CHAPTER 5

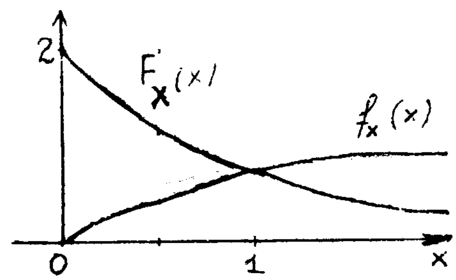
5-1 $\eta = 2\eta_x + 4 = 14 \qquad \sigma_y^2 = 4\sigma_x^2 = 16$

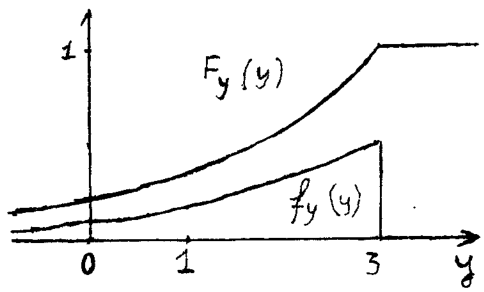
5-2 $\{\underset{\sim}{y} \le y\} = \{-4\underset{\sim}{x} + 3 \le \underset{\sim}{y}\}\ \{\underset{\sim}{x} \le (y-3)/4\}$. Hence

$$F_y(y) = P\left\{\underset{\sim}{x} \ge \frac{3-y}{4}\right\} = 1 - F_x\left(\frac{3-y}{4}\right) \qquad f_y(y) = \frac{1}{4} f_x\left(\frac{3-y}{4}\right)$$

Since $F_x(x) = (1-e^{-2x})U(x)$, this yields

$$F_y(y) = e^{(y-3)/2}U\left(\frac{y-3}{2}\right) \qquad f_y(y) = \frac{1}{2} e^{(y-3)/2}U\left(\frac{y-3}{2}\right)$$

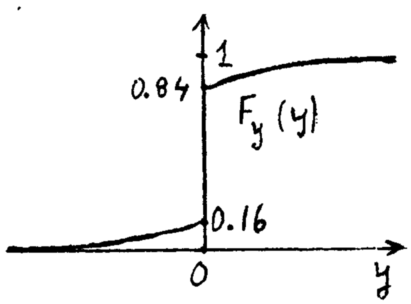


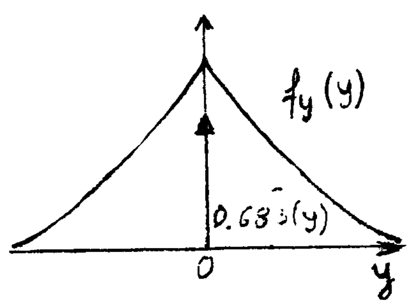


5-3 From Example 5-3 with $F_x = G(x/c)$:

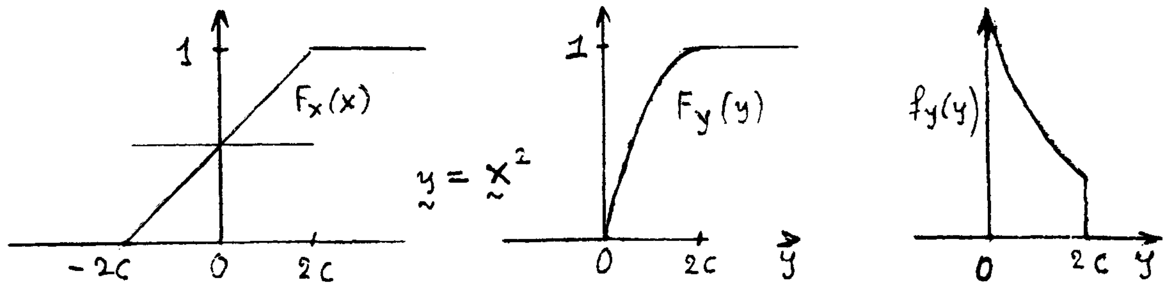
$$f_y(y) - \begin{cases} G(y/c+1) & y \ge 0 \\ G(y/c-1) & y < 0 \end{cases}$$

$$f_y(y) = 0\ 0.68\ \delta(y) + \frac{1}{c\sqrt{2\pi}}\left[e^{-(y+c)^2/2c^2}U(y) + e^{-(y-c)^2/2c^2}U(-y)\right]$$



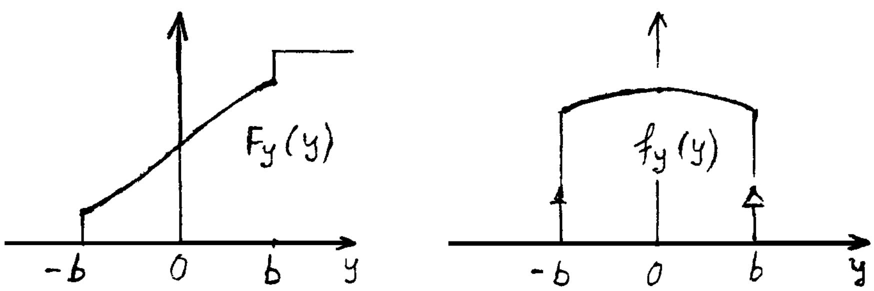


5-4 If $\underset{\sim}{y} = \underset{\sim}{x}^2$ and $F_x(x) = (x+2c)/4c$ for $|x| \le 2c$, then (see Example 5-2) $F_y(y) = \sqrt{y}\,/2c$ and $f_y(y) = 1/4\sqrt{y}$ for $0 < y < 2c$.



5-5 From Example 5-4 with $F_x(x) = G(x/b)$: For $|x| \le b$ $F_u(y) = G(y/b)$ and

$$f_y(y) = 0.16\delta(y+b) + \frac{1}{b\sqrt{2\pi}} e^{-y^2/2b^2} + 0.16\delta(y-b)$$



5-6 The equation $y = -\ell n x$ has a single solution $x = e^{-y}$ for $y > 0$ and no solutions for $y < 0$. Furthermore, $g'(x) = -1/x = -e^{y}$. Hence

$$f_y(y) = \frac{f_x(e^{-y})}{e^{y}} U(y) = e^{-y}U(y)$$

5-7 Clearly, $\underset{\sim}{z} \le z$ iff the number $\underset{\sim}{n}(0,z)$ of the points in the interval (0,z) is at least one. Hence,

$$F_z(z) = P\{\underset{\sim}{z} \le z\} = P\{\underset{\sim}{n}(0,z) > 0\} = 1 - P\{\underset{\sim}{n}(0,z) = 0\}$$

The probability p that a particular point is in the integral (0,z) equals z/100. With n = 200, k = 0, and p = z/100, (3-21) yields $P\{\underset{\sim}{n}(0,z) = 0\} = (1-p)^{200}$. Hence,

(a) $F_z(z) = 1 - \left(1 - \frac{z}{100}\right)^{100}$

(b) From (4-107) it follows thata $F_z(z) \simeq 1 - e^{-2z}$ for z << 100.

5.8

$$Y = \sqrt{X} \quad \Rightarrow \quad x_1 = y^2$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2y}$$

Thus

$$f_Y(y) = \frac{1}{\left|\frac{dy}{dx}\right|} f_X(x_1) = 2y f_X(y^2)$$

$$\frac{2y}{\lambda} e^{-y^2/\lambda} = \begin{cases} \frac{y}{\sigma^2} e^{-y^2/2\sigma^2}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

which represents Rayleigh density function (with $\lambda = 2\sigma^2$).

5-9 For both cases, $f_y(y) = 0$ for $y < 0$.

(a) If $y > 0$ and $|x| = y$, then $x_1 = y$, $x_2 = -y$. Hence

$$f_y(y) = [f_x(y) + f_x(-y)]U(y)$$

(b) If $y > 0$ and $e^{-x}U(x) = y$, then $x = -\ell n y$.

Furthermore, $P\{\underset{\sim}{y} = 0\} = P\{\underset{\sim}{x} \le 0\} = F_x(0)$. Hence

$$f_y(y) = F_x(0)\delta(y) + \frac{1}{y} f_x(-\ell n y)U(y)$$

5-10 (a) If $y \geq 0$ and $(x-1)U(x-1) = y$, then $\{\underset{\sim}{y} \leq y\} = \{\underset{\sim}{x} \leq y+1\}$.
If $y < 0$, then $\{\underset{\sim}{y} < y\} = \{\emptyset\}$

$$F_y(y) = F_x(1+y)U(y) = [1 - e^{-2(y+1)}]U(y)$$

$$f_y(y) = (1 - e^{-2})\delta(y) + 2e^{-2(y+1)}U(y)$$

(b) If $y > 0$ and $y = x^2$, then $\{\underset{\sim}{y} \leq y\} = \{-\sqrt{y} \leq \underset{\sim}{x} \leq \sqrt{y}\}$

$$F_y(y) = F_x(\sqrt{y}) - F_x(-\sqrt{y}) = (1 - e^{-2\sqrt{y}})U(y)$$

$$f_y(y) = \frac{1}{\sqrt{y}} e^{-2\sqrt{y}}U(y)$$

5-11 If $y = \arctan x$, then $\frac{dy}{dx} = \frac{1}{1+x^2}$

$$f_y(y) = (1+x^2)f_x(\tan y) = \frac{1+x^2}{\pi(1+x^2)} = \frac{1}{\pi} \qquad \frac{\pi}{2} < y < \frac{\pi}{2}$$

5-12 (a) If $y = x^3$ then $x = \sqrt[3]{y}$ for any y

$$f_y(y) = \frac{1}{3\sqrt[3]{y^2}} f_x(\sqrt[3]{y}) = \frac{1}{12\pi\sqrt[3]{y^2}}$$

for $|y| < 8\pi^3$ and zero otherwise

(b) If $y = x^4$ and $y > 0$, then $x_1 = \sqrt[4]{y}$ $x_1 = -\sqrt[4]{y}$

$$f_y(y) = \frac{1}{4\sqrt[4]{y^3}} \left[f_x(\sqrt[4]{y}) + f_x(-\sqrt[4]{y})\right] = \frac{1}{8\pi\sqrt[4]{y^3}}$$

for $0 < y < 16\pi^4$ and zero otherwise

(c) If $y = 2\sin(3x + 40°)$ and $|y| < 2$ then $x = x_i$ as shown.

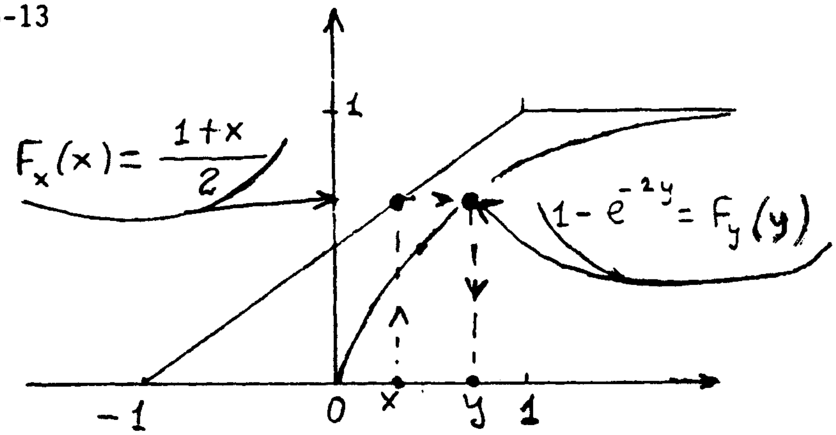
$$\frac{dy}{dx} = \frac{1}{6\sqrt{1 - y^2/4}}$$

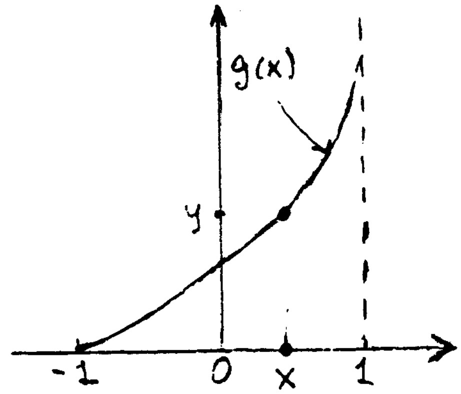
In the interval $(-2\pi, 2\pi)$ there are 12 x_i's. Hence

$$f_y(y) = \frac{1}{3\sqrt{4-y^2}} \sum_i f_x(x_i) = \frac{12}{12\pi\sqrt{4-y^2}} = \frac{1}{\pi\sqrt{4-y^2}}$$

for $|y| < 2$ and zero otherwise.

5-13





As in (5-43) $\qquad F_y[g(x)] = F_x(x)$

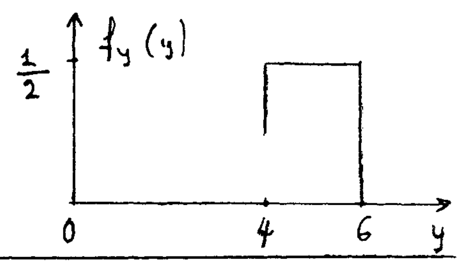
$$\frac{1+x}{2} = 1 - e^{-2y} \qquad y = g(x) = -\frac{1}{2}\ln\frac{1-x}{2}$$

for $|x| < 1$. For $x \leq -1$, $g(x) = 0$; for $x \geq 1$, $g(x) = \infty$.

5-14 (a) $\quad g(x) = 2F_x(x) + 4 \qquad g'(x) = 2f_x(x)$

If $4 < y < 6$ then $y = 2F_x(x) + 4$ has a unique solution x_1 and

$$f_y(y) = \frac{f_x(x_1)}{2f_x(x_1)} = \frac{1}{2}$$



(b) $\quad$ Similarly $g(x) = 2F_x(x) + 8$

5-15 (a) The RV $\underset{\sim}{x}$ takes the values $k = 0,1,\ldots,10$ and

$$P\ x = k\ = p_k = \binom{10}{k}\frac{1}{2^{10}} \qquad 0 \leq k \leq 10$$

$F_x(x)$ is a staircase function with discontinuities at the points $x = k$ and jumps equal to p_k.

(b) The RY $\underset{\sim}{y} = (\underset{\sim}{x} - 3)^2$ takes the values $y = k^2$ for $k = 0,1,\ldots,7$ and probabilities $P\{y = k^2\} = q_k$.

$k =$	0	1	2	3	4	5	6	7
$q_k =$	p_3	p_2+p_4	p_1+p_5	p_0+p_6	p_7	p_8	p_9	p_{10}

5.16

$$X \sim Beta(\alpha, \beta) \quad \text{gives}$$

$$f_X(x) = \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}, \quad 0 < x < 1.$$

$$Y = 1 - X \quad \Rightarrow \quad x_1 = 1 - y, \quad |\,\frac{dy}{dx}\,| = 1$$

$$\Rightarrow F_Y(y) = \frac{1}{|\frac{dy}{dx}|}\, f_X(1-y) = \begin{cases} \frac{1}{B(\beta,\alpha)}\, y^{\beta-1}\,(1-y)^{\alpha-1}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$Y \sim Beta(\beta, \alpha).$$

5.17

$$X \sim \chi^2(n) \Rightarrow$$

$$f_X(x) = \frac{1}{2^{n/2}\Gamma(n/2)}\, x^{n/2-1}\, e^{-x/2}\, U(x)$$

$$y = \sqrt{x} \quad \Rightarrow \quad x_1 = y^2$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

Thus

$$f_Y(y) = 2y f_X(y^2) = \frac{y^{n-1}}{2^{n/2-1}\,\Gamma(n/2)}\, e^{-y^2/2}\, U(y)$$

and it represents the chi-distribution.

5.18

$$X \sim U(0,1)$$

$$Y = -2logX \quad \Rightarrow \quad x_1 = e^{-y/2}$$

$$\frac{dy}{dx} = -\frac{2}{x} = -2e^{y/2}$$

$$f_Y(y) \;= \frac{1}{|\frac{dy}{dx}|}\, f_X(x_1) = \tfrac{1}{2} e^{-y/2}\, U(y)$$

$$\sim \text{ Exponential}(2) \equiv \chi^2(2)$$

5.19

$$f_X(x) = \lambda\, e^{-\lambda x}\, u(x)$$

$$Y = X^{1/\beta} \quad \Rightarrow \quad x_1 = y^{\beta}$$

$$|\frac{dy}{dx}| = \frac{1}{\beta}\, x^{1/\beta - 1} = \frac{1}{\beta}\, y^{1-\beta}$$

$$f_Y(y) = \frac{1}{|\,\frac{dy}{dx}\,|}\, f_X(x_1) = \lambda\, \beta\, y^{\beta-1}\, e^{-\lambda y^{\beta}}\, U(y)$$

and it represents Weibull distribution

5-20 For $|y| < a$ the equation $y = a \sin\omega t$ has infinitely many solutions τ_i; in each interval of length $2\pi/\omega$ there are two such solutions. Furthermore, $y'(t) = \omega\sqrt{a^2-y^2}$

$$\tau_i = \frac{1}{\omega}\sin^{-1}\frac{y}{a} \qquad \tau_{i+2} - \tau_i \quad \frac{2\pi}{\omega} \xrightarrow[\omega\to\infty]{} 0$$

Hence,

$$\frac{1}{\omega\sqrt{a^2-y^2}} \sum_{i=-\infty}^{\infty} f_t(\tau_i) \xrightarrow[\omega\to\infty]{} \frac{1}{\sqrt{a^2-y^2}} \frac{2}{2\pi} \int_{-\infty}^{\infty} f_t(\tau)d\tau = \frac{1}{\pi\sqrt{a^2-y^2}}$$

5-21 If $y > 0$ then

$$F_y(y|\underset{\sim}{x} \geq 0) = F_x(\sqrt{y}|\underset{\sim}{x} \geq 0) + F_x(-\sqrt{y}|\underset{\sim}{x} \geq 0) = F_x(\sqrt{y}|\underset{\sim}{x} \geq 0)$$

$$F_x(\sqrt{y}|x \geq 0) = \frac{P\{0 < x < \sqrt{y}\}}{P\{x \geq 0\}} = \frac{F_x(\sqrt{y}) - F_x(0)}{1 - F_x(0)}$$

$$f_y(y|x \geq 0) = \frac{d}{dy} F_y(\sqrt{y}|x \geq 0) = \frac{f_x(\sqrt{y})}{2\sqrt{y}[1-F_x(0)]}$$

5-22 (a) $\eta_y = a\,\eta_x + b \qquad \sigma_y^2 = E\{[a\,\underset{\sim}{x} + b - (a\,\eta_x + b)]^2\}$

$$\sigma_y^2 = E\{a^2(\underset{\sim}{x} - \eta_x)^2\} = a^2\sigma_x^2$$

(b) $\underset{\sim}{y} = \dfrac{\underset{\sim}{x}-\eta_x}{\sigma_x} \qquad E\{\underset{\sim}{y}\} = 0 \qquad \sigma_y^2 = \dfrac{\sigma_x^2}{\sigma_x^2} = 1$

5-23 If $\underset{\sim}{x}$ has a Rayleigh density, then [see (5-76)]

$$E\{\underset{\sim}{x}^2\} = 2\alpha^2 \qquad E\{\underset{\sim}{x}^4\} = 8\alpha^4$$

If $\underset{\sim}{y} = b + c\underset{\sim}{x}^2$, then

$$E\{\underset{\sim}{y}\} = b + 2\alpha^2 c \qquad E\{\underset{\sim}{y}^2\} = b^2 + 4\alpha^2 b c + 8\alpha^4 c^2$$

$$\sigma_y^2 = E\{\underset{\sim}{y}^2\} - E^2\{\underset{\sim}{y}\} = 4\alpha^4 c^2$$

5-24 $\quad y = 3x^2 \qquad E\{x^2\} = \sigma_x^2 = 4 \qquad E\{x^4\} = 3\sigma_x^4 = 48$

$E\{y\} = 12 \qquad E\{y^2\} = 9 \times 48 = 432 \qquad \sigma_y^2 = 432 - 144 = 288$

If $y > 0$ then $3x^2 = y$ for $x = \pm\sqrt{y/3} \qquad y' = 6x$

$$f_y(y) = \frac{2}{\sqrt{12y}} f_x\left(\sqrt{\frac{y}{3}}\right) = \frac{1}{\sqrt{24\pi y}} e^{-y/24} U(y)$$

5.25

$$X \sim B(n,p) \Rightarrow P(X=k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0,1,2,\cdots n.$$

a)

$$\begin{aligned} E(X) &= \sum_{k=0}^{n} k\, P(X=k) = \sum_{k=1}^{n} k \frac{n!}{k!\,(n-k)!} p^k q^{n-k} \\ &= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} q^{n-k} \\ &= np\,(p+q)^{n-1} = np. \end{aligned}$$

b)

$$\begin{aligned} E[X(X-1)] &= \sum_{k=2}^{n} k\,(k-1) \frac{n!}{k!\,(n-k)!} p^k q^{n-k} \\ &= n(n-1)p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!\,(n-k)!} p^{k-2} q^{n-k} \\ &= n(n-1)p^2\,(p+q)^{n-2} \\ &= n(n-1)p^2 \end{aligned}$$

c)

$$\begin{aligned} E[X(X-1)(X-2)] &= \sum_{k=3}^{n} k(k-1)(k-2) \frac{n!}{k!\,(n-k)!} p^k q^{n-k} \\ &= n(n-1)(n-2)\,p^3 \sum_{k=3}^{n} \frac{(n-3)!}{(k-3)!\,(n-k)!} p^{k-3} q^{n-k} \\ &= n(n-1)(n-2)\,p^3\,(p+q)^{n-3} \\ &= n(n-1)(n-2)\,p^3 \end{aligned}$$

$$E(X^2) = E\left(X(X-1)\right) + E(X) = n^2p^2 + npq$$

$$\begin{aligned} E(X^3) &= E\left(X(X-1)(X-2)\right) + 3E(X^2) - 2E(X) \\ &= n(n-1)(n-2)\,p^3 + 3(n^2p^2 + npq) - 2np \\ &= n^3p^3 + 3n^2p^2q + npq(q-p). \end{aligned}$$

5.26

$$X \sim P(\lambda) \quad \Rightarrow \quad P(X=k) = e^{\lambda}\,\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \cdots$$

a)

$$E(X) = \lambda, \quad \mathrm{Var}(X) = \sigma_X^2 = \lambda$$

From Chebyshev's inequality (5-88)

$$P\left(|X-\mu| < \lambda\right) > 1 - \frac{\sigma^2}{\lambda^2} = 1 - \frac{1}{\lambda}$$

But

$$|X-\mu| < \lambda = |X-\lambda| < \lambda \quad \Rightarrow \quad 0 < X < 2\lambda$$

which gives

$$P(0 < X < 2\lambda) > 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}.$$

b)

$$E\left[X(X-1)\right] = \sum_{k=2}^{\infty} k(k-1)\, e^{-\lambda}\,\frac{\lambda^k}{k!}$$

$$= e^{-\lambda}\,\lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} = e^{-\lambda}\,\lambda^2\, e^{\lambda} = \lambda^2.$$

$$E\left[X(X-1)(X-2)\right] = \sum_{k=3}^{\infty} k(k-1)(k-2)\, e^{-\lambda}\,\frac{\lambda^k}{k!}$$

$$= e^{-\lambda}\,\lambda^3 \sum_{k=3}^{\infty} \frac{\lambda^{k-3}}{(k-3)!} = \lambda^3.$$

5-27 **Follows from (4-74)**

$$E\{\underset{\sim}{x}\} = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{\infty} x \sum_i f(x|A_i) P(A_i)\,dx$$

because $$E\{\underset{\sim}{x}|A_i\} = \int_{-\infty}^{\infty} x f(x|A_i)\,dx$$

5-28 **From (5-89) with** $\alpha = \sqrt{\eta}$:

$$P\{\underset{\sim}{x} \geq \sqrt{\eta}\} \leq \eta/\sqrt{\eta} = \sqrt{\eta}$$

5-29 From (5-86) with $g(x) = x^3$ $g''(x) = 6x$:

$$E\{\underset{\sim}{x}^3\} = \eta^3 + 6\eta \frac{\sigma^2}{2} = 1120$$

5-30 (a) If $y = x^3$, then $x = \sqrt[3]{y}$ $g'(x) = 3x^2 = 3\sqrt[3]{y^2}$

But $f_x(x) = 0.5$ for $10 < x < 12$, i.e., for $10^3 < y < 12^3$

and (5-16) yields

$$f_y(y) = \frac{0.5}{3\sqrt[3]{y^2}} \qquad 10^3 < y < 12^3$$

and zero otherwise.

(b) 1.

$$E\{\underset{\sim}{x}^3\} = 0.5 \int_{10}^{12} x^3 dx = 1342$$

2. With $g(x) = x^3$ $E\{\underset{\sim}{x}\} = 11$ $\sigma_x^2 = 1/3$, (5-86) yields

$$E\{x^3\} \simeq 11^3 + 6 \times 11 \times \frac{1}{6} \simeq 1342$$

5-31 With $g(x)=1/x$, $g''(x)=2/x^3$, $\eta=100$, and $\sigma=3$, (5-55) yields

$$E\left\{\frac{1}{\underset{\sim}{x}}\right\} \simeq \frac{1}{100} + \frac{9}{2} \times \frac{2}{100^3} = 0.010009$$

5-32

$$\frac{\partial|x-a|}{\partial a} = \begin{cases} 1 & x < a \\ -1 & x > a \end{cases} \qquad \text{If } I(a) = E\{|\underset{\sim}{x}-a|\} \text{ then}$$

$$\frac{dI(a)}{da} = E\,\frac{\partial|\underset{\sim}{x}-a|}{\partial a} = 1\,P\{\underset{\sim}{x} < a\} - 1\,P\{\underset{\sim}{x} > a\}$$
$$= 2\,F(a) - 1$$

(a)
$$I(a) = I(m) + \int_m^a I'(\alpha)d\alpha = I(m) + \int_m^a [2\,F(\alpha) - 1]d\alpha$$

$$= E\{|\underset{\sim}{x} - m|\} - 2\int_m^a x\,f(x)dx$$

because

$$\int_m^a F(\alpha)d\alpha = a\,F(a) - m\,F(m) - \int_m^a x f(x)dx$$

$$F(m) = \frac{1}{2} \qquad \int_m^a f(x)dx = F(a) - F(m)$$

(b) $I(a) = E\{|\underset{\sim}{x} - a|\}$ is minimum if

$I'(a) = 2F(a) - 1 = 0$ i.e. if $F(a) = \frac{1}{2}$ $a = m$

5-33

$$E\{|\underset{\sim}{x}|\} = \int_0^\infty xf(x)dx - \int_{-\infty}^0 xf(x)dx$$

$$\eta = E\{\underset{\sim}{x}\} = \int_0^\infty xf(x)dx + \int_{-\infty}^0 xf(x)dx$$

$$\frac{E\{|\underset{\sim}{x}|+\eta\}}{2} = \int_0^\infty xf(x)dx = \frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty x\,e^{-(x-\eta)^2/2\sigma^2}dx$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty (x+\eta)e^{-(x-\eta)^2/2\sigma^2}dx = \frac{1}{\sigma\sqrt{2\pi}}\int_\eta^\infty y\ e^{-y^2/2\sigma^2}dy = \frac{\sigma}{\sqrt{2\pi}}\ e^{-\eta^2/2\sigma^2}$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int_0^\infty e^{-(x-\eta)^2/2\sigma^2}dx = G(\frac{\eta}{\sigma})$$

Multiplying the last line by η and subtracting from the fourth line, we obtain

$$\frac{E\{|x|+\eta\}}{2} = \frac{\sigma}{\sqrt{2\pi}}\ e^{-\eta^2/2\sigma^2} + G(\frac{\eta}{\sigma})$$

5-34 The proof is given in sec 14-3: [see (14-100)].

5-35 (a) Follows from (5-89) (b) $e^{sx} \geq e^{sA}$ iff $x \geq A$ for $s > 0$ and $x \leq A$ for $s < 0$.

5.36 See proof for Lyapunov inequality (Ch.5, Eq.(5-92).)

5-37 (a) If $\Phi(\omega) = e^{-\alpha|\omega|}$ then [see (5-102)]

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha|\omega|}\, e^{j\omega x} d\omega = \frac{1}{\pi}\int_0^{\infty} \cos \omega x\, e^{-\alpha\omega} d\omega = \frac{\alpha}{\pi(\alpha^2+x^2)}$$

(b) If $f(x) = \frac{\alpha}{2} e^{-\alpha|x|}$, then [see (5-94)]

$$\Phi(\omega) = \frac{\alpha}{2}\int_{-\infty}^{\infty} e^{-\alpha|x|} e^{-j\omega x} dx = \alpha\int_0^{\infty} e^{-\alpha x} \cos \omega x dx = \frac{\alpha^2}{\alpha^2+\omega^2}$$

5.38 a) On comparing Eq.(4-34) with Eq.(5-106), Example 5-29, we get

$$X \sim G(\alpha,\beta) \quad \Rightarrow \quad \phi_X(\omega) = (1 - j\beta\omega)^{-\alpha}$$

$$\phi_X'(\omega) = -\alpha(1 - j\beta\omega)^{-(\alpha+1)}\,(-j\beta)$$

so that

$$E(X) = \frac{1}{j}\,\phi_X'(0) = \alpha\,\beta.$$

Similarly

$$\phi_X''(\omega) = j\alpha\beta(\alpha+1)\,(1 - j\beta\,\omega)^{-(\alpha+2)}\,(j\beta)$$

and hence

$$E(X^2) = \frac{1}{j^2}\,\phi_X''(0) = \alpha\beta^2\,(\alpha+1).$$

Thus

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = \alpha\beta^2.$$

b)

$$X \sim \chi^2(n) \quad \Rightarrow \quad \alpha = \frac{n}{2}, \quad \beta = 2$$

in Gamma(α,β). This gives

$$\begin{aligned} \phi_X(\omega) &= (1 - j2\omega)^{-n/2} \\ E(X) &= n \\ \mathrm{Var}(X) &= 2n. \end{aligned}$$

c)

$$X \sim \; B(n,p).$$

From Prob 5-25 (a)-(b)

$$\begin{aligned} E(X) &= np \\ \mathrm{Var}(X) &= E\,(X(X-1)) + E(X) = npq. \end{aligned}$$

and

$$\begin{aligned} \phi_X(\omega) &= \sum_{k=0}^{n} e^{jk\omega}\,P(X=k) \\ &= \sum_{k=0}^{n} \binom{n}{k}\,(p\,e^{j\omega})^k\,q^{n-k} = (pe^{j\omega}+q)^n. \end{aligned}$$

d)

$$X \sim N\,Binomial\,(r, p).$$

From (4-64)

$$\begin{aligned}\phi_X(\omega) &= \sum_{k=0}^{\infty} e^{jk\omega}\, P(X=k) \\ &= \sum_{k=0}^{\infty} \binom{r+k-1}{k} p^r\, (qe^{j\omega})^k \\ &= p^r \sum_{k=0}^{\infty} \binom{-r}{k} (-qe^{j\omega})^k \\ &= p^r\,(1-qe^{j\omega})^{-r}.\end{aligned}$$

5-39

$$\Gamma(z) = \sum_{k=0}^{\infty} p\, q^k z^k = \frac{p}{1-qz} \qquad q = 1-p$$

$$\Gamma'(z) = \frac{pq}{(1-qz)^2} \qquad \Gamma'(1) = \frac{pq}{(1-q)^2} = \frac{p}{q} = \eta_x$$

$$\Gamma''(z) = \frac{2pq^2}{(1-qz)^3} \qquad \Gamma''(1) = \frac{2q^2}{p^2} = m_2 - m_1$$

$$\sigma^2 = m_2 - m_1^2 = 2\frac{q^2}{p^2} + m_1 - m_1^2 = \frac{q}{p^2}$$

5-40

$$\Gamma(z) = p^n \sum_{k=0}^{\infty} \binom{-n}{k} (-q)^k\, z^k = p^n (1-qz)^{-n}$$

(binomial expansion with negative exponent)

$$\Gamma'(z) = \frac{n\,p^n q}{(1-qz)^{n+1}} \qquad \Gamma'(1) = \frac{nq}{p} = \eta_x$$

$$\Gamma''(z) = \frac{n(n+1)p^n q^2}{(1-qz)^{n+2}} \qquad \Gamma''(1) = \frac{n(n+1)q^2}{p^2} = m_2 - m_1$$

$$\sigma_x^2 = \Gamma''(1) + m_1 - m_1^2 = \frac{nq}{p^2}$$

5.41 We have

$$P(X=k)=\binom{k-1}{r-1}p^r\,q^{k-r},\quad k=r,r+1,\cdots$$

Let $k=n+r$ so that

$$\begin{aligned}P(X=n+r) &= \binom{n+r-1}{r-1}p^r\,q^n,\quad n=0,1,2,\cdots\\ &= \frac{(n+r-1)!}{n!\,(r-1)!}\,p^r\,(1-p)^n\\ &= \frac{1}{n!}\,\frac{(n+r-1)(n+r-2)\cdots(r)}{r^n}\,[r(1-p)]^n\,p^r\\ &= \frac{\lambda^n}{n!}\left\{\left(1+\frac{n-1}{r}\right)\left(1+\frac{n-2}{r}\right)\cdots\right\}\left(1-\frac{r(1-p)}{r}\right)^r\\ &= \frac{\lambda^n}{n!}\left\{\prod_{k=1}^n\left(1+\frac{n-k}{r}\right)\right\}\left(1-\frac{\lambda}{r}\right)^r,\end{aligned}$$

where $\lambda=r(1-p)$. Thus

$$\begin{aligned}\lim_{r\to\infty}P(X=n+r) &= \frac{\lambda^n}{n!}\left\{\lim_{r\to\infty}\prod_{k=1}^n\left(1+\frac{n-k}{r}\right)\right\}\lim_{r\to\infty}\left(1-\frac{\lambda}{r}\right)^r\\ &\to \frac{\lambda^n}{n!}e^{-\lambda}\sim P(\lambda).\end{aligned}$$

5-42 $E\{e^{s\mathbf{x}}\} = e^{s\eta} E\{e^{s(\mathbf{x}-\eta)}\} = e^{s\eta} E\left\{\sum_{n=0}^{\infty} \frac{s^n}{n!} (\mathbf{x}-\eta)^n\right\}$

$$= e^{s\eta} \sum_{n=0}^{\infty} \frac{s^n}{n!} \mu_n$$

5-43 If $\Phi(\omega_1) = 0$, then [see also (9-176]

$$\int_{-\infty}^{\infty} (1-e^{j\omega_1 x}) f(x)dx = 0, \text{ hence, } f(x) = \sum_{n=\infty}^{\infty} p_n \delta(x - \frac{2\pi n}{\omega_1})$$

5-44 (a) If $\eta = 0$, then $m_n = \mu_n \qquad \lambda_1 = \eta = 0$

$$\Phi(s) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} s^n \qquad\qquad \Psi(s) = \sum_{n=2}^{\infty} \frac{\lambda_n}{n!} s^n$$

$$1 + \frac{\mu_2}{2!}s^2 + \frac{\mu_3}{3!} s^3 + \frac{\mu_4}{4!} s^4 + \cdots = \exp\left\{\frac{\lambda_2}{2!} s^2 + \frac{\lambda_3}{3!} s^3 + \frac{\lambda_4}{4!} s^4 + \cdots\right\}$$

Expanding the exponential and equating powers of s, we obtain

$$\mu_2 = \lambda_2 \qquad \mu_3 = \lambda_3 \qquad\qquad \frac{\mu_4}{4!} = \frac{\lambda_4}{4!} + \frac{1}{2!} \left(\frac{\lambda_2}{2!}\right)^2$$

(b) If $\mathbf{y}$ is $N(0;\sigma_y)$ then

$$\Psi_y(s) = \frac{\lambda_2}{2} s^2 , \quad \text{hence,} \quad \lambda_n = 0 \quad \text{for} \quad n \geq 3$$

5-45 $$P\{\underset{\sim}{y} = 0\} = P\{\underset{\sim}{x} \le 1\} = p_o + p_1$$

$$P\{\underset{\sim}{y} = k\} = P\{\underset{\sim}{x} = k + 1\} = p_{k+1} \qquad k \ge 1$$

$$\Gamma_y(z) = p_o + p_1 + \sum_{k=1}^{\infty} p_{k+1} z^k = p_o + z^{-1}[\Gamma_x(z) - p_o]$$

$$\eta_y = \sum_{k=1}^{\infty} k\, p_{k+1} = \sum_{r=1}^{\infty} r\, p_r - \sum_{r=1}^{\infty} p_r = \eta_x - 1 + p_o$$

$$E\{\underset{\sim}{y}^2\} = \sum_{k=1}^{\infty} k^2 p_{k+1} = \sum_{r=1}^{\infty} (r-1)^2 p_r = E\{\underset{\sim}{x}^2\} - 2\eta_x + 1 - p_o$$

5-46 $$0 \le E\left\{\left|\sum_{i=1}^{n} a_i e^{j\omega_i \underset{\sim}{x}}\right|^2\right\} = E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* e^{j(\omega_i - \omega_j)\underset{\sim}{x}}\right\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* \Phi(\omega_i - \omega_j)$$

5-47 From the assumptions it follows that

$$g'(-x) = -g'(x) \qquad g''(x) \ge 0 \qquad f(x-\eta) = f(\eta-x)$$

Hence, if $I(a) = E\{g(\underset{\sim}{x}-a)\}$, then

$$I'(a) = -\int_{-\infty}^{\infty} g'(x-a)f(x)dx \qquad I'(\eta) = 0$$

$$I''(a) = \int_{-\infty}^{\infty} g''(x-a)f(x)dx \ge 0 \qquad \text{all } a$$

Hence, $I(a)$ is minimum for $a = \eta$.

5-48 $$f(x,v) = \frac{1}{\sqrt{2\pi v}} e^{-x^2/2v}$$

$$\sqrt{2\pi}\,\frac{\partial f}{\partial v} = \frac{-1 + x^2/v}{2v\,\sqrt{v}}\, e^{-x^2/2v}$$

$$\sqrt{2\pi}\,\frac{\partial^2 f}{\partial x^2} = \frac{-1 + x/.}{v\,\sqrt{v}}\, e^{-x^2/2v}$$

Hence

(see also (6-198) - (6-199)) $$\boxed{\frac{\partial f}{\partial v} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}} \qquad (1)$$

(a) Integrating by parts, using (1) and assuming that $g^{(k)}(x)f(x) \to 0$ as $|x| \to \infty$, $k = 0, 1, 2$, we obtain

$$E\{g''(x)\} = \int_{-\infty}^{\infty} \frac{d^2g}{dx^2} f dx = \int_{-\infty}^{\infty} g \frac{\partial^2 f}{\partial x^2} dx = 2\int_{-\infty}^{\infty} g \frac{\partial f}{\partial v} dx$$

$$= 2\frac{d}{dv}\int_{-\infty}^{\infty} g\, f dx = 2\frac{d}{dv} E\{g(\underset{\sim}{x})\}$$

(b) The moments $\mu_n(u) = E\{\underset{\sim}{x}^n\}$ of $\underset{\sim}{x}$ depend on the variance v of $\underset{\sim}{x}$ and (i) yields

$$\mu_n'(v) = \frac{d}{dv} E\{\underset{\sim}{x}^n\} = \frac{1}{2} E\{n(n-1)\underset{\sim}{x}^{n-2}\} = \frac{n(n-1)}{2}\mu_{n-2}(v)$$

Furthermore, $\mu_n(0) = 0$ because, if $v = 0$, then $\underset{\sim}{x} = 0$.

Hence

$$\mu_n(v) = \frac{n(n-1)}{2}\int_0^v \mu_{n-2}(\beta)d\beta$$

5-49 **The function**

$$\Gamma(e^{j\omega}) = E\{e^{jx\omega}\} = \sum_{k=0}^{\infty} p_k\, e^{jk\omega}$$

is periodic with period 2π and Fourier series coefficients $p_k = E\{x = k\}$.

5.50 The event $\{X = 1\}$ is given by the disjoint union "$TH \cup HT$". Similarly, the event "$X = k$" is given by the union of the disjoint events (k "T"s followed by "H" or k "H"s followed by "T")

$$\text{“}TT\cdots TTH\text{''} \cup \text{“}HH\cdots HHT\text{''}, \qquad k = 1, 2, \cdots$$

Thus

$$\begin{aligned} P(X = k) &= P(\text{“}TT\cdots TH\text{''} \cup \text{“}HH\cdots HT\text{''}) \\ &= P(TT\cdots TH) + P(HH\cdots HT) = q^k p + p^k q, \quad k = 1, 2, \cdots \end{aligned}$$

Also

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} kP(X = k) \\ &= \sum_{k=1}^{\infty} kq^k p + \sum_{k=1}^{\infty} kp^k q = pq\left\{\sum_{k=1}^{\infty} kq^{k-1} + \sum_{k=1}^{\infty} kp^{k-1}\right\} \\ &= pq\left\{\frac{\partial}{\partial q}\sum_{k=1}^{\infty} q^k + \frac{\partial}{\partial p}\sum_{k=1}^{\infty} p^k\right\} = pq\left\{\frac{\partial}{\partial q}\left(\frac{q}{1-q}\right) + \frac{\partial}{\partial p}\left(\frac{p}{1-p}\right)\right\} \\ &= pq\left\{\frac{1}{p^2} + \frac{1}{q^2}\right\} = \frac{p}{q} + \frac{q}{p}. \end{aligned}$$

5.51 (a) When samples are drawn with replacement, probability of each item being defective is given by

$$p = \frac{M}{N} < 1 \quad \text{(constant)}$$

and

$$q = 1 - p = \frac{N-M}{M} < 1$$

represents the constant probability that the chosen item is not defective. In that case (with replacement), there are $\binom{n}{k}$ possible ways of arranging k defective items among n chosen items, and each such arrangement has probability $p^k q^{n-k}$. This gives

$$P(X=k) = \binom{n}{k} p^k\, q^{n-k}, \qquad k = 0, 1, 2, \cdots n$$

which represents the Binomial distribution.
(b) If the samples are drawn without replacement, there are $\binom{M}{k}$ possible ways of choosing k defective item from a total of M defective items, and $\binom{N-M}{n-k}$ possible ways of choosing $n-k$ "good" items from $(N-M)$ "good" items independently. This gives

$$\binom{M}{k}\binom{N-M}{n-k}$$

to be the total number of ways of selecting k defective items and $n-k$ "good" items from a subsample of M and $N-M$ items respectively (favorable ways). But there are a total of $\binom{N}{n}$ ways of selecting n items among N items. This gives

$$P(X=k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}},$$

since $0 \le k \le M$ and $n-k \le N-M, n-k \ge 0$, i.e. $0 \le k \le M, k \le n, k \ge n + M - N$.
(c) From (b)

$$P(X=k) = \frac{M!}{k!(M-k)!}\,\frac{(N-M)!}{(n-k)!(N-M-n+k)!}\,\frac{n!(N-n)!}{N!}$$

$$= \binom{n}{k}\frac{M(M-1)\cdots(M-k+1)}{N(N-1)\cdots(N-k+1)}\,\frac{(N-M)(N-M-1)\cdots(N-M-n+k+1)}{(N-k)(N-k-1)\cdots(N-n+1)}\,\frac{(I}{(I}$$

$$\simeq \binom{n}{k}\left(\tfrac{M}{N}\right)^k\left(\tfrac{N-M}{N}\right)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \cdots n$$

since $N \to \infty, M \to \infty$ such that $M/N \to p$, and $n \ll N$. Thus

$$P(X=k) \to \text{Binomial}(n, p = M/N)$$

under the above conditions.

5.52 (a) Refer to discussions in problem 5.51 (a) if sampleing is done with replacement, then

$$p = \frac{n}{n+m}$$

represents the probability of selecting a white marble on any trial. The event "$X = k$" is given by "$r-1$ white mables among the first $k-1$ trials" followed by "a white marble at the k^{th} trial". But from problem 5.51 (a), the event $r-1$ white mables among the first $k-1$ trials has a binomial distribution whose probability is given by $\binom{k-1}{r-1}p^{r-1}q^{k-r}$. Thus

$$P(X = k) = \binom{k-1}{r-1}p^{r-1}q^{k-r}p = \binom{k-1}{r-1}p^r q^{k-r}, \quad k = r, r+1, \cdots$$

which represents the Negative-binomial distribution

(b) If sampling is done with replacement, then the favarable ways of choosing the white balls are given by:

(i) $\binom{k-1}{r-1}$ ways of selecting $r-1$ white balls among the first $k-1$ trials/balls.

(ii) One ways of selecting (the r^{th}) white ball at the k^{th} trial

(iii) $\binom{m+n-k}{n-r}$ ways of selecting the remaining $n-r$ white balls among the remaining $m+n-k$ balls.

This gives $\binom{k-1}{r-1} \cdot 1 \cdot \binom{m-n-k}{n-r}$ to be the total number of favorable ways of selecting the white balls. Since there are $n+m$ balls there are a total of $\binom{n+m}{n}$ ways of selecting n white balls. This gives

$$P(X = k) = \binom{k-1}{r-1}\frac{\binom{m+n-k}{n-r}}{\binom{n+m}{n}}, \qquad k = r, r+1, \cdots$$

(c) From (b)

$$P(X = k) = \binom{k-1}{r-1}\frac{(m+n-k)!}{(n-r)!\,(m-k+r)!}\frac{n!m!}{(m+n)!}$$

$$= \binom{k-1}{r-1}\left(\frac{n}{m+n}\right)\left(\frac{n-1}{m+n-1}\right)\cdots\left(\frac{n-r+1}{m+n-r+1}\right)\left(\frac{m!(m+n-k)!}{(m+n-r)!\,(m-k+r)!}\right)$$

$$\simeq \binom{k-1}{r-1}\left(\frac{n}{m+n}\right)^r\left(\frac{m}{m+n-r}\right)\left(\frac{m-1}{m+n-r-1}\right)\cdots\left(\frac{m-k+r+1}{m+n-k+1}\right)$$

$$\simeq \binom{k-1}{r-1}\left(\frac{n}{m+n}\right)^r\left(\frac{m}{m+n}\right)^{k-r} \text{ as } m+n \to \infty$$

$$= \binom{k-1}{r-1}p^r q^{k-r}, \qquad k = r, r+1, \cdots, \quad q = 1-p$$

$$\sim \text{NB}\,(r, p = n/(n+m))\,.$$