Chapter 4

4–1. (a)

x	0	1	2	3	4	5
$p_X(x)$	27/50	11/50	6/50	3/50	2/50	1/50

y	0	1	2	3	4
$p_Y(y)$	20/50	15/50	10/50	4/50	1/50

(b)

y	0	1	2	3	4
$p_{Y\|0}(y)$	11/27	8/27	4/27	3/27	1/27

(c)

x	0	1	2	3	4	5
$p_{X\|0}(x)$	11/20	4/20	2/20	1/20	1/20	1/20

4–2. (a)

y	1	2	3	4	5	6	7	8	9
$p_Y(y)$	44/100	26/100	12/100	8/100	4/100	2/100	2/100	1/100	1/100

x	1	2	3	4	5	6
$p_X(x)$	26/100	21/100	17/100	15/100	11/100	10/100

(b)

	y	1	2	3	4	5	6	7	8	9
$x = 1$	$p_{Y\|1}(y)$	10/26	6/26	3/26	2/26	1/26	1/26	1/26	1/26	1/26
$x = 2$	$p_{Y\|2}(y)$	8/21	5/21	3/21	2/21	1/21	1/21	1/21	0	0
$x = 3$	$p_{Y\|3}(y)$	8/17	5/17	2/17	1/17	1/17	0	0	0	0
$x = 4$	$p_{Y\|4}(y)$	7/15	4/15	2/15	1/15	1/15	0	0	0	0
$x = 5$	$p_{Y\|5}(y)$	6/11	3/11	1/11	1/11	0	0	0	0	0
$x = 6$	$p_{Y\|6}(y)$	5/10	3/10	1/10	1/10	0	0	0	0	0

4–3. (a) $\displaystyle\int_0^{10}\int_0^{100} \frac{k}{1000}\,dx_1\,dx_2 = 1 \Rightarrow k = 1$

(b)
$$f_{X_1}(x_1) = \int_0^{10} \frac{1}{1000}\,dx_2 = \frac{1}{100}; \quad 0 \le x_1 \le 100$$
$$= 0; \quad \text{otherwise}$$

$$f_{X_2}(x_2) = \int_0^{100} \frac{1}{1000}\,dx_1 = \frac{1}{1000}x_1\Big|_0^{100} = \frac{1}{10}; \quad 0 \le x_2 \le 10$$
$$= 0; \quad \text{otherwise}$$

(c) $F_{X_1,X_2}(x_1,x_2) = 0; x_1 \leq 0, x_2 \leq 0$

$$
\begin{aligned}
&= 0; x_1 \leq 0, x_2 > 0 \\
&= 0; x_1 > 0, x_2 \leq 0 \\
&= \int_0^{x_1}\int_0^{x_2} \frac{1}{1000}\,dt_2\,dt_1 = \frac{x_1x_2}{1000}; 0 < x_1 < 100, 0 < x_2 < 10 \\
&= \int_0^{x_1}\int_0^{10} \frac{dt_2\,dt_1}{1000} = \frac{x_1}{100}; 0 < x_1 < 100, x_2 \geq 10 \\
&= \int_0^{x_2}\int_0^{100} \frac{dt_1\,dt_2}{1000} = \frac{x_2}{10}; x_1 \geq 100, 0 < x_2 < 10 \\
&= 1; x_1 \geq 100, x_2 \geq 10.
\end{aligned}
$$

4–4. (a) $k\int_2^4\int_0^2 (6 - x_1 - x_2)\,dx_1\,dx_2 = 1 \Rightarrow k = \frac{1}{8}$

(b) $P(X_1 < 1, X_2 < 3) = \int_2^3\int_0^1 \frac{1}{8}(6 - x_1 - x_2)\,dx_1\,dx_2 = \frac{3}{8}$

(c) $P(X_1 + X_2 \leq 4) = \frac{1}{8}\int_2^4\int_0^{4-x_2} (6 - x_1 - x_2)\,dx_1\,dx_2 = \frac{2}{3}$

(d) $f_{X_1}(x_1) = \frac{1}{8}\int_2^4 (6 - x_1 - x_2)\,dx_2 = \frac{1}{8}(6 - 2x_1); 0 \leq x_1 \leq 2$

$P(X_1 < 1.5) = \int_0^{3/2} \frac{1}{8}(6 - 2x_1)\,dx_1 \doteq 0.844$

(e) $f_{X_1}(x_1)$ see (d)

$f_{X_2}(x_2) = \int_0^2 \frac{1}{8}(6 - x_1 - x_2)\,dx_1 = \frac{1}{8}(10 - 2x_2); 2 \leq x_2 \leq 4$

4–5. (a) $P\left(W \leq \frac{2}{3}, Y \leq \frac{1}{2}\right) = \int_0^{2/3}\int_0^1\int_0^{1/2}\int_0^1 16wxyz\,dz\,dy\,dx\,dw = \frac{1}{9}$

(b) $P\left(X \leq \frac{1}{2}, Z \leq \frac{1}{4}\right) = \int_0^1\int_0^{1/2}\int_0^1\int_0^{1/4} 16wxyz\,dz\,dy\,dx\,dw = \frac{1}{64}$

(c) $\int_0^1\int_0^1\int_0^1 16wxyz\,dz\,dy\,dx = 2w; 0 \leq w \leq 1$

$= 0;$ otherwise

Note: x, y, z and w are independent

4–6. From Problem 4–4

$$f_X(x) = \frac{1}{8}(6-2x); 0 \le x \le 2$$

$$f_Y(y) = \frac{1}{8}(10-2y); 2 \le y \le 4$$

$$f_{X|y}(y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{6-x-y}{10-2y}; 0 \le x \le 2, 2 \le y \le 4$$

$$f_{Y|x}(x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{6-x-y}{6-2x}; 2 \le y \le 4, 0 \le x \le 2$$

4–7. $E(Y|X=3) = \sum_{y=1}^{9} y \cdot p_{Y|3}(y) = \frac{33}{17}$

4–8. (a)

x_1	51	52	53	54	55
$p_{X_1}(x_1)$	0.28	0.28	0.22	0.09	0.13

x_2	51	52	53	54	55
$p_{X_2}(x_2)$	0.18	0.15	0.35	0.12	0.20

(b)

x_2	51	52	53	54	55	otherwise
$P_{X_2\|51}(x_2)$	$\frac{6}{28}$	$\frac{7}{28}$	$\frac{5}{28}$	$\frac{5}{28}$	$\frac{5}{28}$	0
$P_{X_2\|52}(x_2)$	$\frac{5}{28}$	$\frac{5}{28}$	$\frac{10}{28}$	$\frac{2}{28}$	$\frac{6}{28}$	0
$P_{X_2\|53}(x_2)$	$\frac{5}{22}$	$\frac{1}{22}$	$\frac{10}{22}$	$\frac{1}{22}$	$\frac{5}{22}$	0
$P_{X_2\|54}(x_2)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0
$P_{X_2\|55}(x_2)$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{5}{13}$	$\frac{3}{13}$	$\frac{3}{13}$	0

$$E(X_2|x_1=51) = \sum_{x_2=51}^{55} x_2 \cdot p_{X_2|51}(x_2) = 52.857$$

$$E(X_2|x_1=52) = \sum_{x_2=51}^{55} x_2 \cdot p_{X_2|52}(x_2) = 52.964$$

$$E(X_2|x_1=53) = \sum_{x_2=51}^{55} x_2 \cdot p_{X_2|53}(x_2) = 53.0$$

$$E(X_2|x_1 = 54) = \sum_{x_2=51}^{55} x_2 \cdot p_{X_2|54}(x_2) = 53.0$$

$$E(X_2|x_1 = 55) = \sum_{x_2=51}^{55} x_2 \cdot p_{X_2|55}(x_2) = 53.461$$

4–9. $$f_{X_1}(x_1) = \int_0^1 6x_1^2 x_2\, dx_2 = 3x_1^2 x_2^2|_{x_2=0}^1 = 3x_1^2; 0 \le x_1 \le 1$$

$$f_{X_2}(x_2) = \int_0^1 6x_1^2 x_2\, dx_1 = 2x_1^3 x_2|_{x_1=0}^1 = 2x_2; 0 \le x_2 \le 1$$

$$f_{X_1|x_2}(x_1) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)} = \frac{6x_1^2 x_2}{2x_2} = 3x_1^2; 0 \le x_1 \le 1$$

$$f_{X_2|x_1}(x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_1}(x_1)} = 2x_2; 0 \le x_2 \le 1$$

$$E(X_1|x_2) = \int_0^1 x_1 f_{X_1|x_2}(x_1)\, dx_1 = \frac{3}{4}$$

$$E(X_2|x_1) = \int_0^1 x_2 f_{X_2|x_1}(x_2)\, dx_2 = \frac{2}{3}$$

4–10. (a) $f_{X_1}(x_1) = 2x_1 e^{-x_1^2}; \quad x_1 \ge 0$

$f_{X_2}(x_2) = 2x_2 e^{-x_2^2}; \quad x_2 \ge 0$

(b) $$f_{X_1|x_2}(x_1) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)} = 2x_1 e^{-x_1^2}; x_1 \ge 0$$

$$f_{X_2|x_1}(x_2) = 2x_2 e^{-x_2^2}; x_2 \ge 0$$

Since $f_{X_1|x_2}(x_1) = f_{X_1}(x_1)$ and $f_{X_2|x_1}(x_2) = f_{X_2}(x_2)$, X_1 and X_2 are independent

(c) $$E(X_1|x_2) = E(X_1) = \int_0^\infty x_1(2x_1 e^{-x_1^2})\, dx_1 = \int_0^\infty 2x_1^2 e^{-x_1^2}\, dx_1$$

Let $u = x_1$, $du = dx_1$, $dv = 2x_1 e^{-x_1^2}$, $v = -e^{-x_1^2}$, and integrate by parts, so

$$E(X_1) = -x_1 e^{-x_1^2}|_0^\infty + \int_0^\infty e^{-x_1^2}\, dx_1 = 0 + \frac{\pi}{4}$$

Similarly, $E(X_2) = \pi/4$.

4–11. $z = xy,\ t = x \Rightarrow x = t,\ y = \dfrac{z}{t}$

$$\begin{vmatrix} \partial x/\partial t & \partial x/\partial z \\ \partial y/\partial t & \partial y/\partial z \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\dfrac{z}{t^2} & \dfrac{1}{t} \end{vmatrix} = \frac{1}{t}$$

so $p(z,t) = g(t)h(z/t)|1/t|$

Thus $\ell(z) = \int_{-\infty}^{\infty} g(t)h(z/t)|1/t|\, dt$

4–12. $a = s_1 \cdot s_2,\ t = s_1$ so $s_2 = \dfrac{a}{t},\ s_1 = t$

Jacobian is $\frac{1}{t}$

And $p(a,t) = (2t)(\frac{1}{8})(\frac{a}{t}) \cdot \frac{1}{t} = \frac{a}{4t}$

$$\therefore \ell(a) = \int_{a/4}^{1} \frac{a}{4t} dt = \frac{a}{4}\, \ell\text{n}(t)\, \Big|_{a/4}^{1}$$

$$= -\frac{a}{4}\, \ell\text{n}\, \frac{a}{4}; 0 < a < 4$$

4–13. $f(x,y) = g(x) \cdot h(y)$

$Z = \dfrac{X}{Y} \quad U = Y \quad y = u \quad x = uz$

$$\begin{vmatrix} \partial x/\partial u & \partial x/\partial z \\ \partial y/\partial u & \partial y/\partial z \end{vmatrix} = \begin{vmatrix} z & u \\ 1 & 0 \end{vmatrix} = u$$

$p(z,u) = g(uz) \cdot h(u)|u|$

$$\ell(z) = \int_{-\infty}^{\infty} g(uz)h(u)|u|\, du$$

4–14. $f(v,i) = 3e^{-3i} \cdot e^{-v}$

$r = \dfrac{v}{i} \quad u = i \quad v = ru \quad i = u$

The Jacobian is u, thus

$$p(r,u) = (e^{-ru})(3e^{-3u})(u) = 3ue^{-u(3+r)}; u \geq 0$$

$$\therefore \ell(r) = 3\int_0^{\infty} ue^{-u(3+r)}\, du = \frac{3}{(3+r)^2}; r \geq 0$$

4–15. $Y = X_1 + X_2 + X_3 + X_4$

$E(Y) = 80, V(Y) = 4(9) = 36$

4–16. If X_1 and X_2 are independent, then

$$p_{X_1,X_2}(x_1,x_2) = p_{X_1}(x_1)p_{X_2}(x_2) \quad \text{for all } x_1, x_2.$$

Thus,

$$p_{X_2|x_1}(x_2) = \frac{p_{X_1,X_2}(x_1,x_2)}{p_{X_1}(x_1)} = \frac{p_{X_1}(x_1)p_{X_2}(x_2)}{p_{X_1}(x_1)} = p_{X_2}(x_2) \quad \text{for all } x_1, x_2.$$

Similarly,

$$p_{X_1|x_2}(x_1) = p_{X_1}(x_1).$$

Now prove the converse. Assume that

$$p_{X_2|x_1}(x_2) = p_{X_2}(x_2) \quad \text{and} \quad p_{X_1|x_2}(x_1) = p_{X_1}(x_1) \quad \text{for all } x_1, x_2.$$

Then

$$p_{X_1,X_2}(x_1,x_2) = p_{X_2|x_1}(x_2)p_{X_1}(x_1) = p_{X_2}(x_2)p_{X_1}(x_1) \quad \text{for all } x_1, x_2.$$

4–17. $X_2 = A + BX_1 \Rightarrow E(X_2) = A + B \cdot E(X_1)$

$$\begin{aligned} V(X_2) = B^2 \cdot V(X_1), E(X_1X_2) &= E(X_1(A + BX_1)) \\ &= AE(X_1) + BE(X_1^2) \end{aligned}$$

So

$$\begin{aligned} \rho^2 &= \frac{[E(X_1X_2) - E(X_1)E(X_2)]^2}{V(X_1) \cdot V(X_2)} \\ &= \frac{[AE(X_1) + BE(X_1^2) - E(X_1)(A + B \cdot E(X_1))]^2}{V(X_1) \cdot B^2 V(X_1)} \\ &= \frac{[BE(X_1^2) - B(E(X_1))^2]^2}{B^2 V(X_1)^2} = \frac{B^2 V(X_1)^2}{B^2 V(X_1)^2} = 1 \end{aligned}$$

$\rho = -1$ if $B < 0$ and $\rho = 1$ if $B > 0$.

4–18. $M_{X_2}(t) = E(e^{tX_2}) = E(e^{t(A+BX_1)}) = E(e^{At} \cdot e^{BtX_1})$
$= e^{At} \cdot E(e^{BtX_1}) = e^{At} \cdot M_{X_1}(Bt)$

4–19. $f_{X_1}(x_1) = \int_{x_1}^{1} 2\,dx_2 = 2(1 - x_1); 0 \le x_1 \le 1$

$$f_{X_2}(x_2) = \int_{0}^{x_2} 2\,dx_1 = 2x_2; 0 \le x_2 \le 1$$

$$E(X_1) = 2\int_0^1 x_1(1 - x_1)\,dx_1 = \frac{1}{3}$$

$$E(X_2) = 2\int_0^1 x_2^2\,dx_2 = \frac{2}{3}$$

$$E(X_1^2) = 2\int_0^1 x_1^2(1 - x_1)\,dx_1 = 2\left[\frac{x_1^3}{3} - \frac{x_1^4}{4}\right]_0^1 = \frac{1}{6}$$

$$E(X_2^2) = 2\int_0^1 x_2^3\,dx_2 = \frac{1}{2}$$

$$V(X_1) = E(X_1^2) - (E(X_1))^2 = \frac{1}{18}$$

$$V(X_2) = E(X_2^2) - (E(X_2))^2 = \frac{1}{18}$$

$$E(X_1X_2) = \int_0^1 \int_0^{1/2} 2x_1x_2\,dx_1\,dx_2 = 2\int_0^1 x_2\left[\frac{x_1^2}{2}\right]_0^{x_2} dx_2$$
$$= \int_0^1 x_2^3\,dx_2 = \frac{1}{4}$$

$$\rho = \frac{E(X_1X_2) - E(X_1)E(X_2)}{\sqrt{V(X_1)V(X_2)}}$$
$$= \frac{\frac{1}{4} - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)}{\sqrt{\left(\frac{1}{18}\right)\left(\frac{1}{18}\right)}} = \frac{1}{2}$$

4–20. $E(U) = A + BE(X_1), E(V) = C + DE(X_2)$

$$V(U) = B^2 \cdot V(X_1), V(V) = D^2 V(X_2)$$

$$E(UV) = E[(A + BX_1)(C + DX_2)]$$

$$= AC + AD \cdot E(X_2) + BC \cdot E(X_1) + BD \cdot E(X_1 X_2)$$

$$\rho_{U,V} = \frac{E(UV) - E(U) \cdot E(V)}{\sqrt{V(U) \cdot V(V)}}$$

$$= \frac{AC + AD \cdot E(X_2) + BC \cdot E(X_1) + BD \cdot E(X_1X_2) - [(A + BE(X_1))(C + DE(X_2))]}{\sqrt{B^2 V(X_1) D^2 V(X_2)}}$$

$$= \frac{BD[E(X_1X_2) - E(X_1)E(X_2)]}{|BD|\sqrt{V(X_1)V(X_2)}} = \frac{BD}{|BD|}\rho_{X_1,X_2}$$

4–21. X and Y are not independent. Note

(a) $P_{X,Y}(5, 4) = 0 \neq p_X(5) \cdot p_Y(4) = \dfrac{1}{50^2}$.

(b)

$$E(X) = 0.9 \qquad E(Y) = 1.02$$
$$E(X^2) = 2.38 \qquad E(Y^2) = 2.14$$
$$V(X) = 1.57 \qquad V(Y) = 1.0996$$
$$E(XY) = 0.74$$

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sqrt{V(X)V(Y)}} = \frac{0.74 - (0.9)(1.02)}{\sqrt{(1.57)(1.0996)}} = -0.135$$

4–22. (a) A sale will take place if $Y \geq X$

(b) $\displaystyle\iint_{x \leq y} f(x, y)\, dx\, dy$

(c) $\displaystyle E(Y|Y \geq x) = \iint_{x \leq y} y f(x, y)\, dx\, dy \Big/ \iint_{x \leq y} f(x, y)\, dx\, dy$

4–23. (a) $\displaystyle f_X(x) = \int_0^{\sqrt{1-x^2}} \frac{2}{\pi} dy = \frac{2}{\pi} y \Big|_0^{\sqrt{1-x^2}} = \frac{2}{\pi}\sqrt{1 - x^2}; -1 < x < +1$

$$f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\pi} dx = \frac{4}{\pi}\sqrt{1 - y^2}; 0 < y < 1$$

(b) $$f_{X|y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{2}{\pi}}{\left(\frac{4}{\pi}\right)\sqrt{1-y^2}} = \frac{1}{2\sqrt{1-y^2}}; -\sqrt{1-y^2} < x < \sqrt{1-y^2}$$

$$f_{Y|x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{2}{\pi}}{\left(\frac{2}{\pi}\right)\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}; 0 < y < \sqrt{1-x^2}$$

(c) $$E(X|y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{x}{2\sqrt{1-y^2}} dx = \frac{1}{2\sqrt{1-y^2}} \left(\frac{x^2}{2} \Big|_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \right) = 0$$

$$E(Y|x) = \int_0^{\sqrt{1-x^2}} \frac{y}{\sqrt{1-x^2}} dy = \frac{1}{\sqrt{1-x^2}} \left(\frac{y^2}{2} \Big|_0^{\sqrt{1-x^2}} \right) = \frac{\sqrt{1-x^2}}{2}$$

4–24. X and Y continuous

$$E(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|y}(x)\, dx = \int_{-\infty}^{\infty} x \frac{f_{X,Y}(x,y)}{f_Y(y)} dx$$

$$= \int_{-\infty}^{\infty} \frac{f_X(x) f_Y(y)}{f_Y(y)} dx = \int_{-\infty}^{\infty} x f_X(x)\, dx = E(X)$$

Change X and Y, reversing roles to show $E(Y|X) = E(Y)$ if X and Y are independent.

4–25. $$E(X|y) = \int_{-\infty}^{\infty} x f_{X|y}(x)\, dx = \int_{-\infty}^{\infty} x \frac{f_{X,Y}(x,y)}{f_Y(y)} dx$$

$$E[E(X|Y)] = \int_{-\infty}^{\infty} E(X|y) \cdot f_Y(y)\, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[x \frac{f_{X,Y}(x,y)}{f_Y(y)} dx \right] f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \right] dx = \int_{-\infty}^{\infty} x f_X(x)\, dx = E(X).$$

Reverse roles of X and Y to show $E(E(Y|X)) = E(Y)$.

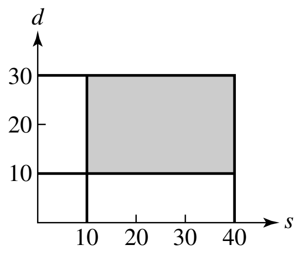
4–26. $w = s + d$, let $y = s - d$

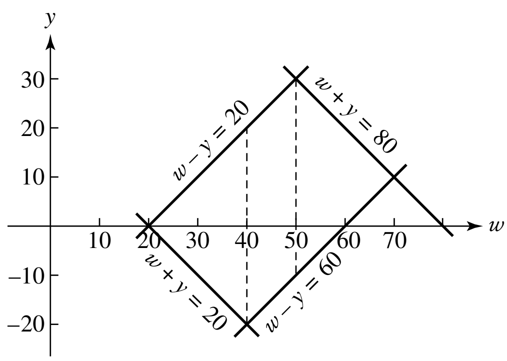
$$s = \frac{w+y}{2}, \quad d = \frac{w-y}{2}$$

$s = 10 \rightarrow w + y = 20$ $\qquad$ $d = 10 \rightarrow w - y = 20$

$s = 40 \rightarrow w + y = 80$ $\qquad$ $d = 30 \rightarrow w - y = 60$

As illustrated





$$\text{The Jacobian } J = \begin{vmatrix} \frac{\partial s}{\partial w} & \frac{\partial s}{\partial y} \\ \frac{\partial d}{\partial w} & \frac{\partial d}{\partial y} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

$$\therefore f_{W,Y}(w,y) = \frac{1}{30} \cdot \frac{1}{20} \cdot \frac{1}{2};\ 10 < \frac{w+y}{2} < 40,\ 10 < \frac{w-y}{2} < 30$$

$$f_W(w) = \int_{20-w}^{w-20} \frac{1}{1200} dy = \frac{w-20}{600}; 20 < w < 40$$

$$= \int_{w-60}^{w-20} \frac{1}{1200} dy = \frac{20}{600}; 40 < w \leq 50$$

$$= \int_{w-60}^{80-w} \frac{1}{1200} dy = \frac{70-w}{600}; 50 < w < 70$$

$$= 0; ow$$

4–27. (a) $f_Y(y) = \int_0^1 (x+y)\, dx = y + \frac{1}{2}; 0 < y < 1$

$$f_{X|y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{x+y}{y+\frac{1}{2}}; 0 < x < 1, 0 < y < 1$$

$$E(X|Y=y) = \int_0^1 x \cdot \frac{(x+y)}{\left(y+\frac{1}{2}\right)} dx = \frac{1}{y+\frac{1}{2}} \left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_0^1$$

$$= \frac{2+3y}{3(1+2y)}; 0 < y < 1.$$

(b) $f_X(x) = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_{y=0}^1 = x + \frac{1}{2}; 0 < x < 1$

$$E(X) = \int_0^1 [x^2 + (x/2)]\,dx = [(x^3/3) + (x^2/4)]_0^1 = \frac{7}{12}$$

(c) $E(Y) = \int_0^1 [y^2 + (y/2)]\,dy = [(y^3/3) + (y^2/4)]_0^1 = \frac{7}{12}$

4–28. (a) $\int_0^\infty \int_0^\infty \frac{k(1+x+y)}{(1+x)^4(1+y)^4} dx\,dy = 1 \Rightarrow k = \frac{9}{2}$

(b) $f_X(x) = \int_0^\infty \frac{(9/2)(1+x+y)\,dy}{(1+x)^4(1+y)^4} = \frac{(9/2)}{(1+x)^4} \int_0^\infty \frac{dy}{(1+y)^4}$

$$+ \frac{(9/2)x}{(1+x)^4} \int_0^\infty \frac{dy}{(1+y)^4} + \frac{(9/2)}{(1+x)^4} \int_0^\infty \frac{y\,dy}{(1+y)^4}$$

$$= \frac{3(3+2x)}{4(1+x)^4}; x > 0$$

4–29. (a) $k \int_0^\infty \int_0^\infty (1+x+y)^{-n}\,dx\,dy = 1, \quad k \int_0^\infty \left[\frac{(1+x+y)^{-n+1}}{(-n+1)}\right]_0^\infty dy = 1$

so $\frac{k}{n-1} \int_0^\infty \frac{dy}{(1+y)^{n-1}} = 1 \Rightarrow \frac{k}{(n-1)(n-2)} = 1 \Rightarrow k = (n-1)(n-2)$

(b) $F_{X,Y}(x,y) = P(X \le x, Y \le y) = k \int_0^x \int_0^y (1+s+t)^{-n}\,dt\,ds$

$$= k \left[\frac{(1+s+y)^{-n+2}}{(-n+2)(-n+1)} - \frac{(1+s)^{-n+2}}{(-n+2)(-n+1)}\right]_{s=0}^x$$

$$= 1 - (1+x)^{2-n} - (1+y)^{2-n} + (1+x+y)^{2-n}; x > 0, y > 0$$

4–30. From Eq. 4–58

$$n \geq \frac{p(1-p)}{\epsilon^2 \cdot \alpha} = \frac{(0.5)(0.5)}{(0.05)^2(0.05)} = 2000$$

4–31. (a) $g(x, y) = (2xe^{-x^2})(2ye^{-y^2})$ so X and Y are independent

(b) X and Y are not independent. For given X, Y may only assume values greater than that value of X.

(c) See Prob. 4–29 with $k = (4-1)(4-2) = 6$ and $n = 4$.

$$f_X(x) = 2(1+x)^{-3}; x > 0$$
$$f_Y(y) = 2(1+y)^{-3}; y > 0$$

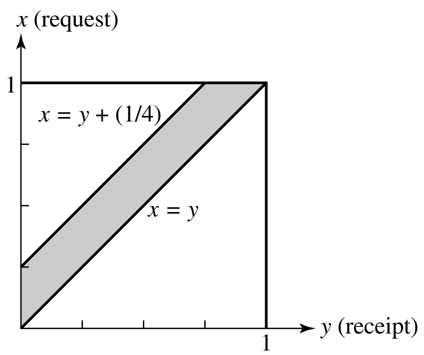
$f_{X,Y}(x, y) \neq f_X(x) \cdot f_Y(y)$ so the variables are not independent.

4–32. The probability is 2/3 since $P(X > Y > Z) = P(X < Y < Z) = \frac{1}{6}$

4–33. (a) $P\left(0 \leq X \leq \frac{1}{2}, 0 \leq Y \leq \frac{1}{2}\right) = \int_0^{1/2} \int_0^{1/2} 1\,dx\,dy = \frac{1}{4}$

(b) $P(X > Y) = \int_0^1 \int_0^x 1\,dy\,dx = \int_0^1 x\,dx = \frac{1}{2}.$

4–34. $P\left(0 \leq Y \leq 1, Y \leq X \leq Y + \frac{1}{4} \leq 1\right) = \frac{7}{32}$



4–35. (a) $F_Z(z) = P(Z \leq z) = P(a + bX \leq z) = P\left(X \leq \frac{z-a}{b}\right)$

$$= F_X\left(\frac{z-a}{b}\right)$$

$$f_Z(z) = f_X\left(\frac{z-a}{b}\right)\left|\frac{1}{b}\right|.$$

(b) $F_Z(z) = P\left(\frac{1}{X} \le z\right) = P\left(X \ge \frac{1}{z}\right) = 1 - F_X\left(\frac{1}{z}\right)$

$f_Z(z) = f_X\left(\frac{1}{z}\right)\frac{1}{z^2}$

(c) $F_Z(z) = P(\ell\text{n}(X) \le Z) = P(e^{\ell n(x)} \le e^z) = P(X \le e^z) \ = \ F_X(e^z)$
$f_Z(z) = f_X(e^z)e^z$

(d) $F_Z(z) = P(e^X \le z) = P(X \le \ell\text{n}\, z) = F_X(\ell\text{n}\, z)$

$f_Z(z) = f_X(\ell\text{n}(z))\left|\frac{1}{z}\right|$

The range space for Z is determined from the range space of X and the definition of the transformation.