CHAPTER 8

8-1 (a) From (8-11) with γ=.95, u=.975, $z_{.975} \simeq 2$, σ=0.1, and n=9 we obtain

$$c = \frac{z_u \sigma}{\sqrt{n}} = 0.066$$

(b) From (8-11) with c=91.01-91=0.05mm:

$$z_u = \frac{c\sqrt{n}}{\sigma} = 1.5 \qquad u = .933 \qquad \gamma = .866$$

8-2 (a) From (8-11) with σ=1 and n=4: $\bar{x} \pm \sigma z_u/\sqrt{n} \simeq 203 \pm 1$mm

(b) From (8-12) with δ=.05: $c = \sigma/\sqrt{n\delta} = 2.236$mm

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8-3 From (8-4)with γ=.9, u=.95: $\bar{x} \pm z_u\sigma/\sqrt{n} = 25{,}000 \pm 1{,}028$ miles

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8-4 We wish to find n such that $P\{|\bar{x}-a|<0.2\} = 0.95$ where $a=E\{\bar{x}\}$.From (8-4) it follows with u=.975 and σ=0.1mm that

$$\frac{z_u \sigma}{\sqrt{n}} \le 0.2, \text{ hence, } n=1$$

8-5 In this problem, x is uniform with $E\{x\}=\theta$ and $\sigma^2=4/3$.We can use, however, the normal approximation for $\bar{x}$ because n=100. With γ=.95, (8-11) yields the interval

$$\bar{x} \pm z_{.975}\, \sigma\sqrt{n} = 30 \pm 0.227$$

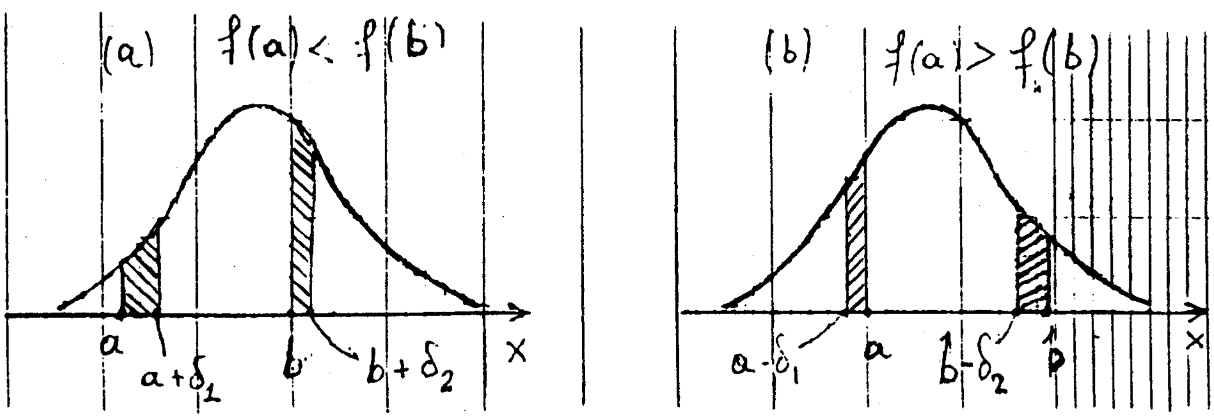
8-6 We shall show that if $f(x)$ is a density with a single maximum and $P\{a<x<b\}=\gamma$, then $b-a$ is minimum if $f(a)=f(b)$. The density $xe^{-x}U(x)$ is a special case. It suffices to show that $b-a$ is not minimum if $f(a)<f(b)$ or $f(a)>f(b)$.

Suppose first that $f(a)<f(b)$ as in figure (a). Clearly. $f'(a)>0$ and $f'(b)<0$, hence, we can find two constants $\delta_1>0$ and $\delta_2>0$ such that $P\{a+\delta_1<x<b+\delta_2\}=\gamma$ and

$$f(a) < f(a+\delta_1) < f(b+\delta_2) < f(b)$$

From this it follows that $\delta_1 > \delta_2$, hence, the length of thenew interval $(a+\delta_1, b+\delta_2)$ is smaller than $b-a$.

If $f(a) > f(b)$, we form the interval $(a-\delta_1, b-\delta_2)$ (Fig. 8-6b) and proceed similarly.



Special case. If $f(x)=xe^{-x}$ then (see Problem 4-9) $F(x)=1-e^{-x}-xe^{-x}$, hence,

$$P\{a < x < b\} = e^{-a} + ae^{-a} - e^{-b} - be^{-b} = .95$$

And since $f(a) = f(b)$, the system

$$ae^{-a} = be^{-b} \qquad e^{-a} - e^{-b} = .95$$

results. Solving, we obtain $a \simeq 0.04 \quad b \simeq 5.75$.

A numerically simpler solution results if we set

$$0.025 = P\{x \le a\} = F(a) \qquad 0.025 = P\{x > b\} = 1 - F(b)$$

as in (9-5). This yields the system

$$0.025 = 1 - e^{-a} - ae^{-a} \qquad 0.025 = e^{-b} + be^{-b}$$

Solving, we obtain a=0.242, b=5.572. However, the length 5.572-0.242=5.33 of the resulting interval is larger than the length 4.75-0.04=4.71 of the optimum interval.

8-7 We start with the general problem: We observe the n samples x_i of an $N(\eta, 10)$ RV x and we wish to predict the value x of x at a future trial in terms of the average $\bar{x}$ of the observations. If η is known, we have an ordinary prediction problem. If it is unknown, we must first estimate it. To do so, we form the RV $w = x - \bar{x}$. This RV is

$N(0,\sigma_w)$ where $\sigma_w^2=\sigma_x^2+\sigma_{\bar{x}}^2=\sigma^2+\sigma^2/n$. With $c=z_{.975}\sigma_w$$ it follows that $P\{|w| < c\}=.95$. Hence

$$P\{\bar{x} - c < x < \bar{x} + c\} = 0.95$$

For n=20 and σ=10 the above yields σ_w=10.25 and c $\simeq$ 20.5. Thus, we can expect with .95 confidence coefficient that our bulb will last at least 80-20.5=59.5 and at most 80+20=100.5 hours.

8-8 The time of arrival of the 40th patient is the sum $x_1 + \cdots + x_n$ of n=39 RVs with exponential distribution. We shall estimate the mean $\eta=1/\theta$ of x in terms of its sample mean $\bar{x}=240/39=6.15$ minutes using two methods. The first is approximate (large n) and is based on (8-11).

<u>Normal approximation</u>. With $\lambda=\eta$ and $z_{.975}/\sqrt{39}=0.315$:

$$P\left\{\frac{\bar{x}}{1.315} < \eta < \frac{\bar{x}}{0.685}\right\} = .95 \qquad 4.68 < \eta < 8.98 \text{ minutes}$$

<u>Exact solution</u>. The RVs $\underset{\sim}{x}_i$ are i.i.d. with exponential distribution. From this and (7-52) it follows that their sum $\underset{\sim}{y}=\underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_n = n\underset{\sim}{\bar{x}}$ has an Erlang distribution:

$$\Phi_y(s) = \frac{\theta^n}{(\theta-s)^n} \qquad f_y(y) = \frac{\theta^n}{(n-1)!} y^{n-1} e^{-\theta y} U(y)$$

and the RV $\underset{\sim}{z}=2\theta\underset{\sim}{y} = 2n\theta\underset{\sim}{\bar{x}}$ has a $\chi^2(2n)$ distribution:

$$f_z(z) = \frac{1}{2\theta} f_y\left(\frac{z}{2\theta}\right)U(z) = \frac{z^{n-1}}{2^n(n-1)!} e^{-z/2} U(z)$$

Hence,

$$P\left\{\chi^2_{\delta/2}(2n) < \frac{2n\bar{x}}{\eta} < \chi^2_{1-\delta/2}(2n)\right\} = \gamma = 1-\delta$$

Since $\chi^2_{.025}(78) = 54.6$, $\chi^2_{.975}(78)=104.4$, and $2n\bar{x}=480$, this yields the interval

$$4.60 < \eta < 8.79 \text{ minutes}$$

8-9 From (8-19) with $\bar{x}=2{,}550/200=12.75$ $n=200$ and $z_u \simeq 2$

$$\lambda^2 - 25.52\,\lambda + 12.75^2 = 0 \qquad \lambda_1 = 12.255 < \lambda < 13.265 = \lambda_2$$

8-10 From (8-21) with $\bar{x}=2{,}080/4000=0.52$, $n=4{,}000$ and $z_u \simeq 2.326$.

$$p_{1,2} \simeq \bar{x} \pm z_u \sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = .52 \pm .018$$

Hence, $.502 < p < .538$.

8-11 (a) In this problem, $\bar{x}=0.40$, $n=900$ and $z_u \simeq 2$. From (8-21): Margin of error

$$\pm\ 100\ z_u\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = \pm\ 3.27\%$$

(b) We wish to find z_u. From (9-21) and Table 1a:

$$100 z_u\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = 2 \qquad z_u = 1.225 \qquad u = .89$$

This yields the confidence coefficient $\gamma = 2u - 1 = .78$

8-12 From (8-21) with $\bar{x}=0.29$ and $z_u=2$:

$$z_u\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = 0.04 \qquad\qquad n > \frac{\bar{x}(1-\bar{x})}{.04^2}\,z^2_u = 515$$

8-13 The problem is to find n such that [see (8-20)] $z_u\sqrt{\frac{p(1-p)}{n}} \le .02$

for every p. Since $z_u \simeq 2$ and $p(1-p) \le 1/4$, this is the case if

$$z_u\sqrt{1/4n} \le .02 \qquad n \ge 2{,}500$$

8-14 From (8-36) with k=1

$$f(p) = \begin{cases} 5 & .4 < p < .6 \\ 0 & \text{otherwise} \end{cases} \qquad P\{k=1\} = 5\int_{.4}^{.6} p\,dp = .5 = \frac{1}{\gamma}$$

$$f_p(p|1) = \begin{cases} 10p & .4 < p < .6 \\ 0 & \text{otherwise} \end{cases} \qquad \hat{p} = 10\int_{.4}^{.6} p^2\,dp = .5067$$

8-15 From Prob. 8-8: $f_{\bar{x}}(\bar{x}|\theta) = \frac{(\theta n)^n}{(n-1)!}\bar{x}^{n-1}e^{-n\theta\bar{x}}$

From (8-32):

$$f_\theta(\theta\,|\bar{x}) = \frac{(c+n\bar{x})^{n+1}}{n!}\theta^n e^{-(c+n\bar{x})\theta}$$

From (8-31):

$$\hat{\theta} = \frac{(c+n\bar{x})^{n+1}}{n!}\int_0^\infty \theta^{n+1}e^{-(c+n\bar{x})\theta}d\theta = \frac{n+1}{c+n\bar{x}}$$

8-16 The sum $n\bar{x}$ is a Poisson RV with mean $n\theta$ (see Prob. 8-8). In the context of Bayesian estimation, this means that

$$f_{\bar{x}}(\bar{x}|\theta) = e^{-n\theta}\frac{(n\theta)^k}{k!} \qquad k = n\bar{x} = 0,1,\ldots$$

Inserting into (8-32), we obtain [see (4-76)]

$$f_\theta(\theta|\bar{x}) = \frac{(n+c)^{n\bar{x}+b+1}}{\Gamma(n\bar{x}+b+a)}\theta^{n\bar{x}+b}e^{-(n+c)\theta}$$

and (8-31) yields

$$\hat{\theta} = \frac{(n+c)^{n\bar{x}+b+1}}{\Gamma(n\bar{x}+b+1)}\frac{\Gamma(n\bar{x}+b+2)}{(n+c)^{n\bar{x}+b+2}} = \frac{n\bar{x}+b+1}{n+c} \xrightarrow[n\to\infty]{} \bar{x}$$

8-17 From (8-17) with $t_{.95}(9)=2.26$

$$\bar{x} \pm \frac{t_u s}{\sqrt{n}} = 90 \pm 3.57 \qquad 86.43 < \eta \cdot 93.57$$

From (8-24) with $\chi^2_{.975}(9)=19.02$, $\chi^2_{.025}(9)=2.70$.

$$\frac{9\times 5^2}{19.02} = 11.83 < \sigma^2 < \frac{9\times 5^2}{2.70} = 83.33 \qquad 3.44 < \sigma < 9.13$$

8-18 The RVs x_i/σ are N(0,1), hence, the sum $z=(x_1^2 + \cdots + x^2_{10})/\sigma^2$ has a $\chi^2(10)$ distribution. This yields

$$P\{\chi^2_{.025}(10) < z < \chi^2_{.975}(10)\} = .95$$

$$\chi^2_{.025}(10) = 3.25 < \frac{4}{\sigma^2} < \chi^2_{.975}(10) = 20.48$$

$$0.442 < \sigma < 1.109$$

8-19 From (8-23) with $n=4, \chi^2_{.025}(4)=0.48$, $\chi^2_{.975}(4)=11.14$

$$n\hat{v} = .1^2 + .15^2 + .05^2 + .04^2 = .0366$$

$$\frac{.0366}{.048} > \sigma^2 > \frac{.0366}{11.14} \qquad 0.276 > \sigma > 0.057$$

8-20 In this problem $n=3$, $x_1+x_2+x_3=9.8$

$$f(x,c) \sim c^4x^3e^{-cx} \qquad f(X,c) = c^{4n}(x_1\ldots x_n)^{3n}e^{-cn\bar{x}}$$

$$\frac{\partial f(X,c)}{\partial c} = \left(\frac{4n}{c} - n\bar{x}\right)f(X,\theta) = 0 \qquad \hat{c} = \frac{4}{\bar{x}} = 1.224$$

8-21 The joint density

$$f(X,c) = c^n e^{-cn(\bar{x}-x_0)} \qquad x_i > x_0$$

has an interior maximum if

$$\frac{\partial f(X,c)}{\partial c} = 0 \qquad \hat{c} = \frac{1}{\bar{x}-x_0}$$

8-22 The probability

$$p = 1 - F_x(200) = e^{-200c}$$

of the event $\{x > 200\}$ is a monoton decreasing function of c. To find the ML estimate $\hat{c}$ of c it suffices to find the ML estimate $\hat{p}$ of p. From Example 8-28 it follows with k=62 and n=80 that

$$\hat{p} = \frac{62}{80} = .775 \text{ hence}$$

$$\hat{c} = -\frac{1}{200} \ln \hat{p} = 0.0013$$

8-23 The samples of x are the integers x_i and the joint density of the RVs x_i equals

$$f(X,\theta) = e^{-n\theta} \prod \frac{\theta^{x_i}}{x_i!} = e^{-n\theta} \frac{\theta^{n\bar{x}}}{\Pi x_i!}$$

Hence, $f(X,\theta)$ is maximum if $-n + n\bar{x}/\theta = 0$. This yields $\hat{\theta} = \bar{x}$

8-24 If $L = \ln f(x,\theta)$ then

$$\frac{\partial L}{\partial \theta} = \frac{1}{f}\frac{\partial f}{\partial \theta} \qquad \frac{\partial^2 L}{\partial \theta^2} = \frac{1}{f}\frac{\partial^2 f}{\partial \theta^2} - \frac{1}{f^2}\left(\frac{\partial f}{\partial \theta}\right)^2 \qquad \frac{\partial^2 L}{\partial \theta^2} + \left(\frac{\partial L}{\partial \theta}\right)^2 = \frac{1}{f}\frac{\partial^2 f}{\partial \theta^2}$$

But

$$E\left\{\frac{1}{f}\frac{\partial^2 f}{\partial \theta^2}\right\} = \int_R \frac{1}{f}\frac{\partial^2 f}{\partial \theta^2} f dX = 0 \quad \text{hence} \quad E\left\{\frac{\partial^2 L}{\partial \theta^2} + \left(\frac{\partial L}{\partial \theta}\right)^2\right\} = 0$$

8-25 (a) From (8-307): Critical region

$$\bar{x} > c = \eta_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}} = 8+2.326 \times \frac{2}{8} = 8.58$$

If η=8.7, then $\eta_q = \frac{8.7-8}{218} = 2.8$

$$\beta(\eta) = G(2.36 - 2.8) = .32$$

(b) We assume that α=.01, β(8.7)=.05 and wish to find n and c.

$$G(z_{1-\alpha}-\eta_q) = \beta \qquad z_{1-\alpha}-\eta_q = z_\beta$$

$$\eta_q = z_{.99} - z_{.05} = 4.97 = \frac{8.7-8}{2/\sqrt{n}}$$

$$n = 129 \qquad c = 8 + \frac{2}{\sqrt{129}} z_{.99} = 8.41$$

8-26 Our objective is to test the composite null hypothesis $\eta > \eta_0 = 28$ against the hypothesis $\eta < \eta_0$. Consider first the simple null hypothesis $\eta = \eta_0 = 28$. In this case, we can use (8-301) with

$$q = \frac{\bar{x}-\eta_0}{s/\sqrt{n}} \qquad \bar{x} = \frac{1}{17}\sum x_i = 27.67 \qquad s^2 = \frac{1}{16}\sum(x_i-\bar{x}) = 17.6$$

This yields s=4.2 and q=-0.33. Since

$$q_u = t_u(n-1) = t_{0.05}(16) = -1.95 < -0.33$$

we conclude that the evidence does not support the rejection of the hypothesis η=28. The resulting OC function $\beta_0(\eta)$ is determined from (9-60c).

If $\eta_0 > 28$, then the corresponding value of q is larger than -0.33. From this it follows that the evidence does not support the hypothesis η_0 for any $\eta_0 > 28$. We note, however, that the corresponding OC function $\beta(\eta)$ is smaller than the function $\beta_0(\eta)$ obtained from (8-301) with $\eta_0 = 28$.

8-27 From (8-297) with $q_u = t_u(n-1)$: Critical region $|\bar{x} - \eta_0| > t_{1-\alpha/2}(n-1)s/\sqrt{n}$

1. $\underline{\alpha = .1}$ $\quad t_{.95}(63) = 1.67 \quad |\bar{x} - 8| > 1.67 \times 1.5/8 = 0.313$

Since $\bar{x} = 7.7$ is in the interval 8 ± 0.317, we accept H_0

2. $\underline{\alpha = .01}$ $\quad t_{.995}(63) = 2.62 \quad |\bar{x} - 8| > 2.62 \times 1.5/8 = 0.49$

Since $\bar{x} = 7.7$ is outside the interval 8 ± 0.49, we reject H_0.

8-28 We assume that the RVs $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are normal and independent. We form the difference $\underset{\sim}{w} = \underset{\sim}{\bar{x}} - \underset{\sim}{\bar{y}}$ of their sample means

$$\underset{\sim}{\bar{x}} = \frac{1}{16} \sum_{i=1}^{16} \underset{\sim}{x}_1 \qquad \underset{\sim}{\bar{y}} = \frac{1}{26} \sum_{i=1}^{26} \underset{\sim}{y}_i$$

and use as test statistic the ratio

$$\underset{\sim}{q} = \frac{\underset{\sim}{w}}{\sigma_w} \qquad \sigma_w^2 = \frac{\sigma_x^2}{16} + \frac{\sigma_y^2}{26}$$

The RV $\underset{\sim}{q}$ is normal with $\sigma_q = 1$ and under hypothesis H_0, $E\{\underset{\sim}{q}\} = 0$. We can, therefore, use (8-307) because $q_u = z_u$. To find q, we must determine σ_w. Since σ_x and σ_y are not specified, we shall use the approximations $\sigma_x \simeq s_x = 1.1$ and $\sigma_y \simeq s_y = 0.9$. This yields

$$\sigma_w^2 \simeq \frac{1.1^2}{16} + \frac{0.9^2}{26} = 0.107 \qquad q = \frac{\bar{x}-\bar{y}}{\sigma_w} = \frac{0.4}{0.327} = 1.223$$

Since $z_{0.95} = 1.645 > 1.223$, we accept H_0.

8-29 (a) In this problem, $n=64$, $k=22$, $p_0 = q_0 = 0.5$

$$q = \frac{k - np_0}{\sqrt{np_0 q_0}} = 2.5 \qquad z_{\alpha/2} = -z_{1-\alpha/2} \simeq -2$$

Since 2.5 is outside the interval (2, -2), we reject the fair coin hypothesis [see (8-313)].

(b) From (8-313) with $n=16$, $p_0 = q_0 = 0.5$:

$$\frac{k_1 - np_0}{\sqrt{np_0 q_0}} = z_{\alpha/2} \qquad \frac{k_2 - np_0}{\sqrt{np_0 q_0}} = -z_{\alpha/2}$$

This yields $k_1 = 8 - 2\times2 = 4$, $k_2 = 8 + 2\times2 = 12$

8-30 We shall use as test statistic the sum

$$\underset{\sim}{q} = \underset{\sim}{x}_1 + \cdots + \underset{\sim}{x}_m \qquad n = 22$$

The critical region of the test is $q<q_\alpha$ where $q=x_1+\cdots+x_n=90$ [see (8-301)]. The RV $\underset{\sim}{q}$ is Poisson distributed with parameter $n\lambda$. Under hypothesis H_0, $\lambda=\lambda_0=5$; hence, $\eta_q=n\lambda_0=110=\sigma_q^2$. To find q_α we shall use the normal approximation. With $\alpha=0.05$ this yields

$$q_\alpha = n\lambda_0 + z_\alpha \sqrt{n\lambda_0} = 90-17.25 = 72.75$$

Since 90 > 72.75, we accapt the hypothesis that $\lambda=5$.

8-31 From (9-75) with n=102 and $p_{0i}=1/6$

$$q = \sum_{i=1}^{6} \frac{(k_i-17)^2}{17} = 2 \qquad \chi^2_{.95}(5) \simeq 11$$

Since 2<11, we accept the fair die hypothesis.

8-32 Uniformly distributed integers from 0 to 9 means that they have the same probability of appearing. With m=10, $p_{01}=.1$, and n=1,000, it follows from (8-325) that

$$q = \sum_{j=0}^{9} \frac{(n_j-100)^2}{100} = 17.76 \qquad \chi^2_{.95}(9) = 16.92$$

Since 17.76 > 16.92, we reject the uniformity hypothesis.

8-33 In this problem

$$f(x,\theta) = e^{-\theta}\frac{\theta^x}{x!} \qquad f(X,\theta) = \frac{e^{-n\theta}\,\theta^{n\bar{x}}}{x_1!\cdots x_n!}$$

$f(X,\theta)$ is maximum for $\theta=\theta_m=\bar{x}$. And since $\theta_{m0}=\theta_0$ we conclude that

$$\lambda(X) = \frac{e^{-n\theta_0}\theta_0^{n\bar{x}}}{e^{-n\bar{x}}\bar{x}^{n\bar{x}}} \qquad w = -2\ ln\ \lambda = 2n(\theta_0-\bar{x}) + \bar{x}\ ln(\bar{x}/\theta_0)$$

With $n=50$, $\theta_0=20$, $\bar{x}=1{,}058/50=21.16$, this yields $w=3$. Since $m_0=1$, $m=1$, and $\chi^2_{.95}(1)=3.84>3$, we accept H_0.

8-34 We form the RVs

$$\underset{\sim}{z} = \sum_{i=1}^{m}\left(\frac{x_i-\eta_x}{\sigma_x}\right)^2 \qquad \underset{\sim}{w} = \sum_{i=1}^{n}\left(\frac{y_i-\eta_y}{\sigma_y}\right)^2$$

These RVs are $\chi^2(m)$ and $\chi^2(n$ respectively. If $\sigma_x=\sigma_y$, then

$$\underset{\sim}{q} = \frac{z/m}{\underset{\sim}{w}/n}$$

Hence (see Prob. 6-23), $\underset{\sim}{q}$ has a Snedecor distribution. To test the hypothesis $\sigma_x=\sigma_y$, we use (8-297) where $q_u=F_u(m,n)$ is the tabulated u percentile of the Snedecor distribution. This yields the following test:

Accept H_o iff $\qquad F_{\alpha/2}(m,n) < q < F_{1-\alpha/2}(m,n)$.

8-35 If $\underset{\sim}{x}$ has a student-t distribution, then $f(-x)=f(x)$, hence (see Prob. 6-75)

$$E\{\underset{\sim}{x}\} = 0 \qquad \sigma_x^2 = E\{\underset{\sim}{x}^2\} = \frac{n}{n-2}$$

8-36 (a) Suppose that the probability $P(A)$ that player A wins a set equals $p=1-q$. He wins the match in five sets if he wins two of the first four sets and the fifth set. Hence, the probability $p_5(A)$ that he wins in five equals $6p^3q^2$. Similarly, the probability $p_5(B)$ that player B wins in five equals $6p^2q^3$. Hence,

$$p_5 = p_5(A) + p_5(B) = 6p^3q^2 + 6p^2q^3 = 6p^2q^2$$

is the probability that the match lasts five sets. If $p=q=1/2$, then $p_5=3/8$.

(b) Suppose now that $P(A) = \underset{\sim}{p}$ is an RV with density $f(p)$. In this case,

$$\underset{\sim}{p}_5 = 6\underset{\sim}{p}^2(1-\underset{\sim}{p}^2)$$

is an RV. We wish to find its best bayesian estimate. Using the MS criterion, we obtain

$$\hat{p}_5 = E\{\underset{\sim}{p}_5\} = \int_0^1 6p^2(1-p^2)f(p)dp$$

If $f(p)=1$, then $\hat{p}_5 = 1/5$.

8-37 Given

$$f_v(v) \sim e^{-v^2/2\sigma^2} \qquad f_\theta(\theta) \sim e^{-(\theta-\theta_o)^2/2\sigma_o^2}$$

To show that

$$f_\theta(\theta|X) \sim e^{-(\theta-\theta_1)^2/2\sigma_1^2}$$

where

$$\frac{1}{\sigma_1^2} \equiv \frac{1}{\sigma_o^2} + \frac{n}{\sigma^2} \qquad \theta_1 \equiv \frac{\sigma_1^2}{\sigma_o^2}\theta_o + \frac{n\sigma_1^2}{\sigma^2}\bar{x}$$

<u>Proof</u>

$$f_x(x|\theta) = f_v(x-\theta) \sim \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\}$$

$$f(X)|\theta \sim \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\theta)^2\right\}$$

Since $\sum(x_i-\theta)^2 = \sum(x_i-\bar{x})^2 + n(\bar{x}-\theta)^2$, we conclude from (8-32) omitting factors that do not depend on θ that

$$f(\theta|X) \sim \exp\left\{-\frac{1}{2}\left[\frac{(\theta-\theta_o)^2}{\sigma_o^2} + \frac{n(\bar{x}-\theta)^2}{\sigma^2}\right]\right\}$$

The above bracket equals

$$\left(\frac{1}{\sigma_o^2} + \frac{n}{\sigma^2}\right)\theta^2 - 2\left(\frac{\theta_o}{\sigma_o^2} + \frac{n\bar{x}}{\sigma^2}\right)\theta + \cdots = \frac{1}{\sigma_1^2}(\theta^2 - 2\theta\theta_1) + \cdots$$

and (i) follows.

8-38 The likelihood function of X equals

$$f(X,\theta) = \frac{1}{(\sqrt{2\pi\theta})^n} \exp\left\{-\frac{1}{2\theta}\sum (x_i-\eta)^2\right\}$$

where $\theta = \sigma^2$ is the unknown parameter. Hence

$$L(X,\theta) = -\frac{n}{2}\ell n\,(2\pi\theta) - \frac{1}{2\theta}\sum (x_i-\eta)^2$$

$$\frac{\partial L(X,\theta)}{\partial\theta} = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum (x_i-\theta)^2 = 0 \qquad \hat{\theta} = \frac{1}{n}\sum (x_i-\eta)^2$$

8-39 The estimators $\hat{\underset{\sim}{\theta}}_1$ and $\hat{\underset{\sim}{\theta}}_2$ have the same variance because otherwise one or the other would not be best. Thus

$$E\{\hat{\underset{\sim}{\theta}}_1\} = E\{\hat{\underset{\sim}{\theta}}_2\} = \theta \qquad \text{var}\,\hat{\underset{\sim}{\theta}}_1 = \text{var}\,\hat{\underset{\sim}{\theta}}_2 = \sigma^2$$

If $\hat{\underset{\sim}{\theta}} = \frac{1}{2}(\hat{\underset{\sim}{\theta}}_1 + \hat{\underset{\sim}{\theta}}_2)$, then

$$E\{\hat{\underset{\sim}{\theta}}\} = \theta \qquad \sigma_{\hat{\theta}}^2 = \frac{1}{2}(\sigma^2 + \sigma^2 + 2r\sigma^2) = \frac{1}{2}(1+r)\sigma^2$$

where σ is the correlation coefficient of $\hat{\underset{\sim}{\theta}}_1$ and $\hat{\underset{\sim}{\theta}}_2$. If r<1 then $\sigma_{\hat{\theta}}<\sigma$ which is impossible. Hence, r=1 and $\hat{\underset{\sim}{\theta}}_1 = \hat{\underset{\sim}{\theta}}_2$ (see Prob. 6-53).

8-40 $k_1+k_2-np_1-np_2 = n-n(p_1+p_2) = 0$; Hence, $|k_1-np_1| = |k_2-np_2|$

$$\frac{(k_1-np_1)^2}{np_1} + \frac{(k_2-np_2)^2}{np_2} = (k_1-np_1)^2\left(\frac{1}{np_1} + \frac{1}{np_2}\right) = \frac{(k_1-np_1)^2}{np_1p_2}$$

8.41 It is given that

$$E\{T(X)\} = \int_{-\infty}^{\infty} T(X)\, f(X;\theta)\, dx = \psi(\theta),$$

so that after differentiating and making use of (8-81) we get

$$\int_{-\infty}^{\infty} T(X)\, \frac{\partial f(X;\theta)}{\partial\theta}\, dx = \psi^{'}(\theta) \qquad (8.41-1)$$

Also using (8-80)

$$\int_{-\infty}^{\infty} \psi(\theta)\, \frac{\partial f(X;\theta)}{\partial\theta}\, dx = 0, \qquad (8.41-2)$$

and the above two expressions give

$$\int_{-\infty}^{\infty} [T(X) - \psi(\theta)]\, \frac{\partial f(X;\theta)}{\partial\theta}\, dx = \psi^{'}(\theta) \qquad (8.41-3)$$

But

$$\frac{\partial f(X;\theta)}{\partial\theta} = \frac{1}{f(X;\theta)}\, \frac{\partial \log f(X;\theta)}{\partial\theta}$$

so that (8.41-3) simplifies to

$$\int_{-\infty}^{\infty} \left[\{T(X) - \psi(\theta)\}\sqrt{f(X;\theta)}\right] \left[\sqrt{f(X;\theta)}\; \frac{\partial \log f(X;\theta)}{\partial\theta}\right] dx = \psi^{'}(\theta)$$

and application of Cauchy-Schwarz inequality as in (8-89)-(8-92), Text gives

$$E\left[\{T(X) - \psi(\theta)\}^2\right] \geq \frac{[\psi^{'}(\theta)]^2}{E\left\{\left(\frac{\partial \log f(X;\theta)}{\partial\theta}\right)^2\right\}}$$