1. Dynkin systems

Definition 1 *A* **dynkin system** *on a set* Ω *is a subset* $\mathcal{D}$ *of the power set* $\mathcal{P}(\Omega)$*, with the following properties:*

$$\begin{array}{ll} (i) & \Omega \in \mathcal{D} \\ (ii) & A, B \in \mathcal{D}, A \subseteq B \;\Rightarrow\; B \setminus A \in \mathcal{D} \\ (iii) & A_n \in \mathcal{D}, A_n \subseteq A_{n+1}, n \geq 1 \;\Rightarrow\; \displaystyle\bigcup_{n=1}^{+\infty} A_n \in \mathcal{D} \end{array}$$

Definition 2 *A* **σ-algebra** *on a set* Ω *is a subset* $\mathcal{F}$ *of the power set* $\mathcal{P}(\Omega)$ *with the following properties:*

$$\begin{array}{ll} (i) & \Omega \in \mathcal{F} \\ (ii) & A \in \mathcal{F} \;\Rightarrow\; A^c \stackrel{\triangle}{=} \Omega \setminus A \in \mathcal{F} \\ (iii) & A_n \in \mathcal{F}, n \geq 1 \;\Rightarrow\; \displaystyle\bigcup_{n=1}^{+\infty} A_n \in \mathcal{F} \end{array}$$

Exercise 1. Let $\mathcal{F}$ be a σ-algebra on Ω. Show that $\emptyset \in \mathcal{F}$, that if $A, B \in \mathcal{F}$ then $A \cup B \in \mathcal{F}$ and also $A \cap B \in \mathcal{F}$. Recall that $B \setminus A = B \cap A^c$ and conclude that $\mathcal{F}$ is also a dynkin system on Ω.

Exercise 2. Let $(\mathcal{D}_i)_{i\in I}$ be an arbitrary family of dynkin systems on Ω, with $I \neq \emptyset$. Show that $\mathcal{D} \stackrel{\triangle}{=} \cap_{i\in I} \mathcal{D}_i$ is also a dynkin system on Ω.

Exercise 3. Let $(\mathcal{F}_i)_{i\in I}$ be an arbitrary family of σ-algebras on Ω, with $I \neq \emptyset$. Show that $\mathcal{F} \stackrel{\triangle}{=} \cap_{i\in I} \mathcal{F}_i$ is also a σ-algebra on Ω.

Exercise 4. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Define:

$$D(\mathcal{A}) \stackrel{\triangle}{=} \{\mathcal{D} \text{ dynkin system on } \Omega : \ \mathcal{A} \subseteq \mathcal{D}\}$$

Show that $\mathcal{P}(\Omega)$ is a dynkin system on Ω, and that $D(\mathcal{A})$ is not empty. Define:

$$\mathcal{D}(\mathcal{A}) \stackrel{\triangle}{=} \bigcap_{\mathcal{D}\in D(\mathcal{A})} \mathcal{D}$$

Show that $\mathcal{D}(\mathcal{A})$ is a dynkin system on Ω such that $\mathcal{A} \subseteq \mathcal{D}(\mathcal{A})$, and that it is the smallest dynkin system on Ω with such property, (i.e. if $\mathcal{D}$ is a dynkin system on Ω with $\mathcal{A} \subseteq \mathcal{D}$, then $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}$).

Definition 3 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call* **dynkin system generated** *by $\mathcal{A}$, the dynkin system on Ω, denoted $\mathcal{D}(\mathcal{A})$, equal to the intersection of all dynkin systems on Ω, which contain $\mathcal{A}$.*

EXERCISE 5. Do exactly as before, but replacing dynkin systems by σ-algebras.

Definition 4 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call* **$\sigma$-algebra generated** *by $\mathcal{A}$, the σ-algebra on Ω, denoted $\sigma(\mathcal{A})$, equal to the intersection of all σ-algebras on Ω, which contain $\mathcal{A}$.*

Definition 5 *A subset $\mathcal{A}$ of the power set $\mathcal{P}(\Omega)$ is called a* **π-system** *on Ω, if and only if it is closed under finite intersection, i.e. if it has the property:*

$$A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$$

Exercise 6. Let $\mathcal{A}$ be a π-system on Ω. For all $A \in \mathcal{D}(\mathcal{A})$, we define:

$$\Gamma(A) \stackrel{\triangle}{=} \{B \in \mathcal{D}(\mathcal{A}) : \; A \cap B \in \mathcal{D}(\mathcal{A})\}$$

1. If $A \in \mathcal{A}$, show that $\mathcal{A} \subseteq \Gamma(A)$
2. Show that for all $A \in \mathcal{D}(\mathcal{A})$, $\Gamma(A)$ is a dynkin system on Ω.
3. Show that if $A \in \mathcal{A}$, then $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(A)$.
4. Show that if $B \in \mathcal{D}(\mathcal{A})$, then $\mathcal{A} \subseteq \Gamma(B)$.
5. Show that for all $B \in \mathcal{D}(\mathcal{A})$, $\mathcal{D}(\mathcal{A}) \subseteq \Gamma(B)$.
6. Conclude that $\mathcal{D}(\mathcal{A})$ is also a π-system on Ω.

Exercise 7. Let $\mathcal{D}$ be a dynkin system on Ω which is also a π-system.

1. Show that if $A, B \in \mathcal{D}$ then $A \cup B \in \mathcal{D}$.

2. Let $A_n \in \mathcal{D}, n \geq 1$. Consider $B_n \stackrel{\triangle}{=} \cup_{i=1}^{n} A_i$. Show that $\cup_{n=1}^{+\infty} A_n = \cup_{n=1}^{+\infty} B_n$.

3. Show that $\mathcal{D}$ is a σ-algebra on Ω.

EXERCISE 8. Let $\mathcal{A}$ be a π-system on Ω. Explain why $\mathcal{D}(\mathcal{A})$ is a σ-algebra on Ω, and $\sigma(\mathcal{A})$ is a dynkin system on Ω. Conclude that $\mathcal{D}(\mathcal{A}) = \sigma(\mathcal{A})$. Prove the theorem:

Theorem 1 (dynkin system) *Let $\mathcal{C}$ be a collection of subsets of Ω which is closed under pairwise intersection. If $\mathcal{D}$ is a dynkin system containing $\mathcal{C}$ then $\mathcal{D}$ also contains the σ-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$.*

2. Caratheodory's Extension

In the following, Ω is a set. Whenever a union of sets is denoted $\uplus$ as opposed to $\cup$, it indicates that the sets involved are pairwise disjoint.

Definition 6 *A* **semi-ring** *on* Ω *is a subset* $\mathcal{S}$ *of the power set* $\mathcal{P}(\Omega)$ *with the following properties:*

$$
\begin{array}{rl}
(i) & \emptyset \in \mathcal{S} \\
(ii) & A, B \in \mathcal{S} \ \Rightarrow \ A \cap B \in \mathcal{S} \\
(iii) & A, B \in \mathcal{S} \ \Rightarrow \ \exists n \geq 0,\ \exists A_i \in \mathcal{S} : \ A \setminus B = \biguplus_{i=1}^{n} A_i
\end{array}
$$

The last property (iii) says that whenever $A, B \in \mathcal{S}$, there is $n \geq 0$ and $A_1, \ldots, A_n$ in $\mathcal{S}$ which are pairwise disjoint, such that $A \setminus B = A_1 \uplus \ldots \uplus A_n$. If $n = 0$, it is understood that the corresponding union is equal to $\emptyset$, (in which case $A \subseteq B$).

Definition 7 *A* **ring** *on Ω is a subset $\mathcal{R}$ of the power set $\mathcal{P}(\Omega)$ with the following properties:*

$$\begin{array}{rl} (i) & \emptyset \in \mathcal{R} \\ (ii) & A, B \in \mathcal{R} \ \Rightarrow \ A \cup B \in \mathcal{R} \\ (iii) & A, B \in \mathcal{R} \ \Rightarrow \ A \setminus B \in \mathcal{R} \end{array}$$

Exercise 1. Show that $A \cap B = A \setminus (A \setminus B)$ and therefore that a ring is closed under pairwise intersection.

Exercise 2.Show that a ring on Ω is also a semi-ring on Ω.

Exercise 3.Suppose that a set Ω can be decomposed as $\Omega = A_1 \uplus A_2 \uplus A_3$ where A_1, A_2 and A_3 are distinct from $\emptyset$ and Ω. Define $\mathcal{S}_1 \stackrel{\triangle}{=} \{\emptyset, A_1, A_2, A_3, \Omega\}$ and $\mathcal{S}_2 \stackrel{\triangle}{=} \{\emptyset, A_1, A_2 \uplus A_3, \Omega\}$. Show that $\mathcal{S}_1$ and $\mathcal{S}_2$ are semi-rings on Ω, but that $\mathcal{S}_1 \cap \mathcal{S}_2$ fails to be a semi-ring on Ω.

Exercise 4. Let $(\mathcal{R}_i)_{i \in I}$ be an arbitrary family of rings on Ω, with $I \neq \emptyset$. Show that $\mathcal{R} \stackrel{\triangle}{=} \cap_{i \in I} \mathcal{R}_i$ is also a ring on Ω.

EXERCISE 5. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Define:

$$R(\mathcal{A}) \stackrel{\triangle}{=} \{\mathcal{R} \text{ ring on } \Omega : \ \mathcal{A} \subseteq \mathcal{R}\}$$

Show that $\mathcal{P}(\Omega)$ is a ring on Ω, and that $R(\mathcal{A})$ is not empty. Define:

$$\mathcal{R}(\mathcal{A}) \stackrel{\triangle}{=} \bigcap_{\mathcal{R} \in R(\mathcal{A})} \mathcal{R}$$

Show that $\mathcal{R}(\mathcal{A})$ is a ring on Ω such that $\mathcal{A} \subseteq \mathcal{R}(\mathcal{A})$, and that it is the smallest ring on Ω with such property, (i.e. if $\mathcal{R}$ is a ring on Ω and $\mathcal{A} \subseteq \mathcal{R}$ then $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{R}$).

Definition 8 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$. We call* **ring generated** *by $\mathcal{A}$, the ring on Ω, denoted $\mathcal{R}(\mathcal{A})$, equal to the intersection of all rings on Ω, which contain $\mathcal{A}$.*

EXERCISE 6. Let $\mathcal{S}$ be a semi-ring on Ω. Define the set $\mathcal{R}$ of all finite unions of pairwise disjoint elements of $\mathcal{S}$, i.e.

$$\mathcal{R} \stackrel{\triangle}{=} \{A : \ A = \uplus_{i=1}^{n} A_i \text{ for some } n \geq 0, A_i \in \mathcal{S}\}$$

(where if $n = 0$, the corresponding union is empty, i.e. $\emptyset \in \mathcal{R}$). Let $A = \uplus_{i=1}^{n} A_i$ and $B = \uplus_{j=1}^{p} B_j \in \mathcal{R}$:

1. Show that $A \cap B = \uplus_{i,j}(A_i \cap B_j)$ and that $\mathcal{R}$ is closed under pairwise intersection.

2. Show that if $p \geq 1$ then $A \setminus B = \cap_{j=1}^{p}(\uplus_{i=1}^{n}(A_i \setminus B_j))$.

3. Show that $\mathcal{R}$ is closed under pairwise difference.

4. Show that $A \cup B = (A \setminus B) \uplus B$ and conclude that $\mathcal{R}$ is a ring on Ω.

5. Show that $\mathcal{R}(\mathcal{S}) = \mathcal{R}$.

Exercise 7. Everything being as before, define:

$$\mathcal{R}' \stackrel{\triangle}{=} \{A : \ A = \cup_{i=1}^{n} A_i \text{ for some } n \geq 0, A_i \in \mathcal{S}\}$$

(We do not require the sets involved in the union to be pairwise disjoint). Using the fact that $\mathcal{R}$ is closed under finite union, show that $\mathcal{R}' \subseteq \mathcal{R}$, and conclude that $\mathcal{R}' = \mathcal{R} = \mathcal{R}(\mathcal{S})$.

Definition 9 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ with $\emptyset \in \mathcal{A}$. We call* **measure** *on $\mathcal{A}$, any map $\mu : \mathcal{A} \to [0, +\infty]$ with the following properties:*

$$(i) \qquad \mu(\emptyset) = 0$$

$$(ii) \qquad A \in \mathcal{A}, A_n \in \mathcal{A} \text{ and } A = \biguplus_{n=1}^{+\infty} A_n \ \Rightarrow \ \mu(A) = \sum_{n=1}^{+\infty} \mu(A_n)$$

The $\uplus$ indicates that we assume the A_n's to be pairwise disjoint in the l.h.s. of (ii). It is customary to say in view of condition (ii) that a measure is *countably additive*.

Exercise 8. If $\mathcal{A}$ is a σ-algebra on Ω explain why property (ii) can be replaced by:

$$(ii)'\ A_n \in \mathcal{A} \text{ and } A = \biguplus_{n=1}^{+\infty} A_n \ \Rightarrow\ \mu(A) = \sum_{n=1}^{+\infty} \mu(A_n)$$

Exercise 9. Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ with $\emptyset \in \mathcal{A}$ and $\mu : \mathcal{A} \to [0, +\infty]$ be a measure on $\mathcal{A}$.

1. Show that if $A_1, \ldots, A_n \in \mathcal{A}$ are pairwise disjoint and the union $A = \uplus_{i=1}^n A_i$ lies in $\mathcal{A}$, then $\mu(A) = \mu(A_1) + \ldots + \mu(A_n)$.

2. Show that if $A, B \in \mathcal{A}$, $A \subseteq B$ and $B \setminus A \in \mathcal{A}$ then $\mu(A) \leq \mu(B)$.

Exercise 10. Let $\mathcal{S}$ be a semi-ring on Ω, and $\mu : \mathcal{S} \to [0, +\infty]$ be a measure on $\mathcal{S}$. Suppose that there exists an extension of μ on $\mathcal{R}(\mathcal{S})$, i.e. a measure $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ such that $\bar{\mu}_{|\mathcal{S}} = \mu$.

1. Let A be an element of $\mathcal{R}(\mathcal{S})$ with representation $A = \uplus_{i=1}^n A_i$ as a finite union of pairwise disjoint elements of $\mathcal{S}$. Show that $\bar{\mu}(A) = \sum_{i=1}^n \mu(A_i)$

2. Show that if $\bar{\mu}' : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ is another measure with $\bar{\mu}'_{|\mathcal{S}} = \mu$, i.e. another extension of μ on $\mathcal{R}(\mathcal{S})$, then $\bar{\mu}' = \bar{\mu}$.

Exercise 11. Let $\mathcal{S}$ be a semi-ring on Ω and $\mu : \mathcal{S} \to [0, +\infty]$ be a measure. Let A be an element of $\mathcal{R}(\mathcal{S})$ with two representations:

$$A = \biguplus_{i=1}^n A_i = \biguplus_{j=1}^p B_j$$

as a finite union of pairwise disjoint elements of $\mathcal{S}$.

1. For $i = 1, \ldots, n$, show that $\mu(A_i) = \sum_{j=1}^p \mu(A_i \cap B_j)$

2. Show that $\sum_{i=1}^n \mu(A_i) = \sum_{j=1}^p \mu(B_j)$

3. Explain why we can define a map $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ as:

$$\bar{\mu}(A) \stackrel{\triangle}{=} \sum_{i=1}^{n} \mu(A_i)$$

4. Show that $\bar{\mu}(\emptyset) = 0$.

Exercise 12. Everything being as before, suppose that $(A_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{R}(\mathcal{S})$, each A_n having the representation:

$$A_n = \biguplus_{k=1}^{p_n} A_n^k \ , \ n \geq 1$$

as a finite union of disjoint elements of $\mathcal{S}$. Suppose moreover that $A = \uplus_{n=1}^{+\infty} A_n$ is an element of $\mathcal{R}(\mathcal{S})$ with representation $A = \uplus_{j=1}^{p} B_j$, as a finite union of pairwise disjoint elements of $\mathcal{S}$.

1. Show that for $j = 1, \ldots, p$, $B_j = \cup_{n=1}^{+\infty} \cup_{k=1}^{p_n} (A_n^k \cap B_j)$ and explain why B_j is of the form $B_j = \uplus_{m=1}^{+\infty} C_m$ for some sequence $(C_m)_{m\geq 1}$ of pairwise disjoint elements of $\mathcal{S}$.

2. Show that $\mu(B_j) = \sum_{n=1}^{+\infty} \sum_{k=1}^{p_n} \mu(A_n^k \cap B_j)$

3. Show that for $n \geq 1$ and $k = 1, \ldots, p_n$, $A_n^k = \uplus_{j=1}^{p}(A_n^k \cap B_j)$

4. Show that $\mu(A_n^k) = \sum_{j=1}^{p} \mu(A_n^k \cap B_j)$

5. Recall the definition of $\bar{\mu}$ of exercise (11) and show that it is a measure on $\mathcal{R}(\mathcal{S})$.

Exercise 13. Prove the following theorem:

Theorem 2 *Let $\mathcal{S}$ be a semi-ring on Ω. Let $\mu : \mathcal{S} \to [0, +\infty]$ be a measure on $\mathcal{S}$. There exists a unique measure $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$ such that $\bar{\mu}_{|\mathcal{S}} = \mu$.*

Definition 10 *We define an* **outer-measure** *on* Ω *as being any map* $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$ *with the following properties:*

$$\begin{array}{rl} (i) & \mu^*(\emptyset) = 0 \\ (ii) & A \subseteq B \ \Rightarrow \ \mu^*(A) \leq \mu^*(B) \\ (iii) & \mu^*\left(\bigcup_{n=1}^{+\infty} A_n\right) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) \end{array}$$

Exercise 14. Show that $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$, where μ^* is an outer-measure on Ω and $A, B \subseteq \Omega$.

Definition 11 *Let* μ^* *be an outer-measure on* Ω*. We define:*

$$\Sigma(\mu^*) \stackrel{\triangle}{=} \{A \subseteq \Omega : \ \mu^*(T) = \mu^*(T \cap A) + \mu^*(T \cap A^c) \ , \ \forall T \subseteq \Omega\}$$

We call $\Sigma(\mu^*)$ *the* **σ-algebra associated** *with the outer-measure* μ^**.*

Note that the fact that $\Sigma(\mu^*)$ is indeed a σ-algebra on Ω, remains to be proved. This will be your task in the following exercises.

Exercise 15. Let μ^* be an outer-measure on Ω. Let $\Sigma = \Sigma(\mu^*)$ be the σ-algebra associated with μ^*. Let $A, B \in \Sigma$ and $T \subseteq \Omega$

1. Show that $\Omega \in \Sigma$ and $A^c \in \Sigma$.
2. Show that $\mu^*(T \cap A) = \mu^*(T \cap A \cap B) + \mu^*(T \cap A \cap B^c)$
3. Show that $T \cap A^c = T \cap (A \cap B)^c \cap A^c$
4. Show that $T \cap A \cap B^c = T \cap (A \cap B)^c \cap A$
5. Show that $\mu^*(T \cap A^c) + \mu^*(T \cap A \cap B^c) = \mu^*(T \cap (A \cap B)^c)$
6. Adding $\mu^*(T \cap (A \cap B))$ on both sides 5., conclude that $A \cap B \in \Sigma$.
7. Show that $A \cup B$ and $A \setminus B$ belong to Σ.

Exercise 16. Everything being as before, let $A_n \in \Sigma, n \geq 1$. Define $B_1 = A_1$ and $B_{n+1} = A_{n+1} \setminus (A_1 \cup \ldots \cup A_n)$. Show that the B_n's are pairwise disjoint elements of Σ and that $\cup_{n=1}^{+\infty} A_n = \uplus_{n=1}^{+\infty} B_n$.

Exercise 17. Everything being as before, show that if $B, C \in \Sigma$ and $B \cap C = \emptyset$, then $\mu^*(T \cap (B \uplus C)) = \mu^*(T \cap B) + \mu^*(T \cap C)$ for any $T \subseteq \Omega$.

Exercise 18.Everything being as before, let $(B_n)_{n \geq 1}$ be a sequence of pairwise disjoint elements of Σ, and let $B \stackrel{\triangle}{=} \uplus_{n=1}^{+\infty} B_n$. Let $N \geq 1$.

1. Explain why $\uplus_{n=1}^{N} B_n \in \Sigma$
2. Show that $\mu^*(T \cap (\uplus_{n=1}^{N} B_n)) = \sum_{n=1}^{N} \mu^*(T \cap B_n)$
3. Show that $\mu^*(T \cap B^c) \leq \mu^*(T \cap (\uplus_{n=1}^{N} B_n)^c)$
4. Show that $\mu^*(T \cap B^c) + \sum_{n=1}^{+\infty} \mu^*(T \cap B_n) \leq \mu^*(T)$, and:
5. $\mu^*(T) \leq \mu^*(T \cap B^c) + \mu^*(T \cap B) \leq \mu^*(T \cap B^c) + \sum_{n=1}^{+\infty} \mu^*(T \cap B_n)$
6. Show that $B \in \Sigma$ and $\mu^*(B) = \sum_{n=1}^{+\infty} \mu^*(B_n)$.
7. Show that Σ is a σ-algebra on Ω, and $\mu^*_{|\Sigma}$ is a measure on Σ.

Theorem 3 *Let $\mu^* : \mathcal{P}(\Omega) \to [0, +\infty]$ be an outer-measure on Ω. Then $\Sigma(\mu^*)$, the so-called σ-*algebra associated *with μ^*, is indeed a σ-algebra on Ω and $\mu^*_{|\Sigma(\mu^*)}$, is a measure on $\Sigma(\mu^*)$.*

EXERCISE 19. Let $\mathcal{R}$ be a ring on Ω and $\mu : \mathcal{R} \to [0, +\infty]$ be a measure on $\mathcal{R}$. For all $T \subseteq \Omega$, define:

$$\mu^*(T) \stackrel{\triangle}{=} \inf \left\{ \sum_{n=1}^{+\infty} \mu(A_n) \ , \ (A_n) \text{ is an } \mathcal{R}\text{-cover of } T \right\}$$

where an $\mathcal{R}$-cover of T is defined as any sequence $(A_n)_{n \geq 1}$ of elements of $\mathcal{R}$ such that $T \subseteq \cup_{n=1}^{+\infty} A_n$. By convention $\inf \emptyset \stackrel{\triangle}{=} +\infty$.

1. Show that $\mu^*(\emptyset) = 0$.

2. Show that if $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$.

3. Let $(A_n)_{n \geq 1}$ be a sequence of subsets of Ω, with $\mu^*(A_n) < +\infty$ for all $n \geq 1$. Given $\epsilon > 0$, show that for all $n \geq 1$, there exists

an $\mathcal{R}$-cover $(A_n^p)^{p\geq 1}$ of A_n such that:

$$\sum_{p=1}^{+\infty} \mu(A_n^p) < \mu^*(A_n) + \epsilon/2^n$$

Why is it important to assume $\mu^*(A_n) < +\infty$.

4. Show that there exists an $\mathcal{R}$-cover (R_k) of $\cup_{n=1}^{+\infty} A_n$ such that:

$$\sum_{k=1}^{+\infty} \mu(R_k) = \sum_{n=1}^{+\infty}\sum_{p=1}^{+\infty} \mu(A_n^p)$$

5. Show that $\mu^*(\cup_{n=1}^{+\infty} A_n) \leq \epsilon + \sum_{n=1}^{+\infty} \mu^*(A_n)$

6. Show that μ^* is an outer-measure on Ω.

Exercise 20. Everything being as before, Let $A \in \mathcal{R}$. Let $(A_n)_{n\geq 1}$ be an $\mathcal{R}$-cover of A and put $B_1 = A_1 \cap A$, and:

$$B_{n+1} \stackrel{\triangle}{=} (A_{n+1} \cap A) \setminus ((A_1 \cap A) \cup \ldots \cup (A_n \cap A))$$

1. Show that $\mu^*(A) \leq \mu(A)$.
2. Show that $(B_n)_{n\geq 1}$ is a sequence of pairwise disjoint elements of $\mathcal{R}$ such that $A = \uplus_{n=1}^{+\infty} B_n$.
3. Show that $\mu(A) \leq \mu^*(A)$ and conclude that $\mu^*_{|\mathcal{R}} = \mu$.

Exercise 21. Everything being as before, Let $A \in \mathcal{R}$ and $T \subseteq \Omega$.

1. Show that $\mu^*(T) \leq \mu^*(T \cap A) + \mu^*(T \cap A^c)$.
2. Let (T_n) be an $\mathcal{R}$-cover of T. Show that $(T_n \cap A)$ and $(T_n \cap A^c)$ are $\mathcal{R}$-covers of $T \cap A$ and $T \cap A^c$ respectively.
3. Show that $\mu^*(T \cap A) + \mu^*(T \cap A^c) \leq \mu^*(T)$.

4. Show that $\mathcal{R} \subseteq \Sigma(\mu^*)$.

5. Conclude that $\sigma(\mathcal{R}) \subseteq \Sigma(\mu^*)$.

EXERCISE 22.Prove the following theorem:

Theorem 4 (caratheodory's extension) *Let $\mathcal{R}$ be a ring on Ω and $\mu : \mathcal{R} \to [0, +\infty]$ be a measure on $\mathcal{R}$. There exists a measure $\mu' : \sigma(\mathcal{R}) \to [0, +\infty]$ such that $\mu'_{|\mathcal{R}} = \mu$.*

EXERCISE 23. Let $\mathcal{S}$ be a semi-ring on Ω. Show that $\sigma(\mathcal{R}(\mathcal{S})) = \sigma(\mathcal{S})$.

EXERCISE 24.Prove the following theorem:

Theorem 5 *Let $\mathcal{S}$ be a semi-ring on Ω and $\mu : \mathcal{S} \to [0, +\infty]$ be a measure on $\mathcal{S}$. There exists a measure $\mu' : \sigma(\mathcal{S}) \to [0, +\infty]$ such that $\mu'_{|\mathcal{S}} = \mu$.*

3. Stieltjes-Lebesgue Measure

Definition 12 *Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ and $\mu : \mathcal{A} \to [0, +\infty]$ be a map. We say that μ is* **finitely additive** *if and only if, given $n \geq 1$:*

$$A \in \mathcal{A}, A_i \in \mathcal{A}, A = \biguplus_{i=1}^{n} A_i \ \Rightarrow \ \mu(A) = \sum_{i=1}^{n} \mu(A_i)$$

We say that μ is **finitely sub-additive** *if and only if, given $n \geq 1$:*

$$A \in \mathcal{A}, A_i \in \mathcal{A}, A \subseteq \bigcup_{i=1}^{n} A_i \ \Rightarrow \ \mu(A) \leq \sum_{i=1}^{n} \mu(A_i)$$

Exercise 1. Let $\mathcal{S} \stackrel{\triangle}{=} \{]a, b] \ , \ a, b \in \mathbf{R}\}$ be the set of all intervals $]a, b]$, defined as $]a, b] = \{x \in \mathbf{R}, a < x \leq b\}$.

1. Show that $]a, b] \cap]c, d] =]a \vee c, b \wedge d]$
2. Show that $]a, b] \setminus]c, d] =]a, b \wedge c] \cup]a \vee d, b]$

3. Show that $c \leq d \Rightarrow b \wedge c \leq a \vee d$.

4. Show that $\mathcal{S}$ is a semi-ring on $\mathbf{R}$.

Exercise 2. Suppose $\mathcal{S}$ is a semi-ring in Ω and $\mu : \mathcal{S} \to [0, +\infty]$ is finitely additive. Show that μ can be extended to a finitely additive map $\bar{\mu} : \mathcal{R}(\mathcal{S}) \to [0, +\infty]$, with $\bar{\mu}_{|\mathcal{S}} = \mu$.

Exercise 3. Everything being as before, Let $A \in \mathcal{R}(\mathcal{S})$, $A_i \in \mathcal{R}(\mathcal{S})$, $A \subseteq \cup_{i=1}^n A_i$ where $n \geq 1$. Define $B_1 = A_1 \cap A$ and for $i = 1, \ldots, n-1$:

$$B_{i+1} \stackrel{\triangle}{=} (A_{i+1} \cap A) \setminus ((A_1 \cap A) \cup \ldots \cup (A_i \cap A))$$

1. Show that $B_1, \ldots, B_n$ are pairwise disjoint elements of $\mathcal{R}(\mathcal{S})$ such that $A = \uplus_{i=1}^n B_i$.

2. Show that for all $i = 1, \ldots, n$, we have $\bar{\mu}(B_i) \leq \bar{\mu}(A_i)$.

3. Show that $\bar{\mu}$ is finitely sub-additive.

4. Show that μ is finitely sub-additive.

Exercise 4. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Let $\mathcal{S}$ be the semi-ring on $\mathbf{R}$, $\mathcal{S} = \{]a, b]\ ,\ a, b \in \mathbf{R}\}$. Define the map $\mu : \mathcal{S} \to [0, +\infty]$ by $\mu(\emptyset) = 0$, and:

$$\forall a \leq b\ ,\ \mu(]a, b]) \stackrel{\triangle}{=} F(b) - F(a) \tag{1}$$

Let $a < b$ and $a_i < b_i$ for $i = 1, \ldots, n$ and $n \geq 1$, with :

$$]a, b] = \biguplus_{i=1}^{n}]a_i, b_i]$$

1. Show that there is $i_1 \in \{1, \ldots, n\}$ such that $a_{i_1} = a$.
2. Show that $]b_{i_1}, b] = \uplus_{i \in \{1,\ldots,n\} \setminus \{i_1\}}]a_i, b_i]$
3. Show the existence of a permutation $(i_1, \ldots, i_n)$ of $\{1, \ldots, n\}$ such that $a = a_{i_1} < b_{i_1} = a_{i_2} < \ldots < b_{i_n} = b$.
4. Show that μ is finitely additive and finitely sub-additive.

Exercise 5. μ being defined as before, suppose $a < b$ and $a_n < b_n$ for $n \geq 1$ with:

$$]a, b] = \biguplus_{n=1}^{+\infty}]a_n, b_n]$$

Given $N \geq 1$, let $(i_1, \ldots, i_N)$ be a permutation of $\{1, \ldots, N\}$ with:

$$a \leq a_{i_1} < b_{i_1} \leq a_{i_2} < \ldots < b_{i_N} \leq b$$

1. Show that $\sum_{k=1}^{N} F(b_{i_k}) - F(a_{i_k}) \leq F(b) - F(a)$.

2. Show that $\sum_{n=1}^{+\infty} \mu(]a_n, b_n]) \leq \mu(]a, b])$

3. Given $\epsilon > 0$, show that there is $\eta \in]0, b-a[$ such that:

$$0 \leq F(a + \eta) - F(a) \leq \epsilon$$

4. For $n \geq 1$, show that there is $\eta_n > 0$ such that:

$$0 \leq F(b_n + \eta_n) - F(b_n) \leq \frac{\epsilon}{2^n}$$

5. Show that $[a+\eta, b] \subseteq \cup_{n=1}^{+\infty}]a_n, b_n + \eta_n[$.

6. Explain why there exist $p \geq 1$ and integers $n_1, \ldots, n_p$ such that:

$$]a+\eta, b] \subseteq \cup_{k=1}^{p}]a_{n_k}, b_{n_k} + \eta_{n_k}]$$

7. Show that $F(b) - F(a) \leq 2\epsilon + \sum_{n=1}^{+\infty} F(b_n) - F(a_n)$

8. Show that $\mu : \mathcal{S} \to [0, +\infty]$ is a measure.

Definition 13 *A* **topology** *on Ω is a subset $\mathcal{T}$ of the power set $\mathcal{P}(\Omega)$, with the following properties:*

$$\begin{array}{rl} (i) & \Omega, \emptyset \in \mathcal{T} \\ (ii) & A, B \in \mathcal{T} \ \Rightarrow \ A \cap B \in \mathcal{T} \\ (iii) & A_i \in \mathcal{T}, \forall i \in I \ \Rightarrow \ \bigcup_{i \in I} A_i \in \mathcal{T} \end{array}$$

Property (iii) of definition (13) can be translated as: *for any family* $(A_i)_{i\in I}$ *of elements of* $\mathcal{T}$*, the union* $\cup_{i\in I}A_i$ *is still an element of* $\mathcal{T}$*.* Hence, a topology on Ω, is a set of subsets of Ω containing Ω and the empty set, which is closed under finite intersection and arbitrary union.

Definition 14 *A* **topological space** *is an ordered pair* $(\Omega, \mathcal{T})$*, where* Ω *is a set and* $\mathcal{T}$ *is a topology on* Ω*.*

Definition 15 *Let* $(\Omega, \mathcal{T})$ *be a topological space. We say that* $A \subseteq \Omega$ *is an* **open set** *in* Ω*, if and only if it is an element of the topology* $\mathcal{T}$*. We say that* $A \subseteq \Omega$ *is a* **closed set** *in* Ω*, if and only if its complement* A^c *is an open set in* Ω*.*

Definition 16 *Let* $(\Omega, \mathcal{T})$ *be a topological space. We define the* **borel** σ**-algebra** *on* Ω*, denoted* $\mathcal{B}(\Omega)$*, as the* σ*-algebra on* Ω*, generated by the topology* $\mathcal{T}$*. In other words,* $\mathcal{B}(\Omega) = \sigma(\mathcal{T})$

Definition 17 *We define the* **usual topology** *on* $\mathbf{R}$*, denoted* $\mathcal{T}_{\mathbf{R}}$*, as the set of all* $U \subseteq \mathbf{R}$ *such that:*

$$\forall x \in U \ , \ \exists \epsilon > 0 \ , \ ]x - \epsilon, x + \epsilon[\subseteq U$$

EXERCISE 6. Show that $\mathcal{T}_{\mathbf{R}}$ is indeed a topology on $\mathbf{R}$.

EXERCISE 7. Consider the semi-ring $\mathcal{S} \stackrel{\triangle}{=} \{]a, b] \ , \ a, b \in \mathbf{R}\}$. Let $\mathcal{T}_{\mathbf{R}}$ be the usual topology on $\mathbf{R}$, and $\mathcal{B}(\mathbf{R})$ be the borel σ-algebra on $\mathbf{R}$.

1. Let $a \leq b$. Show that $]a, b] = \cap_{n=1}^{+\infty}]a, b + 1/n[$.
2. Show that $\sigma(\mathcal{S}) \subseteq \mathcal{B}(\mathbf{R})$.
3. Let U be an open subset of $\mathbf{R}$. Show that for all $x \in U$, there exist $a_x, b_x \in \mathbf{Q}$ such that $x \in]a_x, b_x] \subseteq U$.
4. Show that $U = \cup_{x \in U}]a_x, b_x]$.
5. Show that the set $I \stackrel{\triangle}{=} \{]a_x, b_x] \ , \ x \in U\}$ is countable.

6. Show that U can be written $U = \cup_{i \in I} A_i$ with $A_i \in \mathcal{S}$.

7. Show that $\sigma(\mathcal{S}) = \mathcal{B}(\mathbf{R})$.

Theorem 6 *Let $\mathcal{S}$ be the semi-ring $\mathcal{S} = \{]a,b]\,,\, a,b \in \mathbf{R}\}$. Then, the borel σ-algebra $\mathcal{B}(\mathbf{R})$ on $\mathbf{R}$, is generated by $\mathcal{S}$, i.e. $\mathcal{B}(\mathbf{R}) = \sigma(\mathcal{S})$.*

Definition 18 *A* **measurable space** *is an ordered pair $(\Omega, \mathcal{F})$ where Ω is a set and $\mathcal{F}$ is a σ-algebra on Ω.*

Definition 19 *A* **measure space** *is a triple $(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F})$ is a measurable space and $\mu : \mathcal{F} \to [0, +\infty]$ is a measure on $\mathcal{F}$.*

EXERCISE 8.Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{F}$ such that $A_n \subseteq A_{n+1}$ for all $n \geq 1$, and let $A = \cup_{n=1}^{+\infty} A_n$ (we write $A_n \uparrow A$). Define $B_1 = A_1$ and for all $n \geq 1$, $B_{n+1} = A_{n+1} \setminus A_n$.

1. Show that (B_n) is a sequence of pairwise disjoint elements of $\mathcal{F}$ such that $A = \uplus_{n=1}^{+\infty} B_n$.

2. Given $N \geq 1$ show that $A_N = \uplus_{n=1}^{N} B_n$.

3. Show that $\mu(A_N) \to \mu(A)$ as $N \to +\infty$

4. Show that $\mu(A_n) \leq \mu(A_{n+1})$ for all $n \geq 1$.

Theorem 7 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then if $(A_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{F}$, such that $A_n \uparrow A$, we have $\mu(A_n) \uparrow \mu(A)$*[1].

[1] i.e. the sequence $(\mu(A_n))_{n\geq 1}$ is non-decreasing and converges to $\mu(A)$.

Exercise 9. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{F}$ such that $A_{n+1} \subseteq A_n$ for all $n \geq 1$, and let $A = \cap_{n=1}^{+\infty} A_n$ (we write $A_n \downarrow A$). We assume that $\mu(A_1) < +\infty$.

1. Define $B_n \stackrel{\triangle}{=} A_1 \setminus A_n$ and show that $B_n \in \mathcal{F}, B_n \uparrow A_1 \setminus A$.
2. Show that $\mu(B_n) \uparrow \mu(A_1 \setminus A)$
3. Show that $\mu(A_n) = \mu(A_1) - \mu(A_1 \setminus A_n)$
4. Show that $\mu(A) = \mu(A_1) - \mu(A_1 \setminus A)$
5. Why is $\mu(A_1) < +\infty$ important in deriving those equalities.
6. Show that $\mu(A_n) \to \mu(A)$ as $n \to +\infty$
7. Show that $\mu(A_{n+1}) \leq \mu(A_n)$ for all $n \geq 1$.

Theorem 8 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Then if $(A_n)_{n\geq 1}$ is a sequence of elements of $\mathcal{F}$, such that $A_n \downarrow A$ and $\mu(A_1) < +\infty$, we have $\mu(A_n) \downarrow \mu(A)$.*

Exercise 10. Take $\Omega = \mathbf{R}$ and $\mathcal{F} = \mathcal{B}(\mathbf{R})$. Suppose μ is a measure on $\mathcal{B}(\mathbf{R})$ such that $\mu(]a,b]) = b - a$, for $a < b$. Take $A_n =]n, +\infty[$.

1. Show that $A_n \downarrow \emptyset$.
2. Show that $\mu(A_n) = +\infty$, for all $n \geq 1$.
3. Conclude that $\mu(A_n) \downarrow \mu(\emptyset)$ fails to be true.

Exercise 11. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Show the existence of a measure $\mu : \mathcal{B}(\mathbf{R}) \to [0, +\infty]$ such that:

$$\forall a, b \in \mathbf{R}\ ,\ a \leq b\ ,\ \mu(]a,b]) = F(b) - F(a) \tag{2}$$

Exercise 12. Let μ_1, μ_2 be two measures on $\mathcal{B}(\mathbf{R})$ with property (2). For $n \geq 1$, we define:

$$\mathcal{D}_n \stackrel{\triangle}{=} \{B \in \mathcal{B}(\mathbf{R})\ ,\ \mu_1(B \cap]-n, n]) = \mu_2(B \cap]-n, n])\}$$

1. Show that $\mathcal{D}_n$ is a dynkin system on $\mathbf{R}$.

2. Explain why $\mu_1(]-n,n]) < +\infty$ and $\mu_2(]-n,n]) < +\infty$ is needed when proving 1.

3. Show that $\mathcal{S} \stackrel{\triangle}{=} \{]a,b]\ ,\ a,b \in \mathbf{R}\} \subseteq \mathcal{D}_n$.

4. Show that $\mathcal{B}(\mathbf{R}) \subseteq \mathcal{D}_n$.

5. Show that $\mu_1 = \mu_2$.

6. Prove the following theorem.

Theorem 9 *Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. There exists a unique measure $\mu : \mathcal{B}(\mathbf{R}) \to [0, +\infty]$ such that:*

$$\forall a, b \in \mathbf{R}\ ,\ a \leq b\ ,\ \mu(]a,b]) = F(b) - F(a)$$

Definition 20 *Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. We call* **stieltjes measure** *on $\mathbf{R}$ associated with F, the unique measure on $\mathcal{B}(\mathbf{R})$, denoted dF, such that:*

$$\forall a, b \in \mathbf{R}\ ,\ a \leq b\ ,\ dF(]a,b]) = F(b) - F(a)$$

Definition 21 *We call* **lebesgue measure** *on* $\mathbf{R}$*, the unique measure on* $\mathcal{B}(\mathbf{R})$*, denoted* dx*, such that:*

$$\forall a, b \in \mathbf{R}\ ,\ a \leq b\ ,\ dx(]a, b]) = b - a$$

Exercise 13. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Let $x_0 \in \mathbf{R}$.

1. Show that the limit $F(x_0-) = \lim_{x<x_0, x\to x_0} F(x)$ exists and is an element of $\mathbf{R}$.

2. Show that $\{x_0\} = \cap_{n=1}^{+\infty}]x_0 - 1/n, x_0]$.

3. Show that $\{x_0\} \in \mathcal{B}(\mathbf{R})$

4. Show that $dF(\{x_0\}) = F(x_0) - F(x_0-)$

Exercise 14.Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Let $a \leq b$.

1. Show that $]a, b] \in \mathcal{B}(\mathbf{R})$ and $dF(]a, b]) = F(b) - F(a)$
2. Show that $[a, b] \in \mathcal{B}(\mathbf{R})$ and $dF([a, b]) = F(b) - F(a-)$
3. Show that $]a, b[\in \mathcal{B}(\mathbf{R})$ and $dF(]a, b[) = F(b-) - F(a)$
4. Show that $[a, b[\in \mathcal{B}(\mathbf{R})$ and $dF([a, b[) = F(b-) - F(a-)$

Exercise 15. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Let $\Omega' \subseteq \Omega$. Define:

$$\mathcal{A}_{|\Omega'} \stackrel{\triangle}{=} \{A \cap \Omega' \ , \ A \in \mathcal{A}\}$$

1. Show that if $\mathcal{A}$ is a topology on Ω, $\mathcal{A}_{|\Omega'}$ is a topology on Ω'.
2. Show that if $\mathcal{A}$ is a σ-algebra on Ω, $\mathcal{A}_{|\Omega'}$ is a σ-algebra on Ω'.

Definition 22 *Let Ω be a set, and $\Omega' \subseteq \Omega$. Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. We call* **trace** *of $\mathcal{A}$ on Ω', the subset $\mathcal{A}_{|\Omega'}$ of the power set $\mathcal{P}(\Omega')$ defined by:*

$$\mathcal{A}_{|\Omega'} \stackrel{\triangle}{=} \{A \cap \Omega' \,,\, A \in \mathcal{A}\}$$

Definition 23 *Let $(\Omega, \mathcal{T})$ be a topological space and $\Omega' \subseteq \Omega$. We call* **induced topology** *on Ω', denoted $\mathcal{T}_{|\Omega'}$, the topology on Ω' defined by:*

$$\mathcal{T}_{|\Omega'} \stackrel{\triangle}{=} \{A \cap \Omega' \ ,\ A \in \mathcal{T}\}$$

In other words, the induced topology $\mathcal{T}_{|\Omega'}$ is the trace of $\mathcal{T}$ on Ω'.

Exercise 16.Let $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Let $\Omega' \subseteq \Omega$, and $\mathcal{A}_{|\Omega'}$ be the trace of $\mathcal{A}$ on Ω'. Define:

$$\Gamma \stackrel{\triangle}{=} \{A \in \sigma(\mathcal{A}) \ ,\ A \cap \Omega' \in \sigma(\mathcal{A}_{|\Omega'})\}$$

where $\sigma(\mathcal{A}_{|\Omega'})$ refers to the σ-algebra generated by $\mathcal{A}_{|\Omega'}$ **on** Ω'.

1. Explain why the notation $\sigma(\mathcal{A}_{|\Omega'})$ by itself is ambiguous.

2. Show that $\mathcal{A} \subseteq \Gamma$.

3. Show that Γ is a σ-algebra on Ω.

4. Show that $\sigma(\mathcal{A}_{|\Omega'}) = \sigma(\mathcal{A})_{|\Omega'}$

Theorem 10 *Let $\Omega' \subseteq \Omega$ and $\mathcal{A}$ be a subset of the power set $\mathcal{P}(\Omega)$. Then, the trace on Ω' of the σ-algebra $\sigma(\mathcal{A})$ generated by $\mathcal{A}$, is equal to the σ-algebra on Ω' generated by the trace of $\mathcal{A}$ on Ω'. In other words, $\sigma(\mathcal{A})_{|\Omega'} = \sigma(\mathcal{A}_{|\Omega'})$.*

EXERCISE 17.Let $(\Omega, \mathcal{T})$ be a topological space and $\Omega' \subseteq \Omega$ with its induced topology $\mathcal{T}_{|\Omega'}$.

1. Show that $\mathcal{B}(\Omega)_{|\Omega'} = \mathcal{B}(\Omega')$.

2. Show that if $\Omega' \in \mathcal{B}(\Omega)$ then $\mathcal{B}(\Omega') \subseteq \mathcal{B}(\Omega)$.

3. Show that $\mathcal{B}(\mathbf{R}^+) = \{A \cap \mathbf{R}^+ \ , \ A \in \mathcal{B}(\mathbf{R})\}$.

4. Show that $\mathcal{B}(\mathbf{R}^+) \subseteq \mathcal{B}(\mathbf{R})$.

Exercise 18. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\Omega' \subseteq \Omega$

1. Show that $(\Omega', \mathcal{F}_{|\Omega'})$ is a measurable space.

2. If $\Omega' \in \mathcal{F}$, show that $\mathcal{F}_{|\Omega'} \subseteq \mathcal{F}$.

3. If $\Omega' \in \mathcal{F}$, show that $(\Omega', \mathcal{F}_{|\Omega'}, \mu_{|\Omega'})$ is a measure space, where $\mu_{|\Omega'}$ is defined as $\mu_{|\Omega'} = \mu_{|(\mathcal{F}_{|\Omega'})}$.

Exercise 19. Let $F : \mathbf{R}^+ \to \mathbf{R}$ be a right-continuous, non-decreasing map with $F(0) \geq 0$. Define:

$$\bar{F}(x) \stackrel{\triangle}{=} \begin{cases} 0 & \text{if} \quad x < 0 \\ F(x) & \text{if} \quad x \geq 0 \end{cases}$$

1. Show that $\bar{F} : \mathbf{R} \to \mathbf{R}$ is right-continuous and non-decreasing.

2. Show that $\mu : \mathcal{B}(\mathbf{R}^+) \to [0, +\infty]$ defined by $\mu = d\bar{F}_{|\mathcal{B}(\mathbf{R}^+)}$, is a measure on $\mathcal{B}(\mathbf{R}^+)$ with the properties:

$$\begin{array}{ll} (i) & \mu(\{0\}) = F(0) \\ (ii) & \forall 0 \leq a \leq b\ ,\ \mu(]a, b]) = F(b) - F(a) \end{array}$$

Exercise 20. Define: $\mathcal{C} = \{\{0\}\} \cup \{]a, b]\ ,\ 0 \leq a \leq b\}$

1. Show that $\mathcal{C} \subseteq \mathcal{B}(\mathbf{R}^+)$

2. Let U be open in $\mathbf{R}^+$. Show that U is of the form:

$$U = \bigcup_{i \in I} (\mathbf{R}^+ \cap]a_i, b_i])$$

where I is a countable set and $a_i, b_i \in \mathbf{R}$ with $a_i \leq b_i$.

3. For all $i \in I$, show that $\mathbf{R}^+ \cap]a_i, b_i] \in \sigma(\mathcal{C})$.

4. Show that $\sigma(\mathcal{C}) = \mathcal{B}(\mathbf{R}^+)$

Exercise 21. Let μ_1 and μ_2 be two measures on $\mathcal{B}(\mathbf{R}^+)$ with:

$$\begin{array}{ll} (i) & \mu_1(\{0\}) = \mu_2(\{0\}) = F(0) \\ (ii) & \mu_1(]a,b]) = \mu_2(]a,b]) = F(b) - F(a) \end{array}$$

for all $0 \leq a \leq b$. For $n \geq 1$, we define:

$$\mathcal{D}_n = \{B \in \mathcal{B}(\mathbf{R}^+)\ ,\ \mu_1(B \cap [0,n]) = \mu_2(B \cap [0,n])\}$$

1. Show that $\mathcal{D}_n$ is a dynkin system on $\mathbf{R}^+$ with $\mathcal{C} \subseteq \mathcal{D}_n$, where the set $\mathcal{C}$ is defined as in exercise (20).

2. Explain why $\mu_1([0,n]) < +\infty$ and $\mu_2([0,n]) < +\infty$ is important when proving 1.

3. Show that $\mu_1 = \mu_2$.

4. Prove the following theorem.

Theorem 11 *Let $F : \mathbf{R}^+ \to \mathbf{R}$ be a right-continuous, non-decreasing map with $F(0) \geq 0$. There exists a unique $\mu : \mathcal{B}(\mathbf{R}^+) \to [0, +\infty]$ measure on $\mathcal{B}(\mathbf{R}^+)$ such that:*

$$\begin{array}{ll} (i) & \mu(\{0\}) = F(0) \\ (ii) & \forall 0 \leq a \leq b\ ,\ \mu(]a, b]) = F(b) - F(a) \end{array}$$

Definition 24 *Let $F : \mathbf{R}^+ \to \mathbf{R}$ be a right-continuous, non-decreasing map with $F(0) \geq 0$. We call* **stieltjes measure** *on $\mathbf{R}^+$ associated with F, the unique measure on $\mathcal{B}(\mathbf{R}^+)$, denoted dF, such that:*

$$\begin{array}{ll} (i) & dF(\{0\}) = F(0) \\ (ii) & \forall 0 \leq a \leq b\ ,\ dF(]a, b]) = F(b) - F(a) \end{array}$$

4. Measurability

Definition 25 *Let A and B be two sets, and $f : A \to B$ be a map. Given $A' \subseteq A$, we call* **direct image** *of A' by f the set denoted $f(A')$, and defined by $f(A') = \{f(x) \ : \ x \in A'\}$.*

Definition 26 *Let A and B be two sets, and $f : A \to B$ be a map. Given $B' \subseteq B$, we call* **inverse image** *of B' by f the set denoted $f^{-1}(B')$, and defined by $f^{-1}(B') = \{x \ : \ x \in A \ , \ f(x) \in B'\}$.*

Exercise 1. Let A and B be two sets, and $f : A \to B$ be a bijection from A to B. Let $A' \subseteq A$ and $B' \subseteq B$.

1. Explain why the notation $f^{-1}(B')$ is potentially ambiguous.
2. Show that the inverse image of B' by f is in fact equal to the direct image of B' by f^{-1}.
3. Show that the direct image of A' by f is in fact equal to the inverse image of A' by f^{-1}.

Definition 27 *Let $(\Omega, \mathcal{T})$ and $(S, \mathcal{T}_S)$ be two topological spaces. A map $f : \Omega \to S$ is said to be* **continuous** *if and only if:*

$$\forall B \in \mathcal{T}_S \ , \ f^{-1}(B) \in \mathcal{T}$$

In other words, if and only if the inverse image of any open set in S is an open set in Ω.

We Write $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ *is continuous*, as a way of emphasizing the two topologies $\mathcal{T}$ and $\mathcal{T}_S$ with respect to which f is continuous.

Definition 28 *Let E be a set. A map $d : E \times E \to [0, +\infty[$ is said to be a* **metric** *on E, if and only if:*

$$\begin{array}{ll} (i) & \forall x, y \in E \ , \ d(x,y) = 0 \ \Leftrightarrow \ x = y \\ (ii) & \forall x, y \in E \ , \ d(x,y) = d(y,x) \\ (iii) & \forall x, y, z \in E \ , \ d(x,y) \leq d(x,z) + d(z,y) \end{array}$$

Definition 29 *A* **metric space** *is an ordered pair (E, d) where E is a set, and d is a metric on E.*

Definition 30 *Let (E,d) be a metric space. For all $x \in E$ and $\epsilon > 0$, we define the so-called* **open ball** *in E:*

$$B(x,\epsilon) \stackrel{\triangle}{=} \{y \ : \ y \in E \ , \ d(x,y) < \epsilon\}$$

We call **metric topology** *on E, associated with d, the topology $\mathcal{T}_E^d$ defined by:*

$$\mathcal{T}_E^d \stackrel{\triangle}{=} \{U \subseteq E \ , \forall x \in U, \exists \epsilon > 0, B(x,\epsilon) \subseteq U\}$$

Exercise 2. Let $\mathcal{T}_E^d$ be the metric topology associated with d, where (E,d) is a metric space.

1. Show that $\mathcal{T}_E^d$ is indeed a topology on E.

2. Given $x \in E$ and $\epsilon > 0$, show that $B(x,\epsilon)$ is an open set in E.

Exercise 3. Show that the usual topology on $\mathbf{R}$ is nothing but the metric topology associated with $d(x,y) = |x - y|$.

EXERCISE 4. Let (E, d) and (F, δ) be two metric spaces. Show that a map $f : E \to F$ is continuous, if and only if for all $x \in E$ and $\epsilon > 0$, there exists $\eta > 0$ such that for all $y \in E$:

$$d(x, y) < \eta \quad \Rightarrow \quad \delta(f(x), f(y)) < \epsilon$$

Definition 31 *Let $(\Omega, \mathcal{T})$ and $(S, \mathcal{T}_S)$ be two topological spaces. A map $f : \Omega \to S$ is said to be a* **homeomorphism***, if and only if f is a continuous bijection, such that f^{-1} is also continuous.*

Definition 32 *A topological space $(\Omega, \mathcal{T})$ is said to be* **metrizable***, if and only if there exists a metric d on Ω, such that the associated metric topology coincides with $\mathcal{T}$, i.e. $\mathcal{T}^d_\Omega = \mathcal{T}$.*

Definition 33 *Let (E, d) be a metric space and $F \subseteq E$. We call* **induced metric** *on F, denoted $d_{|F}$, the restriction of the metric d to $F \times F$, i.e. $d_{|F} = d_{|F \times F}$.*

EXERCISE 5. Let (E, d) be a metric space and $F \subseteq E$. We define $\mathcal{T}_F = (\mathcal{T}_E^d)_{|F}$ as the topology on F induced by the metric topology on E. Let $\mathcal{T}'_F = \mathcal{T}_F^{d_{|F}}$ be the metric topology on F associated with the induced metric $d_{|F}$ on F.

1. Show that $\mathcal{T}_F \subseteq \mathcal{T}'_F$.
2. Given $A \in \mathcal{T}'_F$, show that $A = (\cup_{x \in A} B(x, \epsilon_x)) \cap F$ for some $\epsilon_x > 0$, $x \in A$, where $B(x, \epsilon_x)$ denotes the open ball in E.
3. Show that $\mathcal{T}'_F \subseteq \mathcal{T}_F$.

Theorem 12 *Let (E, d) be a metric space and $F \subseteq E$. Then, the topology on F induced by the metric topology, is equal to the metric topology on F associated with the induced metric, i.e. $(\mathcal{T}_E^d)_{|F} = \mathcal{T}_F^{d_{|F}}$.*

Exercise 6. Let $\phi : \mathbf{R} \to]-1, 1[$ be the map defined by:

$$\forall x \in \mathbf{R} \quad , \quad \phi(x) \stackrel{\triangle}{=} \frac{x}{|x| + 1}$$

1. Show that $[-1, 0[$ is not open in $\mathbf{R}$.
2. Show that $[-1, 0[$ is open in $[-1, 1]$.
3. Show that ϕ is a homeomorphism between $\mathbf{R}$ and $]-1, 1[$.
4. Show that $\lim_{x \to +\infty} \phi(x) = 1$ and $\lim_{x \to -\infty} \phi(x) = -1$.

Exercise 7. Let $\bar{\mathbf{R}} = [-\infty, +\infty] = \mathbf{R} \cup \{-\infty, +\infty\}$. Let ϕ be defined as in exercise (6), and $\bar{\phi} : \bar{\mathbf{R}} \to [-1, 1]$ be the map defined by:

$$\bar{\phi}(x) = \left\{ \begin{array}{ccc} \phi(x) & \text{if} & x \in \mathbf{R} \\ 1 & \text{if} & x = +\infty \\ -1 & \text{if} & x = -\infty \end{array} \right.$$

Define:

$$\mathcal{T}_{\bar{\mathbf{R}}} \stackrel{\triangle}{=} \{U \subseteq \bar{\mathbf{R}} \ , \ \bar{\phi}(U) \text{ is open in } [-1, 1]\}$$

1. Show that $\bar{\phi}$ is a bijection from $\bar{\mathbf{R}}$ to $[-1,1]$, and let $\bar{\psi} = \bar{\phi}^{-1}$.
2. Show that $\mathcal{T}_{\bar{\mathbf{R}}}$ is a topology on $\bar{\mathbf{R}}$.
3. Show that $\bar{\phi}$ is a homeomorphism between $\bar{\mathbf{R}}$ and $[-1,1]$.
4. Show that $[-\infty, 2[,\ ]3,+\infty],\ ]3,+\infty[$ are open in $\bar{\mathbf{R}}$.
5. Show that if $\phi' : \bar{\mathbf{R}} \to [-1,1]$ is an arbitrary homeomorphism, then $U \subseteq \bar{\mathbf{R}}$ is open, if and only if $\phi'(U)$ is open in $[-1,1]$.

Definition 34 *The* **usual topology** *on $\bar{\mathbf{R}}$ is defined as:*

$$\mathcal{T}_{\bar{\mathbf{R}}} \stackrel{\triangle}{=} \{U \subseteq \bar{\mathbf{R}}\ ,\ \bar{\phi}(U) \text{ is open in } [-1,1]\}$$

where $\bar{\phi} : \bar{\mathbf{R}} \to [-1,1]$ is defined by $\bar{\phi}(-\infty) = -1$, $\bar{\phi}(+\infty) = 1$ and:

$$\forall x \in \mathbf{R} \quad , \quad \bar{\phi}(x) \stackrel{\triangle}{=} \frac{x}{|x|+1}$$

Exercise 8. Let ϕ and $\bar{\phi}$ be as in exercise (7). Define:

$$\mathcal{T}' \stackrel{\triangle}{=} (\mathcal{T}_{\bar{\mathbf{R}}})_{|\mathbf{R}} \stackrel{\triangle}{=} \{U \cap \mathbf{R}\ ,\ U \in \mathcal{T}_{\bar{\mathbf{R}}}\}$$

1. Recall why $\mathcal{T}'$ is a topology on $\mathbf{R}$.

2. Show that for all $U \subseteq \bar{\mathbf{R}}$, $\phi(U \cap \mathbf{R}) = \bar{\phi}(U) \cap]-1,1[$.

3. Explain why if $U \in \mathcal{T}_{\bar{\mathbf{R}}}$, $\phi(U \cap \mathbf{R})$ is open in $]-1,1[$.

4. Show that $\mathcal{T}' \subseteq \mathcal{T}_{\mathbf{R}}$, (the usual topology on $\mathbf{R}$).

5. Let $U \in \mathcal{T}_{\mathbf{R}}$. Show that $\bar{\phi}(U)$ is open in $]-1,1[$ and $[-1,1]$.

6. Show that $\mathcal{T}_{\mathbf{R}} \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$

7. Show that $\mathcal{T}_{\mathbf{R}} = \mathcal{T}'$, i.e. that the usual topology on $\bar{\mathbf{R}}$ induces the usual topology on $\mathbf{R}$.

8. Show that $\mathcal{B}(\mathbf{R}) = \mathcal{B}(\bar{\mathbf{R}})_{|\mathbf{R}} = \{B \cap \mathbf{R}\ ,\ B \in \mathcal{B}(\bar{\mathbf{R}})\}$

EXERCISE 9. Let $d : \bar{\mathbf{R}} \times \bar{\mathbf{R}} \to [0, +\infty[$ be defined by:

$$\forall (x, y) \in \bar{\mathbf{R}} \times \bar{\mathbf{R}} \quad , \quad d(x, y) = |\phi(x) - \phi(y)|$$

where ϕ is an arbitrary homeomorphism from $\bar{\mathbf{R}}$ to $[-1, 1]$.

1. Show that d is a metric on $\bar{\mathbf{R}}$.
2. Show that if $U \in \mathcal{T}_{\bar{\mathbf{R}}}$, then $\phi(U)$ is open in $[-1, 1]$
3. Show that for all $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ and $y \in \phi(U)$, there exists $\epsilon > 0$ such that:
$$\forall z \in [-1, 1] \ , \ |z - y| < \epsilon \ \Rightarrow \ z \in \phi(U)$$
4. Show that $\mathcal{T}_{\bar{\mathbf{R}}} \subseteq \mathcal{T}^d_{\bar{\mathbf{R}}}$.
5. Show that for all $U \in \mathcal{T}^d_{\bar{\mathbf{R}}}$ and $x \in U$, there is $\epsilon > 0$ such that:
$$\forall y \in \bar{\mathbf{R}} \ , \ |\phi(x) - \phi(y)| < \epsilon \ \Rightarrow \ y \in U$$
6. Show that for all $U \in \mathcal{T}^d_{\bar{\mathbf{R}}}$, $\phi(U)$ is open in $[-1, 1]$.

7. Show that $\mathcal{T}^d_{\bar{\mathbf{R}}} \subseteq \mathcal{T}_{\bar{\mathbf{R}}}$

8. Prove the following theorem.

Theorem 13 *The topological space $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is metrizable.*

Definition 35 *Let $(\Omega, \mathcal{F})$ and (S, Σ) be two measurable spaces. A map $f : \Omega \to S$ is said to be* **measurable** *with respect to $\mathcal{F}$ and Σ, if and only if:*

$$\forall B \in \Sigma \ , \ f^{-1}(B) \in \mathcal{F}$$

We Write $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ *is measurable*, as a way of emphasizing the two σ-algebra $\mathcal{F}$ and Σ with respect to which f is measurable.

Exercise 10. Let $(\Omega, \mathcal{F})$ and (S, Σ) be two measurable spaces. Let S' be a set and $f : \Omega \to S$ be a map such that $f(\Omega) \subseteq S' \subseteq S$. We define Σ' as the trace of Σ on S', i.e. $\Sigma' = \Sigma_{|S'}$.

1. Show that for all $B \in \Sigma$, we have $f^{-1}(B) = f^{-1}(B \cap S')$

2. Show that $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, if and only if $f : (\Omega, \mathcal{F}) \to (S', \Sigma')$ is itself measurable.

3. Let $f : \Omega \to \mathbf{R}^+$. Show that the following are equivalent:

$$\begin{array}{rl} (i) & f : (\Omega, \mathcal{F}) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \text{ is measurable} \\ (ii) & f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \text{ is measurable} \\ (iii) & f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}})) \text{ is measurable} \end{array}$$

Exercise 11. Let $(\Omega, \mathcal{F})$, (S, Σ), (S_1, Σ_1) be three measurable spaces. let $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ and $g : (S, \Sigma) \to (S_1, \Sigma_1)$ be two measurable maps.

1. For all $B \subseteq S_1$, show that $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$

2. Show that $g \circ f : (\Omega, \mathcal{F}) \to (S_1, \Sigma_1)$ is measurable.

Exercise 12. Let $(\Omega, \mathcal{F})$ and (S, Σ) be two measurable spaces. Let $f : \Omega \to S$ be a map. We define:

$$\Gamma \stackrel{\triangle}{=} \{B \in \Sigma \ , \ f^{-1}(B) \in \mathcal{F}\}$$

1. Show that $f^{-1}(S) = \Omega$.
2. Show that for all $B \subseteq S$, $f^{-1}(B^c) = (f^{-1}(B))^c$.
3. Show that if $B_n \subseteq S, n \geq 1$, then $f^{-1}(\cup_{n=1}^{+\infty} B_n) = \cup_{n=1}^{+\infty} f^{-1}(B_n)$
4. Show that Γ is a σ-algebra on S.
5. Prove the following theorem.

Theorem 14 *Let $(\Omega, \mathcal{F})$ and (S, Σ) be two measurable spaces, and $\mathcal{A}$ be a set of subsets of S generating Σ, i.e. such that $\Sigma = \sigma(\mathcal{A})$. Then $f : (\Omega, \mathcal{F}) \to (S, \Sigma)$ is measurable, if and only if:*

$$\forall B \in \mathcal{A} \quad , \quad f^{-1}(B) \in \mathcal{F}$$

Exercise 13. Let $(\Omega, \mathcal{T})$ and $(S, \mathcal{T}_S)$ be two topological spaces. Let $f : \Omega \to S$ be a map. Show that if $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ is continuous, then $f : (\Omega, \mathcal{B}(\Omega)) \to (S, \mathcal{B}(S))$ is measurable.

Exercise 14. We define the following subsets of the power set $\mathcal{P}(\bar{\mathbf{R}})$:

$$\begin{aligned}
\mathcal{C}_1 &\stackrel{\triangle}{=} \{[-\infty, c]\ ,\ c \in \mathbf{R}\} \\
\mathcal{C}_2 &\stackrel{\triangle}{=} \{[-\infty, c[\ ,\ c \in \mathbf{R}\} \\
\mathcal{C}_3 &\stackrel{\triangle}{=} \{[c, +\infty]\ ,\ c \in \mathbf{R}\} \\
\mathcal{C}_4 &\stackrel{\triangle}{=} \{]c, +\infty]\ ,\ c \in \mathbf{R}\}
\end{aligned}$$

1. Show that $\mathcal{C}_2$ and $\mathcal{C}_4$ are subsets of $\mathcal{T}_{\bar{\mathbf{R}}}$.

2. Show that the elements of $\mathcal{C}_1$ and $\mathcal{C}_3$ are closed in $\bar{\mathbf{R}}$.

3. Show that for all $i = 1, 2, 3, 4$, $\sigma(\mathcal{C}_i) \subseteq \mathcal{B}(\bar{\mathbf{R}})$.

4. Let U be open in $\bar{\mathbf{R}}$. Explain why $U \cap \mathbf{R}$ is open in $\mathbf{R}$.

5. Show that any open subset of $\mathbf{R}$ is a countable union of open bounded intervals in $\mathbf{R}$.

6. Let $a < b$, $a, b \in \mathbf{R}$. Show that we have:

$$]a, b[= \bigcup_{n=1}^{+\infty}]a, b - 1/n] = \bigcup_{n=1}^{+\infty} [a + 1/n, b[$$

7. Show that for all $i = 1, 2, 3, 4$, $]a, b[\in \sigma(\mathcal{C}_i)$.

8. Show that for all $i = 1, 2, 3, 4$, $\{\{-\infty\}, \{+\infty\}\} \subseteq \sigma(\mathcal{C}_i)$.

9. Show that for all $i = 1, 2, 3, 4$, $\sigma(\mathcal{C}_i) = \mathcal{B}(\bar{\mathbf{R}})$

10. Prove the following theorem.

Theorem 15 *Let $(\Omega, \mathcal{F})$ be a measurable space, and $f : \Omega \to \bar{\mathbf{R}}$ be a map. The following are equivalent:*

$$
\begin{array}{rl}
(i) & f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}})) \text{ is measurable} \\
(ii) & \forall B \in \mathcal{B}(\bar{\mathbf{R}}) \ , \ \{f \in B\} \in \mathcal{F} \\
(iii) & \forall c \in \mathbf{R} \ , \ \{f \leq c\} \in \mathcal{F} \\
(iv) & \forall c \in \mathbf{R} \ , \ \{f < c\} \in \mathcal{F} \\
(v) & \forall c \in \mathbf{R} \ , \ \{c \leq f\} \in \mathcal{F} \\
(vi) & \forall c \in \mathbf{R} \ , \ \{c < f\} \in \mathcal{F}
\end{array}
$$

Exercise 15. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n \geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$. Let g and h be the maps defined by $g(\omega) = \inf_{n \geq 1} f_n(\omega)$ and $h(\omega) = \sup_{n \geq 1} f_n(\omega)$, for all $\omega \in \Omega$.

1. Let $c \in \mathbf{R}$. Show that $\{c \leq g\} = \cap_{n=1}^{+\infty} \{c \leq f_n\}$.

2. Let $c \in \mathbf{R}$. Show that $\{h \leq c\} = \cap_{n=1}^{+\infty} \{f_n \leq c\}$.

3. Show that $g, h : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.

Definition 36 *Let $(v_n)_{n\geq 1}$ be a sequence in $\bar{\mathbf{R}}$. We define:*

$$u \stackrel{\triangle}{=} \liminf_{n\to+\infty} v_n \stackrel{\triangle}{=} \sup_{n\geq 1}\left(\inf_{k\geq n} v_k\right)$$

and:

$$w \stackrel{\triangle}{=} \limsup_{n\to+\infty} v_n \stackrel{\triangle}{=} \inf_{n\geq 1}\left(\sup_{k\geq n} v_k\right)$$

Then, $u, w \in \bar{\mathbf{R}}$ are respectively called **lower limit** *and* **upper limit** *of the sequence $(v_n)_{n\geq 1}$.*

Exercise 16. Let $(v_n)_{n\geq 1}$ be a sequence in $\bar{\mathbf{R}}$. for $n \geq 1$ we define $u_n = \inf_{k\geq n} v_k$ and $w_n = \sup_{k\geq n} v_k$. Let u and w be the lower limit and upper limit of $(v_n)_{n\geq 1}$, respectively.

1. Show that $u_n \leq u_{n+1} \leq u$, for all $n \geq 1$.

2. Show that $w \leq w_{n+1} \leq w_n$, for all $n \geq 1$.

3. Show that $u_n \to u$ and $w_n \to w$ as $n \to +\infty$.

4. Show that $u_n \leq v_n \leq w_n$, for all $n \geq 1$.

5. Show that $u \leq w$.

6. Show that if $u = w$ then $(v_n)_{n\geq 1}$ converges to a limit $v \in \bar{\mathbf{R}}$, with $u = v = w$.

7. Show that if $a, b \in \mathbf{R}$ are such that $u < a < b < w$ then for all $n \geq 1$, there exist $N_1, N_2 \geq n$ such that $v_{N_1} < a < b < v_{N_2}$.

8. Show that if $a, b \in \mathbf{R}$ are such that $u < a < b < w$ then there exist two strictly increasing sequences of integers $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ such that for all $k \geq 1$, we have $v_{n_k} < a < b < v_{m_k}$.

9. Show that if $(v_n)_{n\geq 1}$ converges to some $v \in \bar{\mathbf{R}}$, then $u = w$.

Theorem 16 *Let $(v_n)_{n\geq 1}$ be a sequence in $\bar{\mathbf{R}}$. Then, the following are equivalent:*

$$(i) \qquad \liminf_{n\to+\infty} v_n = \limsup_{n\to+\infty} v_n$$

$$(ii) \qquad \lim_{n\to+\infty} v_n \ \ \textit{exists in } \bar{\mathbf{R}}.$$

in which case:

$$\lim_{n\to+\infty} v_n = \liminf_{n\to+\infty} v_n = \limsup_{n\to+\infty} v_n$$

Exercise 17. Let $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be two measurable maps, where $(\Omega, \mathcal{F})$ is a measurable space.

1. Show that $\{f < g\} = \cup_{r\in\mathbf{Q}}(\{f < r\} \cap \{r < g\})$.

2. Show that the sets $\{f < g\}, \{f > g\}, \{f = g\}, \{f \leq g\}$, $\{f \geq g\}$ belong to the σ-algebra $\mathcal{F}$.

EXERCISE 18. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$. We define $g = \liminf f_n$ and $h = \limsup f_n$ in the obvious way:

$$\forall \omega \in \Omega\ ,\ g(\omega) \stackrel{\triangle}{=} \liminf_{n\to+\infty} f_n(\omega)$$

$$\forall \omega \in \Omega\ ,\ h(\omega) \stackrel{\triangle}{=} \limsup_{n\to+\infty} f_n(\omega)$$

1. Show that $g, h : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.
2. Show that $g \leq h$, i.e. $\forall \omega \in \Omega\ ,\ g(\omega) \leq h(\omega)$.
3. Show that $\{g = h\} \in \mathcal{F}$.
4. Show that $\{\omega : \omega \in \Omega\ ,\ \lim_{n\to+\infty} f_n(\omega) \text{ exists in } \bar{\mathbf{R}}\} \in \mathcal{F}$.
5. Suppose $\Omega = \{g = h\}$, and let $f(\omega) = \lim_{n\to+\infty} f_n(\omega)$, for all $\omega \in \Omega$. Show that $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

Exercise 19. Let $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be two measurable maps, where $(\Omega, \mathcal{F})$ is a measurable space.

1. Show that $-f, |f|, f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.

2. Let $a \in \bar{\mathbf{R}}$. Explain why the map $a + f$ may not be well defined.

3. Show that $(a+f) : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, whenever $a \in \mathbf{R}$.

4. Show that $(a.f) : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, for all $a \in \bar{\mathbf{R}}$. (Recall the convention $0.\infty = 0$).

5. Explain why the map $f + g$ may not be well defined.

6. Suppose that $f \geq 0$ and $g \geq 0$, i.e. $f(\Omega) \subseteq [0, +\infty]$ and also $g(\Omega) \subseteq [0, +\infty]$. Show that $\{f + g < c\} = \{f < c - g\}$, for all $c \in \mathbf{R}$. Show that $f + g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

7. Show that $f+g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, in the case when f and g take values in $\mathbf{R}$.

8. Show that $1/f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, in the case when $f(\Omega) \subseteq \mathbf{R} \setminus \{0\}$.

9. Suppose that f is $\mathbf{R}$-valued. Show that $\bar{f}$ defined by $\bar{f}(\omega) = f(\omega)$ if $f(\omega) \neq 0$ and $\bar{f}(\omega) = 1$ if $f(\omega) = 0$, is measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.

10. Suppose f and g take values in $\mathbf{R}$. Let $\bar{f}$ be defined as in 9. Show that for all $c \in \mathbf{R}$, the set $\{fg < c\}$ can be expressed as:

$$(\{f>0\} \cap \{g < c/\bar{f}\}) \uplus (\{f<0\} \cap \{g > c/\bar{f}\}) \uplus (\{f = 0\} \cap \{f < c\})$$

11. Show that $fg : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, in the case when f and g take values in $\mathbf{R}$.

Exercise 20. Let $f, g : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be two measurable maps, where $(\Omega, \mathcal{F})$ is a measurable space. Let $\bar{f}, \bar{g}$, be defined by:

$$\bar{f}(\omega) \triangleq \left\{ \begin{array}{rcl} f(\omega) & \text{if} & f(\omega) \notin \{-\infty, +\infty\} \\ 1 & \text{if} & f(\omega) \in \{-\infty, +\infty\} \end{array} \right.$$

$\bar{g}(\omega)$ being defined in a similar way. Consider the partitions of Ω, $\Omega = A_1 \uplus A_2 \uplus A_3 \uplus A_4 \uplus A_5$ and $\Omega = B_1 \uplus B_2 \uplus B_3 \uplus B_4 \uplus B_5$, where $A_1 = \{f \in]0, +\infty[\}$, $A_2 = \{f \in]-\infty, 0[\}$, $A_3 = \{f = 0\}$, $A_4 = \{f = -\infty\}$, $A_5 = \{f = +\infty\}$ and B_1, B_2, B_3, B_4, B_5 being defined in a similar way with g. Recall the conventions $0 \times (+\infty) = 0$, $(-\infty) \times (+\infty) = (-\infty)$, etc...

1. Show that $\bar{f}$ and $\bar{g}$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.

2. Show that all A_i's and B_j's are elements of $\mathcal{F}$.

3. Show that for all $B \in \mathcal{B}(\bar{\mathbf{R}})$:

$$\{fg \in B\} = \biguplus_{i,j=1}^{5} (A_i \cap B_j \cap \{fg \in B\})$$

4. Show that $A_i \cap B_j \cap \{fg \in B\} = A_i \cap B_j \cap \{\bar{f}\bar{g} \in B\}$, in the case when $1 \leq i \leq 3$ and $1 \leq j \leq 3$.

5. Show that $A_i \cap B_j \cap \{fg \in B\}$ is either equal to $\emptyset$ or $A_i \cap B_j$, in the case when $i \geq 4$ or $j \geq 4$.

6. Show that $fg : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

Definition 37 *Let $(\Omega, \mathcal{T})$ be a topological space, and $A \subseteq \Omega$. We call* **closure** *of A in Ω, denoted $\bar{A}$, the set defined by:*

$$\bar{A} \stackrel{\triangle}{=} \{x \in \Omega \ : \ x \in U \in \mathcal{T} \Rightarrow U \cap A \neq \emptyset\}$$

Exercise 21. Let $(E, \mathcal{T})$ be a topological space, and $A \subseteq E$. Let $\bar{A}$ be the closure of A.

1. Show that $A \subseteq \bar{A}$ and that $\bar{A}$ is closed.

2. Show that if B is closed and $A \subseteq B$, then $\bar{A} \subseteq B$.

3. Show that $\bar{A}$ is the smallest closed set in E containing A.

4. Show that A is closed if and only if $A = \bar{A}$.

5. Show that if $(E, \mathcal{T})$ is metrizable, then:

$$\bar{A} = \{x \in E \ : \ \forall\epsilon > 0 \ , \ B(x, \epsilon) \cap A \neq \emptyset\}$$

where $B(x, \epsilon)$ is relative to any metric d such that $\mathcal{T}^d_E = \mathcal{T}$.

Exercise 22. Let (E, d) be a metric space. Let $A \subseteq E$. For all $x \in E$, we define:

$$d(x, A) \stackrel{\triangle}{=} \inf\{d(x, y) \ : \ y \in A\} \stackrel{\triangle}{=} \Phi_A(x)$$

where it is understood that $\inf \emptyset = +\infty$.

1. Show that for all $x \in E$, $d(x, A) = d(x, \bar{A})$.

2. Show that $d(x, A) = 0$, if and only if $x \in \bar{A}$.

3. Show that for all $x, y \in E$, $d(x, A) \leq d(x, y) + d(y, A)$.

4. Show that if $A \neq \emptyset$, $|d(x,A) - d(y,A)| \leq d(x,y)$.

5. Show that $\Phi_A : (E, \mathcal{T}_E^d) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is continuous.

6. Show that if A is closed, then $A = \Phi_A^{-1}(\{0\})$

Exercise 23. Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$, where (E, d) is a metric space. We assume that for all $\omega \in \Omega$, the sequence $(f_n(\omega))_{n\geq 1}$ converges to some $f(\omega) \in E$.

1. Explain why $\liminf f_n$ and $\limsup f_n$ may not be defined in an arbitrary metric space E.

2. Show that $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable, if and only if $f^{-1}(A) \in \mathcal{F}$ for all closed subsets A of E.

3. Show that for all A closed in E, $f^{-1}(A) = (\Phi_A \circ f)^{-1}(\{0\})$, where the map $\Phi_A : E \to \bar{\mathbf{R}}$ is defined as in exercise (22).

4. Show that $\Phi_A \circ f_n : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

5. Show that $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is measurable.

Theorem 17 *Let $(\Omega, \mathcal{F})$ be a measurable space. Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$, where (E, d) is a metric space. Then, if the limit $f = \lim f_n$ exists on Ω, the map $f : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ is itself measurable.*

Definition 38 *The* **usual topology** *on $\mathbf{C}$, the set of complex numbers, is defined as the metric topology associated with $d(z, z') = |z-z'|$.*

Exercise 24. Let $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map, where $(\Omega, \mathcal{F})$ is a measurable space. Let $u = Re(f)$ and $v = Im(f)$. Show that $u, v, |f| : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are all measurable.

Exercise 25. Define the subset of the power set $\mathcal{P}(\mathbf{C})$:

$$\mathcal{C} \stackrel{\triangle}{=} \{]a, b[\times]c, d[\ ,\ a, b, c, d \in \mathbf{R}\}$$

where it is understood that:

$$]a,b[\times]c,d[\ \stackrel{\triangle}{=}\ \{z=x+iy\in\mathbf{C}\ ,\ (x,y)\in]a,b[\times]c,d[\}$$

1. Show that any element of $\mathcal{C}$ is open in $\mathbf{C}$.

2. Show that $\sigma(\mathcal{C})\subseteq\mathcal{B}(\mathbf{C})$.

3. Let $z=x+iy\in\mathbf{C}$. Show that if $|x|<\eta$ and $|y|<\eta$ then we have $|z|<\sqrt{2}\eta$.

4. Let U be open in $\mathbf{C}$. Show that for all $z\in U$, there are rational numbers a_z,b_z,c_z,d_z such that $z\in]a_z,b_z[\times]c_z,d_z[\subseteq U$.

5. Show that U can be written as $U=\cup_{n=1}^{+\infty}A_n$ where $A_n\in\mathcal{C}$.

6. Show that $\sigma(\mathcal{C})=\mathcal{B}(\mathbf{C})$.

7. Let $(\Omega,\mathcal{F})$ be a measurable space, and $u,v:(\Omega,\mathcal{F})\to(\mathbf{R},\mathcal{B}(\mathbf{R}))$ be two measurable maps. Show that $u+iv:(\Omega,\mathcal{F})\to(\mathbf{C},\mathcal{B}(\mathbf{C}))$ is measurable.

5. Lebesgue Integration

In the following, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

Definition 39 *Let $A \subseteq \Omega$. We call* **characteristic function** *of A, the map $1_A : \Omega \to \mathbf{R}$, defined by:*

$$\forall \omega \in \Omega \ , \ 1_A(\omega) \stackrel{\triangle}{=} \left\{ \begin{array}{lll} 1 & \textit{if} & \omega \in A \\ 0 & \textit{if} & \omega \not\in A \end{array} \right.$$

Exercise 1. Given $A \subseteq \Omega$, show that $1_A : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable if and only if $A \in \mathcal{F}$.

Definition 40 *Let $(\Omega, \mathcal{F})$ be a measurable space. We say that a map $s : \Omega \to \mathbf{R}^+$ is a* **simple function** *on $(\Omega, \mathcal{F})$, if and only if s is of the form :*

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{R}^+$ and $A_i \in \mathcal{F}$, for all $i = 1, \ldots, n$.

Exercise 2. Show that $s : (\Omega, \mathcal{F}) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ is measurable, whenever s is a simple function on $(\Omega, \mathcal{F})$.

Exercise 3. Let s be a simple function on $(\Omega, \mathcal{F})$ with representation $s = \sum_{i=1}^n \alpha_i 1_{A_i}$. Consider the map $\phi : \Omega \to \{0,1\}^n$ defined by $\phi(\omega) = (1_{A_1}(\omega), \ldots, 1_{A_n}(\omega))$. For each $y \in s(\Omega)$, pick one $\omega_y \in \Omega$ such that $y = s(\omega_y)$. Consider the map $\psi : s(\Omega) \to \{0,1\}^n$ defined by $\psi(y) = \phi(\omega_y)$.

1. Show that ψ is injective, and that $s(\Omega)$ is a finite subset of $\mathbf{R}^+$.
2. Show that $s = \sum_{\alpha \in s(\Omega)} \alpha 1_{\{s=\alpha\}}$
3. Show that any simple function s can be represented as:

$$s = \sum_{i=1}^n \alpha_i 1_{A_i}$$

where $n \geq 1, \alpha_i \in \mathbf{R}^+, A_i \in \mathcal{F}$ and $\Omega = A_1 \uplus \ldots \uplus A_n$.

Definition 41 *Let $(\Omega, \mathcal{F})$ be a measurable space, and s be a simple function on $(\Omega, \mathcal{F})$. We call* **partition** *of the simple function s, any representation of the form:*

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{R}^+$, $A_i \in \mathcal{F}$ and $\Omega = A_1 \uplus \ldots \uplus A_n$.

Exercise 4. Let s be a simple function on $(\Omega, \mathcal{F})$ with two partitions:

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i} = \sum_{j=1}^{m} \beta_j 1_{B_j}$$

1. Show that $s = \sum_{i,j} \alpha_i 1_{A_i \cap B_j}$ is a partition of s.
2. Recall the convention $0 \times (+\infty) = 0$ and $\alpha \times (+\infty) = +\infty$ if $\alpha > 0$. For all $a_1, \ldots, a_p$ in $[0, +\infty]$, $p \geq 1$ and $x \in [0, +\infty]$, prove the distributive property: $x(a_1 + \ldots + a_p) = xa_1 + \ldots + xa_p$.

3. Show that $\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{j=1}^{m} \beta_j \mu(B_j)$.

4. Explain why the following definition is legitimate.

Definition 42 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and s be a simple function on $(\Omega, \mathcal{F})$. We define the* **integral** *of s with respect to μ, as the sum, denoted $I^\mu(s)$, defined by:*

$$I^\mu(s) \stackrel{\triangle}{=} \sum_{i=1}^{n} \alpha_i \mu(A_i) \in [0, +\infty]$$

where $s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ is any partition of s.

Exercise 5. Let s, t be two simple functions on $(\Omega, \mathcal{F})$ with partitions $s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$ and $t = \sum_{j=1}^{m} \beta_j 1_{B_j}$. Let $\alpha \in \mathbf{R}^+$.

1. Show that $s + t$ is a simple function on $(\Omega, \mathcal{F})$ with partition:

$$s + t = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i + \beta_j) 1_{A_i \cap B_j}$$

2. Show that $I^{\mu}(s+t) = I^{\mu}(s) + I^{\mu}(t)$.

3. Show that αs is a simple function on $(\Omega, \mathcal{F})$.

4. Show that $I^{\mu}(\alpha s) = \alpha I^{\mu}(s)$.

5. Why is the notation $I^{\mu}(\alpha s)$ meaningless if $\alpha = +\infty$ or $\alpha < 0$.

6. Show that if $s \leq t$ then $I^{\mu}(s) \leq I^{\mu}(t)$.

Exercise 6. Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map. For all $n \geq 1$, we define:

$$s_n \stackrel{\triangle}{=} \sum_{k=0}^{n2^n - 1} \frac{k}{2^n} 1_{\{\frac{k}{2^n} \leq f < \frac{k+1}{2^n}\}} + n 1_{\{n \leq f\}} \tag{1}$$

1. Show that s_n is a simple function on $(\Omega, \mathcal{F})$, for all $n \geq 1$.

2. Show that equation (1) is a partition s_n, for all $n \geq 1$.

3. Show that $s_n \leq s_{n+1} \leq f$, for all $n \geq 1$.

4. Show that $s_n \uparrow f$ as $n \to +\infty$[1].

Theorem 18 *Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map, where $(\Omega, \mathcal{F})$ is a measurable space. There exists a sequence $(s_n)_{n\geq 1}$ of simple functions on $(\Omega, \mathcal{F})$ such that $s_n \uparrow f$.*

Definition 43 *Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map, where $(\Omega, \mathcal{F}, \mu)$ is a measure space. We define the* **lebesgue integral** *of f with respect to μ, denoted $\int f d\mu$, as:*

$$\int f d\mu \stackrel{\triangle}{=} \sup\{I^\mu(s) \ : \ s \text{ simple function on } (\Omega, \mathcal{F}) \ , \ s \leq f\}$$

where, given any simple function s on $(\Omega, \mathcal{F})$, $I^\mu(s)$ denotes its integral with respect to μ.

[1] i.e. for all $\omega \in \Omega$, the sequence $(s_n(\omega))_{n\geq 1}$ is non-decreasing and converges to $f(\omega) \in [0, +\infty]$.

Exercise 7. Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map.

1. Show that $\int f d\mu \in [0, +\infty]$.
2. Show that $\int f d\mu = I^\mu(f)$, whenever f is a simple function.
3. Show that $\int g d\mu \leq \int f d\mu$, whenever $g : (\Omega, \mathcal{F}) \to [0, +\infty]$ is non-negative and measurable map with $g \leq f$.
4. Show that $\int (cf) d\mu = c \int f d\mu$, if $0 < c < +\infty$. Explain why both integrals are well defined. Is the equality still true for $c = 0$.
5. For $n \geq 1$, put $A_n = \{f > 1/n\}$, and $s_n = (1/n) 1_{A_n}$. Show that s_n is a simple function on $(\Omega, \mathcal{F})$ with $s_n \leq f$. Show that $A_n \uparrow \{f > 0\}$.
6. Show that $\int f d\mu = 0 \Rightarrow \mu(\{f > 0\}) = 0$.

7. Show that if s is a simple function on $(\Omega, \mathcal{F})$ with $s \leq f$, then $\mu(\{f > 0\}) = 0$ implies $I^{\mu}(s) = 0$.

8. Show that $\int f d\mu = 0 \;\Leftrightarrow\; \mu(\{f > 0\}) = 0$.

9. Show that $\int (+\infty) f d\mu = (+\infty) \int f d\mu$. Explain why both integrals are well defined.

10. Show that $(+\infty)1_{\{f=+\infty\}} \leq f$ and:

$$\int (+\infty)1_{\{f=+\infty\}} d\mu = (+\infty)\mu(\{f = +\infty\})$$

11. Show that $\int f d\mu < +\infty \;\Rightarrow \mu(\{f = +\infty\}) = 0$.

12. Suppose that $\mu(\Omega) = +\infty$ and take $f = 1$. Show that the converse of the previous implication is not true.

Exercise 8. Let s be a simple function on $(\Omega, \mathcal{F})$. let $A \in \mathcal{F}$.

1. Show that $s1_A$ is a simple function on $(\Omega, \mathcal{F})$.

2. Show that for any partition $s = \sum_{i=1}^n \alpha_i 1_{A_i}$ of s, we have:

$$I^\mu(s1_A) = \sum_{i=1}^n \alpha_i \mu(A_i \cap A)$$

3. Let $\nu : \mathcal{F} \to [0, +\infty]$ be defined by $\nu(A) = I^\mu(s1_A)$. Show that ν is a measure on $\mathcal{F}$.

4. Suppose $A_n \in \mathcal{F}, A_n \uparrow A$. Show that $I^\mu(s1_{A_n}) \uparrow I^\mu(s1_A)$.

Exercise 9. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$, such that $f_n \uparrow f$.

1. Recall what the notation $f_n \uparrow f$ means.

2. Explain why $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

3. Let $\alpha = \sup_{n\geq 1} \int f_n d\mu$. Show that $\int f_n d\mu \uparrow \alpha$.

4. Show that $\alpha \leq \int f d\mu$.

5. Let s be any simple function on $(\Omega, \mathcal{F})$ such that $s \leq f$. Let $c \in]0,1[$. For $n \geq 1$, define $A_n = \{cs \leq f_n\}$. Show that $A_n \in \mathcal{F}$ and $A_n \uparrow \Omega$.

6. Show that $cI^\mu(s1_{A_n}) \leq \int f_n d\mu$, for all $n \geq 1$.

7. Show that $cI^\mu(s) \leq \alpha$.

8. Show that $I^\mu(s) \leq \alpha$.

9. Show that $\int f d\mu \leq \alpha$.

10. Conclude that $\int f_n d\mu \uparrow \int f d\mu$.

Theorem 19 (Monotone Convergence) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$ such that $f_n \uparrow f$. Then $\int f_n d\mu \uparrow \int f d\mu$.*

Exercise 10. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $a, b \in [0, +\infty]$.

1. Show that if $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ are two sequences of non-negative and measurable maps such that $f_n \uparrow f$ and $g_n \uparrow g$, then $f_n + g_n \uparrow f + g$.

2. Show that $\int (f+g)d\mu = \int f d\mu + \int g d\mu$.

3. Show that $\int (af + bg)d\mu = a \int f d\mu + b \int g d\mu$.

Exercise 11. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$. Define $f = \sum_{n=1}^{+\infty} f_n$.

1. Explain why $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ is well defined, non-negative and measurable.

2. Show that $\int f d\mu = \sum_{n=1}^{+\infty} \int f_n d\mu$.

Definition 44 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathcal{P}(\omega)$ be a property depending on $\omega \in \Omega$. We say that the property $\mathcal{P}(\omega)$ holds* **μ-almost surely**, *and we write $\mathcal{P}(\omega)$ μ-a.s., if and only if:*

$$\exists N \in \mathcal{F} \ , \ \mu(N) = 0 \ , \ \forall \omega \in N^c, \mathcal{P}(\omega) \text{ holds}$$

Exercise 12. Let $\mathcal{P}(\omega)$ be a property depending on $\omega \in \Omega$, such that $\{\omega \in \Omega : \mathcal{P}(\omega) \text{ holds}\}$ is an element of the σ-algebra $\mathcal{F}$.

1. Show that $\mathcal{P}(\omega)$, μ-a.s. $\Leftrightarrow$ $\mu(\{\omega \in \Omega : \mathcal{P}(\omega) \text{ holds}\}^c) = 0$.
2. Explain why in general, the right-hand side of this equivalence cannot be used to defined μ-almost sure properties.

Exercise 13. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $(A_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{F}$. Show that $\mu(\cup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} \mu(A_n)$.

Exercise 14. Let $(f_n)_{n\geq 1}$ be a sequence of maps $f_n : \Omega \to [0, +\infty]$.

1. Translate formally the statement $f_n \uparrow f$ μ-a.s.

2. Translate formally $f_n \to f$ μ-a.s. and $\forall n, (f_n \leq f_{n+1}$ μ-a.s.)

3. Show that the statements 1. and 2. are equivalent.

Exercise 15. Suppose that $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ are non-negative and measurable with $f = g$ μ-a.s.. Let $N \in \mathcal{F}$, $\mu(N) = 0$ such that $f = g$ on N^c. Explain why $\int f d\mu = \int (f 1_N) d\mu + \int (f 1_{N^c}) d\mu$, all integrals being well defined. Show that $\int f d\mu = \int g d\mu$.

Exercise 16. Suppose $(f_n)_{n\geq 1}$ is a sequence of non-negative and measurable maps such that $f_n \uparrow f$ μ-a.s.. Let $N \in \mathcal{F}$, $\mu(N) = 0$, such that $f_n \uparrow f$ on N^c. Define $\bar{f}_n = f_n 1_{N^c}$ and $\bar{f} = f 1_{N^c}$.

1. Explain why $\bar{f}$ and the $\bar{f}_n$'s are non-negative and measurable.

2. Show that $\bar{f}_n \uparrow \bar{f}$.

3. Show that $\int f_n d\mu \uparrow \int f d\mu$.

EXERCISE 17. Let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$. For $n \geq 1$, we define $g_n = \inf_{k\geq n} f_k$.

1. Explain why the g_n's are non-negative and measurable.

2. Show that $g_n \uparrow \liminf f_n$.

3. Show that $\int g_n d\mu \leq \int f_n d\mu$, for all $n \geq 1$.

4. Show that if $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$ are two sequences in $\bar{\mathbf{R}}$ with $u_n \leq v_n$ for all $n \geq 1$, then $\liminf u_n \leq \liminf v_n$.

5. Show that $\int(\liminf f_n)d\mu \leq \liminf \int f_n d\mu$, and recall why all integrals are well defined.

Theorem 20 (Fatou Lemma) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let $(f_n)_{n\geq 1}$ be a sequence of non-negative and measurable maps $f_n : (\Omega, \mathcal{F}) \to [0, +\infty]$. Then:*

$$\int (\liminf_{n\to+\infty} f_n) d\mu \leq \liminf_{n\to+\infty} \int f_n d\mu$$

Exercise 18. Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map. Let $A \in \mathcal{F}$.

1. Recall what is meant by the induced measure space $(A, \mathcal{F}_{|A}, \mu_{|A})$. Why is it important to have $A \in \mathcal{F}$. Show that the restriction of f to A, $f_{|A} : (A, \mathcal{F}_{|A}) \to [0, +\infty]$ is measurable.

2. We define the map $\mu^A : \mathcal{F} \to [0, +\infty]$ by $\mu^A(E) = \mu(A \cap E)$, for all $E \in \mathcal{F}$. Show that $(\Omega, \mathcal{F}, \mu^A)$ is a measure space.

3. Consider the equalities:

$$\int (f 1_A) d\mu = \int f d\mu^A = \int (f_{|A}) d\mu_{|A} \tag{2}$$

   For each of the above integrals, what is the underlying measure space on which the integral is considered. What is the map being integrated. Explain why each integral is well defined.

4. Show that in order to prove (2), it is sufficient to consider the case when f is a simple function on $(\Omega, \mathcal{F})$.

5. Show that in order to prove (2), it is sufficient to consider the case when f is of the form $f = 1_B$, for some $B \in \mathcal{F}$.

6. Show that (2) is indeed true.

Definition 45 *Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map, where $(\Omega, \mathcal{F}, \mu)$ is a measure space. let $A \in \mathcal{F}$. We call* **partial lebesgue integral** *of f with respect to μ over A, the integral denoted $\int_A f d\mu$, defined as:*

$$\int_A f d\mu \stackrel{\triangle}{=} \int (f 1_A) d\mu = \int f d\mu^A = \int (f_{|A}) d\mu_{|A}$$

where μ^A is the measure on $(\Omega, \mathcal{F})$, $\mu^A = \mu(A \cap \bullet)$, $f_{|A}$ is the restriction of f to A and $\mu_{|A}$ is the restriction of μ to $\mathcal{F}_{|A}$, the trace of $\mathcal{F}$ on A.

Exercise 19. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $\nu : \mathcal{F} \to [0, +\infty]$ be defined by $\nu(A) = \int_A f d\mu$, for all $A \in \mathcal{F}$.

1. Show that ν is a measure on $\mathcal{F}$.

2. Show that:
$$\int g d\nu = \int g f d\mu$$

Theorem 21 *Let $f : (\Omega, \mathcal{F}) \to [0, +\infty]$ be a non-negative and measurable map, where $(\Omega, \mathcal{F}, \mu)$ is a measure space. Let $\nu : \mathcal{F} \to [0, +\infty]$ be defined by $\nu(A) = \int_A f d\mu$, for all $A \in \mathcal{F}$. Then, ν is a measure on $\mathcal{F}$, and for all $g : (\Omega, \mathcal{F}) \to [0, +\infty]$ non-negative and measurable, we have:*
$$\int g d\nu = \int g f d\mu$$

Definition 46 *The* **L^1-spaces** *on a measure space* $(\Omega, \mathcal{F}, \mu)$, *are:*

$$L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \left\{ f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \text{ measurable}, \int |f| d\mu < +\infty \right\}$$

$$L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \left\{ f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C})) \text{ measurable}, \int |f| d\mu < +\infty \right\}$$

EXERCISE 20. Let $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map.

1. Explain why the integral $\int |f| d\mu$ makes sense.
2. Show that $f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable, if $f(\Omega) \subseteq \mathbf{R}$.
3. Show that $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.
4. Show that $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) = \{f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)\ ,\ f(\Omega) \subseteq \mathbf{R}\}$
5. Show that $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{R}$-linear combinations.
6. Show that $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{C}$-linear combinations.

Definition 47 *Let $u : \Omega \to \mathbf{R}$ be a real-valued function defined on a set Ω. We call* **positive part** *and* **negative part** *of u the maps u^+ and u^- respectively, defined as $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$.*

Exercise 21. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Let $u = Re(f)$ and $v = Im(f)$.

1. Show that $u = u^+ - u^-$, $v = v^+ - v^-$, $f = u^+ - u^- + i(v^+ - v^-)$.
2. Show that $|u| = u^+ + u^-$, $|v| = v^+ + v^-$
3. Show that $u^+, u^-, v^+, v^-, |f|, u, v, |u|, |v|$ all lie in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.
4. Explain why the integrals $\int u^+ d\mu, \int u^- d\mu, \int v^+ d\mu, \int v^- d\mu$ are all well defined.
5. We define the integral of f with respect to μ, denoted $\int f d\mu$, as $\int f d\mu = \int u^+ d\mu - \int u^- d\mu + i\left(\int v^+ d\mu - \int v^- d\mu\right)$. Explain why $\int f d\mu$ is a well defined complex number.

6. In the case when $f(\Omega) \subseteq \mathbf{C} \cap [0, +\infty] = \mathbf{R}^+$, explain why this new definition of the integral of f with respect to μ is consistent with the one already known (43) for non-negative and measurable maps.

7. Show that $\int f d\mu = \int u d\mu + i \int v d\mu$ and explain why all integrals involved are well defined.

Definition 48 *Let $f = u + iv \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F}, \mu)$ is a measure space. We define the* **lebesgue integral** *of f with respect to μ, denoted $\int f d\mu$, as:*

$$\int f d\mu \stackrel{\triangle}{=} \int u^+ d\mu - \int u^- d\mu + i \left(\int v^+ d\mu - \int v^- d\mu \right)$$

Exercise 22. Let $f = u + iv \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $A \in \mathcal{F}$.

1. Show that $f 1_A \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

2. Show that $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu^A)$.

3. Show that $f_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, \mu_{|A})$

4. Show that $\int (f 1_A) d\mu = \int f d\mu^A = \int f_{|A} d\mu_{|A}$.

5. Show that 4. is:$\int_A u^+ d\mu - \int_A u^- d\mu + i \left(\int_A v^+ d\mu - \int_A v^- d\mu \right)$.

Definition 49 *Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F}, \mu)$ is a measure space. let $A \in \mathcal{F}$. We call* **partial lebesgue integral** *of f with respect to μ over A, the integral denoted $\int_A f d\mu$, defined as:*

$$\int_A f d\mu \triangleq \int (f 1_A) d\mu = \int f d\mu^A = \int (f_{|A}) d\mu_{|A}$$

where μ^A is the measure on $(\Omega, \mathcal{F})$, $\mu^A = \mu(A \cap \bullet)$, $f_{|A}$ is the restriction of f to A and $\mu_{|A}$ is the restriction of μ to $\mathcal{F}_{|A}$, the trace of $\mathcal{F}$ on A.

Exercise 23. Let $f, g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and let $h = f + g$

1. Show that:
$$\int h^+ d\mu + \int f^- d\mu + \int g^- d\mu = \int h^- d\mu + \int f^+ d\mu + \int g^+ d\mu$$
2. Show that $\int h d\mu = \int f d\mu + \int g d\mu$.
3. Show that $\int (-f) d\mu = -\int f d\mu$
4. Show that if $\alpha \in \mathbf{R}$ then $\int (\alpha f) d\mu = \alpha \int f d\mu$.
5. Show that if $f \leq g$ then $\int f d\mu \leq \int g d\mu$
6. Show the following theorem.

Theorem 22 *For all $f, g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\alpha \in \mathbf{C}$, we have:*
$$\int (\alpha f + g) d\mu = \alpha \int f d\mu + \int g d\mu$$

EXERCISE 24. Let f, g be two maps, and $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, such that:

$$\begin{array}{rl} (i) & \forall \omega \in \Omega\ ,\ \lim_{n\to+\infty} f_n(\omega) = f(\omega) \text{ in } \mathbf{C} \\ (ii) & \forall n \geq 1\ ,\ |f_n| \leq g \\ (iii) & g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \end{array}$$

Let $(u_n)_{n\geq 1}$ be an arbitrary sequence in $\bar{\mathbf{R}}$.

1. Show that $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $f_n \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ for all $n \geq 1$.
2. For $n \geq 1$, define $h_n = 2g - |f_n - f|$. Explain why Fatou lemma (20) can be applied to the sequence $(h_n)_{n\geq 1}$.
3. Show that $\liminf(-u_n) = -\limsup u_n$.
4. Show that if $\alpha \in \mathbf{R}$, then $\liminf(\alpha + u_n) = \alpha + \liminf u_n$.
5. Show that $u_n \to 0$ as $n \to +\infty$ if and only if $\limsup |u_n| = 0$.
6. Show that $\int (2g)d\mu \leq \int (2g)d\mu - \limsup \int |f_n - f| d\mu$

7. Show that $\limsup \int |f_n - f| d\mu = 0$.

8. Conclude that $\int |f_n - f| d\mu \to 0$ as $n \to +\infty$.

Theorem 23 (Dominated Convergence) *Let $(f_n)_{n\geq 1}$ be a sequence of measurable maps $f_n : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ such that $f_n \to f$ in $\mathbf{C}$*[2] *. Suppose that there exists some $g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $|f_n| \leq g$ for all $n \geq 1$. Then $f, f_n \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ for all $n \geq 1$, and:*

$$\lim_{n \to +\infty} \int |f_n - f| d\mu = 0$$

Exercise 25. Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and put $z = \int f d\mu$. Let $\alpha \in \mathbf{C}$, be such that $|\alpha| = 1$ and $\alpha z = |z|$. Put $u = Re(\alpha f)$.

1. Show that $u \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$

2. Show that $u \leq |f|$

[2] i.e. for all $\omega \in \Omega$, the sequence $(f_n(\omega))_{n\geq 1}$ converges to $f(\omega) \in \mathbf{C}$

3. Show that $|\int f d\mu| = \int (\alpha f) d\mu$.

4. Show that $\int (\alpha f) d\mu = \int u d\mu$.

5. Prove the following theorem.

Theorem 24 *Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ where $(\Omega, \mathcal{F}, \mu)$ is a measure space. We have:*

$$\left| \int f d\mu \right| \leq \int |f| d\mu$$

6. Product Spaces

In the following, I is a non-empty set.

Definition 50 *Let $(\Omega_i)_{i\in I}$ be a family of sets, indexed by a non-empty set I. We call* **cartesian product** *of the family $(\Omega_i)_{i\in I}$ the set, denoted $\Pi_{i\in I}\Omega_i$, and defined by:*

$$\prod_{i\in I}\Omega_i \stackrel{\triangle}{=} \{\omega : I \to \cup_{i\in I}\Omega_i \ , \ \omega(i) \in \Omega_i \ , \ \forall i \in I\}$$

In other words, $\Pi_{i\in I}\Omega_i$ is the set of all maps ω defined on I, with values in $\cup_{i\in I}\Omega_i$, such that $\omega(i) \in \Omega_i$ for all $i \in I$.

Theorem 25 (Axiom of choice) *Let $(\Omega_i)_{i\in I}$ be a family of sets, indexed by a non-empty set I. Then, $\Pi_{i\in I}\Omega_i$ is non-empty, if and only if Ω_i is non-empty for all $i \in I$*[1].

[1] When I is finite, this theorem is traditionally derived from other axioms.

Exercise 1.

1. Let Ω be a set and suppose that $\Omega_i = \Omega, \forall i \in I$. We use the notation Ω^I instead of $\Pi_{i\in I}\Omega_i$. Show that Ω^I is the set of all maps $\omega : I \to \Omega$.

2. What are the sets $\mathbf{R}^{\mathbf{R}^+}$, $\mathbf{R}^{\mathbf{N}}$, $[0,1]^{\mathbf{N}}$, $\bar{\mathbf{R}}^{\mathbf{R}}$?

3. Suppose $I = \mathbf{N}^*$. We sometimes use the notation $\Pi_{n=1}^{+\infty}\Omega_n$ instead of $\Pi_{n\in\mathbf{N}^*}\Omega_n$. Let $\mathcal{S}$ be the set of all sequences $(x_n)_{n\geq 1}$ such that $x_n \in \Omega_n$ for all $n \geq 1$. Is $\mathcal{S}$ the same thing as the product $\Pi_{n=1}^{+\infty}\Omega_n$?

4. Suppose $I = \mathbf{N}_n = \{1,\ldots,n\}$, $n \geq 1$. We use the notation $\Omega_1 \times \ldots \times \Omega_n$ instead of $\Pi_{i\in\{1,\ldots,n\}}\Omega_i$. For $\omega \in \Omega_1 \times \ldots \times \Omega_n$, it is customary to write $(\omega_1,\ldots,\omega_n)$ instead of ω, where we have $\omega_i = \omega(i)$. What is your guess for the definition of sets such as $\mathbf{R}^n, \bar{\mathbf{R}}^n, \mathbf{Q}^n, \mathbf{C}^n$.

5. Let E, F, G be three sets. Define $E \times F \times G$.

Definition 51 *Let I be a non-empty set. We say that a family of sets $(I_\lambda)_{\lambda\in\Lambda}$, where $\Lambda \neq \emptyset$, is a* **partition** *of I, if and only if:*

$$\begin{array}{rl} (i) & \forall\lambda\in\Lambda\ ,\ I_\lambda \neq \emptyset \\ (ii) & \forall\lambda,\lambda'\in\Lambda\ ,\ \lambda\neq\lambda' \Rightarrow I_\lambda\cap I_{\lambda'} = \emptyset \\ (iii) & I = \cup_{\lambda\in\Lambda} I_\lambda \end{array}$$

Exercise 2. Let $(\Omega_i)_{i\in I}$ be a family of sets indexed by I, and $(I_\lambda)_{\lambda\in\Lambda}$ be a partition of the set I.

1. For each $\lambda\in\Lambda$, recall the definition of $\Pi_{i\in I_\lambda}\Omega_i$.

2. Recall the definition of $\Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$.

3. Define a *natural* bijection $\Phi : \Pi_{i\in I}\Omega_i \to \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda}\Omega_i)$.

4. Define a *natural* bijection $\psi : \mathbf{R}^p \times \mathbf{R}^n \to \mathbf{R}^{p+n}$, for all $n,p \geq 1$.

Definition 52 *Let* $(\Omega_i)_{i\in I}$ *be a family of sets, indexed by a non-empty set* I*. For all* $i \in I$*, let* $\mathcal{E}_i$ *be a set of subsets of* Ω_i*. We define a* **rectangle** *of the family* $(\mathcal{E}_i)_{i\in I}$*, as any subset* A *of* $\Pi_{i\in I}\Omega_i$*, of the form* $A = \Pi_{i\in I}A_i$ *where* $A_i \in \mathcal{E}_i \cup \{\Omega_i\}$ *for all* $i \in I$*, and such that* $A_i = \Omega_i$ *except for a finite number of indices* $i \in I$*. Consequently, the set of all rectangles, denoted* $\amalg_{i\in I}\mathcal{E}_i$*, is defined as:*

$$\coprod_{i\in I}\mathcal{E}_i \stackrel{\triangle}{=} \left\{\prod_{i\in I}A_i : A_i \in \mathcal{E}_i \cup \{\Omega_i\}\ ,\ A_i \neq \Omega_i \textit{ for finitely many } i \in I\right\}$$

Exercise 3. $(\Omega_i)_{i\in I}$ and $(\mathcal{E}_i)_{i\in I}$ being as above:

1. Show that if $I = \mathbf{N}_n$ and $\Omega_i \in \mathcal{E}_i$ for all $i = 1,\ldots,n$, then $\mathcal{E}_1 \amalg \ldots \amalg \mathcal{E}_n = \{A_1 \times \ldots \times A_n\ :\ A_i \in \mathcal{E}_i\ ,\ \forall i \in I\}$.

2. Let A be a rectangle. Show that there exists a finite subset J of I such that: $A = \{\omega \in \Pi_{i\in I}\Omega_i\ :\ \omega(j) \in A_j\ ,\ \forall j \in J\}$ for some A_j's such that $A_j \in \mathcal{E}_j$, for all $j \in J$.

Definition 53 *Let $(\Omega_i, \mathcal{F}_i)_{i\in I}$ be a family of measurable spaces, indexed by a non-empty set I. We call* **measurable rectangle** *, any rectangle of the family $(\mathcal{F}_i)_{i\in I}$. The set of all measurable rectangles is given by* [2]*:*

$$\coprod_{i\in I} \mathcal{F}_i \stackrel{\triangle}{=} \left\{ \prod_{i\in I} A_i \ : \ A_i \in \mathcal{F}_i \ , \ A_i \neq \Omega_i \text{ for finitely many } i \in I \right\}$$

Definition 54 *Let $(\Omega_i, \mathcal{F}_i)_{i\in I}$ be a family of measurable spaces, indexed by a non-empty set I. We define the* **product σ-algebra** *of $(\mathcal{F}_i)_{i\in I}$, as the σ-algebra on $\Pi_{i\in I}\Omega_i$, denoted $\otimes_{i\in I}\mathcal{F}_i$, and generated by all measurable rectangles, i.e.*

$$\bigotimes_{i\in I} \mathcal{F}_i \stackrel{\triangle}{=} \sigma\left(\coprod_{i\in I} \mathcal{F}_i \right)$$

[2]Note that $\Omega_i \in \mathcal{F}_i$ for all $i \in I$.

Exercise 4.

1. Suppose $I = \mathbf{N}_n$. Show that $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$ is generated by all sets of the form $A_1 \times \ldots \times A_n$, where $A_i \in \mathcal{F}_i$ for all $i = 1, \ldots, n$.

2. Show that $\mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$ is generated by sets of the form $A \times B \times C$ where $A, B, C \in \mathcal{B}(\mathbf{R})$.

3. Show that if $(\Omega, \mathcal{F})$ is a measurable space, $\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}$ is the σ-algebra on $\mathbf{R}^+ \times \Omega$ generated by sets of the form $B \times F$ where $B \in \mathcal{B}(\mathbf{R}^+)$ and $F \in \mathcal{F}$.

Exercise 5. Let $(\Omega_i)_{i \in I}$ be a family of non-empty sets and $\mathcal{E}_i$ be a subset of the power set $\mathcal{P}(\Omega_i)$ for all $i \in I$.

1. Give a generator of the σ-algebra $\otimes_{i \in I} \sigma(\mathcal{E}_i)$ on $\Pi_{i \in I} \Omega_i$.

2. Show that:

$$\sigma\left(\coprod_{i \in I} \mathcal{E}_i\right) \subseteq \bigotimes_{i \in I} \sigma(\mathcal{E}_i)$$

3. Let A be a rectangle of the family $(\sigma(\mathcal{E}_i))_{i\in I}$. Show that if A is not empty, then the representation $A = \Pi_{i\in I} A_i$ with $A_i \in \sigma(\mathcal{E}_i)$ is unique. Define $J_A = \{i \in I : A_i \neq \Omega_i\}$. Explain why J_A is a well-defined finite subset of I.

4. If $A \in \amalg_{i\in I}\sigma(\mathcal{E}_i)$, Show that if $A = \emptyset$, or $A \neq \emptyset$ and $J_A = \emptyset$, then $A \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$.

Exercise 6. Everything being as before, Let $n \geq 0$. We assume that the following induction hypothesis has been proved:

$$A \in \coprod_{i\in I}\sigma(\mathcal{E}_i), A \neq \emptyset, \mathrm{card} J_A = n \;\Rightarrow\; A \in \sigma\left(\coprod_{i\in I}\mathcal{E}_i\right)$$

We assume that A is a non empty measurable rectangle of $(\sigma(\mathcal{E}_i))_{i\in I}$ with $\mathrm{card} J_A = n+1$. Let $J_A = \{i_1, \ldots, i_{n+1}\}$ be an extension of J_A. For all $B \subseteq \Omega_{i_1}$, we define:

$$A^B \stackrel{\triangle}{=} \prod_{i\in I}\bar{A}_i$$

where each $\bar{A}_i$ is equal to A_i except $\bar{A}_{i_1} = B$. We define the set:

$$\Gamma \stackrel{\triangle}{=} \left\{ B \subseteq \Omega_{i_1} \ : \ A^B \in \sigma\left(\coprod_{i\in I} \mathcal{E}_i\right)\right\}$$

1. Show that $A^{\Omega_{i_1}} \neq \emptyset$, $\mathrm{card} J_{A^{\Omega_{i_1}}} = n$ and that $A^{\Omega_{i_1}} \in \amalg_{i\in I}\sigma(\mathcal{E}_i)$.
2. Show that $\Omega_{i_1} \in \Gamma$.
3. Show that for all $B \subseteq \Omega_{i_1}$, we have $A^{\Omega_{i_1}\setminus B} = A^{\Omega_{i_1}} \setminus A^B$.
4. Show that $B \in \Gamma \Rightarrow \Omega_{i_1} \setminus B \in \Gamma$.
5. Let $B_n \subseteq \Omega_{i_1}$, $n \geq 1$. Show that $A^{\cup B_n} = \cup_{n\geq 1} A^{B_n}$.
6. Show that Γ is a σ-algebra on Ω_{i_1}.
7. Let $B \in \mathcal{E}_{i_1}$, and for $i \in I$ define $\bar{B}_i = \Omega_i$ for all i's except $\bar{B}_{i_1} = B$. Show that $A^B = A^{\Omega_{i_1}} \cap (\Pi_{i\in I}\bar{B}_i)$.
8. Show that $\sigma(\mathcal{E}_{i_1}) \subseteq \Gamma$.

9. Show that $A = A^{A_{i_1}}$ and $A \in \sigma(\amalg_{i\in I}\mathcal{E}_i)$.

10. Show that $\amalg_{i\in I}\sigma(\mathcal{E}_i) \subseteq \sigma(\amalg_{i\in I}\mathcal{E}_i)$.

11. Show that $\sigma(\amalg_{i\in I}\mathcal{E}_i) = \otimes_{i\in I}\sigma(\mathcal{E}_i)$.

Theorem 26 *Let $(\Omega_i)_{i\in I}$ be a family of non-empty sets indexed by a non-empty set I. For all $i \in I$, let $\mathcal{E}_i$ be a set of subsets of Ω_i. Then, the product σ-algebra $\otimes_{i\in I}\sigma(\mathcal{E}_i)$ on the cartesian product $\Pi_{i\in I}\Omega_i$ is generated by the rectangles of $(\mathcal{E}_i)_{i\in I}$, i.e. :*

$$\bigotimes_{i\in I}\sigma(\mathcal{E}_i) = \sigma\left(\coprod_{i\in I}\mathcal{E}_i\right)$$

Exercise 7. Let $\mathcal{T}_{\mathbf{R}}$ denote the usual topology in $\mathbf{R}$. Let $n \geq 1$.

1. Show that $\mathcal{T}_{\mathbf{R}} \amalg \ldots \amalg \mathcal{T}_{\mathbf{R}} = \{A_1 \times \ldots \times A_n \ : \ A_i \in \mathcal{T}_{\mathbf{R}}\}$.

2. Show that $\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}) = \sigma(\mathcal{T}_{\mathbf{R}} \amalg \ldots \amalg \mathcal{T}_{\mathbf{R}})$.

3. Define $\mathcal{C}_2 = \{]a_1, b_1] \times \ldots \times]a_n, b_n] \ : \ a_i, b_i \in \mathbf{R}\}$. Show that $\mathcal{C}_2 \subseteq \mathcal{S} \amalg \ldots \amalg \mathcal{S}$, where $\mathcal{S} = \{]a, b] \ : \ a, b \in \mathbf{R}\}$, but that the inclusion is strict.

4. Show that $\mathcal{S} \amalg \ldots \amalg \mathcal{S} \subseteq \sigma(\mathcal{C}_2)$.

5. Show that $\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}) = \sigma(\mathcal{C}_2)$.

Exercise 8. Let Ω and Ω' be two non-empty sets. Let A be a subset of Ω such that $\emptyset \neq A \neq \Omega$. Let $\mathcal{E} = \{A\} \subseteq \mathcal{P}(\Omega)$ and $\mathcal{E}' = \emptyset \subseteq \mathcal{P}(\Omega')$.

1. Show that $\sigma(\mathcal{E}) = \{\emptyset, A, A^c, \Omega\}$.

2. Show that $\sigma(\mathcal{E}') = \{\emptyset, \Omega'\}$.

3. Define $\mathcal{C} = \{E \times F \ , \ E \in \mathcal{E}, F \in \mathcal{E}'\}$ and show that $\mathcal{C} = \emptyset$.

4. Show that $\mathcal{E} \amalg \mathcal{E}' = \{A \times \Omega', \Omega \times \Omega'\}$.

5. Show that $\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}') = \{\emptyset, A \times \Omega', A^c \times \Omega', \Omega \times \Omega'\}$.

6. Conclude that $\sigma(\mathcal{E}) \otimes \sigma(\mathcal{E}') \neq \sigma(\mathcal{C}) = \{\emptyset, \Omega \times \Omega'\}$.

Exercise 9. Let $n \geq 1$ and $p \geq 1$ be two positive integers.

1. Define $\mathcal{F} = \underbrace{\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R})}_{n}$, and $\mathcal{G} = \underbrace{\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R})}_{p}$.
   Explain why $\mathcal{F} \otimes \mathcal{G}$ can be viewed as a σ-algebra on $\mathbf{R}^{n+p}$.

2. Show that $\mathcal{F} \otimes \mathcal{G}$ is generated by sets of the form $A_1 \times \ldots \times A_{n+p}$ where $A_i \in \mathcal{B}(\mathbf{R}), i = 1, \ldots, n+p$.

3. Show that:

$$\underbrace{\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R})}_{n+p} = \underbrace{(\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}))}_{n} \otimes \underbrace{(\mathcal{B}(\mathbf{R}) \otimes \ldots \otimes \mathcal{B}(\mathbf{R}))}_{p}$$

Exercise 10. Let $(\Omega_i, \mathcal{F}_i)_{i \in I}$ be a family of measurable spaces. Let $(I_\lambda)_{\lambda \in \Lambda}$, where $\Lambda \neq \emptyset$, be a partition of I. Let $\Omega = \Pi_{i \in I} \Omega_i$ and $\Omega' = \Pi_{\lambda \in \Lambda}(\Pi_{i \in I_\lambda} \Omega_i)$.

1. Define a *natural* bijection between $\mathcal{P}(\Omega)$ and $\mathcal{P}(\Omega')$.

2. Show that through such bijection, $A = \Pi_{i\in I} A_i \subseteq \Omega$, where $A_i \subseteq \Omega_i$, is identified with $A' = \Pi_{\lambda\in\Lambda}(\Pi_{i\in I_\lambda} A_i) \subseteq \Omega'$.

3. Show that $\amalg_{i\in I}\mathcal{F}_i = \amalg_{\lambda\in\Lambda}(\amalg_{i\in I_\lambda}\mathcal{F}_i)$.

4. Show that $\otimes_{i\in I}\mathcal{F}_i = \otimes_{\lambda\in\Lambda}(\otimes_{i\in I_\lambda}\mathcal{F}_i)$.

Definition 55 *Let Ω be set and $\mathcal{A}$ be a set of subsets of Ω. We call* **topology generated** *by $\mathcal{A}$, the topology on Ω, denoted $\mathcal{T}(\mathcal{A})$, equal to the intersection of all topologies on Ω, which contain $\mathcal{A}$.*

EXERCISE 11. Let Ω be a set and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$.

1. Explain why $\mathcal{T}(\mathcal{A})$ is indeed a topology on Ω.

2. Show that $\mathcal{T}(\mathcal{A})$ is the smallest topology $\mathcal{T}$ such that $\mathcal{A} \subseteq \mathcal{T}$.

3. Show that the metric topology on a metric space (E, d) is generated by the open balls $\mathcal{A} = \{B(x,\epsilon) : x \in E, \epsilon > 0\}$.

Definition 56 *Let $(\Omega_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces, indexed by a non-empty set I. We define the* **product topology** *of $(\mathcal{T}_i)_{i\in I}$, as the topology on $\Pi_{i\in I}\Omega_i$, denoted $\odot_{i\in I}\mathcal{T}_i$, and generated by all rectangles of $(\mathcal{T}_i)_{i\in I}$, i.e.*

$$\bigodot_{i\in I}\mathcal{T}_i \stackrel{\triangle}{=} \mathcal{T}\left(\coprod_{i\in I}\mathcal{T}_i\right)$$

Exercise 12. Let $(\Omega_i, \mathcal{T}_i)_{i\in I}$ be a family of topological spaces.

1. Show that $U \in \odot_{i\in I}\mathcal{T}_i$, if and only if:

$$\forall x \in U\ ,\ \exists V \in \amalg_{i\in I}\mathcal{T}_i\ ,\ x \in V \subseteq U$$

2. Show that $\amalg_{i\in I}\mathcal{T}_i \subseteq \odot_{i\in I}\mathcal{T}_i$.

3. Show that $\otimes_{i\in I}\mathcal{B}(\Omega_i) = \sigma(\amalg_{i\in I}\mathcal{T}_i)$.

4. Show that $\otimes_{i\in I}\mathcal{B}(\Omega_i) \subseteq \mathcal{B}(\Pi_{i\in I}\Omega_i)$.

Exercise 13. Let $n \geq 1$ be a positive integer. For all $x, y \in \mathbf{R}^n$, let:

$$(x, y) \stackrel{\triangle}{=} \sum_{i=1}^{n} x_i y_i$$

and we put $\|x\| = \sqrt{(x, x)}$.

1. Show that for all $t \in \mathbf{R}$, $\|x + ty\|^2 = \|x\|^2 + t^2\|y\|^2 + 2t(x, y)$.
2. From $\|x + ty\|^2 \geq 0$ for all t, deduce that $|(x, y)| \leq \|x\|.\|y\|$.
3. Conclude that $\|x + y\| \leq \|x\| + \|y\|$.

Exercise 14. Let $(\Omega_1, \mathcal{T}_1), \ldots, (\Omega_n, \mathcal{T}_n)$, $n \geq 1$, be metrizable topological spaces. Let $d_1, \ldots, d_n$ be metrics on $\Omega_1, \ldots, \Omega_n$, inducing the topologies $\mathcal{T}_1, \ldots, \mathcal{T}_n$ respectively. Let $\Omega = \Omega_1 \times \ldots \times \Omega_n$ and $\mathcal{T}$ be the product topology on Ω. For all $x, y \in \Omega$, we define:

$$d(x, y) \stackrel{\triangle}{=} \sqrt{\sum_{i=1}^{n} (d_i(x_i, y_i))^2}$$

1. Show that $d : \Omega \times \Omega \to \mathbf{R}^+$ is a metric on Ω.

2. Show that $U \subseteq \Omega$ is open in Ω, if and only if, for all $x \in U$ there are open sets $U_1, \ldots, U_n$ in $\Omega_1, \ldots, \Omega_n$ respectively, such that:
$$x \in U_1 \times \ldots \times U_n \subseteq U$$

3. Let $U \in \mathcal{T}$ and $x \in U$. Show the existence of $\epsilon > 0$ such that:
$$(\forall i = 1, \ldots, n \ d_i(x_i, y_i) < \epsilon) \ \Rightarrow \ y \in U$$

4. Show that $\mathcal{T} \subseteq \mathcal{T}^d_\Omega$.

5. let $U \in \mathcal{T}^d_\Omega$ and $x \in U$. Show that existence of $\epsilon > 0$ such that:
$$x \in B(x_1, \epsilon) \times \ldots \times B(x_n, \epsilon) \subseteq U$$

6. Show that $\mathcal{T}^d_\Omega \subseteq \mathcal{T}$.

7. Show that the product topological space $(\Omega, \mathcal{T})$ is metrizable.

8. For all $x, y \in \Omega$, define:

$$
\begin{aligned}
d'(x,y) &\stackrel{\triangle}{=} \sum_{i=1}^{n} d_i(x_i, y_i) \\
d''(x,y) &\stackrel{\triangle}{=} \max_{i=1,\ldots,n} d_i(x_i, y_i)
\end{aligned}
$$

   Show that d', d'' are metrics on Ω.

9. Show the existence of α', β', α'' and $\beta'' > 0$, such that we have $\alpha' d' \leq d \leq \beta' d'$ and $\alpha'' d'' \leq d \leq \beta'' d''$.

10. Show that d' and d'' also induce the product topology on Ω.

Exercise 15. Let $(\Omega_n, \mathcal{T}_n)_{n\geq 1}$ be a sequence of metrizable topological spaces. For all $n \geq 1$, let d_n be a metric on Ω_n inducing the topology $\mathcal{T}_n$. Let $\Omega = \Pi_{n=1}^{+\infty}\Omega_n$ be the cartesian product and $\mathcal{T}$ be the product topology on Ω. For all $x, y \in \Omega$, we define:

$$d(x,y) \stackrel{\triangle}{=} \sum_{n=1}^{+\infty} \frac{1}{2^n}(1 \wedge d_n(x_n, y_n))$$

1. Show that for all $a, b \in \mathbf{R}^+$, we have $1 \wedge (a+b) \leq 1 \wedge a + 1 \wedge b$.

2. Show that d is a metric on Ω.

3. Show that $U \subseteq \Omega$ is open in Ω, if and only if, for all $x \in U$, there is an integer $N \geq 1$ and open sets $U_1, \ldots, U_N$ in $\Omega_1, \ldots, \Omega_N$ respectively, such that:

$$x \in U_1 \times \ldots \times U_N \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

4. Show that $d(x,y) < 1/2^n \;\Rightarrow\; d_n(x_n, y_n) \leq 2^n d(x,y)$.

5. Show that for all $U \in \mathcal{T}$ and $x \in U$, there exists $\epsilon > 0$ such that $d(x,y) < \epsilon \;\Rightarrow\; y \in U$.

6. Show that $\mathcal{T} \subseteq \mathcal{T}_\Omega^d$.

7. Let $U \in \mathcal{T}_\Omega^d$ and $x \in U$. Show the existence of $\epsilon > 0$ and $N \geq 1$, such that:
$$\sum_{n=1}^{N} \frac{1}{2^n}(1 \wedge d_n(x_n, y_n)) < \epsilon \;\Rightarrow\; y \in U$$

8. Show that for all $U \in \mathcal{T}_\Omega^d$ and $x \in U$, there is $\epsilon > 0$ and $N \geq 1$ such that:
$$x \in B(x_1, \epsilon) \times \ldots \times B(x_N, \epsilon) \times \prod_{n=N+1}^{+\infty} \Omega_n \subseteq U$$

9. Show that $\mathcal{T}_\Omega^d \subseteq \mathcal{T}$.

10. Show that the product topological space $(\Omega, \mathcal{T})$ is metrizable.

Definition 57 *Let $(\Omega, \mathcal{T})$ be a topological space. A subset $\mathcal{H}$ of $\mathcal{T}$ is called a* **countable base** *of $(\Omega, \mathcal{T})$, if and only if $\mathcal{H}$ is at most countable, and has the property:*

$$\forall U \in \mathcal{T} \ , \ \exists \mathcal{H}' \subseteq \mathcal{H} \ , \ U = \bigcup_{V \in \mathcal{H}'} V$$

Exercise 16.

1. Show that $\mathcal{H} = \{]r, q[\ : \ r, q \in \mathbf{Q}\}$ is a countable base of $(\mathbf{R}, \mathcal{T}_{\mathbf{R}})$.

2. Show that if $(\Omega, \mathcal{T})$ is a topological space with countable base, and $\Omega' \subseteq \Omega$, then the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ also has a countable base.

3. Show that $[-1, 1]$ has a countable base.

4. Show that if $(\Omega, \mathcal{T})$ and $(S, \mathcal{T}_S)$ are homeomorphic, then $(\Omega, \mathcal{T})$ has a countable base if and only if $(S, \mathcal{T}_S)$ has a countable base.

5. Show that $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ has a countable base.

Exercise 17. Let $(\Omega_n, \mathcal{T}_n)_{n\geq 1}$ be a sequence of topological spaces with countable base. For $n \geq 1$, Let $\{V_n^k : k \in I_n\}$ be a countable base of $(\Omega_n, \mathcal{T}_n)$ where I_n is a finite or countable set. Let $\Omega = \Pi_{n=1}^{\infty}\Omega_n$ be the cartesian product and $\mathcal{T}$ be the product topology on Ω. For all $p \geq 1$, we define:

$$\mathcal{H}^p \stackrel{\triangle}{=} \left\{ V_1^{k_1} \times \ldots \times V_p^{k_p} \times \prod_{n=p+1}^{+\infty} \Omega_n \ : \ (k_1, \ldots, k_p) \in I_1 \times \ldots \times I_p \right\}$$

and we put $\mathcal{H} = \cup_{p\geq 1}\mathcal{H}^p$.

1. Show that for all $p \geq 1$, $\mathcal{H}^p \subseteq \mathcal{T}$.

2. Show that $\mathcal{H} \subseteq \mathcal{T}$.

3. For all $p \geq 1$, show the existence of an injection $j_p : \mathcal{H}^p \to \mathbf{N}^p$.

4. Show the existence of a bijection $\phi_2 : \mathbf{N}^2 \to \mathbf{N}$.

5. For $p \geq 1$, show the existence of an bijection $\phi_p : \mathbf{N}^p \to \mathbf{N}$.

6. Show that $\mathcal{H}^p$ is at most countable for all $p \geq 1$.

7. Show the existence of an injection $j : \mathcal{H} \to \mathbf{N}^2$.

8. Show that $\mathcal{H}$ is a finite or countable set of open sets in Ω.

9. Let $U \in \mathcal{T}$ and $x \in U$. Show that there is $p \geq 1$ and $U_1, \ldots, U_p$ open sets in $\Omega_1, \ldots, \Omega_p$ such that:

$$x \in U_1 \times \ldots \times U_p \times \prod_{n=p+1}^{+\infty} \Omega_n \subseteq U$$

10. Show the existence of some $V_x \in \mathcal{H}$ such that $x \in V_x \subseteq U$.

11. Show that $\mathcal{H}$ is a countable base of the topological space $(\Omega, \mathcal{T})$.

12. Show that $\otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n) \subseteq \mathcal{B}(\Omega)$.

13. Show that $\mathcal{H} \subseteq \otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$.

14. Show that $\mathcal{B}(\Omega) = \otimes_{n=1}^{+\infty} \mathcal{B}(\Omega_n)$

Theorem 27 *Let $(\Omega_n, \mathcal{T}_n)_{n\geq 1}$ be a sequence of topological spaces with countable base. Then, the product space $(\Pi_{n=1}^{+\infty}\Omega_n, \odot_{n=1}^{+\infty}\mathcal{T}_n)$ has a countable base and:*

$$\mathcal{B}\left(\prod_{n=1}^{+\infty}\Omega_n\right) = \bigotimes_{n=1}^{+\infty}\mathcal{B}(\Omega_n)$$

Exercise 18.

1. Show that if $(\Omega, \mathcal{T})$ has a countable base and $n \geq 1$:

$$\mathcal{B}(\Omega^n) = \underbrace{\mathcal{B}(\Omega) \otimes \ldots \otimes \mathcal{B}(\Omega)}_{n}$$

2. Show that $\mathcal{B}(\bar{\mathbf{R}}^n) = \mathcal{B}(\bar{\mathbf{R}}) \otimes \ldots \otimes \mathcal{B}(\bar{\mathbf{R}})$.

3. Show that $\mathcal{B}(\mathbf{C}) = \mathcal{B}(\mathbf{R}) \otimes \mathcal{B}(\mathbf{R})$.

Definition 58 *We say that a metric space (E,d) is* **separable**, *if and only if there exists a finite or countable dense subset of E, i.e. a finite or countable subset A of E such that $E=\bar{A}$, where $\bar{A}$ is the closure of A in E.*

EXERCISE 19. Let (E,d) be a metric space.

1. Suppose that (E,d) is separable. Let $\mathcal{H}=\{B(x_n,\frac{1}{p}) : n,p\geq 1\}$, where $\{x_n : n\geq 1\}$ is a countable dense subset in E. Show that $\mathcal{H}$ is a countable base of the metric topological space $(E,\mathcal{T}^d_E)$.

2. Suppose conversely that $(E,\mathcal{T}^d_E)$ has a countable base $\mathcal{H}$. For all $V\in\mathcal{H}$ such that $V\neq\emptyset$, take $x_V\in V$. Show that the set $\{x_V\ :\ V\in\mathcal{H}\ ,\ V\neq\emptyset\}$ is at most countable and dense in E.

3. For all $x,y,x',y'\in E$, show that:

$$|d(x,y)-d(x',y')|\leq d(x,x')+d(y,y')$$

4. Let $\mathcal{T}_{E\times E}$ be the product topology on $E \times E$. Show that the map $d : (E \times E, \mathcal{T}_{E\times E}) \to (\mathbf{R}^+, \mathcal{T}_{\mathbf{R}^+})$ is continuous.

5. Show that $d : (E \times E, \mathcal{B}(E \times E)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

6. Show that $d : (E \times E, \mathcal{B}(E) \otimes \mathcal{B}(E)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, whenever (E, d) is a separable metric space.

7. Let $(\Omega, \mathcal{F})$ be a measurable space and $f, g : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E))$ be measurable maps. Show that $\Phi : (\Omega, \mathcal{F}) \to E \times E$ defined by $\Phi(\omega) = (f(\omega), g(\omega))$ is measurable with respect to the product σ-algebra $\mathcal{B}(E) \otimes \mathcal{B}(E)$.

8. Show that if (E, d) is separable, then $\Psi : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ defined by $\Psi(\omega) = d(f(\omega), g(\omega))$ is measurable.

9. Show that if (E, d) is separable then $\{f = g\} \in \mathcal{F}$.

10. Let $(E_n, d_n)_{n\geq 1}$ be a sequence of separable metric spaces. Show that the product space $\Pi_{n=1}^{+\infty} E_n$ is metrizable and separable.

Exercise 20. Prove the following theorem.

Theorem 28 *Let $(\Omega_i, \mathcal{F}_i)_{i\in I}$ be a family of measurable spaces and $(\Omega, \mathcal{F})$ be a measurable space. For all $i \in I$, let $f_i : \Omega \to \Omega_i$ be a map, and define $f : \Omega \to \Pi_{i\in I}\Omega_i$ by $f(\omega) = (f_i(\omega))_{i\in I}$. Then, the map:*

$$f : (\Omega, \mathcal{F}) \to \left(\prod_{i\in I}\Omega_i, \bigotimes_{i\in I}\mathcal{F}_i\right)$$

is measurable, if and only if each $f_i : (\Omega, \mathcal{F}) \to (\Omega_i, \mathcal{F}_i)$ is measurable.

Exercise 21.

1. Let $\phi, \psi : \mathbf{R}^2 \to \mathbf{R}$ with $\phi(x, y) = x + y$ and $\psi(x, y) = x.y$. Show that both ϕ and ψ are continuous.
2. Show that $\phi, \psi : (\mathbf{R}^2, \mathcal{B}(\mathbf{R})\otimes\mathcal{B}(\mathbf{R}))\to(\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ are measurable.
3. Let $(\Omega, \mathcal{F})$ be a measurable space, and $f, g : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be measurable maps. Using the previous results, show that $f+g$ and $f.g$ are measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\mathbf{R})$.

7. Fubini Theorem

Definition 59 *Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. Let $E \subseteq \Omega_1 \times \Omega_2$. For all $\omega_1 \in \Omega_1$, we call ω_1-**section** of E in Ω_2, the set:*

$$E^{\omega_1} \stackrel{\triangle}{=} \{\omega_2 \in \Omega_2 \ : \ (\omega_1, \omega_2) \in E\}$$

Exercise 1. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. Given $\omega_1 \in \Omega_1$, define:

$$\Gamma^{\omega_1} \stackrel{\triangle}{=} \{E \subseteq \Omega_1 \times \Omega_2 \ , \ E^{\omega_1} \in \mathcal{F}_2\}$$

1. Show that for all $\omega_1 \in \Omega_1$, Γ^{ω_1} is a σ-algebra on $\Omega_1 \times \Omega_2$.
2. Show that for all $\omega_1 \in \Omega_1$, $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \Gamma^{\omega_1}$.
3. Show that for all $\omega_1 \in \Omega_1$ and $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we have $E^{\omega_1} \in \mathcal{F}_2$.
4. Show that the map $\omega \to 1_E(\omega_1, \omega)$ is measurable with respect to $\mathcal{F}_2$ and $\mathcal{B}(\bar{\mathbf{R}})$, for all $\omega_1 \in \Omega_1$ and $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

5. Let s be a simple function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. Show that for all $\omega_1 \in \Omega_1$, the map $\omega \to s(\omega_1, \omega)$ is measurable with respect to $\mathcal{F}_2$ and $\mathcal{B}(\bar{\mathbf{R}})$.

6. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty]$ be a non-negative, measurable map. Show that for all $\omega_1 \in \Omega_1$, the map $\omega \to f(\omega_1, \omega)$ is measurable with respect to $\mathcal{F}_2$ and $\mathcal{B}(\bar{\mathbf{R}})$.

7. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ be a measurable map. Show that for all $\omega_1 \in \Omega_1$, the map $\omega \to f(\omega_1, \omega)$ is measurable with respect to $\mathcal{F}_2$ and $\mathcal{B}(\bar{\mathbf{R}})$.

8. Show the following theorem:

Theorem 29 *Let (E, d) be a metric space, and $(\Omega_1, \mathcal{F}_1)$, $(\Omega_2, \mathcal{F}_2)$ be two measurable spaces. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to (E, \mathcal{B}(E))$ be a measurable map . Then for all $\omega_1 \in \Omega_1$, the map $\omega \to f(\omega_1, \omega)$ is measurable with respect to $\mathcal{F}_2$ and $\mathcal{B}(E)$.*

EXERCISE 2. Let $(\Omega_i, \mathcal{F}_i)_{i\in I}$ be a family of measurable spaces with $\text{card}I \geq 2$. Let $f : (\Pi_{i\in I}\Omega_i, \otimes_{i\in I}\mathcal{F}_i) \to (E, \mathcal{B}(E))$ be a measurable map, where (E, d) be a metric space. Let $i_1 \in I$. Put $E_1 = \Omega_{i_1}$, $\mathcal{E}_1 = \mathcal{F}_{i_1}$, $E_2 = \Pi_{i\in I\setminus\{i_1\}}\Omega_i$, $\mathcal{E}_2 = \otimes_{i\in I\setminus\{i_1\}}\mathcal{F}_i$.

1. Explain why f can be viewed as a map defined on $E_1 \times E_2$.

2. Show that $f : (E_1 \times E_2, \mathcal{E}_1 \otimes \mathcal{E}_2) \to (E, \mathcal{B}(E))$ is measurable.

3. For all $\omega_{i_1} \in \Omega_{i_1}$, show that the map $\omega \to f(\omega_{i_1}, \omega)$ defined on $\Pi_{i\in I\setminus\{i_1\}}\Omega_i$ is measurable w.r. to $\otimes_{i\in I\setminus\{i_1\}}\mathcal{F}_i$ and $\mathcal{B}(E)$.

Definition 60 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. $(\Omega, \mathcal{F}, \mu)$ is said to be a* **finite measure space***, or we say that μ is a* **finite measure***, if and only if $\mu(\Omega) < +\infty$.*

Definition 61 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. $(\Omega, \mathcal{F}, \mu)$ is said to be a* **σ-finite measure space**, *or μ a* **σ-finite measure**, *if and only if there exists a sequence $(\Omega_n)_{n\geq 1}$ in $\mathcal{F}$ such that $\Omega_n \uparrow \Omega$ and $\mu(\Omega_n) < +\infty$, for all $n \geq 1$.*

Exercise 3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space.

1. Show that $(\Omega, \mathcal{F}, \mu)$ is σ-finite if and only if there exists a sequence $(\Omega_n)_{n\geq 1}$ in $\mathcal{F}$ such that $\Omega = \uplus_{n=1}^{+\infty} \Omega_n$, and $\mu(\Omega_n) < +\infty$ for all $n \geq 1$.

2. Show that if $(\Omega, \mathcal{F}, \mu)$ is finite, then μ has values in $\mathbf{R}^+$.

3. Show that if $(\Omega, \mathcal{F}, \mu)$ is finite, then it is σ-finite.

4. Let $F : \mathbf{R} \to \mathbf{R}$ be a right-continuous, non-decreasing map. Show that the measure space $(\mathbf{R}, \mathcal{B}(\mathbf{R}), dF)$ is σ-finite, where dF is the stieltjes measure associated with F.

Exercise 4. Let $(\Omega_1, \mathcal{F}_1)$ be a measurable space, and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be a σ-finite measure space. For all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$ and $\omega_1 \in \Omega_1$, define:

$$\Phi_E(\omega_1) \stackrel{\triangle}{=} \int_{\Omega_2} 1_E(\omega_1, x) d\mu_2(x)$$

Let $\mathcal{D}$ be the set of subsets of $\Omega_1 \times \Omega_2$, defined by:

$$\mathcal{D} \stackrel{\triangle}{=} \{E \in \mathcal{F}_1 \otimes \mathcal{F}_2 \ : \ \Phi_E : (\Omega_1, \mathcal{F}_1) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}})) \text{ is measurable}\}$$

1. Explain why for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, the map Φ_E is well defined.
2. Show that $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \mathcal{D}$.
3. Show that if μ_2 is finite, $A, B \in \mathcal{D}$ and $A \subseteq B$, then $B \setminus A \in \mathcal{D}$.
4. Show that if $E_n \in \mathcal{F}_1 \otimes \mathcal{F}_2, n \geq 1$ and $E_n \uparrow E$, then $\Phi_{E_n} \uparrow \Phi_E$.
5. Show that if μ_2 is finite then $\mathcal{D}$ is a dynkin system on $\Omega_1 \times \Omega_2$.
6. Show that if μ_2 is finite, then the map $\Phi_E : (\Omega_1, \mathcal{F}_1) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

7. Let $(\Omega_2^n)_{n\geq 1}$ in $\mathcal{F}_2$ be such that $\Omega_2^n \uparrow \Omega_2$ and $\mu_2(\Omega_2^n) < +\infty$. Define $\mu_2^n = \mu_2^{\Omega_2^n} = \mu_2(\bullet \cap \Omega_2^n)$. For $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we put:
$$\Phi_E^n(\omega_1) \stackrel{\triangle}{=} \int_{\Omega_2} 1_E(\omega_1, x) d\mu_2^n(x)$$
Show that $\Phi_E^n : (\Omega_1, \mathcal{F}_1) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, and:
$$\Phi_E^n(\omega_1) = \int_{\Omega_2} 1_{\Omega_2^n}(x) 1_E(\omega_1, x) d\mu_2(x)$$
Deduce that $\Phi_E^n \uparrow \Phi_E$.

8. Show that the map $\Phi_E : (\Omega_1, \mathcal{F}_1) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable, for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$.

9. Let s be a simple function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. Show that the map $\omega \to \int_{\Omega_2} s(\omega, x) d\mu_2(x)$ is well defined and measurable with respect to $\mathcal{F}_1$ and $\mathcal{B}(\bar{\mathbf{R}})$.

10. Show the following theorem:

Theorem 30 *Let $(\Omega_1, \mathcal{F}_1)$ be a measurable space, and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be a σ-finite measure space. Then for all non-negative and measurable map $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty]$, the map:*

$$\omega \to \int_{\Omega_2} f(\omega, x) d\mu_2(x)$$

is measurable with respect to $\mathcal{F}_1$ and $\mathcal{B}(\bar{\mathbf{R}})$.

Exercise 5. Let $(\Omega_i, \mathcal{F}_i)_{i \in I}$ be a family of measurable spaces, with $\mathrm{card} I \geq 2$. Let $i_0 \in I$, and suppose that μ_0 is a σ-finite measure on $(\Omega_{i_0}, \mathcal{F}_{i_0})$. Show that if $f : (\Pi_{i \in I} \Omega_i, \otimes_{i \in I} \mathcal{F}_i) \to [0, +\infty]$ is a non-negative and measurable map, then:

$$\omega \to \int_{\Omega_{i_0}} f(\omega, x) d\mu_0(x)$$

defined on $\Pi_{i \in I \setminus \{i_0\}} \Omega_i$, is measurable w.r. to $\otimes_{i \in I \setminus \{i_0\}} \mathcal{F}_i$ and $\mathcal{B}(\bar{\mathbf{R}})$.

Exercise 6. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. For all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we define:

$$\mu_1 \otimes \mu_2(E) \stackrel{\triangle}{=} \int_{\Omega_1} \left(\int_{\Omega_2} 1_E(x, y) d\mu_2(y) \right) d\mu_1(x)$$

1. Explain why $\mu_1 \otimes \mu_2 : \mathcal{F}_1 \otimes \mathcal{F}_2 \to [0, +\infty]$ is well defined.

2. Show that $\mu_1 \otimes \mu_2$ is a measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$.

3. Show that if $A \times B \in \mathcal{F}_1 \amalg \mathcal{F}_2$, then:

$$\mu_1 \otimes \mu_2(A \times B) = \mu_1(A)\mu_2(B)$$

Exercise 7. Further to ex. (6), suppose that $\mu : \mathcal{F}_1 \otimes \mathcal{F}_2 \to [0, +\infty]$ is another measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$ with $\mu(A \times B) = \mu_1(A)\mu_2(B)$, for all measurable rectangle $A \times B$. Let $(\Omega_1^n)_{n \geq 1}$ and $(\Omega_2^n)_{n \geq 1}$ be sequences in $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, such that $\Omega_1^n \uparrow \Omega_1$, $\Omega_2^n \uparrow \Omega_2$, $\mu_1(\Omega_1^n) < +\infty$ and $\mu_2(\Omega_2^n) < +\infty$. Define, for all $n \geq 1$:

$$\mathcal{D}_n \stackrel{\triangle}{=} \{E \in \mathcal{F}_1 \otimes \mathcal{F}_2 : \mu(E \cap (\Omega_1^n \times \Omega_2^n)) = \mu_1 \otimes \mu_2(E \cap (\Omega_1^n \times \Omega_2^n))\}$$

1. Show that for all $n \geq 1$, $\mathcal{F}_1 \amalg \mathcal{F}_2 \subseteq \mathcal{D}_n$.

2. Show that for all $n \geq 1$, $\mathcal{D}_n$ is a dynkin system on $\Omega_1 \times \Omega_2$.

3. Show that $\mu = \mu_1 \otimes \mu_2$.

4. Show that $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$ is a σ-finite measure space.

5. Show that for all $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we have:
$$\mu_1 \otimes \mu_2(E) = \int_{\Omega_2} \left(\int_{\Omega_1} 1_E(x, y) d\mu_1(x) \right) d\mu_2(y)$$

Exercise 8. Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, $n \geq 2$. Let $i_0 \in \{1, \ldots, n\}$ and put $E_1 = \Omega_{i_0}$, $E_2 = \Pi_{i \neq i_0} \Omega_i$, $\mathcal{E}_1 = \mathcal{F}_{i_0}$ and $\mathcal{E}_2 = \otimes_{i \neq i_0} \mathcal{F}_i$. Put $\nu_1 = \mu_{i_0}$, and suppose that ν_2 is a σ-finite measure on $(E_2, \mathcal{E}_2)$ such that for all measurable rectangle $\Pi_{i \neq i_0} A_i \in \amalg_{i \neq i_0} \mathcal{F}_i$, we have $\nu_2 \left(\Pi_{i \neq i_0} A_i \right) = \Pi_{i \neq i_0} \mu_i(A_i)$.

1. Show that $\nu_1 \otimes \nu_2$ is a σ-finite measure on the measure space $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$ such that for all measurable rectangles $A_1 \times \ldots \times A_n$, we have:
$$\nu_1 \otimes \nu_2(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$
2. Show by induction the existence of a measure μ on $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, such that for all measurable rectangles $A_1 \times \ldots \times A_n$, we have:
$$\mu(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$
3. Show the uniqueness of such measure, denoted $\mu_1 \otimes \ldots \otimes \mu_n$.
4. Show that $\mu_1 \otimes \ldots \otimes \mu_n$ is σ-finite.
5. Let $i_0 \in \{1, \ldots, n\}$. Show that $\mu_{i_0} \otimes (\otimes_{i \neq i_0} \mu_i) = \mu_1 \otimes \ldots \otimes \mu_n$.

Definition 62 *Let* $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ *be* n σ*-finite measure spaces, with* $n \geq 2$*. We call* **product measure** *of* $\mu_1, \ldots, \mu_n$*, the unique measure on* $\mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$*, denoted* $\mu_1 \otimes \ldots \otimes \mu_n$*, such that for all measurable rectangles* $A_1 \times \ldots \times A_n$ *in* $\mathcal{F}_1 \amalg \ldots \amalg \mathcal{F}_n$*, we have:*

$$\mu_1 \otimes \ldots \otimes \mu_n(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

This measure is itself σ*-finite.*

Exercise 9. Prove that the following definition is legitimate:

Definition 63 *We call* **lebesgue measure** *in* $\mathbf{R}^n$, $n \geq 1$*, the unique measure on* $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$*, denoted* dx*,* dx^n *or* $dx_1 \ldots dx_n$*, such that for all* $a_i \leq b_i$*,* $i = 1, \ldots, n$*, we have:*

$$dx([a_1, b_1] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n} (b_i - a_i)$$

Exercise 10.

1. Show that $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx^n)$ is a σ-finite measure space.

2. For $n, p \geq 1$, show that $dx^{n+p} = dx^n \otimes dx^p$.

Exercise 11. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be σ-finite.

1. Let s be a simple function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$. Show that:

$$\int_{\Omega_1 \times \Omega_2} s d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left(\int_{\Omega_2} s d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} s d\mu_1 \right) d\mu_2$$

2. Show the following:

Theorem 31 (Fubini) *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. Let $f : (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \to [0, +\infty]$ be a non-negative and measurable map. Then:*

$$\int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2 = \int_{\Omega_1} \left(\int_{\Omega_2} f d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} f d\mu_1 \right) d\mu_2$$

Exercise 12. Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, $n \geq 2$. Let $f : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \to [0, +\infty]$ be a non-negative, measurable map. Let σ be a permutation of $\mathbf{N}_n$, i.e. a bijection from $\mathbf{N}_n$ to itself.

1. For all $\omega \in \Pi_{i \neq \sigma(1)} \Omega_i$, define:

$$J_1(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(1)}} f(\omega, x) d\mu_{\sigma(1)}(x)$$

   Explain why $J_1 : (\Pi_{i \neq \sigma(1)} \Omega_i, \otimes_{i \neq \sigma(1)} \mathcal{F}_i) \to [0, +\infty]$ is a well defined, non-negative and measurable map.

2. Suppose $J_k : (\Pi_{i \notin \{\sigma(1), \ldots, \sigma(k)\}} \Omega_i, \otimes_{i \notin \{\sigma(1), \ldots, \sigma(k)\}} \mathcal{F}_i) \to [0, +\infty]$ is a non-negative, measurable map, for $1 \leq k < n - 2$. Define:

$$J_{k+1}(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(k+1)}} J_k(\omega, x) d\mu_{\sigma(k+1)}(x)$$

   and show that:

$$J_{k+1} : (\Pi_{i \notin \{\sigma(1), \ldots, \sigma(k+1)\}} \Omega_i, \otimes_{i \notin \{\sigma(1), \ldots, \sigma(k+1)\}} \mathcal{F}_i) \to [0, +\infty]$$

is also well-defined, non-negative and measurable.

3. Propose a rigorous definition for the following notation:

$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

Exercise 13. Further to ex. (12), Let $(f_p)_{p\geq 1}$ be a sequence of non-negative and measurable maps:

$$f_p : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \to [0, +\infty]$$

such that $f_p \uparrow f$. Define similarly:

$$\begin{aligned} J_1^p(\omega) &\stackrel{\triangle}{=} \int_{\Omega_{\sigma(1)}} f_p(\omega, x) d\mu_{\sigma(1)}(x) \\ J_{k+1}^p(\omega) &\stackrel{\triangle}{=} \int_{\Omega_{\sigma(k+1)}} J_k^p(\omega, x) d\mu_{\sigma(k+1)}(x) \ , \ 1 \leq k < n-2 \end{aligned}$$

1. Show that $J_1^p \uparrow J_1$.

2. Show that if $J_k^p \uparrow J_k$, then $J_{k+1}^p \uparrow J_{k+1}$, $1 \leq k < n-2$.

3. Show that:
$$\int_{\Omega_{\sigma(n)}} \dots \int_{\Omega_{\sigma(1)}} f_p d\mu_{\sigma(1)} \dots d\mu_{\sigma(n)} \uparrow \int_{\Omega_{\sigma(n)}} \dots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \dots d\mu_{\sigma(n)}$$

4. Show that the map $\mu : \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n \to [0, +\infty]$, defined by:
$$\mu(E) = \int_{\Omega_{\sigma(n)}} \dots \int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)} \dots d\mu_{\sigma(n)}$$
is a measure on $\mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$.

5. Show that for all $E \in \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n$, we have:
$$\mu_1 \otimes \dots \otimes \mu_n(E) = \int_{\Omega_{\sigma(n)}} \dots \int_{\Omega_{\sigma(1)}} 1_E d\mu_{\sigma(1)} \dots d\mu_{\sigma(n)}$$

6. Show the following:

Theorem 32 *Let $(\Omega_1, \mathcal{F}_1, \mu_1), \ldots, (\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, with $n \geq 2$. Let $f : (\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n) \to [0, +\infty]$ be a non-negative and measurable map. let σ be a permutation of $\mathbf{N}_n$. Then:*

$$\int_{\Omega_1 \times \ldots \times \Omega_n} f d\mu_1 \otimes \ldots \otimes \mu_n = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

Exercise 14. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define:

$$L^1 \stackrel{\triangle}{=} \{f : \Omega \to \bar{\mathbf{R}}\ ,\ \exists g \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)\ ,\ f = g\ \mu\text{-a.s.}\}$$

1. Show that if $f \in L^1$, then $|f| < +\infty$, μ-a.s.
2. Suppose there exists $A \subseteq \Omega$, such that $A \notin \mathcal{F}$ and $A \subseteq N$ for some $N \in \mathcal{F}$ with $\mu(N) = 0$. Show that $1_A \in L^1$ and 1_A is not measurable with respect to $\mathcal{F}$ and $\mathcal{B}(\bar{\mathbf{R}})$.
3. Explain why if $f \in L^1$, the integrals $\int |f| d\mu$ and $\int f d\mu$ may not be well defined.

4. Suppose that $f : (\Omega, \mathcal{F}) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is a measurable map with $\int |f| d\mu < +\infty$. Show that $f \in L^1$.

5. Show that if $f \in L^1$ and $f = f_1$ μ-a.s. then $f_1 \in L^1$.

6. Suppose that $f \in L^1$ and $g_1, g_2 \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ are such that $f = g_1$ μ-a.s. and $f = g_2$ μ-a.s.. Show that $\int g_1 d\mu = \int g_2 d\mu$.

7. Propose a definition of the integral $\int f d\mu$ for $f \in L^1$ which extends the integral defined on $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

Exercise 15. Further to ex. (14), Let $(f_n)_{n\geq 1}$ be a sequence in L^1, and $f, h \in L^1$, with $f_n \to f$ μ-a.s. and for all $n \geq 1$, $|f_n| \leq h$ μ-a.s..

1. Show the existence of $N_1 \in \mathcal{F}, \mu(N_1) = 0$, such that for all $\omega \in N_1^c$, $f_n(\omega) \to f(\omega)$, and for all $n \geq 1$, $|f_n(\omega)| \leq h(\omega)$.

2. Show the existence of $g_n, g, h_1 \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ and $N_2 \in \mathcal{F}$, $\mu(N_2) = 0$, such that for all $\omega \in N_2^c$, $g(\omega) = f(\omega)$, $h(\omega) = h_1(\omega)$, and for all $n \geq 1$, $g_n(\omega) = f_n(\omega)$.

3. Show the existence of $N \in \mathcal{F}$, $\mu(N) = 0$, such that for all $\omega \in N^c$, $g_n(\omega) \to g(\omega)$, and for all $n \geq 1$, $|g_n(\omega)| \leq h_1(\omega)$.

4. Show that the Dominated Convergence Theorem can be applied to $g_n 1_{N^c}, g 1_{N^c}$ and $h_1 1_{N^c}$.

5. Recall the definition of $\int |f_n - f| d\mu$ when $f, f_n \in L^1$.

6. Show that $\int |f_n - f| d\mu \to 0$.

Exercise 16. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. let f be an element of $L^1_{\mathbf{R}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$.

1. Let $A = \{\omega_1 \in \Omega_1 \ : \ \int_{\Omega_2} |f(\omega_1, x)| d\mu_2(x) < +\infty\}$. Show that $A \in \mathcal{F}_1$ and $\mu_1(A^c) = 0$.

2. Show that $f(\omega_1, .) \in L^1_{\mathbf{R}}(\Omega_2, \mathcal{F}_2, \mu_2)$ for all $\omega_1 \in A$.

3. Show that $\bar{I}(\omega_1) = \int_{\Omega_2} f(\omega_1, x) d\mu_2(x)$ is well defined for all $\omega_1 \in A$. Let I be an arbitrary extension of $\bar{I}$, on Ω_1.

4. Define $J = I1_A$. Show that:
$$J(\omega) = 1_A(\omega)\int_{\Omega_2} f^+(\omega,x)d\mu_2(x) - 1_A(\omega)\int_{\Omega_2} f^-(\omega,x)d\mu_2(x)$$
5. Show that J is $\mathcal{F}_1$-measurable and $\mathbf{R}$-valued.
6. Show that $J \in L^1_{\mathbf{R}}(\Omega_1, \mathcal{F}_1, \mu_1)$ and that $J = I$ μ_1-a.s.
7. Propose a definition for the integral:
$$\int_{\Omega_1}\left(\int_{\Omega_2} f(x,y)d\mu_2(y)\right)d\mu_1(x)$$
8. Show that $\int_{\Omega_1}(1_A\int_{\Omega_2} f^+ d\mu_2)d\mu_1 = \int_{\Omega_1\times\Omega_2} f^+ d\mu_1\otimes\mu_2$.
9. Show that:
$$\int_{\Omega_1}\left(\int_{\Omega_2} f(x,y)d\mu_2(y)\right)d\mu_1(x) = \int_{\Omega_1\times\Omega_2} fd\mu_1\otimes\mu_2$$
10. Prove the following:

Theorem 33 *Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two σ-finite measure spaces. Let $f \in L^1_{\mathbf{C}}(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. Then, the map:*

$$\omega_1 \to \int_{\Omega_2} f(\omega_1, x) d\mu_2(x)$$

is μ_1-almost surely equal to an element of $L^1_{\mathbf{C}}(\Omega_1, \mathcal{F}_1, \mu_1)$ and:

$$\int_{\Omega_1} \left(\int_{\Omega_2} f(x,y) d\mu_2(y) \right) d\mu_1(x) = \int_{\Omega_1 \times \Omega_2} f d\mu_1 \otimes \mu_2$$

Exercise 17. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$,...,$(\Omega_n, \mathcal{F}_n, \mu_n)$ be n σ-finite measure spaces, $n \geq 2$. Let $f \in L^1_{\mathbf{C}}(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n, \mu_1 \otimes \ldots \otimes \mu_n)$. Let σ be a permutation of $\mathbf{N}_n$.

1. For all $\omega \in \Pi_{i \neq \sigma(1)} \Omega_i$, define:

$$J_1(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(1)}} f(\omega, x) d\mu_{\sigma(1)}(x)$$

Explain why J_1 is well defined and equal to an element of $L^1_{\mathbf{C}}(\Pi_{i\neq\sigma(1)}\Omega_i, \otimes_{i\neq\sigma(1)}\mathcal{F}_i, \otimes_{i\neq\sigma(1)}\mu_i)$, $\otimes_{i\neq\sigma(1)}\mu_i$-almost surely.

2. Suppose $1 \leq k < n-2$ and that $\bar{J}_k$ is well defined and equal to an element of:

$$L^1_{\mathbf{C}}(\Pi_{i\not\in\{\sigma(1),\ldots,\sigma(k)\}}\Omega_i, \otimes_{i\not\in\{\sigma(1),\ldots,\sigma(k)\}}\mathcal{F}_i, \otimes_{i\not\in\{\sigma(1),\ldots,\sigma(k)\}}\mu_i)$$

$\otimes_{i\not\in\{\sigma(1),\ldots,\sigma(k)\}}\mu_i$-almost surely. Define:

$$J_{k+1}(\omega) \stackrel{\triangle}{=} \int_{\Omega_{\sigma(k+1)}} \bar{J}_k(\omega, x)d\mu_{\sigma(k+1)}(x)$$

What can you say about J_{k+1}.

3. Show that:

$$\int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

is a well defined complex number. (Propose a definition for it).

4. Show that:

$$\int_{\Omega_{\sigma(n)}} \dots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \dots d\mu_{\sigma(n)} = \int_{\Omega_1 \times \dots \times \Omega_n} f d\mu_1 \otimes \dots \otimes \mu_n$$

8. Jensen inequality

Definition 64 *Let $a, b \in \bar{\mathbf{R}}$, with $a < b$. Let $\phi :]a, b[\to \mathbf{R}$ be an $\mathbf{R}$-valued function. We say that ϕ is a* **convex function**, *if and only if, for all $x, y \in]a, b[$ and $t \in [0, 1]$, we have:*

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

Exercise 1. Let $a, b \in \bar{\mathbf{R}}$, with $a < b$. Let $\phi :]a, b[\to \mathbf{R}$ be a map.

1. Show that $\phi :]a, b[\to \mathbf{R}$ is convex, if and only if for all $x_1, \ldots, x_n$ in $]a, b[$ and $\alpha_1, \ldots, \alpha_n$ in $\mathbf{R}^+$ with $\alpha_1 + \ldots + \alpha_n = 1$, $n \geq 1$, we have:

$$\phi(\alpha_1 x_1 + \ldots + \alpha_n x_n) \leq \alpha_1 \phi(x_1) + \ldots \alpha_n \phi(x_n)$$

2. Show that $\phi :]a, b[\to \mathbf{R}$ is convex, if and only if for all x, y, z with $a < x < y < z < b$ we have:

$$\phi(y) \leq \frac{z-y}{z-x}\phi(x) + \frac{y-x}{z-x}\phi(z)$$

3. Show that $\phi :]a,b[\to \mathbf{R}$ is convex if and only if for all x, y, z with $a < x < y < z < b$, we have:
$$\frac{\phi(y)-\phi(x)}{y-x} \leq \frac{\phi(z)-\phi(y)}{z-y}$$

4. Let $\phi :]a,b[\to \mathbf{R}$ be convex. Let $x_0 \in]a,b[$, and $u, u', v, v' \in]a,b[$ be such that $u < u' < x_0 < v < v'$. Show that for all $x \in]x_0, v[$:
$$\frac{\phi(u')-\phi(u)}{u'-u} \leq \frac{\phi(x)-\phi(x_0)}{x-x_0} \leq \frac{\phi(v')-\phi(v)}{v'-v}$$
and deduce that $\lim_{x\downarrow\downarrow x_0} \phi(x) = \phi(x_0)$

5. Show that if $\phi :]a,b[\to \mathbf{R}$ is convex, then ϕ is continuous.

6. Define $\phi : [0,1] \to \mathbf{R}$ by $\phi(0) = 1$ and $\phi(x) = 0$ for all $x \in]0,1]$. Show that $\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$, $\forall x, y, t \in [0,1]$, but that ϕ fails to be continuous on $[0,1]$.

Definition 65 *Let $(\Omega, \mathcal{T})$ be a topological space. We say that $(\Omega, \mathcal{T})$ is a* **compact topological space** *if and only if, for all family $(V_i)_{i\in I}$ of open sets in Ω, such that $\Omega = \cup_{i\in I} V_i$, there exists a finite subset $\{i_1, \ldots, i_n\}$ of I such that $\Omega = V_{i_1} \cup \ldots \cup V_{i_n}$.*

In short, we say that $(\Omega, \mathcal{T})$ is compact if and only if, from any open covering of Ω, one can extract a finite sub-covering.

Definition 66 *Let $(\Omega, \mathcal{T})$ be a topological space, and $K \subseteq \Omega$. We say that K is a* **compact subset** *of Ω, if and only if the induced topological space $(K, \mathcal{T}_{|K})$ is a compact topological space.*

EXERCISE 2. Let $(\Omega, \mathcal{T})$ be a topological space.

1. Show that if $(\Omega, \mathcal{T})$ is compact, it is a compact subset of itself.

2. Show that $\emptyset$ is a compact subset of Ω.

3. Show that if $\Omega' \subseteq \Omega$ and K is a compact subset of Ω', then K is also a compact subset of Ω.

4. Show that if $(V_i)_{i\in I}$ is a family of open sets in Ω such that $K \subseteq \cup_{i\in I} V_i$, then $K = \cup_{i\in I}(V_i \cap K)$ and $V_i \cap K$ is open in K for all $i \in I$.

5. Show that $K \subseteq \Omega$ is a compact subset of Ω, if and only if for any family $(V_i)_{i\in I}$ of open sets in Ω such that $K \subseteq \cup_{i\in I} V_i$, there is a finite subset $\{i_1, \ldots, i_n\}$ of I such that $K \subseteq V_{i_1} \cup \ldots \cup V_{i_n}$.

6. Show that if $(\Omega, \mathcal{T})$ is compact and K is closed in Ω, then K is a compact subset of Ω.

Exercise 3. Let $a, b \in \mathbf{R}$, $a < b$. Let $(V_i)_{i\in I}$ be a family of open sets in $\mathbf{R}$ such that $[a, b] \subseteq \cup_{i\in I} V_i$. We define A as the set of all $x \in [a, b]$ such that $[a, x]$ can be covered by a finite number of V_i's. Let $c = \sup A$.

1. Show that $a \in A$.

2. Show that there is $\epsilon > 0$ such that $a + \epsilon \in A$.

3. Show that $a < c \leq b$.

4. Show the existence of $i_0 \in I$ and c', c'' with $a < c' < c < c''$, such that $]c', c''] \subseteq V_{i_0}$.

5. Show that $[a, c']$ can be covered by a finite number of V_i's.

6. Show that $[a, c'']$ can be covered by a finite number of V_i's.

7. Show that $b \wedge c'' \leq c$ and conclude that $c = b$.

8. Show that $[a, b]$ is a compact subset of $\mathbf{R}$.

Theorem 34 *Let $a, b \in \mathbf{R}$, $a < b$. The closed interval $[a, b]$ is a compact subset of $\mathbf{R}$.*

Definition 67 *Let $(\Omega, \mathcal{T})$ be a topological space. We say that $(\Omega, \mathcal{T})$ is a* **hausdorff topological space***, if and only if for all $x, y \in \Omega$ with $x \neq y$, there exists open sets U and V in Ω, such that:*

$$x \in U \ , \ y \in V \ , \ U \cap V = \emptyset$$

Exercise 4. Let $(\Omega, \mathcal{T})$ be a topological space.

1. Show that if $(\Omega, \mathcal{T})$ is hausdorff and $\Omega' \subseteq \Omega$, then the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ is itself hausdorff.

2. Show that if $(\Omega, \mathcal{T})$ is metrizable, then it is hausdorff.

3. Show that any subset of $\bar{\mathbf{R}}$ is hausdorff.

4. Let $(\Omega_i, \mathcal{T}_i)_{i\in I}$ be a family of hausdorff topological spaces. Show that the product topological space $\Pi_{i\in I}\Omega_i$ is hausdorff.

Exercise 5. Let $(\Omega, \mathcal{T})$ be a hausdorff topological space. Let K be a compact subset of Ω and suppose there exists $y \in K^c$.

1. Show that for all $x \in K$, there are open sets V_x, W_x in Ω, such that $y \in V_x, x \in W_x$ and $V_x \cap W_x = \emptyset$.

2. Show that there exists a finite subset $\{x_1, \ldots, x_n\}$ of K such that $K \subseteq W^y$ where $W^y = W_{x_1} \cup \ldots \cup W_{x_n}$.

3. Let $V^y = V_{x_1} \cap \ldots \cap V_{x_n}$. Show that V^y is open and $V^y \cap W^y = \emptyset$.

4. Show that $y \in V^y \subseteq K^c$.

5. Show that $K^c = \cup_{y \in K^c} V^y$

6. Show that K is closed in Ω.

Theorem 35 *Let $(\Omega, \mathcal{T})$ be a hausdorff topological space. For all $K \subseteq \Omega$, if K is a compact subset, then it is closed.*

Definition 68 *Let (E, d) be a metric space. For all $A \subseteq E$, we call* **diameter** *of A with respect to d, the element of $\bar{\mathbf{R}}$ denoted $\delta(A)$, defined as $\delta(A) = \sup\{d(x, y) \ : \ x, y \in A\}$, with the convention that $\delta(\emptyset) = -\infty$.*

Definition 69 *Let (E, d) be a metric space, and $A \subseteq E$. We say that A is* **bounded**, *if and only if its diameter is finite, i.e. $\delta(A) < +\infty$.*

EXERCISE 6. Let (E, d) be a metric space. Let $A \subseteq E$.

1. Show that $\delta(A) = 0$ if and only if $A = \{x\}$ for some $x \in E$.

2. Let $\phi : \bar{\mathbf{R}} \to [-1, 1]$ be an increasing homeomorphism. Define $d''(x, y) = |x - y|$ and $d'(x, y) = |\phi(x) - \phi(y)|$, for all $x, y \in \mathbf{R}$. Show that d' is a metric on $\mathbf{R}$ inducing the usual topology on $\mathbf{R}$. Show that $\mathbf{R}$ is bounded with respect to d' but not with respect to d''.

3. Show that if $K \subseteq E$ is a compact subset of E, for all $\epsilon > 0$, there is a finite subset $\{x_1, \ldots, x_n\}$ of K such that:
$$K \subseteq B(x_1, \epsilon) \cup \ldots \cup B(x_n, \epsilon)$$

4. Show that any compact subset of any metrizable topological space $(\Omega, \mathcal{T})$, is bounded with respect to any metric inducing the topology $\mathcal{T}$.

Exercise 7. Suppose K is a closed subset of $\mathbf{R}$ which is bounded with respect to the usual metric on $\mathbf{R}$.

1. Show that there exists $M \in \mathbf{R}^+$ such that $K \subseteq [-M, M]$.
2. Show that K is also closed in $[-M, M]$.
3. Show that K is a compact subset of $[-M, M]$.
4. Show that K is a compact subset of $\mathbf{R}$.
5. Show that any compact subset of $\mathbf{R}$ is closed and bounded.
6. Show the following:

Theorem 36 *A subset of* $\mathbf{R}$ *is compact if and only if it is closed, and bounded with respect to the usual metric on* $\mathbf{R}$.

Exercise 8. Let $(\Omega, \mathcal{T})$ and $(S, \mathcal{T}_S)$ be two topological spaces. Let $f : (\Omega, \mathcal{T}) \to (S, \mathcal{T}_S)$ be a continuous map.

1. Show that if $(W_i)_{i \in I}$ is an open covering of $f(\Omega)$, then the family $(f^{-1}(W_i))_{i \in I}$ is an open covering of Ω.

2. Show that if $(\Omega, \mathcal{T})$ is a compact topological space, then $f(\Omega)$ is a compact subset of $(S, \mathcal{T}_S)$.

Exercise 9.

1. Show that $(\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ is a compact topological space.

2. Show that any compact subset of $\mathbf{R}$ is a compact subset of $\bar{\mathbf{R}}$.

3. Show that a subset of $\bar{\mathbf{R}}$ is compact if and only if it is closed.

4. Let A be a non-empty subset of $\bar{\mathbf{R}}$, and let $\alpha = \sup A$. Show that if $\alpha \neq -\infty$, then for all $U \in \mathcal{T}_{\bar{\mathbf{R}}}$ with $\alpha \in U$, there exists $\beta \in \mathbf{R}$ with $\beta < \alpha$ and $]\beta, \alpha] \subseteq U$. Conclude that $\alpha \in \bar{A}$.

5. Show that if A is a non-empty closed subset of $\bar{\mathbf{R}}$, then we have $\sup A \in A$ and $\inf A \in A$.

6. Consider $A = \{x \in \mathbf{R}\ ,\ \sin(x) = 0\}$. Show that A is closed in $\mathbf{R}$, but that $\sup A \notin A$ and $\inf A \notin A$.

7. Show that if A is a non-empty, closed and bounded subset of $\mathbf{R}$, then $\sup A \in A$ and $\inf A \in A$.

Exercise 10. Let $(\Omega, \mathcal{T})$ be a compact, non-empty topological space. Let $f : (\Omega, \mathcal{T}) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ be a continuous map.

1. Show that if $f(\Omega) \subseteq \mathbf{R}$, the continuity of f with respect to $\mathcal{T}_{\bar{\mathbf{R}}}$ is equivalent to the continuity of f with respect to $\mathcal{T}_{\mathbf{R}}$.

2. Show the following:

Theorem 37 *Let $f : (\Omega, \mathcal{T}) \to (\bar{\mathbf{R}}, \mathcal{T}_{\bar{\mathbf{R}}})$ be a continuous map, where $(\Omega, \mathcal{T})$ is a non-empty topological space. Then, if $(\Omega, \mathcal{T})$ is compact, f attains its maximum and minimum, i.e. there exist $x_m, x_M \in \Omega$, such that:*

$$f(x_m) = \inf_{x \in \Omega} f(x) \ , \ f(x_M) = \sup_{x \in \Omega} f(x)$$

Exercise 11. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$, and differentiable on $]a, b[$, with $f(a) = f(b)$.

1. Show that if $c \in]a, b[$ and $f(c) = \sup_{x \in [a,b]} f(x)$, then $f'(c) = 0$.

2. Show the following:

Theorem 38 (Rolle) *Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$, and differentiable on $]a, b[$, with $f(a) = f(b)$. Then, there exists $c \in]a, b[$ such that $f'(c) = 0$.*

Exercise 12. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on $]a, b[$. Define:

$$h(x) \stackrel{\triangle}{=} f(x) - (x-a)\frac{f(b) - f(a)}{b-a}$$

1. Show that h is continuous on $[a, b]$ and differentiable on $]a, b[$.

2. Show the existence of $c \in]a, b[$ such that:

$$f(b) - f(a) = (b-a)f'(c)$$

Exercise 13. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to \mathbf{R}$ be a map. Let $n \geq 0$. We assume that f is of class C^n on $[a, b]$, and that $f^{(n+1)}$ exists on $]a, b[$. Define:

$$h(x) \stackrel{\triangle}{=} f(b) - f(x) - \sum_{k=1}^{n} \frac{(b-x)^k}{k!} f^{(k)}(x) - \alpha \frac{(b-x)^{n+1}}{(n+1)!}$$

where α is chosen such that $h(a) = 0$.

1. Show that h is continuous on $[a, b]$ and differentiable on $]a, b[$.

2. Show that for all $x \in]a, b[$:

$$h'(x) = \frac{(b-x)^n}{n!}(\alpha - f^{(n+1)}(x))$$

3. Prove the following:

Theorem 39 (Taylor-Lagrange) *Let $a, b \in \mathbf{R}$, $a < b$, and $n \geq 0$. Let $f : [a, b] \to \mathbf{R}$ be a map of class C^n on $[a, b]$ such that $f^{(n+1)}$ exists on $]a, b[$. Then, there exists $c \in]a, b[$ such that:*

$$f(b) - f(a) = \sum_{k=1}^{n} \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

Exercise 14. Let $a, b \in \bar{\mathbf{R}}$, $a < b$ and $\phi :]a,b[\to \mathbf{R}$ be differentiable.

1. Show that if ϕ is convex, then for all $x, y \in]a,b[$, $x < y$, we have:

$$\phi'(x) \leq \phi'(y)$$

2. Show that if $x, y, z \in]a,b[$ with $x < y < z$, there are $c_1, c_2 \in]a,b[$, with $c_1 < c_2$ and:

$$\begin{array}{rcl} \phi(y) - \phi(x) & = & \phi'(c_1)(y-x) \\ \phi(z) - \phi(y) & = & \phi'(c_2)(z-y) \end{array}$$

3. Show conversely that if ϕ' is non-decreasing, then ϕ is convex.

4. Show that $x \to e^x$ is convex on $\mathbf{R}$.

5. Show that $x \to -\ln(x)$ is convex on $]0, +\infty[$.

Definition 70 *we say that a finite measure space* $(\Omega, \mathcal{F}, P)$ *is a* **probability space**, *if and only if* $P(\Omega) = 1$.

Definition 71 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space, and* (S, Σ) *be a measurable space. We call* **random variable** *w.r. to* (S, Σ), *any measurable map* $X : (\Omega, \mathcal{F}) \to (S, \Sigma)$.

Definition 72 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. Let* X *be a non-negative random variable, or an element of* $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, P)$. *We call* **expectation** *of* X, *denoted* $E[X]$, *the integral:*

$$E[X] \stackrel{\triangle}{=} \int_\Omega X dP$$

Exercise 15. Let $a, b \in \bar{\mathbf{R}}$, $a < b$ and $\phi :]a, b[\to \mathbf{R}$ be a convex map. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$ be such that $X(\Omega) \subseteq]a, b[$.

1. Show that $\phi \circ X : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable.
2. Show that $\phi \circ X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$, if and only if $E[|\phi \circ X|] < +\infty$.
3. Show that if $E[X] = a$, then $a \in \mathbf{R}$ and $X = a$ P-a.s.
4. Show that if $E[X] = b$, then $b \in \mathbf{R}$ and $X = b$ P-a.s.
5. Let $m = E[X]$. Show that $m \in]a, b[$.
6. Define:
$$\beta \stackrel{\triangle}{=} \sup_{x \in]a,m[} \frac{\phi(m) - \phi(x)}{m - x}$$
Show that $\beta \in \mathbf{R}$ and that for all $z \in]m, b[$, we have:
$$\beta \leq \frac{\phi(z) - \phi(m)}{z - m}$$

7. Show that for all $x \in]a,b[$, we have $\phi(m) + \beta(x-m) \leq \phi(x)$.

8. Show that for all $\omega \in \Omega$, $\phi(m) + \beta(X(\omega) - m) \leq \phi(X(\omega))$.

9. Show that if $\phi \circ X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$ then $\phi(m) \leq E[\phi \circ X]$.

Theorem 40 (Jensen inequality) *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $a, b \in \bar{\mathbf{R}}$, $a < b$ and $\phi :]a,b[\to \mathbf{R}$ be a convex map. Suppose that $X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$ is such that $X(\Omega) \subseteq]a,b[$ and such that $\phi \circ X \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$. Then:*

$$\phi(E[X]) \leq E[\phi \circ X]$$

9. L^p-spaces, $p \in [1,+\infty]$

In the following, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

EXERCISE 1. Let $f, g : (\Omega, \mathcal{F}) \to [0,+\infty]$ be non-negative and measurable maps. Let $p, q \in \mathbf{R}^+$, such that $1/p + 1/q = 1$.

1. Show that $1 < p < +\infty$ and $1 < q < +\infty$.

2. For all $\alpha \in]0,+\infty[$, we define $\phi^\alpha : [0,+\infty] \to [0,+\infty]$ by:

$$\phi^\alpha(x) \stackrel{\triangle}{=} \begin{cases} x^\alpha & \text{if} \quad x \in \mathbf{R}^+ \\ +\infty & \text{if} \quad x = +\infty \end{cases}$$

   Show that ϕ^α is a continuous map.

3. Define $A = (\int f^p d\mu)^{1/p}$, $B = (\int g^q d\mu)^{1/q}$ and $C = \int fg d\mu$. Explain why A, B and C are well defined elements of $[0,+\infty]$.

4. Show that if $A = 0$ or $B = 0$ then $C \leq AB$.

5. Show that if $A = +\infty$ or $B = +\infty$ then $C \leq AB$.

6. We assume from now on that $0 < A < +\infty$ and $0 < B < +\infty$. Define $F = f/A$ and $G = g/B$. Show that:

$$\int_\Omega F^p d\mu = \int_\Omega G^p d\mu = 1$$

7. Let $a, b \in]0, +\infty[$. Show that:

$$\ln(a) + \ln(b) \leq \ln\left(\frac{1}{p}a^p + \frac{1}{q}b^q\right)$$

and:

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$$

Prove this last inequality for all $a, b \in [0, +\infty]$.

8. Show that for all $\omega \in \Omega$, we have:

$$F(\omega)G(\omega) \leq \frac{1}{p}(F(\omega))^p + \frac{1}{q}(G(\omega))^q$$

9. Show that $C \leq AB$.

Theorem 41 (Hölder's inequality) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $p, q \in \mathbf{R}^+$ be such that $1/p + 1/q = 1$. Then:*

$$\int_\Omega fg d\mu \leq \left(\int_\Omega f^p d\mu\right)^{\frac{1}{p}} \left(\int_\Omega g^q d\mu\right)^{\frac{1}{q}}$$

Theorem 42 (Cauchy-Schwarz's inequality:first) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Then:*

$$\int_\Omega fg d\mu \leq \left(\int_\Omega f^2 d\mu\right)^{\frac{1}{2}} \left(\int_\Omega g^2 d\mu\right)^{\frac{1}{2}}$$

EXERCISE 2. Let $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $p \in]1, +\infty[$. Define $A = (\int f^p d\mu)^{1/p}$ and $B = (\int g^p d\mu)^{1/p}$ and $C = (\int (f+g)^p d\mu)^{1/p}$.

1. Explain why A, B and C are well defined elements of $[0, +\infty]$.

2. Show that for all $a, b \in]0, +\infty[$, we have:

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

   Prove this inequality for all $a, b \in [0, +\infty]$.

3. Show that if $A = +\infty$ or $B = +\infty$ or $C = 0$ then $C \leq A + B$.

4. Show that if $A < +\infty$ and $B < +\infty$ then $C < +\infty$.

5. We assume from now that $A, B \in [0, +\infty[$ and $C \in]0, +\infty[$. Show the existence of some $q \in \mathbf{R}^+$ such that $1/p + 1/q = 1$.

6. Show that for all $a, b \in [0, +\infty]$, we have:

$$(a+b)^p = (a+b).(a+b)^{p-1}$$

7. Show that:

$$\begin{aligned} \int_\Omega f.(f+g)^{p-1} d\mu &\leq AC^{\frac{p}{q}} \\ \int_\Omega g.(f+g)^{p-1} d\mu &\leq BC^{\frac{p}{q}} \end{aligned}$$

8. Show that:

$$\int_\Omega (f+g)^p d\mu \leq C^{\frac{p}{q}}(A+B)$$

9. Show that $C \leq A+B$.

10. Show that the inequality still holds if we assume that $p = 1$.

Theorem 43 (Minkowski's inequality) *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $f, g : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let $p \in [1, +\infty[$. Then:*

$$\left(\int_\Omega (f+g)^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_\Omega f^p d\mu\right)^{\frac{1}{p}} + \left(\int_\Omega g^p d\mu\right)^{\frac{1}{p}}$$

Definition 73 *The* **L^p-spaces**, $p \in [1, +\infty[$, *on* $(\Omega, \mathcal{F}, \mu)$, *are:*

$$L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \left\{ f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \text{ measurable}, \int_\Omega |f|^p d\mu < +\infty \right\}$$

$$L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \left\{ f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C})) \text{ measurable}, \int_\Omega |f|^p d\mu < +\infty \right\}$$

For all $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, *we put:*

$$\|f\|_p \stackrel{\triangle}{=} \left(\int_\Omega |f|^p d\mu \right)^{\frac{1}{p}}$$

Exercise 3. Let $p \in [1, +\infty[$. Let $f, g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

1. Show that $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) = \{ f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \ , \ f(\Omega) \subseteq \mathbf{R} \}$.
2. Show that $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{R}$-linear combinations.
3. Show that $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{C}$-linear combinations.

4. Show that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

5. Show that $\|f\|_p = 0 \;\Leftrightarrow\; f = 0$ μ-a.s.

6. Show that for all $\alpha \in \mathbf{C}$, $\|\alpha f\|_p = |\alpha|.\|f\|_p$.

7. Explain why $(f,g) \to \|f-g\|_p$ is not a metric on $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$

Definition 74 *For all $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ measurable, Let:*

$$\|f\|_\infty \stackrel{\triangle}{=} \inf\{M \in \mathbf{R}^+ \;,\; |f| \leq M \;\; \mu\text{-}a.s.\}$$

The **L^∞-spaces** *on a measure space $(\Omega, \mathcal{F}, \mu)$ are:*

$$L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \{f : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R})) \text{ measurable}, \|f\|_\infty < +\infty\}$$
$$L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} \{f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C})) \text{ measurable}, \|f\|_\infty < +\infty\}$$

Exercise 4. Let $f, g \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

1. Show that $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) = \{f \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)\ ,\ f(\Omega) \subseteq \mathbf{R}\}$.
2. Show that $|f| \leq \|f\|_\infty$ μ-a.s.
3. Show that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.
4. Show that $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{R}$-linear combinations.
5. Show that $L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is closed under $\mathbf{C}$-linear combinations.
6. Show that $\|f\|_\infty = 0 \ \Leftrightarrow\ f = 0$ μ-a.s..
7. Show that for all $\alpha \in \mathbf{C}$, $\|\alpha f\|_\infty = |\alpha|.\|f\|_\infty$.
8. Explain why $(f, g) \rightarrow \|f - g\|_\infty$ is not a metric on $L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$

Definition 75 *Let $p \in [1, +\infty]$. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. For all $\epsilon > 0$ and $f \in L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, we define the so-called* **open ball** *in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$:*

$$B(f, \epsilon) \stackrel{\triangle}{=} \{g : g \in L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu), \|f - g\|_p < \epsilon\}$$

We call **usual topology** *in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, the set $\mathcal{T}$ defined by:*

$$\mathcal{T} \stackrel{\triangle}{=} \{U : U \subseteq L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu), \forall f \in U, \exists \epsilon > 0, B(f, \epsilon) \subseteq U\}$$

Note that if $(f, g) \to \|f - g\|_p$ was a metric, the usual topology in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, would be nothing but the *metric* topology.

EXERCISE 5. Let $p \in [1, +\infty]$. Suppose there exists $N \in \mathcal{F}$ with $\mu(N) = 0$ and $N \neq \emptyset$. Put $f = 1_N$ and $g = 0$

1. Show that $f, g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $f \neq g$.
2. Show that any open set containing f also contains g.
3. Show that $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ are not Hausdorff.

Exercise 6. Show that the usual topology on $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ is induced by the usual topology on $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where $p \in [1, +\infty]$.

Definition 76 *Let $(E, \mathcal{T})$ be a topological space. A sequence $(x_n)_{n\geq 1}$ in E is said to* **converge** *to $x \in E$, and we write $x_n \xrightarrow{\mathcal{T}} x$, if and only if, for all $U \in \mathcal{T}$ such that $x \in U$, there exists $n_0 \geq 1$ such that:*

$$n \geq n_0 \;\Rightarrow\; x_n \in U$$

When $E = L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ or $E = L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, we write $x_n \xrightarrow{L^p} x$.

Exercise 7. Let $(E, \mathcal{T})$ be a topological space and $E' \subseteq E$. Let $\mathcal{T}' = \mathcal{T}_{|E'}$ be the induced topology on E'. Show that if $(x_n)_{n\geq 1}$ is a sequence in E' and $x \in E'$, then $x_n \xrightarrow{\mathcal{T}} x$ is equivalent to $x_n \xrightarrow{\mathcal{T}'} x$.

Exercise 8. Let $f, g, (f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $p \in [1, +\infty]$.

1. Recall what the notation $f_n \to f$ means.

2. Show that $f_n \xrightarrow{L^p} f$ is equivalent to $\|f_n - f\|_p \to 0$.

3. Show that if $f_n \stackrel{L^p}{\to} f$ and $f_n \stackrel{L^p}{\to} g$ then $f = g$ μ-a.s.

EXERCISE 9. Let $p \in [1, +\infty]$. Suppose there exists some $N \in \mathcal{F}$ such that $\mu(N) = 0$ and $N \neq \emptyset$. Find a sequence $(f_n)_{n\geq 1}$ in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and f, g in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $f \neq g$ such that $f_n \stackrel{L^p}{\to} f$ and $f_n \stackrel{L^p}{\to} g$.

Definition 77 *Let $(f_n)_{n\geq 1}$ be a sequence in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F}, \mu)$ is a measure space and $p \in [1, +\infty]$. We say that $(f_n)_{n\geq 1}$ is a* **cauchy sequence**, *if and only if, for all $\epsilon > 0$, there exists $n_0 \geq 1$ such that:*

$$n, m \geq n_0 \;\Rightarrow\; \|f_n - f_m\|_p \leq \epsilon$$

EXERCISE 10. Let $f, (f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $p \in [1, +\infty]$. Show that if $f_n \stackrel{L^p}{\to} f$, then $(f_n)_{n\geq 1}$ is a cauchy sequence.

EXERCISE 11. Let $p \in [1, +\infty]$, and $(f_n)_{n\geq 1}$ be cauchy in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

1. Show the existence of $n_1 \geq 1$ such that:

$$n \geq n_1 \;\Rightarrow\; \|f_n - f_{n_1}\|_p \leq \frac{1}{2}$$

2. Suppose we have found $n_1 < n_2 < \ldots < n_k,\; k \geq 1$, such that:

$$\forall j \in \{1, \ldots, k\}\;,\; n \geq n_j \;\Rightarrow\; \|f_n - f_{n_j}\|_p \leq \frac{1}{2^j}$$

Show the existence of n_{k+1}, $n_k < n_{k+1}$ such that:

$$n \geq n_{k+1} \;\Rightarrow\; \|f_n - f_{n_{k+1}}\|_p \leq \frac{1}{2^{k+1}}$$

3. Show that there exists a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ with:

$$\sum_{k=1}^{+\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty$$

Exercise 12. Let $p \in [1, +\infty]$, and $(f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, with:

$$\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty$$

We define:

$$g \stackrel{\triangle}{=} \sum_{n=1}^{+\infty} |f_{n+1} - f_n|$$

1. Show that $g : (\Omega, \mathcal{F}) \to [0, +\infty]$ is non-negative and measurable.
2. If $p = +\infty$, show that $g \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_\infty$ μ-a.s.
3. If $p \in [1, +\infty[$, show that for all $N \geq 1$, we have:

$$\left\| \sum_{n=1}^{N} |f_{n+1} - f_n| \right\|_p \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$$

4. If $p \in [1, +\infty[$, show that:

$$\left(\int_\Omega g^p d\mu \right)^{\frac{1}{p}} \leq \sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p$$

5. Show that for $p \in [1, +\infty]$, we have $g < +\infty$ μ-a.s.

6. Define $A = \{g < +\infty\}$. Show that for all $\omega \in A$, $(f_n(\omega))_{n \geq 1}$ is a cauchy sequence in $\mathbf{C}$. We denote $z(\omega)$ its limit.

7. Define $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, by:

$$f(\omega) \stackrel{\triangle}{=} \begin{cases} z(\omega) & \text{if} \quad \omega \in A \\ 0 & \text{if} \quad \omega \notin A \end{cases}$$

   Show that f is measurable and $f_n \to f$ μ-a.s.

8. if $p = +\infty$, show that for all $n \geq 1$, $|f_n| \leq |f_1| + g$ and conclude that $f \in L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

9. If $p \in [1, +\infty[$, show the existence of $n_0 \geq 1$, such that:

$$n \geq n_0 \ \Rightarrow \ \int_\Omega |f_n - f_{n_0}|^p d\mu \leq 1$$

   Deduce from Fatou's lemma that $f - f_{n_0} \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

10. Show that for $p \in [1, +\infty]$, $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

11. Suppose that $f_n \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, for all $n \geq 1$. Show the existence of $f \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, such that $f_n \to f$ μ-a.s.

Exercise 13. Let $p \in [1, +\infty]$, and $(f_n)_{n \geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, with:

$$\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty$$

1. Does there exist $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \to f$ μ-a.s.

2. Suppose $p = +\infty$. Show that for all $n < m$, we have:

$$|f_{m+1} - f_n| \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_\infty \ \mu\text{-a.s.}$$

3. Suppose $p = +\infty$. Show that for all $n \geq 1$, we have:

$$\|f - f_n\|_\infty \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_\infty$$

4. Suppose $p \in [1, +\infty[$. Show that for all $n < m$, we have:

$$\left(\int_\Omega |f_{m+1} - f_n|^p d\mu\right)^{\frac{1}{p}} \leq \sum_{k=n}^{m} \|f_{k+1} - f_k\|_p$$

5. Suppose $p \in [1, +\infty[$. Show that for all $n \geq 1$, we have:

$$\|f - f_n\|_p \leq \sum_{k=n}^{+\infty} \|f_{k+1} - f_k\|_p$$

6. Show that for $p \in [1, +\infty]$, we also have $f_n \stackrel{L^p}{\to} f$.

7. Suppose conversely that $g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is such that $f_n \stackrel{L^p}{\to} g$. Show that $f = g$ μ-a.s.. Conclude that $f_n \to g$ μ-a.s..

Theorem 44 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $p \in [1, +\infty]$, and $(f_n)_{n \geq 1}$ be a sequence in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that:*

$$\sum_{n=1}^{+\infty} \|f_{n+1} - f_n\|_p < +\infty$$

Then, there exists $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \to f$ μ-a.s. Moreover, for all $g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, the convergence $f_n \to g$ μ-a.s. and $f_n \stackrel{L^p}{\to} g$ are equivalent.

EXERCISE 14. Let $f, (f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \stackrel{L^p}{\to} f$, where $p \in [1, +\infty]$.

1. Show that there exists a sub-sequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$, with:

$$\sum_{k=1}^{+\infty} \|f_{n_{k+1}} - f_{n_k}\|_p < +\infty$$

2. Show that there exists $g \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_{n_k} \to g$ μ-a.s.

3. Show that $f_{n_k} \stackrel{L^p}{\to} g$ and $g = f$ μ-a.s.

4. Conclude with the following:

Theorem 45 *Let $(f_n)_{n\geq 1}$ be in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \stackrel{L^p}{\to} f$, where $p \in [1, +\infty]$. Then, we can extract a sub-sequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ such that $f_{n_k} \to f$ μ-a.s.*

Exercise 15. Prove the last theorem for $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

Exercise 16. Let $p \in [1, +\infty]$, and $(f_n)_{n\geq 1}$ be cauchy in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

1. Show that there exists a subsequence $(f_{n_k})_{k\geq 1}$ of $(f_n)_{n\geq 1}$ and f belonging to $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, such that $f_{n_k} \stackrel{L^p}{\rightarrow} f$.
2. Using the fact that $(f_n)_{n\geq 1}$ is cauchy, show that $f_n \stackrel{L^p}{\rightarrow} f$.

Theorem 46 *Let $p \in [1, +\infty]$. Let $(f_n)_{n\geq 1}$ be a cauchy sequence in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Then, there exists $f \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that $f_n \stackrel{L^p}{\rightarrow} f$.*

Exercise 17. Prove the last theorem for $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

10. Bounded Linear Functionals in L^2

In the following, $(\Omega, \mathcal{F}, \mu)$ is a measure space.

Definition 78 *Let $(x_n)_{n\geq 1}$ be a sequence in an arbitrary set. We call* **subsequence** *of $(x_n)_{n\geq 1}$, any sequence of the form $(x_{\phi(n)})_{n\geq 1}$, where $\phi : \mathbf{N}^* \to \mathbf{N}^*$ is a strictly increasing map.*

Exercise 1. Let (E, d) be a metric space, with metric topology $\mathcal{T}$. Let $(x_n)_{n\geq 1}$ be a sequence in E. For all $n \geq 1$, let F_n be the closure of the set $\{x_k : k \geq n\}$.

1. Show that for all $x \in E$, $x_n \xrightarrow{\mathcal{T}} x$ is equivalent to:

$$\forall \epsilon > 0 \ , \ \exists n_0 \geq 1 \ , \ n \geq n_0 \ \Rightarrow \ d(x_n, x) \leq \epsilon$$

2. Show that $(F_n)_{n\geq 1}$ is a decreasing sequence of closed sets in E.

3. Show that if $F_n \downarrow \emptyset$, then $(F_n^c)_{n\geq 1}$ is an open covering of E.

4. Show that if $(E, \mathcal{T})$ is compact then $\cap_{n=1}^{+\infty} F_n \neq \emptyset$.

5. Show that if $(E, \mathcal{T})$ is compact, there exists $x \in E$ such that for all $n \geq 1$ and $\epsilon > 0$, we have $B(x, \epsilon) \cap \{x_k \ , \ k \geq n\} \neq \emptyset$.

6. By induction, construct a subsequence $(x_{n_p})_{p \geq 1}$ of $(x_n)_{n \geq 1}$ such that $x_{n_p} \in B(x, 1/p)$ for all $p \geq 1$.

7. Conclude that if $(E, \mathcal{T})$ is compact, any sequence $(x_n)_{n \geq 1}$ in E has a convergent subsequence.

Exercise 2. Let (E, d) be a metric space, with metric topology $\mathcal{T}$. We assume that any sequence $(x_n)_{n \geq 1}$ in E has a convergent subsequence. Let $(V_i)_{i \in I}$ be an open covering of E. For $x \in E$, let:

$$r(x) \stackrel{\triangle}{=} \sup\{r > 0 \ : \ B(x, r) \subseteq V_i \ , \text{ for some } i \in I\}$$

1. Show that $\forall x \in E$, $\exists i \in I$, $\exists r > 0$, such that $B(x, r) \subseteq V_i$.

2. Show that $\forall x \in E$, $r(x) > 0$.

Exercise 3. Further to ex. (2), suppose $\inf_{x\in E} r(x) = 0$.

1. Show that for all $n \geq 1$, there is $x_n \in E$ such that $r(x_n) < 1/n$.

2. Extract a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ converging to some $x^* \in E$. Let $r^* > 0$ and $i \in I$ be such that $B(x^*, r^*) \subseteq V_i$. Show that we can find some $k_0 \geq 1$, such that $d(x^*, x_{n_{k_0}}) < r^*/2$ and $r(x_{n_{k_0}}) \leq r^*/4$.

3. Show that $d(x^*, x_{n_{k_0}}) < r^*/2$ implies that $B(x_{n_{k_0}}, r^*/2) \subseteq V_i$. Show that this contradicts $r(x_{n_{k_0}}) \leq r^*/4$, and conclude that $\inf_{x\in E} r(x) > 0$.

Exercise 4. Further to ex. (3), Let r_0 with $0 < r_0 < \inf_{x\in E} r(x)$. Suppose that E cannot be covered by a finite number of open balls with radius r_0.

1. Show the existence of a sequence $(x_n)_{n\geq 1}$ in E, such that for all $n \geq 1$, $x_{n+1} \notin B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)$.

2. Show that for all $n > m$ we have $d(x_n, x_m) \geq r_0$.

3. Show that $(x_n)_{n\geq 1}$ cannot have a convergent subsequence.

4. Conclude that there exists a finite subset $\{x_1, \ldots, x_n\}$ of E such that $E = B(x_1, r_0) \cup \ldots \cup B(x_n, r_0)$.

5. Show that for all $x \in E$, we have $B(x, r_0) \subseteq V_i$ for some $i \in I$.

6. Conclude that $(E, \mathcal{T})$ is compact.

7. Prove the following:

Theorem 47 *Let $(E, \mathcal{T})$ be a metrizable topological space. Then $(E, \mathcal{T})$ is compact, if and only if for every sequence $(x_n)_{n\geq 1}$ in E, there exists a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$, and some $x \in E$, such that $x_{n_k} \xrightarrow{\mathcal{T}} x$.*

EXERCISE 5. Let $a, b \in \mathbf{R}$, $a < b$ and $(x_n)_{n\geq 1}$ be a sequence in $]a, b[$.

1. Show that $(x_n)_{n\geq 1}$ has a convergent subsequence.

2. Can we conclude that $]a, b[$ is a compact subset of $\mathbf{R}$?

EXERCISE 6. Let $E = [-M, M] \times \ldots \times [-M, M] \subseteq \mathbf{R}^n$, where $n \geq 1$ and $M \in \mathbf{R}^+$. Let $\mathcal{T}_{\mathbf{R}^n}$ be the usual product topology on $\mathbf{R}^n$, and $\mathcal{T}_E = (\mathcal{T}_{\mathbf{R}^n})_{|E}$ be the induced topology on E.

1. Let $(x_p)_{p\geq 1}$ be a sequence in E. Let $x \in E$. Show that $x_p \xrightarrow{\mathcal{T}_E} x$ is equivalent to $x_p \xrightarrow{\mathcal{T}_{\mathbf{R}^n}} x$.

2. Propose a metric on $\mathbf{R}^n$, inducing the topology $\mathcal{T}_{\mathbf{R}^n}$.

3. Let $(x_p)_{p\geq 1}$ be a sequence in $\mathbf{R}^n$. Let $x \in \mathbf{R}^n$. Show that $x_p \xrightarrow{\mathcal{T}_{\mathbf{R}^n}} x$ if and only if, $x_p^i \xrightarrow{\mathcal{T}_{\mathbf{R}}} x^i$ for all $i \in \mathbf{N}_n$.

Exercise 7. Further to ex. (6), suppose $(x_p)_{p\geq 1}$ is a sequence in E.

1. Show the existence of a subsequence $(x_{\phi(p)})_{p\geq 1}$ of $(x_p)_{p\geq 1}$, such that $x^1_{\phi(p)} \stackrel{\mathcal{T}_{[-M,M]}}{\rightarrow} x^1$ for some $x^1 \in [-M, M]$.

2. Explain why the above convergence is equivalent to $x^1_{\phi(p)} \stackrel{\mathcal{T}_{\mathbf{R}}}{\rightarrow} x^1$.

3. Suppose that $1 \leq k \leq n-1$ and $(y_p)_{p\geq 1} = (x_{\phi(p)})_{p\geq 1}$ is a subsequence of $(x_p)_{p\geq 1}$ such that:

$$\forall j = 1, \ldots, k \ , \ x^j_{\phi(p)} \stackrel{\mathcal{T}_{\mathbf{R}}}{\rightarrow} x^j \text{ for some } x^j \in [-M, M]$$

   Show the existence of a subsequence $(y_{\psi(p)})_{p\geq 1}$ of $(y_p)_{p\geq 1}$ such that $y^{k+1}_{\psi(p)} \stackrel{\mathcal{T}_{\mathbf{R}}}{\rightarrow} x^{k+1}$ for some $x^{k+1} \in [-M, M]$.

4. Show that $\phi \circ \psi : \mathbf{N}^* \to \mathbf{N}^*$ is strictly increasing.

5. Show that $(x_{\phi\circ\psi(p)})_{p\geq 1}$ is a subsequence of $(x_p)_{p\geq 1}$ such that:

$$\forall j = 1, \ldots, k+1 \ , \ x^j_{\phi\circ\psi(p)} \stackrel{\mathcal{T}_{\mathbf{R}}}{\rightarrow} x^j \in [-M, M]$$

6. Show the existence of a subsequence $(x_{\phi(p)})_{p\geq 1}$ of $(x_p)_{p\geq 1}$, and $x \in E$, such that $x_{\phi(p)} \stackrel{\mathcal{T}_E}{\to} x$
7. Show that $(E, \mathcal{T}_E)$ is a compact topological space.

Exercise 8. Let A be a closed subset of $\mathbf{R}^n$, $n \geq 1$, which is bounded with respect to the usual metric of $\mathbf{R}^n$.

1. Show that $A \subseteq E = [-M, M]\times\ldots\times[-M, M]$, for some $M \in \mathbf{R}^+$.
2. Show from $E \setminus A = E \cap A^c$ that A is closed in E.
3. Show $(A, (\mathcal{T}_{\mathbf{R}^n})_{|A})$ is a compact topological space.
4. Conversely, let A is a compact subset of $\mathbf{R}^n$. Show that A is closed and bounded.

Theorem 48 *A subset of $\mathbf{R}^n$, $n \geq 1$, is compact if and only if it is closed and bounded (with respect to the usual metric).*

Exercise 9. Let $n \geq 1$. Consider the map:

$$\phi : \left\{ \begin{array}{ccc} \mathbf{C}^n & \to & \mathbf{R}^{2n} \\ (a_1 + ib_1, \ldots, a_n + ib_n) & \to & (a_1, b_1, \ldots, a_n, b_n) \end{array} \right.$$

1. Recall the expressions of the usual metrics $d_{\mathbf{C}^n}$ and $d_{\mathbf{R}^{2n}}$ of $\mathbf{C}^n$ and $\mathbf{R}^{2n}$ respectively.

2. Show that for all $z, z' \in \mathbf{C}^n$, $d_{\mathbf{C}^n}(z, z') = d_{\mathbf{R}^{2n}}(\phi(z), \phi(z'))$.

3. Show that ϕ is a homeomorphism from $\mathbf{C}^n$ to $\mathbf{R}^{2n}$.

4. Show that a subset K of $\mathbf{C}^n$ is compact, if and only if $\phi(K)$ is a compact subset of $\mathbf{R}^{2n}$.

5. Show that K is closed, if and only if $\phi(K)$ is closed.

6. Show that K is bounded, if and only if $\phi(K)$ is bounded.

7. Show that a subset K of $\mathbf{C}^n$ is compact, if and only if it is closed and bounded (with respect to the usual metric).

Definition 79 *Let (E,d) be a metric space. A sequence $(x_n)_{n\geq 1}$ in E, is said to be a* **cauchy sequence** *(relative to the metric d), if and only if, for all $\epsilon > 0$, there exists $n_0 \geq 1$ such that:*

$$n, m \geq n_0 \;\Rightarrow\; d(x_n, x_m) \leq \epsilon$$

Definition 80 *We say that a metric space (E,d) is* **complete***, if and only if, for all $(x_n)_{n\geq 1}$ cauchy sequence in E, there exists $x \in E$ such that $(x_n)_{n\geq 1}$ converges to x.*

Exercise 10.

1. Explain why strictly speaking, given $p \in [1, +\infty]$, definition (77) of Cauchy sequences in $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is not a covered by definition (79).

2. Explain why $L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is not a complete metric space, despite theorem (46) and definition (80).

Exercise 11. Let $(z_k)_{k\geq 1}$ be a Cauchy sequence in $\mathbf{C}^n$, $n \geq 1$, with respect to the usual metric $d(z,z') = \|z - z'\|$, where:

$$\|z\| \stackrel{\triangle}{=} \sqrt{\sum_{i=1}^{n} |z_i|^2}$$

1. Show that the sequence $(z_k)_{k\geq 1}$ is bounded, i.e. that there exists $M \in \mathbf{R}^+$ such that $\|z_k\| \leq M$, for all $k \geq 1$.

2. Define $B = \{z \in \mathbf{C}^n \ , \ \|z\| \leq M\}$. Show that $\delta(B) < +\infty$, and that B is closed in $\mathbf{C}^n$.

3. Show the existence of a subsequence $(z_{k_p})_{p\geq 1}$ of $(z_k)_{k\geq 1}$ such that $z_{k_p} \stackrel{\mathcal{T}_{\mathbf{C}^n}}{\to} z$ for some $z \in B$.

4. Show that for all $\epsilon > 0$, there exists $p_0 \geq 1$ and $n_0 \geq 1$ such that $d(z, z_{k_{p_0}}) \leq \epsilon/2$ and:

$$k \geq n_0 \ \Rightarrow \ d(z_k, z_{k_{p_0}}) \leq \epsilon/2$$

5. Show that $z_k \stackrel{\mathcal{T}_{\mathbf{C}^n}}{\rightarrow} z$.

6. Conclude that $\mathbf{C}^n$ is complete with respect to its usual metric.

7. For which theorem of Tutorial 9 was the completeness of $\mathbf{C}$ used?

Exercise 12. Let $(x_k)_{k\geq 1}$ be a sequence in $\mathbf{R}^n$ such that $x_k \stackrel{\mathcal{T}_{\mathbf{C}^n}}{\rightarrow} z$, for some $z \in \mathbf{C}^n$.

1. Show that $z \in \mathbf{R}^n$.

2. Show that $\mathbf{R}^n$ is complete with respect to its usual metric.

Theorem 49 *For all $n \geq 1$, $\mathbf{C}^n$ and $\mathbf{R}^n$ are complete with respect to their usual metrics.*

Exercise 13. Let (E,d) be a metric space, with metric topology $\mathcal{T}$. Let $F \subseteq E$, and $\bar{F}$ denote the closure of F.

1. Explain why, for all $x \in \bar{F}$ and $n \geq 1$, we have $F \cap B(x, 1/n) \neq \emptyset$.
2. Show that for all $x \in \bar{F}$, there exists a sequence $(x_n)_{n\geq 1}$ in F, such that $x_n \xrightarrow{\mathcal{T}} x$.
3. Show conversely that if there is a sequence $(x_n)_{n\geq 1}$ in F with $x_n \xrightarrow{\mathcal{T}} x$, then $x \in \bar{F}$.
4. Show that F is closed if and only if for all sequence $(x_n)_{n\geq 1}$ in F such that $x_n \xrightarrow{\mathcal{T}} x$ for some $x \in E$, we have $x \in F$.
5. Explain why $(F, \mathcal{T}_{|F})$ is metrizable.
6. Show that if F is complete with respect to the metric $d_{|F\times F}$, then F is closed in E.

7. Let $d_{\bar{\mathbf{R}}}$ be a metric on $\bar{\mathbf{R}}$, inducing the usual topology $\mathcal{T}_{\bar{\mathbf{R}}}$. Show that $d' = (d_{\bar{\mathbf{R}}})_{|\mathbf{R}\times\mathbf{R}}$ is a metric on $\mathbf{R}$, inducing the topology $\mathcal{T}_{\mathbf{R}}$.

8. Find a metric on $[-1,1]$ which induces its usual topology.

9. Show that $\{-1,1\}$ is not open in $[-1,1]$.

10. Show that $\{-\infty,+\infty\}$ is not open in $\bar{\mathbf{R}}$.

11. Show that $\mathbf{R}$ is not closed in $\bar{\mathbf{R}}$.

12. Let $d_{\mathbf{R}}$ be the usual metric of $\mathbf{R}$. Show that $d' = (d_{\bar{\mathbf{R}}})_{|\mathbf{R}\times\mathbf{R}}$ and $d_{\mathbf{R}}$ induce the same topology on $\mathbf{R}$, but that however, $\mathbf{R}$ is complete with respect to $d_{\mathbf{R}}$, whereas it cannot be complete with respect to d'.

Definition 81 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We call* **inner-product** *on $\mathcal{H}$, any map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbf{K}$ with the following properties:*

$$
\begin{array}{rl}
(i) & \forall x, y \in \mathcal{H}\ ,\ \langle x, y \rangle = \overline{\langle y, x \rangle} \\
(ii) & \forall x, y, z \in \mathcal{H}\ ,\ \langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle \\
(iii) & \forall x, y \in \mathcal{H}, \forall \alpha \in \mathbf{K}\ ,\ \langle \alpha x, y \rangle = \alpha \langle x, y \rangle \\
(iv) & \forall x \in \mathcal{H}\ ,\ \langle x, x \rangle \geq 0 \\
(v) & \forall x \in \mathcal{H}\ ,\ (\langle x, x \rangle = 0 \ \Leftrightarrow\ x = 0)
\end{array}
$$

where for all $z \in \mathbf{C}$, $\bar{z}$ denotes the complex conjugate of z. For all $x \in \mathcal{H}$, we call **norm** *of x, denoted $\|x\|$, the number defined by:*

$$\|x\| \stackrel{\triangle}{=} \sqrt{\langle x, x \rangle}$$

EXERCISE 14. Let $\langle \cdot, \cdot \rangle$ be an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$.

1. Show that for all $y \in \mathcal{H}$, the map $x \to \langle x, y \rangle$ is linear.

2. Show that for all $x \in \mathcal{H}$, the map $y \to \langle x, y \rangle$ is linear if $\mathbf{K} = \mathbf{R}$, and conjugate-linear if $\mathbf{K} = \mathbf{C}$.

Exercise 15. Let $\langle \cdot, \cdot \rangle$ be an inner-product on a $\mathbf{K}$-vector space $\mathcal{H}$. Let $x, y \in \mathcal{H}$. Let $A = \|x\|^2$, $B = |\langle x, y \rangle|$ and $C = \|y\|^2$. let $\alpha \in \mathbf{K}$ be such that $|\alpha| = 1$ and:

$$B = \alpha \overline{\langle x, y \rangle}$$

1. Show that $A, B, C \in \mathbf{R}^+$.

2. For all $t \in \mathbf{R}$, show that $\langle x - t\alpha y, x - t\alpha y \rangle = A - 2tB + t^2 C$.

3. Show that if $C = 0$ then $B^2 \leq AC$.

4. Suppose that $C \neq 0$. Show that $P(t) = A - 2tB + t^2 C$ has a minimal value which is in $\mathbf{R}^+$, and conclude that $B^2 \leq AC$.

5. Conclude with the following:

Theorem 50 (Cauchy-Schwarz's inequality:second) *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and $\langle \cdot, \cdot \rangle$ be an inner-product on $\mathcal{H}$. Then, for all $x, y \in \mathcal{H}$, we have:*

$$|\langle x, y \rangle| \leq \|x\|.\|y\|$$

Exercise 16. For all $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we define:

$$\langle f, g \rangle \stackrel{\triangle}{=} \int_\Omega f \bar{g} d\mu$$

1. Use the first cauchy-schwarz inequality (42) to prove that for all $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have $f\bar{g} \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Conclude that $\langle f, g \rangle$ is a well-defined complex number.

2. Show that for all $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have $\|f\|_2 = \sqrt{\langle f, f \rangle}$.

3. Make another use of the first cauchy-schwarz inequality to show that for all $f, g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have:

$$|\langle f, g \rangle| \leq \|f\|_2.\|g\|_2 \tag{1}$$

4. Go through definition (81), and indicate which of the properties $(i)-(v)$ fails to be satisfied by $\langle\cdot,\cdot\rangle$. Conclude that $\langle\cdot,\cdot\rangle$ is not an inner-product on $L^2_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$, and therefore that inequality (*) is not a particular case of the second cauchy-schwarz inequality (50).

5. Let $f,g\in L^2_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$. By considering $\int(|f|+t|g|)^2d\mu$ for $t\in\mathbf{R}$, imitate the proof of the second cauchy-schwarz inequality to show that:
$$\int_\Omega |fg|d\mu \leq \left(\int_\Omega |f|^2d\mu\right)^{\frac{1}{2}}\left(\int_\Omega |g|^2d\mu\right)^{\frac{1}{2}}$$

6. Let $f,g:(\Omega,\mathcal{F})\to[0,+\infty]$ non-negative and measurable. Show that if $\int f^2d\mu$ and $\int g^2d\mu$ are finite, then f and g are μ-almost surely equal to elements of $L^2_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$. Deduce from 5. a new proof of the first Cauchy-Schwarz inequality:
$$\int_\Omega fgd\mu \leq \left(\int_\Omega f^2d\mu\right)^{\frac{1}{2}}\left(\int_\Omega g^2d\mu\right)^{\frac{1}{2}}$$

Exercise 17. Let $\langle\cdot,\cdot\rangle$ be an inner product on a $\mathbf{K}$-vector space $\mathcal{H}$.

1. Show that for all $x, y \in \mathcal{H}$, we have:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle x,y\rangle + \overline{\langle x,y\rangle}$$

2. Using the second cauchy-schwarz inequality (50), show that:

$$\|x+y\| \leq \|x\| + \|y\|$$

3. Show that $d_{\langle\cdot,\cdot\rangle}(x,y) = \|x-y\|$ defines a metric on $\mathcal{H}$.

Definition 82 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and $\langle\cdot,\cdot\rangle$ be an inner-product on $\mathcal{H}$. We call* **norm topology** *on $\mathcal{H}$, denoted $\mathcal{T}_{\langle\cdot,\cdot\rangle}$, the metric topology associated with $d_{\langle\cdot,\cdot\rangle}(x,y) = \|x-y\|$.*

Definition 83 *We call* **hilbert space** *(over $\mathbf{K}$), where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, any ordered pair $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, where $\mathcal{H}$ is a $\mathbf{K}$-vector space, and $\langle\cdot,\cdot\rangle$ is an inner product on $\mathcal{H}$ for which the metric space $(\mathcal{H}, d_{\langle\cdot,\cdot\rangle})$ is complete, where $d_{\langle\cdot,\cdot\rangle}(x,y) = \|x-y\|$.*

Exercise 18. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a hilbert space over $\mathbf{K}$ and let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$, (closed with respect to the norm topology $\mathcal{T}_{\langle\cdot,\cdot\rangle}$). Define $[\cdot,\cdot] = \langle\cdot,\cdot\rangle_{|\mathcal{M}\times\mathcal{M}}$.

1. Show that $[\cdot,\cdot]$ is an inner-product on the $\mathbf{K}$-vector space $\mathcal{M}$.
2. With obvious notations, show that $d_{[\cdot,\cdot]} = (d_{\langle\cdot,\cdot\rangle})_{|\mathcal{M}\times\mathcal{M}}$.
3. Deduce that $\mathcal{T}_{[\cdot,\cdot]} = (\mathcal{T}_{\langle\cdot,\cdot\rangle})_{|\mathcal{M}}$.

Exercise 19. Further to ex. (18), Let $(x_n)_{n\geq 1}$ be a cauchy sequence in $\mathcal{M}$, with respect to the metric $d_{[\cdot,\cdot]}$.

1. Show that $(x_n)_{n\geq 1}$ is a cauchy sequence in $\mathcal{H}$.
2. Explain why there exists $x \in \mathcal{H}$ such that $x_n \stackrel{\mathcal{T}_{\langle\cdot,\cdot\rangle}}{\rightarrow} x$.
3. Explain why $x \in \mathcal{M}$.
4. Explain why we also have $x_n \stackrel{\mathcal{T}_{[\cdot,\cdot]}}{\rightarrow} x$.

5. Explain why $(\mathcal{M}, \langle\cdot,\cdot\rangle_{|\mathcal{M}\times\mathcal{M}})$ is a hilbert space over $\mathbf{K}$.

EXERCISE 20. For all $z, z' \in \mathbf{C}^n$, $n \geq 1$, we define:

$$\langle z, z'\rangle \stackrel{\triangle}{=} \sum_{i=1}^{n} z_i \bar{z}_i'$$

1. Show that $\langle\cdot,\cdot\rangle$ is an inner-product on $\mathbf{C}^n$.
2. Show that the metric $d_{\langle\cdot,\cdot\rangle}$ is equal to the usual metric of $\mathbf{C}^n$.
3. Conclude that $(\mathbf{C}^n, \langle\cdot,\cdot\rangle)$ is a hilbert space over $\mathbf{C}$.
4. Show that $\mathbf{R}^n$ is a closed subset of $\mathbf{C}^n$.
5. Show however that $\mathbf{R}^n$ is not a linear subspace of $\mathbf{C}^n$.
6. Show that $(\mathbf{R}^n, \langle\cdot,\cdot\rangle_{|\mathbf{R}^n\times\mathbf{R}^n})$ is a hilbert space over $\mathbf{R}$.

Definition 84 *Let* $\mathbf{K} = \mathbf{R}$ *or* $\mathbf{C}$. *The* **usual inner-product** *in* $\mathbf{K}^n$, *denoted* $\langle \cdot, \cdot \rangle$, *is defined as:*

$$\forall x, y \in \mathbf{K}^n \ , \ \langle x, y \rangle = \sum_{i=1}^{n} x_i \bar{y_i}$$

Theorem 51 *The spaces* $\mathbf{C}^n$ *and* $\mathbf{R}^n$, $n \geq 1$, *together with their usual inner-products, are hilbert spaces over* $\mathbf{C}$ *and* $\mathbf{R}$ *respectively.*

Definition 85 *Let* $\mathcal{H}$ *be a* $\mathbf{K}$*-vector space, where* $\mathbf{K} = \mathbf{R}$ *or* $\mathbf{C}$. *Let* $\mathcal{C} \subseteq \mathcal{H}$. *We say that* $\mathcal{C}$ *is a* **convex subset** *or* $\mathcal{H}$, *if and only if, for all* $x, y \in \mathcal{C}$ *and* $t \in [0, 1]$, *we have* $tx + (1-t)y \in \mathcal{C}$.

Exercise 21. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a hilbert space over $\mathbf{K}$. Let $\mathcal{C} \subseteq \mathcal{H}$ be a non-empty closed convex subset of $\mathcal{H}$. Let $x_0 \in \mathcal{H}$. Define:

$$\delta_{\min} \stackrel{\triangle}{=} \inf\{\|x - x_0\| \ : \ x \in \mathcal{C}\}$$

1. Show the existence of a sequence $(x_n)_{n\geq 1}$ in $\mathcal{C}$ such that $\|x_n - x_0\| \to \delta_{\min}$.

2. Show that for all $x, y \in \mathcal{H}$, we have:

$$\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - 4\left\|\frac{x+y}{2}\right\|^2$$

3. Explain why for all $n, m \geq 1$, we have:

$$\delta_{\min} \leq \left\|\frac{x_n + x_m}{2} - x_0\right\|$$

4. Show that for all $n, m \geq 1$, we have:

$$\|x_n - x_m\|^2 \leq 2\|x_n - x_0\|^2 + 2\|x_m - x_0\|^2 - 4\delta_{\min}^2$$

5. Show the existence of some $x^* \in \mathcal{H}$, such that $x_n \stackrel{\mathcal{T}_{\langle\cdot,\cdot\rangle}}{\rightarrow} x^*$.

6. Explain why $x^* \in \mathcal{C}$

7. Show that for all $x, y \in \mathcal{H}$, we have $|\ \|x\| - \|y\|\ | \leq \|x - y\|$.

8. Show that $\|x_n - x_0\| \to \|x^* - x_0\|$.

9. Conclude that we have found $x^* \in \mathcal{C}$ such that:
$$\|x^* - x_0\| = \inf\{\|x - x_0\|\ :\ x \in \mathcal{C}\}$$

10. Let y^* be another element of $\mathcal{C}$ with such property. Show that:
$$\|x^* - y^*\|^2 \leq 2\|x^* - x_0\|^2 + 2\|y^* - x_0\|^2 - 4\delta_{\min}^2$$

11. Conclude that $x^* = y^*$.

Theorem 52 *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a hilbert space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{C}$ be a non-empty, closed and convex subset of $\mathcal{H}$. For all $x_0 \in \mathcal{H}$, there exists a unique $x^* \in \mathcal{C}$ such that:*

$$\|x^* - x_0\| = \inf\{\|x - x_0\| \ : \ x \in \mathcal{C}\}$$

Definition 86 *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a hilbert space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{G} \subseteq \mathcal{H}$. We call* **orthogonal** *of $\mathcal{G}$, the subset of $\mathcal{H}$ denoted $\mathcal{G}^\perp$ and defined by:*

$$\mathcal{G}^\perp \stackrel{\triangle}{=} \{\ x \in \mathcal{H} \ : \ \langle x, y\rangle = 0 \ , \ \forall y \in \mathcal{G}\ \}$$

Exercise 22. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a hilbert space over $\mathbf{K}$ and $\mathcal{G} \subseteq \mathcal{H}$.

1. Show that $\mathcal{G}^\perp$ is a linear subspace of $\mathcal{H}$, even if $\mathcal{G}$ isn't.
2. Show that $\phi_y : \mathcal{H} \to K$ defined by $\phi_y(x) = \langle x, y\rangle$ is continuous.
3. Show that $\mathcal{G}^\perp = \cap_{y\in\mathcal{G}}\phi_y^{-1}(\{0\})$.

4. Show that $\mathcal{G}^\perp$ is a closed subset of $\mathcal{H}$, even if $\mathcal{G}$ isn't.

5. Show that $\emptyset^\perp = \{0\}^\perp = \mathcal{H}$.

6. Show that $\mathcal{H}^\perp = \{0\}$.

EXERCISE 23. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a hilbert space over $\mathbf{K}$. Let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$, and $x_0 \in \mathcal{H}$.

1. Explain why there exists $x^* \in \mathcal{M}$ such that:
$$\|x^* - x_0\| = \inf\{\ \|x - x_0\|\ :\ x \in \mathcal{M}\ \}$$

2. Define $y^* = x_0 - x^* \in \mathcal{H}$. Show that for all $y \in \mathcal{M}$ and $\alpha \in \mathbf{K}$:
$$\|y^*\|^2 \leq \|y^* - \alpha y\|^2$$

3. Show that for all $y \in \mathcal{M}$ and $\alpha \in \mathbf{K}$, we have:
$$0 \leq -\alpha\langle y, y^*\rangle - \overline{\alpha\langle y, y^*\rangle} + |\alpha|^2.\|y\|^2$$

4. For all $y \in \mathcal{M} \setminus \{0\}$, taking $\alpha = \overline{\langle y, y^* \rangle} / \|y\|^2$, show that:

$$0 \leq -\frac{|\langle y, y^* \rangle|^2}{\|y\|^2}$$

5. Conclude that $x^* \in \mathcal{M}$, $y^* \in \mathcal{M}^\perp$ and $x_0 = x^* + y^*$.

6. Show that $\mathcal{M} \cap \mathcal{M}^\perp = \{0\}$

7. Show that $x^* \in \mathcal{M}$ and $y^* \in \mathcal{M}^\perp$ with $x_0 = x^* + y^*$, are unique.

Theorem 53 *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a hilbert space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$. Then, for all $x_0 \in \mathcal{H}$, there is a unique decomposition:*

$$x_0 = x^* + y^*$$

where $x^* \in \mathcal{M}$ and $y^* \in \mathcal{M}^\perp$.

Definition 87 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We call* **linear functional**, *any map $\lambda : \mathcal{H} \to \mathbf{K}$, such that for all $x, y \in \mathcal{H}$ and $\alpha \in \mathbf{K}$:*

$$\lambda(x + \alpha y) = \lambda(x) + \alpha\lambda(y)$$

EXERCISE 24. Let λ be a linear functional on a $\mathbf{K}$-hilbert[1] space $\mathcal{H}$.

1. Suppose that λ is continuous at some point $x_0 \in \mathcal{H}$. Show the existence of $\eta > 0$ such that:

$$\forall x \in \mathcal{H}\ ,\ \|x - x_0\| \leq \eta \ \Rightarrow\ |\lambda(x) - \lambda(x_0)| \leq 1$$

   Show that for all $x \in \mathcal{H}$ with $x \neq 0$, we have $|\lambda(\eta x/\|x\|)| \leq 1$.

2. Show that if λ is continuous at x_0, there exits $M \in \mathbf{R}^+$, with:

$$\forall x \in \mathcal{H}\ ,\ |\lambda(x)| \leq M\|x\| \tag{2}$$

[1] Norm vector spaces are introduced later in these tutorials.

3. Show conversely that if (2) holds, λ is continuous everywhere.

Definition 88 *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a hilbert*[2] *space over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let λ be a linear functional on $\mathcal{H}$. Then, the following are equivalent:*

$$\begin{array}{ll} (i) & \lambda : (\mathcal{H}, \mathcal{T}_{\langle\cdot,\cdot\rangle}) \to (K, \mathcal{T}_{\mathbf{K}}) \ \text{ is continuous} \\ (ii) & \exists M \in \mathbf{R}^+ \ , \ \forall x \in \mathcal{H} \ , \ |\lambda(x)| \leq M.\|x\| \end{array}$$

In which case, we say that λ is a **bounded linear functional**.

Exercise 25. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a hilbert space over $\mathbf{K}$. Let λ be a bounded linear functional on $\mathcal{H}$, such that $\lambda(x) \neq 0$ for some $x \in \mathcal{H}$, and define $\mathcal{M} = \lambda^{-1}(\{0\})$.

1. Show the existence of $x_0 \in \mathcal{H}$, such that $x_0 \notin \mathcal{M}$.

2. Show the existence of $x^* \in \mathcal{M}$ and $y^* \in \mathcal{M}^\perp$ with $x_0 = x^* + y^*$.

[2] Norm vector spaces are introduced later in these tutorials.

3. Deduce the existence of some $z \in \mathcal{M}^\perp$ such that $\|z\| = 1$.

4. Show that for all $\alpha \in \mathbf{K} \setminus \{0\}$ and $x \in \mathcal{H}$, we have:

$$\frac{\lambda(x)}{\bar{\alpha}} \langle z, \alpha z \rangle = \lambda(x)$$

5. Show that in order to have:

$$\forall x \in \mathcal{H} \ , \ \lambda(x) = \langle x, \alpha z \rangle$$

it is sufficient to choose $\alpha \in \mathbf{K} \setminus \{0\}$ such that:

$$\forall x \in \mathcal{H} \ , \ \frac{\lambda(x) z}{\bar{\alpha}} - x \in \mathcal{M}$$

6. Show the existence of $y \in \mathcal{H}$ such that:

$$\forall x \in \mathcal{H} \ , \ \lambda(x) = \langle x, y \rangle$$

7. Show the uniqueness of such $y \in \mathcal{H}$.

Theorem 54 *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a hilbert space over $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Let λ be a bounded linear functional on $\mathcal{H}$. Then, there exists a unique $y \in \mathcal{H}$ such that: $\forall x \in \mathcal{H}\ ,\ \lambda(x) = \langle x, y \rangle$.*

Definition 89 *Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We call* **K-vector space***, any set $\mathcal{H}$, together with operators $\oplus$ and $\otimes$ for which there exits an element $0_{\mathcal{H}} \in \mathcal{H}$ such that for all $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbf{K}$, we have:*

$$
\begin{array}{rl}
(i) & 0_{\mathcal{H}} \oplus x = x \\
(ii) & \exists (-x) \in \mathcal{H}\ ,\ (-x) \oplus x = 0_{\mathcal{H}} \\
(iii) & x \oplus (y \oplus z) = (x \oplus y) \oplus z \\
(iv) & x \oplus y = y \oplus x \\
(v) & 1 \otimes x = x \\
(vi) & \alpha \otimes (\beta \otimes x) = (\alpha\beta) \otimes x \\
(vii) & (\alpha + \beta) \otimes x = (\alpha \otimes x) \oplus (\beta \otimes x) \\
(viii) & \alpha \otimes (x \oplus y) = (\alpha \otimes x) \oplus (\alpha \otimes y)
\end{array}
$$

Exercise 26. For all $f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$, define:

$$\mathcal{H} \stackrel{\triangle}{=} \{\ [f]\ :\ f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)\ \}$$

where $[f] = \{g \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) : g = f, \mu\text{-a.s.}\}$. Let $0_{\mathcal{H}} = [0]$, and for all $[f], [g] \in \mathcal{H}$, and $\alpha \in \mathbf{K}$, we define:

$$\begin{aligned} [f] \oplus [g] &\stackrel{\triangle}{=} [f+g] \\ \alpha \otimes [f] &\stackrel{\triangle}{=} [\alpha f] \end{aligned}$$

We assume f, f', g and g' are elements of $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.

1. Show that for $f = g$ μ-a.s. is equivalent to $[f] = [g]$.
2. Show that if $[f] = [f']$ and $[g] = [g']$, then $[f+g] = [f'+g']$.
3. Conclude that $\oplus$ is well-defined.
4. Show that $\otimes$ is also well-defined.
5. Show that $(\mathcal{H}, \oplus, \otimes)$ is a $\mathbf{K}$-vector space.

EXERCISE 27. Further to ex. (26), we define for all $[f], [g] \in \mathcal{H}$:

$$\langle [f], [g] \rangle_{\mathcal{H}} \stackrel{\triangle}{=} \int_\Omega f\bar{g}d\mu$$

1. Show that $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is well-defined.

2. Show that $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is an inner-product on $\mathcal{H}$.

3. Show that $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ is a hilbert space over $\mathbf{K}$.

4. Why is $\langle f, g \rangle \stackrel{\triangle}{=} \int_\Omega f\bar{g}d\mu$ not an inner-product on $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$?

EXERCISE 28. Further to ex. (27), Let $\lambda : L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \to \mathbf{K}$ be a continuous linear functional[3]. Define $\Lambda : \mathcal{H} \to \mathbf{K}$ by $\Lambda([f]) = \lambda(f)$.

[3]As defined in these tutorials, $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ is not a hilbert space (not even a norm vector space). However, both $L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ and $\mathbf{K}$ have natural topologies and it is therefore meaningful to speak of *continuous linear functional*. Note however that we are slightly outside the framework of definition (88).

1. Show the existence of $M \in \mathbf{R}^+$ such that:

$$\forall f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \ , \ |\lambda(f)| \leq M.\|f\|_2$$

2. Show that if $[f] = [g]$ then $\lambda(f) = \lambda(g)$.

3. Show that Λ is a well defined bounded linear functional on $\mathcal{H}$.

4. Conclude with the following:

Theorem 55 *Let $\lambda : L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \to \mathbf{K}$ be a continuous linear functional, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. Then, there exists $g \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$ such that:*

$$\forall f \in L^2_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \ , \ \lambda(f) = \int_\Omega f \bar{g} d\mu$$

11. Complex Measures

In the following, $(\Omega, \mathcal{F})$ denotes an arbitrary measurable space.

Definition 90 *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers. We say that $(a_n)_{n\geq 1}$ has the* **permutation property** *if and only if, for all bijections $\sigma : \mathbf{N}^* \to \mathbf{N}^*$, the series $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges in $\mathbf{C}$*[1]

Exercise 1. Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers.

1. Show that if $(a_n)_{n\geq 1}$ has the permutation property, then the same is true of $(Re(a_n))_{n\geq 1}$ and $(Im(a_n))_{n\geq 1}$.

2. Suppose $a_n \in \mathbf{R}$ for all $n \geq 1$. Show that if $\sum_{k=1}^{+\infty} a_k$ converges:

$$\sum_{k=1}^{+\infty} |a_k| = +\infty \;\Rightarrow\; \sum_{k=1}^{+\infty} a_k^+ = \sum_{k=1}^{+\infty} a_k^- = +\infty$$

[1] which excludes $\pm\infty$ as limit.

Exercise 2. Let $(a_n)_{n\geq 1}$ be a sequence in $\mathbf{R}$, such that the series $\sum_{k=1}^{+\infty} a_k$ converges, and $\sum_{k=1}^{+\infty} |a_k| = +\infty$. Let $A > 0$. We define:

$$N^+ \stackrel{\triangle}{=} \{k \geq 1 \ : \ a_k \geq 0\} \ \ , \ \ N^- \stackrel{\triangle}{=} \{k \geq 1 \ : \ a_k < 0\}$$

1. Show that N^+ and N^- are infinite.

2. Let $\phi^+ : \mathbf{N}^* \to N^+$ and $\phi^- : \mathbf{N}^* \to N^-$ be two bijections. Show the existence of $k_1 \geq 1$ such that:

$$\sum_{k=1}^{k_1} a_{\phi^+(k)} \geq A$$

3. Show the existence of an increasing sequence $(k_p)_{p\geq 1}$ such that:

$$\sum_{k=k_{p-1}+1}^{k_p} a_{\phi^+(k)} \geq A$$

for all $p \geq 1$, where $k_0 = 0$.

4. Consider the permutation $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ defined informally by:

$$(\phi^-(1), \underbrace{\phi^+(1), \ldots, \phi^+(k_1)}, \phi^-(2), \underbrace{\phi^+(k_1+1), \ldots, \phi^+(k_2)}, \ldots)$$

representing $(\sigma(1), \sigma(2), \ldots)$. More specifically, define $k_0^* = 0$ and $k_p^* = k_p + p$ for all $p \geq 1$. For all $n \in \mathbf{N}^*$ and $p \geq 1$ with: [2]

$$k_{p-1}^* < n \leq k_p^* \tag{1}$$

we define:

$$\sigma(n) = \left\{ \begin{array}{lll} \phi^-(p) & \text{if} & n = k_{p-1}^* + 1 \\ \phi^+(n-p) & \text{if} & n > k_{p-1}^* + 1 \end{array} \right. \tag{2}$$

Show that $\sigma : \mathbf{N}^* \to \mathbf{N}^*$ is indeed a bijection.

5. Show that if $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges, there is $N \geq 1$, such that:

$$n \geq N \ , \ p \geq 1 \ \Rightarrow \ \left| \sum_{k=n+1}^{n+p} a_{\sigma(k)} \right| < A$$

[2] Given an integer $n \geq 1$, there exists a unique $p \geq 1$ such that (1) holds.

6. Explain why $(a_n)_{n\geq 1}$ cannot have the permutation property.

7. Prove the following theorem:

Theorem 56 *Let $(a_n)_{n\geq 1}$ be a sequence of complex numbers such that for all bijections $\sigma : \mathbf{N}^* \to \mathbf{N}^*$, the series $\sum_{k=1}^{+\infty} a_{\sigma(k)}$ converges. Then, the series $\sum_{k=1}^{+\infty} a_k$ converges absolutely, i.e.*

$$\sum_{k=1}^{+\infty} |a_k| < +\infty$$

Definition 91 *Let $(\Omega, \mathcal{F})$ be a measurable space and $E \in \mathcal{F}$. We call* **measurable partition** *of E, any sequence $(E_n)_{n\geq 1}$ of pairwise disjoint elements of $\mathcal{F}$, such that $E = \uplus_{n\geq 1} E_n$.*

Definition 92 *We call* **complex measure** *on a measurable space* $(\Omega, \mathcal{F})$ *any map* $\mu : \mathcal{F} \to \mathbf{C}$*, such that for all* $E \in \mathcal{F}$ *and* $(E_n)_{n\geq 1}$ *measurable partition of* E*, the series* $\sum_{n=1}^{+\infty} \mu(E_n)$ *converges to* $\mu(E)$*. The set of all complex measures on* $(\Omega, \mathcal{F})$ *is denoted* $M^1(\Omega, \mathcal{F})$*.*

Definition 93 *We call* **signed measure** *on a measurable space* $(\Omega, \mathcal{F})$*, any complex measure on* $(\Omega, \mathcal{F})$ *with values in* $\mathbf{R}$*.*[3]

EXERCISE 3.

1. Show that a measure on $(\Omega, \mathcal{F})$ may not be a complex measure.
2. Show that for all $\mu \in M^1(\Omega, \mathcal{F})$, $\mu(\emptyset) = 0$.
3. Show that a finite measure on $(\Omega, \mathcal{F})$ is a complex measure with values in $\mathbf{R}^+$, and conversely.

[3] In these tutorials, signed measure may not have values in $\{-\infty, +\infty\}$.

4. Let $\mu \in M^1(\Omega, \mathcal{F})$. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Show that:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| < +\infty$$

5. Let μ be a measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Define:

$$\forall E \in \mathcal{F}\ ,\ \nu(E) \stackrel{\triangle}{=} \int_E f d\mu$$

Show that ν is a complex measure on $(\Omega, \mathcal{F})$.

Definition 94 *Let μ be a complex measure on a measurable space $(\Omega, \mathcal{F})$. We call* **total variation** *of μ, the map $|\mu| : \mathcal{F} \to [0, +\infty]$, defined by:*

$$\forall E \in \mathcal{F}\ ,\ |\mu|(E) \stackrel{\triangle}{=} \sup \sum_{n=1}^{+\infty} |\mu(E_n)|$$

where the 'sup' is taken over all measurable partitions $(E_n)_{n\geq 1}$ of E.

Exercise 4. Let μ be a complex measure on $(\Omega, \mathcal{F})$.

1. Show that for all $E \in \mathcal{F}$, $|\mu(E)| \leq |\mu|(E)$.

2. Show that $|\mu|(\emptyset) = 0$.

Exercise 5. Let μ be a complex measure on $(\Omega, \mathcal{F})$. Let $E \in \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E.

1. Show that there exists $(t_n)_{n\geq 1}$ in $\mathbf{R}$, with $t_n < |\mu|(E_n)$ for all n.

2. Show that for all $n \geq 1$, there exists a measurable partition $(E_n^p)_{p\geq 1}$ of E_n such that:

$$t_n < \sum_{p=1}^{+\infty} |\mu(E_n^p)|$$

3. Show that $(E_n^p)_{n,p\geq 1}$ is a measurable partition of E.

4. Show that for all $N \geq 1$, we have $\sum_{n=1}^{N} t_n \leq |\mu|(E)$.

5. Show that for all $N \geq 1$, we have:

$$\sum_{n=1}^{N} |\mu|(E_n) \leq |\mu|(E)$$

6. Suppose that $(A_p)_{p\geq 1}$ is another arbitrary measurable partition of E. Show that for all $p \geq 1$:

$$|\mu(A_p)| \leq \sum_{n=1}^{+\infty} |\mu(A_p \cap E_n)|$$

7. Show that for all $n \geq 1$:

$$\sum_{p=1}^{+\infty} |\mu(A_p \cap E_n)| \leq |\mu|(E_n)$$

8. Show that:

$$\sum_{p=1}^{+\infty} |\mu(A_p)| \leq \sum_{n=1}^{+\infty} |\mu|(E_n)$$

9. Show that $|\mu| : \mathcal{F} \to [0, +\infty]$ is a measure on $(\Omega, \mathcal{F})$.

Exercise 6. Let $a, b \in \mathbf{R}, a < b$. Let $F \in C^1([a, b]; \mathbf{R})$, and define:

$$\forall x \in [a, b]\ ,\ H(x) \stackrel{\triangle}{=} \int_a^x F'(t)dt$$

1. Show that $H \in C^1([a, b]; \mathbf{R})$ and $H' = F'$.
2. Show that:
$$F(b) - F(a) = \int_a^b F'(t)dt$$
3. Show that:
$$\frac{1}{2\pi} \int_{-\pi/2}^{+\pi/2} \cos\theta d\theta = \frac{1}{\pi}$$
4. Let $u \in \mathbf{R}^n$ and $\tau_u : \mathbf{R}^n \to \mathbf{R}^n$ be the translation $\tau_u(x) = x + u$. Show that the Lebesgue measure dx on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is invariant by translation τ_u, i.e. $dx(\{\tau_u \in B\}) = dx(B)$ for all $B \in \mathcal{B}(\mathbf{R}^n)$.

5. Show that for all $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, and $u \in \mathbf{R}^n$:

$$\int_{\mathbf{R}^n} f(x+u)dx = \int_{\mathbf{R}^n} f(x)dx$$

6. Show that for all $\alpha \in \mathbf{R}$, we have:

$$\int_{-\pi}^{+\pi} \cos^+(\alpha - \theta)d\theta = \int_{-\pi-\alpha}^{+\pi-\alpha} \cos^+ \theta d\theta$$

7. Let $\alpha \in \mathbf{R}$ and $k \in \mathbf{Z}$ such that $k \leq \alpha/2\pi < k+1$. Show:

$$-\pi - \alpha \leq -2k\pi - \pi < \pi - \alpha \leq -2k\pi + \pi$$

8. Show that:

$$\int_{-\pi-\alpha}^{-2k\pi-\pi} \cos^+ \theta d\theta = \int_{\pi-\alpha}^{-2k\pi+\pi} \cos^+ \theta d\theta$$

9. Show that:

$$\int_{-\pi-\alpha}^{+\pi-\alpha} \cos^+ \theta d\theta = \int_{-2k\pi-\pi}^{-2k\pi+\pi} \cos^+ \theta d\theta = \int_{-\pi}^{+\pi} \cos^+ \theta d\theta$$

10. Show that for all $\alpha \in \mathbf{R}$:

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} \cos^+(\alpha - \theta) d\theta = \frac{1}{\pi}$$

Exercise 7. Let $z_1, \ldots, z_N$ be N complex numbers. Let $\alpha_k \in \mathbf{R}$ be such that $z_k = |z_k| e^{i\alpha_k}$, for all $k = 1, \ldots, N$. For all $\theta \in [-\pi, +\pi]$, we define $S(\theta) = \{k = 1, \ldots, N \ : \ \cos(\alpha_k - \theta) > 0\}$.

1. Show that for all $\theta \in [-\pi, +\pi]$, we have:

$$\left| \sum_{k \in S(\theta)} z_k \right| = \left| \sum_{k \in S(\theta)} z_k e^{-i\theta} \right| \geq \sum_{k \in S(\theta)} |z_k| \cos(\alpha_k - \theta)$$

2. Define $\phi : [-\pi, +\pi] \to \mathbf{R}$ by $\phi(\theta) = \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta)$. Show the existence of $\theta_0 \in [-\pi, +\pi]$ such that:

$$\phi(\theta_0) = \sup_{\theta \in [-\pi, +\pi]} \phi(\theta)$$

3. Show that:

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} \phi(\theta) d\theta = \frac{1}{\pi} \sum_{k=1}^{N} |z_k|$$

4. Conclude that:

$$\frac{1}{\pi} \sum_{k=1}^{N} |z_k| \leq \left| \sum_{k \in S(\theta_0)} z_k \right|$$

Exercise 8. Let $\mu \in M^1(\Omega, \mathcal{F})$. Suppose that $|\mu|(E) = +\infty$ for some $E \in \mathcal{F}$. Define $t = \pi(1 + |\mu(E)|) \in \mathbf{R}^+$.

1. Show that there is a measurable partition $(E_n)_{n\geq 1}$ of E, with:

$$t < \sum_{n=1}^{+\infty} |\mu(E_n)|$$

2. Show the existence of $N \geq 1$ such that:

$$t < \sum_{n=1}^{N} |\mu(E_n)|$$

3. Show the existence of $S \subseteq \{1, \ldots, N\}$ such that:

$$\sum_{n=1}^{N} |\mu(E_n)| \leq \pi \left| \sum_{n\in S} \mu(E_n) \right|$$

4. Show that $|\mu(A)| > t/\pi$, where $A = \uplus_{n\in S} E_n$.

5. Let $B = E \setminus A$. Show that $|\mu(B)| \geq |\mu(A)| - |\mu(E)|$.

6. Show that $E = A \uplus B$ with $|\mu(A)| > 1$ and $|\mu(B)| > 1$.

7. Show that $|\mu|(A) = +\infty$ or $|\mu|(B) = +\infty$.

Exercise 9. Let $\mu \in M^1(\Omega, \mathcal{F})$. Suppose that $|\mu|(\Omega) = +\infty$.

1. Show the existence of $A_1, B_1 \in \mathcal{F}$, such that $\Omega = A_1 \uplus B_1$, $|\mu(A_1)| > 1$ and $|\mu|(B_1) = +\infty$.

2. Show the existence of a sequence $(A_n)_{n\geq 1}$ of pairwise disjoint elements of $\mathcal{F}$, such that $|\mu(A_n)| > 1$ for all $n \geq 1$.

3. Show that the series $\sum_{n=1}^{+\infty} \mu(A_n)$ does not converge to $\mu(A)$ where $A = \uplus_{n=1}^{+\infty} A_n$.

4. Conclude that $|\mu|(\Omega) < +\infty$.

Theorem 57 *Let μ be a complex measure on a measurable space $(\Omega, \mathcal{F})$. Then, its total variation $|\mu|$ is a finite measure on $(\Omega, \mathcal{F})$.*

Exercise 10. Show that $M^1(\Omega, \mathcal{F})$ is a $\mathbf{C}$-vector space, with:

$$\begin{aligned} (\lambda + \mu)(E) &\stackrel{\triangle}{=} \lambda(E) + \mu(E) \\ (\alpha\lambda)(E) &\stackrel{\triangle}{=} \alpha.\lambda(E) \end{aligned}$$

where $\lambda, \mu \in M^1(\Omega, \mathcal{F})$, $\alpha \in \mathbf{C}$, and $E \in \mathcal{F}$.

Definition 95 *Let $\mathcal{H}$ be a $\mathbf{K}$-vector space, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We call* **norm** *on $\mathcal{H}$, any map $N : \mathcal{H} \to \mathbf{R}^+$, with the following properties:*

$$\begin{array}{rl} (i) & \forall x \in \mathcal{H}\ ,\ (N(x) = 0\ \Leftrightarrow\ x = 0) \\ (ii) & \forall x \in \mathcal{H}, \forall \alpha \in \mathbf{K}\ ,\ N(\alpha x) = |\alpha| N(x) \\ (iii) & \forall x, y \in \mathcal{H}\ ,\ N(x + y) \leq N(x) + N(y) \end{array}$$

Exercise 11.

1. Explain why $\|.\|_p$ may not be a norm on $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.

2. Show that $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ is a norm, when $\langle \cdot, \cdot \rangle$ is an inner-product.

3. Show that $\|\mu\| \stackrel{\triangle}{=} |\mu|(\Omega)$ defines a norm on $M^1(\Omega, \mathcal{F})$.

Exercise 12. Let $\mu \in M^1(\Omega, \mathcal{F})$ be a signed measure. Show that:

$$\begin{aligned} \mu^+ &\stackrel{\triangle}{=} \frac{1}{2}(|\mu| + \mu) \\ \mu^- &\stackrel{\triangle}{=} \frac{1}{2}(|\mu| - \mu) \end{aligned}$$

are finite measures such that:

$$\mu = \mu^+ - \mu^- \quad , \quad |\mu| = \mu^+ + \mu^-$$

Exercise 13. Let $\mu \in M^1(\Omega, \mathcal{F})$ and $l : \mathbf{R}^2 \to \mathbf{R}$ be a linear map.

1. Show that l is continuous.

2. Show that $l \circ \mu$ is a signed measure on $(\Omega, \mathcal{F})$. [4]

3. Show that all $\mu \in M^1(\Omega, \mathcal{F})$ can be decomposed as:

$$\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$$

where $\mu_1, \mu_2, \mu_3, \mu_4$ are finite measures.

[4] $l \circ \mu$ refers strictly speaking to $l(Re(\mu), Im(\mu))$.

12. Radon-Nikodym Theorem

In the following, $(\Omega, \mathcal{F})$ is an arbitrary measurable space.

Definition 96 *Let μ and ν be two (possibly complex) measures on $(\Omega, \mathcal{F})$. We say that ν is* **absolutely continuous** *with respect to μ, and we write $\nu << \mu$, if and only if, for all $E \in \mathcal{F}$:*

$$\mu(E) = 0 \ \Rightarrow \ \nu(E) = 0$$

Exercise 1. Let μ be a measure on $(\Omega, \mathcal{F})$ and $\nu \in M^1(\Omega, \mathcal{F})$. Show that $\nu << \mu$ is equivalent to $|\nu| << \mu$.

Exercise 2. Let μ be a measure on $(\Omega, \mathcal{F})$ and $\nu \in M^1(\Omega, \mathcal{F})$. Let $\epsilon > 0$. Suppose there exists a sequence $(E_n)_{n\geq 1}$ in $\mathcal{F}$ such that:

$$\forall n \geq 1 \ , \ \mu(E_n) \leq \frac{1}{2^n} \ , \ |\nu(E_n)| \geq \epsilon$$

Define:

$$E \stackrel{\triangle}{=} \limsup_{n\geq 1} E_n \stackrel{\triangle}{=} \bigcap_{n\geq 1} \bigcup_{k\geq n} E_k$$

1. Show that:

$$\mu(E) = \lim_{n \to +\infty} \mu\left(\bigcup_{k \geq n} E_k\right) = 0$$

2. Show that:

$$|\nu|(E) = \lim_{n \to +\infty} |\nu|\left(\bigcup_{k \geq n} E_k\right) \geq \epsilon$$

3. Let λ be a measure on $(\Omega, \mathcal{F})$. Can we conclude in general that:

$$\lambda(E) = \lim_{n \to +\infty} \lambda\left(\bigcup_{k \geq n} E_k\right)$$

4. Prove the following:

Theorem 58 *Let μ be a measure on $(\Omega, \mathcal{F})$ and ν be a complex measure on $(\Omega, \mathcal{F})$. The following are equivalent:*

$$\begin{array}{rl} (i) & \nu << \mu \\ (ii) & |\nu| << \mu \\ (iii) & \forall \epsilon > 0, \exists \delta > 0, \forall E \in \mathcal{F}, \mu(E) \leq \delta \Rightarrow |\nu(E)| < \epsilon \end{array}$$

Exercise 3. Let μ be a measure on $(\Omega, \mathcal{F})$ and $\nu \in M^1(\Omega, \mathcal{F})$ such that $\nu << \mu$. Let $\nu_1 = Re(\nu)$ and $\nu_2 = Im(\nu)$.

1. Show that $\nu_1 << \mu$ and $\nu_2 << \mu$.
2. Show that $\nu_1^+, \nu_1^-, \nu_2^+, \nu_2^-$ are absolutely continuous w.r. to μ.

Exercise 4. Let μ be a finite measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Let S be a closed subset of $\mathbf{C}$. We assume that for all $E \in \mathcal{F}$ such that $\mu(E) > 0$, we have:

$$\frac{1}{\mu(E)} \int_E f d\mu \ \in \ S$$

1. Show the existence of a sequence (D_n) of closed discs in $\mathbf{C}$ with:

$$S^c = \bigcup_{n=1}^{+\infty} D_n$$

   Let $\alpha_n \in \mathbf{C}$, $r_n > 0$ be such that $D_n = \{z \in \mathbf{C} : |z - \alpha_n| \leq r_n\}$.

2. Suppose $\mu(E_n) > 0$ for some $n \geq 1$, where $E_n = \{f \in D_n\}$. Show that:

$$\left| \frac{1}{\mu(E_n)} \int_{E_n} f d\mu - \alpha_n \right| \leq \frac{1}{\mu(E_n)} \int_{E_n} |f - \alpha_n| d\mu \leq r_n$$

3. Show that for all $n \geq 1$, $\mu(\{f \in D_n\}) = 0$.

4. Prove the following:

Theorem 59 *Let μ be a finite measure on $(\Omega, \mathcal{F})$, $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Let S be a closed subset of $\mathbf{C}$ such that for all $E \in \mathcal{F}$ with $\mu(E) > 0$, we have:*

$$\frac{1}{\mu(E)} \int_E f d\mu \ \in \ S$$

Then, $f \in S$ μ-a.s.

Exercise 5. Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$. Let $(E_n)_{n \geq 1}$ be a sequence in $\mathcal{F}$ such that $E_n \uparrow \Omega$ and $\mu(E_n) < +\infty$ for all $n \geq 1$. Define $w : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ as:

$$w \stackrel{\triangle}{=} \sum_{n=1}^{+\infty} \frac{1}{2^n} \frac{1}{1 + \mu(E_n)} 1_{E_n}$$

1. Show that for all $\omega \in \Omega$, $0 < w(\omega) \leq 1$.

2. Show that $w \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

Exercise 6. Let μ be a σ-finite measure on $(\Omega,\mathcal{F})$ and ν be a finite measure on $(\Omega,\mathcal{F})$, such that $\nu << \mu$. Let $w \in L^1_{\mathbf{R}}(\Omega,\mathcal{F},\mu)$ be such that $0 < w \leq 1$. We define $\bar{\mu} = \int w d\mu$, i.e.

$$\forall E \in \mathcal{F}\ ,\ \bar{\mu}(E) \stackrel{\triangle}{=} \int_E w d\mu$$

1. Show that $\bar{\mu}$ is a finite measure on $(\Omega,\mathcal{F})$.
2. Show that $\phi = \nu + \bar{\mu}$ is also a finite measure on $(\Omega,\mathcal{F})$.
3. Show that for all $f \in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$, we have $f \in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\nu)$, $fw \in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)$, and:
$$\int_\Omega f d\phi = \int_\Omega f d\nu + \int_\Omega f w d\mu$$
4. Show that for all $f \in L^2_{\mathbf{C}}(\Omega,\mathcal{F},\phi)$, we have:
$$\int_\Omega |f| d\nu \leq \int_\Omega |f| d\phi \leq \left(\int_\Omega |f|^2 d\phi\right)^{\frac{1}{2}} (\phi(\Omega))^{\frac{1}{2}}$$

5. Show that $L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$, and for $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$:
$$\left| \int_\Omega f d\nu \right| \leq \sqrt{\phi(\Omega)}.\|f\|_2$$

6. Show the existence of $g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$ such that:
$$\forall f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi) \ , \ \int_\Omega f d\nu = \int_\Omega fg d\phi \tag{1}$$

7. Show that for all $E \in \mathcal{F}$ such that $\phi(E) > 0$, we have:
$$\frac{1}{\phi(E)} \int_E g d\phi \ \in \ [0, 1]$$

8. Show the existence of $g \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$ such that $g(\omega) \in [0, 1]$ for all $\omega \in \Omega$, and (1) still holds.

9. Show that for all $f \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$, we have:
$$\int_\Omega f(1-g) d\nu = \int_\Omega fgw d\mu$$

10. Show that for all $n \geq 1$ and $E \in \mathcal{F}$,

$$f \stackrel{\triangle}{=} (1 + g + \ldots + g^n)1_E \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, \phi)$$

11. Show that for all $n \geq 1$ and $E \in \mathcal{F}$,

$$\int_E (1 - g^{n+1}) d\nu = \int_E g(1 + g + \ldots + g^n) w d\mu$$

12. Define:

$$h \stackrel{\triangle}{=} gw \left(\sum_{n=0}^{+\infty} g^n \right)$$

Show that if $A = \{0 \leq g < 1\}$, then for all $E \in \mathcal{F}$:

$$\nu(E \cap A) = \int_E h d\mu$$

13. Show that $\{h = +\infty\} = A^c$ and conclude that $\mu(A^c) = 0$.

14. Show that for all $E \in \mathcal{F}$, we have $\nu(E) = \int_E h d\mu$.

15. Show that if μ is σ-finite on $(\Omega, \mathcal{F})$, and ν is a finite measure on $(\Omega, \mathcal{F})$ such that $\nu << \mu$, there exists $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$, such that $h \geq 0$ and:

$$\forall E \in \mathcal{F} \ , \ \nu(E) = \int_E h d\mu$$

16. Prove the following:

Theorem 60 (Radon-Nikodym:1) *Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$. let ν be a complex measure on $(\Omega, \mathcal{F})$ such that $\nu << \mu$. Then, there exists some $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ such that:*

$$\forall E \in \mathcal{F} \ , \ \nu(E) = \int_E h d\mu$$

If ν is a signed measure on $(\Omega, \mathcal{F})$, we can assume $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. If ν is a finite measure on $(\Omega, \mathcal{F})$, we can assume $h \geq 0$.

Exercise 7. Let $f = u + iv \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, such that:

$$\forall E \in \mathcal{F} \ , \ \int_E f d\mu = 0$$

where μ is a measure on $(\Omega, \mathcal{F})$.

1. Show that:
$$\int_\Omega u^+ d\mu = \int_{\{u \geq 0\}} u d\mu$$

2. Show that $f = 0$ μ-a.s.

3. State and prove some uniqueness property in theorem (60).

Exercise 8. Let μ and ν be two σ-finite measures on $(\Omega, \mathcal{F})$ such that $\nu << \mu$. Let $(E_n)_{n\geq 1}$ be a sequence in $\mathcal{F}$ such that $E_n \uparrow \Omega$ and $\nu(E_n) < +\infty$ for all $n \geq 1$. We define:

$$\forall n \geq 1 \ , \ \nu_n \stackrel{\triangle}{=} \nu^{E_n} \stackrel{\triangle}{=} \nu(E_n \cap \cdot)$$

1. Show that there exists $h_n \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ with $h_n \geq 0$ and:

$$\forall E \in \mathcal{F}\ ,\ \nu_n(E) = \int_E h_n d\mu \tag{2}$$

for all $n \geq 1$.

2. Show that for all $E \in \mathcal{F}$,

$$\int_E h_n d\mu \leq \int_E h_{n+1} d\mu$$

3. Show that for all $n, p \geq 1$,

$$\mu(\{h_n - h_{n+1} > \frac{1}{p}\}) = 0$$

4. Show that $h_n \leq h_{n+1}$ μ-a.s.

5. Show the existence of a sequence $(h_n)_{n\geq 1}$ in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that $0 \leq h_n \leq h_{n+1}$ for all $n \geq 1$ and with (2) still holding.

6. Let $h = \sup_{n\geq 1} h_n$. Show that:

$$\forall E \in \mathcal{F}\ ,\ \nu(E) = \int_E h d\mu \tag{3}$$

7. Show that for all $n \geq 1$, $\int_{E_n} h d\mu < +\infty$.

8. Show that $h < +\infty$ μ-a.s.

9. Show there exists $h : (\Omega, \mathcal{F}) \to \mathbf{R}^+$ measurable, while (3) holds.

10. Show that for all $n \geq 1$, $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu^{E_n})$.

Theorem 61 (Radon-Nikodym:2) *Let μ and ν be two σ-finite measures on $(\Omega, \mathcal{F})$ such that $\nu << \mu$. There exists a measurable map $h : (\Omega, \mathcal{F}) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ such that:*

$$\forall E \in \mathcal{F}\ ,\ \nu(E) = \int_E h d\mu$$

Exercise 9. Let $h, h' : (\Omega, \mathcal{F}) \to [0, +\infty]$ be two non-negative and measurable maps. Let μ be a σ-finite measure on $(\Omega, \mathcal{F})$. We assume:

$$\forall E \in \mathcal{F} \ , \ \int_E h d\mu = \int_E h' d\mu$$

Let $(E_n)_{n \geq 1}$ be a sequence in $\mathcal{F}$ with $E_n \uparrow \Omega$ and $\mu(E_n) < +\infty$ for all $n \geq 1$. We define $F_n = E_n \cap \{h \leq n\}$ for all $n \geq 1$.

1. Show that for all n and $E \in \mathcal{F}$, $\int_E h d\mu^{F_n} = \int_E h' d\mu^{F_n} < +\infty$.

2. Show that for all $n, p \geq 1$, $\mu(F_n \cap \{h > h' + 1/p\}) = 0$.

3. Show that for all $n \geq 1$, $\mu(\{F_n \cap \{h \neq h'\}) = 0$.

4. Show that $\mu(\{h \neq h'\} \cap \{h < +\infty\}) = 0$.

5. Show that $h = h'$ μ-a.s.

6. State and prove some uniqueness property in theorem (61).

Exercise 10. Take $\Omega = \{*\}$ and $\mathcal{F} = \mathcal{P}(\Omega) = \{\emptyset, \{*\}\}$. Let μ be the measure on $(\Omega, \mathcal{F})$ defined by $\mu(\emptyset) = 0$ and $\mu(\{*\}) = +\infty$. Let $h, h' : (\Omega, \mathcal{F}) \to [0, +\infty]$ be defined by $h(*) = 1 \neq 2 = h'(*)$. Show that we have:

$$\forall E \in \mathcal{F} \ , \ \int_E h d\mu = \int_E h' d\mu$$

Explain why this does not contradict the previous exercise.

Exercise 11. Let μ be a complex measure on $(\Omega, \mathcal{F})$.

1. Show that $\mu << |\mu|$.

2. Show the existence of some $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ such that:

$$\forall E \in \mathcal{F} \ , \ \mu(E) = \int_E h d|\mu|$$

3. If μ is a signed measure, can we assume $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$?

Exercise 12. Further to ex. (11), define $A_r = \{|h| < r\}$ for all $r > 0$.

1. Show that for all measurable partition $(E_n)_{n\geq 1}$ of A_r:

$$\sum_{n=1}^{+\infty} |\mu(E_n)| \leq r|\mu|(A_r)$$

2. Show that $|\mu|(A_r) = 0$ for all $0 < r < 1$.

3. Show that $|h| \geq 1$ $|\mu|$-a.s.

4. Suppose that $E \in \mathcal{F}$ is such that $|\mu|(E) > 0$. Show that:

$$\left| \frac{1}{|\mu|(E)} \int_E h d|\mu| \right| \leq 1$$

5. Show that $|h| \leq 1$ $|\mu|$-a.s.

6. Prove the following:

Theorem 62 *For all complex measure μ on $(\Omega, \mathcal{F})$, there exists h belonging to $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ such that $|h| = 1$ and:*

$$\forall E \in \mathcal{F}\ ,\ \mu(E) = \int_E hd|\mu|$$

If μ is a signed measure on $(\Omega, \mathcal{F})$, we can assume $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$.

EXERCISE 13. Let $A \in \mathcal{F}$, and $(A_n)_{n\geq 1}$ be a sequence in $\mathcal{F}$.

1. Show that if $A_n \uparrow A$ then $1_{A_n} \uparrow 1_A$.
2. Show that if $A_n \downarrow A$ then $1_{A_n} \downarrow 1_A$.
3. Show that if $1_{A_n} \to 1_A$, then for all $\mu \in M^1(\Omega, \mathcal{F})$:

$$\mu(A) = \lim_{n\to+\infty} \mu(A_n)$$

Exercise 14. Let μ be a measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$.

1. Show that $\nu = \int f d\mu \in M^1(\Omega, \mathcal{F})$.

2. Let $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\nu|)$ be such that $|h| = 1$ and $\nu = \int h d|\nu|$. Show that for all $E, F \in \mathcal{F}$:

$$\int_E f 1_F d\mu = \int_E h 1_F d|\nu|$$

3. Show that if $g : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is bounded and measurable:

$$\forall E \in \mathcal{F} \ , \ \int_E fg d\mu = \int_E hg d|\nu|$$

4. Show that:

$$\forall E \in \mathcal{F} \ , \ |\nu|(E) = \int_E f\bar{h} d\mu$$

5. Show that for all $E \in \mathcal{F}$,

$$\int_E Re(f\bar{h}) d\mu \geq 0 \quad , \quad \int_E Im(f\bar{h}) d\mu = 0$$

6. Show that $f\bar{h} \in \mathbf{R}^+$ μ-a.s.

7. Show that $f\bar{h} = |f|$ μ-a.s.

8. Prove the following:

Theorem 63 *Let μ be a measure on $(\Omega, \mathcal{F})$ and $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$. Then, $\nu = \int f d\mu$ defined by:*

$$\forall E \in \mathcal{F} \ , \ \nu(E) \stackrel{\triangle}{=} \int_E f d\mu$$

is a complex measure on $(\Omega, \mathcal{F})$ with total variation:

$$\forall E \in \mathcal{F} \ , \ |\nu|(E) = \int_E |f| d\mu$$

Exercise 15. Let $\mu \in M^1(\Omega, \mathcal{F})$ be a signed measure. Suppose that $h \in L^1_{\mathbf{R}}(\Omega, \mathcal{F}, |\mu|)$ is such that $|h| = 1$ and $\mu = \int h d|\mu|$. Define $A = \{h = 1\}$ and $B = \{h = -1\}$.

1. Show that for all $E \in \mathcal{F}$, $\mu^+(E) = \int_E \frac{1}{2}(1+h)d|\mu|$.
2. Show that for all $E \in \mathcal{F}$, $\mu^-(E) = \int_E \frac{1}{2}(1-h)d|\mu|$.
3. Show that $\mu^+ = \mu^A = \mu(A \cap \cdot\,)$.
4. Show that $\mu^- = -\mu^B = -\mu(B \cap \cdot\,)$.

Theorem 64 (Hahn Decomposition) *Let μ be a signed measure on $(\Omega, \mathcal{F})$. There exist $A, B \in \mathcal{F}$, such that $A \cap B = \emptyset$, $\Omega = A \uplus B$ and for all $E \in \mathcal{F}$, $\mu^+(E) = \mu(A \cap E)$ and $\mu^-(E) = -\mu(B \cap E)$.*

Definition 97 *Let μ be a complex measure on $(\Omega, \mathcal{F})$. We define:*

$$L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \stackrel{\triangle}{=} L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$$

and for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, the **lebesgue integral** *of f with respect to μ, is defined as:*

$$\int f d\mu \stackrel{\triangle}{=} \int f h d|\mu|$$

where $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ is such that $|h| = 1$ and $\mu = \int h d|\mu|$.

Exercise 16. Let μ be a complex measure on $(\Omega, \mathcal{F})$.

1. Show that for all $f : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ measurable:

$$f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \;\Leftrightarrow\; \int |f| d|\mu| < +\infty$$

2. Show that for $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $\int f d\mu$ is unambiguously defined.

3. Show that for all $E \in \mathcal{F}$, $1_E \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\int 1_E d\mu = \mu(E)$.

4. Show that if μ is a finite measure, then $|\mu| = \mu$.

5. Show that if μ is a finite measure, definition (97) of integral and space $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is consistent with that already known for measures.

6. Show that $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ is a **C**-vector space and that:

$$\int (f + \alpha g) d\mu = \int f d\mu + \alpha \int g d\mu$$

   for all $f, g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$ and $\alpha \in \mathbf{C}$.

7. Show that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have:

$$\left| \int f d\mu \right| \leq \int |f| d|\mu|$$

Exercise 17. Let $\mu, \nu \in M^1(\Omega, \mathcal{F})$, let $\alpha \in \mathbf{C}$.

1. Show that $|\alpha\nu| = |\alpha|.|\nu|$
2. Show that $|\mu + \nu| \le |\mu| + |\nu|$
3. Show that $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu + \alpha\nu)$
4. Show that for all $E \in \mathcal{F}$:
$$\int 1_E d(\mu + \alpha\nu) = \int 1_E d\mu + \alpha \int 1_E d\nu$$
5. Show that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu)$:
$$\int f d(\mu + \alpha\nu) = \int f d\mu + \alpha \int f d\nu$$

Exercise 18. Let $\mu = \mu_1 + i\mu_2 \in M^1(\Omega, \mathcal{F})$.

1. Show that $|\mu_1| \leq |\mu|$ and $|\mu_2| \leq |\mu|$.

2. Show that $|\mu| \leq |\mu_1| + |\mu_2|$.

3. Show that $L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) = L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2)$.

4. Show that:

$$\begin{array}{rcl} L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1) & = & L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1^+) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_1^-) \\ L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2) & = & L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2^+) \cap L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu_2^-) \end{array}$$

5. Show that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$:

$$\int f d\mu = \int f d\mu_1^+ - \int f d\mu_1^- + i\left(\int f d\mu_2^+ - \int f d\mu_2^-\right)$$

Exercise 19. Let $\mu \in M^1(\Omega, \mathcal{F})$. Let $A \in \mathcal{F}$. Let $h \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, |\mu|)$ be such that $|h| = 1$ and $\mu = \int h d|\mu|$. Recall that $\mu^A = \mu(A \cap \cdot\,)$ and $\mu_{|A} = \mu_{|(\mathcal{F}_{|A})}$ where $\mathcal{F}_{|A} = \{A \cap E\ ,\ E \in \mathcal{F}\} \subseteq \mathcal{F}$.

1. Show that we also have $\mathcal{F}_{|A} = \{E :\ E \in \mathcal{F}\ ,\ E \subseteq A\}$.

2. Show that $\mu^A \in M^1(\Omega, \mathcal{F})$ and $\mu_{|A} \in M^1(A, \mathcal{F}_{|A})$.

3. Let $E \in \mathcal{F}$ and $(E_n)_{n \geq 1}$ be a measurable partition of E. Show:
$$\sum_{n=1}^{+\infty} |\mu^A(E_n)| \leq |\mu|^A(E)$$

4. Show that we have $|\mu^A| \leq |\mu|^A$.

5. Let $E \in \mathcal{F}$ and $(E_n)_{n \geq 1}$ be a measurable partition of $A \cap E$. Show that:
$$\sum_{n=1}^{+\infty} |\mu(E_n)| \leq |\mu^A|(A \cap E)$$

6. Show that $|\mu^A|(A^c) = 0$.

7. Show that $|\mu^A| = |\mu|^A$.

8. Let $E \in \mathcal{F}_{|A}$ and $(E_n)_{n\geq 1}$ be an $\mathcal{F}_{|A}$-measurable partition of E. Show that:
$$\sum_{n=1}^{+\infty} |\mu_{|A}(E_n)| \leq |\mu|_{|A}(E)$$

9. Show that $|\mu_{|A}| \leq |\mu|_{|A}$.

10. Let $E \in \mathcal{F}_{|A} \subseteq \mathcal{F}$ and $(E_n)_{n\geq 1}$ be a measurable partition of E. Show that $(E_n)_{n\geq 1}$ is also an $\mathcal{F}_{|A}$-measurable partition of E, and conclude:
$$\sum_{n=1}^{+\infty} |\mu(E_n)| \leq |\mu_{|A}|(E)$$

11. Show that $|\mu_{|A}| = |\mu|_{|A}$.

12. Show that $\mu^A = \int h d|\mu^A|$.

13. Show that $h_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, |\mu_{|A}|)$ and $\mu_{|A} = \int h_{|A} d|\mu_{|A}|$.

14. Show that for all $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, we have:

    $f1_A \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu^A)$, $f_{|A} \in L^1_{\mathbf{C}}(A, \mathcal{F}_{|A}, \mu_{|A})$

    and:

$$\int f1_A d\mu = \int f d\mu^A = \int f_{|A} d\mu_{|A}$$

Definition 98 *Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where μ is a complex measure on $(\Omega, \mathcal{F})$. let $A \in \mathcal{F}$. We call* **partial lebesgue integral** *of f with respect to μ over A, the integral denoted $\int_A f d\mu$, defined as:*

$$\int_A f d\mu \stackrel{\triangle}{=} \int (f1_A) d\mu = \int f d\mu^A = \int (f_{|A}) d\mu_{|A}$$

where μ^A is the complex measure on $(\Omega, \mathcal{F})$, $\mu^A = \mu(A \cap \cdot)$, $f_{|A}$ is the restriction of f to A and $\mu_{|A}$ is the restriction of μ to $\mathcal{F}_{|A}$, the trace of $\mathcal{F}$ on A.

Exercise 20. Prove the following:

Theorem 65 *Let $f \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, where μ is a complex measure on $(\Omega, \mathcal{F})$. Then, $\nu = \int f d\mu$ defined as:*

$$\forall E \in \mathcal{F} \ , \ \nu(E) \stackrel{\triangle}{=} \int_E f d\mu$$

is a complex measure on $(\Omega, \mathcal{F})$, with total variation:

$$\forall E \in \mathcal{F} \ , \ |\nu|(E) = \int_E |f| d|\mu|$$

Moreover, for all measurable map $g : (\Omega, \mathcal{F}) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, we have:

$$g \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \nu) \ \Leftrightarrow \ gf \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$$

and when such condition is satisfied:

$$\int g d\nu = \int g f d\mu$$

Exercise 21. Let $(\Omega_1, \mathcal{F}_1), \ldots, (\Omega_n, \mathcal{F}_n)$ be n measurable spaces, where $n \geq 2$. Let $\mu_1 \in M^1(\Omega_1, \mathcal{F}_1)$, ..., $\mu_n \in M^1(\Omega_n, \mathcal{F}_n)$. For all $i \in \mathbf{N}_n$, let h_i belonging to $L^1_{\mathbf{C}}(\Omega_i, \mathcal{F}_i, |\mu_i|)$ be such that $|h_i| = 1$ and $\mu_i = \int h_i d|\mu_i|$. For all $E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$, we define:

$$\mu(E) \stackrel{\triangle}{=} \int_E h_1 \ldots h_n d|\mu_1| \otimes \ldots \otimes |\mu_n|$$

1. Show that $\mu \in M^1(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$
2. Show that for all measurable rectangle $A_1 \times \ldots \times A_n$:
$$\mu(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$
3. Prove the following:

Theorem 66 *Let $\mu_1, \ldots, \mu_n$ be n complex measures on measurable spaces $(\Omega_1, \mathcal{F}_1), \ldots, (\Omega_n, \mathcal{F}_n)$ respectively, where $n \geq 2$. There exists a unique complex measure $\mu_1 \otimes \ldots \otimes \mu_n$ on $(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n)$ such that for all measurable rectangle $A_1 \times \ldots \times A_n$, we have:*

$$\mu_1 \otimes \ldots \otimes \mu_n(A_1 \times \ldots \times A_n) = \mu_1(A_1) \ldots \mu_n(A_n)$$

Exercise 22. Further to theorem (66),

1. Show that $|\mu_1 \otimes \ldots \otimes \mu_n| = |\mu_1| \otimes \ldots \otimes |\mu_n|$.
2. Show that $\|\mu_1 \otimes \ldots \otimes \mu_n\| = \|\mu_1\| \ldots \|\mu_n\|$.
3. Show that for all $E \in \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n$:
$$\mu_1 \otimes \ldots \otimes \mu_n(E) = \int_E h_1 \ldots h_n d|\mu_1 \otimes \ldots \otimes \mu_n|$$
4. Let $f \in L^1_{\mathbf{C}}(\Omega_1 \times \ldots \times \Omega_n, \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_n, \mu_1 \otimes \ldots \otimes \mu_n)$. Show:
$$\int f d\mu_1 \otimes \ldots \otimes \mu_n = \int f h_1 \ldots h_n d|\mu_1| \otimes \ldots \otimes |\mu_n|$$

5. let σ be a permutation of $\{1,\ldots,n\}$. Show that:

$$\int f d\mu_1 \otimes \ldots \otimes \mu_n = \int_{\Omega_{\sigma(n)}} \ldots \int_{\Omega_{\sigma(1)}} f d\mu_{\sigma(1)} \ldots d\mu_{\sigma(n)}$$

13. Regular Measure

In the following, $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$.

Definition 99 *Let $(\Omega, \mathcal{F})$ be a measurable space. We say that a map $s : \Omega \to \mathbf{C}$ is a* **complex simple function** *on $(\Omega, \mathcal{F})$, if and only if it is of the form:*

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{C}$ and $A_i \in \mathcal{F}$ for all $i \in \mathbf{N}_n$. The set of all complex simple functions on $(\Omega, \mathcal{F})$ is denoted $S_{\mathbf{C}}(\Omega, \mathcal{F})$. The set of all $\mathbf{R}$-valued complex simple functions in $(\Omega, \mathcal{F})$ is denoted $S_{\mathbf{R}}(\Omega, \mathcal{F})$.

Recall that a simple function on $(\Omega, \mathcal{F})$, as defined in (40), is just a non-negative element of $S_{\mathbf{R}}(\Omega, \mathcal{F})$.

Exercise 1. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty[$.

1. Suppose $s : \Omega \to \mathbf{C}$ is of the form

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{C}$, $A_i \in \mathcal{F}$ and $\mu(A_i) < +\infty$ for all $i \in \mathbf{N}_n$. Show that $s \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{C}}(\Omega, \mathcal{F})$.

2. Show that any $s \in S_{\mathbf{C}}(\Omega, \mathcal{F})$ can be written as:

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1,\ \alpha_i \in \mathbf{C} \setminus \{0\},\ A_i \in \mathcal{F}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$.

3. Show that any $s \in L^p_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{C}}(\Omega, \mathcal{F})$ is of the form:

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{C}$, $A_i \in \mathcal{F}$ and $\mu(A_i) < +\infty$, for all $i \in \mathbf{N}_n$.

4. Show that $L^\infty_{\mathbf{C}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{C}}(\Omega, \mathcal{F}) = S_{\mathbf{C}}(\Omega, \mathcal{F})$.

Exercise 2. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty[$. Let f be a non-negative element of $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$.

1. Show the existence of a sequence $(s_n)_{n\geq 1}$ of non-negative functions in $L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{R}}(\Omega, \mathcal{F})$ such that $s_n \uparrow f$.

2. Show that:
$$\lim_{n\to+\infty} \int |s_n - f|^p d\mu = 0$$

3. Show that there exists $s \in L^p_{\mathbf{R}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{R}}(\Omega, \mathcal{F})$ such that $\|f - s\|_p \leq \epsilon$, for all $\epsilon > 0$.

4. Show that $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{K}}(\Omega, \mathcal{F})$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.

EXERCISE 3. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let f be a non-negative element of $L^\infty_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$. For all $n \geq 1$, we define:

$$s_n \stackrel{\triangle}{=} \sum_{k=0}^{n2^n-1} \frac{k}{2^n} 1_{\{k/2^n \leq f < (k+1)/2^n\}} + n1_{\{n \leq f\}}$$

1. Show that for all $n \geq 1$, s_n is a simple function.
2. Show there exists $n_0 \geq 1$ and $N \in \mathcal{F}$ with $\mu(N) = 0$, such that:
$$\forall \omega \in N^c \ , \ 0 \leq f(\omega) < n_0$$
3. Show that for all $n \geq n_0$ and $\omega \in N^c$, we have:
$$0 \leq f(\omega) - s_n(\omega) < \frac{1}{2^n}$$
4. Conclude that:
$$\lim_{n \to +\infty} \|f - s_n\|_\infty = 0$$
5. Show the following:

Theorem 67 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p \in [1, +\infty]$. Then, $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu) \cap S_{\mathbf{K}}(\Omega, \mathcal{F})$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{F}, \mu)$.*

Exercise 4. Let $(\Omega, \mathcal{T})$ be a metrizable topological space, and μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. We define Σ as the set of all $B \in \mathcal{B}(\Omega)$ such that for all $\epsilon > 0$, there exist F closed and G open in Ω, with:

$$F \subseteq B \subseteq G \ , \ \mu(G \setminus F) \leq \epsilon$$

Given a metric d on $(\Omega, \mathcal{T})$ inducing the topology $\mathcal{T}$, we define:

$$d(x, A) \stackrel{\triangle}{=} \inf\{d(x, y) : \ y \in A\}$$

for all $A \subseteq \Omega$ and $x \in \Omega$.

1. Show that $x \to d(x, A)$ from Ω to $\bar{\mathbf{R}}$ is continuous for all $A \subseteq \Omega$.
2. Show that if F is closed in Ω, $x \in F$ is equivalent to $d(x, F) = 0$.

Exercise 5. Further to exercise (4), we assume that F is a closed subset of Ω. For all $n \geq 1$, we define:

$$G_n \stackrel{\triangle}{=} \{x \in \Omega : \ d(x, F) < \frac{1}{n}\}$$

1. Show that G_n is open for all $n \geq 1$.
2. Show that $G_n \downarrow F$.
3. Show that $F \in \Sigma$.
4. Was it important to assume that μ is finite?
5. Show that $\Omega \in \Sigma$.
6. Show that if $B \in \Sigma$, then $B^c \in \Sigma$.

Exercise 6. Further to exercise (5), let $(B_n)_{n\geq 1}$ be a sequence in Σ. Define $B = \cup_{n=1}^{+\infty} B_n$ and let $\epsilon > 0$.

1. Show that for all n, there is F_n closed and G_n open in Ω, with:

$$F_n \subseteq B_n \subseteq G_n \ , \ \mu(G_n \setminus F_n) \leq \frac{\epsilon}{2^n}$$

2. Show the existence of some $N \geq 1$ such that:

$$\mu\left(\left(\bigcup_{n=1}^{+\infty} F_n\right) \setminus \left(\bigcup_{n=1}^{N} F_n\right)\right) \leq \epsilon$$

3. Define $G = \cup_{n=1}^{+\infty} G_n$ and $F = \cup_{n=1}^{N} F_n$. Show that F is closed, G is open and $F \subseteq B \subseteq G$.

4. Show that:

$$G \setminus F \subseteq G \setminus \left(\bigcup_{n=1}^{+\infty} F_n\right) \uplus \left(\bigcup_{n=1}^{+\infty} F_n\right) \setminus F$$

5. Show that:

$$G \setminus \left(\bigcup_{n=1}^{+\infty} F_n \right) \subseteq \bigcup_{n=1}^{+\infty} G_n \setminus F_n$$

6. Show that $\mu(G \setminus F) \leq 2\epsilon$.

7. Show that Σ is a σ-algebra on Ω, and conclude that $\Sigma = \mathcal{B}(\Omega)$.

Theorem 68 *Let $(\Omega, \mathcal{T})$ be a metrizable topological space, and μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Then, for all $B \in \mathcal{B}(\Omega)$ and $\epsilon > 0$, there exist F closed and G open in Ω such that:*

$$F \subseteq B \subseteq G \ , \ \mu(G \setminus F) \leq \epsilon$$

Definition 100 *Let $(\Omega, \mathcal{T})$ be a topological space. We denote $C^b_{\mathbf{K}}(\Omega)$ the $\mathbf{K}$-vector space of all* **continuous, bounded** *maps $\phi : \Omega \to \mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$.*

Exercise 7. Let $(\Omega, \mathcal{T})$ be a metrizable topological space with some metric d. Let μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$ and F be a closed subset of Ω. For all $n \geq 1$, we define $\phi_n : \Omega \to \mathbf{R}$ by:

$$\forall x \in \Omega \ , \ \phi_n(x) \stackrel{\triangle}{=} 1 - 1 \wedge (nd(x, F))$$

1. Show that for all $p \in [1, +\infty]$, we have $C^b_{\mathbf{K}}(\Omega) \subseteq L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.
2. Show that for all $n \geq 1$, $\phi_n \in C^b_{\mathbf{R}}(\Omega)$.
3. Show that $\phi_n \to 1_F$.
4. Show that for all $p \in [1, +\infty[$, we have:

$$\lim_{n \to +\infty} \int |\phi_n - 1_F|^p d\mu = 0$$

5. Show that for all $p \in [1, +\infty[$ and $\epsilon > 0$, there exists $\phi \in C^b_{\mathbf{R}}(\Omega)$ such that $\|\phi - 1_F\|_p \leq \epsilon$.

6. Let $\nu \in M^1(\Omega, \mathcal{B}(\Omega))$. Show that $C^b_{\mathbf{C}}(\Omega) \subseteq L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), \nu)$ and:

$$\nu(F) = \lim_{n \to +\infty} \int \phi_n d\nu$$

7. Prove the following:

Theorem 69 *Let $(\Omega, \mathcal{T})$ be a metrizable topological space and μ, ν be two complex measures on $(\Omega, \mathcal{B}(\Omega))$ such that:*

$$\forall \phi \in C^b_{\mathbf{R}}(\Omega) \; , \; \int \phi d\mu = \int \phi d\nu$$

Then $\mu = \nu$.

Exercise 8. Let $(\Omega, \mathcal{T})$ be a metrizable topological space and μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $s \in S_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega))$ be a complex simple function:

$$s = \sum_{i=1}^{n} \alpha_i 1_{A_i}$$

where $n \geq 1$, $\alpha_i \in \mathbf{C}$, $A_i \in \mathcal{B}(\Omega)$ for all $i \in \mathbf{N}_n$. Let $p \in [1, +\infty[$.

1. Show that given $\epsilon > 0$, for all $i \in \mathbf{N}_n$ there is a closed subset F_i of Ω such that $F_i \subseteq A_i$ and $\mu(A_i \setminus F_i) \leq \epsilon$. Let:
$$s' \stackrel{\triangle}{=} \sum_{i=1}^{n} \alpha_i 1_{F_i}$$
2. Show that:
$$\|s - s'\|_p \leq \left(\sum_{i=1}^{n} |\alpha_i| \right) \epsilon^{\frac{1}{p}}$$
3. Conclude that given $\epsilon > 0$, there exists $\phi \in C^b_{\mathbf{C}}(\Omega)$ such that:
$$\|\phi - s\|_p \leq \epsilon$$
4. Prove the following:

Theorem 70 *Let $(\Omega, \mathcal{T})$ be a metrizable topological space and μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Then, for all $p \in [1, +\infty[$, $C^b_{\mathbf{K}}(\Omega)$ is dense in $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.*

Definition 101 *A topological space $(\Omega, \mathcal{T})$ is said to be* **σ-compact** *if and only if, there exists a sequence $(K_n)_{n\geq 1}$ of compact subsets of Ω such that $K_n \uparrow \Omega$.*

EXERCISE 9. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space, with metric d. Let Ω' be open in Ω. For all $n \geq 1$, we define:

$$F_n \stackrel{\triangle}{=} \{x \in \Omega : \ d(x, (\Omega')^c) \geq 1/n\}$$

Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$.

1. Show that for all $n \geq 1$, F_n is closed in Ω.

2. Show that $F_n \uparrow \Omega'$.

3. Show that $F_n \cap K_n \uparrow \Omega'$.

4. Show that $F_n \cap K_n$ is closed in K_n for all $n \geq 1$.

5. Show that $F_n \cap K_n$ is a compact subset of Ω' for all $n \geq 1$

6. Prove the following:

Theorem 71 *Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Then, for all Ω' open subset of Ω, the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ is itself metrizable and σ-compact.*

Definition 102 *Let $(\Omega, \mathcal{T})$ be a topological space and μ be a measure on $(\Omega, \mathcal{B}(\Omega))$. We say that μ is* **locally finite**, *if and only if, every $x \in \Omega$ has an open neighborhood of finite μ-measure, i.e.*

$$\forall x \in \Omega\ ,\ \exists U \in \mathcal{T}\ ,\ x \in U\ ,\ \mu(U) < +\infty$$

Definition 103 *Let μ be a measure on a topological space $(\Omega, \mathcal{T})$. We say that μ is* **inner-regular**, *if and only if, for all $B \in \mathcal{B}(\Omega)$:*

$$\mu(B) = \sup\{\mu(K) : \ K \subseteq B \ , \ K \ \mathit{compact}\}$$

We say that μ is **outer-regular**, *if and only if, for all $B \in \mathcal{B}(\Omega)$:*

$$\mu(B) = \inf\{\mu(G) : \ B \subseteq G \ , \ G \ \mathit{open}\}$$

We say that μ is **regular** *if it is both inner and outer-regular.*

Exercise 10. Let $(\Omega, \mathcal{T})$ be a topological space and μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let K be a compact subset of Ω.

1. Show the existence of open sets $V_1, \ldots, V_n$ with $\mu(V_i) < +\infty$ for all $i \in \mathbf{N}_n$ and $K \subseteq V_1 \cup \ldots \cup V_n$, where $n \geq 1$.

2. Conclude that $\mu(K) < +\infty$.

Exercise 11. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$. Let $B \in \mathcal{B}(\Omega)$. We define $\alpha = \sup\{\mu(K) : \ K \subseteq B \ , \ K \text{ compact}\}$.

1. Show that given $\epsilon > 0$, there exists F closed in Ω such that $F \subseteq B$ and $\mu(B \setminus F) \leq \epsilon$.
2. Show that $F \setminus (K_n \cap F) \downarrow \emptyset$.
3. Show that $K_n \cap F$ is closed in K_n.
4. Show that $K_n \cap F$ is compact.
5. Conclude that given $\epsilon > 0$, there exists K compact subset of Ω such that $K \subseteq F$ and $\mu(F \setminus K) \leq \epsilon$.
6. Show that $\mu(B) \leq \mu(K) + 2\epsilon$.
7. Show that $\mu(B) \leq \alpha$ and conclude that μ is inner-regular.

Exercise 12. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$. Let $B \in \mathcal{B}(\Omega)$, and $\alpha \in \mathbf{R}$ be such that $\alpha < \mu(B)$.

1. Show that $\mu(K_n \cap B) \uparrow \mu(B)$.
2. Show the existence of $K \subseteq \Omega$ compact, with $\alpha < \mu(K \cap B)$.
3. Let $\mu^K = \mu(K \cap \cdot\,)$. Show that μ^K is a finite measure, and conclude that $\mu^K(B) = \sup\{\mu^K(K^*) : K^* \subseteq B\ ,\ K^* \text{ compact}\}$.
4. Show the existence of a compact subset K^* of Ω, such that $K^* \subseteq B$ and $\alpha < \mu(K \cap K^*)$.
5. Show that K^* is closed in Ω.
6. Show that $K \cap K^*$ is closed in K.
7. Show that $K \cap K^*$ is compact.

8. Show that $\alpha < \sup\{\mu(K') : K' \subseteq B , K' \text{ compact}\}$.

9. Show that $\mu(B) \leq \sup\{\mu(K') : K' \subseteq B , K' \text{ compact}\}$.

10. Conclude that μ is inner-regular.

Exercise 13. Let $(\Omega, \mathcal{T})$ be a metrizable topological space.

1. Show that $(\Omega, \mathcal{T})$ is separable if and only if it has a countable base.

2. Show that if $(\Omega, \mathcal{T})$ is compact, for all $n \geq 1$, Ω can be covered by a finite number of open balls with radius $1/n$.

3. Show that if $(\Omega, \mathcal{T})$ is compact, then it is separable.

Exercise 14. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space with metric d. Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$.

1. For all $n \geq 1$, give a metric on K_n inducing the topology $\mathcal{T}_{|K_n}$.
2. Show that $(K_n, \mathcal{T}_{|K_n})$ is separable. Let $(x_n^p)_{p\geq 1}$ be a countable dense family of $(K_n, \mathcal{T}_{|K_n})$.
3. Show that $(x_n^p)_{n,p\geq 1}$ is a countable dense family of $(\Omega, \mathcal{T})$, and conclude with the following:

Theorem 72 *Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Then, $(\Omega, \mathcal{T})$ is separable and has a countable base.*

Exercise 15. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $\mathcal{H}$ be a countable base of $(\Omega, \mathcal{T})$. We define $\mathcal{H}' = \{V \in \mathcal{H} : \mu(V) < +\infty\}$.

1. Show that for all U open in Ω and $x \in U$, there is U_x open in Ω such that $x \in U_x \subseteq U$ and $\mu(U_x) < +\infty$.

2. Show the existence of $V_x \in \mathcal{H}$ such that $x \in V_x \subseteq U_x$.

3. Conclude that $\mathcal{H}'$ is a countable base of $(\Omega, \mathcal{T})$.

4. Show the existence of a sequence $(V_n)_{n\geq 1}$ of open sets in Ω with:

$$\Omega = \bigcup_{n=1}^{+\infty} V_n \ , \ \mu(V_n) < +\infty \ , \ \forall n \geq 1$$

Exercise 16. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $(V_n)_{n\geq 1}$ a sequence of open subsets of Ω such that:

$$\Omega = \bigcup_{n=1}^{+\infty} V_n \ , \ \mu(V_n) < +\infty \ , \ \forall n \geq 1$$

Let $B \in \mathcal{B}(\Omega)$ and $\alpha = \inf\{\mu(G) : \ B \subseteq G \ , \ G \text{ open}\}$.

1. Given $\epsilon > 0$, show that there exists G_n open in Ω such that $B \subseteq G_n$ and $\mu^{V_n}(G_n \setminus B) \leq \epsilon/2^n$, where $\mu^{V_n} = \mu(V_n \cap \cdot\,)$.
2. Let $G = \cup_{n=1}^{+\infty}(V_n \cap G_n)$. Show that G is open in Ω, and $B \subseteq G$.
3. Show that $G \setminus B = \cup_{n=1}^{+\infty} V_n \cap (G_n \setminus B)$.
4. Show that $\mu(G) \leq \mu(B) + \epsilon$.
5. Show that $\alpha \leq \mu(B)$.
6. Conclude that is μ outer-regular.
7. Show the following:

Theorem 73 *Let μ be a locally finite measure on a metrizable and σ-compact topological space $(\Omega, \mathcal{T})$. Then, μ is regular, i.e.:*

$$\begin{aligned} \mu(B) &= \sup\{\mu(K) :\ K \subseteq B\ ,\ K \textit{ compact}\} \\ &= \inf\{\mu(G) :\ B \subseteq G\ ,\ G \textit{ open}\} \end{aligned}$$

for all $B \in \mathcal{B}(\Omega)$.

EXERCISE 17. Show the following:

Theorem 74 *Let Ω be an open subset of $\mathbf{R}^n$, where $n \geq 1$. Any locally finite measure on $(\Omega, \mathcal{B}(\Omega))$ is regular.*

Definition 104 *We call* **strongly σ-compact** *topological space, a topological space $(\Omega, \mathcal{T})$, for which there exists a sequence $(V_n)_{n\geq 1}$ of open sets with compact closure, such that $V_n \uparrow \Omega$.*

Definition 105 *We call* **locally compact** *topological space, a topological space $(\Omega, \mathcal{T})$, for which every $x \in \Omega$ has an open neighborhood with compact closure, i.e. such that:*

$$\forall x \in \Omega \ , \ \exists U \in \mathcal{T} : \ x \in U \ , \ \bar{U} \text{ is compact}$$

Exercise 18. Let $(\Omega, \mathcal{T})$ be a σ-compact and locally compact topological space. Let $(K_n)_{n\geq 1}$ be a sequence of compact subsets of Ω such that $K_n \uparrow \Omega$.

1. Show that for all $n \geq 1$, there are open sets $V_1^n, \ldots, V_{p_n}^n$, $p_n \geq 1$, such that $K_n \subseteq V_1^n \cup \ldots \cup V_{p_n}^n$ and $\bar{V}_1^n, \ldots, \bar{V}_{p_n}^n$ are compact subsets of Ω.

2. Define $W_n = V_1^n \cup \ldots \cup V_{p_n}^n$ and $V_n = \cup_{k=1}^n W_k$, for $n \geq 1$. Show that $(V_n)_{n\geq 1}$ is a sequence of open sets in Ω with $V_n \uparrow \Omega$.

3. Show that $\bar{W}_n = \bar{V}_1^n \cup \ldots \cup \bar{V}_{p_n}^n$ for all $n \geq 1$.

4. Show that $\bar{W}_n$ is compact for all $n \geq 1$.

5. Show that $\bar{V}_n$ is compact for all $n \geq 1$

6. Conclude with the following:

Theorem 75 *A topological space $(\Omega, \mathcal{T})$ is strongly σ-compact, if and only if it is σ-compact and locally compact.*

Exercise 19. Let $(\Omega, \mathcal{T})$ be a topological space and Ω' be an open subset of Ω. Let $A \subseteq \Omega'$. We denote $\bar{A}^{\Omega'}$ the closure of A in the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$, and $\bar{A}$ its closure in Ω.

1. Show that $A \subseteq \Omega' \cap \bar{A}$.

2. Show that $\Omega' \cap \bar{A}$ is closed in Ω'.

3. Show that $\bar{A}^{\Omega'} \subseteq \Omega' \cap \bar{A}$.

4. Let $x \in \Omega' \cap \bar{A}$. Show that if $x \in U' \in \mathcal{T}_{|\Omega'}$, then $A \cap U' \neq \emptyset$.

5. Show that $\bar{A}^{\Omega'} = \Omega' \cap \bar{A}$.

Exercise 20. Let (Ω, d) be a metric space.

1. Show that for all $x \in \Omega$ and $\epsilon > 0$, we have:
$$\overline{B(x,\epsilon)} \subseteq \{y \in \Omega : \ d(x,y) \leq \epsilon\}$$
2. Take $\Omega = [0, 1/2[\cup \{1\}$. Show that $B(0,1) = [0, 1/2[$.
3. Show that $[0, 1/2[$ is closed in Ω.
4. Show that $\overline{B(0,1)} = [0, 1/2[$.
5. Conclude that $\overline{B(0,1)} \neq \{y \in \Omega : \ |y| \leq 1\} = \Omega$.

Exercise 21. Let (Ω, d) be a locally compact metric space. Let Ω' be an open subset of Ω. Let $x \in \Omega'$.

1. Show there exists U open with compact closure, such that $x \in U$.
2. Show the existence of $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U \cap \Omega'$.

3. Show that $\overline{B(x,\epsilon/2)} \subseteq \bar{U}$.

4. Show that $\overline{B(x,\epsilon/2)}$ is closed in $\bar{U}$.

5. Show that $\overline{B(x,\epsilon/2)}$ is a compact subset of Ω.

6. Show that $\overline{B(x,\epsilon/2)} \subseteq \Omega'$.

7. Let $U' = B(x,\epsilon/2) \cap \Omega' = B(x,\epsilon/2)$. Show $x \in U' \in \mathcal{T}_{|\Omega'}$, and:

$$\bar{U'}^{\Omega'} = \overline{B(x,\epsilon/2)}$$

8. Show that the induced topological space Ω' is locally compact.

9. Prove the following:

Theorem 76 *Let $(\Omega, \mathcal{T})$ be a metrizable and strongly σ-compact topological space. Then, for all Ω' open subset of Ω, the induced topological space $(\Omega', \mathcal{T}_{|\Omega'})$ is itself metrizable and strongly σ-compact.*

Definition 106 *Let $(\Omega, \mathcal{T})$ be a topological space, and $\phi : \Omega \to \mathbf{C}$ be a map. We call* **support** *of ϕ, the closure of the set $\{\phi \neq 0\}$, i.e.:*

$$supp(\phi) \stackrel{\triangle}{=} \overline{\{\omega \in \Omega : \ \phi(\omega) \neq 0\}}$$

Definition 107 *Let $(\Omega, \mathcal{T})$ be a topological space. We denote $C^c_{\mathbf{K}}(\Omega)$ the $\mathbf{K}$-vector space of all* **continuous** *map with* **compact support** *$\phi : \Omega \to \mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$.*

Exercise 22. Let $(\Omega, \mathcal{T})$ be a topological space.

1. Show that $0 \in C^c_{\mathbf{K}}(\Omega)$.
2. Show that $C^c_{\mathbf{K}}(\Omega)$ is indeed a $\mathbf{K}$-vector space.
3. Show that $C^c_{\mathbf{K}}(\Omega) \subseteq C^b_{\mathbf{K}}(\Omega)$.

Exercise 23. let (Ω, d) be a locally compact metric space. let K be a compact subset of Ω, and G be open in Ω, with $K \subseteq G$.

1. Show the existence of open sets $V_1, \ldots, V_n$ such that:
$$K \subseteq V_1 \cup \ldots \cup V_n$$
and $\bar{V}_1, \ldots, \bar{V}_n$ are compact subsets of Ω.

2. Show that $V = (V_1 \cup \ldots \cup V_n) \cap G$ is open in Ω, and $K \subseteq V \subseteq G$.

3. Show that $\bar{V} \subseteq \bar{V}_1 \cup \ldots \cup \bar{V}_n$.

4. Show that $\bar{V}$ is compact.

5. We assume $K \neq \emptyset$ and $V \neq \Omega$, and we define $\phi : \Omega \to \mathbf{R}$ by:
$$\forall x \in \Omega\ ,\ \phi(x) \stackrel{\triangle}{=} \frac{d(x, V^c)}{d(x, V^c) + d(x, K)}$$

6. Show that ϕ is well-defined and continuous.

7. Show that $\{\phi \neq 0\} = V$.

8. Show that $\phi \in C^c_{\mathbf{R}}(\Omega)$.

9. Show that $1_K \leq \phi \leq 1_G$.

10. Show that if $K = \emptyset$, there is $\phi \in C^c_{\mathbf{R}}(\Omega)$ with $1_K \leq \phi \leq 1_G$.

11. Show that if $V = \Omega$ then Ω is compact.

12. Show that if $V = \Omega$, there $\phi \in C^c_{\mathbf{R}}(\Omega)$ with $1_K \leq \phi \leq 1_G$.

Theorem 77 *Let $(\Omega, \mathcal{T})$ be a metrizable and locally compact topological space. Let K be a compact subset of Ω, and G be an open subset of Ω such that $K \subseteq G$. Then, there exists $\phi \in C^c_{\mathbf{R}}(\Omega)$, real-valued continuous map with compact support, such that:*

$$1_K \leq \phi \leq 1_G$$

EXERCISE 24. Let $(\Omega, \mathcal{T})$ be a metrizable and strongly σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let $B \in \mathcal{B}(\Omega)$ be such that $\mu(B) < +\infty$. Let $p \in [1, +\infty[$.

1. Show that $C^c_{\mathbf{K}}(\Omega) \subseteq L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$.

2. Let $\epsilon > 0$. Show the existence of K compact and G open, with:

$$K \subseteq B \subseteq G \ , \ \mu(G \setminus K) \leq \epsilon$$

3. Where did you use the fact that $\mu(B) < +\infty$?

4. Show the existence of $\phi \in C^c_{\mathbf{R}}(\Omega)$ with $1_K \leq \phi \leq 1_G$.

5. Show that:

$$\int |\phi - 1_B|^p d\mu \leq \mu(G \setminus K)$$

6. Conclude that for all $\epsilon > 0$, there exists $\phi \in C^c_{\mathbf{R}}(\Omega)$ such that:

$$\|\phi - 1_B\|_p \leq \epsilon$$

7. Let $s \in S_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega)) \cap L^p_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), \mu)$. Show that for all $\epsilon > 0$, there exists $\phi \in C^c_{\mathbf{C}}(\Omega)$ such that $\|\phi - s\|_p \leq \epsilon$.

8. Prove the following:

Theorem 78 *Let* $(\Omega, \mathcal{T})$ *be a metrizable and strongly* σ*-compact topological space*[1]*. Let* μ *be a locally finite measure on* $(\Omega, \mathcal{B}(\Omega))$*. Then, for all* $p \in [1, +\infty[$*, the space* $C^c_{\mathbf{K}}(\Omega)$ *of* $\mathbf{K}$*-valued, continuous maps with compact support, is dense in* $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$*.*

Exercise 25. Prove the following:

Theorem 79 *Let* Ω *be an open subset of* $\mathbf{R}^n$*, where* $n \geq 1$*. Then, for any locally finite measure* μ *on* $(\Omega, \mathcal{B}(\Omega))$ *and* $p \in [1, +\infty[$*,* $C^c_{\mathbf{K}}(\Omega)$ *is dense in* $L^p_{\mathbf{K}}(\Omega, \mathcal{B}(\Omega), \mu)$*.*

[1] i.e. a metrizable topological space for which there exists a sequence $(V_n)_{n\geq 1}$ of open sets with compact closure, such that $V_n \uparrow \Omega$.

14. Maps of Finite Variation

Definition 108 *We call* **total variation** *of a map* $b : \mathbf{R}^+ \to \mathbf{C}$ *the map* $|b| : \mathbf{R}^+ \to [0, +\infty]$ *defined as:*

$$\forall t \in \mathbf{R}^+ \ , \ |b|(t) \stackrel{\triangle}{=} |b(0)| + \sup \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

where the 'sup' is taken over all finite $t_0 \leq \ldots \leq t_n$ *in* $[0, t]$, $n \geq 1$. *We say that* b *is of* **finite variation**, *if and only if:*

$$\forall t \in \mathbf{R}^+ \ , \ |b|(t) < +\infty$$

We say that b *is of* **bounded variation**, *if and only if:*

$$\sup_{t \in \mathbf{R}^+} |b|(t) < +\infty$$

Warning: The notation $|b|$ can be misleading: it can refer to the map $t \to |b(t)|$(modulus), or to the map $t \to |b|(t)$ (total variation).

EXERCISE 1. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be non-decreasing with $a(0) \geq 0$.

1. Show that $|a| = a$ and a is of finite variation.
2. Show that the limit $a(\infty) = \lim_{t\to+\infty} a(t)$ exists in $\bar{\mathbf{R}}$.
3. Show that a is of bounded variation if and only if $a(\infty) < +\infty$.

EXERCISE 2. Let $b = b_1 + ib_2 : \mathbf{R}^+ \to \mathbf{C}$ be a map.

1. Show that $|b_1| \leq |b|$ and $|b_2| \leq |b|$.
2. Show that $|b| \leq |b_1| + |b_2|$.
3. Show that b is of finite variation if and only if b_1, b_2 are.
4. Show that b is of bounded variation if and only if b_1, b_2 are.
5. Show that $|b|(0) = |b(0)|$.

Exercise 3. Let $b : \mathbf{R}^+ \to \mathbf{R}$ be continuous and differentiable, such that b' is bounded on each compact interval. Show that b is of finite variation.

Exercise 4. Show that if $b : \mathbf{R}^+ \to \mathbf{C}$ is of class C^1, i.e. continuous and differentiable with continuous derivative, then b is of finite variation.

Exercise 5. Let $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ be a measurable map, with $\int_0^t |f(s)|ds < +\infty$ for all $t \in \mathbf{R}^+$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ defined by:

$$\forall t \in \mathbf{R}^+ \ , \ b(t) \stackrel{\triangle}{=} \int_{\mathbf{R}^+} f 1_{[0,t]} ds$$

1. Show that b is of finite variation and:

$$\forall t \in \mathbf{R}^+ \ , \ |b|(t) \leq \int_0^t |f(s)|ds$$

2. Show that $f \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), ds) \ \Rightarrow b$ is of bounded variation.

Exercise 6. Show that if $b, b' : \mathbf{R}^+ \to \mathbf{C}$ are maps of finite variation, and $\alpha \in \mathbf{C}$, then $b + \alpha b'$ is also a map of finite variation. Prove the same result when the word 'finite' is replaced by 'bounded'.

Exercise 7. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map. For all $t \in \mathbf{R}^+$, let $\mathcal{S}(t)$ be the set of all finite subsets A of $[0, t]$, with $\mathrm{card} A \geq 2$. For all $A \in \mathcal{S}(t)$, we define:

$$S(A) \stackrel{\triangle}{=} \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

where it is understood that $t_0, \ldots, t_n$ are such that:

$$t_0 < t_1 < \ldots < t_n \text{ and } A = \{t_0, \ldots, t_n\} \subseteq [0, t]$$

1. Show that for all $t \in \mathbf{R}^+$, if $s_0 \leq \ldots \leq s_p$ ($p \geq 1$) is a finite sequence in $[0, t]$, then if:

$$S \stackrel{\triangle}{=} \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})|$$

either $S = 0$ or $S = S(A)$ for some $A \in \mathcal{S}(t)$.

2. Conclude that:

$$\forall t \in \mathbf{R}^+ \ , \ |b|(t) = |b(0)| + \sup\{S(A) : \ A \in \mathcal{S}(t)\}$$

3. Let $A \in \mathcal{S}(t)$ and $s \in [0, t]$. Show that $S(A) \leq S(A \cup \{s\})$.

4. Let $A, B \in \mathcal{S}(t)$. Show that:

$$A \subseteq B \ \Rightarrow \ S(A) \leq S(B)$$

5. Show that if $t_0 \leq \ldots \leq t_n$, $n \geq 1$, and $s_0 \leq \ldots \leq s_p$, $p \geq 1$, are finite sequence in $[0, t]$ such that:

$$\{t_0, \ldots, t_n\} \subseteq \{s_0, \ldots, s_p\}$$

then:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq \sum_{j=1}^{p} |b(s_j) - b(s_{j-1})|$$

Exercise 8. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be of finite variation. Let $s, t \in \mathbf{R}^+$, with $s \leq t$. We define:

$$\delta \stackrel{\triangle}{=} \sup \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

where the 'sup' is taken over all finite $t_0 \leq \ldots \leq t_n$, $n \geq 1$, in $[s, t]$.

1. let $s_0 \leq \ldots \leq s_p$ and $t_0 \leq \ldots \leq t_n$ be finite sequences in $[0, s]$ and $[s, t]$ respectively, where $n, p \geq 1$. Show that:

$$\sum_{j=1}^{p} |b(s_j) - b(s_{j-1})| + |b(t_0) - b(s_p)| + \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

   is less or equal than $|b|(t) - |b(0)|$.

2. Show that $\delta \leq |b|(t) - |b|(s)$.

3. Let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0, t]$, where $n \geq 1$, and

suppose that $s = t_j$ for some $0 < j < n$. Show that:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(s) - |b(0)| + \delta \tag{1}$$

4. Show that inequality (1) holds, for all $t_0 \leq \ldots \leq t_n$ in $[0, t]$.

5. Prove the following:

Theorem 80 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map of finite variation. Then, for all $s, t \in \mathbf{R}^+$, $s \leq t$, we have:*

$$|b|(t) - |b|(s) = \sup \sum_{i=1}^{n} |b(t_i) - b(t_{i-1})|$$

where the 'sup' is taken over all finite $t_0 \leq \ldots \leq t_n$, $n \geq 1$, in $[s, t]$.

Exercise 9. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map of finite variation. Show that $|b|$ is non-decreasing with $|b|(0) \geq 0$, and $||b|| = |b|$.

Definition 109 *Let $b : \mathbf{R}^+ \to \mathbf{R}$ be a map of finite variation. Let:*

$$\begin{aligned} |b|^+ &\stackrel{\triangle}{=} \frac{1}{2}(|b| + b) \\ |b|^- &\stackrel{\triangle}{=} \frac{1}{2}(|b| - b) \end{aligned}$$

$|b|^+$, $|b|^-$ *are respectively the* **positive**, **negative variation** *of b.*

Exercise 10. Let $b : \mathbf{R}^+ \to \mathbf{R}$ be a map of finite variation.

1. Show that $|b| = |b|^+ + |b|^-$ and $b = |b|^+ - |b|^-$.
2. Show that $|b|^+(0) = b^+(0) \geq 0$ and $|b|^-(0) = b^-(0) \geq 0$.
3. Show that for all $s, t \in \mathbf{R}^+$, $s \leq t$, we have:

$$|b(t) - b(s)| \leq |b|(t) - |b|(s)$$

4. Show that $|b|^+$ and $|b|^-$ are non-decreasing.

Exercise 11. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be of finite variation. Show the existence of $b_1, b_2, b_3, b_4 : \mathbf{R}^+ \to \mathbf{R}^+$, non-decreasing with $b_i(0) \geq 0$, such that $b = b_1 - b_2 + i(b_3 - b_4)$. Show conversely that if $b : \mathbf{R}^+ \to \mathbf{C}$ is a map with such decomposition, then it is of finite variation.

Exercise 12. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation, and $x_0 \in \mathbf{R}^+$.

1. Show that $|b|(x_0+) = \lim_{t \downarrow\downarrow x_0} |b|(t) = \inf_{x_0 < t} |b|(t) \in \mathbf{R}$.
2. Show that $|b|(x_0) \leq |b|(x_0+)$.
3. Given $\epsilon > 0$, show the existence of $y_0 \in \mathbf{R}^+$, $x_0 < y_0$, such that:

$$
\begin{array}{lcl}
u \in]x_0, y_0] & \Rightarrow & |b(u) - b(x_0)| \leq \epsilon/2 \\
u \in]x_0, y_0] & \Rightarrow & |b|(y_0) - |b|(u) \leq \epsilon/2
\end{array}
$$

EXERCISE 13. Further to exercise (12), let $t_0 \leq \ldots \leq t_n$, $n \geq 1$, be a finite sequence in $[0, y_0]$, such that $x_0 = t_j$ for some $0 < j < n-1$. We choose j to be the maximum index satisfying this condition, so that $x_0 < t_{j+1} \leq y_0$.

1. Show that $\sum_{i=1}^{j} |b(t_i) - b(t_{i-1})| \leq |b|(x_0) - |b(0)|$.

2. Show that $|b(t_{j+1}) - b(t_j)| \leq \epsilon/2$.

3. Show that $\sum_{i=j+2}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(y_0) - |b|(t_{j+1}) \leq \epsilon/2$.

4. Show that for all finite sequence $t_0 \leq \ldots \leq t_n$, $n \geq 1$, in $[0, y_0]$:
$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(x_0) - |b(0)| + \epsilon$$

5. Show that $|b|(y_0) \leq |b|(x_0) + \epsilon$.

6. Show that $|b|(x_0+) \leq |b|(x_0)$ and that $|b|$ is right-continuous.

Exercise 14. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a left-continuous map of finite variation, and let $x_0 \in \mathbf{R}^+ \setminus \{0\}$.

1. Show that $|b|(x_0-) = \lim_{t\uparrow\uparrow x_0} |b|(t) = \sup_{t<x_0} |b|(t) \in \mathbf{R}$.

2. Show that $|b|(x_0-) \leq |b|(x_0)$.

3. Given $\epsilon > 0$, show the existence of $y_0 \in]0, x_0[$, such that:

$$\begin{aligned} u \in [y_0, x_0[\quad &\Rightarrow \quad |b(x_0) - b(u)| \leq \epsilon/2 \\ u \in [y_0, x_0[\quad &\Rightarrow \quad |b|(u) - |b|(y_0) \leq \epsilon/2 \end{aligned}$$

Exercise 15. Further to exercise (14), let $t_0 \leq \ldots \leq t_n$, $n \geq 1$, be a finite sequence in $[0, x_0]$, such that $y_0 = t_j$ for some $0 < j < n-1$, and $x_0 = t_n$. We denote $k = \max\{i : \ j \leq i \ , \ t_i < x_0\}$.

1. Show that $j \leq k \leq n-1$ and $t_k \in [y_0, x_0[$.

2. Show that $\sum_{i=1}^{j} |b(t_i) - b(t_{i-1})| \leq |b|(y_0) - |b(0)|$.

3. Show that $\sum_{i=j+1}^{k} |b(t_i) - b(t_{i-1})| \leq |b|(t_k) - |b|(y_0) \leq \epsilon/2$, where if $j = k$, the corresponding sum is zero.

4. Show that $\sum_{i=k+1}^{n} |b(t_i) - b(t_{i-1})| = |b(x_0) - b(t_k)| \leq \epsilon/2$.

5. Show that for all finite sequence $t_0 \leq \ldots \leq t_n$, $n \geq 1$, in $[0, x_0]$:

$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |b|(y_0) - |b(0)| + \epsilon$$

6. Show that $|b|(x_0) \leq |b|(y_0) + \epsilon$.

7. Show that $|b|(x_0) \leq |b|(x_0-)$ and that $|b|$ is left-continuous.

8. Prove the following:

Theorem 81 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map of finite variation. Then:*

$$\begin{array}{rcl} b \textit{ is right-continuous} & \Rightarrow & |b| \textit{ is right-continuous} \\ b \textit{ is left-continuous} & \Rightarrow & |b| \textit{ is left-continuous} \\ b \textit{ is continuous} & \Rightarrow & |b| \textit{ is continuous} \end{array}$$

Exercise 16. Let $b : \mathbf{R}^+ \to \mathbf{R}$ be an $\mathbf{R}$-valued map of finite variation.

1. Show that if b is right-continuous, then so are $|b|^+$ and $|b|^-$.

2. State and prove similar results for left-continuity and continuity.

Exercise 17. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. Show the existence of $b_1, b_2, b_3, b_4 : \mathbf{R}^+ \to \mathbf{R}^+$, right-continuous and non-decreasing maps with $b_i(0) \geq 0$, such that:

$$b = b_1 - b_2 + i(b_3 - b_4)$$

Exercise 18. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map. Let $t \in \mathbf{R}^+$. For all $p \geq 1$, we define:

$$S_p \stackrel{\triangle}{=} |b(0)| + \sum_{k=1}^{2^p} |b(kt/2^p) - b((k-1)t/2^p)|$$

1. Show that for all $p \geq 1$, $S_p \leq S_{p+1}$ and define $S = \sup_{p\geq 1} S_p$.

2. Show that $S \leq |b|(t)$.

Exercise 19. Further to exercise (18), let $t_0 < \ldots < t_n$ be a finite sequence of distinct elements of $[0, t]$. Let $\epsilon > 0$.

1. Show that for all $i = 0, \ldots, n$, there exists $p_i \geq 1$ and $q_i \in \{0, 1, \ldots, 2^{p_i}\}$ such that:

$$0 \leq t_0 \leq \frac{q_0 t}{2^{p_0}} < t_1 \leq \frac{q_1 t}{2^{p_1}} < \ldots < t_n \leq \frac{q_n t}{2^{p_n}} \leq t$$

and:

$$|b(t_i) - b(q_i t/2^{p_i})| \leq \epsilon \ , \ \forall i = 0, \ldots, n$$

2. Show the existence of $p \geq 1$, and $k_0, \ldots, k_n \in \{0, \ldots, 2^p\}$ with:

$$0 \leq t_0 \leq \frac{k_0 t}{2^p} < t_1 \leq \frac{k_1 t}{2^p} < \ldots < t_n \leq \frac{k_n t}{2^p} \leq t$$

and:

$$|b(t_i) - b(k_i t/2^p)| \leq \epsilon \ , \ \forall i = 0, \ldots, n$$

3. Show that:
$$\sum_{i=1}^{n} |b(k_i t/2^p) - b(k_{i-1} t/2^p)| \leq S_p - |b(0)|$$

4. Show that:
$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq S - |b(0)| + 2n\epsilon$$

5. Show that:
$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq S - |b(0)|$$

6. Conclude that $|b|(t) \leq S$.

7. Prove the following:

Theorem 82 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous or left-continuous. Then, for all $t \in \mathbf{R}^+$:*

$$|b|(t) = |b(0)| + \lim_{n \to +\infty} \sum_{k=1}^{2^n} |b(kt/2^n) - b((k-1)t/2^n)|$$

Exercise 20. Let $b : \mathbf{R}^+ \to \mathbf{R}^+$ be defined by $b = 1_{\mathbf{Q}^+}$. Show that:

$$+\infty = |b|(1) \neq \lim_{n \to +\infty} \sum_{k=1}^{2^n} |b(k/2^n) - b((k-1)/2^n)| = 0$$

Exercise 21. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of bounded variation.

1. Let $b = b_1 + ib_2$. Explain why $d|b_1|^+, d|b_1|^-, d|b_2|^+$ and $d|b_2|^-$ are all well-defined measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$.

2. Is this still true, if b is right-continuous of finite variation?

3. Show that $d|b_1|^+, d|b_1|^-, d|b_2|^+$ and $d|b_2|^-$ are finite measures.

4. Let $db = d|b_1|^+ - d|b_1|^- + i(d|b_2|^+ - d|b_2|^-)$. Show that db is a well-defined complex measure on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$.

5. Show that $db(\{0\}) = b(0)$.

6. Show that for all $s, t \in \mathbf{R}^+$, $s \leq t$, $db(]s,t]) = b(t) - b(s)$.

7. Show that $\lim_{t\to+\infty} b(t)$ exists in $\mathbf{C}$. We denote $b(\infty)$ this limit.

8. Show that $db(\mathbf{R}^+) = b(\infty)$.

9. Proving the uniqueness of db, justify the following:

Definition 110 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of bounded variation. There exists a unique complex measure db on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$, such that:*

$$\begin{array}{ll} (i) & db(\{0\}) = b(0) \\ (ii) & \forall s,t \in \mathbf{R}^+\ s \leq t\ ,\ db(]s,t]) = b(t) - b(s) \end{array}$$

db is called the **complex stieltjes measure** *associated with b.*

Exercise 22. Show that if $a : \mathbf{R}^+ \to \mathbf{R}^+$ is right-continuous, non-decreasing with $a(0) \geq 0$ and $a(\infty) < +\infty$, then definition (110) of da coincide with the already known definition (24).

Exercise 23. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Let $b = b_1 + ib_2$. Explain why $d|b_1|^+, d|b_1|^-, d|b_2|^+$ and $d|b_2|^-$ are all well-defined measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$.

2. Why is it in general impossible to define:

$$db \stackrel{\triangle}{=} d|b_1|^+ - d|b_1|^- + i(d|b_2|^+ - d|b_2|^-)$$

Warning: it does not make sense to write something like 'db', unless b is either right-continuous, non-decreasing and $b(0) \geq 0$, or b is a right-continuous map of bounded variation.

Exercise 24. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a map. For all $T \in \mathbf{R}^+$, we define $b^T : \mathbf{R}^+ \to \mathbf{C}$ as $b^T(t) = b(T \wedge t)$ for all $t \in \mathbf{R}^+$.

1. Show that for all $T \in \mathbf{R}^+$, $|b^T| = |b|^T$.

2. Show that if b is of finite variation, then for all $T \in \mathbf{R}^+$, b^T is of bounded variation, and we have $|b^T|(\infty) = |b|(T) < +\infty$.

3. Show that if b is right-continuous and of finite variation, for all $T \in \mathbf{R}^+$, db^T is the unique complex measure on $\mathbf{R}^+$, with:

$$\begin{array}{ll} (i) & db^T(\{0\}) = b(0) \\ (ii) & \forall s,t \in \mathbf{R}^+, s \le t \ , \ db^T(]s,t]) = b(T \wedge t) - b(T \wedge s) \end{array}$$

4. Show that if b is $\mathbf{R}$-valued and of finite variation, for all $T \in \mathbf{R}^+$, we have $|b^T|^+ = (|b|^+)^T$ and $|b^T|^- = (|b|^-)^T$.

5. Show that if b is right-continuous and of bounded variation, for all $T \in \mathbf{R}^+$, we have $db^T = db^{[0,T]} = db([0,T] \cap \cdot\,)$

6. Show that if b is right-continuous, non-decreasing with $b(0) \ge 0$, for all $T \in \mathbf{R}^+$, we have $db^T = db^{[0,T]} = db([0,T] \cap \cdot\,)$

Exercise 25. Let μ, ν be two finite measures on $\mathbf{R}^+$, such that:

$$
\begin{array}{ll}
(i) & \mu(\{0\}) \leq \nu(\{0\}) \\
(ii) & \forall s,t \in \mathbf{R}^+, s \leq t\ ,\ \mu(]s,t]) \leq \nu(]s,t])
\end{array}
$$

We define $a, c : \mathbf{R}^+ \to \mathbf{R}^+$ by $a(t) = \mu([0,t])$ and $c(t) = \nu([0,t])$.

1. Show that a and c are right-continuous, non-decreasing with $a(0) \geq 0$ and $c(0) \geq 0$.
2. Show that $da = \mu$ and $dc = \nu$.
3. Show that $a \leq c$.
4. Define $b : \mathbf{R}^+ \to \mathbf{R}^+$ by $b = c - a$. Show that b is right-continuous, non-decreasing with $b(0) \geq 0$.
5. Show that $da + db = dc$.
6. Conclude with the following:

Theorem 83 *Let μ, ν be two finite measures on $(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ with:*

$$\begin{array}{ll} (i) & \mu(\{0\}) \leq \nu(\{0\}) \\ (ii) & \forall s,t \in \mathbf{R}^+, s \leq t\ ,\ \mu(]s,t]) \leq \nu(]s,t]) \end{array}$$

Then $\mu \leq \nu$, i.e. for all $B \in \mathcal{B}(\mathbf{R}^+)$, $\mu(B) \leq \nu(B)$.

Exercise 26. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of bounded variation.

1. Show that $|db|(\{0\}) = |b(0)| = d|b|(\{0\})$. Let $s, t \in \mathbf{R}^+$, $s \leq t$.
2. Let $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[s,t]$. Show:
$$\sum_{i=1}^{n} |b(t_i) - b(t_{i-1})| \leq |db|(]s,t])$$
3. Show that $|b|(t) - |b|(s) \leq |db|(]s,t])$.
4. Show that $d|b| \leq |db|$.

5. Show that $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|) \subseteq L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), d|b|)$.

6. Show that $\mathbf{R}^+$ is metrizable and strongly σ-compact.

7. Show that $C^c_{\mathbf{C}}(\mathbf{R}^+)$, $C^b_{\mathbf{C}}(\mathbf{R}^+)$ are dense in $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$.

8. Let $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$. Given $\epsilon > 0$, show the existence of $\phi \in C^b_{\mathbf{C}}(\mathbf{R}^+)$ such that $\int |\phi - h||db| \leq \epsilon$.

9. Show that $|\int h db| \leq |\int \phi db| + \epsilon$.

10. Show that:

$$\left| \int |\phi| d|b| - \int |h| d|b| \right| \leq \int |\phi - h| d|b| \leq \int |\phi - h||db|$$

11. Show that $\int |\phi| d|b| \leq \int |h| d|b| + \epsilon$.

12. For all $n \geq 1$, we define:

$$\phi_n \stackrel{\triangle}{=} \phi(0) 1_{\{0\}} + \sum_{k=0}^{n2^n - 1} \phi(k/2^n) 1_{]k/2^n, (k+1)/2^n]}$$

Show there is $M \in \mathbf{R}^+$, such that $|\phi_n(x)| \leq M$ for all x and n.

13. Using the continuity of ϕ, show that $\phi_n \to \phi$.

14. Show that $\lim \int \phi_n db = \int \phi db$.

15. Show that $\lim \int |\phi_n| d|b| = \int |\phi| d|b|$.

16. Show that for all $n \geq 1$:

$$\int \phi_n db = \phi(0)b(0) + \sum_{k=0}^{n2^n-1} \phi(k/2^n)(b((k+1)/2^n) - b(k/2^n))$$

17. Show that $|\int \phi_n db| \leq \int |\phi_n| d|b|$ for all $n \geq 1$.

18. Show that $|\int \phi db| \leq \int |\phi| d|b|$.

19. Show that $|\int h db| \leq \int |h| d|b| + 2\epsilon$.

20. Show that $|\int h db| \leq \int |h| d|b|$ for all $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$.

21. Let $B \in \mathcal{B}(\mathbf{R}^+)$ and $h \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$ be such that $|h| = 1$ and $db = \int h|db|$. Show that $|db|(B) = \int_B \bar{h} db$.

22. Conclude that $|db| \leq d|b|$.

Exercise 27. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Show that for all $T \in \mathbf{R}^+$, $|db^T| = d|b^T|$ and $d|b^T| = d|b|^T$.

2. Show that $d|b|^T = d|b|^{[0,T]} = d|b|([0,T] \cap \cdot\,)$, and conclude:

Theorem 84 *If $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of bounded variation, the total variation of its associated complex stieltjes measure, is equal to the stieltjes measure associated with its total variation, i.e.*

$$|db| = d|b|$$

If $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation, then for all $T \in \mathbf{R}^+$, b^T defined by $b^T(t) = b(T \wedge t)$, is right-continuous of bounded variation, and we have $|db^T| = d|b|([0,T] \cap \cdot\,)$.

Definition 111 *Let $b : \mathbf{R}^+ \to E$ be a map, where E is a topological space. We say that b is* **cadlag** *with respect to E, if and only if b is right-continuous, and the limit:*

$$b(t-) = \lim_{s \uparrow\uparrow t} b(s)$$

exists in E, for all $t \in \mathbf{R}^+ \setminus \{0\}$. In the case when $E = \mathbf{C}$, given b cadlag, we define $b(0-) = 0$, and for all $t \in \mathbf{R}^+$:

$$\Delta b(t) \stackrel{\triangle}{=} b(t) - b(t-)$$

Exercise 28. Let $b : \mathbf{R}^+ \to E$ be cadlag, where E is a topological space. Suppose b has values in $E' \subseteq E$.

1. Explain why b may not be cadlag with respect to E'.
2. Show that b is cadlag with respect to $\bar{E}'$.
3. Show that $b : \mathbf{R}^+ \to \mathbf{R}$ is cadlag $\Leftrightarrow$ it is cadlag w.r. to $\mathbf{C}$.

Exercise 29.

1. Show that if $b : \mathbf{R}^+ \to \mathbf{C}$ is cadlag, then b is continuous with $b(0) = 0$ if and only if $\Delta b(t) = 0$ for all $t \in \mathbf{R}^+$.

2. Show that if $a : \mathbf{R}^+ \to \mathbf{R}^+$ is right-continuous, non-decreasing with $a(0) \geq 0$, then a is cadlag (w.r. to $\mathbf{R}$) with $\Delta a \geq 0$.

3. Show that any linear combination of cadlag maps is itself cadlag.

4. Show that if $b : \mathbf{R}^+ \to \mathbf{C}$ is a right-continuous map of finite variation, then b is cadlag.

5. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Show that $da(\{t\}) = \Delta a(t)$ for all $t \in \mathbf{R}^+$.

6. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of bounded variation. Show that $db(\{t\}) = \Delta b(t)$ for all $t \in \mathbf{R}^+$.

7. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. Let $T \in \mathbf{R}^+$. Show that:

$$\forall t \in \mathbf{R}^+ \ , \ b^T(t-) = \left\{ \begin{array}{ccc} b(t-) & \text{if} & t \leq T \\ b(T) & \text{if} & T < t \end{array} \right.$$

Show that $\Delta b^T = (\Delta b)1_{[0,T]}$, and $db^T(\{t\}) = \Delta b(t)1_{[0,T]}(t)$.

Exercise 30. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a cadlag map and $T \in \mathbf{R}^+$.

1. Show that if $t \to b(t-)$ is not bounded on $[0,T]$, there exists a sequence $(t_n)_{n\geq 1}$ in $[0,T]$ such that $|b(t_n)| \to +\infty$.

2. Suppose from now on that b is not bounded on $[0,T]$. Show the existence of a sequence $(t_n)_{n\geq 1}$ in $[0,T]$, such that $t_n \to t$ for some $t \in [0,T]$, and $|b(t_n)| \to +\infty$.

3. Define $R = \{n \geq 1 : \ t \leq t_n\}$ and $L = \{n \geq 1 : \ t_n < t\}$. Show that R and L cannot be both finite.

4. Suppose that R is infinite. Show the existence of $n_1 \geq 1$, with:
$$t_{n_1} \in [t, t+1[\cap[0, T]$$

5. If R is infinite, show there is $n_1 < n_2 < \ldots$ such that:
$$t_{n_k} \in [t, t+\frac{1}{k}[\cap[0, T] \ , \ \forall k \geq 1$$

6. Show that $|b(t_{n_k})| \not\to +\infty$.

7. Show that if L is infinite, then $t > 0$ and there is an increasing sequence $n_1 < n_2 < \ldots$, such that:
$$t_{n_k} \in]t - \frac{1}{k}, t[\cap[0, T] \ , \ \forall k \geq 1$$

8. Show that: $|b(t_{n_k})| \not\to +\infty$.

9. Prove the following:

Theorem 85 *Let* $b : \mathbf{R}^+ \to \mathbf{C}$ *be a cadlag map. Let* $T \in \mathbf{R}^+$*. Then* b *and* $t \to b(t-)$ *are bounded on* $[0, T]$*, i.e. there exists* $M \in \mathbf{R}^+$ *such that:*

$$|b(t)| \vee |b(t-)| \leq M \ , \ \forall t \in [0, T]$$

15. Stieltjes Integration

Definition 112 $b : \mathbf{R}^+ \to \mathbf{C}$ *is right-continuous of finite variation. The* **stieltjes $\mathbf{L^1}$-spaces** *associated with b are defined as:*

$$L^1_{\mathbf{C}}(b) \stackrel{\triangle}{=} \left\{ f : \mathbf{R}^+ \to \mathbf{C} \text{ measurable}, \int |f| d|b| < +\infty \right\}$$

$$L^{1,loc}_{\mathbf{C}}(b) \stackrel{\triangle}{=} \left\{ f : \mathbf{R}^+ \to \mathbf{C} \text{ measurable}, \int_0^t |f| d|b| < +\infty, \forall t \in \mathbf{R}^+ \right\}$$

Warning : In these tutorials, $\int_0^t \ldots$ refers to $\int_{[0,t]} \ldots$, i.e. the domain of integration is always $[0, t]$, not $]0, t]$, $[0, t[$, or $]0, t[$.

EXERCISE 1. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Propose a definition for $L^1_{\mathbf{R}}(b)$ and $L^{1,\text{loc}}_{\mathbf{R}}(b)$.
2. Is $L^1_{\mathbf{C}}(b)$ the same thing as $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), d|b|)$?
3. Is it meaningful to speak of $L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), |db|)$?

4. Show that $L^1_{\mathbf{C}}(b) = L^1_{\mathbf{C}}(|b|)$ and $L^{1,\mathrm{loc}}_{\mathbf{C}}(b) = L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$.

5. Show that $L^1_{\mathbf{C}}(b) \subseteq L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$.

Exercise 2. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. For all $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$, we define $f.a : \mathbf{R}^+ \to \mathbf{C}$ as:

$$f.a(t) \stackrel{\triangle}{=} \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

1. Explain why $f.a : \mathbf{R}^+ \to \mathbf{C}$ is a well-defined map.

2. Let $t \in \mathbf{R}^+$, $(t_n)_{n\geq 1}$ be a sequence in $\mathbf{R}^+$ with $t_n \downarrow\downarrow t$. Show:

$$\lim_{n\to+\infty} \int f 1_{[0,t_n]} da = \int f 1_{[0,t]} da$$

3. Show that $f.a$ is right-continuous.

4. Let $t \in \mathbf{R}^+$ and $t_0 \leq \ldots \leq t_n$ be a finite sequence in $[0,t]$. Show:

$$\sum_{i=1}^{n} |f.a(t_i) - f.a(t_{i-1})| \leq \int_{]0,t]} |f| da$$

5. Show that $f.a$ is a map of finite variation with:

$$|f.a|(t) \leq \int_0^t |f| da \ , \ \forall t \in \mathbf{R}^+$$

Exercise 3. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^1_{\mathbf{C}}(a)$.

1. Show that $f.a$ is a right-continuous map of bounded variation.

2. Show $d(f.a)([0,t]) = \nu([0,t])$, for all $t \in \mathbf{R}^+$, where $\nu = \int f da$.

3. Prove the following:

Theorem 86 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^1_{\mathbf{C}}(a)$. The map $f.a : \mathbf{R}^+ \to \mathbf{C}$ defined by:*

$$f.a(t) \stackrel{\triangle}{=} \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

is a right-continuous map of bounded variation, and its associated complex stieltjes measure is given by $d(f.a) = \int f da$, i.e.

$$d(f.a)(B) = \int_B f da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

Exercise 4. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^{1,\mathrm{loc}}_{\mathbf{R}}(a)$, $f \geq 0$.

1. Show $f.a$ is right-continuous, non-decreasing with $f.a(0) \geq 0$.
2. Show $d(f.a)([0, t]) = \mu([0, t])$, for all $t \in \mathbf{R}^+$, where $\mu = \int f da$.
3. Prove that $d(f.a)([0, T] \cap \cdot\,) = \mu([0, T] \cap \cdot\,)$, for all $T \in \mathbf{R}^+$.

4. Prove with the following:

Theorem 87 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^{1,loc}_{\mathbf{R}}(a)$, $f \geq 0$. The map $f.a : \mathbf{R}^+ \to \mathbf{R}^+$ defined by:*

$$f.a(t) \stackrel{\triangle}{=} \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

is right-continuous, non-decreasing with $(f.a)(0) \geq 0$, and its associated stieltjes measure is given by $d(f.a) = \int f da$, i.e.

$$d(f.a)(B) = \int_B f da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

Exercise 5. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$ and $T \in \mathbf{R}^+$.

1. Show that $\int |f| 1_{[0,T]} da = \int |f| da^{[0,T]} = \int |f| da^T$.

2. Show that $f1_{[0,T]} \in L^1_{\mathbf{C}}(a)$ and $f \in L^1_{\mathbf{C}}(a^T)$.

3. Show that $(f.a)^T = f.(a^T) = (f1_{[0,T]}).a$.

4. Show that for all $B \in \mathcal{B}(\mathbf{R}^+)$:

$$d(f.a)^T(B) = \int_B f da^T = \int_B f1_{[0,T]} da$$

5. Explain why it does not in general make sense to write:

$$d(f.a)^T = d(f.a)([0,T] \cap \cdot\,)$$

6. Show that for all $B \in \mathcal{B}(\mathbf{R}^+)$:

$$|d(f.a)^T|(B) = \int_B |f|1_{[0,T]} da\ ,\ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

7. Show that $|d(f.a)^T| = d|f.a|([0,T] \cap \cdot\,)$

8. Show that for all $t \in \mathbf{R}^+$
$$|f.a|(t) = (|f|.a)(t) = \int_0^t |f|da$$

9. Show that $f.a$ is of bounded variation if and only if $f \in L^1_{\mathbf{C}}(a)$.

10. Show that $\Delta(f.a)(0) = f(0)\Delta a(0)$.

11. Let $t > 0$, $(t_n)_{n\geq 1}$ be a sequence in $\mathbf{R}^+$ with $t_n \uparrow\uparrow t$. Show:
$$\lim_{n\to+\infty} \int f1_{[0,t_n]}da = \int f1_{[0,t[}da$$

12. Show that $\Delta(f.a)(t) = f(t)\Delta a(t)$ for all $t \in \mathbf{R}^+$.

13. Show that if a is continuous with $a(0) = 0$, then $f.a$ is itself continuous with $(f.a)(0) = 0$.

14. Prove with the following:

Theorem 88 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f \in L^{1,loc}_{\mathbf{C}}(a)$. The map $f.a : \mathbf{R}^+ \to \mathbf{C}$ defined by:*

$$f.a(t) \stackrel{\triangle}{=} \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

is right-continuous of finite variation, and we have $|f.a| = |f|.a$, i.e.

$$|f.a|(t) = \int_0^t |f| da \ , \ \forall t \in \mathbf{R}^+$$

In particular, $f.a$ is of bounded variation if and only if $f \in L^1_{\mathbf{C}}(a)$. Furthermore, we have $\Delta(f.a) = f\Delta a$.

Exercise 6. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation.

1. Prove the equivalence between the following:

$$(i) \qquad d|b| << da$$

$$(ii) \qquad |db^T| << da \ , \ \forall T \in \mathbf{R}^+$$
$$(iii) \qquad db^T << da \ , \ \forall T \in \mathbf{R}^+$$

2. Does it make sense in general to write $db << da$?

Definition 113 *Let $a:\mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b:\mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. We say that b is* **absolutely continuous** *with respect to a, and we write $b << a$, if and only if, one of the following holds:*

$$(i) \qquad d|b| << da$$
$$(ii) \qquad |db^T| << da \ , \ \forall T \in \mathbf{R}^+$$
$$(iii) \qquad db^T << da \ , \ \forall T \in \mathbf{R}^+$$

In other words, b is absolutely continuous w.r. to a, if and only if the stieltjes measure associated with the total variation of b is absolutely continuous w.r. to the stieltjes measure associated with a.

Exercise 7. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation, absolutely continuous w.r. to a, i.e. with $b << a$.

1. Show that for all $T \in \mathbf{R}^+$, there exits $f_T \in L^1_{\mathbf{C}}(a)$ such that:
$$db^T(B) = \int_B f_T da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$
2. Suppose that $T, T' \in \mathbf{R}^+$ and $T \leq T'$. Show that:
$$\int_B f_T da = \int_{B \cap [0,T]} f_{T'} da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$
3. Show that $f_T = f_{T'} 1_{[0,T]}$ da-a.s.
4. Show the existence of a sequence $(f_n)_{n \geq 1}$ in $L^1_{\mathbf{C}}(a)$, such that for all $1 \leq n \leq p$, $f_n = f_p 1_{[0,n]}$ and:
$$\forall n \geq 1 \ , \ db^n(B) = \int_B f_n da \ , \ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

5. We define $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ by:

$$\forall t \in \mathbf{R}^+ \ , \ f(t) \stackrel{\triangle}{=} f_n(t) \text{ for any } n \geq 1 : \ t \in [0, n]$$

   Explain why f is unambiguously defined.

6. Show that for all $B \in \mathcal{B}(\mathbf{C})$, $\{f \in B\} = \cup_{n=1}^{+\infty}[0, n] \cap \{f_n \in B\}$.

7. Show that $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ is measurable.

8. Show that $f \in L^{1,\text{loc}}_{\mathbf{C}}(a)$ and that we have:

$$b(t) = \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

9. Prove the following:

Theorem 89 *Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. Then, b is absolutely continuous w.r. to a, i.e. $d|b| << da$, if and only if there exists $f \in L^{1,loc}_{\mathbf{C}}(a)$ such that $b = f.a$, i.e.*

$$b(t) = \int_0^t f da \ , \ \forall t \in \mathbf{R}^+$$

If b is $\mathbf{R}$-valued, we can assume that $f \in L^{1,loc}_{\mathbf{R}}(a)$.
If b is non-decreasing with $b(0) \geq 0$, we can assume that $f \geq 0$.

Exercise 8. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. Let $f, g \in L^{1,\text{loc}}_{\mathbf{C}}(a)$ be such that $f.a = g.a$, i.e.:

$$\int_0^t f da = \int_0^t g da \ , \ \forall t \in \mathbf{R}^+$$

1. Show that for all $T \in \mathbf{R}^+$ and $B \in \mathcal{B}(\mathbf{R}^+)$:

$$d(f.a)^T(B) = \int_B f1_{[0,T]}da = \int_B g1_{[0,T]}da$$

2. Show that for all $T \in \mathbf{R}^+$, $f1_{[0,T]} = g1_{[0,T]}$ da-a.s.

3. Show that $f = g$ da-a.s.

Exercise 9. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Show the existence of $h \in L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$ such that $b = h.|b|$.

2. Show that for all $B \in \mathcal{B}(\mathbf{R}^+)$ and $T \in \mathbf{R}^+$:

$$db^T(B) = \int_B hd|b|^T = \int_B h|db^T|$$

3. Show that $|h| = 1$ $|db^T|$-a.s. for all $T \in \mathbf{R}^+$.

4. Show that for all $T \in \mathbf{R}^+$, $d|b|([0,T] \cap \{|h| \neq 1\}) = 0$.

5. Show that $|h| = 1$ $d|b|$-a.s.

6. Prove the following:

Theorem 90 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. There exists $h \in L^{1,loc}_{\mathbf{C}}(|b|)$ such that $|h| = 1$ and $b = h.|b|$, i.e.*

$$b(t) = \int_0^t hd|b| \ , \ \forall t \in \mathbf{R}^+$$

Definition 114 *$b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation. For all $f \in L^1_{\mathbf{C}}(b)$, the* **stieltjes integral** *of f with respect to b, is defined as:*

$$\int fdb \stackrel{\triangle}{=} \int fhd|b|$$

where $h \in L^{1,loc}_{\mathbf{C}}(|b|)$ is such that $|h| = 1$ and $b = h.|b|$.

Warning : the notation $\int f db$ of definition (114) is controversial and potentially confusing: 'db' is not in general a complex measure on $\mathbf{R}^+$, unless b is of bounded variation.

Exercise 10. $b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation.

1. Show that if $f \in L^1_{\mathbf{C}}(b)$, then $\int fhd|b|$ is well-defined.

2. Explain why, given $f \in L^1_{\mathbf{C}}(b)$, $\int f db$ is unambiguously defined.

3. Show that if b is right-continuous, non-decreasing with $b(0) \geq 0$, definition (114) of $\int f db$ coincides with that of an integral w.r. to the stieltjes measure db.

4. Show that if b is a right-continuous map of bounded variation, definition (114) of $\int f db$ coincides with that of an integral w.r. to the complex stieltjes measure db.

Exercise 11. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. For all $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$, we define $f.b : \mathbf{R}^+ \to \mathbf{C}$ as:

$$f.b(t) \stackrel{\triangle}{=} \int_0^t f db\ ,\ \forall t \in \mathbf{R}^+$$

1. Explain why $f.b : \mathbf{R}^+ \to \mathbf{C}$ is a well-defined map.
2. If b is right-continuous, non-decreasing with $b(0) \geq 0$, show this definition of $f.b$ coincides with that of theorem (88).
3. Show $f.b = (fh).|b|$, where $h \in L^{1,\mathrm{loc}}_{\mathbf{C}}(|b|)$, $|h| = 1$, $b = h.|b|$.
4. Show that $f.b : \mathbf{R}^+ \to \mathbf{C}$ is right-continuous of finite variation, with $|f.b| = |f|.|b|$, i.e.
$$|f.b|(t) = \int_0^t |f|d|b|\ ,\ \forall t \in \mathbf{R}^+$$
5. Show that $f.b$ is of bounded variation if and only if $f \in L^1_{\mathbf{C}}(b)$.

6. Let $t > 0$, $(t_n)_{n\geq 1}$ be a sequence in $\mathbf{R}^+$ such that $t_n \uparrow\uparrow t$. Show:
$$\lim_{n\to+\infty} \int fh1_{[0,t_n]}d|b| = \int fh1_{[0,t[}d|b|$$

7. Show that $\Delta(f.b) = f\Delta b$.

8. Show that if b is continuous with $b(0) = 0$, then $f.b$ is itself continuous with $(f.b)(0) = 0$.

9. Prove the following:

Theorem 91 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. For all $f \in L^{1,loc}_{\mathbf{C}}(b)$, the map $f.b : \mathbf{R}^+ \to \mathbf{C}$ defined by:*

$$f.b(t) \stackrel{\triangle}{=} \int_0^t f db \ , \ \forall t \in \mathbf{R}^+$$

is right-continuous of finite variation, and we have $|f.b| = |f|.|b|$, i.e.

$$|f.b|(t) = \int_0^t |f|d|b| \ , \ \forall t \in \mathbf{R}^+$$

In particular, $f.b$ is of bounded variation if and only if $f \in L^1_{\mathbf{C}}(b)$. Furthermore, we have $\Delta(f.b) = f\Delta b$.

Exercise 12. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. Let $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$ and $T \in \mathbf{R}^+$.

1. Show that $\int |f|1_{[0,T]}d|b| = \int |f|d|b|^{[0,T]} = \int |f|d|b^T|$.
2. Show that $f1_{[0,T]} \in L^1_{\mathbf{C}}(b)$ and $f \in L^1_{\mathbf{C}}(b^T)$.

3. Show $b^T = h.|b^T|$, where $h \in L^{1,\text{loc}}_{\mathbf{C}}(|b|)$, $|h| = 1$, $b = h.|b|$.

4. Show that $(f.b)^T = f.(b^T) = (f 1_{[0,T]}).b$

5. Show that $d|f.b|(B) = \int_B |f|d|b|$ for all $B \in \mathcal{B}(\mathbf{R}^+)$.

6. Let $g : \mathbf{R}^+ \to \mathbf{C}$ be a measurable map. Show the equivalence:
$$g \in L^{1,\text{loc}}_{\mathbf{C}}(f.b) \;\Leftrightarrow\; gf \in L^{1,\text{loc}}_{\mathbf{C}}(b)$$

7. Show that $d(f.b)^T(B) = \int_B fhd|b^T|$ for all $B \in \mathcal{B}(\mathbf{R}^+)$.

8. Show that $db^T = \int hd|b^T|$ and conclude that:
$$d(f.b)^T(B) = \int_B f db^T \ ,\ \forall B \in \mathcal{B}(\mathbf{R}^+)$$

9. Let $g \in L^{1,\text{loc}}_{\mathbf{C}}(f.b)$. Show that $g \in L^1_{\mathbf{C}}((f.b)^T)$ and:
$$\int g 1_{[0,t]} d(f.b)^T = \int g f 1_{[0,t]} db^T \ ,\ \forall t \in \mathbf{R}^+$$

10. Show that $g.\big((f.b)^T\big) = (gf).(b^T)$.

11. Show that $(g.(f.b))^T = ((gf).b)^T$.

12. Show that $g.(f.b) = (gf).b$

13. Prove the following:

Theorem 92 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. For all $f \in L^{1,loc}_{\mathbf{C}}(b)$ and $g : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$ measurable map, we have the equivalence:*

$$g \in L^{1,loc}_{\mathbf{C}}(f.b) \ \Leftrightarrow \ gf \in L^{1,loc}_{\mathbf{C}}(b)$$

and when such condition is satisfied, $g.(f.b) = (fg).b$, i.e.

$$\int_0^t g d(f.b) = \int_0^t gf db \ , \ \forall t \in \mathbf{R}^+$$

Exercise 13. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation. let $f, g \in L^{1,\text{loc}}_{\mathbf{C}}(b)$ and $\alpha \in \mathbf{C}$. Show that $f + \alpha g \in L^{1,\text{loc}}_{\mathbf{C}}(b)$, and:

$$(f + \alpha g).b = f.b + \alpha g.b$$

Exercise 14. Let $b, c : \mathbf{R}^+ \to \mathbf{C}$ be two right-continuous maps of finite variations. Let $f \in L^{1,\text{loc}}_{\mathbf{C}}(b) \cap L^{1,\text{loc}}_{\mathbf{C}}(c)$ and $\alpha \in \mathbf{C}$.

1. Show that for all $T \in \mathbf{R}^+$, $d(b + \alpha c)^T = db^T + \alpha dc^T$.
2. Show that for all $T \in \mathbf{R}^+$, $d|b + \alpha c|^T \leq d|b|^T + |\alpha| d|c|^T$.
3. Show that $d|b + \alpha c| \leq d|b| + |\alpha| d|c|$.
4. Show that $f \in L^{1,\text{loc}}_{\mathbf{C}}(b + \alpha c)$.
5. Show $d(f.(b + \alpha c))^T(B) = \int_B f d(b + \alpha c)^T$ for all $B \in \mathcal{B}(\mathbf{R}^+)$.
6. Show that $d(f.(b + \alpha c))^T = d(f.b)^T + \alpha d(f.c)^T$.

7. Show that $(f.(b+\alpha c))^T = (f.b)^T + \alpha(f.c)^T$

8. Show that $f.(b+\alpha c) = f.b + \alpha(f.c)$.

Exercise 15. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation.

1. Show that $d|b| \leq d|b_1| + d|b_2|$, where $b = b_1 + ib_2$.

2. Show that $d|b_1| \leq d|b|$ and $d|b_2| \leq d|b|$.

3. Show that $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$, if and only if:

$$f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(|b_1|^+) \cap L^{1,\mathrm{loc}}_{\mathbf{C}}(|b_1|^-) \cap L^{1,\mathrm{loc}}_{\mathbf{C}}(|b_2|^+) \cap L^{1,\mathrm{loc}}_{\mathbf{C}}(|b_2|^-)$$

4. Show that if $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(b)$, for all $t \in \mathbf{R}^+$:

$$\int_0^t f db = \int_0^t f d|b_1|^+ - \int_0^t f d|b_1|^- + i\left(\int_0^t f d|b_2|^+ - \int_0^t f d|b_2|^-\right)$$

Exercise 16. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. We define $c : \mathbf{R}^+ \to [0, +\infty]$ as:

$$c(t) \stackrel{\triangle}{=} \inf\{s \in \mathbf{R}^+ : \ t < a(s)\} \ , \ \forall t \in \mathbf{R}^+$$

where it is understood that $\inf \emptyset = +\infty$. Let $s, t \in \mathbf{R}^+$.

1. Show that $t < a(s) \ \Rightarrow \ c(t) \leq s$.
2. Show that $c(t) < s \ \Rightarrow \ t < a(s)$.
3. Show that $c(t) \leq s \ \Rightarrow \ t < a(s+\epsilon) \ , \ \forall \epsilon > 0$.
4. Show that $c(t) \leq s \ \Rightarrow \ t \leq a(s)$.
5. Show that $c(t) < +\infty \ \Leftrightarrow \ t < a(\infty)$.
6. Show that c is non-decreasing.
7. Show that if $t_0 \in [a(\infty), +\infty[$, c is right-continuous at t_0.

8. Suppose $t_0 \in [0, a(\infty)[$. Given $\epsilon > 0$, show the existence of $s \in \mathbf{R}^+$, such that $c(t_0) \leq s < c(t_0) + \epsilon$ and $t_0 < a(s)$.

9. Show that $t \in [t_0, a(s)[\;\Rightarrow\; c(t_0) \leq c(t) \leq c(t_0) + \epsilon$.

10. Show that c is right-continuous.

11. Show that if $a(\infty) = +\infty$, then c is a map $c : \mathbf{R}^+ \to \mathbf{R}^+$ which is right-continuous, non-decreasing with $c(0) \geq 0$.

12. We define $\bar{a}(s) = \inf\{t \in \mathbf{R}^+ : \; s < c(t)\}$ for all $s \in \mathbf{R}^+$. Show that for all $s, t \in \mathbf{R}^+$, $s < c(t) \;\Rightarrow\; a(s) \leq t$.

13. Show that $a \leq \bar{a}$.

14. Show that for all $s, t \in \mathbf{R}^+$ and $\epsilon > 0$:

$$a(s + \epsilon) \leq t \;\Rightarrow\; s < s + \epsilon \leq c(t)$$

15. Show that for all $s, t \in \mathbf{R}^+$ and $\epsilon > 0$, $a(s + \epsilon) \leq t \;\Rightarrow\; \bar{a}(s) \leq t$.

16. Show that $\bar{a} \leq a$ and conclude that:

$$a(s) = \inf\{t \in \mathbf{R}^+ :\ s < c(t)\}$$

EXERCISE 17. Let $f : \mathbf{R}^+ \to \bar{\mathbf{R}}$ be a non-decreasing map. Let $\alpha \in \mathbf{R}$. We define:

$$x_0 \stackrel{\triangle}{=} \sup\{x \in \mathbf{R}^+ :\ f(x) \leq \alpha\}$$

1. Explain why $x_0 = -\infty$ if and only if $\{f \leq \alpha\} = \emptyset$.

2. Show that $x_0 = +\infty$ if and only if $\{f \leq \alpha\} = \mathbf{R}^+$.

3. We assume from now on that $x_0 \neq \pm\infty$. Show that $x_0 \in \mathbf{R}^+$.

4. Show that if $f(x_0) \leq \alpha$ then $\{f \leq \alpha\} = [0, x_0]$.

5. Show that if $\alpha < f(x_0)$ then $\{f \leq \alpha\} = [0, x_0[$.

6. Conclude that $f : (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+)) \to (\bar{\mathbf{R}}, \mathcal{B}(\bar{\mathbf{R}}))$ is measurable.

Exercise 18. Let $a : \mathbf{R}^+ \to \mathbf{R}^+$ be right-continuous, non-decreasing with $a(0) \geq 0$. We define $c : \mathbf{R}^+ \to [0, +\infty]$ as:

$$c(t) \stackrel{\triangle}{=} \inf\{s \in \mathbf{R}^+ : \ t < a(s)\} \ , \ \forall t \in \mathbf{R}^+$$

1. Let $f : \mathbf{R}^+ \to [0, +\infty]$ be non-negative and measurable. Show $(f \circ c)1_{\{c<+\infty\}}$ is well-defined, non-negative and measurable.
2. Let $t, u \in \mathbf{R}^+$, and ds be the lebesgue measure on $\mathbf{R}^+$. Show:
$$\int_0^{a(t)} (1_{[0,u]} \circ c)1_{\{c<+\infty\}} ds \leq \int 1_{[0,a(t\wedge u)]} 1_{\{c<+\infty\}} ds$$
3. Show that:
$$\int_0^{a(t)} (1_{[0,u]} \circ c)1_{\{c<+\infty\}} ds \leq a(t \wedge u)$$
4. Show that:
$$a(t \wedge u) = \int_0^{a(t)} 1_{[0,a(u)[} ds = \int_0^{a(t)} 1_{[0,a(u)[} 1_{\{c<+\infty\}} ds$$

5. Show that:

$$a(t \wedge u) \leq \int_0^{a(t)} (1_{[0,u]} \circ c) 1_{\{c<+\infty\}} ds$$

6. Show that:

$$\int_0^t 1_{[0,u]} da = \int_0^{a(t)} (1_{[0,u]} \circ c) 1_{\{c<+\infty\}} ds$$

7. Define:

$$\mathcal{D}_t \stackrel{\triangle}{=} \left\{ B \in \mathcal{B}(\mathbf{R}^+) : \int_0^t 1_B da = \int_0^{a(t)} (1_B \circ c) 1_{\{c<+\infty\}} ds \right\}$$

Show that $\mathcal{D}_t$ is a dynkin system on $\mathbf{R}^+$, and $\mathcal{D}_t = \mathcal{B}(\mathbf{R}^+)$.

8. Show that if $f : \mathbf{R}^+ \to [0, +\infty]$ is non-negative measurable:

$$\int_0^t f da = \int_0^{a(t)} (f \circ c) 1_{\{c<+\infty\}} ds \ , \ \forall t \in \mathbf{R}^+$$

9. Let $f : \mathbf{R}^+ \to \mathbf{C}$ be measurable. Show that $(f \circ c)1_{\{c<+\infty\}}$ is itself well-defined and measurable.

10. Show that if $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(a)$, then for all $t \in \mathbf{R}^+$, we have:
$$(f \circ c)1_{\{c<+\infty\}}1_{[0,a(t)]} \in L^1_{\mathbf{C}}(\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+), ds)$$
and furthermore:
$$\int_0^t f da = \int_0^{a(t)} (f \circ c)1_{\{c<+\infty\}} ds$$

11. Show that we also have:
$$\int_0^t f da = \int (f \circ c)1_{[0,a(t)[} ds$$

12. Conclude with the following:

Theorem 93 *Let* $a : \mathbf{R}^+ \to \mathbf{R}^+$ *be right-continuous, non-decreasing with* $a(0) \geq 0$. *We define* $c : \mathbf{R}^+ \to [0, +\infty]$ *as:*

$$c(t) \stackrel{\triangle}{=} \inf\{s \in \mathbf{R}^+ : \ t < a(s)\} \ , \ \forall t \in \mathbf{R}^+$$

Then, for all $f \in L^{1,loc}_{\mathbf{C}}(a)$, *we have:*

$$\int_0^t f da = \int_0^{a(t)} (f \circ c) 1_{\{c<+\infty\}} ds \ , \ \forall t \in \mathbf{R}^+$$

where ds *is the lebesgue measure on* $\mathbf{R}^+$.

16. Differentiation

Definition 115 *Let $(\Omega, \mathcal{T})$ be a topological space. A map $f : \Omega \to \bar{\mathbf{R}}$ is said to be* **lower-semi-continuous** *(l.s.c), if and only if:*

$$\forall \lambda \in \mathbf{R}\ ,\ \{\lambda < f\} \text{ is open}$$

We say that f is **upper-semi-continuous** *(u.s.c), if and only if:*

$$\forall \lambda \in \mathbf{R}\ ,\ \{f < \lambda\} \text{ is open}$$

Exercise 1. Let $f : \Omega \to \bar{\mathbf{R}}$ be a map, where Ω is a topological space.

1. Show that f is l.s.c if and only if $\{\lambda < f\}$ is open for all $\lambda \in \bar{\mathbf{R}}$.
2. Show that f is u.s.c if and only if $\{f < \lambda\}$ is open for all $\lambda \in \bar{\mathbf{R}}$.
3. Show that every open set U in $\bar{\mathbf{R}}$ can be written:

$$U = V^+ \cup V^- \cup \bigcup_{i \in I}]\alpha_i, \beta_i[$$

for some index set I, $\alpha_i, \beta_i \in \mathbf{R}$, $V^+ = \emptyset$ or $V^+ =]\alpha, +\infty]$, $(\alpha \in \mathbf{R})$ and $V^- = \emptyset$ or $V^- = [-\infty, \beta[$, $(\beta \in \mathbf{R})$.

4. Show that f is continuous if and only if it is both l.s.c and u.s.c.

5. Let $u : \Omega \to \mathbf{R}$ and $v : \Omega \to \bar{\mathbf{R}}$. Let $\lambda \in \mathbf{R}$. Show that:

$$\{\lambda < u + v\} = \bigcup_{\substack{(\lambda_1, \lambda_2) \in \mathbf{R}^2 \\ \lambda_1 + \lambda_2 = \lambda}} \{\lambda_1 < u\} \cap \{\lambda_2 < v\}$$

6. Show that if both u and v are l.s.c, then $u + v$ is also l.s.c.

7. Show that if both u and v are u.s.c, then $u + v$ is also u.s.c.

8. Show that if f is l.s.c, then αf is l.s.c, for all $\alpha \in \mathbf{R}^+$.

9. Show that if f is u.s.c, then αf is u.s.c, for all $\alpha \in \mathbf{R}^+$.

10. Show that if f is l.s.c, then $-f$ is u.s.c.

11. Show that if f is u.s.c, then $-f$ is l.s.c.

12. Show that if V is open in Ω, then $f = 1_V$ is l.s.c.

13. Show that if F is closed in Ω, then $f = 1_F$ is u.s.c.

Exercise 2. Let $(f_i)_{i\in I}$ be an arbitrary family of maps $f_i : \Omega \to \bar{\mathbf{R}}$, defined on a topological space Ω.

1. Show that if all f_i's are l.s.c, then $f = \sup_{i\in I} f_i$ is l.s.c.

2. Show that if all f_i's are u.s.c, then $f = \inf_{i\in I} f_i$ is u.s.c.

Exercise 3. Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$. Let f be an element of $\in L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$, such that $f \geq 0$.

1. Let $(s_n)_{n\geq 1}$ be a sequence of simple functions on $(\Omega, \mathcal{B}(\Omega))$ such that $s_n \uparrow f$. Define $t_1 = s_1$ and $t_n = s_n - s_{n-1}$ for all $n \geq 2$. Show that t_n is a simple function on $(\Omega, \mathcal{B}(\Omega))$, for all $n \geq 1$.

2. Show that f can be written as:

$$f = \sum_{n=1}^{+\infty} \alpha_n 1_{A_n}$$

where $\alpha_n \in \mathbf{R}^+ \setminus \{0\}$ and $A_n \in \mathcal{B}(\Omega)$, for all $n \geq 1$.

3. Show that $\mu(A_n) < +\infty$, for all $n \geq 1$.

4. Show that there exist K_n compact and V_n open in Ω such that:

$$K_n \subseteq A_n \subseteq V_n \quad , \quad \mu(V_n \setminus K_n) \leq \frac{\epsilon}{\alpha_n 2^{n+1}}$$

for all $\epsilon > 0$ and $n \geq 1$.

5. Show the existence of $N \geq 1$ such that:

$$\sum_{n=N+1}^{+\infty} \alpha_n \mu(A_n) \leq \frac{\epsilon}{2}$$

6. Define $u = \sum_{n=1}^{N} \alpha_n 1_{K_n}$. Show that u is u.s.c.

7. Define $v = \sum_{n=1}^{+\infty} \alpha_n 1_{V_n}$. Show that v is l.s.c.

8. Show that we have $0 \leq u \leq f \leq v$.

9. Show that we have:

$$v = u + \sum_{n=N+1}^{+\infty} \alpha_n 1_{K_n} + \sum_{n=1}^{+\infty} \alpha_n 1_{V_n \setminus K_n}$$

10. Show that $\int v d\mu \leq \int u d\mu + \epsilon < +\infty$.

11. Show that $u \in L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$.

12. Explain why v may fail to be in $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$.

13. Show that v is μ-a.s. equal to an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$.

14. Show that $\int (v - u) d\mu \leq \epsilon$.

15. Prove the following:

Theorem 94 (Vitali-Caratheodory) *Let $(\Omega, \mathcal{T})$ be a metrizable and σ-compact topological space. Let μ be a locally finite measure on $(\Omega, \mathcal{B}(\Omega))$ and f be an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$. Then, for all $\epsilon > 0$, there exist maps $u, v : \Omega \to \bar{\mathbf{R}}$, which are μ-a.s. equal to elements of $L^1_{\mathbf{R}}(\Omega, \mathcal{B}(\Omega), \mu)$, such that $u \leq f \leq v$, u is u.s.c, v is l.s.c, and furthermore:*

$$\int (v - u) d\mu \leq \epsilon$$

Definition 116 *We call* **connected topological space***, a topological space $(\Omega, \mathcal{T})$, for which the only subsets of Ω which are both open and closed, are Ω and $\emptyset$.*

Exercise 4. Let $(\Omega, \mathcal{T})$ be a topological space.

1. Show that $(\Omega, \mathcal{T})$ is connected if and only if whenever $\Omega = A \uplus B$ where A, B are disjoint open sets, we have $A = \emptyset$ or $B = \emptyset$.

2. Show that $(\Omega, \mathcal{T})$ is connected if and only if whenever $\Omega = A \uplus B$ where A, B are disjoint closed sets, we have $A = \emptyset$ or $B = \emptyset$.

Definition 117 *Let $(\Omega, \mathcal{T})$ be a topological space, and $A \subseteq \Omega$. We say that A is a* **connected subset** *of Ω, if and only if the induced topological space $(A, \mathcal{T}_{|A})$ is connected.*

Exercise 5. Let A be open and closed in $\mathbf{R}$, with $A \neq \emptyset$ and $A^c \neq \emptyset$.

1. Let $x \in A^c$. Show that $A \cap [x, +\infty[$ or $A \cap]-\infty, x]$ is non-empty.

2. Suppose $B = A \cap [x, +\infty[\neq \emptyset$. Show that B is closed and that we have $B = A \cap]x, +\infty[$. Conclude that B is also open.

3. Let $b = \inf B$. Show that $b \in B$ (and in particular $b \in \mathbf{R}$).

4. Show the existence of $\epsilon > 0$ such that $]b - \epsilon, b + \epsilon[\subseteq B$.

5. Conclude with the following:

Theorem 95 *The topological space* $(\mathbf{R}, \mathcal{T}_{\mathbf{R}})$ *is connected.*

Exercise 6. Let $(\Omega, \mathcal{T})$ be a topological space and $A \subseteq \Omega$ be a connected subset of Ω. Let B be a subset of Ω such that $A \subseteq B \subseteq \bar{A}$. We assume that $B = V_1 \uplus V_2$ where V_1, V_2 are disjoint open sets in B.

1. Show there is U_1, U_2 open in Ω, with $V_1 = B \cap U_1$, $V_2 = B \cap U_2$.

2. Show that $A \cap U_1 = \emptyset$ or $A \cap U_2 = \emptyset$.

3. Suppose that $A \cap U_1 = \emptyset$. Show that $\bar{A} \subseteq U_1^c$.

4. Show then that $V_1 = B \cap U_1 = \emptyset$.

5. Conclude that B and $\bar{A}$ are both connected subsets of Ω.

Exercise 7. Prove the following:

Theorem 96 *Let $(\Omega, \mathcal{T})$, $(\Omega', \mathcal{T}')$ be two topological spaces, and f be a continuous map, $f : \Omega \to \Omega'$. If $(\Omega, \mathcal{T})$ is connected, then $f(\Omega)$ is a connected subset of Ω'.*

Definition 118 *Let $A \subseteq \bar{\mathbf{R}}$. We say that A is an* **interval**, *if and only if for all $x, y \in A$ with $x \leq y$, we have $[x, y] \subseteq A$, where:*

$$[x, y] \stackrel{\triangle}{=} \{z \in \bar{\mathbf{R}} \ : \ x \leq z \leq y\}$$

Exercise 8. Let $A \subseteq \bar{\mathbf{R}}$.

1. If A is an interval, and $\alpha = \inf A$, $\beta = \sup A$, show that:

$$]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta]$$

2. Show that A is an interval if and only if, it is of the form $[\alpha, \beta]$, $[\alpha, \beta[$, $]\alpha, \beta]$ or $]\alpha, \beta[$, for some $\alpha, \beta \in \bar{\mathbf{R}}$.

3. Show that an interval of the form $]-\infty, \alpha[$, where $\alpha \in \mathbf{R}$, is homeomorphic to $]-1, \alpha'[$, for some $\alpha' \in \mathbf{R}$.

4. Show that an interval of the form $]\alpha, +\infty[$, where $\alpha \in \mathbf{R}$, is homeomorphic to $]\alpha', 1[$, for some $\alpha' \in \mathbf{R}$.

5. Show that an interval of the form $]\alpha, \beta[$, where $\alpha, \beta \in \mathbf{R}$ and $\alpha < \beta$, is homeomorphic to $]-1, 1[$.

6. Show that $]-1, 1[$ is homeomorphic to $\mathbf{R}$.

7. Show an non-empty open interval in $\mathbf{R}$, is homeomorphic to $\mathbf{R}$.

8. Show that an open interval in $\mathbf{R}$, is a connected subset of $\mathbf{R}$.

9. Show that an interval in $\mathbf{R}$, is a connected subset of $\mathbf{R}$.

Exercise 9. Let $A \subseteq \mathbf{R}$ be a non-empty connected subset of $\mathbf{R}$, and $\alpha = \inf A$, $\beta = \sup A$. We assume there exists $x_0 \in A^c \cap]\alpha, \beta[$.

1. Show that $A \cap]x_0, +\infty[$ or $A \cap]-\infty, x_0[$ is empty.
2. Show that if $A \cap]x_0, +\infty[= \emptyset$, then β cannot be $\sup A$.
3. Show that $]\alpha, \beta[\subseteq A \subseteq [\alpha, \beta]$.
4. Show the following:

Theorem 97 *For all $A \subseteq \mathbf{R}$, A is a connected subset of $\mathbf{R}$, if and only if A is an interval.*

Exercise 10. Prove the following:

Theorem 98 *Let $f : \Omega \to \mathbf{R}$ be a continuous map, where $(\Omega, \mathcal{T})$ is a connected topological space. Let $a, b \in \Omega$ such that $f(a) \leq f(b)$. Then, for all $z \in [f(a), f(b)]$, there exists $x \in \Omega$ such that $z = f(x)$.*

Exercise 11. Let $a, b \in \mathbf{R}$, $a < b$, and $f : [a, b] \to \mathbf{R}$ be a map such that $f'(x)$ exists for all $x \in [a, b]$.

1. Show that $f' : ([a, b], \mathcal{B}([a, b])) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ is measurable.

2. Show that $f' \in L^1_{\mathbf{R}}([a, b], \mathcal{B}([a, b]), dx)$ is equivalent to:
$$\int_a^b |f'(t)|dt < +\infty$$

3. We assume from now on that $f' \in L^1_{\mathbf{R}}([a, b], \mathcal{B}([a, b]), dx)$. Given $\epsilon > 0$, show the existence of $g : [a, b] \to \bar{\mathbf{R}}$, almost surely equal to an element of $L^1_{\mathbf{R}}([a, b], \mathcal{B}([a, b]), dx)$, such that $f' \leq g$ and g is l.s.c, with:
$$\int_a^b g(t)dt \leq \int_a^b f'(t)dt + \epsilon$$

4. By considering $g + \alpha$ for some $\alpha > 0$, show that without loss of generality, we can assume that $f' < g$ with the above inequality still holding.

5. We define the complex measure $\nu = \int g dx \in M^1([a,b], \mathcal{B}([a,b]))$. Show that:

$$\forall \epsilon' > 0 \ , \ \exists \delta > 0 \ , \ \forall E \in \mathcal{B}([a,b]) \ , \ dx(E) \leq \delta \ \Rightarrow \ |\nu(E)| < \epsilon'$$

6. For all $\eta > 0$ and $x \in [a,b]$, we define:

$$F_\eta(x) \stackrel{\triangle}{=} \int_a^x g(t)dt - f(x) + f(a) + \eta(x-a)$$

Show that $F_\eta : [a,b] \to \mathbf{R}$ is a continuous map.

7. η being fixed, let $x = \sup F_\eta^{-1}(\{0\})$. Show that $x \in [a,b]$ and $F_\eta(x) = 0$.

8. We assume that $x \in [a,b[$. Show the existence of $\delta > 0$ such that for all $t \in]x, x+\delta[\cap[a,b]$, we have:

$$f'(x) < g(t) \quad \text{and} \quad \frac{f(t)-f(x)}{t-x} < f'(x) + \eta$$

9. Show that for all $t \in]x, x+\delta[\cap [a,b]$, we have $F_\eta(t) > F_\eta(x) = 0$.

10. Show that there exists t_0 such that $x < t_0 < b$ and $F_\eta(t_0) > 0$.

11. Show that if $F_\eta(b) < 0$ then x cannot be $\sup F_\eta^{-1}(\{0\})$.

12. Conclude that $F_\eta(b) \geq 0$, even if $x = b$.

13. Show that $f(b) - f(a) \leq \int_a^b f'(t)dt$, and conclude:

Theorem 99 (Fundamental Calculus) *Let $a, b \in \mathbf{R}$, $a < b$, and $f : [a,b] \to \mathbf{R}$ be a map which is differentiable at every point of $[a,b]$, and such that:*

$$\int_a^b |f'(t)|dt < +\infty$$

Then, we have:

$$f(b) - f(a) = \int_a^b f'(t)dt$$

EXERCISE 12. Let $\alpha > 0$, and $k_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ defined by $k_\alpha(x) = \alpha x$.

1. Show that $k_\alpha : (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n)) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is measurable.
2. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$, we have:
$$dx(\{k_\alpha \in B\}) = \frac{1}{\alpha^n} dx(B)$$
3. Show that for all $\epsilon > 0$ and $x \in \mathbf{R}^n$:
$$dx(B(x,\epsilon)) = \epsilon^n dx(B(0,1))$$

Definition 119 *Let μ be a complex measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, $n \geq 1$, with total variation $|\mu|$. We call* **maximal function** *of μ, the map $M\mu : \mathbf{R}^n \to [0, +\infty]$, defined by:*

$$\forall x \in \mathbf{R}^n \ , \ (M\mu)(x) \stackrel{\triangle}{=} \sup_{\epsilon > 0} \frac{|\mu|(B(x,\epsilon))}{dx(B(x,\epsilon))}$$

where $B(x,\epsilon)$ is the open ball in $\mathbf{R}^n$, of center x and radius ϵ, with respect to the usual metric of $\mathbf{R}^n$.

Exercise 13. Let μ be a complex measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

1. Let $\lambda \in \mathbf{R}$. Show that if $\lambda < 0$, then $\{\lambda < M\mu\} = \mathbf{R}^n$.

2. Show that if $\lambda = 0$, then $\{\lambda < M\mu\} = \mathbf{R}^n$ if $\mu \neq 0$, and $\{\lambda < M\mu\}$ is the empty set if $\mu = 0$.

3. Suppose $\lambda > 0$. Let $x \in \{\lambda < M\mu\}$. Show the existence of $\epsilon > 0$ such that $|\mu|(B(x,\epsilon)) = tdx(B(x,\epsilon))$, for some $t > \lambda$.

4. Show the existence of $\delta > 0$ such that $(\epsilon + \delta)^n < \epsilon^n t/\lambda$.

5. Show that if $y \in B(x,\delta)$, then $B(x,\epsilon) \subseteq B(y,\epsilon+\delta)$.

6. Show that if $y \in B(x,\delta)$, then:

$$|\mu|(B(y,\epsilon+\delta)) \geq \frac{\epsilon^n t}{(\epsilon+\delta)^n} dx(B(y,\epsilon+\delta)) > \lambda dx(B(y,\epsilon+\delta))$$

7. Conclude that $B(x,\delta) \subseteq \{\lambda < M\mu\}$, and that the maximal function $M\mu : \mathbf{R}^n \to [0,+\infty]$ is l.s.c, and therefore measurable.

Exercise 14. Let $B_i = B(x_i, \epsilon_i)$, $i = 1, \ldots, N$, $N \geq 1$, be a finite collection of open balls in $\mathbf{R}^n$. Assume without loss of generality that $\epsilon_N \leq \ldots \leq \epsilon_1$. We define a sequence (J_k) of sets by $J_0 = \{1, \ldots, N\}$ and for all $k \geq 1$:

$$J_k \stackrel{\triangle}{=} \begin{cases} J_{k-1} \cap \{j : \ j > i_k \ , \ B_j \cap B_{i_k} = \emptyset\} & \text{if } J_{k-1} \neq \emptyset \\ \emptyset & \text{if } J_{k-1} = \emptyset \end{cases}$$

where we have put $i_k = \min J_{k-1}$, whenever $J_{k-1} \neq \emptyset$.

1. Show that if $J_{k-1} \neq \emptyset$ then $J_k \subset J_{k-1}$ (strict inclusion), $k \geq 1$.

2. Let $p = \min\{k \geq 1 : \ J_k = \emptyset\}$. Show that p is well-defined.

3. Let $S = \{i_1, \ldots, i_p\}$. Explain why S is well defined.

4. Suppose that $1 \leq k < k' \leq p$. Show that $i_{k'} \in J_k$.

5. Show that $(B_i)_{i \in S}$ is a family of pairwise disjoint open balls.

6. Let $i \in \{1, \ldots, N\} \setminus S$, and define k_0 to be the minimum of the set $\{k \in \mathbf{N}_p : \ i \notin J_k\}$. Explain why k_0 is well-defined.

7. Show that $i \in J_{k_0-1}$ and $i_{k_0} \leq i$.

8. Show that $B_i \cap B_{i_{k_0}} \neq \emptyset$.

9. Show that $B_i \subseteq B(x_{i_{k_0}}, 3\epsilon_{i_{k_0}})$.

10. Conclude that there exists a subset S of $\{1, \ldots, N\}$ such that $(B_i)_{i \in S}$ is a family of pairwise disjoint balls, and:

$$\bigcup_{i=1}^{N} B(x_i, \epsilon_i) \subseteq \bigcup_{i \in S} B(x_i, 3\epsilon_i)$$

11. Show that:

$$dx\left(\bigcup_{i=1}^{N} B(x_i, \epsilon_i)\right) \leq 3^n \sum_{i \in S} dx(B(x_i, \epsilon_i))$$

EXERCISE 15. Let μ be a complex measure on $\mathbf{R}^n$. Let $\lambda > 0$ and K be a non-empty compact subset of $\{\lambda < M\mu\}$.

1. Show that K can be covered by a finite collection $B_i = B(x_i, \epsilon_i)$, $i = 1, \ldots, N$ of open balls, such that:

$$\forall i = 1, \ldots, N \ , \ \lambda dx(B_i) < |\mu|(B_i)$$

2. Show the existence of $S \subseteq \{1, \ldots, N\}$ such that:

$$dx(K) \leq 3^n \lambda^{-1} |\mu| \left(\bigcup_{i \in S} B(x_i, \epsilon_i) \right)$$

3. Show that $dx(K) \leq 3^n \lambda^{-1} \|\mu\|$

4. Conclude with the following:

Theorem 100 *Let μ be a complex measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, $n \geq 1$, with maximal function $M\mu$. Then, for all $\lambda \in \mathbf{R}^+ \setminus \{0\}$, we have:*

$$dx(\{\lambda < M\mu\}) \leq 3^n \lambda^{-1} \|\mu\|$$

Definition 120 *Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, and μ be the complex measure $\mu = \int f dx$ on $\mathbf{R}^n$, $n \geq 1$. We call* **maximal function** *of f, denoted Mf, the maximal function $M\mu$ of μ.*

Exercise 16. Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$.

1. Show that for all $x \in \mathbf{R}^n$:

$$(Mf)(x) = \sup_{\epsilon > 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f| dx$$

2. Show that for all $\lambda > 0$, $dx(\{\lambda < Mf\}) \leq 3^n \lambda^{-1} \|f\|_1$.

Definition 121 *Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$. We say that $x \in \mathbf{R}^n$ is a* **lebesgue point** *of f, if and only if we have:*

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)| dy = 0$$

Exercise 17. Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$.

1. Show that if f is continuous at $x \in \mathbf{R}^n$, then x is a Lebesgue point of f.

2. Show that if $x \in \mathbf{R}^n$ is a Lebesgue point of f, then:

$$f(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} f(y)dy$$

Exercise 18. Let $n \geq 1$ and $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$. For all $\epsilon > 0$ and $x \in \mathbf{R}^n$, we define:

$$(T_\epsilon f)(x) \stackrel{\triangle}{=} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)|dy$$

and we put, for all $x \in \mathbf{R}^n$:

$$(Tf)(x) \stackrel{\triangle}{=} \limsup_{\epsilon \downarrow\downarrow 0} (T_\epsilon f)(x) \stackrel{\triangle}{=} \inf_{\epsilon > 0} \sup_{u \in]0,\epsilon[} (T_u f)(x)$$

1. Given $\eta > 0$, show the existence of $g \in C^c_{\mathbf{C}}(\mathbf{R}^n)$ such that:
$$\|f - g\|_1 \leq \eta$$

2. Let $h = f - g$. Show that for all $\epsilon > 0$ and $x \in \mathbf{R}^n$:
$$(T_\epsilon h)(x) \leq \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |h| dx + |h(x)|$$

3. Show that $Th \leq Mh + |h|$.

4. Show that for all $\epsilon > 0$, we have $T_\epsilon f \leq T_\epsilon g + T_\epsilon h$.

5. Show that $Tf \leq Tg + Th$.

6. Using the continuity of g, show that $Tg = 0$.

7. Show that $Tf \leq Mh + |h|$.

8. Show that for all $\alpha > 0$, $\{2\alpha < Tf\} \subseteq \{\alpha < Mh\} \cup \{\alpha < |h|\}$.

9. Show that $dx(\{\alpha < |h|\}) \leq \alpha^{-1}\|h\|_1$.

10. Conclude that for all $\alpha > 0$ and $\eta > 0$, there is $N_{\alpha,\eta} \in \mathcal{B}(\mathbf{R}^n)$ such that $\{2\alpha < Tf\} \subseteq N_{\alpha,\eta}$ and $dx(N_{\alpha,\eta}) \leq \eta$.

11. Show that for all $\alpha > 0$, there exists $N_\alpha \in \mathcal{B}(\mathbf{R}^n)$ such that $\{2\alpha < Tf\} \subseteq N_\alpha$ and $dx(N_\alpha) = 0$.

12. Show there is $N \in \mathcal{B}(\mathbf{R}^n)$, $dx(N) = 0$, such that $\{Tf > 0\} \subseteq N$.

13. Conclude that $Tf = 0$, $dx-$a.s.

14. Conclude with the following:

Theorem 101 *Let $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, $n \geq 1$. Then, dx-almost surely, any $x \in \mathbf{R}^n$ is a lebesgue points of f, i.e.*

$$dx\text{-}a.s.\ ,\ \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} |f(y) - f(x)| dy = 0$$

Exercise 19. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $\Omega' \in \mathcal{F}$. We define $\mathcal{F}' = \mathcal{F}_{|\Omega'}$ and $\mu' = \mu_{|\mathcal{F}'}$. For all map $f : \Omega' \to [0, +\infty]$ (or $\mathbf{C}$), we define $\tilde{f} : \Omega \to [0, +\infty]$ (or $\mathbf{C}$), by:

$$\tilde{f}(\omega) \triangleq \begin{cases} f(\omega) & \text{if} \quad \omega \in \Omega' \\ 0 & \text{if} \quad \omega \notin \Omega' \end{cases}$$

1. Show that $\mathcal{F}' \subseteq \mathcal{F}$ and conclude that μ' is therefore a well-defined measure on $(\Omega', \mathcal{F}')$.

2. Let $A \in \mathcal{F}'$ and $1'_A$ be the characteristic function of A defined on Ω'. Let 1_A be the characteristic function of A defined on Ω. Show that $\tilde{1}'_A = 1_A$.

3. Let $f : (\Omega', \mathcal{F}') \to [0, +\infty]$ be a non-negative and measurable map. Show that $\tilde{f} : (\Omega, \mathcal{F}) \to [0, +\infty]$ is also non-negative and measurable, and that we have:

$$\int_{\Omega'} f d\mu' = \int_{\Omega} \tilde{f} d\mu$$

4. Let $f \in L^1_{\mathbf{C}}(\Omega', \mathcal{F}', \mu')$. Show that $\tilde{f} \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, \mu)$, and:

$$\int_{\Omega'} f d\mu' = \int_{\Omega} \tilde{f} d\mu$$

Definition 122 *Let $b : \mathbf{R}^+ \to \mathbf{C}$ be a right-continuous map of finite variation. We say that b is* **absolutely continuous**, *if and only if it is absolutely continuous with respect to $a(t) = t$.*

EXERCISE 20. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be right-continuous of finite variation.

1. Show that b is absolutely continuous, if and only if there is $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(t)$ such that $b(t) = \int_0^t f(s)ds$, for all $t \in \mathbf{R}^+$.

2. Show that b absolutely continuous $\Rightarrow$ b continuous with $b(0) = 0$.

Exercise 21. Let $b : \mathbf{R}^+ \to \mathbf{C}$ be an absolutely continuous map. Let $f \in L^{1,\mathrm{loc}}_{\mathbf{C}}(t)$ be such that $b = f.t$. For all $n \geq 1$, we define $f_n : \mathbf{R} \to \mathbf{C}$ by:

$$f_n(t) \stackrel{\triangle}{=} \begin{cases} f(t)1_{[0,n]}(t) & \text{if} \quad t \in \mathbf{R}^+ \\ 0 & \text{if} \quad t < 0 \end{cases}$$

1. Let $n \geq 1$. Show $f_n \in L^1_{\mathbf{C}}(\mathbf{R}, \mathcal{B}(\mathbf{R}), dx)$ and for all $t \in [0, n]$:

$$b(t) = \int_0^t f_n dx$$

2. Show the existence of $N_n \in \mathcal{B}(\mathbf{R})$ such that $dx(N_n) = 0$, and for all $t \in N_n^c$, t is a Lebesgue point of f_n.

3. Show that for all $t \in \mathbf{R}$, and $\epsilon > 0$:

$$\frac{1}{\epsilon} \int_t^{t+\epsilon} |f_n(s) - f_n(t)| ds \leq \frac{2}{dx(B(t,\epsilon))} \int_{B(t,\epsilon)} |f_n(s) - f_n(t)| ds$$

4. Show that for all $t \in N_n^c$, we have:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{1}{\epsilon} \int_t^{t+\epsilon} f_n(s)ds = f_n(t)$$

5. Show similarly that for all $t \in N_n^c$, we have:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{1}{\epsilon} \int_{t-\epsilon}^{t} f_n(s)ds = f_n(t)$$

6. Show that for all $t \in N_n^c \cap [0, n[$, $b'(t)$ exists and $b'(t) = f(t)$.[1]

7. Show the existence of $N \in \mathcal{B}(\mathbf{R}^+)$, such that $dx(N) = 0$, and:

$$\forall t \in N^c \ , \ b'(t) \text{ exists with } b'(t) = f(t)$$

8. Conclude with the following:

[1] $b'(0)$ being a r.h.s derivative only.

Theorem 102 *A map $b : \mathbf{R}^+ \to \mathbf{C}$ is absolutely continuous, if and only if there exists $f \in L^{1,loc}_{\mathbf{C}}(t)$ such that:*

$$\forall t \in \mathbf{R}^+ \ , \ b(t) = \int_0^t f(s)ds$$

in which case, b is almost surely differentiable with $b' = f$ dx-a.s.

17. Image Measure

In the following, $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$. We denote $\mathcal{M}_n(\mathbf{K})$, $n \geq 1$, the set of all $n \times n$-matrices with $\mathbf{K}$-valued entries. We recall that for all $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$, M is identified with the linear map $M : \mathbf{K}^n \to \mathbf{K}^n$ uniquely determined by:

$$\forall j = 1, \ldots, n \ , \ Me_j \stackrel{\triangle}{=} \sum_{i=1}^{n} m_{ij} e_i$$

where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbf{K}^n$, i.e. $e_i \stackrel{\triangle}{=} (0, ., \overbrace{1}^{i}, ., 0)$.

EXERCISE 1. For all $\alpha \in \mathbf{K}$, let $H_\alpha \in \mathcal{M}_n(\mathbf{K})$ be defined by:

$$H_\alpha \stackrel{\triangle}{=} \begin{pmatrix} \alpha & & & \\ & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix}$$

i.e. by $H_\alpha e_1 = \alpha e_1$, $H_\alpha e_j = e_j$, for all $j \geq 2$. For $k, l \in \{1, \ldots, n\}$, we define the matrix $\Sigma_{kl} \in \mathcal{M}_n(\mathbf{K})$ by $\Sigma_{kl} e_k = e_l$, $\Sigma_{kl} e_l = e_k$ and $\Sigma_{kl} e_j = e_j$, for all $j \in \{1, \ldots, n\} \setminus \{k, l\}$. If $n \geq 2$, we define the matrix $U \in \mathcal{M}_n(\mathbf{K})$ by:

$$U \stackrel{\triangle}{=} \left(\begin{array}{cccc} 1 & 0 & & \\ 1 & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{array} \right)$$

i.e. by $Ue_1 = e_1 + e_2$, $Ue_j = e_j$ for all $j \geq 2$. If $n = 1$, we put $U = 1$. We define $\mathcal{N}_n(\mathbf{K}) = \{H_\alpha : \ \alpha \in \mathbf{K}\} \cup \{\Sigma_{kl} : \ k, l = 1, \ldots, n\} \cup \{U\}$, and $\mathcal{M}'_n(\mathbf{K})$ to be the set of all finite products of elements of $\mathcal{N}_n(\mathbf{K})$:

$$\mathcal{M}'_n(\mathbf{K}) \stackrel{\triangle}{=} \{M \in \mathcal{M}_n(\mathbf{K}) : M = Q_1. \ldots .Q_p \, , \, p \geq 1 \, , \, Q_j \in \mathcal{N}_n(\mathbf{K}) \, , \, \forall j\}$$

We shall prove that $\mathcal{M}_n(\mathbf{K}) = \mathcal{M}'_n(\mathbf{K})$.

1. Show that if $\alpha \in \mathbf{K} \setminus \{0\}$, H_α is non-singular with $H_\alpha^{-1} = H_{1/\alpha}$

2. Show that if $k, l = 1, \ldots, n$, Σ_{kl} is non-singular with $\Sigma_{kl}^{-1} = \Sigma_{kl}$.

3. Show that U is non-singular, and that for $n \geq 2$:

$$U^{-1} = \begin{pmatrix} 1 & 0 & & \\ -1 & 1 & 0 & \\ & 0 & \ddots & \\ & & & 1 \end{pmatrix}$$

4. Let $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$. Let $R_1, \ldots, R_n$ be the rows of M:

$$M \triangleq \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Show that for all $\alpha \in \mathbf{K}$:

$$H_\alpha.M = \begin{pmatrix} \alpha R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}$$

Conclude that *multiplying M by H_α from the left, amounts to multiplying the first row of M by α.*

5. Show that *multiplying M by H_α from the right, amounts to multiplying the first column of M by α.*

6. Show that *multiplying M by Σ_{kl} from the left, amounts to swapping the rows R_l and R_k.*

7. Show that *multiplying M by Σ_{kl} from the right, amounts to swapping the columns C_l and C_k.*

8. Show that *multiplying* M *by* U^{-1} *from the left* ($n \geq 2$), *amounts to subtracting* R_1 *to* R_2, i.e.:

$$U^{-1}.\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 - R_1 \\ \vdots \\ R_n \end{pmatrix}$$

9. Show that *multiplying* M *by* U^{-1} *from the right (for* $n \geq 2$*), amounts to subtracting* C_2 *to* C_1.

10. Define $U' = \Sigma_{12}.U^{-1}.\Sigma_{12}$, $(n \geq 2)$. Show that multiplying M by U' from the right, amounts to subtracting C_1 to C_2.

11. Show that if $n = 1$, then indeed we have $\mathcal{M}_1(\mathbf{K}) = \mathcal{M}'_1(\mathbf{K})$.

Exercise 2. Further to exercise (1), we now assume that $n \geq 2$, and make the induction hypothesis that $\mathcal{M}_{n-1}(\mathbf{K}) = \mathcal{M}'_{n-1}(\mathbf{K})$.

1. Let $O_n \in \mathcal{M}_n(\mathbf{K})$ be the matrix with all entries equal to zero. Show the existence of $Q'_1, \ldots, Q'_p \in \mathcal{N}_{n-1}(\mathbf{K})$, $p \geq 1$, such that:
$$O_{n-1} = Q'_1 . \ldots . Q'_p$$
2. For $k = 1, \ldots, p$, we define $Q_k \in \mathcal{M}_n(\mathbf{K})$, by:
$$Q_k \stackrel{\triangle}{=} \left(\begin{array}{cccc} & & & 0 \\ & Q'_k & & \vdots \\ & & & 0 \\ 0 & \ldots & 0 & 1 \end{array}\right)$$

Show that $Q_k \in \mathcal{N}_n(\mathbf{K})$, and that we have:

$$\Sigma_{1n}.Q_1.\ldots.Q_p.\Sigma_{1n} = \left(\begin{array}{cccc} 1 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & O_{n-1} & \\ 0 & & & \end{array}\right)$$

3. Conclude that $O_n \in \mathcal{M}'_n(\mathbf{K})$.

4. We now consider $M = (m_{ij}) \in \mathcal{M}_n(\mathbf{K})$, $M \neq O_n$. We want to show that $M \in \mathcal{M}'_n(\mathbf{K})$. Show that for some $k, l \in \{1, \ldots, n\}$:

$$H^{-1}_{m_{kl}}.\Sigma_{1k}.M.\Sigma_{1l} = \left(\begin{array}{cccc} 1 & * & \ldots & * \\ * & & & \\ \vdots & & * & \\ * & & & \end{array}\right)$$

5. Show that if $H^{-1}_{m_{kl}}.\Sigma_{1k}.M.\Sigma_{1l} \in \mathcal{M}'_n(\mathbf{K})$, then $M \in \mathcal{M}'_n(\mathbf{K})$. Conclude that without loss of generality, in order to prove that

M lies in $\mathcal{M}'_n(\mathbf{K})$ we can assume that $m_{11} = 1$.

6. Let $i = 2, \ldots, n$. Show that if $m_{i1} \neq 0$, we have:

$$H_{m_{i1}}^{-1}.\Sigma_{2i}.U^{-1}.\Sigma_{2i}.H_{1/m_{i1}}^{-1}.M = \begin{pmatrix} 1 & * & \ldots & * \\ * & & & \\ 0 & \leftarrow i & * & \\ * & & & \end{pmatrix}$$

7. Conclude that without loss of generality, we can assume that $m_{i1} = 0$ for all $i \geq 2$, i.e. that M is of the form:

$$M = \begin{pmatrix} 1 & * & \ldots & * \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}$$

8. Show that in order to prove that $M \in \mathcal{M}'_n(\mathbf{K})$, without loss of

generality, we can assume that M is of the form:

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & M' & \\ 0 & & & \end{pmatrix}$$

9. Prove that $M \in \mathcal{M}'_n(\mathbf{K})$ and conclude with the following:

Theorem 103 *Given $n \geq 2$, any $n \times n$-matrix with values in $\mathbf{K}$ is a finite product of matrices Q of the following types:*

$$\begin{array}{ll} (i) & Qe_1 = \alpha e_1 \ , \ Qe_j = e_j \ , \ \forall j = 2, \ldots, n \ , \ (\alpha \in \mathbf{K}) \\ (ii) & Qe_l = e_k \ , \ Qe_k = e_l \ , \ Qe_j = e_j \ , \ \forall j \neq k, l \ , \ (k, l \in \mathbf{N}_n) \\ (iii) & Qe_1 = e_1 + e_2 \ , \ Qe_j = e_j \ , \ \forall j = 2, \ldots, n \end{array}$$

where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbf{K}^n$.

Definition 123 *Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are two measurable spaces. Given a measure μ (possibly complex) on $(\Omega, \mathcal{F})$, we call* **distribution** *of X under μ, or* **law** *of X under μ, or* **image measure** *of μ by X, the measure (possibly complex) denoted μ^X or $X(\mu)$ on $(\Omega', \mathcal{F}')$, defined by:*

$$\forall B \in \mathcal{F}' \ , \ \mu^X(B) \stackrel{\triangle}{=} \mu(\{X \in B\}) = \mu(X^{-1}(B))$$

Exercise 3. Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ are two measurable spaces.

1. Show that if μ is a measure on $(\Omega, \mathcal{F})$, μ^X is a well-defined measure on $(\Omega', \mathcal{F}')$.

2. Show that if μ is a complex measure on $(\Omega, \mathcal{F})$, μ^X is a well-defined complex measure on $(\Omega', \mathcal{F}')$.

3. Let $B \in \mathcal{F}'$. Show that if $(E_n)_{n\geq 1}$ is a measurable partition of B, then $(X^{-1}(E_n))_{n\geq 1}$ is a measurable partition of $X^{-1}(B)$.

4. Show that if μ is a complex measure on $(\Omega, \mathcal{F})$, then $|\mu^X| \leq |\mu|^X$.

5. Let $Y : (\Omega', \mathcal{F}') \to (\Omega'', \mathcal{F}'')$ be a measurable map, where $(\Omega'', \mathcal{F}'')$ is another measurable space. Show that for all (possibly complex) measure μ on $(\Omega, \mathcal{F})$, we have:

$$Y(X(\mu)) = (Y \circ X)(\mu) = (\mu^X)^Y = \mu^{(Y \circ X)}$$

Definition 124 *Let μ be a measure (possibly complex) on $\mathbf{R}^n$, $n \geq 1$. We say that μ is* **invariant by translation***, if and only if for all $a \in \mathbf{R}^n$, and associated translation mapping $\tau_a : \mathbf{R}^n \to \mathbf{R}^n$ defined by $\tau_a(x) = a + x$, we have $\tau_a(\mu) = \mu$.*

EXERCISE 4. Let μ be a measure (possibly complex) on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

1. Show that $\tau_a : (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n)) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is measurable.

2. Show $\tau_a(\mu)$ is therefore a well-defined (possibly complex) measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, for all $a \in \mathbf{R}^n$.

3. Show that $\tau_a(dx) = dx$ for all $a \in \mathbf{R}^n$.

4. Show the lebesgue measure on $\mathbf{R}^n$ is invariant by translation.

Exercise 5. Let $k_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ be defined by $k_\alpha(x) = \alpha x$, $\alpha > 0$.

1. Show that $k_\alpha : (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n)) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ is measurable.

2. Show that $k_\alpha(dx) = \alpha^{-n} dx$.

Exercise 6. Show the following:

Theorem 104 (Integral Projection 1) *Let $X : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$ be a measurable map, where $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ are measurable spaces. Let μ be a measure on $(\Omega, \mathcal{F})$. Then, for all $f : (\Omega', \mathcal{F}') \to [0, +\infty]$ non-negative and measurable, we have:*

$$\int_\Omega f \circ X d\mu = \int_{\Omega'} f dX(\mu)$$

Exercise 7. Show the following:

Theorem 105 (Integral Projection 2) *Let* $X:(\Omega,\mathcal{F})\to(\Omega',\mathcal{F}')$ *be a measurable map, where* $(\Omega,\mathcal{F})$, $(\Omega',\mathcal{F}')$ *are measurable spaces. Let* μ *be a measure on* $(\Omega,\mathcal{F})$. *Then, for all* $f:(\Omega',\mathcal{F}')\to(\mathbf{C},\mathcal{B}(\mathbf{C}))$ *measurable, we have the equivalence:*

$$f\circ X\in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)\ \Leftrightarrow\ f\in L^1_{\mathbf{C}}(\Omega',\mathcal{F}',X(\mu))$$

in which case, we have:

$$\int_\Omega f\circ X d\mu=\int_{\Omega'} f dX(\mu)$$

Exercise 8. Further to theorem (105), suppose μ is in fact a complex measure on $(\Omega,\mathcal{F})$. Show that:

$$\int_{\Omega'}|f|d|X(\mu)|\leq\int_\Omega|f\circ X|d|\mu| \tag{1}$$

Conclude with the following:

Theorem 106 (Integral Projection 3) *Let $X:(\Omega,\mathcal{F})\to(\Omega',\mathcal{F}')$ be a measurable map, where $(\Omega,\mathcal{F})$, $(\Omega',\mathcal{F}')$ are measurable spaces. Let μ be a complex measure on $(\Omega,\mathcal{F})$. Then, for all measurable map $f:(\Omega',\mathcal{F}')\to(\mathbf{C},\mathcal{B}(\mathbf{C}))$, we have:*

$$f\circ X\in L^1_{\mathbf{C}}(\Omega,\mathcal{F},\mu)\ \Rightarrow\ f\in L^1_{\mathbf{C}}(\Omega',\mathcal{F}',X(\mu))$$

and when the left-hand side of this implication is satisfied:

$$\int_\Omega f\circ X d\mu=\int_{\Omega'} f dX(\mu)$$

Exercise 9. Let $X:(\Omega,\mathcal{F})\to(\mathbf{R}^n,\mathcal{B}(\mathbf{R}^n))$ be a measurable map with distribution $\mu=X(P)$, where $(\Omega,\mathcal{F},P)$ is a probability space.

1. Show that X is integrable, if and only if:

$$\int_{-\infty}^{+\infty}|x|d\mu(x)<+\infty$$

2. Show that if X is integrable, then:

$$E[X] = \int_{-\infty}^{+\infty} x d\mu(x)$$

3. Show that:

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 d\mu(x)$$

Exercise 10. Let μ be a locally finite measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, which is invariant by translation. For all $a = (a_1, \ldots, a_n) \in (\mathbf{R}^+)^n$, we define $Q_a = [0, a_1[\times \ldots \times [0, a_n[$, and in particular $Q = Q_{(1,\ldots,1)} = [0, 1[^n$.

1. Show that $\mu(Q_a) < +\infty$ for all $a \in (\mathbf{R}^+)^n$, and $\mu(Q) < +\infty$.

2. Let $p = (p_1, \ldots, p_n)$ where $p_i \geq 1$ is an integer for all i's. Show:

$$Q_p = \biguplus_{\substack{k=(k_1,\ldots,k_n)\\ 0 \leq k_i < p_i}} [k_1, k_1 + 1[\times \ldots \times [k_n, k_n + 1[$$

3. Show that $\mu(Q_p) = p_1 \ldots p_n \mu(Q)$.

4. Let $q_1, \ldots, q_n \geq 1$ be n positive integers. Show that:

$$Q_p = \biguplus_{\substack{k=(k_1,\ldots,k_n)\\ 0\leq k_i<q_i}} [\frac{k_1 p_1}{q_1}, \frac{(k_1+1)p_1}{q_1}[\times \ldots \times [\frac{k_n p_n}{q_n}, \frac{(k_n+1)p_n}{q_n}[$$

5. Show that $\mu(Q_p) = q_1 \ldots q_n \mu(Q_{(p_1/q_1,\ldots,p_n/q_n)})$

6. Show that $\mu(Q_r) = r_1 \ldots r_n \mu(Q)$, for all $r \in (\mathbf{Q}^+)^n$.

7. Show that $\mu(Q_a) = a_1 \ldots a_n \mu(Q)$, for all $a \in (\mathbf{R}^+)^n$.

8. Show that $\mu(B) = \mu(Q) dx(B)$, for all $B \in \mathcal{C}$, where:

$$\mathcal{C} \stackrel{\triangle}{=} \{[a_1, b_1[\times \ldots \times [a_n, b_n[\ ,\ a_i, b_i \in \mathbf{R}\ ,\ a_i \leq b_i\ ,\ \forall i \in \mathbf{N}^n\}$$

9. Show that $B(\mathbf{R}^n) = \sigma(\mathcal{C})$.

10. Show that $\mu = \mu(Q)dx$, and conclude with the following:

Theorem 107 *Let μ be a locally finite measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$. If μ is invariant by translation, then there exists $\alpha \in \mathbf{R}^+$ such that:*

$$\mu = \alpha dx$$

Exercise 11. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection.

1. Show that T and T^{-1} are continuous.

2. Show that for all $B \subseteq \mathbf{R}^n$, the inverse image $T^{-1}(B) = \{T \in B\}$ coincides with the direct image:

$$T^{-1}(B) \stackrel{\triangle}{=} \{y : \ y = T^{-1}(x) \text{ for some } x \in B\}$$

3. Show that for all $B \subseteq \mathbf{R}^n$, the direct image $T(B)$ coincides with the inverse image $(T^{-1})^{-1}(B) = \{T^{-1} \in B\}$.

4. Let $K \subseteq \mathbf{R}^n$ be compact. Show that $\{T \in K\}$ is compact.

5. Show that $T(dx)$ is a locally finite measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

6. Let τ_a be the translation of vector $a \in \mathbf{R}^n$. Show that:
$$T \circ \tau_{T^{-1}(a)} = \tau_a \circ T$$

7. Show that $T(dx)$ is invariant by translation.

8. Show the existence of $\alpha \in \mathbf{R}^+$, such that $T(dx) = \alpha dx$. Show that such constant is unique, and denote it by $\Delta(T)$.

9. Show that $Q = T([0,1]^n) \in \mathcal{B}(\mathbf{R}^n)$ and that we have:
$$\Delta(T)dx(Q) = T(dx)(Q) = 1$$

10. Show that $\Delta(T) \neq 0$.

11. Let $T_1, T_2 : \mathbf{R}^n \to \mathbf{R}^n$ be two linear bijections. Show that:
$$(T_1 \circ T_2)(dx) = \Delta(T_1)\Delta(T_2)dx$$
and conclude that $\Delta(T_1 \circ T_2) = \Delta(T_1)\Delta(T_2)$.

Exercise 12. Let $\alpha \in \mathbf{R} \setminus \{0\}$. Let $H_\alpha : \mathbf{R}^n \to \mathbf{R}^n$ be the linear bijection uniquely defined by $H_\alpha(e_1) = \alpha e_1$, $H_\alpha(e_j) = e_j$ for $j \geq 2$.

1. Show that $H_\alpha(dx)([0,1]^n) = |\alpha|^{-1}$.
2. Conclude that $\Delta(H_\alpha) = |\det H_\alpha|^{-1}$.

Exercise 13. Let $k, l \in \mathbf{N}_n$and $\Sigma : \mathbf{R}^n \to \mathbf{R}^n$ be the linear bijection uniquely defined by $\Sigma(e_k) = e_l$, $\Sigma(e_l) = e_k$, $\Sigma(e_j) = e_j$, for $j \neq k, l$.

1. Show that $\Sigma(dx)([0,1]^n) = 1$.
2. Show that $\Sigma.\Sigma = I_n$. (Identity mapping on $\mathbf{R}^n$).
3. Show that $|\det \Sigma| = 1$.
4. Conclude that $\Delta(\Sigma) = |\det \Sigma|^{-1}$.

Exercise 14. Let $n \geq 2$ and $U : \mathbf{R}^n \to \mathbf{R}^n$ be the linear bijection uniquely defined by $U(e_1) = e_1 + e_2$ and $U(e_j) = e_j$ for $j \geq 2$. Let $Q = [0,1[^n$.

1. Show that:

$$U^{-1}(Q) = \{x \in \mathbf{R}^n : \ 0 \leq x_1 + x_2 < 1 \ , \ 0 \leq x_i < 1 \ , \ \forall i \neq 2\}$$

2. Define:

$$\begin{aligned} \Omega_1 &\stackrel{\triangle}{=} U^{-1}(Q) \cap \{x \in \mathbf{R}^n : \ x_2 \geq 0\} \\ \Omega_2 &\stackrel{\triangle}{=} U^{-1}(Q) \cap \{x \in \mathbf{R}^n : \ x_2 < 0\} \end{aligned}$$

   Show that $\Omega_1, \Omega_2 \in \mathcal{B}(\mathbf{R}^n)$.

3. Let τ_{e_2} be the translation of vector e_2. Draw a picture of Q, Ω_1, Ω_2 and $\tau_{e_2}(\Omega_2)$ in the case when $n = 2$.

4. Show that if $x \in \Omega_1$, then $0 \leq x_2 < 1$.

5. Show that $\Omega_1 \subseteq Q$.

6. Show that if $x \in \tau_{e_2}(\Omega_2)$, then $0 \leq x_2 < 1$.

7. Show that $\tau_{e_2}(\Omega_2) \subseteq Q$.

8. Show that if $x \in Q$ and $x_1 + x_2 < 1$ then $x \in \Omega_1$.

9. Show that if $x \in Q$ and $x_1 + x_2 \geq 1$ then $x \in \tau_{e_2}(\Omega_2)$.

10. Show that if $x \in \tau_{e_2}(\Omega_2)$ then $x_1 + x_2 \geq 1$.

11. Show that $\tau_{e_2}(\Omega_2) \cap \Omega_1 = \emptyset$.

12. Show that $Q = \Omega_1 \uplus \tau_{e_2}(\Omega_2)$.

13. Show that $dx(Q) = dx(U^{-1}(Q))$.

14. Show that $\Delta(U) = 1$.

15. Show that $\Delta(U) = |\det U|^{-1}$.

Exercise 15. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection, $(n \geq 1)$.

1. Show the existence of linear bijections $Q_1, \ldots, Q_p : \mathbf{R}^n \to \mathbf{R}^n$, $p \geq 1$, with $T = Q_1 \circ \ldots \circ Q_p$, $\Delta(Q_i) = |\det Q_i|^{-1}$ for all $i \in \mathbf{N}_p$.

2. Show that $\Delta(T) = |\det T|^{-1}$.

3. Conclude with the following:

Theorem 108 *Let $n \geq 1$ and $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection. Then, the image measure $T(dx)$ of the lebesgue measure on $\mathbf{R}^n$ is:*

$$T(dx) = |\det T|^{-1} dx$$

Exercise 16. Let $f : (\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2)) \to [0, +\infty]$ be a non-negative and measurable map. Let $a, b, c, d \in \mathbf{R}$ such that $ad - bc \neq 0$. Show that:

$$\int_{\mathbf{R}^2} f(ax + by, cx + dy) dx dy = |ad - bc|^{-1} \int_{\mathbf{R}^2} f(x, y) dx dy$$

Exercise 17. Let $T : \mathbf{R}^n \to \mathbf{R}^n$ be a linear bijection. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$, we have $T(B) \in \mathcal{B}(\mathbf{R}^n)$ and:

$$dx(T(B)) = |\det T| dx(B)$$

Exercise 18. Let V be a linear subspace of $\mathbf{R}^n$ and $p = \dim V$. We assume that $1 \leq p \leq n-1$. Let $u_1, \ldots, u_p$ be an orthonormal basis of V, and $u_{p+1}, \ldots, u_n$ be such that $u_1, \ldots, u_n$ is an orthonormal basis of $\mathbf{R}^n$. For $i \in \mathbf{N}_n$, Let $\phi_i : \mathbf{R}^n \to \mathbf{R}$ be defined by $\phi_i(x) = \langle u_i, x\rangle$.

1. Show that all ϕ_i's are continuous.

2. Show that $V = \bigcap_{j=p+1}^n \phi_j^{-1}(\{0\})$.

3. Show that V is a closed subset of $\mathbf{R}^n$.

4. Let $Q = (q_{ij}) \in \mathcal{M}_n(\mathbf{R})$ be the matrix uniquely defined by $Qe_j = u_j$ for all $j \in \mathbf{N}_n$, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbf{R}^n$. Show that for all $i, j \in \mathbf{N}_n$:

$$\langle u_i, u_j\rangle = \sum_{k=1}^n q_{ki}q_{kj}$$

5. Show that $Q^T.Q = Q.Q^T = I_n$ and conclude that $|\det Q| = 1$.

6. Show that $dx(\{Q \in V\}) = dx(V)$.

7. Show that $\{Q \in V\} = \text{span}(e_1, \ldots, e_p)$.[1]

8. For all $m \geq 1$, we define:

$$E_m \stackrel{\triangle}{=} \overbrace{[-m,m] \times \ldots \times [-m,m]}^{n-1} \times \{0\}$$

   Show that $dx(E_m) = 0$ for all $m \geq 1$.

9. Show that $dx(\text{span}(e_1, \ldots, e_{n-1})) = 0$.

10. Conclude with the following:

Theorem 109 *Let $n \geq 1$. Any linear subspace V of $\mathbf{R}^n$ is a closed subset of $\mathbf{R}^n$. Moreover, if $\dim V \leq n-1$, then $dx(V) = 0$.*

[1] i.e. the linear subspace of $\mathbf{R}^n$ generated by $e_1, \ldots, e_p$.

18. The Jacobian Formula

In the following, $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$.

Definition 125 *We call* $\mathbf{K}$-**normed space**, *an ordered pair* (E, N), *where* E *is a* $\mathbf{K}$*-vector space, and* $N : E \to \mathbf{R}^+$ *is a norm on* E.

See definition (89) for *vector space*, and definition (95) for *norm*.

Exercise 1. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a $\mathbf{K}$-hilbert space, and $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

1. Show that $\|\cdot\|$ is a norm on $\mathcal{H}$.
2. Show that $(\mathcal{H}, \|\cdot\|)$ is a $\mathbf{K}$-normed space.

Exercise 2. Let $(E, \|\cdot\|)$ be a $\mathbf{K}$-normed space:

1. Show that $d(x, y) = \|x - y\|$ defines a metric on E.
2. Show that for all $x, y \in E$, we have $|\,\|x\| - \|y\|\,| \leq \|x - y\|$.

Definition 126 *Let $(E, \|\cdot\|)$ be a $\mathbf{K}$-normed space, and d be the metric defined by $d(x,y) = \|x-y\|$. We call* **norm topology** *on E, denoted $\mathcal{T}_{\|\cdot\|}$, the topology on E associated with d.*

Exercise 3. Let E, F be two $\mathbf{K}$-normed spaces, and $l : E \to F$ be a linear map. Show that the following are equivalent:

(i) l is continuous (w.r. to the norm topologies)

(ii) l is continuous at $x = 0$.

(iii) $\exists K \in \mathbf{R}^+ \ , \ \forall x \in E \ , \ \|l(x)\| \leq K\|x\|$

(iv) $\sup\{\|l(x)\| : \ x \in E \ , \ \|x\| = 1\} < +\infty$

Definition 127 *Let E, F be $\mathbf{K}$-normed spaces. The $\mathbf{K}$-vector space of all* **continuous linear maps** *$l : E \to F$ is denoted $\mathcal{L}_{\mathbf{K}}(E, F)$.*

Exercise 4. Show that $\mathcal{L}_{\mathbf{K}}(E, F)$ is indeed a $\mathbf{K}$-vector space.

EXERCISE 5. Let E, F be $\mathbf{K}$-normed spaces. Given $l \in \mathcal{L}_{\mathbf{K}}(E,F)$, let:

$$\|l\| \stackrel{\triangle}{=} \sup\{\|l(x)\| : \ x \in E \ , \ \|x\| = 1\} < +\infty$$

1. Show that:

$$\|l\| = \sup\{\|l(x)\| : \ x \in E \ , \ \|x\| \leq 1\}$$

2. Show that:

$$\|l\| = \sup\left\{\frac{\|l(x)\|}{\|x\|} : \ x \in E \ , \ x \neq 0\right\}$$

3. Show that $\|l(x)\| \leq \|l\|.\|x\|$, for all $x \in E$.

4. Show that $\|l\|$ is the smallest $K \in \mathbf{R}^+$, such that:

$$\forall x \in E \ , \ \|l(x)\| \leq K\|x\|$$

5. Show that $l \to \|l\|$ is a norm on $\mathcal{L}_{\mathbf{K}}(E,F)$.

6. Show that $(\mathcal{L}_{\mathbf{K}}(E,F), \|\cdot\|)$ is a $\mathbf{K}$-normed space.

Definition 128 *Let E, F be* **R***-normed spaces and U be an open subset of E. We say that a map $\phi : U \to F$ is* **differentiable** *at some $a \in U$, if and only if there exists $l \in \mathcal{L}_{\mathbf{R}}(E, F)$ such that, for all $\epsilon > 0$, there exists $\delta > 0$, such that for all $h \in E$:*

$$\|h\| \leq \delta \ \Rightarrow \ a + h \in U \ \textit{and} \ \|\phi(a+h) - \phi(a) - l(h)\| \leq \epsilon \|h\|$$

EXERCISE 6. Let E, F be two **R**-normed spaces, and U be open in E. Let $\phi : U \to F$ be a map and $a \in U$.

1. Suppose that $\phi : U \to F$ is differentiable at $a \in U$, and that $l_1, l_2 \in \mathcal{L}_{\mathbf{R}}(E, F)$ satisfy the requirement of definition (128). Show that for all $\epsilon > 0$, there exists $\delta > 0$ such that:

$$\forall h \in E \ , \ \|h\| \leq \delta \ \Rightarrow \ \|l_1(h) - l_2(h)\| \leq \epsilon \|h\|$$

2. Conclude that $\|l_1 - l_2\| = 0$ and finally that $l_1 = l_2$.

Definition 129 *Let E, F be $\mathbf{R}$-normed spaces and U be an open subset of E. Let $\phi : U \to F$ be a map and $a \in U$. If ϕ is differentiable at a, we call* **differential** *of ϕ at a, the unique element of $\mathcal{L}_{\mathbf{R}}(E, F)$, denoted $d\phi(a)$, satisfying the requirement of definition (128). If ϕ is differentiable at all points of U, the map $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E, F)$ is also called the differential of ϕ.*

Definition 130 *Let E, F be $\mathbf{R}$-normed spaces and U be an open subset of E. A map $\phi : U \to F$ is said to be of* **class** C^1*, if and only if $d\phi(a)$ exists for all $a \in U$, and the differential $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E, F)$ is a continuous map.*

EXERCISE 7. Let E, F be two $\mathbf{R}$-normed spaces and U be open in E. Let $\phi : U \to F$ be a map, and $a \in U$.

1. Show that ϕ differentiable at $a \Rightarrow \phi$ continuous at a.

2. If ϕ is of class C^1, explain with respect to which topologies the differential $d\phi : U \to \mathcal{L}_{\mathbf{R}}(E, F)$ is said to be continuous.

3. Show that if ϕ is of class C^1, then ϕ is continuous.

4. Suppose that $E = \mathbf{R}$. Show that for all $a \in U$, ϕ is differentiable at $a \in U$, if and only if the derivative:

$$\phi'(a) \stackrel{\triangle}{=} \lim_{t \neq 0, t \to 0} \frac{\phi(a+t) - \phi(a)}{t}$$

exists in F, in which case $d\phi(a) \in \mathcal{L}_{\mathbf{R}}(\mathbf{R}, F)$ is given by:

$$\forall t \in \mathbf{R}\ ,\ d\phi(a)(t) = t.\phi'(a)$$

In particular, $\phi'(a) = d\phi(a)(1)$.

Exercise 8. Let E, F, G be three $\mathbf{R}$-normed spaces. Let U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps such that $\phi(U) \subseteq V$. We assume that ϕ is differentiable at $a \in U$, and we put:

$$l_1 \stackrel{\triangle}{=} d\phi(a) \in \mathcal{L}_{\mathbf{R}}(E, F)$$

We assume that ψ is differentiable at $\phi(a) \in V$, and we put:

$$l_2 \stackrel{\triangle}{=} d\psi(\phi(a)) \in \mathcal{L}_{\mathbf{R}}(F, G)$$

1. Explain why $\psi \circ \phi : U \to G$ is a well-defined map.

2. Given $\epsilon > 0$, show the existence of $\eta > 0$ such that:

$$\eta(\eta + \|l_1\| + \|l_2\|) \leq \epsilon$$

3. Show the existence of $\delta_2 > 0$ such that for all $h_2 \in F$ with $\|h_2\| \leq \delta_2$, we have $\phi(a) + h_2 \in V$ and:

$$\|\psi(\phi(a) + h_2) - \psi \circ \phi(a) - l_2(h_2)\| \leq \eta \|h_2\|$$

4. Show that if $h_2 \in F$ and $\|h_2\| \leq \delta_2$, then for all $h \in E$, we have:

$$\|\psi(\phi(a) + h_2) - \psi \circ \phi(a) - l_2 \circ l_1(h)\| \leq \eta \|h_2\| + \|l_2\|.\|h_2 - l_1(h)\|$$

5. Show the existence of $\delta > 0$ such that for all $h \in E$ with $\|h\| \leq \delta$, we have $a + h \in U$ and $\|\phi(a+h) - \phi(a) - l_1(h)\| \leq \eta \|h\|$, together with $\|\phi(a+h) - \phi(a)\| \leq \delta_2$.

6. Show that if $h \in E$ is such that $\|h\| \leq \delta$, then $a + h \in U$ and:

$$\begin{aligned}\|\psi \circ \phi(a+h) - \psi \circ \phi(a) - l_2 \circ l_1(h)\| &\leq \eta\|\phi(a+h) - \phi(a)\| + \eta\|l_2\|.\|h\| \\ &\leq \eta(\eta + \|l_1\| + \|l_2\|)\|h\| \\ &\leq \epsilon\|h\|\end{aligned}$$

7. Show that $l_2 \circ l_1 \in \mathcal{L}_{\mathbf{R}}(E, G)$

8. Conclude with the following:

Theorem 110 *Let E, F, G be three $\mathbf{R}$-normed spaces, U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps such that $\phi(U) \subseteq V$. Let $a \in U$. Then, if ϕ is differentiable at $a \in U$, and ψ is differentiable at $\phi(a) \in V$, then $\psi \circ \phi$ is differentiable at $a \in U$, and furthermore:*

$$d(\psi \circ \phi)(a) = d\psi(\phi(a)) \circ d\phi(a)$$

Exercise 9. Let E, F, G be three $\mathbf{R}$-normed spaces. Let U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps of class C^1 such that $\phi(U) \subseteq V$.

1. For all $(l_1, l_2) \in \mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$, we define:

$$\begin{array}{rcl} N_1(l_1, l_2) & \stackrel{\triangle}{=} & \|l_1\| + \|l_2\| \\ N_2(l_1, l_2) & \stackrel{\triangle}{=} & \sqrt{\|l_1\|^2 + \|l_2\|^2} \\ N_\infty(l_1, l_2) & \stackrel{\triangle}{=} & \max(\|l_1\|, \|l_2\|) \end{array}$$

   Show that N_1, N_2, N_∞ are all norms on $\mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$.

2. Show they induce the product topology on $\mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$.

3. We define the map $H : \mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F) \to \mathcal{L}_{\mathbf{R}}(E, G)$ by:

$$\forall (l_1, l_2) \in \mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F) \ , \ H(l_1, l_2) \stackrel{\triangle}{=} l_1 \circ l_2$$

   Show that $\|H(l_1, l_2)\| \leq \|l_1\|.\|l_2\|$, for all l_1, l_2.

4. Show that H is continuous.

5. We define $K : U \to \mathcal{L}_{\mathbf{R}}(F, G) \times \mathcal{L}_{\mathbf{R}}(E, F)$ by:

$$\forall a \in U \ , \ K(a) \stackrel{\triangle}{=} (d\psi(\phi(a)), d\phi(a))$$

   Show that K is continuous.

6. Show that $\psi \circ \phi$ is differentiable on U.

7. Show that $d(\psi \circ \phi) = H \circ K$.

8. Conclude with the following:

Theorem 111 *Let E, F, G be three $\mathbf{R}$-normed spaces, U be open in E and V be open in F. Let $\phi : U \to F$ and $\psi : V \to G$ be two maps of class C^1 such that $\phi(U) \subseteq V$. Then, $\psi \circ \phi : U \to G$ is of class C^1.*

Exercise 10. Let E be an $\mathbf{R}$-normed space. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to E$ and $g : [a, b] \to \mathbf{R}$ be two continuous maps which are differentiable at every points of $]a, b[$. We assume that:

$$\forall t \in]a, b[\ ,\ \|f'(t)\| \leq g'(t)$$

1. Given $\epsilon > 0$, we define $\phi_\epsilon : [a, b] \to \mathbf{R}$ by:

$$\phi_\epsilon(t) \stackrel{\triangle}{=} \|f(t) - f(a)\| - g(t) + g(a) - \epsilon(t - a)$$

   for all $t \in [a, b]$. Show that ϕ_ϵ is continuous.

2. Define $E_\epsilon = \{t \in [a, b] : \phi_\epsilon(t) \leq \epsilon\}$, and $c = \sup E_\epsilon$. Show that:

$$c \in [a, b] \text{ and } \phi_\epsilon(c) \leq \epsilon$$

3. Show the existence of $h > 0$, such that:

$$\forall t \in [a, a + h[\cap[a, b]\ ,\ \phi_\epsilon(t) \leq \epsilon$$

4. Show that $c \in]a, b]$.

5. Suppose that $c \in]a,b[$. Show the existence of $t_0 \in]c,b]$ such that:
$$\left\| \frac{f(t_0)-f(c)}{t_0-c} \right\| \leq \|f'(c)\| + \epsilon/2 \text{ and } g'(c) \leq \frac{g(t_0)-g(c)}{t_0-c} + \epsilon/2$$

6. Show that $\|f(t_0)-f(c)\| \leq g(t_0)-g(c)+\epsilon(t_0-c)$.

7. Show that $\|f(c)-f(a)\| \leq g(c)-g(a)+\epsilon(c-a)+\epsilon$.

8. Show that $\|f(t_0)-f(a)\| \leq g(t_0)-g(a)+\epsilon(t_0-a)+\epsilon$.

9. Show that c cannot be the supremum of E_ϵ unless $c=b$.

10. Show that $\|f(b)-f(a)\| \leq g(b)-g(a)+\epsilon(b-a)+\epsilon$.

11. Conclude with the following:

Theorem 112 *Let E be an $\mathbf{R}$-normed space. Let $a, b \in \mathbf{R}$, $a < b$. Let $f : [a, b] \to E$ and $g : [a, b] \to \mathbf{R}$ be two continuous maps which are differentiable at every point of $]a, b[$, and such that:*

$$\forall t \in]a, b[\ ,\ \|f'(t)\| \leq g'(t)$$

Then:

$$\|f(b) - f(a)\| \leq g(b) - g(a)$$

Definition 131 *Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. For all $i = 1, \ldots, n$, we say that ϕ has an ith* **partial derivative** *at $a \in U$, if and only if the limit:*

$$\frac{\partial \phi}{\partial x_i}(a) \stackrel{\triangle}{=} \lim_{h \neq 0, h \to 0} \frac{\phi(a + he_i) - \phi(a)}{h}$$

exists in E, where $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbf{R}^n$.

EXERCISE 11. Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space.

1. Suppose ϕ is differentiable at $a \in U$. Show that for all $i \in \mathbf{N}_n$:

$$\lim_{h \neq 0, h \to 0} \frac{1}{\|he_i\|} \|\phi(a + he_i) - \phi(a) - d\phi(a)(he_i)\| = 0$$

2. Show that for all $i \in \mathbf{N}_n$, $\frac{\partial \phi}{\partial x_i}(a)$ exists, and:

$$\frac{\partial \phi}{\partial x_i}(a) = d\phi(a)(e_i)$$

3. Conclude with the following:

Theorem 113 *Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. Then, if ϕ is differentiable at $a \in U$, for all $i = 1, \ldots, n$, $\frac{\partial \phi}{\partial x_i}(a)$ exists and we have:*

$$\forall h \stackrel{\triangle}{=} (h_1, \ldots, h_n) \in \mathbf{R}^n \ , \ d\phi(a)(h) = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(a) h_i$$

Exercise 12. Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space.

1. Show that if ϕ is differentiable at $a, b \in U$, then for all $i \in \mathbf{N}_n$:

$$\left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\| \leq \|d\phi(b) - d\phi(a)\|$$

2. Conclude that if ϕ is of class C^1 on U, then $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on U, for all $i \in \mathbf{N}_n$.

Exercise 13. Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. We assume that $\frac{\partial \phi}{\partial x_i}$ exists on U, and is continuous at $a \in U$, for all $i \in \mathbf{N}_n$. We define $l : \mathbf{R}^n \to E$ by:

$$\forall h \stackrel{\triangle}{=} (h_1, \ldots, h_n) \in \mathbf{R}^n \ , \ l(h) \stackrel{\triangle}{=} \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(a) h_i$$

1. Show that $l \in \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, E)$.

2. Given $\epsilon > 0$, show the existence of $\eta > 0$ such that for all $h \in \mathbf{R}^n$ with $\|h\| < \eta$, we have $a + h \in U$, and:

$$\forall i = 1, \ldots, n \ , \ \left\| \frac{\partial \phi}{\partial x_i}(a+h) - \frac{\partial \phi}{\partial x_i}(a) \right\| \leq \epsilon$$

3. Let $h = (h_1, \ldots, h_n) \in \mathbf{R}^n$ be such that $\|h\| < \eta$. $(e_1, \ldots, e_n)$ being the canonical basis of $\mathbf{R}^n$, we define $k_0 = a$ and for $i \in \mathbf{N}_n$:

$$k_i \stackrel{\triangle}{=} a + \sum_{j=1}^{i} h_i e_i$$

Show that $k_0, \ldots, k_n \in U$, and that we have:

$$\phi(a+h) - \phi(a) - l(h) = \sum_{i=1}^{n} \left(\phi(k_{i-1} + h_i e_i) - \phi(k_{i-1}) - h_i \frac{\partial \phi}{\partial x_i}(a) \right)$$

4. Let $i \in \mathbf{N}_n$. Assume that $h_i > 0$. We define $f_i : [0, h_i] \to E$ by:

$$\forall t \in [0, h_i] \ , \ f_i(t) \stackrel{\triangle}{=} \phi(k_{i-1} + te_i) - \phi(k_{i-1}) - t \frac{\partial \phi}{\partial x_i}(a)$$

Show f_i is well-defined, $f_i'(t)$ exists for all $t \in [0, h_i]$, and:

$$\forall t \in [0, h_i]\ ,\ f_i'(t) = \frac{\partial \phi}{\partial x_i}(k_{i-1} + te_i) - \frac{\partial \phi}{\partial x_i}(a)$$

5. Show f_i is continuous on $[0, h_i]$, differentiable on $]0, h_i[$, with:

$$\forall t \in]0, h_i[\ ,\ \|f_i'(t)\| \leq \epsilon$$

6. Show that:

$$\left\| \phi(k_{i-1} + h_i e_i) - \phi(k_{i-1}) - h_i \frac{\partial \phi}{\partial x_i}(a) \right\| \leq \epsilon |h_i|$$

7. Show that the previous inequality still holds if $h_i \leq 0$.

8. Conclude that for all $h \in \mathbf{R}^n$ with $\|h\| < \eta$, we have:

$$\|\phi(a + h) - \phi(a) - l(h)\| \leq \epsilon \sqrt{n} \|h\|$$

9. Prove the following:

Theorem 114 *Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. If, for all $i \in \mathbf{N}_n$ $\frac{\partial \phi}{\partial x_i}$ exists on U and is continuous at $a \in U$, then ϕ is differentiable at $a \in U$.*

EXERCISE 14. Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. We assume that for all $i \in \mathbf{N}_n$, $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on U.

1. Show that ϕ is differentiable on U.

2. Show that for all $a, b \in U$ and $h \in \mathbf{R}^n$:

$$\|(d\phi(b) - d\phi(a))(h)\| \leq \left(\sum_{i=1}^{n} \left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\|^2\right)^{1/2} \|h\|$$

3. Show that for all $a, b \in U$:

$$\|d\phi(b) - d\phi(a)\| \leq \left(\sum_{i=1}^{n} \left\| \frac{\partial \phi}{\partial x_i}(b) - \frac{\partial \phi}{\partial x_i}(a) \right\|^2\right)^{1/2}$$

4. Show that $d\phi : U \to \mathcal{L}_{\mathbf{R}}(\mathbf{R}^n, E)$ is continuous.

5. Prove the following:

Theorem 115 *Let $n \geq 1$ and U be open in $\mathbf{R}^n$. Let $\phi : U \to E$ be a map, where E is an $\mathbf{R}$-normed space. Then, ϕ is of class C^1 on U, if and only if for all $i = 1, \ldots, n$, $\frac{\partial \phi}{\partial x_i}$ exists and is continuous on U.*

Exercise 15. Let E, F be two $\mathbf{R}$-normed spaces and $l \in \mathcal{L}_{\mathbf{R}}(E, F)$. Let U be open in E and $l_{|U}$ be the restriction of l to U. Show that $l_{|U}$ is of class C^1 on U, and that we have:

$$\forall x \in U \ , \ d(l_{|U})(x) = l$$

Exercise 16. Let $E_1,\ldots,E_p$, $(p\geq 1)$, be p **R**-normed spaces. Let $E=E_1\times\ldots\times E_p$. For all $x=(x_1,\ldots,x_p)\in E$, we define:

$$\begin{aligned}
\|x\|_1 &\stackrel{\triangle}{=} \sum_{i=1}^{p}\|x_i\| \\
\|x\|_2 &\stackrel{\triangle}{=} \sqrt{\sum_{i=1}^{p}\|x_i\|^2} \\
\|x\|_\infty &\stackrel{\triangle}{=} \max_{i=1,\ldots,p}\|x_i\|
\end{aligned}$$

1. Show that $\|.\|_1, \|.\|_2$ and $\|.\|_\infty$ are all norms on E.
2. Show $\|.\|_1, \|.\|_2$ and $\|.\|_\infty$ induce the product topology on E.
3. Conclude that E is also an **R**-normed space, and that the norm topology on E is exactly the product topology on E.

Exercise 17. Let $E, F_1, \ldots, F_p$, $(p \geq 1)$ be $p+1$ $\mathbf{R}$-normed spaces, U be open in E, $F = F_1 \times \ldots \times F_p$ and $\phi : U \to F$ be a map.

1. For $i = 1, \ldots, p$, let $p_i : F \to F_i$ be the canonical projection. Show that $p_i \in \mathcal{L}_{\mathbf{R}}(F, F_i)$. We put $\phi_i = p_i \circ \phi$.

2. For $i = 1, \ldots, p$, let $u_i : F_i \to F$ be defined by:

$$\forall x_i \in F_i \;,\; u_i(x_i) \stackrel{\triangle}{=} (0, \ldots, \overbrace{x_i}^{i}, \ldots, 0)$$

   Show that $u_i \in \mathcal{L}_{\mathbf{R}}(F_i, F)$ and $\phi = \sum_{i=1}^{p} u_i \circ \phi_i$.

3. Show that if ϕ is differentiable at $a \in U$, then for all $i = 1, \ldots, p$, $\phi_i : U \to F_i$ is differentiable at $a \in U$ and $d\phi_i(a) = p_i \circ d\phi(a)$.

4. Show that if for all $i = 1, \ldots, p$, ϕ_i is differentiable at $a \in U$, then ϕ is differentiable at $a \in U$ and:

$$d\phi(a) = \sum_{i=1}^{p} u_i \circ d\phi_i(a)$$

5. Suppose that ϕ is differentiable at $a, b \in U$. We assume that F is given the norm $\|(x_1, \ldots, x_p)\|_2 = \sqrt{\sum_{i=1}^p \|x_i\|^2}$. Show that for all $i \in \mathbf{N}_p$:

$$\|d\phi_i(b) - d\phi_i(a)\| \leq \|d\phi(b) - d\phi(a)\|$$

6. Show that:

$$\|d\phi(b) - d\phi(a)\| \leq \sqrt{\sum_{i=1}^p \|d\phi_i(b) - d\phi_i(a)\|^2}$$

7. Show that ϕ is of class $C^1 \Leftrightarrow \phi_i$ is of class C^1 for all $i \in \mathbf{N}_p$.

8. Explain why this conclusion would still hold, if F were given the norm $\|.\|_1$ or $\|.\|_\infty$ instead of $\|.\|_2$.

9. Conclude with theorem (116)

Theorem 116 *Let* $E, F_1, \ldots, F_p$, $(p \geq 1)$, *be* $p+1$ $\mathbf{R}$*-normed spaces and* U *be open in* E. *Let* F *be the* $\mathbf{R}$*-normed space* $F = F_1 \times \ldots \times F_p$ *and* $\phi = (\phi_1, \ldots, \phi_p) : U \to F$ *be a map. Then,* ϕ *is differentiable at* $a \in U$, *if and only if* $d\phi_i(a)$ *exists for all* $i \in \mathbf{N}_p$, *in which case:*

$$\forall h \in E \ , \ d\phi(a)(h) = (d\phi_1(a)(h), \ldots, d\phi_p(a)(h))$$

Also, ϕ *is of class* C^1 *on* $U \Leftrightarrow \phi_i$ *is of class* C^1 *on* U, *for all* $i \in \mathbf{N}_p$.

Theorem 117 *Let* $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbf{R}^n$ *be a map, where* $n \geq 1$ *and* U *is open in* $\mathbf{R}^n$. *We assume that* ϕ *is differentiable at* $a \in U$. *Then, for all* $i, j = 1, \ldots, n$, $\frac{\partial \phi_i}{\partial x_j}(a)$ *exists, and we have:*

$$d\phi(a) = \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1}(a) & \ldots & \frac{\partial \phi_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial \phi_n}{\partial x_1}(a) & \ldots & \frac{\partial \phi_n}{\partial x_n}(a) \end{pmatrix}$$

Moreover, ϕ *is of class* C^1 *on* U, *if and only if for all* $i, j = 1, \ldots, n$, $\frac{\partial \phi_i}{\partial x_j}$ *exists and is continuous on* U.

Exercise 18. Prove theorem (117)

Definition 132 *Let* $\phi = (\phi_1, \ldots, \phi_n) : U \to \mathbf{R}^n$ *be a map, where* $n \geq 1$ *and* U *is open in* $\mathbf{R}^n$*. We assume that* ϕ *is differentiable at* $a \in U$*. We call* **jacobian** *of* ϕ *at* a*, denoted* $J(\phi)(a)$*, the determinant of the differential* $d\phi(a)$ *at* a*, i.e.*

$$J(\phi)(a) = \det \begin{pmatrix} \frac{\partial \phi_1}{\partial x_1}(a) & \ldots & \frac{\partial \phi_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial \phi_n}{\partial x_1}(a) & \ldots & \frac{\partial \phi_n}{\partial x_n}(a) \end{pmatrix}$$

Definition 133 *Let* $n \geq 1$ *and* Ω*,* Ω' *be open in* $\mathbf{R}^n$*. A bijection* $\phi : \Omega \to \Omega'$ *is called a* C^1**-diffeomorphism** *between* Ω *and* Ω'*, if and only if* $\phi : \Omega \to \mathbf{R}^n$ *and* $\phi^{-1} : \Omega' \to \mathbf{R}^n$ *are both of class* C^1*.*

EXERCISE 19. Let Ω and Ω' be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism, $\psi = \phi^{-1}$, and I_n be the identity mapping of $\mathbf{R}^n$.

1. Explain why $J(\psi) : \Omega' \to \mathbf{R}$ and $J(\phi) : \Omega \to \mathbf{R}$ are continuous.
2. Show that $d\phi(\psi(x)) \circ d\psi(x) = I_n$, for all $x \in \Omega'$.
3. Show that $d\psi(\phi(x)) \circ d\phi(x) = I_n$, for all $x \in \Omega$.
4. Show that $J(\psi)(x) \neq 0$ for all $x \in \Omega'$.
5. Show that $J(\phi)(x) \neq 0$ for all $x \in \Omega$.
6. Show that $J(\psi) = 1/(J(\phi) \circ \psi)$ and $J(\phi) = 1/(J(\psi) \circ \phi)$.

Definition 134 *Let $n \geq 1$ and $\Omega \in \mathcal{B}(\mathbf{R}^n)$, be a borel set in $\mathbf{R}^n$. We define the* **lebesgue measure** *on Ω, denoted $dx_{|\Omega}$, as the restriction to $\mathcal{B}(\Omega)$ of the lebesgue measure on $\mathbf{R}^n$, i.e the measure on $(\Omega, \mathcal{B}(\Omega))$ defined by:*

$$\forall B \in \mathcal{B}(\Omega)\ ,\ dx_{|\Omega}(B) \stackrel{\triangle}{=} dx(B)$$

Exercise 20. Show that $dx_{|\Omega}$ is a well-defined measure on $(\Omega, \mathcal{B}(\Omega))$.

Exercise 21. Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$. Let $a \in \Omega'$. We assume that $d\psi(a) = I_n$, (identity mapping on $\mathbf{R}^n$), and given $\epsilon > 0$, we denote:

$$B(a, \epsilon) \stackrel{\triangle}{=} \{x \in \mathbf{R}^n :\ \|a - x\| < \epsilon\}$$

where $\|.\|$ is the usual norm in $\mathbf{R}^n$.

1. Why are $dx_{|\Omega'}$, $\phi(dx_{|\Omega})$ well-defined measures on $(\Omega', \mathcal{B}(\Omega'))$.

2. Show that for $\epsilon > 0$ sufficiently small, $B(a, \epsilon) \in \mathcal{B}(\Omega')$.

3. Show that it makes sense to investigate whether the limit:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a, \epsilon))}{dx_{|\Omega'}(B(a, \epsilon))}$$

   does exists in $\mathbf{R}$.

4. Given $r > 0$, show the existence of $\epsilon_1 > 0$ such that for all $h \in \mathbf{R}^n$ with $\|h\| \leq \epsilon_1$, we have $a + h \in \Omega'$, and:

$$\|\psi(a+h) - \psi(a) - h\| \leq r\|h\|$$

5. Show for all $h \in \mathbf{R}^n$ with $\|h\| \leq \epsilon_1$, we have $a + h \in \Omega'$, and:

$$\|\psi(a+h) - \psi(a)\| \leq (1+r)\|h\|$$

6. Show that for all $\epsilon \in]0, \epsilon_1[$, we have $B(a, \epsilon) \subseteq \Omega'$, and:

$$\psi(B(a,\epsilon)) \subseteq B(\psi(a), \epsilon(1+r))$$

7. Show that $d\phi(\psi(a)) = I_n$.

8. Show the existence of $\epsilon_2 > 0$ such that for all $k \in \mathbf{R}^n$ with $\|k\| \leq \epsilon_2$, we have $\psi(a) + k \in \Omega$, and:

$$\|\phi(\psi(a)+k) - a - k\| \leq r\|k\|$$

9. Show for all $k \in \mathbf{R}^n$ with $\|k\| \leq \epsilon_2$, we have $\psi(a) + k \in \Omega$, and:

$$\|\phi(\psi(a) + k) - a\| \leq (1 + r)\|k\|$$

10. Show for all $\epsilon \in]0, \epsilon_2(1 + r)[$, we have $B(\psi(a), \frac{\epsilon}{1+r}) \subseteq \Omega$, and:

$$B(\psi(a), \frac{\epsilon}{1 + r}) \subseteq \{\phi \in B(a, \epsilon)\}$$

11. Show that if $B(a, \epsilon) \subseteq \Omega'$, then $\psi(B(a, \epsilon)) = \{\phi \in B(a, \epsilon)\}$.

12. Show if $0 < \epsilon < \epsilon_0 = \epsilon_1 \wedge \epsilon_2(1 + r)$, then $B(a, \epsilon) \subseteq \Omega'$, and:

$$B(\psi(a), \frac{\epsilon}{1 + r}) \subseteq \{\phi \in B(a, \epsilon)\} \subseteq B(\psi(a), \epsilon(1 + r))$$

13. Show that for all $\epsilon \in]0, \epsilon_0[$:

$$\begin{array}{ll} (i) & dx(B(\psi(a), \frac{\epsilon}{1 + r})) = (1 + r)^{-n} dx_{|\Omega'}(B(a, \epsilon)) \\ (ii) & dx(B(\psi(a), \epsilon(1 + r))) = (1 + r)^n dx_{|\Omega'}(B(a, \epsilon)) \\ (iii) & dx(\{\phi \in B(a, \epsilon)\}) = \phi(dx_{|\Omega})(B(a, \epsilon)) \end{array}$$

14. Show that for all $\epsilon \in]0, \epsilon_0[$, $B(a, \epsilon) \subseteq \Omega'$, and:

$$(1+r)^{-n} \leq \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} \leq (1+r)^n$$

15. Conclude that:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = 1$$

Exercise 22. Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$. Let $a \in \Omega'$. We put $A = d\psi(a)$.

1. Show that $A : \mathbf{R}^n \to \mathbf{R}^n$ is a linear bijection.
2. Define $\Omega'' = A^{-1}(\Omega)$. Show that this definition does not depend on whether $A^{-1}(\Omega)$ is viewed as inverse , or direct image.
3. Show that Ω'' is an open subset of $\mathbf{R}^n$.

4. We define $\tilde{\phi} : \Omega'' \to \Omega'$ by $\tilde{\phi}(x) = \phi \circ A(x)$. Show that $\tilde{\phi}$ is a C^1-diffeomorphism with $\tilde{\psi} = \tilde{\phi}^{-1} = A^{-1} \circ \psi$.

5. Show that $d\tilde{\psi}(a) = I_n$.

6. Show that:
$$\lim_{\epsilon \downarrow\downarrow 0} \frac{\tilde{\phi}(dx_{|\Omega''})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = 1$$

7. Let $\epsilon > 0$ with $B(a,\epsilon) \subseteq \Omega'$. Justify each of the following steps:
$$\begin{aligned}
\tilde{\phi}(dx_{|\Omega''})(B(a,\epsilon)) &= dx_{|\Omega''}(\{\tilde{\phi} \in B(a,\epsilon)\}) && (1)\\
&= dx(\{\tilde{\phi} \in B(a,\epsilon)\}) && (2)\\
&= dx(\{x \in \Omega'' :\ \phi \circ A(x) \in B(a,\epsilon)\}) && (3)\\
&= dx(\{x \in \Omega'' : A(x) \in \phi^{-1}(B(a,\epsilon))\}) && (4)\\
&= dx(\{x \in \mathbf{R}^n : A(x) \in \phi^{-1}(B(a,\epsilon))\}) && (5)\\
&= A(dx)(\{\phi \in B(a,\epsilon)\}) && (6)\\
&= |\det A|^{-1} dx(\{\phi \in B(a,\epsilon)\}) && (7)
\end{aligned}$$

$$= |\det A|^{-1} dx_{|\Omega}(\{\phi \in B(a,\epsilon)\}) \tag{8}$$

$$= |\det A|^{-1} \phi(dx_{|\Omega})(B(a,\epsilon)) \tag{9}$$

8. Show that:

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = |\det A|$$

9. Conclude with the following:

Theorem 118 *Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$. Then, for all $a \in \Omega'$, we have:*

$$\lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(a,\epsilon))}{dx_{|\Omega'}(B(a,\epsilon))} = |J(\psi)(a)|$$

where $J(\psi)(a)$ is the jacobian of ψ at a, $B(a,\epsilon)$ is the open ball in $\mathbf{R}^n$, and $dx_{|\Omega}$, $dx_{|\Omega'}$ are the lebesgue measures on Ω and Ω' respectively.

Exercise 23. Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$.

1. Let $K \subseteq \Omega'$ be a compact subset of Ω' such that $dx_{|\Omega'}(K) = 0$. Given $\epsilon > 0$, show the existence of V open in Ω', such that $K \subseteq V \subseteq \Omega'$, and $dx_{|\Omega'}(V) \leq \epsilon$.

2. Explain why V is also open in $\mathbf{R}^n$.

3. Show that $M = \sup_{x \in K} \|d\psi(x)\| < +\infty$.

4. For all $x \in K$, show there is $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subseteq V$, and for all $h \in \mathbf{R}^n$ with $\|h\| \leq 3\epsilon_x$, we have $x + h \in \Omega'$, and:
$$\|\psi(x+h) - \psi(x)\| \leq (M+1)\|h\|$$

5. Show that for all $x \in K$, $B(x, 3\epsilon_x) \subseteq \Omega'$, and:
$$\psi(B(x, 3\epsilon_x)) \subseteq B(\psi(x), 3(M+1)\epsilon_x)$$

6. Show that $\psi(B(x, 3\epsilon_x)) = \{\phi \in B(x, 3\epsilon_x)\}$, for all $x \in K$.

7. Show the existence of $\{x_1, \ldots, x_p\} \subseteq K$, $(p \geq 0)$, such that:
$$K \subseteq B(x_1, \epsilon_{x_1}) \cup \ldots \cup B(x_p, \epsilon_{x_p})$$
8. Show the existence of $S \subseteq \{1, \ldots, p\}$ such that the $B(x_i, \epsilon_{x_i})$'s are pairwise disjoint for $i \in S$, and:
$$K \subseteq \bigcup_{i \in S} B(x_i, 3\epsilon_{x_i})$$
9. Show that $\{\phi \in K\} \subseteq \cup_{i \in S} B(\psi(x_i), 3(M+1)\epsilon_{x_i})$.
10. Show that $\phi(dx_{|\Omega})(K) \leq \sum_{i \in S} 3^n (M+1)^n dx(B(x_i, \epsilon_{x_i}))$.
11. Show that $\phi(dx_{|\Omega})(K) \leq 3^n (M+1)^n dx(V)$.
12. Show that $\phi(dx_{|\Omega})(K) \leq 3^n (M+1)^n \epsilon$.
13. Conclude that $\phi(dx_{|\Omega})(K) = 0$.
14. Show that $\phi(dx_{|\Omega})$ is a locally finite measure on $(\Omega', \mathcal{B}(\Omega'))$.

15. Let $B \in \mathcal{B}(\Omega')$ be such that $dx_{|\Omega'}(B) = 0$. Show that:

$$\phi(dx_{|\Omega})(B) = \sup\{\phi(dx_{|\Omega})(K) :\ K \subseteq B\ ,\ K \text{ compact }\}$$

16. Show that $\phi(dx_{|\Omega})(B) = 0$.

17. Conclude with the following:

Theorem 119 *Let $n \geq 1$, $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$, and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism. Then, the image measure $\phi(dx_{|\Omega})$, by ϕ of the lebesgue measure on Ω, is absolutely continuous with respect to $dx_{|\Omega'}$, the lebesgue measure on Ω', i.e.:*

$$\phi(dx_{|\Omega}) << dx_{|\Omega'}$$

Exercise 24. Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$.

1. Explain why there exists a sequence $(V_p)_{p\geq 1}$ of open sets in Ω', such that $V_p \uparrow \Omega'$ and for all $p \geq 1$, the closure of V_p in Ω', i.e. $\bar{V}_p^{\Omega'}$, is compact.

2. Show that each V_p is also open in $\mathbf{R}^n$, and that $\bar{V}_p^{\Omega'} = \bar{V}_p$.

3. Show that $\phi(dx_{|\Omega})(V_p) < +\infty$, for all $p \geq 1$.

4. Show that $dx_{|\Omega'}$ and $\phi(dx_{|\Omega})$ are two σ-finite measures on Ω'.

5. Show there is $h : (\Omega', \mathcal{B}(\Omega')) \to (\mathbf{R}^+, \mathcal{B}(\mathbf{R}^+))$ measurable, with:
$$\forall B \in \mathcal{B}(\Omega')\ ,\ \phi(dx_{|\Omega})(B) = \int_B h dx_{|\Omega'}$$

6. For all $p \geq 1$, we define $h_p = h1_{V_p}$, and we put:
$$\forall x \in \mathbf{R}^n\ ,\ \tilde{h}_p(x) \stackrel{\triangle}{=} \left\{ \begin{array}{lll} h_p(x) & \text{if} & x \in \Omega' \\ 0 & \text{if} & x \notin \Omega' \end{array} \right.$$

Show that:

$$\int_{\mathbf{R}^n} \tilde{h}_p dx = \int_{\Omega'} h_p dx_{|\Omega'} = \phi(dx_{|\Omega})(V_p) < +\infty$$

and conclude that $\tilde{h}_p \in L^1_{\mathbf{R}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$.

7. Show the existence of some $N \in \mathcal{B}(\mathbf{R}^n)$, such that $dx(N) = 0$ and for all $x \in N^c$ and $p \geq 1$, we have:

$$\tilde{h}_p(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{1}{dx(B(x,\epsilon))} \int_{B(x,\epsilon)} \tilde{h}_p dx$$

8. Put $N' = N \cap \Omega'$. Show that $N' \in \mathcal{B}(\Omega')$ and $dx_{|\Omega'}(N') = 0$.

9. Let $x \in \Omega'$ and $p \geq 1$ be such that $x \in V_p$. Show that if $\epsilon > 0$ is such that $B(x,\epsilon) \subseteq V_p$, then $dx(B(x,\epsilon)) = dx_{|\Omega'}(B(x,\epsilon))$, and:

$$\int_{B(x,\epsilon)} \tilde{h}_p dx = \int_{\mathbf{R}^n} 1_{B(x,\epsilon)} \tilde{h}_p dx = \int_{\Omega'} 1_{B(x,\epsilon)} h_p dx_{|\Omega'}$$

10. Show that:

$$\int_{\Omega'} 1_{B(x,\epsilon)} h_p dx_{|\Omega'} = \int_{\Omega'} 1_{B(x,\epsilon)} h dx_{|\Omega'} = \phi(dx_{|\Omega})(B(x,\epsilon))$$

11. Show that for all $x \in \Omega' \setminus N'$, we have:

$$h(x) = \lim_{\epsilon \downarrow\downarrow 0} \frac{\phi(dx_{|\Omega})(B(x,\epsilon))}{dx_{|\Omega'}(B(x,\epsilon))}$$

12. Show that $h = |J(\psi)|\ dx_{|\Omega'}$-a.s. and conclude with the following:

Theorem 120 *Let $n \geq 1$ and $\Omega,\ \Omega'$ be open in $\mathbf{R}^n$. Let $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism and $\psi = \phi^{-1}$. Then, the image measure by ϕ of the lebesgue measure on Ω, is equal to the measure on $(\Omega', \mathcal{B}(\Omega'))$ with density $|J(\psi)|$ with respect to the lebesgue measure on Ω', i.e.:*

$$\phi(dx_{|\Omega}) = \int |J(\psi)| dx_{|\Omega'}$$

Exercise 25. Prove the following:

Theorem 121 (Jacobian Formula 1) *Let $n \geq 1$ and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism where Ω, Ω' are open in $\mathbf{R}^n$. Let $\psi = \phi^{-1}$. Then, for all $f : (\Omega', \mathcal{B}(\Omega')) \to [0, +\infty]$ non-negative and measurable:*

$$\int_{\Omega} f \circ \phi \, dx_{|\Omega} = \int_{\Omega'} f|J(\psi)|dx_{|\Omega'}$$

and:

$$\int_{\Omega} (f \circ \phi)|J(\phi)|dx_{|\Omega} = \int_{\Omega'} f dx_{|\Omega'}$$

Exercise 26. Prove the following:

Theorem 122 (Jacobian Formula 2) *Let $n \geq 1$ and $\phi : \Omega \to \Omega'$ be a C^1-diffeomorphism where Ω, Ω' are open in $\mathbf{R}^n$. Let $\psi = \phi^{-1}$. Then, for all measurable map $f : (\Omega', \mathcal{B}(\Omega')) \to (\mathbf{C}, \mathcal{B}(\mathbf{C}))$, we have the equivalence:*

$$f \circ \phi \in L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), dx_{|\Omega}) \;\Leftrightarrow\; f|J(\psi)| \in L^1_{\mathbf{C}}(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'})$$

in which case:

$$\int_\Omega f \circ \phi \, dx_{|\Omega} = \int_{\Omega'} f|J(\psi)| dx_{|\Omega'}$$

and, furthermore:

$$(f \circ \phi)|J(\phi)| \in L^1_{\mathbf{C}}(\Omega, \mathcal{B}(\Omega), dx_{|\Omega}) \;\Leftrightarrow\; f \in L^1_{\mathbf{C}}(\Omega', \mathcal{B}(\Omega'), dx_{|\Omega'})$$

in which case:

$$\int_\Omega (f \circ \phi)|J(\phi)| dx_{|\Omega} = \int_{\Omega'} f dx_{|\Omega'}$$

EXERCISE 27. Let $f:\mathbf{R}^2 \to [0,+\infty]$, with $f(x,y) = \exp(-(x^2+y^2)/2)$.

1. Show that:

$$\int_{\mathbf{R}^2} f(x,y)dxdy = \left(\int_{-\infty}^{+\infty} e^{-u^2/2}du\right)^2$$

2. Define:

$$\begin{aligned}\Delta_1 &\stackrel{\triangle}{=} \{(x,y) \in \mathbf{R}^2 : \ x > 0 \ , \ y > 0\} \\ \Delta_2 &\stackrel{\triangle}{=} \{(x,y) \in \mathbf{R}^2 : \ x < 0 \ , \ y > 0\}\end{aligned}$$

and let Δ_3 and Δ_4 be the other two open quarters of $\mathbf{R}^2$. Show:

$$\int_{\mathbf{R}^2} f(x,y)dxdy = \int_{\Delta_1 \cup \ldots \cup \Delta_4} f(x,y)dxdy$$

3. Let $Q : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $Q(x,y) = (-x,y)$. Show that:

$$\int_{\Delta_1} f(x,y)dxdy = \int_{\Delta_2} f \circ Q^{-1}(x,y)dxdy$$

4. Show that:

$$\int_{\mathbf{R}^2} f(x,y)dxdy = 4\int_{\Delta_1} f(x,y)dxdy$$

5. Let $D_1 =]0,+\infty[\times]0,\pi/2[\subseteq \mathbf{R}^2$, and define $\phi : D_1 \to \Delta_1$ by:

$$\forall (r,\theta) \in D_1\ ,\ \phi(r,\theta) \stackrel{\triangle}{=} (r\cos\theta, r\sin\theta)$$

Show that ϕ is a bijection and that $\psi = \phi^{-1}$ is given by:

$$\forall (x,y) \in \Delta_1\ ,\ \psi(x,y) = (\sqrt{x^2+y^2}, \arctan(y/x))$$

6. Show that ϕ is a C^1-diffeomorphism, with:

$$\forall (r,\theta) \in D_1\ ,\ d\phi(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

and:

$$\forall (x,y) \in \Delta_1\ ,\ d\psi(x,y) = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

7. Show that $J(\phi)(r,\theta) = r$, for all $(r,\theta) \in D_1$.

8. Show that $J(\psi)(x,y) = 1/(\sqrt{x^2+y^2})$, for all $(x,y) \in \Delta_1$.

9. Show that:
$$\int_{\Delta_1} f(x,y)dxdy = \frac{\pi}{2}$$

10. Prove the following:

Theorem 123 *We have:*
$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-u^2/2} du = 1$$

19. Fourier Transform

Exercise 1. We define the maps $\psi : \mathbf{R}^2 \to \mathbf{C}$ and $\phi : \mathbf{R} \to \mathbf{C}$:

$$\forall (u,x) \in \mathbf{R}^2 \ , \ \psi(u,x) \stackrel{\triangle}{=} e^{iux - x^2/2}$$

$$\forall u \in \mathbf{R} \ , \ \phi(u) \stackrel{\triangle}{=} \int_{-\infty}^{+\infty} \psi(u,x)dx$$

1. Show that for all $u \in \mathbf{R}$, the map $x \to \psi(u,x)$ is measurable.
2. Show that for all $u \in \mathbf{R}$, we have:

   $$\int_{-\infty}^{+\infty} |\psi(u,x)|dx = \sqrt{2\pi} < +\infty$$

   and conclude that ϕ is well defined.
3. Let $u \in \mathbf{R}$ and $(u_n)_{n\geq 1}$ be a sequence in $\mathbf{R}$ converging to u. Show that $\phi(u_n) \to \phi(u)$ and conclude that ϕ is continuous.

4. Show that:
$$\int_0^{+\infty} xe^{-x^2/2}dx = 1$$

5. Show that for all $u \in \mathbf{R}$, we have:
$$\int_{-\infty}^{+\infty} \left|\frac{\partial\psi}{\partial u}(u,x)\right| dx = 2 < +\infty$$

6. Let $a, b \in \mathbf{R},\ a < b$. Show that:
$$e^{ib} - e^{ia} = \int_a^b ie^{ix}dx$$

7. Let $a, b \in \mathbf{R},\ a < b$. Show that:
$$|e^{ib} - e^{ia}| \leq |b - a|$$

8. Let $a, b \in \mathbf{R},\ a \neq b$. Show that for all $x \in \mathbf{R}$:
$$\left|\frac{\psi(b,x) - \psi(a,x)}{b - a}\right| \leq |x|e^{-x^2/2}$$

9. Let $u \in \mathbf{R}$ and $(u_n)_{n\geq 1}$ be a sequence in $\mathbf{R}$ converging to u, with $u_n \neq u$ for all n. Show that:

$$\lim_{n\to+\infty} \frac{\phi(u_n) - \phi(u)}{u_n - u} = \int_{-\infty}^{+\infty} \frac{\partial \psi}{\partial u}(u,x)dx$$

10. Show that ϕ is differentiable with:

$$\forall u \in \mathbf{R}\ ,\ \phi'(u) = \int_{-\infty}^{+\infty} \frac{\partial \psi}{\partial u}(u,x)dx$$

11. Show that ϕ is of class C^1.

12. Show that for all $(u,x) \in \mathbf{R}^2$, we have:

$$\frac{\partial \psi}{\partial u}(u,x) = -u\psi(u,x) - i\frac{\partial \psi}{\partial x}(u,x)$$

13. Show that for all $u \in \mathbf{R}$:

$$\int_{-\infty}^{+\infty} \left|\frac{\partial \psi}{\partial x}(u,x)\right| dx < +\infty$$

14. Let $a, b \in \mathbf{R}$, $a < b$. Show that for all $u \in \mathbf{R}$:

$$\psi(u,b) - \psi(u,a) = \int_a^b \frac{\partial \psi}{\partial x}(u,x)dx$$

15. Show that for all $u \in \mathbf{R}$:

$$\int_{-\infty}^{+\infty} \frac{\partial \psi}{\partial x}(u,x)dx = 0$$

16. Show that for all $u \in \mathbf{R}$:

$$\phi'(u) = -u\phi(u)$$

Exercise 2. Let $\mathcal{S}$ be the set of functions defined by:

$$\mathcal{S} \stackrel{\triangle}{=} \{h : \ h \in C^1(\mathbf{R}, \mathbf{R}) \ , \ \forall u \in \mathbf{R} \ , \ h'(u) = -uh(u)\}$$

1. Let ϕ be as in ex. (1). Show that $Re(\phi)$ and $Im(\phi)$ lie in $\mathcal{S}$.

2. Given $h \in \mathcal{S}$, we define $g : \mathbf{R} \to \mathbf{R}$, by:

$$\forall u \in \mathbf{R}\ ,\ g(u) \stackrel{\triangle}{=} h(u)e^{u^2/2}$$

Show that g is of class C^1 with $g' = 0$.

3. Let $a, b \in \mathbf{R},\ a < b$. Show the existence of $c \in]a, b[$, such that:

$$g(b) - g(a) = g'(c)(b - a)$$

4. Conclude that for all $h \in \mathcal{S}$, we have:

$$\forall u \in \mathbf{R}\ ,\ h(u) = h(0)e^{-u^2/2}$$

5. Prove the following:

Theorem 124 *For all $u \in \mathbf{R}$, we have:*

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iux - x^2/2} dx = e^{-u^2/2}$$

Definition 135 *Let $\mu_1, \ldots, \mu_p$ be complex measures on $\mathbf{R}^n$,*[1] *where $n, p \geq 1$. We call* **convolution** *of $\mu_1, \ldots, \mu_p$, denoted $\mu_1 \star \ldots \star \mu_p$, the image measure of the product measure $\mu_1 \otimes \ldots \otimes \mu_p$ by the measurable map $S : (\mathbf{R}^n)^p \to \mathbf{R}^n$ defined by:*

$$S(x_1, \ldots, x_p) \stackrel{\triangle}{=} x_1 + \ldots + x_p$$

In other words, $\mu_1 \star \ldots \star \mu_p$ is the complex measure on $\mathbf{R}^n$, defined by:

$$\mu_1 \star \ldots \star \mu_p \stackrel{\triangle}{=} S(\mu_1 \otimes \ldots \otimes \mu_p)$$

Exercise 3. Let μ, ν be complex measures on $\mathbf{R}^n$.

1. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:

$$\mu \star \nu(B) = \int_{\mathbf{R}^n \times \mathbf{R}^n} 1_B(x+y) d\mu \otimes \nu(x, y)$$

[1] An obvious shortcut to saying 'complex measures on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$'.

2. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:

$$\mu \star \nu(B) = \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} 1_B(x+y) d\mu(x) \right) d\nu(y)$$

3. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:

$$\mu \star \nu(B) = \int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^n} 1_B(x+y) d\nu(x) \right) d\mu(y)$$

4. Show that $\mu \star \nu = \nu \star \mu$.

5. Let $f : \mathbf{R}^n \to \mathbf{C}$ be bounded and measurable. Show that:

$$\int_{\mathbf{R}^n} f d\mu \star \nu = \int_{\mathbf{R}^n \times \mathbf{R}^n} f(x+y) d\mu \otimes \nu(x,y)$$

Exercise 4. Let μ, ν be complex measures on $\mathbf{R}^n$. Given $B \subseteq \mathbf{R}^n$ and $x \in \mathbf{R}^n$, we define $B - x = \{y \in \mathbf{R}^n \ , \ y + x \in B\}$.

1. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$, $B - x \in \mathcal{B}(\mathbf{R}^n)$.

2. Show $x \to \mu(B-x)$ is measurable and bounded, for $B \in \mathcal{B}(\mathbf{R}^n)$.

3. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:

$$\mu \star \nu(B) = \int_{\mathbf{R}^n} \mu(B - x) d\nu(x)$$

4. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:

$$\mu \star \nu(B) = \int_{\mathbf{R}^n} \nu(B - x) d\mu(x)$$

Exercise 5. Let μ_1, μ_2, μ_3 be complex measures on $\mathbf{R}^n$.

1. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:

$$\mu_1 \star (\mu_2 \star \mu_3)(B) = \int_{\mathbf{R}^n \times \mathbf{R}^n} 1_B(x+y) d\mu_1 \otimes (\mu_2 \star \mu_3)(x,y)$$

2. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$:

$$\int_{\mathbf{R}^n} 1_B(x+y) d\mu_2 \star \mu_3(y) = \int_{\mathbf{R}^n \times \mathbf{R}^n} 1_B(x+y+z) d\mu_2 \otimes \mu_3(y,z)$$

3. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$:

$$\mu_1 \star (\mu_2 \star \mu_3)(B) = \int_{\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n} 1_B(x+y+z) d\mu_1 \otimes \mu_2 \otimes \mu_3(x,y,z)$$

4. Show that $\mu_1 \star (\mu_2 \star \mu_3) = \mu_1 \star \mu_2 \star \mu_3 = (\mu_1 \star \mu_2) \star \mu_3$

Definition 136 *Let $n \geq 1$ and $a \in \mathbf{R}^n$. We define $\delta_a : \mathcal{B}(\mathbf{R}^n) \to \mathbf{R}^+$:*

$$\forall B \in \mathcal{B}(\mathbf{R}^n) \ , \ \delta_a(B) \stackrel{\triangle}{=} 1_B(a)$$

δ_a is called the **dirac probability measure** *on $\mathbf{R}^n$, centered in a.*

Exercise 6. Let $n \geq 1$ and $a \in \mathbf{R}^n$.

1. Show that δ_a is indeed a probability measure on $\mathbf{R}^n$.
2. Show for all $f : \mathbf{R}^n \to [0, +\infty]$ non-negative and measurable:

$$\int_{\mathbf{R}^n} f d\delta_a = f(a)$$

3. Show if $f : \mathbf{R}^n \to \mathbf{C}$ is measurable, $f \in L^1_C(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), \delta_a)$ and:

$$\int_{\mathbf{R}^n} f d\delta_a = f(a)$$

4. Show that for any complex measure μ on $\mathbf{R}^n$:

$$\mu \star \delta_0 = \delta_0 \star \mu = \mu$$

5. Let $\tau_a(x) = a + x$ define the translation of vector a in $\mathbf{R}^n$. Show that for any complex measure μ on $\mathbf{R}^n$:

$$\mu \star \delta_a = \delta_a \star \mu = \tau_a(\mu)$$

Exercise 7. Let $n \geq 1$ and μ, ν be complex measures on $\mathbf{R}^n$. We assume that $\nu << dx$, i.e. that ν is absolutely continuous with respect to the lebesgue measure on $\mathbf{R}^n$.

1. Show there is $f \in L^1_C(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, such that $\nu = \int f dx$.

2. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$, we have:

$$\mu \star \nu(B) = \int_{\mathbf{R}^n} \nu(B - x) d\mu(x)$$

3. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$:

$$\nu(B - x) = \int_{\mathbf{R}^n} 1_B(y) f(y - x) dy$$

4. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$ the map:

$$(x, y) \to 1_B(y) f(y - x)$$

lies in $L^1_{\mathbf{C}}(\mathbf{R}^n \times \mathbf{R}^n, \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^n), \mu \otimes dy)$.

5. Show that for all $B \in \mathcal{B}(\mathbf{R}^n)$, we have:

$$\mu \star \nu(B) = \int_B \left(\int_{\mathbf{R}^n} f(y - x) d\mu(x) \right) dy$$

6. Given $y \in \mathbf{R}^n$, we define:

$$g(y) \stackrel{\triangle}{=} \int_{\mathbf{R}^n} f(y - x) d\mu(x)$$

Show that $g(y)$ is well-defined for dy-almost all $y \in \mathbf{R}^n$.

7. Define an element $\bar{g}$ of $L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, with $g = \bar{g}\ dx - a.s.$

8. Show that $\mu \star \nu$ is absolutely continuous w.r. to the lebesgue measure on $\mathbf{R}^n$, with density g.

Theorem 125 *Let μ, ν be two complex measures on $\mathbf{R}^n$, $n \geq 1$. If $\nu << dx$, i.e. ν is absolutely continuous with respect to the lebesgue measure on $\mathbf{R}^n$, with density $f \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, then the convolution $\mu \star \nu = \nu \star \mu$ is itself absolutely continuous with respect to the lebesgue measure on $\mathbf{R}^n$, with density:*

$$g(y) = \int_{\mathbf{R}^n} f(y - x) d\mu(x)\ ,\ dy - a.s.$$

In other words, $\mu \star \nu = \nu \star \mu = \int g dx$.

Exercise 8. Further to theorem (125), show that if $\mu = \int h dx$ for some $h \in L^1_{\mathbf{C}}(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n), dx)$, then:

$$g(y) = \int_{\mathbf{R}^n} f(y - x) h(x) dx\ ,\ dy - a.s.$$

Definition 137 *Let μ be a complex measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, $n \geq 1$. We call* **fourier transform** *of μ, the map $\mathcal{F}\mu : \mathbf{R}^n \to \mathbf{C}$ defined by:*

$$\forall u \in \mathbf{R}^n \ , \ \mathcal{F}\mu(u) \stackrel{\triangle}{=} \int_{\mathbf{R}^n} e^{i\langle u,x\rangle} d\mu(x)$$

where $\langle \cdot, \cdot \rangle$ is the usual inner-product in $\mathbf{R}^n$.

Exercise 9. Further to definition (137):

1. Show that $\mathcal{F}\mu$ is well-defined.

2. Show that $\mathcal{F}\mu \in C^b_{\mathbf{C}}(\mathbf{R}^n)$, i.e $\mathcal{F}\mu$ is continuous and bounded.

3. Show that for all $a, u \in \mathbf{R}^n$, we have $\forall u \in \mathbf{R}^n \ , \ \mathcal{F}\delta_a(u) = e^{i\langle u,a\rangle}$.

4. Let μ be the probability measure on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ defined by:

$$\forall B \in \mathcal{B}(\mathbf{R}) \ , \ \mu(B) \stackrel{\triangle}{=} \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2} dx$$

   Show that $\mathcal{F}\mu(u) = e^{-u^2/2}$, for all $u \in \mathbf{R}$.

Exercise 10. Let $\mu_1, \ldots, \mu_p$ be complex measures on $\mathbf{R}^n$, $n, p \geq 1$.

1. Show that for all $u \in \mathbf{R}^n$, we have:

$$\mathcal{F}(\mu_1 \star \ldots \star \mu_p)(u) = \int_{(\mathbf{R}^n)^p} e^{i\langle u, x_1 + \ldots + x_p \rangle} d\mu_1 \otimes \ldots \otimes \mu_p(x_1, \ldots, x_p)$$

2. Show that $\mathcal{F}(\mu_1 \star \ldots \star \mu_p) = \Pi_{j=1}^p \mathcal{F}\mu_j$.

Exercise 11. Let $n \geq 1$, $\sigma > 0$ and $g_\sigma : \mathbf{R}^n \to \mathbf{R}^+$ defined by:

$$\forall x \in \mathbf{R}^n \ , \ g_\sigma(x) \stackrel{\triangle}{=} \frac{1}{(2\pi)^{\frac{n}{2}} \sigma^n} e^{-\|x\|^2 / 2\sigma^2}$$

1. Show that:

$$\int_{\mathbf{R}^n} g_\sigma(x) dx = 1$$

2. Show that for all $u \in \mathbf{R}^n$, we have:

$$\int_{\mathbf{R}^n} g_\sigma(x) e^{i\langle u, x \rangle} dx = e^{-\sigma^2 \|u\|^2 / 2}$$

3. Show that $P_\sigma = \int g_\sigma dx$ is a probability on $\mathbf{R}^n$ with fourier transform:
$$\forall u \in \mathbf{R}^n \ , \ \mathcal{F}P_\sigma(u) = e^{-\sigma^2\|u\|^2/2}$$

4. Show that for all $x \in \mathbf{R}^n$, we have:
$$g_\sigma(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i\langle x,u\rangle - \sigma^2\|u\|^2/2}du$$

Exercise 12. Further to ex. (11), let μ be a complex measure on $\mathbf{R}^n$.

1. Show that $\mu \star P_\sigma = \int \phi_\sigma dx$ where:
$$\phi_\sigma(x) = \int_{\mathbf{R}^n} g_\sigma(x-y)d\mu(y) \ , \ dx - a.s.$$

2. Show that we also have:
$$\phi_\sigma(x) = \int_{\mathbf{R}^n} g_\sigma(y-x)d\mu(y) \ , \ dx - a.s.$$

3. Show that:
$$\phi_\sigma(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n} e^{i\langle y-x,u\rangle - \sigma^2\|u\|^2/2}du\right)d\mu(y)\ ,\ dx-a.s.$$

4. Show that:
$$\phi_\sigma(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{-i\langle x,u\rangle - \sigma^2\|u\|^2/2}(\mathcal{F}\mu)(u)du$$

5. Show that if μ,ν are two complex measures on $\mathbf{R}^n$ such that $\mathcal{F}\mu = \mathcal{F}\nu$, then for all $\sigma > 0$, we have $\mu \star P_\sigma = \nu \star P_\sigma$.

Definition 138 *Let $(\Omega,\mathcal{T})$ be a topological space. Let $(\mu_k)_{k\geq 1}$ be a sequence of complex measures on $(\Omega,\mathcal{B}(\Omega))$. We say that the sequence $(\mu_k)_{k\geq 1}$* **narrowly converges** *to a complex measure μ on $(\Omega,\mathcal{B}(\Omega))$, and we write $\mu_k \to \mu$ narrowly, if and only if:*
$$\forall f \in C^b_{\mathbf{R}}(\Omega)\ ,\ \lim_{k\to+\infty}\int f d\mu_k = \int f d\mu$$

Exercise 13. Further to definition (138):

1. Show that $\mu_k \to \mu$ narrowly, is equivalent to:

$$\forall f \in C^b_{\mathbf{C}}(\Omega)\ ,\ \lim_{k\to+\infty} \int f d\mu_k = \int f d\mu$$

2. Show that if $(\Omega, \mathcal{T})$ is metrizable and ν is a complex measure on $(\Omega, \mathcal{B}(\Omega))$ such that $\mu_k \to \mu$ and $\mu_k \to \nu$ narrowly, then $\mu = \nu$.

Theorem 126 *On a metrizable topological space, the narrow limit when it exists, of any sequence of complex measures, is unique.*

Exercise 14.

1. Show that on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$, we have $\delta_{1/n} \to \delta_0$ narrowly.

2. Show there is $B \in \mathcal{B}(\mathbf{R})$, such that $\delta_{1/n}(B) \not\to \delta_0(B)$.

Exercise 15. Let $n \geq 1$. Given $\sigma > 0$, let P_σ be the probability measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ defined as in ex. (11). Let $(\sigma_k)_{k\geq 1}$ be a sequence in $\mathbf{R}^+$ such that $\sigma_k > 0$ and $\sigma_k \to 0$.

1. Show that for all $f \in C^b_{\mathbf{R}}(\mathbf{R}^n)$, we have:

$$\int_{\mathbf{R}^n} f(x) g_{\sigma_k}(x) dx = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} f(\sigma_k x) e^{-\|x\|^2/2} dx$$

2. Show that for all $f \in C^b_{\mathbf{R}}(\mathbf{R}^n)$, we have:

$$\lim_{k \to +\infty} \int_{\mathbf{R}^n} f(x) g_{\sigma_k}(x) dx = f(0)$$

3. Show that $P_{\sigma_k} \to \delta_0$ narrowly.

Exercise 16. Let μ, ν be two complex measures on $\mathbf{R}^n$. Let $(\nu_k)_{k\geq 1}$ be a sequence of complex measures on $\mathbf{R}^n$, which narrowly converges to ν. Let $f \in C^b_{\mathbf{R}}(\mathbf{R}^n)$, and $\phi : \mathbf{R}^n \to \mathbf{R}$ be defined by:

$$\forall y \in \mathbf{R}^n \ , \ \phi(y) \stackrel{\triangle}{=} \int_{\mathbf{R}^n} f(x+y)d\mu(x)$$

1. Show that:

$$\int_{\mathbf{R}^n} f d\mu \star \nu_k = \int_{\mathbf{R}^n \times \mathbf{R}^n} f(x+y) d\mu \otimes \nu_k(x,y)$$

2. Show that:

$$\int_{\mathbf{R}^n} f d\mu \star \nu_k = \int_{\mathbf{R}^n} \phi d\nu_k$$

3. Show that $\phi \in C^b_{\mathbf{C}}(\mathbf{R}^n)$.

4. Show that:

$$\lim_{k\to+\infty} \int_{\mathbf{R}^n} \phi d\nu_k = \int_{\mathbf{R}^n} \phi d\nu$$

5. Show that:

$$\lim_{k\to+\infty}\int_{\mathbf{R}^n} f d\mu\star\nu_k = \int_{\mathbf{R}^n} f d\mu\star\nu$$

6. Show that $\mu\star\nu_k\to\mu\star\nu$ narrowly.

Theorem 127 *Let μ,ν be two complex measures on $\mathbf{R}^n$, $n\geq 1$. Let $(\nu_k)_{k\geq 1}$ be a sequence of complex measures on $\mathbf{R}^n$. Then:*

$$\nu_k\to\nu \textit{ narrowly } \Rightarrow \quad \mu\star\nu_k\to\mu\star\nu \textit{ narrowly}$$

Exercise 17. Let μ,ν be two complex measures on $\mathbf{R}^n$, such that $\mathcal{F}\mu=\mathcal{F}\nu$. For all $\sigma>0$, let P_σ be the probability measure on $(\mathbf{R}^n,\mathcal{B}(\mathbf{R}^n))$ as defined in ex. (11). Let $(\sigma_k)_{k\geq 1}$ be a sequence in $\mathbf{R}^+$ such that $\sigma_k>0$ and $\sigma_k\to 0$.

1. Show that $\mu\star P_{\sigma_k}=\nu\star P_{\sigma_k}$, for all $k\geq 1$.

2. Show that $\mu\star P_{\sigma_k}\to\mu\star\delta_0$ narrowly.

3. Show that $(\mu \star P_{\sigma_k})_{k\geq 1}$ narrowly converges to both μ and ν.

4. Prove the following:

Theorem 128 *Let μ, ν be two complex measures on $\mathbf{R}^n$. Then:*

$$\mathcal{F}\mu = \mathcal{F}\nu \quad \Rightarrow \quad \mu = \nu$$

i.e. the fourier transform is an injective mapping on $M^1(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

Definition 139 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Given $n \geq 1$, and a measurable map $X : (\Omega, \mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, the mapping ϕ_X defined as:*

$$\forall u \in \mathbf{R}^n \ , \ \phi_X(u) \stackrel{\triangle}{=} E[e^{i\langle u, X\rangle}]$$

is called the **characteristic function**[2] *of the random variable X.*

[2] Do not confuse with the *characteristic function* 1_A of a set A, definition (39).

Exercise 18. Further to definition (139):

1. Show that ϕ_X is well-defined, bounded and continuous.

2. Show that we have:

$$\forall u \in \mathbf{R}^n \ , \ \phi_X(u) = \int_{\mathbf{R}^n} e^{i\langle u,x\rangle} dX(P)(x)$$

3. Show ϕ_X is the fourier transform of the image measure $X(P)$.

4. Show the following:

Theorem 129 *Let $X, Y : (\Omega, \mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, $n \geq 1$, be two random variables on a probability space $(\Omega, \mathcal{F}, P)$. If X and Y have the same characteristic functions, i.e.*

$$\forall u \in \mathbf{R}^n \ , \ E[e^{i\langle u,X\rangle}] = E[e^{i\langle u,Y\rangle}]$$

then X and Y have the same distributions, i.e.

$$\forall B \in \mathcal{B}(\mathbf{R}^n) \ , \ P(\{X \in B\}) = P(\{Y \in B\})$$

Definition 140 *Let $n \geq 1$. Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, we define the modulus of α, denoted $|\alpha|$, by $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Given $x \in \mathbf{R}^n$ and $\alpha \in \mathbf{N}^n$, we put:*

$$x^\alpha \stackrel{\triangle}{=} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

where it is understood that $x_j^{\alpha_j} = 1$ whenever $\alpha_j = 0$. Given a map $f : U \to \mathbf{C}$, where U is an open subset of $\mathbf{R}^n$, we denote $\partial^\alpha f$ the $|\alpha|$-th partial derivative, when it exists:

$$\partial^\alpha f \stackrel{\triangle}{=} \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$$

Note that $\partial^\alpha f = f$, whenever $|\alpha| = 0$. Given $k \geq 0$, we say that f is of **class** *C^k, if and only if for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \leq k$, $\partial^\alpha f$ exists and is continuous on U.*

EXERCISE 19. Explain why def. (140) is consistent with def. (130).

Exercise 20. Let μ be a complex measure on $\mathbf{R}^n$, and $\alpha \in \mathbf{N}^n$, with:

$$\int_{\mathbf{R}^n} |x^\alpha| d|\mu|(x) < +\infty \tag{1}$$

Let $x^\alpha \mu$ the complex measure on $\mathbf{R}^n$ defined by $x^\alpha \mu = \int x^\alpha d\mu$.

1. Explain why the above integral (1) is well-defined.

2. Show that $x^\alpha \mu$ is a well-defined complex measure on $\mathbf{R}^n$.

3. Show that the total variation of $x^\alpha \mu$ is given by:

$$\forall B \in \mathcal{B}(\mathbf{R}^n)\ ,\ |x^\alpha \mu|(B) = \int_B |x^\alpha| d|\mu|(x)$$

4. Show that the fourier transform of $x^\alpha \mu$ is given by:

$$\forall u \in \mathbf{R}^n\ ,\ \mathcal{F}(x^\alpha \mu)(u) = \int_{\mathbf{R}^n} x^\alpha e^{i\langle u,x\rangle} d\mu(x)$$

Exercise 21. Let μ be a complex measure on $\mathbf{R}^n$. Let $\beta \in \mathbf{N}^n$ with $|\beta| = 1$, and:

$$\int_{\mathbf{R}^n} |x^\beta| d|\mu|(x) < +\infty$$

Let $x^\beta \mu$ be the complex measure on $\mathbf{R}^n$ defined as in ex. (20).

1. Show that there is $j \in \mathbf{N}_n$ with $x^\beta = x_j$ for all $x \in \mathbf{R}^n$.

2. Show that for all $u \in \mathbf{R}^n$, $\frac{\partial \mathcal{F}\mu}{\partial u_j}(u)$ exists and that we have:

$$\frac{\partial \mathcal{F}\mu}{\partial u_j}(u) = i \int_{\mathbf{R}^n} x_j e^{i\langle u,x\rangle} d\mu(x)$$

3. Conclude that $\partial^\beta \mathcal{F}\mu$ exists and that we have:

$$\partial^\beta \mathcal{F}\mu = i\mathcal{F}(x^\beta \mu)$$

4. Explain why $\partial^\beta \mathcal{F}\mu$ is continuous.

EXERCISE 22. Let μ be a complex measure on $\mathbf{R}^n$. Let $k \geq 0$ be an integer. We assume that for all $\alpha \in \mathbf{N}^n$, we have:

$$|\alpha| \leq k \;\Rightarrow\; \int_{\mathbf{R}^n} |x^\alpha| d|\mu|(x) < +\infty \tag{2}$$

In particular, if $|\alpha| \leq k$, the measure $x^\alpha \mu$ of ex. (20) is well-defined. We claim that for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \leq k$, $\partial^\alpha \mathcal{F}\mu$ exists, and:

$$\partial^\alpha \mathcal{F}\mu = i^{|\alpha|}\mathcal{F}(x^\alpha \mu)$$

1. Show that if $k = 0$, then the property is obviously true. We assume the property is true for some $k \geq 0$, and that the above integrability condition (2) holds for $k + 1$.

2. Let $\alpha' \in \mathbf{N}^n$ be such that $|\alpha'| \leq k + 1$. Explain why if $|\alpha'| \leq k$, then $\partial^{\alpha'} \mathcal{F}\mu$ exists, with:

$$\partial^{\alpha'} \mathcal{F}\mu = i^{|\alpha'|}\mathcal{F}(x^{\alpha'} \mu)$$

3. We assume that $|\alpha'| = k + 1$. Show the existence of $\alpha, \beta \in \mathbf{N}^n$ such that $\alpha + \beta = \alpha'$, $|\alpha| = k$ and $|\beta| = 1$.

4. Explain why $\partial^\alpha \mathcal{F}\mu$ exists, and:
$$\partial^\alpha \mathcal{F}\mu = i^{|\alpha|}\mathcal{F}(x^\alpha \mu)$$

5. Show that:
$$\int_{\mathbf{R}^n} |x^\beta| d|x^\alpha \mu|(x) < +\infty$$

6. Show that $\partial^\beta \mathcal{F}(x^\alpha \mu)$ exists, with:
$$\partial^\beta \mathcal{F}(x^\alpha \mu) = i\mathcal{F}(x^\beta (x^\alpha \mu))$$

7. Show that $\partial^\beta(\partial^\alpha \mathcal{F}\mu)$ exists, with:
$$\partial^\beta(\partial^\alpha \mathcal{F}\mu) = i^{|\alpha|+1}\mathcal{F}(x^\beta(x^\alpha \mu))$$

8. Show that $x^\beta(x^\alpha \mu) = x^{\alpha'}\mu$.

9. Conclude that the property is true for $k+1$.

10. Show the following:

Theorem 130 *Let μ be a complex measure on $\mathbf{R}^n$, $n \geq 1$. Let $k \geq 0$ be an integer such that for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \leq k$, we have:*

$$\int_{\mathbf{R}^n} |x^\alpha| d|\mu|(x) < +\infty$$

Then, the fourier transform $\mathcal{F}\mu$ is of class C^k on $\mathbf{R}^n$, and for all $\alpha \in \mathbf{N}^n$ with $|\alpha| \leq k$, we have:

$$\forall u \in \mathbf{R}^n \ , \ \partial^\alpha \mathcal{F}\mu(u) = i^{|\alpha|} \int_{\mathbf{R}^n} x^\alpha e^{i\langle u, x\rangle} d\mu(x)$$

In particular:

$$\int_{\mathbf{R}^n} x^\alpha d\mu(x) = i^{-|\alpha|} \partial^\alpha \mathcal{F}\mu(0)$$

20. Gaussian Measures

$\mathcal{M}_n(\mathbf{R})$ is the set of all $n \times n$-matrices with real entries, $n \geq 1$.

Definition 141 *A matrix $M \in \mathcal{M}_n(\mathbf{R})$ is said to be* **symmetric**, *if and only if $M = M^t$. M is* **orthogonal**, *if and only if M is non-singular and $M^{-1} = M^t$. If M is symmetric, we say that M is* **non-negative**, *if and only if:*

$$\forall u \in \mathbf{R}^n \ , \ \langle u, Mu \rangle \geq 0$$

Theorem 131 *Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$, $n \geq 1$, be a symmetric and non-negative real matrix. There exist $\lambda_1, \ldots, \lambda_n \in \mathbf{R}^+$ and $P \in \mathcal{M}_n(\mathbf{R})$ orthogonal matrix, such that:*

$$\Sigma = P.\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.P^t$$

In particular, there exists $A \in \mathcal{M}_n(\mathbf{R})$ such that $\Sigma = A.A^t$.

As a rare exception, theorem (131) is given without proof.

Exercise 1. Given $n \geq 1$ and $M \in \mathcal{M}_n(\mathbf{R})$, show that we have:

$$\forall u, v \in \mathbf{R}^n \ , \ \langle u, Mv \rangle = \langle M^t u, v \rangle$$

Exercise 2. Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative matrix. Let μ_1 be the probability measure on $\mathbf{R}$:

$$\forall B \in \mathcal{B}(\mathbf{R}) \ , \ \mu_1(B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2} dx$$

Let $\mu = \mu_1 \otimes \ldots \otimes \mu_1$ be the product measure on $\mathbf{R}^n$. Let $A \in \mathcal{M}_n(\mathbf{R})$ be such that $\Sigma = A.A^t$. We define the map $\phi : \mathbf{R}^n \to \mathbf{R}^n$ by:

$$\forall x \in \mathbf{R}^n \ , \ \phi(x) \stackrel{\triangle}{=} Ax + m$$

1. Show that μ is a probability measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

2. Explain why the image measure $P = \phi(\mu)$ is well-defined.

3. Show that P is a probability measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

4. Show that for all $u \in \mathbf{R}^n$:

$$\mathcal{F}P(u) = \int_{\mathbf{R}^n} e^{i\langle u,\phi(x)\rangle} d\mu(x)$$

5. Let $v = A^t u$. Show that for all $u \in \mathbf{R}^n$:

$$\mathcal{F}P(u) = e^{i\langle u,m\rangle - \|v\|^2/2}$$

6. Show the following:

Theorem 132 *Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. There exists a unique complex measure on $\mathbf{R}^n$, denoted $N_n(m,\Sigma)$, with fourier transform:*

$$\mathcal{F}N_n(m,\Sigma)(u) \stackrel{\triangle}{=} \int_{\mathbf{R}^n} e^{i\langle u,x\rangle} dN_n(m,\Sigma)(x) = e^{i\langle u,m\rangle - \frac{1}{2}\langle u,\Sigma u\rangle}$$

for all $u \in \mathbf{R}^n$. Furthermore, $N_n(m,\Sigma)$ is a probability measure.

Definition 142 *Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. The probability measure $N_n(m, \Sigma)$ on $\mathbf{R}^n$ defined in theorem (132) is called the n-dimensional* **gaussian measure** *or* **normal distribution**, *with mean $m \in \mathbf{R}^n$ and covariance matrix Σ.*

Exercise 3. Let $n \geq 1$ and $m \in \mathbf{R}^n$. Show that $N_n(m, 0) = \delta_m$.

Exercise 4. Let $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. Let $A \in \mathcal{M}_n(\mathbf{R})$ be such that $\Sigma = A.A^t$. A map $p : \mathbf{R}^n \to \mathbf{C}$ is said to be a *polynomial*, if and only if, it is a finite linear complex combination of maps $x \to x^\alpha$,[1] for $\alpha \in \mathbf{N}^n$.

1. Show that for all $B \in \mathcal{B}(\mathbf{R})$, we have:

$$N_1(0,1)(B) = \frac{1}{\sqrt{2\pi}} \int_B e^{-x^2/2} dx$$

[1]See definition (140).

2. Show that:
$$\int_{-\infty}^{+\infty} |x| dN_1(0,1)(x) < +\infty$$

3. Show that for all integer $k \geq 1$:
$$\frac{1}{\sqrt{2\pi}} \int_0^{+\infty} x^{k+1} e^{-x^2/2} dx = \frac{k}{\sqrt{2\pi}} \int_0^{+\infty} x^{k-1} e^{-x^2/2} dx$$

4. Show that for all integer $k \geq 0$:
$$\int_{-\infty}^{+\infty} |x|^k dN_1(0,1)(x) < +\infty$$

5. Show that for all $\alpha \in \mathbf{N}^n$:
$$\int_{\mathbf{R}^n} |x^\alpha| dN_1(0,1) \otimes \ldots \otimes N_1(0,1)(x) < +\infty$$

6. Let $p : \mathbf{R}^n \to \mathbf{C}$ be a polynomial. Show that:
$$\int_{\mathbf{R}^n} |p(x)| dN_1(0,1) \otimes \ldots \otimes N_1(0,1)(x) < +\infty$$

7. Let $\phi : \mathbf{R}^n \to \mathbf{R}^n$ be defined by $\phi(x) = Ax + m$. Explain why the image measure $\phi(N_1(0,1) \otimes \ldots \otimes N_1(0,1))$ is well-defined.

8. Show that $\phi(N_1(0,1) \otimes \ldots \otimes N_1(0,1)) = N_n(m, \Sigma)$.

9. Show if $\beta \in \mathbf{N}^n$ and $|\beta| = 1$, then $x \to \phi(x)^\beta$ is a polynomial.

10. Show that if $\alpha' \in \mathbf{N}^n$ and $|\alpha'| = k+1$, then $\phi(x)^{\alpha'} = \phi(x)^\alpha \phi(x)^\beta$ for some $\alpha, \beta \in \mathbf{N}^n$ such that $|\alpha| = k$ and $|\beta| = 1$.

11. Show that the product of two polynomials is a polynomial.

12. Show that for all $\alpha \in \mathbf{N}^n$, $x \to \phi(x)^\alpha$ is a polynomial.

13. Show that for all $\alpha \in \mathbf{N}^n$:
$$\int_{\mathbf{R}^n} |\phi(x)^\alpha| dN_1(0,1) \otimes \ldots \otimes N_1(0,1)(x) < +\infty$$

14. Show the following:

Theorem 133 *Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. Then, for all $\alpha \in \mathbf{N}^n$, the map $x \to x^\alpha$ is integrable with respect to the gaussian measure $N_n(m, \Sigma)$:*

$$\int_{\mathbf{R}^n} |x^\alpha| dN_n(m, \Sigma)(x) < +\infty$$

Exercise 5. Let $m \in \mathbf{R}^n$. Let $\Sigma = (\sigma_{ij}) \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. Let $j, k \in \mathbf{N}_n$. Let ϕ be the fourier transform of the gaussian measure $N_n(m, \Sigma)$, i.e.:

$$\forall u \in \mathbf{R}^n \ , \ \phi(u) \stackrel{\triangle}{=} e^{i\langle u, m\rangle - \frac{1}{2}\langle u, \Sigma u\rangle}$$

1. Show that:

$$\int_{\mathbf{R}^n} x_j dN_n(m, \Sigma)(x) = i^{-1} \frac{\partial \phi}{\partial u_j}(0)$$

2. Show that:
$$\int_{\mathbf{R}^n} x_j dN_n(m,\Sigma)(x) = m_j$$

3. Show that:
$$\int_{\mathbf{R}^n} x_j x_k dN_n(m,\Sigma)(x) = i^{-2}\frac{\partial^2 \phi}{\partial u_j \partial u_k}(0)$$

4. Show that:
$$\int_{\mathbf{R}^n} x_j x_k dN_n(m,\Sigma)(x) = \sigma_{jk} - m_j m_k$$

5. Show that:
$$\int_{\mathbf{R}^n} (x_j - m_j)(x_k - m_k) dN_n(m,\Sigma)(x) = \sigma_{jk}$$

Theorem 134 *Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma = (\sigma_{ij}) \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. Let $N_n(m, \Sigma)$ be the gaussian measure with mean m and covariance matrix Σ. Then, for all $j, k \in \mathbf{N}_n$, we have:*

$$\int_{\mathbf{R}^n} x_j dN_n(m, \Sigma)(x) = m_j$$

and:

$$\int_{\mathbf{R}^n} (x_j - m_j)(x_k - m_k) dN_n(m, \Sigma)(x) = \sigma_{jk}$$

Definition 143 *Let $n \geq 1$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X : (\Omega, \mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ be a measurable map. We say that X is an n-dimensional* **gaussian** *or* **normal vector***, if and only if its distribution is a gaussian measure, i.e. $X(P) = N_n(m, \Sigma)$ for some $m \in \mathbf{R}^n$ and $\Sigma \in \mathcal{M}_n(\mathbf{R})$ symmetric and non-negative real matrix.*

Exercise 6. Show the following:

Theorem 135 *Let $n \geq 1$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X : (\Omega, \mathcal{F}) \to \mathbf{R}^n$ be a measurable map. Then X is a gaussian vector, if and only if there exist $m \in \mathbf{R}^n$ and $\Sigma \in \mathcal{M}_n(\mathbf{R})$ symmetric and non-negative real matrix, such that:*

$$\forall u \in \mathbf{R}^n \ , \ E[e^{i\langle u,X\rangle}] = e^{i\langle u,m\rangle - \frac{1}{2}\langle u,\Sigma u\rangle}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner-product on $\mathbf{R}^n$.

Definition 144 *Let $X : (\Omega, \mathcal{F}) \to \bar{\mathbf{R}}$ (or $\mathbf{C}$) be a random variable on a probability space $(\Omega, \mathcal{F}, P)$. We say that X is* **integrable**, *if and only if we have $E[|X|] < +\infty$. We say that X is* **square-integrable**, *if and only if we have $E[|X|^2] < +\infty$.*

Exercise 7. Further to definition (144), suppose X is $\mathbf{C}$-valued.

1. Show X is integrable if and only if $X \in L^1_{\mathbf{C}}(\Omega, \mathcal{F}, P)$.
2. Show X is square-integrable, if and only if $X \in L^2_{\mathbf{C}}(\Omega, \mathcal{F}, P)$.

Exercise 8. Further to definition (144), suppose X is $\bar{\mathbf{R}}$-valued.

1. Show that X is integrable, if and only if X is P-almost surely equal to an element of $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, P)$.

2. Show that X is square-integrable, if and only if X is P-almost surely equal to an element of $L^2_{\mathbf{R}}(\Omega, \mathcal{F}, P)$.

Exercise 9. Let $X, Y : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be two square-integrable random variables on a probability space $(\Omega, \mathcal{F}, P)$.

1. Show that both X and Y are integrable.

2. Show that XY is integrable

3. Show that $(X - E[X])(Y - E[Y])$ is a well-defined and integrable.

Definition 145 *Let* $X, Y : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ *be two square-integrable random variables on a probability space* $(\Omega, \mathcal{F}, P)$*. We define the* **covariance** *between* X *and* Y*, denoted* $cov(X, Y)$*, as:*

$$cov(X, Y) \stackrel{\triangle}{=} E[(X - E[X])(Y - E[Y])]$$

We say that X *and* Y *are* **uncorrelated** *if and only if* $cov(X, Y) = 0$*. If* $X = Y$*,* $cov(X, Y)$ *is called the* **variance** *of* X*, denoted* $var(X)$*.*

Exercise 10. Let X, Y be two square integrable, real random variable on a probability space $(\Omega, \mathcal{F}, P)$.

1. Show that $cov(X, Y) = E[XY] - E[X]E[Y]$.
2. Show that $var(X) = E[X^2] - E[X]^2$.
3. Show that $var(X + Y) = var(X) + 2cov(X, Y) + var(Y)$
4. Show that X and Y are uncorrelated, if and only if:

$$var(X + Y) = var(X) + var(Y)$$

Exercise 11. Let X be an n-dimensional normal vector on some probability space $(\Omega, \mathcal{F}, P)$, with law $N_n(m, \Sigma)$, where $m \in \mathbf{R}^n$ and $\Sigma = (\sigma_{ij}) \in \mathcal{M}_n(\mathbf{R})$ is a symmetric and non-negative real matrix.

1. Show that each coordinate $X_j : (\Omega, \mathcal{F}) \to \mathbf{R}$ is measurable.
2. Show that $E[|X^\alpha|] < +\infty$ for all $\alpha \in \mathbf{N}^n$.
3. Show that for all $j = 1, \ldots, n$, we have $E[X_j] = m_j$.
4. Show that for all $j, k = 1, \ldots, n$, we have $cov(X_j, X_k) = \sigma_{jk}$.

Theorem 136 *Let X be an n-dimensional normal vector on a probability space $(\Omega, \mathcal{F}, P)$, with law $N_n(m, \Sigma)$. Then, for all $\alpha \in \mathbf{N}^n$, X^α is integrable. Moreover, for all $j, k \in \mathbf{N}_n$, we have:*

$$E[X_j] = m_j$$

and:

$$cov(X_j, X_k) = \sigma_{jk}$$

where $(\sigma_{ij}) = \Sigma$.

Exercise 12. Show the following:

Theorem 137 *Let $X : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be a real random variable on a probability space $(\Omega, \mathcal{F}, P)$. Then, X is a normal random variable, if and only if it is square integrable, and:*

$$\forall u \in \mathbf{R}\ ,\ E[e^{iuX}] = e^{iuE[X] - \frac{1}{2}u^2 var(X)}$$

Exercise 13. Let X be an n-dimensional normal vector on a probability space $(\Omega, \mathcal{F}, P)$, with law $N_n(m, \Sigma)$. Let $A \in \mathcal{M}_{d,n}(\mathbf{R})$ be an $d \times n$ real matrix, $(n, d \geq 1)$. Let $b \in \mathbf{R}^n$ and $Y = AX + b$.

1. Show that $Y : (\Omega, \mathcal{F}) \to (\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$ is measurable.
2. Show that the law of Y is $N_d(Am + b, A.\Sigma.A^t)$
3. Conclude that Y is an $\mathbf{R}^d$-valued normal random vector.

Theorem 138 *Let X be an n-dimensional normal vector with law $N_n(m, \Sigma)$ on a probability space $(\Omega, \mathcal{F}, P)$, $(n \geq 1)$. Let $d \geq 1$ and $A \in \mathcal{M}_{d,n}(\mathbf{R})$ be an $d \times n$ real matrix. Let $b \in \mathbf{R}^d$. Then, $Y = AX + b$ is an d-dimensional normal vector, with law:*

$$Y(P) = N_d(Am + b, A.\Sigma.A^t)$$

Exercise 14. Let $X : (\Omega, \mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ be a measurable map, where $(\Omega, \mathcal{F}, P)$ is a probability space. Show that if X is a gaussian vector, then for all $u \in \mathbf{R}^n$, $\langle u, X \rangle$ is a normal random variable.

Exercise 15. Let $X : (\Omega, \mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ be a measurable map, where $(\Omega, \mathcal{F}, P)$ is a probability space. We assume that for all $u \in \mathbf{R}^n$, $\langle u, X \rangle$ is a normal random variable.

1. Show that for all $j = 1, \ldots, n$, X_j is integrable.
2. Show that for all $j = 1, \ldots, n$, X_j is square integrable.
3. Explain why given $j, k = 1, \ldots, n$, $cov(X_j, X_k)$ is well-defined.

4. Let $m \in \mathbf{R}^n$ be defined by $m_j = E[X_j]$, and $u \in \mathbf{R}^n$. Show:

$$E[\langle u, X\rangle] = \langle u, m\rangle$$

5. Let $\Sigma = (cov(X_i, X_j))$. Show that for all $u \in \mathbf{R}^n$, we have:

$$var(\langle u, X\rangle) = \langle u, \Sigma u\rangle$$

6. Show that Σ is a symmetric and non-negative $n \times n$ real matrix.

7. Show that for all $u \in \mathbf{R}^n$:

$$E[e^{i\langle u,X\rangle}] = e^{iE[\langle u,X\rangle]-\frac{1}{2}var(\langle u,X\rangle)}$$

8. Show that for all $u \in \mathbf{R}^n$:

$$E[e^{i\langle u,X\rangle}] = e^{i\langle u,m\rangle-\frac{1}{2}\langle u,\Sigma u\rangle}$$

9. Show that X is a normal vector.

10. Show the following:

Theorem 139 *Let* $X : (\Omega, \mathcal{F}) \to (\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$ *be a measurable map on a probability space* $(\Omega, \mathcal{F}, P)$. *Then,* X *is an* n*-dimensional normal vector, if and only if, any linear combination of its coordinates is itself normal, or in other words* $\langle u, X \rangle$ *is normal for all* $u \in \mathbf{R}^n$.

EXERCISE 16. Let $(\Omega, \mathcal{F}) = (\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2))$ and μ be the probability on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ defined by $\mu = \frac{1}{2}(\delta_0 + \delta_1)$. Let $P = N_1(0,1) \otimes \mu$, and $X, Y : (\Omega, \mathcal{F}) \to (\mathbf{R}, \mathcal{B}(\mathbf{R}))$ be the canonical projections defined by $X(x,y) = x$ and $Y(x,y) = y$.

1. Show that P is a probability measure on $(\Omega, \mathcal{F})$.

2. Explain why X and Y are measurable.

3. Show that X has the distribution $N_1(0,1)$.

4. Show that $P(\{Y = 0\}) = P(\{Y = 1\}) = \frac{1}{2}$.

5. Show that $P^{(X,Y)} = P$.

6. Show for all $\phi : (\mathbf{R}^2, \mathcal{B}(\mathbf{R}^2)) \to \mathbf{C}$ measurable and bounded:

$$E[\phi(X,Y)] = \frac{1}{2}(E[\phi(X,0)] + E[\phi(X,1)])$$

7. Let $X_1 = X$ and X_2 be defined as:

$$X_2 \stackrel{\triangle}{=} X1_{\{Y=0\}} - X1_{\{Y=1\}}$$

   Show that $E[e^{iuX_2}] = e^{-u^2/2}$ for all $u \in \mathbf{R}$.

8. Show that $X_1(P) = X_2(P) = N_1(0,1)$.

9. Explain why $cov(X_1, X_2)$ is well-defined.

10. Show that X_1 and X_2 are uncorrelated.

11. Let $Z = \frac{1}{2}(X_1 + X_2)$. Show that:

$$\forall u \in \mathbf{R}\ ,\ E[e^{iuZ}] = \frac{1}{2}(1 + e^{-u^2/2})$$

12. Show that Z cannot be gaussian.

13. Conclude that although X_1, X_2 are normally distributed, (and even uncorrelated), (X_1, X_2) is not a gaussian vector.

EXERCISE 17. Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. Let $A \in \mathcal{M}_n(\mathbf{R})$ be such that $\Sigma = A.A^t$. We assume that Σ is non-singular. We define $p_{m,\Sigma} : \mathbf{R}^n \to \mathbf{R}^+$ by:

$$\forall x \in \mathbf{R}^n \ , \ p_{m,\Sigma}(x) \stackrel{\triangle}{=} \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(\Sigma)}} e^{-\frac{1}{2}\langle x-m, \Sigma^{-1}(x-m)\rangle}$$

1. Explain why $\det(\Sigma) > 0$.

2. Explain why $\sqrt{\det(\Sigma)} = |\det(A)|$.

3. Explain why A is non-singular.

4. Let $\phi : \mathbf{R}^n \to \mathbf{R}^n$ be defined by:

$$\forall x \in \mathbf{R}^n \ , \ \phi(x) \stackrel{\triangle}{=} A^{-1}(x-m)$$

Show that for all $x \in \mathbf{R}^n$, $\langle x-m, \Sigma^{-1}(x-m)\rangle = \|\phi(x)\|^2$.

5. Show that ϕ is a C^1-diffeomorphism.

6. Show that $\phi(dx) = |\det(A)|dx$.

7. Show that:

$$\int_{\mathbf{R}^n} p_{m,\Sigma}(x)dx = 1$$

8. Let $\mu = \int p_{m,\Sigma}dx$. Show that:

$$\forall u \in \mathbf{R}^n \ , \ \mathcal{F}\mu(u) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} e^{i\langle u, Ax+m\rangle - \|x\|^2/2} dx$$

9. Show that the fourier transform of μ is therefore given by:

$$\forall u \in \mathbf{R}^n \ , \ \mathcal{F}\mu(u) = e^{i\langle u,m\rangle - \frac{1}{2}\langle u, \Sigma u\rangle}$$

10. Show that $\mu = N_n(m, \Sigma)$.

11. Show that $N_n(m, \Sigma) << dx$, i.e. that $N_n(m, \Sigma)$ is absolutely continuous w.r. to the lebesgue measure on $\mathbf{R}^n$.

Exercise 18. Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. We assume that Σ is singular. Let $u \in \mathbf{R}^n$ be such that $\Sigma u = 0$ and $u \neq 0$. We define:

$$B \stackrel{\triangle}{=} \{x \in \mathbf{R}^n \ , \ \langle u, x \rangle = \langle u, m \rangle\}$$

Given $a \in \mathbf{R}^n$, let $\tau_a : \mathbf{R}^n \to \mathbf{R}^n$ be the translation of vector a.

1. Show $B = \tau_{-m}^{-1}(u^{\perp})$, where $u^{\perp}$ is the orthogonal of u in $\mathbf{R}^n$.

2. Show that $B \in \mathcal{B}(\mathbf{R}^n)$.

3. Explain why $dx(u^{\perp}) = 0$. Is it important to have $u \neq 0$?

4. Show that $dx(B) = 0$.

5. Show that $\phi : \mathbf{R}^n \to \mathbf{R}$ defined by $\phi(x) = \langle u, x \rangle$, is measurable.

6. Explain why $\phi(N_n(m, \Sigma))$ is a well-defined probability on $\mathbf{R}$.

7. Show that for all $\alpha \in \mathbf{R}$, we have:

$$\mathcal{F}\phi(N_n(m, \Sigma))(\alpha) = \int_{\mathbf{R}^n} e^{i\alpha\langle u,x\rangle} dN_n(m, \Sigma)(x)$$

8. Show that $\phi(N_n(m, \Sigma))$ is the dirac distribution on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$ centered on $\langle u, m \rangle$, i.e. $\phi(N_n(m, \Sigma)) = \delta_{\langle u,m \rangle}$.

9. Show that $N_n(m, \Sigma)(B) = 1$.

10. Conclude that $N_n(m, \Sigma)$ cannot be absolutely continuous with respect to the lebesgue measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$.

11. Show the following:

Theorem 140 *Let $n \geq 1$ and $m \in \mathbf{R}^n$. Let $\Sigma \in \mathcal{M}_n(\mathbf{R})$ be a symmetric and non-negative real matrix. Then, the gaussian measure $N_n(m,\Sigma)$ is absolutely continuous with respect to the lebesgue measure on $(\mathbf{R}^n, \mathcal{B}(\mathbf{R}^n))$, if and only if Σ is non-singular, in which case for all $B \in \mathcal{B}(\mathbf{R}^n)$, we have:*

$$N_n(m,\Sigma)(B) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(\Sigma)}} \int_B e^{-\frac{1}{2}\langle x-m, \Sigma^{-1}(x-m)\rangle} dx$$