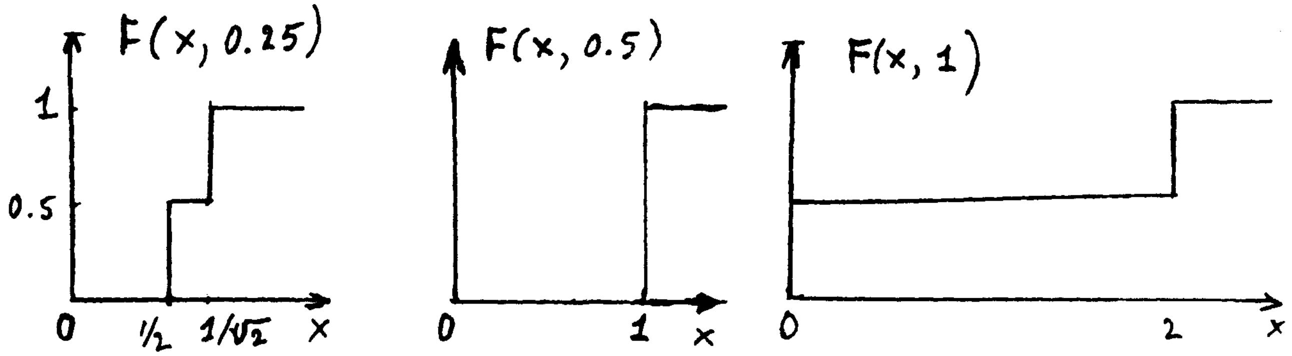
9-1 (a) $E\{x(t)\} = t + 0.5\sin\pi t$

$$x(t, \text{heads}) = \sin\pi t = \begin{cases} 1/\sqrt{2} & t = 0.25 \\ 1 & t = 0.5 \\ 0 & t = 1 \end{cases} \qquad x(t, \text{tails}) = 2t = \begin{cases} 0.5 \\ 1 \\ 2 \end{cases}$$



9-2

$$x(t) = e^{at}$$

$$\eta(t) = \int_{-\infty}^{\infty} e^{at} f_a(a)da \qquad R(t_1,t_2) = \int_{-\infty}^{\infty} e^{at_1} e^{at_2} f_a(a)da$$

From (5-16) with $x = g(a) = e^{ta}$ $\quad g'(a) = t\,e^{ta} = tx$

$$f(x,t) = \frac{1}{x|t|} f_a\left(\frac{1}{t}\ln x\right)U(x)$$

9-3 As we know, $E\{\underset{\sim}{x}(t)\} = \lambda t$ and var $\underset{\sim}{x}(t) = \lambda^2 t^2$ [see (9-18)]. But $E\{\underset{\sim}{x}(9) = 6\}$ by assumption, hence, $\lambda = 2/3$

(a) $E\{\underset{\sim}{x}(8)\} = 24$ var $\underset{\sim}{x}^2(t) = 24^2$

(b) The RV $\underset{\sim}{x}(2)$ is Poisson distributed with parameter $2\lambda = 6$. Hence,

$$P\{\underset{\sim}{x}(2) \le 3\} = e^{-2\lambda} \sum_{k=0}^{3} \frac{(2\lambda)^k}{k!}$$

(c) The RVs $\underset{\sim}{z} = \underset{\sim}{x}(2)$ and $\underset{\sim}{w} = \underset{\sim}{x}(4) - \underset{\sim}{x}(2)$ are independent and Poisson distributed with parameter 2λ. Hence,

$$P\{\underset{\sim}{z}=k\} = e^{-2\lambda}\frac{(2\lambda)^k}{k!} \qquad P\{\underset{\sim}{z} = k, \underset{\sim}{w} = m\} = e^{-4\lambda}\frac{(2\lambda)^k}{k!}\frac{(2\lambda)^m}{m!}$$

$$P\{\underset{\sim}{x}(4) \le 5 \mid \underset{\sim}{x}(2) \le 3\} = \frac{P\{z\le 3,\ w\le 5-z\}}{P\{\underset{\sim}{z}\le 3\}} \qquad P\{\underset{\sim}{z} \le 3\} = \sum_{k=0}^{3} p\{\underset{\sim}{z}=k\}$$

$$P\{\underset{\sim}{z} \le 3, \underset{\sim}{w} \le 5 - \underset{\sim}{z}\} = \sum_{k=0}^{3}\ \sum_{m=0}^{5-k} P\{\underset{\sim}{z} = k, \underset{\sim}{w} = m\}$$

9-4 $$\underset{\sim}{x}(t) = U(t - \underset{\sim}{c}) \qquad \underset{\sim}{y}(t) = \delta(t - \underset{\sim}{c}) = \underset{\sim}{x}'(t)$$

For t_1 or $t_2 < 0$, $R(t_1,t_2) = 0$; for t_1 and $t_2 > T$, $R(t_1,t_2) = 1$. Otherwise,

$$R_x(t_1,t_2) = \frac{1}{T}\min(t_1,t_2) \qquad \frac{\partial R_x}{\partial t_1} = \frac{1}{T}U(t_1 - t_2) \qquad -\frac{\partial^2 R_x}{\partial t_1 \partial t_2} = \frac{1}{T}\delta(t_1 - t_2)$$

From this and (9-105) it follows that $T R_y(t_1 - t_2) = \delta(t_1 - t_2)$ for $0 < t_1, t_2 < T$ and 0 otherwise.

9-5 $\underset{\sim}{a} - \underset{\sim}{b}t = 0$ iff $t = \underset{\sim}{t}_1 = \underset{\sim}{a}/\underset{\sim}{b}$. Setting $\sigma_1 = \sigma_2 = \sigma$ and $r = 0$ in (6-63), we obtain

$$P\{0 < \underset{\sim}{t}_1 < T\} = \frac{1}{2} + \frac{1}{\pi}\text{arc tan}\,T - \left(\frac{1}{2} + \frac{1}{\pi}\text{arc tan}\,0\right)$$

9-6 **The equations**

$$\mathbf{w}''(t) = \mathbf{v}(t)U(t) \qquad \mathbf{w}(0) = \mathbf{w}'(0) = 0$$

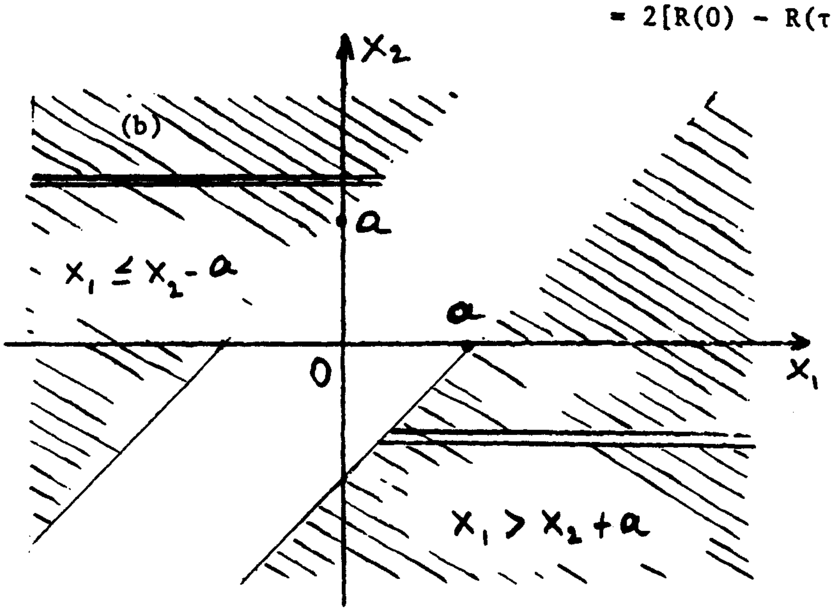
specify a system with input $\mathbf{v}(t)U(t)$ and impulse response $h(t) = t\,U(t)$. Hence [see (9-100)]

$$E\{\mathbf{w}^2(t)\} = q(t)U(t) * t^2 U(t) = \int_0^t (t-\tau)^2 q(\tau)\,d\tau$$

9-7 (a) From (5-88) with $\mathbf{x} = \mathbf{x}(t+\tau) - \mathbf{x}(t)$, and (8-101):

$$P\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)| \geq a\} \leq \frac{E\{[\mathbf{x}(t+\tau) - \mathbf{x}(t)]^2\}}{a^2}$$

$$= 2[R(0) - R(\tau)]/a^2$$



The above probability equals the mass in the regions (shaded) $x_2 - x_1 > a$ and $x_2 - x_1 < -a$

Hence,

$$P\{|\mathbf{x}(t+\tau) - \mathbf{x}(t)| \geq a\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x_2 - a} f(x_1,x_2;\tau)\,dx_1 dx_2 + \int_{-\infty}^{\infty}\int_{x_2+a}^{\infty} f(x_1,x_2;\tau)\,dx_1 dx_2$$

9-8 (a) The RV $\underset{\sim}{x}(t)$ is normal with zero mean and variance $E\{\underset{\sim}{x}^2(t)\} = R(0)=4$, hence it is N(0,2) and $P\{\underset{\sim}{x}(t)\le 3\} = F(3) = G(1.5) = 0.933$

(b) $E\{[\underset{\sim}{x}(t+1) - \underset{\sim}{x}(t-1)]\} = 2[R(0)-R(2)] = 8(1-e^{-4})$

9-9 If $\underset{\sim}{x}(t) = \underset{\sim}{c}e^{j(\omega t+\theta)}$ and $\eta_c = 0$ then

$$\eta_x(t) = \eta_c e^{j(\omega t+\theta)} = 0 \qquad R_{xx}(t+\tau,t) = \sigma_c^2 e^{j\omega\tau}$$

hence, $\underset{\sim}{x}(t)$ is WSS. We shall prove the converse:

If the process $\underset{\sim}{x}(t) = \underset{\sim}{c}\,w(t)$ is WSS, then $\eta_c=0$ and $w(t) = e^{j(\omega t+\theta)}$ within a constant factor.

<u>Proof</u> $\eta_x(t) = \eta_c w(t)$ is independent of t; hence, $\eta_c=0$. The function $R_{xx}(t_1,t_2) = \sigma_c^2 w(t_1)w^*(t_2)$ depends only on $\tau=t_1-t_2$; hence, $w(t+\tau)w^*(t)=g(\tau)$. With $\tau=0$ this yields

$$|w(t)|^2 = g(0) = \text{constant} \qquad w(t) = ae^{j\phi(t)}$$

$$w(t+\tau)w^*(t) = a^2 e^{j[\phi(t+\tau)-\phi(t)]}$$

Hence the difference $\phi(t+\tau)-\phi(t)$ depends only on τ:

$$\phi(t+\tau)-\phi(t) = f(\tau) \qquad \text{(i)}$$

From this it follows that, if $\phi(t)$ is continuous then, $\phi(t)$ is a linear function of t. To simplify the proof, we shall assume that $\phi(t)$ is differentiable. Differentiating with respect to t, we obtain $\phi'(t+\tau) = \phi'(t)$ for every τ. With t=0 this yields

$$\phi'(\tau) = \phi'(0) = \text{constant} \qquad \phi(\tau) = a\tau+b$$

9-10 We shall show that if $\underset{\sim}{x}(t)$ is a normal process with zero mean and $\underset{\sim}{z}(t) = \underset{\sim}{x}^2(t)$, then $C_{zz}(\tau) = 2C_{xx}^2(\tau)$.

From (7-61): If the RVs $\underset{\sim}{x}_k$ are normal and $E\{\underset{\sim}{x}_k\}=0$, then

$$E\{x_1x_2x_3x_4\} = E\{x_1x_2\}\,E\{x_3x_4\} + E\{x_1x_3\}\,E\{x_2x_4\} + E\{x_1x_4\}\,E\{x_2x_3\}$$

With $x_1 = x_2 = x(t+\tau)$ and $x_3 = x_4 = x(t)$, we conclude that the autocorrelation of $z(t)$ equals

$$E\{x^2(t+\tau)x^2(t)\} = E^2\{x^2(t+\tau)\} + 2E^2\,\{x(t+\tau)x(t)\} = R_{xx}^2(0) + 2R_{xx}^2(\tau)$$

And since $R_{xx}(\tau) = C_{xx}(\tau)$, and $E\{z(t)\} = R_{xx}(0)$, the above yields

$$C_{zz}(\tau) = R_{zz}(\tau) - E^2\{z(t)\} = 2C_{xx}^2(\tau)$$

9-11 $y''(t) + 4y'(t) + 13y(t) = x(t)$ all t

The process $y(t)$ is the response of a system with input $x(t) = 26 + \nu(t)$ and

$$H(s) = \frac{1}{s^2+45+13} \qquad h(t) = \frac{1}{3}\,e^{-2t}\sin 3tU(t)$$

Since $\eta_x = 26$, this yields $\eta_y = \eta_x H(0) = 2$. The centered process $\tilde{y}(t) = y(t) - \eta_y$ is the response due to $\nu(t)$. Hence [see (9-100)]

$$E\{\tilde{y}^2(t)\} = q\int_0^\infty h^2(t)dt = \frac{10}{104}$$

With b=4 and c=13 it follows that (see Example 9-276)

$$R_{yy}(\tau) = \frac{10}{104}e^{-2|\tau|}\left(\cos 3\tau - \frac{2}{3}\sin 3|\tau|\right) + 4$$

If ν is normal, then $y(t)$ is normal with mean 2 and variance $R_{yy}(0) - 4 = 10/104$; hence,

$$P\{y(t) \le 3\} = G\left(\frac{3-2}{0.31}\right) = G(3.24)$$

9-12 $E\{y(t)\} = 0 \qquad R_{yy}(t_1,t_2) = \dfrac{R_{xx}(t_1,t_2)}{f(t_1)f(t_2)} = w(t_1-t_2)$

$$E\{z(t)\} = 0 \qquad R_{zz}(t_1,t_2) = \frac{R_{xx}(t_1,t_2)}{\sqrt{q(t_1)}\,\sqrt{q(t_2)}} = \delta(t_1-t_2)$$

because $q(t_1)\delta(t_1-t_2) = \sqrt{q(t_1)}\,\sqrt{q(t_2)}\,\delta(t_1-t_2)$.

9-13 From (9-181) and the identity $4ab \leq (a+b)^2$ it follows that

$$|R_{xy}(\tau)|^2 \leq R_{xx}(0)R_{yy}(0) \leq \frac{1}{4}[R_{xx}(0)+R_{yy}(0)]^2$$

9-14 Clearly (stationarity assumption)

$$E\{|\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)|^2\} = E\{|\underset{\sim}{x}(0) - \underset{\sim}{y}(0)|^2\} = 0$$

Furthermore,

$$E\{\underset{\sim}{x}(t+\tau)[\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)]\} = R_{xx}(\tau) - R_{xy}(\tau)$$

and [see (9-177)]

$$|E\{\underset{\sim}{x}(t+\tau)[\underset{\sim}{x}^*(t) - \underset{\sim}{y}^*(t)]\}|^2 \leq E\{|\underset{\sim}{x}(t+\tau)|^2\}E\{|\underset{\sim}{x}^*(t)-\underset{\sim}{y}^*(t)|^2\} = 0$$

Hence, $R_{xx}(\tau) - R_{xy}(\tau) = 0$; similarly, $R_{yy}(\tau) = R_{xy}(\tau)$

9-15 $E\{|\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)|^2\} = E\{[\underset{\sim}{x}(t+\tau) - \underset{\sim}{x}(t)][\underset{\sim}{x}^*(t+\tau) - \underset{\sim}{x}^*(t)]\}$

$= R(0) - R(\tau) - R^*(\tau) + R(0) = 2R(0) - 2\,\underline{\text{Re}}\,R(\tau)$

9-16 From $\Phi(1) = \Phi(2) = 0$ it follows that

$$E\{\cos\underset{\sim}{\phi}\} = E\{\sin\underset{\sim}{\phi}\} = E\{\cos 2\underset{\sim}{\phi}\} = E\{\sin 2\underset{\sim}{\phi}\} = 0$$

Hence, $E\{\underset{\sim}{x}(t)\} = \cos\omega t\, E\{\cos\underset{\sim}{\phi}\} - \sin\omega t\, E\{\sin\underset{\sim}{\phi}\} = 0$

and as in Example 9-14

$$2\cos[\omega(t+\tau)+\underset{\sim}{\phi}]\cos(\omega t+\underset{\sim}{\phi}) = \cos\omega\tau + \cos(2\omega t+\omega\tau+2\underset{\sim}{\phi})$$

$$2R_x(\tau) = \cos\omega\tau$$

If $\underset{\sim}{\phi}$ is uniform in $(-\pi,\pi)$, then

$$\Phi(\lambda) = \frac{\sin\pi\omega}{\pi\omega} \qquad \Phi(1) = \Phi(2) = 0$$

9-17 (a) $\underset{\sim}{x}(t_1)\underset{\sim}{x}(t_2) = [\underset{\sim}{x}(t_1) - \underset{\sim}{x}(0)][\underset{\sim}{x}(t_2) - \underset{\sim}{x}(t_1) + \underset{\sim}{x}(t_1) - x(0)]$

$$R(t_1,t_2) = E\{[\underset{\sim}{x}(t_1) - \underset{\sim}{x}(0)]^2\} = E\{\underset{\sim}{x}^2(t_1)\} = R(t_1,t_1)$$

(b) If $t_1 + \varepsilon < t_2$, then $R_y(t_1,t_2) = 0$; if

$t_1 < t_2 < t_1 + \varepsilon$ then

$$E\{[\underset{\sim}{x}(t_1+\varepsilon) - \underset{\sim}{x}(t_1)][\underset{\sim}{x}(t_2+\varepsilon) - \underset{\sim}{x}(t_2)]\} = q(t_1 + \varepsilon - t_2)$$

Hence, $\varepsilon^2 R_y(\tau) = q(\varepsilon - |\tau|)$ for $|\tau| = |t_2 - t_1| \le \varepsilon$

9-18

$$E\{\underset{\sim}{x}(t)\underset{\sim}{y}(t)\} = \int_{-\infty}^{\infty} E\{\underset{\sim}{x}(t)\underset{\sim}{x}(t-\tau)\}h(\tau)d\tau$$

$$= \int_{-\infty}^{\infty} R_{xx}(t,t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} q(t)\delta(\tau)h(\tau)d\tau = h(0)q(t)$$

9-19 As in Prob. 5-14, $g(x) = 6 + 3\,F_x(x)$. In this case,

$E\{\underset{\sim}{x}^2(t)\} = 4$, hence, $\underset{\sim}{x}(t)$ is $N(0,2)$ and $F_x(x) = G(x/2)$

9-20 $\underset{\sim}{x}(t)$ is SSS, hence, $P\{x(t) \le y\} = F_x(y)$ does not depend on t. The RVs $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{x}(t)$ are independent, hence, [see (6-238)]

$$F_y(y) = P\{\underset{\sim}{x}(t - \underset{\sim}{\varepsilon}) \le y \mid \underset{\sim}{\varepsilon} = \varepsilon\} = P\{\underset{\sim}{x}(t-\varepsilon) \le y \mid \underset{\sim}{\varepsilon} = \varepsilon\}$$

$$= P\{\underset{\sim}{x}(t-\varepsilon) < y\} = F_x(y)$$

is independent of t. Similarly for higher order distributions.

9-21 $E\{\underset{\sim}{x}(t)\} = \eta =$ constant, hence, [see (9-102)] $E\{\underset{\sim}{x}'(t)\} = 0$

Furthermore, $R_{xx}(-\tau) = R_{xx}(\tau)$. hence, $R'_{xx}(0) = 0$ and (10-97) yields

$$E\{\underset{\sim}{x}(t)\underset{\sim}{x}'(t)\} = R_{xx'}(0) = 0$$

9-22 (a) $E\{\underset{\sim}{z}\,\underset{\sim}{w}\} = R_x(2) = 4e^{-4}$ $\qquad E\{\underset{\sim}{z}^2\} = E\{\underset{\sim}{w}^2\} = R_x(0) = 4$

$$E\{(\underset{\sim}{z} + \underset{\sim}{w})^2\} = R_x(0) + R_x(0) + 2R_x(2) = 8(1 + e^{-4})$$

(b) $\underset{\sim}{z}$ is $N(0,2)$ $\qquad P\{\underset{\sim}{z} < 1\} = F_z(1) = G(1/2)$

$r_{zw} = e^{-4}$, $\qquad f_{zw}(z,w)\ :\ N(0,0;2,2;e^{-4})$

9-23 The RV $\underset{\sim}{x}'(t)$ is normal with zero mean and variance

$$E\{|\underset{\sim}{x}'(t)|^2\} = R_{x'x'}(0) = -R''(0)$$

Hence, $P\{\underset{\sim}{x}'(t) \le a\} = F_{x'}(a) = G\left[a/\sqrt{|R''(0)|}\right]$

9-24 The function arc sin x is odd, hence, it can be expanded into a sine series in the interval (-1,1):

$$\alpha(x) \equiv \text{arc sin } x = \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad |x| \le 1$$

$$b_n = \int_{-1}^{1} \alpha(x)\sin n\pi x\,dx = -\frac{1}{n\pi}\int_{-1}^{1} \alpha(x)\,d\cos n\pi x$$

$$= -\left.\frac{\alpha(x)\cos n\pi x}{n\pi}\right|_{-1}^{1} + \frac{1}{n\pi}\int_{-1}^{1} \cos n\pi x\,d\alpha(x)$$

$$= -\frac{\cos n\pi}{n} + \frac{1}{n\pi}\int_{-\pi/2}^{\pi/2} \cos(n\pi \sin x)\,dx$$

and the result follows because [see (9-81)]

$$R_y(\tau) = \frac{2}{\pi}\text{ arc sin }\frac{R_x(\tau)}{R_x(0)} \qquad J_o(z) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(z \sin x)\,dx$$

9-25 As we know [see (5-100) and (6-193)]

$$E\{e^{j\omega\mathbf{x}(t)}\} = \exp\{-\frac{R(0)}{2}\ \omega^2\}$$

$$E\{e^{j[\omega_1\mathbf{x}(t+\tau)+\omega_2\mathbf{x}(t)]}\} = \exp\{-\frac{1}{2}[R(0)\omega_1^2 + 2R(\tau)\omega_1\omega_2 + R(0)\omega_2^2]\}$$

Hence, with $j\omega = a$

$$E\{I\, e^{a\mathbf{x}(t)}\} = \exp\{\frac{a^2}{2} R_x(0)\}\, I$$

$$E\{I\, e^{a\mathbf{x}(t+\tau)}\, I\, e^{a\mathbf{x}(t)}\} = I^2\exp\{a\,[R_x(0) + R_x(\tau)]\}$$

9-26 (a) $R_y(\tau) = a^2E\{\mathbf{x}[c(t+\tau)]\mathbf{x}(ct)\} = a^2R(c\tau)$

(b) If $\mathbf{z}_\varepsilon(t) = \sqrt{\varepsilon}\ \mathbf{x}(\varepsilon t)$ then $R_{z_\varepsilon}(\tau) = \varepsilon R_x(\varepsilon\tau)$ [as in (a)].

If $\delta > 0$ is sufficiently small and $\phi(t)$ is continuous at the origin, then

$$\int_{-\delta}^{\delta} R_{z_\varepsilon}(\tau)\phi(\tau)d\tau \simeq \phi(0)\int_{-\delta}^{\delta} \varepsilon R_x(\varepsilon\tau)\, d\tau$$

$$= \phi(0)\int_{-\varepsilon\delta}^{\varepsilon\delta} R(\tau)d\tau \xrightarrow[\varepsilon\to\infty]{} \phi(0)\int_{-\infty}^{\infty} R(\tau)d\tau = q\ \phi(0)$$

Hence, $R_{z_\varepsilon}(\tau) \longrightarrow q\ \delta(\tau)$ as $\varepsilon \to \infty$.

9-27

$$\underset{\sim}{y}(t) = \int_{t-T}^{t} \underset{\sim}{x}(\tau)h(t-\tau)d\tau$$

Hence, $\underset{\sim}{y}(t_1)$ and $\underset{\sim}{y}(t_2)$ depend linearly on the values of $\underset{\sim}{x}(t)$ in the intervals $(t_1 - T,\ t_1)$ and $(t_2 - T,\ t_2)$ respectively. If $|t_1 - t_2| > T$ then these intervals do not overlap and since $E\{\underset{\sim}{x}(\tau_1)\underset{\sim}{x}(\tau_2)\} = 0$ for $\tau_1 \neq \tau_2$, it follows that $E\{\underset{\sim}{y}(t_1)\underset{\sim}{y}(t_2)\} = 0$.

9-28 (a)

$$I(t) = E\left\{\int_0^t \int_0^t h(t,\alpha)\underset{\sim}{x}(\alpha)h(t,\beta)\underset{\sim}{x}(\beta)\ d\alpha d\beta\right\}$$

$$= \int_0^t \int_0^t h(t,\alpha)h(t,\alpha)q(\alpha)\delta(\alpha-\beta)d\alpha d\beta = \int_0^t h^2(t,\alpha)q(\alpha)d\alpha$$

(b) If $\underset{\sim}{y}'(t) + c(t)\underset{\sim}{y}(t) = \underset{\sim}{x}(t)$, then $\underset{\sim}{y}(t)$ is the output of a linear time-varying system as in (a) with impulse response $h(t,\alpha)$ such that

$$\frac{\partial h(t,\alpha)}{\partial t} + c(t)h(t,\alpha) = \delta(t-\alpha) \qquad h(\alpha^-,\alpha) = 0$$

or equivalently

$$\frac{\partial h(t,\alpha)}{\partial t} + c(t)h(t,\alpha) = 0 \qquad t > 0 \quad h(\alpha^+,\alpha) = 1$$

This yields

$$h(t,\alpha) = e^{-\int_\alpha^t c(\tau)d\tau}$$

Hence, if

$$I(t) = \int_0^t h^2(t,\alpha)q(\alpha)d\alpha \quad \text{then} \quad I'(t) + 2c(t)I(t) = q(t)$$

because the impulse response of this equation equals

$$e^{-2\int_\alpha^t c(\tau)d\tau} = h^2(t,\alpha)$$

9-29 (a) If $\underset{\sim}{y}'(t) + 2\underset{\sim}{y}(t) = \underset{\sim}{x}(t)$, then $\underset{\sim}{y}(t) = \underset{\sim}{x}(t)*h(t)$

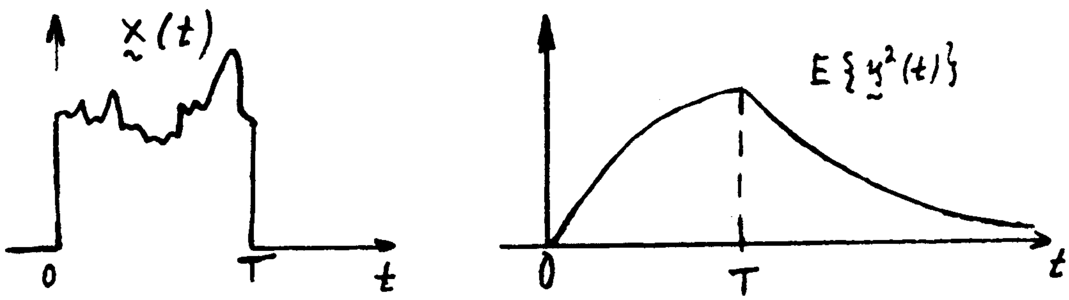
where $h(t) = e^{-2t}U(t)$ and with $q(t) = 5$, (10-90) yields

$$E\{\underset{\sim}{y}^2(t)\} = 5*e^{-4t}U(t) = 5\int_0^\infty e^{-4\tau}d\tau = \frac{5}{4}$$

(b) As in (a) with $q(t) = 5U(t)$. Hence, for $t > 0$

$$E\{\underset{\sim}{y}^2(t)\} = 5U(t)*e^{-4t}U(t) = 5\int_0^t e^{-4\tau}d\tau = \frac{5}{4}(1 - e^{-4t})$$

9-30



From (9-90) with $q(t) = N[U(t) - U(t - T)]$

$$E\{\underset{\sim}{y}^2(t)\} = \begin{cases} AN\int_0^t e^{-2\alpha(t-\tau)}d\tau = \frac{AN}{2\alpha}(1-e^{-2\alpha t}) & 0 \le t < T \\ AN\int_0^T e^{-2\alpha(t-\tau)}d\tau = \frac{AN}{2\alpha}(e^{2\alpha T} - 1)e^{-2\alpha t} & t > T \end{cases}$$

9-31 Since $\underset{\sim}{x}(t)$ is WSS, the moments of S equal the moments of

$$\underset{\sim}{z} = \int_{-5}^{5} \underset{\sim}{x}(t)dt$$

Hence, (see Fig. 9-5)

$$E\{\underset{\sim}{s}^2\} = \int_{-5}^{5}\int_{-5}^{5} R_x(t_1-t_2)dt_1dt_2 = \int_{-10}^{10}(10-|\tau|)R_x(\tau)d\tau$$

$$E\{\underset{\sim}{s}\} = 80 \qquad \sigma_s^2 = 2\int_0^{10}(10-\tau)10e^{-2\tau}d\tau$$

9-32 $\underset{\sim}{y}(t) = \underset{\sim}{x}(t)*h(t) \qquad h(t) = e^{-2t}U(t)$

(a) $E\{\underset{\sim}{y}^2(t)\} = 5*e^{-4t}\,U(t) = 5/4$

$$R_{xy}(t_1,t_2) = 5\,\delta(t_1-t_2)*e^{-2t_2}\,U(t_2) = 5e^{-2(t_2-t_1)}U(t_2-t_1)$$

$$R_{yy}(t_1,t_2) = 5e^{-2(t_2-t_1)}U(t_2-t_1)*e^{-2t_1}\,U(t_1)$$

$$= \frac{5}{4}\,e^{-2|t_1-t_2|}$$

The first equation follows from (9-100) with $q(t) = 5$; the second from (9-94) with $R_{xx}(t_1,t_2) = 5\delta(t_1-t_2)$, and the third from (9-96).

(b) With $R_{xx}(t_1,t_2) = 5\delta(t_1-t_2)U(t_1)U(t_2)$, (9-94) and (9-96) yield the following: For t_1 or $t_2 < 0$, $R_{xy}(t_1,t_2) = R_{yy}(t_1,t_2) = 0$. For $0 < t_1 < t_2$

$$R_{xy}(t_1,t_2) = 5\delta(t_1-t_2)*e^{-2t_2} = 5\,e^{-2t_2}$$

$$R_{yy}(t_1,t_2) = \int_0^{t_1} 5\,e^{-2(t_1-\tau)}\,e^{-2(t_1-\tau)}\,d\tau = \frac{5}{4}\,e^{-2(t_2-t_1)}(1-e^{-4t_1})$$

9-33 $$\int_{-\infty}^{\infty} e^{-\alpha\tau^2} e^{-s\tau} d\tau = e^{s^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha(\tau + s/2\alpha)^2} d\tau = \sqrt{\frac{\pi}{\alpha}}\, e^{s^2/4\alpha}$$

This yields

$$e^{-\alpha\tau^2} \longleftrightarrow \sqrt{\frac{\pi}{\alpha}}\, e^{-\omega^2/4\alpha}$$

$$e^{-\alpha\tau^2} \cos \omega_o\tau \longleftrightarrow \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \left[e^{-(\omega-\omega_o)^2/4\alpha} + e^{-(\omega+\omega_o)^2/4\alpha} \right]$$

9-34 $$G(x_1,x_2;\omega) = \int_{-\infty}^{\infty} f(x_1,x_2;\tau) e^{-j\omega\tau} d\tau$$

$$R(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{x}(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2\, f(x_1,x_2;\tau)dx_1dx_2$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} e^{-j\omega\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2\, f(x_1,x_2;\tau)dx_1\, dx_2 d\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 \int_{-\infty}^{\infty} e^{-j\omega\tau}\, f(x_1,x_2;\tau)d\tau dx_1dx_2$$

9-35 The process $\mathbf{y}(t) = \mathbf{x}(t+a) - \mathbf{x}(t-a)$ is the output of a system with input $\mathbf{x}(t)$ and system function

$$H(\omega) = e^{ja\omega} - e^{-ja\omega} = 2j \sin a\omega$$

Hence [see (9-150)]

$$S_y(\omega) = 4 \sin^2 a\omega\, S_x(\omega) = (2 - e^{j2a\omega} - e^{-j2a\omega})S_x(\omega)$$

$$R_y(\tau) = 2\, R_x(\tau) - R_x(\tau+2a) - R_x(\tau-2a)$$

9-36 Since $S(\omega) \geq 0$, we conclude with (9-136) that

$$R(0) - R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)(1 - \cos \omega\tau)\, d\omega$$

$$\geq \frac{1}{8\pi} \int_{-\infty}^{\infty} S(\omega)(1 - \cos 2\omega\tau)\, d\omega = \frac{1}{4}[R(0) - R(2\tau)]$$

and the result follows for $n = 1$. Repeating the above, we obtain the general result.

9-37 **From** (6-197)

$$E\{x^2(t+\tau)x^2(t)\} = E\{x^2(t+\tau)\}E\{x^2(t)\} + 2E^2\{x^2(t+\tau)x^2(t)\}$$

Hence,

$$R_y(\tau) = R_x^2(0) + 2R_x^2(\tau) = I^2(1 + e^{-2\alpha|\tau|} + e^{-2\alpha|\tau|}\cos 2\beta\tau)$$

$$S_y(\omega) = \left[2\pi\delta(\omega) + \frac{4\alpha}{4\alpha^2 + \omega^2} + \frac{2\alpha}{4\alpha^2 + (\omega - 2\beta)^2} + \frac{2\alpha}{4\alpha^2 + (\omega + 2\beta)^2}\right]$$

Furthermore,

$$\eta_y = E\{x^2(t)\} = R_x(0) \qquad C_y(\tau) = 2R_x^2(\tau)$$

9-38

$$\int_{-\infty}^{\infty} S(\omega)\left|\sum_i a_i e^{j\omega\tau_i}\right|^2 d\omega = \int_{-\infty}^{\infty} S(\omega)\sum_{i,k} a_i a_k^* e^{j\omega(\tau_i - \tau_k)} d\omega$$

$$= \sum_{i,k} a_i a_k^* R(\tau_i - \tau_k) \geq 0$$

9-39 (a) $S(s) = \dfrac{1}{1+s^4} = \dfrac{1}{(s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1)}$

A special case of example 9-27b with $b = \sqrt{2}$, $c = 1$. Hence,

$$R(\tau) = \frac{1}{2\sqrt{2}} e^{-|\tau|/\sqrt{2}} \left(\cos\frac{\tau}{\sqrt{2}} + \sin\frac{|\tau|}{\sqrt{2}}\right)$$

(b) From the pair $e^{-2|\tau|} \leftrightarrow 4/(4+\omega^2)$ and the convolution theorem it follows that

$$e^{-2|\tau|} * e^{-2|\tau|} \leftrightarrow \frac{16}{(4+\omega^2)^2}$$

Hence, for $\tau > 0$

$$16\, R(\tau) = \int_{\infty}^{\infty} e^{-2|x|}\, e^{-2|\tau-x|} dx = \int_{-\infty}^{0} e^{2x}\, e^{-2(\tau-x)} dx$$

$$+ \int_{0}^{\tau} e^{-2x}\, e^{-2(\tau-x)}\, dx + \int_{\tau}^{\infty} e^{-2x}\, e^{2(\tau-x)}\, dx = \frac{1}{2} e^{-2\tau}(1+2\tau)$$

And since $R(-\tau) = R(\tau)$, the above yields

$$e^{-2|\tau|}\,\frac{1+2|\tau|}{32} \longleftrightarrow \frac{1}{(4+\omega^2)^2}$$

9-40 $H^*(-s^*)\Big|_{s=j\omega} = H^*(j\omega)$ $\qquad H^*(1/z^*)\Big|_{z=e^{j\omega T}} = H^*(e^{j\omega T})$

Hence

$H(s)H^*(-s^*)\Big|_{s=j\omega} = |H(j\omega)|^2$ $\qquad H(z)H^*(1/z^*)\Big|_{z=j\omega T} = |H(e^{j\omega T})|^2$

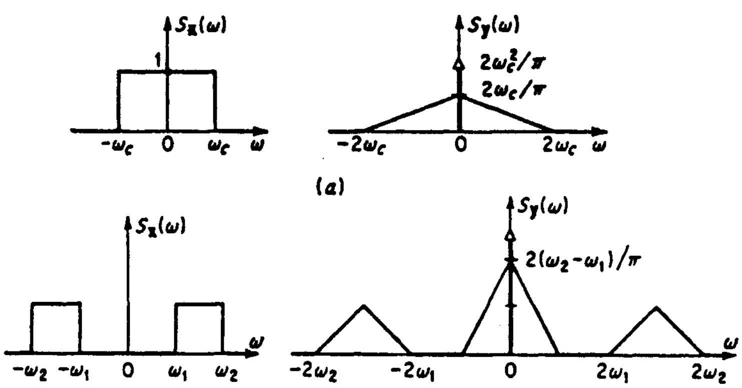
9-41 From (6-197)

$$R_y(\tau) = E\{x^2(t+\tau)x^2(t)\}$$

$$= E\{x^2(t+\tau)\}E\{x^2(t)\} + 2\ E^2\{x(t+\tau)x(t)\} = R_x^2(0) + 2\ R_x^2(\tau)$$

From the above and the frequency convolution theorem it follows that

$$S_y(\omega) = 2\pi R_x^2(0)\delta(\omega) + \frac{1}{\pi} S_x(\omega) * S_x(\omega)$$



9-42 $\quad y(t) = 2x(t) + 3x'(t) \qquad \eta_x = 5 \qquad C_{xx}(\tau) = 4e^{-2|\sigma|}$

The process $y(t)$ is the output of the system $H(s) = 2+3s$ with input $x(t)$. Hence, $\eta_y=5H(0)=10$

$$S_{yy}{}^c(\omega) = S_{xx}{}^c(\omega)|2+3j\omega|^2 = \frac{16}{4+\omega^2}(4+9\omega^2) = 144 - \frac{512}{4+\omega^2} = S_{yy}(\omega) - 2\pi\eta_y{}^2\delta(\omega)$$

9-43 (a) $y'(t) + 3y(t) = x(t)$, $R_{xx}(\tau) = 5\delta(\tau)$. The process $y(t)$ is the output of the system

$$H(s) = \frac{1}{s+3} \qquad h(t) = e^{-3t}U(t)$$

Hence, [see (9-100) and (9-150)]

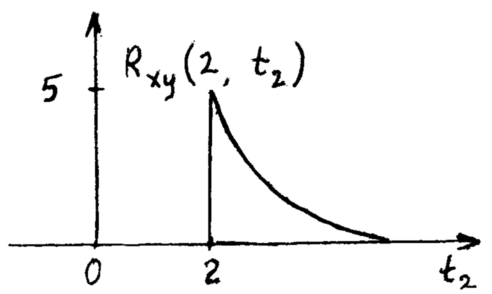
$$E\{y^2(t)\} = 5\int_0^{\infty} e^{-6t}dt = \frac{5}{6}$$

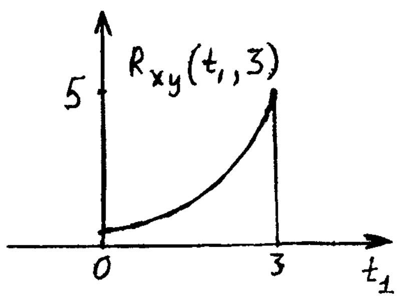
$$S_{yy}(\omega) = \frac{5}{\omega^2+9} \qquad R_{yy}(\tau) = \frac{5}{6}e^{-3|\tau|}$$

(b) As in Example 9-18:

$$E\{y^2(t)\} = 5\int_0^{t} e^{-6\alpha}d\alpha = \frac{5}{6}(1-e^{-6t}) \qquad t > 0$$

$$R_{xy}(t_1,t_2) = 5e^{-2|t_2-t_1|}U(t_1)U(t_2)U(t_2-t_1)$$





9-44 We shall show that: If $x(t)$ is a complex process with autocorrelation $R(\tau)$ and $|R(\tau_1)|=R(0)$ for some τ_1, then $R(\tau)=e^{j\omega_o\tau}w(\tau)$ where $w(\tau)$ is a periodic function with period τ_1. Furthermore, the process $y(t) = e^{-j\omega_o t}x(t)$ is MS periodic.

<u>Proof</u> Clearly, $R(\tau_1) = R(0)e^{j\phi}$. With $\omega_o = \phi/\tau_1$,

$$R_{yy}(\tau) = E\{x(t+\tau)e^{-j\omega_o(t+\tau)}x^*(t)e^{j\omega_o t}\} = R(\tau)e^{-j\omega\tau}$$

Hence, $R_{yy}(\tau_1)=e^{-j\omega_o\tau_1}R(\tau_1) = R(0) = R_{yy}(0)$. From this and (10-168) it follows that the function $w(\tau) = R_{yy}(\tau)$ is periodic.

9-45 (a) The cross spectrum $S_{\check{x}x}(\omega) = -j\,\mathrm{sgn}\,\omega\, S_{xx}(\omega)$ is an odd function. Hence,

$$E\{x(t)\check{x}'(t)\} = \frac{-j}{2\pi}\int_{\infty}^{\infty} \mathrm{sgn}\,\omega S_{xx}(\omega)\,d\omega = 0$$

(b) The process $\check{\check{x}}(t)$ is the output of the system

$$(-j\,\mathrm{sgn}\,\omega)(-j\,\mathrm{sgn}\,\omega) = -1$$

with input $x(t)$. Hence, $\check{\check{x}}(t) = -x(t)$.

9-46 In general

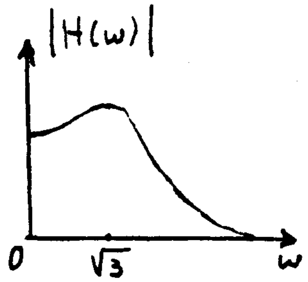
$$E\{y^2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)|H(\omega)|^2 d\omega$$

$$\le |H(\omega_m)|^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,d\omega = E\{x^2(t)\}|H(\omega_m)|^2$$

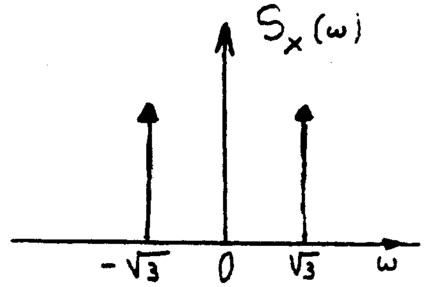
where $|H(\omega_m)|$ is the maximum of $|H(\omega)|$. In our case,

$$|H(\omega)|^2 = \frac{1}{(5-\omega)^2 + 4\omega^2} \quad \text{is maximum for } \omega = \sqrt{3}$$

and $|H(\omega_m)|^2 = 1/16$. Hence $E\{y^2(t)\} \le 10/16$ with equality if $R_x(10) = 10\cos\sqrt{3}\,\tau$ (Fig. b).



(a)



(b)

9-47 If $R_x(\tau) = e^{j\omega_o\tau}$, then $S_x(\omega) = 2\pi\delta(\omega-\omega_o)$, hence, the integral of $S_x(\omega)$ equals zero in any interval not including the point $\omega = \omega_o$. From (9-182) it follows that the same is true for the integral of $S_{xy}(\omega)$. This shows that $S_{xy}(\omega)$ is a line at $\omega = \omega_o$ for any $y(t)$.

9-48 (a) As in (9-147) and (9-149)

$$R_{yx}(\tau) = R_{xx}(\tau) * h(\tau) = \int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma)} h(\gamma)\, d\gamma = e^{j\alpha\tau} H(\alpha)$$

$$R_{yy}(\tau) = R_{xx}(\tau) * \rho(\tau) = \int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma)} \rho(\gamma)\, d\gamma = e^{j\alpha\tau} |H(\alpha)|^2$$

(b) As in (9-94) and (9-95)

$$R_{yx}(t_1,t_2) = e^{-j\beta t_2} \int_{-\infty}^{\infty} e^{j\alpha(t_1-\gamma)} h(\gamma)\, d\gamma = e^{j(\alpha t_1-\beta t_2)} H(\alpha)$$

$$R_{yy}(t_1,t_2) = e^{-j\alpha t_1} H(\alpha) \int_{-\infty}^{\infty} e^{-j\beta(t_2-\gamma)} h(\gamma)\, d\gamma = e^{j(\alpha t_1-\beta t_2)} H(\alpha) H^*(\beta)$$

because $h(t)$ is real and $H(-\beta) = H^*(\beta)$.

9-49 If $S_{xx}(\omega) S_{yy}(\omega) \equiv 0$ then $S_{xx}(\omega) = 0$ or $S_{yy}(\omega) = 0$ in any interval (a,b). From this and (10-168) it follows that the integral of $S_{xy}(\omega)$ in any interval equals zero, hence, $S_{xy}(\omega) \equiv 0$.

9-50 This is the discrete-time version of theorem (9-162). From (9-163)

$$E^2\{(\underset{\sim}{x}[n+m+1] - \underset{\sim}{x}[n+m])\underset{\sim}{x}[n]\} \le E\{|\underset{\sim}{x}[n+m+1] - \underset{\sim}{x}[n+m]|^2\}E\{|\underset{\sim}{x}[n]|^2\}$$

$$(R[m+1] - R[m])^2 \le 2(R[0] - R[1])R[0] = 0$$

Hence, $R[m+1] = R[m]$ for any m.

9-51 We shall show that

$$2\frac{R^2[1]}{R[0]} - R[0] \le R[2] \le R[0] \qquad \text{(i)}$$

The covariance matrix of the RVs $\underset{\sim}{x}[n]$, $\underset{\sim}{x}[n+1]$, and $\underset{\sim}{x}[n+2]$ is nonnegative [see (7-29)]:

$$\begin{vmatrix} R[0] & R[1] & R[2] \\ R[1] & R[0] & R[1] \\ R[2] & R[1] & R[0] \end{vmatrix} \ge 0$$

This yields

$$R[0]R^2[2] - 2R^2[1]R[2] - R^3[0] + 2R[0]R^2[1] \le 0$$

The above is a quadratic in R[2] with roots

$$R[0] \text{ and } -R[0] + 2R^2[1]/R[0]$$

Since it is nonpositive, R[2] must be between the roots as in (i)

9-52 If $\underset{\sim}{x}[n] = Ae^{jn\underset{\sim}{\omega}T}$ then

$$R_x[m] = A^2E\{e^{j(m+n)\underset{\sim}{\omega}T}e^{-jn\underset{\sim}{\omega}T}\} = A^2\int_{-\sigma}^{\sigma} e^{jm\omega T} f(\omega)d\omega$$

But [see (9-194)]

$$R[m] = \frac{1}{2\sigma}\int_{\sigma}^{\sigma} S_x(\omega)e^{jm\omega T}d\omega$$

hence, $A^2f(\omega) = S_x(\omega)/2\sigma$

9-53 (a) If $\underset{\sim}{y}(0) = \underset{\sim}{y}'(0) = 0$, then $\underset{\sim}{y}(t)$ is the output of a system with input $\underset{\sim}{x}(t)U(t)$ and impulse response $h(t)$ such that

$$h''(t) + 7h'(t) + 10h(t) = \delta(t) \qquad h(0^-) = h'(0^-) = 0$$

$$h(t) = \frac{1}{3}(e^{-2t} - e^{-5t})U(t)$$

and with $q(t) = 5\,U(t)$, (9-100) yields

$$E\{\underset{\sim}{y}^2(t)\} = \frac{5}{9}\int_0^t (e^{-2\tau} - e^{-5\tau})^2 d\tau$$

(b) If $\underset{\sim}{y}[-1] = \underset{\sim}{y}[-2] = 0$, then $\underset{\sim}{y}[n]$ is the output of a system with input $\underset{\sim}{x}[n]U[n]$ and delta response $h[n]$ such that

$$8h[n] - 6h[n-1] + h[n-2] = \delta[n] \qquad h[-1] = h[-2] = 0$$

$$h[n] = \left(\frac{1}{2^{n+2}} - \frac{1}{2^{2n+3}}\right) U[n]$$

and with $q[n] = 5\,U[n]$, (10-176) yields

$$E\{\underset{\sim}{y}^2[n]\} = 5\sum_{k=0}^{n}\left(\frac{1}{2^{k+2}} - \frac{1}{2^{2k+3}}\right)^2$$

9-54

$$\underset{\sim}{y}[n] = \underset{\sim}{x}[n]*h[n] \qquad h[n] = 2^{-n}U[n]$$

(a)

$$E\{\underset{\sim}{y}^2[n]\} = 5*2^{-2n}U[n] = 0$$

$$R_{xy}[m_1,m_2] = 5\delta[m_1-m_2]*2^{-m_2}U[m_2] = 5\,2^{-(m_2-m_1)}U[m_2-m_1]$$

$$R_{yy}[m_1,m_2] = 5\times 2^{-(m_2-m_1)}U[m_2-m_1]*2^{-m_1}U[m_1]$$

$$= \frac{20}{3}\times 2^{-|m_1-m_2|}$$

The first equation follows from (9-190) with $q[n] = 5$; the second and third from (9-191) with $R_{xx}[m_1,m_2] = 5\,\delta[m_1-m_2]$.

(b) With $R_{xx}[m_1,m_2] = 5\,\delta[m_1-m_2]U[m_1]U[m_2]$, Prob. 9-25a yields the following: For m_1 or $m_2 < 0$, $R_{xy}[m_1,m_2] = R_{yy}[m_1,m_2] = 0$. For $0 < m_1 < m_2$

$$R_{xy}[m_1,m_2] = 5\,\delta[m_1-m_2]*2^{-m_2} = 5\times 2^{-m_2}$$

$$R_{yy}[m_1,m_2] = \sum_{k=0}^{m_1} 5\times 2^{-(m_2-k)}\,2^{-(m_1-k)} = \frac{5}{3}\,2^{-(m_2-m_1)}(4 - 2^{-2m_1})$$

9-55

(a) $R_x[m_1,m_2] = q[m_1]\delta[m_1-m_2]$

$$E\{\underset{\sim}{s}^2\} = \sum_{n=0}^{N} \sum_{k=0}^{N} a_n a_k \, E\{\underset{\sim}{x}[n]\underset{\sim}{x}[k]\}$$

$$= \sum_{n=0}^{N} \sum_{k=0}^{N} a_n a_k \, q[n]\delta[n-k] = \sum_{n=0}^{N} a_n^2 \, q[n]$$

(b) $R_x(t_1,t_2) = q(t_1)\delta(t_1-t_2)$

$$E\{s^2\} = \int_0^T \int_0^T a(t)a(\tau) \; E\{x(t)x(\tau)\}d\tau dt$$

$$= \int_0^T \int_0^T a(t)a(\tau)q(t)\delta(t-\tau)d\tau dt = \int_0^T a^2(t)q(t)dt$$
