CHAPTER 4

4-1 From the evenness of $f(x)$: $1 - F(x) = F(-x)$.

From the definition of x_u: $u = F(x_u)$, $1 - u = F(x_{1-u})$. Hence

$$1 - u = 1 - F(x_u) = F(-x_u) = F(x_{1-u}) \qquad -x_u = x_{1-u}$$

4-2 From the symmetry of $f(x)$: $1 - F(\eta+a) = F(\eta-a)$. Hence [see (4-8)]

$$P\{\eta-a < \tilde{x} < \eta+a\} = F(\eta+a) - F(\eta-a) = 2F(\eta+a) - 1$$

This yields

$$1-\alpha = 2F(\eta+a) - 1 \qquad F(\eta+a) = 1 - \alpha/2 \qquad \eta+a = x_{1-\alpha/2}$$

$$F(a-\eta) = \alpha/2 \qquad a-\eta = x_{\alpha/2}$$

4-3 (a) In a linear interpolation:

$$x_u \simeq x_a + \frac{x_b-x_a}{u_b-u_a}(u-u_a) \qquad \text{for } x_a < x_u < x_b$$

From Table 4-1 page 106

$$z_{0.9} \simeq 1.25 + \frac{0.00565}{0.00885} \times 0.05 = 1.2819$$

Proceeding simiplarly, we obtain

$u =$	0.9	0.925	0.95	0.975	0.99
$z_u =$	1.282	1.440	1.645	1.960	2.327

(b) If $\tilde{z}$ is such that $\tilde{x} = \eta + \sigma\tilde{z}$ then $\tilde{z}$ is N(0,1) and $G(z) = F_x(\eta+\sigma z)$. Hence,

$$u = G(z_u) = F_x(\eta+\sigma z_u) = F_x(x_u) \qquad x_u = \eta+\sigma z_u$$

4-4 $p_k - 2G(k) = 1 = 2\,\text{erf}k$

(a) From Table 4-1

$k =$	1	2	3
$p_k =$	0.6827	0.9545	0.9973

(b) From Table 3-1 with linear interpolation:

$p_k =$	0.9	0.99	0.999
$k =$	1.282	2.32	3.090

(c) $P\{\eta - z_u\sigma < \underset{\sim}{x} < \eta + z_u\sigma\} = 2G(z_u) - 1 = \gamma$

Hence, $G(z_u) = (1+\gamma)/2 \qquad u = (1+\gamma)/2$

4-5 (a) $F(x) = x$ for $0 \le x \le 1$; hence, $u = F(x_u) \doteq x_u$

(b) $F(x) = 1 - e^{-2x}$ for $x \ge 0$; hence, $u = 1 - e^{-2x_u}$

$$x_u = -\frac{1}{2}\ell n(1-u)$$

$u =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$x_u =$	0.0527	0.1116	0.1783	0.2554	0.3466	0.4581	0.6020	0.847	1.1513

4-6 Percentage of units between 96 and 104 ohms equals 100p where $p = P\{96 < \underset{\sim}{R} < 104\} = F(104) - F(96)$

(a) $F(R) = 0.1(R-95)$ for $95 \le R \le 105$. Hence,

$p = 0.1(104-95) - 0.1(96-95) = 0.8$

(b) $p = G(2.5) - G(-2.5) = 0.9876$

4-7 From (4-34), with $\alpha = 2$ and $\beta = 1/\lambda$ we get $f(x) = c^2 xe^{-cx}U(x)$

$$F(x) = c^2 \int_0^x ye^{-cy}dy = 1 - e^{-cx} - cxe^{-cx}$$

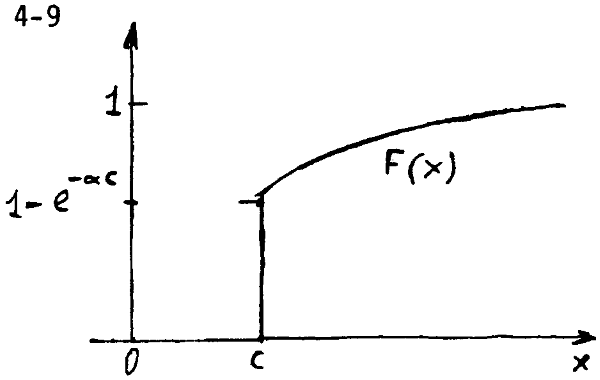
4-8 $\{(\underset{\sim}{x} - 10)^2 < 4\} = \{8 < \underset{\sim}{x} < 12\}$

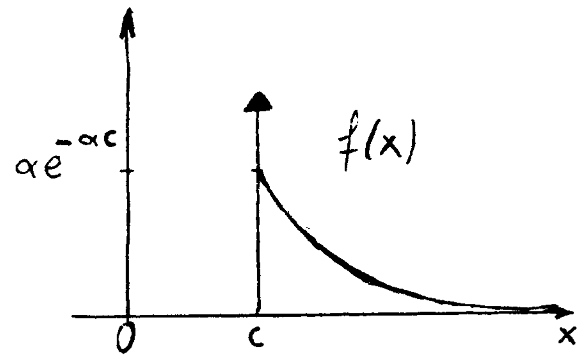
$$P\{(\underset{\sim}{x} - 10)^2 < 4\} = G(12 - 10) - G(8 - 10) = 0.954$$

$$f(x|(\underset{\sim}{x} - 10)^2 < 4\} = \frac{f(x)}{P\{8 < \underset{\sim}{x} < 12\}} = \frac{1}{0.954\sqrt{2\pi}} e^{-\frac{(x-10)^2}{2}}$$

for $8 < x < 12$ and zero otherwise

4-9





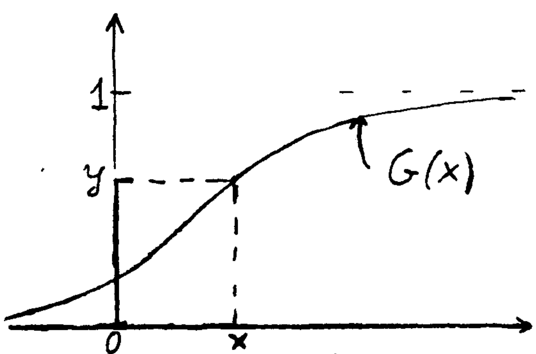
$$F(x) = (1 - e^{-\alpha x})U(x-c) \qquad f(x) = (1 - e^{-\alpha c})\delta(x-c) + e^{-\alpha x}U(x-c)$$

4-10 (a) $P\{1 \leq x \leq 2\} = G(\frac{2}{2}) - G(\frac{1}{2}) = 0.1499$

(b) $P\{1 \leq \underset{\sim}{x} \leq 2|\underset{\sim}{x} \geq 1\} = \dfrac{G(1) - G(0.5)}{1 - G(0.5)} = \dfrac{0.1499}{0.3085} = 0.4857$

because $\{1 \leq \underset{\sim}{x} \leq 2, \underset{\sim}{x} \geq 1\} = \{1 \leq \underset{\sim}{x} \leq 2\}$

4-11



$x = G^{-1}(y) = H(y)$

If $\underset{\sim}{x}(t_1) \leq x$

then

$t_i \leq y = G(x)$

Hence,

$$P\{\underset{\sim}{x} \leq x\} = P\{t_i \leq y\} = y = G(x)$$

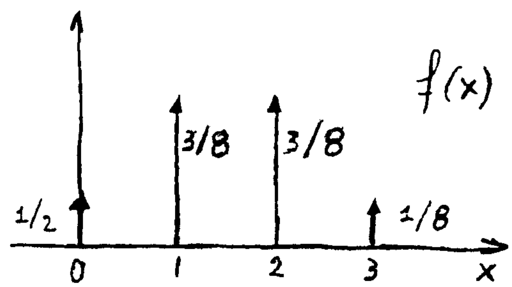
4-12 (a) $P\{\underset{\sim}{x} < 1024\} = G(\frac{1024 - 1000}{20}) = G(1.2) = 0.8849$

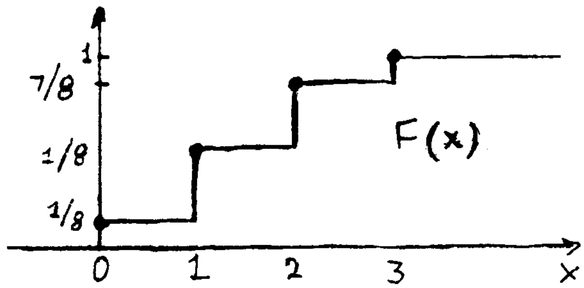
(b) $P\{\underset{\sim}{x} < 1024 | \underset{\sim}{x} > 961\} = \frac{P\{961 < \underset{\sim}{x} < 1024\}}{P\{\underset{\sim}{x} > 961\}}$

$= \frac{G(1.2) - G(-1.95)}{1 - G(-1.95)} = 0.8819$

(c) $P\{31 < \sqrt{\underset{\sim}{x}} \leq 32\} = P\{961 < \underset{\sim}{x} \leq 1024\} = 0.8593$

4-13 $P\{\underset{\sim}{x} = 0\} = \frac{1}{8}$ $P\{\underset{\sim}{x} = 1\} = \frac{3}{8}$ $P\{\underset{\sim}{x} = 2\} = \frac{3}{8}$ $P\{\underset{\sim}{x} = 3\} = \frac{1}{8}$





4-14 (a) 1. $f_x(x) = \frac{1}{2^{900}} \sum_{k=0}^{900} \binom{900}{k} \delta(x-k)$

2. $f_x(x) = \frac{1}{15\sqrt{2\pi}} \sum_{k=0}^{900} e^{-(k-450)^2/450} \delta(x-k)$

(b) $P\{435 \leq x \leq 460\} = G(\frac{10}{15}) - G(-\frac{15}{15}) = 0.5888$

4-15 If $x > b$ then $\{\underset{\sim}{x} \leq x\} = S$ $F(x) = 1$

If $x < a$ then $\{\underset{\sim}{x} \leq x\} = \{\emptyset\}$ $F(x) = 0$

4-16 If $\underset{\sim}{y}(\zeta_i) \leq w$, then $\underset{\sim}{x}(\zeta_i) \leq w$ because $\underset{\sim}{x}(\zeta_i) \leq \underset{\sim}{y}(\zeta_i)$.

Hence,

$\{\underset{\sim}{y} \leq w\} \subset \{\underset{\sim}{x} \leq w\}$ $P\{\underset{\sim}{y} \leq w\} \leq P\{\underset{\sim}{x} \leq w\}$

Therefore $F_y(w) \leq F_x(w)$

4-17 From (4-80)

$$f(x) = kx\, e^{-\int_0^x kt\,dt} = kx\, e^{-kx^2/2}$$

4-18 It follows from (2-41) with

$$A_1 = \{\underset{\sim}{x} \le x\} \qquad A_2 = \{\underset{\sim}{x} > x\}$$

4-19 It follows from

$$F_x(x|A) = \frac{P\{\underset{\sim}{x} \le x, A\}}{P(A)} \qquad P\{A|\underset{\sim}{x} \le x\} = \frac{P\{\underset{\sim}{x} \le x, A\}}{P\{\underset{\sim}{x} \le x\}}$$

4-20 We replace in (4-80) all probabilities with conditional probabilities assuming $\{\underset{\sim}{x} \le x_o\}$. This yields

$$\int_{-\infty}^{\infty} P(A|\underset{\sim}{x} = x, \underset{\sim}{x} \le x_o) f(x|\underset{\sim}{x} \le x_o)dx = P(A|\underset{\sim}{x} \le x_o)$$

But $f(x|\underset{\sim}{x} \le x_o) = 0$ for $x > x_o$ and

$\{\underset{\sim}{x} = x, \underset{\sim}{x} \le x_o\} = \{\underset{\sim}{x} = x\}$ for $x \le x_o$. Hence,

$$\int_{-\infty}^{x_o} P(A|\underset{\sim}{x} = x) f(x|\underset{\sim}{x} \le x_o)dx = P(A|\underset{\sim}{x} \le x_o)$$

Writing a similar equation for $P(B|\underset{\sim}{x} \le x_o)$ we conclude that, if $P(A|\underset{\sim}{x} = x) = P(B|\underset{\sim}{x} = x)$ for $x \le x_o$, then $P(A|\underset{\sim}{x} \le x_o) = P(B|\underset{\sim}{x} \le x_o)$

4-21 (a) Clearly, f(p) = 1 for $0 \le p \le 1$ and 0 otherwise; hence

$$P\{0.3 \le \underset{\sim}{p} \le 0.7\} = \int_{0.3}^{0.7} dp = 0.4$$

(b) We wish to find the conditional probability $P\{0.3 \le \underset{\sim}{p} \le 0.7 | A\}$ where A = {6 heads in 10 tosses}. Clearly $P\{A|\underset{\sim}{p}=p\} = p^6(1-p)^4$. Hence, [see (4-81)]

$$f(p|A) = \frac{p^6(1-p)^4}{\int_0^1 p^6(1-p)^4 dp} = \frac{p^6(1-p)^4}{4329\times10^{-7}}$$

This yields

$$P\{0.3 \le \underset{\sim}{p} \le 0.7|A\} = \int_{0.3}^{0.7} f(p|A)dp = \frac{10^7}{4329}\int_{0.3}^{0.7} p^6(1-p)^4 dp = 0.768$$

4-22 (a) In this problem, f(p) = 5 for $0.4 \le \underset{\sim}{p} \le 0.6$ and zero otherwise; hence [see(4-82)]

$$P(H) = 5\int_{0.4}^{0.6} p\,dp = 0.5$$

(b) With A = {60 heads in 100 tosses} it follows from (4-82) that

$$f(p|A) = p^{60}(1-p)^{40} \Big/ \int_{0.4}^{0.6} p^{60}(1-p)^{40}dp$$

for $0.4 \le p \le 0.6$ and 0 otherwise. Replacing f(p) by f(p|A) in (4-82), we obtain

$$P(H|A) = \int_{0.4}^{0.6} p f(p|A)dp = 0.56$$

4-23 $n = 900 \qquad p = q = 0.5 \qquad np = 450 \qquad \sqrt{npq} = 15$

$$k_1 = 420 \qquad k_2 = 465 \qquad \frac{k_2 - np}{\sqrt{npq}} = 1 \qquad \frac{k_1 - np}{\sqrt{npq}} = -2$$

$$P\{420 \le k \le 465\} = G(1) - [1 - G(-2)] = G(1) + G(2) - 1 = 0.819$$

4-24 For a fair coin $\sqrt{npq} = \sqrt{n}/2$. If

$k_1 = 0.49n$ and $k_2 = 0.52n$ then

$$\frac{k_2 - np}{\sqrt{npq}} = \frac{0.52n - n/2}{\sqrt{n}/2} = 0.04\sqrt{n} \qquad \frac{k_1 - np}{\sqrt{npq}} = -0.02\sqrt{n}$$

$$P\{k_1 \le k \le k_2\} = G(0.04\sqrt{n}) + G(0.02\sqrt{n}) - 1 \ge 0.9$$

From Table 4-1 (page 106) it follows that

$$0.02\sqrt{n} > 1.3 \qquad n > 65^2$$

4-25

(a) Assume n = 1,000 (Note correction to the problem)

$P(A) = 0.6 \qquad np = 600 \qquad npq = 240 \qquad k_2 = 650 \qquad k_1 = 550$

$$\frac{k_2 - np}{\sqrt{npq}} = \frac{50}{\sqrt{240}} = 3.23 \qquad\qquad \frac{k_1 - np}{\sqrt{npq}} = -3.23$$

$P\{550 \leq k \leq 650\} = 2G(3.23) - 1 = 0.999$

(b) $P\{0.59n \leq k \leq 0.61n\} = 2G\left(\frac{0.01n}{\sqrt{0.24n}}\right) - 1$

$$= 2G\left(\sqrt{\frac{n}{2400}}\right) - 1 = 0.476$$

Hence, (Table 3-1) $n \simeq 9220$

4-26 With a = 0, b = T/4 it follows that

$p = 1-e^{-1/4} = 0.22 \qquad np = 220 \qquad npq = 171.6 \qquad k_2 = 100$

$$\frac{k_2 - np}{\sqrt{npq}} = -9.16 \quad \text{and (4-100) yields}$$

$P\{0 \leq k \leq 100\} \simeq G(-9.16) \simeq 0.$

4-27 The event

A = {k heads show at the first n tossings but not earlier}

occurs iff the following two events occur

B = {k-1 heads show at the first n-1 tossing}

C = {heads show at the nth tossing}

And since these two events are independent and

$$P(B) = \binom{n-1}{k-1} p^{k-1} q^{n-1-(k-1)} \qquad\qquad P(C) = p$$

we conclude that

$$P(A) = P(B)P(C) = \binom{n-1}{k-1} p^k q^{n-k}$$

4-28 $$-\frac{d}{dx}\left(\frac{1}{x}e^{-x^2/2}\right) = \left(1+\frac{1}{x^2}\right)e^{-x^2/2} > e^{-x^2/2}$$

Multiplying by $1/\sqrt{2\pi}$ and integrating from x to ∞, we obtain

$$\frac{1}{x\sqrt{2\pi}}e^{-x^2/2} > \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\zeta^2/2}\,d\zeta = 1 - G(x)$$

because

$$\frac{1}{x}e^{-x^2/2} \xrightarrow[x\to\infty]{} 0$$

The first inequality follows similarly because

$$-\frac{d}{dx}\left[\left(\frac{1}{x}-\frac{1}{x^3}\right)e^{-x^2/2}\right] = \left(1-\frac{3}{x^4}\right)e^{-x^2/2} < e^{-x^2/2}$$

4-29 If $P(A) = p$ then $P(\overline{A}) = 1-p$. Clearly $P_1 = 1-Q_1$ where Q_1 equals the probability that A does not occur at all. If $pn \ll 1$, then $Q_1 = (1-p)^n \simeq 1 - np \quad P_1 \simeq p$

4-30 With p = 0.02, n = 100, k = 3, it follows from (4-107) that the unknown probability equals

$$\binom{100}{3}(0.02)^3(0.98)^{97} \simeq \frac{2^3}{3!}e^{-2} = \frac{4}{3}e^{-2}$$

4-31 With $n = 3$, $r = 3$, $k_1 = 2$, $k_2 = 2$, $k_3 = 1$, $p_1 = p_2 = p_3 = 1/6$, it follows from (4-102) that the unknown probability equals

$$\frac{5!}{1!2!2!}\,\frac{1}{6^6} = 0.00386$$

4-32 With $r = 2$, $k_1 = k$, $k_2 = n-k$, $p_1 = p$, $p_2 = 1-p = q$, we obtain

$$k_1 - np_1 = k - np \qquad k_2 - np_2 = n-k-nq = np - k$$

Hence, the bracket in (4-103) equals

$$\frac{(k_1-np_1)^2}{np_1} + \frac{(k_2-np_2)^2}{np_2} = \frac{(k-np)^2}{n}\left(\frac{1}{p}+\frac{1}{q}\right) = \frac{(k-np)^2}{npq}$$

as in (4-90).

4-33 $P(M) = 2/36$ $\quad P(\overline{M}) = 34/36$. The events M and $\overline{M}$ form a partition, hence, [see (2-41)]

$$P(A) = P(A|M)P(M) + P(A|\overline{M})P(\overline{M}) \qquad \text{(i)}$$

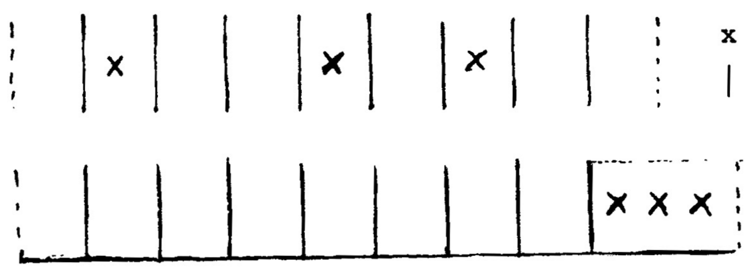
Clearly, $P(A|M) = 1$ because, if M occurs at first try, X wins. The probability that X wins after the first try equals $P(A|\overline{M})$. But in the experiment that starts at the second rolling, the first player is Y and the probability that he wins equals $P(\overline{A}) = 1-p$. Hence, $P(A|\overline{M}) = P(\overline{A}) = 1-p$. And since $P(M) = 1/18$ $P(\overline{M}) = 17/18$ (i) yields

$$p = \frac{1}{18} + (1-p)\,\frac{17}{18} \qquad p = \frac{18}{35}$$

4-34

(a) Each of the n particles can be placed in any one of the m boxes. There are n particles, hence, the number of possibilities equals $N - m^n$. In the m preselected boxes, the particles can be placed in $N_A = n!$ ways (all permutations of n objects). Hence $p = n!/m^n$.

(b)



x n particles

| m-1 interior walls

All possibilities are obtained by permuting the n+m-1 objects consisting of the m-1 interior walls with and n particles. The (m-1)! permutations of the walls and the n! permutations of the particles must count as one. Hence

$$N = \frac{(n+m-1)!}{n!\,(m-1)!} \qquad N_A = 1$$

(c) Suppose that S is a set consisting of the m boxes. Each placing of the particles specifies a subset of S consisting of n elements (box). The number of such subsets equals $\binom{m}{n}$ (see Prob. 2-26). Hence,

$$N = \binom{m}{n} \qquad N_A = 1$$

4-35 If $k_1 + k_2 << n$, then $k_3, \simeq n$ and

$$k_3(p_1 + p_2) = [n-(k_1 + k_2)](p_1 + p_2) \simeq n(p_1 + p_2)$$

$$p_3 = 1 - (p_1 + p_2) \simeq e^{-(p_1+p_2)} \qquad p_3^{k_3} \simeq e^{-n(p_1+p_2)}$$

$$\frac{n!}{k_1!k_2!k_3!} = \frac{n(n-1)\cdots(n-k_3+1)}{k_1!k_2!} \simeq \frac{n^{k_1+k_2}}{k_1!k_2!}$$

Hence,

$$\frac{n!}{k_1!k_2!k_3!} p_1^{k_1} p_2^{k_2} p_3^{k_3} \simeq e^{-np_1}\frac{(np_1)^{k_1}}{k_1!} e^{-np_2}\frac{(np_2)^{k_2}}{k_2!}$$

4-36 The probability p that a particular point is in the interval (0,2) equals 2/100. (a) From (3-13) it follows that the probability p_1 that only one out of the 200 points is in the interval (0,2) equals

$$p_1 = \binom{200}{1} \times 0.02 \times 0.09^{199}$$

(b) With $np = 200 \times 0.02 = 4$ and $k = 1$, (3-41) yields $p_1 \simeq e^{-4} \times 4 = 0.073$
