CHAPTER 11

11-1 $$S_x(z) = \frac{5-2(z+1/z)}{10-3(z+1/z)} = \frac{2}{3} + \frac{5/9}{10/3 - (z+1/z)}$$

$$R[m] = \frac{2}{3} + \frac{5}{18}\,3^{-|m|} \qquad \Gamma(z) = \frac{3z-1}{2z-1}$$

11-2 $$S_x(s) = \frac{s^4+64}{s^4-10s^2+9} = \frac{s^2+4s+8}{s^2+4s+3}\;\frac{s^2-4s+8}{s^2-4s+3}$$

$$L(s) = \frac{s^2+4s+8}{s^2+4s+3}$$

11-3 First proof

$$\underline{s}[n] = \sum_{k=0}^{\infty} \ell[n]\underline{i}[n-k] \qquad E\{\underline{x}^2[n]\} = \sum_{k=0}^{\infty} \ell^2[k]$$

Second proof

$$S(z) = L(z)L(1/z) \qquad R[m] = \ell[m] * \ell[-m] = \sum_{k=0}^{\infty} \ell[k]\ell[k-m]$$

$$R[0] = \sum_{k=0}^{\infty} \ell^2[k]$$

11-4 (a) This is a special case of (11-22) and (11-23).

(b) From (a) it follows that

$$R''_{yx}(\tau) + 3\,R'_{yx}(\tau) + 2\,R_{yx}(\tau) = q\delta(\tau)$$

Since $R_{xx}(\tau) = 0$ for $\tau < 0$, the above shows that

$$R_{yx}(\tau) = 0 \text{ for } \tau \leq 0^- \qquad R'_{yx}(0^-) = 0$$

Furthermore,

$$S_{yx}(s) = \frac{q}{s^2 + 3s + 2}$$

hence (initial value theorem)

$$R_{yx}(0^+) = \lim_{s\to\infty} s\ S_{yx}(s) = 0 \qquad R'_{yx}(0^+) = \lim_{s\to\infty} s^2\ S_{yx}(s) = q$$

Similarly,

$$R''_{yy}(\tau) + 3\,R'_{yy}(\tau) + 2\,R_{yy}(\tau) = R_{xy}(\tau) = R_{yx}(-\tau) = 0 \quad \text{for } \tau > 0$$

$$S_{yy}(s) = \frac{q}{(s^2 + 3s + 2)(s^2 - 3s + 2)} = \frac{qs/12 + q/4}{s^2 + 3s + 2} + \frac{-qs/12 + q/4}{s^2 - 3s + 2}$$

$$S^+_{yy}(s) = \frac{qs/12 + q/4}{s^2 + 3s + 2}$$

$$R^+_{yy}(0^+) = R_{yy}(0) = \lim_{s\to\infty} s^2\ S^+_{yy}(s) = \frac{q}{12}$$

$$R'_{yy}(0) = \lim_{s\to\infty} s\ [s\ S^+_{yy}(s) - \frac{q}{12}] = 0$$

11-5 $$S_x(z) = S_s(z) + S_\nu(z) = \frac{1}{D(z)} + q = \frac{1 + qD(z)}{D(z)}$$

If $R_s[m] = 2^{-|m|}$ and $S_\nu(z) = 5$, then (see Example 9-31)

$$S_s(z) = \frac{1.5}{2.5 - (z^{-1} + z)} \qquad S_x(z) = \frac{5 - 14z^{-1} + 5z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$$

11-6 The process

$$\underset{\sim}{y}[n] = \frac{1}{n}\sum_{k=1}^{n} \underset{\sim}{x}(nT + kT)$$

is the output of a system with input $\underset{\sim}{x}[n]$ and system function

$$H(z) = \frac{1}{n}\sum_{k=1}^{n} z^k$$

Furthermore, $\underset{\sim}{s} = \underset{\sim}{y}[0]$ and

$$n^2\left|H(e^{j\omega T})\right|^2 = \left|\sum_{k=1}^{n} e^{jk\omega T}\right|^2$$

$$= \left|\frac{e^{j\omega T} - e^{j(n+1)\omega T}}{1 - e^{j\omega T}}\right|^2 = \frac{\sin^2 n\omega T/2}{\sin^2 \omega T/2}$$

Hence [see (9-51)]

$$E\{\underset{\sim}{s}^2\} = R_y[0] = \frac{1}{2\pi n^2}\int_{-\infty}^{\infty} S_x(\omega)\,\frac{\sin^2 n\omega T/2}{\sin^2 \omega T/2}\,d\omega$$

11-7 Since $R(t_1,t_2) = e^{-c|t_1-t_2|}$, (12-58) yields

$$\int_{-a}^{t_1} e^{-c(t_1-t_2)}\phi(t_2)dt_2 + \int_{t_1}^{a} e^{c(t_1-t_2)}\phi(t_2)dt_2 = \lambda\phi(t_1) \qquad \text{(i)}$$

Differentiating twice and using (i) we obtain (omitting details)

$$\lambda\phi''(t) + (2c - \lambda c^2)\phi(t) = 0$$

Hence,

$$\phi(t) = \beta\cos\omega t \quad \text{and} \quad \phi(t) = \beta'\cos\omega' t$$

To determine ω, we insert into (i). This yields

$$\frac{2c}{c^2+\omega^2} + \frac{\omega\sin a\omega - c\cos a\omega}{c^2+\omega^2}\; e^{-ac}\,(e^{ct} + e^{-ct}) = 2c\,\lambda\cos\omega t$$

This yields

$$\omega_n \sin a\omega_n - c \cos a\omega_n = 0 \qquad \lambda_n = \frac{2c}{c^2 + \omega_n^2}$$

The constants β_n are determined from (normalization)

$$1 = \int_{-a}^{a} \beta_n^2 \cos^2 \omega_n t \, dt \qquad \beta_n^2 = \frac{1}{a + c\lambda_n}$$

Similarly for $\beta_n' \sin \omega_n' t$.

11-8 As in (9-60)

$$E\{|X_T(\omega)|^2\} = \int_{-T/2}^{T/2} R(t_1 - t_2) e^{-j\omega(t_1 - t_2)} dt_1 dt_2$$

$$= \int_{-T}^{T} (T - |\tau|) R(\tau) \, e^{-j\omega\tau} d\tau$$

Differentiating with respect to T and using the fact that if

$$\phi(t) = \int_{-t}^{t} f(x;t) dx$$

then

$$\frac{d\phi(t)}{dt} = f(t;t) - f(-t,t) + \int_{-t}^{t} \frac{\partial f}{\partial t}(x,t) dx$$

we obtain

$$\frac{\partial E\{|X_T(\omega)|^2\}}{\partial T} = \int_{-T}^{T} R(\tau) e^{-j\omega\tau} d\tau = E\{\frac{\partial}{\partial T} |X_T(\omega)|^2\}$$

The above approaches $S(\omega)$ as $T \to \infty$.

11-9 $$E\{\underset{\sim}{X}(\omega)\} = \int_{-a}^{a} 5\cos 3t\, e^{-j\omega t} dt = \frac{5\sin a(\omega-3)}{\omega-3} + \frac{5\sin a(\omega+3)}{\omega+3}$$

Var. $\underset{\sim}{X}(\omega) = 2qa = 4a.$

11-10 $$E\{\underset{\sim}{X}(u)\underset{\sim}{X}(v)\} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sigma_n^2 \delta[n-k]\ e^{-j(nu-kv)T}$$

$$= \sum_{n=-\infty}^{\infty} \sigma_n^2\ e^{-jn(u-v)T}$$

11-11 Shifting the origin, we set

$$\underset{\sim}{c}_n = \frac{1}{T}\int_{-T/2}^{T/2} \underset{\sim}{x}(t)e^{-jn\omega_o t}dt \qquad \beta_n(\alpha) = \frac{1}{T}\int_{-T/2}^{T/2} R(\tau-\alpha)e^{-jn\omega_o\tau}d\tau$$

(a) We shall show that if

$$\hat{\underset{\sim}{x}}(t) = \sum_{n=-\infty}^{\infty} \underset{\sim}{c}_n e^{jn\omega_o t} \text{ then } E\{|\underset{\sim}{x}(t)-\hat{\underset{\sim}{x}}(t)|^2\} = 0 \text{ for } |t| < T/2 \qquad \text{(i)}$$

<u>Proof</u> $$E\{\underset{\sim}{c}_n\underset{\sim}{x}^*(\alpha)\} = \frac{1}{T}\int_{-T/2}^{T/2} E\{\underset{\sim}{x}(t)\underset{\sim}{x}^*(\alpha)\}e^{-jn\omega_o t}dt = \beta_n(\alpha)$$

The functions $\beta_n(\alpha)$ are the coefficients of the Fourier expansion of $R(\tau-\alpha)$:

$$R(\tau-\alpha) = \sum_{n=-\infty}^{\infty} \beta_n(\alpha)e^{jn\omega_o\tau} \qquad |\tau| < T/2 \qquad \text{(ii)}$$

Hence

$$E\{\hat{\underset{\sim}{x}}(t)\underset{\sim}{x}^*(t)\} = \sum_{n=-\infty}^{\infty} E\{\underset{\sim}{c}_n\underset{\sim}{x}^*(t)\}^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} \beta_n(t)e^{jn\omega_o t}$$

From (ii) it follows with $\tau = \alpha = t$ that the last sum equals R(0). Similarly, $E\{\hat{x}^*(t)x(t)\} =$ R(0) and (i) results.

(b) $$E\{c_n c_m^*\} = \frac{1}{T}\int_{-T/2}^{T/2} E\{c_n x^*(t)\}e^{jn\omega_o t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \beta_n(t)e^{jn\omega_o t}dt$$

(c) If T is sufficiently large, then

$$T\beta_n(\alpha) = \int_{-T/2}^{T/2} R(\tau-\alpha)e^{-jn\omega_o\tau}d\tau \simeq S(n\omega_o)e^{-jn\omega_o\alpha}$$

$$E\{c_n c_m^*\} = \frac{S(n\omega_o)}{T^2}\int_{-T/2}^{T/2} e^{j(m-n)\omega_o\alpha}d\alpha = \begin{cases} S(n\omega_o)/T & m=n \\ 0 & m\neq n \end{cases}$$

Thus, for large T, the coefficients c_n of an arbitrary WSS process are nearly orthogonal.

11-12 $$E\{x(t_1)x^*(t_2)\} = \frac{1}{4\pi^2} E\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{X(u)X^*(v)\}e^{j(ut_1-vt_2)}dudv\right.$$

$$= \frac{1}{4\pi^2} E\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(u)\delta(u-v)e^{j(ut_1-vt_2)}dudv\right. = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} Q(u)e^{ju(t_1-t_2)}du$$

This depends only on $\tau = t_1 - t_2$:

$$R_{xx}(\tau) = \frac{1}{x_2}\int_{-\infty}^{\infty} Q(u)e^{ju\tau}du \qquad S_{xx}(\omega) = \frac{Q(\omega)}{2\pi}$$

11-13 Equations (11-79) can be written in the following form:

$$E\{A(u)A(v)\} = Q(u)\delta(u-v) = E\{B(u)B(v)\} \qquad E\{A(u)B(v)\} = 0$$

for $u \geq 0$, $v \geq 0$. We shall show that if the above is true and $E\{A(\omega)\} = E\{B(\omega)\} = 0$, then the process

$$x(t) = \frac{1}{\pi}\int_0^{\infty}\left[A(\omega)\cos\omega t - B(\omega)\sin\omega t\right] d\omega$$

is WSS.

<u>Proof</u> Clearly, $E\{x(t)\} = 0$ and

$E\{\underset{\sim}{x}(t+\tau)\underset{\sim}{x}(t)\}$

$$= \frac{1}{\pi^2}\int_0^\infty\int_0^\infty E\{\underset{\sim}{A}(u)\cos u(t+\tau) - \underset{\sim}{B}(u)\sin u(t+\tau)\}\,[\underset{\sim}{A}(v)\cos vt - B(v)\sin vt]\,du\,dv$$

$$= \frac{1}{\pi}\int_0^\infty\int_0^\infty Q(u)\delta(u-v)\,[\cos u(t+\tau)\cos vt + \sin u(t+\tau)\sin v(t)\,du\,dv\,]\,dv\,du$$

$$= \frac{1}{\pi^2}\int_0^\infty Q(u)\,[\cos u(t+\tau)\cos u\,t + \sin u(t+\tau)\sin ut]\,du$$

$$= \frac{1}{\pi^2}\int_0^\infty Q(u)\cos u\tau\,du$$

From this and (9-136) it follows that $\underset{\sim}{x}(t)$ is WSS with $S_{xx}(\omega) = Q(\omega)/\pi$.

11-14 $E\{\underset{\sim}{v}(t)\} = 0$ $E\{\underset{\sim}{X}_T(\omega)\} = \int_{-T}^{T} f(t)e^{-j\omega t}dt$

The above integral is the transform of the product $f(t)p_T(t)$, hence (frequency convolution theorem), it equals $F(\omega)*\sin T\omega/\pi\omega$.

$$\text{Var}\,\underset{\sim}{X}_T(\omega) = E\left\{\,\left|\int_{-T}^{T}\underset{\sim}{\nu}(t)e^{-j\omega t}dt\right|^2\right\}$$

The integral is the transform of the nonstationary white noise $\underset{\sim}{\nu}(t)p_T(t)$. The autocorrelation of this process equals $q(t_1)\delta(t_1-t_2)$ where $q(t) = qp_T(t)$. Hence, [see (11-69)]

$$\text{Var}\,\underset{\sim}{X}_T(\omega) = Q(0) = \int_{-T}^{T} q\,dt = 2qT$$
