CHAPTER 2

2-1 We use De Morgan's law:

(a) $\overline{\bar{A}+\bar{B}} + \overline{\bar{A}+B} = AB + A\bar{B} = A(B+\bar{B}) = A$

(b) $(A+B)(\overline{AB}) = (A+B)(\bar{A}+\bar{B}) = A\bar{B} + B\bar{A}$

because $A\bar{A} = \{\emptyset\}$ $\quad B\bar{B} = \{\emptyset\}$

2-2 If $A = \{2 \le x \le 5\}$ $\quad B = \{3 \le x \le 6\}$ $\quad S = \{-\infty < x < \infty\}$ then

$A+B = \{2 \le x \le 6\}$ $\quad AB = \{3 \le x \le 5\}$

$$(A+B)(\overline{AB}) = \{2 \le x \le 6\}\ [\{x<3\} + \{x>5\}]$$
$$= \{2 \le x < 3\} + \{5 < x \le 6\}$$

2-3 If $AB = \{\emptyset\}$ then $A \subset \bar{B}$ hence

$$P(A) \le P(\bar{B})$$

2-4 (a) $P(A) = P(AB) + P(A\bar{B})$ $\qquad P(B) = P(AB) + P(\bar{A}B)$

If, therefore, $P(A) = P(B) = P(AB)$ then

$P(A\bar{B}) = 0 \qquad P(\bar{A}B) = 0$ $\qquad$ hence

$$P(\bar{A}B + A\bar{B}) = P(\bar{A}B) + P(A\bar{B}) = 0$$

(b) If $P(A) = P(B) = 1$ then $1 = P(A) \le P(A+B)$ hence

$$1 = P(A+B) = P(A) + P(B) - P(AB) = 2 - P(AB)$$

This yields $P(AB) = 1$

2-5 From (2-13) it follows that

$$P(A+B+C) = P(A) + P(B+C) - P[A(B+C)]$$
$$P(B+C) = P(B) + P(C) - P(BC)$$
$$P[A(B+C)] = P(AB) + P(AC) - P(ABC)$$

because $ABAC = ABC$. Combining, we obtain the desired result.

Using induction, we can show similarly that

$$P(A_1 + A_2 + \cdots + A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$
$$- P(A_1A_2) - \cdots - P(A_{n-1}A_n)$$
$$+ P(A_1A_2A_3) + \cdots + P(A_{n-2}A_n)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\pm P(A_1A_2\cdots A_n)$$

2-6 Any subset of S contains a countable number of elements, hence, it can be written as a countable union of elementary events. It is therefore an event.

2-7 Forming all unions, intersections, and complements of the sets {1} and {2,3}, we obtain the following sets:

{∅}, {1}, {4}, {2,3}, {1,4}, {1,2,3}, {2,3,4}, {1,2,3,4}

2-8 If $A \subset B, P(A) = 1/4$, and $P(B) = 1/3$, then

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

2-9

$$P(A|BC)P(B|C) = \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(C)} = \frac{P(ABC)}{P(C)} = P(AB|C)$$

$$P(A|BC)P(B|C)P(C) = \frac{P(ABC)}{P(BC)}\frac{P(BC)}{P(C)}P(C) = P(ABC)$$

2-10 We use induction. The formula is true for $n=2$ because $P(A_1A_2) = P(A_2|A_1)P(A_1)$. Suppose that it is true for n. Since

$$P(A_{n+1}A_n \cdots A_1) = P(A_{n+1}|A_n \cdots A_2A_1)P(A_1 \cdots A_n)$$

we conclude that it must be true for $n+1$.

2-11 First solution. The total number of m element subsets equals $\binom{n}{m}$ (see Probl. 2-26). The total number of m element subsets containing ζ_o equals $\binom{n-1}{m-1}$. Hence

$$p = \binom{n}{m}\Big/\binom{n-1}{m-1} = \frac{m}{n}$$

Second solution. Clearly, $P\{\zeta_o|A_m\} = m/n$ is the probability that ζ_o is in a specific A_m. Hence (total probability)

$$p = \sum P\{\zeta_o|A_m\}p(A_m) = \frac{m}{n}\sum P(A_m) = \frac{m}{n}$$

where the summation is over all sets A_m.

2-12 (a) $P\{6 \le t \le 8\} = \frac{2}{10}$

(b) $P\{6 \le t \le 8 | t > 5\} = \frac{P\{6 \le t \le 8\}}{P\{t > 5\}} = \frac{2}{5}$

2-13 From (2-27) it follows that

$$P\{t_o \le t \le t_o + t_1 | t \ge t_o\} = \int_{t_o}^{t_o + t_1} \alpha(t)dt \Big/ \int_{t_o}^{\infty} \alpha(t)dt$$

$$P\{t \le t_1\} = \int_0^{t_1} \alpha(t)dt$$

Equating the two sides and setting $t_1 = t_0 + \Delta t$ we obtain

$$\alpha(t_o) \Big/ \int_{t_o}^{\infty} \alpha(t)dt = \alpha(0)$$

for every t_o. Hence,

$$-\ln \int_{t_o}^{\infty} \alpha(t)dt = \alpha(0)t_o \qquad \int_{t_o}^{\infty} \alpha(t)dt = e^{-\alpha(0)t_o}$$

Differentiating the setting $c = \alpha(0)$, we conclude that

$$\alpha(t_o) = c\,e^{ct} \qquad P\{t \le t_1\} = 1 - e^{-ct_1}$$

2-14 If A and B are independent, then $P(AB) = P(A)P(B)$. If they are mutually exclusive, then $P(AB) = 0$. Hence, A and B are mutually exclusive and independent iff $P(A)P(B) = 0$.

2-15 Clearly, $A_1 = A_1A_2 + A_1\bar{A}_2$ hence

$$P(A_1) = P(A_1A_2) + P(A_1\bar{A}_2)$$

If the events A_1 and $\bar{A}_2$ are independent, then

$$P(A_1\bar{A}_2) = P(A_1) - P(A_1A_2) = P(A_1) - P(A_1)P(A_2)$$
$$= P(A_1)[1 - P(A_2)] = P(A_1)P(\bar{A}_2)$$

hence, the events A_1 and $\bar{A}_2$ are independent. Furthermore, S is independent with any A because SA = A. This yields

$$P(SA) = P(A) = P(S)P(A)$$

Hence, the theorem is true for $n = 2$. To prove it in general we use induction: Suppose that A_{n+1} is independent of $A_1, \ldots, A_n$. Clearly, A_{n+1} and $\bar{A}_{n+1}$ are independent of $B_1, \ldots, B_n$. Therefore

$$P(B_1 \cdots B_n A_{n+1}) = P(B_1 \cdots B_n)P(A_{n+1})$$
$$P(B_1 \cdots B_n \bar{A}_{n+1}) = P(B_1 \cdots B_n)P(\bar{A}_{n+1})$$

2.16 The desired probabilities are given by (a)

$$\frac{\binom{m-1}{k-1}}{\binom{n}{k}}$$

(b)

$$\frac{\binom{m}{k}}{\binom{n}{k}}$$

2.17 Let A_1, A_2 and A_3 represent the events

$A_1 =$ *"ball numbered less than or equal to m is drawn"*
$A_2 =$ *"ball numbered m is drawn"*
$A_3 =$ *"ball numbered greater than m is drawn"*

$$
\begin{aligned}
P(A_1 \text{ occurs } n_1 = k-1, \quad & A_2 \text{ occurs } n_2 = 1 \text{ and } A_3 \text{ occurs } n_3 = 0) \\
&= \frac{(n_1+n_2+n_3)!}{n_1!\, n_2!\, n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3} \\
&= \frac{k!}{(k-1)!} \left(\frac{m}{n}\right)^{k-1} \left(\frac{1}{n}\right) \\
&= \frac{k}{n} \left(\frac{m}{n}\right)^{k-1}
\end{aligned}
$$

2.18 All cars are equally likely so that the first car is selected with probability $p = 1/3$. This gives the desired probability to be

$$\binom{10}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^7 = 0.26$$

2.19 $P\{\text{"drawing a white ball "}\} = \dfrac{m}{m+n}$
$P(\text{"atleat one white ball in } k \text{ trials "})$

$$
\begin{aligned}
&= 1 - P(\text{"all black balls in } k \text{ trials"}) \\
&= 1 - \frac{\binom{n}{k}}{\binom{m+n}{k}}
\end{aligned}
$$

2.20 Let $D = 2r$ represent the penny diameter. So long as the center of the penny is at a distance of r away from any side of the square, the penny will be entirely inside the square. This gives the desired probability to be

$$\frac{(1-2r)^2}{1} = \left(1 - \frac{3}{4}\right)^2 = \frac{1}{16}.$$

2.21 Refer to Example 3.14.

(a) Using (3.39), we get

$$P(\text{"}all\ one-digit\ numbers\text{"}) = \frac{\binom{9}{6}\binom{42}{0}}{\binom{51}{6}} = 5 \times 10^{-6}.$$

(b)

$$P(\text{"}two\ one\text{–}digit\ and\ four\ two\text{–}digit\ numbers\text{"}) = \frac{\binom{9}{2}\binom{42}{4}}{\binom{51}{6}} = 0.224.$$

2-22 The number of equations of the form $P(A_iA_k) = P(A_i)P(A_k)$ equals $\binom{n}{2}$. The number of equations involving r sets equals $\binom{n}{r}$. Hence the total number N of such equations equals

$$N = \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n}$$

And since

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = (1+1)^n = 2^n$$

we conclude that

$$N = 2^n - \binom{n}{0} - \binom{n}{1} = 2^n - 1 - n$$

2-23 We denote by B_1 and B_2 respectively the balls in boxes 1 and 2 and by R the set of red balls. We have (assumption)

$$P(B_1) = P(B_2) = 0.5 \qquad P(R|B_1) = 0.999 \qquad P(R|B_2) = 0.001$$

Hence (Bayes' theorem)

$$P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R|B_1)P(B_1) + P(R|B_2)P(B_2)} = \frac{0.999}{0.999 + 0.001} = 0.999$$

2-24 We denote by B_1 and B_2 respectively the ball in boxes 1 and 2 and by D all pairs of defective parts. We have (assumption)

$$P(B_1) = P(B_2) = 0.5$$

To find $P(D|B_1)$ we proceed as in Example 2-10:

First solution. In box B_1 there are 1000×999 pairs. The number of pairs with both elements defective equals 100×99. Hence,

$$P(D|B_1) = \frac{100 \times 99}{1000 \times 999}$$

Second solution. The probability that the first bulb selected from B_1 is defective equals 100/1000. The probability that the second is defective assuming the first was effective equals 99/999. Hence,

$$P(D|B_1) = \frac{100}{1000} \times \frac{99}{999}$$

We similarly find

$$P(D|B_2) = \frac{100}{2000} \times \frac{99}{1999}$$

(a) $P(D) = P(D|B_1)P(B_1) + P(D|B_2)P(B_2) = 0.0062$

(b) $P(B_1|D) = \dfrac{P(D|B_1)P(B_1)}{P(D)} = 0.80$

2-25 Reasoning as in Example 2-13, we conclude that the probability that the bus and the train meet equals

$$(10 + x)60 - \frac{10^2}{2} - \frac{x^2}{2}$$

Equating with 0.5, we find $x = 60 - 10\sqrt{11}$.

2-26 We wish to show that the number $N_n(k)$ of the element subsets of S equals

$$N_n(k) = \frac{n(n-1) \cdots (n-k+1)}{1 \cdot 2 \cdots k}$$

This is true for $k = 1$ because the number of 1-element subsets equals n. Using induction in k, we shall show that

$$N_n(k+1) = N_n(k)\,\frac{n-k}{k+1} \qquad 1 < k < n \qquad \text{(i)}$$

We attach to each k-element subset of S one of the remaining $n-k$ elements of S. We, then, form $N_n(k)(n-k)$ $k+1$-element subsets. However, these subsets are not all different. They form groups each of which has $k+1$ identical elements. We must, therefore, divide by $k+1$.

2-27 In this experiment we have 8 outcomes. Each outcome is a selection of a particular coin and a specific sequence of heads or tails; for example fhh is the outcome "we selected the fair coin and we observed hh". The event F = {the selected coin is fair} consists of the four outcomes fhh, fht, fth and fhh. Its complement $\overline{F}$ is the selection of the two-headead coin. The event HH = {heads at both tosses} consists of two outcomes. Clearly,

$$P(F) = P(\overline{F}) = \frac{1}{2} \qquad P(HH|F) = \frac{1}{4} \qquad P(HH|\overline{F}) = 1$$

Our problem is to find P(F|HH). From (2-41) and (2-43) it follows that

$$P(HH) = P(HH|F)P(F) + P(HH|\overline{F})P(\overline{F}) = \frac{5}{8}$$

$$P(F|HH) = \frac{P(HH|F)P(F)}{P(HH)} = \frac{1/4 \times 1/2}{5/8} = \frac{1}{5}$$
