Chapter 19

19–1. Let n denote the number of coin flips. Then the number of heads observed is $X \sim \text{Bin}(n, 0.5)$. Therefore, we can expect to see about $n/2$ heads over the long term.

19–2. If $\hat{\pi}_n$ denotes the estimator for π after n darts have been thrown, then it is easy to see that $\hat{\pi}_n \sim (4/n)\text{Bin}(n, \pi/4)$. Then $E(\hat{\pi}_n) = \pi$, and we can expect to see the estimator converge towards π as n becomes large.

19–3. By the Law of the Unconscious Statistician,

$$\begin{aligned} E(\hat{I}_n) &= \frac{b-a}{n} E\left(\sum_{i=1}^{n} f(a + (b-a)U_i) \right) \\ &= (b-a)E[f(a + (b-a)U_i)] \\ &= (b-a) \int_0^1 f(a + (b-a)u) \cdot 1 \, du \\ &= I \end{aligned}$$

19–4. (a) The exact answer is $\Phi(2) - \Phi(0) = 0.4772$. The $n = 1000$ result will tend to be closer than the $n = 10$.

(b) We can instead integrate over $\int_0^4$, say, since $\int_4^{10} \approx 0$. This strategy will prevent the "waste" of observations on the trivial tail region.

(c) The exact answer is 0.

19–5. (a)

customer	arrival time	begin service	service time	depart time	wait
1	3	3.0	6.0	9.0	0.0
2	4	9.0	5.5	14.5	5.0
3	6	14.5	4.0	18.5	8.5
4	7	18.5	1.0	19.5	11.5
5	13	19.5	2.5	22.0	6.5
6	14	22.0	2.0	24.0	8.0
7	20	24.0	2.0	26.0	4.0
8	25	26.0	2.5	28.5	1.0
9	28	28.5	4.0	32.5	0.5
10	30	32.5	2.5	35.0	2.5

Time	Event	Customers in System
3	Cust 1 arrival	1
4	Cust 2 arrival	1 2
6	Cust 3 arrival	1 2 3
7	Cust 4 arrival	1 2 3 4
9	Cust 1 depart	2 3 4
13	Cust 5 arrival	2 3 4 5
14	Cust 6 arrival	2 3 4 5 6
14.5	Cust 2 depart	3 4 5 6
18.5	Cust 3 depart	4 5 6
19.5	Cust 4 depart	5 6
20	Cust 7 arrival	5 6 7
22	Cust 5 depart	6 7
24	Cust 6 depart	7
25	Cust 8 arrival	7 8
26	Cust 7 depart	8
28	Cust 9 arrival	8 9
28.5	Cust 8 depart	9
30	Cust 10 arrival	9 10
32.5	Cust 9 depart	10
35	Cust 10 depart	

Thus, the last customer leaves at time 35.

(b) The average waiting time for the 10 customers is 4.75.

(c) The maximum number of customers in the system is 5 (between times 14 and 14.5).

(d) The average number of customers over the first 30 minutes is calculated by adding up all of the customer minutes from the second table — one customer from times 3 to 4, two customers from times 4 to 6, etc.

$$\frac{1}{30}\int_0^{30} L(t)\,dt \;=\; \frac{79.5}{30} \;=\; 2.65$$

19–6. The following table gives a history for this (S, s) inventory system.

day	initial stock	customer order	end stock	reorder?	lost orders
1	20	10	10	no	0
2	10	6	4	yes	0
3	20	11	9	no	0
4	9	3	6	yes	0
5	20	20	0	yes	0
6	20	6	14	no	0
7	14	8	6	no	0

By using $s = 6$, we had no lost orders.

19–7. (a) Here is the complete table for the generator.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
X_i	0	1	16	15	12	13	2	11	8	9	14	7	4	5	10	3	0

Thus, $U_1 = 1/16$, $U_2 = 6/16$.

(b) Yes (see the table).

(c) Since the generator cycles, we have $X_0 = X_{16} = \cdots = X_{144} = 0$. Then $X_{145} = 1$, $X_{146} = 6$, $\ldots$, $X_{150} = 2$.

19–8. (a) Using the algorithm given in Example 19–8 of the text, we find that $X_1 = 1422014746$ and $X_2 = 456328559$. Since $U_i = X_i/(2^{31} - 1)$, we have $U_1 = 0.6622$, $U_2 = 0.2125$.

19–9. (a) $X = -(1/2)\ell n(1 - U)$.

(b) $X = -(1/2)\ell n(0.025) = 0.693$.

19–10. (a) $Z = \Phi^{-1}(0.25) = -0.6745$.

(b) $X = \mu + \sigma Z = 1 + 3Z = -1.0235$.

19–11. $f(x) = |x/4|$, $-2 < x < 2$.

(a) If $-2 < x < 0$, then $F(x) = \displaystyle\int_{-2}^{x} -\frac{t}{4}\,dt = \frac{1}{2} - \frac{x^2}{8}$.

If $0 < x < 2$, then $F(x) = \displaystyle\frac{1}{2} + \int_{0}^{x} \frac{t}{4}\,dt = \frac{1}{2} + \frac{x^2}{8}$.

Thus, for $0 < U < 1/2$, we set $F(X) = \frac{1}{2} - \frac{X^2}{8} = U$.

Solving, we get $X = -\sqrt{4 - 8U}$.

For $1/2 < U < 1$, we set $F(X) = \frac{1}{2} + \frac{X^2}{8} = U$.

Solving this time, we get $X = \sqrt{8U - 4}$.

Recap:

$$X = \begin{cases} -\sqrt{4-8U}, & 0 < U < 1/2 \\ \sqrt{8U-4} & 1/2 < U < 1 \end{cases}$$

(b) $X = \sqrt{8(0.6) - 4} = 0.894$.

19–12. (a)

x	$p(x)$	$F(x)$	U
−2.5	0.35	0.35	[0,0.35)
1.0	0.25	0.60	[0.35,0.60)
10.5	0.40	1.0	[0.60,1.0)

(b) $U = 0.86$ yields $X = 10.5$.

19–13. (a) $F(X) = 1 - e^{-(X/\alpha)^\beta} = U$.

Solving for X, we obtain $X = \alpha[-\ell\text{n}(1 - U)]^{1/\beta}$.

(b) $X = (1.5)[-\ell\text{n}(0.34)]^{1/2} = 1.558$.

19–14. We have

$$Z_1 = \sqrt{-2\ell\text{n}(0.45)}\,\cos(2\pi(0.12)) = 0.921$$

and

$$Z_2 = \sqrt{-2\ell\text{n}(0.45)}\,\sin(2\pi(0.12)) = 0.865.$$

19–15. $\sum_{i=1}^{12} U_i - 6 = 1.07.$

19–16. Since the X_i's are IID exponential(λ) random variables, we know that their m.g.f. is

$$M_{X_i}(t) \;=\; \frac{\lambda}{\lambda - t}, \quad t < \lambda, \quad i = 1, 2 \ldots, n.$$

Then the m.g.f. of $Y = \sum_{i=1}^{n} X_i$ is

$$M_Y(t) \;=\; \prod_{i=1}^{n} M_{X_i}(t) \;=\; \left(\frac{\lambda}{\lambda - t}\right)^n.$$

We will be done as soon as we can show that this m.g.f. matches that corresponding to the p.d.f. from Equation (19–4), namely,

$$\begin{aligned}
M_Y(t) &= \int_0^\infty e^{ty} \lambda^n e^{-\lambda y} y^{n-1}/(n-1)!\, dy \\
&= \frac{\lambda^n}{(n-1)!} \int_0^\infty e^{-(\lambda - t)y}\, y^{n-1}\, dy \\
&= \frac{\lambda^n}{(n-1)!} \int_0^\infty e^{-u} \left(\frac{u}{\lambda - t}\right)^{n-1} \frac{du}{\lambda - t} \\
&= \frac{\lambda^n}{(\lambda - t)^n (n-1)!} \int_0^\infty e^{-u}\, u^{n-1}\, du \\
&= \frac{\lambda^n}{(\lambda - t)^n (n-1)!} \Gamma(n) \\
&= \left(\frac{\lambda}{\lambda - t}\right)^n.
\end{aligned}$$

Since both versions of $M_Y(t)$ match, that means that the two versions of Y must come from the same distribution — and we are done.

19–17. $X = -\frac{1}{\lambda}\, \ell\mathrm{n}\left(\prod_{i=1}^{n} U_i\right) = -\frac{1}{3}\, \ell\mathrm{n}((0.73)(0.11)) = 0.841.$

19–18. (a) To get a Bernoulli(p) random variable X_i, simply set

$$X_i \;=\; \begin{cases} 1, & \text{if } U_i \le p \\ 0, & \text{if } U_i > p \end{cases}$$

(Note that there are other allocations of the uniforms that will do the trick just as well.)

(b) Suppose $X_1, X_2, \ldots, X_n$ are IID Bernoulli's, generated according to (a). To get a Binomial(n,p), let $Y = \sum_{i=1}^n X_i$.

19–19. Suppose success (S) on trial i corresponds to $U_i \le 0.25$, and failure (F) corresponds to $U_i > 0.25$. Then, from the sequence of uniforms in Problem 19–15, we have $FFFFS$, i.e., we require $X = 5$ trials before observing the first success.

19–20. The grand sample mean is

$$\bar{Z}_b \;=\; \frac{1}{b}\sum_{i=1}^{b} Z_i \;=\; 4,$$

while

$$\hat{V}_B \;=\; \frac{1}{b-1}\sum_{i=1}^{b}(Z_i - \bar{Z}_b)^2 \;=\; 1.$$

So the 90% batch means CI for μ is

$$\begin{aligned}
\mu &\in \bar{Z}_b \pm t_{\alpha/2,b-1}\sqrt{\hat{V}_B/b} \\
&= 4 \pm t_{0.05,2}\sqrt{1/3} \\
&= 4 \pm 2.92\sqrt{1/3} \\
&= 4 \pm 1.686
\end{aligned}$$

19–21. The 90% confidence interval is of the form

$$\begin{aligned}
[-2.5,\, 3.5] &= \bar{X} \pm t_{\alpha/2,b-1}\, y \\
&= 0.5 \pm t_{0.05,4}\, y \\
&= 0.5 \pm 2.132\, y
\end{aligned}$$

Since the half-length of the CI is 3, we must have that $y = 3/2.132 = 1.407$.

The 95% CI will therefore be of the form

$$\bar{X} \pm t_{0.025,4}\, y \;=\; 0.5 \pm (2.776)(1.407) \;=\; 0.5 \pm 3.91 \;=\; [-3.41,\, 4.41].$$

19–23. The 95% batch means CI for μ is

$$\begin{aligned}\mu &\in \bar{Z}_b \pm t_{\alpha/2,b-1}\sqrt{\hat{V}_B/b} \\ &= 100 \pm t_{0.025,4}\sqrt{250/5} \\ &= 100 \pm 2.776\sqrt{50} \\ &= 100 \pm 19.63\end{aligned}$$

19–25. (a) (i) Both exponential(1).

(ii) $Cov(U_i, 1-U_i) = -V(U_i) = -1/12$.

(iii) Yes.

(b) After a little algebra,

$$\begin{aligned}V((\bar{X}_n + \bar{Y}_n)/2) &= \frac{1}{4}\left[V(\bar{X}_n) + V(\bar{Y}_n) + 2Cov(\bar{X}_n, \bar{Y}_n)\right] \\ &= \frac{1}{2}\left[V(\bar{X}_n) + Cov(\bar{X}_n, \bar{Y}_n)\right] \\ &= \frac{1}{2n}\left[V(X_i) + Cov(X_i, Y_i)\right] \\ &\le \frac{1}{2n}V(X_i) \\ &= V(\bar{X}_{2n}).\end{aligned}$$

So the variance decreases compared to $V(\bar{X}_{2n})$.

(c) You get zero, which is the correct answer.

19–26. (a) $E(C) = E(\bar{X}) - E[k(Y - E(Y))] = \mu - [k(E(Y) - E(Y))] = \mu$.

(b) $V(C) = V(\bar{X}) + k^2V(Y) - 2k\,Cov(\bar{X}, Y)$. Comment: It would be nice, in terms of minimizing variance, if $k\,Cov(\bar{X}, Y) > 0$.

(c)

$$\frac{d}{dk}V(C) \;=\; 2kV(Y) - 2\,Cov(\bar{X}, Y) \;=\; 0.$$

This implies that the critical (minimizing) point is

$$k \;=\; \frac{Cov(\bar{X}, Y)}{V(Y)}.$$

Thus, the optimal variance is

$$\begin{aligned} V(C) &= V(\bar{X}) + \left(\frac{Cov(\bar{X},Y)}{V(Y)}\right)^2 V(Y) - 2\left(\frac{Cov(\bar{X},Y)}{V(Y)}\right) Cov(\bar{X},Y) \\ &= V(\bar{X}) - \frac{Cov^2(\bar{X},Y)}{V(Y)}. \end{aligned}$$

19–27. They are exponential(1).

19–28. They should look normal.

19–29. You should have a bivariate normal distribution (with correlation 0), centered at zero with symmetric tails in all directions.