3.1 (a) $P(\text{A } \textit{occurs atleast twice in n trials})$

$$= 1 - P(\text{A } \textit{never occurs in n trials}) - P(\text{A } \textit{occurs once in n trials})$$

$$= 1 - (1-p)^n - np(1-p)^{n-1}$$

(b) $P(\text{A } \textit{occurs atleast thrice in n trials})$

$$= 1 - P(\text{A never occurs in } n \text{ trials}) - P(\text{A occurs once in } n \text{ trials})$$
$$-P(\text{A occurs twice in } n \text{ trials})$$

$$= 1 - (1-p)^n - np(1-p)^{n-1} - \frac{n(n-1)}{2} p^2 (1-p)^{n-2}$$

3.2

$$P(doublesix) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

$P(\text{"double six atleast three times in n trials"})$

$$= 1 - \binom{50}{0}\left(\frac{1}{36}\right)^0\left(\frac{35}{36}\right)^{50} - \binom{50}{1}\left(\frac{1}{36}\right)\left(\frac{35}{36}\right)^{49} - \binom{50}{2}\left(\frac{1}{36}\right)^2\left(\frac{35}{36}\right)^{48}$$

$$= 0.162$$

3–3 If A = {seven}, then

$$P(A) = \frac{6}{36} \qquad P(\bar{A}) = \frac{5}{6}$$

If the dice are tossed 10 times, then the probability that $\bar{A}$ will occur 10 times equals $(5/6)^{10}$. Hence, the probability p that {seven} will show at least once equals

$$1 - (5/6)^{10}$$

3–4 If k is the number of heads, then

$$P\{\text{even}\} = P\{k = 0\} + P\{k = 2\} + \cdots$$
$$= q^n + \binom{n}{2} p^2 q^{n-2} + \binom{n}{4} p^4 q^{n-4} + \cdots$$

But

$$1 = (q + q)^n = q^n + \binom{n}{1} p\, q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \cdots$$
$$(p - q)^n = q^n - \binom{n}{1} p\, q^{n-1} + \binom{n}{2} p^2\, q^{n-2} - \cdots$$

Adding, we obtain

$$1 + (p - q)^n = 2\, P\{\text{even}\}$$

3–5 In this experiment, the total number of outcomes is the number $\binom{N}{n}$ of ways of picking n out of N objects. The number of ways of picking k out of the K good components equals $\binom{K}{k}$ and the number of ways of picking n-k out of the N-K defective components equals $\binom{N-K}{n-k}$. Hence, the number of ways of picking k good components and n-k deafective components equals $\binom{K}{k}\binom{N-K}{n-k}$. From this and (2-25) it follows that

$$p = \binom{K}{k}\binom{N-K}{n-k} \Big/ \binom{N}{n}$$

3.6 (a)

$$p_1 = 1 - \left(\frac{5}{6}\right)^6 = 0.665$$

(b)

$$1 - \left(\frac{5}{6}\right)^{12} - \binom{12}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{11} = 0.619$$

(c)

$$1 - \left(\frac{5}{6}\right)^{18} - \binom{18}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{17} - \binom{18}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{16} = 0.597$$

3.7 (a) Let n represent the number of wins required in 50 games so that the net gain or loss *does not* exceed \$1. This gives the net gain to be

$$-1 < n - \frac{50 - n}{4} < 1$$

$$16 < n < 17.3$$

$$n = 17$$

$$P(\text{net gain } \textit{does not} \text{ exceed \$1}) = \binom{50}{17}\left(\frac{1}{4}\right)^{17}\left(\frac{3}{4}\right)^{33} = 0.432$$

$$P(\text{net gain or loss exceeds \$1}) = 1 - 0.432 = 0.568$$

(b) Let n represent the number of wins required so that the net gain or loss *does not* exceed \$5. This gives

$$-5 < n - \frac{(50 - n)}{2} < 5$$

$$13.3 < n < 20$$

$$P(\text{net gain } \textit{does not} \text{ exceed \$5}) = \sum_{n = 14}^{19}\binom{50}{n}\left(\frac{1}{4}\right)^{n}\left(\frac{3}{4}\right)^{50-n} = 0.349$$

$$P(\text{net gain or loss exceeds \$5}) = 1 - 0.349 = 0.651$$

3.8 Define the events

A=" r successes in n Bernoulli trials"

B="success at the i^{th} Bernoulli trial"

C="$r-1$ successes in the remaining $n-1$ Bernoulli trials excluding the i^{th} trial"

$$P(A) = \binom{n}{r} p^r \, q^{n-r}$$

$$P(B) = p$$

$$P(C) = \binom{n-1}{r-1} p^{r-1} \, q^{n-r}$$

We need

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(BC)}{P(A)} = \frac{P(B)\,P(C)}{P(A)} = \frac{r}{n}.$$

3.9 There are $\binom{52}{13}$ ways of selecting 13 cards out of 52 cards. The number of ways to select 13 cards of any suit (out of 13 cards) equals $\binom{13}{13} = 1$. Four such (mutually exclusive) suits give the total number of favorable outcomes to be 4. Thus the desired probability is given by

$$\frac{4}{\binom{52}{13}} = 6.3 \times 10^{-12}$$

3.10 Using the hint, we obtain

$$p\,(N_{k+1} - N_k) = q\,(N_k - N_{k-1}) - 1$$

Let

$$M_{k+1} = N_{k+1} - N_k$$

so that the above iteration gives

$$\begin{aligned} M_{k+1} &= \frac{q}{p}\,M_k - \frac{1}{p} \\ &= \begin{cases} \left(\frac{q}{p}\right)\,M_1 - \dfrac{1}{p-q}\,\left\{1 - (\frac{q}{p})^i\right\}, & p \neq q \\ M_1 - \frac{k}{p}, & p = q \end{cases} \end{aligned}$$

This gives

$$\begin{aligned} N_i &= \sum_{k=0}^{i-1} M_{k+1} \\ &= \begin{cases} \left(M_1 + \dfrac{1}{p-q}\right) \sum\limits_{k=0}^{i-1} \left(\frac{q}{p}\right)^k - \dfrac{i}{p-q}, & p \neq q \\ iM_1 - \dfrac{i(i-1)}{2p}, & p = q \end{cases} \end{aligned}$$

where we have used $N_o = 0$. Similarly $N_{a+b} = 0$ gives

$$M_1 + \frac{1}{p-q} = \frac{a+b}{p-q} \cdot \frac{1-q/p}{1-(q/p)^{a+b}}.$$

Thus

$$N_i = \begin{cases} \dfrac{a+b}{p-q} \cdot \dfrac{1-(q/p)^i}{1-(q/p)^{a+b}} - \dfrac{i}{p-q}, & p \neq q \\ i(a+b-i), & p = q \end{cases}$$

which gives for $i = a$

$$N_a = \begin{cases} \frac{a+b}{p-q} \cdot \frac{1-(q/p)^a}{1-(q/p)^{a+b}} - \frac{a}{p-q}, & p \neq q \\ ab, & p = q \end{cases}$$

$$= \begin{cases} \frac{b}{2p-1} - \frac{a+b}{2p-1} \cdot \frac{1-(q/p)^b}{1-(q/p)^{a+b}} - \frac{a}{p-q}, & p \neq q \\ ab, & p = q \end{cases}$$

3.11

$$P_n = pP_{n+\alpha} + qP_{n-\beta}$$

Arguing as in (3.43), we get the corresponding iteration equation

$$P_n = P_{n+\alpha} + qP_{n-\beta}$$

and proceed as in Example 3.15.

3.12 Suppose one bet on $k = 1, 2, \cdots, 6$.
Then

$$p_1 = P(k \textit{ appears on one dice}) = \binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2$$

$$p_2 = P(k \textit{ appear on two dice}) = \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)$$

$$p_3 = P(k \textit{ appear on all the tree dice}) = \left(\frac{1}{6}\right)^3$$

$$p_0 = P(k \textit{ appear none}) = \left(\frac{5}{6}\right)^3$$

Thus, we get

$$\text{Net gain} = 2p_1 + 3p_2 + 4p_3 - p_0 = 0.343.$$