Chapter 5

5–1. The probability mass function is

x	$p(x)$
0	$(1-p)^4$
1	$4p(1-p)^3$
2	$6p^2(1-p)^2$
3	$4p^3(1-p)$
4	p^4
otherwise	0

5–2.

$$\begin{aligned} P(X \geq 5) &= \sum_{x=5}^{6} \binom{6}{x} (0.95)^x (0.05)^{6-x} \\ &= 6(0.95)^5(0.05) + (0.95)^6 \\ &= 0.9672 \end{aligned}$$

5–3. Assume independence and let W represent the number of orders received.

$$\begin{aligned} P(W \geq 4) &= \sum_{w=4}^{12} \binom{12}{w} (0.5)^w (0.5)^{12-w} \\ &= (0.5)^{12} \sum_{w=4}^{12} \binom{12}{w} \\ &= 1 - (0.5)^{12} \sum_{w=0}^{3} \binom{12}{w} \ = \ 0.9270 \end{aligned}$$

5–4. Assume customer decisions are independent.

$$P(X \geq 10) \ = \ 1 - P(X \leq 9) \ = \ 1 - \sum_{x=0}^{9} \binom{20}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{20-x} \ = \ 0.0918$$

5–5.

$$P(X > 2) \ = \ 1 - P(X \leq 2) \ = \ 1 - \sum_{x=0}^{2} \binom{50}{x} (0.02)^x (0.98)^{50-x} \ = \ 0.0784.$$

5–6.

$$\begin{aligned} M_X(t) &= E[e^{tX}] = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x} \\ &= (pe^t + q)^n, \quad \text{where } q = 1-p \end{aligned}$$

$$E[X] = M_X'(0) = [n(pe^t+q)^{n-1}pe^t]_{t=0} = np$$

$$\begin{aligned} E[X^2] &= M_X''(0) = np[e^t(n-1)(pe^t+q)^{n-2}(pe^t) + (pe^t+q)^{n-1}e^t]_{t=0} \\ &= (np)^2 - np^2 + np \end{aligned}$$

$$V(X) = E[X^2] - (E[X])^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq$$

5–7.

$$\begin{aligned} P(\hat{p} \leq 0.03) &= P\left(\frac{X}{100} \leq 0.03\right) = P(X \leq 3) \\ &= \sum_{x=0}^{3} \binom{100}{3} (0.01)^x (0.99)^{100-x} = 0.9816 \end{aligned}$$

5–8.

$$\begin{aligned} P(\hat{p} > p + \sqrt{pq/n}) &= P\left(\hat{p} > 0.07 + \sqrt{(0.07)(0.93)/200}\right) \\ &= P(X > 200(0.088)) \\ &= P(X > 17.6) \\ &= 1 - P(X \leq 17.6) \\ &= 1 - \sum_{x=0}^{17} \binom{200}{x} (0.07)^x (0.93)^{200-x} \\ &= 0.1649 \end{aligned}$$

By two standard deviations,

$$\begin{aligned} P(\hat{p} > p + 2\sqrt{pq/n}) &= P(\hat{p} > 0.106) \\ &= P(X > 21.2) \\ &= 1 - P(X \leq 21) \\ &= 1 - \sum_{x=0}^{21} \binom{200}{x} (0.07)^x (0.93)^{200-x} \\ &= 0.0242 \end{aligned}$$

By three standard deviations,

$$\begin{aligned} P(\hat{p} > p + 3\sqrt{pq/n}) &= P(X > 24.8) \\ &= 1 - P(X \leq 24) \\ &= 1 - \sum_{x=0}^{24} \binom{200}{x} (0.07)^x (0.93)^{200-x} \\ &= 0.0036 \end{aligned}$$

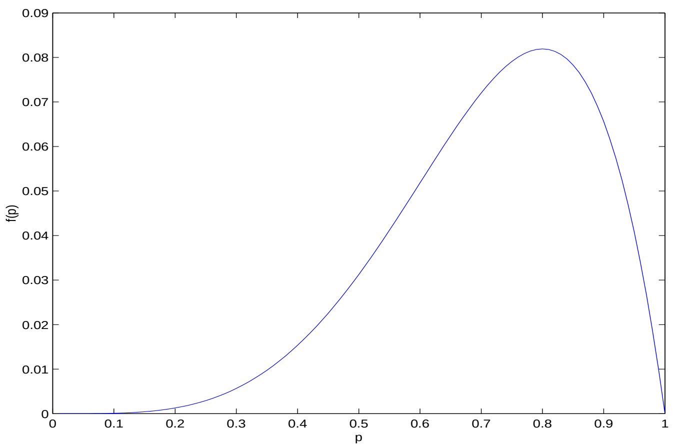
5–9. $P(X = 5) = (0.95)^4(0.05) = 0.0407$

5–10. A: Successful on first three calls, B: Unsuccessful on fourth call

$$P(B|A) = P(B) = 0.90 \quad \text{if } A \text{ and } B \text{ are independent}$$

5–11.

$$\begin{aligned} P(X = 5) &= p^4(1-p) = f(p) \\ \frac{df(p)}{dp} &= 5p^4 - 4p^3 = 0 \Rightarrow p = \frac{4}{5} \end{aligned}$$



5–12. (a) X = trials required up to and including the first sale

$$\begin{aligned} C(X) &= 1000X + 3000(X-1) \\ &= 4000X - 3000 \end{aligned}$$

$$E[C(X)] = 4000E[X] - 3000 = 4000\left(\frac{1}{0.1}\right) - 3000 = 37000$$

(b) Since \$37000 > \$15000, the trips should not be undertaken.

(c)

$$\begin{aligned} P(C(X) > 100000) &= P(4000X - 3000 > 100000) \\ &= P\left(X > \frac{103000}{4000}\right) \doteq P(X > 25.75) \\ &= 1 - P(X \le 25) \\ &= 1 - \sum_{x=1}^{25} (0.90)^{x-1}(0.10) \end{aligned}$$

5–13.

$$M_X(t) = \frac{pe^t}{1 - qe^t}, \quad \text{where } q = 1 - p$$

$$\begin{aligned} E[X] &= M_X'(0) = \left[\frac{(1-qe^t)(pe^t) + (pe^t)(qe^t)}{(1-qe^t)^2}\right]_{t=0} \\ &= \frac{(1-q)p + pq}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} \end{aligned}$$

$$\begin{aligned} E[X^2] &= M_X''(0) = \left[\frac{(1-qe^t)^2(pe^t) + 2pe^t(1-qe^t)(qe^t)}{(1-qe^t)^4}\right]_{t=0} \\ &= \frac{(1+q)p}{(1-q)^3} = \frac{1+q}{p^2} \end{aligned}$$

$$V(X) = \frac{q}{p^2}$$

5–14.

$$\begin{aligned} P(X \leq 2) &= P(X = 1) + P(X = 2) = 0.8 + (0.2)(0.8) = 0.96 \\ P(X \leq 3) &= \sum_{x=1}^{3} (0.2)^{x-1}(0.8) = 0.992 \end{aligned}$$

5–15.

$$P(X = 36) = (0.95)^{35}(0.05) = 0.0083$$

5–16. (a)

$$P(X = 8) = \binom{7}{2}(0.1)^2(0.9)^5 = 0.0124$$

(b)

$$P(X > 8) = \sum_{x=9}^{\infty} \binom{x-1}{2}(0.1)^2(0.9)^{x-3}$$

5–17.

$$P(X = 4) = \binom{3}{1}(0.8)^2(0.2)^2 = 0.0768$$

$$P(X < 4) = P(X = 2) + P(X = 3) = (0.8)^2 + \binom{2}{1}(0.8)^2(0.2) = 0.896$$

5–18. Suppose $X_1, X_2, \ldots, X_r$ are independent geometric random variables, each with parameter p. X_1 is the number of trials to first success, X_2 the number of trials from first to the second, etc. Let

$$X = X_1 + X_2 + \ldots + X_r$$

The moment generating function for the geometric is $\frac{pe^t}{1-qe^t}$, so

$$M_X(f) = \prod_{i=1}^{r} M_{X_i}(t) = \left[\frac{pe^t}{1-qe^t}\right]^r$$

$$E[X] = M_X'(t)|_{t=0} = \frac{r}{p}$$

We could also have obtained this result as follows.

$$E[X] = \sum_{i=1}^{r} E[X_i] = r\left(\frac{1}{p}\right) = \frac{r}{p}$$

Continuing,

$$V(X) = \left[M_X''(t)|_{t=0}\right] - \left(\frac{r}{p}\right)^2 = \frac{rq}{p^2}$$

We could also have obtained this result as follows.

$$V(X) = \sum_{i=1}^{r} V(X_i) = \frac{rq}{p^2}$$

5–19.

$$E[X] = \frac{r}{p} = \frac{5}{0.8} = 6.25, \quad V(X) = \frac{rq}{p^2} = \frac{(5)(0.2)}{(0.8)^2} = 1.5625$$

5–20. $X =$ Mission number on which 4th hit occurs.

$$p(x) = \begin{cases} \binom{x-1}{3}(0.8)^4(0.2)^{x-4} & x = 4, 5, 6, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$P(X \leq 7) = \sum_{x=4}^{7} \binom{x-1}{3}(0.8)^4(0.2)^{x-4}$$

5–21. $(X, Y, Z) \sim$ multinomial $(n = 3, p_1 = 0.4, p_2 = 0.3, p_3 = 0.3)$. The probability that one company receives all orders is

$$\begin{aligned} & P(3,0,0) + P(0,3,0) + P(0,0,3) \\ = \; & \frac{3!}{3!0!0!}(0.4)^3(0.3)^0(0.3)^0 + \frac{3!}{0!3!0!}(0.4)^0(0.3)^3(0.3)^0 + \frac{3!}{0!0!3!}(0.4)^0(0.3)^0(0.3)^3 \\ = \; & 0.4^3 + 0.3^3 + 0.3^3 \; \doteq \; 0.118 \end{aligned}$$

5–22. (a) $(X_1, X_2, X_3, X_4) \sim$ multinomial $(n = 5, p_1 = p_2 = p_3 = p_4 = \frac{1}{4})$. Therefore, $P(5,0,0,0) + P(0,5,0,0) + P(0,0,5,0) + P(0,0,0,5)$ is the probability that one company gets all five. That is,

$$4\left[\frac{5!}{5!0!0!0!}\right]\left(\frac{1}{4}\right)^5\left(\frac{1}{4}\right)^0\left(\frac{1}{4}\right)^0\left(\frac{1}{4}\right)^0 = \frac{1}{256}$$

(b)

$$1 - \left[4(60)\left(\frac{1}{4}\right)^5\right] \doteq 0.7656$$

5–23.

$$P(Y_1 = 4, Y_2 = 1, Y_3 = 3, Y_4 = 2) = \frac{10!}{4!1!3!2!}(0.2)^4(0.2)^1(0.2)^3(0.4)^2 \doteq 0.005$$

5–24.

$$P(Y_1 = 0, Y_2 = 0, Y_3 = 0, Y_4 = 10) = \frac{10!}{0!0!0!10!}(0.2)^0(0.2)^0(0.2)^0(0.4)^{10}$$

$$P(Y_1 = 5, Y_2 = 0, Y_3 = 0, Y_4 = 5) = \frac{10!}{5!0!0!5!}(0.2)^5(0.2)^0(0.2)^0(0.4)^5$$

5–25. (a)

$$P(X \le 2) = \sum_{x=0}^{2} \frac{\binom{4}{x}\binom{21}{5-x}}{\binom{24}{5}} \doteq 0.98$$

(b)

$$P(X \le 2) = \sum_{x=0}^{2} \binom{5}{x}\left(\frac{4}{25}\right)^x\left(\frac{21}{25}\right)^{5-x} \doteq 0.97$$

5–26. The approximation improves as $\frac{n}{N}$ decreases. $n = 5, N = 100$ is a better condition than $n = 5, N = 25$.

5–27. we want the smallest n such that

$$P(X \geq 1) \;=\; 1 - \frac{\binom{7}{0}\binom{18}{n}}{\binom{25}{n}} \;\geq\; 0.95 \qquad \Leftrightarrow \qquad \frac{\binom{18}{n}}{\binom{25}{n}} \;\leq\; 0.05.$$

By trial and error, we find that $n = 8$ does the job.

We could instead use the binomial approximation; now we want n such that

$$0.05 \;\geq\; \binom{n}{0}\left(\frac{7}{25}\right)^0\left(\frac{18}{25}\right)^n \;=\; \left(\frac{18}{25}\right)^n.$$

We find that $n \doteq 9$.

5–28.

$$M_X(t) \;=\; E[e^{tX}] \;=\; \sum_{x=0}^{\infty} e^{tx}\frac{e^{-c}c^x}{x!} \;=\; e^{-c}\sum_{x=0}^{\infty}\frac{(ce^t)^x}{x!} \;=\; e^{-c}e^{ce^t} \;=\; e^{c(e^t-1)}$$

5–29.

$$P(X < 10) \;=\; P(X \leq 9) \;=\; \sum_{x=0}^{9}\frac{e^{-25}(25)^x}{x!} \;\doteq\; 0.0002$$

5–30.

$$\begin{aligned} P(X > 20) = P(X \geq 21) \;&=\; \sum_{x=21}^{\infty}\frac{e^{-10}(10)^x}{x!} \\ &=\; 1 - P(X \leq 20) = 1 - \sum_{x=0}^{20}\frac{e^{-10}(10)^x}{x!} \\ &=\; 0.002 \end{aligned}$$

5–31.

$$\begin{aligned} P(X > 5) \;&=\; P(X \geq 6) \\ &=\; 1 - P(X \leq 5) = 1 - \sum_{x=0}^{5}\frac{e^{-4}4^x}{x!} \\ &\doteq\; 0.2149 \end{aligned}$$

5–32. Mean count rate $= (1-p)c$. Therefore,

$$P(Y_t = y) = \frac{e^{-[(1-p)c]t}[(1-p)ct]^y}{y!} \qquad y = 0, 1, 2, \ldots$$

5–33. Using a Poisson model,

$$\begin{aligned} P(X \leq 3) &= \sum_{x=0}^{3} \frac{e^{-\lambda}\lambda^x}{x!} \qquad \lambda = 15000(0.002) = 30 \\ P(X \geq 5) &= \sum_{x=5}^{\infty} \frac{e^{-30}(30)^x}{x!} = 1 - \sum_{x=0}^{4} \frac{e^{-30}(30)^x}{x!} \end{aligned}$$

5–34. $Y =$ Number of requests.

(a)
$$P(Y > 3) = 1 - P(Y \leq 3) = 1 - \sum_{y=0}^{3} \frac{e^{-2}2^y}{y!}$$

(b)
$$E[Y] = c = 2$$

(c)
$$P(Y \leq y) \geq 0.9 \quad \text{so } y = 4 \text{ and } P(Y \leq 4) = 0.9473$$

(d) $X =$ Number serviced.

y	x	$p(x)$	$xp(x)$
0	0	e^{-2}	0
1	1	$2e^{-2}$	$2e^{-2}$
2	2	$2e^{-2}$	$4e^{-2}$
3 or more	3	$1-5e^{-2}$	$3-15e^{-2}$

$$E[X] = 1.78$$

(e) Let $M =$ number of crews going to central stores. Then $M = Y - X$

$$E[M] = E[Y] - E[X] = 2 - 1.78 = 0.22$$

5–35. Using a Poisson model,

$$P(X < 3) = P(X \leq 2) = \sum_{x=0}^{2} \frac{e^{-2.5}(2.5)^x}{x!} \doteq 0.544$$

5–36. Let Y = No. Boarding
Let X = No. Recorded

Y	0	1	2	3	4	5	6	7	8	9	≥ 10
X	0	1	2	3	4	5	6	7	8	9	10

$$p_X(x) = \frac{e^{-c}c^x}{x!}, \qquad x = 0, 1, 2, \ldots, 9$$
$$= \sum_{i=10}^{\infty} \frac{e^{-c}c^i}{i!} = 1 - \sum_{i=0}^{9} \frac{e^{-c}c^i}{i!}, \qquad x = 10$$

5–37. (a) Let X denote the number of errors on 50 pages. Then

$$X \sim \text{Binomial}(5, \frac{50}{200}) = \text{Binomial}(5, 1/4).$$

This implies that

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{5}{0}(1/4)^0(3/4)^5 = 0.763.$$

(b) Now $X \sim \text{Binomial}(5, \frac{n}{200})$, where n is the number of pages sampled.

We want the smallest n such that

$$\sum_{i=3}^{5} \binom{5}{i} \left(\frac{n}{200}\right)^i \left(\frac{200-n}{200}\right)^{5-i} \geq 0.90$$

By trial and error, we find that $n = 151$ does the trick.

We could also have done this problem using a Poisson approximation. For (a), we would use $\lambda = 0.025$ errors / page with 50 pages. Then $c = 50(0.025) = 1.25$, and we would eventually obtain $P(X \geq 1) = 1 - \frac{e^{-1.25}(1.25)^0}{0!} \doteq 0.7135$, which is a bit off of our exact answer. For (b), we would take $c = n(0.025)$, eventually yielding $n = 160$ after trial and error.

5–38.

$$P(X = 0) = \frac{e^{-c}c^0}{0!} \quad \text{with } c = 10000(0.0001) = 1,$$
$$P(X = 0) = e^{-1} = 0.3679$$

and

$$P(X \geq 2) = 1 - P(X \leq 1) = 0.265$$

5–39. X $\sim$ Poisson with $\alpha = 10(0.1) = 0.10$

$$P(X \geq 2) = 1 - P(X \leq 1) = 0.0047$$

5–40. Kendall and Stuart state: “the liability of individuals to accident varies.” That is, the individuals who compose a population have different degrees of accident proneness.

5–41. Use Table XV and scaling by 10^{-5}.

(a) From Col. 3 of Table XV,

Realization 1	Realization 2
$u_1 = 0.01536 < 0.5 \Rightarrow x_1 = 1$	$u_1 = 0.63661 > 0.5 \Rightarrow x_1 = 0$
$u_2 = 0.25595 < 0.5 \Rightarrow x_2 = 1$	$u_2 = 0.53342 > 0.5 \Rightarrow x_2 = 0$
$u_3 = 0.22527 < 0.5 \Rightarrow x_3 = 1$	$u_3 = 0.88231 > 0.5 \Rightarrow x_3 = 0$
$u_4 = 0.06243 < 0.5 \Rightarrow x_4 = 1$	$u_4 = 0.48235 < 0.5 \Rightarrow x_4 = 1$
$u_5 = 0.81837 > 0.5 \Rightarrow x_5 = 0$	$u_5 = 0.52636 > 0.5 \Rightarrow x_5 = 0$
$u_6 = 0.11008 < 0.5 \Rightarrow x_6 = 1$	$u_6 = 0.87529 > 0.5 \Rightarrow x_6 = 0$
$u_7 = 0.56420 > 0.5 \Rightarrow x_7 = 0$	$u_7 = 0.71048 > 0.5 \Rightarrow x_7 = 0$
$u_8 = 0.05463 < 0.5 \Rightarrow x_8 = 1$	$u_8 = 0.51821 > 0.5 \Rightarrow x_8 = 0$
$x = 6$	$x = 1$

Continue to get three more realizations.

(b) Use Col. 4 of Table XV ($p = 0.4$).
Realization 1

$$u_1 = 0.02011 \leq 0.4 \Rightarrow x = 1$$

Realization 2

$$\begin{aligned} u_1 &= 0.85393 > 0.4 \\ u_2 &= 0.97265 > 0.4 \\ u_3 &= 0.61680 > 0.4 \\ u_4 &= 0.16656 < 0.4 \Rightarrow x = 4 \end{aligned}$$

Realization 3

$$\begin{aligned} u_1 &= 0.42751 > 0.4 \\ u_2 &= 0.69994 > 0.4 \\ u_3 &= 0.07972 < 0.4 \Rightarrow x = 3 \end{aligned}$$

Continue to get seven more realizations of X.

(c) $\lambda t = c = 0.15$, $e^{-0.15} = 0.8607$. Using Col. 6 of Table XV,

Realization	u_i	product	$< e^{-0.15}$?	x
#1	$u_1 = 0.91646$	0.91646	No	
	$u_2 = 0.89198$	0.81746	Yes	$x = 1$
#2	$u_1 = 0.64809$	0.64809	Yes	$x = 0$
#3	$u_1 = 0.16376$	0.16376	Yes	$x = 0$
#4	$u_1 = 0.91782$	0.91782	No	
	$u_2 = 0.53498$	0.49102	Yes	$x = 1$
#5	$u_1 = 0.31016$	0.31016	Yes	$x = 0$

5–42. $X \sim$ Geometric with $p = 1/6$.

$y = x^{1/3}$

Using Col. 5 of Table XV, we obtain the following realizations.

1

$$\begin{aligned} u_1 &= 0.81647 > 1/6 \\ u_2 &= 0.30995 > 1/6 \\ u_3 &= 0.76393 > 1/6 \\ u_4 &= 0.07856 < 1/6 \ \Rightarrow\ x = 4, y = 1.587 \end{aligned}$$

2

$$u_1 = 0.06121 < 1/6 \ \Rightarrow\ x = 4, y = 1$$

Continue to get additional realizations.