Chapter 2

2–1. $R_X = \{0,1,2,3,4\}, P(X=0) = \dfrac{\binom{4}{0}\binom{48}{5}}{\binom{52}{5}}$

$$P(X=1) = \frac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}, \qquad P(X=2) = \frac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}},$$

$$P(X=3) = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}, \qquad P(X=4) = \frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}}$$

2–2. $\mu = 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{12} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{12} = \frac{26}{12}$

$$\sigma^2 = \left[0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{3} + 3^2 \cdot \frac{1}{12} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{12}\right] - \left(\frac{26}{12}\right)^2 = \frac{83}{36}$$

2–3. $\int_0^\infty ce^{-x}\,dx = 1 \Rightarrow c = 1,$ so

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \int_0^\infty xe^{-x}\,dx = -xe^{-x}|_0^\infty + \int_0^\infty e^{-x}\,dx = 1$$

$$\begin{aligned} \sigma^2 = \int_0^\infty x^2 e^{-x}\,dx - 1^2 &= \left[-x^2e^{-x}|_0^\infty + \int_0^\infty 2xe^{-x}\,dx\right] - 1 \\ &= \left[-2xe^{-x}|_0^\infty + \int_0^\infty 2e^{-x}\,dx\right] - 1 \\ &= 2 - 1 = 1 \end{aligned}$$

2–4. $F_T(t) = P_T(T \leq t) = 1 - e^{-ct}; t \geq 0$

$$\therefore f_T(t) = F_T'(t) = \begin{cases} ce^{-ct} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

2–5. (a) Yes

(b) No, since $G_X(\infty) \neq 1$ and $G_X(b) \ngeq G_X(a)$ if $b \geq a$

(c) Yes

2–6. (a) $f_X(x) = F'_X(x) = e^{-x}; 0 < x < \infty$
$= 0; x \leq 0$

(c) $h_X(x) = H'_X(x) = e^{x}; -\infty < x \leq 0$
$= 0; x > 0$

2–7. Both are since $p_X(x) \geq 0$, all x; and $\Sigma_{\text{all } x}\ p_X(x) = 1$

2–8. The probability mass function is

x	$p_X(x)$
-1	$\frac{1}{5}$
0	$\frac{1}{10}$
$+1$	$\frac{2}{5}$
$+2$	$\frac{3}{10}$
ow	0

$$E(X) = \left(-1 \cdot \frac{1}{5}\right) + \left(0 \cdot \frac{1}{10}\right) + \left(1 \cdot \frac{2}{5}\right) + \left(2 \cdot \frac{3}{10}\right) = \frac{4}{5}$$

$$V(X) = \left((-1)^2 \cdot \frac{1}{5}\right) + \left(0^2 \cdot \frac{1}{10}\right) + \left(1^2 \cdot \frac{2}{5}\right) + \left(2^2 \cdot \frac{3}{10}\right) - \left(\frac{4}{5}\right)^2 = \frac{29}{25}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{5}, & -1 \le x < 0 \\ \frac{3}{10}, & 0 \le x < 1 \\ \frac{7}{10}, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

2–9. $P(X < 30) = P(X \le 29) = \sum_{x=0}^{29} \frac{e^{-20}(20)^x}{x!} = 0.978$

2–10. (a) $P_X(x) = F_X(x) - F_X(x-1)$

$$= \left[1 - \left(\frac{1}{2}\right)^{x+1}\right] - \left[1 - \left(\frac{1}{2}\right)^{x}\right]$$

$$= \left(\frac{1}{2}\right)^{x} - \left(\frac{1}{2}\right)^{x+1}$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^{x}; x = 0, 1, 2, \ldots$$

$$= 0; ow$$

(b) $P_X(0 < X \le 8) = F_X(8) - F_X(0) = 0.498$

(c) $P_X(X \text{ even}) = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k = \frac{2}{3}$

2–11. (a) $\int_0^2 kx\,dx + \int_2^4 k(4-x)\,dx = 1 \Rightarrow k = \frac{1}{4}$

and $f_X(x) \ge 0$ for $k = \frac{1}{4}$

(b) $\mu = \int_0^2 \frac{1}{4}x^2\,dx + \int_2^4 \frac{1}{4}(4x - x^2)\,dx = 2$

$$\sigma^2 = \int_0^2 \frac{1}{4}x^3\,dx + \int_2^4 \frac{1}{4}(4x^2 - x^3)\,dx - 2^2 \;=\; \frac{2}{3}$$

(c) $F_X(x) = 0; \quad x < 0$

$$= \int_0^x \frac{1}{4}t\,dt = \frac{x^2}{8}; \quad 0 \le x < 2$$

$$= \int_0^2 \frac{1}{4}x\,dx + \int_2^x \frac{1}{4}(4-t)\,dt = -1 + x - \frac{x^2}{8}; \quad 2 \le x < 4$$

$$= 1; \quad x > 4$$

2–12. (a) $k\left[\int_0^a x\,dx + \int_a^{2a}(2a - x)\,dx\right] = 1 \Rightarrow k = \frac{1}{a^2}$

(b) $F_X(x) = 0; x < 0$

$$= \int_0^x kt\,dt = k\left(\frac{x^2}{2}\right); 0 \le x < a$$

$$= \int_0^a kx\,dx + \int_a^x k(2a - t)\,dt = k\left(\frac{a^2}{2}\right) + k\left[2a(x-a) + \frac{a^2 - x^2}{2}\right]$$

for $a \le x \le 2a$

$$= 1; x > 2a$$

(c) $\mu = k\left[\int_0^a x^2\,dx + \int_a^{2a}(2ax - x^2)\,dx\right] = a$

$$\sigma^2 = \frac{a^2}{6}$$

2–13. From Chebyshev's inequality $1 - \frac{1}{k^2} = 0.75 \Rightarrow k = 2$ with $\mu = 2$, $\sigma = \sqrt{2}$, and the interval is $[14 - 2\sqrt{2}, 14 + 2\sqrt{2}]$.

2–14. (a) $\int_{-1}^0 kt^2\,dt = 1 \Rightarrow k = 3$

(b) $\mu = \int_{-1}^{0} 3t^3\,dt = 3\left[\left.\frac{t^4}{4}\right|_{-1}^{0}\right] = -\frac{3}{4}$

$$\sigma^2 = \int_{-1}^{0} 3t^4\,dt - \left(-\frac{3}{4}\right)^2 = 3\left(\left.\frac{t^5}{5}\right|_{-1}^{0}\right) - \frac{9}{16} = \frac{3}{80}$$

(c) $F_T(t) = 0; \quad t < -1$

$$= \int_{-1}^{t} 3u^2\,du = t^3 + 1; \quad -1 \le t \le 0$$

$$= 1; \quad t > 0$$

2–15. (a) $k\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = 1 \Rightarrow k = \frac{8}{7}$

(b) $\mu = \frac{8}{7}\left[1 \cdot \left(\frac{1}{2}\right) + 2 \cdot \left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3\right] = \frac{11}{7}$

$$\sigma^2 = \frac{8}{7}\left[1^2 \cdot \left(\frac{1}{2}\right) + 2^2 \cdot \left(\frac{1}{2}\right)^2 + 3^2 \cdot \left(\frac{1}{2}\right)^3\right] - \left(\frac{11}{7}\right)^2 = \frac{26}{49}$$

(c) $F_X(x) = 0; \quad x < 1$

$$= \frac{8}{14}; \quad 1 \le x < 2$$

$$= \frac{12}{14}; \quad 2 \le x < 3$$

$$= 1; \quad x \ge 3$$

2–16. $\sum_{n=0}^{\infty} kr^n = 1 => k = 1 - r$

2–17. Using Chebychev's inequality, $1 - \frac{1}{k^2} = 0.99 \Rightarrow k = 10$, $\mu = 2$, $\sigma = \sqrt{0.4}$, so the interval is $[2 - 10(0.6324)];\ 2 + 10(0.6324)]$. The letters should be mailed 8.3 days before delivery date required.

2–18. (a) $\mu_A = 1000(0.2) + 1050(0.3) + 1100(0.1) + 1150(0.3) + 1200(0.05) + 1350(0.05)$
$= 1097.5$
$\mu_B = 1135$

(b) Assume independence so $\mu_{A|B=1130} = \mu_A = 1097.5$

(c) With independence, $P(A = k \text{ and } B = k) = P(A = k) \cdot P(B = k)$. So

$$\sum_k P(A=k)P(B=k) = (0.1)(0.2) + (0.3)(0.1) + (0.1)(0.3) + (0.3)(0.3) + (0.05)(0.1) + (0.05)(0.1) = 0.18$$

2–19. $p(x_i) = \dfrac{x_i - 1}{36}; \quad x_i = 2, 3, \ldots, 6$

$= \dfrac{13 - x_i}{36}; \quad x_i = 7, 8, \ldots, 12$

so

x_i	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

2–20. $\mu = 7$, $\sigma^2 = \dfrac{105}{18}$

2–21. (a) $F_X(x) = 0; \quad x < 0$

$= x^2/9; \quad 0 \le x < 3$

$= 1; \quad x \ge 3$

(b) $\mu = 2$, $\sigma^2 = \dfrac{1}{2}$

(c) $\mu_3' = \dfrac{54}{5}$

(d) $m = \dfrac{3}{\sqrt{2}}$

2–22. $\mu = 0$, $\sigma^2 = 25$, $\sigma = 5$
$P[|X - \mu| \ge k\sigma] = P[|X| \ge 5k] = 0$ if $k > 1$ and $= 1$, $0 < k \le 1$.
From Chebychev's inequality, the upper bound is $\frac{1}{k^2}$.

2–23. $F(x) = 0; x < 0$

$$F(x) = \int_0^x \left(\frac{u}{t^2}\right) e^{-(u^2/2t^2)}\, du = \int_0^{x^2/2t^2} e^{-v}\, dv$$

$= 1 - e^{-x^2/2t^2}; x \ge 0$

2–24. $F(x) = \int_0^x \frac{1}{\sigma\pi} \frac{du}{\{1 + ((u-\mu)^2/\sigma^2)\}}; -\infty < x \leq \infty$

Let $t = \frac{u-\mu}{\sigma}$, $dt = \frac{1}{\sigma} du$ and

$$F(x) = \int_0^{\frac{x-u}{\sigma}} \frac{1}{\pi} \cdot \frac{dt}{1+t^2} = \frac{1}{\pi} \tan^{-1}\left(\frac{x-u}{\sigma}\right); -\infty < x < \infty$$

2–25. $\int_0^{\pi/2} k \sin y \, dy = 1 \Rightarrow k[-\cos y|_0^{\pi/2}] = 1 \Rightarrow k = 1$

$$\mu = \int_0^{\pi/2} y \sin y \, dy = \sin\left(\frac{\pi}{2}\right) = 1$$

2–26. Assume X continuous

$$\begin{aligned}
\mu_k &= \int_{-\infty}^{\infty} (x-\mu)^k f_X(x)\, dx = \int_{-\infty}^{\infty} \left[\sum_{j=0}^{k} \binom{k}{j} (-\mu^j) x^{k-j}\right] f_X(x)\, dx \\
&= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mu^j \int_{-\infty}^{\infty} x^{k-j} f_X(x)\, dx \\
&= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \mu^j \mu'_{k-j}
\end{aligned}$$