Chapter 18

18–1. (a) $\lambda = 2$ arrivals/hr, $\mu = 3$ services/hr, $\rho = 2/3$.

$$P(n > 5) = 1 - P(n \le 5) = 1 - \sum_{j=0}^{5} \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^j = 0.088$$

(b) $L = \dfrac{\rho}{1-\rho} = 2$, $L_q = \dfrac{\lambda^2}{\mu(\mu - \lambda)} = 4/3$

(c) $W = \dfrac{1}{\mu - \lambda} = 1$ hour

18–2. (a) State 0 = rain; State 1 = clear.

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.9 \end{bmatrix}$$

(b)

$$\begin{aligned} p_0 &= 0.7p_0 + 0.1p_1 \\ 1 &= p_0 + p_1 \end{aligned}$$

This implies $p_0 = 1/4$, $p_1 = 3/4$.

(c) To find $p_{11}^{(3)}$, note that

$$P^{(3)} = P^3 = \begin{bmatrix} 0.412 & 0.588 \\ 0.196 & 0.804 \end{bmatrix}$$

Thus, $p_{11}^{(3)} = 0.804$.

(d) $f_{10}^{(2)} = p_{10}^{(2)} - f_{10}^{(1)} p_{10} = 0.16 - (0.1)(0.1) = 0.15$

(e) $\mu_0 = 1/p_0 = 4$ days

18–3.

$$P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

$$P^{\infty} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

18–4. (a)

$$P = \begin{bmatrix} 1-2\lambda\Delta t & 2\lambda\Delta t & 0 \\ \mu\Delta t & 1-(\mu+\frac{3}{2}\lambda)\Delta t & \frac{3}{2}\lambda\Delta t \\ 0 & \mu\Delta t & 1-\mu\Delta t \end{bmatrix}$$

(b)

$$\begin{aligned} -2\lambda p_0 + \mu p_1 &= 0 \\ 2\lambda p_0 - (\mu + \frac{3}{2}\lambda)p_1 + \mu p_2 &= 0 \\ \frac{3}{2}\lambda p_1 - \mu p_2 &= 0 \\ p_0 + p_1 + p_2 &= 1 \end{aligned}$$

Solving yields

$$\begin{aligned} p_0 &= \frac{\mu^2}{\mu^2 + 2\lambda\mu + 3\lambda^2} \\ p_1 &= \frac{2\lambda\mu}{\mu^2 + 2\lambda\mu + 3\lambda^2} \\ p_2 &= \frac{3\lambda^2}{\mu^2 + 2\lambda\mu + 3\lambda^2} \end{aligned}$$

18–5. (a) $p_{ii} = 1$ implies that State 1 is an absorbing state, so States 0 and 3 are absorbing.

(b) $p_0 = 1$. The system never leaves State 0.

(c) $b_{10} = \frac{2}{3} \cdot 1 + \frac{1}{6}b_{10} + \frac{1}{6}b_{20}$

$b_{20} = \frac{2}{3}b_{10} + \frac{1}{6}b_{20}$

$b_{10} = \frac{20}{21},\ b_{20} = \frac{16}{21}$

$b_{13} = \frac{1}{6}b_{13} + \frac{1}{6}b_{23}$

$b_{23} = \frac{2}{3}b_{13} + \frac{1}{6}b_{23} + \frac{1}{6} \cdot 1$

$b_{13} = \frac{1}{21},\ b_{23} = \frac{5}{21}$

$$p_0 = \frac{1}{2} \cdot \frac{20}{21} + \frac{1}{2} \cdot \frac{16}{21} = \frac{18}{21}$$

(d) $$p_0 = \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot \frac{20}{21} + \frac{1}{4} \cdot \frac{16}{21} + \frac{1}{4} \cdot 1 = \frac{39}{42}$$

18–6. (a)

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

(b)

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 & 0 \\ 0 & 0.7 & 0 & 0.3 & 0 \\ 0 & 0 & 0.7 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} b_{10} &= 0.7(1) + 0.3b_{20} \\ b_{20} &= 0.7b_{10} + 0.3b_{20} \\ b_{30} &= 0.7b_{20} \end{aligned}$$

$$\begin{aligned} b_{10} &= 0.953 \\ b_{20} &= 0.845 \\ b_{30} &= 0.591 \end{aligned}$$

$$\begin{aligned} b_{34} &= 0.7b_{24} + 0.3(1) \\ b_{24} &= 0.7b_{14} + 0.3b_{34} \\ b_{14} &= 0.3b_{24} \end{aligned}$$

$$\begin{aligned} b_{14} &= 0.0465 \\ b_{24} &= 0.155 \\ b_{34} &= 0.408 \end{aligned}$$

18–7. (a)

$$P = \begin{bmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{bmatrix}$$

$$A = [1\ 0\ 0\ 0]$$

(b,c) The transition matrix P is said to be *doubly stochastic* since each row and each column add to one. For such matrices, it is easy to show that the steady-state probabilities are all equal. Thus, for $p = q = 1/2$ or $p = 4/5$, $q = 1/5$, we have $p_1 = p_2 = p_3 = p_4 = 1/4$.

18–9. $\lambda = 1/10$, $\mu = 1/3$, $\rho = 3/10$.

(a) $P(\text{wait}) = 1 - p_0 = \rho = 3/10$.

(b) $L_q = \dfrac{\lambda^2}{\mu(\mu-\lambda)} = 9/70$.

(c) $W_q = 3 = \dfrac{\lambda}{\mu(\mu-\lambda)} \;\Rightarrow\; \lambda = 1/6 =$ criteria for adding service.

(d) $P(W_q > 0) = \displaystyle\int_{10}^{\infty} \lambda(1-\rho)\, e^{-(\mu-\lambda)w_q}\, dw_q = \frac{3}{10}\, e^{-7/3} \doteq 0.03$.

(e) $P(W > 10) = e^{-\frac{1}{3}(\frac{7}{10})(0.10)} \doteq 0.10$.

(f) $\rho = 1 - p_0 = 3/10$.

18–10. $\lambda = 15$, $\mu = 27$, $N = 3$, $\rho = \dfrac{15}{27} = \dfrac{5}{9}$.

(a) $L = \dfrac{5}{9}\left[\dfrac{1 - 4(\frac{5}{9})^3 + 3(\frac{5}{9})^4}{\frac{4}{5}[1 - (\frac{5}{9})]^4}\right] \doteq 0.83$.

(b) $\left(\dfrac{5}{9}\right)^3 \left[\dfrac{\frac{4}{9}}{1 - (\frac{5}{9})^4}\right] \doteq 0.084$.

(c) Since both the number in the system and the number in the queue are the same when the system is empty.

18–11. The general steady-state equations for the $M/M/s$ queueing system are

$$\begin{aligned}
\rho &= \lambda/(s\mu) \\
p_0 &= \left\{\left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!}\right] + \left[\frac{(s\rho)^s}{(s!)(1-\rho)}\right]\right\}^{-1} \\
L &= s\rho + \frac{(s\rho)^{s+1}p_0}{s(s!)(1-\rho)^2} \\
L_q &= \frac{(s\rho)^{s+1}p_0}{s(s!)(1-\rho)^2} \\
W &= L/\lambda \\
W_q &= W - \frac{1}{\mu}
\end{aligned}$$

In this problem, we have $\lambda = 0.0416$, $\mu = 0.025$.

(a) $\rho = 0.555$ for $s = 3$.

(b) Plugging into the appropriate equation above, we find that $p_0 = 0.1732$.

Then $L_q = 0.3725$.

And finally, $W = W_q + \frac{1}{\mu} = \frac{L_q}{0.0416} + 40 = 48.96$ min

(c) $\rho = 0.832$ for $s = 2$.

Plugging into the appropriate equation above, we find that $p_0 = 0.0917$.

Then $L_q = 3.742$.

And finally, $W = \frac{L_q}{0.0416} + 40 = 129.96$ min.

18–12. $\lambda = 18$, $\mu = 30$, $\phi = 0.6$, $\rho = 0.6/s$.

$$p_0 = \left[\sum_{j=0}^{s-1} \frac{\phi^j}{j!} + \frac{\phi^s}{s!(1-\rho)}\right]^{-1}.$$

$s = 1 \;\Rightarrow\; p_0 = 0.4$.

$s = 2 \;\Rightarrow\; p_0 = 0.5384$

and $P(\text{wait}) = 1 - p_0 - p_1 = 1 - 0.5384 - \dfrac{(0.6)^1}{1!}(0.5384) = 0.14.$

18–13. $\lambda = 50,\ \mu = 20,\ \phi = 2.5,\ \rho = 2.5/s.$

(a)

$$C_j = \frac{\lambda_{j-1}\lambda_{j-2}\cdots\lambda_0}{\mu_j\mu_{j-1}\cdots\mu_1},$$

where

$$\lambda_j = \begin{cases} \lambda, & j \le s \\ 0, & j > s \end{cases}$$

and

$$\mu_j = \begin{cases} j\mu, & j \le s \\ s\mu, & j > s \end{cases}$$

Therefore,

$$C_j = \frac{(\lambda/\mu)^j}{j!}, \quad j = 0, 1, \ldots, s.$$

$$p_j = \begin{cases} \frac{(\lambda/\mu)^j}{j!} p_0 & j = 0, 1, \ldots, s \\ 0 & \text{otherwise} \end{cases},$$

where

$$p_0 = \left[\sum_{j=0}^{s} \frac{(\lambda/\mu)^j}{j!}\right]^{-1}$$

(b) $p_s = \dfrac{(\lambda/\mu)^s}{s!}\left[\displaystyle\sum_{j=0}^{s} \frac{(\lambda/\mu)^j}{j!}\right]^{-1} \le 0.05$

Try $s = 5 \;\Rightarrow\; p_s = 0.065$

$s = 6 \;\Rightarrow\; \rho = 0.416,\ p_s = 0.0354$ (which works).

(c) $p_6 = 0.0354$

(d) Utilization would change from 41.6% to $100\rho = 100(0.5/6) = 8.33\%$.

(e) $\phi = 50/12 = 4.166$. $p_6 = 0.377$ fraction getting busy tone.