CHAPTER 7

7-1 $$0 \leq P\{x_1 < \mathbf{x} \leq x_2,\ y_1 < \mathbf{y} \leq y_2,\ z_1 < \mathbf{z} \leq z_2\} =$$

$$= P\{\mathbf{x} \leq x_2,\ y_1 < \mathbf{y} \leq y_2,\ z_1 < \mathbf{z} \leq z_2\} - P\{\mathbf{x} \leq x_1,\ y_1 < \mathbf{y} \leq y_2,\ z_1 < \mathbf{z} \leq z_2\} =$$

$$= P\{\mathbf{x} \leq x_2,\ \mathbf{y} \leq y_2,\ z_1 < \mathbf{z} \leq z_2\} - P\{\mathbf{x} \leq x_2,\ \mathbf{y} \leq y_1,\ z_1 < \mathbf{z} \leq z_2\}$$

$$- P\{\mathbf{x} \leq x_1,\ \mathbf{y} \leq y_2,\ z_1 < \mathbf{z} \leq z_2\} + P\{\mathbf{x} \leq x_1,\ \mathbf{y} \leq y_1,\ z_1 < \mathbf{z} \leq z_2\} =$$

$$= P\{\mathbf{x} \leq x_2,\ \mathbf{y} \leq y_2,\ \mathbf{z} \leq z_2\} - P\{\mathbf{x} \leq x_2,\ \mathbf{y} \leq y_2,\ \mathbf{z} \leq z_1\}$$

$$- P\{\mathbf{x} \leq x_2,\ \mathbf{y} \leq y_1,\ \mathbf{z} \leq z_2\} + P\{\mathbf{x} \leq x_2,\ \mathbf{y} \leq y_1,\ \mathbf{z} \leq z_1\}$$

$$- P\{\mathbf{x} \leq x_1,\ \mathbf{y} \leq y_2,\ \mathbf{z} \leq z_2\} + P\{\mathbf{x} \leq x_1,\ \mathbf{y} \leq y_2,\ \mathbf{z} \leq z_1\}$$

$$+ P\{\mathbf{x} \leq x_1,\ \mathbf{y} \leq y_1,\ \mathbf{z} \leq z_2\} - P\{\mathbf{x} \leq x_1,\ \mathbf{y} \leq y_1,\ \mathbf{z} \leq z_1\}$$

7-2 $P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 1,\ \mathbf{x}_C = 1\} = P(ABC) = 1/4$

$P\{\mathbf{x}_A = 1\} = P(A) = 1/2$ $\qquad$ $P\{\mathbf{x}_B = 1\} = P(B) = 1/2$

$P\{\mathbf{x}_C = 1\} = P(C) = 1/2$ hence

$$P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 1,\ \mathbf{x}_C = 1\} \neq P\{\mathbf{x}_A = 1\}P\{\mathbf{x}_B = 1\}P\{\mathbf{x}_C = 1\}$$

hence $\mathbf{x}_A$, $\mathbf{x}_B$, $\mathbf{x}_C$ are not independent. But

$P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 1\} = P(AB) = 1/4 = P\{\mathbf{x}_A = 1\}P\{\mathbf{x}_B = 1\}$

Similarly for any other combination, e.g.,
since $P(A) = P(AB) + P(A\bar{B})$, we conclude that
$P(A\bar{B}) = 1/2 - 1/4 = 1/4$ $\qquad$ $P(\bar{B}) = 1 - P(B) = 1/2$
$P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 0\} = P(A\bar{B}) = 1/4$
$P\{\mathbf{x}_B = 0\} = P(\bar{B}) = 1/2$ hence
$P\{\mathbf{x}_A = 1,\ \mathbf{x}_B = 0\} = P\{\mathbf{x}_A = 1\}P\{\mathbf{x}_B = 0\}$

7-3 If x, y, z are independent in pairs, then

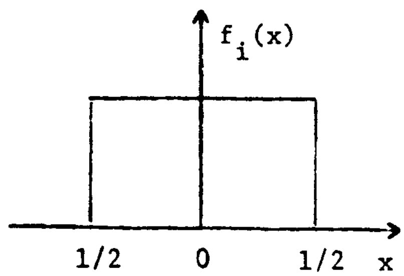
$$r_{xy} = r_{xz} = r_{yz} = 0$$

and (7-60) yields (we assume $\eta_x = \eta_y = \eta_z = 0$)

$$\Phi(\omega_1, \omega_2, \omega_3) = \exp\left\{-\frac{1}{2}\left(\sigma_1^2\omega_1^2 + \sigma_2^2\omega_2^2 + \sigma_3^2\omega_3^2\right)\right\}$$

$$f(x_1, x_2, x_3) = f(x_1)f(x_2)f(x_3)$$

7-4 $x = x_1 + x_2 + x_3$. To determine $E\{x^4\}$ we shall use char. functions



$$\Phi_i(\omega) = \int_{-1/2}^{1/2} e^{j\omega x}dx = \frac{2\sin(\omega/2)}{\omega}$$

$$\Phi(\omega) = \left[\frac{2\sin(\omega/2)}{\omega}\right]^3 = \left(1 - \frac{\omega^2}{24} + \frac{\omega^4}{1920} - \cdots\right)^3$$

The coefficient of ω^4 in this expansion equals

$$\frac{13}{1920} \text{ hence } \frac{1}{4!}\frac{d^4\Phi(0)}{d\omega^4} = \frac{13}{1920}$$

and [see (5-103)]

$$E\{x^4\} = m_4 = \frac{13 \times 4!}{1920} = \frac{13}{80}$$

7-5 (a) The joint density $f(x,y)$ has circular symmetry because

$$f(x,y) = \int_{-\infty}^{\infty} f(\sqrt{x^2 + y^2 + z^2})\, dz$$

depends only on $x^2 + y^2$. The same holds for $f(x,z)$ and $f(y,z)$. And since the RVs $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ are independent, they must be normal [see (6-29)].

(b) From (a) it follows that the RVs $\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z$ are $N(0; \sqrt{kT/m})$. With $\sigma^2 = kT/m$ and $n = 3$ it follows from (7-62)-(7-63) and (5-25) that

$$f_v(v) = \sqrt{\frac{2m^3}{\pi k^3 T^3}}\; v^2\, e^{-mv^2/2kT}\, U(v)$$

$$E\{\mathbf{v}\} = 2\sqrt{\frac{2kT}{\pi m}} \qquad E\{\mathbf{v}^{2n}\} = 1 \times 3 \cdots (2n+1)\left(\frac{kT}{m}\right)^n$$

7-6 From Prob. 6-52: $\mathbf{y} = a\mathbf{x} + b$, $\mathbf{z} = c\mathbf{y} + d$, hence,

$$\mathbf{z} = A\mathbf{x} + B \qquad \eta_z = A\eta_x + B \qquad \sigma_z = A\sigma_x$$

$$E\{(\mathbf{z} - \eta_z)(\mathbf{x} - \eta_x)\} = E\{A(\mathbf{x} - \eta_x)(\mathbf{x} - \eta_x)\} = A\sigma_x^2 = \sigma_x \sigma_z$$

7-7 It follows from (6-241) with $g_1(x) = x$, $g_2(y) = y$ if we replace all densities with conditional densities assuming $\mathbf{x}_3$.

7-8 **Reasoning as in** (7-82), **we conclude that**

$E\{[\underset{\sim}{y} - (a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_1)]^2\}$ **is minimum if**

$$E\{[\underset{\sim}{y} - (a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2)]\underset{\sim}{x}_i\} = 0 \qquad i = 1,2$$

With $R_{oi} = E\{\underset{\sim}{y}\,\underset{\sim}{x}_i\}$, $R_{ij} = E\{\underset{\sim}{x}_i\underset{\sim}{x}_j\}$, **the above yields**

$$R_{01} = a_1R_{11} + a_2R_{12} \qquad\qquad R_{02} = a_1R_{12} + a_2R_{22}$$

But $\hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1\} = A\,\underset{\sim}{x}_1 \qquad\qquad A = R_{01}/R_{11} = a_1 + a_2R_{12}/R_{11}$

$$\hat{E}\{\hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1,x_2\}|\underset{\sim}{x}_1\} = \hat{E}\{a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2|\underset{\sim}{x}_1\}$$

$$= a_1\underset{\sim}{x}_1 + a_2\hat{E}\{\underset{\sim}{x}_2|\underset{\sim}{x}_1\} = \left(a_1 + a_2\frac{R_{12}}{R_{11}}\right)\underset{\sim}{x}_1 = A\,\underset{\sim}{x}_1$$

7-9 **As in Probl.** 6-51

$E^2\{\underset{\sim}{x}_i\underset{\sim}{x}_j\} \le E^2\{\underset{\sim}{x}_i\}E^2\{\underset{\sim}{x}_j\} = M^2 \qquad\qquad |E\{\underset{\sim}{x}_i\underset{\sim}{x}_j\}| \le M$

$$E\{\underset{\sim}{s}^2|\underset{\sim}{n} = n\} = E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\underset{\sim}{x}_i\underset{\sim}{x}_j\right\} \le Mn^2$$

Hence [see (6-240)]

$$E\{\underset{\sim}{s}^2\} = E\{E\{\underset{\sim}{s}^2|\underset{\sim}{n}\}\} < E\{M\underset{\sim}{n}^2\}$$

7-10 As we know,

$$1 + x + \cdots + x^n + \cdots = \frac{1}{1 - x} \qquad |x| < 1$$

Differentiating, we obtain

$$1 + 2x + \cdots + n x^{n-1} + \cdots = \sum_{k=1}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2} \qquad (i)$$

The RV $\underset{\sim}{x}_1$ equals the number of tosses until heads shows for the first time, Hence, $\underset{\sim}{x}_1$ takes the values 1,2,... with $P\{\underset{\sim}{x}_1 = k\} = pq^{k-1}$. Hence, [see (3-12) and (i)]

$$E\{\underset{\sim}{x}_1\} = \sum_{k=1}^{\infty} k\, P\{\underset{\sim}{x}_1 = k\} = \sum_{k=1}^{\infty} k\, p\, q^{k-1} = \frac{p}{(1-q)^2} = \frac{1}{p}$$

Starting the count after the first head shows, we conclude that the RV $\underset{\sim}{x}_2 - \underset{\sim}{x}_1$ has the same statistics as the RV $\underset{\sim}{x}_1$. Hence,

$$E\{\underset{\sim}{x}_2 - \underset{\sim}{x}_1\} = E\{\underset{\sim}{x}_1\} \qquad E\{\underset{\sim}{x}_2\} = 2E\{\underset{\sim}{x}_1\} = \frac{2}{p}$$

Reasoning similarly, we conclude that

$E\{\underset{\sim}{x}_n - \underset{\sim}{x}_{n-1}\} = E\{\underset{\sim}{x}_1\}$. Hence (induction)

$$E\{\underset{\sim}{x}_n\} = E\{\underset{\sim}{x}_{n-1}\} + E\{\underset{\sim}{x}_1\} = \frac{n-1}{p} + \frac{1}{p} = \frac{n}{p}$$

7-11 If n accidents occur in a day, the probability that m of them will be fatal equals $\binom{n}{m} p^m q^{n-m}$ for $m \le n$ and zero for $m > n$. Hence,

$$P\{\underset{\sim}{m} = m \mid \underset{\sim}{n} = n\} = \begin{cases} 0 & m > n \\ \binom{n}{m} p^m q^{n-m} & m \le n \end{cases}$$

This yields

$$E\{e^{j\omega \underset{\sim}{m}} \mid \underset{\sim}{n} = n\} = \sum_{m=0}^{n} e^{j\omega m} \binom{n}{m} p^m q^{n-m} = (p e^{j\omega} + q)^n$$

But

$$P\{\underset{\sim}{n} = n\} = e^{-a} \frac{a^n}{n!} \qquad n = 0,1,\ldots$$

Hence,

$$E\{e^{j\omega \underset{\sim}{m}}\} = E\{E\{e^{j\omega \underset{\sim}{m}} \mid \underset{\sim}{n}\}\} = E\{(p\,e^{j\omega} + q)^{\underset{\sim}{n}}\}$$

$$\sum_{n=o}^{\infty} (p\,e^{j\omega} + q)^n\, e^{-a} \frac{a^n}{n!} = e^{a(p\,e^{j\omega} + q)} e^{-a} = e^{a\,p\,(e^{j\omega} - 1)}$$

This shows that the RV $\underset{\sim}{m}$ is Poisson distributed with parameter $a\,p$ [see (5-119)].

7-12 We shall determine first the conditional distribution

$$F_s(s \mid \underset{\sim}{n} = n) = \frac{P\{\underset{\sim}{s} \le s,\ \underset{\sim}{n} = n\}}{P\{\underset{\sim}{n} = n\}}$$

The event $\{\underset{\sim}{s} < s,\ \underset{\sim}{n} = n\}$ consists of all outcomes such that $\underset{\sim}{n} = n$ and $\sum_{k=1}^{n} \underset{\sim}{x}_k \le s$. Since the RV $\underset{\sim}{n}$ is independent of the RVs $\underset{\sim}{x}_k$, this yields

$$F_s(s \mid \underset{\sim}{n} = n) = P\{\sum_{k=1}^{n} \underset{\sim}{x}_k \le s\}P\{\underset{\sim}{n} = n\}/P\{\underset{\sim}{n} = n\}$$

From the above and the independence of the RVs $\underset{\sim}{x}_k$ it follows that [see (7-51)]

$$f_s(s \mid \underset{\sim}{n} = n) = f_1(s) * f_2(s) * \cdots * f_n(s)$$

Setting $A_k = \{\underset{\sim}{n} = k\}$ in (4-74), we obtain

$$f_s(s) = \sum_k p_k\, [f_1(s) * \cdots * f_k(s)]$$

7-13 From the independence of the RVs n and x_i it follows that

$$E\{e^{sy} | n = k\} = E\{e^{s(x_1 + \cdots + x_k)}\}$$

$$= E\{e^{sx_1}\} \cdots E\{e^{sx_k}\} = \Phi_x^k(s)$$

Hence,

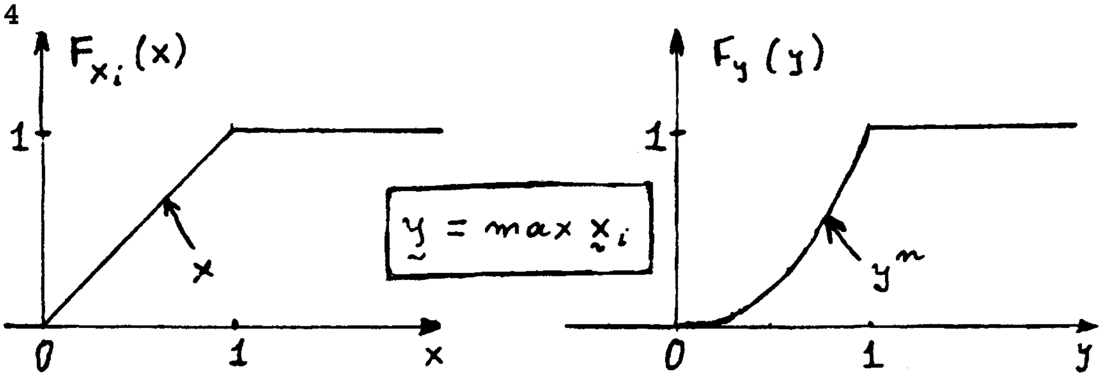
$$\Phi_y(s) = E\{e^{sy}\} = E\{E\{e^{sy} | n\}\} = E\{\Phi_x^n(s)\}$$

$$= \Gamma_n[\Phi_x(s)] \text{ because } E\{z^n\} = \Gamma_n(z)$$

Special case. If n is Poisson with parameter a, then [see (5-119)]

$$\Gamma_n(z) = e^{az - a} \qquad \Phi_y(s) = e^{a\Phi_x(s) - a}$$

7-14



$$\{y \le y\} = \{x_1 \le y, x_2 \le y, \ldots, x_n \le y\}$$

From the independence of x_i and the abcve it follows that

$$F_y(y) = P\{y \le y\} = P\{x_1 \le y\} \cdots P\{x_n \le y\}$$

$$= F_1(y) \cdots F_n(y)$$

where $F_i(y) = y$ for $0 \le y \le 1$.

7-15 The RV $\underset{\sim}{x}$ is defined in the space S. The set

$$C = \{z < \underset{\sim}{z} \le z + dz,\ w < \underset{\sim}{w} \le w + dz\} \qquad z > w$$

is an event in the space S_n of repeated trials and its probability equals

$$P(C) = f_{zw}(z,w)dzdw$$

We introduce the events

$$D_1 = \{\underset{\sim}{x} \le w\} \qquad D_2 = \{w < \underset{\sim}{x} \le w + dw\} \qquad D_3 = \{w + dw < \underset{\sim}{x} \le z\}$$

$$D_4 = \{z < \underset{\sim}{x} \le z + dz\} \qquad D_5 = \{z + dz < \underset{\sim}{x}\}$$

These events form a partition of S and their probabilities $p_i = P(D_i)$ equal

$$F_x(w) \qquad f_x(w)dw \qquad F_x(z)-F_x(w+dw) \qquad f_z(z)dz \qquad 1-F_x(z+dz)$$

respectively. The event C occurs iff the smallest of the RVs $\underset{\sim}{x}_i$ is in the interval (w, w+dw), the largest is in the interval (z, z+dz), and, consequently, all others are between w+dw and z. This is the case iff D_1 does not occur at all, D_2 occurs once, D_3 occurs n-2 times, D_4 occurs once, and D_5 does not occur at all. With

$$k_1=0 \qquad k_2=1 \qquad k_3=n-2 \qquad k_4=1 \qquad k_5=0$$

it follows from (4-102) that

$$P(C) = \frac{n!}{(n-2)!} p_2\, p_3^{n-2}\, p_4 = n(n-1)f_x(w)dw\,[f_x(z)-F_x(w+dw)]^{n-1}f_x(z)dz$$

for z > w, and 0 otherwise.

7-16 If $\underset{\sim}{z}$ is $N(\eta,1)$ then

$$E\{e^{s\underset{\sim}{z}^2}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{sz}e^{-(z-\eta)^2/2}dz$$

$$sz^2 - \frac{(z-\eta)^2}{2} = \left(s - \frac{1}{2}\right)\left(z - \frac{\eta}{1-2s}\right)^2 + \frac{\eta^2 s}{1-1s}$$

Since

$$\frac{1}{\sqrt{2\pi}}\int_{-\eta}^{\infty} e^{-a(z-b)^2} dz = \frac{1}{\sqrt{2a}}$$

the above yields

$$E\{e^{s\underset{\sim}{z}^2}\} = \frac{1}{\sqrt{2(1/2-S)}} \exp\left\{\frac{\eta^2 S}{1-2S}\right\}$$

$$\Phi_w(s) = \frac{1}{\sqrt{1-2s}} \exp\left\{\frac{\eta_1 s}{1-2s}\right\} \cdots \frac{1}{\sqrt{1-2s}} \exp\left\{\frac{\eta_n s}{1-2s}\right\}$$

7-17 We wish to show that the RVs

$$\bar{\underset{\sim}{x}} = \frac{1}{n}\sum_{i=1}^{n} \underset{\sim}{x}_i \qquad \underset{\sim}{s}^2 = \frac{1}{n-1}\sum_{i=1}^{n} (\underset{\sim}{x}_i - \bar{x})^2$$

are independent. Since $\underset{\sim}{s}^2$ is a function of the n RVs $\underset{\sim}{x}_i - \bar{\underset{\sim}{x}}$, it suffices to show that each of these RVs is independent of $\bar{\underset{\sim}{x}}$. We assume for simplicity that $E\{\underset{\sim}{x}_i\}=0$. Clearly,

$$E\{\underset{\sim}{x}_i\bar{x}\} = \frac{1}{n} E\{\underset{\sim}{x}_i^2\} = \frac{\sigma^2}{n} \qquad E\{\bar{\underset{\sim}{x}}\bar{\underset{\sim}{x}}\} = \frac{1}{n^2}\sum_{i=1}^{n} \underset{\sim}{x}_i^2 = \frac{\sigma^2}{n}$$

because $E\{\underset{\sim}{x}_i\underset{\sim}{x}_j\}=0$ for $i \neq i$. Hence,

$$E\{(\underset{\sim}{x}_i - \bar{\underset{\sim}{x}})\bar{\underset{\sim}{x}}\} = 0$$

Thus, the RVs $\underset{\sim}{x}_i - \bar{\underset{\sim}{x}}$ and $\bar{\underset{\sim}{x}}$ are orthogonal; and since they are jointly normal, they are independent.

7-18 Since $\eta_s = \alpha_0 + \alpha_1\eta_1 + \alpha_2\eta_2$ [see (7-87)], the mean of the error

$$\underset{\sim}{\varepsilon} = \underset{\sim}{s} - (\alpha_0 + \alpha_1\underset{\sim}{x}_1 + \alpha_2\underset{\sim}{x}_2) = (\underset{\sim}{s} - \eta_s) - [\alpha_1(\underset{\sim}{x}_1 - \eta_1) + \alpha_2(\underset{\sim}{x}_2 - \eta_2)]$$

is zero. Furthermore, $\underset{\sim}{\varepsilon}$ is orthogonal to $\underset{\sim}{x}_i$, hence, it is also orthogonal to $\underset{\sim}{x}_i - \eta_i$.

7-19 From the orthogonality principle:

$$\hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1,\underset{\sim}{x}_2\} = a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2 \qquad \underset{\sim}{y} - \{a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2\} \perp \underset{\sim}{x}_1,\underset{\sim}{x}_2$$

$$\hat{E}\{y|x_1\} = A\underset{\sim}{x}_1 \qquad y - A\underset{\sim}{x}_1 \perp \underset{\sim}{x}_1$$

Hence

$$\underset{\sim}{y} - (a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2) - (\underset{\sim}{y} - A\underset{\sim}{x}_1) = a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2 - A\underset{\sim}{x}_1 \perp \underset{\sim}{x}_1$$

From this it follows that

$$\hat{E}\{a_1\underset{\sim}{x}_1 + a_2\underset{\sim}{x}_2|\underset{\sim}{x}_1\} = A\underset{\sim}{x}_1$$

$$\hat{E}\{\hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1,\underset{\sim}{x}_2\}|\underset{\sim}{x}_1\} = \hat{E}\{\underset{\sim}{y}|\underset{\sim}{x}_1\}$$

7-20 The event $\{\mathbf{x} \le x\}$ occurs if there is at least one point in the interval $(0,x)$; the event $\{\mathbf{y} \le y\}$ occurs if all the points are in the interval $(0,y)$:

$$A_x = \{\text{at least one point in } (0,x)\} = \{\mathbf{x} \le x\}$$

$$B_y = \{\text{no points in } (y,1)\}$$

$$= \{\text{all points in } (0,y)\} = \{\mathbf{y} \le y\}$$

Hence, for $0 \le x \le 1$, $0 \le y \le 1$

$$F_x(x) = P(A_x) = 1 - P(\bar{A}_x) = 1 - (1 - x)^n$$

$$F_y(y) = P(B_y) = y^n$$

Furthermore,

$$\{\mathbf{x} \le x, \mathbf{y} \le y\} = A_x B_y \qquad\qquad A_x B_y + \bar{A}_x B_y = B_y$$

If $x \le y$ then

$$\bar{A}_x B_y = \{\text{all points in } (x,y)\}$$

$$P(\bar{A}_x B_y) = (y - x)^n$$

If $x > y$, then $\bar{A}_x B_y = \{\emptyset\}$. Hence

$$F_{xy}(x,y) = P(A_x B_y) = \begin{cases} y^n - (y-x)^n & x \le y \\ y^n & x > y \end{cases}$$

7-21 Suppose that $E\{x_i\} = 0$, $E\{x_i^2\} = \sigma^2$, $E\{x_i^4\} = \mu_4$

If $A = \sum_{i=1}^{n} x_i^2$, then $E\{A\} = n\sigma^2$

$$E\{A^2\} = \sum_{i,j=1}^{n} E\{x_i^2 x_j^2\} = n\mu_4 + (n^2 - n)\sigma^4$$

because

$$E\{x_i^2 x_j^2\} = \begin{cases} \mu_4 & i = j \\ \sigma^4 & i \neq j \end{cases}$$

Furthermore

$$E\{\bar{x}^2 x_j^2\} = \frac{1}{n^2} E\left\{\left(\sum_{i=1}^{n} x_i\right)^2 x_j^2\right\} = \frac{1}{n^2}[\mu_4 + (n-1)\sigma^4]$$

$$E\{\bar{x}^2 A\} = \frac{1}{n}[\mu_4 + (n-1)\sigma^4]$$

$$E\{\bar{x}^4\} = \frac{1}{n^4} E\left\{\left(\sum_{i=1}^{n} x_i\right)^4\right\} = \frac{1}{n^4}[n\mu_4 + 3n(n-1)\sigma^4]$$

because

$$E\{x_i x_j x_k x_r\} = \begin{cases} \mu_4 & i = j = k = r \quad \text{[n such terms]} \\ \sigma^4 & i = j \neq k = r \quad \text{[3n(n-1) such terms]} \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $(n-1)\bar{V} = \sum_{i=1}^{n}(x_i - \bar{x})^2 = A - n\bar{x}^2$, $E\{\bar{V}\} = \sigma^2$. Hence

$$(n-1)^2 E\{V^2\} = E\{A^2\} - 2nE\{\bar{x}^2 A\} + n^2 E\{\bar{x}^4\}$$

$$= n\mu_4 + (n^2 - n)\sigma^4 - 2[\mu_4 + (n-1)\sigma^4] + \frac{1}{n}[\mu_4 + 3(n-1)\sigma^4]$$

This yields

$$E\{\bar{V}^2\} = \frac{\mu_4}{n} + \frac{n^2 - 2n + 3}{n(n-1)}\sigma^4 = \sigma^4 + \sigma_{\bar{V}}^2$$

<u>Note</u> If the RVs x_i are $N(0,\sigma^2)$, then $\mu_4 = 3\sigma^4$

$$\sigma_{\bar{V}}^2 = \frac{1}{n}\left(3\sigma^4 - \frac{n-3}{n-1}\sigma^4\right) = \frac{2}{n-1}\sigma^4$$

7-22 From Prob. 6-49:

$$E\{|x_{2i} - x_{2i-1}|\} = \frac{2\sigma}{\sqrt{\pi}} \qquad E\{|x_{2i} - x_{2i-1}|^2\} = 2\sigma^2$$

Hence,

$$E\{|x_{2i} - x_{2i-1}||x_{2j} - x_{2j-1}|\} = \begin{cases} 2\sigma^2 & i = j \\ 4\sigma^2/\pi & i \neq j \end{cases}$$

$$E\{z\} = \frac{\sqrt{\pi}}{2n}\frac{2\sigma n}{\sqrt{\pi}} = \sigma$$

$$E\{z^2\} = \frac{\pi}{4n^2}[2n\sigma^2 + \frac{4\sigma^2}{\pi}(n^2 - n)]$$

$$\sigma_z^2 = \frac{\pi}{2n}\sigma^2 + (1 - \frac{1}{n})\sigma^2 - \sigma^2 = \frac{\pi - 2}{2n}\sigma^2$$

7-23 If $R^{-1} = \begin{bmatrix} a_{11} \cdots a_{1n} \\ \\ a_{n1} \cdots a_{nn} \end{bmatrix}$ then $\sum_j a_{ij}R_{ji} = 1$

Hence,

$$E\{XR^{-1}X^t\} = E\{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} x_j\}$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}R_{j1} = \sum_{i=1}^{n} 1 = n$$

7-24 The density $f_z(z)$ of the sum $z = x_1 + \cdots + x_n$ tends to a normal curve with variance $\sigma_1^2 + \cdots + \sigma_n^2 \to \infty$ as $n \to \infty$ (we assume $\sigma_i > c > 0$). Hence, $f_z(z)$ tends to a constant in any interval of length 2π. The result follows as in (5-37) and Prob. 5-20.

7-25 Since $a_n - a \to 0$, we conclude that

$$E\{(\underset{\sim}{x}_n - a)^2\} = E\{[(\underset{\sim}{x}_n - a_n) + (a_n - a)]^2\}$$

$$= E\{(\underset{\sim}{x}_n - a_n)^2\} + 2(a_n - a)E\{\underset{\sim}{x}_n - a_n\} + (a_n - a)^2 \to 0$$

as $n \to \infty$.

7-26 If $E\{\underset{\sim}{x}_n \underset{\sim}{x}_m\} \to a$ as $n,m \to \infty$, then, given $\varepsilon > 0$, we can find a number n_o such that

$$E\{\underset{\sim}{x}_n \underset{\sim}{x}_n\} = a + \theta(n,m) \qquad |\theta| < \varepsilon \qquad \text{if } n,m > 0$$

Hence,

$$E\{(\underset{\sim}{x}_n - \underset{\sim}{x}_m)^2\} = E\{\underset{\sim}{x}_n^2\} + E\{\underset{\sim}{x}_m^2\} - 2E\{\underset{\sim}{x}_n \underset{\sim}{x}_m\}$$

$$= a + \theta_1 + a + \theta_2 - 2(a + \theta_3) = \theta_1 + \theta - 2\theta_3$$

and since $|\theta_1 + \theta_2 - 2\theta_3| < 4\,\varepsilon$ for any ε, it follows that

$E\{(\underset{\sim}{x}_n - \underset{\sim}{x}_m)^2\} \to 0$, hence (Cauchy) $\underset{\sim}{x}_n$ tends to a limit.

<u>Conversely</u> If $\underset{\sim}{x}_n \to \underset{\sim}{x}$ in the MS sense, then

$E\{(\underset{\sim}{x}_n - \underset{\sim}{x})^2\} \to 0$. Furthermore,

$$E\{\underset{\sim}{x}_n^2\} \to E\{\underset{\sim}{x}^2\} \qquad E\{\underset{\sim}{x}\,\underset{\sim}{x}_n\} \to E\{\underset{\sim}{x}^2\}$$

because (see Prob. 6-51)

$$E^2\{\underset{\sim}{x}_n^2 - \underset{\sim}{x}^2\} = E^2\{(\underset{\sim}{x}_n - \underset{\sim}{x})(\underset{\sim}{x}_n + \underset{\sim}{x})\}$$

$$\le E\{(\underset{\sim}{x}_n - \underset{\sim}{x})^2\}E\{(\underset{\sim}{x}_n + \underset{\sim}{x})^2\} \to 0$$

$$E^2\{\underset{\sim}{x}(\underset{\sim}{x}_n - \underset{\sim}{x})\} \le E\{\underset{\sim}{x}^2\}E\{(\underset{\sim}{x}_n - \underset{\sim}{x})^2\} \to 0$$

Similarly, $E\{(\underset{\sim}{x}_n - \underset{\sim}{x})(\underset{\sim}{x}_m - \underset{\sim}{x})\} \to 0$. Hence,

$$E\{\underset{\sim}{x}_n \underset{\sim}{x}_m\} + E\{\underset{\sim}{x}^2\} - E\{\underset{\sim}{x}\,\underset{\sim}{x}_n\} - E\{\underset{\sim}{x}\,\underset{\sim}{x}_m\} \to 0$$

Combining, we conclude that $E\{\underset{\sim}{x}_n \underset{\sim}{x}_m\} \to E\{\underset{\sim}{x}^2\}$.

7-27 $$E\{\underset{\sim}{x}_k\} = 0 \qquad E\{\underset{\sim}{x}_k^2\} = \sigma_k^2$$

$$E\left\{\left(\sum_{k=n_1}^{n_2} \underset{\sim}{x}_k\right)^2\right\} = \sum_{k=n_1}^{n_2} E\{\underset{\sim}{x}_k^2\}$$

If $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, then given $\varepsilon > 0$, we can find n_o such that $\sum_{k=n+1}^{n+m} \sigma_k^2 < \varepsilon$

for any m and $n > n_o$. Thus

$$E\{(\underset{\sim}{y}_{n+m} - \underset{\sim}{y}_n)^2\} = E\left\{\left(\sum_{k=n+1}^{n+m} \underset{\sim}{x}_k\right)^2\right\} = \sum_{k=n+1}^{n+m} \sigma_k^2 < \varepsilon$$

This shows that (Cauchy), $\underset{\sim}{y}_k$ converges in the MS sense. The proof of the converse is similar.

7-28 If $f_i(x) = c\,e^{-cx}U(x)$ then $\Phi_i(s) = \dfrac{c}{c-s}$

$$\Phi(s) = \Phi_1(s) \cdots \Phi_n(s) = \frac{c^n}{(c-s)^n}$$

Hence (see Example 5-29) $f(x) = \dfrac{c^n x^{n-1}}{(n-1)!} e^{-cx}\,U(x)$

7-29 From Prob. 7-28 it follows that f(x) is the density of the sum $\underset{\sim}{x} = \underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n$. Furthermore,

$$E\{\underset{\sim}{x}\} = \frac{n}{c} \qquad \sigma_x^2 = \frac{n}{c^2}$$

From the central limit theorem it follows, therefore, that for large n, the Erlang density is nearly equal to a normal curve with mean n/c and variance n/c^2.

7-30 $E\{r_i\} = 500$ $\sigma_i^2 = 50^2/3$

$r = r_1 + r_2 + r_3 + r_4$ $E\{r\} = 2{,}000$ $\sigma_r^2 = 10^4/3$

Thus, r is approximately $N(2000;10^2/\sqrt{3})$

$P\{1900 \leq r \leq 2100\} = 2\,G\,(\frac{100\sqrt{3}}{100}) - 1 = 0.9169.$

7-31 The RVs x_i are independent with (see Prob. 5-37)

$$f_i(x) = \frac{c_i}{\pi(c_i^2 + x^2)} \qquad \Phi_i(\omega) = e^{-c_i|\omega|}$$

In that case, (7-104) does not hold because

$$\int_{-\infty}^{\infty} x^{\alpha} f(x)dx = \frac{c_i}{\pi} \int_{-\infty}^{\infty} \frac{x^{\alpha}}{c_i^2 + x^2} dx = \infty \qquad \alpha > 2$$

In fact, the density of $x = x_1 + \cdots + x_n$ is Cauchy with parameter $c = c_1 + \cdots + c_n$ because

$$\Phi(\omega) = e^{-c_1|\omega|} \cdots e^{-c_n|\omega|} = e^{-(c_1 + \cdots + c_n)|\omega|}$$

7-32 In this problem, $\sigma_z^2 = E\{|z|^2\} = E\{x^2 + y^2\} = 2\sigma^2$

$$f_z(x) = f_x(x)f_y(y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} = \frac{1}{2\pi\sigma_z^2} e^{-|z|^2/\sigma_z^2}$$

$$\Phi_z(\Omega) = \Phi_x(u)\Phi_y(v) = \exp\left\{ -\frac{1}{2}\sigma^2(u^2+v^2)\right\} = \exp\left\{ -\frac{1}{4}\sigma_z^2|\Omega|^2\right\}$$
