Chapter 7

7–1. (a) $P(0 \leq Z \leq 2) = \Phi(2) - \Phi(0) = 0.97725 - 0.5 = 0.47725$

(b) $P(-1 \leq Z \leq 1) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.68268$

(c) $P(Z \leq 1.65) = \Phi(1.65) = 0.95053$

(d) $P(Z \geq -1.96) = \Phi(1.96) = 0.9750$

(e) $P(|Z| \geq 1.5) = 2[1 - \Phi(1.5)] = 0.1336$

(f) $P(-1.9 \leq Z \leq 2) = \Phi(2) - \Phi(-1.9) = \Phi(2) - [1 - \Phi(1.9)] = 0.94853$

(g) $P(Z \leq 1.37) = 0.91465$

(h) $P(|Z| \leq 2.57) = 2\Phi(2.57) - 1 = 0.98984$

7–2. $X \sim N(10, 9)$.

(a) $P(X \leq 8) = \Phi\left(\frac{8 - 10}{3}\right) = \Phi\left(-\frac{2}{3}\right) = 0.2525$

(b) $P(X \geq 12) = 1 - \Phi\left(\frac{12 - 10}{3}\right) = 1 - \Phi\left(\frac{2}{3}\right) = 0.2525$

(c) $P(2 \leq X \leq 10) = \Phi\left(\frac{10 - 10}{3}\right) - \Phi\left(\frac{2 - 10}{3}\right) = 0.5 - \Phi(-2.67) = 0.496$

7–3. From Table II of the Appendix

(a) $c = 1.56$

(b) $c = 1.96$

(c) $c = 2.57$

(d) $c = -1.645$

7–4. $P(Z \geq Z_\alpha) = \alpha \;\Rightarrow\; \Phi(Z_\alpha) = 1 - \alpha$.

(a) $Z_{0.025} = 1.96$

(b) $Z_{0.005} = 2.57$

(c) $Z_{0.05} = 1.645$

(d) $Z_{0.0014} = 2.99$

7–5. $X \sim N(80, 100)$.

(a) $P(X \leq 100) = \Phi\left(\dfrac{100-80}{10}\right) = \Phi(2) = 0.97725$

(b) $P(X \leq 80) = \Phi\left(\dfrac{80-80}{10}\right) = 0.5$

(c) $P(75 \leq X \leq 100) = \Phi\left(\dfrac{100-80}{10}\right) - \Phi\left(\dfrac{75-80}{10}\right) = \Phi(2) - \Phi(-0.5) = 0.97725 - 0.30854 = 0.66869$

(d) $P(X \geq 75) = 1 - \Phi\left(\dfrac{75-80}{10}\right) = 1 - \Phi(-0.5) = \Phi(0.5) = 0.69146$

(e) $P(|X - 80| \leq 19.6) = \Phi(1.96) - \Phi(-1.96) = 0.95$

7–6. (a) $P(X > 680) = 1 - \Phi\left(\dfrac{680-600}{60}\right) = 1 - \Phi(1.33) = 0.09176$

(b) $P(X \leq 550) = \Phi\left(\dfrac{550-600}{60}\right) = \Phi(-5/6) = 1 - \Phi(5/6) = 0.20327$

7–7. $P(X > 500) = 1 - \Phi\left(\dfrac{500-485}{30}\right) = 1 - \Phi(0.5) = 0.30854$, i.e., 30.854%

7–8. (a) $P(X \geq 28.5) = 1 - \Phi\left(\dfrac{28.5-30}{1.1}\right) = 1 - \Phi(-1.36) = \Phi(1.36) = 0.91308$

(b) $P(X \leq 31) = \Phi\left(\dfrac{31-30}{1.1}\right) = 0.819$

(c) $P(|X - 30| > 2) = 1 - \left[\Phi\left(\dfrac{2}{1.1}\right) - \Phi\left(-\dfrac{2}{1.1}\right)\right] = 1 - [0.96485 - 0.03515] = 0.0703$

7–9. $X \sim N(2500, 5625)$. Then $P(X < \ell) = 0.05$ implies that

$$P\left(Z < \frac{\ell - 2500}{75}\right) = 0.05$$

or

$$\frac{\ell - 2500}{75} = -1.645.$$

Thus, $\ell = 2376.63$ is the lower specification limit.

7–10.

$$\begin{aligned}
M_X(t) &= E(e^{tX}) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx \\
&= \frac{\sigma}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t(y\sigma+\mu)}\, e^{-y^2/2}\, dy \quad (\text{letting } y = (x-\mu)/\sigma) \\
&= \frac{e^{\mu t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(y^2-2\sigma t y)/2}\, dy \\
&= \frac{e^{\mu t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(y^2-2\sigma t y+\sigma^2t^2-\sigma^2t^2)/2}\, dy \\
&= \frac{e^{\mu t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(y-\sigma t)^2/2} e^{\sigma^2t^2/2}\, dy \\
&= e^{\mu t+(1/2)\sigma^2t^2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-w^2/2}\, dw \quad (\text{letting } w = y-\sigma t) \\
&= e^{\mu t+(1/2)\sigma^2t^2},
\end{aligned}$$

since the integral term equals 1.

7–11.

$$\begin{aligned}
F_Y(y) &= P(aX+b \le y) = P\left(X \le \frac{y-b}{a}\right) \\
&= \Phi\left(\frac{\frac{y-b}{a}-\mu}{\sigma}\right) \\
&= \Phi\left(\frac{y-b-a\mu}{a\sigma}\right) = \Phi\left(\frac{y-(a\mu+b)}{a\sigma}\right)
\end{aligned}$$

This implies that $Y \sim N(a\mu+b, a^2\sigma^2)$.

7–12. $X \sim N(12, (0.02)^2)$.

(a) $P(X > 12.05) = 1 - \Phi\left(\frac{12.05-12}{0.02}\right) = 1 - \Phi(2.5) = 0.00621$

(b)

$$\begin{aligned}
&P(X > c) = 0.9 \\
&\Rightarrow \quad 1 - \Phi\left(\frac{c - 12}{0.02}\right) = 0.9 \\
&\Rightarrow \quad \Phi\left(\frac{c - 12}{0.02}\right) = 0.1 \\
&\Rightarrow \quad \frac{c - 12}{0.02} = -1.28 \\
&\Rightarrow \quad c = 12 - 0.0256 = 11.97
\end{aligned}$$

(c)

$$\begin{aligned}
P(11.95 \le X \le 12.05) &= \Phi\left(\frac{12.05 - 12}{0.02}\right) - \Phi\left(\frac{11.95 - 12}{0.02}\right) \\
&= \Phi(2.5) - \Phi(-2.5) = 0.9876
\end{aligned}$$

7–13. $X \sim N(\mu, (0.1)^2)$.

(a) Take $\mu = 7.0$. Then

$$\begin{aligned}
&P(X > 7.2) + P(X < 6.8) \\
&= 1 - \Phi\left(\frac{7.2 - 7}{0.1}\right) + \Phi\left(\frac{6.8 - 7}{0.1}\right) \\
&= 1 - \Phi(2) + \Phi(-2) \\
&= 1 - 0.97725 + 0.02275 = 0.0455
\end{aligned}$$

(b)

$$\begin{aligned}
&1 - \Phi\left(\frac{7.2 - 7.05}{0.1}\right) + \Phi\left(\frac{6.8 - 7.05}{0.1}\right) \\
&= 1 - \Phi(1.5) + \Phi(-2.5) \\
&= 1 - 0.93319 + 0.00621 = 0.07302
\end{aligned}$$

(c)

$$\begin{aligned}
&\Phi\left(\frac{7.2 - 7.25}{0.1}\right) - \Phi\left(\frac{6.8 - 7.25}{0.1}\right) \\
&= \Phi(-0.5) - \Phi(-4.5) \\
&\doteq 1 - \Phi(0.5) = 0.3085
\end{aligned}$$

(d)

$$\Phi\left(\frac{7.2-6.75}{0.1}\right)-\Phi\left(\frac{6.8-6.75}{0.1}\right) \doteq 1-\Phi(0.5) = 0.3085$$

7–14. $X \sim N(50,25)$, $Y \sim N(45,6.25)$.

If $Y \geq X$, i.e., if $Y - X \geq 0$, a transaction will occur.

Let $W = Y - X \sim N(-5, 31.25)$.

$$P(W>0) = P\left(Z \geq \frac{0+5}{5.59}\right) = 1-\Phi(0.89) = 0.1867.$$

7–15. \$9.00 = revenue / capacitor, k = manufacturing cost for process A, $2k$ = manufacturing cost for process B. The profits are

$$P_A^* = \begin{cases} 9-k & \text{if } 1000 \leq X \leq 5000 \\ 9-k-3 & \text{otherwise} \end{cases}$$

$$P_B^* = \begin{cases} 9-2k & \text{if } 1000 \leq X \leq 5000 \\ 9-2k-3 & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} E(P_A^*) &= (9-k)P(1000 \leq X \leq 5000) + (6-k)[1-P(1000 \leq X \leq 5000)] \\ &= (9-k)0.9544 + (6-k)0.0456 = 8.8632 - k \\ E(P_B^*) &= (9-2k)P(1000 \leq X \leq 5000) + (6-2k)[1-P(1000 \leq X \leq 5000)] \\ &= (9-2k)(1) + (6-k)(0) = 9-2k \end{aligned}$$

Since $E(P_A^*) < E(P_B^*)$ when $k < 0.1368$, use process B; When $k \geq 0.1368$, use process A.

7–16. The profit P is

$$P = \begin{cases} C & \text{if } 6 \leq X \leq 8 \\ -R_1 & \text{if } X < 6 \\ -R_2 & \text{if } X > 8 \end{cases}$$

$$
\begin{aligned}
E(P) &= CP(6 \le X \le 8) - R_1 P(X < 6) - R_2 P(X > 8) \\
&= C[\Phi(8-\mu) - \Phi(6-\mu)] - R_1\Phi(6-\mu) - R_2[1-\Phi(8-\mu)] \\
&= (C+R_2)\Phi(8-\mu) - (C+R_1)\Phi(6-\mu) - R_2
\end{aligned}
$$

Then

$$
\frac{dE(P)}{d\mu} = -(C+R_2)\Phi(8-\mu) + (C+R_1)\Phi(6-\mu) = 0,
$$

or

$$
\frac{C+R_2}{C+R_1} = \frac{\Phi(6-\mu)}{\Phi(8-\mu)} = e^{14-2\mu}
$$

Thus,

$$
\mu = 7 - \frac{1}{2}\,\ell\text{n}\left(\frac{C+R_2}{C+R_1}\right).
$$

7–17. If $R_1 = R_2 = R$, then $\mu = 7 - 0.5\,\ell\text{n}(1) = 7$, which is the midpoint of the interval [6,8].

7–18. $\mu = 7 - \frac{1}{2}\,\ell\text{n}\left(\frac{12}{10}\right) = 6.909.$

7–19. $X \sim N(70, 16)$.

(a) We have

$$
\begin{aligned}
P(62 \le X \le 72) &= \Phi\left(\frac{72-70}{4}\right) - \Phi\left(\frac{62-70}{4}\right) \\
&= \Phi(0.5) - \Phi(-2) \\
&= 0.69146 - 0.02275 = 0.66871.
\end{aligned}
$$

(b) $c = 1.96\sigma = 7.84$.

(c) $(9)(0.66871) = 6.018$.

7-20. $E(Y) \;=\; E\left(\sum_{i=1}^{n} X_i\right) \;=\; \sum_{i=1}^{n} E(X_i) \;=\; n\mu$, so

$$E(Z_n) \;=\; \frac{E(Y) - n\mu}{\sqrt{\sigma^2 n}} \;=\; 0$$

$$V(Y) \;=\; V\left(\sum_{i=1}^{n} X_i\right) \;=\; \sum_{i=1}^{n} V(X_i) \;=\; n\sigma^2$$

$$V(Z_n) \;=\; \frac{V(Y)}{n\sigma^2} \;=\; 1$$

7–21. $E(\bar{X}) \;=\; E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \;=\; \frac{1}{n}\sum_{i=1}^{n} E(X_i) \;=\; \frac{n\mu}{n} \;=\; \mu$

$$V(\bar{X}) \;=\; V\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \;=\; \frac{1}{n^2}\sum_{i=1}^{n} V(X_i) \;=\; \frac{n\sigma^2}{n^2} \;=\; \frac{\sigma^2}{n}$$

7–22. $X_1 \sim N(1.25, 0.0009)$ and $X_2 \sim N(1.2, 0.0016)$.

$Y = X_1 - X_2$, $E(Y) = 0.05$, $V(Y) = 0.0025$.

$Y \sim N(0.05, 0.0025)$

$$P(Y < 0) \;=\; \Phi\left(\frac{0 - 0.05}{0.05}\right) \;=\; \Phi(-1) \;=\; 1 - \Phi(1) \;=\; 0.15866.$$

7–23. $X_i \sim N(2, 0.04)$, $i = 1, 2, 3$, and $Y = X_1 + X_2 + X_3 \sim N(6, 0.12)$.

Then

$$\begin{aligned} P(5.7 < Y < 6.3) \;&=\; \Phi\left(\frac{6.3 - 6.0}{0.3464}\right) - \Phi\left(\frac{5.7 - 6.0}{0.3464}\right) \\ &=\; \Phi(0.866) - \Phi(-0.866) \;=\; 0.6156. \end{aligned}$$

7–24. $E(Y) \;=\; E(X_1) + 2E(X_2) + E(X_3) + E(X_4) \;=\; 4 + 2(4) + 2 + 3 \;=\; 17.$

With independence,

$$V(Y) \;=\; V(X_1) + 2^2 V(X_2) + V(X_3) + V(X_4) \;=\; 3 + 4(4) + 4 + 2 \;=\; 25.$$

$$\begin{aligned} P(15 \le Y \le 20) &= \Phi\left(\frac{20-17}{5}\right) - \Phi\left(\frac{15-17}{5}\right) \\ &= \Phi(0.6) - \Phi(-0.4) \\ &= 0.72575 - 0.34458 \;=\; 0.38117. \end{aligned}$$

7–25. $E(X_i) = 0$, $V(X_i) = 1/12$.

$$Y \;=\; \sum_{i=1}^{50} X_i, \quad E(Y) \;=\; 0, \quad V(Y) \;=\; 50/12$$

$$P(Y > 5) \;=\; 1 - \Phi\left(\frac{5-0}{\sqrt{50/12}}\right) \;=\; 1 - \Phi(2.45) \;=\; 0.00714.$$

7–26. $E(X_i) = 1$, $V(X_i) = 0.0001$, $i = 1, 2, \ldots, 100$.

$$Y \;=\; \sum_{i=1}^{100} X_i.$$

Assuming that the X_i's are independent, we use the central limit theorem to approximate the distribution of $Y \sim N(1000, 0.01)$. Then

$$P(Y > 102) \;=\; P\left(Z > \frac{102-100}{0.1}\right) \;=\; 1 - \Phi(20) \;\doteq\; 0.$$

7–27. $\bar{X} \sim N(11.9, 0.0025)$ and $n = 9$. Thus, $\mu = 11.9$, $\sigma^2/n = \sigma^2/9 = 0.0025$, so $\sigma^2 = 0.0225$. All of this implies that $X \sim N(11.9, 0.0225)$. Then

$$P(11.8 < X < 12.2) \;=\; \Phi(2) - \Phi(-0.67) \;=\; 0.7258,$$

so that there are 27.4% defective.

If $\mu = 12$, then

$$P(11.8 < X < 12.2) \;=\; \Phi(1.33) - \Phi(-1.33) \;=\; 0.8164,$$

or 18.4% defective. This is the optimal value of the mean.

7–28. $Y = \sum_{i=1}^{n} E(X_i)$, where X_i is the travel time between pair i.

$$\begin{aligned} E(Y) &= \sum_{i=1}^{n} E(X_i) = 30 \\ V(Y) &= \sum_{i=1}^{n} V(X_i) \\ &= (0.4)^2 + (0.6)^2 + (0.3)^2 + (1.2)^2 + (0.9)^2 + (0.4)^2 + (0.4)^2 = 3.18. \end{aligned}$$

Thus,

$$P(Y \le 32) = \Phi\left(\frac{32-30}{\sqrt{3.18}}\right) = \Phi(1.12) = 0.86864.$$

7–29. $p = 0.08$, $n = 200$, $np = 16$, $\sqrt{npq} = 3.84$.

(a) $P(X \le 16) = \Phi\left(\frac{16.5-16}{3.84}\right) = \Phi(0.13) = 0.55172.$

(b) $\Phi\left(\frac{15.5-16}{3.84}\right) - \Phi\left(\frac{14.5-16}{3.84}\right) = \Phi(-0.13) - \Phi(-0.391) = 0.1.$

(c) $\Phi\left(\frac{20.5-16}{3.84}\right) - \Phi\left(\frac{11.5-16}{3.84}\right) = \Phi(1.17) - \Phi(-1.17) = 0.758.$

(d) $\Phi\left(\frac{14.5-16}{3.84}\right) - \Phi\left(\frac{13.5-16}{3.84}\right) = \Phi(-0.391) - \Phi(-0.651) = 0.09.$

7–30. $P(0.05 \le \hat{p} \le 0.15) = 0.95$ implies that

$$P\left(\frac{0.05-0.10}{\sqrt{0.09/n}} \le Z \le \frac{0.15-0.10}{\sqrt{0.09/n}}\right) = \Phi\left(\frac{0.05}{0.3/\sqrt{n}}\right) - \Phi\left(\frac{-0.05}{0.3/\sqrt{n}}\right) = 0.95$$

$$\Rightarrow 2\,\Phi\left(\frac{0.05}{0.3/\sqrt{n}}\right) = 1.95$$

$$\Rightarrow \Phi(0.167\sqrt{n}) = 0.9750$$

$$\Rightarrow 0.167\sqrt{n} = 1.96$$

$$\Rightarrow n = 139$$

7–31. $Z_1 = \sqrt{-2\,\ell\text{n}(u_1)} \cdot \cos(2\pi u_2)$, $Z_2 = \sqrt{-2\,\ell\text{n}(u_1)} \cdot \sin(2\pi u_2)$. Note that the sine and cosine calculations are carried out in radians.

Obtain uniforms from Col. 2.

u_1	u_2	z_1	z_2
0.15011	0.46573	−1.902	0.416
0.48360	0.93093	1.093	−0.507
0.39975	0.06907	1.229	0.569

These results give

$$\begin{aligned}
x_1 &= 100 + 2(-1.902) = 96.196 \\
x_2 &= 100 + 2(0.416) = 100.832 \\
x_3 &= 100 + 2(1.093) = 102.186 \\
x_4 &= 100 + 2(-0.507) = 98.986 \\
x_5 &= 100 + 2(1.229) = 102.458 \\
x_6 &= 100 + 2(0.569) = 101.138
\end{aligned}$$

7–32. Calculate Z_1 and Z_2 as in Problem 7–31, obtaining uniforms from Col. 4.

u_1	u_2	z_1	z_2
0.02011	0.08539	2.402	1.429
0.97265	0.61680	−0.175	−0.158
0.16656	0.42751	−1.700	0.833

These results give realizations of X_1.

x_1	$3x_1$
$10 + 1.732(2.402) = 14.161$	42.483
$10 + 1.732(1.429) = 12.475$	37.425
$10 + 1.732(-0.175) = 9.697$	29.091
$10 + 1.732(-0.158) = 9.726$	29.179
$10 + 1.732(-1.700) = 7.056$	21.167
$10 + 1.732(0.833) = 11.443$	34.328

Meanwhile, we use Col. 5 to calculate realizations of X_2.

x_2	$-2x_2$
$20(0.81647) = 16.329$	-32.658
$20(0.30995) = 6.199$	-12.398
$20(0.76393) = 15.279$	-30.558
$20(0.07856) = 1.571$	-3.142
$20(0.06121) = 1.224$	-2.448
$20(0.27756) = 5.551$	-11.102

Finally, we get the six realizations of Y,

$$\begin{aligned} y_1 &= 9.825 \\ y_2 &= 25.027 \\ y_3 &= -1.467 \\ y_4 &= 26.039 \\ y_5 &= 18.719 \\ y_6 &= 23.226 \end{aligned}$$

7–33. Using the z_i realizations from Problem 7–31,

$$\begin{aligned} (-1.092)^2 &= 3.618 \\ (0.416)^2 &= 0.173 \\ (1.093)^2 &= 1.195 \\ (-0.507)^2 &= 0.257 \\ (1.229)^2 &= 1.510 \end{aligned}$$

7–34. Let $z_1, z_2, \ldots, z_n$ be realizations of $N(0,1)$ r.v.'s.

$y_i = \mu_Y + \sigma z_i$, $i = 1, 2, \ldots, n$.

$x_i = e^{y_i}$, $i = 1, 2, \ldots, n$.

7–35. Generate a pair z_1, z_2 of $N(0,1)$ r.v.'s.

Let $x_1 = \mu_1 + \sigma_1 z_1$ and $x_2 = \mu_2 + \sigma_2 z_2$. Thus, $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, 2$.

Let $y_1 = \sqrt{x_1}/x_2^2$.

Repeat this procedure for as many realizations as desired.

7–36. This is a normal distribution truncated on the right, with p.d.f.

$$\begin{aligned} f(x) &= \frac{1}{\Phi(\frac{r-\mu}{\sigma})\,\sigma\,\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \quad \text{if } -\infty < x \le r \\ &= 0 \quad \text{if } x > r \end{aligned}$$

For our problem, $r = 2600$, $\mu = 2500$, and $\sigma = 50$. Now, after a bit of calculus,

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x)\,dx \\ &= \mu - \frac{\sigma}{\Phi(\frac{r-\mu}{\sigma})\,\sqrt{2\pi}} \exp\left[-\frac{(r-\mu)^2}{2\sigma^2}\right] \\ &= 2500 - \frac{50}{0.9772\,\sqrt{2\pi}} \exp\left[-\frac{(2600-2500)^2}{2(50)^2}\right] \\ &= 2497.24 \end{aligned}$$

7–37. $E(X) = e^{62.5}$, $V(X) = e^{125}(e^{25} - 1)$, median$(X) = e^{50}$, mode$(X) = e^{25}$

7–38. W is lognormal with parameters 17.06 and 7.0692, or $\ell\text{n}(W) \sim N(17.06, 7.0692)$.

Thus, $P(L \le W \le R) = 0.90$ implies $P(\ell\text{n}(L) \le \ell\text{n}(W) \le \ell\text{n}(R)) = 0.90$, or

$$\Phi\left(\frac{\ell\text{n}(R) - 17.06}{\sqrt{7.0692}}\right) - \Phi\left(\frac{\ell\text{n}(L) - 17.06}{\sqrt{7.0692}}\right) = 0.90.$$

Assuming that the interval $[\ell\text{n}(L), \ell\text{n}(R)]$ is symmetric about 17.06, we obtain $\ell\text{n}(L) = 17.06 - c$ and $\ell\text{n}(R) = 17.06 + c$, so that

$$\Phi\left(\frac{c}{2.6588}\right) - \Phi\left(\frac{-c}{2.6588}\right) = 0.90.$$

This means that $\dfrac{c}{2.6588} = 1.645$, or $c = 4.374$.

$\ell\text{n}(L) = 12.69$, or $L = 324486.8$ and
$\ell\text{n}(R) = 21.43$, or $R = 2027359410$

7–39. $Y \sim N(\mu, \sigma^2)$, $Y = \ell\text{n}(X)$, or $X = e^Y$. The function e^y is strictly increasing in y; thus, from Theorem 3–1,

$$f(x) \;=\; \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ell\text{n}(x) - \mu)^2}{2\sigma^2}\right]; \quad x \geq 0.$$

7–41. $X_1 \sim N(2000, 2500)$, $X_2 \sim N(0.10, 0.01)$, $\rho = 0.87$.

$$\begin{aligned} E(X_1|x_2) &= \mu_1 + \rho(\sigma_1/\sigma_2)(x_2 - \mu_2) \\ &= 2000 + (0.87)(50/0.1)(0.098 - 0.10) \\ &= 1999.13 \end{aligned}$$

$$V(X_1|x_2) \;=\; \sigma_1^2(1 - \rho^2) \;=\; 2500(1 - 0.7569) \;=\; 607.75$$

$$\begin{aligned} P(X_1 > 1950|x_2 = 0.098) &= P\left(Z > \frac{1950 - 1999.13}{\sqrt{607.75}}\right) \\ &= 1 - \Phi(-1.993) \;=\; 0.9769 \end{aligned}$$

7–42. $X_1 \sim N(75, 25)$, $X_2 \sim N(83, 16)$, $\rho = 0.8$.

$$\begin{aligned} E(X_2|x_1) &= \mu_2 + \rho(\sigma_2/\sigma_1)(x_1 - \mu_1) \\ &= 83 + (0.8)(4/5)(80 - 75) \\ &= 86.2 \end{aligned}$$

$$V(X_2|x_1) \;=\; \sigma_2^2(1 - \rho^2) \;=\; 16(1 - 0.64) \;=\; 5.76$$

$$P(X_2 > 80|x_1 = 80) \;=\; P\left(Z > \frac{80 - 86.2}{\sqrt{5.76}}\right) \;=\; 0.9951$$

7–43. (a) $f(x_1, x_2) = k$ implies that

$$k = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

For a selected value of k, the quantity in brackets assumes a value, say c; thus,

$$\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - c = 0,$$

which is a quadratic in $x_1 - \mu_1$ and $x_2 - \mu_2$. If we write the general second-degree equation as

$$Ay_1^2 + By_1y_2 + Cy_2^2 + Dy_1 + Ey_2 + F = 0,$$

we can determine the nature of the curve from the second-order terms. In particular, if $B^2 - 4AC < 0$, the curve is an ellipse. In any case,

$$B^2 - 4AC = \left(\frac{2\rho}{\sigma_1\sigma_2}\right)^2 - \frac{4}{\sigma_1^2\sigma_2^2} = \frac{4(\rho^2-1)}{\sigma_1^2\sigma_2^2} < 0,$$

the last inequality a result of the fact that $\rho^2 < 1$ (for $\rho \neq 0$). Thus, we have an ellipse.

(b) Let $\sigma_1^2 = \sigma_2^2 = \sigma^2$ and $\rho = 0$. Then the equation of the curve becomes

$$\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - c = 0,$$

which is a circle with center (μ_1, μ_2) and radius $\sigma\sqrt{c}$.

7–44.

$$\begin{aligned} F(r) &= P(R \le r) \\ &= P\left(\sqrt{X_1^2 + X_2^2} \le r\right) \\ &= P(X_1^2 + X_2^2 \le r^2) \\ &= \int\!\!\int_A \frac{1}{2\pi\sigma^2} \exp\left[-\frac{(t_1^2 + t_2^2)}{2\sigma^2}\right] dt_1\, dt_2, \end{aligned}$$

where $A = \{(x_1, x_2) : x_1^2 + x_2^2 \le r^2\}$.

Let $x_1 = \rho\cos(\theta)$ and $x_2 = \rho\sin(\theta)$. Then

$$\begin{aligned} P(R \le r) &= \int_0^r \int_0^{2\pi} \frac{\rho}{2\pi\sigma^2}\exp(-\rho^2/2\sigma^2)\,d\theta\,d\rho \\ &= 1 - \exp(-r^2/2\sigma^2) \end{aligned}$$

Thus,

$$f(r) = (r/\sigma^2)\exp(-r^2/2\sigma^2); \quad r > 0.$$

7–45. Using the fact that $\sum_{i=1}^n X_i^2$ has a χ_n^2 distribution, we obtain

$$f(r) = \frac{r^{n-1}e^{-r^2/2}}{2^{(n-2)/2}\Gamma(n/2)}; \quad r \ge 0$$

7–46. Let $Y_1 = X_1/X_2$ and $Y_2 = X_2$, with $X_2 \neq 0$. Then the Jacobian is

$$|J| = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = |y_2|$$

So we have

$$f(y_1, y_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}|y_2|\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{y_1y_2}{\sigma_1}\right)^2 - \frac{2\rho y_1 y_2^2}{\sigma_1\sigma_2} + \left(\frac{y_2}{\sigma_2}\right)^2\right]\right\}$$

So the marginal is

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{-\infty}^{\infty} f(y_1, y_2)\,dy_2 \\ &= \frac{\sqrt{1-\rho^2}}{\pi\sigma_1\sigma_2}\left[\left(\frac{y_1}{\sigma_1} - \frac{\rho}{\sigma_2}\right)^2 + \frac{1-\rho^2}{\sigma_2^2}\right]^{-1}; \quad -\infty < y_1 < \infty \end{aligned}$$

When $\rho = 0$ and $\sigma_1^2 = \sigma_2^2 = \sigma^2$, the distribution becomes

$$f_{Y_1}(y_1) = \frac{1}{\pi(1+y_1^2)}; \quad -\infty < y_1 < \infty,$$

also known as the *Cauchy* distribution.

7–47. The CDF is

$$\begin{aligned} F_Y(y) &= P(Y \le y) \;=\; P(Z^2 \le y) \\ &= P(-\sqrt{y} \le Z \le \sqrt{y}) \\ &= 2\int_0^{\sqrt{y}} \frac{1}{2\pi} e^{-z^2/2}\, dz \end{aligned}$$

Take $z = \sqrt{u}$ so that $dz = (2\sqrt{u})^{-1}\, du$. Then

$$F_Y(y) \;=\; \int_0^y \frac{1}{2\pi} u^{(1/2)-1} e^{-u/2}\, du$$

and so, by Leibniz' rule,

$$f(y) \;=\; \frac{1}{2\pi}\, y^{-1/2} e^{-y/2}, \quad y > 0.$$

7–48. The CDF is

$$\begin{aligned} F_Y(y) &= P(Y \le y) \;=\; P\left(\sum_{i=1}^n X_i^2 \le y\right) \\ &= \int\int \cdots \int_A (2\pi)^{-n/2} \exp\left[-\frac{1}{2}\sum_{i=1}^n x_i^2\right] dx_1\, dx_2 \cdots dx_n, \end{aligned}$$

where

$$A \;=\; \left\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^n x_i^2 \le y\right\}.$$

Transform to polar coordinates:

$$\begin{aligned} x_1 &= y^{1/2} \cos(\theta_1) \\ x_2 &= y^{1/2} \sin(\theta_1) \cos(\theta_2) \\ x_3 &= y^{1/2} \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) \\ &\vdots \\ x_{n-1} &= y^{1/2} \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{n-2}) \cos(\theta_{n-1}) \\ x_n &= y^{1/2} \sin(\theta_1) \sin(\theta_2) \cdots \sin(\theta_{n-2}) \sin(\theta_{n-1}) \end{aligned}$$

The Jacobian of this transformation is

$$\begin{vmatrix} \frac{\partial x_1}{\partial y} & \frac{\partial x_1}{\partial \theta_1} & \cdots & \frac{\partial x_1}{\partial \theta_{n-1}} \\ \frac{\partial x_2}{\partial y} & \frac{\partial x_2}{\partial \theta_1} & \cdots & \frac{\partial x_2}{\partial \theta_{n-1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_n}{\partial y} & \frac{\partial x_n}{\partial \theta_1} & \cdots & \frac{\partial x_n}{\partial \theta_{n-1}} \end{vmatrix}$$

or, after a little algebra,

$$\begin{vmatrix} \frac{\cos(\theta_1)}{2\sqrt{y}} & -\sqrt{y}\sin(\theta_1) & 0 & \cdots & 0 \\ \frac{\sin(\theta_1)\cos(\theta_2)}{2\sqrt{y}} & \sqrt{y}\cos(\theta_1)\cos(\theta_2) & -\sqrt{y}\cos(\theta_1)\sin(\theta_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\sin(\theta_1)\cdots\sin(\theta_{n-1})}{2\sqrt{y}} & \sqrt{y}\cos(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-1}) & \cdots & \cdots & \sqrt{y}\sin(\theta_1)\sin(\theta_2)\cdots\cos(\theta_{n-1}) \end{vmatrix}$$

In other words, $J = (1/2)y^{(n/2)-1}|\Delta_n|$, where Δ_n is an $n \times n$ matrix obtained by taking out $(2\sqrt{y})^{-1}$ from the first column and $\sqrt{y}$ from the last $n-1$ columns. Expanding this determinant with respect to the last column, we have

$$\begin{aligned} |\Delta_n| &= \sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{n-2})|\Delta_{n-1}| \\ &= \sin^{n-2}(\theta_1)\sin^{n-3}(\theta_2)\cdots\sin(\theta_{n-2}) \end{aligned}$$

This transformation gives variables whose limits are much easier. In the region covered by A, we have $0 \leq \theta_i \leq \pi$ for $i = 1, 2, \ldots, n-2$, and $0 < \theta_{n-1} < 2\pi$. Thus,

$$\begin{aligned} & P\left(\sum_{i=1}^{n} X_i^2 \leq y^*\right) \\ &= \int\int\cdots\int_A \frac{1}{(2\pi)^{n/2}}\,\frac{1}{2}\,y^{(n/2)-1}e^{-y/2}|\Delta_n|\,dy\,d\theta_{n-1}\cdots d\theta_1 \\ &= \frac{1}{2(2\pi)^{n/2}}\int_0^{y^*} y^{(n/2)-1}e^{-y/2}\,dy\int_0^{2\pi} d\theta_{n-1}\int_0^{\pi}\sin(\theta_{n-2})\,d\theta_{n-2}\cdots\int_0^{\pi}\sin^{n-2}(\theta_1)\,d\theta_1 \\ &= K\int_0^{y^*} y^{(n/2)-1}e^{-y/2}\,dy \;\equiv\; F(y^*) \end{aligned}$$

Thus,

$$f(y) = F'(y) = Ky^{(n/2)-1}e^{-y/2}, \quad y \geq 0.$$

To evaluate K, use $K\int_0^\infty f(y)\,dy = 1$. This finally gives

$$f(y) \;=\; \frac{1}{2^{n/2}\Gamma(n/2)} y^{(n/2)-1} e^{-y/2}, \quad y \geq 0.$$

7–49. For $x \geq 0$,

$$\begin{aligned} F(x) &= P(|X| \leq x) \;=\; P(-x \leq X \leq x) \\ &= \int_{-x}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt \\ &= 2\int_0^x \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt, \end{aligned}$$

so that

$$\begin{aligned} f(x) &= \frac{2}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad x > 0 \\ &= 0, \quad \text{otherwise} \end{aligned}$$