CHAPTER 10

10-1

(a) If $\underset{\sim}{x}(t)$ is a Poisson process as in Fig. 9-3a, then for a fixed t, $\underset{\sim}{x}(t)$ is a Poisson RV with parameter λt. Hence [see (5-119)] its characteristic function equals $\exp\{\lambda t(e^{j\omega}-1)\}$.

(b) If $\underset{\sim}{x}(t)$ is a Wiener process then $f(x,t)$ is $N(0,\sqrt{\alpha t})$. Hence [see (5-100)] its first order characteristic function equals $\exp\{-\alpha t\omega^2/2\}$.

10-2 For large $\underset{\sim}{t}$, $\underset{\sim}{x}(t)$ and $\underset{\sim}{y}(t)$ can be approximated by two independent Wiener processes as in (10-52):

$$f_x(x,t) = \frac{1}{\sqrt{2\pi\alpha t}}\, e^{-x^2/2\alpha t} \qquad f_y(y,t) = \frac{1}{\sqrt{2\pi\alpha t}}\, e^{-y^2/2\alpha t}$$

Hence, $\underset{\sim}{z}(t)$ has a Rayleigh density [see (6-70)]. [<u>Note</u>. Exactly, $\underset{\sim}{z}(t)$ is a discrete-type RV taking the values $s\sqrt{m^2+n^2}$ where m and n are integers]. The product $f_z(z,t)dz$ equals approximately the probability that $\underset{\sim}{z}(t)$ is between z and $z+dz$ provided that $dz \gg T$.

10-3 The voltage $\underset{\sim}{v}(t)$ is the output of a system with input $\underset{\sim}{n}_e(t)$ and system function

$$H_1(s) = \frac{1}{LCs^2 + RCs + 1}$$

Hence,

$$S_v(\omega) = S_{n_e}(\omega)\,|H_1(j\omega)|^2 = \frac{2kTR}{(1-\omega^2 LC)^2 + R^2C^2\omega^2}$$

Furthermore,

$$Z_{ab}(s) = \frac{R + Ls}{LCs^2 + RCs + 1} \qquad \underline{Re}\; Z_{ab}(j\omega) = \frac{R}{(1-\omega^2 LC)^2 + R^2C^2\omega^2}$$

in agreement with (10-75).

The current $\underset{\sim}{i}(t)$ is the output of a system with input $\underset{\sim}{n}_e(t)$ and system function

$$H_2(s) = \frac{1}{R + Ls}$$

Hence,

$$S_i(\omega) = S_{n_e}(\omega)\,|H_2\;(j\omega)|^2 = \frac{2kTR}{R^2 + \omega^2 L^2}$$

Furthermore (short circuit admittance)

$$Y_{ab}(s) = \frac{1}{R + LS} \qquad \underline{Re} Y_{ab}(j\omega) = \frac{2kTR}{R^2 + L^2\omega^2}$$

in agreement with (10-78).

10-4 The equation $m\underset{\sim}{x}''(t) + f\underset{\sim}{x}'(t) = \underset{\sim}{F}(t)$ specifies a system with

$$H(s) = \frac{1}{ms^2 + fs} \qquad h(t) = \frac{1}{f}(1 - e^{-ft/m})U(t)$$

and (9-100) yields

$$E\{\underset{\sim}{x}^2(t)\} = \frac{2kTf}{f^2}\int_0^t (1 - e^{-2\alpha\tau})^2\, d\tau \qquad \alpha = \frac{f}{2m}$$

10-5 As in Example 12-2, a and b are such that

$$\underset{\sim}{x}(\tau) - a\,\underset{\sim}{x}(0) - b\underset{\sim}{v}(0) \perp \underset{\sim}{x}(0),\ \underset{\sim}{v}(0)$$

This yields

$$R_{xx}(\tau) = aR_{xx}(0) + b\,R_{xv}(0)$$

$$R_{xv}(\tau) = aR_{xv}(0) + b\,R_{vv}(0) \qquad \text{(i)}$$

where [see (10-163)]

$$R_{xx}(\tau) = A\,e^{-\alpha\tau}(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta\tau) \qquad \tau > 0$$

$$R_{xv}(\tau) = -R'_{xx}(\tau) = A\ e^{-\alpha\tau}(\sin\beta\tau)\,\frac{\alpha^2+\beta^2}{\beta}$$

$$R_{vv}(\tau) = R'_{xv}(\tau) = A\ e^{-\alpha\tau}(\cos\beta\tau - \frac{\alpha}{\beta}\sin\beta\tau)\,\frac{\alpha^2+\beta^2}{\beta^2}$$

Inserting into (i) and solving, we obtain

$$a = e^{-\alpha\tau}(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta\tau)$$

$$b = \frac{1}{\beta}e^{-\alpha\tau}\ \sin\beta\tau$$

Finally,

$$P = E\{[\underset{\sim}{x}(t) - a\,\underset{\sim}{x}(0) - b\,\underset{\sim}{v}(0)]\underset{\sim}{x}(t)\} = R_{xx}(0) - a\,R_{xx}(t) - b\,R_{xv}(t)$$

$$= \frac{2kTf}{m^2}\left[1 - e^{-2\alpha t}(1 + \frac{2\alpha^2}{\beta}\sin^2\beta t + \frac{\alpha}{\beta}\sin 2\beta t)\right]$$

10-6 If $\underset{\sim}{x}(t) = \underset{\sim}{w}(t^2)$ then [see (10-70)]

$$R_x(t_1,t_2) = E\{\underset{\sim}{w}(t_1^2)\underset{\sim}{w}(t_2^2)\} = \alpha t_1^2$$

If $\underset{\sim}{y}(t) = \underset{\sim}{w}^2(t)$ then [see (6-197)]

$$R_y(t_1,t_2) = E\{\underset{\sim}{w}^2(t_1)\underset{\sim}{w}^2(t_2)\}$$

$$= E\underset{\sim}{w}^2(t_1)E\{\underset{\sim}{w}^2(t_2)\} + 2\,E^2\{\underset{\sim}{w}(t_1)\underset{\sim}{w}(t_2)\} = \alpha^2 t_1 t_2 + 2\alpha^2 t_1^2$$

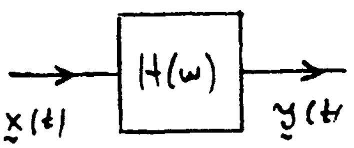
10-7 From (10-112):

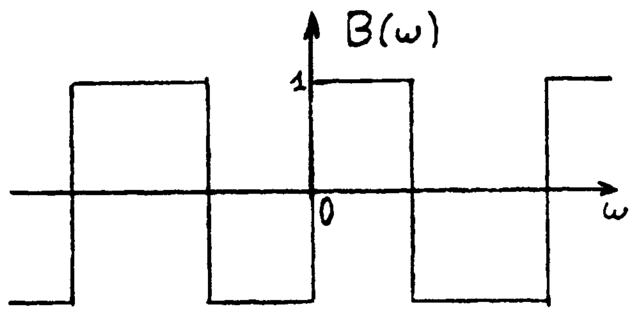
$$\eta_s = 3\int_0^{10} 2\,dt = 60 \qquad \sigma_s^2 = 3\int_0^{10} 4dt = 120 \qquad E\{\underset{\sim}{s}^2\} = 3720$$

$\underset{\sim}{s}(7) = 0$ if there are no points in the interval (7-10, 7). The number of points in this interval is a Poission RV with parameter $10\lambda = 30$. Hence, $P\{\underset{\sim}{s}(7) = 0\} = e^{-30}$.

10-8

$H(\omega) = j\,B(\omega)$





From the assumption: $S_{xx}(\omega) = S_{yy}(\omega) \qquad S_{xy}(-\omega) = -S_{xy}(\omega)$

From (9-148): $S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2 \qquad S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega)$

Combining, we obtain

$$|H(\omega)|^2 = 1 \qquad H(-\omega) = -H(\omega)$$

Since h(t) is real, the second equation yields $H(\omega) = jB(\omega)$ and from the first it follows that

$$|B(\omega)| = 1$$

as in the figure.

10-9 With $\underset{\sim}{i}(t) = \underset{\sim}{a}(t)$, $\underset{\sim}{q}(t) = \underset{\sim}{b}(t)$, (11-63) yields

$$S_i(\omega) = S_q(\omega) \qquad S_{iq}(\omega) = -S_{qi}(\omega) = S_{qi}(-\omega)$$

Hence [see (11-75) and (11-82)],

$$S_w(\omega) = 2\,S_i(\omega) + 2j\,S_{qi}(\omega)$$

$$S_w(-\omega) = 2\,S_i(\omega) - 2j\,S_{qi}(\omega)$$

Adding and subtracting, we obtain

$$4\,S_i(\omega) = S_w(\omega) + S_w(-\omega) \qquad 4j\,S_{iq}(\omega) = S_w(-\omega) - S_w(\omega)$$

10-10 From (10-133)

$$\underset{\sim}{x}(t) = \underset{\sim}{Re}\,[\underset{\sim}{w}(t)e^{j\omega_0 t}]$$

$$\underset{\sim}{x}(t-\tau) = \underset{\sim}{Re}[\underset{\sim}{w}_\tau(t)e^{j\omega_0 t}] = \underset{\sim}{Re}[\underset{\sim}{w}(t-\tau)e^{j\omega_0(t-\tau)}]$$

$$\underset{\sim}{w}_\tau(t) = \underset{\sim}{w}(t-\tau)e^{-j\omega_0\tau}$$

10-11 $R_x''(\tau) \longleftrightarrow -\omega^2 S_x(\omega)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 S_x(\omega)d\omega = -R_x''(0)$$

and with ω_0 the optimum carrier frequency,(10-150) yields

$$E\{|\underset{\sim}{w}'(t)|^2\} = \frac{M}{2\pi} = -2R_x''(0) - 2\omega_0^2 R_x(0)$$

10-12 From the stationarity of the process $\underset{\sim}{x}(t)\cos\omega t + \underset{\sim}{y}(t)\sin\omega t$ it follows that [see (10-130)]

$$C_{xx}(\tau) = C_{yy}(\tau) \qquad C_{xy} = -C_{yx}(\tau) \tag{i}$$

Using these identities, we shall express the joint density f(X,Y) of the 2n RVs

$$\underset{\sim}{X} = [\underset{\sim}{x}(t_1), \ldots, \underset{\sim}{x}(t_n)] \qquad \underset{\sim}{Y} = [\underset{\sim}{y}(t_1), \ldots, \underset{\sim}{y}(t_n)]$$

in terms of the covariance matrix C_{ZZ} of the complex vector $\underset{\sim}{Z} = \underset{\sim}{X} + j\underset{\sim}{Y}$. From (i) it follows that

$$E\{x(t_i)\underset{\sim}{x}(t_j)\} = E\{\underset{\sim}{y}(t_i)\underset{\sim}{y}(t_j)\} \qquad E\{\underset{\sim}{x}(t_i)\underset{\sim}{y}(t_j)\} = -E\{\underset{\sim}{y}(t_i)\underset{\sim}{x}(t_j)\}$$

This yields

$C_{XX} = C_{YY}$, and $C_{XY} = -C_{YX}$; hence, f(X,Y) is given by (8-62).

10-13 The signal $\underset{\sim}{c}(t) = f(t)$ is an extreme case of a cyclostationary process as in (10-178) with

$$h(t) = \begin{cases} f(t) & 0 \le t < T \\ 0 & \text{otherwise} \end{cases} \quad \longleftrightarrow \quad H(\omega) = \int_0^T f(t)e^{-j\omega t}dt$$

and $\underset{\sim}{c}_m = 1$, $R[m] = 1$. Hence [see (10A-2)]

$$\sum_{m=-\infty}^{\infty} R_m e^{-jm\omega T} = \sum_{m=-\infty}^{\infty} e^{-jm\omega T} = T \sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T} m)$$

From the above and (10-180) it follows that the process $\underset{\sim}{x}(t) = f(t - \underset{\sim}{\theta})$ is stationary with power spectrum

$$S(\omega) = \left| \int_0^T f(t)e^{-j\omega t}dt \right|^2 \sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T} m)$$

10-14

The process

$$\underline{y}_N(t) = \underline{x}(t+\tau) - \sum_{n=-N}^{N} \underline{x}(t+nT)\,\frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}$$

is the output of a system with input $\underline{x}(t)$ and system function

$$H_N(\omega) = e^{j\omega\tau} - \sum_{n=-N}^{N} \frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}\, e^{jnT\omega}$$

Furthermore, $\underline{\varepsilon}_N(\tau) = \underline{y}_N(0)$, hence [see (9-153)]

$$E\{\underline{\varepsilon}_N^2(\tau)\} = E\{\underline{y}^2(0)\} = \frac{1}{2\pi}\int_{\infty}^{\infty} S(\omega)\,|H_N(\omega)|^2 d\omega \qquad \text{(i)}$$

The function $H_N(\omega)$ is the truncation error in the Fourier series expansion of $e^{j\omega\tau}$ in the interval $(-\sigma,\sigma)$. Hence, for $N > N_0$

$$|H_N(\omega)| < \varepsilon \qquad\qquad |\omega| < \sigma$$

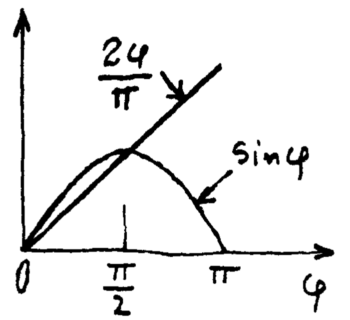
From this and (i) it follows that, if $S(\omega) = 0$ for $|\omega| < \sigma$, then

$$E\{\underline{\varepsilon}_N^2(\tau)\} = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\,|H_N(\omega)|^2 d\omega < \varepsilon\, R(0) \qquad N > N_0$$

10-15 [see after (10-195)]

$$R(0) - R(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)(1-\cos\omega\tau)\,d\omega$$

$$\le \frac{\tau^2}{4\pi}\int_{-\sigma}^{\sigma} \omega^2 S(\omega)\,d\omega = \frac{-\tau^2}{2} R''(0)$$



Furthermore, since

$$\sin\phi \ge \frac{2\phi}{\pi} \qquad 0 \le \phi \le \frac{\pi}{2}$$

we obtain

$$R(0) - R(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\, 2\sin^2\frac{\omega\tau}{2}\, d\omega$$

$$\ge \frac{2\tau^2}{\pi^2}\frac{1}{2\pi}\int_{-\sigma}^{\sigma} \omega^2 S(\omega)\,d\omega = \frac{-2\tau^2}{\pi^2} R''(0)$$

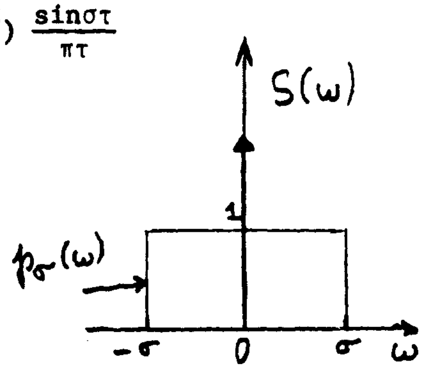
10-16 With $T = \pi/\sigma$

$$R(mT) = E\{\mathbf{x}(nT+mT)\mathbf{x}(nT)\} = \begin{cases} I & m = 0 \\ \eta^2 & m \ne 0 \end{cases}$$

Hence [see (10-196)]

$$R(\tau) = \sum_{m=-\infty}^{\infty} R(mT)\frac{\sin\sigma(\tau-mT)}{\sigma(\tau-mT)} = \eta^2 + (I-\eta^2)\frac{\sin\sigma\tau}{\pi\tau}$$

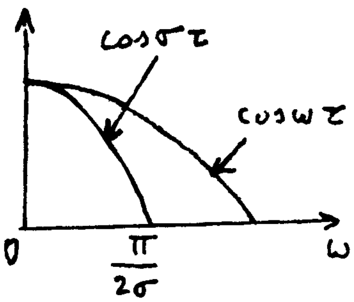
$$S(\omega) = 2\pi\eta^2\delta(\omega) + 2\pi(I-\eta^2)p_\sigma(\omega)$$



10-17 Given $E\{\underset{\sim}{x}(n+m)\underset{\sim}{x}(n)\} = N\delta[m]$

This is a special case of Prob. 10-16 with $\eta = 0$, $I = N$.

10-18 If $|\tau| < \pi/2\sigma$, then



$$\cos \omega\tau \geq \cos \sigma\tau \qquad |\omega| \leq \sigma$$

$$R(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\cos\omega\tau\, d\omega$$

$$\geq \frac{\cos\sigma\tau}{2\pi}\int_{-\sigma}^{\sigma} S(\omega)\, d\omega = R(0)\cos\sigma\tau$$

10-19 From (10-133) with $c = \sigma$

$$P_1(\omega,\tau) + j\omega\, P_2(\omega,\tau) = 1$$

$$P_1(\omega,\tau) + j(\omega+\tau)P_2(\omega,\tau) = e^{j\sigma\tau}$$

Hence,

$$P_1(\omega,\tau) = 1 - \frac{\omega}{\sigma}(e^{j\sigma\tau} - 1) \qquad P_2(\omega,\tau) = \frac{1}{j\sigma}(e^{j\sigma\tau} - 1)$$

Inserting into (11-141), we obtain

$$p_1(\tau) = \frac{4\sin^2(\sigma\tau/2)}{\sigma^2\tau^2} \qquad p_2(\tau) = \frac{4\sin^2(\sigma\tau/2)}{\sigma^2\tau}$$

and with $t = 0$, the desired result follows from (10-206) because $\bar{T} = 2T$ and

$$\sin^2\frac{\sigma(\tau-2nT)}{2} = \sin^2\left(\frac{\sigma\tau}{2} - n\pi\right) = \sin^2\frac{\sigma\tau}{2}$$

10-20 As in (10-213)

$$\underset{\sim}{P}(\omega) = \frac{1}{\lambda} \int_{-a}^{a} \cos\omega t \; \underset{\sim}{z}(t) \cos\omega_c t \, dt$$

$$E\{\underset{\sim}{P}(\omega)\} = \int_{-a}^{a} \cos\omega t \cos\omega_c t \, dt$$

$$\sigma^2_{P(\omega)} = \frac{1}{\lambda} \int_{-a}^{a} \cos^2\omega_c t_2 \cos^2 \omega t_2 \, dt_2$$

10-21 We shall show that if

$$\underset{\sim}{X}_c(\omega) = \frac{1}{\lambda} \sum_{|t_i|<c} \underset{\sim}{x}(t_i) e^{-j\omega t_i} = \frac{1}{\lambda} \int_{-a}^{a} \underset{\sim}{x}(t) \underset{\sim}{z}(t) e^{-j\omega t} dt$$

where $\underset{\sim}{z}(t) = \sum \delta(t-\underset{\sim}{t}_i)$ is a Poisson impulse train, then

$$E\{|\underset{\sim}{X}_c(\omega)|^2\} \simeq 2cS_x(\omega) + \frac{2c}{\lambda} R_x(0)$$

<u>Proof</u>

Since $R_z(\tau) = \lambda^2 + \lambda\delta(\tau)$, it follows that

$$E\left\{|\underset{\sim}{X}_c(\omega)|^2\right\} = \frac{1}{\lambda^2} \int_{-c}^{c} \int_{-c}^{c} R_x(t_1-t_2) e^{-j\omega(t_1-t_2)} dt_1 dt_2$$

$$= \int_{-c}^{c} e^{j\omega t_2} \int_{-c}^{c} R_x(t_1-t_2) e^{-j\omega t_1} dt_1 dt_2 + \frac{1}{\lambda} \int_{-c}^{c} R_x(0) dt_2$$

If $\int_{-\infty}^{\infty} |R_x(\tau)| < \infty$ then for sufficient large c, the inner integral on the right is nearly equal to $S_x(\omega)^{-j\omega t_2}$ and (i) follows.

10-22 $E\{\underset{\sim}{z}(t)\} = g(t) \qquad E\{\underset{\sim}{w}(t)\} = g(t) - g(T)t/T = g(t)$

$$\underset{\sim}{w}(t) = (1 - \frac{t}{T}) \int_0^t \underset{\sim}{x}(\alpha)d\alpha - \frac{t}{T}\int_t^T \underset{\sim}{x}(\alpha)d\alpha$$

The above two integrals are uncorrelated because $\underset{\sim}{n}(t)$ is white noise. Hence, as in Example 9-5

$$\sigma_w^2 = (1 - \frac{t}{T})^2 Nt + \frac{t^2}{T^2} N(T - t) = Nt(1 - \frac{t}{T})$$

Note The above shows that the information that $g(T) = 0$ can be used to improve the estimate of $g(t)$. Indeed, if we use $\underset{\sim}{w}(t)$ instead of $\underset{\sim}{z}(t)$ for the estimate of $g(t)$ in terms of the data $\underset{\sim}{x}(t)$, the variance is reduced from Nt to $Nt(1 - t/T)$.

10-23 (a) Since $|\sum_i a_i b_i| \le \sum |a_i||b_i|$, it suffices to assume that the numbers a_i and b_i are real. The quadratic

$$I(z) = \sum_i (a_i - z\,b_i)^2 = z^2 \sum_i b_i^2 - 2z \sum_i a_i b_i + \sum_i a_i^2$$

is nonnegative for every real z, hence, its discriminant cannot be positive. This yields (i).

(b) With $f[n]$ and $R_\nu[m] = S_0\delta[m]$ as in Prob. 10-24a (white noise)

$$y_f[n_0] = \sum h[n] f[n_0 - n] \qquad \underset{\sim}{y}_\nu[n] = \sum h[n] \underset{\sim}{\nu}[n]$$

$$E\{\underset{\sim}{y}_\nu^2[n]\} = S_0\,\rho[0] = S_0 \sum |h[n]|^2$$

[see (9-213)] And (i) yields

$$\frac{y_f^2[n_0]}{E\{\underset{\sim}{y}_\nu^2[n]\}} = \frac{|\sum h[n] f[n_0 - n]|^2}{S_0 \sum h^2[n]} \le \frac{1}{S_0} \sum |h[n]|^2$$

with equality iff $h[n] = kf^*[n_0 - n]$.

10-24 (a) Given F(z) and $S_\nu(\omega) = S_0 \equiv$ constant . The z transform of $y_f[n]$ equals F(z)H(z). Hence, [see (9-109)]

$$y_f[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega T})H(e^{j\omega T})e^{jn\omega T}d\omega$$

$$\frac{y_f^2[n]}{E\{y_\nu^2[n]\}} = \frac{\left|\int_{-\pi}^{\pi} F(e^{j\omega T})H(e^{j\omega T})d\omega\right|^2}{S_0\int_{-\pi}^{\pi}\left|H(e^{j\omega T})\right|^2 d\omega}$$

$$\le \frac{1}{S_0}\int_{\pi}^{\pi}\left|F(e^{j\omega T})\right|^2 d\omega$$

The last inequality follows from Schwarz's inequality with equality iff

$$H(e^{j\omega T}) = kF^*(e^{j\omega T}) = kF(e^{-j\omega T}),\ \text{i.e., iff } H(z) = kF(z^{-1})$$

(b) Given arbitrary $R_\nu[m]$, F(z), and the form of H(z) (FIR); to find the coefficients a_m of H(z). In this case

$$y_f[n] = a_0 f[n] + a_1 f[n-1] + \cdots + a_N f[n-N]$$

$$y_\nu[n] = a_0 \nu[n] + a_1 \nu[n-1] + \cdots + a_N \nu[n-N]$$

To maximize the signal-to-noise ratio it suffices to minimize

$$E\{y_\nu^2[n]\} = \sum_{k,r=0}^{N} a_k a_r R_\nu[k-r]$$

subject to the constraint that the sum

$$y_f[0] = a_0 f[0] + a_1 f[-1] + \cdots + a_N f[-N]$$

is constant. With λ a constant (Lagrange multiplier), we minimize the sum

$$I = \sum_{k,r=0}^{N} a_k a_r R\,[k-r] - \lambda\left[\sum_{k=0}^{N} a_k f[-k] - y_f[0]\right]$$

this yields the system

$$\frac{\partial I}{\partial a_k} = 0 = \sum_{r=0}^{N}\left[a_r R_\nu[k-r] - \lambda f[-k]\right] \qquad k = 0,\ldots,N$$

whose solution yields a_k.

10-25 $B = A|H(\omega_0)| = \dfrac{A}{\sqrt{\alpha^2+\omega_0^2}}$ $\qquad S_{y_n}(\omega) = \dfrac{N}{\alpha^2+\omega^2}$

$$R_{y_n}(\tau) = \frac{N}{2\alpha}\; e^{-\alpha|\tau|} \qquad E\{\underset{\sim}{y}_n^2(t)\} = R_{y_n}(0) = \frac{N}{2\alpha}$$

$$\frac{B^2}{E\{\underset{\sim}{y}_n^2(t)\}} = \frac{2A^2}{N}\,\frac{\alpha}{\alpha^2+\omega_0^2}$$

Max. if $\alpha = \omega_0$

10-26 Since $H(\omega)$ is determined within a constant factor, we can sssume that the response $y_f(t_o)$ of the optimum $H(\omega)$ due to f(t) is constant:

$$y_f(t_o) = \sum_{i=0}^{m} a_i\, f(t_o - iT) = c \qquad \text{(i)}$$

Our problem is to minimize the variance

$$V = E\{\underset{\sim}{y}_\nu^2(t)\} = \sum_{n=0}^{m} a_n \sum_{i=0}^{m} a_i\, R(nT - iT) \qquad \text{(ii)}$$

of $\underset{\sim}{y}_\nu(t)$ subject to the constraint (i). This yields the system

$$\frac{\partial V}{\partial a_n} = \sum_{i=0}^{m} a_i\, R(nT-iT) - kf(t_o - nT) = 0$$

where k is a constant (lagrange multiplier). With a_n so determined, we conclude from (ii) that

$$V = \sum_{n=0}^{m} ka_n\, f(t_o - nT) = ky_f(t_o) \qquad r^2 = \frac{y_f^2(t_o)}{ky_f(t_o)}$$

10-27 $R_{yyy}(\mu,\nu) = E\{\underset{\sim}{x}(t+\mu)+c[\,[\underset{\sim}{x}(t+\nu)+c]\,[\underset{\sim}{x}(t)+c]\} = R(\mu,\nu) + cR(\mu) + cR(\nu) + cR(\mu-\nu) + c^3$

because $E\{\underset{\sim}{x}(t)\} = 0$. Furthermore,

$$R(\mu) \leftrightarrow 2\pi S(u)\delta(v) \qquad R(\nu) = 2\pi\delta(u)S(v) \qquad c^3 \leftrightarrow 4\pi^2\delta(u)\delta(v)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(\mu-\nu)e^{-j(u\mu+v\nu)}d\mu d\nu = \int_{-\infty}^{\infty} R(\tau)e^{-ju\tau}d\tau \int_{-\infty}^{\infty} e^{-j(u+v)\nu}d\nu = 2\pi S(u)\delta(u+v)$$

10-28 We shall use the equations $E\{\tilde{x}(t)\} = 0$, $E\{\tilde{x}^2(t)\} = \lambda t$. Suppose that $t_1 < t_2 < t_3$. Clearly,

$$\tilde{x}(t_2) = \tilde{x}(t_1) + [\tilde{x}(t_2) - \tilde{x}(t_1)]$$

$$\tilde{x}(t_3) = \tilde{x}(t_1) + [\tilde{x}(t_2) - \tilde{x}(t_1)] + [\tilde{x}(t_3) - \tilde{x}(t_2)] \qquad \text{(i)}$$

Inserting into the product $\tilde{x}(t_1)\tilde{x}(t_2)\tilde{x}(t_3)$ and using the identity $E\{\tilde{x}(t_i) - \tilde{x}(t_j)\} = 0$ and the independence of the three terms on the right of (i), we obtain

$$E\{x(t_1)x(t_2)x(t_3)\} = E\{x^3(t_1)\} = \lambda t_1 = \lambda \min(t_1, t_2, t_3)$$

Since $\tilde{z}(t) = \tilde{x}'(t)$, we conclude from (9-120)-(9-122) that

$$R_{\tilde{z}\tilde{z}\tilde{z}}(t_1,t_2,t_3) = \frac{\partial^3 R_{xxx}(t_1,t_2,t_3)}{\partial t_1 \partial t_2 \partial t_3} = \lambda \frac{\partial^3 \min(t_1,t_2,t_3)}{\partial t_1 \partial t_2 \partial t_3}$$

It suffices therefore to show that the right side equals $\lambda\delta(t_1-t_2)\delta(t_1-t_2)$. This is a consequence of the following:

$$\frac{\partial \min(t_1,t_2,t_3)}{\partial t_3} = t_1 U(t_2-t_1)\delta(t_3-t_1) + t_2\, U(t_1-t_2)\,\delta(t_3-t_2)$$

$$+ U(t_1-t_3)U(t_2-t_3) - t_3\delta(t_1-t_3)U(t_2-t_3) - t_3 U(t_1-t_3)\delta(t_2-t_3)$$

$$= U(t_1-t_3)U(t_2-t_3)$$

because $t_i\delta(t_i-t_j) = t_j\delta(t_j-t_i)$. Hence,

$$\frac{\partial^2 \min(t_1,t_2,t_3)}{\partial t_2 \partial t_3} = U(t_1-t_3)\delta(t_2-t_3) \qquad \frac{\partial^2 \min(t_1,t_2,t_3)}{\partial t_1 \delta t_2 \partial t_3} = \delta(t_1-t_2)\delta(t_1-t_3)$$

10-29 See outline given in text.