to obtain the right-hand-side of (272); braiding now corresponds to $\Omega$ (applied to the second and third representation), and in order to recover the right-hand-side of (274), we have to apply the inverse of $F$ again. Thus we find [40]

$$B(\epsilon) = F^{-1}\left(\mathbb{1} \otimes \Omega(-\epsilon)\right)F. \tag{277}$$

The consistency conditions that have to be satisfied by $B$ and $F$ can therefore be formulated in terms of $F$ and $\Omega$. In essence, there are two non-trivial identities, the *pentagon* identity, and the *hexagon* identity. The former can be obtained by considering sequences of fusing identities in a 5-point function, and is explictly given as [40]

$$F_{23}F_{12}F_{23} = P_{23}F_{13}F_{12}, \tag{278}$$

where $F_{12}$ acts on the first two representation spaces, *etc.*, and $P_{23}$ is the permutation matrix that exchanges the second and third representation space. The hexagon identity can be derived by considering a sequence of transformations involving $F$ and $\Omega$ in a 4-point function [40]

$$F(\Omega(\epsilon) \otimes \mathbb{1})F = (\mathbb{1} \otimes \Omega(\epsilon))\, F\, (\mathbb{1} \otimes \Omega(\epsilon)). \tag{279}$$

It was shown by Moore & Seiberg [40] using category theory that all relations that arise from comparing different expansions of an arbitrary $n$-point function on the sphere are a consequence of the pentagon and hexagon identity. This is a deep result which allows us, at least in principle (and ignoring problems of convergence, *etc.*), to construct all $n$-point functions of the theory from the three-point functions. Indeed, the three-point functions determine in essence the chiral vertex operators, and by composing these operators as above, we can construct a basis for an arbitrary $n$-point function. If $F$ and

$\Omega$ satisfy the pentagon and hexagon identities, the resulting space will be independent of the particular expansion we used.

Actually, Moore & Seiberg also solved the problem for the case of arbitrary $n$-point functions on an arbitrary surface of genus $g$. (The proof in [40] is not quite complete; see however [167].) In this case there are three additional consistency conditions that originate from considering correlation functions on the torus and involve the modular transformation matrix $S$ (see [40] for more details). As was also shown in [40] this extended set of relations implies Verlinde's formula.

## 5.5. Quantum Groups

It was observed in [40] that every (compact) group $G$ gives rise to matrices $F$ and $B$ (or $\Omega$) that satisfy the polynomial equations: let us denote by $\{R_i\}$ the set of irreducible representations of $G$. Every tensor product of two irreducible representations can be decomposed into irreducibles,

$$R_i \otimes R_j = \oplus_k V_{ij}^k \otimes R_k \,, \tag{280}$$

where the vector space $V_{ij}^k$ can be identified with the space of intertwining operators,

$$\begin{pmatrix} k \\ i\ j \end{pmatrix} : R_i \otimes R_j \rightarrow R_k \,, \tag{281}$$

and $\dim(V_{ij}^k)$ is the multiplicity with which $R_k$ appears in the tensor product of $R_i$ and $R_j$. There exist natural isomorphisms between representations,

$$\hat{\Omega} : R_i \otimes R_j \cong R_j \otimes R_i \tag{282}$$

$$\hat{F} : (R_i \otimes R_j) \otimes R_k \cong R_i \otimes (R_j \otimes R_k) \,, \tag{283}$$

and they induce isomorphisms on the space of intertwining operators,

$$\Omega : V_{ij}^k \cong V_{ji}^k \tag{284}$$

$$F : \ \oplus_r V_{ij}^r \otimes V_{rk}^l \cong \oplus_s V_{is}^l \otimes V_{jk}^s \,. \tag{285}$$

The pentagon commutative diagram

$$\begin{array}{ccccc} R_1 \otimes (R_2 \otimes (R_3 \otimes R_4)) & \overset{\hat{F}}{\rightarrow} & (R_1 \otimes R_2) \otimes (R_3 \otimes R_4) & \overset{\hat{F}}{\rightarrow} & ((R_1 \otimes R_2) \otimes R_3) \otimes R_4 \\ \downarrow {\scriptstyle (1 \otimes \hat{F})} & & & & \downarrow {\scriptstyle (\hat{F} \otimes 1)} \\ R_1 \otimes ((R_2 \otimes R_3) \otimes R_4) & & \overset{\hat{F}}{\longrightarrow} & & (R_1 \otimes (R_2 \otimes R_3)) \otimes R_4 \end{array}$$

then implies the pentagon identity for $F$ (278), while the hexagon identity follows from

$$\begin{array}{ccccc} R_1 \otimes (R_2 \otimes R_3) & \overset{\hat{F}}{\rightarrow} & (R_1 \otimes R_2) \otimes R_3 & \overset{\hat{\Omega}}{\rightarrow} & R_3 \otimes (R_1 \otimes R_2) \\ \downarrow {\scriptstyle (1 \otimes \hat{\Omega})} & & & & \downarrow {\scriptstyle \hat{F}} \\ R_1 \otimes (R_3 \otimes R_2) & \overset{\hat{F}}{\rightarrow} & (R_1 \otimes R_3) \otimes R_2 & \overset{\hat{\Omega} \otimes 1}{\rightarrow} & (R_3 \otimes R_1) \otimes R_2 \,. \end{array}$$

Thus the representation ring of a compact group gives rise to a solution of the polynomial relations; the fusion rules are then identified as $N_{ij}^k = \dim(V_{ij}^k)$.

The $F$ and $B$ matrices that are associated to a chiral conformal field theory are, however, usually not of this form. Indeed, for compact groups we have $\Omega^2 = 1$ since the tensor product is symmetric, but because of (276) this would require that the conformal weights of all highest weight states are half-integer which is not the case for most conformal field theories of interest. For a general conformal field theory the relation $\Omega^2 = 1$ is replaced by $\Omega(+)\Omega(-) = 1$; this is a manifestation of the fact that the fields of a (two-dimensional) conformal field theory obey braid group statistics rather than permutation group statistics [168, 169].

On the other hand, many conformal field theories possess a 'classical limit' [40] in which the conformal dimensions tend to zero, and in this limit the $F$ and $B$ matrices come from compact groups. This suggests [40,170] that the actual $F$ and $B$ matrices of a chiral conformal field theory can be thought of as being associated to the representation theory of a *quantum group* [171–174], a certain deformation of a group (for reviews on quantum groups see [175, 176]). Indeed, the chiral conformal field theory of the WZNW model associated to the affine algebra $\hat{su}(2)$ at level $k$ has the same $F$ and $B$ matrices [177, 178] as the quantum group $U_q(sl(2))$ [179] at $q = e^{i\pi/(k+2)}$ [180]. Similar relations have also been found for the WZNW models associated to the other groups [181], and the minimal models [160, 170].

These observations suggest that chiral conformal field theories may have a hidden quantum group symmetry. Various attempts have been made to realise the relevant quantum group generators in terms of the chiral conformal field theory [182–185] but no clear picture has emerged so far. A different proposal has been put forward in [166] following [165] (see also [186, 187]). According to this idea the quantum group symmetry acts naturally on a certain (finite-dimensional) truncation of the anti-chiral representation space, namely the special subspace; these anti-chiral degrees of freedom arise naturally in an operator formulation of the chiral theory.

The actual quantum symmetries that arise for rational theories are typically quantum groups at roots of unity; tensor products of certain representations of such quantum groups are then not completely reducible [188], and in order to obtain a structure as in (280), it is necessary to truncate the tensor products in a suitable way. The resulting symmetry structure is then more correctly described as a quasi-Hopf algebra [189,190]. In fact, the underlying structure of a chiral conformal field theory *must* be a quasi-Hopf algebra (rather than a normal quantum group) whenever the *quantum dimensions* are not integers: to each representation $\mathcal{H}_i$ we can associate (because of the Perron-Frobenius theorem) a unique positive real number $d_i$, the quantum dimension, so that

$$d_i d_j = \sum_k N_{ij}^k d_k \,, \tag{286}$$

where $N_{ij}^k$ are the fusion rules; if the symmetry structure of the conformal field theory is described by a quantum group, the choice $d_i = \dim(R_i)$ satisfies (286), and thus each

quantum dimension must be a positive integer. It follows from (265) that

$$d_i = \frac{S_{i0}}{S_{00}} \tag{287}$$

satisfies (286); for unitary theories this expression is a positive number, and thus coincides with the quantum dimension. For most theories of interest this number is not an integer for all $i$.

It has been shown in [191] (see also [192]) that for every rational chiral conformal field theory, a weak quasi-triangular quasi-Hopf algebra exists that reproduces the fusing and the braiding matrices of the conformal field theory. This quasi-Hopf algebra is however not unique; for every choice of positive integers $D_i$ satisfying

$$D_i D_j \geq \sum_k N_{ij}^k D_k \tag{288}$$

such a quasi-Hopf algebra can be constructed. As we have seen above, the dimensions of the special subspaces $d_i^s$ satisfy this inequality (235) provided that they are finite; this gives rise to one preferred such quasi-Hopf algebra in this case [166, 187].

The quantum groups at roots of unity also play a central rôle in the various knot invariants that have been constructed starting with the work of Jones [193–195]. These have also a direct relation to the braiding matrices of conformal field theories [168, 169] and can be interpreted in terms of $2 + 1$ dimensional Chern-Simons theory [196, 197].

## 6. Conclusions and Outlook

Let us conclude this review with a summary of general problems in conformal field theory that deserve, in our opinion, further work.

*1. The local theory*: It is generally believed that to every modular invariant partition function of tensor products of representations of a chiral algebra, a consistent local theory can be defined. Unfortunately, only very few local theories have been constructed in detail, and there are virtually no general results (see however [198]).

Recently it has been realised that the operator product expansion coefficients in a boundary conformal field theory can be expressed in terms of certain elements of the fusing matrix $F$ [32–34]. Since there exists a close relation between the operator product expansion coefficients of the boundary theory and those of the bulk theory, this may open the way for a general construction of local conformal field theories.

*2. Algebraic formulae for fusing and braiding matrices*: Essentially all of the structure of conformal field theory can be described in terms of the representation theory of certain algebraic structures. However, in order to obtain the fusing and braiding matrices that we discussed above, it is necessary to analyse the *analytical* properties of correlation functions, in particular their monodromy matrices. If it is indeed true that the whole structure is determined by the algebraic data of the theory, a direct (representation theoretic) expression should exist for these matrices as well.

*3. Finite versus rational*: As we have explained in this article (and as is indeed illustrated by the appendix), there are different conditions that guarantee that different aspects of the theory are well-behaved in some sense. Unfortunately, it is not clear at the moment what the precise logical relation between the different conditions are, and which of them is crucial in distinguishing between theories that are tractable, and those that are less so. In this context it would also be very interesting to understand under which conditions the correlation functions (of representations of the meromorphic conformal field theory) do not contain logarithmic branch cuts.

*4. Existence of higher correlation functions*: It is generally believed that the higher correlation functions of representations of a finite conformal field theory define analytic functions that have appropriate singularities and branch cuts. This is actually a crucial assumption in the definition of the fusing and braiding matrices, and therefore in the derivation of the polynomial relations of Moore & Seiberg (from which Verlinde's formula can be derived). It would be interesting to prove this in general.

*5. Higher genus*: Despite recent advances in our understanding of the theory on higher genus Riemann surfaces [27–29], a completely satisfactory treatment for the case of a general conformal field theory is not available at present.

## Appendix A. Definitions of Rationality

### Appendix A.1. Zhu's Definition

According to [16], a meromorphic conformal field theory is *rational* if it has only finitely many irreducible highest weight representations. The Fock space of each of these representations has finite-dimensional weight spaces, *i.e.* for each eigenvalue of $L_0$, the corresponding eigenspace is finite-dimensional. Furthermore, each finitely generated representation is a direct sum of these irreducible representations.

If a meromorphic conformal field theory is rational in this sense, Zhu's algebra is a semi-simple complex algebra, and therefore finite-dimensional [16, 199]. Zhu has also conjectured that every such theory satisfies the $C_2$ criterion, *i.e.* the condition that the quotient space (218) is finite-dimensional.

If a meromorphic conformal field theory is rational in this sense and satisfies the $C_2$ condition then the characters of its representations define a representation of the modular group $SL(2, \mathbb{Z})$ [16].

### Appendix A.2. The DLM Definitions

Dong, Li & Mason call a representation *admissible* if it satisfies the representation criterion (*i.e.* if the corresponding amplitudes satisfy the condition (186)), and if it possesses a decomposition of the form $\oplus_{n=0}^{\infty} M_{n+\lambda}$, where $\lambda$ is fixed and $V_n(\psi)M_\mu \subset M_{\mu-n}$ for $\mu = \lambda + m$ for some $m$. A meromorphic conformal field theory is then called *rational* if every admissible representation can be decomposed into irreducible admissible representations.

If a meromorphic conformal field theory is rational in this sense, then Zhu's algebra is a semi-simple complex algebra (and hence finite-dimensional), every irreducible admissible representation is an irreducible representation for which each $M_\mu$ is finite-dimensional and an eigenspace of $L_0$, and the number of irreducible representations is finite [199].

This definition of rationality therefore implies Zhu's notion of rationality, but it is not clear whether the converse is true.

It has been conjectured by Dong & Mason [149] that the finite-dimensionality of Zhu's algebra implies rationality (in either sense). This is not true as has been demonstrated by the counterexample of Gaberdiel & Kausch [151].

## *Appendix A.3. Physicists Definition*

Physicists call a meromorphic conformal field theory rational if it has finitely many irreducible highest weight representations. Each of these has a Fock space with finite-dimensional $L_0$ eigenspaces, and the characters of these representations form a representation of the modular group $SL(2, \mathbb{Z})$. (Sometimes this last condition is not imposed.)

If a meromorphic conformal field theory is rational in the sense of Zhu and satisfies the $C_2$ condition, then it is rational in the above sense.

The notion of rationality is also sometimes applied to the whole conformal field theory: a conformal field theory is called rational if its meromorphic and anti-meromorphic conformal subtheories are rational.

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