Chapter 9

9–1. Since

$$f(x_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right],$$

we have

$$\begin{aligned} f(x_1, x_2, \ldots, x_5) &= \prod_{i=1}^{5} f(x_i) \\ &= \prod_{i=1}^{5} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right] \\ &= \left(\frac{1}{\sigma^2 2\pi}\right)^{5/2} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{5}(x_i-\mu)^2\right] \end{aligned}$$

9–2. Since

$$f(x_i) = \lambda e^{-\lambda x_i},$$

we have

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= \prod_{i=1}^{n} f(x_i) \\ &= \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \\ &= \lambda^n \exp\left[-\lambda \sum_{i=1}^{n} x_i\right] \end{aligned}$$

9–3. Since $f(x_i) = 1$, we have

$$f(x_1, x_2, x_3, x_4) = \prod_{i=1}^{4} f(x_i) = 1$$

9–4. The joint probability function for X_1 and X_2 is

$$\begin{aligned}
p_{X_1,X_2}(0,0) &= \frac{\binom{N-M}{0}\binom{M}{2}}{\binom{N}{2}} \\
p_{X_1,X_2}(0,1) &= \frac{\binom{N-M}{1}\binom{M}{1}}{2\binom{N}{2}} \\
p_{X_1,X_2}(1,0) &= \frac{\binom{N-M}{1}\binom{M}{1}}{2\binom{N}{2}} \\
p_{X_1,X_2}(1,1) &= \frac{\binom{N-M}{2}\binom{M}{0}}{\binom{N}{2}}
\end{aligned}$$

Of course,

$$\begin{aligned}
p_{X_1}(x_1) &= \sum_{x_2=0}^{1} p_{X_1,X_2}(x_1,x_2) \qquad \text{and} \\
p_{X_2}(x_2) &= \sum_{x_1=0}^{1} p_{X_1,X_2}(x_1,x_2)
\end{aligned}$$

So $p_{X_1}(0) = M/N$, $p_{X_1}(1) = 1-(M/N)$, $p_{X_2}(0) = M/N$, $p_{X_2}(1) = 1-(M/N)$.

Thus, X_1 and X_2 are not independent since

$$p_{X_1,X_2}(0,0) \ \neq \ p_{X_1}(0)p_{X_2}(0)$$

9–5. $N(\mu, \sigma^2/n) = N(5, 0.00125)$

9–6. $\sigma/\sqrt{n} = 0.1/\sqrt{8} = 0.0353$

9–7. Use estimated standard error $S/\sqrt{n}$.

9–8. $N(-5, 0.22)$

9–9. The standard error of $\bar{X}_1 - \bar{X}_2$ is

$$\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} = \sqrt{\frac{(1.5)^2}{25} + \frac{(2.0)^2}{30}} = 0.473$$

9–10. $Y = \bar{X}_1 - \bar{X}_2$ is a linear combination of the 55 variables X_{ij}, $i = 1, j = 1, 2, \ldots, 25$, $i = 2, j = 1, 2, \ldots, 30$. As such, we would expect Y to be very nearly normal with mean $\mu_Y = -0.5$ and variance $(0.473)^2 = 0.223$.

9–11. $N(0, 1)$

9–12. $N(\hat{p}, \hat{p}(1-\hat{p})/n)$

9–13. $se(\hat{p}) = \sqrt{p(1-p)/n}$, $\widehat{se}(\hat{p}) = \sqrt{\hat{p}(1-\hat{p})/n}$,

9–14.

$$\begin{aligned} M_X(t) &= E(e^{tX}) = \int_0^\infty e^{tx} \frac{1}{2^{n/2}\Gamma(n/2)} x^{(n/2)-1} e^{-x/2}\, dx \\ &= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty x^{(n/2)-1} e^{-x[(1/2)-t]}\, dx \end{aligned}$$

This integral converges if $1/2 > t$.

Let $u = x[(1/2) - t]$. Then $dx = [(1/2) - t]^{-1} du$. Thus,

$$\begin{aligned} M_X(t) &= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty \frac{u^{(n/2)-1}}{[(1/2)-t]^{(n/2)-1}} e^{-u} \frac{1}{[(1/2)-t]}\, du \\ &= \frac{1}{2^{n/2}\Gamma(n/2)[(1-2t)/2]^{n/2}} \int_0^\infty u^{(n/2)-1} e^{-u}\, du \\ &= \frac{1}{(1-2t)^{n/2}}, \quad t < 1/2, \end{aligned}$$

since $\Gamma(n/2) = \displaystyle\int_0^\infty u^{(n/2)-1} e^{-u}\, du$.

9–15. First of all,

$$\begin{aligned} M'_X(t) &= n(1-2t)^{-(n/2)-1} \\ M''_X(t) &= n(n+2)(1-2t)^{-(n/2)-2} \end{aligned}$$

Then

$$\begin{aligned} E(X) &= M'_X(0) \;=\; n \\ E(X^2) &= M''_X(0) \;=\; n(n+2) \\ V(X) &= E(X^2)-[E(X)]^2 \;=\; 2n \end{aligned}$$

9–16. Let $T = Z/\sqrt{\chi^2_n/n} = Z\sqrt{n/\chi^2_n}$. Now

$E(T) = E(Z)E(\sqrt{n/\chi^2_n}\,) = 0$, because $E(Z) = 0$.

$V(T) = E(T^2)$, because $E(T) = 0$. Thus,

$V(T) = E[Z^2(n/\chi^2_n)] = E(Z^2)E(n/\chi^2_n)$.

Note that $E(Z^2) = V(Z) = 1$, so that

$$\begin{aligned} V(T) &= E(n/\chi^2_n) \\ &= \int_0^\infty \frac{(n/s)}{2^{n/2}\Gamma(n/2)} s^{(n/2)-1}\, e^{-s/2}\, ds \\ &= \frac{n}{2^{n/2}\Gamma(n/2)} \int_0^\infty s^{(n/2)-2}\, e^{-s/2}\, ds \\ &= \frac{n}{2^{(n/2)-1}\Gamma(n/2)} \int_0^\infty (2u)^{(n/2)-2}\, e^{-u}\, du \\ &= \frac{n\Gamma(\frac{n}{2}-1)}{2\Gamma(\frac{n}{2})}, \quad \text{if } n > 2 \\ &= \frac{n\Gamma(\frac{n}{2}-1)}{2(\frac{n}{2}-1)\Gamma(\frac{n}{2}-1)}, \quad \text{if } n > 2 \\ &= \frac{n}{n-2}, \quad \text{if } n > 2 \end{aligned}$$

9–17. $E(F_{m,n}) = E[(\chi_m^2/m)/(\chi_n^2/n)] = E(\chi_m^2/m)E(n/\chi_n^2)$.

$E(\chi_m^2/m) = (1/m)E(\chi_m^2) = 1$.

From Problem 9–16, we have $E(n/\chi_n^2) = n/(n-2)$.

Therefore, $E(F_{m,n}) = n/(n-2)$, if $n > 2$.

To find $V(F_{m,n})$, let $X \sim \chi_m^2$ and $Y \sim \chi_n^2$. Then

$E(F_{m,n}^2) = (n/m)^2 E(X^2)E(1/Y^2)$.

Since $E(X^2) = V(X) + [E(X)]^2$ and $X \sim \chi_m^2$, we have $E(X^2) = 2m + m^2$. Now

$$\begin{aligned}
E(1/Y^2) &= \int_0^\infty \frac{(1/y^2)}{2^{n/2}\Gamma(n/2)} y^{(n/2)-1}\, e^{-y/2}\, dy \\
&= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty y^{(n/2)-3}\, e^{-y/2}\, dy \\
&= \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty 2(2u)^{(n/2)-3}\, e^{-u}\, du \\
&= \frac{1}{(n-2)(n-4)}, \quad \text{if } n > 4
\end{aligned}$$

Thus,

$$\begin{aligned}
V(F_{m,n}) &= (n/m)^2 E(X^2)E(1/Y^2) - \left(\frac{n}{n-2}\right)^2 \\
&= \frac{n^2(2m+m^2)}{m^2(n-2)(n-4)} - \frac{n^2}{(n-2)^2} = \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}
\end{aligned}$$

9–18. $X_{(1)}$ is greater than t if and only if every observation is greater than t. Then

$$\begin{aligned}
P(X_{(1)} > t) &= P(X_1 > t, X_2 > t, \ldots, X_n > t) \\
&= P(X_1 > t)P(X_2 > t) \cdots P(X_n > t) \\
&= P(X > t)P(X > t) \cdots P(X > t) \\
&= [1 - F(t)]^n
\end{aligned}$$

So $F_{X(1)}(t) = 1 - P(X_{(1)} > t) = 1 - [1 - F(t)]^n$.

If X is continuous, then so is $X_{(1)}$; so

$$f_{X_{(1)}}(t) \;=\; F'_{X(1)}(t) \;=\; n[1 - F(t)]^{n-1} f(t)$$

Similarly,

$$\begin{aligned} F_{X(n)}(t) &= P(X_{(n)} \le t) \\ &= P(X_1 \le t, X_2 \le t, \ldots, X_n \le t) \\ &= P(X_1 \le t)P(X_2 \le t)\cdots P(X_n \le t) \\ &= P(X \le t)P(X \le t)\cdots P(X \le t) \\ &= [F(t)]^n \end{aligned}$$

Since $X_{(n)}$ is continuous,

$$f_{X_{(n)}}(t) \;=\; F'_{X(n)}(t) \;=\; n[F(t)]^{n-1} f(t)$$

9–19.

$$F(t) \;=\; \begin{cases} 0 & t < 0 \\ 1 - p & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}$$

Then

$$P(X_{(n)} = 1) \;=\; F_{X_{(n)}}(1) - F_{X_{(n)}}(0) \;=\; [F(1)]^n - [F(0)]^n \;=\; 1 - (1-p)^n$$

$$P(X_{(1)} = 0) \;=\; 1 - [1 - F(0)]^n \;=\; 1 - [1 - (1-p)]^n \;=\; 1 - p^n$$

9–20.

$$f_{X_{(1)}}(t) \;=\; n\left[1 - \Phi\left(\frac{t-\mu}{\sigma}\right)\right]^{n-1} \frac{1}{\sigma\sqrt{2\pi}}\,\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

$$f_{X_{(n)}}(t) \;=\; n\left[\Phi\left(\frac{t-\mu}{\sigma}\right)\right]^{n-1} \frac{1}{\sigma\sqrt{2\pi}}\,\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

9–21. $f(t) = \lambda e^{-\lambda t}, \quad t > 0$

$F(t) = 1 - e^{-\lambda t}$

$F_{X_{(1)}}(t) = 1 - [1 - F(t)]^n = 1 - [1 - (1 - e^{-\lambda t})]^n = 1 - e^{-n\lambda t}$

$f_{X_{(1)}}(t) = n\lambda e^{-n\lambda t}, \quad t > 0$

$F_{X_{(n)}}(t) = [F(t)]^n = (1 - e^{-\lambda t})^n$

$f_{X_{(n)}}(t) = n(1 - e^{-\lambda t})^{n-1}\lambda e^{-\lambda t}, \quad t > 0$

9–22. $f_{X_{(n)}}(X_{(n)}) = n[F(X_{(n)})]^{n-1} f(X_{(n)})$

Treat $F(X_{(n)})$ as a random variable giving the fraction of objects in the population having values of $X \leq X_{(n)}$.

Let $Y = F(X_{(n)})$. Then $dy = f(X_{(n)})dx_{(n)}$, and thus $f(y) = ny^{n-1}$, $0 \leq y \leq 1$.

This gives

$$E(Y) = \int_0^1 ny^n \, dy = \frac{n}{n+1}.$$

Similarly, $f_{X_{(1)}}(X_{(1)}) = n[1 - F(X_{(1)})]^{n-1} f(X_{(1)})$

Treat $F(X_{(1)})$ as a random variable giving the fraction of objects in the population having values of $X \leq X_{(1)}$.

Let $Y = F(X_{(1)})$. Then $dy = f(X_{(1)})dx_{(1)}$, and thus $f(y) = n(1-y)^{n-1}$, $0 \leq y \leq 1$.

This gives

$$E(Y) = \int_0^1 ny(1-y)^{n-1} \, dy$$

The family of Beta distributions is defined by p.d.f.'s of the form

$$g(x) \;=\; \begin{cases} [\beta(r,s)]^{-1}x^{r-1}(1-x)^{s-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\beta(r,s) = \Gamma(r)\Gamma(s)/\Gamma(r+s)$.

Thus,

$$\begin{aligned} E(Y) &= n\int_0^1 y(1-y)^{n-1}\,dy \;=\; n\beta(2,n) \\ &= \frac{n\Gamma(2)\Gamma(n)}{\Gamma(n+2)} \;=\; \frac{n!1!}{(n+1)!} \;=\; \frac{1}{n+1} \end{aligned}$$

9–23. (a) 2.73
(b) 11.34
(c) 34.17
(d) 20.48

9–24. (a) 2.228
(b) 0.687
(c) 1.813

9–25. (a) 1.63
(b) 2.85
(c) 0.241
(d) 0.588