CHAPTER 6

6.1 (a) Define

$$Z = X + Y$$

Note that both X and Y positive random variables hence (use Eq. (6-45))

$$\begin{aligned} f_Z(z) &= \int_0^z f_{XY}(z-y, y)dy = \int_0^z e^{-(z-y+y)}dy \\ &= z\, e^{-z}U(z). \end{aligned}$$

(b)

$$Z = X - Y$$

Z ranges over the entire real axis for the random variables X and Y (see Eq. (6-55))

$$F_Z(z) = \begin{cases} \int_0^\infty \int_0^{z+y} f_{XY}(x, y)dx\, dy, & z > 0 \\ \int_{-z}^\infty \int_0^{z+y} f_{XY}(x, y)dx\, dy, & z < 0 \end{cases}$$

Differrentiation gives

$$f_Z(z) = \begin{cases} \int_0^\infty f_{XY}(z+y, y)\, dy, & z > 0 \\ \int_{-z}^\infty f_{XY}(z+y, y)\, dy, & z < 0 \end{cases}$$

$$f_Z(z) = \begin{cases} \int_0^\infty e^{-(z+y+y)}\, dy = e^{-z}\int_0^\infty e^{-2y}\, dy = \frac{1}{2}\, e^{-z}, & z > 0 \\ \int_{-z}^\infty e^{-(z+y+y)}\, dy = e^{-z}\int_{-z}^\infty e^{-2y}\, dy = \frac{1}{2}\, e^{z}, & z < 0 \end{cases}$$

or

$$f_Z(z) = \frac{1}{2}\, e^{-|z|}, \qquad -\infty \leq z \leq \infty.$$

(c)

$$Z = XY.$$

$$\begin{aligned} F_Z(z) &= P\{Z \leq z\} = P\{XY \leq z\} \\ &= \int_0^\infty \int_0^{z/y} f_{XY}(x, y)dx\, dy \end{aligned}$$

or (see Eq. (6-148))

$$f_Z(z) = \int_0^\infty \frac{1}{y} f_{XY}(\frac{z}{y}, y)dy = \int_0^\infty \frac{1}{y} e^{-((z/y)+y)}dy$$

(d)

$$Z = X/Y$$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{\tfrac{X}{Y} \le z\} \\ &= \int_0^\infty \int_0^{yz} f_{XY}(x,y)dx\,dy \end{aligned}$$

(use Eq. (6-60))

$$\begin{aligned} f_Z(z) &= \int_0^\infty y f_{XY}(yz,y)dy = \int_0^\infty y e^{y(z+1)}dy = \int_0^\infty y e^{(1+z)y} \\ &= \left[y\frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty + \left(\frac{1}{1+z}\right)\int_0^\infty e^{(1+z)y}dy \\ &= \left(\frac{1}{1+z}\right)\left[\frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty = \frac{1}{(1+z)^2}\,U(z) \end{aligned}$$

(e)

$$Z = \min\,(X,Y)$$

$$\begin{aligned} F_Z(z) &= P\{\min\,(X,Y) \le z\} \\ &= 1 - P\{Z > z, Y > z\} \\ &= 1 - [1 - F_X(z)]\,[1 - F_Y(z)] \\ &= F_X(z) + F_Y(z) - F_X(z)\,F_Y(z) \end{aligned}$$

(see Eq. (6-81))

$$f_Z(z) = f_X(z) + f_Y(z) - F_X(z) f_Y(z) - f_X(z) F_Y(z).$$

We have

$$f_X(z) = f_Y(z) = e^{-z}\,U(z)$$

so that

$$F_X(z) = \int_0^z e^{-x}dx = (1 - e^{-z})\,U(z) = F_Y(z)$$

$$\begin{aligned} f_Z(z) &= [e^{-z} + e^{-z} - 2(1 - e^{-z})\,e^{-z}]U(z) \\ &= 2e^{-z}\,[1 - 1 + e^{-z}]\,U(z) \\ &= 2e^{-2z}\,U(z) \sim \text{Exponential (2)}. \end{aligned}$$

(f)

$$Z = \max(X,Y)$$

$$\begin{aligned} F_Z(z) &= P\{\max(X,Y) \le z\} = P\{X \le z, Y \le z\} \\ &= P\{X \le z\}\,P\{Y \le z\} = F_X(z)\,F_Y(z) \end{aligned}$$

$$\begin{aligned} f_Z(z) &= F_X(z)\,f_Y(z) + f_X(z)\,F_Y(z) \\ &= e^{-z}\,(1 - e^{-z}) + e^{-z}\,(1 - e^{-z}) \\ &= 2e^{-z}\,(1 - e^{-z})\,U(z) \end{aligned}$$

(g)

$$Z = \frac{\min(X,Y)}{\max(X,Y)}, \quad 0 < z < 1$$

$$\begin{aligned}
F_Z(z) &= P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \bigcap ((X \le Y) \bigcup (X > Y))\right\} \\
&= P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \bigcap (X \le Y)\right\} + P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \bigcap (X > Y)\right\} \\
&= P\{\tfrac{X}{Y} \le z, X \le Y\} + P\{\tfrac{Y}{X} \le z, X > Y\} \\
&= P\{X \le Yz, X \le Y\} + P\{Y \le Xz, X > Y\} \\
&= \int_0^\infty \int_0^{yz} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_0^{xz} f_{XY}(x,y)\,dy\,dx
\end{aligned}$$

$$\begin{aligned}
f_Z(z) &= \int_0^\infty y\, f_{XY}(yz,y)dy + \int_0^\infty x f_{XY}(x,xz)dx \\
&= \int_0^\infty y\, f_{XY}(yz,y)dy + \int_0^\infty y\, f_{XY}(y,yz)dy \\
&= \int_0^\infty y\left(e^{-(yz+y)} + e^{-(y+yz)}\right) dy \\
&= 2\int_0^\infty ye^{-y(1+z)}dz = \begin{cases} \dfrac{2}{(1+z)^2}, & 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

6.2

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \frac{1}{a^2}, \qquad 0 < x \le a, \quad 0 < y \le a$$

(a)

$$F_Z(z) = P\left\{\frac{X}{Y} \le z\right\} = P\{X \le zY\}$$

(i) $z < 1$

$$\begin{aligned}
F_Z(z) &= P\{X \le zY\} \\
&= \int_0^a \int_0^{zy} \tfrac{1}{a} \cdot \tfrac{1}{a}\, dx\, dy = \tfrac{z}{2}, \quad z \le 1
\end{aligned}$$

(ii) $z \ge 1$

$$\begin{aligned}
F_Z(z) &= P\{X \le zY\} \\
&= 1 - \int_0^a \int_0^{x/z} \tfrac{1}{a} \cdot \tfrac{1}{a}\, dy\, dx \\
&= 1 - \int_0^1 \tfrac{x}{z}\, dx = 1 - \tfrac{1}{2z} \quad z > 1
\end{aligned}$$

$$f_Z(z) = \begin{cases} \tfrac{1}{2}, & z \le 1 \\ \dfrac{1}{2z^2}, & z > 1 \end{cases}$$

(b)

$$\begin{aligned}
F_Z(z) &= P(Z \le z) = P\left\{\frac{Y}{X+Y} \le z\right\} \\
&= P\left\{\frac{X}{Y} \ge \frac{1}{z} - 1\right\} = 1 - P\left(\frac{X}{Y} \le \frac{1-z}{z}\right) \\
&= \begin{cases} \frac{1}{2}\left(\frac{z}{1-z}\right), & 0 < z \le 1/2 \\ 1 - \frac{1}{2}\left(\frac{1-z}{z}\right), & 1/2 < z < 1 \end{cases}
\end{aligned}$$

$$f_Z(z) = \begin{cases} \dfrac{1}{2(1-z)^2}, & 0 < z \le 1/2 \\ \dfrac{1}{2z^2}, & 1/2 < z < 1 \end{cases}$$

(c)

$$\begin{aligned}
F_Z(z) &= P\{Z \le z\} = P\{|X-Y| \le z\} \\
&= P\{\{|X-Y| \le z\} \textstyle\bigcap (X \ge Y)\} + P\{\{|X-Y| \le z\} \textstyle\bigcap (X < Y)\} \\
&= P\{X - Y \le z, X \ge Y\} + P\{Y - X \le z, X < Y\} \\
&= \int_0^\infty \int_y^{y+z} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_x^{x+z} f_{XY}(x,y)\,dy\,dx \\
&= \int_0^\infty \int_y^{y+z} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_y^{y+z} f_{XY}(y,x)\,dx\,dy \\
&= \int_0^\infty \int_y^{y+z} \{f_{XY}(x,y) + f_{XY}(y,x)\}\,dx\,dy.
\end{aligned}$$

In general

$$\begin{aligned}
f_Z(z) &= \int_0^\infty \frac{d}{dz} \int_y^{y+z} f_{XY}(x,y) + f_{XY}(y,x)\,dx\,dy \\
&= \int_0^\infty \{f_{XY}(y+z,y) + f_{XY}(y,y+z)\}\,dy.
\end{aligned}$$

Here

$$X \sim U(0,a), \qquad Y \sim U(0,a)$$

$$F_Z(z) = 1 - \frac{1}{a^2} \cdot 2 \cdot \frac{(a-z)^2}{2} = 1 - \left(1 - \frac{z}{a}\right)^2$$

and

$$f_Z(z) = \frac{2}{a}\left(1 - \frac{z}{a}\right) \qquad 0 \le z \le a.$$

6.3

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{X+Y \le z\} \\ &= \tfrac{1}{2} - \tfrac{z^2}{2}, \quad -1 < z < 0, \end{aligned}$$

(which represents the area below the line $X + Y = z$.)

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = P\{X+Y \le z\} \\ &= \tfrac{1}{2} + \tfrac{z^2}{2}, \quad 0 \le z < 1 \end{aligned}$$

$$f_Z(z) = \begin{cases} -z, & -1 \le z < 0 \\ z, & 0 \le z < 1 \end{cases}$$

6.4

$$Z = X - Y$$

For $z < 0$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} \\ &= \int_0^{(1+z)/2} \int_{x-z}^{1-x} f_{XY}(x,y)\,dy\,dx = \int_0^{(1+z)/2} \int_{x-z}^{1-x} 6x\,dy\,dx \\ &= \int_0^{(1+z)/2} 6x\,[y]_{x-z}^{1-x}\,dx = \int_0^{(1+z)/2} 6x(1-x-x+z)\,dx \\ &= 6\left[(1+z)\frac{x^2}{2} - \frac{2x^3}{3}\right]_0^{(1+z)/2} = 6\left[\frac{(1+z)^3}{8} - \frac{(1+z)^3}{12}\right] \\ &= \frac{(1+z)^3}{4}, \quad z \le 0. \end{aligned}$$

For $z > 0$

$$\begin{aligned} F_Z(z) &= P\{Z \le z\} = 1 - P\{Z > z\} \\ &= 1 - \int_0^{(1-z)/2} \int_{z+y}^{1-y} f_{XY}(x,y)\,dx\,dy = 1 - \int_0^{(1-z)/2} \int_{z+y}^{1-y} 6x\,dy \\ &= 1 - \int_0^{(1-z)/2} \left[\frac{6x^2}{2}\right]_{z+y}^{1-y} dy = 1 - 3\int_0^{(1-z)/2} \left[(1-y)^2 - (z-y)^2\right] dy \\ &= 1 - 3\,(1+z)\left[\frac{(1-z)^2}{2} - \frac{(1-z)^2}{4}\right] = 1 - \tfrac{3}{4}\,(1+z)(1-z)^2 \quad z \le 0. \end{aligned}$$

$$f_Z(z) = \begin{cases} \tfrac{3}{4}\,(1-z)(1+3z), & 0 \le z \le 1 \\ \tfrac{3}{4}\,(1+z)^2, & -1 < z < 0 \end{cases}$$

6.5 (a) See Example 6-15 for solutions
(b) See Example 6-14 for solutions
(c)

$$U = X - Y \sim N(0, 2\sigma^2)$$

since linear combinations of jointly Gaussian random variables are Gaussian random variables (see Eq. (6-120) Text.). Here $Var(U) = Var(X) + Var(Y) = 2\sigma^2$.

6.6

$$Z = XY$$

$$\begin{aligned} F_Z(z) &= P(XY \leq z) = 1 - P(XY > z) \\ &= 1 - \int_z^1 \int_{z/y}^1 f_{XY}(x,y)\,dx\,dy \\ f_Z(z) &= 1 + \int_z^1 \frac{1}{y} f_{XY}(z/y,y)\,dy = 1 + \int_z^1 \left\{\frac{2}{y} - \frac{2z}{y^2}\right\} dy \\ &= 1 - 2\ln z + 2z, \quad 0 \leq z \leq 1 \end{aligned}$$

6.7 (a)

$$Z_1 = X + Y$$

$$F_{Z_1}(z) = P(X+Y \leq z) = \begin{cases} \displaystyle\int_0^z \int_0^{z-y} f_{XY}(x,y)\,dx\,dy, & 0 < z < 1 \\ \displaystyle 1 - \int_{z-1}^1 \int_{z-y}^1 f_{XY}(x,y)\,dx\,dy, & 1 < z < 2 \end{cases}$$

$$f_{Z_1}(z) = \begin{cases} \displaystyle\int_0^z f_{XY}(z-y,y)\,dy, & 0 < z < 1 \\ \displaystyle\int_{z-1}^1 f_{XY}(z-y,y)\,dy, & 1 < z < 2 \end{cases}$$

$$= \begin{cases} z^2, & 0 < z < 1 \\ z(2-z), & 1 < z < 2 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$Z_2 = XY$$

$$F_{Z_2}(z) = P(XY \leq z) = 1 - \int_z^1 \int_{z/y}^1 f_{XY}(x,y)\,dx\,dy$$

$$\begin{aligned} f_{Z_2}(z) &= \int_z^1 \frac{1}{y} f_{XY}(z/y,y)\,dy = \int_z^1 \frac{1}{y}\left(\frac{z}{y} + y\right) dy \\ &= 2(1-z), \quad 0 < z < 1 \end{aligned}$$

(c)

$$Z_3 = \frac{Y}{X}$$

$$F_{Z_3}(z) = P(Y/X \leq z) = \begin{cases} \displaystyle\int_0^1 \int_0^{zx} f_{XY}(x,y)\,dy\,dx, & 0 < z < 1 \\ \displaystyle 1 - \int_0^1 \int_0^{y/z} f_{XY}(x,y)\,dx\,dy, & z > 1 \end{cases}$$

$$f_{Z_3}(z) = \begin{cases} \int_0^1 x\, f_{XY}(x, zx)\, dx, & 0 < z < 1 \\ \int_0^1 \frac{y}{z^2}\, f_{XY}(y/z, y)\, dy, & z > 1 \end{cases}$$

$$= \begin{cases} \frac{1+z}{3}, & 0 < z < 1 \\ \frac{1+z}{3z^3}, & z > 1 \end{cases}$$

(d)

$$Z_4 = Y - X$$

$$F_{Z_4}(z) = P(Y - X \leq z) = \begin{cases} 1 - \int_z^1 \int_0^{y-z} f_{XY}(x, y)\, dx\, dy & 0 < z < 1 \\ \int_0^{z+1} \int_{y-z}^1 f_{XY}(x, y)\, dx\, dy, & -1 < z < 0 \end{cases}$$

$$f_{Z_4}(z) = \begin{cases} \int_z^1 f_{XY}(y - z, y)\, dy, & 0 < z < 1 \\ \int_0^{z+1} f_{XY}(y - z, y)\, dy, & -1 < z < 0 \end{cases}$$

$$= \begin{cases} 1 - z, & 0 < z < 1 \\ 1 + z, & -1 < z < 0 \end{cases} = 1 - |z|, \quad |z| < 1$$

6.8

$$F_Z(z) = P(X + Y \leq z)$$

$$= \begin{cases} \int_0^{z/3} \int_{2y}^{z-y} f_{XY}(x, y)\, dx\, dy = \frac{z^2}{6}, & 0 < z < 2 \\ 1 - \int_{2z/3}^2 \int_{z-x}^{x/2} f_{XY}(x, y)\, dy\, dx = 2z - \frac{z^2}{3} - 2, & 2 < z < 3 \end{cases}$$

Thus

$$f_Z(z) = \begin{cases} \int_0^{z/3} f_{XY}(z - y, y)\, dy & 0 < z < 2 \\ \int_{2z/3}^2 f_{XY}(x, z - x)\, dx & 2 < z < 3 \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{1}{3}\, z, & 0 < z < 2 \\ 2 - \frac{2z}{3}, & 2 < z < 3 \\ 0, & \text{otherwise} \end{cases}$$

6.9 (a)

$$Z = \frac{X}{Y}, \qquad z \geq 1$$

$$F_Z(z) = P(X \leq Yz) = \int_0^1 \int_{x/z}^{x} f_{XY}(x,y)\,dy\,dx$$

$$f_Z(z) = \int_0^1 \frac{x}{z^2}\, f_{XY}(x, x/z)\,dx = \frac{1}{z^2}, \quad z \geq 1$$

(b)

$$W = XY$$

$$\begin{aligned} F_W(w) &= P(W \leq w) = P(XY \leq w) = 1 - P(XY > w) \\ &= 1 - \int_{\sqrt{w}}^{1} \int_{w/x}^{x} f_{XY}(x,y)\,dy\,dx \end{aligned}$$

Hence

$$\begin{aligned} f_W(w) &= \int_{\sqrt{w}}^{1} \tfrac{1}{x}\, f_{XY}(x, w/x)\,dx = \int_{\sqrt{w}}^{1} \tfrac{2}{x}\,dx \\ &= \ln(1/w), \quad 0 < w \leq 1 \end{aligned}$$

6.10 (a)

$$Z = X + Y$$

$$F_Z(z) = \int_0^{z/2} \int_x^{2-x} f_{XY}(x,y)\,dx = \frac{z^2}{4}, \quad 0 < z < 2$$

$$f_Z(z) = \frac{z}{2}, \qquad 0 < z < 2$$

(b)

$$W = X - Y$$

$$F_W(w) = \frac{1}{2}\,(2+w)\,(1 + \frac{w}{2}) = \left(1 + \frac{w}{2}\right)^2$$

$$f_W(w) = \begin{cases} 1 + \frac{w}{2}, & -2 < w < 0 \\ 0, & \text{otherwise} \end{cases}$$

6.11 (a) The charaterristic function of $X+Y$ is given by

$$\begin{aligned}\phi_{X+Y}(\omega) &= \phi_X(\omega)\,\phi_Y(\omega) = \tfrac{1}{(1-j\omega\beta)^\alpha}\cdot\tfrac{1}{(1-j\omega\beta)^\alpha}\\ &= \tfrac{1}{(1-j\omega\beta)^{2\alpha}} \sim \text{Gamma}\,(2\alpha,\beta)\end{aligned}$$

(b)

$$f_{XY}(x,y) = f_X(x)\,f_Y(y) = \frac{(xy)^{(\alpha-1)}}{(\Gamma(\alpha)\beta^\alpha)^2}\; e^{(x+y)/\beta}, \quad x>0,\; y>0$$

Let

$$Z = \frac{X}{Y}$$

Using (Eq. 6-60) we get

$$\begin{aligned}f_Z(z) &= \int_0^\infty y\,\frac{(y^2 z)^{(\alpha-1)}}{(\Gamma(\alpha)\beta^\alpha)^2}\; e^{-(1+z)y/\beta}\,dy\\ &= \frac{z^{(\alpha-1)}}{(\Gamma(\alpha)\beta^\alpha)^2}\int_0^\infty y^{(2\alpha-1)}\,e^{-(1+z)y/\beta}\,dy\\ &= \frac{z^{(\alpha-1)}}{(\Gamma(\alpha))^2\;\beta^{2\alpha}}\,\frac{\beta^{(2\alpha-1)}}{(1+z)^{2\alpha-1}}\,\frac{\beta}{(1+z)}\int_0^\infty u^{2\alpha-1}\,e^{-u}\,du\\ &= \frac{(\Gamma(2\alpha))\;z^{\alpha-1}}{(\Gamma(\alpha))^2\,(1+z)^{2\alpha}}, \quad z>0\end{aligned}$$

(see also Example 6-27 for the answer).
(c)

$$W = \frac{X}{X+Y} = \frac{X/Y}{X/Y+1} = \frac{Z}{Z+1}$$

$$F_W(w) = P\left(\frac{Z}{Z+1}\le w\right) = P\left(Z\le\frac{w}{1-w}\right) = F_Z\left(\frac{w}{1-w}\right)$$

This gives

$$\begin{aligned}f_W(w) &= \frac{1}{(1-w)^2}\,f_Z\left(\frac{w}{1-w}\right)\\ &= \frac{\Gamma(2\alpha)}{(\Gamma(\alpha))^2}\,w^{\alpha-1}\,(1-w)^{\alpha-1}\\ &\sim \;\mathit{Beta}\,(\alpha,\alpha)\end{aligned}$$

where we have used results from (b) above.

6.12

$$X\sim U(0,1), \quad Y\sim U(0,1), \quad X,Y \text{ are independent, and}$$

$$U = X+Y, \qquad V = X-Y \;\Rightarrow\; |v|<u<2.$$

U and V have one pair of solutions given by

$$x_1 = \frac{u+v}{2},\, y_1 = \frac{u-v}{2}.$$

Also the Jacobian is given by

$$J = \begin{vmatrix} 1 & 1\\ 1 & -1\end{vmatrix} = -2$$

so that

$$f_{UV}(u,v) = \frac{1}{|J|}\,f_{XY}(x_1,y_1) = \frac{1}{2}, \quad 0<|v|<u<2$$

6.13

$$f_{XY}(x,y) = \frac{xy}{\sigma^4} e^{-(x^2+y^2)/2\sigma^2}, \quad x, y \geq 0$$

$$Z = \frac{X}{Y}$$

$$F_Z(z) = P(Z \leq z) = P(X/Y \leq z) = \int_0^\infty \int_0^{zy} f_{XY}(x,y)\, dx\, dy.$$

This gives the density function of z to be

$$\begin{aligned} f_Z(z) &= \int_0^\infty y\, f_{XY}(zy, y)\, dy = \int_0^\infty \frac{zy^3}{\sigma^4} e^{-(z^2y^2+y^2)/2\sigma^2}\, dy \\ &= \frac{z}{\sigma^4}\int_0^\infty y^3\, e^{-y^2(z^2+1)/2\sigma^2}\, dy \quad \text{Let, } t = y^2(z^2+1)/2\sigma^2 \\ &= \frac{2z}{(z^2+1)^2}\int_0^\infty t\, e^{-t}\, dt = \frac{2z}{(z^2+1)^2}, \quad 0 \leq z \leq \infty. \end{aligned}$$

6-14 $\quad \underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y} \qquad f_z(z) = f_x(z) * f_y(z)$

For $z > 0$

$$c^2 z\, e^{-cz} = \int_0^z c\, e^{-c(z-y)}\, f_y(y)\, dy$$

$$c\, z = \int_0^z e^{cy}\, f_y(y)\, dy \qquad c = e^{cz} f_y(z)$$

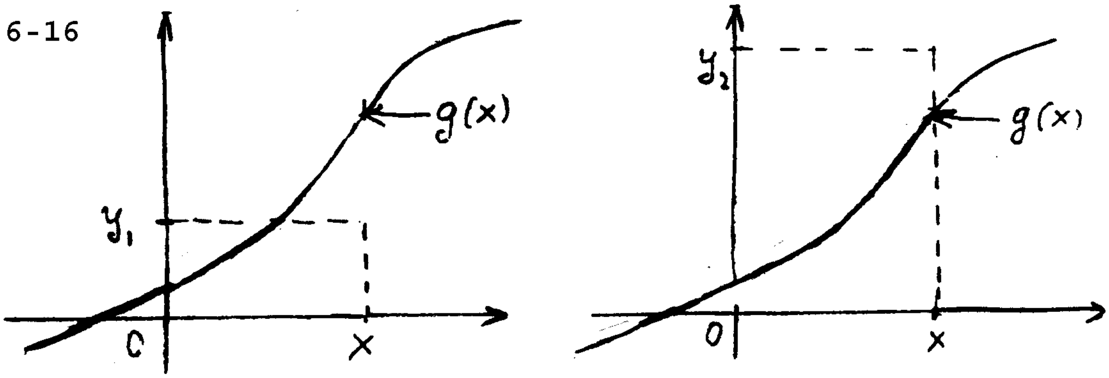
(differentiation). Hence, $f_y(z) = c\, e^{-cz}$; and zero for $z < 0$.

6-15

$$f_z(z) = \int_{-\infty}^{\infty} f_x(x) f_y(z-x)\, dx = \int_{z-1}^{z} f_x(x)\, dx = F_x(z) - F_x(z-1)$$

because $f_y(z-x) = 1$ for $z-1 < x < z$ and zero otherwise.

6-16



All probability masses are on the line $y = g(x)$.

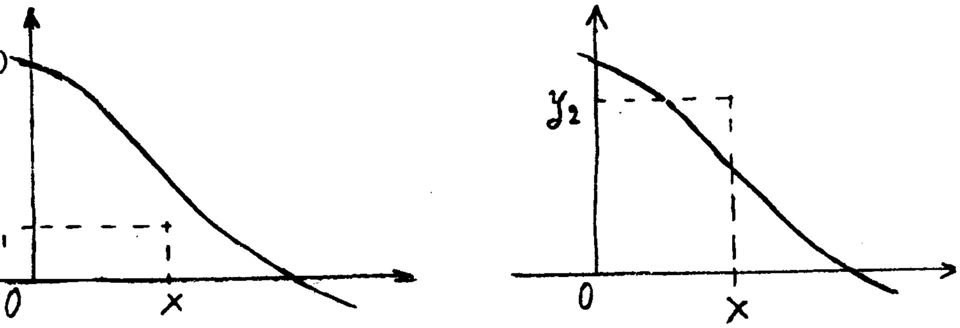
(a) If $y = y_1 < g(x)$ then

$$F(x,y) = P\{\mathbf{x} \le x,\ \mathbf{y} \le y_1\} = P\{\mathbf{y} \le y_1\} = F_y(y_1).$$

If $y = y_2 > g(x)$ then

$$F(x,y) = P\{\mathbf{x} \le x,\ \mathbf{y} \le y_2\} = P\{\mathbf{x} \le x\} = F_x(x)$$

(b)



If $y = y_1 < g(x)$ then

$$F(x,y) = P\{\mathbf{x} \le x,\ \mathbf{y} \le y_1\} = 0$$

If $y = y_2 > g(x)$ then

$$F(x,y) = P\{\mathbf{x} \le x,\ \mathbf{y} \le y_2\} = P\{\mathbf{x} \le x\} - P\{\mathbf{y} > y_2\}$$

$$= F_x(x) - [1 - F_y(y_2)]$$

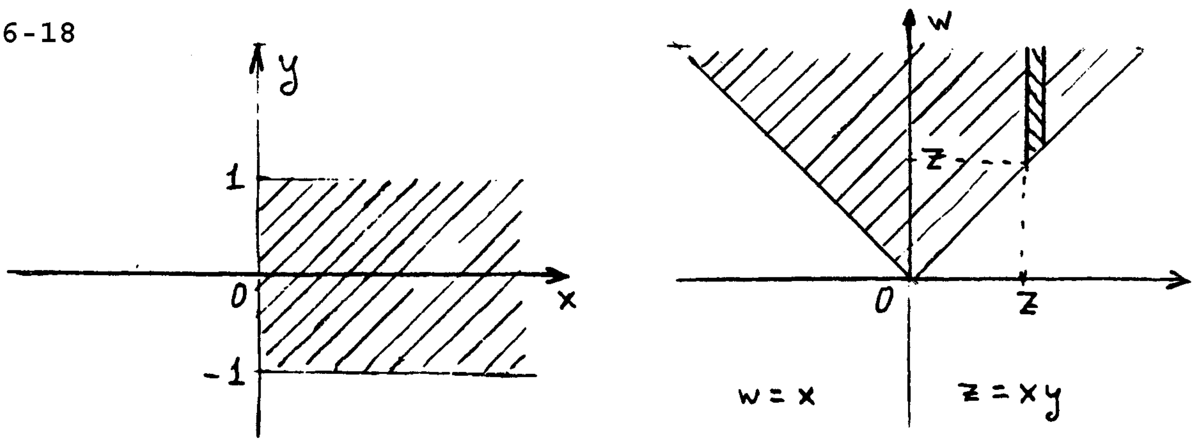
6-17 (a) If $\underset{\sim}{z} = 2\underset{\sim}{x} + 3\underset{\sim}{y}$ then $E\{\underset{\sim}{z}\} = 0$ $\qquad \sigma_z^2 = 4\sigma_x^2 + 9\sigma_y^2 = 5^2$

Hence, $\underset{\sim}{z}$ is $N(0;\sqrt{52})$

(b) If $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y}$, then from (6-63) with $\sigma_1 = \sigma_2 = 2$, $r = 0$

$$F_z(z) = \frac{1}{2} + \frac{1}{\pi}\arctan z \qquad f_z(z) = \frac{1}{\pi(1+z^2)}$$

6-18



$$f_{zw}(z,w) = \frac{1}{|x|} f_{xy}(x,y) \qquad x = w \qquad y = z/w$$

The function $f_{zw}(z,w)$ is different from zero in the shaded areas shown. Hence, with $w^2 - z^2 = s^2$

$$f_z(z) = \frac{1}{\pi\alpha^2}\int_{|z|}^{\infty} e^{-w^2/2\alpha^2} \frac{dw}{\sqrt{1 - z^2/w^2}}$$

$$= \frac{1}{\pi\alpha^2}\int_0^{\infty} e^{-(z^2+s^2)/2\alpha^2}\, ds = \frac{1}{\alpha\sqrt{2\pi}} e^{-z^2/2\alpha^2}$$

6-19 (a) $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y}$ $\quad \underset{\sim}{w} = \underset{\sim}{y}$ $\quad J = 1/y$

$$f_z(z) = \int_{-\infty}^{\infty} |w| f_x(zw) f_y(w)\, dw \qquad z > 0$$

$$= \frac{z}{\alpha^2\beta^2} \int_0^{\infty} w^3\, e^{-cw^2} dw = \frac{z}{2\alpha^2\beta^2 c^2} \qquad c = \frac{z^2}{2\alpha^2} + \frac{1}{2\beta^2}$$

$$= \frac{2\alpha^2}{\beta^2} \frac{z}{(z^2 + \alpha^2/\beta^2)^2} \qquad \text{for } z > 0 \text{ and zero otherwise}$$

(b) $$F_z(z) = \int_0^z \frac{2\alpha^2 z\, dz}{\beta^2 (z^2 + \alpha^2/\beta^2)^2} = \frac{\alpha 2}{\beta^2} \int_{\alpha^2/\beta^2}^{z^2 + \alpha^2/\beta^2} \frac{dt}{t^2}$$

$$= \frac{z^2}{z^2 + \alpha^2/\beta^2} = P\{\underset{\sim}{z} \le z\} = P\{\underset{\sim}{x} \le zy\}$$

6-20 1. The density of $2\underset{\sim}{x}$ equals $\frac{1}{2} f_x(\frac{x}{2})$. Hence, if $\underset{\sim}{z} = 2\underset{\sim}{x} + \underset{\sim}{y}$, then

$$f_z(z) = \int_0^z \frac{\alpha}{2} e^{-\alpha x/2} \beta e^{-\beta(z-x)} dx = \frac{\alpha\beta}{\alpha - 2\beta}(e^{-\beta z} - e^{-\alpha z/2}) U(z)$$

2. The density of $\underset{\sim}{y}$ equals $f_y(-y)$. Hence, if $\underset{\sim}{z} = \underset{\sim}{x} - \underset{\sim}{y}$, then

$$f_z(z) = f_x(z) * f_y(-z)$$

$$= \alpha\beta \begin{cases} \int_z^\infty e^{-\alpha x} e^{-\beta(x-z)} dx = \frac{\alpha\beta}{\alpha+\beta} e^{-\alpha z} & z > 0 \\ \int_0^\infty e^{-\alpha x} e^{-\beta(x-z)} dx = \frac{\alpha\beta}{\alpha+\beta} e^{\beta z} & z < 0 \end{cases}$$

3. $\underset{\sim}{z} = \underset{\sim}{x}/\underset{\sim}{y}$ $\qquad$ $\underset{\sim}{w} = \underset{\sim}{y}$ $\qquad$ $J = 1/y$

$$f_z(z) = \alpha\beta \int_0^\infty w e^{-\alpha z w} e^{-\beta w} dw = \frac{\alpha\beta}{(\alpha z + \beta)^2} U(z)$$

4. $\underset{\sim}{z} = \max(\underset{\sim}{x}, \underset{\sim}{y})$ $\qquad$ $F_z(z) = F_{xy}(z,z) = F_x(z)F_y(z)$

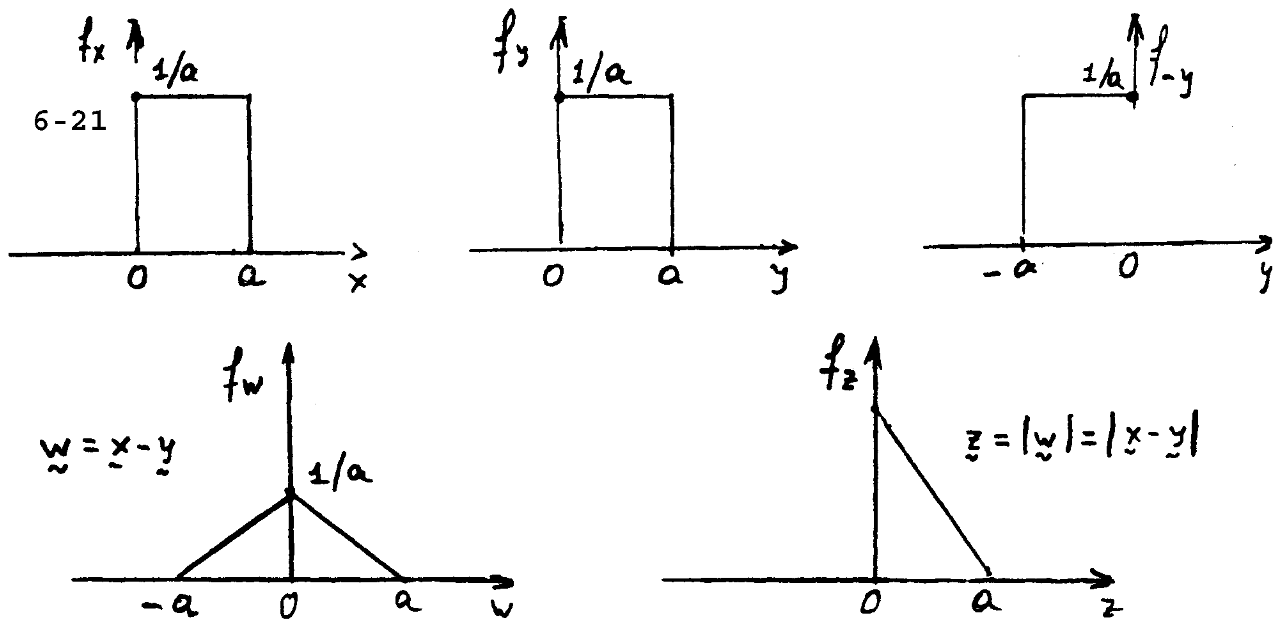
$$f_z(z) = f_x(z)F_y(z) + f_y(z)F_x(z)$$

$$= \left[\alpha e^{-\alpha z}(1 - e^{-\beta z}) + \beta e^{-\beta z}(1 - e^{-\alpha z})\right] U(z)$$

5. $\underset{\sim}{z} = \min(\underset{\sim}{x}, \underset{\sim}{y})$ $\qquad$ $F_z(z) = F_x(z) + F_y(z) - F_x(z)F_y(z)$

$$f_z(z) = f_x(z)[1 - F_y(z)] + f_y(z)[1 - F_x(z)] = (\alpha + \beta) e^{-(\alpha+\beta)z} U(z)$$

6-21



6-22 (a) $$\alpha y^2 + \beta(x-y)^2 = (\alpha+\beta)\left(y - \frac{\beta x}{\alpha+\beta}\right)^2 + \frac{\alpha\beta}{\alpha+\beta}x^2$$

$$e^{-\alpha x^2} * e^{-\beta x^2} = \int_{-\infty}^{\infty} e^{-\alpha y^2 - \beta(x-y)^2}\,dy$$

$$= e^{-\alpha\beta x^2/(\alpha+\beta)} \int_{-\infty}^{\infty} e^{-(\alpha+\beta)\left(y - \frac{\beta x}{\alpha+\beta}\right)^2} dy = \sqrt{\frac{\pi}{\alpha+\beta}}\, e^{\frac{-\alpha\beta x^2}{\alpha+\beta}}$$

(b) $$\frac{\alpha/\pi}{x^2+\alpha^2} * \frac{\beta/\pi}{x^2+\beta^2} = \frac{\alpha\beta}{\pi^2}\int_{-\infty}^{\infty} \frac{dy}{(y^2+\alpha^2)[(x-y)^2+\beta^2]} = \frac{(\alpha+\beta)/\pi}{x^2+(\alpha+\beta)^2}$$

Characteristic functions lead to a simpler derivation of the above [see (6-192)]

6-23 We introduce the auxiliary variable w=y. The Jacobian of the transformation z=nx/my, w=y equals n/my. Since x=mzw/n, y=w and the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent, (6-113) yields

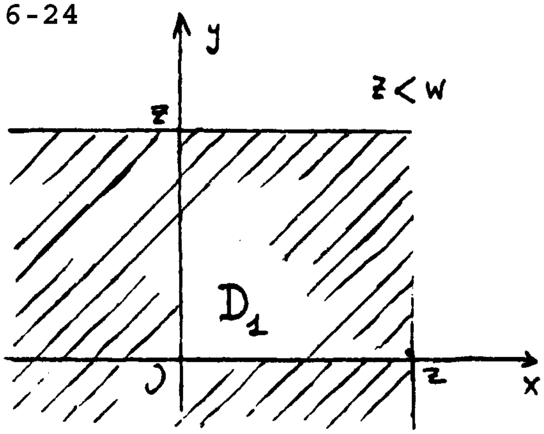
$$f_{zw}(z,w) = \frac{mw}{n} f_x\left(\frac{m}{n}zw\right) f_y(w) \sim w(zw)^{m/2-1}e^{-mzw/2}w^{n/2-1}e^{-w/2}$$

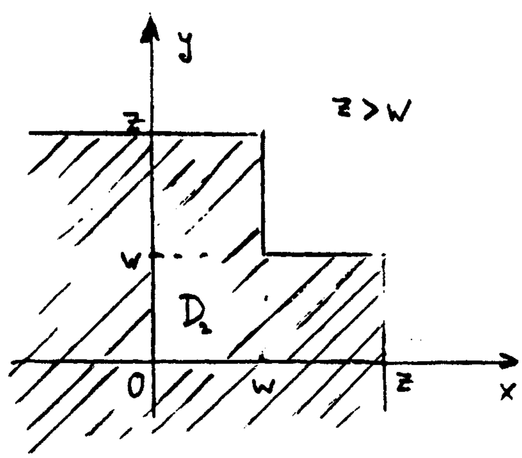
for z>0, w>0 and 0 othrwise. Integrating with respect to w, we obtain

$$f_z(z) \sim z^{m/2-1}\int_0^\infty w^{(m+n)/2-1}\exp\left\{-\frac{w}{2}\left(1+\frac{m}{n}z\right)\right\}dw$$

$$\sim \frac{z^{m/2-1}}{(1+mz/n)^{(m+n/2)}}\int_0^\infty q^{(m+n)/2}e^{-q}dq$$

6-24





If $z \le w$ then

$$P\{\underset{\sim}{z} \le z, \underset{\sim}{w} \le w\} = P\{\underset{\sim}{z} \le z\} = P\{(\underset{\sim}{x},\underset{\sim}{y}) \in D_1\} = F_{xy}(z,z)$$

If $z > w$ then

$$P\{\underset{\sim}{z} \le z, \underset{\sim}{w} \le w\} = P\{(\underset{\sim}{x},\underset{\sim}{y}) \in D_2\}$$

$$= F_{xy}(z,w) + F_{xy}(w,z) - F_{xy}(w,w)$$

6.25

$$X \sim \text{Exponential}(\lambda), \quad Y \sim \text{Exponential}(\lambda)$$

X and Y are independent so that

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = \frac{1}{\lambda^2}\, e^{-(x+y)/\lambda}\, U(x)\, U(y)$$

$$Z = X + Y$$

$$\phi_Z(\omega) = \phi_X(\omega)\, \phi_Y(\omega) \frac{1}{(1 - j\omega\lambda)^2}$$

$$Z \sim \text{ Gamma } (2, \lambda)$$

This gives

$$f_Z(z) = \frac{z}{\lambda^2}\, e^{-z/\lambda} U(z)$$

$$P(Z > 2\lambda) = \int_{2\lambda}^{\infty} \frac{z}{\lambda^2}\, e^{-z/\lambda}\, dz = \int_{2}^{\infty} x e^{-x}\, dx = 3e^{-2} = 0.406$$

Let,

$$W = Y - X$$

Then

$$P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} f_W(w) dw$$

Notice that $F_W(w)$ is given by (6-55).
For $w > 0$, this gives

$$\begin{aligned} f_W(w) &= \int_0^{\infty} \frac{1}{\lambda^2}\, e^{-(w+2y)/\lambda}\, dy = \frac{1}{\lambda^2}\, e^{-w/\lambda} \int_0^{\infty} e^{-2y/\lambda}\, dy \\ &= \frac{1}{2\lambda}\, e^{-w/\lambda}, \quad w > 0 \end{aligned}$$

Hence

$$P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} \frac{1}{2\lambda}\, e^{-w/\lambda}\, dw = \frac{1}{2e}$$

6.26 (a)

$$\begin{aligned} R &= W - Z \\ &= \max(\mathrm{X}, \mathrm{Y}) - \min(\mathrm{X}, \mathrm{Y}) \\ &= \begin{cases} X - Y, & X \geq Y \\ Y - X, & X < Y \end{cases} \end{aligned}$$

$$\begin{aligned} F_R(r) &= P\{R \leq r\} \\ &= P\{R \leq r, X \geq Y\} + P\{R \leq r, X < Y\} \\ &= P\{X - Y \leq r, X \geq Y\} + P\{Y - X \leq r, X < Y\} \\ &= 1 - 2\frac{(1-r)^2}{2} = 1 - (1-r)^2, \quad 0 \leq r \leq 1 \end{aligned}$$

$$f_R(r) = \begin{cases} 2(1-r), & 0 \leq r \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} S &= W + Z \\ &= \max(\mathrm{X}, \mathrm{Y}) + \min(\mathrm{X}, \mathrm{Y}) = X + Y \end{aligned}$$

Case 1: $0 < s < 1$

$$F_S(s) = P\{S \leq s\} = P\{X + Y \leq s\} = \frac{s^2}{2}, \quad 0 < s < 1$$

Case 2: $1 \leq s \leq 2$

$$F_S(s) = P\{S \leq s\} = P\{X + Y \leq s\} = 1 - \frac{(2-s)^2}{2}, \quad 1 \leq s \leq 2$$

$$F_S(s) = \begin{cases} s, & 0 \leq s \leq 1 \\ (2-s), & 1 \leq s \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

6.27 (a) X, Y are independent, identically distributed exponential random variables.

$$Z = \frac{Y}{\max(X,Y)} = \begin{cases} \frac{Y}{X}, & X \geq Y \\ 1, & X < Y \end{cases} \quad \Rightarrow \; 0 < z \leq 1.$$

$0 < z < 1$

$$\begin{aligned} F_Z(z) &= P(Z \leq z) = P\left\{\tfrac{Y}{X} \leq z, \, X > Y\right\} \\ &= P\{Y \leq Xz, \, X > Y\} = \int_0^\infty \int_0^{xz} f_{XY}(x,y)\, dy\, dx \end{aligned}$$

$$f_Z(z) = \int_0^\infty x\, f_{XY}(x, xz)\, dx = \int_0^\infty \frac{x}{\lambda^2}\, e^{-(1+z)x/\lambda}\, dx = \frac{1}{(1+z)^2}, \quad 0 < z < 1.$$

Also

$$P(Z = 1) = P(X < Y) = \int_0^\infty \int_0^y \frac{1}{\lambda^2}\, e^{-(x+y)/\lambda}\, dx\, dy = \frac{1}{2}$$

(b)

$$W = \frac{X}{\min(X, 2Y)} = \begin{cases} \frac{X}{2Y}, & X \geq 2Y \\ 1, & X < 2Y \end{cases} \quad \Rightarrow \; 1 \leq w < \infty$$

$$F_W(w) = P(X \leq 2Yw, \, X > 2Y) = \int_0^\infty \int_{2y}^{2wy} f_{XY}(x,y)\, dx\, dy$$

This gives

$$\begin{aligned} f_W(w) &= \int_0^\infty 2y\, f_{XY}(2wy, y) dy = \int_0^\infty \frac{2y}{\lambda^2}\, e^{-(1+2w)y/\lambda} dy \\ &= \frac{2}{(1+2w)^2}, \quad w > 1 \end{aligned}$$

Also

$$P(W = 1) = P(X < 2Y) = \int_0^\infty \int_0^{2y} \frac{1}{\lambda^2}\, e^{-(x+y)/\lambda}\, dx\, dy = \frac{2}{3}$$

Note that the p.d.f. of Z as well as W has an impulse at $z = 1$ and $w = 1$ respectively.

6.28 X, Y are independent identically distributed exponential random variables.

$$Z = \frac{X}{X+Y}$$

$$\begin{aligned} F_Z(z) &= P\left(\frac{X}{X+Y} \le z\right) = P\left(\frac{X}{Y} \le \frac{z}{1-z}\right) \\ &= P\left\{X \le \frac{zY}{1-z}\right\} = \int_0^\infty \int_0^{(zy)/(1-z)} f_{XY}(x,y)\,dx\,dy \end{aligned}$$

$$\begin{aligned} f_Z(z) &= \int_0^\infty \frac{y}{(1-z)^2}\, f_{XY}(zy/(1-z), y)\,dy \\ &= \frac{1}{(1-z)^2}\int_0^\infty y\,\frac{1}{\lambda^2}\, e^{-(z/(1-z)+1)(y/\lambda)}\,dy \\ &= \frac{1}{(1-z)^2}\int_0^\infty \frac{y}{\lambda^2}\, e^{-[y/(1-z)\lambda]}\,dy \\ &= \int_0^\infty u\, e^{-u}\,du = 1, \quad 0 < z < 1 \\ &\Rightarrow \frac{X}{X+Y} \sim U(0,1) \end{aligned}$$

6.29 Let

$$f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} U(x), \quad f_Y(y) = \frac{1}{\lambda} e^{-y/\lambda} U(y).$$

$$Z = \min(X,Y)$$

$$W = \max(X,Y) - \min(X,Y)$$

$$Z = \begin{cases} Y, & X \ge Y \\ X, & X < Y \end{cases}$$

$$W = \begin{cases} X - Y, & X \ge Y \\ Y - X, & X < Y \end{cases}$$

$Z = \min(X,Y)$. See Example 6-18, Eq.(6-82) for solution. From there (replace λ by $1/\lambda$ in (6-82))

$$f_Z(z) = \frac{2}{\lambda} e^{-2z/\lambda} U(z).$$

$$\begin{aligned} F_W(w) &= P(X - Y \le w, X \ge Y) + P(Y - X \le w, X < Y) \\ &= \int_0^\infty \int_y^{y+w} f_{XY}(x,y)\,dx\,dy \\ &+ \int_0^\infty \int_x^{x+w} f_{XY}(x,y)\,dy\,dx, \quad w > 0 \end{aligned}$$

This gives

$$\begin{aligned} F_W(w) &= \int_0^\infty f_{XY}(y+w, y)\,dy + \int_0^\infty f_{XY}(x, x+w)\,dx \\ &= 2\int_0^\infty \frac{1}{\lambda^2}\, e^{(2y+w)/\lambda}\,dy \\ &= \frac{2}{\lambda^2}\, e^{-w/\lambda}\, \frac{e^{-2y/\lambda}}{-2/\lambda}\Bigg|_0^\infty = \frac{1}{\lambda} e^{-w/\lambda}, \quad w > 0 \end{aligned}$$

Also

$$\begin{aligned} F_{ZW}(z,w) &= P\{Z \le z, W \le w\} \\ &= P\{Y \le z, X - Y \le w, X \ge Y\} \\ &+ P\{X \le z, Y - X \le w, X < Y\} \\ &= \int_0^z \int_y^{y+w} f_{XY}(x,y)\,dx\,dy + \int_0^z \int_x^{x+w} f_{XY}(x,y)\,dy\,dx \end{aligned}$$

Repeated use of (6-39)-(6-40) gives

$$\begin{aligned} f_{ZW}(z,w) &= f_{XY}(z+w,z) + f_{XY}(z,z+w) \\ &= \frac{2}{\lambda^2}\,e^{-(2z+w)/\lambda} = \frac{2}{\lambda}\,e^{-2z/\lambda}\frac{1}{\lambda}\,e^{-w/\lambda} \\ &= f_Z(z)\,f_W(w) \end{aligned}$$

Thus Z and W are independent exponential random variables.

6.30 (a) Let

$$U = X + Y, \qquad 0 < u < 2\beta.$$

The probability density function of U can be computed as in (6-48)-(6-50). Using Fig. 6-11, for $0 < u \le \beta$, we have

$$F_U(u) = \int_0^u \int_0^{u-x} f_{XY}(x,y)\,dy\,dx$$

which gives

$$\begin{aligned} f_U(u) &= \int_0^u f_{XY}(x,u-x)dx = \alpha^2\beta^{-2\alpha}\int_0^u x^{\alpha-1}\,(u-x)^{\alpha-1}\,dx \\ &= \alpha^2\beta^{-2\alpha}u^{2\alpha-1}\int_0^1 y^{\alpha-1}\,(1-y)^{\alpha-1}\,dy \\ &= B(\alpha,\alpha)\,\alpha^2\beta^{-2\alpha}u^{2\alpha-1} \quad 0 < u \le \beta \end{aligned}$$

where we have substituted $y = ux$ and made use of the beta function defied in (4-49)-(4-51). Similarly for $\beta < u \le 2\beta$, we get (see (6-49))

$$F_U(u) = 1 - \int_{u-\beta}^{\beta} \int_{u-x}^{\beta} f_{XY}(x,y)\,dydx$$

and hence

$$\begin{aligned} f_U(u) &= \int_{u-\beta}^{\beta} f_{XY}(x,u-x)dx = \alpha^2\beta^{-2\alpha}\int_{u-\beta}^{\beta} x^{\alpha-1}(u-x)^{\alpha-1}\,dx \\ &= \alpha^2\beta^{-2\alpha}u^{2\alpha-1}\int_{1-\beta/u}^{\beta/u} y^{\alpha-1}(1-y)^{\alpha-1}\,dy, \quad \beta < u \le 2\beta \end{aligned}$$

(b)

$$Z = \min(X,Y), \qquad W = \max(X,Y)$$

We can proceed as in Example 6-21 to complete this problem. From (6-92) and (6-93), we get

$$F_{ZW}(z,w) = \begin{cases} F_{XY}(z,w) + F_{XY}(w,z) - F_{XY}(z,z), & w \ge z \\ F_{XY}(w,w), & w < z \end{cases}$$

which gives

$$f_{ZW}(z,w) = f_X(z)f_Y(w) + f_X(w)f_Y(z), \quad 0 < z \le w < \beta$$

$$f_{ZW}(z,w) = \begin{cases} 2\alpha^2\beta^{-2\alpha}z^{\alpha-1}w^{\alpha-1}, & 0 < z \le w < \beta \\ 0, & \text{otherwise} \end{cases}$$

check:

$$\begin{aligned} \int_0^\beta \int_0^w f_{ZW}(z,w)dzdw &= 2\alpha^2\beta^{-2\alpha}\int_0^\beta w^{\alpha-1}\left(\left.\frac{z^\alpha}{\alpha}\right|_0^w\right)dw \\ &= 2\alpha\beta^{-2\alpha}\int_0^\beta w^{2\alpha-1}dw = 1 \end{aligned}$$

Note: Z and W are not independent random variables, since

$$f_Z(z) = 2\alpha\beta^{-2\alpha}\, z^{\alpha-1}\,(\beta^\alpha - z^\alpha), \quad 0 < z < \beta$$

and

$$f_W(w) = 2\alpha\beta^{-2\alpha}\, w^{2\alpha-1}, \quad 0 < w < \beta$$

(c) Let

$$V = \frac{Z}{W} = \frac{\min(X,Y)}{\max(X,Y)} = \begin{cases} \frac{Y}{X}, & X \ge Y \\ \frac{X}{Y}, & X < Y \end{cases}$$

and

$$W = \max(X,Y) = \begin{cases} X, & X \ge Y \\ Y, & X < Y \end{cases}$$

For $0 < v < 1,\ 0 < w < \beta$

$$\begin{aligned} F_{VW}(v,w) &= P(V \le v, W \le w) \\ &= P\{V \le v, W \le w,\ (X \ge Y)\bigcup(X < Y)\} \\ &= P\{Y \le Xv,\ X \le w,\ X \ge Y\} \\ &+ P\{X < Yv,\ Y \le w,\ X < Y\} \\ &= \int_0^w \int_0^{xv} f_{XY}(x,y)\,dy\,dx + \int_0^w \int_0^{yv} f_{XY}(x,y)\,dx\,dy \end{aligned}$$

Hence

$$
\begin{aligned}
f_{VW}(v,w) &= \frac{\partial^2 F_{VW}(v,w)}{\partial v\,\partial w} \\
&= \frac{\partial}{\partial v}\left\{\int_0^{vw} f_{XY}(w,y)\,dy + \int_0^{vw} f_{XY}(x,w)\,dx\right\} \\
&= w\,\{f_{XY}(w,vw) + f_{XY}(vw,w)\} \\
&= 2\alpha^2\,\beta^{-2\alpha}\,w^{2\alpha-1}\,v^{\alpha-1}, \qquad 0<v<1,\ 0<w<\beta
\end{aligned}
$$

Hence

$$f_V(v) = \int_0^{\beta} f_{VW}(v,w)dw = \alpha\,v^{\alpha-1}, \quad 0<v<1$$

$$f_W(w) = \int_0^{1} f_{VW}(v,w)dv = 2\alpha\,\beta^{-2\alpha}\,w^{2\alpha-1}, \quad 0<w<\beta$$

and

$$f_{VW}(v,w) = f_V(v)\,f_W(w).$$

Thus V and W are independent random variables.

6.31 (a) Solved in Examples 6-27 and 6-12.
(b) Solved in Example 6-27.
(c)

$$Z = X+Y, \qquad W = \frac{X}{X+Y}$$

$$x_1 = zw, \qquad y_1 = z - x_1 = z(1-w)$$

$$J = \begin{vmatrix} 1 & 1 \\ \dfrac{y}{(x+y)^2} & -\dfrac{x}{(x+y)^2} \end{vmatrix} = \frac{1}{x+y} = \frac{1}{z}$$

$$
\begin{aligned}
f_{ZW}(z,w) &= \frac{z}{\alpha^{m+n}\Gamma(m)\Gamma(n)}\,(zw)^{m-1}\,\{z(1-w)\}^{n-1} \\
&= \left(\frac{z^{m+n-1}}{\alpha^{m+n}\Gamma(\alpha+\beta)}\;e^{-z/\alpha}\right)\left(\frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\;w^{m-1}\,(1-w)^{n-1}\right) \\
&= f_Z(z)\,f_W(w)
\end{aligned}
$$

Thus Z and W are independent random variables.

6.32 (a)

$$Z = \frac{X}{|Y|}, \qquad W = \frac{|X|}{|Y|} = |Z|$$

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(X \le |Y|z) = \int_{-\infty}^{\infty} \int_0^{|y|z} f_{XY}(x,y)\,dx\,dy \\ &= 2\int_0^{\infty} |y| f_{XY}(|y|z, y) dy = \frac{2}{2\pi\sigma^2} \int_0^{\infty} y e^{-(z^2+1)y^2/2\sigma^2}\,dy \\ &= \frac{1/\pi}{1+z^2}, \qquad -\infty < z < \infty \end{aligned}$$

Thus Z is a Cauchy random variable. Interestingly, the random variable X/Y is also a Cauchy random variable (see Example 6-11).

$$W = |Z|$$

so that

$$\begin{aligned} F_W(w) &= P(W \le w) = P(|Z| \le w) \\ &= P(-w < Z < w) = F_Z(w) - F_Z(-w) \end{aligned}$$

and hence

$$f_W(w) = f_Z(w) + f_Z(-w) = \frac{2/\pi}{1+w^2}, \quad w > 0.$$

(b)

$$U = X + Y \sim N(0,2)$$
$$V = X^2 + Y^2 \sim \text{Exponential } (2)$$

(see Example 6-14). Here U, V are *not* independent, since

$$J(x,y) = \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = -2(x-y) = 2\sqrt{2v - u^2}$$

and

$$\begin{aligned} f_{UV}(u,v) &= \frac{1}{2\sqrt{2v-u^2}} \frac{1}{2\pi\sigma^2} e^{-v/2\sigma^2} \\ &\ne f_U(u)\, f_V(v), \qquad -\infty < u < \infty, \;\; v > 0. \end{aligned}$$

6.33

$$Z = X + Y, \qquad W = X - Y$$

are jointly normal random variables. Hence if they are uncorrelated, then they are also independent.

$$\begin{aligned} Cov(Z,W) &= E[(Z - \mu_Z)(W - \mu_W)] \\ &= E\left[\{(X - \mu_X) + (Y - \mu_Y)\}\{(X - \mu_X) - (Y - \mu_Y)\}\right] \\ &= \text{Var}(X) - \text{Var}(Y) = \sigma_X^2 - \sigma_Y^2. \end{aligned}$$

The random variables Z and W are uncorrelated implies that $Cov(Z,W) = 0$. Hence $\sigma_X^2 = \sigma_Y^2$ is the necessary and sufficient condition for the independence of $X + Y$ and $X - Y$.

6.34 (a)-(b) Let

$$R = \sqrt{X^2 + Y^2}, \qquad \theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

Form Example 6-22, R and θ are independent random variables with joint p.d.f. as in (6-128). (see (6-131)). In term of R and θ, we have $X = R\cos\theta, Y = R\sin\theta$ and hence we obtain

$$U = \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}} = R\cos 2\theta$$

$$V = \frac{2XY}{\sqrt{X^2 + Y^2}} = R\sin 2\theta$$

This gives

$$J = \begin{vmatrix} \cos 2\theta & -2r\sin 2\theta \\ \sin 2\theta & 2r\cos 2\theta \end{vmatrix} = 2r = 2\sqrt{u^2 + v^2}$$

$$r = \sqrt{u^2 + v^2}, \quad \theta_1 = \frac{1}{2}\tan^{-1}\left(\frac{v}{u}\right), \quad 2\theta_2 = \pi + 2\theta_1.$$

There are two sets of solutions (r, θ_1) and (r, θ_2). Substituting into (6-128) we get

$$\begin{aligned} f_{UV}(u,v) &= \frac{1}{|J|}\{f_{r,\theta}(r,\theta_1) + f_{r,\theta}(r,\theta_2)\} = \frac{2}{|J|} f_{r,\theta}(r,\theta_1) \\ &= \frac{2}{2\sqrt{u^2+v^2}} \frac{\sqrt{u^2+v^2}}{2\pi\sigma^2} e^{-(u^2+v^2)/2\sigma^2} \\ &= \frac{1}{2\pi\sigma^2} e^{-(u^2+v^2)/2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-u^2/2\sigma^2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-v^2/2\sigma^2} \\ &= f_U(u) f_V(v) \end{aligned}$$

Thus U and V are independent normal random variables. Hence it follows that $U = \dfrac{X^2 - Y^2}{\sqrt{X^2+Y^2}}$ and $V/2 = \dfrac{XY}{\sqrt{X^2+Y^2}}$ are independent random variables.

(c)

$$\begin{aligned} Z &= \frac{(X-Y)^2 - 2Y^2}{\sqrt{X^2+Y^2}} = \frac{(X^2 - Y^2) - 2XY}{\sqrt{X^2+Y^2}} \\ &= \frac{X^2 - Y^2}{\sqrt{X^2+Y^2}} - \frac{2XY}{\sqrt{X^2+Y^2}} \\ &= U - V \sim N(0, 2\sigma^2). \end{aligned}$$

6.35 (a) $Z \sim F(m,n)$ is given by (6-157) Let

$$Y = \frac{1}{Z}$$

Then

$$\begin{aligned} F_Y(y) &= \frac{1}{|dy/dz|}\, f_Z(1/y) \\ &= \frac{1}{y^2}\,\frac{(m/n)^{m/2}}{\beta(m/2,n/2)}\,\frac{1}{y^{m/2-1}}\,\frac{1}{(1+m/ny)^{m+n/2}} \\ &= \frac{(n/m)^{n/2}}{\beta(n/2,m/2)}\, y^{n/2-1}\left(1+\frac{n}{my}\right)^{-(m+n)/2} \\ &\sim F(n,m). \end{aligned}$$

(b)

$$W = \frac{Zm}{Zm+n}$$

$$\begin{aligned} F_W(w) &= P(W \le w) = P\left(\frac{Zm}{Zm+n} \le w\right) \\ &= P\left(Z \le \frac{nw}{m(1-w)}\right) = F_Z\left(\frac{nw}{m(1-w)}\right) \end{aligned}$$

which gives

$$\begin{aligned} f_W(w) &= \frac{n}{m(1-w)^2}\, f_Z\left(\frac{nw}{m(1-w)}\right) \\ &= \frac{n}{m(1-w)^2}\,\frac{(m/n)^{m/2}}{\beta(m/2,n/2)}\left(\frac{nw}{m(1-w)}\right)^{m/2-1}\left(1+\frac{w}{(1-w)}\right)^{-(m+n)/2} \\ &= \frac{1}{\beta(m/2,n/2)}\, w^{m/2-1}\,(1-w)^{n/2-1}, \quad 0<w<1. \end{aligned}$$

Thus W has Beta distribution.

6.36

$$Z = X+Y > 0, \qquad W = X-Y > 0$$

$$x_1 = \frac{z+w}{2}, \qquad y_1 = \frac{z-w}{2}$$

is the only solution. Moreover

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

so that

$$f_{ZW}(z,w) = \frac{1}{|J|} f_{XY}(x_1,y_1) = \frac{1}{2}\, e^{-(z+w)/2}, \quad 0<w<z<\infty$$

$$\begin{aligned} F_Z(z) &= \int_0^z f_{ZW}(z,w)\,dw = \frac{1}{2}\, e^{-z/2}\,\left.\frac{e^{-w/2}}{(-1/2)}\right|_0^z \\ &= \frac{1}{2}\, e^{-z/2}\,\left.\frac{e^{-w/2}}{(-1/2)}\right|_0^z = e^{-z/2}(1-e^{-z/2}), \quad z>0 \end{aligned}$$

6.37

$$Z = X + Y > 0, \qquad W = \frac{Y}{X} > 1$$

$$y = xw, \quad x(1+w) = z, \quad x_1 = \frac{z}{1+w}, \quad y_1 = \frac{zw}{1+w}$$

is the only solution. Also

$$J = \begin{vmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{x+y}{x^2} = \frac{(1+w)^2}{z}$$

This gives

$$\begin{aligned} f_{ZW}(z,w) &= \frac{1}{|J|}\, f_{XY}(x_1, y_1) \\ &= \frac{z}{(1+w)^2}\, 2\, e^{-z}, \quad z > 0,\ w > 1 \\ &= z\, e^{-z}\, \frac{2}{(1+w)^2} = f_Z(z)\, f_W(w) \end{aligned}$$

since

$$\begin{aligned} f_Z(z) &= \int_1^\infty f_{ZW}(z,w)\, dw \\ &= 2ze^{-z} \int_1^\infty \frac{1}{(1+w)^2}\, dw = z\, e^{-z}, \quad z > 0 \end{aligned}$$

and

$$\begin{aligned} f_w(w) &= \int_0^\infty f_{ZW}(z,w)\, dz \\ &= \frac{2}{(1+w)^2} \int_0^\infty ze^{-z}\, dz = \frac{2}{(1+w)^2}, \quad w > 1. \end{aligned}$$

Thus Z and W are independent random variables.

6-38 $\underset{\sim}{z} = \underset{\sim}{x}\,\underset{\sim}{y}$ $\qquad \underset{\sim}{y} = \cos(\omega t + \underset{\sim}{\theta})$

$\underset{\sim}{w} = \underset{\sim}{y}$ $\qquad J = |y|$ $\qquad f_y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}} & |y| < 1 \\ 0 & |y| > 1 \end{cases}$

The RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are independent. Hence,

$$f_{zw}(z,w) = \frac{1}{|w|} f_x\left(\frac{z}{w}\right) f_y(w)$$

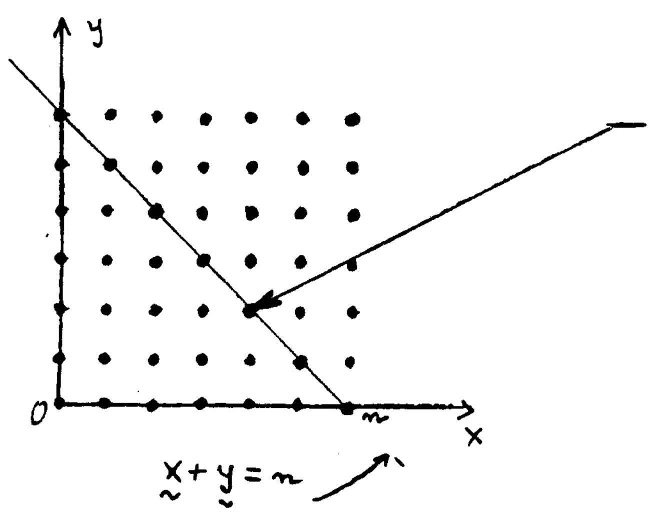
$$f_z(z) = \frac{1}{\pi}\int_{-1}^{1} \frac{f_x(z/w)}{|w|\sqrt{1-w^2}}\,dw = \frac{1}{\pi}\int_{|x|>z} \frac{f_x(x)}{\sqrt{x^2-z^2}}\,dx$$

6-39 $\underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{s}$ $\qquad \underset{\sim}{s} = a\cos\underset{\sim}{y}$

$f_z(z) = f_x(z) * f_s(z)$ $\qquad f_s(s) = \begin{cases} \dfrac{1}{\pi\sqrt{a^2-s^2}} & |s| < a \\ 0 & |s| > a \end{cases}$

$$f_z(z) = \frac{1}{\pi\sigma\sqrt{2\pi}}\int_{-a}^{a} \frac{e^{-(z-s)^2/2\sigma^2}}{\sqrt{a^2-s^2}}\,ds = \frac{1}{\pi\sigma\sqrt{2\pi}}\int_{-\pi}^{\pi} e^{-(z-a\cos y)^2/2\sigma^2}\,dy$$

6-40



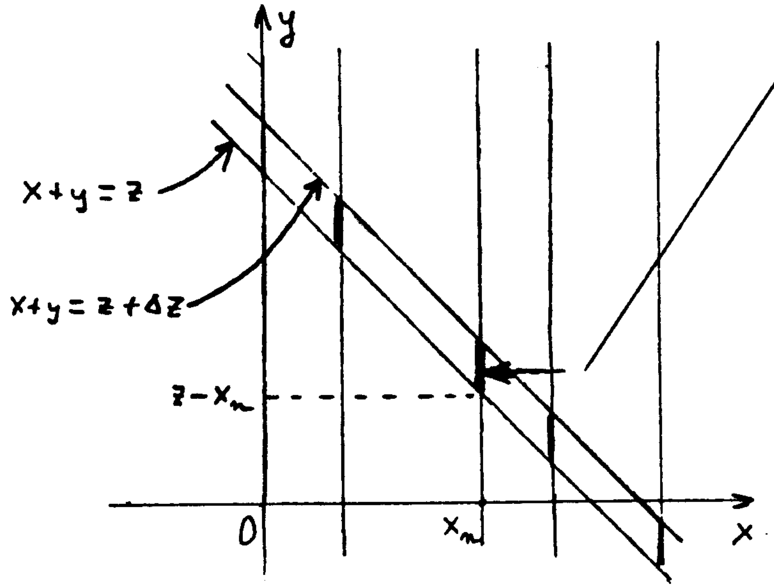
Point masses

$P\{\underset{\sim}{x} = k,\ \underset{\sim}{y} = n-k\} = a_k b_{n-k}$

$$\{\underset{\sim}{z} = n\} = \sum_{k=0}^{n} \{\underset{\sim}{x} = k,\ \underset{\sim}{y} = n-k\}$$

$$P\{\underset{\sim}{z} = n\} = \sum_{k=0}^{n} P\{\underset{\sim}{x} = k,\ \underset{\sim}{y} = n-k\}$$

6-41 (a) **Line masses**

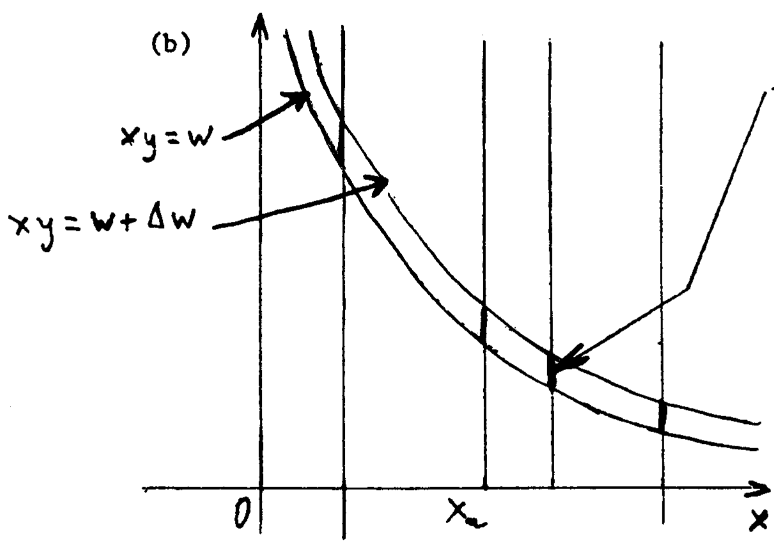


$$P\{x = x_n,\ z - x_n < y \le z - x_n + \Delta z\} = p_n f_y(z - x_n)\Delta z$$

$$\{z < z \le z + \Delta z\} = \sum_n \{x = x_n,\ z - x_n < y \le z - x_n + \Delta z\}$$

$$f_z(z)\Delta z = \sum_n p_n f_y(z - x_n)\Delta z$$

(b)



$$P\{x = x_n,\ \frac{w}{x_n} < y \le \frac{w+\Delta w}{x_n}\} = p_n f_y\left(\frac{w}{x_n}\right)\frac{\Delta w}{x_n}$$

$$\{w < w \le w + \Delta w\} = \sum_n \{x = x_n,\ \frac{w}{x_n} < y \le \frac{w+\Delta w}{x_n}\}$$

$$f_w(w)\Delta w = \sum_n p_n f_y\left(\frac{w}{x_n}\right)\Delta w$$

6.42 X, Y are independent geometric random variables. Thus

$$\begin{aligned} P\{X=k, Y=m\} &= P\{X=k\}\, P\{Y=m\} \\ &= (pq^k)\,(pq^m) = p^2 q^{k+m}, \qquad k, m = 0, 1, 2, \cdots \end{aligned}$$

(a) Let

$$Z = X + Y$$

$$\begin{aligned} P\{Z=n\} &= P\{X+Y=n\} = \textstyle\sum_k P\{X=k, Y=n-k\} \\ &= \textstyle\sum_{k=0}^{n} P\{X=k, Y=n-k\} \\ &= \textstyle\sum_{k=0}^{n} P\{X=k\}\, P\{Y=n-k\} \\ &= \textstyle\sum_{k=0}^{n} pq^k\, pq^{n-k} = \sum_{k=0}^{n} p^2\, q^n \\ &= (n+1)\, p^2\, q^n, \quad n = 0, 1, 2, \cdots \end{aligned}$$

(b) Let

$$W = X - Y$$

Case 1: $W \geq 0 \Rightarrow X \geq Y$. Thus for $m \geq 0$

$$\begin{aligned} P\{W=m\} &= P\{X-Y=m\} = \textstyle\sum_{k=0}^{\infty} P\{X=m+k, Y=k\} \\ &= \textstyle\sum_{k=0}^{\infty} P\{X=m+k, Y=k\} \\ &= \textstyle\sum_{k=0}^{\infty} P\{X=m+k\}\, P\{Y=k\} \\ &= \textstyle\sum_{k=0}^{\infty} (pq^{m+k})\,(pq^k) = p^2 q^m \sum_{k=0}^{\infty} q^{2k} \\ &= p^2 q^m\,(1+q^2+q^4+\cdots) = \frac{p^2 q^m}{(1-q^2)} \\ &= \frac{pq^m}{1+q}, \qquad m = 0, 1, 2, \cdots \qquad (1) \end{aligned}$$

Case 2: $W < 0 \Rightarrow X < Y$. Thus for $m < 0$

$$\begin{aligned} P\{W=m\} &= P\{X-Y=m\} = \textstyle\sum_k P\{X=k, Y=k-m\} \\ &= \textstyle\sum_{k=0}^{\infty} P\{X=k, Y=k-m\} \\ &= \textstyle\sum_{k=0}^{\infty} P\{X=k\}\, P\{Y=k-m\} \\ &= \textstyle\sum_{k=0}^{\infty} (pq^k)\,(pq^{k-m}) = p^2 q^{-m} \sum_{k=0}^{\infty} q^{2k} \\ &= \frac{p^2 q^{-m}}{(1-q^2)} = \frac{pq^{-m}}{1+q}, \quad m = -1, -2, \cdots \qquad (2) \end{aligned}$$

Thus combining (1) and (2) we can write

$$P\{W=m\} = \frac{pq^{|m|}}{1+q}, \qquad m = 0, \pm 1, \pm 2, \cdots$$

6.43 We have X and Y are independent and $P(X=k)=P(Y=k)=p_k$. Also

$$
\begin{aligned}
P(X=k|X+Y=k) &= \frac{P(X=k, Y=0)}{P(X+Y=k)} \\
&= \frac{p_k p_0}{\sum\limits_{i=0}^{k} p_i p_{k-i}} = \frac{1}{k+1}. \qquad (1)
\end{aligned}
$$

Also

$$
\begin{aligned}
&P(X=k-1|X+Y=k) \\
&= \frac{P(X=k-1, Y=1)}{P(X+Y=k)} \frac{p_{k-1}p_1}{\sum_{i=0}^{k} p_i p_{k-i}} = \frac{1}{k+1}. \qquad (2)
\end{aligned}
$$

From (1) and (2),

$$
\frac{p_k}{p_{k-1}} = \frac{p_1}{p_0} \Rightarrow p_k = \lambda p_{k-1} = \lambda^k p_0
$$

where $\lambda \triangleq p_1/p_0$. Since $\sum_{k=0}^{\infty} p_k = 1$, we must have $\lambda < 1$, and this gives

$$
\sum_{k=0}^{\infty} p_k = \frac{p_0}{1-\lambda} = 1 \rightarrow p_0 = 1-\lambda.
$$

Thus

$$
p_k = p_0\, \lambda^k = (1-\lambda)\lambda^k, \quad k = 0, 1, 2, \cdots, \; 0 < \lambda < 1
$$

represents a geometric distribution. Thus X and Y are geometric random variables.

6.44 The moment generating functions of X and Y are given by (see (5-117))

$$
\Gamma_X(z) = (pz+q)^n, \qquad \Gamma_Y(z) = (pz+q)^n
$$

Also

$$
\Gamma_{X+Y}(z) = E[z^{X+Y}] = \Gamma_X(z)\Gamma_Y(z) = (pz+q)^{2n} \sim \text{Binomial}\,(2n, p)
$$

6.45 (a) Let

$$Z = \min(X, Y), \qquad W = X - Y$$

$$\begin{aligned}
&P\{Z = k, W = m\} \\
&= P\{\min(X,Y) = k, X - Y = m\} \\
&= P\{(\min(X,Y) = k, X - Y = m) \bigcap (X \geq Y \bigcup X < Y)\} \\
&= P\{Y = k, X - Y = m, X \geq Y\} + P\{X = k, X - Y = m, X < Y\} \\
&= P\{X = m + k, Y = k, X \geq Y\} + P\{X = k, Y = k - m, X < Y\}
\end{aligned}$$

Note that $k \geq 0$, and m takes both positive, zero and negative values. Hence

$$\begin{aligned}
P\{Z = k, W = m\} &= \begin{cases} P\{X = k + m, Y = k, X \geq Y\}, & k \geq 0, \ m \geq 0 \\ P\{X = k, Y = k - m, X < Y\}, & k \geq 0, \ m < 0 \end{cases} \\
&= \begin{cases} pq^{k+m}\, pq^{k}, & k \geq 0, \ m \geq 0 \\ pq^{k}\, pq^{k-m}, & k \geq 0, \ m < 0 \end{cases}
\end{aligned}$$

$$P\{Z = k, W = m\} = p^2 q^{2k+|m|}, \qquad k = 0, 1, 2, \cdots, \quad m = 0, \pm 1, \pm 2, \cdots$$

Also

$$\begin{aligned}
P\{Z = k\} &= \sum_{m=-\infty}^{\infty} P\{Z = k, W = m\} \\
&= p^2 q^{2k} \sum_{m=-\infty}^{\infty} q^{|m|} = p^2 q^{2k} \left(1 + 2\sum_{m=1}^{\infty} q^m\right) \\
&= p^2 q^{2k} \left(1 + \frac{2q}{p}\right) = p(1+q)q^{2k}, \qquad k = 0, 1, 2, \cdots
\end{aligned}$$

and

$$\begin{aligned}
P\{W = m\} &= \sum_{k=0}^{\infty} P\{Z = k, W = m\} \\
&= p^2 q^{|m|} \sum_{k=0}^{\infty} q^{2k} \\
&= \frac{p}{1+q} q^{|m|}, \qquad m = 0, \pm 1, \pm 2, \cdots
\end{aligned}$$

Note that

$$P\{Z = k, W = m\} = P\{Z = k\}\, P\{W = m\}$$

and hence Z and W are independent random variables.
(b) Let

$$Z = \min(X, Y), \quad W = \max(X, Y) - \min(X, Y)$$

Proceeding as in (a), we obtain

$$\begin{aligned} &P\{Z = k, W = m\} \\ &= P(Y = k, X - Y = m, X \geq Y) + P(X = k, Y - X = m, X < Y) \\ &= P(X = k + m, Y = k, X \geq Y) + P(X = k, Y = k + m, X < Y) \\ &= \begin{cases} pq^{k+m}\, pq^k + pq^k\, pq^{k+m}, & k = 0, 1, 2, \cdots,\ m = 1, 2, \cdots \\ pq^{k+m}\, pq^k, & k = 0, 1, 2, \cdots,\ m = 0 \end{cases} \\ &= \begin{cases} 2p^2 q^{2k+m}, & k = 0, 1, 2, \cdots,\ m = 1, 2, \cdots \\ p^2 q^{2k}, & k = 0, 1, 2, \cdots,\ m = 0 \end{cases} \end{aligned}$$

This gives

$$\begin{aligned} P\{Z = k\} &= \sum_{m=0}^{\infty} P\{Z = k, W = m\} \\ &= p^2 q^{2k} \left(1 + 2\sum_{m=1}^{\infty} q^m\right) = p^2 q^{2k} \left(1 + \frac{2q}{p}\right) \\ &= p(1 + q) q^{2k}, \quad k = 0, 1, 2, \cdots \end{aligned}$$

Also

$$\begin{aligned} P\{W = m\} &= \sum_{k=0}^{\infty} P\{Z = k, W = m\} \\ &= \begin{cases} \dfrac{p}{1 + q}, & m = 0 \\ \dfrac{2p}{1 + q}\, q^m, & m = 1, 2, \cdots \end{cases} \end{aligned}$$

Notice that

$$P\{Z = k, W = m\} = P\{Z = k\}\, P\{W = m\}$$

and hence Z and W are also indepedndent random variables in this case also.

6.46 The moment generating function of X and Y are given by (see (5-119))

$$\Gamma_X(z) = e^{\lambda_1(z-1)}, \qquad \Gamma_Y(z) = e^{\lambda_2(z-1)}$$

Also

$$\Gamma_{X+Y}(z) = \Gamma_X(z)\Gamma_Y(z) = e^{(\lambda_1+\lambda_2)(z-1)}$$

so that

$$Z \sim P(\lambda_1 + \lambda_2)$$

Thus

$$P(X+Y=k) = e^{-(\lambda_1+\lambda_2)}\,\frac{(\lambda_1+\lambda_2)^k}{k!}$$

and

$$\begin{aligned}
&P(X=k|X+Y=n)\\
&= \frac{P(X=k, X+Y=n)}{P(X+Y=n)} = \frac{P(X=k)P(Y=n-k)}{P(X+Y=n)}\\
&= \frac{e^{-\lambda_1}(\lambda_1^k/k!)\,e^{-\lambda_2}(\lambda_2^{n-k}/(n-k)!)}{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n/n!}\\
&= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}, \quad k = 0, 1, 2, \cdots n\\
&\sim \text{Binomial}(n,p), \text{ where } p = \frac{\lambda_1}{\lambda_1+\lambda_2}.
\end{aligned}$$

See also (6-222). From there the converse is also true (proceed as in Example 6-43).

6-47

$$C = \begin{bmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \qquad \Delta = \sigma_1^2\sigma_2^2(1-r^2)$$

$$C^{-1} = \begin{bmatrix} \dfrac{1}{(1-r^2)\sigma_1^2} & \dfrac{r}{(1-r^2)\sigma_1\sigma_2} \\ \dfrac{r}{(1-r^2)\sigma_1\sigma_2} & \dfrac{1}{(1-r^2)\sigma^2} \end{bmatrix}$$

$$XC^{-1}X^t = \frac{1}{(1-r^2)}\left(\frac{x_1^2}{\sigma_1^2} - 2r\,\frac{x_1x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right)$$

6-48

$$\{\mathbf{x}\,\mathbf{y} < 0\} = \{\mathbf{x} < 0,\ \mathbf{y} > 0\} + \{\mathbf{x} > 0,\ \mathbf{y} < 0\}$$

$$P\{\mathbf{x}\,\mathbf{y} < 0\} = F_x(0)[1 - F_y(0)] + [1 - F_x(0)]F_y(0)$$

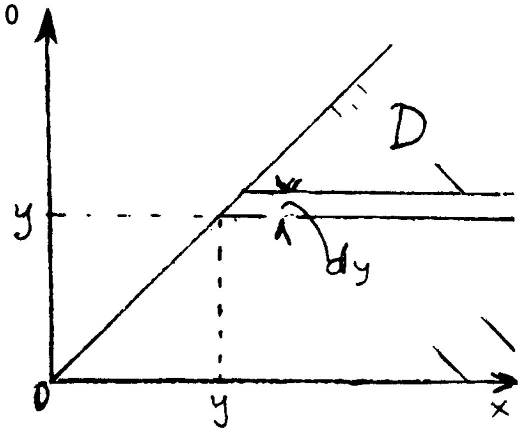
$$F_x(0) = 1 - G\left(\frac{\eta_x}{\sigma_x}\right) \qquad F_y(0) = 1 - G\left(\frac{\eta_y}{\sigma_y}\right)$$

6-49 If $\underset{\sim}{w} = \underset{\sim}{x} - \underset{\sim}{y}$, then $E\{\underset{\sim}{w}\} = 0 \qquad \sigma_w^2 = \sigma_x^2 + \sigma_y^2 = 2\sigma^2$

Thus, $\underset{\sim}{w} = 1$, $N(0; \sigma\sqrt{2})$ and [see (5-74)]

$$E\{\underset{\sim}{z}\} = E\{|\underset{\sim}{w}|\} = \sqrt{2}\,\sigma\sqrt{\frac{2}{\pi}} \qquad E\{\underset{\sim}{z}^2\} = E\{\underset{\sim}{w}^2\} = 2\sigma^2$$

6-50



$$E\{\underset{\sim}{z}\} = \iint_D (x-y)f(x,y)\,dxdy$$

$$= \int_0^\infty \int_y^\infty (x-y)e^{-x}\,e^{-y}\,dxdy = \frac{1}{2}$$

6-51 Since $|E\{\underset{\sim}{x}\,\underset{\sim}{y}\}| \le E\{|\underset{\sim}{x}||\underset{\sim}{y}|\}$, we can assume that the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are real

(a) $D \le E\{[z\,\underset{\sim}{x} - \underset{\sim}{y}]^2\} = z^2\,E\{\underset{\sim}{x}^2\} - 2z\,E\{\underset{\sim}{x}\,\underset{\sim}{y}\} + E\{\underset{\sim}{y}^2\}$

The above is a non-negative quadratic in z for any z. Hence, its discriminant is non-positive.

(b) Using (a), we obtain

$$E\{\underset{\sim}{x}^2\} + E\{\underset{\sim}{y}^2\} + 2\sqrt{E\{\underset{\sim}{x}^2\}E\{\underset{\sim}{y}^2\}}$$

$$\ge E\{\underset{\sim}{x}^2\} + E\{\underset{\sim}{y}^2\} + 2\,E\{\underset{\sim}{x}\,\underset{\sim}{y}\} = E\{(\underset{\sim}{x} + \underset{\sim}{y})^2\}$$

6-52 If $r_{xy} = 1$ then

$$E^2\{(\underset{\sim}{x} - \eta_x)(\underset{\sim}{y} - \eta_y)\} = E\{(\underset{\sim}{x} - \eta_x)^2\}E\{(\underset{\sim}{y} - \eta_y)^2\}$$

i.e., the discriminant of the quadratic

$$E\{[z(\underset{\sim}{x} - \eta_x) - (\underset{\sim}{y} - \eta_y)]^2\}$$

is zero. This is possible only if the quadratic is zero for some $z = z_o$. This shows that $z(\underset{\sim}{x} - \eta_x) - (\underset{\sim}{y} - \eta_y) = 0$ in the MS sense.

6-53 If $E\{x\} = E\{y^2\} = E\{x\ y\}$, then

$$E\{(x - y)^2\} = E\{x^2\} + E\{y^2\} - 2\ E\{x\ y\} = 0.$$

Hence, $x = y$ in the MS sense.

6-54 If x has a Cauchy density, then (Prob. 5-31)

$$E\{e^{j\omega x}\} = e^{-\alpha|\omega|} \qquad E\{e^{j\omega k x}\} = e^{-\alpha k|\omega|}$$

Hence, [see (6-240)]

$$\Phi_z(\omega) = E\{e^{j\omega n x}\} = E\{E\{e^{j\omega n x}|n\}\} =$$

$$\sum_{k=0}^{\infty} E\{e^{j\omega k x}\}e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty} e^{-\alpha k|\omega|}\frac{k}{k!} = e^{-\lambda}e^{-\lambda e^{-\alpha|\omega|}}$$

6.55 If $X = k$, then

$$Y = n - k$$

and

$$Z = X - Y = 2X - n,$$

where Z takes the values $-n, -(n-2), \cdots n-2, n$.

$$\begin{aligned} P\{Z = z\} &= P\{2X - n = z\}P\{X = \tfrac{n+z}{2}\} \\ &= \binom{n}{n + z/2} p^{(n+z)/2}\ q^{(n-z)/2}. \end{aligned}$$

Also

$$E(Z) = E[2X - n] = 2np - n = n(2p - 1).$$

$$\text{Var}(Z) = E[(z - \mu_z)^2] = 4E[(X - np)^2] = 4\text{Var}(X) = 4npq$$

6.56 (a)

$$\begin{aligned}\phi_Z(\omega) &= E[e^{j\omega Z}] = E[e^{j\omega(aX+bY+c)}] \\ &= \phi_X(a\omega)\,\phi_Y(b\omega)e^{j\omega c} = e^{j\omega c-(a^2\sigma_1^2+b^2\sigma_2^2)\omega^2/2}\end{aligned}$$

(see (5-100)).
(b) On comparing with (5-100) we obtain

$$Z \sim \mathrm{N}(c, a^2\,\sigma_1^2 + b^2\,\sigma_2^2)$$

(c)

$$E[Z] = c, \qquad \mathrm{Var}(Z) = a^2\,\sigma_1^2 + b^2\,\sigma_2^2$$

6.57

$$P(X=k|Y=n) = \binom{n}{k} p_1^k\, q_1^{n-k}, \quad k = 0, 1, 2, \cdots n$$

$$E[e^{j\omega X}|Y=n] = \sum_{k=0}^{n} e^{j\omega k}\, P(X=k|Y=n) = \left(p_1 e^{j\omega} + q_1\right)^n$$

use (5-117). Also

$$\begin{aligned}\phi_X(\omega) &= E[e^{j\omega X}] = E\left\{E[e^{j\omega X}|Y=n]\right\} \\ &= \sum\nolimits_{n=0}^{M} E[e^{j\omega X}|Y=n]\,P(Y=n) \\ &= \sum\nolimits_{n=0}^{\infty} (p_1 e^{j\omega} + q_1)^n \binom{M}{n} p_2^n\, q_2^{M-n} \\ &= \sum\nolimits_{n=0}^{M} \binom{M}{n} [p_2(p_1\, e^{j\omega} + q_1)]^n\, q_2^{M-n} \\ &= (p_2 p_1 e^{j\omega} + q_1 p_2 + q_2)^M\end{aligned}$$

But

$$1 - p_1 p_2 = 1 - (1-q_1)(1-q_2) = q_1 p_2 + q_2$$

Hence

$$\phi_X(\omega) = \left(p e^{j\omega} + q\right)^M$$

where $p = p_1 p_2$. Thus

$$X \sim \mathrm{Binomial}(M, p_1 p_2).$$

6.58

$$\int\int f_{XY}(x,y)\,dx\,dy = \int_0^1 \int_x^1 kx\,dy\,dx = k\int_0^1 x(1-x)\,dx$$

$$\frac{k}{6} = 1 \quad \Rightarrow \quad k = 6.$$

$$f_X(x) = \int_x^1 6x\,dy = 6x(1-x), \quad 0 < x < 1.$$

$$f_Y(y) = \int_0^y 6x\,dy = 3y^2, \quad 0 < y < 1.$$

$$E[X] = \int_0^1 x\,f_X(x)\,dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{2}.$$

$$E[X^2] = \int_0^1 x^2\,f_X(x)\,dx = 6\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{3}{10}.$$

$$\mathrm{Var}(X) = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}.$$

$$E[Y] = \int_0^1 y\,f_Y(y)\,dy = 3\left.\frac{y^4}{4}\right|_0^1 = \frac{3}{4}.$$

$$E[Y^2] = \int_0^1 y^2\,f_Y(y)\,dy = 3\left.\frac{y^5}{5}\right|_0^1 = \frac{3}{5}.$$

$$\mathrm{Var}(Y) = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

$$\begin{aligned} E[XY] &= \int\int xy\,f_{XY}(x,y)\,dy\,dx \\ &= \int_0^1\int_x^1 xy\,6x\,dy\,dx = \int_0^1 3x^2\,(1-x^2)\,dx \\ &= 3\left(\tfrac{x^3}{3} - \tfrac{x^5}{5}\right)\Big|_0^1 = 3\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{2}{5} \end{aligned}$$

$$\begin{aligned} Cov(X,Y) &= E(XY) - E(X)E(Y) \\ &= \tfrac{2}{5} - \tfrac{1}{2}\tfrac{3}{4} = \tfrac{1}{40} \end{aligned}$$

6.59 (a)

$$\begin{aligned} \phi_{X,Y}(\omega_1,\omega_2) &= E[e^{j(\omega_1 X + \omega_2 Y)}] \\ &= E[e^{j\omega_1 X}]\;E[e^{j\omega_2 Y}] = \phi_X(\omega_1)\,\phi_Y(\omega_2) \\ &= e^{\lambda(e^{j\omega_1}-1)}\;e^{(j\mu\omega_2 - \sigma^2\omega_2^2/2)} \end{aligned}$$

(b)

$$\begin{aligned} \phi_Z(\omega) &= E[e^{j\omega Z}] \\ &= E[e^{j\omega(X+Y)}] = \phi_{X,Y}(\omega,\omega) \\ &= e^{\{\lambda(e^{j\omega}-1) + (j\mu\omega - \sigma^2\omega^2/2)\}} \end{aligned}$$

6.60 (a)

$$Z = \min(X, Y)$$

From Example 6-18, we have

$$f_Z(z) = 2\lambda\, e^{-2\lambda z}, \qquad z \geq 0$$

and hence

$$E[Z] = E[\min(X, Y)] = \frac{1}{2\lambda}$$

(b)

$$\begin{aligned}
E[\max(2X, Y)] &= \int\int \max(2x, y)\, f_{XY}(x, y)\, dx\, dy \\
&= \int\int_{2x \geq y} 2x\, f_{XY}(x, y)\, dx\, dy + \int\int_{2x < y} y\, f_{XY}(x, y)\, dx\, dy \\
&= \int_0^\infty \int_0^{2x} 2x\, \lambda^2\, e^{-\lambda x}\, e^{-\lambda y}\; dy\, dx + \int_0^\infty \int_0^{y/2} y\, \lambda^2\, e^{-\lambda x}\, e^{-\lambda y}\; dx\, dy \\
&= \lambda \int_0^\infty 2x\, e^{-\lambda x}(1 - e^{-2\lambda x})\, dx + \lambda \int_0^\infty y\, e^{-\lambda y}(1 - e^{-\lambda y/2})\, dy \\
&= 2\lambda \int_0^\infty \left(x e^{-\lambda x} + 2x e^{-2\lambda x} - 3x e^{-3\lambda x} \right) dx \\
&= \frac{2}{\lambda} \int_0^\infty \left(u e^{-u} + 2u e^{-2u} - 3u e^{-3u} \right) du \\
&= \frac{2}{\lambda} \left(1 + \frac{2}{4} - \frac{3}{9} \right) = \frac{7}{3\lambda}.
\end{aligned}$$

6.61 (a)

$$Z = X - Y \quad \rightarrow \quad -1 < z < 1.$$

$z > 0$

$$\begin{aligned}
F_Z(z) &= P(X - Y \leq z) = 1 - P(X - Y > z) \\
&= 1 - \int_0^{(1-z)/2} \int_{y+z}^{1-y} f_{XY}(x, y)\, dx\, dy \\
&= 1 - \int_0^{(1-z)/2} \left(\int_{y+z}^{1-y} 6x dx \right) dy \\
&= 1 - 3 \int_0^{(1-z)/2} \{(1 - z^2) - 2(1 + z)y\}\, dy \\
&= 1 - \frac{3}{4}(1 + z)(1 - z)^2, \quad z \geq 0.
\end{aligned}$$

$z < 0$

$$\begin{aligned}
F_Z(z) &= P(X - Y \leq z) \\
&= \int_0^{(1+z)/2} \int_{x-z}^{1-x} 6x\, dy\, dx = \int_0^{(1+z)/2} 6x\,(1 + z - 2x)\, dx \\
&= \frac{(1+z)^3}{4}, \quad z < 0.
\end{aligned}$$

This gives

$$f_Z(z) = \begin{cases} \frac{3}{4}(1-z)(1+3z), & 0 < z < 1 \\ \dfrac{3(1+z)^2}{4}, & -1 < z < 0 \end{cases}$$

(b)

$$f_X(x) = \int_0^{1-x} 6x\,dy = 6x(1-x), \quad 0 < x < 1$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{F_X(x)} = \frac{1}{1-x}, \quad 0 < y \le 1-x$$

(c)

$$W = X + Y$$

we have

$$F_W(w) = P(X+Y \le w) = \int_0^w \left(\int_0^{w-x} 6x\,dy\right) dx = w^3,$$

and

$$f_W(w) = \int_0^w 6x dx = 3w^2, \quad 0 < w < 1$$

$$E[W] = \frac{3}{4}$$

$$E[W^2] = \frac{3}{5}$$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(W) = E(W^2) - (E(W))^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.$$

6.62

$$X = \frac{1}{Z}.$$

where Z represents a Chi-square random variable. Thus (see (4-39))

$$f_Z(z) = \frac{z^{-1/2}}{\sqrt{2}\Gamma(1/2)}\, e^{-z/2} = \frac{z^{-1/2}}{\sqrt{2\pi}}\, e^{-z/2}$$

or

$$f_X(x) = \frac{1}{\left|\frac{dx}{dz}\right|} f_Z(1/x) = \frac{1}{x^2}\,\frac{x^{1/2}}{\sqrt{2\pi}}\, e^{-1/2x} = \frac{1}{\sqrt{2\pi}x^{3/2}}\, e^{-1/2x}, \quad x > 0$$

Also it is given that

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi x}}\, e^{-y^2/2x}$$

so that

$$f_{XY}(x,y) = f_{Y|X}(y|x)\, f_X(x) = \frac{1}{2\pi x^2}\, e^{-(1+y^2)/2x}$$

and hence

$$\begin{aligned} f_Y(y) &= \int_0^\infty f_{XY}(x,y)\,dx \\ &= \frac{1}{2\pi}\int_0^\infty \frac{1}{x^2}\, e^{-(1+y^2)/2x}\, dx \\ &= \frac{1}{2\pi}\,\frac{2}{1+y^2}\int_0^\infty e^{-u}\,du = \frac{1/\pi}{1+y^2}, \quad -\infty < y < \infty. \end{aligned}$$

Thus Y represents a Cauchy random variable.

6.63 (a) For any two random variables X and Y we have

$$\begin{aligned}\sigma_{X+Y}^2 &= \mathrm{Var}(X+Y) = E[\{(X-\mu_X)+(Y-\mu_Y)\}^2] \\ &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y) = \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y\rho_{XY} \\ &\leq (\sigma_X + \sigma_Y)^2\end{aligned}$$

since $|\rho_{XY}| \leq 1$. Thus

$$\sigma_{X+Y} \leq \sigma_X + \sigma_Y,$$

and hence it easily follows that

$$\frac{\sigma_{X+Y}}{\sigma_X + \sigma_Y} \leq 1.$$

(However, (b) is not so easy!)
(b) We shall prove this result in three parts by making use of Holder's inequality.
(i) **Holder's inequality:** The function $\log x$ is concave, for $0 < \alpha < 1$, and hence we have

$$\log[\alpha x_1 + (1-\alpha)x_2] \geq \alpha \log x_1 + (1-\alpha)\log x_2$$

or

$$x_1^\alpha x_2^{1-\alpha} \leq \alpha x_1 + (1-\alpha)x_2, \quad 0 < \alpha < 1. \qquad (6.63-1)$$

Let

$$x_1 = |x|^p, \quad \alpha = \frac{1}{p}, \text{ so that } 1-\alpha = 1 - \frac{1}{p} \triangleq \frac{1}{q}, \quad x_2 = |y|^q \quad (6.63-2)$$

so that (6.63-1) becomes

$$|xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}, \quad p > 1, \qquad (6.63-3)$$

the Holder's inequality. From (6.63-2), note that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, \quad q > 1 \qquad (6.63-4)$$

(ii) Define

$$x = X\left(E\{|X|^p\}\right)^{-1/p}, \quad y = Y\left(E\{|Y|^q\}\right)^{-1/q}$$

where p and q are as in (6.63-4). Substituting these into the Holder's inequality in (6.63-3), we get

$$\begin{aligned}|XY| \leq\ & p^{-1}\,|X|^p\,\left(E\{|X|^p\}\right)^{1/p-1}\left(E\{|Y|\}\right)^{1/q} \\ & + q^{-1}\,|Y|^q\,\left(E\{|Y|^q\}\right)^{1/q-1}\left(E\{|X|^p\}\right)^{1/p}. \end{aligned} \qquad (6.63-5)$$

Taking expected values on both sides of (6.63-5), we get

$$E\{|XY|\} \leq \left(E\{|X|^p\}\right)^{1/p}\left(E\{|Y|^q\}\right)^{1/q} \qquad (6.63-6)$$

which represents the generalization of the Cauchy-Schwarz inequality. (Note $p = q = 2$ corresponds to Cauchy-Schwarz inequality)

(iii) To prove the desired inequality, notice that

$$\begin{aligned}|X+Y|^p &= |X+Y||X+Y|^{p-1} \\ &\leq |X||X+Y|^{p-1} + |Y||X+Y|^{p-1}, \quad p > 1\end{aligned}$$

and taking expected values on both sides we get

$$E\{|X+Y|^p\} \leq E\{|X||X+Y|^{p-1}\} + E\{|Y||X+Y|^{p-1}\}. \quad (6.63-7)$$

Applying (6.63-6) to each term on the right side of (6.63-7) we get

$$E\{|X||X+Y|^{p-1}\} \leq \left(E\{|X|^p\}\right)^{1/p}\left(E\{|X+Y|^{(p-1)q}\}\right)^{1/q} \quad (6.63-8)$$

and

$$E\{|Y||X+Y|^{p-1}\} \leq \left(E\{|Y|^p\}\right)^{1/p}\left(E\{|X+Y|^{(p-1)q}\}\right)^{1/q} \quad (6.63-9)$$

Using (6.63-8) and (6.63-9) together with $(p-1)q = p$ in (6.63-7) we get

$$E\{|X+Y|^p\} \leq \left[\left(E\{|X|^p\}\right)^{1/p} + \left(E\{|Y|^p\}\right)^{1/p}\right]\cdot\left(E\{|X+Y|^p\}\right)^{1/q}$$

or for $p > 1$

$$\left(E\{|X+Y|^p\}\right)^{1/p} \leq \left(E\{|X|^p\}\right)^{1/p} + \left(E\{|Y|^p\}\right)^{1/p}.$$

the desired inequality. Since $p = 1$ follows trivially, we get

$$\frac{\left(E\{|X+Y|^p\}\right)^{1/p}}{\left(E\{|X|^p\}\right)^{1/p} + \left(E\{|Y|^p\}\right)^{1/p}} \leq 1, \quad p \geq 1.$$

6.64 (a) See Example 6-41. From there

$$E(Y|X=x) = \mu_Y + \frac{\rho_{XY}\sigma_Y(x-\mu_X)}{\sigma_X}$$

(b) Similarly

$$f_{X|Y}(X|Y=y) \sim N(\mu, \sigma^2)$$

where

$$\mu = \mu_X + \frac{\rho_{XY}\sigma_X(y-\mu_Y)}{\sigma_Y}$$

and

$$\sigma^2 = \sigma_X^2(1-\rho_{XY}^2).$$

Since

$$E(X^2|Y=y) = \text{Var}(X|Y=y) + (E[X|Y=y])^2$$

we obtain

$$E(X^2|Y=y) = \sigma^2 + \mu^2$$

6.65 (a) See footnote 4, Chapter 8, Page 337. From there (or directly) we have

$$\text{Var}(X|Y) \triangleq E(X^2|Y) - (E\{X|Y\})^2$$

$$\text{Var}\,(E\{X|Y\}) \triangleq E\,[E\{X|Y\}]^2 - (E[E\{X|Y\}])^2$$

so that

$$\begin{aligned} E[\text{Var}(X|Y)] + \text{Var}\,(E\{X|Y\}) &= E\,[E\{X^2|Y\}] - (E[E\{X|Y\}])^2 \\ &= E(X^2) - [E(X)]^2 = \text{Var}(X) \qquad (1) \end{aligned}$$

or

$$\text{Var}(X) \geq E[\text{Var}\{X|Y\}]$$

Also

$$\text{Var}(X) \geq \text{Var}[E\{X|Y\}]$$

(b) See (1).

6.66

$$Z = aX + (1-a)Y, \qquad 0 < a < 1$$

$$\sigma_Z^2 = \mathrm{Var}(Z) = a^2\sigma_1^2 + (1-a)^2\sigma_2^2$$

$$\frac{\partial \sigma_Z^2}{\partial a} = 2a\sigma_1^2 + 2(1-a)(-1)\sigma_2^2 = 0$$

or

$$a(\sigma_1^2 + \sigma_2^2) = \sigma_2^2$$

$$a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} < 1$$

minimizes $\mathrm{Var}(Z)$.

6-67 From (6-240)

$$E\{g(\underset{\sim}{x},\underset{\sim}{y})\} = E\{E\{g(\underset{\sim}{x},\underset{\sim}{y})\,|\,\underset{\sim}{y}\}\} = E\{g(x_n,\underset{\sim}{y})P\{\underset{\sim}{x} = x_n\}\} \ .$$

From (4-74) with $A_n = \{\underset{\sim}{x} = x_n\}$

$$f_z(z) = \sum_n f_z(z|\underset{\sim}{x} = x_n)P\{\underset{\sim}{x} = x_n\}$$

6-68 (a) The conditional density $f(y|x)$ is $N(rx;\sigma\sqrt{1-r^2})$ [see (7-42)]. Hence

$$E\{f_y(\underset{\sim}{y}|x)\} = \int_{-\infty}^{\infty} f_y(y|x)f_y(y)dy$$

$$= \frac{1}{2\pi\sigma^2\sqrt{1-r^2}} \int_{-\infty}^{\infty} \exp\left\{\frac{-(y-rx)^2}{2\sigma^2(1-r^2)}\right\} \exp\left\{\frac{-y^2}{2\sigma^2}\right\} dy = \frac{1}{\sigma\sqrt{2\pi(2-r^2)}} \exp\left\{\frac{-r^2x^2}{2\sigma^2(2-r^2)}\right\}$$

(b) From (6-241) it follows that

$$E\{f_x(\underset{\sim}{x})f_y(\underset{\sim}{y})\} = E\{f_x(\underset{\sim}{x})E\{f_y(\underset{\sim}{y}|\underset{\sim}{x})\}\} = \int_{-\infty}^{\infty} f_x(x)\ E\{f_y(y|x)\}f_x(x)dx$$

$$= \frac{1}{2\pi\sigma^3\sqrt{2\pi(2-r^2)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2}{\sigma^2}\right\} \exp\left\{\frac{-r^2x^2}{2\sigma^2(2-r^2)}\right\} dx = \frac{1}{2\pi\sigma^2\sqrt{4-r^2}}$$

6-69 We shall use (6-64) and Price's theorem (10-94):

$$\frac{\partial E\{|xy|\}}{\partial \mu} = E\left\{\frac{d|x|}{dx}\frac{d|y|}{\partial y}\right\} = E\{\text{sgn } x \text{ sgn } y\}$$

$$= P\{xy > 0\} - P\{xy < 0\} = \frac{2\alpha}{\pi} = \frac{2}{\pi} \text{arc sin} \frac{\mu}{\sigma_1\sigma_2}$$

If $\mu = 0$, then the RVs x and y are independent, hence,

$$E\{|x y|\}\Big|_{\mu=0} = E\{|x|\}E\{|y|\} = \frac{2}{\pi}\sigma_1\sigma_2$$

[see (5-74)]. Integrating (i) and using the above, we obtain

$$E\{|x y|\} = \frac{2}{\pi}\int_0^{\mu} \text{arc sin} \frac{C}{\sigma_1\sigma_2}\, dC + \frac{2}{\pi}\sigma_1\sigma_2 = \frac{2\sigma_1\sigma_2}{\pi}(\cos\alpha + \alpha\sin\alpha)$$

6-70 From Example 6-41

$$f(y|x) : N(\eta_2 + \frac{r\sigma_2}{\sigma_1}x ; \sigma_2\sqrt{1 - r^2}) = N(4+x;\ \sqrt{3})$$

$$f(x|y) : N(\eta_1 + \frac{r\sigma_1}{\sigma_2}y ; \sigma_1\sqrt{1 - r^2}) = N(3+\frac{y}{4};\ \sqrt{3}/2)$$

6-71 The mass density in the square $|x|\le 1$, $|y|\le 1$ of the xy plane equals 1/4; hence, $P\{r \le 1\} = \pi/4$ and $P\{r \le r\} = \pi r^2/4$ for $r<1$. This yields

$$P\{r\le r, r\le 1\} = \begin{cases} P\{r\le r\} = \pi r^2/4 & r\le 1 \\ P\{r\le 1\} = \pi/4 & r>1 \end{cases}$$

$$F_r(r|M) = \frac{P\{r\le r, M\}}{P(M)} = \begin{cases} r^2 & r\le 1 \\ 1 & r>1 \end{cases} \qquad f_r(r|m) = \begin{cases} 2r, & r<+1 \\ 0 & \text{otherwise} \end{cases}$$

6-72 $\underset{\sim}{z} = \underset{\sim}{x} + \underset{\sim}{y}$ $\underset{\sim}{w} = \underset{\sim}{x}$ $f_{xz}(x,z) = f_{xy}(x, z-x)$

If $f_{xy}(x,y) = f_x(x)f_y(y)$, then

$$f_z(z|x) = \frac{f_{xz}(x,z)}{f_x(x)} = f_y(z-x)$$

6-73 The system $z = F_x(x)$ $w = F_y(y|x)$ has a solution only if $z \le z \le 1$ and $0 \le w \le 1$. Furthermore,

$$\frac{\partial z}{\partial x} = f_x(x) \qquad \frac{\partial z}{\partial y} = 0$$

$$J = f_x(x)f_y(y|x)$$

$$\frac{\partial w}{\partial x} \qquad \frac{\partial w}{\partial y} = f_y(y|x)$$

$$f_{zw}(z,w) = \frac{f_{xy}(x,y)}{f_x(x)f_y(y|x)} = 1 \quad \text{for} \quad 0 \le z,w \le 1$$

6-74 We introduce the events C_r = {we selected the rth coin} and A_k = {heads in a specific order}. From the assumptions it follows that

$$P(C_r) = \frac{1}{m} \qquad P(A_k|C_r) = p_r^k(1-p_r)^{n-k}$$

We wish to find the probability $P(C_r|A_k)$. The events C_r form a pertition; hence,

$$P(C_r|A_k) - \frac{\frac{1}{m}P(a_k|C_r)}{\frac{1}{m}\sum_{i-1}^{m} P(A_k|C_i)}$$

6-75 We wish to show that

$$E\{x^2\} = \frac{n}{n-1}$$

From page 207: $x^2 = ny^2/z$ where y is N(0,1) and z is $\chi^2(n)$. Hence, $E\{y^2\} = 1$ and (also (4-35) and (4-39))

$$E\left\{\frac{1}{z}\right\} = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^\infty z^{n/2-2}e^{-z/2}dz = \frac{2^{m/2-1}\Gamma(n/2-1)}{2^{n/2}\Gamma(n/2)}$$

From this and the independence of y and z it follows that

$$E\{x^2\} = n\, E\{y^2\}\, E\left\{\frac{1}{z}\right\} = \frac{n}{n-2}$$

6-76 From (6-222) :

$$R_x(x) = \exp\left\{-\int_0^x \beta_x(t)dt\right\} = \exp\left\{-k\int_0^x \beta_y(t)dt\right\} = R_y^k(t)$$

6-77 From (5-89) it follows with $x = |z|^2$ and $\alpha = \varepsilon^2$ that

$$E\{|z|^2 > \varepsilon^2\} \le \frac{E\{|z|^2\}}{\varepsilon^2}$$

for any z. And the result follows with $z = x - y$.

6-78 $E\{U(a-\underset{\sim}{x})\} = \int_{-\infty}^{\infty} U(a-x)f(x)dx = \int_{-\infty}^{a} f(x)dx = F_x(a)$

$$E\{U(b-\underset{\sim}{y})\} = F_y(b)$$

$$E\{U(a-\underset{\sim}{x})U(b-\underset{\sim}{y}) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)dxdy = F_{xy}(a,b)$$

Hence

$$F_{xy}(a,b) = F_x(a)F_y(b)$$

6-79 From Example 6-38

$$E\{\underset{\sim}{y}|\underset{\sim}{x} \le 0\} = \int_{-\infty}^{\infty} y\ f_y(y|\underset{\sim}{x} \le 0)dy = \frac{1}{F_x(0)} \int_{-\infty}^{\infty} y\ \frac{\partial F(0,y)}{\partial y}\ dy$$

From (7-41) and (7-57)

$$\int_{-\infty}^{\infty} E\{\underset{\sim}{y}|x\}f_x(x)dx = \int_{-\infty}^{\infty} y \int_{-\infty}^{0} f(x,y)dxdy = \int_{-\infty}^{\infty} y\ \frac{\partial F(0,y)}{\partial y}\ dy$$
