Chapter 3

3–1. (a)

y	$p_Y(y)$
0	0.6
20	0.3
80	0.1
ow	0

(b) $E(Y) = \sum_y y \cdot p_Y(y) = 14$

$V(Y) = \sum_y y^2 \cdot p_Y(y) - (14)^2 = 564$

3–2. Let Profit $= P = 10 + 2X$

$$P(P \le p) = P(10 + 2X \le p) = P\left(X \le \frac{p-10}{2}\right) = \int_0^{\frac{p-10}{2}} \frac{x}{18}dx$$

$$F_P(p) = \left.\frac{x^2}{36}\right|_0^{\frac{p-10}{2}} = \frac{1}{144}(p^2 - 20p + 100)$$

$$f_P(p) = \frac{1}{72}(p-10); 10 \le p \le 22$$

$$= 0; ow$$

3–3. (a) $P(T < 1) = 1 - e^{-1/4} = 0.221$

(b) $E[P] = 200 - 200P(T < 1) = 155.80$

3–4. (a)

x	$p_X(x)$	$y = 2000(12 - x)$	$p_Y(y)$
10	0.1	4000	0.1
11	0.3	2000	0.3
12	0.4	0	0.4
13	0.1	−2000	0.1
14	0.1	−4000	0.1
ow	0	ow	0

(b) $E(X) = 10(0.1) + 11(0.3) + 12(0.4) + 13(0.1) + 14(0.1) = 11.8$ days

$$V(X) = 10^2(0.1) + 11^2(0.3) + 12^2(0.4) + 13^2(0.1) + 14^2(0.1) - (11.8)^2$$
$$= 1.16 \text{ days}^2$$

$$E(Y) = (4000)(0.1) + (2000)(0.3) + 0(0.4) + (-2000)(0.1) + (-4000)(0.1) = \$400$$

$$V(Y) = (4000)^2(0.1) + (2000)^2(0.3) + 0^2(0.4) + (-2000)^2(0.1) + (-4000)^2(0.1) - 400^2 = 4{,}640{,}000(\$^2)$$

3–5. $F_Z(z) = P(Z \le z) = P(X^2 \le z) = P(|X| \le \sqrt{z})$
$= P(0 \le X \le \sqrt{z}) = \int_0^{\sqrt{z}} 2xe^{-x^2}\,dx$

Let $u = x^2$, $du = 2x\,dx$, so

$$F_Z(z) = \int_0^z e^{-u}\,du = 1 - e^z$$

$$f_Z(z) = e^{-z};\, z \ge 0$$
$$= 0;\, ow$$

3–6. (a) $E(D_i) = \sum_{d=0}^{9} d \cdot p_{D_i}(d) = \frac{1}{10}(1+2+3+4+5+6+7+8+9) = 4.5$

(b) $V(D_i) = \sum_{d=0}^{9} d^2 \cdot p_{D_i}(d) - (4.5)^2$

$$= \frac{1}{10}[1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2] - (20.25) = 8.25$$

(c)

d	y
0	4
1	3
2	2
3	1
4	0
5	0
6	1
7	2
8	3
9	4

y	$p_Y(y)$
0	0.2
1	0.2
2	0.2
3	0.2
4	0.2
ow	0

$$E(Y) = \frac{2}{10}(1+2+3+4) = 2$$

$$V(Y) = \frac{2}{10}(1^2+2^2+3^2+4^2) - 4 = 2$$

3–7. $R = \text{Revenue/Gal}$

$$\begin{aligned} R &= 0.92; A < 0.7 \\ &= 0.98; A \geq 0.7 \end{aligned}$$

$$\begin{aligned} E(R) &= 0.92\ P(A < 0.7) + 0.98\ P(A \geq 0.7) \\ &= 0.92(0.7) + 0.98(0.3) = 93.8\text{¢/gal.} \end{aligned}$$

3–8. $$M_X(t) = \int_\beta^\infty e^{tx}\frac{1}{\theta}e^{-\frac{1}{\theta}(x-\beta)}\,dx = \frac{1}{\theta}e^{\beta/\theta}\int_\beta^\infty e^{-x(\frac{1}{\theta}-t)}\,dx$$

$$= \frac{1}{\theta}\left(\frac{1}{\theta} - t\right)^{-1} e^{\beta t},\ \text{for } \tfrac{1}{\theta} - t > 0$$

$$M_X'(t) = \frac{1}{\theta}\left(\frac{1-\theta t}{\theta}\right)^{-2} e^{\beta t}\left[\frac{1-\theta t}{\theta}\beta + 1\right]$$

$$M_X''(t) = \frac{1}{\theta}\left(\frac{1-\theta t}{\theta}\right)^{-3} e^{\beta t}\left[\left(\frac{1-\theta t}{\theta}\right)^2\beta^2 + \left(\frac{1-\theta t}{\theta}\right)\beta + 2 + \left(\frac{1-\theta t}{\theta}\right)\beta\right]$$

$$E(X) = M_X'(0) = \beta + \theta$$

$$V(X) = M_X''(0) - (\beta+\theta)^2 = \theta^2$$

3–9. (a) $$F_Y(y) = P(2X^2 \leq y) = P\left[-\sqrt{\frac{y}{2}} \leq X \leq +\sqrt{\frac{y}{2}}\right]$$

$$= P\left(0 \leq X \leq \sqrt{\frac{y}{2}}\right)$$

$$= \int_0^{\sqrt{\frac{y}{2}}} e^{-x}\,dx = -e^{-x}\Big|_0^{\sqrt{\frac{y}{2}}} = 1 - e^{-\sqrt{\frac{y}{2}}}$$

$$f_Y(y) = F_Y'(y) = \frac{1}{4}\left(\frac{y}{2}\right)^{-1/2} \cdot e^{-(y/2)^{1/2}}; y > 0$$

$$= 0; \text{otherwise}$$

(b) $F_V(v) = P(X^{1/2} \le v) = P(X \le v^2)$

$$= \int_0^{v^2} e^{-x}\,dx = -e^{-x}|_0^{v^2} = 1 - e^{-v^2}$$

$$f_V(v) = F'_V(v) = 2ve^{-v^2}; v > 0$$
$$= 0; \text{otherwise}$$

(c) $F_U(u) = P(\ell\text{n}(X) \le u) = P(X \le e^u) = \int_0^{e^u} e^{-x}\,dx$

$$= 1 - e^{-e^u}$$

$$f_U(u) = F'_U(u) = e^{-(e^u - u)}; u > 0$$
$$= 0; \text{otherwise}$$

3–10. Note that as stated $Y > y_0 \Rightarrow$ signal read (not $Y > |y_0|$), so

$$P_Y(Y > y) = \int_{\tan^{-1}y}^{\pi/2} \frac{1}{2\pi}dx + \int_{\tan^{-1}y+\pi}^{3\pi/2} \frac{1}{2\pi}dx$$
$$F_Y(y) = 1 - P_Y(Y > y)$$
$$= \frac{1}{2} + \frac{1}{\pi}\tan^{-1}y; y \ge 0$$
$$f_Y(y) = \frac{1}{\pi}\left(\frac{1}{1+y^2}\right); -\infty < y < +\infty$$

Note the symmetry in $(\frac{1}{1+y^2})$.

3–11. Let

$$S = \text{Stock Level}$$
$$L(X,S) = 0.5(X - S), X > S$$
$$= 0.25(S - X), X \le S$$
$$E(L(X,S)) = \int_{10^6}^{S} 0.25(S - X)\cdot 10^6\,dx + \int_S^{2\cdot 10^6} 0.5(X - S)\cdot 10^{-6}\,dx$$
$$\frac{dE[L(X,S)]}{dS} = \frac{6}{8}10^{-6}S - \frac{5}{4} = 0 \Rightarrow S = \frac{5}{3}10^6$$

3–12. $$\mu_G = E(G) = \int_0^4 g f_G(g)\, dg = \int_0^4 \frac{g^2}{8} dg = 8/3.$$

$$E(G^2) = \int_0^4 \frac{g^3}{8} dg = 8.$$

$$\sigma_G^2 = V(G) = E(G^2) - (E(G))^2 = 8/9.$$

$$H(G) = (3 + 0.05G)^2$$

$$H(\mu_G) = (3 + 0.05\mu_G)^2$$

$$H'(\mu_G) = (0.1)(3 + 0.05\mu_G)$$

$$H''(\mu_G) = 0.005.$$

Thus,

$$\mu_A \approx H(\mu_G) + \frac{1}{2}H''(\mu_G)\sigma_G^2 = H(8/3) + \frac{1}{2}(0.005)\frac{8}{9} = 9.82$$

and

$$\sigma_A^2 \approx (H'(\mu_G))^2 \sigma_G^2 = 0.082.$$

3–13. (a) $$\begin{aligned} F_Y(y) &= P(4 - X^2 \le y) = P(X^2 \ge 4 - y) \\ &= P(X \le -\sqrt{4-y}) + P(X \ge \sqrt{4-y}) \\ &= 0 + \int_{\sqrt{4-y}}^2 dx = 2 - \sqrt{4-y} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{1}{2}(4-y)^{-1/2}; 0 \le y \le 3 \\ &= 0; \text{otherwise} \end{aligned}$$

(b) $$\begin{aligned} F_Y(y) &= P(e^x \le y) = P(X \le \ell\text{n}(y)) \\ &= \int_1^{\ell\text{n}(y)} dx = \ell\text{n}(y) - 1 \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \frac{1}{y}; e^1 \le y \le e^2 \\ &= 0; \text{otherwise} \end{aligned}$$

3–14.
$$
\begin{aligned}
y &= \frac{3}{(1+x)^2}\\
x &= \sqrt{3}y^{-1/2} - 1\\
\frac{dx}{dy} &= -\frac{\sqrt{3}}{2}y^{-3/2}\\
\left|\frac{dx}{dy}\right| &= \frac{\sqrt{3}}{2}y^{-3/2}\\
f_Y(y) &= f_X(x)\left|\frac{dx}{dy}\right|\\
&= f_X(\sqrt{3}y^{-1/2} - 1)\frac{\sqrt{3}}{2}y^{-3/2}\\
&= \exp[-\sqrt{3}y^{-1/2} + 1]\frac{\sqrt{3}}{2}y^{-3/2}; \quad 0 \le y \le 3
\end{aligned}
$$

3–15. With equal probability $p_X(1) = \cdots = p_X(6) = \frac{1}{6}$,

$$
\begin{aligned}
M_X(t) &= \sum_{x=1}^{6}\left(\frac{1}{6}\right)e^{tx}\\
E(X) &= M'_X(0) = \frac{7}{2}\\
V(X) &= M''_X(0) - [M'_X(0)]^2 = \frac{35}{12}
\end{aligned}
$$

3–16. (a) Using the substitution $y = bx^2$, we obtain

$$
\begin{aligned}
1 &= \int_0^\infty ax^2e^{-bx^2}\,dx\\
&= \int_0^\infty a\,\frac{y}{b}\,e^{-y}\frac{dy}{2\sqrt{by}}\\
&= \frac{a}{2b^{3/2}}\int_0^\infty y^{(3/2)-1}e^{-y}\,dy\\
&= \frac{a}{2b^{3/2}}\,\Gamma(\frac{3}{2}) \;=\; \frac{a}{2b^{3/2}}\,\frac{1}{2}\,\Gamma(\frac{1}{2}) \;=\; \frac{a\sqrt{\pi}}{4b^{3/2}}
\end{aligned}
$$

Thus,

$$
a = \frac{4b^{3/2}}{\sqrt{\pi}}.
$$

(b) Similarly,

$$\mu_X = E(X) = \int_0^\infty ax^3 e^{-bx^2}\,dx = \frac{a}{2b^2}\,\Gamma(2) = \frac{2}{\sqrt{\pi b}}$$

and

$$E(X^2) = \frac{a}{2b^{5/2}}\,\Gamma\left(\frac{5}{2}\right) = \frac{3}{2b}.$$

These facts imply that

$$\sigma_X^2 = E(X^2) - [E(X)]^2 = \frac{3}{2b} - \frac{4}{\pi b} = \frac{3\pi - 8}{2\pi b}.$$

Now we have

$$\begin{aligned} H(x) &= 18x^2 \\ H(\mu_X) &= 18\mu_X^2 = 18 \cdot \frac{4}{\pi b} = \frac{72}{\pi b} \\ H'(\mu_X) &= 36\mu_X = 36 \cdot \frac{2}{\sqrt{\pi b}} = \frac{72}{\sqrt{\pi b}} \\ H''(\mu_X) &= 36 \end{aligned}$$

Then

$$\begin{aligned} \mu_Y &\doteq H(\mu_X) + \frac{1}{2}H''(\mu_X) \cdot \sigma_X^2 \\ &= \frac{72}{\pi b} + \frac{1}{2} \cdot 36 \cdot \frac{3\pi - 8}{2\pi b} = \frac{27}{b} \end{aligned}$$

and

$$\begin{aligned} \sigma_Y^2 &\doteq (H'(\mu_X))^2 \sigma_X^2 \\ &= \left(\frac{72}{\sqrt{\pi b}}\right)^2 \left(\frac{3\pi - 8}{2\pi b}\right) = \frac{2592(3\pi - 8)}{\pi^2 b^2} \end{aligned}$$

3–17. $E(Y) = 1, V(Y) = 1 \qquad H(Y) = \sqrt{Y^2 + 36}$

$$E(X) \cong H(\mu_Y) + \frac{1}{2}H''(\mu_Y)\sigma_Y^2$$

$$V(X) \cong [H'(\mu_Y)]^2\sigma_Y^2$$

$$H'(y) = y(y^2 + 36)^{-1/2}, \qquad H''(y) = 36(y^2 + 36)^{-3/2}$$

$$E(X) \cong (37)^{1/2} + \frac{1}{2}(36)(37)^{-3/2} \doteq 6.16$$

$$V(X) \cong (37)^{-1} = 0.027027$$

3–18. $E(P) = \int_0^1 (1+3r)6r(1-r)\,dr = \int_0^1 (6r + 12r^2 - 18r^3)\,dr \;=\; \dfrac{5}{2}$

Since $H(r) = 1 + 3r$ is increasing,

$$f_P(p) = 6\left(\frac{p-1}{3}\right)\left(1 - \frac{p-1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{9}(5p - p^2 - 4); 1 < p < 4$$
$$= 0; \text{otherwise}$$

3–19. $M_X(t) = \int_0^\infty e^{tx} \cdot 4xe^{-2x}\,dx$

$$= \int_0^\infty 4xe^{x(t-2)}\,dx = \left(1 - \frac{t}{2}\right)^{-2}$$

$E(X) = M_X'(0) = 1$

$V(X) = M_X''(0) - [M_X'(0)]^2 = \dfrac{1}{2}$

3–20. $V = \dfrac{\pi \cdot X^2}{4} \cdot 1$

$$E(V) = \frac{\pi}{4}E(X^2) = \frac{\pi}{4}[V(X) + (E(X))^2]$$
$$= \frac{\pi}{4} \cdot [25 \cdot 10^{-6} + 2^2] = 3.14162$$

3–21. $M_Y(t) = E(e^{tY}) = E(e^{t(aX+b)})$

$$= e^{tb}E(e^{(at)X})$$
$$= e^{tb}M_X(at)$$

3–22. (a) $\int_0^1 k(1-x)^{a-1}x^{b-1}\,dx = 1.$

Let $\Gamma(p) = \int_0^\infty u^{p-1}e^{-u}\,du$ define the gamma function. Integrating by parts and applying L'Hôspital's Rule, it can be shown that $\Gamma(p+1) = p \cdot \Gamma(p)$.

Make a variable change $u = v^2$ so that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Working with $\Gamma(p)$, let $u = v^2$ so $du = 2v\,dv$ and

$$\Gamma(p) = 2\int_0^\infty (v^2)^{p-1} e^{-v^2} v\,dv \;=\; 2\int_0^\infty v^{2p-1} e^{-v^2}\,dv.$$

$$\text{Then } \Gamma(a)\cdot\Gamma(b) = 4\int_0^\infty s^{2a-1} e^{-s^2}\,ds \int_0^\infty t^{2b-1} e^{-t^2}\,dt$$

$$= 4\int_0^\infty \int_0^\infty s^{2a-1} t^{2b-1} e^{-(s^2+t^2)}\,ds\,dt$$

Let $s = \rho\cos\theta$, $t = \rho\sin\theta$, so the Jacobian $= \rho$

$$\Gamma(a)\cdot\Gamma(b) = 4\int_0^\infty \int_0^{\pi/2} (\rho\cos\theta)^{2a-1} (\rho\sin\theta)^{2b-1} e^{-\rho^2} \rho\,d\theta\,d\rho$$

$$= 4\int_0^\infty \rho^{2a+2b-1} e^{-\rho^2}\,d\rho \int_0^{\pi/2} (\cos\theta)^{2a-1} (\sin\theta)^{2b-1}\,d\theta$$

Substitute $\rho^2 = y$ in the first integral and $\sin^2\theta = x$ in the second.

$$\Gamma(a)\Gamma(b) = \int_0^\infty y^{a+b-1} e^{-y}\,dy \int_0^1 x^{b-1}(1-x)^{a-1}\,dx$$

$$= \Gamma(a+b)\int_0^1 x^{b-1}(1-x)^{a-1}\,dx$$

so $\displaystyle\int_0^1 x^{b-1}(1-x)^{a-1}\,dx = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)} \Rightarrow k = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)}$

(b) $E(X^k) = \dfrac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \displaystyle\int_0^1 x^{b+k-1}(1-x)^{a-1}$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \int_0^1 x^{(b+k)-1}(1-x)^{a-1}\,dx$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} \cdot \frac{\Gamma(a)\Gamma(b+k)}{\Gamma(a+b+k)}.$$

Since $\Gamma(1) = 1$ we obtain $\Gamma(p+1) = p!$ for p a positive integer.

Then $E(X) = \dfrac{(a+b-1)!}{(a-1)!(b-1)!} = \dfrac{b}{a+b}$

$$E(X^2) = \frac{[(a+b)-1]!}{(a-1)!(b-1)!} \cdot \frac{(a-1)!(b+1)!}{(a+b+1)!} = \frac{b(b+1)}{(a+b+1)(a+b)}$$

so $V(X) = \dfrac{ab}{(a+b)^2(a+b+1)}$

3–23. $M_X(t) = \dfrac{1}{2} + \dfrac{1}{4}e^t + \dfrac{1}{8}e^{2t} + \dfrac{1}{8}e^{3t}$

$$M'_X(t) = \frac{1}{4}e^t + \frac{1}{4}e^{2t} + \frac{3}{8}e^{3t}$$

$$M''_X(t) = \frac{1}{4}e^t + \frac{1}{2}e^{2t} + \frac{9}{8}e^{3t}$$

(a) $E(X) = M'_X(0) = \dfrac{7}{8}$

$$V(Y) = M''_X(0) - [M'_X(0)]^2 = \frac{15}{8} - \left(\frac{7}{8}\right)^2 = \frac{71}{64}$$

(b)

x	y	y	$p_Y(y)$
0	4	0	$\frac{1}{8}$
1	1	1	$\frac{3}{8}$
2	0	4	$\frac{1}{2}$
3	1	ow	0
ow	0		

$$
\begin{aligned}
F_Y(y) &= 0; y < 0 \\
&= \frac{1}{8}; 0 \le y < 1 \\
&= \frac{1}{2}; 1 \le y < 4 \\
&= 1; y \ge 4
\end{aligned}
$$

3–24. $\mu_3 = E(X - \mu_1')^3$

$= E(X^3) - 3\mu_1' \cdot E(X^2) + 3\mu_1' \cdot E(X) - (\mu_1')^3$

$= \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3.$

A r.v. is symmetric about a if $f(x+a) = f(-x+a)$ and we assume X continuous. Since $X - a$ and $a - X$ have the same p.d.f., $E(X-a) = E(a-X)$ or $E(X) - a = a - E(X)$, so $E(X) = a = \mu_1'$. Now,

$$
\begin{aligned}
E(X-\mu_1')^r = E(X-a)^r &= \int_{-\infty}^{\infty} (x-a)^r f(x)\,dx \\
&= \int_{-\infty}^{a} (x-a)^r f(x)\,dx + \int_{0}^{\infty} (x-a)^r f(x)\,dx \\
&= \int_{-\infty}^{0} y^r f(y+a)\,dy + \int_{0}^{\infty} y^r f(y+a)\,dy \\
&= -\int_{-\infty}^{0} (1-y)^r f(-y+a)\,dy + \int_{0}^{\infty} y^r f(y+a)\,dy \\
&= \int_{0}^{\infty} (-y)^r f(y+a)\,dy + \int_{0}^{\infty} y^r f(y+a)\,dy \\
&= (-1)^r \int_{0}^{\infty} y^r f(y+a)\,dy + \int_{0}^{\infty} y^r f(y+a)\,dy \\
&= 0 \text{ for odd } r.
\end{aligned}
$$

Thus $E(X - \mu_1')^3 = 0$.

3–25. If $\int_{-\infty}^{\infty} |x|^r f(x)\, dx = k < \infty$, then $\int_{-\infty}^{\infty} |x|^n f(x)\, dx < \infty$, where $0 \le n < r$.

Proof: If $|x| \le 1$, then $|x|^n \le 1$ and if $|x| > 1$, then $|x|^n \le |x|^r$.

Then

$$\begin{aligned}\int_{-\infty}^{\infty} |x|^n f(x)\, dx &= \int_{|x|\le 1} |x|^n \cdot f(x)\, dx + \int_{|x|>1} |x|^n \cdot f(x)\, dx \\ &\le \int_{|x|\le 1} 1 f(x)\, dx + \int_{|x|>1} |x|^r f(x)\, dx \le 1 + k < \infty\end{aligned}$$

3–26.
$$M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) \Rightarrow \psi_X(t) = \mu t + \frac{\sigma^2 t^2}{2}$$

$$\begin{aligned}d\psi_X(t)/dt|_{t=0} &= [\mu + \sigma^2 t]_{t=0} = \mu \\ d^2\psi_X(t)\, dt^2|_{t=0} &= \sigma^2|_{t=0} = \sigma^2 \\ d^r\psi_X(t)/dt^r|_{t=0} &= 0; r \ge 3\end{aligned}$$

3–27. From Table XV, using the first column with scaling: $u_1 = 0.10480$, $u_2 = 0.22368$, $u_3 = 0.24130$, $u_4 = 0.42167$, $u_5 = 0.37570$, $u_6 = 0.77921, \ldots, u_{20} = 0.07056$.

$$\begin{aligned}F_X(x) &= 0; \quad x < 0 \\ &= 0.5; \quad 0 \le x < 1 \\ &= 0.75; \quad 1 \le x < 2 \\ &= 0.875; \quad 2 \le x < 3 \\ &= 1; \quad x \ge 3\end{aligned}$$

So, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$, $x_5 = 0$, $x_6 = 2, \ldots, x_{20} = 0$

3–28. The c.d.f. is

$$F_T(t) \;=\; 1 - e^{-t/4}$$

Thus, we set

$$t \;=\; -4\ell\text{n}(1 - F_T(t)) \;=\; -4\ell\text{n}(1-u)$$

From Table XV, using the second column with scaling: $u_1 = 0.15011$, $u_2 = 0.46573$, $u_3 = 0.48360, \ldots, u_{10} = 0.36257$

So,

$$
\begin{aligned}
t_1 &= -4\ell\text{n}(0.84989) = 0.6506 \\
t_2 &= -4\ell\text{n}(0.53427) = 2.5074 \\
&\vdots \\
t_{10} &= -4\ell\text{n}(0.63143) = 1.8391
\end{aligned}
$$