Ramanujan's Notebooks

Part III

Bruce C. Berndt

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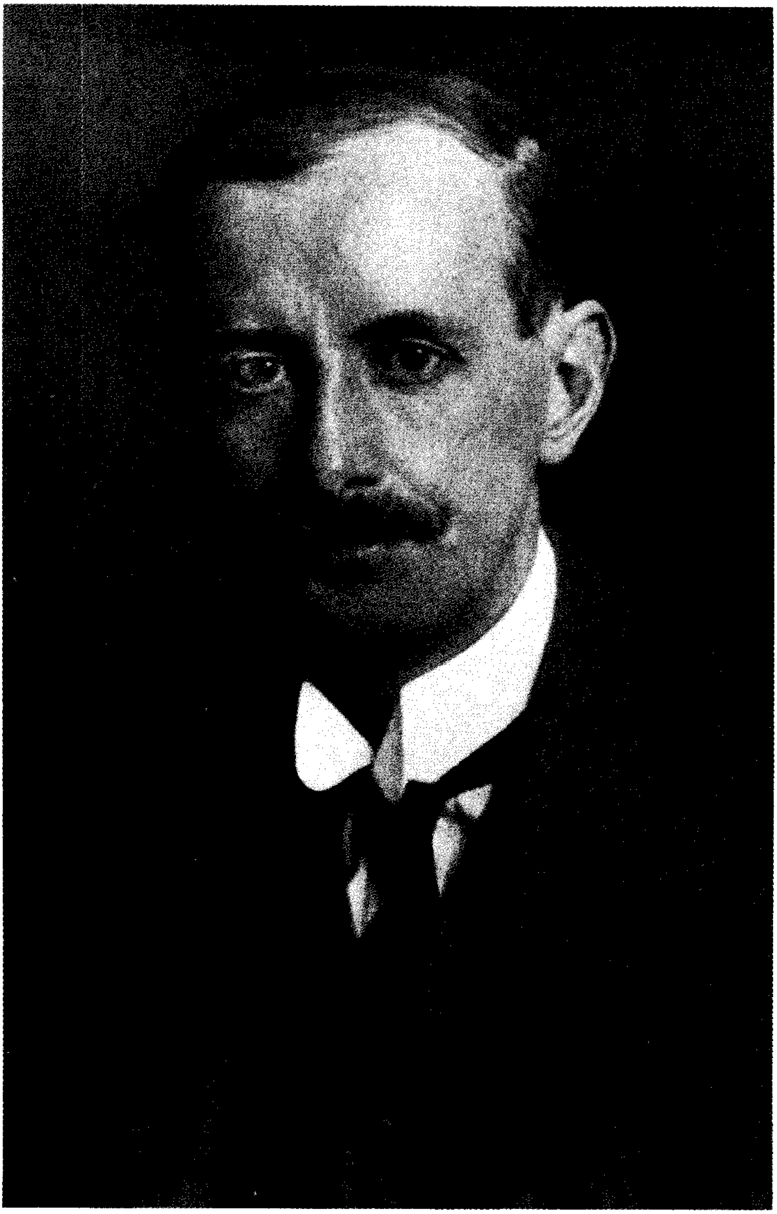
Dedicated to
S. Janaki Ammal
(Mrs. Ramanujan)

S. Janaki Ammal
Photograph by B. Berndt, 1987

A significant portion of G. N. Watson's research was influenced by Ramanujan. No less than thirty of Watson's published papers were motivated by assertions made by Ramanujan in his letters to G. H. Hardy and in his notebooks. Beginning in about 1928, Watson invested at least ten years to the editing of Ramanujan's notebooks. He never completed the task, but fortunately his efforts have been preserved. Through the suggestion of R. A. Rankin and the generosity of Mrs. Watson, all material pertaining to the notebooks compiled by Watson was donated to the library of Trinity College, Cambridge. These notes were invaluable to the author in the preparation of this book. In particular, many proofs in Chapters 19–21 are due to Watson. We are grateful to the Master and Fellows of Trinity College, Cambridge for providing us a copy of Watson's notes. For an engaging biography of Watson, see Rankin's paper [1].



G. N. Watson
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Preface

During the years 1903–1914, Ramanujan recorded most of his mathematical discoveries without proofs in notebooks. Although many of his results were already in the literature, more were not. Almost a decade after Ramanujan's death in 1920, G. N. Watson and B. M. Wilson began to edit his notebooks but never completed the task. A photostat edition, with no editing, was published by the Tata Institute of Fundamental Research in Bombay in 1957.

This book is the third of five volumes devoted to the editing of Ramanujan's notebooks. Part I, published in 1985, contains an account of Chapters 1–9 in the second notebook as well as a description of Ramanujan's quarterly reports. Part II, published in 1989, comprises accounts of Chapters 10–15 in Ramanujan's second notebook. In this volume, we examine Chapters 16–21 in the second notebook. For many of the results that are known, we provide references in the literature where proofs may be found. Otherwise, we give complete proofs. Most of the theorems in these six chapters have not previously been proved in print. Parts IV and V will contain accounts of the 100 pages of unorganized material at the end of the second notebook, the thirty-three pages of unorganized results comprising the third notebook, and those results in the first notebook not recorded by Ramanujan in the second or third notebooks. The second notebook is chiefly a much enlarged and somewhat more organized edition of the first notebook.

Urbana, Illinois *Bruce C. Berndt*
May, 1990

Contents

Introduction

> In der Theorie der Thetafunctionen ist es leicht, eine beliebig grosse Menge von Relationen aufzustellen, aber die Schwierigkeit beginnt da, wo es sich darum handelt, aus diesem Labyrinth von Formeln einen Ausweg zu finden.
>
> G. Frobenius

The content of this volume is more unified than those of the first two volumes of our attempts to provide proofs of the many beautiful theorems bequeathed to us by Ramanujan in his notebooks. Theta-functions provide the binding glue that blends Chapters 16–21 together. Although we provide proofs here for all of Ramanujan's formulas, in many cases, we have been unable to find the roads that led Ramanujan to his discoveries. It is hoped that others will attempt to discover the pathways that Ramanujan took on his journey through his luxuriant labyrinthine forest of enchanting and alluring formulas.

We first briefly review the content of Chapters 16–21. Although theta-functions play the leading role, several other topics make appearances as well.

Some of Ramanujan's most famous theorems are found in Chapter 16. The chapter begins with basic hypergeometric series and some q-continued fractions. In particular, a generalization of the Rogers–Ramanujan continued fraction and a finite version of the Rogers–Ramanujan continued fraction are found. Entry 7 offers an identity from which the Rogers–Ramanujan identities (found in Section 38) can be deduced as limiting cases, a fact that evidently Ramanujan failed to notice. The material on q-series ends with Ramanujan's celebrated ${}_1\psi_1$ summation. After stating the Jacobi triple product identity, which is a corollary of Ramanujan's ${}_1\psi_1$ summation, Ramanujan commences his work on theta-functions. Several of his results are classical and well known, but Ramanujan offers many interesting new results, especially in Sections 33–35. For an enlightening discussion of Ramanujan's contributions to basic hypergeometric series, as well as to hypergeometric series, see R. Askey's survey paper [8].

Chapter 17 begins with Ramanujan's development of some of the basic theory of elliptic functions highlighted by Entry 6, which provides the basic inversion formula relating theta-functions with elliptic integrals and hypergeometric functions. Section 7 offers many beautiful theorems on elliptic integrals. The following sections are devoted to a catalogue of formulas for the most well-known theta-functions and for Ramanujan's Eisenstein series, L, M, and N, evaluated at different powers of the argument. These formulas are of central importance in proving modular equations in Chapters 19–21.

Several topics are examined in Chapter 18, although most attention is given to the Jacobian elliptic functions. Approximations to π and the perimeter of an ellipse are found. More problems in geometry are discussed in this chapter than in any other chapter. The chapter ends with Ramanujan's initial findings about modular equations.

Chapters 19 and 20 are devoted to modular equations and associated theta-function identities. Most of the results in these two chapters are new and show Ramanujan at his very best. It is here that our proofs undoubtedly often stray from the paths followed by Ramanujan.

Chapter 21 occupies only 4 pages and is the shortest chapter in the second notebook. The content is not unlike that of the previous two chapters, but here the emphasis is on formulas for the series L, M, and N.

Since Ramanujan's death in 1920, there has been much speculation on the sources from which Ramanujan first learned about elliptic functions. In commenting on Ramanujan's paper [2] in Ramanujan's Collected Papers [10], L. J. Mordell writes "It would be extremely interesting to know if and how much Ramanujan is indebted to other writers." Mordell then conjectures that Ramanujan might have studied either Greenhill's [1] or Cayley's [1] books on elliptic functions. Greenhill's book can be found in the library at the Government College of Kumbakonam, but we have been unable to ascertain for certain if this book was in the library when Ramanujan lived in Kumbakonam. Hardy [3, p. 212] remarks that these two books were in the library at the University of Madras, where Ramanujan held a scholarship for nine months before departing for England. Hardy then quotes Littlewood's thoughts: "a sufficient, and I think necessary, explanation would be that Greenhill's very odd and individual Elliptic Functions was his text-book." Mordell, Hardy, and Littlewood surmised that Greenhill's book served as Ramanujan's source of knowledge partly because Greenhill's development avoids the theory of functions of a complex variable, a subject thought to have been never learned by Ramanujan. In particular, the double periodicity of elliptic functions is not mentioned by Greenhill until page 254. In the unorganized portions of the second notebook and in the third notebook, there is some evidence that Ramanujan knew a few facts about complex function theory. (See Berndt's book [11].) However, Ramanujan's development of the theory of elliptic functions did not need or depend on complex function theory.

Ramanujan also never mentions double periodicity. Because Cayley's book contains several sections on modular equations, it is reasonable to conjecture that this book might have been one of Ramanujan's sources of learning.

The origins of Ramanujan's knowledge of elliptic functions are probably not very important, since Ramanujan's development of the subject is uniquely and characteristically his own without a trace of influence by any other author. Ramanujan does not even use the standard notations for elliptic integrals and any of the classical elliptic functions. The content of Ramanujan's initial efforts overlaps with some of Jacobi's findings in his famous *Fundamenta Nova* [1], [2]. However, it is unlikely that Ramanujan had access to this work. Moreover, while the Jacobian elliptic functions were central in Jacobi's development, they play a far more minor role in Ramanujan's theory. (Our proofs in the pages that follow undoubtedly employ the Jacobian elliptic functions more than Ramanujan did.) Both Jacobi and Ramanujan extensively utilized theta-functions, but the evolution of Ramanujan's theory is quite different from that of Jacobi. The classical, general theta-function $\vartheta_3(z, q)$ may be defined by

$$\vartheta_3(z, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2inz}, \tag{I1}$$

where $|q| < 1$ and z is any complex number. Ramanujan's general theta-function $f(a, b)$ is given by

$$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{I2}$$

where $|ab| < 1$. The generalities of (I1) and (I2) are the same. To see this, set $a = q \exp(2iz)$ and $b = q \exp(2iz)$. For many purposes, the definition (I1) is superior. However, for Ramanujan's interests and theory, (I2) is definitely the preferred definition and was strongly instrumental in helping Ramanujan discover many new theorems in the subject.

Upon studying Ramanujan's development of the theory of modular equations in Chapters 18–21, we now are able to understand more clearly the rationale for Ramanujan's introduction of "modular equations" in Sections 15 and 16 of Chapter 15 of his second notebook [9], which we have previously described in Part II [9]. Before returning to this material, we need to define the generalized hypergeometric function ${}_{p+1}F_p$ by

$${}_{p+1}F_p(\alpha_1, \alpha_2, \ldots, \alpha_{p+1}; \beta_1, \beta_2, \ldots, \beta_p; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_{p+1})_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_p)_n} \frac{z^n}{n!},$$

where p is a nonnegative integer, $\alpha_1, \alpha_2, \ldots, \alpha_{p+1}$, $\beta_1, \beta_2, \ldots, \beta_p$ are complex numbers, $|z| < 1$, and

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1),$$

for each nonnegative integer n.

Ramanujan begins his study of "modular equatons" in Chapter 15 by defining

$$F(x) := (1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} x^n = {}_1F_0(\tfrac{1}{2}; x), \qquad |x| < 1. \tag{I3}$$

He then states the trivial identity

$$F\left(\frac{2t}{1+t}\right) = (1+t)F(t^2). \tag{I4}$$

After setting $\alpha = 2t/(1+t)$ and $\beta = t^2$, Ramanujan offers the "modular equation of degree 2,"

$$\beta(2-\alpha)^2 = \alpha^2, \tag{I5}$$

which is readily verified. The factor $(1+t)$ in (I4) is called the multiplier. He then derives some modular equations of higher degree and offers some general remarks. We emphasize that this definition of modular equation has no connection with any of the standard definitions, but we shall draw some parallels shortly.

There are many definitions of a modular equation in the literature. See Ramanathan's paper [10] or our expository introduction to Ramanujan's modular equations [7] for discussions of some of these alternative definitions. We now give the definition of a modular equation that Ramanujan employed and the one that we shall use in the sequel. First, the complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{\pi}{2}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2), \tag{I6}$$

where $0 < k < 1$ and where the series representation in (I6) is found by expanding the integrand in a binomial series and integrating termwise. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', ℓ, and ℓ', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{I7}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and ℓ which is implied by (I7). Ramanujan writes his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = \ell^2$. We shall often say that β has degree n. As we shall see in Section 6 of Chapter 17, modular equations can alternatively be expressed as identities involving theta-functions. In fact, often one first proves a theta-function identity and then transcribes it into an equivalent modular equation by using the formulas in Entries 10–12 in Chapter 17. Ramanujan undoubtedly used this procedure in proving most of his modular equations, and we shall proceed in the same fashion. The multiplier m for a modular equation of degree n is defined by

$$m = \frac{K}{L}. \tag{I8}$$

Ramanujan also established many "mixed" modular equations in which four distinct moduli appear. See the introduction of Chapter 20 for the definition of "mixed" modular equation.

For those not familiar with modular equations, these definitions may appear to be arbitrary and unmotivated. The raison d'etre can be found in the first six sections of Chapter 17. In particular, we note that the base q in the classical theory of elliptic functions is defined by $q = \exp(-\pi K'/K)$. Often one seeks relations among theta-functions where the arguments appearing are q and q^n, for some interger n. Further motivation can be found in two survey articles (Berndt [7], [8]).

Before offering some historical remarks about modular equations, we point out the analogies between Ramanujan's definition of a "modular equation" in Chapter 15 and the standard definition arising from (I7) that we have given above. The function $F(x)$ in (I3) is an analogue of $K(k)$ in (I6). Note that if one of the parameters $\frac{1}{2}$ of ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; k^2)$ in (I6) is replaced by 1, then this hypergeometric function reduces to ${}_1F_0(\frac{1}{2}; k^2)$, which appears in (I6) with $x = k^2$. Observe that (I5) is a relation between the "moduli" α and β. Furthermore, note that the multiplier $1 + \sqrt{\beta}$ in (I4) is analogous to the multiplier defined in (I8).

One could argue, as we did in [7], that the theory of modular equations began in 1771 and 1775 with the appearance of J. Landen's two papers [1], [2] in which Landen's transformation was introduced. Strictly speaking, the theory commenced when A. M. Legendre [2] derived a modular equation of degree 3 in 1825 and C. G. J. Jacobi established modular equations of degrees 3 and 5 in his *Fundamenta Nova* [1], [2] in 1829. Subsequently, in the century that followed, contributions were made by many mathematicians including C. Guetzlaff, L. A. Sohncke, H. Schröter, L. Schläfli, F. Klein, A. Hurwitz, E. Fiedler, A. Cayley, R. Fricke, R. Russell, and H. Weber. Classical texts containing much material on modular equations include those of Enneper [1], Weber [2], [3], Klein [2], [3], and Fricke [3]. Enneper's book [1] and Hanna's paper [1] contain many references to the literature. As we shall see in the remainder of this book, Ramanujan's contributions in the area of modular equations are immense. He discovered many of the classical modular equations found by the aforementioned authors, but he derived many more new ones as well. With little or no exaggeration, we suggest that perhaps Ramanujan found more modular equations than all of his predecessors discovered together. After approximately a half century of dormancy, modular equations have become prominent once again. They arise in the theory of elliptic curves, in the hard hexagon models of lattice gases (Joyce [1]), and in algorithms for the rapid calculation of π (J. M. Borwein [1]; J. M. and P. B. Borwein [1]–[6]; J. M. Borwein, P. B. Borwein, and D. H. Bailey [1]). H. Cohn [1]–[8] and Cohn and J. Deutsch [1] have returned to the classical viewpoints but with a more modern approach and with computer algebra. Further references and applications of modular equations are discussed in our expository survey paper [7]. A briefer and more elementary introduction to modular equations has been given by us in [8]. T. Kondo and T. Tasaka [1], [2], G. Köhler [1], [2], and I. J. Zucker [3] have recently discovered some new beautiful theta-function identities in the spirit of those arising in the theory of modular equations.

Many algebraic, analytic, and elementary methods have been devised to prove modular equations. Except for H. Schröter, we have not found the methods of others helpful in proving Ramanujan's modular equations. Watson (Hardy [3, p. 220]) has declared that "when dealing with Ramanujan's modular equations generally, it has always seemed to me that knowledge of other people's work is a positive disadvantage in that it tends to put one off the shortest track."

In attempting to establish Ramanujan's modular equations, we have utilized three approaches. The first relies on the theory of theta-functions and frequently employs Schröter's formulas, first established in his dissertation [1] in 1854. Schröter's primary theorem is a formula representing a product of theta-functions as a linear combination of products of other theta-functions. Schröter's formulas can be found in the books of Hardy [3, p. 219], Tannery and Molk [1, pp. 163–167], Enneper [1, p. 142], and J. M. and P. B. Borwein [2, p. 111], as well as in a recent paper by Kondo and Tasaka [1]. In our applications, we need to slightly modify Schröter's formulas and obtain related representations for $f(a, b)f(c, d) \pm f(-a, -b)f(-c, -d)$. All of the requisite formulas are proved in detail in Section 36 of Chapter 16. Schröter [1]–[4] utilized his formulas to find several modular equations, although, except for his thesis [1], he never published complete proofs of his results. Ramanujan, to our knowledge, has not explicitly stated Schröter's formulas in any of his published papers, notebooks, or unpublished manuscripts. However, it seems clear, from the theory of theta-functions and modular equations that he did develop, that Ramanujan must have been aware of these formulas or at least of the principles that yield the many special cases that Ramanujan doubtless used. However, Schröter's formulas are applicable in only a small minority of instances. We conjecture that Ramanujan possessed other general formulas or procedures involving theta-functions that are unknown to us. In particular, we think that he had derived a formula involving quotients of theta-functions that he did not record in his notebooks and that we have been unable to find elsewhere in the literature as well. Watson [5, p. 150] asserted that "a prolonged study of his modular equations has convinced me that he was in possession of a general formula by means of which modular equations can be constructed in almost terrifying numbers." Watson then intimates that Ramanujan's "general formula" is, in fact, Schröter's most general formula. However, as pointed out above, Schröter's formulas cannot be used in most instances. Further efforts should be made in attempting to discover Ramanujan's analytical methods.

The second method exploits previously derived modular equations and may involve a heavy dosage of elementary algebra. The primary idea is to find parametric representations for α and β which are then employed along with elementary algebra to verify a given modular equation. Ramanujan probably used such methods, especially for small values of the degree n. The algebraic difficulties normally increase very rapidly with n. Some of our algebraic proofs are very tedious, and it is doubtful that Ramanujan would have employed such drudgery. Ramanujan, with his great skills in spotting algebraic relationships, could undoubtedly discover modular equations using algebraic manipulation, but, particularly in Chapters 19 and 20, the reader will see that some of the proofs presented here could not have been accomplished without knowing the modular equation in advance.

Our third method employs the theory of modular forms. In some ways, this represents the best approach. First, the theory of modular forms provides the theoretical basis which explains why certain identities among theta-functions exist. Second, this approach usually does not become too much more complicated with increasing n, and so proofs remain comparatively short, after the requisite theory has been developed. The primary disadvantage to this method is that the modular equation must be known in advance, and so, as in the second approach, the proofs are more properly called verifications. The principal idea is to show that the multiplier systems of certain modular forms agree and that the coefficients in the expansion of a certain modular form are equal to zero up to a certain prescribed point. We then can conclude that the modular form must identically be equal to zero. This approach has been used by A. J. Biagioli [1], S. Raghavan [1], [2], Raghavan and S. S. Rangachari [1], and R. J. Evans [1] in establishing several of Ramanujan's theta-function identities. It might be argued that Ramanujan used a variant of this method by comparing coefficients in the expansions of theta-functions. This is extremely doubtful, however, because Ramanujan would not have discovered the identities by this procedure.

An earlier version of Chapter 16, coauthored with C. Adiga, S. Bhargava, and G. N. Watson, was published in "Chapter 16 of Ramanujan's second notebook: Theta-functions and q-series," *Memoirs of the American Mathematical Society*, vol. 53, no. 315, 1985. The revised version appears here by permission of the American Mathematical Society. A substantial majority of the theorems and proofs appearing in Chapters 17–21 have not heretofore appeared in print. B. C. Berndt, A. J. Biagioli, and J. M. Purtilo [1]–[3] have proved some of Ramanujan's modular equations in journals commemorating the centenary of Ramanujan's birth. A brief description of Ramanujan's work on Eisenstein series in Chapter 21 was given by us in [10]. Some of Ramanujan's work on modular equations has also been examined by K. G. Ramanathan [9], [10], V. R. Thiruvenkatachar and K. Venkatachaliengar [1], and K. Venkatachaliengar [1].

To help readers find modular equations of certain degrees, we offer a table indicating the chapter and sections where the desired modular equations may be found.

Degree	Chapter	Sections
3	19	5, 7
5	19	11, 13
7	19	18, 19
	20	21
11	20	7
13	20	8
15	20	21
17	20	12
19	20	16
23	20	15
31	20	22
47	20	23
71	20	23
3, 9	20	3
5, 25	19	15
3, 5, 15	20	11
3, 7, 21	20	13
3, 9, 27	20	5
3, 11, 33	20	14
3, 13, 39	20	19, 21
3, 21, 63	20	20
3, 29, 87	20	24
5, 7, 35	20	18, 19
5, 11, 55	20	19, 21
5, 19, 95	20	20
5, 27, 135	20	24
7, 9, 63	20	19, 21
7, 17, 119	20	20
7, 25, 175	20	24
9, 15, 135	20	20
9, 23, 207	20	24
11, 13, 143	20	20
11, 21, 231	20	24
13, 19, 247	20	24
15, 17, 255	20	24

Each of Chapters 16–20 in the second notebook contains 12 pages, while Chapter 21 has only 4 pages. The number of theorems, corollaries, and examples found in each chapter is listed in the following table.

Chapter	Number of Results
16	134
17	162
18	135
19	185
20	173
21	45
Total	834

Many of the theorems that Ramanujan communicated in his letters of January 16, 1913 and February 27, 1913 to G. H. Hardy may be found in Chapters 16–21. We list these results in the following table.

Location in Collected Papers	Location in Notebooks
p. xxviii, (1)	Chapter 16, Entry 15 and corollary, Entry 39 (i)
p. xxviii, (6)	Chapter 20, Entry 20 (i)
p. xxix, (15)	Chapter 18, Corollary in Section 12
p. xxix, (20) (i), (v)	Chapter 20, Entries 11 (i), (ii), (xiv)
p. xxix, (21)	Chapter 20, Entry 19 (iii)
p. 350, (3)	Chapter 18, Entry 12 (ii)
p. 353, (20) (ii), (iii), (iv), (vi)	Chapter 20, Entries 11 (iii), (iv), (v), (xv)
p. 353, (21)	Chapter 20, Entry 19 (iii)
p. 353, (22)	Chapter 20, Entry 24 (i)

A few of Ramanujan's published papers and questions posed to readers of the *Journal of the Indian Mathematical Society* have their origins in Chapters 16–21 of the second notebook. In some cases, only a small portion of the paper actually arises from material in the notebooks. The following table lists those papers and the corresponding locations in the notebooks.

Paper	Location in Notebooks
Squaring the circle	Chaper 18, Entry 20 (i)
Modular equations and approximations to π	Chapter 18, Entry 3, Corollary in Section 3; Chapter 21
Question 584	Chapter 16, Entries 38 (i), (ii)
Some definite integrals	Chapter 16, Entry 14
Question 662	Chapter 19, Entry 7 (iv) (first part)
On certain arithmetical functions	Chapter 16, Section 35; Chapter 17, Entry 13
Question 755	Chapter 18, Corollary (ii) of Section 19
Proof of certain identities in combinatory analysis	Chapter 16, Entries 38 (i), (ii)

In the sequel, equation numbers refer to equations in the same chapter, unless another chapter is indicated. Unless otherwise stated, page numbers refer to pages in the pagination of the Tata Institute's publication of Ramanujan's second notebook [9]. Page numbers unattended by any reference number always refer to Ramanujan's second notebook. Parts I and II refer to the author's accounts [5] and [9], respectively, of Ramanujan's notebooks.

We mention some standard notations that will be used in the sequel. The rational integers, the rational numbers, the real numbers, and the complex numbers are denoted by $\mathbb{I}$, Q, $\mathbb{R}$, and $\mathscr{C}$, respectively. The residue of a meromorphic function f at a pole α is denoted by R_α, if the identity of the function f is understood.

I am very grateful to many mathematicians for the proofs and suggestions that they have supplied. I am most indebted to G. N. Watson for the notes that he compiled on Chapters 16–21. In particular, many of the proofs in Chapters 19–21 are due to Watson. F. J. Dyson [1, p. 7] has affirmed that "Watson was chief gardener in the 1930's and worked hard to develop and elucidate Ramanujan's ideas." Evidently, Watson was very careful about whom he would permit to stroll through this garden. However, through the extensive notes that he left behind, he has allowed me to view many of the flowers in the garden, and I am very appreciative.

I owe special thanks to the following mathematicians. C. Adiga and S. Bhargava made many contributions in their coauthoring an earlier version of Chapter 16 with me. The quality of Chapter 16 has greatly been enhanced by the many suggestions offered by R. A. Askey. A. J. Biagioli and J. M. Purtilo provided invaluable and necessary help in the theory of modular forms and MACSYMA, respectively. R. J. Evans [1] furnished beautiful proofs of some of Ramanujan's most intractable theta-function identities, and we have reproduced in the sequel much of his paper. L. Jacobsen has contributed several helpful remarks and suggestions on continued fractions.

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The author bears the responsibility for all errors and wishes to be notified of such, whether they be minor or serious.

Most of the manuscript for this book was typed by Dee Wrather, and I thank her for her very accurate and rapid typing.

The figures in Chapters 19 and 20 were drawn by Jonathan Manton using the graphics of Mathematica.

A perusal of the references at the conclusion of this book indicates that several are obscure. Nancy Anderson, the mathematics librarian at the University of Illinois, helped to unearth many of these, and I owe her special thanks.

Lastly, I express my deep gratitude to James Vaughn and the Vaughn Foundation, and to the National Science Foundation for their financial support during several summers.

CHAPTER 16

q-Series and Theta-Functions

In Chapter 16, Ramanujan develops two closely related topics, q-series and theta-functions. The first 17 sections are devoted primarily to q-series, while the latter 22 sections constitute a very thorough development of the theory of theta-functions.

Ramanujan begins by stating some mostly familiar theorems in the theory of q-series. In particular, Ramanujan rediscovered some of Heine's famous theorems including his q-analogue of Gauss' theorem. However, several results appear to be new. Perhaps most noteworthy in this respect are the continued fractions in Sections 10–13. (Entry 10 is not a q-continued fraction and is more properly placed in Chapter 12 among other theorems of this type.) Entry 13 was later generalized by Ramanujan in his "lost notebook" [11]. Entry 16 is a "finite" form of what is now generally known as the "Rogers–Ramanujan continued fraction" and was first established in print by Hirschhorn [1] in 1972 while being unaware that the result is found in Ramanujan's notebooks.

As is to be expected, Ramanujan's findings in the theory of theta-functions contain many of their classical properties. In particular, he rediscovered several theorems found in Jacobi's epic *Fundamenta Nova* [1], [2]. In Entry 27, Ramanujan records transformation formulas for the modular transformation: $T(\tau) = -1/\tau$. He did not discover more general transformation formulas. In Entry 19, Ramanujan gives the famous Jacobi triple product identity of which he made numerous applications. Because several of our proofs employ Watson's quintuple product identity, it would seem that Ramanujan had discovered it. Indeed, the quintuple product identity can be found in Ramanujan's "lost notebook" [11]. Results in the last part of Chapter 16 indicate that Ramanujan had found Schröter's formulas [1]. Although Ramanujan does not give these formulas in their most general form, he does offer several special cases and deductions from them.

But more importantly, Ramanujan discovered several new and deep theorems in the theory of theta-functions. For example, the beautiful theorems in Sections 33–35 appear to be new, as well as Entry 38(iv) and the corollaries in Section 37.

In closing our brief survey of the content of Chapter 16, we would like to mention that this chapter contains four results that are due originally to Ramanujan and for which he is justly famous. Entry 14 offers Ramanujan's q-analogue of the beta-function. The evaluation of this integral was first recorded by Ramanujan in [4], [10, p. 57]. There are now at least four distinct verifications. In Entry 17 we find "Ramanujan's ${}_1\psi_1$ summation." Several proofs, including a new one offered here, now exist. Ramanujan found many applications for his ${}_1\psi_1$ summation, including a proof of Jacobi's triple product identity. The remarkable Rogers–Ramanujan identities are found in Entries 38(i), (ii), and the "Rogers–Ramanujan continued fraction" in Entry 38(iii). It might be remarked that this continued fraction is the only continued fraction proved in Ramanujan's published papers. However, he did submit several formulas containing continued fractions to the problems section of the *Journal of the Indian Mathematical Society.* Also, Ramanujan's letters to Hardy contain many beautiful theorems on continued fractions.

We conclude our introduction with several remarks on notation. For those reading this book in conjunction with the notebooks, it seems best to retain Ramanujan's notation $f(a, b)$ for the theta-functions (see (18.1)). We remark that $f(a, b) = \vartheta_3(z, \tau)$, where $ab = e^{2\pi i\tau}$, $a/b = e^{4iz}$, and $\vartheta_3(z, \tau)$ denotes the classical theta-function in the notation of Whittaker and Watson [1]. Most of the results in the sequel are, in fact, more easily stated in the notation $f(a, b)$ rather than in the notation $\vartheta_3(z, \tau)$. Ramanujan uses x to denote his primary variable. Since q is almost universally used today instead of x, we have adopted the more standard designation. It is assumed throughout the sequel that $|q| < 1$. As usual, for any complex number a, we write

$$(a)_k := (a; q)_k := (1 - a)(1 - aq)(1 - aq^2)\cdots(1 - aq^{k-1})$$

and

$$(a)_\infty := (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k).$$

Ramanujan writes $\prod(-a, x)$ for $(a)_\infty$, where $x = q$. The basic hypergeometric series ${}_{s+1}\varphi_s$ is defined by

$${}_{s+1}\varphi_s\left[\begin{matrix} a_1, a_2, \ldots, a_{s+1} \\ b_1, b_2, \ldots, b_s \end{matrix}; x\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_{s+1})_k}{(b_1)_k(b_2)_k\cdots(b_s)_k} \frac{x^k}{(q)_k}, \tag{0.1}$$

where $|x| < 1$ and $a_1, a_2, \ldots, a_{s+1}$, $b_1, b_2, \ldots, b_s$ are arbitrary, except that, of course, $(b_j)_k \neq 0$, $1 \leq j \leq s$, $0 \leq k < \infty$. If s is "small," we shall write ${}_{s+1}\varphi_s(a_1, \ldots, a_{s+1}; b_1, \ldots, b_s; x)$ in place of the notation at the left side of (0.1). Finally, to denote the dependence on the base q, we may write ${}_{s+1}\varphi_s(a_1, \ldots, a_{s+1}; b_1, \ldots, b_s; q; x)$.

Entry 1. *Let q be real with $|q| < 1$, and suppose that a and x are any complex numbers. Let the principal branches of $(1-a)^x$ and $(1-q)^x$ be chosen. Then*

(i)
$$\lim_{q\to 1} \frac{(a)_\infty}{(aq^x)_\infty} = (1-a)^x,$$

(ii)
$$\lim_{q\to 1} \frac{(q)_\infty}{(1-q)^x(q^{x+1})_\infty} = \Gamma(x+1),$$

(iii)
$$(a)_\infty = \prod_{k=0}^{n-1} (aq^k; q^n)_\infty,$$

and

(iv)
$$(a)_\infty = \frac{(a; \sqrt{q})_\infty}{(a\sqrt{q}; q)_\infty}.$$

Proof. First assume that $|a| < 1$. Apply (2.1) below with a and t replaced by aq^x and q^{-x}, respectively. Hence,

$$\lim_{q\to 1} \frac{(a)_\infty}{(aq^x)_\infty} = 1 + \sum_{k=1}^{\infty} \frac{(-x)(-x+1)\cdots(-x+k-1)}{k!} a^k = (1-a)^x,$$

by the binomial theorem. The general result follows by analytic continuation.

The following proof of (ii) is due to R. W. Gosper, Write

$$\lim_{q\to 1} \frac{(q)_\infty}{(1-q)^x(q^{x+1})_\infty} = \lim_{q\to 1} \prod_{k=1}^{\infty} \frac{1-q^k}{1-q^{k+x}} \left(\frac{1-q^{k+1}}{1-q^k}\right)^x$$
$$= \prod_{k=1}^{\infty} \frac{k}{k+x} \left(\frac{k+1}{k}\right)^x = \Gamma(x+1).$$

Identity (iii) follows easily by regrouping the factors on the left side. To prove (iv), let $n = 2$ in (iii) and replace q by $\sqrt{q}$.

The q-gamma function $\Gamma_q(x)$ is defined by

$$\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1-q)^{1-x}. \tag{1.1}$$

Thus, Entry 1(ii) may be rewritten in the form

$$\lim_{q\to 1-} \Gamma_q(x+1) = \Gamma(x+1).$$

Gosper's proof of Entry 1(ii) may also be found in Andrews' monograph [18, p. 109]. Our proofs of Entries 1(i), (ii) are not completely rigorous, because limits were taken without justification under the summation and product signs, respectively. T. H. Koornwinder [1] has indeed justified these formal processes and provided rigorous proofs.

Ramanujan's proof of Entry 2 below can be found in his paper [4] [10, pp. 57–58].

Entry 2. *If* $|q|, |a| < 1$, *then*

$$\frac{(-b)_\infty}{(a)_\infty} = \sum_{k=0}^{\infty} \frac{(-b/a)_k a^k}{(q)_k}.$$

The earliest known reference for Entry 2, the q-binomial theorem, is the work of Rothe [1]. Entry 2 was also discovered by Cauchy [1], [2, pp. 42–50] and has been attributed to him, Euler, Gauss, and Heine. If we put $-b = at$, Entry 2 may be written in the form

$$\frac{(at)_\infty}{(a)_\infty} = \sum_{k=0}^{\infty} \frac{(t)_k a^k}{(q)_k}. \tag{2.1}$$

Entry 3. *If a is arbitrary and* $|q| < 1$, *then*

$$\frac{1}{(aq)_\infty} = \sum_{k=0}^{\infty} \frac{a^k q^{k^2}}{(q)_k (aq)_k}. \tag{3.1}$$

Entry 3 is normally attributed to Cauchy [1], [2, pp. 42–50]. However, (3.1) can be found in Jacobi's *Fundamenta Nova* [1], [2, p. 232] published 14 years earlier. We defer a proof of Entry 3 until Section 9 where a generalization will be proved.

Entry 4. *If* $|abc| < 1$, *then*

$$\frac{(ab)_\infty (ac)_\infty}{(a)_\infty (abc)_\infty} = \sum_{k=0}^{\infty} \frac{(1/b)_k (1/c)_k (abc)^k}{(q)_k (a)_k}.$$

Entry 4 is a famous result of Heine [1] and is the q-analogue of Gauss' summation of the ordinary hypergeometric series. For a proof of Entry 4, see Andrews' text [9, p. 20].

Observe that if we replace b by $1/t$, c by $1/c$, and then a by atc and lastly put $c = 0$, we obtain (2.1). Letting b and c tend to 0 and replacing a by aq in Entry 4, we deduce Entry 3.

As Askey [8, p. 69] has observed, Ramanujan formulates his discoveries on basic hypergeometric series to emphasize the symmetry of the *value* of the sum, of which the statements of Entries 4 and 5, for example, attest.

Entry 5. *If* $|q|, |abcd| < 1$, *then*

$$\sum_{k=0}^{\infty} \frac{(a/q)_k (1/b)_k (1/c)_k (1/d)_k (1 - aq^{2k-1})(abcd)^k}{(ab)_k (ac)_k (ad)_k (q)_k (1 - a/q)}$$
$$= \frac{(a)_\infty (abc)_\infty (abd)_\infty (acd)_\infty}{(ab)_\infty (ac)_\infty (ad)_\infty (abcd)_\infty}.$$

Entry 5 was first found by L. J. Rogers [2, p. 29] (where $v - \sqrt{q}$ should be replaced by $1 - v\sqrt{q}$ in one factor). It is a limiting case of a more general identity found by F. H. Jackson [1]. Another proof of Jackson's theorem has been given by Andrews and Askey [1].

PROOF. The aforementioned identity of Jackson is the q-analogue of Dougall's theorem and is given by (Bailey [4, p. 67])

$$\begin{aligned}&{}_8\varphi_7\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, q^{-N} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq^{N+1}\end{matrix}; q\right] \\ &\qquad = \frac{(aq)_N(aq/cd)_N(aq/bd)_N(aq/bc)_N}{(aq/b)_N(aq/c)_N(aq/d)_N(aq/bcd)_N},\end{aligned} \tag{5.1}$$

where N is a positive integer and $a^2q^{N+1} = bcde$. Observe that

$$\frac{(q\sqrt{a})_k(-q\sqrt{a})_k}{(\sqrt{a})_k(-\sqrt{a})_k} = \frac{1 - aq^{2k}}{1 - a}. \tag{5.2}$$

We now let N tend to ∞ and e tend to 0 in (5.1). Since

$$\lim_{e\to 0} \frac{(e)_k(q^{-N})_k}{(aq/e)_k(aq^{N+1})_k} = \lim_{e\to 0} \frac{(e)_k(a^2q/bcde)_k}{(aq/e)_k(bcde/a)_k} = \left(\frac{a}{bcd}\right)^k,$$

we find that

$$\begin{aligned}&\sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k(d)_k(1 - aq^{2k})}{(aq/b)_k(aq/c)_k(aq/d)_k(q)_k(1 - a)}\left(\frac{aq}{bcd}\right)^k \\ &\qquad = \frac{(aq)_\infty(aq/cd)_\infty(aq/bd)_\infty(aq/bc)_\infty}{(aq/b)_\infty(aq/c)_\infty(aq/d)_\infty(aq/bcd)_\infty}.\end{aligned} \tag{5.3}$$

Replacing a by a/q, b by $1/b$, c by $1/c$, and d by $1/d$ in (5.3), we deduce Entry 5 at once.

Entry 6. *If* $|a|, |c|, |q| < 1$, *then*

$${}_2\varphi_1(b/a, c; d; a) = \frac{(b)_\infty(c)_\infty}{(a)_\infty(d)_\infty}\,{}_2\varphi_1(d/c, a; b; c).$$

This beautiful theorem is due to Heine [1], and a simple proof based on Entry 2 may be found in Andrews' book [9, p. 19]. In the other direction, setting $c = d$ in Entry 6, we obtain Entry 2.

Applying Entry 6 three times, we find that

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k(q)_k} t^k &= \frac{(b)_\infty(at)_\infty}{(c)_\infty(t)_\infty} \sum_{k=0}^{\infty} \frac{(c/b)_k(t)_k}{(at)_k(q)_k} b^k \\ &= \frac{(b)_\infty(at)_\infty}{(c)_\infty(t)_\infty}\,\frac{(c/b)_\infty(bt)_\infty}{(at)_\infty(b)_\infty} \sum_{k=0}^{\infty} \frac{(b)_k(abt/c)_k}{(bt)_k(q)_k}\left(\frac{c}{b}\right)^k \\ &= \frac{(c/b)_\infty(bt)_\infty}{(c)_\infty(t)_\infty}\,\frac{(abt/c)_\infty(c)_\infty}{(c/b)_\infty(bt)_\infty} \sum_{k=0}^{\infty} \frac{(c/a)_k(c/b)_k}{(c)_k(q)_k}\left(\frac{abt}{c}\right)^k \\ &= \frac{(abt/c)_\infty}{(t)_\infty} \sum_{k=0}^{\infty} \frac{(c/a)_k(c/b)_k}{(c)_k(q)_k}\left(\frac{abt}{c}\right)^k,\end{aligned} \tag{6.1}$$

which we use below.

Entry 7. *If* $|q| < 1$, *then*

$$\sum_{k=0}^{\infty} \frac{(a)_k(d/b)_k(d/c)_k(d/q)_k(1-dq^{2k-1})(bc/a)^k q^{k(k-1)}}{(b)_k(c)_k(d/a)_k(q)_k(1-d/q)}$$

$$= \frac{(a)_\infty(d)_\infty}{(b)_\infty(c)_\infty} \sum_{k=0}^{\infty} \frac{(b/a)_k(c/a)_k}{(d/a)_k(q)_k} a^k.$$

Proof. By a theorem of Watson [2] (Bailey [4, p. 69]),

$$ {}_8\varphi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, c, d, e, f, q^{-N} \\ \sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f, aq^{N+1} \end{matrix}; \frac{a^2q^{N+2}}{cdef}\right]$$

$$= \frac{(aq)_\infty(aq^{N+1}/f)_\infty(aq^{N+1}/e)_\infty(aq/ef)_\infty}{(aq/e)_\infty(aq/f)_\infty(aq^{N+1})_\infty(aq^{N+1}/ef)_\infty} {}_4\varphi_3\left[\begin{matrix} aq/cd, e, f, q^{-N} \\ efq^{-N}/a, aq/c, aq/d \end{matrix}; q\right], \tag{7.1}$$

where N is a positive integer. Short calculations show that

$$\lim_{N\to\infty} \frac{q^{Nk}(q^{-N})_k}{(aq^{N+1})_k} = (-1)^k q^{k(k-1)/2}$$

and

$$\lim_{N\to\infty} \frac{(q^{-N})_k}{(efq^{-N}/a)_k} = \left(\frac{a}{ef}\right)^k.$$

Letting N tend to ∞ in (7.1) and using the calculations above as well as (5.2), we find that

$$\sum_{k=0}^{\infty} \frac{(a)_k(c)_k(d)_k(e)_k(f)_k(1-aq^{2k})(-a^2/cdef)^k q^{k(k+3)/2}}{(aq/c)_k(aq/d)_k(aq/e)_k(aq/f)_k(q)_k(1-a)}$$

$$= \frac{(aq)_\infty(aq/ef)_\infty}{(aq/e)_\infty(aq/f)_\infty} {}_3\varphi_2\left[\begin{matrix} aq/cd, e, f \\ aq/c, aq/d \end{matrix}; \frac{aq}{ef}\right]. \tag{7.2}$$

We next let d tend to ∞ in (7.2). Since

$$\lim_{d\to\infty} \frac{(d)_k d^{-k}}{(aq/d)_k} = (-1)^k q^{k(k-1)/2},$$

we find that

$$\sum_{k=0}^{\infty} \frac{(a)_k(c)_k(e)_k(f)_k(1-aq^{2k})(a^2/cef)^k q^{k(k+1)}}{(aq/c)_k(aq/e)_k(aq/f)_k(q)_k(1-a)}$$

$$= \frac{(aq)_\infty(aq/ef)_\infty}{(aq/e)_\infty(aq/f)_\infty} \sum_{k=0}^{\infty} \frac{(e)_k(f)_k}{(aq/c)_k(q)_k}\left(\frac{aq}{ef}\right)^k. \tag{7.3}$$

Replacing a, c, e, and f by d/q, a, d/b, and d/c, respectively, in (7.3), we find that

$$\sum_{k=0}^{\infty} \frac{(a)_k(d/b)_k(d/c)_k(d/q)_k(1-dq^{2k-1})(bc/a)^k q^{k(k-1)}}{(b)_k(c)_k(d/a)_k(q)_k(1-d/q)}$$

$$= \frac{(d)_\infty(bc/d)_\infty}{(b)_\infty(c)_\infty} \sum_{k=0}^{\infty} \frac{(d/b)_k(d/c)_k}{(d/a)_k(q)_k}\left(\frac{bc}{d}\right)^k$$

$$= \frac{(d)_\infty (bc/d)_\infty}{(b)_\infty (c)_\infty} \frac{(a)_\infty}{(bc/d)_\infty} \sum_{k=0}^{\infty} \frac{(b/a)_k (c/a)_k}{(d/a)_k (q)_k} a^k,$$

where we have applied (6.1). This completes the proof of Entry 7.

An important application of Entry 7 will be made in Section 38.

We now prove a lemma from which Entries 8 and 9 will follow as limiting cases.

Lemma. *For* $|de/abc|, |e/a|, |q| < 1$,

$${}_3\varphi_2(a, b, c; d, e; de/abc) = \frac{(e/a)_\infty (de/bc)_\infty}{(e)_\infty (de/abc)_\infty} {}_3\varphi_2(a, d/b, d/c; d, de/bc; e/a). \qquad (8.1)$$

PROOF. Using Entry 2 and (6.1), we find that, for $|a|, |e/a|, |de/abc| < 1$,

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k (q)_k} \left(\frac{de}{abc}\right)^k = \frac{(a)_\infty}{(e)_\infty} \sum_{k=0}^{\infty} \frac{(b)_k (c)_k (eq^k)_\infty}{(d)_k (q)_k (aq^k)_\infty} \left(\frac{de}{abc}\right)^k$$

$$= \frac{(a)_\infty}{(e)_\infty} \sum_{k=0}^{\infty} \frac{(b)_k (c)_k}{(d)_k (q)_k} \left(\frac{de}{abc}\right)^k \sum_{m=0}^{\infty} \frac{(e/a)_m}{(q)_m} (aq^k)^m$$

$$= \frac{(a)_\infty}{(e)_\infty} \sum_{m=0}^{\infty} \frac{(e/a)_m}{(q)_m} a^m \sum_{k=0}^{\infty} \frac{(b)_k (c)_k}{(d)_k (q)_k} \left(\frac{deq^m}{abc}\right)^k$$

$$= \frac{(a)_\infty}{(e)_\infty} \sum_{m=0}^{\infty} \frac{(e/a)_m}{(q)_m} a^m \frac{(eq^m/a)_\infty}{(deq^m/abc)_\infty} \sum_{k=0}^{\infty} \frac{(d/b)_k (d/c)_k}{(d)_k (q)_k} \left(\frac{eq^m}{a}\right)^k$$

$$= \frac{(a)_\infty (e/a)_\infty}{(e)_\infty (de/abc)_\infty} \sum_{k=0}^{\infty} \frac{(d/b)_k (d/c)_k}{(d)_k (q)_k} \left(\frac{e}{a}\right)^k \sum_{m=0}^{\infty} \frac{(de/abc)_m}{(q)_m} (aq^k)^m$$

$$= \frac{(a)_\infty (e/a)_\infty}{(e)_\infty (de/abc)_\infty} \sum_{k=0}^{\infty} \frac{(d/b)_k (d/c)_k}{(d)_k (q)_k} \left(\frac{e}{a}\right)^k \frac{(deq^k/bc)_\infty}{(aq^k)_\infty},$$

by Entry 2 again. The restriction $|a| < 1$ may now be removed by analytic continuation, and the lemma easily follows.

Entry 8. *If* $|a|, |q| < 1$, *then*

$$\frac{(a)_\infty}{(b)_\infty} \sum_{k=0}^{\infty} \frac{(c)_k (b/a)_k}{(d)_k (q)_k} a^k = \sum_{k=0}^{\infty} \frac{(-1)^k (b/a)_k (d/c)_k (ac)^k q^{k(k-1)/2}}{(b)_k (d)_k (q)_k}.$$

PROOF. Let b tend to ∞ in (8.1). Then replace a, c, and e by b/a, d/c, and b, respectively, to achieve the desired result.

Observe that Entry 8 is a q-extension of Pfaff's transformation (Bailey [4, p. 10])

$${}_2F_1(a, b; c; x) = (1 - x)^{-a} {}_2F_1(a, c - b; c; -x/(1 - x)).$$

Entry 9. *If* $|q| < 1$, *then*

$$(aq)_\infty \sum_{k=0}^{\infty} \frac{b^k q^{k^2}}{(q)_k (aq)_k} = \sum_{k=0}^{\infty} \frac{(-1)^k (b/a)_k a^k q^{k(k+1)/2}}{(q)_k}.$$

FIRST PROOF. Letting a and b tend to ∞ in (8.1), we deduce that

$$\sum_{k=0}^{\infty} \frac{(c)_k q^{k^2-k}}{(d)_k (e)_k (q)_k} \left(\frac{de}{c}\right)^k = \frac{1}{(e)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (d/c)_k e^k q^{k(k-1)/2}}{(d)_k (q)_k}.$$

Next, let $d = bc/a$ and $e = aq$. Letting c tend to 0, we complete the proof of Entry 9.

SECOND PROOF. Using Entry 6 twice, we find that

$$\begin{aligned} {}_2\varphi_1(a, b; c; x) &= \frac{(ax)_\infty (b)_\infty}{(x)_\infty (c)_\infty} {}_2\varphi_1(c/b, x; ax; b) \\ &= \frac{(bx)_\infty (c/b)_\infty}{(x)_\infty (c)_\infty} {}_2\varphi_1(abx/c, b; bx; c/b). \end{aligned}$$

Now replace x by x/ab and let a and b tend to ∞. This yields

$$\sum_{k=0}^{\infty} \frac{x^k q^{k^2-k}}{(q)_k (c)_k} = \frac{1}{(c)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (x/c)_k c^k q^{k(k-1)/2}}{(q)_k}.$$

Replacing x by bq and c by aq, we obtain Entry 9.

V. Ramamani [1] has given a proof of Entry 9 by obtaining two functional relations for the right side. Andrews [10] has shown that Entry 9 is a limiting case of an identity due to Rogers. V. Ramamani and K. Venkatachaliengar [1] have established Entry 9 by showing that it is a limiting case of Heine's transformation, Entry 6. A generalization of Entry 9 has been discovered by Bhargava and Adiga [3]. H. M. Srivastava [1] subsequently established an equivalent form of their result. The particular case $a = -1$ of Entry 9 was posed as a problem by Carlitz [2].

Observe that if we let $a = b$, then Entry 9 reduces to Entry 3.

Corollary (i). *If* $|q| < 1$, *then*

$$(q)_\infty \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q)_k^2} = \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)/2}.$$

PROOF. Put $a = 1$ and $b = q$ in Entry 9.

Corollary (ii). *If* $|q| < 1$, *then*

$$(q; q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{k(2k+1)}}{(q)_{2k}} = \sum_{k=0}^{\infty} (-1)^k q^{k^2}.$$

PROOF. Replace q by q^2 in Entry 9 and then set $a = 1/q$ and $b = q$.

Entry 10. *Let* x, ℓ, m, *and* n *denote complex numbers. Define*

$$P=\frac{\Gamma(\frac{1}{2}(x+\ell-m+n+1))\Gamma(\frac{1}{2}(x+\ell-m-n+1))\Gamma(\frac{1}{2}(x-\ell+m+n+1))\Gamma(\frac{1}{2}(x-\ell+m-n+1))}{\Gamma(\frac{1}{2}(x-\ell-m+n+1))\Gamma(\frac{1}{2}(x-\ell-m-n+1))\Gamma(\frac{1}{2}(x+\ell+m+n+1))\Gamma(\frac{1}{2}(x+\ell+m-n+1))}.$$

Then, if either ℓ, m, *or* n *is an integer or if* $\operatorname{Re} x > 0$,

$$\frac{1-P}{1+P}=\frac{2\ell m x}{x^2+\ell^2+m^2-n^2-1+}\;\frac{4(x^2-1^2)(\ell^2-1^2)(m^2-1^2)}{3(x^2+\ell^2+m^2-n^2-5)}$$
$$+\;\frac{4(x^2-2^2)(\ell^2-2^2)(m^2-2^2)}{5(x^2+\ell^2+m^2-n^2-13)}$$
$$+\cdots+\;\frac{4(x^2-(k-1)^2)(\ell^2-(k-1)^2)(m^2-(k-1)^2)}{(2k-1)(x^2+\ell^2+m^2-n^2-2k^2+2k-1)}+\cdots.$$

PROOF. We apply Entry 40 from Chapter 12 in Ramanujan's second notebook [9, p. 163] (Part II [9, pp. 151–152]), which was initially proved by Watson [6]. Let

$$R=\Pi\Gamma(\tfrac{1}{2}(\alpha\pm\beta\pm\gamma\pm\delta\pm\varepsilon+1)),$$

where the product contains eight gamma functions and where the argument of each gamma function contains an even number of minus signs. Let

$$Q=\Pi\Gamma(\tfrac{1}{2}(\alpha\pm\beta\pm\gamma\pm\delta\pm\varepsilon+1)),$$

where the product contains eight gamma functions and where the argument of each gamma function contains an odd number of minus signs. Suppose that at least one of the parameters β, γ, δ, ε is equal to a nonzero integer. Then

$$\frac{1-Q/R}{1+Q/R}$$
$$=\frac{8\alpha\beta\gamma\delta\varepsilon}{1\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\varepsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2-1)^2-2^2\}}$$
$$+\;\frac{64(\alpha^2-1^2)(\beta^2-1^2)(\gamma^2-1^2)(\delta^2-1^2)(\varepsilon^2-1^2)}{3\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\varepsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2-5)^2-6^2\}}$$
$$+\;\frac{64(\alpha^2-2^2)(\beta^2-2^2)(\gamma^2-2^2)(\delta^2-2^2)(\varepsilon^2-2^2)}{5\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\varepsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2-13)^2-14^2\}}+\cdots. \tag{10.1}$$

In (10.1), let $\alpha = x$, $\beta = n - \varepsilon$, $\gamma = \ell$, and $\delta = m$, where ε is a positive integer. In the quotient Q/R of 16 gamma functions, we observe that eight are independent of ε and eight depend on ε. The quotient that is independent of ε is precisely equal to P, while the quotient that depends on ε is equal to

$$\frac{\Gamma(\frac{1}{2}(x+\ell+m+n-2\varepsilon+1))\Gamma(\frac{1}{2}(x+\ell+m-n+2\varepsilon+1))\Gamma(\frac{1}{2}(x-\ell-m-n+2\varepsilon+1))\Gamma(\frac{1}{2}(x-\ell-m+n-2\varepsilon+1))}{\Gamma(\frac{1}{2}(x-\ell+m+n-2\varepsilon+1))\Gamma(\frac{1}{2}(x-\ell+m-n+2\varepsilon+1))\Gamma(\frac{1}{2}(x+\ell-m-n+2\varepsilon+1))\Gamma(\frac{1}{2}(x+\ell-m+n-2\varepsilon+1))}.$$

By Stirling's formula, the quotient above tends to 1 as ε tends to ∞. Hence,

$$\lim_{\varepsilon\to\infty} \frac{1 - Q/R}{1 + Q/R} = \frac{1 - P}{1 + P}. \tag{10.2}$$

We next examine the right side of (10.1) as ε tends to ∞. An elementary calculation shows that

$$\begin{aligned} &2(x^4 + (n-\varepsilon)^4 + \ell^4 + m^4 + \varepsilon^4 + 1) - (x^2 + (n-\varepsilon)^2 + \ell^2 + m^2 + \varepsilon^2 \\ &\qquad - (2j^2 + 2j + 1))^2 \\ &\qquad = 4(n^2 - x^2 - \ell^2 - m^2 + 2j^2 + 2j + 1)\varepsilon^2 + O(\varepsilon), \end{aligned}$$

as ε tends to ∞, where $0 \le j < \infty$. Hence, the continued fraction on the right side of (10.1) is equal to

$$\frac{-8x\ell m\varepsilon^2 + O(\varepsilon)}{1\{4(n^2 - x^2 - \ell^2 - m^2 + 1)\varepsilon^2 + O(\varepsilon)\}} \\ + \frac{64(x^2 - 1^2)(\ell^2 - 1^2)(m^2 - 1^2)\varepsilon^4 + O(\varepsilon^3)}{3\{4(n^2 - x^2 - \ell^2 - m^2 + 5)\varepsilon^2 + O(\varepsilon)\}} \\ + \frac{64(x^2 - 2^2)(\ell^2 - 2^2)(m^2 - 2^2)\varepsilon^4 + O(\varepsilon^3)}{5\{4(n^2 - x^2 - \ell^2 - m^2 + 13)\varepsilon^2 + O(\varepsilon)\}} + \cdots,$$

as ε tends to ∞. Successively dividing the numerators and denominators above by $-4\varepsilon^2$ and letting ε tend to ∞, we find that the foregoing continued fraction tends termwise to

$$\frac{2x\ell m}{x^2 + \ell^2 + m^2 - n^2 - 1} + \frac{4(x^2 - 1^2)(\ell^2 - 1^2)(m^2 - 1^2)}{3(x^2 + \ell^2 + m^2 - n^2 - 5)} \\ + \frac{4(x^2 - 2^2)(\ell^2 - 2^2)(m^2 - 2^2)}{5(x^2 + \ell^2 + m^2 - n^2 - 13)} + \cdots. \tag{10.3}$$

Combining this with (10.2), we complete the proof of Entry 10, except for an examination of the convergence of (10.3).

If either ℓ, m, or n is an integer, then the continued fraction terminates, and the limiting process is easily justified. If none of these parameters is an integer, then the convergence for $\operatorname{Re} x > 0$ follows from an application of the uniform parabola theorem. The details are similar to those in proving Entry 35 of Chapter 12 [9, p. 135] (Part II [9, pp. 156–158]). We refer the reader to Jacobsen's paper [1, pp. 427–429] for all of these necessary details.

The following beautiful theorem has some resemblance to Entry 33 in Chapter 12 [9, p. 149] (Part II [9, p. 155]). Our first proof below has also been given by Ramanathan [6].

Entry 11. *Suppose that either q, a, and b are complex numbers with $|q| < 1$, or q, a, and b are complex numbers with $a = bq^m$ for some integer m. Then*

$$\frac{(-a)_\infty(b)_\infty - (a)_\infty(-b)_\infty}{(-a)_\infty(b)_\infty + (a)_\infty(-b)_\infty} = \frac{a-b}{1-q+}\,\frac{(a-bq)(aq-b)}{1-q^3}\;+\;\frac{q(a-bq^2)(aq^2-b)}{1-q^5}\;+\cdots.$$

We give two proofs for the case $a \neq bq^m$ for all integers m.

First Proof. We employ Heine's [1] continued fraction for a quotient of two contiguous basic hypergeometric series, namely, for $|q| < 1$ and $|z| < 1$,

$$\frac{{}_2\varphi_1(\alpha, \beta q; \gamma q; q; z)}{{}_2\varphi_1(\alpha, \beta; \gamma; q; z)} = \frac{1}{1+}\,\frac{a_1}{1+}\,\frac{a_2}{1+}\,\frac{a_3}{1+\cdots}, \tag{11.1}$$

where

$$a_{2k} = -\frac{z\alpha q^{k-1}(1-\beta q^k)(1-\gamma q^k/\alpha)}{(1-\gamma q^{2k-1})(1-\gamma q^{2k})}, \qquad k \geq 1,$$

and

$$a_{2k+1} = -\frac{z\beta q^k(1-\alpha q^k)(1-\gamma q^k/\beta)}{(1-\gamma q^{2k})(1-\gamma q^{2k+1})}, \qquad k \geq 0.$$

Now replace α, β, γ, q, and z by bq/a, b/a, q, q^2, and a^2, respectively. Then

$$a_{2k} = -q^{2k-1}\frac{(a-bq^{2k})(b-aq^{2k})}{(1-q^{4k-1})(1-q^{4k+1})}, \qquad k \geq 1,$$

and

$$a_{2k+1} = -q^{2k}\frac{(a-bq^{2k+1})(b-aq^{2k+1})}{(1-q^{4k+1})(1-q^{4k+3})}, \qquad k \geq 0.$$

In summary,

$$a_k = q^{k-1}\frac{(a-bq^k)(aq^k-b)}{(1-q^{2k-1})(1-q^{2k+1})}, \qquad k \geq 1.$$

It follows from (11.1) that, for $|a| < 1$,

$$\frac{a-b}{1-q}\,\frac{{}_2\varphi_1(bq/a, bq^2/a; q^3; q^2; a^2)}{{}_2\varphi_1(bq/a, b/a; q; q^2; a^2)} = \frac{a-b}{1-q+}\,\frac{(a-bq)(aq-b)}{1-q^3}\;+\;\frac{q(a-bq^2)(aq^2-b)}{1-q^5}\;+\cdots. \tag{11.2}$$

Letting

$$A = {}_2\varphi_1(bq/a, b/a; q; q^2; a^2) \quad \text{and} \quad B = \frac{a-b}{1-q}\,{}_2\varphi_1(bq/a, bq^2/a; q^3; q^2; a^2),$$

we observe that, for $|a| < 1$,

$$A \pm B = 1 \pm \frac{a-b}{1-q} + \frac{(a-b)(a-bq)}{(1-q)(1-q^2)} \pm \frac{(a-b)(a-bq)(a-bq^2)}{(1-q)(1-q^2)(1-q^3)} + \cdots$$

$$= \frac{(\pm b)_\infty}{(\pm a)_\infty},$$

by Entry 2. It follows that

$$\frac{(b)_\infty/(a)_\infty - (-b)_\infty/(-a)_\infty}{(b)_\infty/(a)_\infty + (-b)_\infty/(-a)_\infty} = \frac{A+B-(A-B)}{A+B+(A-B)} = \frac{B}{A}. \tag{11.3}$$

Combining (11.2) and (11.3), we complete the proof for $|a| < 1$.

If $|q| < 1$, the given continued fraction is easily seen to be equivalent to a continued fraction of the form $\mathrm{K}(c_k/1)$, where c_k tends to 0 locally uniformly with respect to either a, b, or q, and where we have used a familiar notation $\mathrm{K}(a_k/b_k)$ for a continued fraction with kth partial numerator a_k and kth partial denominator b_k, $k \geq 1$. By analytic continuation, equality holds for all a, b, and q such that $a \neq bq^m$ for all integers m (Jacobsen [1, pp. 418, 435]).

SECOND PROOF. For $|q| < 1$ and $0 < |a| < 1$, define the sequence $\{P_m\}$ by

$$P_0 = \sum_{n=0}^{\infty} \frac{(b/a)_{2n}}{(q)_n} a^{2n} \tag{11.4}$$

and

$$P_m = \frac{1}{(q)_{2m-1}} \sum_{n=0}^{\infty} \frac{(bq^m/a)_{2n}}{(q^{2m})_{2n}} (q^{2n+2}; q^2)_{m-1} a^{2n}, \qquad m \geq 1. \tag{11.5}$$

Then $\{P_m\}$ satisfies the recurrence relation

$$P_m = (1-q^{2m+1})P_{m+1} + q^m(a-bq^{m+1})(aq^{m+1}-b)P_{m+2}, \qquad m \geq 0, \tag{11.6}$$

since by (11.4) and (11.5), $P_0 - (1-q)P_1 = (a-bq)(aq-b)P_2$ and, for $m \geq 1$,

$$P_m - (1-q^{2m+1})P_{m+1} = \frac{(q^2;q^2)_{m-1}}{(q)_{2m-1}} + \frac{1}{(q)_{2m-1}} \sum_{n=1}^{\infty} \frac{(bq^m/a)_{2n}}{(q^{2m})_{2n}} (q^{2n+2}; q^2)_{m-1} a^{2n}$$

$$- \frac{1-q^{2m+1}}{(q)_{2m+1}} (q^2;q^2)_m - \frac{1-q^{2m+1}}{(q)_{2m+1}} \sum_{n=1}^{\infty} \frac{(bq^{m+1}/a)_{2n}}{(q^{2m+2})_{2n}}$$

$$\times (q^{2n+2}; q^2)_m a^{2n}$$

$$= \frac{1}{(q)_{2m}} \sum_{n=1}^{\infty} \frac{(bq^{m+1}/a)_{2n-1}}{(q^{2m+2})_{2n-2}} (q^{2n+2}; q^2)_{m-1}$$

$$\times\, a^{2n}\left\{\frac{1-bq^m/a}{1-q^{2m+1}}-\frac{1-bq^{m+2n}/a}{1-q^{2m+2n+1}}\right\}$$

$$=\frac{1}{(q)_{2m}}\sum_{n=0}^{\infty}\frac{(bq^{m+2}/a)_{2n}}{(q^{2m+2})_{2n}}(q^{2n+4};q^2)_{m-1}$$

$$\times\, a^{2n+2}\frac{q^m\left(q^{m+1}-\dfrac{b}{a}\right)(1-q^{2n+2})\left(1-\dfrac{b}{a}q^{m+1}\right)}{(1-q^{2m+1})(1-q^{2m+2n+3})}$$

$$=q^m(a-bq^{m+1})(aq^{m+1}-b)\frac{1}{(q)_{2m+3}}\sum_{n=0}^{\infty}\frac{(bq^{m+2}/a)_{2n}}{(q^{2m+4})_{2n}}$$

$$\times\,(q^{2n+2};q^2)_{m+1}a^{2n}$$

$$=q^m(a-bq^{m+1})(aq^{m+1}-b)P_{m+2}.$$

Hence, by (11.6),

$$\frac{P_0}{P_1}=1-q+\frac{(a-bq)(aq-b)}{P_1/P_2}$$

$$=1-q+\frac{(a-bq)(aq-b)}{1-q^3}\,\underset{+}{}\,\frac{q(a-bq^2)(aq^2-b)}{P_2/P_3},$$

and so on. The continued fraction thus generated is limit periodic. Indeed, the kth partial numerator approaches 0 and the kth partial denominator approaches 1 as k tends to ∞. Hence, the continued fraction converges to P_0/P_1 if $\lim_{m\to\infty}P_m/P_{m+1}\neq -1$. From the definition (11.5) of P_m, we observe that $\lim_{m\to\infty}P_m/P_{m+1}=1$. Thus, for $|q|<1$ and $0<|a|<1$, we conclude that

$$\frac{P_0}{P_1}=1-q+\frac{(a-bq)(aq-b)}{1-q^3}\,\underset{+}{}\,\frac{q(a-bq^2)(aq^2-b)}{1-q^5}\,\underset{+}{}\cdots.$$

Therefore the continued fraction in this entry converges to

$$\frac{a-b}{P_0/P_1}=\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(b/a)_{2n+1}}{(q)_{2n+1}}a^{2n+1}}{\displaystyle\sum_{n=0}^{\infty}\frac{(b/a)_{2n}}{(q)_{2n}}a^{2n}}=\frac{(b)_\infty/(a)_\infty-(-b)_\infty/(-a)_\infty}{(b)_\infty/(a)_\infty+(-b)_\infty/(-a)_\infty},$$

where the last equality follows from Entry 2. Multiplying the numerator and denominator of this last expression by $(a)_\infty(-a)_\infty$, we complete the second proof of Entry 11 for $|q|<1$, $0<|a|<1$, and $a\neq bq^m$. The result follows for all complex a by analytic continuation.

It remains to prove Entry 11 for $a=bq^m$, when the continued fraction terminates. In such a case, both sides are rational functions of q and b. We refer the reader to L. Jacobsen's paper [1, p. 435] for complete details.

The second proof we have given is a simplification, due to Jacobsen, of that given in the monograph by Adiga, Berndt, Bhargava, and Watson [1].

Entry 12. *Suppose that a, b, and q are complex numbers with* $|ab| < 1$ *and* $|q| < 1$ *or that* $a = b^{2m+1}$ *for some integer m. Then*

$$\frac{(a^2q^3; q^4)_\infty(b^2q^3; q^4)_\infty}{(a^2q; q^4)_\infty(b^2q; q^4)_\infty} = \frac{1}{1 - ab} + \frac{(a - bq)(b - aq)}{(1 - ab)(q^2 + 1)} + \frac{(a - bq^3)(b - aq^3)}{(1 - ab)(q^4 + 1)} + \cdots. \tag{12.1}$$

The following proof is due to L. Jacobsen [1]. It supplants the more complicated original proof found in the monograph by Adiga, Berndt, Bhargava, and Watson [1]. Jacobsen's proof, although it has some time-consuming calculations, has two advantages. First, by using Vitali's theorem, she [1, p. 424] shows that it is sufficient to establish (12.1) for a certain discrete set of values of a as well as for a limit point of this set. In fact, the continued fractions on this set are terminating. Second, the proof uses three other entries from this chapter, Entries 2, 8, and 15.

Lemma 12.1. *For any complex numbers a and q and nonnegative integer n,*

$$(a)_n = \sum_{k=0}^{n} (-1)^k \frac{(q^{n+1-k})_k}{(q)_k} q^{(k^2-k)/2} a^k. \tag{12.2}$$

Proof. Since

$$(q^{n+1-k})_k = (-1)^k (q^{-n})_k q^{nk+(k-k^2)/2}, \qquad k \geq 0,$$

the right side of (12.2) may be written in the form

$$\sum_{k=0}^{n} \frac{(q^{-n})_k (aq^n)^k}{(q)_k}.$$

Applying (2.1) with $t = q^{-n}$ and a replaced by aq^n, we complete the proof.

Proof of Entry 12. Following Jacobsen [1, p. 425], we first establish (12.1) for $a = bq^{2m+1}$, where m is any fixed nonnegative integer and $|ab| = |b^2q^{2m+1}| < 1$. Since (12.1) is trivial when $bq = 0$, we assume that $bq \neq 0$ in the sequel. Observe that the left side of (12.1) equals

$$\frac{(b^2q^{4m+5}; q^4)_\infty(b^2q^3; q^4)_\infty}{(b^2q^{4m+3}; q^4)_\infty(b^2q; q^4)_\infty} = \frac{(b^2q^3; q^4)_m}{(b^2q; q^4)_{m+1}}$$

$$= \frac{\displaystyle\sum_{k=0}^{m} (-1)^k \frac{(q^{4(m+1-k)}; q^4)_k}{(q^4; q^4)_k} q^{2k^2}(b^2q)^k}{\displaystyle\sum_{k=0}^{m+1} (-1)^k \frac{(q^{4(m+2-k)}; q^4)_k}{(q^4; q^4)_k} q^{2k^2-2k}(b^2q)^k}, \tag{12.3}$$

by Lemma 12.1.

When $a = bq^{2m+1}$, the right side of (12.1) is the terminating continued fraction

$$\frac{1}{1-b^2q^{2m+1}} - \frac{b^2q(1-q^{2m})(1-q^{2m+2})}{(1-b^2q^{2m+1})(1+q^2)} - \frac{b^2q^3(1-q^{2m-2})(1-q^{2m+4})}{(1-b^2q^{2m+1})(1+q^4)} - \cdots - \frac{b^2q^{2m-1}(1-q^2)(1-q^{4m})}{(1-b^2q^{2m+1})(1+q^{2m})}$$
$$= \frac{1}{1-b^2q^{2m+1}} + \frac{d_1b^2q}{1-b^2q^{2m+1}} + \frac{d_2b^2q}{1-b^2q^{2m+1}} + \cdots + \frac{d_mb^2q}{1-b^2q^{2m+1}}, \tag{12.4}$$

where

$$d_1 = -\frac{(1-q^{2m})(1-q^{2m+2})}{1+q^2}$$

and

$$d_n = -\frac{q^{2n-2}(1-q^{2m-2n+2})(1-q^{2m+2n})}{(1+q^{2n})(1+q^{2n-2})},$$

for $2 \le n \le m$.

Let

$$F_0 = \sum_{k=0}^{\infty} (-1)^k \frac{(q^{4(m+2-k)}; q^4)_k}{(q^4; q^4)_k} q^{2k^2-2k}(b^2q)^k$$

and

$$F_n = \sum_{k=0}^{\infty} (-1)^k \frac{(q^{4(m+2-n-k)}; q^4)_k(-q^2; q^2)_k(-q^{2m-2k+2}; q^2)_k}{(q^4; q^4)_k(-q^{2n}; q^2)_k(-q^{2m-2n-2k+4}; q^2)_k} q^{2k(k-1+n)}(b^2q)^k,$$

where $1 \le n \le m + 1$. By a somewhat tedious calculation, it can be shown that, for $n = 0, 1, 2, \ldots, m - 1$, F_n satisfies the recurrence relation

$$F_n = (1 - q^{2m+1}b^2)F_{n+1} + d_{n+1}b^2qF_{n+2}. \tag{12.5}$$

It should be pointed out that the verification for $n = 0$ must be accomplished separately from the cases $1 \le n \le m - 1$, because for $n = 0$ the definitions of F_n and d_{n+1} are different from those when $n > 0$. From (12.5),

$$\begin{aligned}\frac{F_0}{F_1} &= 1 - b^2q^{2m+1} + \frac{d_1b^2q}{F_1/F_2}\\ &= 1 - b^2q^{2m+1} + \frac{d_1b^2q}{1-b^2q^{2m+1}} + \frac{d_2b^2q}{F_2/F_3}\\ &= 1 - b^2q^{2m+1} + \frac{d_1b^2q}{1-b^2q^{2m+1}} + \cdots + \frac{d_mb^2q}{F_m/F_{m+1}}\\ &= 1 - b^2q^{2m+1} + \frac{d_1b^2q}{1-b^2q^{2m+1}} + \cdots + \frac{d_mb^2q}{1-b^2q^{2m+1}},\end{aligned}$$

since $F_m = 1 - b^2q^{2m+1}$ and $F_{m+1} = 1$. Hence, the continued fraction in (12.4) equals F_1/F_0. But this is exactly the expression that we found in (12.3) for the left side of (12.1). Hence, we have proved (12.1) for the set $\{a = bq^{2m+1}: 0 \le m < \infty\}$.

This set has a limit point $a = 0$. Next, we must establish (12.1) for $a = 0$. In this case, (12.1) reduces to the equality

$$\frac{(b^2q^3; q^4)_\infty}{(b^2q; q^4)_\infty} = \frac{1}{1} - \frac{b^2q}{1 + q^2} - \frac{b^2q^3}{1 + q^4} - \frac{b^2q^5}{1 + q^6} - \cdots. \tag{12.6}$$

However, by applying Entry 15 below with $a = -1$, q replaced by q^2, and b replaced by $-b^2/q$, we find, after some simplification, that

$$\frac{\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{(b^2q^3)^k(q^4)^{(k^2-k)/2}}{(q^4; q^4)_k}}{\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{(b^2q)^k(q^4)^{(k^2-k)/2}}{(q^4; q^4)_k}} = \frac{1}{1} - \frac{b^2q}{1 + q^2} - \frac{b^2q^3}{1 + q^4} - \frac{b^2q^5}{1 + q^6} - \cdots. \tag{12.7}$$

By letting n tend to ∞ in Lemma 12.1 with q replaced by q^4 and $a = b^2q^3$ and b^2q, respectively, we find that the left side of (12.7) equals

$$\frac{(b^2q^3; q^4)_\infty}{(b^2q; q^4)_\infty}.$$

(Alternatively, we may draw the same conclusion by applying Entry 8 with q replaced by q^4, $b = 0$, $c = 1$, $d = 0$, and a replaced by b^2q^3 and b^2q, respectively.) Hence, we have shown that (12.7) reduces to (12.6), as desired.

Next, if we write the continued fraction of (12.1) in the form

$$\mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_k}{b_k} = \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{c_k}{1},$$

where $c_k = a_k/b_{k-1}b_k$, $k \ge 1$, and $b_0 = 1$, we must show, for $k > m + 1$ and m sufficiently large, that the elements c_{k+1} lie in the parabolic region

$$\{z: |z| - \operatorname{Re} z \le \tfrac{1}{2}\} \tag{12.8}$$

(Jacobsen [1, pp. 417, 424]). A brief calculation yields

$$c_{k+1} = \frac{b^2q^{2m+1}}{(1 - b^2q^{2m+1})^2} \frac{(1 - q^{2k-2m-2})(1 - q^{2m+2k})}{(1 + q^{2k})(1 + q^{2k-2})}.$$

Clearly, for $k > m + 1$ and m sufficiently large, c_{k+1} lies in the region (12.8).

Lastly, we must examine the convergence of the continued fraction in (12.1). Now, as k tends to ∞,

$$c_{k+1} = \frac{(a - bq^{2k-1})(b - aq^{2k-1})}{(1 - ab)^2(q^{2k} + 1)(q^{2k-2} + 1)}$$

tends to $ab/(1 - ab)^2$ locally uniformly with respect to a, b, and q. For convergence, we need that $ab/(1 - ab)^2 \notin (-\infty, -\frac{1}{4}]$, and this is so if and only if $|ab| < 1$. This completes the proof.

We have not described here the entire theoretical background for the proof above; we refer to Jacobsen's paper [1] for the remainder of the theory.

If $|ab| > 1$, the continued fraction in (12.1) is equivalent to

$$\frac{-1/(ab)}{1-1/(ab)+}\;\frac{(1/a-q/b)(1/b-q/a)}{(1-1/(ab))(q^2+1)}\;+\;\frac{(1/a-q^3/b)(1/b-q^3/a)}{(1-1/(ab))(q^4+1)}\;+\cdots,$$

which, by Entry 12, converges to

$$-\frac{1}{ab}\frac{(q^3/a^2;q^4)_\infty(q^3/b^2;q^4)_\infty}{(q/a^2;q^4)_\infty(q/b^2;q^4)_\infty},$$

which, in general, does not equal the left side of (12.1). Since the continued fraction diverges for $|ab| = 1$, the hypotheses on a and b in Entry 12 cannot be relaxed.

If $|ab| < 1$ and $|q| > 1$, the continued fraction in (12.1) converges to

$$\frac{(a^2/q^3;1/q^4)_\infty(b^2/q^3;1/q^4)_\infty}{(a^2/q;1/q^4)_\infty(b^2/q;1/q^4)_\infty}$$

which furthermore evinces the beautiful symmetry in this wonderful entry.

Entry 13. *If* $|q| < 1$, *then*

$$\sum_{k=0}^{\infty}(-a)^k q^{k(k+1)/2} = \frac{1}{1+}\;\frac{aq}{1+}\;\frac{a(q^2-q)}{1}\;+\;\frac{aq^3}{1}\;+\;\frac{a(q^4-q^2)}{1}\;+\cdots. \tag{13.1}$$

PROOF. Let

$$f(b,a) = (aq)_\infty \sum_{k=0}^{\infty}\frac{b^k q^{k^2}}{(aq)_k(q)_k}. \tag{13.2}$$

Then it is easy to verify that

$$f(b,a) = f(b,aq) - aqf(bq,aq) \tag{13.3}$$

and

$$f(b,a) = f(bq,a) + bqf(bq^2,aq). \tag{13.4}$$

From (13.3) and (13.4),

$$\begin{aligned} f(bq,a) &= f(bq^2,a) + bq^2 f(bq^3,aq) \\ &= f(bq^2,aq) + (bq^2-aq)f(bq^3,aq). \end{aligned}$$

Using (13.4), the equality above, and iteration, we find that

$$\frac{f(b,a)}{f(bq,a)} = 1 + \cfrac{bq}{\cfrac{f(bq,a)}{f(bq^2,aq)}} = 1 + \cfrac{bq}{1 + \cfrac{bq^2-aq}{\cfrac{f(bq^2,aq)}{f(bq^3,aq)}}}$$

$$= 1 + \frac{bq}{1+}\;\frac{bq^2-aq}{1}\;+\;\frac{bq^3}{1}\;+\;\cfrac{bq^4-aq^2}{\cfrac{f(bq^4,aq^2)}{f(bq^5,aq^2)}}$$

$$= 1 + \frac{bq}{1 +} \frac{bq^2 - aq}{1} \frac{bq^3}{+ 1 +} \frac{bq^4 - aq^2}{1} + \cdots. \qquad (13.5)$$

This continued fraction is of the form

$$1 + \mathrm{K}\left(\frac{ac_n + bd_n}{1}\right),$$

where c_n and d_n tend to 0 as n tends to ∞. The continued fraction therefore converges. Furthermore, since also $f(bq^{2n}, aq^n)/f(bq^{2n+1}, aq^n) = 1 + O(q^{2n+1})$ as n tends to ∞, the continued fraction converges to $f(b, a)/f(bq, a)$.

Now set $b = a$ in (13.5) and employ Entry 9. Observe that, by Entry 9, $f(a, a) = 1$. Taking the reciprocal of both sides of (13.5) we deduce (13.1).

Entry 13 is originally due to Eisenstein [1], [2, pp. 35–39]. However, the special case $a = 1$ can be found in an entry of Gauss' diary [1, p. 68], dated February 16, 1797. See also J. J. Gray's paper [1, p. 114], which provides a translation of Gauss' diary. A generalization of Entry 13 appears in Ramanujan's "lost notebook" [11]. For proofs of this more general theorem, see papers of Andrews [10, Eq. (1.4)], Hirschhorn [3], and Ramanathan [4]. Bhargava and Adiga [1] have established, with a unified approach, some continued fraction expansions in Ramanujan's "lost notebook," including those mentioned above as well as a related continued fraction of Hirschhorn [2]. R. Y. Denis [5] and N. A. Bhagirathi [1] have proved some very general continued fractions for certain basic bilateral hypergeometric series which include Entry 13 as special cases.

After stating Entry 13, Ramanujan gives formulas for the denominator D_n of the nth convergent of (13.1), namely,

$$D_{2n} = \sum_{k=0}^{n} \frac{a^k q^{nk}}{(q)_k} \prod_{j=1}^{k} (1 - q^{n+1-j}), \qquad n \geq 1, \qquad (13.6)$$

and

$$D_{2n+1} = \sum_{k=0}^{n} \frac{a^k q^{(n+1)k}}{(q)_k} \prod_{j=1}^{k} (1 - q^{n+1-j}), \qquad n \geq 0. \qquad (13.7)$$

To prove (13.6) and (13.7), we employ a familiar recursion formula for denominators (Wall [1, p. 15]) along with induction on n.

First, from (13.1), it is obvious that $D_1 = 1$ and $D_2 = 1 + aq$, which are in agreement with (13.7) and (13.6), respectively. Proceeding by induction and using the aforementioned recursion formula, we find that

$$\begin{aligned} D_{2n} &= D_{2n-1} + aq^{2n-1} D_{2n-2} \\ &= \sum_{k=0}^{n-1} \frac{a^k q^{nk}}{(q)_k} \prod_{j=1}^{k} (1 - q^{n-j}) + aq^{2n-1} \sum_{k=0}^{n-1} \frac{a^k q^{(n-1)k}}{(q)_k} \prod_{j=1}^{k} (1 - q^{n-j}) \\ &= 1 + \sum_{k=1}^{n} \frac{a^k q^{nk}}{(q)_k} \prod_{j=1}^{k} (1 - q^{n-j}) + aq^{2n-1} \sum_{k=1}^{n} \frac{a^{k-1} q^{(n-1)(k-1)}}{(q)_{k-1}} \prod_{j=1}^{k-1} (1 - q^{n-j}) \end{aligned}$$

$$= 1 + \sum_{k=1}^{n} \frac{a^k q^{nk}}{(q)_k} \{(1 - q^{n-k}) + (q^{n-k} - q^n)\} \prod_{j=1}^{k-1} (1 - q^{n-j})$$

$$= \sum_{k=0}^{n} \frac{a^k q^{nk}}{(q)_k} \prod_{j=0}^{k-1} (1 - q^{n-j}).$$

Replacing j by $j - 1$, we complete the proof of (13.6).

The proof of (13.7), which begins with the recursion formula

$$D_{2n+1} = D_{2n} + a(q^{2n} - q^n)D_{2n-1},$$

is very similar to the proof of (13.6), and so we omit the details.

Entry 14. *If* $n < 1$ *and* $0 < a < q^{1-n}$, *then*

$$\int_0^\infty \frac{(-at)_\infty}{t^n(-t)_\infty} dt = \frac{\pi}{\sin(\pi n)} \frac{(a)_\infty (q^n)_\infty}{(q)_\infty (aq^{n-1})_\infty}. \tag{14.1}$$

This beautiful integral formula was stated by Ramanujan in his paper [4], [10, p. 57]. Acknowledging that he did not possess a rigorous proof, Ramanujan confessed: "My own proofs of the above results make use of a general formula, the truth of which depends on conditions which I have not yet investigated completely. A direct proof depending on Cauchy's theorem will be found in Mr. Hardy's note which follows this paper." (That paper is [1], [2, pp. 594–597].) The special case $a = 0$ of (14.1) is found in Ramanujan's quarterly reports. To see how Ramanujan "proved" (14.1), consult Hardy's book [3, p. 194] or Berndt's account of the quarterly reports [5].

Askey [2] has found a simple proof of Entry 14 and has demonstrated why (14.1) is a q-analogue of the beta function. Recalling the definition of the q-gamma function in (1.1), we may rewrite (14.1) in the form

$$\int_0^\infty t^{\alpha-1} \frac{(-tq^{\alpha+\beta})_\infty}{(-t)_\infty} dt = \frac{\Gamma(\alpha)\Gamma(1-\alpha)\Gamma_q(\beta)}{\Gamma_q(1-\alpha)\Gamma_q(\alpha+\beta)}, \tag{14.2}$$

where $\alpha, \beta > 0$ (Askey [2], [8]). It is thus clear that (14.2) is an extension of the beta function

$$\int_0^\infty \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

R. L. Lamphere [1] has found a very elementary proof of (14.1).

For further q-beta integrals, see papers by Askey [1], [3], [5], [6], Andrews and Askey [3], Askey and Roy [1], and Rahman [1] as well as Andrews' monograph [18]. An asymptotic expansion for $\Gamma_q(x)$, uniform in q as q tends to $1-$, that is, an analogue of Stirling's formula, has been proved by N. Koblitz [1] and by D. S. Moak [2], who [1] has also studied orthogonal polynomials with respect to weight functions akin to the integrand in (14.1). More general work in this direction has been accomplished by Askey and Wilson [1], where a plethora of references may be found.

Entry 15. *If* $|q| < 1$, *then*

$$\frac{\displaystyle\sum_{k=0}^{\infty} \frac{b^k q^{k^2}}{(aq)_k (q)_k}}{\displaystyle\sum_{k=0}^{\infty} \frac{b^k q^{k(k+1)}}{(aq)_k (q)_k}} = 1 + \frac{bq}{1 - aq +} \frac{bq^2}{1 - aq^2 +} \frac{bq^3}{1 - aq^3 +} \cdots.$$

PROOF. Let $f(b, a)$ be defined by (13.2). Replacing a by aq in (13.4) and adding the result to (13.3), we find that

$$f(b, a) = (1 - aq)f(bq, aq) + bqf(bq^2, aq^2).$$

Replacing a by aq^{n-1} and b by bq^n, we may rewrite the previous equality in the form

$$\frac{f(bq^n, aq^{n-1})}{f(bq^{n+1}, aq^n)} = 1 - aq^n + \frac{bq^{n+1}}{\dfrac{f(bq^{n+1}, aq^n)}{f(bq^{n+2}, aq^{n+1})}}, \qquad n \geq 1. \tag{15.1}$$

Using (13.4), (15.1), and iteration, we deduce that

$$\frac{f(b, a)}{f(bq, a)} = 1 + \frac{bq}{\dfrac{f(bq, a)}{f(bq^2, aq)}} = 1 + \frac{bq}{1 - aq +} \frac{bq^2}{\dfrac{f(bq^2, aq)}{f(bq^3, aq^2)}}$$

$$= 1 + \frac{bq}{1 - aq +} \frac{bq^2}{1 - aq^2 +} \frac{bq^3}{1 - aq^3 +} \cdots.$$

That this continued fraction converges and that it converges, indeed, to $f(b, a)/f(bq, a)$ follow as in the proof of Entry 13. If $b \neq 0$ and $aq^n = 1$ for some positive integer n, then equality holds with the convention that we take the limit of both sides as a tends to $1/q^n$. If $b = 0$ and $aq^n = 1$, we interpret both sides as equaling 1. This completes the proof.

Corollary. *If* $|q| < 1$, *then*

$$\frac{\displaystyle\sum_{k=0}^{\infty} \frac{a^k q^{k(k+1)}}{(q)_k}}{\displaystyle\sum_{k=0}^{\infty} \frac{a^k q^{k^2}}{(q)_k}} = \frac{1}{1 +} \frac{aq}{1 +} \frac{aq^2}{1 +} \frac{aq^3}{1 +} \cdots.$$

PROOF. Set $a = 0$ in Entry 15 and then replace b by a. The corollary now readily follows.

The continued fractions of both Entry 15 and its corollary were mentioned by Ramanujan [10, p. xxviii] in his second letter to Hardy. The corollary was established earlier by Rogers [1, p. 328, Eq. (4)] and then later by Watson [3]. The special case $a = 1$ is Entry 38(iii) and is discussed in detail in Section 38.

Ramamani [1] has given a similar proof of Entry 15 by obtaining functional relations for the function

$$\sum_{k=0}^{\infty} \frac{(-1)^k (b/a)_k a^k q^{k(k+1)/2}}{(q)_k}$$

and using Entry 9. Entry 9 is not used in our proof. Entry 16 below provides a finite version of Entry 15. Thus, an alternative proof of Entry 15 is obtained by letting n tend to ∞ in Entry 16. Hirschhorn [1], [4] has given a proof of Entry 16; perhaps our proof is somewhat simpler.

Entry 16. *For each positive integer n, let*

$$\mu = \mu_n(a, q) = \sum_{k=0}^{[(n+1)/2]} \frac{a^k q^{k^2} (q)_{n-k+1}}{(q)_k (q)_{n-2k+1}}$$

and

$$\nu = \nu_n(a, q) = \sum_{k=0}^{[n/2]} \frac{a^k q^{k(k+1)} (q)_{n-k}}{(q)_k (q)_{n-2k}}.$$

Then

$$\frac{\mu}{\nu} = 1 + \frac{aq}{1} + \frac{aq^2}{1} + \cdots + \frac{aq^n}{1}.$$

PROOF. For each nonnegative integer r, define

$$F_r = \sum_{k=0}^{\infty} \frac{a^k q^{k(r+k)} (q)_{n-r-k+1}}{(q)_k (q)_{n-r-2k+1}}.$$

Observe that $F_0 = \mu$, $F_1 = \nu$, $F_n = 1$, and $F_{n-1} = 1 + aq^n$. A straightforward calculation shows that

$$F_r - F_{r+1} = aq^{r+1} F_{r+2}, \qquad r \geq 0. \tag{16.1}$$

Using (16.1), iteration, and the special cases pointed out above, we find that

$$\begin{aligned} \frac{\mu}{\nu} = \frac{F_0}{F_1} &= 1 + \frac{aq}{F_1/F_2} = 1 + \frac{aq}{1} + \frac{aq^2}{F_2/F_3} \\ &= 1 + \frac{aq}{1} + \frac{aq^2}{1} + \cdots + \frac{aq^{n-1}}{F_{n-1}/F_n} \\ &= 1 + \frac{aq}{1} + \frac{aq^2}{1} + \cdots + \frac{aq^{n-1}}{1} + \frac{aq^n}{1}, \end{aligned}$$

which is the required result.

Entry 17 offers another famous discovery of Ramanujan known as "Ramanujan's summation of the ${}_1\psi_1$." It was first brought before the mathematical world by Hardy [3, pp. 222, 223] who described it as "a remarkable formula with many parameters." Hardy did not supply a proof but indicated that a proof could be constructed from the q-binomial theorem. The first published proofs appear to be by W. Hahn [1] and M. Jackson [1] in 1949 and 1950, respectively. Other proofs have been given by Andrews [2], [3], Andrews and Askey [2], Askey [2], Ismail [1], Fine [1, pp. 19–20], and Mimachi [1]. The short proof of Entry 17 that we offer below has been motivated by Askey's paper [2] and has been discovered independently by K. Venkatachaliengar [1]. See also his monograph with V. R. Thiruvenkatachar [1]. Askey [4] has discussed our proof along with a "proof" of a "false theorm" to illustrate certain pitfalls in formally manipulating Laurent series.

We emphasize that Entry 17 is an extremely useful result, and several applications of it will be made in the sequel. Fine [1] and Bhargava and Adiga [5] have employed Entry 17 in their work on sums of squares. For a connection between Entries 14 and 17, see Askey's paper [2]. Further applications of Entry 17 have been made by Andrews [10], [18, Chap. 5], Askey [3], [5], and Moak [1]. A generalization of Entry 17 has been found by Andrews [12, Theorem 6].

As we shall see in Section 19, the Jacobi triple product identity is a special case of Ramanujan's ${}_1\psi_1$ summation. In 1972, I. Macdonald [1] found multidimensional analogues of the Jacobi triple product identity, which can also be considered as analogues of Entry 17, and which are now called the Macdonald identities. One of the Macdonald identities is, in fact, the quintuple product identity, discussed in detail in Section 38. More elementary proofs of some of Macdonald's identities have been found by S. Milne [1]. These considerations partly motivated Milne [2]–[4] to develop multiple sum generalizations of Ramanujan's ${}_1\psi_1$ sum. R. Gustafson [1] has found further analogues of the ${}_1\psi_1$ summation. Lastly, we mention that D. Stanton [1] has developed an elementary approach to the Macdonald identities.

Entry 17. *Suppose that* $|\beta q| < |z| < 1/|\alpha q|$. *Then*

$$1+\sum_{k=1}^{\infty}\frac{(1/\alpha;q^2)_k(-\alpha q)^k}{(\beta q^2;q^2)_k}z^k+\sum_{k=1}^{\infty}\frac{(1/\beta;q^2)_k(-\beta q)^k}{(\alpha q^2;q^2)_k}z^{-k}$$
$$=\left\{\frac{(-qz;q^2)_\infty(-q/z;q^2)_\infty}{(-\alpha qz;q^2)_\infty(-\beta q/z;q^2)_\infty}\right\}\left\{\frac{(q^2;q^2)_\infty(\alpha\beta q^2;q^2)_\infty}{(\alpha q^2;q^2)_\infty(\beta q^2;q^2)_\infty}\right\}. \quad (17.1)$$

PROOF. Let $f(z)$ denote the former expression in curly brackets on the right side of (17.1). Since $f(z)$ is analytic in the annulus, $|\beta q| < |z| < 1/|\alpha q|$, we may set

$$f(z)=\sum_{k=-\infty}^{\infty}c_k z^k, \qquad |\beta q|<|z|<1/|\alpha q|.$$

From the definition of f, it is easy to see that

$$(\beta+qz)f(q^2z)=(1+\alpha qz)f(z),$$

provided that also $|\beta q| < |q^2 z|$. Thus, in the sequel we assume that $|\beta/q| < |z| < 1/|\alpha q|$.

Equating coefficients of z^k, $-\infty < k < \infty$, on both sides, we find that

$$\beta q^{2k} c_k + q^{2k-1} c_{k-1} = c_k + \alpha q c_{k-1}. \tag{17.2}$$

Hence,

$$c_k = -\frac{\alpha q(1 - q^{2k-2}/\alpha)c_{k-1}}{1 - \beta q^{2k}}, \qquad 1 \le k < \infty,$$

and

$$c_{-k} = -\frac{\beta q(1 - q^{2k-2}/\beta)c_{-k+1}}{1 - \alpha q^{2k}}, \qquad 1 \le k < \infty,$$

where, to get the latter equality, we replaced k by $1 - k$ in (17.2). Iterating the last two equalities, we deduce that, respectively,

$$c_k = \frac{(-\alpha q)^k (1/\alpha; q^2)_k c_0}{(\beta q^2; q^2)_k}, \qquad 1 \le k < \infty, \tag{17.3}$$

and

$$c_{-k} = \frac{(-\beta q)^k (1/\beta; q^2)_k c_0}{(\alpha q^2; q^2)_k}, \qquad 1 \le k < \infty.$$

Examining (17.1), we see that, to complete the proof, it suffices to show that

$$c_0 = \frac{(\alpha q^2; q^2)_\infty (\beta q^2; q^2)_\infty}{(q^2; q^2)_\infty (\alpha\beta q^2; q^2)_\infty}. \tag{17.4}$$

Now let $\varphi(z)$ and $\psi(z)$ denote, respectively, the two infinite series on the left side of (17.1). Now $f(z)$ has a simple pole at $z = -1/\alpha q$, and since $\psi(z)$ is analytic for $|z| > |\beta q|$, we find that

$$\lim_{z \to -1/\alpha q} (1 + \alpha q z) f(z) = \lim_{z \to -1/\alpha q} (1 + \alpha q z)\varphi(z) = \lim_{n \to \infty} \frac{(-1)^n c_n}{(\alpha q)^n}, \tag{17.5}$$

by Abel's theorem. Using the definition of $f(z)$ and (17.3), we may rewrite (17.5) in the form

$$\frac{(1/\alpha; q^2)_\infty (\alpha q^2; q^2)_\infty}{(q^2; q^2)_\infty (\alpha\beta q^2; q^2)_\infty} = \frac{(1/\alpha; q^2)_\infty c_0}{(\beta q^2; q^2)_\infty}.$$

Equality (17.4) obviously follows, and so the proof of Entry 17 is complete for $|\beta/q| < |z| < 1/|\alpha q|$. By analytic continuation, (17.1) is valid for $|\beta q| < |z| < 1/|\alpha q|$.

Entry 17 can be reformulated in a more compact setting. We first extend the definition of $(c; q)_k$ by defining

$$(c)_k = (c; q)_k = \frac{(c; q)_\infty}{(cq^k; q)_\infty},$$

for *every* integer k. In Entry 17, now replace α, β, and z by $1/a$, b/q^2, and $-az/q$, respectively. Lastly, replace q^2 by q. Then (17.1) can be written in the form

$$\sum_{k=-\infty}^{\infty} \frac{(a)_k}{(b)_k} z^k = \frac{(az)_\infty(q/az)_\infty(q)_\infty(b/a)_\infty}{(z)_\infty(b/az)_\infty(b)_\infty(q/a)_\infty}, \tag{17.6}$$

where $|b/a| < |z| < 1$.

For another proof of (17.4), see the monograph of Thiruvenkatachar and Venkatachaliengar [1].

Corollary. *If* $|nq| < |z| < 1/|nq|$, *then*

$$1 + \sum_{k=1}^{\infty} \frac{(1/n; q^2)_k(-nq)^k(z^k + z^{-k})}{(nq^2; q^2)_k}$$
$$= \frac{(-qz; q^2)_\infty(-q/z; q^2)_\infty(q^2; q^2)_\infty(n^2q^2; q^2)_\infty}{(-nqz; q^2)_\infty(-nq/z; q^2)_\infty(nq^2; q^2)_\infty^2}.$$

PROOF. Set $\alpha = \beta = n$ in Entry 17.

The remainder of Chapter 16 is devoted to the theta-function

$$f(a, b) = 1 + \sum_{k=1}^{\infty} (ab)^{k(k-1)/2}(a^k + b^k) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2}b^{k(k-1)/2}, \tag{18.1}$$

where $|ab| < 1$. If we set $a = qe^{2iz}$, $b = qe^{-2iz}$, and $q = e^{\pi i\tau}$, where z is complex and $\mathrm{Im}(\tau) > 0$, then $f(a, b) = \vartheta_3(z, \tau)$, where $\vartheta_3(z, \tau)$ denotes one of the classical theta-functions in its standard notation (Whittaker and Watson [1, p. 464]). Thus, all of Ramanujan's theorems on $f(a, b)$ may be reformulated in terms of $\vartheta_3(z, \tau)$. It seems preferable, however, to retain Ramanujan's notation. Not only will the reader find it easier to follow our presentation in conjunction with Ramanujan's, but Ramanujan's theorems are more simply and elegantly stated in his notation.

Entry 18. *We have*

(i) $$f(a, b) = f(b, a),$$

(ii) $$f(1, a) = 2f(a, a^3),$$

(iii) $$f(-1, a) = 0,$$

and, if n is an integer,

(iv) $$f(a, b) = a^{n(n+1)/2}b^{n(n-1)/2}f(a(ab)^n, b(ab)^{-n}).$$

Ramanujan remarks that (iv) is approximately true when n is not an integer. We have not been able to give a mathematically precise formulation of this statement. Repeated use of (iv) will be made in the sequel.

PROOF. First, (i) is obvious.

Second,

$$
\begin{aligned}
f(1, a) &= 2 + \sum_{k=1}^{\infty} a^{k(k+1)/2} + \sum_{k=2}^{\infty} a^{k(k-1)/2} \\
&= 2\left(1 + \sum_{k=1}^{\infty} a^{k(k+1)/2}\right) \\
&= 2\left(1 + \left\{\sum_{\substack{k=1 \\ k \text{ even}}}^{\infty} + \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty}\right\} a^{k(k+1)/2}\right) \\
&= 2\left(1 + \sum_{k=1}^{\infty} a^{k(2k+1)} + \sum_{k=1}^{\infty} a^{k(2k-1)}\right) \\
&= 2\left(1 + \sum_{k=1}^{\infty} a^{k(k-1)/2}(a^3)^{k(k+1)/2} + \sum_{k=1}^{\infty} a^{k(k+1)/2}(a^3)^{k(k-1)/2}\right) \\
&= 2f(a, a^3).
\end{aligned}
$$

Third,

$$
f(-1, a) = \sum_{k=2}^{\infty} (-1)^{k(k+1)/2} a^{k(k-1)/2} + \sum_{k=1}^{\infty} (-1)^{k(k-1)/2} a^{k(k+1)/2} = 0,
$$

upon the replacement of k by $k + 1$ in the first sum on the right side.

Fourth, replacing k by $k + n$ on the far right side of (18.1), we find that

$$
\begin{aligned}
f(a, b) &= \sum_{k=-\infty}^{\infty} a^{(k+n)(k+n+1)/2} b^{(k+n)(k+n-1)/2} \\
&= a^{n(n+1)/2} b^{n(n-1)/2} \sum_{k=-\infty}^{\infty} a^{k(k+2n+1)/2} b^{k(k+2n-1)/2} \\
&= a^{n(n+1)/2} b^{n(n-1)/2} \sum_{k=-\infty}^{\infty} \{a(ab)^n\}^{k(k+1)/2} \{b(ab)^{-n}\}^{k(k-1)/2},
\end{aligned}
$$

which completes the proof of (iv).

Entry 19. *We have*

$$
f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.
$$

PROOF. In Entry 17, let $qz = a$, $q/z = b$, and $\alpha = \beta = 0$.

In the notebooks [9, Vol. 2, p. 197], Ramanujan informs us how he proved Entry 19 by remarking: "This result can be got like XVI. 17 Cor. or as follows. We see from iv. that if $a(ab)^n$ or $b(ab)^n$ be equal to -1 then $f(a, b) = 0$ and also if $(ab)^n = 1$, $f(a, b)\{1 - (a/b)^{n/2}\} = 0$ and hence $f(a, b) = 0$. Therefore $(-a; ab)_\infty$, $(-b; ab)_\infty$, and $(ab; ab)_\infty$ are the factors of $f(a, b)$." (We have slightly altered Ramanujan's notation.) The product and series in Entry 19 converge only when $|ab| < 1$, but there is even a more serious objection to Ramanujan's argument. It is not clear that the *only* factors of $f(a, b)$ are $(-a; ab)_\infty$, $(-b; ab)_\infty$, and $(ab; ab)_\infty$.

Entry 19 is Jacobi's famous triple product identity, established in his *Fundamenta Nova* [1], [2] but, in fact, first proved by Gauss [3, p. 464]. See the texts of Andrews [9, pp. 21, 22] and Hardy and Wright [1, pp. 282, 283] for other proofs.

Entry 20. *If* $\alpha\beta = \pi$, $\mathrm{Re}(\alpha^2) > 0$, *and n is any complex number, then*

$$\sqrt{\alpha} f(e^{-\alpha^2+n\alpha}, e^{-\alpha^2-n\alpha}) = e^{n^2/4}\sqrt{\beta} f(e^{-\beta^2+in\beta}, e^{-\beta^2-in\beta}).$$

Entry 20 is a formulation of the classical transformation formula for the theta-function $\vartheta_3(z, \tau)$ (Whittaker and Watson [1, p. 475]). This entry is also recorded in Chapter 14 [9, Vol. 2, p. 169, Entry 7]. A proof via the Poisson summation formula is sketched in our book [9, p. 253].

Entry 21. *If* $|q|, |a|, |b| < 1$, *then*

$$\mathrm{Log}(-a; q)_\infty = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}a^k}{k(1-q^k)} \tag{21.1}$$

and

$$\mathrm{Log} f(a, b) = \mathrm{Log}(ab; ab)_\infty + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(a^k + b^k)}{k(1 - a^k b^k)}. \tag{21.2}$$

PROOF. For $|q|, |a| < 1$,

$$\begin{aligned}\mathrm{Log}(-a; q) &= \sum_{n=0}^{\infty} \mathrm{Log}(1 + aq^n) = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(aq^n)^k}{k} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}a^k}{k} \sum_{n=0}^{\infty} (q^k)^n = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}a^k}{k(1-q^k)}.\end{aligned}$$

Equality (21.2) follows immediately from Entry 19 and (21.1).

Entry 22. *If* $|q| < 1$, *then*

(i) $$\varphi(q) := f(q, q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (-q^2; q^2)_\infty},$$

(ii) $$\psi(q) := f(q, q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty},$$

(iii) $$\begin{aligned} f(-q) := f(-q, -q^2) &= \sum_{k=0}^{\infty} (-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty} (-1)^k q^{k(3k+1)/2} \\ &= (q; q)_\infty, \end{aligned}$$

and

(iv) $\chi(q) := (-q; q^2)_\infty.$

Observe that $\varphi(q) = \vartheta_3(0, \tau)$, where $q = e^{\pi i\tau}$. If $q = e^{2\pi i\tau}$, then $f(-q) = e^{-\pi i\tau/12}\eta(\tau)$, where $\eta(\tau)$ denotes the classical Dedekind eta-function. Equality (iii) is a statement of Euler's famous pentagonal number theorem [1], [5]. See Andrews' book [9, pp. 9–12, 14] for an elementary proof and further references. Note that (iv) is only a *definition* of $\chi(q)$.

PROOF OF (i). The first equality follows immediately from the definition (18.1) of $f(a, b)$.

From Entry 19,

$$f(q, q) = (-q; q^2)^2_\infty (q^2; q^2)_\infty. \tag{22.1}$$

Now,

$$\begin{aligned}(-q; q^2)_\infty &= \prod_{n=1}^\infty (1 + q^{2n-1}) = \prod_{n=1}^\infty \frac{1 + q^n}{1 + q^{2n}} \\ &= \prod_{n=1}^\infty \frac{1 - q^{2n}}{(1 - q^n)(1 + q^{2n})} = \frac{1}{(q; q^2)_\infty(-q^2; q^2)_\infty},\end{aligned} \tag{22.2}$$

which is a famous identity of Euler. Substituting (22.2) into (22.1), we complete the proof of (i).

Observe that (22.2) may be rewritten in the form

$$(-q; q)_\infty = \frac{1}{(q; q^2)_\infty}. \tag{22.3}$$

The equality (22.3) is the analytic equivalent of Euler's famous theorem: the number of partitions of a positive integer n into distinct parts is equal to the number of partitions of n into odd parts.

Using (22.3) in Entry 22(i), we derive the useful representations

$$\varphi(-q) = (q; q)_\infty (q; q^2)_\infty = \frac{(q; q)_\infty}{(-q; q)_\infty}. \tag{22.4}$$

PROOF OF (ii). For $|q| < 1$,

$$\begin{aligned}f(q, q^3) &= 1 + \sum_{k=1}^\infty q^{2k(2k-1)/2} + \sum_{k=1}^\infty q^{2k(2k+1)/2} \\ &= \sum_{\substack{k=1 \\ k \text{ odd}}}^\infty q^{k(k+1)/2} + \sum_{\substack{k=0 \\ k \text{ even}}}^\infty q^{k(k+1)/2} \\ &= \sum_{k=0}^\infty q^{k(k+1)/2},\end{aligned}$$

which proves the first equality.

By Entry 19,

$$\begin{aligned} f(q, q^3) &= (-q; q^4)_\infty (-q^3; q^4)_\infty (q^4; q^4)_\infty \\ &= (-q; q^2)_\infty (-q^2; q^2)_\infty (q^2; q^2)_\infty \\ &= \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \end{aligned}$$

by (22.2).

Proof of (iii). The first equality follows immediately from the definition (18.1) of $f(a, b)$.

By Entry 19,

$$f(-q, -q^2) = (q; q^3)_\infty (q^2; q^3)_\infty (q^3; q^3)_\infty = (q; q)_\infty.$$

Entry 23. *For* $|q| < 1$,

(i) $$\operatorname{Log} \varphi(q) = 2 \sum_{k=1}^{\infty} \frac{q^{2k-1}}{(2k-1)(1+q^{2k-1})},$$

(ii) $$\operatorname{Log} \psi(q) = \sum_{k=1}^{\infty} \frac{q^k}{k(1+q^k)},$$

(iii) $$\operatorname{Log} f(-q) = -\sum_{k=1}^{\infty} \frac{q^k}{k(1-q^k)},$$

(iv) $$\operatorname{Log} \chi(q) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^k}{k(1-q^{2k})},$$

and

(v) $$\frac{\psi(q)}{\varphi(q)} = \frac{(-q^2; q^2)_\infty}{(-q; q^2)_\infty}.$$

Proof. Equalities (i)–(iv) follow easily from (i)–(iv), respectively, of Entry 22. Since the proofs are very similar, we prove only (i). Thus, by Entry 22(i), for $|q| < 1$,

$$\begin{aligned} \operatorname{Log} \varphi(q) &= \sum_{n=1}^{\infty} \operatorname{Log} \frac{1+q^{2n-1}}{1-q^{2n-1}} - \sum_{n=1}^{\infty} \operatorname{Log} \frac{1+q^{2n}}{1-q^{2n}} \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \operatorname{Log} \frac{1+q^n}{1-q^n} \\ &= 2 \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{n=1}^{\infty} \frac{(q^n)^{2k-1}}{2k-1} \\ &= 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} \sum_{n=1}^{\infty} (-1)^{n-1} (q^{2k-1})^n \\ &= 2 \sum_{k=1}^{\infty} \frac{q^{2k-1}}{(2k-1)(1+q^{2k-1})}. \end{aligned}$$

Finally, (v) is an immediate consequence of Entries 22(i), (ii).

Example.

$$\frac{11}{10}\frac{1111}{1110}\frac{111111}{111110}\cdots = 1.101001000100001\ldots.$$

PROOF. Put $q = \frac{1}{10}$ in Entry 22(ii), and the desired result readily follows.

Entry 24. *We have*

(i)
$$\frac{f(q)}{f(-q)} = \frac{\psi(q)}{\psi(-q)} = \frac{\chi(q)}{\chi(-q)} = \sqrt{\frac{\varphi(q)}{\varphi(-q)}},$$

(ii)
$$f^3(-q) = \varphi^2(-q)\psi(q) = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2},$$

(iii)
$$\chi(q) = \frac{f(q)}{f(-q^2)} = \sqrt[3]{\frac{\varphi(q)}{\psi(-q)}} = \frac{\varphi(q)}{f(q)} = \frac{f(-q^2)}{\psi(-q)},$$

(iv)
$$f^3(-q^2) = \varphi(-q)\psi^2(q), \quad \textit{and} \quad \chi(q)\chi(-q) = \chi(-q^2).$$

PROOF OF (i). Using Entries 22(iii), (ii), (iv), and (i), respectively, we find that each of the given ratios is equal to $(-q; q^2)_\infty/(q; q^2)_\infty$.

PROOF OF (ii). By Entries 22(i) and (ii),

$$\begin{aligned}\varphi^2(-q)\psi(q) &= \frac{(q^2; q^2)^3_\infty(q; q^2)_\infty}{(-q; q)^2_\infty}\\ &= (q^2; q^2)^3_\infty(q; q^2)^3_\infty\\ &= (q; q)^3_\infty = f^3(-q),\end{aligned}$$

by Entry 22(iii), where in the penultimate line we used (22.3).

The second equality in (ii) is another famous theorem of Jacobi [1], [2] and is a limiting case of his triple product identity. We refer to the well-known book of Hardy and Wright [1, p. 285] for a proof. An elegant generalization of Jacobi's identity has been discovered by Bhargava, Adiga, and Somashekara [4].

PROOF OF (iii). Each of the five displayed expressions is equal to $(-q; q^2)_\infty$. For the first, second, and fifth, this claim follows immediately from Entry 22. For the third, (22.2) must also be used. Lastly, by Entry 22,

$$\begin{aligned}\frac{\varphi(q)}{f(q)} &= \frac{(-q; q^2)_\infty(q^2; q^2)_\infty}{(q; q^2)_\infty(-q^2; q^2)_\infty(-q; -q)_\infty}\\ &= \frac{(-q; q^2)_\infty(q^2; q^2)_\infty}{(q; q^2)_\infty(-q^2; q^2)_\infty(q^2; q^2)_\infty(-q; q^2)_\infty} = (-q; q^2)_\infty,\end{aligned}$$

by (22.2).

PROOF OF (iv). By Entry 22,

$$\varphi(-q)\psi^2(q) = \frac{(q^2;q^2)_\infty^3}{(-q;q^2)_\infty(-q^2;q^2)_\infty(q;q^2)_\infty} = (q^2;q^2)_\infty^3 = f^3(-q^2),$$

where we have used (22.2) again.

Lastly, by Entry 22(iv),

$$\chi(q)\chi(-q) = (-q;q^2)_\infty(q;q^2)_\infty = (q^2;q^4)_\infty = \chi(-q^2).$$

Entry 25. *We have*

(i) $\varphi(q) + \varphi(-q) = 2\varphi(q^4),$

(ii) $\varphi(q) - \varphi(-q) = 4q\psi(q^8),$

(iii) $\varphi(q)\varphi(-q) = \varphi^2(-q^2), \qquad \psi(q)\psi(-q) = \psi(q^2)\varphi(-q^2),$

(iv) $\varphi(q)\psi(q^2) = \psi^2(q),$

(v) $\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4),$

(vi) $\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2),$

and

(vii) $\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2).$

PROOF OF (i). By Entry 22(i),

$$\varphi(q) + \varphi(-q) = 2 + 2\sum_{k=1}^{\infty}(q^{k^2} + (-q)^{k^2})$$
$$= 2 + 4\sum_{k=1}^{\infty} q^{4k^2} = 2\varphi(q^4).$$

PROOF OF (ii). By Entries 22(i) and (ii),

$$\varphi(q) - \varphi(-q) = 4\sum_{k=1}^{\infty} q^{(2k-1)^2}$$
$$= 4q\sum_{k=1}^{\infty} (q^8)^{k(k-1)/2} = 4q\psi(q^8).$$

PROOF OF (iii). The first equality is a ready consequence of Entry 22(i). By Entries 22(i) and (ii) and (22.2),

$$\psi(q^2)\varphi(-q^2) = \frac{(q^4;q^4)_\infty^2(q^2;q^4)_\infty}{(q^2;q^4)_\infty(-q^2;q^4)_\infty(-q^4;q^4)_\infty}$$
$$= (q^4;q^4)_\infty^2(q^2;q^4)_\infty$$
$$= (q^4;q^4)_\infty(q^2;q^2)_\infty.$$

On the other hand, by Entry 22(ii) and (22.2),

$$\psi(q)\psi(-q) = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty(-q;q^2)_\infty}$$
$$=(q^2;q^2)_\infty^2(-q^2;q^2)_\infty = (q^2;q^2)_\infty(q^4;q^4)_\infty.$$

Hence, the second equality in (iii) follows.

PROOF OF (iv). By Entries 22(i) and (ii),

$$\varphi(q)\psi(q^2) = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty(q^4;q^4)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty(q^2;q^4)_\infty}$$
$$= \frac{(-q;q^2)_\infty(q^2;q^2)_\infty^2(-q^2;q^2)_\infty}{(q;q^2)_\infty^2(-q^2;q^2)_\infty(-q;q^2)_\infty} = \psi^2(q).$$

PROOF OF (v). Multiply equalities (i) and (ii) and then employ equality (iv).

PROOF OF (vi). By Entry 22(i),

$$\varphi^2(q) + \varphi^2(-q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+n^2} + \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+n^2}$$
$$= 2 \sum_{\substack{m,n=-\infty \\ m+n \text{ even}}}^{\infty} q^{m^2+n^2}$$
$$=2 \sum_{j,k=-\infty}^{\infty} q^{2(j^2+k^2)} = 2\varphi^2(q^2),$$

where we set $m + n = 2j$ and $m - n = 2k$.

PROOF OF (vii). Multiply equalities (v) and (vi) together and use (iv).

Corollary. *If*

$$\frac{1-t}{1+t} = \frac{\varphi^2(-q)}{\varphi^2(q)}, \tag{25.1}$$

then

$$1 - t^2 = \frac{\varphi^4(-q^2)}{\varphi^4(q^2)}.$$

PROOF. Let λ denote the right side of (25.1). By Entries 25(iii) and (vi),

$$\frac{\varphi^4(-q^2)}{\varphi^4(q^2)} = \left\{\frac{2\varphi(q)\varphi(-q)}{\varphi^2(q) + \varphi^2(-q)}\right\}^2 = \frac{4\lambda}{(\lambda+1)^2} = 1 - t^2,$$

after an elementary algebraic computation.

We have given above a slightly more explicit version of the corollary than did Ramanujan.

In the next section, we write, for brevity,

$$G(q) = q^{(m-n)^2/8(m+n)} f(q^m, q^n), \qquad m, n > 0.$$

In Entry 26 and its two corollaries, we quote from the notebooks [9, Vol. 2, p. 198].

Entry 26. *$G(q)$ is a perfect, complete, pure, double series of $\frac{1}{2}$ a degree.*

Corollary (i). *$\varphi(q)$, $\sqrt[8]{q}\psi(q)$, and $\sqrt[24]{q}f(q)$ are complete series of $\frac{1}{2}$ a degree.*

Corollary (ii). *$\chi(q)/\sqrt[24]{q}$ is a complete series of 0 degree.*

Proof of Entry 26. The definitions of "perfect," "complete," "pure," "double," and "degree" are found in Section 10 of Chapter 15 (pp. 186, 187) (Part II [9, pp. 320–321]).

Apply the Euler–Maclaurin summation formula (Whittaker and Watson [1, p. 128]) to the function

$$\begin{aligned} g(x) &:= q^{(m-n)^2/8(m+n)+mx(x+1)/2+nx(x-1)/2} \\ &= q^{\{2(m+n)x+(m-n)\}^2/8(m+n)} \end{aligned}$$

on the interval $-\infty < x < \infty$. The series

$$-\tfrac{1}{2}g(-\infty)h - \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} g^{(2k-1)}(-\infty)h^{2k},$$

where B_j denotes the jth Bernoulli number, terminates and, in fact, is identically equal to zero. Thus, according to Ramanujan, $G(q) = \sum_{k=-\infty}^{\infty} g(k)$ is perfect and complete, respectively.

Next, $G(q)$ is pure because the coefficients (which are $\equiv 1$) are homogeneous.

As with Example 6, Section 10 of Chapter 15, which is the special case $m = n = 1$ here, Ramanujan evidently intends "double" series to mean "bilateral" series in this example.

Lastly, M. E. H. Ismail has suggested to us that the degree of a series should be equal to the order of an appropriate singularity on the boundary of convergence of the series. This does not seem to be the definition given rather hazily by Ramanujan. But Ismail's interpretation is more feasible for some of Ramanujan's examples. By the Poisson summation formula (Knopp [1, p. 40]),

$$\begin{aligned} G(q) &= \sum_{k=-\infty}^{\infty} e^{-\pi t(k+(m-n)/2(m+n))^2} \\ &= \frac{1}{\sqrt{t}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/t + \pi i k(m-n)/(m+n)}, \end{aligned}$$

where $e^{-\pi t} = q^{(m+n)/2}$. Thus, $G(q)$ has a singularity at $t = 0$ of degree $\frac{1}{2}$.

PROOF OF COROLLARY (i). The results follow immediately from Entries 22 and 26.

PROOF OF COROLLARY (ii). This claim follows at once from Corollary (i) and the fact $\chi(q) = \varphi(q)/f(q)$ from Entry 24(iii).

Entry 27. *It is assumed that α and β are such that the modulus of each exponential argument below is less than 1. If $\alpha\beta = \pi$, then*

(i) $$\sqrt{\alpha}\varphi(e^{-\alpha^2}) = \sqrt{\beta}\varphi(e^{-\beta^2})$$

and

(ii) $$2\sqrt{\alpha}\psi(e^{-2\alpha^2}) = \sqrt{\beta}e^{\alpha^2/4}\varphi(-e^{-\beta^2}).$$

If $\alpha\beta = \pi^2$, then

(iii) $$e^{-\alpha/12}\sqrt[4]{\alpha}f(-e^{-2\alpha}) = e^{-\beta/12}\sqrt[4]{\beta}f(-e^{-2\beta}),$$

(iv) $$e^{-\alpha/24}\sqrt[4]{\alpha}f(e^{-\alpha}) = e^{-\beta/24}\sqrt[4]{\beta}f(e^{-\beta}),$$

and

(v) $$e^{\alpha/24}\chi(e^{-\alpha}) = e^{\beta/24}\chi(e^{-\beta}).$$

PROOF OF (i). Set $n = 0$ in Entry 20.

PROOF OF (ii). Set $n = \alpha$ in Entry 20 and observe that $f(1, e^{-2\alpha^2}) = 2\psi(e^{-2\alpha^2})$ by Entries 18(ii) and 22(ii).

PROOF OF (iii). By Entries 27(i), 27(ii), and 25(iii),

$$\begin{aligned}2\alpha\varphi(e^{-\alpha^2})\psi(e^{-2\alpha^2}) &= \beta e^{\alpha^2/4}\varphi(e^{-\beta^2})\varphi(-e^{-\beta^2})\\ &= \beta e^{\alpha^2/4}\varphi^2(-e^{-2\beta^2}),\end{aligned}$$

where $\alpha\beta = \pi$. Multiplying both sides by $\psi(e^{-2\beta^2})$ and using Entry 24(ii), we find that

$$2\alpha\varphi(e^{-\alpha^2})\psi(e^{-2\alpha^2})\psi(e^{-2\beta^2}) = \beta e^{\alpha^2/4}f^3(-e^{-2\beta^2}).$$

Interchanging α and β, we deduce that

$$2\beta\varphi(e^{-\beta^2})\psi(e^{-2\alpha^2})\psi(e^{-2\beta^2}) = \alpha e^{\beta^2/4}f^3(-e^{-2\alpha^2}).$$

Divide the former equality by the latter and use Entry 27(i) to conclude that

$$\sqrt{\frac{\alpha}{\beta}} = \frac{\beta e^{\alpha^2/4}f^3(-e^{-2\beta^2})}{\alpha e^{\beta^2/4}f^3(-e^{-2\alpha^2})}.$$

Taking the cube root of both sides and replacing α^2 and β^2 by α and β, respectively, we complete the proof.

PROOF OF (iv). Replace α^2 and β^2 by α and β, respectively, in Entry 27(i) and multiply the resulting equality by Entry 27(iii) to get

$$\sqrt{\alpha}e^{-\alpha/12}\varphi(e^{-\alpha})f(-e^{-2\alpha}) = \sqrt{\beta}e^{-\beta/12}\varphi(e^{-\beta})f(-e^{-2\beta}). \tag{27.1}$$

Using the equality $f^2(q) = \varphi(q)f(-q^2)$ from Entry 24(iii) and then taking the square root of both sides of (27.1), we complete the proof.

PROOF OF (v). In Entry 27(i) replace α^2 and β^2 by α and β, respectively, and then divide that equality by the equality of Entry 27(iv) to obtain

$$e^{\alpha/24}\frac{\varphi(e^{-\alpha})}{f(e^{-\alpha})} = e^{\beta/24}\frac{\varphi(e^{-\beta})}{f(e^{-\beta})},$$

where $\alpha\beta = \pi^2$. Using the equality $\chi(q) = \varphi(q)/f(q)$ from Entry 24(iii), we complete the proof.

Of course, Entry 27(i) is the transformation formula for $\vartheta_3(0, \tau) = \vartheta_3(\tau)$, and Entry 27(iii) gives the transformation formula for $\eta(\tau)$. Several proofs exist for each of these transformation formulas. Moreover, Entries 27(i) and (iii) are only special cases of more general transformation formulas under modular transformations. See Knopp's book [1, Chap. 3] for a full discussion of these transformation formulas. A unified approach to the transformation formulas of $\eta(\tau)$, $\vartheta_3(\tau)$, the other classical theta-functions, and many generalizations has been presented by Berndt [1], [2], [4].

Entry 28. *If $p = ab$ and n is any natural number, then*

$$\prod_{k=1}^{n} f(ap^{k-1}, bp^{n-k}) = \frac{f(a, b)f^n(-p^n)}{f(-p)}.$$

PROOF. Using Entries 19, 22(iii), and 1(iii), we find that

$$\begin{aligned}\frac{f(a, b)f^n(-p^n)}{f(-p)} &= (-a; p)_\infty(-b; p)_\infty(p^n; p^n)_\infty^n \\ &= \left\{\prod_{k=1}^{n} (-ap^{k-1}; p^n)_\infty\right\}\left\{\prod_{k=1}^{n} (-bp^{k-1}; p^n)_\infty\right\}(p^n; p^n)_\infty^n \\ &= \prod_{k=1}^{n} (-ap^{k-1}; p^n)_\infty(-bp^{n-k}; p^n)_\infty(p^n; p^n)_\infty \\ &= \prod_{k=1}^{n} f(ap^{k-1}, bp^{n-k}).\end{aligned}$$

Corollary. *We have*

$$f(-q^2, -q^3)f(-q, -q^4) = f(-q)f(-q^5)$$

and

$$f(-q, -q^6)f(-q^2, -q^5)f(-q^3, -q^4) = f(-q)f^2(-q^7).$$

After the last equality, Ramanujan [9, p. 199] remarks "and so on." By these words, he implies that

$$\prod_{k=1}^{n} f(-q^k, -q^{2n+1-k}) = f(-q)f^{n-1}(-q^{2n+1}), \tag{28.1}$$

where n is any positive integer. The corollary records the cases $n = 2, 3$ of (28.1).

We shall now establish (28.1). Employing Entries 19, 1(iii), and 22(iii), we find that

$$\prod_{k=1}^{n} f(-q^k, -q^{2n+1-k}) = \prod_{k=1}^{n} \{(q^k; q^{2n+1})_\infty (q^{2n+1-k}; q^{2n+1})_\infty (q^{2n+1}; q^{2n+1})_\infty\}$$
$$= (q)_\infty (q^{2n+1}; q^{2n+1})_\infty^{n-1}$$
$$= f(-q) f^{n-1}(-q^{2n+1}).$$

Entry 29. *If $ab = cd$, then*

(i) $$f(a, b)f(c, d) + f(-a, -b)f(-c, -d) = 2f(ac, bd)f(ad, bc)$$

and

(ii) $$f(a, b)f(c, d) - f(-a, -b)f(-c, -d)$$
$$= 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right)f\left(\frac{b}{d}, \frac{d}{b}abcd\right).$$

Many of the identities of Entry 25 above and Entry 30 below are instances of Entry 29. Formula (ii) was discussed by Hardy [3, p. 223] who also briefly sketched a proof. Since the proofs of (i) and (ii) are similar, we give only the proof of (i). Less elementary proofs of Entries 29(i), (ii) may be found in the treatise of Tannery and Molk [1].

Proof of (i). Letting $p = ab = cd$, we see that

$$f(a, b)f(c, d) = \sum_{m,n=-\infty}^{\infty} p^{(m^2+n^2)/2-(m+n)/2} a^m c^n.$$

Thus, setting $m - n = 2j$ and $m + n = 2k$, we find that

$$f(a, b)f(c, d) + f(-a, -b)f(-c, -d)$$
$$= \sum_{\substack{m,n=-\infty \\ m+n \text{ even}}}^{\infty} p^{(m^2+n^2)/2-(m+n)/2} a^m c^n$$
$$= 2 \sum_{j,k=-\infty}^{\infty} p^{j^2+k^2-k} a^{j+k} c^{k-j}$$
$$= 2 \sum_{j,k=-\infty}^{\infty} p^{k(k-1)} (ac)^k p^{j(j+1)} (bc)^{-j}$$
$$= 2f(ac, bd)f(ad, bc).$$

Several of the identities of Entry 25 are special cases of the formulas in Entry 30.

Entry 30. *We have*

(i) $$f(a, ab^2)f(b, a^2b) = f(a, b)\psi(ab),$$

(ii) $$f(a, b) + f(-a, -b) = 2f(a^3b, ab^3),$$

(iii) $$f(a, b) - f(-a, -b) = 2af\left(\frac{b}{a}, \frac{a}{b}a^4b^4\right),$$

(iv) $$f(a, b)f(-a, -b) = f(-a^2, -b^2)\varphi(-ab),$$

(v) $$f^2(a, b) + f^2(-a, -b) = 2f(a^2, b^2)\varphi(ab),$$

and

(vi) $$f^2(a, b) - f^2(-a, -b) = 4af\left(\frac{b}{a}, \frac{a}{b}a^2b^2\right)\psi(a^2b^2).$$

In the proofs below, we set $p = ab$.

Proof of (i). Using in turn Entries 28, 24(iv), and 24(iii), we deduce that

$$\begin{aligned} f(a, ab^2)f(b, a^2b) &= \frac{f(a, b)f^2(-p^2)}{f(-p)} \\ &= \frac{f(a, b)\varphi(-p)\psi^2(p)}{f(-p)f(-p^2)} = f(a, b)\psi(p). \end{aligned}$$

Proof of (ii). Using the definition (18.1) of $f(a, b)$, we find that

$$\begin{aligned} f(a, b) + f(-a, -b) &= 2\sum_{\substack{k=-\infty \\ k \text{ even}}}^{\infty} p^{k(k-1)/2}a^k \\ &= 2\sum_{k=-\infty}^{\infty} p^{k(2k-1)}a^{2k} \\ &= 2\sum_{k=-\infty}^{\infty} (p^4)^{k(k-1)/2}(a^3b)^k \\ &= 2f(a^3b, ab^3). \end{aligned}$$

Proof of (iii). Proceeding as in the proof above, we have

$$\begin{aligned} f(a, b) - f(-a, -b) &= 2\sum_{\substack{k=-\infty \\ k \text{ odd}}}^{\infty} p^{k(k-1)/2}a^k \\ &= 2\sum_{k=-\infty}^{\infty} p^{k(2k+1)}a^{2k+1} \\ &= 2a\sum_{k=-\infty}^{\infty} (p^4)^{k(k+1)/2}(a/b)^k \\ &= 2af\left(\frac{b}{a}, \frac{a}{b}p^4\right). \end{aligned}$$

Proof of (iv). By Entries 19 and 22(i) and (22.3),

$$f(a, b)f(-a, -b) = (a^2; p^2)_\infty(b^2; p^2)_\infty(p; p)_\infty^2$$
$$= f(-a^2, -b^2)\frac{(p; p)_\infty^2}{(p^2; p^2)_\infty} = f(-a^2, -b^2)\varphi(-p).$$

Alternatively, if we set $c = -a$ and $d = -b$ in Entry 29(i), we easily obtain the desired result.

Proof of (v). Putting $c = a$ and $d = b$ in Entry 29(i), we easily achieve the sought result.

Proof of (vi). Set $c = a$ and $d = b$ in Entry 29(ii) and use Entries 18(ii) and 22(ii).

Corollary. *If $ab = cd$, then*

$$f(a, b)f(c, d)f(an, b/n)f(cn, d/n)$$
$$-f(-a, -b)f(-c, -d)f(-an, -b/n)f(-cn, -d/n)$$
$$= 2af(c/a, ad)f(d/an, acn)f(n, ab/n)\psi(ab).$$

Proof. For brevity, set

$$\alpha = f(a, b)f(c, d), \qquad \alpha' = f(-a, -b)f(-c, -d),$$
$$\beta = f(an, b/n)f(cn, d/n), \qquad \beta' = f(-an, -b/n)f(-cn, -d/n),$$

and

$$L = \alpha\beta - \alpha'\beta'.$$

By Entries 29(i) and (ii), we readily find that

$$\alpha + \alpha' = 2f(ac, bd)f(ad, bc),$$
$$\alpha - \alpha' = 2af\left(\frac{b}{c}, \frac{c}{b}abcd\right)f\left(\frac{b}{d}, \frac{d}{b}abcd\right),$$
$$\beta + \beta' = 2f\left(acn^2, \frac{bd}{n^2}\right)f(ad, bc),$$

and

$$\beta - \beta' = 2anf\left(\frac{b}{cn^2}, \frac{cn^2}{b}abcd\right)f\left(\frac{b}{d}, \frac{d}{b}abcd\right).$$

Substituting these in the obvious identity

$$2(\alpha\beta - \alpha'\beta') = (\alpha + \alpha')(\beta - \beta') + (\alpha - \alpha')(\beta + \beta')$$

and using Entries 29(i) and (ii), we find that

$$\begin{aligned}
L &= af(ad, bc)f\left(\frac{b}{d}, \frac{d}{b}abcd\right)\left\{2nf(ac, bd)f\left(\frac{b}{cn^2}, \frac{cn^2}{b}abcd\right)\right. \\
&\quad \left. + 2f\left(\frac{b}{c}, \frac{c}{b}abcd\right)f\left(acn^2, \frac{bd}{n^2}\right)\right\} \\
&= af(ad, bc)f\left(\frac{b}{d}, \frac{d}{b}abcd\right)\left\{\left[f\left(n, \frac{cd}{n}\right)f\left(acn, \frac{b}{cn}\right)\right.\right. \\
&\quad \left. - f\left(-n, -\frac{cd}{n}\right)f\left(-acn, -\frac{b}{cn}\right)\right] + \left[f\left(n, \frac{cd}{n}\right)f\left(acn, \frac{b}{cn}\right)\right. \\
&\quad \left.\left. + f\left(-n, -\frac{cd}{n}\right)f\left(-acn, -\frac{b}{cn}\right)\right]\right\} \\
&= 2af(ad, bc)f\left(\frac{b}{d}, \frac{d}{b}abcd\right)f\left(n, \frac{cd}{n}\right)f\left(acn, \frac{b}{cn}\right).
\end{aligned}$$

If we apply Entry 30(i) and use the hypothesis $ab = cd$, we find that the equality above may be written

$$L = 2af(c/a, ad)\psi(ab)f\left(n, \frac{ab}{n}\right)f\left(\frac{d}{an}, acn\right),$$

which is what we wanted to prove.

Entry 31. *Let* $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ *and* $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$ *for each integer* n. *Then*

$$f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{31.1}$$

Ramanujan writes Entry 31 in the form

$$\begin{aligned}
f(U_1, V_1) &= f(U_n, V_n) + U_1 f\left(\frac{V_{n-1}}{U_1}, \frac{U_{n+1}}{U_1}\right) + V_1 f\left(\frac{U_{n-1}}{V_1}, \frac{V_{n+1}}{V_1}\right) \\
&\quad + U_2 f\left(\frac{V_{n-2}}{U_2}, \frac{U_{n+2}}{U_2}\right) + V_2 f\left(\frac{U_{n-2}}{V_2}, \frac{V_{n+2}}{V_2}\right) + \cdots,
\end{aligned} \tag{31.2}$$

where the sum on the right side evidently contains n terms. However, by Entry 18(iv), for $r \geq 1$,

$$\begin{aligned}
V_r f\left(\frac{U_{n-r}}{V_r}, \frac{V_{n+r}}{V_r}\right) &= U_{n-r} f\left(\frac{U_{n-r}^2 V_{n+r}}{V_r^3}, \frac{V_r}{U_{n-r}}\right) \\
&= U_{n-r} f\left(\frac{U_{2n-r}}{U_{n-r}}, \frac{V_r}{U_{n-r}}\right).
\end{aligned} \tag{31.3}$$

This shows that the sums on the right sides of (31.1) and (31.2) are equal.

PROOF. We have

$$\sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right) = \sum_{k=-\infty}^{\infty} \sum_{r=0}^{n-1} U_r \left(\frac{U_{n+r}}{U_r}\right)^{k(k+1)/2} \left(\frac{V_{n-r}}{U_r}\right)^{k(k-1)/2}$$
$$= \sum_{k=-\infty}^{\infty} \sum_{r=0}^{n-1} U_r^{1-k^2} U_{n+r}^{k(k+1)/2} V_{n-r}^{k(k-1)/2}$$
$$= \sum_{k=-\infty}^{\infty} \sum_{r=0}^{n-1} a^{(nk+r)(nk+r+1)/2} b^{(nk+r)(nk+r-1)/2}$$
$$= \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2} = f(U_1, V_1).$$

Bhargava and Adiga [4] have given a slightly different proof of Entry 31.

Corollary. *We have*

(i)
$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15})$$
$$= \varphi(q^{25}) + 2qf(q^{15}, q^{35}) + 2q^4 f(q^5, q^{45})$$

and

(ii)
$$\psi(q) = f(q^3, q^6) + q\psi(q^9)$$
$$= f(q^6, q^{10}) + qf(q^2, q^{14})$$
$$= f(q^{10}, q^{15}) + qf(q^5, q^{20}) + q^3\psi(q^{25})$$
$$= f(q^{15}, q^{21}) + q\psi(q^9) + q^3 f(q^3, q^{33}).$$

PROOF. The two equalities of part (i) follow from Entry 31 and (31.3) by setting $a = b = q$ and $n = 3, 5$, respectively.

The four equalities in part (ii) follow from Entry 31 by setting $(a, b, n) = (1, q, 3), (q, q^3, 2), (1, q, 5)$, and $(q, q^3, 3)$, respectively.

Parts (i) and (ii) above are, in fact, special cases of the following general formulas:

$$\varphi(q) = \varphi(q^{n^2}) + \sum_{r=1}^{n-1} q^{r^2} f(q^{n(n-2r)}, q^{n(n+2r)})$$

and

$$\psi(q) = \frac{1}{2} \sum_{r=0}^{n-1} q^{r(r-1)/2} f(q^{n(n-2r+1)/2}, q^{n(n+2r-1)/2})$$
$$= \sum_{r=0}^{n-1} q^{r(2r-1)} f(q^{n(2n-4r+1)}, q^{n(2n+4r-1)}),$$

where n is any positive integer. These three formulas are obtained from Entry 31 by setting $(a, b) = (q, q), (1, q)$, and (q, q^3), respectively.

Example (i). *We have*

$$\frac{\varphi^2(q)}{\varphi^2(-q)}+\frac{\varphi^2(r)}{\varphi^2(-r)}+\frac{\varphi^2(s)}{\varphi^2(-s)}+\frac{\varphi^2(q)\varphi^2(r)\varphi^2(s)}{\varphi^2(-q)\varphi^2(-r)\varphi^2(-s)}$$
$$=4\frac{\varphi^2(q^2)\varphi^2(r^2)\varphi^2(s^2)}{\varphi^2(-q)\varphi^2(-r)\varphi^2(-s)}+256qrs\frac{\psi^2(q^4)\psi^2(r^4)\psi^2(s^4)}{\varphi^2(-q)\varphi^2(-r)\varphi^2(-s)}. \tag{31.4}$$

PROOF. By Entries 25(vi) and (v), the right side of (31.4) can be written

$$\frac{1}{2}\left\{1+\frac{\varphi^2(q)}{\varphi^2(-q)}\right\}\left\{1+\frac{\varphi^2(r)}{\varphi^2(-r)}\right\}\left\{1+\frac{\varphi^2(s)}{\varphi^2(-s)}\right\}$$
$$-\frac{1}{2}\left\{1-\frac{\varphi^2(q)}{\varphi^2(-q)}\right\}\left\{1-\frac{\varphi^2(r)}{\varphi^2(-r)}\right\}\left\{1-\frac{\varphi^2(s)}{\varphi^2(-s)}\right\}.$$

Upon simplification, the expression above yields the left side of (31.4).

Example (ii). *We have*

$$\frac{1}{\varphi(q^4)}=\frac{1}{\varphi(q)\pm\varphi(q^2)}+\frac{1}{\varphi(-q)\pm\varphi(q^2)} \tag{31.5}$$

and

$$\frac{1}{\varphi(-q^2)}=\frac{1}{\varphi(-q^2)\pm\varphi(q)}+\frac{1}{\varphi(-q^2)\pm\varphi(-q)}. \tag{31.6}$$

PROOF. It is easy to see that the equation

$$\frac{1}{A}=\frac{1}{C\pm B}+\frac{1}{D\pm B} \tag{31.7}$$

is satisfied if

$$2A=C+D \quad\text{and}\quad 2A^2-B^2=CD. \tag{31.8}$$

Take $A=\varphi(q^4)$, $B=\varphi(q^2)$, $C=\varphi(q)$, and $D=\varphi(-q)$. By Entries 25(i), (iii), and (vi), equalities (31.8) are seen to be satisfied for these choices of A, B, C, and D. Hence, (31.5) follows at once from (31.7).

Second, the equality

$$\frac{1}{E}=\frac{1}{E\pm C}+\frac{1}{E\pm D} \tag{31.9}$$

is seen to be satisfied if $CD=E^2$. Now by Entry 25(iii), this equality is satisfied when $C=\varphi(q)$, $D=\varphi(-q)$, and $E=\varphi(-q^2)$. Hence, (31.6) follows from (31.9).

Example (iii). *For each natural number n, the coefficient of q^n in the expansion of $[q/(1-q)]\psi(q^2)$ is the integer nearest to $\sqrt{n}$.*

PROOF. By Entry 22(ii),

$$\frac{q}{1-q}\psi(q^2) = \sum_{j=1}^{\infty} q^j \sum_{k=1}^{\infty} q^{k(k-1)}.$$

Thus, the coefficient of q^n is equal to the number of powers $q^{k(k-1)}$ such that $k(k-1) < n$. In other words, the coefficient of q^n is the unique integer k such that

$$(k-\tfrac{1}{2})^2 < k(k-1)+1 \le n \le k(k+1) < (k+\tfrac{1}{2})^2.$$

Clearly, from the inequalities above, k is the nearest integer to $\sqrt{n}$.

Example (iv). *We have*

$$\varphi(-q) + \varphi(q^2) = 2\frac{f^2(q^3, q^5)}{\psi(q)}$$

and

$$\varphi(-q) - \varphi(q^2) = -2q\frac{f^2(q, q^7)}{\psi(q)}.$$

PROOF. Putting $a = q$, $b = q^3$, and $c = d = q^2$ in Entries 29(i) and (ii), we find that

$$\psi(q)\varphi(q^2) + \psi(-q)\varphi(-q^2) = 2f^2(q^3, q^5)$$

and

$$\psi(q)\varphi(q^2) - \psi(-q)\varphi(-q^2) = 2qf^2(q, q^7).$$

These equalities reduce to the desired identities on using the fact

$$\psi(-q)\varphi(-q^2) = \varphi(-q)\psi(q), \tag{31.10}$$

which is deducible from Entries 25(iii) and (iv).

Example (v). *We have* $f(q, q^5) = \psi(-q^3)\chi(q)$.

PROOF. By Entries 19, 22(ii), and 22(iv),

$$\begin{aligned} f(q, q^5) &= (-q; q^6)_\infty(-q^5; q^6)_\infty(q^6; q^6)_\infty \\ &= (-q; q^2)_\infty \frac{(q^6; q^6)_\infty}{(-q^3; q^6)_\infty} = \chi(q)\psi(-q^3). \end{aligned}$$

Entry 32. *We have*

(i) $$\frac{\varphi'(q)}{\varphi(q)} - \frac{\psi'(q)}{\psi(q)} = \frac{1-\varphi^4(-q)}{8q},$$

(ii)
$$\frac{\psi'(q)}{\psi(q)} - 2q\frac{\psi'(q^2)}{\psi(q^2)} = \frac{1-\varphi^4(-q)}{8q},$$

(iii)
$$\frac{\varphi'(q)}{\varphi(q)} + \frac{\varphi'(-q)}{\varphi(-q)} = \frac{\varphi^4(q)-\varphi^4(-q)}{4q},$$

and

(iv)
$$\frac{\varphi'(q)}{\varphi(q)} - \frac{\varphi'(-q)}{\varphi(-q)} = -4q\frac{\varphi'(-q^2)}{\varphi(-q^2)}.$$

Proof of (i), (ii). Using first Entry 25(iv) and then Entry 23(ii), we find that

$$\begin{aligned}\frac{\varphi'(q)}{\varphi(q)} - \frac{\psi'(q)}{\psi(q)} &= \frac{\psi'(q)}{\psi(q)} - 2q\frac{\psi'(q^2)}{\psi(q^2)}\\ &= \frac{d}{dq}\sum_{k=1}^{\infty}\frac{q^k}{k(1+q^k)} - 2\frac{d}{dq}\sum_{k=1}^{\infty}\frac{q^{2k}}{2k(1+q^{2k})}\\ &= \frac{d}{dq}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^k}{k(1+q^k)}\\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^{k-1}}{(1+q^k)^2} = \frac{1-\varphi^4(-q)}{8q},\end{aligned}$$

by (33.5).

Proof of (iii). Proceeding in the same manner as above, with the help of Entry 23(i) and (33.5), we find that

$$\begin{aligned}\frac{\varphi'(q)}{\varphi(q)} + \frac{\varphi'(-q)}{\varphi(-q)} &= 2\frac{d}{dq}\sum_{k=1}^{\infty}\left\{\frac{q^{2k-1}}{(2k-1)(1+q^{2k-1})} + \frac{q^{2k-1}}{(2k-1)(1-q^{2k-1})}\right\}\\ &= 2\frac{d}{dq}\sum_{k=1}^{\infty}\left\{\frac{(-1)^{k-1}q^k}{k(1+q^k)} + \frac{q^k}{k(1+(-q)^k)}\right\}\\ &= 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^{k-1}}{(1+q^k)^2} + 2\sum_{k=1}^{\infty}\frac{q^{k-1}}{(1+(-q)^k)^2}\\ &= \frac{1-\varphi^4(-q)}{4q} + \frac{\varphi^4(q)-1}{4q},\end{aligned}$$

which concludes the proof of (iii).

Proof of (iv). The desired equality is an obvious consequence of Entry 25(iii).

Entry 33(i). *If* $|q| < 1$ *and* θ *is real, then*

$$\operatorname{Log}\left[\frac{1+2\sum_{k=1}^{\infty}q^{k^2}\cos(k\theta)}{f(-q^2)}\right] = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^k}{k(1-q^{2k})}\cos(k\theta).$$

Proof. In (21.2), set $a = qe^{i\theta}$ and $b = qe^{-i\theta}$, and then employ Entry 22(iii).

Entry 33(ii). *If* $|q| < 1$ *and* n *is real, then*

$$\frac{1}{4}\operatorname{Log}\left[\frac{\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^{k(k-1)/2}\sin(2k-1)n}{\sin n}}{\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}(2k-1)q^{k(k-1)/2}}\right] = \sum_{k=1}^{\infty}\frac{q^k\sin^2(kn)}{k(1-q^k)}. \tag{33.1}$$

Proof. Letting $z = e^{2in}$ and using Entry 19, we find that

$$\begin{aligned}(zq;q)_\infty\left(\frac{q}{z};q\right)_\infty(q;q)_\infty &= \frac{1}{1-1/z}f(-zq,-1/z)\\ &= \frac{1}{1-1/z}\sum_{k=-\infty}^{\infty}(-1)^k q^{k(k+1)/2}z^k\\ &= \sum_{k=1}^{\infty}(-1)^{k-1}q^{k(k-1)/2}\frac{z^{k-1/2}-z^{-k+1/2}}{z^{1/2}-z^{-1/2}}\\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^{k(k-1)/2}\sin(2k-1)n}{\sin n}.\end{aligned} \tag{33.2}$$

By letting n tend to 0 in (33.2), or by employing Entry 24(ii) along with Entry 22(iii), we see that

$$(q;q)_\infty^3 = \sum_{k=1}^{\infty}(-1)^{k-1}(2k-1)q^{k(k-1)/2}.$$

Thus, if Q is the quotient in large parentheses on the left side of (33.1), we have shown that

$$Q = \frac{(e^{2in}q;q)_\infty(e^{-2in}q;q)_\infty}{(q;q)_\infty^2}.$$

Taking the logarithm of Q above and using (21.1), we find that

$$\begin{aligned}\frac{1}{4}\operatorname{Log} Q &= \frac{1}{4}\sum_{k=1}^{\infty}\frac{q^k}{k(1-q^k)}(-e^{2ink}-e^{-2ink}+2)\\ &= \sum_{k=1}^{\infty}\frac{q^k\sin^2(kn)}{k(1-q^k)},\end{aligned}$$

which completes the proof.

Entry 33(iii). *If* $|q| < 1$ *and* n *is real, then*

$$1 + 4\sum_{k=1}^{\infty}\frac{q^k\cos(kn)}{1+q^{2k}} = \varphi^2(-q^2)\frac{1+2\displaystyle\sum_{k=1}^{\infty}q^{k^2}\cos(kn)}{1+2\displaystyle\sum_{k=1}^{\infty}(-1)^k q^{k^2}\cos(kn)}. \tag{33.3}$$

PROOF. Observe that the right side of (33.3) can be written

$$\varphi^2(-q^2)\frac{f(zq, q/z)}{f(-zq, -q/z)} = \frac{(-zq; q^2)_\infty(-q/z; q^2)_\infty(q^2; q^2)_\infty^2}{(zq; q^2)_\infty(q/z; q^2)_\infty(-q^2; q^2)_\infty^2}, \tag{33.4}$$

where we have employed Entry 19 and (22.4). If we now replace n by -1 and z by e^{in} in the corollary to Entry 17, we find that the right side of (33.4) is equal to

$$1 + 2\sum_{k=1}^{\infty}\frac{(-1; q^2)_k q^k \cos(kn)}{(-q^2; q^2)_k} = 1 + 4\sum_{k=1}^{\infty}\frac{q^k \cos(kn)}{1 + q^{2k}},$$

and this completes the proof.

Corollary. *If* $|a|, |b| < 1$, *then*

$$\varphi^2(-ab)\frac{f(a, b)}{f(-a, -b)} = 1 + 2\sum_{k=1}^{\infty}\frac{a^k + b^k}{1 + a^k b^k}.$$

PROOF. Put $qe^{in} = a$ and $qe^{-in} = b$ in Entry 33(iii).

Entry 33(iii) essentially gives the Fourier series of the elliptic function dn u, where $u = Kn/\pi$, and where K denotes the complete elliptic integral of the first kind.

We now derive a useful consequence of Entry 33(iii). Replacing n by $\pi - n$ in (33.3) and multiplying the two results together, we find that

$$\varphi^4(-q^2) = \left(1 + 4\sum_{k=1}^{\infty}\frac{q^k \cos(kn)}{1 + q^{2k}}\right)\left(1 + 4\sum_{k=1}^{\infty}\frac{(-q)^k \cos(kn)}{1 + q^{2k}}\right).$$

Form the product of the two series on the right side and then integrate both sides with respect to n over the interval $[-\pi, \pi]$. Since the set of functions $\{\cos(kn)\}$, $0 \le k < \infty$, is orthogonal on $[-\pi, \pi]$, we deduce that

$$\varphi^4(-q^2) = 1 + 8\sum_{k=1}^{\infty}\frac{(-1)^k q^{2k}}{(1 + q^{2k})^2}, \tag{33.5}$$

a result due to Jacobi [1], [2].

Formulas for $\varphi^{2n}(q)$, $1 \le n \le 12$, similar to that found above, have been derived by Ramamani [1].

Entry 34(i). *If* $|q| < 1$ *and* n *is real, then*

$$\text{Log}\left[\frac{\varphi^2(q)}{1 + 4\cos n\sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^{2k-1}\cos(2k-1)n}{1 - q^{2k-1}}}\right] = 4\sum_{k=1}^{\infty}\frac{(-1)^{k-1}q^k \sin^2(kn)}{k(1 + q^k)}. \tag{34.1}$$

PROOF. Letting $\alpha = \beta = q$ in Entry 17, we see that

$$\frac{(-zq; q^2)_\infty(-q/z; q^2)_\infty(q^2; q^2)_\infty(q^4; q^2)_\infty}{(-zq^2; q^2)_\infty(-q^2/z; q^2)_\infty(q^3; q^2)_\infty^2}$$

$$= 1 \sum_{k=1}^{\infty} \frac{(1/q; q^2)_k(-1)^k q^{2k}}{(q^3; q^2)_k}(z^k + z^{-k})$$

$$= 1 - \frac{(1 - 1/q)q^2}{1 - q^3}(z + z^{-1}) + \sum_{k=2}^{\infty} \frac{(1 - 1/q)(1 - q)(-1)^k q^{2k}}{(1 - q^{2k-1})(1 - q^{2k+1})}(z^k + z^{-k})$$

$$= 1 + \frac{(1 - q)q}{1 - q^3}(z + z^{-1})$$

$$+ \sum_{k=2}^{\infty} \frac{(1 - 1/q)(1 - q)(-1)^k q^{2k}}{1 - q^2}\left\{\frac{1}{1 - q^{2k-1}} - \frac{q^2}{1 - q^{2k+1}}\right\}(z^k + z^{-k})$$

$$= 1 + \frac{(1 - q)q}{1 - q^3}(z + z^{-1}) + \frac{1 - q}{1 + q}\sum_{k=2}^{\infty} \frac{(-1)^{k-1}q^{2k-1}}{1 - q^{2k-1}}(z^k + z^{-k})$$

$$+ \frac{1 - q}{1 + q}\sum_{k=3}^{\infty} \frac{(-1)^{k-1}q^{2k-1}}{1 - q^{2k-1}}(z^{k-1} + z^{-k+1})$$

$$= 1 + \frac{1 - q}{1 + q}\left\{\sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^{2k-1}}{1 - q^{2k-1}}(z^k + z^{-k})\right.$$

$$\left. + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}q^{2k-1}}{1 - q^{2k-1}}(z^{k-1} + z^{-k+1})\right\}$$

$$= 1 + \frac{1 - q}{1 + q}\left\{-\frac{2q}{1 - q} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^{2k-1}(z^{k-1/2} + z^{-k+1/2})(z^{1/2} + z^{-1/2})}{1 - q^{2k-1}}\right\}$$

$$= \frac{1 - q}{1 + q}\left\{1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^{2k-1}(z^{k-1/2} + z^{-k+1/2})(z^{1/2} + z^{-1/2})}{1 - q^{2k-1}}\right\}.$$

With $z = e^{2in}$ and the use of Entry 22(i), we may rewrite the foregoing conclusion in the form

$$\varphi^2(q)\frac{F(n)}{F(0)} = 1 + 4\cos n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^{2k-1}\cos(2k - 1)n}{1 - q^{2k-1}}, \tag{34.2}$$

where

$$F(n) = \frac{(-zq; q^2)_\infty(-q/z; q^2)_\infty}{(-zq^2; q^2)_\infty(-q^2/z; q^2)_\infty}.$$

Comparing (34.1) and (34.2), we see that it suffices to show that

$$\operatorname{Log}\frac{F(n)}{F(0)} = 4\sum_{k=1}^{\infty} \frac{(-1)^k q^k \sin^2(kn)}{k(1 + q^k)}.$$

To show the equality above, it suffices to show that

$$\operatorname{Log} F(n) = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} q^k \cos(2kn)}{k(1+q^k)}.$$

The proof of this is quite straightforward and much like the proofs of Entries 21 and 23, and so we omit it.

Entry 34(ii). *If* $|q| < 1$ *and* n *is real, then*

$$\frac{1}{8} \operatorname{Log} \left[\frac{\varphi^2(q)}{1 + 4 \sum_{k=1}^{\infty} \frac{q^k \cos(2kn)}{1+q^{2k}}} \right] = \sum_{k=1}^{\infty} \frac{q^{2k-1} \sin^2(2k-1)n}{(2k-1)(1-q^{4k-2})}. \tag{34.3}$$

PROOF. Putting $\alpha = \beta = -1$ and $z = e^{2in}$ in Entry 17, we find that

$$\begin{aligned} 1 + 4 \sum_{k=1}^{\infty} \frac{q^k \cos(2kn)}{1+q^{2k}} &= \frac{(-zq; q^2)_\infty(-q/z; q^2)_\infty(q^2; q^2)_\infty^2}{(zq; q^2)_\infty(q/z; q^2)_\infty(-q^2; q^2)_\infty^2} \\ &= \varphi^2(q) \frac{G(n)}{G(0)}, \end{aligned} \tag{34.4}$$

where we have used Entry 22(i) and where

$$G(n) = \frac{(-zq; q^2)_\infty(-q/z; q^2)_\infty}{(zq; q^2)_\infty(q/z; q^2)_\infty}.$$

Comparing (34.3) and (34.4), we see that it suffices to show that

$$\operatorname{Log} G(n) = 4 \sum_{k=1}^{\infty} \frac{q^{2k-1} \cos\{2(2k-1)n\}}{(2k-1)(1-q^{4k-2})}.$$

Like the calculation of Log $F(n)$ in the previous proof, the proof of the equality above is quite straightforward.

Ramanujan now states two "corollaries." We have not been able to discern why the appellation "corollary" has been given to these two results. Moreover, the "corollaries" are incorrect. We give two corrected versions of each corollary. First, we prove versions where the "right sides" are corrected; second, we establish reformulations when the "left sides" are corrected. Most likely, the first versions are what Ramanujan had in mind. Our first proof below uses Watson's quintuple product identity, which has been rediscovered several times and which appears in the literature in several guises. We provide a thorough discussion of this identity at the end of Section 38, after giving a proof based on one of Schröter's formulas which we develop in Section 36.

Corollary (i) (First Version). *If* $|q| < 1$ *and* $z = e^{2in}$, *where* n *is real, then*

$$\frac{1}{4}\operatorname{Log}\left[\frac{\sum_{k=-\infty}^{\infty} q^{k(3k-2)}\sin\{2(3k-1)n\}}{\sin(2n)\sum_{k=-\infty}^{\infty}(3k-1)q^{k(3k-2)}}\right] = \sum_{k=1}^{\infty}\frac{q^k\sin^2(kn)}{k(1-q^{2k})} + \sum_{k=1}^{\infty}\frac{q^{4k}\sin^2(2kn)}{k(1-q^{4k})}. \tag{34.5}$$

Proof. By Watson's quintuple product identity (38.9),

$$\begin{aligned}(q^2;q^2)_\infty(zq;q^2)_\infty(q/z;q^2)_\infty(z^2;q^4)_\infty(q^4/z^2;q^4)_\infty &= \sum_{k=-\infty}^{\infty} q^{3k^2+k}(z^{3k}q^{-3k} - z^{-3k-1}q^{3k+1})\\ &= z\left\{\sum_{k=-\infty}^{\infty} q^{3k^2-2k}z^{3k-1} - \sum_{k=-\infty}^{\infty} q^{3k^2-2k}z^{-3k+1}\right\}\\ &= 2iz\sum_{k=-\infty}^{\infty} q^{3k^2-2k}\sin\{2(3k-1)n\},\end{aligned}$$

where we obtained the penultimate equality by replacing k by $k-1$ in the previous latter summands. Thus, upon dividing both sides above by $1 - z^2$, we find that

$$\frac{F(n)}{F(0)} = \frac{\sum_{k=-\infty}^{\infty} q^{3k^2-2k}\sin\{2(3k-1)n\}}{\sin(2n)\sum_{k=-\infty}^{\infty}(3k-1)q^{3k^2-2k}}, \tag{34.6}$$

where

$$F(n) = (qz;q^2)_\infty(q/z;q^2)_\infty(q^4z^2;q^4)_\infty(q^4/z^2;q^4)_\infty.$$

Now a straightforward calculation yields

$$\operatorname{Log} F(n) = -2\sum_{k=1}^{\infty}\frac{q^k\cos(2kn)}{k(1-q^{2k})} - 2\sum_{k=1}^{\infty}\frac{q^{4k}\cos(4kn)}{k(1-q^{4k})},$$

so that

$$\frac{1}{4}\operatorname{Log}\frac{F(n)}{F(0)} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{q^k(1-\cos(2kn))}{k(1-q^{2k})} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{q^{4k}(1-\cos(4kn))}{k(1-q^{4k})}. \tag{34.7}$$

Combining (34.6) and (34.7), we readily arrive at (34.5).

Corollary (i) (Second Version). *If* $|q| < 1$ *and* $z = e^{2in}$, *where* n *is real, then*

$$\frac{1}{4} \operatorname{Log} \left[\frac{\sum_{k=1}^{\infty} (-q)^{k(k-1)/2} \cos(2k-1)n}{\psi(-q) \cos n} \right] = \sum_{k=1}^{\infty} \frac{q^k \sin^2(kn)}{k(1+q^k)} + \sum_{k=1}^{\infty} \frac{q^{4k} \sin^2(2kn)}{k(1-q^{4k})}. \tag{34.8}$$

PROOF. Let

$$P(z) = \frac{(zq; q^2)_\infty (q/z; q^2)_\infty (z^2q^4; q^4)_\infty (q^4/z^2; q^4)_\infty (q^2; q^2)_\infty^2}{(zq^2; q^2)_\infty (q^2/z; q^2)_\infty (q; q^2)_\infty^2 (q^4; q^4)_\infty^2}.$$

Then a straightforward, but rather lengthy, calculation shows that

$$\operatorname{Log} P(z) = 4 \sum_{k=1}^{\infty} \frac{q^k \sin^2(kn)}{k(1+q^k)} + 4 \sum_{k=1}^{\infty} \frac{q^{4k} \sin^2(2kn)}{k(1-q^{4k})}. \tag{34.9}$$

Using the factorization

$$(z^2q^4; q^4)_\infty = (zq^2; q^2)_\infty (-zq^2; q^2)_\infty,$$

a similar factorization for $(q^4/z^2; q^4)_\infty$, and Entry 19, we find that

$$\begin{aligned} P(z) &= \frac{(zq; q^2)_\infty (q/z; q^2)_\infty (-zq^2; q^2)_\infty (-q^2/z; q^2)_\infty (q^2; q^2)_\infty^2}{(q; q^2)_\infty^2 (q^4; q^4)_\infty^2} \\ &= \frac{f(-zq, -q/z) f(z, q^2/z)}{(1+z)(q; q^2)_\infty^2 (q^4; q^4)_\infty^2} \qquad (34.10) \\ &= \frac{\psi(-q) f(-q/z, z)}{(1+z)(q; q^2)_\infty^2 (q^4; q^4)_\infty^2}, \end{aligned}$$

where we have used Entry 30(i) with $a = -q/z$ and $b = z$.

Set

$$F(n) = \frac{f(-q/z, z)}{1-z}, \tag{34.11}$$

and observe that, by Entry 22(ii) and (22.2),

$$\begin{aligned} \frac{\psi(-q)}{(q; q^2)_\infty^2 (q^4; q^4)_\infty^2} &= \frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty (q; q^2)_\infty^2 (-q^2; q^2)_\infty^2 (q^2; q^2)_\infty^2} \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} = \frac{1}{\psi(-q)}. \end{aligned}$$

Hence, (34.10) may be written

$$P(z) = \frac{F(n)}{\psi(-q)}. \tag{34.12}$$

By (34.8), (34.9), and (34.12), it suffices to show that

$$F(n) = \sec n \sum_{k=1}^{\infty} (-q)^{k(k-1)/2} \cos(2k-1)n. \tag{34.13}$$

But, by (34.11),

$$\begin{aligned} F(n) &= \frac{z^{-1/2}}{z^{1/2}+z^{-1/2}} \sum_{k=-\infty}^{\infty} (-q)^{k(k-1)/2} z^k \\ &= \frac{1}{z^{1/2}+z^{-1/2}} \sum_{k=1}^{\infty} (-q)^{k(k-1)/2} (z^{k-1/2} + z^{-k+1/2}) \end{aligned}$$

from which (34.13) is apparent. This completes the proof.

Corollary (ii) (First Version). *If* $|q| < 1$ *and* $z = e^{2in}$, *where* n *is real, then*

$$\frac{1}{4} \operatorname{Log} \left[\frac{\sum_{k=-\infty}^{\infty} q^{(3k^2+k)/2} \sin(6k+1)n}{\sin n \sum_{k=-\infty}^{\infty} (6k+1) q^{(3k^2+k)/2}} \right] = \sum_{k=1}^{\infty} \frac{q^k \sin^2(kn)}{k(1-q^k)} + \sum_{k=1}^{\infty} \frac{q^k \sin^2(2kn)}{k(1-q^{2k})}. \tag{34.14}$$

Proof. Applying Watson's quintuple product identity (38.9), we find that

$$\begin{aligned} &(q;q)_\infty (qz;q)_\infty (1/z;q)_\infty (qz^2;q^2)_\infty (q/z^2;q^2)_\infty \\ &= \sum_{k=-\infty}^{\infty} q^{(3k^2+k)/2} (z^{3k} - z^{-3k-1}) \\ &= 2iz^{-1/2} \sum_{k=-\infty}^{\infty} q^{(3k^2+k)/2} \sin(6k+1)n. \end{aligned}$$

Dividing both sides by $(1 - 1/z)$, we deduce that

$$\frac{G(n)}{G(0)} = \frac{\sum_{k=-\infty}^{\infty} q^{(3k^2+k)/2} \sin(6k+1)n}{\sin n \sum_{k=-\infty}^{\infty} (6k+1) q^{(3k^2+k)/2}}, \tag{34.15}$$

where

$$G(n) = (qz;q)_\infty (q/z;q)_\infty (qz^2;q^2)_\infty (q/z^2;q^2)_\infty.$$

Proceeding in the same fashion as in the first proof of Corollary (i), we find that

$$\frac{1}{4} \operatorname{Log} \frac{G(n)}{G(0)} = \sum_{k=1}^{\infty} \frac{q^k \sin^2(kn)}{k(1-q^k)} + \sum_{k=1}^{\infty} \frac{q^k \sin^2(2kn)}{k(1-q^{2k})}. \tag{34.16}$$

Taking (34.15) and (34.16) together, we deduce (34.14).

Corollary (ii) (Second Version). *If* $|q| < 1$ *and* $z = e^{2in}$, *where n is real, then*

$$\tfrac{1}{4}\operatorname{Log}\left(\frac{4\cos n}{\varphi^2(-q)}\sum_{k=-\infty}^{\infty}\frac{(-1)^k q^k\cos(2k+1)n}{1+2q^{2k}\cos(2n)+q^{4k}}\right) = \sum_{k=1}^{\infty}\frac{q^k\sin^2(kn)}{k(1-q^k)} + \sum_{k=1}^{\infty}\frac{q^k\sin^2(2kn)}{k(1+q^k)}. \tag{34.17}$$

PROOF. Let

$$Q(z) = \frac{(zq;q)_\infty(q/z;q)_\infty(z^2q;q^2)_\infty(q/z^2;q^2)_\infty(q^2;q^2)_\infty^2}{(z^2q^2;q^2)_\infty(q^2/z^2;q^2)_\infty(q;q)_\infty^2(q;q^2)_\infty^2}.$$

Then an elementary calculation yields

$$\operatorname{Log} Q(z) = 4\sum_{k=1}^{\infty}\frac{q^k\sin^2(kn)}{k(1-q^k)} + 4\sum_{k=1}^{\infty}\frac{q^k\sin^2(2kn)}{k(1+q^k)}. \tag{34.18}$$

Put

$$G(n) = (q;q)_\infty^2(q;q^2)_\infty^2 Q(z).$$

Using the factorization

$$(z^2q^2;q^2)_\infty = (zq;q)_\infty(-zq^2;q^2)_\infty(-zq;q^2)_\infty$$

and a similar factorization for $(q^2/z^2;q^2)_\infty$, we find that

$$G(n) = \frac{(z^2q;q^2)_\infty(q/z^2;q^2)_\infty(q^2;q^2)_\infty^2}{(-zq^2;q^2)_\infty(-zq;q^2)_\infty(-q^2/z;q^2)_\infty(-q/z;q^2)_\infty}.$$

Employing Entry 17 with z, α, and β replaced by $-z^2$, $-1/z$, and $-z$, respectively, we deduce that

$$\begin{aligned} G(n) &= 1 + \sum_{k=1}^{\infty}\frac{(-1)^k q^k(1+z)z^k}{1+zq^{2k}} + \sum_{k=1}^{\infty}\frac{(-1)^k q^k(1+1/z)z^{-k}}{1+q^{2k}/z} \\ &= 1 + (z^{1/2}+z^{-1/2}) \\ &\quad\times\sum_{k=1}^{\infty}\frac{(-1)^k q^k\{(z^{k+1/2}+z^{-k-1/2})+q^{2k}(z^{k-1/2}+z^{-k+1/2})\}}{(1+zq^{2k})(1+q^{2k}/z)} \\ &= (z^{1/2}+z^{-1/2})\sum_{k=-\infty}^{\infty}\frac{(-1)^k q^k(z^{k+1/2}+z^{-k-1/2})}{(1+zq^{2k})(1+q^{2k}/z)} \\ &= 4\cos n\sum_{k=-\infty}^{\infty}\frac{(-1)^k q^k\cos(2k+1)n}{1+2q^{2k}\cos(2n)+q^{4k}}. \end{aligned} \tag{34.19}$$

Next, by (22.4),

$$Q(z) = \frac{G(n)}{(q;q)_\infty^2(q;q^2)_\infty^2} = \frac{G(n)}{\varphi^2(-q)}. \tag{34.20}$$

Putting (34.19) into (34.20) and combining the result with (34.18), we complete the proof.

For other results in the spirit of those in Section 34, see two papers by Rogers [1], [2].

Recall that the Bernoulli numbers B_n, $0 \le n < \infty$, are defined by (Gradshteyn and Ryzhik [1, p. 1076])

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n, \qquad |x| < 2\pi.$$

Recall also that the Euler numbers E_{2n}, $0 \le n < \infty$, are given by (Gradshteyn and Ryzhik [1, p. 1078])

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n}, \qquad |x| < \pi/2. \tag{35.1}$$

These conventions for the Bernoulli and Euler numbers are different from those used by Ramanujan.

Entry 35(i). *For each positive integer n, let*

$$P_n = -\frac{B_n}{2n} + \sum_{k=1}^{\infty} \frac{k^{n-1} q^k}{1 - q^k},$$

where B_n denotes the nth *Bernoulli number. If n is any nonnegative integer, let*

$$Q_n = \frac{1}{n+1} \frac{\sum_{k=1}^{\infty} (-1)^{k+1} (2k-1)^{n+1} q^{k(k-1)/2}}{\sum_{k=1}^{\infty} (-1)^{k+1} (2k-1) q^{k(k-1)/2}}.$$

Then for each positive integer n,

$$\tfrac{1}{2} Q_{2n} = -\sum_{k=1}^{n} \binom{2n-1}{2k-1} 2^{2k} P_{2k} Q_{2n-2k}. \tag{35.2}$$

PROOF. We shall write Entry 33(ii) in the form

$$\begin{aligned} L &:= \tfrac{1}{2} \operatorname{Log} \left\{ \frac{\sum_{k=1}^{\infty} (-1)^{k-1} q^{k(k-1)/2} (\sin(2k-1)\theta)/\theta}{\sum_{k=1}^{\infty} (-1)^{k-1} (2k-1) q^{k(k-1)/2}} \right\} \\ &= \tfrac{1}{2} \operatorname{Log}\left(\frac{\sin\theta}{\theta}\right) + \sum_{k=1}^{\infty} \frac{q^k (1 - \cos(2k\theta))}{k(1 - q^k)} =: R, \end{aligned} \tag{35.3}$$

and equate coefficients of like powers of θ. Expanding $\sin(2k-1)\theta$ in its Maclaurin series and then inverting the order of summation, we readily find that

$$L = \tfrac{1}{2}\,\mathrm{Log}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} Q_{2n}\theta^{2n}\right). \tag{35.4}$$

On the other hand, using a familiar expansion for $\mathrm{Log}((\sin\theta)/\theta)$, which, in fact, Ramanujan derived in Chapter 5 (p. 52) (Part I [5, p. 122, Entry 16]), we find that, for $|\theta| < \pi$,

$$\begin{aligned} R &= \frac{1}{2}\sum_{j=1}^{\infty} \frac{(-1)^j B_{2j}(2\theta)^{2j}}{(2j)(2j)!} + \sum_{j=1}^{\infty} \frac{(-1)^{j+1}2^{2j}}{(2j)!} \sum_{k=1}^{\infty} \frac{k^{2j-1}q^k}{1-q^k}\theta^{2j} \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}2^{2j}}{(2j)!}\left(-\frac{B_{2j}}{4j} + \sum_{k=1}^{\infty} \frac{k^{2j-1}q^k}{1-q^k}\right)\theta^{2j} \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}2^{2j}}{(2j)!} P_{2j}\theta^{2j}. \end{aligned} \tag{35.5}$$

Using (35.4) and (35.5) in (35.3), we deduce that, for $|\theta| < \pi$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} Q_{2n}\theta^{2n} = \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1}2^{2j+1}}{(2j)!} P_{2j}\theta^{2j}\right).$$

Differentiating both sides with respect to θ, we find that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n Q_{2n}}{(2n-1)!}\theta^{2n-1} &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}2^{2j+1}}{(2j-1)!} P_{2j}\theta^{2j-1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} Q_{2k}\theta^{2k} \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \sum_{j=1}^{n} \frac{2^{2j+1}P_{2j}Q_{2n-2j}}{(2j-1)!(2n-2j)!}\theta^{2n-1}. \end{aligned}$$

Equating coefficients of θ^{2n-1}, $n \geq 1$, on both sides, we readily deduce (35.2).

In particular, if $n = 1$ in (35.2), we deduce that

$$\tfrac{1}{2}Q_2 = -4P_2.$$

If, as usual, $\sigma(k)$ denotes the sum of the positive divisors of k, we find, by equating coefficients of q^m, that the foregoing equality is equivalent to the arithmetical identities

$$24\sum_{k=1}^{n} (-1)^{n-k+1}(2k-1)\sigma\left(\frac{n(n-1)}{2} - \frac{k(k-1)}{2}\right) = (2n-1)^3,$$

if $m = n(n-1)/2$, $n \geq 1$, and

$$\sum_{\substack{k\geq 1 \\ k(k-1)/2 \leq m}} (-1)^k(2k-1)\sigma\left(m - \frac{k(k-1)}{2}\right) = 0,$$

otherwise. Here, $\sigma(0) = -\frac{1}{24}$. These last two identities were first established by Halphen [1] in 1877. Otherwise, Entry 35(i) and its general associated arithmetical identity appear to be new.

Entry 35(ii). *For each positive integer n, let*

$$P_n = \frac{(1-2^n)B_n}{2n} + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}k^{n-1}q^k}{1+q^k},$$

where B_n denotes the nth *Bernoulli number. For each nonnegative integer n, define*

$$Q_{2n} = \frac{\frac{1}{4}E_{2n} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-1)^{2n}q^{2k-1}}{1-q^{2k-1}}}{\frac{1}{4} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^{2k-1}}{1-q^{2k-1}}},$$

where E_{2n} denotes the $2n$th *Euler number. If n is any positive integer, then*

$$\tfrac{1}{2}Q_{2n} = \sum_{k=1}^{n} \binom{2n-1}{2k-1} 2^{2k} P_{2k} Q_{2n-2k}. \tag{35.6}$$

Proof. Write Entry 34(i) in the form

$$\begin{aligned} L &:= -\mathrm{Log}\left[\frac{\sec\theta + 4\sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^{2k-1}\cos(2k-1)\theta}{1-q^{2k-1}}}{\varphi^2(q)}\right] \\ &= -\mathrm{Log}(\sec\theta) + 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^k(1-\cos(2k\theta))}{k(1+q^k)} =: R. \end{aligned} \tag{35.7}$$

As in the previous proof, we expand both sides in powers of θ and equate coefficients.

Using (35.1), we find that, for $|\theta| < \pi/2$,

$$\begin{aligned} &\sec\theta + 4\sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^{2k-1}\cos(2k-1)\theta}{1-q^{2k-1}} \\ &\quad = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!}\theta^{2n} + 4\sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^{2k-1}}{1-q^{2k-1}} \sum_{n=0}^{\infty} \frac{(-1)^n(2k-1)^{2n}}{(2n)!}\theta^{2n} \\ &\quad = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{4}E_{2n} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-1)^{2n}q^{2k-1}}{1-q^{2k-1}}\right)\theta^{2n}. \end{aligned} \tag{35.8}$$

Putting $n = 0$ in Entry 34(i), we find that

$$\varphi^2(q) = 1 + 4\sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^{2k-1}}{1-q^{2k-1}}. \tag{35.9}$$

Combining (35.7)–(35.9), we conclude that

$$L = -\mathrm{Log}\left(\sum_{n=0}^{\infty} \frac{(-1)^n Q_{2n}}{(2n)!}\theta^{2n}\right). \tag{35.10}$$

On the other hand, using a well-known expansion for $\mathrm{Log}(\sec\theta)$, which, in fact, Ramanujan found in Chapter 5 (p. 52) (Part I [5, p. 64, Entry 17]), we find that

$$\begin{aligned} R &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}2^{2j}(1-2^{2j})B_{2j}}{(2j)(2j)!}\theta^{2j} \\ &\quad + \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j+1}}{(2j)!} \sum_{k=1}^{\infty} \frac{(-1)^k k^{2j-1} q^k}{1+q^k}\theta^{2j} \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}2^{2j+1}}{(2j)!}\left(\frac{(1-2^{2j})B_{2j}}{4j} + \sum_{k=1}^{\infty} \frac{(-1)^k k^{2j-1} q^k}{1+q^k}\right)\theta^{2j} \qquad (35.11) \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}2^{2j+1}P_{2j}}{(2j)!}\theta^{2j}. \end{aligned}$$

Hence, putting (35.10) and (35.11) in (35.7), we deduce that

$$\sum_{n=0}^{\infty} \frac{(-1)^n Q_{2n}}{(2n)!}\theta^{2n} = \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j+1} P_{2j}}{(2j)!}\theta^{2j}\right).$$

Differentiating both sides with respect to θ, we find that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^n Q_{2n}}{(2n-1)!}\theta^{2n-1} &= \sum_{j=1}^{\infty} \frac{(-1)^j 2^{2j+1} P_{2j}}{(2j-1)!}\theta^{2j-1} \sum_{k=0}^{\infty} \frac{(-1)^k Q_{2k}}{(2k)!}\theta^{2k} \\ &= \sum_{n=1}^{\infty} (-1)^n \sum_{j=1}^{n} \frac{2^{2j+1} P_{2j} Q_{2n-2j}}{(2j-1)!(2n-2j)!}\theta^{2n-1}. \end{aligned}$$

Equating coefficients of θ^{2n-1}, $n \geq 1$, on both sides, we easily deduce (35.6).

Ramanujan (p. 202) has an erroneous factor of $(-1)^{n-k}$ in the sum on the right side of (35.6).

In order to state some arithmetical deductions from (35.6), we need to define the divisor sums

$$\sigma_\nu^*(r) = \sum_{(2k-1)|r} (-1)^{k-1}(2k-1)^\nu$$

and

$$\tilde{\sigma}_\nu(r) = \sum_{k|r} (-1)^{k+r/k} k^\nu.$$

The case $n = 1$ of (35.6) gives the equality $Q_2 = 8P_2$ from which we may deduce the curious formula

$$\sigma_2^*(n) = 8 \sum_{k=0}^{n} \sigma_0^*(k)\tilde{\sigma}_1(n-k), \qquad (35.12)$$

where n is any nonnegative integer, $\sigma_0^*(0) = \frac{1}{4} = -\sigma_2^*(0)$, and $\tilde{\sigma}_1(0) = -\frac{1}{8}$. Not only is (35.6) new, but even the arithmetically equivalent special case (35.12) does not appear to have been given in the literature before we mentioned it in [6].

Define

$$L = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k},$$

$$M = 1 + 240 \sum_{k=1}^{\infty} \frac{k^3q^k}{1-q^k},$$

and

$$N = 1 - 504 \sum_{k=1}^{\infty} \frac{k^5q^k}{1-q^k}.$$

Ramanujan remarks that

$$S_n := \sum_{k=1}^{\infty} (-1)^{k+1}(2k-1)^{2n+1}q^{k(k-1)/2}, \qquad n \geq 1,$$

can always be expressed in terms of L, M, and N. Now by Entry 35(i), S_n can be written as a polynomial in $P_2, P_4, \ldots, P_{2n}$, $n \geq 1$. In an epic paper, Ramanujan [6], [10, pp. 136–162] proved that P_{2k}, $k \geq 1$, can be expressed as a polynomial in L, M, and N. (In [6], L, M, and N are denoted by P, Q, and R, respectively.) Hence, S_n can be represented as a polynomial in L, M, and N.

Examples. *Let Q_n, $n \geq 2$, be defined as in Entry 35(i). Let L, M, and N be defined as above. Then*

(i) $$3Q_2 = L,$$

(ii) $$5Q_4 = \frac{5L^2 - 2M}{3},$$

and

(iii) $$7Q_6 = \frac{35L^3 - 42ML + 16N}{9}.$$

PROOF. The three desired equalities follow by putting $n = 1, 2$, and 3, respectively, in Entry 35(i). The calculations are straightforward.

Note that Entries 36(i), (ii) below reduce to Entries 29(i), (ii), respectively, if $p = 1$.

Entry 36. *If $p = ab/cd$, then*

(i) $$S := \tfrac{1}{2}\{f(a, b)f(c, d) + f(-a, -b)f(-c, -d)\}$$
$$= \sum_{k=-\infty}^{\infty} (ad)^{k(k+1)/2}(bc)^{k(k-1)/2} f(acp^k, bd/p^k)$$

and

(ii)
$$D := \tfrac{1}{2}\{f(a, b)f(c, d) - f(-a, -b)f(-c, -d)\}$$
$$= \sum_{k=-\infty}^{\infty} a^{2k+1}(ad)^{k(k-1)/2}(bc)^{k(k+1)/2} f\left(\frac{c}{ap^k}, \frac{ap^k}{c} abcd\right).$$

PROOF. We prove just (i), since the proof of (ii) is similar. Putting $n - m = 2k$ and $n + m = 2j$, $-\infty < j, k < \infty$, we find that

$$S = \sum_{\substack{m,n=-\infty \\ m+n \text{ even}}}^{\infty} (cd)^{(m^2+n^2-m-n)/2} a^m c^n p^{m(m-1)/2}$$
$$= \sum_{j,k=-\infty}^{\infty} (cd)^{j^2+k^2-j} a^{j-k} c^{j+k} p^{(j^2+k^2-j+k)/2-jk}$$
$$= \sum_{k=-\infty}^{\infty} (ad)^{k(k-1)/2}(bc)^{k(k+1)/2} \sum_{j=-\infty}^{\infty} \left(\frac{ac}{p^k}\right)^{j(j+1)/2} (bdp^k)^{j(j-1)/2},$$

which, upon the replacement of k by $-k$, is seen to equal the desired result.

We now state and prove some very general and useful formulas originally due to H. Schröter in his dissertation [1]. These formulas will be employed in Sections 37 and 38. But even more importantly, Schröter's formulas will be utilized in proving many of Ramanujan's modular equations, especially in Chapter 20. Ramanujan evidently never stated these general formulas in his writings. However, from the many special cases that he clearly had proved, he at least possessed the ideas needed to prove the general formulas.

Put

$$a = Aq^{\mu+\nu}, \quad b = q^{\mu+\nu}/A, \quad c = Bq^{\mu-\nu}, \quad \text{and} \quad d = q^{\mu-\nu}/B,$$

where μ and ν are integers such that $\mu > \nu \geq 0$. Then

$$p = q^{4\nu}, \quad abcd = q^{4\mu}, \quad \text{and} \quad \frac{ad}{bc} = A^2/B^2.$$

Entry 36(i) now takes the form

$$S = \sum_{k=-\infty}^{\infty} \left(\frac{A}{B}\right)^k q^{2\mu k^2} f\left(ABq^{2\mu+4\nu k}, \frac{q^{2\mu-4\nu k}}{AB}\right).$$

Now let $k = \mu n + m$, $-\infty < n < \infty$, $0 \leq m \leq \mu - 1$. Thus,

$$S = \sum_{m=0}^{\mu-1} \sum_{n=-\infty}^{\infty} \left(\frac{A}{B}\right)^{\mu n+m} q^{2\mu(\mu n+m)^2} f\left(ABq^{2\mu+4\nu m+4\mu\nu n}, \frac{q^{2\mu-4\nu m-4\mu\nu n}}{AB}\right).$$

Next, apply Entry 18(iv) with

$$a = ABq^{2\mu+4\nu m}, \qquad b = q^{2\mu-4\nu m}/AB,$$

and n replaced by νn. Hence,

$$S = \sum_{m=0}^{\mu-1} \sum_{n=-\infty}^{\infty} \left(\frac{A}{B}\right)^{\mu n+m} q^{2\mu(\mu n+m)^2}(AB)^{-\nu n} q^{-2\mu\nu^2 n^2 - 4\nu^2 mn}$$
$$\times f\left(ABq^{2\mu+4\nu m}, \frac{q^{2\mu-4\nu m}}{AB}\right)$$
$$= \sum_{m=0}^{\mu-1} \left(\frac{A}{B}\right)^m q^{2\mu m^2} \sum_{n=-\infty}^{\infty} \frac{A^{n(\mu-\nu)}}{B^{n(\mu+\nu)}} q^{2\mu(\mu^2-\nu^2)n+4mn(\mu^2-\nu^2)}$$
$$\times f\left(ABq^{2\mu+4\nu m}, \frac{q^{2\mu-4\nu m}}{AB}\right).$$

In summary, we have shown that

$$S = \tfrac{1}{2}\{f(Aq^{\mu+\nu}, q^{\mu+\nu}/A)f(Bq^{\mu-\nu}, q^{\mu-\nu}/B)$$
$$+ f(-Aq^{\mu+\nu}, -q^{\mu+\nu}/A)f(-Bq^{\mu-\nu}, -q^{\mu-\nu}/B)\}$$
$$= \sum_{m=0}^{\mu-1} \left(\frac{A}{B}\right)^m q^{2\mu m^2} f\left(\frac{A^{\mu-\nu}}{B^{\mu+\nu}} q^{(2\mu+4m)(\mu^2-\nu^2)}, \frac{B^{\mu+\nu}}{A^{\mu-\nu}} q^{(2\mu-4m)(\mu^2-\nu^2)}\right)$$
$$\times f\left(ABq^{2\mu+4\nu m}, \frac{q^{2\mu-4\nu m}}{AB}\right). \tag{36.1}$$

We now examine Entry 36(ii) under the same substitutions as above. Thus, letting $k = \mu n + m$, $-\infty < n < \infty$, $0 \le m \le \mu - 1$, we find that

$$D = \sum_{k=-\infty}^{\infty} (Aq^{\mu+\nu})^{2k+1} \left(\frac{B}{A}\right)^k q^{2\mu k^2} f\left(\frac{B}{A} q^{-4\nu k-2\nu}, \frac{A}{B} q^{4\nu k+2\nu+4\mu}\right)$$
$$= A \sum_{m=0}^{\mu-1} \sum_{n=-\infty}^{\infty} (AB)^{\mu n+m} q^{(\mu+\nu)(2\mu n+2m+1)+2\mu(\mu n+m)^2}$$
$$\times f\left(\frac{A}{B} q^{4\nu(\mu n+m)+2\nu+4\mu}, \frac{B}{A} q^{-4\nu(\mu n+m)-2\nu}\right).$$

Apply Entry 18(iv) with

$$a = \frac{A}{B} q^{4\mu+2\nu+4\nu m}, \qquad b = \frac{B}{A} q^{-2\nu-4\nu m},$$

and with n replaced νn. Therefore,

$$D = A \sum_{m=0}^{\mu-1} \sum_{n=-\infty}^{\infty} (AB)^{\mu n+m} q^{(\mu+\nu)(2\mu n+2m+1)+2\mu(\mu n+m)^2} \left(\frac{A}{B}\right)^{-\nu n}$$
$$\times q^{-2\mu\nu^2 n^2-(2\mu+2\nu+4\nu m)\nu n} f\left(\frac{A}{B} q^{4\mu+2\nu+4\nu m}, \frac{B}{A} q^{-2\nu-4\nu m}\right)$$
$$= A \sum_{m=0}^{\mu-1} (AB)^m q^{(2m+1)(\mu+\nu)+2\mu m^2} \sum_{n=-\infty}^{\infty} A^{(\mu-\nu)n} B^{(\mu+\nu)n}$$
$$\times q^{2\mu(\mu^2-\nu^2)n^2+2(2m+1)(\mu^2-\nu^2)n} f\left(\frac{A}{B} q^{4\mu+2\nu+4\nu m}, \frac{B}{A} q^{-2\nu-4\nu m}\right).$$

In summary, we have shown that

$$\begin{aligned}D &= \tfrac{1}{2}\{f(Aq^{\mu+\nu}, q^{\mu+\nu}/A)f(Bq^{\mu-\nu}, q^{\mu-\nu}/B)\\&\quad - f(-Aq^{\mu+\nu}, -q^{\mu+\nu}/A)f(-Bq^{\mu-\nu}, -q^{\mu-\nu}/B)\}\\&= A\sum_{m=0}^{\mu-1}(AB)^m q^{(2m+1)(\mu+\nu)+2\mu m^2}\\&\quad\times f\left(A^{\mu-\nu}B^{\mu+\nu}q^{(2\mu+4m+2)(\mu^2-\nu^2)}, \frac{q^{(2\mu-4m-2)(\mu^2-\nu^2)}}{A^{\mu-\nu}B^{\mu+\nu}}\right)\\&\quad\times f\left(\frac{A}{B}q^{4\mu+2\nu+4\nu m}, \frac{B}{A}q^{-2\nu-4\nu m}\right).\end{aligned} \tag{36.2}$$

We now record a couple of special cases of (36.1). Letting $A = B = 1$ in (36.1), we find that

$$\begin{aligned}&\tfrac{1}{2}\{\varphi(q^{\mu+\nu})\varphi(q^{\mu-\nu}) + \varphi(-q^{\mu+\nu})\varphi(-q^{\mu-\nu})\}\\&\quad= \sum_{m=0}^{\mu-1} q^{2\mu m^2} f(q^{(2\mu+4m)(\mu^2-\nu^2)}, q^{(2\mu-4m)(\mu^2-\nu^2)})f(q^{2\mu+4\nu m}, q^{2\mu-4\nu m}).\end{aligned} \tag{36.3}$$

Next, putting $A = q^{\mu+\nu}$ and $B = q^{\mu-\nu}$ in (36.1) and using Entries 18(ii), (iii), we find that

$$\begin{aligned}&2\psi(q^{2\mu+2\nu})\psi(q^{2\mu-2\nu})\\&\quad= \sum_{m=0}^{\mu-1} q^{2\mu m^2+2\nu m} f(q^{(2\mu+4m)(\mu^2-\nu^2)}, q^{(2\mu-4m)(\mu^2-\nu^2)})f(q^{4\mu+4\nu m}, q^{-4\nu m}).\end{aligned} \tag{36.4}$$

Adding Entries 30(ii) and (iii) yields

$$f(a, b) = f(a^3b, ab^3) + af\left(\frac{b}{a}, \frac{a}{b}a^4b^4\right).$$

Putting $a = q^{2\nu m+\mu/2}$ and $b = q^{-2\nu m+\mu/2}$, we see that

$$\begin{aligned}&f(q^{2\nu m+\mu/2}, q^{-2\nu m+\mu/2})\\&\quad= f(q^{4\nu m+2\mu}, q^{-4\nu m+2\mu}) + q^{2\nu m+\mu/2}f(q^{-4\nu m}, q^{4\nu m+4\mu}).\end{aligned} \tag{36.5}$$

Multiplying (36.4) by $q^{\mu/2}$, adding the resulting equality to (36.3), and using (36.5), we deduce that

$$\begin{aligned}&\tfrac{1}{2}\{\varphi(q^{\mu+\nu})\varphi(q^{\mu-\nu}) + \varphi(-q^{\mu+\nu})\varphi(-q^{\mu-\nu})\} + 2q^{\mu/2}\psi(q^{2\mu+2\nu})\psi(q^{2\mu-2\nu})\\&\quad= \sum_{m=0}^{\mu-1} q^{2\mu m^2} f(q^{(2\mu+4m)(\mu^2-\nu^2)}, q^{(2\mu-4m)(\mu^2-\nu^2)})\\&\qquad\times \{f(q^{2\mu+4\nu m}, q^{2\mu-4\nu m}) + q^{2\nu m+\mu/2}f(q^{4\mu+4\nu m}, q^{-4\nu m})\}\\&\quad= \sum_{m=0}^{\mu-1} q^{2\mu m^2} f(q^{(2\mu+4m)(\mu^2-\nu^2)}, q^{(2\mu-4m)(\mu^2-\nu^2)})f(q^{2\nu m+\mu/2}, q^{-2\nu m+\mu/2}).\end{aligned} \tag{36.6}$$

Looking back at the proofs of (36.1) and (36.2), we observe that we can replace m by $m + j\mu$ for any integer j and not alter the summands on the right sides of (36.1) and (36.2). Note that (36.3) and (36.6) also remain unchanged if m is replaced my $-m$. Finally, observe that, with the use of Entry 18(iv), we may replace m by $-m$ on the right side of (36.4) as well. These observations are useful in simplifying these formulas somewhat.

To illustrate the remarks above, consider (36.4). Replacing q^2 by q, we deduce that

$$\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = \varphi(q^{\mu(\mu^2-\nu^2)})\psi(q^{2\mu}) + \sum_{m=1}^{(\mu-1)/2} q^{\mu m^2-\nu m} f(q^{(\mu+2m)(\mu^2-\nu^2)}, q^{(\mu-2m)(\mu^2-\nu^2)}) f(q^{2\nu m}, q^{2\mu-2\nu m}), \quad (36.7)$$

if μ is odd, and

$$\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = \varphi(q^{\mu(\mu^2-\nu^2)})\psi(q^{2\mu}) + \sum_{m=1}^{\mu/2-1} q^{\mu m^2-\nu m} f(q^{(\mu+2m)(\mu^2-\nu^2)}, q^{(\mu-2m)(\mu^2-\nu^2)}) f(q^{2\nu m}, q^{2\mu-2\nu m}) + q^{\mu^3/4-\mu\nu/2}\psi(q^{2\mu(\mu^2-\nu^2)}) f(q^{\mu\nu}, q^{2\mu-\mu\nu}), \quad (36.8)$$

if μ is even.

As a second illustration, set $A = q^{\mu+\nu}$ and $B = q^{-(\mu-\nu)}$ in the "difference" formula (36.2). Replacing q^2 by q, employing Entries 18(ii)–(iv), and using the remarks above, we deduce that

$$\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = q^{\mu^3/4-\mu/4}\psi(q^{2\mu(\mu^2-\nu^2)}) f(q^{\mu+\mu\nu}, q^{\mu-\mu\nu}) + \sum_{m=0}^{(\mu-3)/2} q^{\mu m(m+1)} f(q^{(\mu+2m+1)(\mu^2-\nu^2)}, q^{(\mu-2m-1)(\mu^2-\nu^2)}) \times f(q^{\mu+\nu+2\nu m)}, q^{\mu-\nu-2m}), \quad (36.9)$$

if μ is odd, while

$$\psi(q^{\mu+\nu})\psi(q^{\mu-\nu}) = \sum_{m=0}^{\mu/2-1} q^{\mu m(m+1)} f(q^{(\mu+2m+1)(\mu^2-\nu^2)}, q^{(\mu-2m-1)(\mu^2-\nu^2)}) f(q^{\mu+\nu+2\nu m}, q^{\mu-\nu-2m}), \quad (36.10)$$

if μ is even.

The next formula will be particularly useful in Chapter 20. Let μ be an even positive integer and suppose that ω is an odd positive integer such that $(\mu, \omega) = 1$ and $2\mu - \omega^2 > 0$. Then

$$\Omega := \tfrac{1}{2}\{f(q^{2\mu-\omega^2}A, q^{2\mu-\omega^2}/A) f(qB, q/B) + f(-q^{2\mu-\omega^2}A, -q^{2\mu-\omega^2}/A) f(-qB, -q/B)\} = \sum_{m=0}^{\mu-1} q^{4m^2} B^{-2m} f\left(\frac{A^\omega}{B^{2\mu-\omega^2}} q^{(2\mu-\omega^2)(2\mu+4m)}, \frac{B^{2\mu-\omega^2}}{A^\omega} q^{(2\mu-\omega^2)(2\mu-4m)}\right) \times f(AB^\omega q^{2\mu-4\omega m}, A^{-1}B^{-\omega} q^{2\mu+4\omega m}). \quad (36.11)$$

The restriction $(\mu, \omega) = 1$ is not strictly necessary. However, its removal would cause complicated modifications in the formula (36.11). For brevity, the expression $q^{2\mu-\omega^2}$ will be replaced by Q when convenient.

To prove (36.11), first let $a = qB$, $b = q/B$, and $n = \omega$ in Entry 31. Then apply Entry 18(iv) with $n = r$. Accordingly,

$$f(qB, q/B) = \sum_{r=0}^{\omega-1} q^{r^2}B^r f(q^{\omega^2+2r\omega}B^\omega, q^{\omega^2-2r\omega}B^{-\omega})$$
$$= \sum_{r=0}^{\omega-1} q^{(\omega-1)^2r^2}B^{-(\omega-1)r} f(q^{\omega^2-2\omega(\omega-1)r}B^\omega, q^{\omega^2+2\omega(\omega-1)r}B^{-\omega}).$$

Now substitute this sum and the corresponding sum for $f(-qB, -q/B)$ into the left side of (36.11). Recall that ω is odd so that $(\omega - 1)r$ is even. We then apply the "sum" formula (36.1). In this application, A and μ remain unchanged, but B is replaced by $B^\omega q^{-2\omega(\omega-1)r}$ and v is replaced by $\mu - \omega^2$. Now (36.1) was derived under the assumption that $0 \le v < \mu$. However, by a similar argument, we can conclude that (36.1) is valid for all integers v such that $|v| < \mu$. Thus, our application is valid provided that $2\mu - \omega^2 > 0$. Hence,

$$\Omega = \frac{1}{2}\sum_{r=0}^{\omega-1} q^{(\omega-1)^2r^2}B^{-(\omega-1)r}\{f(QA, Q/A)f(q^{\omega^2-2\omega(\omega-1)r}B^\omega, q^{\omega^2+2\omega(\omega-1)r}B^{-\omega})$$
$$+ f(-QA, -Q/A)f(-q^{\omega^2-2\omega(\omega-1)r}B^\omega, -q^{\omega^2+2\omega(\omega-1)r}B^{\omega})\}$$
$$= \sum_{r=0}^{\omega-1}\sum_{m=0}^{\mu-1} q^{(\omega-1)^2r^2}B^{-(\omega-1)r}q^{2\mu m^2}\left(\frac{Aq^{2\omega(\omega-1)r}}{B^\omega}\right)^m$$
$$\times f\left(\frac{A^{\omega^2}}{B^{\omega(2\mu-\omega^2)}}Q^{(2\mu+4m)\omega^2+2\omega(\omega-1)r}, \frac{B^{\omega(2\mu-\omega^2)}}{A}Q^{(2\mu-4m)\omega^2-2\omega(\omega-1)r}\right)$$
$$\times f\left(AB^\omega q^{2\mu+4m\mu-4\omega^2m-2\omega(\omega-1)r}, \frac{1}{AB^\omega}q^{2\mu-4m\mu+4\omega^2m+2\omega(\omega-1)r}\right)$$
$$= \sum_{r=0}^{\omega-1}\sum_{m=0}^{\mu-1} q^{4\{\omega m+(\omega-1)r/2\}^2}B^{-2\{\omega m+(\omega-1)r/2\}}$$
$$\times f\left(\frac{A^{\omega^2}}{B^{\omega(2\mu-\omega^2)}}Q^{2\mu\omega^2+4\omega\{\omega m+(\omega-1)r/2\}}, \frac{B^{\omega(2\mu-\omega^2)}}{A^{\omega^2}}Q^{2\mu\omega^2-4\omega\{\omega m+(\omega-1)r/2\}}\right)$$
$$\times f\left(AB^\omega q^{2\mu-4\omega\{\omega m+(\omega-1)r/2\}}, \frac{1}{AB^\omega}q^{2\mu+4\omega\{\omega m+(\omega-1)r/2\}}\right),$$

where in this last step we applied Entry 18(iv) with $n = m$.

In each expression on the right side above, m and r occur only in the expression $\omega m + \frac{1}{2}(\omega - 1)r$. For each pair (m, r), apply the division algorithm to deduce that

$$\omega m + \tfrac{1}{2}(\omega - 1)r = q\mu + n, \qquad 0 \le n \le \mu - 1,$$

for a certain nonnegative integer q. Suppose r is fixed and m varies from 0 to

$\mu - 1$. We claim that n then assumes all values from 0 to $\mu - 1$ in some order. To that end, let n and n' correspond to the distinct values m and m'. Then

$$n - n' \equiv \omega(m - m')(\bmod \mu).$$

But $|m - m'| < \mu$ and $(\omega, \mu) = 1$. Thus, $\mu \nmid (n - n')$; that is, $n \neq n'$, and our claim is established.

On the other hand, suppose we consider two different values of r, say r and r', and select values of m which yield the same n. This is possible by the conclusion of the preceding paragraph. Let q and q' be the corresponding quotients. Then

$$\tfrac{1}{2}(\omega - 1)(r - r') \equiv \mu(q - q')(\bmod \omega).$$

Now $\frac{1}{2}(\omega - 1)$ and ω cannot have a factor in common, and $0 < |r - r'| < \omega$. Thus, $\omega \nmid (q - q')$. In other words, q and q' cannot differ by a multiple of ω. Hence, as r assumes the values from 0 to $\omega - 1$, the ω values of q differ from multiples of ω by the numbers $0, 1, 2, \ldots, \omega - 1$ in some order.

Now each term of the last double sum above remains unaltered if a multiple of $\mu\omega$, say $l\mu\omega$, is added to the argument $\omega m + \frac{1}{2}(\omega - 1)r$. To see this, apply Entry 18(iv) with $n = l$ and $n = \omega^2 l$, respectively, to the pair of theta-functions in each summand. Consequently, when we effect the substitution $\omega m + \frac{1}{2}(\omega - 1)r = n + q\mu$, n runs from 0 to $\mu - 1$, and it can be assumed that q runs from 0 to $\omega - 1$. Thus, replacing the index q by s, to avoid a conflict in notation, we have proved that

$$\begin{aligned}
\Omega = &\sum_{s=0}^{\omega-1} \sum_{n=0}^{\mu-1} q^{4(n+s\mu)^2} B^{-2(n+s\mu)} \\
&\times f\left(\frac{A^{\omega^2}}{B^{\omega(2\mu-\omega^2)}} Q^{2\mu\omega^2+4\omega(n+s\mu)}, \frac{B^{\omega(2\mu-\omega^2)}}{A^{\omega^2}} Q^{2\mu\omega^2-4\omega(n+s\mu)}\right) \\
&\times f\left(AB^{\omega} q^{2\mu-4\omega(n+s\mu)}, \frac{1}{AB^{\omega}} q^{2\mu+4\omega(n+s\mu)}\right) \\
= &\sum_{n=0}^{\mu-1} q^{4n^2} B^{-2n} f\left(AB^{\omega} q^{2\mu-4\omega n}, \frac{1}{AB^{\omega}} q^{2\mu+4\omega n}\right) \\
&\times \sum_{s=0}^{\omega-1} \frac{A^{\omega s}}{B^{(2\mu-\omega^2)s}} q^{(2\mu-\omega^2)(2\mu s+4n)s} \\
&\times f\left(\frac{A^{\omega^2}}{B^{\omega(2\mu-\omega^2)}} Q^{2\mu\omega^2+4\omega(n+s\mu)}, \frac{B^{\omega(2\mu-\omega^2)}}{A^{\omega^2}} Q^{2\mu\omega^2-4\omega(n+s\mu)}\right),
\end{aligned}$$

where we have applied Entry 18(iv) with $n = \omega s$. Lastly, apply Entry 31 with $n = \omega$,

$$a = \frac{A^{\omega}}{B^{2\mu-\omega^2}} Q^{2\mu+4m},$$

and

$$b = \frac{B^{2\mu-\omega^2}}{A^\omega} Q^{2\mu-4m}.$$

We find that the inner sum above reduces to

$$f\left(\frac{A^\omega}{B^{2\mu-\omega^2}} Q^{2\mu+4m}, \frac{B^{2\mu-\omega^2}}{A^\omega} Q^{2\mu-4m}\right).$$

This then completes the proof of (36.11).

We now record some special instances of (36.11) that will be useful in establishing certain modular equations in Chapter 20. In each case, of course, μ is even, ω is odd, and $(\mu, \omega) = 1$.

First, if $A = B = 1$, then

$$\tfrac{1}{2}\{\varphi(q^{2\mu-\omega^2})\varphi(q) + \varphi(-q^{2\mu-\omega^2})\varphi(-q)\}$$
$$= \sum_{m=0}^{\mu-1} q^{4m^2} f(q^{(2\mu-\omega^2)(2\mu+4m)}, q^{(2\mu-\omega^2)(2\mu-4m)}) f(q^{2\mu-4\omega m}, q^{2\mu+4\omega m}). \quad (36.12)$$

Second, let $A = q^{2\mu-\omega^2}$ and $B = q^\omega$. By Entry 18(iv) with $n = (\omega - 1)/2$,

$$f(q^{\omega+1}, q^{-\omega+1}) = q^{-(\omega^2-1)/4} f(q^2, 1) = 2q^{-(\omega^2-1)/4}\psi(q^2).$$

Hence,

$$2q^{\mu/2-(\omega^2-1)/4}\psi(q^{4\mu-2\omega^2})\psi(q^2)$$
$$= \sum_{m=0}^{\mu-1} q^{4m^2+\mu/2-2\omega m} f(q^{(2\mu-\omega^2)(2\mu+4m)}, q^{(2\mu-\omega^2)(2\mu-4m)}) f(q^{4\mu-4\omega m}, q^{4\omega m}).$$
$$(36.13)$$

Adding (36.12) and (36.13), we find that

$$\tfrac{1}{2}\{\varphi(q^{2\mu-\omega^2})\varphi(q) + \varphi(-q^{2\mu-\omega^2})\varphi(-q)\} + 2q^{\mu/2-(\omega^2-1)/4}\psi(q^{4\mu-2\omega^2})\psi(q^2)$$
$$= \sum_{m=0}^{\mu-1} q^{4m^2} f(q^{(2\mu-\omega^2)(2\mu+4m)}, q^{(2\mu-\omega^2)(2\mu-4m)}) \{ f(q^{2\mu-4\omega m}, q^{2\mu+4\omega m})$$
$$+ q^{\mu/2-2\omega m} f(q^{4\mu-4\omega m}, q^{4\omega m})\}$$
$$= \sum_{m=0}^{\mu-1} q^{4m^2} f(q^{(2\mu-\omega^2)(2\mu+4m)}, q^{(2\mu-\omega^2)(2\mu-4m)}) f(q^{\mu/2-2\omega m}, q^{\mu/2+2\omega m}),$$
$$(36.14)$$

where we have utilized Entry 31 with $a = q^{\mu/2-2\omega m}$, $b = q^{\mu/2+2\omega m}$, and $n = 2$.

An indicated above, (36.1) and (36.2) are due to Schröter [1] who did not publish his proofs outside his thesis. However, he did write three short papers [2], [3], [4] in which he took special cases of his general formulas to establish certain modular equations. Proofs of Schröter's formulas may be found in the books of Tannery and Molk [1, pp. 163–167] and Enneper [1, pp. 470ff]. A more recent proof has been given by T. Kondo and T. Tasaka [1].

An elegant generalization of Schröter's work has been discovered by R. Blecksmith, J. Brillhart, and I. Gerst [2, Theorem 2]. We translate their formula into Ramanujan's notation. It will be convenient, however, to put

$$f_0(a, b) = f(a, b) \quad \text{and} \quad f_1(a, b) = f(-a, -b).$$

Theorem. *Let a, b, c, and d denote complex numbers such that $|ab|$, $|cd| < 1$. Suppose that there exist positive integers α, β, and m such that*

$$(ab)^{\beta} = (cd)^{\alpha(m-\alpha\beta)}.$$

Let $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$. Then

$$\begin{aligned} f_{\varepsilon_1}(a, b) f_{\varepsilon_2}(c, d) &= \sum_{r\in R} (-1)^{\varepsilon_2 r} c^{r(r+1)/2} d^{r(r-1)/2} f_{\delta_1}\left(\frac{a(cd)^{\alpha(\alpha+1-2r)/2}}{c^{\alpha}}, \frac{b(cd)^{\alpha(\alpha+1+2r)/2}}{d^{\alpha}}\right) \\ &\times f_{\delta_2}\left(\frac{(a/b)^{\beta/2}(cd)^{(m-\alpha\beta)(m+1+2r)/2}}{d^{m-\alpha\beta}}, \frac{(b/a)^{\beta/2}(cd)^{(m-\alpha\beta)(m+1-2r)/2}}{c^{m-\alpha\beta}}\right), \end{aligned} \tag{36.15}$$

where R is a complete residue system (mod m),

$$\delta_1 = \begin{cases} 0, & \text{if } \varepsilon_1 + \alpha\varepsilon_2 \text{ is even,} \\ 1, & \text{if } \varepsilon_1 + \alpha\varepsilon_2 \text{ is odd,} \end{cases}$$

and

$$\delta_2 = \begin{cases} 0, & \text{if } \varepsilon_1\beta + \varepsilon_2(m - \alpha\beta) \text{ is even,} \\ 1, & \text{if } \varepsilon_1\beta + \varepsilon_2(m - \alpha\beta) \text{ is odd.} \end{cases}$$

Letting $\alpha = \beta = 1$ and $m = 2$ in (36.15), we obtain an equality that is also achieved by adding Entries 29(i) and (ii).

Blecksmith, Brillhart, and Gerst [1], [2] have employed theta-functions in proving theorems on the parity of partition functions while also obtaining some elegant new identities for theta-functions. Using Ramanujan's theory of theta-functions, Bhargava, Adiga, and Somashekara [3] and Blecksmith, Brillhart, and Gerst [3] have given alternate proofs of these theta-function identities.

Entry 37. *We have*

(i) $$\tfrac{1}{2}\{\varphi(a)\varphi(b) + \varphi(-a)\varphi(-b)\} = \varphi(ab) + 2\sum_{k=1}^{\infty} (ab)^{k^2} f\left(ab\frac{a^{2k}}{b^{2k}}, ab\frac{b^{2k}}{a^{2k}}\right),$$

(ii) $$\tfrac{1}{2}\{\varphi(a)\varphi(b) - \varphi(-a)\varphi(-b)\} = 2\sum_{k=1}^{\infty} a^{k^2} b^{(k-1)^2} f\left(a^2b^2\frac{a^{2k-1}}{b^{2k-1}}, \frac{b^{2k-1}}{a^{2k-1}}\right),$$

and

(iii) $$\psi(a)\psi(b) = \psi(ab) + \sum_{k=1}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2} f\left(ab\frac{a^k}{b^k}, \frac{b^k}{a^k}\right).$$

PROOF OF (i). In Entry 36(i), put $b = a$ and $d = c$, and then replace c by b.

PROOF OF (ii). In Entry 36(ii), put $b = a$ and $d = c$, and then replace c by b. Replacing k by $k - 1$, we find that

$$\begin{aligned}&\tfrac{1}{2}\{\varphi(a)\varphi(b) - \varphi(-a)\varphi(-b)\}\\&= \left(\sum_{k=-\infty}^{0} + \sum_{k=1}^{\infty}\right) a^{k^2} b^{(k-1)^2} f\left(\frac{b^{2k-1}}{a^{2k-1}}, a^2 b^2 \frac{a^{2k-1}}{b^{2k-1}}\right).\end{aligned} \tag{37.1}$$

In the first sum on the right side, replace k by $1 - k$ and apply Entry 18(iv) with $n = 1$. Hence,

$$\begin{aligned}&\sum_{k=-\infty}^{0} a^{k^2} b^{(k-1)^2} f\left(\frac{b^{2k-1}}{a^{2k-1}}, a^2 b^2 \frac{a^{2k-1}}{b^{2k-1}}\right)\\&= \sum_{k=1}^{\infty} a^{(1-k)^2} b^{k^2} f\left(\frac{a^{2k-1}}{b^{2k-1}}, a^2 b^2 \frac{b^{2k-1}}{a^{2k-1}}\right) \qquad (37.2)\\&= \sum_{k=1}^{\infty} a^{(1-k)^2} b^{k^2} \frac{a^{2k-1}}{b^{2k-1}} f\left(a^2 b^2 \frac{a^{2k-1}}{b^{2k-1}}, \frac{b^{2k-1}}{a^{2k-1}}\right).\end{aligned}$$

Simplifying the right side of (37.2) and then substituting it into (37.1), we complete the proof.

PROOF OF (iii). In Entry 36(i), let $a = c = 1$ and then replace d by a. Using Entries 18(ii), 18(iii), and 22(ii), we deduce that

$$\begin{aligned}2\psi(a)\psi(b) &= \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2} f\left(\frac{b^k}{a^k}, ab\frac{a^k}{b^k}\right)\\&= 2\psi(ab) + \left(\sum_{k=-\infty}^{-1} + \sum_{k=1}^{\infty}\right) a^{k(k+1)2} b^{k(k-1)/2} f\left(\frac{b^k}{a^k}, ab\frac{a^k}{b^k}\right).\end{aligned}$$

Replacing k by $-k$ in the former sum on the right side above and employing Entry 18(iv) with $n = 1$, we readily complete the proof in the same fashion as in the proof of Entry 37(ii).

Corollary. *We have*

(i) $\psi(q^3)\psi(q^{13}) - \psi(-q^3)\psi(-q^{13}) = q^3\{\psi(q)\psi(q^{39}) + \psi(-q)\psi(-q^{39})\}$,

(ii) $\psi(q^5)\psi(q^{11}) - \psi(-q^5)\psi(-q^{11}) = q^5\{\psi(q)\psi(q^{55}) + \psi(-q)\psi(-q^{55})\}$,

and

(iii) $\psi(q^7)\psi(q^9) - \psi(-q^7)\psi(-q^9) = q^6\{\psi(q)\psi(q^{63}) - \psi(-q)\psi(-q^{63})\}$.

PROOF OF (i). In (36.8), set $\mu = 8$ and $\nu = 5$ to obtain the equality

$$\begin{aligned}\psi(q^{13})\psi(q^3) &= \varphi(q^{312})\psi(q^{16}) + q^{108}\psi(q^{624}) f(q^{40}, q^{-24})\\&\quad + \sum_{m=1}^{3} q^{8m^2-5m} f(q^{312+78m}, q^{312-78m}) f(q^{10m}, q^{16-10m}).\end{aligned}$$

Replace q by $-q$ and subtract the two formulas to get

$$\psi(q^3)\psi(q^{13}) - \psi(-q^3)\psi(-q^{13})$$
$$= 2q^3 f(q^{390}, q^{234}) f(q^{10}, q^6) + 2q^{57} f(q^{546}, q^{78}) f(q^{30}, q^{-14}). \qquad (37.3)$$

Now apply Entry 18(iv) with $a = q^{14}$, $b = q^2$, and $n = 1$ to find that

$$f(q^{14}, q^2) = q^{14} f(q^{30}, q^{-14}).$$

Employing this in (37.3) yields

$$\psi(q^3)\psi(q^{13}) - \psi(-q^3)\psi(-q^{13})$$
$$= 2q^3 f(q^{234}, q^{390}) f(q^6, q^{10}) + 2q^{43} f(q^{78}, q^{546}) f(q^2, q^{14}). \quad (37.4)$$

Now from Corollary (ii) in Section 31, we see that

$$\psi(\pm q) = f(q^6, q^{10}) \pm q f(q^2, q^{14})$$

and

$$\psi(\pm q^{39}) = f(q^{234}, q^{390}) \pm q^{39} f(q^{78}, q^{546}).$$

Using these equalities in (37.4), we conclude that

$$\psi(q^3)\psi(q^{13}) - \psi(-q^3)\psi(-q^{13})$$
$$= \tfrac{1}{2}q^3(\{\psi(q) + \psi(-q)\}\{\psi(q^{39}) + \psi(-q^{39})\}$$
$$+ \{\psi(q) - \psi(-q)\}\{\psi(q^{39}) - \psi(-q^{39})\}),$$

from which the desired result readily follows.

PROOF OF (ii). Letting $\mu = 8$ and $\nu = 3$ in (36.8), we find that

$$\psi(q^{11})\psi(q^5) = \varphi(q^{440})\psi(q^{16}) + q^{116}\psi(q^{880}) f(q^{24}, q^{-8})$$
$$+ \sum_{m=1}^{3} q^{8m^2-3m} f(q^{440+110m}, q^{440-110m}) f(q^{6m}, q^{16-6m}).$$

Changing the sign of q and subtracting the two equations, we find that

$$\psi(q^5)\psi(q^{11}) - \psi(-q^5)\psi(-q^{11})$$
$$= 2q^5 f(q^{550}, q^{330}) f(q^6, q^{10}) + 2q^{63} f(q^{770}, q^{110}) f(q^{18}, q^{-2})$$
$$= 2q^5 f(q^{330}, q^{550}) f(q^6, q^{10}) + 2q^{61} f(q^{110}, q^{770}) f(q^2, q^{14}), \qquad (37.5)$$

where we have applied Entry 18(iv) with $a = q^2$, $b = q^{14}$, and $n = 1$. By Corollary (ii) in Section 31,

$$\psi(\pm q) = f(q^6, q^{10}) \pm q f(q^2, q^{14})$$

and

$$\psi(\pm q^{55}) = f(q^{330}, q^{550}) \pm q^{55} f(q^{110}, q^{770}).$$

Using these equalities in (37.5), we arrive at the desired result.

PROOF OF (iii). In (36.8), let $\mu = 8$ and $\nu = 1$. Replacing q by $-q$ and subtracting the two equalities, we find that

$$\psi(q^7)\psi(q^9) - \psi(-q^7)\psi(-q^9) = 2q^7 f(q^{378}, q^{630}) f(q^2, q^{14}) + 2q^{69} f(q^{126}, q^{882}) f(q^6, q^{10}). \tag{37.6}$$

By Corollary (ii) in Section 31,

$$\psi(\pm q) = f(q^6, q^{10}) \pm q f(q^2, q^{14})$$

and

$$\psi(\pm q^{63}) = f(q^{378}, q^{630}) \pm q^{63} f(q^{126}, q^{882}).$$

Using these equalities in (37.6), we readily complete the proof.

Example. *We have*

$$\psi(q)\psi(q^5) - \psi(-q)\psi(-q^5) = 2q \frac{\varphi(-q^6)\varphi(-q^{120})}{\chi(-q^2)\chi(-q^{40})} + 4q^{15}\psi(q^6)\psi(q^{120}).$$

PROOF. In (36.7), let $\mu = 3$ and $\nu = 2$ to find that

$$\psi(q^4)\psi(q) = \varphi(q^{15})\psi(q^6) + q f(q^{25}, q^5) f(q^4, q^2).$$

Replacing q by $-q$ and subtracting the two formulas, we deduce that

$$\psi(q)\psi(q^5) - \psi(-q)\psi(-q^5) = \psi(q^6)\{\varphi(q^{15}) - \varphi(-q^{15})\} + q f(q^2, q^4)\{f(q^5, q^{25}) + f(-q^5, -q^{25})\}.$$

Applying Entries 25(ii), 19, and 30(ii), we obtain

$$\begin{aligned}\psi(q)\psi(q^5) - \psi(-q)\psi(-q^5) &= 4q^{15}\psi(q^6)\psi(q^{120}) + 2q(-q^2; q^6)_\infty(-q^4; q^6)_\infty(q^6; q^6)_\infty f(q^{40}, q^{80}) \\ &= 4q^{15}\psi(q^6)\psi(q^{120}) + 2q(-q^2; q^2)_\infty \frac{(q^6; q^6)_\infty}{(-q^6; q^6)_\infty}(-q^{40}; q^{40})_\infty \\ &\quad \times \frac{(q^{120}; q^{120})_\infty}{(-q^{120}; q^{120})_\infty}.\end{aligned} \tag{37.7}$$

By (22.4),

$$\varphi(-q) = \frac{(q; q)_\infty}{(-q; q)_\infty},$$

and by Entry 22(iv) and (22.3),

$$\chi(-q) = (q; q^2)_\infty = \frac{1}{(-q; q)_\infty}.$$

Using each of these equalities twice in (37.7), we achieve the desired result.

Entries 38(i), (ii). *For* $|q| < 1$,

$$\frac{f(-q^5)}{f(-q, -q^4)} = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k}$$

and

$$\frac{f(-q^5)}{f(-q^2, -q^3)} = \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q)_k}.$$

Entries 38(i) and (ii) constitute the famous Rogers–Ramanujan identities. For the early history of these fascinating identities, consult Ramanujan's Collected Papers [10, pp. 344–346], Hardy's book [3, pp. 90–99], or Andrews' book [9, Chap. 7]. Briefly, for several years after Ramanujan initially discovered these identities, he was unable to supply proofs. In fact, he [3] submitted the identities to the problem section of the *Journal of the Indian Mathematical Society.* In 1916, while browsing through past issues of the *Proceedings of the London Mathematical Society*, Ramanujan discovered a proof of the identities in a paper by Rogers [1]. Stimulated by the renewed interest in his work, Rogers [4] published another proof. Ramanujan soon found his own proof, and Rogers devised still another proof. Hardy thereupon arranged for these two proofs to be published together (Ramanujan [8], [10, pp. 214–215]; Rogers [5]). At about the same time that Ramanujan unearthed Rogers' work, I. J. Schur also independently discovered Entries 38(i), (ii) and published two proofs [1], [3, Vol. 2, pp. 117–136] as well as proofs of similar results [2], [3, Vol. 3, pp. 43–50]. Watson's paper [2] gave still another proof of the Rogers–Ramanujan identities.

We would now like to point out that although Ramanujan discovered the Rogers–Ramanujan identities in India and it took several years before he found a proof, these identities are limiting cases of Entry 7. In fact, the proof is not much different from Watson's proof. We are grateful to R. A. Askey for informing us that the Rogers–Ramanujan identities are deducible from Entry 7.

If we let c, e, and f tend to ∞ in (7.3), we find that

$$\sum_{k=0}^{\infty} \frac{(-1)^k (a)_k (1 - aq^{2k}) a^{2k} q^{5k^2/2 - k/2}}{(q)_k (1 - a)} = (aq)_\infty \sum_{k=0}^{\infty} \frac{a^k q^{k^2}}{(q)_k}.$$

Letting $a = 1$ yields

$$\begin{aligned}(q)_\infty \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k} &= 1 + \sum_{k=1}^{\infty} (-1)^k (1 + q^k) q^{5k^2/2 - k/2} \\ &= f(-q^2, -q^3).\end{aligned}$$

If we now apply the Jacobi triple product identity (Entry 19) to $f(-q^2, -q^3)$, we obtain the first Rogers–Ramanujan identity, Entry 38(i). Similarly, letting

$a = q$ above, we deduce that

$$(q)_\infty \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(q)_k} = \sum_{k=0}^{\infty} (-1)^k(1 - q^{2k+1})q^{5k^2/2+3k/2}$$
$$= f(-q, -q^4).$$

To obtain this last equality, apply the distributive law in the penultimate line and replace k by $k - 1$ in the second sum. Applying Entry 19 to $f(-q, -q^4)$, we obtain the second Rogers–Ramanujan identity, Entry 38(ii).

Today, there exist many proofs of the Rogers–Ramanujan identities as well as considerable generalizations. It would be impossible here to list all of these proofs and generalizations, and so we describe only a selected sampling of papers and conclude with a description of sources where more complete bibliographies may be found.

After the work of Rogers and Ramanujan, some of the most important earlier papers were written by Slater [1], Alder [1], and Gordon [1]. Other generalizations have been found by Andrews [5], [15], Bressoud [1], Denis [4], Milne [5], Paule [1], Verma [1], and Verma and Jain [1], [2]. Using primarily the q-binomial theorem, Bressoud [2] has developed an especially elegant and simple proof of the Rogers–Ramanujan identities, which has been reproduced by J. and P. Borwein in their book [2]. Emphasizing operators and explicit solutions of functional equations, Ehrenpreis [1] has developed a new approach to Rogers–Ramanujan identities. His paper also contains a discussion of several other proofs. For an enlightening exposition of Rogers' first proof of the Rogers–Ramanujan identities and their connections with certain q-orthogonal polynomials, see Askey's paper [7]. Andrews [14] has also discussed the importance of Rogers' work and has reproduced two fascinating letters of Rogers. To see how computer algebra can be an aid in proving the Rogers–Ramanujan identities, see Andrews' paper [20]. Proofs of the Rogers–Ramanujan identities employ the Jacobi triple product identity at some stage, except for one proof found by Andrews [19]. In the 1980s, R. J. Baxter discovered the Rogers–Ramanujan identities and several beautiful analogues in his work on the hard hexagon model. For descriptions of this work, see Baxter's book [1] as well as papers by Baxter [2], Andrews [11], and Andrews, Baxter, and Forrester [1]. An interesting proof of the Rogers–Ramanujan identities motivated by their discovery and occurrence in the solution of the hard hexagon model has been given by Andrews and Baxter [1]. A. K. Agarwal and Andrews [1] have proved some Rogers–Ramanujan identities for certain partitions with "*n* copies of *n*," which also have applications to the hard hexagon model. For many years a purely combinatorial proof of the Rogers–Ramanujan identities was sought. Finally, in 1981, a bijective proof of the identities was devised by Garsia and Milne [1]. Relying heavily on the work of Garsia and Milne, Bressoud and Zeilberger [1] found a simpler bijective proof. Lepowsky and Wilson [1] have proved the Rogers–Ramanujan identities within the setting of Euclidean Lie algebras.

We have listed only a minority of the proofs and generalizations of the Rogers–Ramanujan identities found by Andrews. His survery papers [4]–[6], [8], [21], book [9], and monograph [18] provide references for many original papers on this subject, including his own important contributions. Particularly recommended is Andrews' paper [21], which provides a classification and discussion of almost all of the known proofs of the Rogers–Ramanujan identities. Askey's paper [7] also offers many references. Another survey paper has been written by Verma [2].

Entry 38(iii). *For* $|q| < 1$,

$$\frac{f(-q, -q^4)}{f(-q^2, -q^3)} = \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\cdots}.$$

PROOF. Set $a = 1$ in the corollary to Entry 15. Using Entries 38(i), (ii), we complete the proof.

Entry 38(iii) is another famous theorem of Ramanujan and is generally known as "Ramanujan's continued fraction" or as the "Rogers–Ramanujan continued fraction." As pointed out in Section 15, the first proof of Entry 38(iii) was given by Rogers [1]. Ramanujan's proof is found in his paper [8], [10, pp. 214–215]. Shortly thereafter, Rogers [6] gave another proof. Although the continued fraction was mentioned in Ramanujan's [10, p. xxviii] first letter to Hardy, the equality of Entry 38(iii) was not. Ramanujan eventually found several generalizations and ramifications of his continued fraction which he recorded in his "lost notebook" [11], in the unorganized pages of his second notebook, and in his third notebook. For an account of some of these developments, see two papers by Andrews [10], [13], several papers by Ramanathan [1], [2], [4]–[6], [9], and a paper by Andrews, Berndt, Jacobsen, and Lamphere [1]. After the work of Rogers and Ramanujan, no significant generalizations were found until Selberg [1], [2, pp. 1–23] published his first paper in 1936. In addition to papers cited in Sections 15 and 16 and in our discussion of the Rogers–Ramanujan identities above, further generalizations and related work may be found in papers by Carlitz [1], Carlitz and Scoville [1], Gordon [2], Hirschhorn [3], [6], Al-Salam and Ismail [1], Bhargava and Adiga [1], [2], Bhargava, Adiga, and Somashekara [1], [2], Bhargava [1], Churchhouse [1], Denis [1]–[3], Verma, Denis, and Rao [1], Singh [1], and Hovstad [1].

The Rogers–Ramanujan continued fraction has combinatorial interpretations, a fact first recognized by G. Szekeres [1]. We mention one such combinatorial interpretation, discussed by A. M. Odlyzko and H. S. Wilf [1].

An (n, k) fountain is an arrangement of n coins in rows such that there are exactly k coins in the bottom row, and such that each coin in a higher row touches exactly two coins in the next lower row. Let $f(n, k)$ denote the number of (n, k) fountains, and put $f(n) = \sum_{k=1}^{n} f(n, k)$. Thus, $f(1) = 1$,

$f(2) = 1$, $f(3) = 2$, $f(4) = 3$, $f(5) = 5$, $f(6) = 9$, $f(7) = 15$, and so on. Then $\{f(n)\}$ has the generating function

$$1 + \sum_{n=1}^{\infty} f(n)x^n = \frac{1}{1} - \frac{x}{1} - \frac{x^2}{1} - \frac{x^3}{1} - \cdots.$$

Similar interpretations have been examined by Glasser, Privman, and Švrakić [1] and Privman and Švrakić [1].

For further combinatorial interpretations of the Rogers–Ramanujan continued fraction, see papers by Flajolet [1] and Andrews [13].

Entry 38(iv). *For* $|q| < 1$,

$$f^2(-q^2, -q^3) - q^{2/5}f^2(-q, -q^4) = f(-q)\{f(-q^{1/5}) + q^{1/5}f(-q^5)\}. \tag{38.1}$$

We establish the following formulation of the quintuple product identity from which we deduce Entry 38(iv).

Theorem (Quintuple Product Identity). *For* $|q| < 1$,

$$f(B^3q, q^5/B^3) - B^2f(q/B^3, B^3q^5) = f(-q^2)\frac{f(-B^2, -q^2/B^2)}{f(Bq, q/B)}. \tag{38.2}$$

PROOF. In (36.1), set $\mu = 3$ and $\nu = 1$ and replace q^2 by q. Recalling also the remarks made after (36.6), we find that

$$\begin{aligned}&\tfrac{1}{2}\{f(Aq^2, q^2/A)f(Bq, q/B) + f(-Aq^2, -q^2/A)f(-Bq, -q/B)\}\\ &= \sum_{m=-1}^{1}\left(\frac{A}{B}\right)^m q^{3m^2} f\left(\frac{A^2}{B^4}q^{24+16m}, \frac{B^4}{A^2}q^{24-16m}\right) f\left(ABq^{3+2m}, \frac{q^{3-2m}}{AB}\right).\end{aligned} \tag{38.3}$$

Make the same substitutions in (36.2) but also replace A by $1/A$. Accordingly,

$$\begin{aligned}&\tfrac{1}{2}\{f(q^2/A, Aq^2)f(Bq, q/B) - f(-q^2/A, -Aq^2)f(-Bq, -q/B)\}\\ &= A^{-1}\sum_{m=-1}^{1}\left(\frac{B}{A}\right)^m q^{3m^2+4m+2} f\left(\frac{B^4}{A^2}q^{32+16m}, \frac{A^2}{B^4}q^{16-16m}\right)\\ &\quad\times f\left(\frac{q^{7+2m}}{AB}, ABq^{-1-2m}\right).\end{aligned} \tag{38.4}$$

Note that

$$f\left(\frac{A^2}{B^4}q^{24+16m}, \frac{B^4}{A^2}q^{24-16m}\right) = f\left(\frac{B^4}{A^2}q^{32+16m}, \frac{A^2}{B^4}q^{16-16m}\right),$$

when $A = B^2q^2$. Giving A this value and subtracting (38.4) from (38.3), we deduce that

$$\begin{aligned}
&f(-B^2q^4, -1/B^2)f(-Bq, -q/B)\\
&\quad= f(q^{28}, q^{20})\{f(B^3q^5, q/B^3) - (1/B^2)f(q^5/B^3, B^3q)\}\\
&\qquad+ f(q^{44}, q^4)q^5\{Bf(B^3q^7, 1/B^3q) - (1/B^3)f(q^7/B^3, B^3/q)\}.
\end{aligned} \tag{38.5}$$

We now apply Entry 18(iv) twice. First, letting $a = 1/qB^3$, $b = B^3q^7$, and $n = 1$, we deduce that

$$f(B^3q^7, 1/B^3q) = \frac{1}{B^3q}f(q^5/B^3, B^3q).$$

Second, putting $a = B^3/q$, $b = q^7/B^3$, and $n = 1$ yields

$$f(q^7/B^3, B^3/q) = \frac{B^3}{q}f(B^3q^5, q/B^3).$$

Using the two foregoing equalities in (38.5), we find that

$$\begin{aligned}
&f(-B^2q^4, -1/B^2)f(-Bq, -q/B)\\
&\quad= \{f(q^{38}, q^{20}) - q^4f(q^{44}, q^4)\}\{f(B^3q^5, q/B^3) - (1/B^2)f(q^5/B^3, B^3q)\}\\
&\quad= f(-q^{-4}, -q^8)\{f(B^3q^5, q/B^3) - (1/B^2)f(q^5/B^3, B^3q)\},
\end{aligned} \tag{38.6}$$

where we have applied Entry 31 with $a = -q^4$, $b = -q^8$, and $n = 2$.

Next, replace B by $1/B$ in (38.6) and successively employ Entries 30(iv), 30(i), and 24(iii) to conclude that

$$\begin{aligned}
&f(q^5/B^3, B^3q) - B^2f(q/B^3, B^3q^5)\\
&\quad= \frac{f(-B^2, -q^4/B^2)f(-Bq, -q/B)}{f(-q^4)}\\
&\quad= \frac{f(-B^2, -q^4/B^2)f(-B^2q^2, -q^2/B^2)\varphi(-q^2)}{f(-q^4)f(Bq, q/B)}\\
&\quad= \frac{f(-B^2, -q^2/B^2)\psi(q^2)\varphi(-q^2)}{f(-q^4)f(Bq, q/B)}\\
&\quad= \frac{f(-q^2)f(-B^2, -q^2/B^2)}{f(Bq, q/B)},
\end{aligned}$$

which completes the proof.

PROOF OF ENTRY 38(iv). In Entry 31, let $a = -q$, $b = -q^2$, and $n = 5$. Using also Entry 18(iv) three times, we find that

$$\begin{aligned}
f(-q) = &-qf(-q^{25}) + \{f(-q^{35}, -q^{40}) - q^{15}f(-q^{-10}, -q^{85})\}\\
&-q^2\{f(-q^{20}, -q^{55}) - q^5f(-q^5, -q^{70})\}.
\end{aligned} \tag{38.7}$$

We now apply (38.2) twice. In each case, replace q by $q^{25/2}$, and then let $B = -q^{15/2}$ and $-q^{5/2}$, respectively. Hence, (38.7) becomes

$$f(-q) + qf(-q^{25}) = f(-q^{25})\left\{\frac{f(-q^{15}, -q^{10})}{f(-q^{20}, -q^5)} - q^2\frac{f(-q^5, -q^{20})}{f(-q^{15}, -q^{10})}\right\}.$$

Replace q by $q^{1/5}$ and multiply both sides by $f(-q)$. Using the corollary to Entry 28, we deduce that

$$\begin{aligned} &f(-q)\{f(-q^{1/5}) + q^{1/5}f(-q^5)\} \\ &= f(-q)f(-q^5)\left\{\frac{f(-q^2, -q^3)}{f(-q, -q^4)} - q^{2/5}\frac{f(-q, -q^4)}{f(-q^2, -q^3)}\right\} \\ &= f^2(-q^2, -q^3) - q^{2/5}f^2(-q, -q^4), \end{aligned}$$

which is precisely (38.1).

As Ramanathan [8] has pointed out, the quintuple product identity can be found in Ramanujan's "lost notebook" [11] in the form

$$\frac{f(-x^2, -\lambda x)f(-\lambda x^3)}{f(-x, -\lambda x^2)} = f(-\lambda^2 x^3, -\lambda x^6) + xf(-\lambda, -\lambda^2 x^9). \quad (38.8)$$

To see that (38.2) and (38.8) are equivalent, set $\lambda x^3 = q^2$ and $x = -q/B$. Then (38.8) takes the form

$$\begin{aligned} \frac{f(-q^2/B^2, -B^2)f(-q^2)}{f(q/B, qB)} &= f(B^3q, q^5/B^3) - \frac{q}{B}f(B^3/q, q^7/B^3) \\ &= f(B^3q, q^5/B^3) - B^2f(B^3q^5, q/B^3), \end{aligned}$$

by an application of Entry 18(iv).

Next, we put (38.8) in a form that is perhaps more common and that legitimizes the designation "quintuple product identity." Let $\lambda x^3 = q$. By Entry 22(iii) and the Jacobi triple product identity, Entry 19, the left side of (38.8) equals

$$\begin{aligned} \frac{(q;q)_\infty(x^2;q)_\infty(q/x^2;q)_\infty}{(x;q)_\infty(q/x;q)_\infty} &= \frac{(q;q)_\infty(x^2;q)_\infty(q/x^2;q)_\infty(-x;q)_\infty(-q/x;q)_\infty}{(x^2;q^2)_\infty(q^2/x^2;q^2)_\infty} \\ &= (q;q)_\infty(x^2q;q^2)_\infty(q/x^2;q^2)_\infty(-x;q)_\infty(-q/x;q)_\infty. \end{aligned}$$

On the other hand, from the definition of $f(a, b)$, the right side of (38.8) is readily seen to equal

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} x^{3n}(1 + xq^n).$$

Lastly, replacing x by $-1/z$, we summarize our calculations above with a more familiar form of the quintuple product identity,

$$\sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2}(z^{3n} - z^{-3n-1}) = (q;q)_\infty(zq;q)_\infty(1/z;q)_\infty(z^2q;q^2)_\infty(q/z^2;q^2)_\infty. \quad (38.9)$$

The quintuple product identity has a long history, and it is difficult to assign priority to it. In one form, it was probably known to Weierstrass, for in H. A. Schwarz's book [1, p. 47], published in 1893, the quintuple product identity is written in terms of Weierstrass sigma functions. In R. Fricke's book [2, pp. 432–433], the quintuple product identity is presented in terms of theta-functions. Watson's name is associated with the quintuple product identity because in 1929 he [3] proved it en route to establishing (39.1) below. W. N. Bailey [1], who was familiar with Watson's work, found a proof in 1951. Shortly thereafter in 1952, D. B. Sears [1] showed that the quintuple product identity followed easily from some work he had done a year earlier. In the course of proving a conjecture of Dyson, A. O. L. Atkin and P. Swinnerton-Dyer [1] established the quintuple product identity in 1954 without realizing its prior occurrence in the literature. The identity was rediscovered in 1961 by B. Gordon [1]. L. J. Mordell [2], attributing the result to Gordon, gave another proof shortly thereafter. In 1970, M. V. Subbarao and M. Vidyasagar [1] found a proof. In 1972, L. Carlitz [3] discovered two proofs and, in the same year, in collaboration with Subbarao, published still another proof [1]. Andrews [7] showed that the quintuple product identity is a consequence of Bailey's summation of a well poised ${}_6\psi_6$. In 1988, employing the Jacobi triple product identity, M. Hirschhorn [5] established a significant generalization of the quintuple product identity. A year later, Blecksmith, Brillhart, and Gerst [2, p. 307] pointed out that the quintuple product identity is a special case of their theorem, which we related in Section 36. Lastly, in 1990, R. J. Evens [1] used complex function theory to give a short, elegant proof of the quintuple product identity that is completely unlike previous proofs.

Entry 39. *If* $\alpha, \beta > 0$ *and* $\alpha\beta = \pi^2$, *then*

(i)
$$\left\{\frac{\sqrt{5}+1}{2} + \frac{e^{-2\alpha/5}}{1\,+}\,\frac{e^{-2\alpha}}{1\,+}\,\frac{e^{-4\alpha}}{1\,+}\cdots\right\} \times \left\{\frac{\sqrt{5}+1}{2} + \frac{e^{-2\beta/5}}{1\,+}\,\frac{e^{-2\beta}}{1\,+}\,\frac{e^{-4\beta}}{1\,+}\cdots\right\} = \frac{5+\sqrt{5}}{2}$$

and

(ii)
$$\left\{\frac{\sqrt{5}-1}{2} + \frac{e^{-\alpha/5}}{1\,-}\,\frac{e^{-\alpha}}{1\,+}\,\frac{e^{-2\alpha}}{1\,-}\cdots\right\} \times \left\{\frac{\sqrt{5}-1}{2} + \frac{e^{-\beta/5}}{1\,-}\,\frac{e^{-\beta}}{1\,+}\,\frac{e^{-2\beta}}{1\,-}\cdots\right\} = \frac{5-\sqrt{5}}{2}.$$

Jacobsen [1, p. 435] has shown that, in fact, Entries 39(i), (ii) are valid for all complex numbers α and β with $\alpha\beta = \pi^2$, Re $\alpha > 0$, and Re $\beta > 0$.

Formula (i) was communicated by Ramanujan [10, p. xxviii] in his first letter to Hardy and was first proved in print by Watson [4]. Ramanathan [1], [4] has proved both (i) and (ii) and has established additional theorems of this type. We shall give below a proof of (ii) which is different from the proofs of both Watson and Ramanathan but which possesses features of both proofs. It seems likely that our proof is close to that found by Ramanujan. Our proof of (i) is very similar, and we give only a brief sketch of it.

Proof of (ii). Let

$$F(e^{-\alpha}) = \frac{e^{-\alpha/5}}{1} - \frac{e^{-\alpha}}{1} + \frac{e^{-2\alpha}}{1} - \frac{e^{-3\alpha}}{1} + \cdots.$$

Then employing Entries 38(iii), (iv) and the corollary to Entry 28, we find that

$$\begin{aligned}
&\frac{1}{F(e^{-\alpha})} - F(e^{-\alpha}) + 1 \\
&= \frac{e^{\alpha/5} f(-e^{-2\alpha}, e^{-3\alpha})}{f(e^{-\alpha}, -e^{-4\alpha})} - \frac{e^{-\alpha/5} f(e^{-\alpha}, -e^{-4\alpha})}{f(-e^{-2\alpha}, e^{-3\alpha})} + 1 \\
&= \frac{e^{\alpha/5} f^2(-e^{-2\alpha}, e^{-3\alpha}) - e^{-\alpha/5} f^2(e^{-\alpha}, -e^{-4\alpha}) + f(e^{-\alpha}, -e^{-4\alpha}) f(-e^{-2\alpha}, e^{3\alpha})}{f(e^{-\alpha}, -e^{-4\alpha}) f(-e^{-2\alpha}, e^{-3\alpha})} \\
&= \frac{e^{\alpha/5} f(e^{-\alpha})\{f(e^{-\alpha/5}) - e^{-\alpha/5} f(e^{-5\alpha})\} + f(e^{-\alpha}) f(e^{-5\alpha})}{f(e^{-\alpha}) f(e^{-5\alpha})} \\
&= \frac{e^{\alpha/5} f(e^{-\alpha/5})}{f(e^{-5\alpha})}.
\end{aligned} \tag{39.1}$$

Thus,

$$\left\{\frac{1}{F(e^{-\alpha})} - F(e^{-\alpha}) + 1\right\}\left\{\frac{1}{F(e^{-\beta})} - F(e^{-\beta}) + 1\right\} = \frac{e^{(\alpha+\beta)/5} f(e^{-\alpha/5}) f(e^{-\beta/5})}{f(e^{-5\alpha}) f(e^{-5\beta})}. \tag{39.2}$$

In Entry 27(iv), replace α by $\alpha/5$ and β by 5β to deduce that

$$e^{-\alpha/120}(\alpha/5)^{1/4} f(e^{-\alpha/5}) = e^{-5\beta/24}(5\beta)^{1/4} f(e^{-5\beta}),$$

where $\alpha\beta = \pi^2$. Using this equality and a similar equality with the roles of α and β reversed, we find from (39.2) that

$$\left\{\frac{1}{F(e^{-\alpha})} - F(e^{-\alpha}) + 1\right\}\left\{\frac{1}{F(e^{-\beta})} - F(e^{-\beta}) + 1\right\} = 5. \tag{39.3}$$

For brevity, set $A = F(e^{-\alpha})$ and $B = F(e^{-\beta})$. Then (39.3) takes the form

$$(A^2 - A - 1)(B^2 - B - 1) = 5AB,$$

or, after a brief calculation,

$$\{AB - \tfrac{1}{2}(A + B) - 1\}^2 = \tfrac{5}{4}(A + B)^2. \tag{39.4}$$

Suppose that

$$AB - \tfrac{1}{2}(A + B) - 1 > 0. \tag{39.5}$$

Then, from (39.4),

$$AB - \tfrac{1}{2}(A + B) - 1 = \frac{\sqrt{5}}{2}(A + B),$$

since $A, B > 0$. After some elementary manipulation, we find that

$$(2A - \sqrt{5} - 1)(2B - \sqrt{5} - 1) = 10 + 2\sqrt{5}.$$

By (39.1), $A, B < (\sqrt{5} + 1)/2$, and so, since $A, B > 0$, the left side above is no greater than

$$(\sqrt{5} + 1)^2 = 6 + 2\sqrt{5} < 10 + 2\sqrt{5}.$$

Since this is an obvious contradiction, our assumption (39.5) is incorrect, and we must conclude that

$$AB - \tfrac{1}{2}(A + B) - 1 = -\frac{\sqrt{5}}{2}(A + B).$$

After some elementary algebraic manipulation, the foregoing equality may be written in the form

$$\tfrac{1}{4}(2A + \sqrt{5} - 1)(2B + \sqrt{5} - 1) = \frac{5 - \sqrt{5}}{2},$$

which is equality (ii).

PROOF OF (i). Define

$$F(e^{-\alpha}) = \frac{e^{-\alpha/5}}{1} + \frac{e^{-\alpha}}{1} + \frac{e^{-2\alpha}}{1} + \frac{e^{-3\alpha}}{1} + \cdots.$$

Then, proceeding as above, we can show that

$$\frac{1}{F(e^{-\alpha})} - F(e^{-\alpha}) - 1 = \frac{e^{\alpha/5} f(-e^{-\alpha/5})}{f(-e^{-5\alpha})}.$$

Putting $A = F(e^{-2\alpha})$ and $B = F(e^{-2\beta})$, we find, with the use of Entry 27(iii), that

$$(A^2 + A - 1)(B^2 + B - 1) = 5AB,$$

which is equivalent to

$$\{AB + \tfrac{1}{2}(A + B) - 1\}^2 = \tfrac{5}{4}(A + B)^2.$$

The remainder of the proof is parallel to that of (ii).

Corollary. *We have*

(i)
$$\frac{e^{-\pi/5}}{1}\,\frac{}{-}\,\frac{e^{-\pi}}{1}\,\frac{}{+}\,\frac{e^{-2\pi}}{1}\,\frac{}{-}\cdots=\sqrt{\frac{5-\sqrt{5}}{2}}-\frac{\sqrt{5}-1}{2}$$

and

(ii)
$$\frac{e^{-2\pi/5}}{1}\,\frac{}{+}\,\frac{e^{-2\pi}}{1}\,\frac{}{+}\,\frac{e^{-4\pi}}{1}\,\frac{}{+}\cdots=\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2}.$$

PROOF. Let x denote the continued fraction on the left side of (i). Putting $\alpha = \beta = \pi$ in Entry 39(ii), we observe, after simplification, that x satisfies the equation

$$x^2 + (\sqrt{5} - 1)x - 1 = 0.$$

Solving this equation and observing that $x > 0$, we easily obtain the desired result.

In a similar fashion, Corollary (ii) follows from Entry 39(i).

Corollaries (i) and (ii) are both found in Ramanujan's [10, p. xxvii] first letter to Hardy. Ramanathan [1], [2], [4], [5], [9] has not only proved Corollaries (i) and (ii) but has established several additional beautiful results of this sort.

Some of the proofs in this chapter appear in the doctoral dissertation of C. Adiga [1] at the University of Mysore.

CHAPTER 17

Fundamental Properties of Elliptic Functions

Chapter 17 is almost entirely devoted to the theory of elliptic functions. The groundwork was prepared in the sections on theta-functions in Chapter 16. In the present chapter, Ramanujan introduces Jacobian elliptic functions and elliptic integrals. It is interesting that Ramanujan does not use the classical notation and terminology from the theory of elliptic functions and integrals. In Section 6, we identify the functions and parameters employed by Ramanujan with the more familiar notations in the theory of elliptic functions.

Much of Chapter 17 concentrates on various types of infinite series that can be evaluated in terms of parameters that arise frequently and naturally in the theory of elliptic functions and integrals. Many of Ramanujan's identities involving infinite series may be derived from theorems found in Jacobi's *Fundamenta Nova* [1], [2]. In particular, the Fourier series of the Jacobian elliptic functions can be utilized to establish many of Ramanujan's findings. Generally, however, we prefer to employ theta-functions, as did Ramanujan.

It is difficult to assess how many results in this chapter are original with Ramanujan. Perhaps a majority of the formulas in Chapter 17 cannot be found in print. However, if they are not in the literature, most can be derived without too much difficulty from published results. In particular, as indicated above, many of the results in Chapter 17 can be deduced from Jacobi's *Fundamenta Nova* [1], [2].

We conclude this introduction with a few remarks about notation. As usual, put

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}, \tag{0.1}$$

where a is any complex number and k is a nonnegative integer. The generalized hypergeometric series ${}_pF_q$ is defined by

$$ {}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q\end{matrix}; x\right] = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k(\beta_2)_k \cdots (\beta_q)_k} \frac{x^k}{k!}, \tag{0.2} $$

where p and q are nonnegative integers and $\alpha_1, \alpha_2, \ldots, \alpha_p$, $\beta_1, \beta_2, \ldots, \beta_q$ are complex numbers. If the number of parameters is "small," we use the notation ${}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; x)$ instead of that on the left side of (0.2). If $x = 1$, we omit the argument. In this chapter, $p = q + 1$, and so ${}_pF_q$ converges for $|x| < 1$ always, and if $\operatorname{Re}(\alpha_1 + \alpha_2 + \cdots + \alpha_{q+1}) < \operatorname{Re}(\beta_1 + \beta_2 + \cdots + \beta_q)$, ${}_{q+1}F_q$ converges for $x = 1$ as well.

As is customary in the theory of q-series, we also utilize the notations

$$ (a)_n := (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k) \tag{0.3} $$

and

$$ (a)_\infty := (a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k), $$

where $|q| < 1$ here and throughout the sequel. The notations (0.1) and (0.3) evidently conflict. However, the context will immediately make it clear whether (0.1) or (0.3) is being used.

Finally, if $\psi(z) = \Gamma'(z)/\Gamma(z)$, recall that (e.g., see Whittaker and Watson's text [1, p. 247])

$$ \psi(z) = -\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+z}\right), \tag{0.4} $$

where γ denotes Euler's constant. Formula (0.4) will frequently be used in the sequel, often without comment.

Entry 1. *Let n and x be real numbers with $0 \le x < 1$. Then*

$$ \int_0^{\pi/2} \frac{\cos\{(1-2n)\sin^{-1}(\sqrt{x}\sin\varphi)\}}{\sqrt{1 - x\sin^2\varphi}}\, d\varphi = \frac{\pi}{2}\, {}_2F_1(1-n, n; 1; x). $$

Proof. By Entry 35(iii) in Chapter 11 of Ramanujan's second notebook (Part II [9, p. 99]),

$$ (1 - x^2)^{-1/2} \cos(2n \sin^{-1} x) = {}_2F_1(\tfrac{1}{2} + n, \tfrac{1}{2} - n; \tfrac{1}{2}; x^2), $$

where n is arbitrary and $0 \le x < 1$. Replace $2n$ by $1 - 2n$ and x by $\sqrt{x}\sin\varphi$ and then integrate both sides over $0 \le \varphi \le \pi/2$. Accordingly, upon inverting the order of integration and summation, we find that

$$ \begin{aligned} &\int_0^{\pi/2} \frac{\cos\{(1-2n)\sin^{-1}(\sqrt{x}\sin\varphi)\}}{\sqrt{1 - x\sin^2\varphi}}\, d\varphi \\ &\quad = \sum_{k=0}^{\infty} \frac{(1-n)_k(n)_k}{(\frac{1}{2})_k k!} x^k \int_0^{\pi/2} \sin^{2k}\varphi\, d\varphi \end{aligned} $$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(1-n)_k (n)_k \Gamma(\frac{1}{2}) \Gamma(k+\frac{1}{2})}{(\frac{1}{2})_k k!\, \Gamma(k+1)} x^k,$$

from which the desired conclusion readily follows.

We now transcribe Entry 1 into perhaps more familiar forms. Put $\sin\theta = \sqrt{x}\sin\varphi$. Then a brief calculation gives

$$\int_0^{\sin^{-1}\sqrt{x}} \frac{\cos(2n-1)\theta}{\sqrt{x-\sin^2\theta}}\, d\theta = \frac{\pi}{2}\, {}_2F_1(1-n, n; 1; x). \tag{1.1}$$

By a result of Murphy, which can be found in Whittaker and Watson's treatise [1, p. 312], ${}_2F_1(1-n, n; 1; x) = P_{-n}(1-2x)$, where P_n denotes the nth Legendre function. Thus, (1.1) may be written in the form

$$\int_0^{\sin^{-1}\sqrt{x}} \frac{\cos(2n-1)\theta}{\sqrt{x-\sin^2\theta}}\, d\theta = \frac{\pi}{2} P_{-n}(1-2x). \tag{1.2}$$

In particular, setting $n = \frac{1}{2}$, we deduce that

$$\int_0^{\sin^{-1}\sqrt{x}} \frac{d\theta}{\sqrt{x-\sin^2\theta}} = \frac{\pi}{2} P_{-1/2}(1-2x).$$

Thus, we have a representation of the Legendre function $P_{-1/2}$ in terms of an elliptic integral of the first kind. This result is due to Kleiber [1, p. 10]. Lastly, if we replace $1-2x$ by $\cos\alpha$, n by $-n$, and θ by $\varphi/2$ in (1.2), we further find that

$$\frac{\sqrt{2}}{\pi} \int_0^{\alpha} \frac{\cos(n+\frac{1}{2})\varphi}{\sqrt{\cos\varphi - \cos\alpha}}\, d\varphi = P_n(\cos\alpha),$$

which is known as the Mehler–Dirichlet integral (Whittaker and Watson [1, p. 315]).

Corollary (i). *For any real number n,*

$$\int_0^{\pi/2} \frac{\cos\left\{(1-2n)\sin^{-1}\left(\frac{\sin\varphi}{\sqrt{2}}\right)\right\}}{\sqrt{1-\frac{1}{2}\sin^2\varphi}}\, d\varphi = \frac{\pi^{3/2}}{2\Gamma(1-\frac{1}{2}n)\Gamma(\frac{1}{2}+\frac{1}{2}n)}. \tag{1.3}$$

Proof. Letting $x = \frac{1}{2}$ in Entry 1, we find that the left side of (1.3) is equal to $(\pi/2)\, {}_2F_1(1-n, n; 1; \frac{1}{2})$. Recall now the following theorem of Gauss, which may be found in Bailey's tract [4, p. 11] and which was rediscovered by Ramanujan in Chapter 10, Entry 34 (Part II [9, p. 42]). If a and b are arbitrary, then

$${}_2F_1(a, b; \tfrac{1}{2}(a+b+1); \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)}. \tag{1.4}$$

In particular,

$$ {}_2F_1(1-n, n; 1; \tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})}{\Gamma(1-\frac{1}{2}n)\Gamma(\frac{1}{2}+\frac{1}{2}n)}, $$

and so the result follows.

Corollary (ii). *For* $0 \le x < 1$, *let*

$$ u_x = \int_0^{\pi/2} \frac{\cos\{(1-2n)\sin^{-1}(\sqrt{x}\sin\varphi)\}}{\sqrt{1-x\sin^2\varphi}}\, d\varphi. $$

Then, for $0 < x < 1$ *and nonintegral* n,

$$ \exp\left(-\frac{\pi}{\sin(\pi n)}\frac{u_{1-x}}{u_x}\right) = x\exp(\psi(n)+\psi(1-n)+2\gamma) $$
$$ \times \{1 + (2n^2-2n+1)x + (1-\tfrac{7}{2}(n-n^2)+\tfrac{13}{4}(n-n^2)^2)x^2 + \cdots\}, $$

where, as usual, $\psi(x) = \Gamma'(x)/\Gamma(x)$ *and* γ *denotes Euler's constant.*

Proof. We turn to Corollary 1 of Entry 25 in Chapter 11 (Part II [9, p. 77]). In that corollary, let $n = 0$ and then replace a and b by $n-1$ and $-n$, respectively. Using Entry 1 as well, we find that, for $0 < x < 1$,

$$ -\frac{2}{\sin(\pi n)}u_{1-x} $$
$$ = \sum_{k=0}^{\infty} \frac{(n)_k(1-n)_k}{(k!)^2}\{\psi(n+k)+\psi(1-n+k)-2\psi(k+1)+\operatorname{Log} x\}x^k $$
$$ = \frac{2}{\pi}u_x\{\operatorname{Log} x + \psi(n) + \psi(1-n) - 2\psi(1)\} $$
$$ + \sum_{k=1}^{\infty} \frac{(n)_k(1-n)_k}{(k!)^2}\{\psi(n+k)-\psi(n)+\psi(1-n+k)-\psi(1-n) $$
$$ -2\psi(k+1)+2\psi(1)\}x^k. $$

Since $\psi(1) = -\gamma$, this may be written in the form

$$ -\frac{\pi}{\sin(\pi n)}\frac{u_{1-x}}{u_x} = \operatorname{Log} x + \psi(n) + \psi(1-n) + 2\gamma $$
$$ + \frac{\pi}{2u_x}\sum_{k=1}^{\infty}\frac{(n)_k(1-n)_k}{(k!)^2}\{\psi(n+k)-\psi(n)+\psi(1-n+k) $$
$$ -\psi(1-n)-2\psi(k+1)-2\gamma\}x^k. \tag{1.5} $$

Exponentiating the equality above, we find that

$$\exp\left(-\frac{\pi}{\sin(\pi n)}\frac{u_{1-x}}{u_x}\right) = x\exp(\psi(n)+\psi(1-n)+2\gamma)e^w, \tag{1.6}$$

where, with the use of (0.4),

$$w = \frac{n(1-n)\left\{\frac{1}{n}+\frac{1}{1-n}-2\right\}x + \frac{n(n+1)(1-n)(2-n)}{4}\left\{\frac{1}{n}+\frac{1}{n+1}+\frac{1}{1-n}+\frac{1}{2-n}-3\right\}x^2+\cdots}{1+n(1-n)x+\cdots}$$

$$= \frac{(2n^2-2n+1)x - \frac{1}{4}(3n^4-6n^3+n^2+2n-2)x^2+\cdots}{1+(n-n^2)x+\cdots}$$

$$= (2n^2-2n+1)x + (\tfrac{5}{4}n^4 - \tfrac{5}{2}n^3 + \tfrac{11}{4}n^2 - \tfrac{3}{2}n + \tfrac{1}{2})x^2 + \cdots.$$

Expanding $\exp w$ and putting the result in (1.6), we complete the proof.

For $0 < x < 1$, let

$$F(x) = \exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right). \tag{2.1}$$

We can extend this definition to $x = 0$ and $x = 1$, because it is easily seen that

$$\lim_{x\to 0+} F(x) = 0 \tag{2.2}$$

and

$$\lim_{x\to 1-} F(x) = 1. \tag{2.3}$$

The function $F(x)$ was briefly examined in Section 27 of Chapter 11.

Entry 2(i). *If* $0 < x < 1$, *then*

$$F(x) = \frac{x}{16}\exp\left(4\frac{\sum_{k=1}^{\infty}\frac{(\frac{1}{2})_k^2}{(k!)^2}\sum_{j=1}^{k}\frac{1}{2j(2j-1)}x^k}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right).$$

Entry 2(i) was stated as a corollary and proved in Section 26 of Chapter 11 (Part II [9, pp. 78–79]).

The next result is very characteristic of Ramanujan, and we quote him exactly. (We need to assume that $x \geq 0$ below.)

Entry 2(ii).

$$F(1-e^{-x}) = \frac{x}{10+\sqrt{36+x^2}} \quad \textit{very nearly}. \tag{2.4}$$

Proof. By (2.2) and (2.3) it is readily seen that the left- and right-hand sides of (2.4) agree at $x = 0$ and $x = \infty$.

Calculating the Maclaurin series of the right side of (2.4), we find that, for $|x| < 8$,

$$\frac{x}{10+\sqrt{36+x^2}} = \frac{x}{16} - \frac{x^3}{3072} + \frac{280x^5}{2^{19}\cdot 3^3\cdot 5} + \cdots. \qquad (2.5)$$

On the other hand, replacing x by $x/8$ in Entry 2(vii) below, we deduce that, for $x \geq 0$,

$$F(1-e^{-x}) = \frac{x}{16} - \frac{x^3}{3072} + \frac{279x^5}{2^{19}\cdot 3^3\cdot 5} + \cdots. \qquad (2.6)$$

Comparing (2.5) and (2.6), we complete the proof of Ramanujan's excellent approximation for $x \geq 0$ and x small.

Observe, from (2.5) and (2.6), that the coefficients of x^5 differ by only $1/(2^{19}\cdot 3^3\cdot 5)$. Thus, Ramanujan's approximation is uncannily accurate.

H. Waadeland has communicated to us a very plausible explanation for Ramaunjan's approximation. Replacing x^2 by t in (2.6), we arrive at

$$xF(1-e^{-x}) = \frac{t}{16} - \frac{t^2}{3072} + \frac{93t^2}{2^{19}\cdot 3^2\cdot 5} + \cdots$$

$$= \frac{\frac{t}{16}}{1} + \frac{\frac{t}{192}}{1} + \frac{\frac{53t}{2^9\cdot 3\cdot 5}}{1} + \cdots.$$

Now,

$$\frac{53}{2^9\cdot 3\cdot 5} = \frac{1}{144.905660\ldots}.$$

Ramanujan liked highly composite numbers. Thus, replacing 144.905660 by 144 and replacing all subsequent numerators in the continued fraction above by $t/144$, we find that

$$xF(1-e^{-x}) \approx \frac{\frac{t}{16}}{1} + \frac{\frac{t}{192}}{1} + \frac{\frac{t}{144}}{1} + \frac{\frac{t}{144}}{1} + \frac{\frac{t}{144}}{1} + \cdots$$

$$= \cfrac{t/16}{1 + \cfrac{t/192}{\frac{1}{2}(1+\sqrt{1+t/36})}}$$

$$= \frac{t}{10+\sqrt{36+t}}.$$

Entry 2(iii). *For* $0 < x < 1$,

$$\operatorname{Log} F(x) \operatorname{Log} F(1-x) = \pi^2.$$

PROOF. This result follows at once from the definition of F in (2.1).

Entry 2(iv). *We have*

$$F(1-x)+F(1-1/x)=0.$$

Proof. Using Entries 30 and 32(ii) in Chapter 11 (Part II [9, pp. 87, 92]), we find that

$$\begin{aligned}&\frac{d}{dx}(\operatorname{Log} F(1-x)-\operatorname{Log} F(1-1/x))\\&=\frac{d}{dx}\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}+\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1/x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-1/x)}\right)\\&=-\frac{1}{{}_2F_1{}^2(\frac{1}{2},\frac{1}{2};1;1-x)x(x-1)}+\frac{1}{{}_2F_1{}^2(\frac{1}{2},\frac{1}{2};1;1-1/x)(x-1)}\\&=-\frac{1}{{}_2F_1{}^2(\frac{1}{2},\frac{1}{2};1;1-x)x(x-1)}+\frac{1}{{}_2F_1{}^2(\frac{1}{2},\frac{1}{2};1;1-x)x(x-1)}\\&=0.\end{aligned}$$

Thus, for some constant c,

$$\frac{F(1-x)}{F(1-1/x)}=c.$$

By using Entry 2(i), we easily see that

$$\lim_{x\to 1}\frac{F(1-x)}{F(1-1/x)}=-1.$$

Hence, $c=-1$, and the proof is complete.

Entry 2(v). *We have*

$$F(x^2)=F\left(\frac{4x}{(1+x)^2}\right)^2.$$

Proof. In our proof of Entry 32(iii) in Chapter 11 [9, p. 93], we showed that

$${}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\frac{4x}{(1+x)^2}\right)=(1+x)\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;x^2). \tag{2.7}$$

Replacing x by $(1-x)/(1+x)$, we find that

$${}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;1-x^2)=\frac{2}{1+x}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;1-\frac{4x}{(1+x)^2}\right). \tag{2.8}$$

Dividing (2.8) by (2.7), we arrive at

$$\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x^2)}=2\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\frac{4x}{(1+x)^2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{4x}{(1+x)^2}\right)}.$$

Multiplying the equality above by $-\pi$ and exponentiating, we deduce the desired result.

In a note following Entry 2(v), Ramanujan describes a very ingenious algorithm for calculating the power series expansion of $F(2x/(1+x))$, which we now relate in detail.

First, by Entry 2(iv),

$$F\left(\frac{-2x}{1-x}\right)=F\left(1-\frac{1+x}{1-x}\right)=-F\left(1-\frac{1-x}{1+x}\right)=-F\left(\frac{2x}{1+x}\right).$$

Thus, since $F(2x/(1+x))$ is an odd function of x, we can write

$$F\left(\frac{2x}{1+x}\right)=\sum_{k=1}^{n}a_k x^{2k-1}+O(x^{2n+1}) \tag{2.9}$$

in a neighborhood of the origin (in fact, for $|x|<\frac{1}{3}$). Setting $y^2=2x/(1+x)$, we find that, in a neighborhood of $y=0$,

$$\begin{aligned}F(y^2)&=\sum_{k=1}^{n}a_k\frac{y^{4k-2}}{(2-y^2)^{2k-1}}+O(y^{4n+2})\\&=\sum_{k=1}^{2n}b_k y^{2k}+O(y^{4n+2}),\end{aligned}$$

where b_{2m-1} and b_{2m} are expressible in terms of $a_1, a_2, \ldots, a_m$ for each m, $1\le m\le n$. Hence, by Entry 2(v), in some neighborhood of $y=0$,

$$\begin{aligned}F\left(\frac{4y}{(1+y)^2}\right)&=\left(\sum_{k=1}^{2n}b_k y^{2k}+O(y^{4n+2})\right)^{1/2}\\&=\sum_{k=1}^{2n}c_k y^{2k-1}+O(y^{4n+1}),\end{aligned} \tag{2.10}$$

where c_m is expressible in terms of $b_1, \ldots, b_m$, $1\le m\le 2n$. Hence, c_{2m-1} and c_{2m} are expressible in terms of $a_1, \ldots, a_m$ for each m, $1\le m\le n$. Next, set $x=2y/(1+y^2)$. Then

$$y=\frac{x}{1+\sqrt{1-x^2}} \quad\text{and}\quad \frac{2x}{1+x}=\frac{4y}{(1+y)^2}.$$

Hence, for $|x|$ sufficiently small, (2.10) becomes

$$\begin{aligned}F\left(\frac{2x}{1+x}\right)&=\sum_{k=1}^{2n}c_k\left(\frac{x}{1-\sqrt{1-x^2}}\right)^{2k-1}+O(x^{4n+1})\\&=\sum_{k=1}^{2n}d_k x^{2k-1}+O(x^{4n+1}),\end{aligned} \tag{2.11}$$

where each d_m, $1\le m\le 2n$, is expressible in terms of $c_1, \ldots, c_m$. Thus, since d_{2m-1} is expressible in terms of $c_1, \ldots, c_{2m-1}$ and d_{2m} is expressible in terms of $c_1, \ldots, c_{2m}$, we deduce that d_{2m-1} and d_{2m} are expressible in terms of $a_1, \ldots,$

a_m. But, comparing (2.9) and (2.11), we see that $a_k = d_k$, $k \geq 1$. In conclusion, we have therefore shown that a_{2m-1} and a_{2m} can be determined from $a_1, \ldots, a_m$.

Entry 2(vi) illustrates the algorithm described above.

Entry 2(vi). *If* $|x| < \frac{1}{3}$, *then*

$$F\left(\frac{2x}{1+x}\right) = \frac{1}{2^3}x + \frac{5}{2^7}x^3 + \frac{369}{2^{14}}x^5 + \frac{4097}{2^{18}}x^7 + \frac{1594895}{2^{27}}x^9 + \cdots.$$

PROOF. From Entry 2(i), it is clear that $a_1 = \frac{1}{8}$, in the notation (2.9). Thus, we write, for $|x| < \frac{1}{3}$,

$$F\left(\frac{2x}{1+x}\right) = \frac{1}{8}x + O(x^3).$$

Setting $y^2 = 2x/(1+x)$, we find that

$$F(y^2) = \frac{1}{16}y^2 + \frac{1}{32}y^4 + O(y^6).$$

Hence, by Entry 2(v),

$$F\left(\frac{4y}{(1+y)^2}\right) = \left\{\frac{1}{16}y^2 + \frac{1}{32}y^4 + O(y^6)\right\}^{1/2}$$
$$= \frac{1}{4}y + \frac{1}{16}y^3 + O(y^5).$$

Setting $x = 2y/(1+y^2)$, we find from the equality above that

$$F\left(\frac{2x}{1+x}\right) = \frac{1}{4}\frac{x}{1+\sqrt{1-x^2}} + \frac{1}{16}\left(\frac{x}{1+\sqrt{1-x^2}}\right)^3 + O(x^5) \quad (2.12)$$

for $|x| < \frac{1}{3}$. Now, by Corollary 1, Section 14 of Chapter 3 (Part I [5, p. 71]),

$$\left(\frac{2}{1+\sqrt{1-4a}}\right)^n = 1 + na + n\sum_{k=2}^{\infty}\frac{\Gamma(n+2k)a^k}{\Gamma(n+k+1)k!}, \quad (2.13)$$

where n is any real number and $|a| < \frac{1}{4}$. Hence, for $|x| < 1$,

$$\frac{2}{1+\sqrt{1-x^2}} = 1 + \frac{x^2}{4} + \frac{x^4}{8} + \frac{5x^6}{64} + \frac{7x^8}{128} + \cdots \quad (2.14)$$

and

$$\frac{8}{(1+\sqrt{1-x^2})^3} = 1 + \frac{3x^2}{4} + \frac{9x^4}{16} + \frac{7x^6}{16} + \cdots. \quad (2.15)$$

Putting (2.14) and (2.15) in (2.12), we arrive at

$$F\left(\frac{2x}{1+x}\right) = \frac{1}{8}x + \frac{5}{128}x^3 + O(x^5).$$

Repeating the procedure above, but with $n = 2$ in the algorithm, we have

$$F(y^2) = \frac{1}{16}y^2 + \frac{1}{32}y^4 + \frac{21}{1024}y^6 + \frac{31}{2048}y^8 + O(y^{10})$$

and

$$F\left(\frac{4y}{(1+y)^2}\right) = \frac{1}{4}y + \frac{1}{16}y^3 + \frac{17}{512}y^5 + \frac{45}{2048}y^7 + O(y^9).$$

By (2.13),

$$\frac{32}{(1+\sqrt{1-x^2})^5} = 1 + \frac{5x^2}{4} + \frac{5x^4}{4} + \cdots \tag{2.16}$$

and

$$\frac{128}{(1+\sqrt{1-x^2})^7} = 1 + \frac{7x^2}{4} + \cdots. \tag{2.17}$$

Thus, from (2.14)–(2.17),

$$F\left(\frac{2x}{1+x}\right) = \frac{1}{8}x + \frac{5}{128}x^3 + \frac{369}{2^{14}}x^5 + \frac{4097}{2^{18}}x^7 + O(x^9).$$

We repeat this procedure once more, but with $n = 3$. Accordingly, we find that

$$F(y^2) = \frac{1}{16}y^2 + \frac{1}{32}y^4 + \frac{21}{2^{10}}y^6 + \frac{31}{2^{11}}y^8 + \frac{6257}{2^{19}}y^{10} + O(y^{12}),$$

from which we deduce that

$$F\left(\frac{4y}{(1+y)^2}\right) = \frac{1}{4}y + \frac{1}{16}y^3 + \frac{17}{2^9}y^5 + \frac{45}{2^{11}}y^7 + \frac{4239}{2^{18}}y^9 + O(y^{11}).$$

Finally, using (2.14)–(2.17), we find that the coefficient of x^9 in the power series expansion of $F(2x/(1+x))$ is $1594895/2^{27}$. This completes the proof.

Entry 2(vii). *For* $x \geq 0$,

$$F(1 - e^{-8x}) = \frac{1}{2}x - \frac{1}{6}x^3 + \frac{31}{240}x^5 - \frac{661}{5040}x^7 + \frac{219677}{1451520}x^9 + \cdots.$$

PROOF. We shall apply Entry 2(vi) with x replaced by $\tanh(4x)$. Since

$$\frac{2\tanh(4x)}{1+\tanh(4x)} = 1 - e^{-8x},$$

we find that

$$F(1 - e^{-8x}) = \frac{1}{8}\tanh(4x) + \frac{5}{128}\tanh^3(4x) + \frac{369}{2^{14}}\tanh^5(4x)$$
$$+ \frac{4097}{2^{18}}\tanh^7(4x) + \frac{1594895}{2^{27}}\tanh^9(4x) + \cdots. \tag{2.18}$$

Now, for $|x| < \pi/2$ (Gradshteyn and Ryzhik [1, p. 35]),

$$\tanh(4x) = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}(4x)^{2k-1}$$

$$= 4x - \frac{2^6}{3}x^3 + \frac{2^{11}}{15}x^5 - \frac{2^{14}17}{3^2\cdot 5\cdot 7}x^7 + \frac{2^{19}31}{3^4\cdot 5\cdot 7}x^9 + \cdots, \tag{2.19}$$

where B_n, $0 \le n < \infty$, denotes the nth Bernoulli number. Substituting (2.19) into (2.18), we achieve the desired expansion after a rather tedious computation.

Example 1. *We have*

$$F(0) = 0, \qquad F(\tfrac{1}{2}) = e^{-\pi}, \qquad F(1) = 1,$$
$$F((\sqrt{2}-1)^2) = e^{-\pi\sqrt{2}}, \quad \textit{and} \quad F((\sqrt{2}-1)^4) = e^{-2\pi}.$$

Proof. The values for $F(0)$ and $F(1)$ were previously observed in (2.2) and (2.3), respectively. The value for $F(\frac{1}{2})$ follows immediately from (2.1).

In (2.7), put $x = \sqrt{2} - 1$ to find that

$$ {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-(\sqrt{2}-1)^2) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 2(\sqrt{2}-1))$$
$$= \sqrt{2}\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; (\sqrt{2}-1)^2).$$

The proposed value for $F((\sqrt{2}-1)^2)$ follows immediately from (2.1) and the extremal equality above.

Lastly, set $x = (\sqrt{2}-1)^2$ in Entry 2(v) to get

$$e^{-\pi} = F(\tfrac{1}{2}) = F\left(\frac{4(\sqrt{2}-1)^2}{(1+(\sqrt{2}-1)^2)^2}\right) = F((\sqrt{2}-1)^4)^{1/2}.$$

Squaring the extremal equality above finishes the proof.

Example 2. *If* $0 \le x < 1$, *then*

$$F\left(1 - \exp\left(\frac{-8x}{1-x^2}\right)\right) = \frac{1}{2}x + \frac{1}{3}x^3 + \frac{31}{240}x^5 + \frac{37}{2520}x^7 + \frac{5981}{1451520}x^9 + \cdots.$$

Proof. In Entry 2(vii), replace x by $x/(1-x^2)$ to find that, for $0 \le x < 1$,

$$F\left(1 - \exp\left(\frac{-8x}{1-x^2}\right)\right) = \frac{1}{2}\frac{x}{1-x^2} - \frac{1}{6}\frac{x^3}{(1-x^2)^3} + \frac{31}{240}\frac{x^5}{(1-x^2)^5}$$
$$- \frac{661}{5040}\frac{x^7}{(1-x^2)^7} + \frac{219677}{1451520}\frac{x^9}{(1-x^2)^9} + \cdots.$$

Expanding $(1-x^2)^{-2n-1}$, $0 \le n \le 4$, in Maclaurin series and collecting coefficients of like powers, we complete the proof.

For $|q| < 1$, let

$$\varphi(q) = \sum_{k=-\infty}^{\infty} q^{k^2}, \tag{3.1}$$

one of the classical theta-functions studied by Ramanujan in Chapter 16.

Lemma. *For* $|q| < 1$,

$${}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}\right) = \frac{\varphi^2(q)}{\varphi^2(q^2)}\, {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - \frac{\varphi^4(-q^2)}{\varphi^4(q^2)}\right).$$

PROOF. By Entry 32(iii) in Chapter 11 (Part II [9, p. 93]),

$${}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - \left(\frac{1-x}{1+x}\right)^2\right) = (1+x)\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x^2). \tag{3.2}$$

Now if

$$\frac{1-x}{1+x} = \frac{\varphi^2(-q)}{\varphi^2(q)}, \tag{3.3}$$

then

$$1 - x^2 = \frac{\varphi^4(-q^2)}{\varphi^4(q^2)}, \tag{3.4}$$

by the corollary in Section 25 of Chapter 16. An elementary calculation now shows that

$$1 + x = \frac{\varphi^2(-q^2)\varphi(q)}{\varphi^2(q^2)\varphi(-q)}.$$

By Entry 25(iii) in Chapter 16,

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2). \tag{3.5}$$

Hence,

$$1 + x = \frac{\varphi^2(q)}{\varphi^2(q^2)}. \tag{3.6}$$

Substituting (3.3), (3.4), and (3.6) into (3.2), we complete the proof.

Entry 3. *If* $|q| < 1$, *then*

$$\varphi^2(q) = {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}\right).$$

PROOF. Iterate the identity of the foregoing lemma a total of m times. If $n = 2^m$, we then find that

$$ {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;1-\frac{\varphi^4(-q)}{\varphi^4(q)}\right)=\frac{\varphi^2(q)}{\varphi^2(q^n)}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;1-\frac{\varphi^4(-q^n)}{\varphi^4(q^n)}\right). \tag{3.7} $$

Now let n tend to ∞. Since $\varphi(-q^n)$ and $\varphi(q^n)$ tend to 1, the desired result follows.

The proof of the lemma and Entry 3 are very briefly sketched by Ramanujan (p. 206). Proofs in the latter half of the second notebook are very rare indeed.

In his sketch, Ramanujan seems to claim that

$$ {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\left(\frac{1-x}{1+x}\right)^2\right)=\tfrac{1}{2}(1+x)\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;x^2). $$

Since the right side is analytic at $x = 0$ while the left side is not, the proposed identity is false. Fortunately, there is no evidence that Ramanujan actually used this claim.

Lemma. *For* $|q| < 1$,

$$ {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\frac{\varphi^4(-q)}{\varphi^4(q)}\right)=\frac{\varphi^2(q)}{2\varphi^2(q^2)}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\frac{\varphi^4(-q^2)}{\varphi^4(q^2)}\right). $$

PROOF. We shall employ (2.7) with $x = \varphi^2(-q)/\varphi^2(q)$. By Entry 25(vi) in Chapter 16,

$$ \varphi^2(q)+\varphi^2(-q)=2\varphi^2(q^2). \tag{4.1} $$

Using (3.5) and (4.1), we readily find that

$$ \frac{4x}{(1+x)^2}=\frac{\varphi^4(-q^2)}{\varphi^4(q^2)} \tag{4.2} $$

and

$$ 1+x=\frac{2\varphi^2(q^2)}{\varphi^2(q)}. \tag{4.3} $$

Substituting (4.2) and (4.3) into (2.7), we complete the proof.

Entry 4(i). *If m is a nonnegative integer and $n = 2^m$, then*

$$ F\left(\frac{\varphi^4(-q)}{\varphi^4(q)}\right)=F\left(\frac{\varphi^4(-q^n)}{\varphi^4(q^n)}\right)^n. $$

PROOF. Iterate the identity in the previous lemma m times to obtain the equality

$$ {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\frac{\varphi^4(-q)}{\varphi^4(q)}\right)=\frac{\varphi^2(q)}{n\varphi^2(q^n)}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\frac{\varphi^4(-q^n)}{\varphi^4(q^n)}\right). $$

Combine this equality with (3.7) to deduce that

$$\frac{{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;1-\dfrac{\varphi^4(-q)}{\varphi^4(q)}\right)}{{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\dfrac{\varphi^4(-q)}{\varphi^4(q)}\right)} = n\frac{{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;1-\dfrac{\varphi^4(-q^n)}{\varphi^4(q^n)}\right)}{{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\dfrac{\varphi^4(-q^n)}{\varphi^4(q^n)}\right)}. \tag{4.4}$$

Multiplying both sides by $-\pi$ and exponentiating, we complete the proof.

Entry 4(ii). *If* $n = 2^m$, *as above, then*

$$F\left(1-\frac{\varphi^4(-q)}{\varphi^4(q)}\right)^n = F\left(1-\frac{\varphi^4(-q^n)}{\varphi^4(q^n)}\right).$$

Proof. This follows immediately from (4.4).

The proofs of the results in Section 4 were also sketched by Ramanujan. It was doubtless the importance of the inversion formula in Section 5 below which led Ramanujan to include sketches of the results in Sections 3 and 4 in his notebooks.

Entry 5 (Inversion Formula). *For* $|q| < 1$,

$$F\left(1-\frac{\varphi^4(-q)}{\varphi^4(q)}\right) = q.$$

Proof. We shall let n tend to ∞ in Entry 4(ii).

As x tends to 0, by Entry 2(i), $F(x) \sim x/16$. Thus, if $\xi_n = \varphi^4(-q^n)/\varphi^4(q^n)$,

$$\lim_{n\to\infty} \sqrt[n]{F(1-\xi_n)} = \lim_{n\to\infty} \sqrt[n]{1-\xi_n}.$$

Let

$$\psi(q) = \sum_{k=0}^{\infty} q^{k(k+1)/2}, \qquad |q| < 1, \tag{5.1}$$

another classical theta-function studied by Ramanujan in Chapter 16. By Entry 25(vii) in Chapter 16,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\operatorname{Log}(1-\xi_n) &= \lim_{n\to\infty} \frac{1}{n}\operatorname{Log}\left(\frac{\varphi^4(q^n)-\varphi^4(-q^n)}{\varphi^4(q^n)}\right)\\ &= \lim_{n\to\infty} \frac{1}{n}\operatorname{Log}\left(\frac{16q^n\psi^4(q^{2n})}{\varphi^4(q^n)}\right)\\ &= \operatorname{Log} q + \lim_{n\to\infty} \frac{1}{n}\operatorname{Log}\left(\frac{\psi^4(q^{2n})}{\varphi^4(q^n)}\right)\\ &= \operatorname{Log} q.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \sqrt[n]{F(1-\xi_n)} = q,$$

and applying Entry 4(ii), we complete the proof.

Entry 6. *In the notations* (2.1) *and* (3.1),

$$\varphi^2(F(x)) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x). \tag{6.1}$$

If furthermore

$$z = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x) \tag{6.2}$$

and

$$y = \pi \frac{{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-x)}{{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x)}, \tag{6.3}$$

then

$$\varphi(e^{-y}) = \sqrt{z}. \tag{6.4}$$

We remark that the notations (6.2)–(6.4) will be used extensively in the sequel.

Proof. By Entries 3 and 5, respectively,

$$\varphi^2(F(x)) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-u) \tag{6.5}$$

and

$$F(1-u) = F(x),$$

where $u = u(x) = \varphi^4(-F(x))/\varphi^4(F(x))$. Thus,

$$\frac{{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; u)}{{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-u)} = \frac{{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-x)}{{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x)}. \tag{6.6}$$

From (6.6) we would like to deduce that

$${}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-u) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x) \tag{6.7}$$

and thus deduce (6.1) from (6.5) and (6.7).

Suppose that (6.7) is not true. Then there are values x_0 and $u_0 = u(x_0)$ such that

$${}_2F_1(1-u_0) := {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-u_0) \neq {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x_0) =: {}_2F_1(x_0). \tag{6.8}$$

Assume, without loss of generality, that ${}_2F_1(1-u_0) < {}_2F_1(x_0)$. Then, by (6.6), ${}_2F_1(u_0) < {}_2F_1(1-x_0)$. Now ${}_2F_1(x)$ is increasing on (0, 1). Thus, $1-u_0 < x_0$ and $u_0 < 1-x_0$. These two inequalities are incompatible. Hence, (6.8) is invalid, and so (6.7) is established to complete the proof of (6.1).

From the definitions (2.1) and (6.3), $F(x) = e^{-y}$. Using this fact, we easily see that (6.1) and (6.4) are equivalent.

At this juncture, we should identify the quantities x, y, and z with the customary parameters in the theory of elliptic functions. The complete elliptic integral of the first kind is defined by (Whittaker and Watson [1, pp. 499, 500])

$$K := K(k) := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \tfrac{1}{2}\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) = \tfrac{1}{2}\pi\varphi^2(q). \tag{6.9}$$

Here k, $0 < k < 1$, is the *modulus* of K. To obtain the second representation for K, expand the integrand in a binomial series and integrate termwise. (See Part II [9, p. 79].) The last equality is one of the most fundamental results in the theory of elliptic functions and follows from (6.2) and (6.4). Ramanujan does not use the universal notation k and sets $x = k^2$. Later, when deriving modular equations, Ramanujan puts $\alpha = k^2$. The *complementary modulus* k' is defined by $k' = \sqrt{1 - k^2}$. From Entry 3, (6.9), and the monotonicity of $\varphi^2(q)$ for $0 < q < 1$, we see that

$$\sqrt{1 - x} = k' = \frac{\varphi^2(-q)}{\varphi^2(q)}. \tag{6.10}$$

From (6.2) and (6.9),

$$z = \frac{2}{\pi} K. \tag{6.11}$$

Also from (6.3) (Whittaker and Watson [1, p. 486]),

$$q = e^{-y} = e^{-\pi K'/K}. \tag{6.12}$$

Ramanujan uses the notation x instead of q, which is universally employed today, and so we use q as well.

The following corollary is the famous inversion formula for the theta-function φ. This formula is also found in Section 7 of Chapter 14, p. 169, and in Entry 27(i) of Chapter 16, p. 199. The following, perhaps new and novel, proof is obviously the one which Ramanujan found at this point and is different from either of his two previous proofs.

Corollary. *Let $\alpha, \beta > 0$ with $\alpha\beta = \pi$. Then*

$$\sqrt{\alpha}\varphi(e^{-\alpha^2}) = \sqrt{\beta}\varphi(e^{-\beta^2}).$$

PROOF. Let $y = \alpha^2$. Since $F(x) = e^{-y}$,

$$F(1 - x) = e^{-\pi^2/y} = e^{-\pi^2/\alpha^2} = e^{-\beta^2}. \tag{6.13}$$

From (6.1) and (6.3),

$$\frac{\varphi^2(F(1 - x))}{\varphi^2(F(x))} = \frac{y}{\pi} = \frac{\alpha}{\beta}. \tag{6.14}$$

Using (6.13) in (6.14), we complete the proof.

Example (i).
$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(\frac{3}{4})}.$$

PROOF. By Example 1 in Section 2, $F(\frac{1}{2}) = e^{-\pi}$. Therefore by Entry 6,
$$\varphi^2(e^{-\pi}) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}).$$
But, by (1.4),
$${}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}) = \frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})}, \tag{6.15}$$
and the desired result follows.

Example (ii).
$$\varphi(e^{-\pi\sqrt{2}}) = \frac{\Gamma(\frac{9}{8})}{\Gamma(\frac{5}{4})}\sqrt{\frac{\Gamma(\frac{1}{4})}{2^{1/4}\pi}}.$$

PROOF. Recall from Example 1 in Section 2 that $F((\sqrt{2}-1)^2) = e^{-\pi\sqrt{2}}$. Thus, by Entry 6,
$$\varphi^2(e^{-\pi\sqrt{2}}) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; (\sqrt{2}-1)^2). \tag{6.16}$$
In order to evaluate ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; (\sqrt{2}-1)^2)$, we invoke Entry 33(iv) of Chapter 11 (Part II [9, p. 95]),
$${}_2F_1\left(\tfrac{1}{4}, \tfrac{3}{4}; 1; \frac{4x}{(1+x)^2}\right) = \sqrt{1+x}\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x).$$
Letting $x = (\sqrt{2}-1)^2$ and using (1.4), the duplication theorem, and the reflection principle, we find that
$$\begin{aligned}
\sqrt{4-2\sqrt{2}}\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; (\sqrt{2}-1)^2)
&= {}_2F_1(\tfrac{1}{4}, \tfrac{3}{4}; 1; \tfrac{1}{2}) \\
&= \frac{\sqrt{\pi}}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})} \\
&= \frac{2^{1/4}\Gamma(\frac{9}{8})}{\Gamma(\frac{5}{4})\Gamma(\frac{7}{8})} \\
&= \frac{\Gamma(\frac{9}{8})\Gamma(\frac{1}{8})\sqrt{2-\sqrt{2}}}{\pi 2^{3/4}\Gamma(\frac{5}{4})} \\
&= \frac{\Gamma^2(\frac{9}{8})2^{9/4}\sqrt{2-\sqrt{2}}}{\pi\Gamma(\frac{5}{4})} \\
&= \frac{\Gamma^2(\frac{9}{8})}{\Gamma^2(\frac{5}{4})}\,\frac{\sqrt{4-2\sqrt{2}}\,\Gamma(\frac{1}{4})}{2^{1/4}\pi}.
\end{aligned}$$
Substituting this into (6.16), we complete the proof.

Ramanujan (p. 207) inadvertently omitted the factor $\sqrt{\Gamma(\frac{1}{4})/2^{1/4}}$ in his formulation.

Example (iii). $$\varphi(e^{-2\pi}) = \frac{\sqrt{2+\sqrt{2}}}{2}\frac{\pi^{1/4}}{\Gamma(\frac{3}{4})}.$$

PROOF. By Example 1 in Section 2, $F((\sqrt{2}-1)^4) = e^{-2\pi}$. Thus, by Entry 6,

$$\varphi^2(e^{-2\pi}) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; (\sqrt{2}-1)^4). \tag{6.17}$$

To evaluate ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; (\sqrt{2}-1)^4)$, we shall employ (3.2) in which we set $x = (\sqrt{2}-1)^2$. Thus,

$$\begin{aligned} {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; (\sqrt{2}-1)^4) &= \frac{1}{4-2\sqrt{2}}\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}) \\ &= \frac{2+\sqrt{2}}{4}\frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})}, \end{aligned}$$

by (1.4). Substituting this in (6.17), we complete the proof.

Example (iv). $$\sum_{k=1}^{\infty} (k^2\pi - \tfrac{1}{4})e^{-\pi k^2} = \tfrac{1}{8}.$$

PROOF. In the corollary above, differentiate both sides with respect to α to get

$$\begin{aligned} &\frac{1}{2\sqrt{\alpha}}\varphi(e^{-\alpha^2}) + 2\sqrt{\alpha}\sum_{k=1}^{\infty}(-2\alpha k^2)e^{-\alpha^2 k^2} \\ &\quad = -\frac{\pi}{\alpha^2}\left(\frac{1}{2\sqrt{\beta}}\varphi(e^{-\beta^2}) + 2\sqrt{\beta}\sum_{k=1}^{\infty}(-2\beta k^2)e^{-\beta^2 k^2}\right). \end{aligned}$$

Letting $\alpha = \beta = \sqrt{\pi}$ and simplifying, we obtain the desired result.

Section 7 consists of a large collection of results on elliptic integrals. As we shall see, some are quite elementary, others are less elementary but known, and perhaps a couple may be new. Throughout Section 7, we tacitly assume that the parameter x is chosen so that the integrals exist, and that all upper limits on integrals do not exceed $\pi/2$ so that all changes of variables are valid. Proofs of some of the results in Section 7 have also been given by Thiruvenkatachar and Venkatachaliengar [1].

Entry 7(i). *If* $\sin\alpha = \sqrt{x}\sin\beta$, *then*

$$\int_0^{\alpha}\frac{d\varphi}{\sqrt{x-\sin^2\varphi}} = \int_0^{\beta}\frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}.$$

PROOF. In the former integral, make the change of variable $\sin\varphi = \sqrt{x}\sin\theta$.

Since

$$\frac{d\varphi}{d\theta} = \frac{\sqrt{x}\cos\theta}{\sqrt{1 - x\sin^2\theta}} \quad \text{and} \quad \sqrt{x - \sin^2\varphi} = \sqrt{x}\cos\theta,$$

the desired result follows at once.

Entry 7(ii). *If* $\tan\alpha = \sqrt{1-x}\tan\beta$, *then*

$$\int_0^\alpha \frac{d\varphi}{\sqrt{1 - x\cos^2\varphi}} = \int_0^\beta \frac{d\varphi}{\sqrt{1 - x\sin^2\varphi}}.$$

PROOF. In the former integral, make the change of variable $\tan\varphi = \sqrt{1-x}\tan\theta$. Then elementary calculations give

$$1 - x\cos^2\varphi = \frac{1-x}{1 - x\sin^2\theta} \quad \text{and} \quad \frac{d\varphi}{d\theta} = \frac{\sqrt{1-x}}{1 - x\sin^2\theta}.$$

The result now follows.

Entry 7(iii). *If* $\tan\alpha = \sqrt{1-b}\tan\beta$, *then*

$$\int_0^\alpha \frac{d\varphi}{\sqrt{1 - \dfrac{a-b}{1-b}\sin^2\varphi}} = \sqrt{1-b}\int_0^\beta \frac{d\varphi}{\sqrt{(1 - a\sin^2\varphi)(1 - b\sin^2\varphi)}}.$$

PROOF. In the former integral, put $\tan\varphi = \sqrt{1-b}\tan\theta$. Then

$$1 - \frac{a-b}{1-b}\sin^2\varphi = \frac{1 - a\sin^2\theta}{1 - b\sin^2\theta} \quad \text{and} \quad \frac{d\varphi}{d\theta} = \frac{\sqrt{1-b}}{1 - b\sin^2\theta}.$$

The sought result now follows.

Entry 7(iv). *If* $\tan\alpha = \sqrt{1+x}\tan\beta$, *then*

$$\int_0^\alpha \frac{d\varphi}{\sqrt{1 + x\cos 2\varphi}} = \int_0^\beta \frac{d\varphi}{\sqrt{1 - x^2\sin^4\varphi}}.$$

PROOF. In the former integral, put $\tan\varphi = \sqrt{1+x}\tan\theta$. Then elementary calculations give

$$1 + x\cos 2\varphi = \frac{(1+x)(1 - x\sin^2\theta)}{1 + x\sin^2\theta} \quad \text{and} \quad \frac{d\varphi}{d\theta} = \frac{\sqrt{1+x}}{1 + x\sin^2\theta}.$$

The desired result immediately follows.

The next result is a degenerate form of the addition theorem.

Entry 7(v). *If* $\cot\alpha\cot\beta = \sqrt{1-x}$, *then*

$$\int_0^\alpha \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}} + \int_0^\beta \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}} = \frac{\pi}{2}\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;x).$$

PROOF. Noting (6.11), we see that the proposed formula may be written

$$\int_0^\alpha \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}} = \int_\beta^{\pi/2} \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}.$$

In the former integral, make the change of variable $\cot\varphi = \sqrt{1-x}\tan\theta$. The equality above follows very easily from calculations similar to those in the foregoing entries.

Entry 7(vi). *If* $\cot\alpha\tan(\beta/2) = \sqrt{1-x\sin^2\alpha}$, *then*

$$2\int_0^\alpha \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}} = \int_0^\beta \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}.$$

PROOF. Although a proof may be given along the same lines as the previous proofs, we alternatively observe that Entry 7(vi) is a special case of the converse of Entry 7(viii), (a) below. (In fact, the conditions (a)–(d) in Entry 7(viii) are both necessary and sufficient.) To see this, replace β and γ by α and β, respectively, in this converse theorem.

In fact, Entry 7(vi) is the classical duplication formula. The next result is known as Jacobi's imaginary transformation. See Cayley's text [1, p. 68].

Entry 7(vii). *If* $\alpha = \operatorname{Log}(\tan(\pi/4 + \beta/2))$, *then*

$$\int_0^{\alpha i} \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}} = i\int_0^\beta \frac{d\varphi}{\sqrt{1-(1-x)\sin^2\varphi}}.$$

PROOF. On the left side, let $\sin\varphi = i\tan\theta$, or $\varphi = -i\operatorname{Log}((1-\sin\theta)/\cos\theta)$. Elementary calculations give

$$1 - x\sin^2\varphi = 1 + x\tan^2\theta \quad\text{and}\quad \frac{d\varphi}{d\theta} = i\sec\theta.$$

Upon substitution in the integral on the left side, we see that it remains to show that the given hypothesis is equivalent to $\alpha = \operatorname{Log}(\cos\beta/(1-\sin\beta))$. This is an elementary exercise with trigonometric identities.

Entry 7(viii) offers the addition theorem under four different sets of hypotheses. Let

$$u = \int_0^\alpha \frac{d\varphi}{\sqrt{1 - x\sin^2\varphi}}, \quad v = \int_0^\beta \frac{d\varphi}{\sqrt{1 - x\sin^2\varphi}},$$

$$\text{and} \quad w = \int_0^\gamma \frac{d\varphi}{\sqrt{1 - x\sin^2\varphi}}.$$

Then the addition theorem states that

$$u + v = w. \tag{7.1}$$

In Entry 7(viii), Ramanujan assumes (7.1) and derives four implications. As intimated in the proof of Entry 7(vi), the steps in the proofs are reversible. Thus, each of the four conditions below implies (7.1). We remark that formulation (c) below is Legendre's canonical form of the addition theorem.

It will be convenient to use the theory of the Jacobian elliptic functions as set forth, for example, in the texts of Whittaker and Watson [1, Chap. 22] or Cayley [1, Chap. 4]. In particular, heavy use will be made of the many identities connecting the Jacobian functions sn, cn, and dn.

Entry 7(viii). *If* (7.1) *holds, then*

(a) $$\tan\tfrac{1}{2}\gamma = \frac{\sin\alpha\sqrt{1 - x\sin^2\beta} + \sin\beta\sqrt{1 - x\sin^2\alpha}}{\cos\alpha + \cos\beta},$$

(b) $$\gamma = \tan^{-1}(\tan\alpha\sqrt{1 - x\sin^2\beta}) + \tan^{-1}(\tan\beta\sqrt{1 - x\sin^2\alpha}),$$

(c) $$\cot\alpha\cot\beta = \frac{\cos\gamma}{\sin\alpha\sin\beta} + \sqrt{1 - x\sin^2\gamma},$$

and

(d) $$\frac{\sqrt{\sin s\sin(s-\alpha)\sin(s-\beta)\sin(s-\gamma)}}{\sin\alpha\sin\beta\sin\gamma} = \frac{\sqrt{x}}{2},$$

where $2s = \alpha + \beta + \gamma$.

Proof of (a). From the theory of elliptic functions, it suffices to show that

$$\frac{\operatorname{sn} w}{1 + \operatorname{cn} w} = \frac{\operatorname{sn} u\operatorname{dn} v + \operatorname{sn} v\operatorname{dn} u}{\operatorname{cn} u + \operatorname{cn} v}.$$

Setting $w = u + v$, employing the addition theorems for $\operatorname{sn}(u + v)$ and $\operatorname{cn}(u + v)$ (see Cayley [1, p. 63]), cross-multiplying, and simplifying, we can establish the required identity.

Proof of (b). The proposed identity may be put in the form

$$\tan\gamma = \frac{\tan\alpha\sqrt{1 - x\sin^2\beta} + \tan\beta\sqrt{1 - x\sin^2\alpha}}{1 - \tan\alpha\tan\beta\sqrt{(1 - x\sin^2\alpha)(1 - x\sin^2\beta)}}.$$

Thus, in terms of elliptic functions, we must show that

$$\frac{\operatorname{sn}(u+v)}{\operatorname{cn}(u+v)} = \frac{\dfrac{\operatorname{sn} u}{\operatorname{cn} u}\operatorname{dn} v + \dfrac{\operatorname{sn} v}{\operatorname{cn} v}\operatorname{dn} u}{1 - \dfrac{\operatorname{sn} u \operatorname{sn} v}{\operatorname{cn} u \operatorname{cn} v}\operatorname{dn} u \operatorname{dn} v}.$$

This identity is an immediate consequence of the addition theorems for $\operatorname{sn}(u+v)$ and $\operatorname{cn}(u+v)$.

PROOF OF (c). The third equality is equivalent to

$$\frac{\operatorname{cn} u}{\operatorname{sn} u}\frac{\operatorname{cn} v}{\operatorname{sn} v} = \frac{\operatorname{cn}(u+v)}{\operatorname{sn} u \operatorname{sn} v} + \operatorname{dn}(u+v).$$

Using the addition theorems for $\operatorname{cn}(u+v)$ and $\operatorname{dn}(u+v)$ (Cayley [1, p. 63]), we easily complete the proof.

PROOF OF (d). To the identity of part (c),

$$\cos\alpha\cos\beta = \cos\gamma + \sin\alpha\sin\beta\sqrt{1 - x\sin^2\gamma},$$

add $\pm\sin\alpha\sin\beta$ to both sides to obtain, after some manipulation,

$$-2\sin\tfrac{1}{2}(\alpha\mp\beta+\gamma)\sin\tfrac{1}{2}(\alpha\mp\beta-\gamma) = \sin\alpha\sin\beta(\pm1 + \sqrt{1 - x\sin^2\gamma}).$$

Multiply these two equalities together and use the definition of s. The proposed identity readily follows.

Entry 7(ix). *If* $|x| < 1$, *then*

$$\frac{\pi}{2}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1 + x\sin\varphi}} = \int_0^{\pi/2}\frac{\cos^{-1}(x\sin^2\varphi)\,d\varphi}{\sqrt{1 - x^2\sin^4\varphi}}.$$

We shall provide two proofs, neither of which is completely satisfactory, because they are in the nature of verifications.

FIRST PROOF. Expanding $(1 + x\sin\varphi)^{-1/2}$ in a binomial series, we find that

$$\begin{aligned}\frac{\pi}{2}\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1 + x\sin\varphi}} &= \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{1}{2})_k x^k}{k!}\int_0^{\pi/2}\sin^k\varphi\,d\varphi \\ &= \frac{\pi}{4}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2}k+\frac{1}{2})x^k}{k!\,\Gamma(\frac{1}{2}k+1)}. \qquad (7.2)\end{aligned}$$

Next, set

$$y = \frac{\cos^{-1} u}{\sqrt{1 - u^2}}.$$

We want to find the Maclaurin series for y. An elementary calculation shows that y satisfies the initial value problem

$$(1-u^2)y' - uy + 1 = 0, \qquad y(0) = \pi/2.$$

Solving this problem by customary power series methods, we find that

$$\frac{\cos^{-1} u}{\sqrt{1-u^2}} = y = \frac{\pi}{2}\sum_{k=0}^{\infty} \frac{(2k)!u^{2k}}{2^{2k}(k!)^2} - \sum_{k=0}^{\infty} \frac{2^{2k}(k!)^2 u^{2k+1}}{(2k+1)!}, \tag{7.3}$$

where $|u| < 1$. Hence,

$$\begin{aligned}\int_0^{\pi/2} &\frac{\cos^{-1}(x\sin^2\varphi)}{\sqrt{1-x^2\sin^4\varphi}}\,d\varphi \\ &= \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(2k)!x^{2k}}{2^{2k}(k!)^2}\int_0^{\pi/2}\sin^{4k}\varphi\,d\varphi - \sum_{k=0}^{\infty}\frac{2^{2k}(k!)^2x^{2k+1}}{(2k+1)!}\int_0^{\pi/2}\sin^{4k+2}\varphi\,d\varphi \\ &= \frac{\pi}{4}\sum_{k=0}^{\infty}\frac{\Gamma(2k+\frac{1}{2})\Gamma(\frac{1}{2})x^{2k}}{2^{2k}(k!)^2} - \sum_{k=0}^{\infty}\frac{2^{2k-1}(k!)^2\Gamma(2k+\frac{3}{2})\Gamma(\frac{1}{2})x^{2k+1}}{\{(2k+1)!\}^2}. \end{aligned}\tag{7.4}$$

Comparing (7.2) with (7.4), we see that it suffices to show that

$$\frac{\Gamma(k+\frac{1}{2})}{(2k)!} = \frac{\Gamma(\frac{1}{2})}{2^{2k}k!}, \qquad k \geq 0,$$

and

$$\frac{\pi}{4\Gamma(k+\frac{3}{2})} = \frac{2^{2k-1}k!\Gamma(\frac{1}{2})}{(2k+1)!}, \quad k \geq 0.$$

Both equalities are immediate consequences of the duplication theorem, and the proof is complete.

Second Proof. Expanding $(1 + x\sin\theta\sin^2\varphi)^{-1}$ in a geometric series, we readily find that

$$\int_0^{\pi/2}\int_0^{\pi/2}\frac{d\theta\,d\varphi}{1+x\sin\theta\sin^2\varphi} = \frac{\pi}{4}\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2}k+\frac{1}{2})x^k}{k!\Gamma(\frac{1}{2}k+1)}.$$

Thus, from (7.2), we have shown that

$$\frac{\pi}{2}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1+x\sin\theta}} = \int_0^{\pi/2}\int_0^{\pi/2}\frac{d\theta\,d\varphi}{1+x\sin\theta\sin^2\varphi}.$$

Comparing this with Entry 7(ix), we observe that it suffices to show that

$$\int_0^{\pi/2}\frac{d\theta}{1+u\sin\theta} = \frac{\cos^{-1}u}{\sqrt{1-u^2}}, \qquad |u| < 1. \tag{7.5}$$

By expanding the left side of (7.5) in powers of u and comparing the result with (7.3), we may deduce the evaluation (7.5).

Glasser [1] has constructed tables of elliptic integrals from which Entry 7(ix) can be deduced from Table 1, formula (10).

Entry 7(x). *If* $|x| < 1$, *then*

$$\int_0^{\pi/2}\int_0^{\pi/2} \frac{d\theta\, d\varphi}{\sqrt{(1-x\sin^2\theta)(1-x\sin^2\theta\sin^2\varphi)}} = \left(\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-x\sin^4\varphi}}\right)^2.$$

PROOF. Expanding the integrand in a binomial series and integrating termwise, we find that

$$\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-x\sin^2\theta\sin^2\varphi}} = \frac{\pi}{2}\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^2}{(k!)^2} x^k \sin^{2k}\theta. \tag{7.6}$$

Proceeding in a similar manner and using the calculation above, we further find that

$$\begin{aligned}\int_0^{\pi/2}\int_0^{\pi/2} &\frac{d\theta\, d\varphi}{\sqrt{(1-x\sin^2\theta)(1-x\sin^2\theta\sin^2\varphi)}} \\ &= \frac{\pi^2}{4}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(\frac{1}{2})_k^2(\frac{1}{2})_j(\frac{1}{2})_{j+k}}{(k!)^2 j!(j+k)!} x^{j+k} \\ &= \frac{\pi^2}{4}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{(\frac{1}{2})_k^2(\frac{1}{2})_{n-k}}{(k!)^2(n-k)!}\right)\frac{(\frac{1}{2})_n}{n!}x^n.\end{aligned} \tag{7.7}$$

In the same fashion,

$$\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-x\sin^4\varphi}} = \frac{\pi}{2}\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k(\frac{1}{2})_{2k}}{k!(2k)!}x^k.$$

Hence,

$$\left(\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-x\sin^4\varphi}}\right)^2 = \frac{\pi^2}{4}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\frac{1}{2})_k(\frac{1}{2})_{2k}(\frac{1}{2})_{n-k}(\frac{1}{2})_{2n-2k}}{k!(2k)!(n-k)!(2n-2k)!}x^n. \tag{7.8}$$

Comparing (7.7) and (7.8), we see that it remains to show that

$$\begin{aligned}\sum_{k=0}^{n} &\frac{(\frac{1}{2})_k(\frac{1}{2})_{2k}(\frac{1}{2})_{n-k}(\frac{1}{2})_{2n-2k}}{k!(2k)!(n-k)!(2n-2k)!} \\ &= \frac{(\frac{1}{2})_n}{n!}\sum_{k=0}^{n} \frac{(\frac{1}{2})_k^2(\frac{1}{2})_{n-k}}{(k!)^2(n-k)!}.\end{aligned} \tag{7.9}$$

Using the elementary relations

$$(a)_{n-k} = \frac{(-1)^k(a)_n}{(-a-n+1)_k} \quad \text{and} \quad (a)_{2k} = 2^{2k}(\tfrac{1}{2}a)_k(\tfrac{1}{2}a+\tfrac{1}{2})_k, \tag{7.10}$$

we find, after some calculation and simplification, that (7.9) is equivalent to the identity

$$ {}_4F_3\left[\begin{matrix}\frac14, \frac34, -n, -n\\ \frac14 - n, \frac34 - n, 1\end{matrix}\right] = \frac{(\frac12)_n(1)_{2n}}{(\frac12)_{2n}(1)_n}\,{}_3F_2\left[\begin{matrix}\frac12, \frac12, -n\\ 1, \frac12 - n\end{matrix}\right]. \tag{7.11} $$

In order to prove (7.11), we shall combine some results from Chapter 11 with a formula connecting two terminating ${}_4F_3$'s, one of which is Saalschützian. First, from our book (Berndt [9, p. 98, lines 13, 17]), we deduce that

$$ {}_3F_2\left[\begin{matrix}\frac12, \frac12, -n\\ 1, \frac12 - n\end{matrix}\right] = \frac{(\frac34)_n^2}{(\frac12)_n(1)_n}\,{}_4F_3\left[\begin{matrix}\frac14, \frac14, -\frac12 - n, -n\\ \frac12, \frac14 - n, \frac14 - n\end{matrix}\right]. \tag{7.12} $$

From Bailey's tract [4, p. 56],

$$ {}_4F_3\left[\begin{matrix}x, y, z, -n\\ u, v, w\end{matrix}\right] = \frac{(v-z)_n(w-z)_n}{(v)_n(w)_n}\,{}_4F_3\left[\begin{matrix}u-x, u-y, z, -n\\ 1-v+z-n, 1-w+z-n, u\end{matrix}\right], \tag{7.13} $$

where $u + v + w = x + y + z - n + 1$. Let $x = -\frac12 - n$, $y = z = \frac14$, $u = w = \frac14 - n$, and $v = \frac12$. Then, from (7.12) and (7.13), we find that

$$ {}_3F_2\left[\begin{matrix}\frac12, \frac12, -n\\ 1, \frac12 - n\end{matrix}\right] = \frac{(\frac34)_n(\frac14)_n}{(\frac12)_n^2}\,{}_4F_3\left[\begin{matrix}\frac14, \frac34, -n, -n\\ \frac14 - n, \frac34 - n, 1\end{matrix}\right]. \tag{7.14} $$

Using (7.10), we can easily show that (7.11) and (7.14) are equivalent, and so the proof is complete.

Entry 7(xi). *If* $|x| < 1$, *then*

$$ \int_0^{\pi/2}\int_0^{\pi/2} \frac{x \sin\varphi \, d\theta \, d\varphi}{\sqrt{(1 - x^2\sin^2\varphi)(1 - x^2\sin^2\theta\sin^2\varphi)}} $$
$$ = \int_0^{\pi/2}\int_0^{\sin^{-1}x} \frac{d\theta\, d\varphi}{\sqrt{1 - x^2\sin^2\varphi - \sin^2\theta\cos^2\varphi}} $$
$$ = \tfrac12\left(\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1 - \frac12(1+x)\sin^2\varphi}}\right)^2 - \tfrac12\left(\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1 - \frac12(1-x)\sin^2\varphi}}\right)^2. $$

PROOF. The first equality is elementary, while the second is somewhat more recondite.

On the right side of the first equality, let $\sin\theta = x\cos\psi/\sqrt{1 - x^2\sin^2\psi}$. The limits $\theta = 0$, $\sin^{-1} x$ are sent into $\psi = \pi/2$, 0, respectively. Elementary calculations show that

$$ \frac{d\theta}{d\psi} = -\frac{x\sqrt{1 - x^2}\sin\psi}{1 - x^2\sin^2\psi} $$

and

$$ 1 - x^2\sin^2\varphi - \sin^2\theta\cos^2\varphi = \frac{(1 - x^2)(1 - x^2\sin^2\varphi\sin^2\psi)}{1 - x^2\sin^2\psi}. $$

Using these equalities in the integral on the right side of the first equality in Entry 7(xi), we complete the proof.

To prove the second equality in Entry 7(xi), we first expand by using (7.6) and then expand again via the binomial series to find that

$$\begin{aligned}
I &:= \int_0^{\pi/2}\int_0^{\pi/2} \frac{x\sin\varphi\, d\theta\, d\varphi}{\sqrt{(1-x^2\sin^2\varphi)(1-x^2\sin^2\theta\sin^2\varphi)}} \\
&= \frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k^2 x^{2k+1}}{(k!)^2}\sum_{j=0}^{\infty}\frac{(\frac{1}{2})_j x^{2j}}{j!}\int_0^{\pi/2}\sin^{2j+2k+1}\varphi\, d\varphi \\
&= \frac{\pi}{2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(\frac{1}{2})_k^2(\frac{1}{2})_j(j+k)!}{(k!)^2 j!(\frac{3}{2})_{j+k}}x^{2j+2k+1} \\
&= \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(\frac{1}{2})_k^2(\frac{1}{2})_{n-k}}{(k!)^2(n-k)!}\right)\frac{n!}{(\frac{3}{2})_n}x^{2n+1} \\
&= \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{(\frac{3}{2})_n}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, -n\\ 1, \frac{1}{2}-n\end{matrix}\right]x^{2n+1},
\end{aligned}$$

where we have used (7.10). Employing two results from Chapter 11 of our book (Berndt [9, p. 98, line 17; p. 97, Entry 34(iii)]), we deduce that

$$\begin{aligned}
I &= \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{n!}{(\frac{3}{2})_n}\,{}_3F_2\left[\begin{matrix}\frac{1}{2}, \frac{1}{2}, -n\\ 1, 1\end{matrix}\right]x^{2n+1} \\
&= \frac{\pi^2}{8}\{{}_2F_1{}^2(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}(1+x)) - {}_2F_1{}^2(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}(1-x))\}.
\end{aligned}$$

Using (6.11), we finish the proof.

Our proofs of Entries 7(x) and 7(xi) are undoubtedly not those found by Ramanujan. However, our proofs do depend on results from Chapter 11, and so possibly Ramanujan might have started with these theorems on hypergeometric series and then was led to elliptic integrals.

Entry 7(xii). *If* $(1 + x\sin^2\alpha)\sin\beta = (1+x)\sin\alpha$, *then*

$$(1+x)\int_0^{\alpha}\frac{d\varphi}{\sqrt{1-x^2\sin^2\varphi}} = \int_0^{\beta}\frac{d\varphi}{\sqrt{1-\dfrac{4x}{(1+x)^2}\sin^2\varphi}}.$$

PROOF. In the integrand on the right side, make the substitution $\sin\varphi = (1+x)\sin\theta/(1+x\sin^2\theta)$. Then elementary calculations yield

$$\frac{d\varphi}{d\theta} = \frac{(1+x)(1-x\sin^2\theta)}{\sqrt{1-x^2\sin^2\theta}\,(1+x\sin^2\theta)}$$

and

$$\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\varphi}=\frac{1-x\sin^2\theta}{1+x\sin^2\theta}.$$

The desired result now follows.

Entry 7(xii) is known as Gauss' transformation, while Entry 7(xiii), which is very similar in appearance, is called Landen's transformation [1], [2].

Entry 7(xiii). *If* $x \sin \alpha = \sin(2\beta - \alpha)$, *then*

$$(1+x)\int_0^\alpha \frac{d\varphi}{\sqrt{1-x^2\sin^2\varphi}} = 2\int_0^\beta \frac{d\varphi}{\sqrt{1-\dfrac{4x}{(1+x)^2}\sin^2\varphi}}.$$

PROOF. In the latter integral, let $x \sin\theta = \sin(2\varphi - \theta)$, or $\varphi = \frac{1}{2}(\sin^{-1}(x\sin\theta) + \theta)$. Then

$$2\frac{d\varphi}{d\theta}=\frac{x\cos\theta+\sqrt{1-x^2\sin^2\theta}}{\sqrt{1-x^2\sin^2\theta}}$$

and

$$\sqrt{1-\frac{4x}{(1+x)^2}\sin^2\varphi}=\frac{x\cos\theta+\sqrt{1-x^2\sin^2\theta}}{1+x},$$

and the proof is complete.

In his "lost notebook" [11], Ramanujan recorded many deep results on elliptic integrals. See a paper by S. Raghavan and S. S. Rangachari [1] for proofs of several of the these beautiful theorems.

Much, of course, has been written about elliptic integrals. The most complete tables have been compiled by Byrd and Friedman [1]. Other sources are tables of Gradshteyn and Ryzhik [1] and Glasser [1].

In the sequel, we shall be rearranging the terms of absolutely convergent double series. To describe the different rearrangements, we employ the terminology of MacMahon [1, pp. 26–32]. Let us set forth the terms of a double series $\sum_{m,n=1}^{\infty} a_{mn}$ in an array

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots. \\ \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

The two most common methods of summation are by *rows* and by *columns*. It sometimes will be convenient firstly to sum the first row, secondly to sum the remainder of the first column, thirdly to sum the remainder of the second row, then to sum the remainder of the second column, and so on. This is called the *row–column* method of summation. Similarly, we may want first to sum the first column, then the remainder of the first row, then the remainder of the second column, and so on. This is called the *column–row* method of summation. It is occasionally convenient, especially when $a_{mn} = a_{nm}$, $1 \le m, n < \infty$, to sum first all elements in the first row and first column, next to sum the remaining elements in the second row and second column, and so on. We designate this procedure the *Clausen* transformation.

In Chapter 16, Ramanujan studied the general theta-function

$$f(a, b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \qquad |ab| < 1.$$

Recall from Entry 22 of Chapter 16 that $\varphi(q) = f(q, q)$, $\psi(q) = f(q, q^3)$, and

$$f(-q) = f(-q, -q^2). \tag{8.1}$$

We shall quote extensively from Chapter 16 in the sequel.

Entry 8. *We have*

(i) $$\varphi^2(q) = 1 + 4 \sum_{k=0}^{\infty} \frac{(-1)^k q^{2k+1}}{1 - q^{2k+1}},$$

(ii) $$\varphi^4(q) = 1 + 8 \sum_{k=1}^{\infty} \frac{kq^k}{1 + (-q)^k},$$

(iii) $$\varphi(q)\varphi(q^2) = 1 - 2 \sum_{k=1}^{\infty} \frac{(-1)^{k(k+1)/2} q^{2k-1}}{1 - q^{2k-1}},$$

(iv) $$\varphi(q)\varphi(q^3) = 1 + 2 \sum_{k=1}^{\infty} \left(\frac{k}{3}\right) \frac{q^k}{1 + (-q)^k},$$

where $(k/3)$ *denotes the Legendre symbol,*

(v) $$\varphi^2(-q) = 1 + 4 \sum_{k=1}^{\infty} \frac{(-1)^k q^{k(k+1)/2}}{1 + q^k},$$

(vi) $$\psi(q)\varphi(q^2) = \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)/2} \frac{1 + q^{2k+1}}{1 - q^{2k+1}},$$

(vii) $$\psi^2(q) = \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)} \frac{1 + q^{2k+1}}{1 - q^{2k+1}},$$

(viii) $$\sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k} = \sum_{k=1}^{\infty} (-1)^{k+1} q^{k(k+1)/2} \frac{1 + q^k}{(1 - q^k)^2},$$

(ix) $$\varphi^2(-q)f(-q) = \sum_{k=-\infty}^{\infty} (6k + 1) q^{(3k^2+k)/2},$$

(x) $$\psi(q^2)f^2(-q) = \sum_{k=-\infty}^{\infty} (3k+1)q^{3k^2+2k},$$

(xi) $$f(-q)f(-q^2) = \varphi(-q)\psi(q),$$

and

(xii) $$\frac{f(-q)}{f(-q^4)} = \frac{\varphi(-q^2)}{\psi(q)}.$$

PROOF OF (i). In Entry 33(iii) of Chapter 16, let $n = \pi/2$ and replace q^2 by $-q$. We immediately find that

$$\varphi^2(q) = 1 + 4\sum_{k=1}^{\infty} \frac{q^k}{1+q^{2k}}. \tag{8.2}$$

Expand $1/(1+q^{2k})$, $1 \le k < \infty$, in a geometric series so that we obtain a double series above. Summing this double series by columns, we find that

$$\sum_{k=1}^{\infty} \frac{q^k}{1+q^{2k}} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}q^{2k-1}}{1-q^{2k-1}}.$$

This completes the proof of (i), which is due to Jacobi [1], [2].

Bhargava and Adiga [5] have used Entry 17 in Chapter 16 to give simple proofs of Entries 8(i), (ii).

PROOF OF (ii). In (33.5) of Chapter 16, replace q^2 by $-q$ to find that

$$\varphi^4(q) = 1 + 8\sum_{k=1}^{\infty} \frac{q^k}{(1+(-q)^k)^2}.$$

Writing the series on the right side as a double series and summing by columns, we complete the proof of Entry 8(ii), which again is due to Jacobi [1], [2].

PROOF OF (iii). Applying the corollary in Section 33, Chapter 16 with $a = q$ and $b = q^3$, we find that

$$\frac{f(q,q^3)}{f(-q,-q^3)}\varphi^2(-q^4) = 1 + 2\sum_{k=1}^{\infty} \frac{q^k+q^{3k}}{1+q^{4k}}. \tag{8.3}$$

Using Entries 25(iii) and 24(i) in Chapter 16, we deduce that

$$\frac{f(q,q^3)}{f(-q,-q^3)}\varphi^2(-q^4) = \frac{\psi(q)}{\psi(-q)}\varphi(q^2)\varphi(-q^2)$$

$$= \varphi(q^2)\sqrt{\frac{\varphi(q)}{\varphi(-q)}}\sqrt{\varphi(q)\varphi(-q)} = \varphi(q^2)\varphi(q). \tag{8.4}$$

Writing the series on the right side of (8.3) as a double series, we find that it is represented by the array

$$\begin{array}{cccccccc} q & q^3 & -q^5 & -q^7 & q^9 & q^{11} & \cdots \\ q^2 & q^6 & -q^{10} & -q^{14} & q^{18} & q^{22} & \cdots \\ q^3 & q^9 & -q^{15} & -q^{21} & q^{27} & q^{33} & \cdots. \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Summing by columns and using (8.4) also in (8.3), we complete the proof.

PROOF OF (iv). Putting $a = q$ and $b = -q^2$ in the corollary of Section 33 of Chapter 16, we find that

$$\varphi^2(q^3)\frac{f(q, -q^2)}{f(-q, q^2)} = 1 + 2\sum_{k=1}^{\infty} \frac{q^k + (-q^2)^k}{1 + (-q)^{3k}}.$$

Using (8.1) and Entries 30(iv) and 24(iii) in Chapter 16, we have

$$\varphi^2(q^3)\frac{f(q, -q^2)}{f(-q, q^2)} = \frac{f^2(q)\varphi^2(q^3)}{f(-q^2)\varphi(q^3)} = \varphi(q)\varphi(q^3).$$

Summing by columns, we deduce that

$$\sum_{k=1}^{\infty} \frac{q^k + (-q^2)^k}{1 + (-q)^{3k}} = \sum_{k=1}^{\infty} \left(\frac{k}{3}\right)\frac{q^k}{1 + (-q)^k}.$$

The desired result now follows.

PROOF OF (v). From (8.2),

$$\varphi^2(-q) = 1 + 4\sum_{k=1}^{\infty} \frac{(-q)^k}{1 + q^{2k}}.$$

Summing by the column–row method, we complete the proof of this result, originally discovered by Jacobi [1], [2, p. 187].

PROOFS OF (vi), (vii). From the same corollary in Section 33 of Chapter 16,

$$\varphi^2(ab)\frac{f(-a, b)}{f(a, -b)} = 1 + 2\sum_{k=1}^{\infty} \frac{(-a)^k + b^k}{1 + (-ab)^k}.$$

Reversing the roles of a and b and subtracting the two equalities yields

$$\varphi^2(ab)\left\{\frac{f(a, -b)}{f(-a, b)} - \frac{f(-a, b)}{f(a, -b)}\right\} = 4\sum_{k=1}^{\infty} \frac{a^{2k-1} - b^{2k-1}}{1 - (ab)^{2k-1}}.$$

Applying Entries 30(iv) and (vi) in Chapter 16, we deduce that

$$\begin{aligned} a\frac{f(-b/a, -a^3b)}{f(-a^2, -b^2)}\varphi(ab)\psi(a^2b^2) &= \sum_{k=1}^{\infty} \frac{a^{2k-1} - b^{2k-1}}{1 - (ab)^{2k-1}} \\ &= \sum_{k=1}^{\infty} \left(\frac{a^k b^{k-1}}{1 - a^{2k}b^{2k-2}} - \frac{a^{k-1}b^k}{1 - a^{2k-2}b^{2k}}\right), \end{aligned} \tag{8.5}$$

where we have summed by columns.

Letting $b = a^3$ in (8.5) and employing Entry 25(iv) of Chapter 16, we arrive at

$$\psi^2(a^4) = \varphi(a^4)\psi(a^8) = \sum_{k=0}^{\infty} \frac{a^{4k}}{1 - a^{8k+2}} - \sum_{k=0}^{\infty} \frac{a^{4k+2}}{1 - a^{8k+6}}. \tag{8.6}$$

Replacing a by ia, we get

$$\psi^2(a^4) = \varphi(a^4)\psi(a^8) = \sum_{k=0}^{\infty} \frac{a^{4k}}{1 + a^{8k+2}} + \sum_{k=0}^{\infty} \frac{a^{4k+2}}{1 + a^{8k+6}}.$$

Adding the latter two equalities and replacing a^4 by q, we arrive at

$$\psi^2(q) = \varphi(q)\psi(q^2) = \sum_{k=0}^{\infty} \frac{q^k}{1 - q^{4k+1}} - \sum_{k=0}^{\infty} \frac{q^{3k+2}}{1 - q^{4k+3}}. \tag{8.7}$$

Applying Clausen's transformation to each series on the right side, we conclude that

$$\begin{aligned}\psi^2(q) &= \sum_{k=0}^{\infty} q^{2k(2k+1)} \frac{1 + q^{4k+1}}{1 - q^{4k+1}} - \sum_{k=0}^{\infty} q^{(2k+1)(2k+2)} \frac{1 + q^{4k+3}}{1 - q^{4k+3}} \\ &= \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)} \frac{1 + q^{2k+1}}{1 - q^{2k+1}},\end{aligned}$$

which is Entry 8(vii).

Next, replace q by $\pm\sqrt{q}$ in (8.7) and add the two equalities to get, by Entry 25(i),

$$\begin{aligned}\psi(q)\varphi(q^2) &= \tfrac{1}{2}\psi(q)\{\varphi(\sqrt{q}) + \varphi(-\sqrt{q})\} \\ &= \tfrac{1}{2}\sum_{k=0}^{\infty}\left\{\frac{q^{k/2}}{1 - q^{2k+1/2}} + \frac{(-1)^k q^{k/2}}{1 + q^{2k+1/2}}\right\} - \tfrac{1}{2}\sum_{k=0}^{\infty}\left\{\frac{q^{3k/2+1}}{1 - q^{2k+3/2}} + \frac{(-1)^k q^{3k/2+1}}{1 + q^{2k+3/2}}\right\} \\ &= \tfrac{1}{2}\sum_{k=0}^{\infty}\left\{\frac{q^k}{1 - q^{4k+1/2}} + \frac{q^k}{1 + q^{4k+1/2}}\right\} + \tfrac{1}{2}\sum_{k=0}^{\infty}\left\{\frac{q^{k+1/2}}{1 - q^{4k+5/2}} - \frac{q^{k+1/2}}{1 + q^{4k+5/2}}\right\} \\ &\quad - \tfrac{1}{2}\sum_{k=0}^{\infty}\left\{\frac{q^{3k+1}}{1 - q^{4k+3/2}} + \frac{q^{3k+1}}{1 + q^{4k+3/2}}\right\} - \tfrac{1}{2}\sum_{k=0}^{\infty}\left\{\frac{q^{3k+5/2}}{1 - q^{4k+7/2}} - \frac{q^{3k+5/2}}{1 + q^{4k+7/2}}\right\} \\ &= \sum_{k=0}^{\infty} \frac{q^k}{1 - q^{8k+1}} + \sum_{k=0}^{\infty} \frac{q^{5k+3}}{1 - q^{8k+5}} - \sum_{k=0}^{\infty} \frac{q^{3k+1}}{1 - q^{8k+3}} - \sum_{k=0}^{\infty} \frac{q^{7k+6}}{1 - q^{8k+7}} \\ &= \sum_{k=0}^{\infty} q^{2k(4k+1)} \frac{1 + q^{8k+1}}{1 - q^{8k+1}} + \sum_{k=0}^{\infty} q^{(2k+1)(4k+3)} \frac{1 + q^{8k+5}}{1 - q^{8k+5}} \\ &\quad - \sum_{k=0}^{\infty} q^{(2k+1)(4k+1)} \frac{1 + q^{8k+3}}{1 - q^{8k+3}} - \sum_{k=0}^{\infty} q^{(2k+2)(4k+3)} \frac{1 + q^{8k+7}}{1 - q^{8k+7}} \\ &= \sum_{k=0}^{\infty} (-1)^k q^{k(k+1)/2} \frac{1 + q^{2k+1}}{1 - q^{2k+1}},\end{aligned}$$

where, in the penultimate equality, we transformed the series by Clausen's method. Thus, (vi) is established.

PROOF OF (viii). Consider the series

$$\sum_{k=1}^{\infty}\left\{\frac{2q^{2k-1}}{(1-q^{2k-1})^2}-\frac{(2k-1)q^{2k-1}}{1-q^{2k-1}}\right\}$$
$$=\sum_{k=1}^{\infty}(-1)^{k+1}q^{k(k+1)/2}\frac{1+q^k}{(1-q^k)^2},$$

where we have transformed the series on the left side by the row–column method. On the other hand, summing by columns, we get

$$\sum_{k=1}^{\infty}\frac{q^{2k-1}}{(1-q^{2k-1})^2}=\sum_{k=1}^{\infty}\frac{kq^k}{1-q^{2k}}.$$

Hence, combining these equalities, we deduce that

$$\sum_{k=1}^{\infty}(-1)^{k+1}q^{k(k+1)/2}\frac{1+q^k}{(1-q^k)^2}$$
$$=2\sum_{k=1}^{\infty}\frac{kq^k}{1-q^{2k}}-\sum_{k=1}^{\infty}\frac{(2k-1)q^{2k-1}}{1-q^{2k-1}}$$
$$=2\sum_{k=1}^{\infty}k\left(\frac{q^k}{1-q^k}-\frac{q^{2k}}{1-q^{2k}}\right)-\sum_{k=1}^{\infty}\frac{(2k-1)q^{2k-1}}{1-q^{2k-1}}$$
$$=\sum_{k=1}^{\infty}\frac{2kq^k}{1-q^k}-\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}$$
$$=\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k},$$

which completes the proof.

PROOF OF (ix). From the proof of the first version of Corollary (ii), Section 34 of Chapter 16,

$$(q;q)_\infty(qz;q)_\infty(q/z;q)_\infty(qz^2;q^2)_\infty(q/z^2;q^2)_\infty$$
$$=\sum_{k=-\infty}^{\infty}q^{(3k^2+k)/2}\frac{\sin\{(6k+1)n\}}{\sin n},$$

where $z=\exp(2in)$. Letting n tend to 0, we find that

$$\sum_{k=-\infty}^{\infty}(6k+1)q^{(3k^2+k)/2}=(q;q)_\infty^3(q;q^2)_\infty^2=\varphi^2(-q)f(-q),$$

by Entry 22(iii) and (22.4) in Chapter 16.

PROOF OF (x). From the proof of the first version of Corollary (i), Section 34 of Chapter 16,

$$(q^2; q^2)_\infty(qz; q^2)_\infty(q/z; q^2)_\infty(q^4z^2; q^4)_\infty(q^4/z^2; q^4)_\infty$$
$$= \sum_{k=-\infty}^{\infty} q^{3k^2+2k}\frac{\sin\{2(3k+1)n\}}{\sin(2n)},$$

where $z = \exp(2in)$. Letting n tend to 0 yields

$$\sum_{k=-\infty}^{\infty} (3k+1)q^{3k^2+2k} = (q^2; q^2)_\infty(q; q^2)_\infty^2(q^4; q^4)_\infty^2$$
$$= \frac{(q^4; q^4)_\infty}{(q^2; q^4)_\infty}(q; q)_\infty^2$$
$$= \psi(q^2)f^2(-q),$$

by Entries 22(ii), (iii) in Chapter 16.

Proof of (xi). This result is contained in Entry 24(iii) of Chapter 16.

Proof of (xii). Using first Entry 8(xi) above and then Entries 25(iii), (iv) in Chapter 16, we find that

$$\frac{f(-q)}{f(-q^4)} = \frac{\varphi(-q)\psi(q)}{\varphi(-q^2)\psi(q^2)} = \frac{\varphi^2(-q^2)}{\varphi(q)\psi(-q)}$$
$$= \frac{\varphi^2(-q^2)\psi(q^2)}{\psi^2(q)\psi(-q)} = \frac{\varphi(-q^2)}{\psi(q)}.$$

Example.

$$\psi(q^2)f^2(-q) + 2q\psi(q^8)f^2(-q^4) = \varphi^2(-q^8)f(-q^8).$$

We give two proofs. The first and shorter proof is probably the one that Ramanujan had. The second proof shows that the result is not as deep as the first proof indicates.

First Proof. By Entry 8(x),

$$\psi(q^2)f^2(-q) + 2q\psi(q^8)f^2(-q^4)$$
$$= \sum_{k=-\infty}^{\infty} (3k+1)q^{3k^2+2k} + 2q\sum_{k=-\infty}^{\infty} (3k+1)q^{12k^2+8k}$$
$$= \sum_{k=-\infty}^{\infty} (6k+1)q^{12k^2+4k} + \sum_{k=-\infty}^{\infty} (-6k-2)q^{12k^2+8k+1}$$
$$+ 2q\sum_{k=-\infty}^{\infty} (3k+1)q^{12k^2+8k}$$
$$= \sum_{k=-\infty}^{\infty} (6k+1)q^{12k^2+4k}$$
$$= \varphi^2(-q^8)f(-q^8),$$

by Entry 8(ix).

SECOND PROOF. By (3.1) and (5.1), we first observe that

$$\varphi(-q) + 2q\psi(q^8) = 1 + 2\sum_{k=1}^{\infty}(-1)^k q^{k^2} + 2\sum_{k=0}^{\infty} q^{(2k+1)^2}$$
$$= 1 + 2\sum_{k=1}^{\infty} q^{4k^2} = \varphi(q^4).$$

Multiplying both sides by $f^2(-q^4)$ then gives us

$$\varphi(-q)f^2(-q^4) + 2q\psi(q^8)f^2(-q^4) = \varphi(q^4)f^2(-q^4). \tag{8.8}$$

By using the product formulas from Entry 22 of Chapter 16, we can easily show that

$$\varphi(-q)f^2(-q^4) = \psi(q^2)f^2(-q)$$

and

$$\varphi(q^4)f^2(-q^4) = \varphi^2(-q^8)f(-q^8).$$

Substituting these equalities into (8.8), we finish the second proof.

Entry 9. *Recall that x, y, and z are related by* (2.1) *and* (6.2)–(6.4). *Then*

(i) $$\frac{dy}{dx} = -\frac{1}{x(1-x)z^2},$$

(ii) $$\frac{dz}{dx} = \frac{\int_0^z z\,dx}{4x(1-x)},$$

(iii) $$z\int_0^x \left\{\int_0^u t^{n-1}z(t)\,dt\right\}\frac{du}{u(1-u)z^2(u)} = \frac{x^n}{n^2}\,{}_3F_2\left[\begin{matrix} n+\frac{1}{2}, n+\frac{1}{2}, 1 \\ n+1, n+1 \end{matrix}; x\right],$$

where $n > 0$, *and*

(iv) $$1 - 24\sum_{k=1}^{\infty}\frac{k}{e^{2ky}-1} = (1-2x)z^2 + 6x(1-x)z\frac{dz}{dx}.$$

PROOF OF (i). This formula is the special case $n = \frac{1}{2}$ of the corollary in Section 30 of Chapter 11 (Part II [9, p. 88]).

PROOF OF (ii). By L'Hospital's rule,

$$\lim_{x\to 0}\frac{\int_0^z z\,dx}{4x(1-x)} = \frac{1}{4}.$$

Also, $z'(0) = \frac{1}{4}$. Thus, in order to prove that (ii) holds, it suffices to show that the derivatives of both sides of (ii) are equal. Multiplying both sides of (ii) by $4x(1-x)$ and then differentiating the resulting equality, we find that

$$4(1-x)z' - 4xz' + 4x(1-x)z'' = z. \tag{9.1}$$

But (see Bailey's tract [4, p. 1]), this is precisely the hypergeometric differential equation satisfied by $z = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$, and so the proof is complete. (In Entry

31(i) of Chapter 11, Ramanujan states an equivalent form of the hypergeometric differential equation.)

PROOF OF (iii). This result is a special case of Entry 31(ii) in Chapter 11 (Part II [9, p. 88]). In particular, set $\alpha = \beta = \frac{1}{2}$ and $\gamma = \delta = 1$ and replace n by $n + 1$ to obtain the present result.

PROOF OF (iv). In the derivation below, we employ the following results from Chapter 16: Entry 22(iii), namely, $f(-q) = (q; q)_\infty$, Entry 24(iv), Entry 25(iv), and Entry 25(vii). We also use (6.10) and (6.4). Accordingly, we find that

$$\begin{aligned}
1 - 24 \sum_{k=1}^{\infty} \frac{k}{e^{2ky} - 1} &= 1 - 12 \frac{d}{dy} \sum_{k=1}^{\infty} \operatorname{Log}(1 - e^{-2ky}) \\
&= 1 - 12 \frac{d}{dy} \operatorname{Log} \prod_{k=1}^{\infty} (1 - e^{-2ky}) \\
&= -4 \frac{d}{dy} \operatorname{Log}\{e^{-y/4} f^3(-e^{-2y})\} \\
&= -4 \frac{d}{dy} \operatorname{Log}\{e^{-y/4} \varphi(-e^{-y}) \psi^2(e^{-y})\} \\
&= -4 \frac{d}{dy} \operatorname{Log}\{e^{-y/4} \varphi(-e^{-y}) \varphi(e^{-y}) \psi(e^{-2y})\} \\
&= -4 \frac{d}{dy} \operatorname{Log}\{\tfrac{1}{2} \varphi(-e^{-y}) \varphi(e^{-y}) (\varphi^4(e^{-y}) - \varphi^4(-e^{-y}))^{1/4}\} \\
&= -4 \frac{d}{dy} \operatorname{Log}\{\tfrac{1}{2} \varphi(-e^{-y}) \varphi^2(e^{-y}) x^{1/4}\} \\
&= -4 \frac{d}{dy} \operatorname{Log}\{\tfrac{1}{2} (1 - x)^{1/4} z^{3/2} x^{1/4}\} \\
&= -\frac{d}{dy} \operatorname{Log}\{(1 - x) x z^6\} \\
&= x(1 - x) z^2 \frac{d}{dx} \operatorname{Log}\{(1 - x) x z^6\} \\
&= -x z^2 + (1 - x) z^2 + 6x(1 - x) z \frac{dz}{dx},
\end{aligned}$$

which completes the proof. Note that in the penultimate line we employed Entry 9(i).

In the notation of Section 9 of Chapter 15,

$$L(e^{-2y}) = 1 - 24 \sum_{k=1}^{\infty} \frac{k}{e^{2ky} - 1}. \tag{9.2}$$

Thus, we have shown that

$$L(e^{-2y}) = (1-2x)z^2 + 6x(1-x)z\frac{dz}{dx}. \tag{9.3}$$

Example. *For* $y > 0$,

$$16^{11}e^{-11y} = x^{11}(1 + \tfrac{11}{2}x + \tfrac{1111}{64}x^2 + \tfrac{111111}{2688}x^3 + \cdots).$$

This example is more properly placed in Section 2. It is not clear why Ramanujan was led to examine $\exp(-11y)$.

Proof. Replacing x by $x/(2-x)$ in Entry 2(vi), we find that, for $|x| < 1$,

$$F(x) = \frac{x}{8(2-x)} + \frac{5}{128}\left(\frac{x}{2-x}\right)^3 + O(x^5)$$

or

$$16F(x) = x + \tfrac{1}{2}x^2 + \tfrac{21}{64}x^3 + \tfrac{31}{128}x^4 + O(x^5).$$

Using (2.1) and raising each side to the 11th power, we arrive at the desired formula after a moderate amount of calculation.

Although the results in Entries 10–12 are easy to prove, their importance cannot be overestimated, for we shall utilize them many times in proving Ramanujan's modular equations in Chapters 19–21.

Entry 10. *If* x, y, *and* z *are related by* (6.2)–(6.4), *then*

(i) $$\varphi(e^{-y}) = \sqrt{z},$$

(ii) $$\varphi(-e^{-y}) = \sqrt{z}(1-x)^{1/4},$$

(iii) $$\varphi(-e^{-2y}) = \sqrt{z}(1-x)^{1/8},$$

(iv) $$\varphi(e^{-2y}) = \sqrt{z}(\tfrac{1}{2}(1+\sqrt{1-x}))^{1/2},$$

(v) $$\varphi(e^{-4y}) = \tfrac{1}{2}\sqrt{z}(1+(1-x)^{1/4}),$$

(vi) $$\varphi(e^{-y/2}) = \sqrt{z}(1+\sqrt{x})^{1/2},$$

(vii) $$\varphi(-e^{-y/2}) = \sqrt{z}(1-\sqrt{x})^{1/2},$$

(viii) $$\varphi(e^{-y/4}) = \sqrt{z}(1+x^{1/4}),$$

and

(ix) $$\varphi(-e^{-y/4}) = \sqrt{z}(1-x^{1/4}).$$

Proof. Part (i) repeats (6.4).

Part (ii) follows from (6.10) and part (i).

For (iii)–(v), we employ the identities

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2) \tag{10.1}$$

and

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), \tag{10.2}$$

found in Chapter 16, Entries 25(iii) and (vi), respectively. (These identities were also established by Jacobi [1, 2, Section 37].)

Part (iii) follows at once from (10.1) and parts (i) and (ii).

Part (iv) is an immediate consequence of (10.2) and parts (i) and (ii).

Part (v) arises from (10.2) and parts (iii) and (iv).

Using (10.1) and (10.2), we readily can show that

$$\varphi(q) \pm \varphi(-q) = \sqrt{2}\{\varphi^2(q^2) \pm \varphi^2(-q^2)\}^{1/2}.$$

Hence,

$$\varphi(\pm q) = \frac{1}{\sqrt{2}}(\{\varphi^2(q^2) + \varphi^2(-q^2)\}^{1/2} \pm \{\varphi^2(q^2) - \varphi^2(-q^2)\}^{1/2}), \tag{10.3}$$

which will be used to establish parts (vi)–(ix).

To prove both (vi) and (vii), we use (10.3) along with parts (i) and (ii).

Lastly, parts (viii) and (ix) follow from (10.3) with the help of parts (vi) and (vii).

Entry 11. *Recall that* $\psi(q)$ *is defined by* (5.1). *Then*

(i) $$\psi(e^{-y}) = \sqrt{\tfrac{1}{2}z}(xe^y)^{1/8},$$

(ii) $$\psi(-e^{-y}) = \sqrt{\tfrac{1}{2}z}\{x(1-x)e^y\}^{1/8},$$

(iii) $$\psi(e^{-2y}) = \tfrac{1}{2}\sqrt{z}(xe^y)^{1/4},$$

(iv) $$\psi(e^{-4y}) = \tfrac{1}{2}\sqrt{\tfrac{1}{2}z}\{(1-\sqrt{1-x})e^y\}^{1/2},$$

(v) $$\psi(e^{-8y}) = \tfrac{1}{4}\sqrt{z}\{1-(1-x)^{1/4}\}e^y,$$

(vi) $$\psi(e^{-y/2}) = \sqrt{z}\{\tfrac{1}{2}(1+\sqrt{x})\}^{1/4}(xe^y)^{1/16},$$

(vii) $$\psi(-e^{-y/2}) = \sqrt{z}\{\tfrac{1}{2}(1-\sqrt{x})\}^{1/4}(xe^y)^{1/16},$$

(viii) $$\psi(e^{-y/4}) = \sqrt{z}(1+x^{1/4})^{1/2}\{\tfrac{1}{2}(1+\sqrt{x})\}^{1/8}(xe^y)^{1/32},$$

and

(ix) $$\psi(-e^{-y/4}) = \sqrt{z}(1-x^{1/4})^{1/2}\{\tfrac{1}{2}(1+\sqrt{x})\}^{1/8}(xe^y)^{1/32}.$$

Proof. Our proofs depend on employing the following formulas from Entry 25 of Chapter 16:

$$\psi(q^2) = \tfrac{1}{2}q^{-1/4}(\varphi^4(q) - \varphi^4(-q))^{1/4}, \tag{11.1}$$

$$\psi(q^8) = \frac{1}{4q}(\varphi(q) - \varphi(-q)), \tag{11.2}$$

and

$$\psi(q) = \sqrt{\varphi(q)\psi(q^2)}, \tag{11.3}$$

in conjunction with Entry 10.

Part (i) follows from (11.1) and Entries 10(vi), (vii).

To prove (iii), employ (11.1) along with Entries 10(i), (ii).

Part (ii) follows on using (11.3), Entry 10(ii), and Entry 11(iii).

Using (11.2) along with Entries 10(vi), (vii), we may deduce (iv).

Use (11.2) and Entries 10(i), (ii) to easily deduce (v).

To establish (vi), use (11.3), Entry 10(vi), and Entry 11(i).

The proof of (vii) is identical with that of (vi), except that Entry 10(vii) is used instead of Entry 10(vi).

To prove (viii), employ (11.3) along with Entry 10(viii) and Entry 11(vi).

The proof of (ix) is identical with that of (viii), except that Entry 10(ix) is used instead of Entry 10(viii).

Entry 12. *Let f be defined by* (8.1) *and recall from Entry* 22 *in Chapter* 16 *the definition*

$$\chi(q) = (-q; q^2)_\infty. \tag{12.1}$$

Then

(i) $$f(e^{-y}) = \sqrt{z}2^{-1/6}\{x(1-x)e^y\}^{1/24},$$

(ii) $$f(-e^{-y}) = \sqrt{z}2^{-1/6}(1-x)^{1/6}(xe^y)^{1/24},$$

(iii) $$f(-e^{-2y}) = \sqrt{z}2^{-1/3}\{x(1-x)e^y\}^{1/12},$$

(iv) $$f(-e^{-4y}) = \sqrt{z}4^{-1/3}(1-x)^{1/24}(xe^y)^{1/6},$$

(v) $$\chi(e^{-y}) = 2^{1/6}\{x(1-x)e^y\}^{-1/24},$$

(vi) $$\chi(-e^{-y}) = 2^{1/6}(1-x)^{1/12}(xe^y)^{-1/24},$$

and

(vii) $$\chi(-e^{-2y}) = 2^{1/3}(1-x)^{1/24}(xe^y)^{-1/12}.$$

PROOF. We employ the relations

$$f^3(-q) = \varphi^2(-q)\psi(q), \tag{12.2}$$

$$f^3(-q^2) = \varphi(-q)\psi^2(q), \tag{12.3}$$

and

$$\chi(q) = \frac{\varphi(q)}{f(q)}, \tag{12.4}$$

which are contained in Entries 24(ii), (iv), and (iii), respectively, in Chapter 16.

The proof of (i) uses (12.2) and Entries 10(i) and 11(ii).
To prove (ii), use (12.2) and Entries 10(ii) nad 11(i).
Employ (12.3) and Entries 10(ii) and 11(i) to prove (iii).
To prove (iv), use (12.3) and Entries 10(iii) and 11(iii).
Use (12.4) and Entries 10(i) and 12(i) to establish (v).
Part (vi) follows from (12.4) and Entries 10(ii) and 12(ii).
To prove (vii), employ (12.4) and Entries 10(iii) and 12(iii).

Before proceeding further, we describe three procedures in the theory of elliptic functions by which "new" formulas can be produced from "old" formulas.

Consider a formula of the form

$$\Omega(x, e^{-y}, z) = 0, \tag{13.1}$$

and suppose that x', y', and z' is another set of parameters such that

$$\Omega(x', e^{-y'}, z') = 0$$

and

$$x = \frac{4\sqrt{x'}}{(1+\sqrt{x'})^2}.$$

Solving for x', we find that

$$x' = \left(\frac{2 - x - 2\sqrt{1-x}}{x}\right)^2 = \left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2. \tag{13.2}$$

From Entry 2(v),

$$e^{-y} = F(x) = F\left(\frac{4\sqrt{x'}}{(1+\sqrt{x'})^2}\right) = \sqrt{F(x')} = e^{-y'/2},$$

that is, $y' = 2y$. From (2.7),

$$\begin{aligned} z = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x) &= {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{4\sqrt{x'}}{(1+\sqrt{x'})^2}\right) \\ &= (1+\sqrt{x'})\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x') = (1+\sqrt{x'})z'. \end{aligned}$$

Solving for z', with the aid of (13.2), we find that

$$z' = \tfrac{1}{2}z(1+\sqrt{1-x}).$$

Hence, given the formula (13.1), we can deduce the formula

$$\Omega\left(\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2, e^{-2y}, \tfrac{1}{2}z(1+\sqrt{1-x})\right) = 0.$$

This process is called *obtaining a formula by duplication.*

By reversing the transformation, we obtain the formula

$$\Omega\left(\frac{4\sqrt{x}}{(1+\sqrt{x})^2}, e^{-y/2}, (1+\sqrt{x})z\right) = 0.$$

We designate this process as *obtaining a formula by dimidiation.* These two processes are equivalent to Landen's transformation.

Next, let

$$x' = \frac{x}{x-1} \quad \text{or} \quad x = \frac{x'}{x'-1}.$$

Replacing x by $1-x$ in Entry 2(iv), we may deduce that

$$F(x) + F(x') = 0.$$

Hence,

$$e^{-y} = F(x) = -F(x') = -e^{-y'}.$$

By Entry 32(ii) in Chapter 11 (Part II [9, p. 92]),

$$\begin{aligned} z' = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x') &= {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{x}{x-1}\right) \\ &= \sqrt{1-x}\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x) = \sqrt{1-x}\, z. \end{aligned}$$

In conclusion, given (13.1), we can deduce the formula

$$\Omega\left(\frac{x}{x-1}, -e^{-y}, z\sqrt{1-x}\right) = 0.$$

This process is called *obtaining a formula by change of sign* and is due to Jacobi.

We have previously defined the function L in (9.2). Now define M and N by

$$M(q) = 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k}$$

and

$$N(q) = 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1-q^k},$$

where $|q| < 1$. These two Eisenstein series along with L were extensively studied by Ramanujan in Chapter 15 and in his paper [6], [10, pp. 136–162].

Results akin to those in the next few sections have been used by Ramanathan [3] in proving some results of Ramanujan in his first notebook, "lost notebook" [11], and letters to Hardy.

Entry 13. *Let L, M, and N be defined as above. Then*

(i) $$M(e^{-2y}) = z^4(1 - x + x^2),$$

(ii) $$N(e^{-2y}) = z^6(1 + x)(1 - \tfrac{1}{2}x)(1 - 2x),$$

(iii) $$M(e^{-y}) = z^4(1 + 14x + x^2),$$

(iv) $$N(e^{-y}) = z^6(1 + x)(1 - 34x + x^2),$$

(v) $$M(e^{-4y}) = z^4(1 - x + \tfrac{1}{16}x^2),$$

(vi) $$N(e^{-4y}) = z^6(1 - \tfrac{1}{2}x)(1 - x - \tfrac{1}{32}x^2),$$

(vii) "*if x is changed to $((1 - \sqrt{1 - x})/(1 + \sqrt{1 - x}))^2$ then y is changed to $2y$,*"

(viii) $$2L(e^{-2y}) - L(e^{-y}) = 1 + 24 \sum_{k=1}^{\infty} \frac{k}{e^{ky} + 1} = z^2(1 + x),$$

(ix) $$2L(e^{-4y}) - L(e^{-2y}) = 1 + 24 \sum_{k=1}^{\infty} \frac{k}{e^{2ky} + 1} = z^2(1 - \tfrac{1}{2}x),$$

(x) $$2M(e^{-2y}) - M(e^{-y}) = 1 - 240 \sum_{k=1}^{\infty} \frac{k^3}{e^{ky} + 1} = z^4(1 - 16x + x^2),$$

(xi) $$2N(e^{-2y}) - N(e^{-y}) = 1 + 504 \sum_{k=1}^{\infty} \frac{k^5}{e^{ky} + 1} = z^6(1 + x)(1 + 29x + x^2),$$

(xii) $$2M(e^{-4y}) - M(e^{-2y}) = 1 - 240 \sum_{k=1}^{\infty} \frac{k^3}{e^{2ky} + 1} = z^4(1 - x - \tfrac{7}{8}x^2),$$

and

(xiii) $$2N(e^{-4y}) - N(e^{-2y}) = 1 + 504 \sum_{k=1}^{\infty} \frac{k^5}{e^{2ky} + 1} = z^6(1 - \tfrac{1}{2}x)(1 - x + \tfrac{31}{16}x^2).$$

PROOF OF (i). From Section 13 in Chapter 15 (Part II [9, p. 330]),

$$t\frac{dL(t)}{dt} = \frac{L^2(t) - M(t)}{12}.$$

Thus, by the chain rule,

$$\frac{dL(e^{-2y})}{dy} = \frac{M(e^{-2y}) - L^2(e^{-2y})}{6}.$$

Moreover, by Entry 9(i),

$$\frac{dL(e^{-2y})}{dx} = -\frac{1}{x(1 - x)z^2}\frac{dL(e^{-2y})}{dy}.$$

Hence,

$$-x(1 - x)z^2\frac{dL(e^{-2y})}{dx} = \frac{M(e^{-2y}) - L^2(e^{-2y})}{6}. \tag{13.3}$$

Thus we see that we can determine $M(e^{-2y})$ from (9.3) and (13.3).

Using (9.3) and the hypergeometric differential equation (9.1), we find, upon a direct calculation, that

$$\frac{dL(e^{-2y})}{dx} = -2x(1-x)\left\{z\frac{d^2z}{dx^2} - 3\left(\frac{dz}{dx}\right)^2\right\}. \tag{13.4}$$

Thus, from (9.3), (13.3), and (13.4),

$$M(e^{-2y}) = 12x^2(1-x)^2z^2\left\{z\frac{d^2z}{dx^2} - 3\left(\frac{dz}{dx}\right)^2\right\} + \left\{(1-2x)z^2 + 6x(1-x)z\frac{dz}{dx}\right\}^2.$$

Upon simplifying with the use of the hypergeometric equation (9.1), we reach the desired conclusion.

PROOF OF (ii). The proof is similar to that of (i). From Section 13 of Chapter 15 (Part II [9, p. 330]),

$$t\frac{dM}{dt} = \frac{LM(t) - N(t)}{3}.$$

By the chain rule and Entry 9(i), this equality may be written in the form

$$-3x(1-x)z^2\frac{dM(e^{-2y})}{dx} = 2N(e^{-2y}) - 2LM(e^{-2y}).$$

Solving for $N(e^{-2y})$ and using (9.3) and Entry 13(i), we readily deduce part (ii).

PROOF OF (iii). Apply the process of dimidiation to Entry 13(i).

PROOF OF (iv). Apply the process of dimidiation to Entry 13(ii).

PROOF OF (v). Apply the process of duplication to Entry 13(i).

PROOF OF (vi). Apply the process of duplication to Entry 13(ii).

PROOF OF (vii). This is just Ramanujan's statement of the principle of duplication.

PROOF OF (viii). An elementary calculation shows that

$$2L(e^{-2y}) - L(e^{-y}) = 1 + 24\sum_{k=1}^{\infty}\frac{k}{e^{ky}+1}. \tag{13.5}$$

Since we know the value of $L(e^{-2y})$ from (9.3), it remains to determine $L(e^{-y})$. Our proof is similar to that of Entry 9(iv).

Using Entries 24(ii), 25(iv), and 25(vii) from Chapter 16 and (6.4) and (6.10), or Entries 10(i), (ii), we find that

$$
\begin{aligned}
L(e^{-y}) &= -8\frac{d}{dy}\operatorname{Log}\{e^{-y/8}f^3(-e^{-y})\} \\
&= -8\frac{d}{dy}\operatorname{Log}\{e^{-y/8}\varphi^2(-e^{-y})\psi(e^{-y})\} \\
&= -8\frac{d}{dy}\operatorname{Log}\{e^{-y/8}\varphi^2(-e^{-y})\varphi^{1/2}(e^{-y})\psi^{1/2}(e^{-2y})\} \\
&= -8\frac{d}{dy}\operatorname{Log}\left\{\frac{1}{\sqrt{2}}\varphi^2(-e^{-y})\varphi^{1/2}(e^{-y})(\varphi^4(e^{-y}) - \varphi^4(-e^{-y}))^{1/8}\right\} \\
&= -8\frac{d}{dy}\operatorname{Log}\{\varphi^2(-e^{-y})\varphi(e^{-y})x^{1/8}\} \\
&= -8\frac{d}{dy}\operatorname{Log}\{(1-x)^{1/2}z^{3/2}x^{1/8}\} \\
&= x(1-x)z^2\frac{d}{dx}\operatorname{Log}\{(1-x)^4z^{12}x\} \\
&= (1-5x)z^2 + 12x(1-x)z\frac{dz}{dx}.
\end{aligned}
$$

Using this last equality along with (9.3) in (13.5), we complete the proof.

PROOF OF (ix). Apply the principle of duplication to Entry 13(viii).

PROOF OF (x). An elementary calculation shows that

$$1 - 240\sum_{k=1}^{\infty}\frac{k^3}{e^{ky}+1} = 2M(e^{-2y}) - M(e^{-y}). \tag{13.6}$$

Using parts (i) and (iii), we finish the proof.

PROOF OF (xi). An elementary calculation gives

$$1 + 504\sum_{k=1}^{\infty}\frac{k^5}{e^{ky}+1} = 2N(e^{-2y}) - N(e^{-y}). \tag{13.7}$$

Now use parts (ii) and (iv).

PROOF OF (xii). Use (13.6) with y replaced by $2y$ and then use parts (i) and (v).

PROOF OF (xiii). Employ (13.7) with y replaced by $2y$ and then use parts (ii) and (vi).

Entry 14. *We have*

(i) $$1 - 8\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k}{e^{ky}+1} = z^2(1-x),$$

(ii) $$1 + 16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^3}{e^{ky}+1} = z^4(1-x^2),$$

(iii) $$1 - 8 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^5}{e^{ky}+1} = z^6(1-x)(1-x+x^2),$$

(iv) $$17 + 32 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^7}{e^{ky}+1} = z^8(1-x^2)(17-32x+17x^2),$$

(v) $$1 - 16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^3}{e^{ky}-1} = z^4(1-x)^2,$$

(vi) $$1 + 8 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^5}{e^{ky}-1} = z^6(1-x)(1-x^2),$$

(vii) $$17 - 32 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^7}{e^{ky}-1} = z^8(1-x)^2(17-2x+17x^2),$$

(viii) $$31 + 8 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^9}{e^{ky}-1} = z^{10}(1-x)(1-x^2)(31-46x+31x^2),$$

(ix) $$1 - 16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^3}{e^{2ky}-1} = z^4(1-x),$$

(x) $$1 + 8 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^5}{e^{2ky}-1} = z^6(1-x)(1-\tfrac{1}{2}x),$$

(xi) $$17 - 32 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^7}{e^{2ky}-1} = z^8(1-x)(17-17x+2x^2),$$

and

(xii) "*if* x *is changed to* $-x/(1-x)$, *then* e^{-y} *is changed to* $-e^{-y}$."

PROOF OF (i). An elementary calculation gives

$$3\left(1 - 8 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k}{e^{ky}+1}\right) = 4\left(1 + 24 \sum_{k=1}^{\infty} \frac{k}{e^{2ky}+1}\right) - \left(1 + 24 \sum_{k=1}^{\infty} \frac{k}{e^{ky}+1}\right).$$

Apply Entries 13(viii), (ix) to complete the proof.

PROOF OF (ii). By an elementary calculation,

$$15\left(1 + 16 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^3}{e^{ky}+1}\right) = 16\left(1 - 240 \sum_{k=1}^{\infty} \frac{k^3}{e^{2ky}+1}\right) - \left(1 - 240 \sum_{k=1}^{\infty} \frac{k^3}{e^{ky}+1}\right).$$

Now use Entries 13(x), (xii) to complete the proof.

PROOF OF (iii). An elementary calculation yields

$$63\left(1-8\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^5}{e^{ky}+1}\right)$$
$$=64\left(1+504\sum_{k=1}^{\infty}\frac{k^5}{e^{2ky}+1}\right)-\left(1+504\sum_{k=1}^{\infty}\frac{k^5}{e^{ky}+1}\right).$$

Using Entries 13(xi), (xiii), we finish the proof.

PROOF OF (iv). From Entry 12(ii) in Chapter 15 (Part II [9, p. 326]),

$$M^2(e^{-2y})=1+480\sum_{k=1}^{\infty}\frac{k^7}{e^{2ky}-1}. \tag{14.1}$$

An easy calculation gives

$$2M^2(e^{-2y})-M^2(e^{-y})=1-480\sum_{k=1}^{\infty}\frac{k^7}{e^{ky}+1}.$$

Thus,

$$255+480\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^7}{e^{ky}+1}$$
$$=256\{2M^2(e^{-4y})-M^2(e^{-2y})\}-\{2M^2(e^{-2y})-M^2(e^{-y})\}.$$

Employing Entries 13(i), (iii), and (v), we complete the proof.

Ramanujan (p. 212) inadvertently wrote $17-32x+x^2$ instead of $17-32x+17x^2$ on the right side of (iv).

PROOF OF (v). A routine calculation yields

$$16M(e^{-2y})-M(e^{-y})=15\left(1-16\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^3}{e^{ky}-1}\right).$$

Using Entries 13(i), (iii), we reach the desired conclusion.

PROOF OF (vi). A simple calculation gives

$$64N(e^{-2y})-N(e^{-y})=63\left(1+8\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^5}{e^{ky}-1}\right).$$

Now use Entries 13(ii), (iv).

PROOF OF (vii). By (14.1),

$$256M^2(e^{-2y})-M^2(e^{-y})=15\left(17-32\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k^7}{e^{ky}-1}\right).$$

Applying Entries 13(i), (iii), we finish the proof.

PROOF OF (viii). By Entry 12(iii) of Chapter 15 (Part II [9, p. 326]),

$$M(e^{-2y})N(e^{-2y}) = 1 - 264 \sum_{k=1}^{\infty} \frac{k^9}{e^{2ky}-1},$$

and so

$$1024M(e^{-2y})N(e^{-2y}) - M(e^{-y})N(e^{-y})$$
$$= 33\left(31 + 8 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}k^9}{e^{ky}-1}\right).$$

Using Entries 13(i)–(iv), we complete the proof.

PROOFS OF (ix), (x), (xi). Apply the principle of duplication to Entries (v), (vi), and (vii) to deduce (ix), (x), and (xi), respectively.

PROOF OF (xii). This is an enunciation of the principle of change of sign.

Entry 15. *We have*

(i) $$\sum_{k=1}^{\infty} \frac{k^3}{\sinh(ky)} = \tfrac{1}{8}z^4x,$$

(ii) $$\sum_{k=1}^{\infty} \frac{k^5}{\sinh(ky)} = \tfrac{1}{8}z^6x(1+x),$$

(iii) $$\sum_{k=1}^{\infty} \frac{k^7}{\sinh(ky)} = \tfrac{1}{8}z^8x(1+\tfrac{13}{2}x+x^2),$$

(iv) $$\sum_{k=1}^{\infty} \frac{k^9}{\sinh(ky)} = \tfrac{1}{8}z^{10}x(1+x)(1+29x+x^2),$$

(v) $$\sum_{k=1}^{\infty} \frac{k^3}{\sinh(2ky)} = \tfrac{1}{128}z^4x^2,$$

(vi) $$\sum_{k=1}^{\infty} \frac{k^5}{\sinh(2ky)} = \tfrac{1}{128}z^6x^2(1-\tfrac{1}{2}x),$$

(vii) $$\sum_{k=1}^{\infty} \frac{k^7}{\sinh(2ky)} = \tfrac{1}{128}z^8x^2(1-x+\tfrac{17}{32}x^2),$$

(viii) $$\sum_{k=1}^{\infty} \frac{k^9}{\sinh(2ky)} = \tfrac{1}{128}z^{10}x^2(1-\tfrac{1}{2}x)(1-x+\tfrac{31}{16}x^2),$$

(ix) $$\sum_{k=0}^{\infty} \frac{2k+1}{\sinh(2k+1)y} = \tfrac{1}{8}z^2x,$$

(x) $$\sum_{k=0}^{\infty} \frac{(2k+1)^3}{\sinh(2k+1)y} = \tfrac{1}{8}z^4x(1-\tfrac{1}{2}x),$$

(xi) $$\sum_{k=0}^{\infty} \frac{(2k+1)^5}{\sinh(2k+1)y} = \tfrac{1}{8}z^6x(1-x+x^2),$$

(xii) $$\sum_{k=0}^{\infty} \frac{(2k+1)^7}{\sinh(2k+1)y} = \tfrac{1}{8}z^8 x(1-\tfrac{1}{2}x)(1-x+\tfrac{17}{2}x^2),$$

(xiii) $$\sum_{k=0}^{\infty} \frac{2k+1}{\sinh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^2\sqrt{x},$$

(xiv) $$\sum_{k=0}^{\infty} \frac{(2k+1)^3}{\sinh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^4(1+x)\sqrt{x},$$

(xv) $$\sum_{k=0}^{\infty} \frac{(2k+1)^5}{\sinh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^6(1+14x+x^2)\sqrt{x},$$

and

(xvi) $$\sum_{k=0}^{\infty} \frac{(2k+1)^7}{\sinh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^8(1+x)(1+134x+x^2)\sqrt{x}.$$

PROOF OF (i). Observe that

$$\sum_{k=1}^{\infty} \frac{k^3}{e^{ky}-e^{-ky}} = \tfrac{1}{240}\{M(e^{-y}) - M(e^{-2y})\}.$$

Now use Entries 13(i), (iii).

PROOF OF (ii). The sum to be evaluated is equal to $-\frac{1}{252}\{N(e^{-y}) - N(e^{-2y})\}$. Employ Entries 13(ii), (iv) to complete the proof.

PROOF OF (iii). The sum to be determined is equal to $\frac{1}{240}\{M^2(e^{-y}) - M^2(e^{-2y})\}$. Now use Entries 13(i), (iii).

PROOF OF (iv). The sum to be evaluated is equal to $-\frac{1}{132}\{M(e^{-y})N(e^{-y}) - M(e^{-2y})N(e^{-2y})\}$. Apply Entries 13(i)–(iv) to complete the proof.

PROOFS OF (v)–(viii). Apply the process of duplication to (i)–(iv) to obtain (v)–(viii), respectively.

PROOF OF (ix). By a straightforward calculation,

$$\sum_{k=0}^{\infty} \frac{2k+1}{e^{(2k+1)y} - e^{-(2k+1)y}}$$
$$= \tfrac{1}{24}\{L(e^{-2y}) - L(e^{-y})\} - \tfrac{1}{12}\{L(e^{-4y}) - L(e^{-2y})\}$$
$$= \tfrac{1}{24}\{2L(e^{-2y}) - L(e^{-y})\} - \tfrac{1}{24}\{2L(e^{-4y}) - L(e^{-2y})\}.$$

Use Entries 13(viii), (ix) to complete the proof.

PROOFS OF (x)–(xii). Trivially,

$$\sum_{k=0}^{\infty} \frac{(2k+1)^3}{\sinh\{\frac{1}{2}(2k+1)y\}} = \sum_{k=1}^{\infty} \frac{k^3}{\sinh(ky)} - \sum_{k=1}^{\infty} \frac{(2k)^3}{\sinh(2ky)}.$$

Use Entries 15(ii), (vi) to complete the proof of (x).

The proofs of (xi) and (xii) are similar.

PROOFS OF (xii)–(xvi). Apply the principle of dimidiation to (ix)–(xii) in order to obtain (xiii)–(xvi), respectively.

Entry 16. *We have*

(i) $$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)}{\cosh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^2\sqrt{x(1-x)},$$

(ii) $$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^3}{\cosh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^4(1-2x)\sqrt{x(1-x)},$$

(iii) $$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^5}{\cosh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^6\{1-16x(1-x)\}\sqrt{x(1-x)},$$

(iv) $$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^7}{\cosh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^8(1-2x)\{1-136x(1-x)\}\sqrt{x(1-x)},$$

(v) $$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^9}{\cosh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^{10}\{1-1232x(1-x) + 7936x^2(1-x)^2\}\sqrt{x(1-x)},$$

(vi) $$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^{11}}{\cosh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^{12}(1-2x)\{1-11072x(1-x) + 176896x^2(1-x)^2\}\sqrt{x(1-x)},$$

(vii) $$\sum_{k=0}^{\infty} (-1)^k \tan^{-1} e^{-(2k+1)y/2} = \tfrac{1}{4}\sin^{-1}\sqrt{x},$$

(viii) $$\sum_{k=0}^{\infty} (-1)^k \tan^{-1} e^{-(2k+1)y/4} = \tfrac{1}{2}\tan^{-1} x^{1/4},$$

(ix) $$\sum_{k=0}^{\infty} \frac{1}{\cosh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z\sqrt{x},$$

(x) $$\sum_{k=0}^{\infty} \frac{(2k+1)^2}{\cosh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^3\sqrt{x},$$

(xi) $$\sum_{k=0}^{\infty} \frac{(2k+1)^4}{\cosh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^5(1+4x)\sqrt{x},$$

(xii) $$\sum_{k=0}^{\infty} \frac{(2k+1)^6}{\cosh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^7\{1+11(4x)+(4x)^2\}\sqrt{x},$$

and

(xiii) $$\sum_{k=0}^{\infty} \frac{(2k+1)^8}{\cosh\{\frac{1}{2}(2k+1)y\}} = \tfrac{1}{2}z^9\{1+102(4x)+57(4x)^2+(4x)^3\}\sqrt{x}.$$

The appearance here of formulas 16(v) and (vi) is rather mysterious, because the intermediate results needed to prove (v) and (vi) are not given by Ramanujan. In particular, series with summands involving eleventh powers in them have not heretofore been considered by Ramanujan in this chapter. Of course, we could establish (v) and (vi) by first deriving the aforementioned ancillary formulas. However, we proceed in an entirely different fashion and use the Fourier series of the Jacobian elliptic function sn. In fact, most of the results in Sections 13–17 could similarly be established by employing the Fourier series of the appropriate elliptic function.

Entry 16(vii) is originally due to Jacobi [1], [2, p. 164] who remarked "quae inter formulas elegantissimas censeri debet." Likewise, Entry 16(viii) is an elegant, beautiful formula.

PROOFS OF (i)–(iv). We obtain the sought formulas by the process of change of sign in Entries 15(xiii)–(xvi), respectively. Observe that, under this procedure,

$$\sum_{k=0}^{\infty} \frac{(2k+1)^n}{e^{(2k+1)y/2} - e^{-(2k+1)y/2}} \to i \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^n}{e^{(2k+1)y/2} + e^{-(2k+1)y/2}}, \tag{16.1}$$

where, here, $n = 1, 3, 5, 7$. The four desired formulas follow without difficulty from (16.1).

PROOFS OF (v), (vi). We use the notations (6.9) and (6.10).

By a theorem of Hermite, which may be found in Cayley's book [1, p. 56], for $|u| < K'$,

$$\begin{aligned} \operatorname{sn} u = u &- (1 + k^2)\frac{u^3}{3!} + (1 + 14k^2 + k^4)\frac{u^5}{5!} \\ &- (1 + 135k^2 + 135k^4 + k^6)\frac{u^7}{7!} \\ &+ (1 + 1228k^2 + 5478k^4 + 1228k^6 + k^8)\frac{u^9}{9!} \\ &- (1 + 11069k^2 + 165826k^4 + 165826k^6 + 11069k^8 + k^{10})\frac{u^{11}}{11!} + \cdots. \end{aligned} \tag{16.2}$$

On the other hand, by a result of Jacobi [1], [2, p. 165], which may be found in Whittaker and Watson's treatise [1, p. 510], for $u = 2Kt/\pi$ and $|u| < K'$,

$$\begin{aligned} \operatorname{sn} u &= \frac{2\pi}{Kk} \sum_{n=0}^{\infty} \frac{q^{(2n+1)/2} \sin(2n+1)t}{1 - q^{2n+1}} \\ &= \frac{2\pi}{Kk} \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j+1}}{(2j+1)!} \sum_{n=0}^{\infty} \frac{(2n+1)^{2j+1} q^{(2n+1)/2}}{1 - q^{2n+1}} \\ &= \frac{\pi}{Kk} \sum_{j=0}^{\infty} \frac{(-1)^j (\pi u/2K)^{2j+1}}{(2j+1)!} \sum_{n=0}^{\infty} \frac{(2n+1)^{2j+1}}{\sinh\{\frac{1}{2}(2n+1)y\}}. \end{aligned} \tag{16.3}$$

Equating coefficients of u^9 and u^{11} in (16.2) and (16.3), we deduce that, respectively,

$$\sum_{n=0}^{\infty} \frac{(2n+1)^9}{\sinh\{\frac{1}{2}(2n+1)y\}} = \tfrac{1}{2}z^{10}\sqrt{x}(1 + 1228x + 5478x^2 + 1228x^3 + x^4) \tag{16.4}$$

and

$$\sum_{n=0}^{\infty} \frac{(2n+1)^{11}}{\sinh\{\frac{1}{2}(2n+1)y\}} = \tfrac{1}{2}z^{12}\sqrt{x}(1 + 11069x + 165826x^2 + 165826x^3 + 11069x^4 + x^5). \tag{16.5}$$

Formulas (v) and (vi) are now obtained from (16.4) and (16.5), respectively, by the process of change of sign.

PROOF OF (vii). Integrate Entry 16(i) over $[0, y]$. On the left side, we find that

$$\int_0^y \sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)}{\cosh\{\frac{1}{2}(2k+1)y\}}\,dy = 2\sum_{k=0}^{\infty}(-1)^k(2k+1)\int_0^y \frac{e^{-(2k+1)y/2}}{1+e^{-(2k+1)y}}\,dy$$
$$= 4\sum_{k=0}^{\infty}(-1)^{k+1}\tan^{-1} e^{-(2k+1)y/2}. \tag{16.6}$$

Using Entry 9(i) and making two changes of variables, we find that on the right side we get

$$\tfrac{1}{2}\int_0^y z^2\sqrt{x(1-x)}\,dy = -\tfrac{1}{2}\int_0^y \frac{1}{\sqrt{x(1-x)}}\frac{dx}{dy}\,dy$$
$$= -\tfrac{1}{2}\int_0^x \frac{dx}{\sqrt{x(1-x)}}$$
$$= -\int_0^{\sqrt{x}} \frac{du}{\sqrt{1-u^2}} = -\sin^{-1}\sqrt{x}. \tag{16.7}$$

Combining (16.6) and (16.7), we complete the proof.

PROOF OF (viii). Apply the principle of dimidiation to Entry 16(vii). Examining the right side of (viii), we see that we are required to show that

$$\tfrac{1}{2}\sin^{-1}\left(\frac{2x^{1/4}}{1+\sqrt{x}}\right) = \tan^{-1} x^{1/4}.$$

We leave the verification of this equality as a straightforward exercise for the reader.

PROOF OF (ix). Apply the process of duplication to Entry 17(i) to find that

$$1 + 2\sum_{k=1}^{\infty} \frac{1}{\cosh(2ky)} = \tfrac{1}{2}(1 + \sqrt{1-x})z.$$

Subtract this formula from Entry 17(i) to get

$$\sum_{k=0}^{\infty} \frac{1}{\cosh(2k+1)y} = \tfrac{1}{4}z(1 - \sqrt{1-x}).$$

Applying the principle of dimidiation, we obtain the desired formula.

PROOF OF (x). Apply the process of duplication to Entry 17(ii) to deduce that

$$4\sum_{k=1}^{\infty} \frac{k^2}{\cosh(2ky)} = \tfrac{1}{16}z^3x(1 - \sqrt{1-x}).$$

Subtract this formula from Entry 17(ii) and find that

$$\sum_{k=0}^{\infty} \frac{(2k+1)^2}{\cosh(2k+1)y} = \tfrac{1}{16}z^3x(1 + \sqrt{1-x}).$$

Employing the principle of dimidiation, we complete the proof of (x).

PROOF OF (xi). Applying the process of duplication to Entry 17(iii), we find that

$$16\sum_{k=1}^{\infty} \frac{k^4}{\cosh(2ky)} = \tfrac{1}{16}z^5(1 + \sqrt{1-x})(\tfrac{5}{4}x^2 - 2x + 2 - \sqrt{1-x} - (1-x)^{3/2}).$$

Subtract the formula above from Entry 17(iii) and obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(2k+1)^4}{\cosh(2k+1)y} &= \tfrac{1}{2}z^5\left(\frac{x}{4} + \left(\frac{x}{4}\right)^2\right) - \tfrac{1}{16}z^5(1 + \sqrt{1-x}) \\ &\quad \times (\tfrac{5}{4}x^2 - 2x + 2 - \sqrt{1-x} - (1-x)^{3/2}).\end{aligned}$$

Using the principle of dimidiation, we achieve the proposed formula.

PROOF OF (xii). The proof is similar to the foregoing proofs. We first apply the principle of duplication to Entry 17(iv). The formula so obtained and Entry 17(iv) then give

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(2k+1)^6}{\cosh(2k+1)y} &= \tfrac{1}{2}z^7\left(\frac{x}{4} + 11\left(\frac{x}{4}\right)^2 + \left(\frac{x}{4}\right)^3\right) \\ &\quad - \tfrac{1}{256}(1 + \sqrt{1-x})z^7\{16x^2(1 + \sqrt{1-x})^2 \\ &\quad + 44x^2(1 - \sqrt{1-x})^2 + (1 - \sqrt{1-x})^6\}.\end{aligned}$$

The proposed formula now follows by dimidiation.

PROOF OF (xiii). The proof is like previous proofs. Applying the principle of duplication to Entry 17(v) and combining the result with Entry 17(v), we deduce that

$$\sum_{k=0}^{\infty} \frac{(2k+1)^8}{\cosh(2k+1)y} = \tfrac{1}{2}z^9\left(\frac{x}{4} + 57\left(\frac{x}{4}\right)^2 + 102\left(\frac{x}{4}\right)^3 + \left(\frac{x}{4}\right)^4\right)$$
$$- \tfrac{1}{1024}(1+\sqrt{1-x})z^9(64x^2(1+\sqrt{1-x})^4 + 912x^4$$
$$+ 408x^2(1-\sqrt{1-x})^4 + (1-\sqrt{1-x})^8).$$

Using dimidiation, we obtain the desired formula.

Entry 17. *We have*

(i) $$1 + 2\sum_{k=1}^{\infty} \frac{1}{\cosh(ky)} = z,$$

(ii) $$\sum_{k=1}^{\infty} \frac{k^2}{\cosh(ky)} = \tfrac{1}{8}z^3 x,$$

(iii) $$\sum_{k=1}^{\infty} \frac{k^4}{\cosh(ky)} = \tfrac{1}{2}z^5\left(\frac{x}{4} + \left(\frac{x}{4}\right)^2\right),$$

(iv) $$\sum_{k=1}^{\infty} \frac{k^6}{\cosh(ky)} = \tfrac{1}{2}z^7\left(\frac{x}{4} + 11\left(\frac{x}{4}\right)^2 + \left(\frac{x}{4}\right)^3\right),$$

(v) $$\sum_{k=1}^{\infty} \frac{k^8}{\cosh(ky)} = \tfrac{1}{2}z^9\left(\frac{x}{4} + 57\left(\frac{x}{4}\right)^2 + 102\left(\frac{x}{4}\right)^3 + \left(\frac{x}{4}\right)^4\right),$$

(vi) $$1 + 4\sum_{k=0}^{\infty} \frac{(-1)^k}{e^{(2k+1)y} - 1} = z,$$

(vii) $$1 - 4\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^2}{e^{(2k+1)y} - 1} = z^3(1-x),$$

(viii) $$5 + 4\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^4}{e^{(2k+1)y} - 1} = z^5(5-x)(1-x),$$

and

(ix) $$61 - 4\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^6}{e^{(2k+1)y} - 1} = z^7(1-x)(61-46x+x^2).$$

PROOFS OF (i)–(v). From Cayley's book [1, p. 57], for $|u| < K'$,

$$\operatorname{dn} u = 1 - k^2\frac{u^2}{2!} + k^2(4+k^2)\frac{u^4}{4!} - k^2(16+44k^2+k^4)\frac{u^6}{6!}$$
$$+ k^2(64+912k^2+408k^4+k^6)\frac{u^8}{8!} + \cdots. \tag{17.1}$$

(We have corrected a slight misprint; Cayley has written u instead of 1 for the first term on the right side.) Also, from Whittaker and Watson's text [1, p. 511], for $u = 2Kt/\pi$ and $|u| < K'$,

$$\begin{aligned}\operatorname{dn} u &= \frac{\pi}{2K} + \frac{2\pi}{K}\sum_{k=1}^{\infty}\frac{q^k\cos(2kt)}{1+q^{2k}} \\ &= \frac{1}{z} + \frac{2}{z}\sum_{k=1}^{\infty}\frac{1}{\cosh(ky)}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}\left(\frac{2ku}{z}\right)^{2j} \\ &= \frac{1}{z} + \frac{2}{z}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}\left(\frac{2u}{z}\right)^{2j}\sum_{k=1}^{\infty}\frac{k^{2j}}{\cosh(ky)},\end{aligned} \tag{17.2}$$

where the notations (6.9)–(6.12) have been utilized. (In fact, this Fourier series was essentially derived by Ramanujan in Entry 33(iii) of Chapter 16.) Equating coefficients of u^{2n}, $0 \le n \le 4$, in (17.1) and (17.2), we arrive at (i)–(v), respectively.

Proof of (vi). This result is identical with Entry 8(i), since $z = \varphi^2(e^{-y})$.

Proofs of (vii)–(ix). These results are obtained from Entry 35(ii) in Chapter 16 by setting $n = 1, 2$, and 3 there in turn. In the notation of that theorem, we need to determine P_2, P_4, and P_6. But these are constant multiples of the series in Entries 14(i)–(iii), respectively.

Examples. *Recall that* $\varphi(q)$ *and* $\psi(q)$ *are defined by* (3.1) *and* (5.1), *respectively. Then*

(i) $$\varphi^8(q) = 1 + 16\sum_{k=1}^{\infty}\frac{k^3q^k}{1-(-q)^k},$$

(ii) $$q\psi^8(q) = \sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^{2k}},$$

(iii) $$q\psi^4(q^2) = \sum_{k=0}^{\infty}\frac{(2k+1)q^{2k+1}}{1-q^{4k+2}},$$

(iv) $$\psi^2(q^2) = \sum_{k=0}^{\infty}\frac{q^k}{1+q^{2k+1}},$$

(v) $$\varphi^2(q)\psi^4(q) = \sum_{k=0}^{\infty}\frac{(2k+1)^2q^k}{1+q^{2k+1}},$$

and

(vi) $$\sum_{k=1}^{\infty}\frac{k^9q^k}{1-q^{2k}} = q\psi^8(q)\left(1 + 504\sum_{k=1}^{\infty}\frac{k^5q^k}{1+q^k}\right).$$

Proof of (i). From Entry 10(ii),

$$\varphi^8(-q) = z^4(1-x)^2.$$

The desired result now follows from Entry 14(v) after replacing q by $-q$.

Proof of (ii). From Entry 11(i),

$$e^{-y}\psi^8(e^{-y}) = \tfrac{1}{16}z^4x. \tag{17.3}$$

Using Entry 15(i), we complete the proof.

PROOF OF (iii). By Entry 11(iii),

$$e^{-y}\psi^4(e^{-2y}) = \tfrac{1}{16}z^2x.$$

The desired result now follows from Entry 15(ix).

PROOF OF (iv). The proposed formula follows from the equality immediately above and Entry 16(ix).

PROOF OF (v). By Entry 16(x), (6.4), (6.10), (11.1), and (11.3),

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(2k+1)^2 q^k}{1+q^{2k+1}} &= \tfrac{1}{4}q^{-1/2}\varphi^4(q)\{\varphi^4(q) - \varphi^4(-q)\}^{1/2} \\ &= \varphi^4(q)\psi^2(q^2) \\ &= \varphi^2(q)\psi^4(q).\end{aligned}$$

PROOF OF (vi). The desired formula follows at once from Entries 13(xi) and 15(iv) along with (17.3).

Entry 18. *If n is a positive integer, then*

(i) $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\cosh\{\frac{1}{2}(2k+1)\pi\sqrt{3}\}} = \frac{\pi}{24},$$

(ii) $$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\cosh\{\frac{1}{2}(2k+1)\pi/\sqrt{3}\}} = \frac{5\pi}{24},$$

and

(iii) $$\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^{6n-1}}{\cosh\{\frac{1}{2}(2k+1)\pi\sqrt{3}\}} = 0 = \sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^{6n-1}}{\cosh\{\frac{1}{2}(2k+1)\pi/\sqrt{3}\}}.$$

Formula (i) and the first part of (iii) were, in fact, first established by Cauchy [3, p. 317]. Rao and Ayyar [1] and Riesel [1] each rediscovered the first part of (iii). Proofs of (i) and the first part of (iii) can also be found in Berndt's paper [4, Corollaries 7.2, 7.6]. Zucker [2] has derived both (i) and (ii), while Ling [3] has proved both parts of (iii). Zucker [1] has also established the first part of (iii) for $n = 1$.

Ramanujan probably established Entry 18 via partial fractions. For example, if ω is a primitive cube root of unity, then, for each nonnegative integer n,

$$\frac{u^{6n}}{\cos u \cos(\omega u) \cos(\omega^2 u)}$$
$$= 12 \sum_{k=0}^{\infty} \frac{(-1)^k \{(k+\frac{1}{2})\pi\}^{6n+5}}{\{(k+\frac{1}{2})^6 \pi^6 - u^6\} \cosh\{(k+\frac{1}{2})\pi\sqrt{3}\}}, \tag{18.1}$$

after a somewhat lengthy, but routine, calculation. Letting u tend to 0, we deduce (i) for $n = 0$ and the first part of (iii) for $n > 0$.

Entry 18 may seem a bit out of place in relation to the remainder of this chapter. However, Ramanujan probably chose to place these formulas at this juncture because of their obvious connection with Entry 16(iii). Ramanujan returns to these sums in Chapter 18; in particular, see Section 10.

In Chapter 14, Ramanujan established several additional results in the same spirit as Entry 18. For many other results of this type and for numerous references to the literature, see Berndt's papers [3], [4] and book [9].

Examples. *Recall that $\chi(q)$ is defined by* (12.1).

(i) *If*

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^7 q^k}{1+q^{2k+1}} = 0,$$

then

$$\chi(q) = 2^{1/4} q^{1/24} \quad \textit{or} \quad 2^{1/4}(34q)^{1/24}.$$

(ii) *If*

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^9 q^k}{1+q^{2k+1}} = 0,$$

then

$$\chi(q) = 2^{1/4}\{(154 \pm 6\sqrt{645})q\}^{1/24}.$$

(iii) *If*

$$\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^{11} q^k}{1+q^{2k+1}} = 0,$$

then

$$\chi(q) = 2^{1/4} q^{1/24} \quad \textit{or} \quad 2^{1/4}(4q)^{1/24} \quad \textit{or} \quad 2^{1/4}(2764q)^{1/24}.$$

Proof of (i). Consider Entry 16(iv). If the left-hand side is equal to 0, then, since $0 < x < 1$, either $x = \frac{1}{2}$ or $x(1-x) = \frac{1}{136}$. The offered conclusion now follows from Entry 12(v).

Proof of (ii). Turn to Entry 16(v). We see that the left side vanishes when $\{x(1-x)\}^{-1} = 616 \pm 24\sqrt{645}$. By Entry 12(v), we may complete the proof.

PROOF OF (iii). The left side of Entry 16(vi) vanishes if and only if $x = \frac{1}{2}$ or $\{x(1-x)\}^{-1} = 16$ or 11056. The desired results now follow from Entry 12(v).

We do not know Ramanujan's motivation in deriving the previous examples.

In conclusion, we remark that Ling [1]–[3], Zucker [1], [2], and Schoissengeier [1] have evaluated several series like those found in the latter sections of Chapter 17 in terms of parameters in the theory of elliptic functions.

Many of the results in Chapter 17 were independently proved by S. Bhargava and C. Adiga and can be found in Adiga's thesis [1].

CHAPTER 18

The Jacobian Elliptic Functions

In Chapter 18, Ramanujan continues his development of the theory of elliptic functions begun in Chapter 16 with the theory of theta-functions and continued in Chapter 17 with an introduction to elliptic integrals and the compilation of a large catalog of series that can be evaluated in terms of elliptic function parameters. This chapter contains further series identities depending on the theory of elliptic functions. Such results are considerably fewer in number here than in Chapter 17 and generally are more difficult to prove. In particular, see Sections 4–7.

Chapter 18 also contains Ramanujan's introduction to the Jacobian elliptic functions sn, cn, and dn, although we have already used knowledge of these functions to prove some of Ramanujan's results on elliptic integrals in Chapter 17. In contrast to Jacobi [1], [2] and other writers, Ramanujan introduces these functions in Section 14 via their Fourier series. He derives only a handful of the basic facts about Jacobian elliptic functions and terminates his development rather early, but he also obtains some results apparently not in the literature. One of Ramanujan's most interesting results is the very unusual identity

$$\left\{1+2\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{\cosh(n\pi)}\right\}^{-2}+\left\{1+2\sum_{n=1}^{\infty}\frac{\cosh(n\theta)}{\cosh(n\pi)}\right\}^{-2}=\frac{2\Gamma^4(\frac{3}{4})}{\pi},$$

which, at first appearance, does not seem to have any connection with Jacobian elliptic functions. However, this highly intriguing formula arises from elementary Jacobian elliptic function identities. As in other aspects of his development of the theory of elliptic functions, Ramanujan does not use any of the historical or standard notations for Jacobian elliptic functions.

Three sections (12, 13, and 22) are concerned with continued fractions that arise in the theory of elliptic functions. Most of these results are connected with the work of T. J. Stieltjes and L. J. Rogers. Undoubtedly, the most interesting result is a corollary in Section 12. If $F(\alpha, \beta)$ is a certain continued fraction, then

$$F(\tfrac{1}{2}(\alpha + \beta), \sqrt{\alpha\beta}) = \tfrac{1}{2}\{F(\alpha, \beta) + F(\beta, \alpha)\},$$

or, in words, F determined at the arithmetic and geometric means of α and β is equal to the arithmetic mean of $F(\alpha, \beta)$ and $F(\beta, \alpha)$.

Two sections, 23 and 24, contain beautiful new theorems on theta-functions.

As in Chapter 17, the parameters x, y, and z designate the principal parameters in Ramanujan's study. See Entry 6 of Chapter 17 for the meanings of x, y, and z and the latter part of Section 6 for the relationships of x, y, and z with the more standard notations in the theory of elliptic functions. Because Chapter 18 also contains material not particularly related to elliptic functions, we list here those sections that are entirely devoted to elliptic functions: 1, 2, 4–7, 11–18, and 22. In these sections, x, y, and z *always* have the meanings indicated above. In other sections, x, y, and z denote generic variables. There should be no cause for confusion.

Evidently, Ramanujan was greatly intrigued with the problem of approximating the perimeter of an ellipse. Two long sections, 3 and 19, are primarily devoted to this topic. Ramanujan's approximations are very accurate, and those in Section 19 have a rather unusual character. Ramanujan also finds several approximations to π, some arising out of geometrical considerations. Approximations to π and some geometrical problems are considered in Sections 3, 20, and 24.

Sections 8–10 and 21 are devoted to partial fraction expansions that superficially resemble series connected with elliptic functions found in Chapter 17 and elsewhere in Chapter 18. In these sections, we proceed in the standard manner via the residue calculus. We calculate the principal parts of a certain meromorphic function f and conclude by the Mittag–Leffler theorem that f is equal to the sum of its principal parts plus an entire function g. In every instance in this chapter, it is easily shown that $g(z) \equiv 0$ by letting z tend to ∞. This aspect of the proof is always tacitly assumed in the sequel. Lastly, R_{z_0} denotes the residue of f at a pole z_0.

Entry 1. *Recall from Section* 9 *of Chapter* 15 *the definition*

$$L(e^{-y}) = 1 - 24 \sum_{n=1}^{\infty} \frac{n}{e^{ny} - 1}.$$

Then

$$\begin{aligned} {}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; x) &= z(1 - x) + \int_0^x z\, dx \\ &= \frac{z}{3}(1 + x) + \frac{2}{3z} L(e^{-2y}). \end{aligned}$$

PROOF. Recall from Entry 6 of Chapter 17 that

$$z = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x). \tag{1.1}$$

Elementary calculations then show that

$$z(1-x) = 1 + \sum_{n=1}^{\infty} \frac{(\frac{1}{2})_{n-1}^2}{(n!)^2}(\tfrac{1}{4} - n)x^n \tag{1.2}$$

and

$$\int_0^x z\,dx = \sum_{n=1}^{\infty} \frac{(\frac{1}{2})_{n-1}^2}{n!(n-1)!}x^n. \tag{1.3}$$

Adding (1.2) and (1.3), we readily deduce the first equality of Entry 1.

By Entries 9(ii) and 9(iv) in Chapter 17,

$$z(1-x) + \int_0^x z\,dx = z(1-x) + 4x(1-x)\frac{dz}{dx} \tag{1.4}$$

and

$$\frac{2}{3z}L(e^{-2y}) = \tfrac{2}{3}(1-2x)z + 4x(1-x)\frac{dz}{dx}. \tag{1.5}$$

Substituting (1.5) in (1.4) and simplifying, we easily achieve the second equality in Entry 1.

Entry 2. *In the same notation as Entry* 1,

$$\begin{aligned} {}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}; 1; x) &= z(1-x) + \frac{1}{2}\int_0^x z\,dx \\ &= \frac{z}{3}(2-x) + \frac{1}{3z}L(e^{-2y}). \end{aligned}$$

PROOF. The proofs of these two equalities follow precisely along the same lines as the proofs of the corresponding two equalities in Entry 1.

The formulas in Entries 1 and 2 were presumably suggested by the formulas for the perimeter of an ellipse in Entry 3.

Entry 3. *Consider the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with eccentricity $e = (1/a)\sqrt{a^2 - b^2}$. *If* $L = L(a, b)$ *denotes the perimeter of this ellipse, then*

$$L = 2\pi a\,{}_2F_1(\tfrac{1}{2}, -\tfrac{1}{2}; 1; e^2) \tag{3.1}$$

$$= \pi(a+b)\,{}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; t) \tag{3.2}$$

$$= \pi\{3(a+b) - \sqrt{(a+3b)(3a+b)}\} \quad \textit{nearly} \tag{3.3}$$

$$= \pi(a+b)\left\{1 + \frac{3t}{10 + \sqrt{4-3t}}\right\} \quad \textit{very nearly}, \tag{3.4}$$

where, in (3.2) *and* (3.4), $t = ((a-b)/(a+b))^2$.

The appendants "nearly" and "very nearly" are quoted from the second notebook (p. 217).

Formula (3.1) is due to Maclaurin [1] in 1742.

The second formula, (3.2), is obtainable from (3.1) by using Landen's transformation. We have not been able to find (3.2) explicitly in the work of Landen. In fact, it appears that (3.2) is originally due to J. Ivory [1] in 1796. Ivory's paper is rather unusual in that it begins with a letter to the editor, John Playfair. Ivory informs us in his letter that he was led to this theorem by the study of mutual disturbances of planets. Evidently then the editor considered it fair play to print Ivory's letter. Ivory's proof of (3.2) is quite ingenious and since it is unlikely to be known to many, we give it below.

The approximation (3.3) is due to Ramanujan and was rediscovered by Fergestad in 1951. (See papers by Selmer [1] and Stubban [1].) Both (3.3) and (3.4) are stated without proof in even more precise forms near the end of Ramanujan's paper [2], [10, p. 39], where he indicates that the formulas were discovered empirically.

However, Jacobsen and Waadeland [1], [2] have offered a very plausible explanation of Ramanujan's approximation (3.4). Wirte

$${}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; t) = 1 + \frac{t}{4(1+w)}. \tag{3.5}$$

Then

$$w = \frac{1}{3}\left\{\frac{-\frac{3}{16}t}{1} + \frac{-\frac{3}{16}t}{1} + \frac{-\frac{3}{16}t}{1} + \frac{-\frac{11}{48}t}{1} + \cdots\right\}. \tag{3.6}$$

If each numerator is replaced by $-3t/16$, then we obtain the approximation

$$w \approx \tfrac{1}{12}(-2 + \sqrt{4-3t}),$$

which immediately yields the approximation (3.4). Since Ramanujan's ability to represent analytic functions as continued fractions is unparalleled in mathematical history, it seems likely that Ramanujan's formula (3.4) had its source here. Jacobsen and Waadeland [1], [2] have found a similar argument for (3.3).

Many approximations to $L(a, b)$ have appeared in the literature. The approximations

$$L(a, b) \approx \pi(a + b)$$

and

$$L(a, b) \approx 2\pi\sqrt{ab},$$

given by Kepler [1, pp. 401, 402] in 1609, are perhaps the first to appear in the literature. As might be expected, the relative sizes of a and b determine the nature of the estimates. Most approximations, including Ramanujan's, are best when a and b are somewhat close in size. Almkvist [1] and Almkvist and Berndt [1] have described several such approximations when t is "small" and discussed their accuracy. Two of the approximations that combine simplicity and accuracy are

$$L \approx 2\pi\left(\frac{a^{3/2} + b^{3/2}}{2}\right)^{2/3},$$

given by Muir [1] in 1883, and

$$L \approx \pi(a + b)\left\{1 + \frac{1}{8}\left(\frac{a - b}{a + b}\right)^2\right\}^2,$$

published by Nyvoll [1] in 1978. Other approximations of this sort have been found by Euler [3], [6, pp. 357–370], Peano [1], Sipos (see a paper by Woyciechowsky [1]), and Selmer [1].

The perimeter of an ellipse is intimately connected with the arithmetic–geometric mean. See the papers by Almkvist [1] and Almkvist and Berndt [1] for this relationship. These authors also describe some of Gauss' beautiful contributions and how they relate to the modern day calculation of π.

PROOF OF (3.1). Parameterizing the given ellipse by $x = a \cos \varphi$, $y = b \sin \varphi$, $0 \le \varphi \le 2\pi$, we find from elementary calculus that

$$\begin{aligned} L &= 4 \int_0^{\pi/2} (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)^{1/2}\, d\varphi \\ &= 4a \int_0^{\pi/2} (1 - e^2 \cos^2 \varphi)^{1/2}\, d\varphi \\ &= 4a \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})_n}{n!} e^{2n} \int_0^{\pi/2} \cos^{2n} \varphi\, d\varphi \\ &= 2\pi a\, {}_2F_1(\tfrac{1}{2}, -\tfrac{1}{2}; 1; e^2). \end{aligned} \tag{3.7}$$

FIRST PROOF OF (3.2). Take a special case

$${}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}; 1; 4u(1 + u)^{-2}) = (1 + u)^{-1}\, {}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; u^2)$$

of Landen's transformation (see Erdélyi's compendium [1, p. 111, formula (5)]) and set $u = (a - b)/(a + b)$. After simplification, we deduce (3.2).

IVORY'S PROOF OF (3.2). Using (3.7), we find that

$$L = 2a \int_0^\pi \left\{1 - \frac{a^2 - b^2}{2a^2}(1 - \cos(2\varphi))\right\}^{1/2} d\varphi$$

$$= \int_0^\pi \{2(a^2 + b^2) + 2(a^2 - b^2)\cos(2\varphi)\}^{1/2} d\varphi$$

$$= (a+b)\int_0^\pi \left\{1 + 2\left(\frac{a-b}{a+b}\right)\cos(2\varphi) + \left(\frac{a-b}{a+b}\right)^2\right\}^{1/2} d\varphi$$

$$= (a+b)\int_0^\pi \left\{1 + \frac{a-b}{a+b}e^{2i\varphi}\right\}^{1/2}\left\{1 + \frac{a-b}{a+b}e^{-2i\varphi}\right\}^{1/2} d\varphi$$

$$= (a+b)\sum_{m=0}^\infty \frac{(-1)^m(-\frac{1}{2})_m}{m!}\left(\frac{a-b}{a+b}\right)^m$$

$$\times \sum_{n=0}^\infty \frac{(-1)^n(-\frac{1}{2})_n}{n!}\left(\frac{a-b}{a+b}\right)^n \int_0^\pi e^{2i(m-n)\varphi}\, d\varphi$$

$$= \pi(a+b)\,{}_2F_1\left(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; \left(\frac{a-b}{a+b}\right)^2\right).$$

PROOF OF (3.3). For brevity, define the coefficients α_n, $n \geq 0$, by

$${}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; t) = 1 + \frac{1}{4}t + \frac{1}{4^3}t^2 + \frac{1}{4^4}t^3 + \frac{25}{4^7}t^4 + \frac{49}{4^8}t^5 + \cdots$$

$$= \sum_{n=0}^\infty \alpha_n t^n. \tag{3.8}$$

Next, after some rearrangement, define the coefficients β_n, $n \geq 0$, by

$$3(a+b) - \sqrt{(a+3b)(3a+b)} = (a+b)\left\{3 - \left(\frac{(a+3b)(3a+b)}{(a+b)^2}\right)^{1/2}\right\}$$

$$= (a+b)\left\{3 - \left(4 - \left(\frac{a-b}{a+b}\right)^2\right)^{1/2}\right\}$$

$$= (a+b)\{3 - \sqrt{4-t}\}$$

$$= (a+b)\left(1 + \frac{1}{4}t + \frac{1}{4^3}t^2 + \frac{1}{2\cdot 4^4}t^3 + \frac{5}{4^7}t^4 + \cdots\right)$$

$$= (a+b)\sum_{n=0}^\infty \beta_n t^n, \tag{3.9}$$

where $|t| < 4$. Comparing (3.8) and (3.9), we see that $\alpha_n = \beta_n$, $0 \leq n \leq 2$, $\alpha_3 = 2\beta_3$, and $\alpha_4 = 5\beta_4$. Thus, the approximation (3.3) differs from $L/\{\pi(a+b)\}$ by only about $t^3/2^9$. It furthermore appears that $\alpha_n > \beta_n$ for $n \geq 3$. We prove this in the next theorem.

Theorem 1. *For $n \geq 3$, $\beta_n \leq \alpha_n/2^{n-2}$.*

PROOF. From (3.8) and (3.9), for $n \geq 1$,

$$\frac{\alpha_{n+1}}{\alpha_n} = \frac{(2n-1)^2}{(2n+2)^2} \quad \text{and} \quad \frac{\beta_{n+1}}{\beta_n} = \frac{1}{8}\left(\frac{2n-1}{n+1}\right).$$

Thus,

$$\frac{\beta_{n+1}/\beta_n}{\alpha_{n+1}/\alpha_n} = \frac{n+1}{2(2n-1)} \leq \frac{1}{2},$$

if $n \geq 2$. Proceeding by induction, we deduce that

$$\frac{\beta_{n+1}}{\alpha_{n+1}} \leq \frac{1}{2}\frac{\beta_n}{\alpha_n} \leq \frac{1}{2^{n-1}},$$

for $n \geq 2$, and the proof is completed.

PROOF OF (3.4). Define the coefficients γ_n, $n \geq 0$, by

$$1 + \frac{3t}{10 + \sqrt{4-3t}} = 1 + \frac{1}{4}t + \frac{1}{4^3}t^2 + \frac{1}{4^4}t^3 + \frac{25}{4^7}t^4 + \frac{95}{2\cdot 4^8}t^5 + \cdots$$

$$= \sum_{n=0}^{\infty} \gamma_n t^n, \tag{3.10}$$

where $|t| < \frac{4}{3}$. Comparing (3.8) and (3.10), we find that $\alpha_n = \gamma_n$ for $n \leq 4$, while $\gamma_5 = \frac{95}{98}\alpha_5$. Thus, Ramanujan's approximation is amazingly accurate, with the error being about $3t^5/2^{17}$.

We are very grateful to G. Almkvist and R. A. Askey who each provided the following proof of an analogue of Theorem 1.

Theorem 2. *Let α_n and γ_n, $n \geq 0$, be defined by* (3.8) *and* (3.10), *respectively. Then for $n \geq 5$, $\gamma_n < \alpha_n$.*

PROOF. We have, for $|t| < \frac{4}{3}$,

$$\sum_{n=0}^{\infty} \gamma_n t^n = 1 + \frac{t\{10 - 2\sqrt{1-3t/4}\}}{32+t}$$

$$= 1 + \frac{t}{16}\sum_{j=0}^{\infty}\left(\frac{-t}{32}\right)^j\left\{5 - \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\left(\frac{-3t}{4}\right)^k\right\}$$

$$= 1 + \frac{t}{16}\sum_{n=0}^{\infty}\left(\frac{-t}{32}\right)^n(5 - d_n),$$

where, for $n \geq 0$,

$$d_n = \sum_{k=0}^{n}\binom{\frac{1}{2}}{k}(24)^k.$$

The terms comprising d_n alternate in sign for $k \geq 1$ and are increasing in absolute value as k increases. For $n \geq 3$, it is easily seen that

$$\left|\binom{\frac{1}{2}}{n-1}(24)^{n-1} - \binom{\frac{1}{2}}{n-2}(24)^{n-2}\right| > 5.$$

It follows that for $n \geq 4$,

$$\gamma_n \leq \frac{1}{16}\frac{1}{(32)^{n-1}}\left|\binom{\frac{1}{2}}{n-1}\right|(24)^{n-1}.$$

Since

$$\alpha_n = \frac{(-\frac{1}{2})_n^2}{(1)_n^2} = \frac{1}{2^{4n}(2n-1)^2}\binom{2n}{n}^2, \qquad n \geq 0,$$

it follows that

$$u_n := \frac{\gamma_n}{\alpha_n} \leq \frac{n(2n-1)3^n}{6(2n-3)\binom{2n}{n}},$$

for $n \geq 4$. It is easily seen that u_n is a decreasing function of n for $n \geq 4$. A short calculation shows that $u_6 = \frac{27}{28}$. Hence, since we have already shown that $\gamma_5 < \alpha_5$, the proof of Theorem 2 is complete.

Suppose that we let $A(t)$ be an approximation for $L(a, b)/\{\pi(a+b)\}$, where $t = ((a-b)/(a+b))^2$. By (3.2),

$$A(t) - \frac{L(a, b)}{\pi(a+b)} = A(t) - {}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; t). \tag{3.11}$$

The first nonzero term in the power series of the right side of (3.11) gives an indication of the accuracy of the approximation $A(t)$. Let us say that $A(t)$ is of order n if the leading power in (3.11) is the nth. Thus, (3.8) and (3.9) show that, for (3.3), $A(t)$ is of order 3, while for (3.4), we see, from (3.8) and (3.10), that $A(t)$ is of order 5. The aforementioned approximations of Kepler, Euler, Sipos, Peano, and Muir are of orders 1, 1, 2, 2, and 2, respectively. Selmer [1] found an approximation of order 3 and two of order 4. See the paper by Almkvist and Berndt [1] for more details.

M. B. Villarino [1] has examined Ramanujan's second approximation in closer detail and has proved that

$$-(0.000512272\ldots)t^5 = -\left(\frac{4}{\pi} - \frac{14}{11}\right)t^5$$
$$< A(t) - \frac{L(a, b)}{\pi(a+b)} < -\frac{3}{2^{17}}t^5 = -(0.000022888\ldots)t^5,$$

if $a \neq b$.

Example.

(i) $$\pi = 3.14159\,26535\,89793\,23846\,26434,$$

(ii) $$\operatorname{Log} 10 = 2.30258\,50929\,94045\,68401\,8,$$

(iii) $$e^{-\pi} = 0.04321\,39182\,63772\,25,$$

(iv) $$e^{\pi/2} = 4.81047\,73809\,65351\,65547\,3.$$

It is not clear whether Ramanujan recorded these values on finding them in books or calculated the values himself. Ramanujan was acquainted with few books in India. We have examined those of which we know he had knowledge, and we are unable to find any of these decimal expansions, which, indeed, are correct.

Gauss [2, p. 427] recorded π to 100 decimal places and Log 10 to 50 decimal places but doubtless took his values from Wolfram's tables [1]. However, Gauss himself calculated $e^{-\pi}$ to 50 decimal places [2, p. 428] and $e^{\pi/2}$ to 34 places [2, p. 431]. Abramowitz and Stegun [1, pp. 2,3] give these four decimal expansions, although they record less digits than Ramanujan for (i) and (iv). The calculations of Gauss, Ramanujan, and Abramowitz and Stegun are in agreement.

Corollary. *According to Ramanujan,*

$$\pi = \frac{355}{113}\left(1 - \frac{0.0003}{3533}\right) \quad \textit{very nearly.}$$

and

$$\pi = (97\tfrac{1}{2} - \tfrac{1}{11})^{1/4} \quad \textit{nearly.}$$

In fact,

$$\frac{355}{113}\left(1 - \frac{0.0003}{3533}\right) = 3.14159\,26535\,89794\,32.$$

A comparison of this expansion with that of π given above shows that this approximation is greater than π by about 10^{-15}. Ramanujan also gives this approximation in his paper [2] and says that he found it by taking the reciprocal of $1 - 113\pi/355$ [10, p. 35].

The second approximation

$$(97\tfrac{1}{2} - \tfrac{1}{11})^{1/4} = 3.14159\,26526\,2$$

is less than π by about 10^{-9}. Ramanujan [2], [10, p. 35] informs us that he empirically discovered this approximation. However, N. D. Mermin [1] has suggested how Ramanujan might have discovered this approximation and why it is so accurate. The simple continued fraction expansion for π^4 is given by

$$\pi^4 = 97 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{16{,}539+}\,\frac{1}{1+}\cdots.$$

If we truncate the continued fraction after the fourth partial quotient, we obtain the approximation $\pi^4 \approx 97\frac{9}{22}$.

Entry 4. *Let x and y be as in Section* 6 *of Chapter 17. Then*

$$\frac{\sqrt{x}}{2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^2}{(n!)^2}\frac{x^n}{2n+1}=\sum_{n=0}^{\infty}(-1)^n(2n+1)\operatorname{Log}\frac{1+e^{-(2n+1)y/2}}{1-e^{-(2n+1)y/2}}.$$

PROOF. First observe that

$$\begin{aligned}S &:= \sum_{n=0}^{\infty}(-1)^n(2n+1)\operatorname{Log}\frac{1+e^{-(2n+1)y/2}}{1-e^{-(2n+1)y/2}}\\ &= \int_y^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^2}{e^{(2n+1)y/2}-e^{-(2n+1)y/2}}\,dy\\ &= \int_y^{\infty}\left(\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^2}{e^{(2n+1)y/2}-1}-\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^2}{e^{(2n+1)y}-1}\right)dy.\end{aligned}$$

The second series on the far right side is evaluated in Entry 17(vii) of Chapter 17, while the determination of the first series can be gotten from the second by the process of dimidiation. Thus,

$$\begin{aligned}S &= \frac{1}{4}\int_y^{\infty}(1-z^3(1-x)(1-\sqrt{x})+z^3(1-x)-1)\,dy\\ &= \frac{1}{4}\int_y^{\infty}z^3(1-x)\sqrt{x}\,dy.\end{aligned}$$

Employing Entry 9(i) in Chapter 17 to make a change of variable and then using (1.1), we deduce that

$$S=\frac{1}{4}\int_0^x\frac{z}{\sqrt{x}}\,dx=\frac{1}{4}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^2}{(n!)^2}\int_0^x x^{n-1/2}\,dx,$$

and the desired result follows.

Entry 5. *Let x and y be as given in Entry* 4. *Then*

$$\operatorname{Log}\frac{16}{x}-\sum_{n=1}^{\infty}\frac{(\frac{1}{2})_n^2}{(n!)^2}\frac{x^n}{n}=y-4\sum_{n=0}^{\infty}(-1)^n(2n+1)\operatorname{Log}(1-e^{-(2n+1)y}).$$

PROOF. First we observe that

$$\begin{aligned}T &:= 4\sum_{n=0}^{\infty}(-1)^n(2n+1)\operatorname{Log}(1-e^{-(2n+1)y})\\ &= -4\int_y^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^2}{e^{(2n+1)y}-1}\,dy.\end{aligned}$$

Just as in the last proof, we employ Entries 17(vii) and 9(i) in Chapter 17 and (1.1) to find that

$$\begin{aligned} T &= \int_y^\infty (z^3(1-x)-1)\,dy \\ &= \int_0^x \left(\frac{z}{x}+\frac{dy}{dx}\right)dx \\ &= \int_0^x \frac{d}{dx}(y+\operatorname{Log} x)\,dx + \int_0^x \frac{z-1}{x}\,dx \\ &= y+\operatorname{Log} x - \lim_{x\to 0+}(y+\operatorname{Log} x) + \sum_{n=1}^\infty \frac{(\frac{1}{2})_n^2}{(n!)^2}\int_0^x x^{n-1}\,dx. \end{aligned}$$

Now from Entry 2(i) in Chapter 17, we may easily deduce that

$$y + \operatorname{Log} x = \operatorname{Log} 16 + o(1)$$

as x tends to $0+$. Using this in (5.1), we readily deduce the desired formula.

Entry 6. *With x, y, and z as in Section* 6 *of Chapter* 17,

$$\sum_{n=0}^\infty \frac{1}{(2n+1)^2\cosh\{\frac{1}{2}(2n+1)y\}} = \frac{\sqrt{x}}{2z}\,{}_3F_2(1,1,1;\tfrac{3}{2},\tfrac{3}{2};x).$$

PROOF. An elementary calculation yields

$$\frac{d^2}{dy^2}\left(\frac{1}{e^{(2n+1)y/2}+e^{-(2n+1)y/2}}\right) = \frac{(2n+1)^2}{4}\,\frac{e^{(2n+1)y}+e^{-(2n+1)y}-6}{(e^{(2n+1)y/2}+e^{-(2n+1)y/2})^3}.$$

Thus,

$$\begin{aligned} U &:= \sum_{n=0}^\infty \frac{1}{(2n+1)^2\cosh\{\frac{1}{2}(2n+1)y\}} \\ &= \frac{1}{2}\int_y^\infty\int_y^\infty \sum_{n=0}^\infty \frac{e^{(2n+1)y}+e^{-(2n+1)y}-6}{(e^{(2n+1)y/2}+e^{-(2n+1)y/2})^3}\,dy\,dy. \end{aligned} \tag{6.1}$$

If $u = \exp(-\frac{1}{2}y)$, the series in the integrand above can be written in the form

$$\sum_{n=0}^\infty \frac{u^{2n+1}-6u^{3(2n+1)}+u^{5(2n+1)}}{(1+u^{4n+2})^3}.$$

We now expand $(1+u^{4n+2})^{-3}$ in a binomial series. The resulting double series can be represented by the array

$$\begin{array}{llll} u-6u^3+u^5 & -3u^2(u-6u^3+u^5) & 6u^4(u-6u^3+u^5) & \cdots \\ u^3-6u^9+u^{15} & -3u^6(u^3-6u^9+u^{15}) & 6u^{12}(u^3-6u^9+u^{15}) & \cdots \\ u^5-6u^{15}+u^{25} & -3u^{10}(u^5-6u^{15}+u^{25}) & 6u^{20}(u^5-6u^{15}+u^{25}) & \cdots. \\ \vdots & \vdots & \vdots & \end{array}$$

Arranging this array in ascending powers yields

$$\begin{array}{lllll} u & -9u^3 & 25u^5 & -49u^7 & \cdots \\ u^3 & -9u^9 & 25u^{15} & -49u^{21} & \cdots \\ u^5 & -9u^{15} & 25u^{25} & -49u^{35} & \cdots. \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

Summing the new array by columns, we obtain the sum

$$\frac{u}{1-u^2} - \frac{3^2u^3}{1-u^6} + \frac{5^2u^5}{1-u^{10}} - \frac{7^2u^7}{1-u^{14}} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)^2}{e^{(2n+1)y/2} - e^{-(2n+1)y/2}}.$$

Hence, using this in (6.1), we find that

$$U = \frac{1}{2}\int_y^{\infty}\int_y^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)^2}{e^{(2n+1)y/2} - e^{-(2n+1)y/2}}\, dy\, dy.$$

The sum in the integrand appeared in our proof of Entry 4, and so using our calculations therefrom, we find that

$$U = \frac{1}{8}\int_y^{\infty}\int_y^{\infty} z^3(1-x)\sqrt{x}\, dy\, dy. \tag{6.2}$$

Now Entry 9(iii) in Chapter 17 can be written in the form

$$z\int_y^{\infty}\int_y^{\infty} x^n(1-x)z^3\, dy\, dy = \frac{x^n}{n^2}\, {}_3F_2(n+\tfrac{1}{2}, n+\tfrac{1}{2}, 1; n+1, n+1; x),$$

where $n > 0$. Setting $n = \frac{1}{2}$ and substituting the result in (6.2), we complete the proof.

Before stating Entry 7, we offer a remark about transforming a formula of the sort

$$\Omega(x, y, z) = 0$$

into a "new" formula. Suppose that we replace x by $1 - x$. Then by (6.3) and (6.2) in Chapter 17, y and z are transformed into π^2/y and yz/π, respectively. We therefore obtain a new formula

$$\Omega(1 - x, \pi^2/y, yz/\pi) = 0.$$

This process is equivalent to *Jacobi's imaginary transformation.*

Entry 7. *Recall that Catalan's constant C is defined by*

$$C = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

Then

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2(e^{(2n+1)y}+1)}$$

$$= \tfrac{1}{2}C - \frac{\pi y}{16} + \frac{\sqrt{1-x}}{4z}\,{}_3F_2(1, 1, 1; \tfrac{3}{2}, \tfrac{3}{2}; 1-x).$$

PROOF. Apply Jacobi's imaginary transformation to Entry 6 to discover that

$$\frac{\sqrt{1-x}\,\pi}{2yz}\,{}_3F_2(1, 1, 1; \tfrac{3}{2}, \tfrac{3}{2}; 1-x)$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \cosh\{\frac{1}{2}(2n+1)\pi^2/y\}}. \tag{7.1}$$

Next, we expand

$$f(w) = \frac{\tan w}{w \cosh(\pi w/y)}$$

into partial fractions. The function f has simple poles at 0, $iy(n+\frac{1}{2})$, and $(n+\frac{1}{2})\pi$, $-\infty < n < \infty$. First, we find that

$$R_{iy(n+1/2)} = \frac{i(-1)^{n+1}\tanh(n+\frac{1}{2})y}{\pi(n+\frac{1}{2})},$$

for each integer n. An elementary calculation then shows that the sum of the two principal parts corresponding to the poles $w = iy(n+\frac{1}{2})$ and $w = -iy(n+\frac{1}{2})$, $0 \le n < \infty$, is equal to

$$\frac{2(-1)^n y \tanh(n+\frac{1}{2})y}{\pi(y^2(n+\frac{1}{2})^2 + w^2)}. \tag{7.2}$$

Next,

$$R_{(n+1/2)\pi} = -\frac{1}{(n+\frac{1}{2})\pi \cosh\{\pi^2(n+\frac{1}{2})/y\}},$$

for each integer n. Another brief calculation shows that the sum of the two principal parts associated with the poles $(n+\frac{1}{2})\pi$ and $-(n+\frac{1}{2})\pi$, $0 \le n < \infty$, is equal to

$$\frac{2}{\cosh\{\pi^2(n+\frac{1}{2})/y\}((n+\frac{1}{2})^2\pi^2 - w^2)}. \tag{7.3}$$

Hence, from (7.2) and (7.3), we deduce that

$$f(w) = \frac{2y}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \tanh(n+\frac{1}{2})y}{y^2(n+\frac{1}{2})^2 + w^2}$$
$$+ 2 \sum_{n=0}^{\infty} \frac{1}{\cosh\{\pi^2(n+\frac{1}{2})/y\}((n+\frac{1}{2})^2\pi^2 - w^2)}.$$

Letting w tend to 0, we find that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \cosh\{\pi^2(n+\frac{1}{2})/y\}}$$
$$= \frac{\pi^2}{8} - \frac{\pi}{4y} \sum_{n=0}^{\infty} \frac{(-1)^n \tanh(n+\frac{1}{2})y}{(n+\frac{1}{2})^2}$$
$$= \frac{\pi^2}{8} - \frac{\pi}{y} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \left\{1 - \frac{2}{e^{(2n+1)y}+1}\right\}$$
$$= \frac{\pi^2}{8} - \frac{\pi}{y} C + \frac{2\pi}{y} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2(e^{(2n+1)y}+1)}. \tag{7.4}$$

Substituting (7.4) into (7.1), we complete the proof.

Although the following example appears in Section 7, it does not appear to be closely connected with Entry 7.

Example. *Let x and y be as above. Then*

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)\sinh\{\frac{1}{2}(2n+1)y\}} = \frac{1}{4} \operatorname{Log}\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right).$$

PROOF. By an elementary calculation,

$$V := \sum_{n=0}^{\infty} \frac{1}{(2n+1)\sinh\{\frac{1}{2}(2n+1)y\}}$$
$$= \int_y^{\infty} \sum_{n=0}^{\infty} \frac{e^{(2n+1)y/2} + e^{-(2n+1)y/2}}{(e^{(2n+1)y/2} - e^{-(2n+1)y/2})^2}\, dy. \tag{7.5}$$

We wish to transform the series in the integrand. If $u = \exp(-\frac{1}{2}y)$, this series can be written

$$\sum_{n=0}^{\infty} \frac{u^{2n+1} + u^{6n+3}}{(1-u^{4n+2})^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (m+1)\{u^{(2n+1)(2m+1)} + u^{(2n+1)(2m+3)}\}$$
$$= \sum_{m=0}^{\infty} \frac{m+1}{u^{-(2m+1)} - u^{2m+1}} + \sum_{m=0}^{\infty} \frac{m}{u^{-(2m+1)} - u^{2m+1}}$$
$$= \sum_{m=0}^{\infty} \frac{2m+1}{2\sinh\{\frac{1}{2}(2m+1)y\}}.$$

Putting this representation into (7.5) and utilizing Entries 15(xiii) and 9(i) in

Chapter 17, we find that

$$\begin{aligned} V &= \frac{1}{2}\int_y^\infty \sum_{n=0}^\infty \frac{2n+1}{\sinh\{\frac{1}{2}(2n+1)y\}}\,dy \\ &= \frac{1}{4}\int_y^\infty z^2\sqrt{x}\,dy \\ &= \frac{1}{4}\int_0^x \frac{dx}{\sqrt{x}(1-x)} \\ &= \frac{1}{4}\int_0^x \frac{d}{dx}\operatorname{Log}\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)dx \\ &= \tfrac{1}{4}\operatorname{Log}\left(\frac{1+\sqrt{x}}{1+\sqrt{x}}\right). \end{aligned}$$

This completes the proof.

Entry 8. *Let θ be real. If $|\theta| < \pi$, then*

(i) $$\sum_{n=0}^\infty (-1)^n \frac{\cos(2n+1)\theta + 2\cos\{\frac{1}{2}(2n+1)\theta\}\cosh\{\frac{1}{2}(2n+1)\sqrt{3}\theta\}}{(2n+1)\cosh\{\frac{1}{2}(2n+1)\pi\sqrt{3}\}} = \frac{\pi}{8},$$

and if $|\theta| < \pi/2$, then

(ii) $$\sum_{n=0}^\infty (-1)^n \frac{\cos(2n+1)\theta(\cos(2n+1)\theta + \cosh\{(2n+1)\sqrt{3}\theta\})}{(2n+1)\cosh\{\frac{1}{2}(2n+1)\pi\sqrt{3}\}} = \frac{\pi}{12},$$

(iii) $$\sum_{n=0}^\infty (-1)^n \frac{\sin(2n+1)\theta(\cos(2n+1)\theta - \cosh\{(2n+1)\sqrt{3}\theta\})}{(2n+1)^4\cosh\{\frac{1}{2}(2n+1)\pi\sqrt{3}\}} = -\frac{\pi}{12}\theta^3,$$

and

(iv) $$\sum_{n=0}^\infty (-1)^n \frac{\cos(2n+1)\theta(\cos(2n+1)\theta + \cosh\{(2n+1)\sqrt{3}\theta\}}{(2n+1)^7\cosh\{\frac{1}{2}(2n+1)\pi\sqrt{3}\}} = \frac{\pi^7}{11520} - \frac{\pi\theta^6}{180}.$$

These beautiful series evaluations apparently have not been given previously in the literature. As the proofs below make clear, even more general theorems undoubtedly exist.

Proof of (i). Let $\omega = \exp(2\pi i/3)$ and define

$$f(z) = \frac{\cos(2\theta z) + \cos(2\omega\theta z) + \cos(2\omega^2\theta z)}{\cos(\pi z)\cos(\pi\omega z)\cos(\pi\omega^2 z)}.$$

We expand $f(z)$ into partial fractions.

First, $f(z)$ has simple poles at $z = n + \frac{1}{2}$, $-\infty < n < \infty$. After an elementary calculation and simplification, we find that

$$R_{n+1/2} = 2(-1)^{n+1} f(n, \theta),$$

where

$$f(n, \theta) = \frac{\cos(2n+1)\theta + 2\cos\{\frac{1}{2}(2n+1)\theta\}\cosh\{\frac{1}{2}(2n+1)\sqrt{3}\theta\}}{\pi \cosh\{\frac{1}{2}(2n+1)\pi\sqrt{3}\}}.$$

The sum of the two principal parts corresponding to the poles $n + \frac{1}{2}$ and $-(n + \frac{1}{2})$, $0 \le n < \infty$, is thus equal to

$$\frac{4(-1)^{n+1}(n+\frac{1}{2}) f(n, \theta)}{z^2 - (n+\frac{1}{2})^2}. \tag{8.1}$$

Second, f has simple poles at $z = \omega(n + \frac{1}{2})$, $-\infty < n < \infty$, with

$$R_{\omega(n+1/2)} = 2(-1)^{n+1} \omega f(n, \theta).$$

The sum of the two principal parts corresponding to the poles $\omega(n + \frac{1}{2})$ and $-\omega(n + \frac{1}{2})$, $0 \le n < \infty$, is equal to

$$\frac{4(-1)^{n+1}(n+\frac{1}{2})\omega^2 f(n, \theta)}{z^2 - \omega^2(n+\frac{1}{2})^2}. \tag{8.2}$$

Lastly, f has simple poles at $z = \omega^2(n + \frac{1}{2})$, $-\infty < n < \infty$, with

$$R_{\omega^2(n+1/2)} = 2(-1)^{n+1} \omega^2 f(n, \theta).$$

The sum of the two principal parts associated with the simple poles $\omega^2(n + \frac{1}{2})$ and $-\omega^2(n + \frac{1}{2})$, $0 \le n < \infty$, is equal to

$$\frac{4(-1)^{n+1}(n+\frac{1}{2})\omega f(n, \theta)}{z^2 - \omega(n+\frac{1}{2})^2}. \tag{8.3}$$

Hence, from (8.1)–(8.3), we deduce that

$$f(z) = \frac{12}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(n+\frac{1}{2})^5(\cos(2n+1)\theta + 2\cos\{\frac{1}{2}(2n+1)\theta\}\cosh\{\frac{1}{2}(2n+1)\sqrt{3}\theta\})}{\cosh\{\frac{1}{2}(2n+1)\pi\sqrt{3}\}(z^6 - (n+\frac{1}{2})^6)}.$$

Letting $z = 0$ above, we deduce Entry 8(i).

Proof of (ii). Since the proofs of (ii) and (iii) are similar to the proof of (i), we are brief in our details.

Consider

$$g(z) := \frac{1 + \cos(4\theta z) + \cos(4\omega\theta z) + \cos(4\omega^2\theta z)}{\cos(\pi z)\cos(\pi\omega z)\cos(\pi\omega^2 z)}.$$

Of course, g has the same simple poles as f in the proof of (i). Thus, by calculations similar to those above, we deduce the partial fraction expansion

$$g(z)=\frac{24}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(n+\frac{1}{2})^5\cos(2n+1)\theta(\cos(2n+1)\theta+\cosh\{(2n+1)\sqrt{3}\theta\})}{\cosh\{\frac{1}{2}(2n+1)\pi\sqrt{3}\}(z^6-(n+\frac{1}{2})^6)}. \tag{8.4}$$

Putting $z = 0$ yields the desired result.

Proof of (iii). We calculate the partial fraction decomposition of

$$h(z) := \frac{\sin(4\theta z) + \sin(4\omega\theta z) + \sin(4\omega^2\theta z)}{z^3\cos(\pi z)\cos(\pi\omega z)\cos(\pi\omega^2 z)}.$$

The function h has the same simple poles as f and g in the proofs above. By calculations like those above, we find that

$$h(z)=\frac{24}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(n+\frac{1}{2})^2\sin(2n+1)\theta(\cos(2n+1)\theta-\cosh\{(2n+1)\sqrt{3}\theta\})}{\cosh\{\frac{1}{2}(2n+1)\pi\sqrt{3}\}(z^6-(n+\frac{1}{2})^6)}.$$

Letting z tend to 0 above, we deduce Entry 8(iii).

Proof of (iv). Expanding both sides of (8.4) in powers of z, we find that, for $|z| < \frac{1}{2}$,

$$\left(4 - 3\frac{(4\theta z)^6}{6!} + \cdots\right)\left(1 + \frac{(\pi z)^6}{15} + \cdots\right)$$
$$= \frac{24}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\theta(\cos(2n+1)\theta + \cosh\{(2n+1)\sqrt{3}\theta\})}{(n+\frac{1}{2})\cosh\{\frac{1}{2}(2n+1)\pi\sqrt{3}\}}$$
$$\times\left(1 + \frac{z^6}{(n+\frac{1}{2})^6} + \cdots\right).$$

Equating coefficients of z^6 on both sides, we deduce the desired result.

Entry 9. *If* $z \neq \pm\omega^j(2n + 1)$, $0 \leq j \leq 2$, $0 \leq n < \infty$, *where* $\omega = \exp(2\pi i/3)$, *then*

$$\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^5}{\cosh\{\frac{1}{2}(2n+1)\pi\sqrt{3}\}((2n+1)^6 - z^6)}$$
$$= \frac{\pi}{12}\frac{1}{\cos(\frac{1}{2}\pi z)(\cos(\frac{1}{2}\pi z) + \cosh(\frac{1}{2}\pi\sqrt{3}z))}.$$

Proof. In (18.1) of Chapter 17, let $n = 0$ and $u = \frac{1}{2}\pi z$. After some elementary manipulation we achieve the desired result.

Example. *For each complex number z,*

(i) $$\tfrac{1}{2}\cos(\tfrac{1}{2}z)(\cos(\tfrac{1}{2}z) + \cosh(\tfrac{1}{2}\sqrt{3}z)) = 1 + \frac{3}{4}\sum_{n=1}^{\infty}\frac{(-1)^n z^{6n}}{(6n)!} = \prod_{n=0}^{\infty}\left(1 - \frac{z^6}{(2n+1)^6\pi^6}\right)$$

and

(ii) $$\tfrac{1}{2}\sin(\tfrac{1}{2}z)(\cos(\tfrac{1}{2}z) - \cosh(\tfrac{1}{2}\sqrt{3}z)) = -\frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)^n z^{6n+3}}{(6n+3)!} = -\frac{z^3}{8}\prod_{n=1}^{\infty}\left(1 - \frac{z^6}{(2n\pi)^6}\right).$$

Proof. First we verify the elementary identity

$$\tfrac{1}{2}\cos(\tfrac{1}{2}z)(\cos(\tfrac{1}{2}z) + \cosh(\tfrac{1}{2}\sqrt{3}z)) = \tfrac{1}{4}(1 + \cos z + \cos(\omega z) + \cos(\omega^2 z)).$$

The first equality in (i) now easily follows. To prove the second equality of (i), use the elementary identity

$$\tfrac{1}{2}\cos(\tfrac{1}{2}z)(\cos(\tfrac{1}{2}z) + \cosh(\tfrac{1}{2}\sqrt{3}z)) = \cos(\tfrac{1}{2}z)\cos(\tfrac{1}{2}\omega z)\cos(\tfrac{1}{2}\omega^2 z)$$

along with the familiar infinite product representation for $\cos z$.

Similarly, part (ii) follows easily from the elementary identities

$$\tfrac{1}{4}(\sin z + \sin(\omega z) + \sin(\omega^2 z)) = \tfrac{1}{2}\sin(\tfrac{1}{2}z)(\cos(\tfrac{1}{2}z) - \cosh(\tfrac{1}{2}\sqrt{3}z)) = -\sin(\tfrac{1}{2}z)\sin(\tfrac{1}{2}\omega z)\sin(\tfrac{1}{2}\omega^2 z).$$

Entry 10. *If* $z \neq \pm\omega^j(2n+1)$, $0 \le j \le 2$, $0 \le n < \infty$, *where* $\omega = \exp(2\pi i/3)$, *then*

$$\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^5}{\cosh\{\frac{1}{2}(2n+1)\pi/\sqrt{3}\}((2n+1)^6 - z^6)} = \frac{\pi}{12}\,\frac{4\cosh\left(\dfrac{\pi z}{2\sqrt{3}}\right)\left(\cos(\frac{1}{2}\pi z) + \cosh\left(\dfrac{\pi z}{2\sqrt{3}}\right)\right) - 3}{\cos(\frac{1}{2}\pi z)(\cos(\frac{1}{2}\pi z) + \cosh(\frac{1}{2}\pi\sqrt{3}z))}.$$

Proof. For each nonnegative integer m, we expand

$$f(z) := \frac{z^{6m}\left(\cosh\left(\frac{\pi z}{\sqrt{3}}\right) + \cosh\left(\frac{\pi\omega z}{\sqrt{3}}\right) + \cosh\left(\frac{\pi\omega^2 z}{\sqrt{3}}\right) - \frac{1}{2}\right)}{\cos(\frac{1}{2}\pi z)\cos(\frac{1}{2}\pi\omega z)\cos(\frac{1}{2}\pi\omega^2 z)}$$

$$= \frac{z^{6m}\left(4\cosh\left(\frac{\pi z}{2\sqrt{3}}\right)\left(\cos(\frac{1}{2}\pi z) + \cosh\left(\frac{\pi z}{2\sqrt{3}}\right)\right) - 3\right)}{\cos(\frac{1}{2}\pi z)(\cos(\frac{1}{2}\pi z) + \cosh(\frac{1}{2}\pi\sqrt{3}z))}$$

into partial fractions. The function $f(z)$ has simple poles at $z = \omega^j(2n+1)$, $j = 0, 1, 2$, $-\infty < n < \infty$. After a somewhat lengthy but elementary calculation, we find that

$$f(z) = \frac{12}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)^{6m+5}}{\cosh\{\frac{1}{2}(2n+1)\pi/\sqrt{3}\}((2n+1)^6 - z^6)}. \tag{10.1}$$

Setting $m = 0$, we complete the proof.

Clearly, (10.1) is an analogue of (18.1) in Chapter 17.

Example. *Under the same hypotheses as Entry* 10,

$$z^3\sum_{n=0}^{\infty}\frac{(2n+1)^2}{(2n+1)^6 - z^6}$$
$$= \frac{\pi}{12}\,\frac{\cosh(\frac{1}{2}\pi\sqrt{3}z) - \cos(\frac{1}{2}\pi z)}{\cosh(\frac{1}{2}\pi\sqrt{3}z) + \cos(\frac{1}{2}\pi z)}\tan(\tfrac{1}{2}\pi z).$$

Proof. In a more symmetric form, this example may be written

$$\frac{\pi\tan(\frac{1}{2}\pi z)\tan(\frac{1}{2}\pi\omega z)\tan(\frac{1}{2}\pi\omega^2 z)}{12z^3} = \sum_{n=0}^{\infty}\frac{(2n+1)^2}{(2n+1)^6 - z^6}. \tag{10.2}$$

We expand the left side of (10.2) into partial fractions. We first find that

$$R_{(2n+1)\omega^j} = -\frac{\omega^j}{6(2n+1)^3}\tan(\tfrac{1}{2}(2n+1)\pi\omega)\tan(\tfrac{1}{2}(2n+1)\pi\omega^2),$$

where $j = 0, 1, 2$, $-\infty < n < \infty$. Straightforward calculations then give (10.2).

Example. $$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^7\cosh\{\frac{1}{2}(2n+1)\pi\sqrt{3}\}} = \frac{\pi^7}{23040}.$$

Proof. From (18.1) of Chapter 17,

$$\frac{1}{\cos z\cos(\omega z)\cos(\omega^2 z)} = 24\pi^5\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^5}{\cosh\{\frac{1}{2}(2n+1)\pi\sqrt{3}\}((2n+1)^6\pi^6 - (2z)^6)},$$

where $\omega = \exp(2\pi i/3)$. Expand both sides in powers of z and equate coefficients of z^6 on both sides to achieve the proposed formula. (The calculations are very similar to those needed in the proof of Entry 8(iv).)

The last example is found in Ramanujan's [10, p. 350] first letter to Hardy and was first established in print by Watson [1] who employed contour integration in his proof.

Entry 11.

(i) *If* $|\mathrm{Im}\,\theta| < \pi$, *then*

$$\left\{1 + 2\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{\cosh(n\pi)}\right\}^{-2} + \left\{1 + 2\sum_{n=1}^{\infty} \frac{\cosh(n\theta)}{\cosh(n\pi)}\right\}^{-2} = \frac{2\Gamma^4(\frac{3}{4})}{\pi}.$$

(ii) *Let* x', y', *and* z' *denote the parameters associated with the complementary modulus* k'. *If* $|\mathrm{Im}\,\theta| < y/2$, *then*

$$\sum_{n=0}^{\infty} \frac{\cos(2n+1)\theta}{\cosh\{\frac{1}{2}(2n+1)y\}} \sum_{n=0}^{\infty} \frac{\cosh(2n+1)\theta}{\cosh\{\frac{1}{2}(2n+1)y'\}} = \tfrac{1}{4}zz'\sqrt{xx'}.$$

Proof of (i). Using the Fourier series of the Jacobian elliptic function dn (Whittaker and Watson [1, p. 511]),

$$\mathrm{dn}\left(\frac{K\theta}{\pi}\right) = \frac{\pi}{2K}\left(1 + 2\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{\cosh(ny)}\right), \qquad |\mathrm{Im}\,\theta| < \pi, \tag{11.1}$$

we may rewrite Entry 11(i) in the form

$$\left\{\frac{2K}{\pi}\,\mathrm{dn}\left(\frac{K\theta}{\pi}\right)\right\}^{-2} + \left\{\frac{2K}{\pi}\,\mathrm{dn}\left(\frac{Ki\theta}{\pi}\right)\right\}^{-2} = \frac{2\Gamma^4(\frac{3}{4})}{\pi}, \tag{11.2}$$

where $y = \pi$, $x = \frac{1}{2}$, and by (6.15) in Chapter 17, $K = \frac{1}{2}\pi^{3/2}/\Gamma^2(\frac{3}{4})$. Thus, by (11.2), it suffices to prove that

$$\mathrm{dn}^{-2}\left(\frac{K\theta}{\pi}\right) + \mathrm{dn}^{-2}\left(\frac{Ki\theta}{\pi}\right) = 2. \tag{11.3}$$

Using Jacobi's imaginary transformation (Whittaker and Watson [1, p. 505])

$$\mathrm{dn}(iu, k) = \mathrm{dc}(u, k') = \frac{\mathrm{dn}(u, k)}{\mathrm{cn}(u, k)},$$

since $k' = k = 1/\sqrt{2}$ here, we find that

$$\mathrm{dn}^{-2}\left(\frac{K\theta}{\pi}\right) + \mathrm{dn}^{-2}\left(\frac{Ki\theta}{\pi}\right) = \frac{\mathrm{cn}^2(K\theta/\pi) + 1}{\mathrm{dn}^2(K\theta/\pi)}$$

$$= \frac{2 - \mathrm{sn}^2\,(K\theta/\pi)}{\mathrm{dn}^2\,(K\theta/\pi)} = \frac{2\,\mathrm{dn}^2\,(K\theta/\pi)}{\mathrm{dn}^2\,(K\theta/\pi)} = 2,$$

where we have employed elementary identities for the Jacobian elliptic functions (Whittaker and Watson [1, p. 493]). (See also Section 14 below.) Thus, (11.3) is established to complete the proof of Entry 11(i).

Entry 11(i) is a fascinating identity even though it is not particularly deep, as the proof shows. One wonders how Ramanujan ever discovered this most unusual and beautiful formula.

PROOF OF (ii). Using the Fourier series for cn,

$$\operatorname{cn}\left(\frac{2K\theta}{\pi}\right) = \frac{\pi}{Kk}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\theta}{\cosh\{\frac{1}{2}(2n+1)y\}}, \qquad |\operatorname{Im}\theta| < \tfrac{1}{2}y, \tag{11.4}$$

and Jacobi's imaginary transformation (Whittaker and Watson [1, pp. 511, 505]), we find that the left side of Entry 11(ii) may be written

$$\frac{Kk}{\pi}\operatorname{cn}\left(\frac{2K\theta}{\pi}\right)\frac{K'k'}{\pi}\operatorname{cn}\left(\frac{2K'i\theta}{\pi}\right) = \frac{KK'kk'}{\pi^2} = \tfrac{1}{4}zz'\sqrt{xx'}.$$

Entry 12. *If $n > 0$, then*

(i) $$\frac{1}{2} + \sum_{j=1}^{\infty}\frac{\operatorname{sech}(jy)}{1+(jn)^2} = \frac{z}{2+}\;\frac{(nz)^2x}{2}\;+\;\frac{(2nz)^2}{2}\;+\;\frac{(3nz)^2x}{2}\;+\;\frac{(4nz)^2}{2}\;+\cdots$$

and

(ii) $$2\sum_{j=0}^{\infty}\frac{\operatorname{sech}\{\frac{1}{2}(2j+1)y\}}{1+(2j+1)^2n^2}$$
$$= \frac{z\sqrt{x}}{1}\;+\;\frac{(nz)^2}{1}\;+\;\frac{(2nz)^2x}{1}\;+\;\frac{(3nz)^2}{1}\;+\;\frac{(4nz)^2x}{1}\;+\cdots.$$

PROOF OF (i). Using (11.1) and integrating termwise, we find that

$$\frac{1}{n}\int_0^{\infty} e^{-2u/(nz)}\operatorname{dn} u\,du = \frac{1}{2} + \frac{2}{nz}\sum_{j=1}^{\infty}\operatorname{sech}(jy)\int_0^{\infty} e^{-2u/(nz)}\cos\frac{2ju}{z}\,du$$
$$= \frac{1}{2} + \sum_{j=1}^{\infty}\frac{\operatorname{sech}(jy)}{1+(jn)^2}. \tag{12.1}$$

On the other hand, by a theorem of Stieltjes [1], [2, pp. 184–200] that was rediscovered by Rogers [3],

$$\frac{1}{n}\int_0^{\infty} e^{-2u/(nz)}\operatorname{dn} u\,du = \frac{z}{2+}\;\frac{(nz)^2x}{2}\;+\;\frac{(2nz)^2}{2}\;+\;\frac{(3nz)^2x}{2}\;+\;\frac{(4nz)^2}{2}\;+\cdots. \tag{12.2}$$

Combining (12.1) and (12.2), we complete the proof.

The continued fraction (ii) appears in Ramanujan's [10, p. 350] first letter to Hardy and was first established in print by Preece [1]. The following corollary is found in Ramanujan's [10, pp. xxix] second letter to Hardy and was proved by Preece [2]. The proof that we give is much different from that of Preece and is undoubtedly similar to the one Ramanujan must have found.

Corollary. *For* $n > 0$, Re $\alpha > 0$, *and* Re $\beta > 0$, *define*

$$F(\alpha, \beta) = \frac{\alpha}{n+}\,\frac{\beta^2}{n+}\,\frac{(2\alpha)^2}{n}\,+\,\frac{(3\beta)^2}{n}\,+\,\frac{(4\alpha)^2}{n}\,+\cdots.$$

Then

$$F\left(\frac{\alpha+\beta}{2}, \sqrt{\alpha\beta}\right) = \tfrac{1}{2}\{F(\alpha, \beta) + F(\beta, \alpha)\}. \tag{12.3}$$

What a marvelous theorem! In words, the continued fraction F evaluated at the arithmetic and geometric means of α and β is equal to the arithmetic mean of $F(\alpha, \beta)$ and $F(\beta, \alpha)$.

Proof. First, let $\beta > 1$ and choose α such that $0 < \alpha < \beta$ and

$$\beta = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha^2/\beta^2).$$

Thus, in Entry 12(ii), we set $x = \alpha^2/\beta^2$ and $z = \beta$. Also replace n by $1/n$. We then find that

$$\frac{2}{n}\sum_{j=0}^{\infty}\frac{\operatorname{sech}\{\frac{1}{2}(2j+1)y\}}{1+\{(2j+1)/n\}^2} = F(\alpha, \beta).$$

In Entry 12(i), make the same substitutions for α and β, but replace n by $2/n$. Accordingly, we discover that

$$\frac{2}{n}\left(\frac{1}{2} + \sum_{j=1}^{\infty}\frac{\operatorname{sech}(jy)}{1+(2j/n)^2}\right) = F(\beta, \alpha). \tag{12.4}$$

Hence,

$$\frac{1}{n}\left(\frac{1}{2} + \sum_{j=1}^{\infty}\frac{\operatorname{sech}(\frac{1}{2}jy)}{1+(j/n)^2}\right) = \tfrac{1}{2}\{F(\alpha, \beta) + F(\beta, \alpha)\}. \tag{12.5}$$

Now the left side above appears, by (12.4), to be $F(\beta, \alpha)$, except that y is replaced by $\frac{1}{2}y$ and n by $2n$. Thus, we apply the process of dimidiation described in Section 13 of Chapter 17. Since x is transformed into $4\sqrt{x}/(1+\sqrt{x})^2$, we see that α^2/β^2 is replaced by $4\alpha\beta/(\alpha+\beta)^2$. Also, z is transformed into $(1+\sqrt{x})z$, and so β is replaced by $\alpha+\beta$. Combining these two changes, we see that α is replaced by $2\sqrt{\alpha\beta}$. Thus, from (12.4), after a little manipulation,

$$\frac{1}{n}\left(\frac{1}{2} + \sum_{j=1}^{\infty}\frac{\operatorname{sech}(\frac{1}{2}jy)}{1+(j/n)^2}\right) = F\left(\frac{\alpha+\beta}{2}, \sqrt{\alpha\beta}\right).$$

Combining this with (12.5), we complete the proof for $0 < \alpha < \beta < 1$.

By symmetry, we see that (12.3) also holds for $0 < \beta < \alpha < 1$. Now each of the three continued fractions in (12.3) converges to an analytic function of α and β for $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$. Thus, by analytic continuation, (12.3) is valid for $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$.

Entry 13. *If $n > 0$, then*

(i)
$$2\sum_{j=0}^{\infty}\frac{(-1)^j \operatorname{csch}\{\frac{1}{2}(2j+1)y\}}{1+(2j+1)^2n^2} = \frac{z\sqrt{x}}{1}\,+\,\frac{(1-x)(nz)^2}{1}\,-\,\frac{x(2nz)^2}{1}\,+\,\frac{(1-x)(3nz)^2}{1}\,-\cdots,$$

(ii)
$$2\sum_{j=0}^{\infty}\frac{(-1)^j(2j+1)\operatorname{sech}\{\frac{1}{2}(2j+1)y\}}{1+(2j+1)^2n^2} = \frac{z^2\sqrt{x(1-x)}}{1+(nz)^2(1-2x)}\,+\,\frac{2^2(2^2-1)x(1-x)(nz)^4}{1+(3nz)^2(1-2x)}$$
$$+\,\frac{4^2(4^2-1)x(1-x)(nz)^4}{1+(5nz)^2(1-2x)}\,+\cdots,$$

and

(iii)
$$2\sum_{j=0}^{\infty}\frac{(2j+1)\operatorname{csch}\{\frac{1}{2}(2j+1)y\}}{1+(2j+1)^2n^2} = \frac{z^2\sqrt{x}}{1+(nz)^2(1+x)}\,-\,\frac{2^2(2^2-1)x(nz)^4}{1+(3nz)^2(1+x)}\,-\,\frac{4^2(4^2-1)x(nz)^4}{1+(5nz)^2(1+x)}\,-\cdots,$$

where, in (i) *and* (ii), $0 < x < 1/\sqrt{2}$.

Proof of (i). Recall the Fourier series of the Jacobian elliptic function $\operatorname{cd} u$ (Whittaker and Watson [1, p. 511])

$$\tfrac{1}{2}z\sqrt{x}\operatorname{cd}(zu) = \sum_{j=0}^{\infty}\frac{(-1)^j\cos(2j+1)u}{\sinh\{\frac{1}{2}(2j+1)y\}}, \qquad |\operatorname{Im} u| < \tfrac{1}{2}y.$$

It follows that

$$\begin{aligned} z\sqrt{x}\int_0^{\infty} e^{-u/n}\operatorname{cd}(zu)\,du &= 2\sum_{j=0}^{\infty}\frac{(-1)^j}{\sinh\{\frac{1}{2}(2j+1)y\}}\int_0^{\infty} e^{-u/n}\cos(2j+1)u\,du \\ &= 2n\sum_{j=0}^{\infty}\frac{(-1)^j\operatorname{csch}\{\frac{1}{2}(2j+1)y\}}{1+(2j+1)^2n^2}. \end{aligned} \tag{13.1}$$

Next, from Jacobi's *Fundamenta Nova* [1], [2, p. 147],

$$\operatorname{cd} u = \operatorname{cn}(k'u, ik'/k).$$

Thus,

$$z\sqrt{x}\int_0^\infty e^{-u/n}\,\mathrm{cd}(zu)\,du = \frac{k}{k'}\int_0^\infty e^{-u/(nzk')}\,\mathrm{cn}(u, ik/k')\,du. \tag{13.2}$$

Now Stieltjes [1], [2] and Rogers [3] have shown that (see also Perron's book [1, p. 220])

$$\begin{aligned}
&\frac{k}{k'}\int_0^\infty e^{-u/(nzk')}\,\mathrm{cn}(u, ik/k')\,du \\
&\quad = \frac{k}{k'}\left(\frac{nzk'}{1}\,+\,\frac{(nzk')^2}{1}\,+\,\frac{(2nzk')^2(ik/k')^2}{1}\right. \\
&\qquad \left.+\,\frac{(3nzk')^2}{1}\,+\,\frac{(4nzk')^2(ik/k')^2}{1}\,+\cdots\right) \\
&\quad = \frac{nzk}{1}\,+\,\frac{(nzk')^2}{1}\,-\,\frac{(2nzk)^2}{1}\,+\,\frac{(3nzk')^2}{1}\,-\,\frac{(4nzk)^2}{1}\,+\cdots.
\end{aligned} \tag{13.3}$$

To ensure the convergence of this continued fraction, by a theorem in Perron's book [1, p. 53, Satz 2.16], we must require that $k/k' < 1$ or $x < 1/\sqrt{2}$. Taking (13.1)–(13.3) together, we complete the proof.

Proof of (ii). We employ the Fourier expansion of the Jacobian elliptic function sd u (Whittaker and Watson [1, p. 511]),

$$z\sqrt{x(1-x)}\,\mathrm{sd}(zu) = 2\sum_{j=0}^\infty \frac{(-1)^j \sin(2j+1)u}{\cosh\{\frac{1}{2}(2j+1)y\}}, \qquad |\mathrm{Im}\, u| < \tfrac{1}{2}y.$$

Differentiating with respect to u, we find that

$$z\sqrt{x(1-x)}\,\frac{d}{du}\,\mathrm{sd}(zu) = 2\sum_{j=0}^\infty \frac{(-1)^j(2j+1)\cos(2j+1)u}{\cosh\{\frac{1}{2}(2j+1)y\}}.$$

Hence, integrating by parts and integrating termwise, we arrive at

$$\begin{aligned}
\frac{z\sqrt{x(1-x)}}{n}\int_0^\infty e^{-u/n}\,\mathrm{sd}(zu)\,du &= z\sqrt{x(1-x)}\int_0^\infty e^{-u/n}\frac{d}{du}\,\mathrm{sd}(zu)\,du \\
&= 2\sum_{j=0}^\infty \frac{(-1)^j(2j+1)}{\cosh\{\frac{1}{2}(2j+1)y\}}\int_0^\infty e^{-u/n}\cos(2j+1)\,u\,du \\
&= 2n\sum_{j=0}^\infty \frac{(-1)^j(2j+1)\,\mathrm{sech}\{\frac{1}{2}(2j+1)y\}}{1+(2j+1)^2n^2}.
\end{aligned} \tag{13.4}$$

From Jacobi's work [1], [2, p. 147],

$$\mathrm{sd}\,u = \mathrm{sn}(k'u, ik/k').$$

Thus,

$$\frac{z\sqrt{x(1-x)}}{n}\int_0^\infty e^{-u/n}\,\mathrm{sd}(zu)\,du = \frac{1}{n}\sqrt{\frac{x}{1-x}}\int_0^\infty e^{-u/(nzk')}\,\mathrm{sn}(u, ik/k')\,du. \tag{13.5}$$

By another result of Stieltjes [1], [2], and Rogers [3],

$$\int_0^\infty e^{-u/(nzk')}\,\mathrm{sn}(u, ik/k')\,du$$

$$= \frac{1}{(nzk')^{-2} + (1 + (ik/k')^2) -} \; \frac{2^2(2^2-1)(ik/k')^2}{(nzk')^{-2} + 3^2(1 + (ik/k')^2)}$$

$$- \frac{4^2(4^2-1)(ik/k')^2}{(nzk')^2 + 5^2(1 + (ik/k')^2)} - \cdots$$

$$= \frac{n^2z^2(1-x)}{1 + (nz)^2(1-2x) +} \; \frac{2^2(2^2-1)n^4z^4x(1-x)}{1 + (3nz)^2(1-2x)}$$

$$+ \frac{4^2(4^2-1)n^4z^4x(1-x)}{1 + (5nz)^2(1-2x)} + \cdots. \tag{13.6}$$

The assurance that this continued fraction converges is guaranteed by a theorem in Perron's text [1, p. 47, Satz 2.11]. Equalities (13.4)–(13.6) now imply the sought result.

Proof of (iii). Using the Fourier series (Whittaker and Watson [1, p. 511])

$$z\sqrt{x}\,\mathrm{sn}(zu) = 2\sum_{j=0}^{\infty} \frac{\sin(2j+1)u}{\sinh\{\frac{1}{2}(2j+1)y\}}, \qquad |\mathrm{Im}\, u| < \tfrac{1}{2}y, \tag{13.7}$$

we find upon an integration by parts and integrating termwise that

$$\frac{z\sqrt{x}}{n}\int_0^\infty e^{-u/n}\,\mathrm{sn}(zu)\,du = z\sqrt{x}\int_0^\infty e^{-u/n}\frac{d}{du}\,\mathrm{sn}(zu)\,du$$

$$= 2\sum_{j=0}^{\infty}\frac{2j+1}{\sinh\{\frac{1}{2}(2j+1)y\}}\int_0^\infty e^{-u/n}\cos(2j+1)u\,du$$

$$= 2n\sum_{j=0}^{\infty}\frac{(2j+1)\,\mathrm{csch}\{\frac{1}{2}(2j+1)y\}}{1 + (2j+1)^2n^2}. \tag{13.8}$$

On the other hand, by the same result of Stieltjes and Rogers that we used in (13.6),

$$\int_0^\infty e^{-u/(nz)}\,\mathrm{sn}\,u\,du = \frac{1}{(nz)^{-2} + 1 + x -} \; \frac{2^2(2^2-1)x}{(nz)^{-2} + 3^2(1+x)}$$

$$- \frac{4^2(4^2-1)x}{(nz)^{-2} + 5^2(1+x)} - \cdots$$

$$= \frac{(nz)^2}{1 + (nz)^2(1+x) -} \; \frac{2^2(2^2-1)x(nz)^4}{1 + (3nz)^2(1+x)}$$

$$- \frac{4^2(4^2-1)x(nz)^4}{1 + (5nz)^2(1+x)} - \cdots. \tag{13.9}$$

The desired result now follows from (13.8) and (13.9).

For additional proofs of the continued fractions of Stieltjes and Rogers employed in the proofs above, see the paper by Flajolet and Francon [1], where combinatorial applications are given. Further work on combinatorial implications of continued fractions of the Jacobian elliptic functions can be found in Flajolet's paper [1]. Generalizations of some of the Stieltjes–Rogers continued fractions have been discovered by D. V. and G. V. Chudnovsky [1, pp. 30–31].

Corollary. *If $n > 0$, then*

$$\sum_{j=0}^{\infty} \frac{(-1)^j(2j+1)\operatorname{sech}\{\frac{1}{2}(2j+1)\pi\}}{1+(2j+1)^2n^2}$$
$$= \frac{1}{4}\left(\frac{\mu^2}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{1\cdot 3(n\mu)^4}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{6\cdot 10(n\mu)^4}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{15\cdot 21(n\mu)^4}{1}\,\genfrac{}{}{0pt}{}{}{+}\cdots\right),$$

where $\mu = \sqrt{\pi}/\Gamma^2(3/4)$.

Proof. Set $y = \pi$ in Entry 13(ii). Then $x = \frac{1}{2}$ and, by (6.15) in Chapter 17, $z = \mu$. The corollary now easily follows.

In Section 14, Ramanujan defines three functions S, C, and C_1, for real θ, by

$$S = S(\theta) = \sum_{j=0}^{\infty} \frac{\sin(2j+1)\theta}{\sinh\{\frac{1}{2}(2j+1)y\}},$$

$$C = C(\theta) = \sum_{j=0}^{\infty} \frac{\cos(2j+1)\theta}{\cosh\{\frac{1}{2}(2j+1)y\}},$$

and

$$C_1 = C_1(\theta) = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{\cos(2j\theta)}{\cosh(jy)}.$$

Now, in fact, Ramanujan has replaced θ by 2θ on each right side above. But in all subsequent work after Section 14, Ramanujan employs the definitions that we have given. We have actually already encountered these functions in (13.7), (11.4), and (11.1), respectively. More precisely,

$$S = \tfrac{1}{2}z\sqrt{x}\,\operatorname{sn}(z\theta), \quad C = \tfrac{1}{2}z\sqrt{x}\,\operatorname{cn}(z\theta), \quad \text{and} \quad C_1 = \tfrac{1}{2}z\,\operatorname{dn}(z\theta). \tag{14.1}$$

Entry 14. *If C, S, and C_1 are defined as above, then*

$$C^2 + S^2 = \tfrac{1}{4}xz^2, \tag{14.2}$$

$$C_1^2 + S^2 = \tfrac{1}{4}z^2, \tag{14.3}$$

$$CS = \sum_{j=1}^{\infty} \frac{j\sin(2j\theta)}{\cosh(jy)}, \tag{14.4}$$

$$2CS + \frac{dC_1}{d\theta} = 0, \tag{14.5}$$

$$2C_1S + \frac{dC}{d\theta} = 0, \tag{14.6}$$

and

$$2CC_1 = \frac{dS}{d\theta}. \tag{14.7}$$

Furthermore, define φ, $0 \le \varphi < 2\pi$, *by*

$$C = \tfrac{1}{2}z\sqrt{x}\cos\varphi \quad \textit{and} \quad S = \tfrac{1}{2}z\sqrt{x}\sin\varphi. \tag{14.8}$$

Then

$$C_1 = \tfrac{1}{2}z\sqrt{1 - x\sin^2\varphi}, \tag{14.9}$$

$$z\cos\varphi\sqrt{1 - x\sin^2\varphi} = \frac{d\sin\varphi}{d\theta} = \cos\varphi\frac{d\varphi}{d\theta}, \tag{14.10}$$

and

$$\theta = \frac{1}{z}\int_0^{\varphi}\frac{d\varphi}{\sqrt{1 - x\sin^2\varphi}}. \tag{14.11}$$

The formulas (14.2) and (14.3) are simply translations of the fundamental formulas (Whittaker and Watson [1, p. 493])

$$\text{cn}^2 u + \text{sn}^2 u = 1 \quad \text{and} \quad \text{dn}^2 u + k^2 \text{sn}^2 u = 1,$$

respectively. These formulas are generally proved by utilizing representations for cn u, sn u, and dn u in terms of theta-functions. However, Ramanujan probably used Fourier series. Thus, we proceed by finding the Fourier series for C^2, S^2, and C_1^2. In fact, Jacobi [1], [2, p. 196] found the Fourier series for C^2 and S^2 but not for C_1^2. More elegant derivations of these Fourier series as well as the Fourier series for C_1^2 have been found by Glaisher [2], and since Glaisher's work is not particularly well known, we present it.

Kiper [1] has derived the Fourier series for higher powers of the Jacobian elliptic functions, while Langebartel [1] has developed Fourier expansions for several rational functions of Jacobian elliptic functions.

It should be remarked that the two formulas in (14.8) are compatible because of (14.2).

Equality (14.11) represents Ramanujan's form of the inversion of the elliptic integral of the first kind.

Proofs. From the definition of S, we see that

$$-4S^2 = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{e^{2i(m+n+1)\theta}}{\sinh((m+\frac{1}{2})y)\sinh((n+\frac{1}{2})y)}.$$

The coefficient of $\exp(2in\theta)$, $1 \le n < \infty$, in this double series is seen to be

$$\sum_{m=-\infty}^{\infty} \frac{1}{\sinh((m+\frac{1}{2})y)\sinh((n-m-\frac{1}{2})y)}$$
$$= \frac{1}{\sinh(ny)} \sum_{m=-\infty}^{\infty} \{\coth((m+\tfrac{1}{2})y) - \coth((m-n+\tfrac{1}{2})y)\}$$
$$= \frac{1}{\sinh(ny)} \lim_{M\to\infty} \sum_{m=-M-1}^{M} \{\coth((m+\tfrac{1}{2})y) - \coth((m-n+\tfrac{1}{2})y)\}$$
$$= \frac{1}{\sinh(ny)} \lim_{M\to\infty} \left\{ \sum_{m=M-n+1}^{M} \coth((m+\tfrac{1}{2})y) - \sum_{m=-M-n-1}^{-M-2} \coth((m+\tfrac{1}{2})y) \right\}$$
$$= \frac{2n}{\sinh(ny)}.$$

A similar calculation shows that the coefficient of $\exp(-2in\theta)$, $1 \le n < \infty$, is equal to the coefficient of $\exp(2in\theta)$. Hence,

$$S^2 = \frac{1}{4} \sum_{m=-\infty}^{\infty} \operatorname{csch}^2((m+\tfrac{1}{2})y) - \sum_{n=1}^{\infty} \frac{n\cos(2n\theta)}{\sinh(ny)}. \tag{14.12}$$

The derivation for C^2 is similar. From the definition of C,

$$4C^2 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{e^{2i(m+n+1)\theta}}{\cosh((m+\frac{1}{2})y)\cosh((n+\frac{1}{2})y)}.$$

The coefficient of $\exp(2in\theta)$, $1 \le n < \infty$, in this double series is equal to

$$\sum_{m=-\infty}^{\infty} \frac{1}{\cosh((m+\frac{1}{2})y)\cosh((n-m-\frac{1}{2})y)}$$
$$= \frac{1}{\sinh(ny)} \sum_{m=-\infty}^{\infty} \{\tanh((m+\tfrac{1}{2})y) - \tanh((m-n+\tfrac{1}{2})y)\}$$
$$= \frac{2n}{\sinh(ny)},$$

by the same type of reasoning as that used above. The coefficient of $\exp(-2in\theta)$, $1 \le n < \infty$, is the same as that for $\exp(2in\theta)$. Hence,

$$C^2 = \frac{1}{4} \sum_{m=-\infty}^{\infty} \operatorname{sech}^2((m+\tfrac{1}{2})y) + \sum_{n=1}^{\infty} \frac{n\cos(2n\theta)}{\sinh(ny)}. \tag{14.13}$$

Third,

$$4C_1^2 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{e^{2i(m+n)\theta}}{\cosh(my)\cosh(ny)}.$$

The coefficient of $\exp(2in\theta)$, $1 \le n < \infty$, in this double series is found to be

$$\sum_{m=-\infty}^{\infty} \frac{1}{\cosh(my)\cosh(n-m)y}$$
$$= \frac{1}{\sinh(ny)} \sum_{m=-\infty}^{\infty} \{\tanh(my) - \tanh(m-n)y\}$$
$$= \frac{2n}{\sinh(ny)}.$$

The coefficient of $\exp(-2in\theta)$ is the same as that for $\exp(2in\theta)$, $1 \le n < \infty$. Therefore,

$$C_1^2 = \frac{1}{4} + \frac{1}{2}\sum_{m=1}^{\infty} \operatorname{sech}^2(my) + \sum_{n=1}^{\infty} \frac{n\cos(2n\theta)}{\sinh(ny)}. \tag{14.14}$$

It is evident from (14.12)–(14.14) that $C^2 + S^2$ and $C_1^2 + S^2$ are independent of θ. Now $S(0) = 0$, and by Entries 16(ix) and 17(i) in Chapter 17, respectively, $C(0) = \frac{1}{2}z\sqrt{x}$ and $C_1(0) = \frac{1}{2}z$. The formulas (14.2) and (14.3) now follow immediately.

We next prove (14.4). First, observe that

$$4iCS = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \frac{e^{2i(m+n+1)\theta}}{\cosh((m+\frac{1}{2})y)\sinh((n+\frac{1}{2})y)}.$$

The coefficient of $\exp(2in\theta)$, $1 \le n < \infty$, in this double series is equal to

$$\sum_{m=-\infty}^{\infty} \frac{1}{\cosh((m+\frac{1}{2})y)\sinh((n-m-\frac{1}{2})y)}$$
$$= \frac{1}{\cosh(ny)} \sum_{m=-\infty}^{\infty} \{\coth((n-m-\tfrac{1}{2})y) + \tanh((m+\tfrac{1}{2})y\}$$
$$= \frac{1}{\cosh(ny)} \lim_{M\to\infty} \sum_{m=-M-1}^{M} \{\coth((n-m-\tfrac{1}{2})y) + \tanh((m+\tfrac{1}{2})y)\}.$$

In the last sum, all of the hyperbolic tangent terms cancel as well as all but $2n$ of the hyperbolic cotangent terms. Each of these $2n$ surviving terms tends to 1 as M tends to ∞. Hence, the coefficient of $\exp(2in\theta)$ is equal to $2n \operatorname{sech}(ny)$. It is now not difficult to see that the coefficient of $\exp(-2in\theta)$ is equal to $-2n \operatorname{sech}(ny)$. The Fourier series (14.4) is now immediately evident.

Equality (14.5) is a direct consequence of (14.4) and the definition of C_1.

Upon differentiating (14.2) and (14.3) with respect to θ, we deduce that

$$C\frac{dC}{d\theta} = C_1\frac{dC_1}{d\theta}.$$

Thus, (14.6) follows from (14.5) and the equality above.

Again, from (14.2),

$$\frac{dS}{d\theta} = -\frac{C}{S}\frac{dC}{d\theta}.$$

Using the equality above and (14.6), we deduce (14.7).

Equality (14.9) follows immediately from (14.3) and the representation for S in (14.8).

From (14.7)–(14.9),

$$\tfrac{1}{2}z^2\sqrt{x}\cos\varphi\sqrt{1-x\sin^2\varphi} = 2CC_1 = \frac{dS}{d\theta} = \tfrac{1}{2}z\sqrt{x}\frac{d\sin\varphi}{d\theta}.$$

The equalities in (14.10) are now obvious.

The important result (14.11) follows readily from (14.10) and the fact that θ and φ vanish together.

Since $S(0) = 0$, we may deduce from (14.12) that

$$\sum_{n=-\infty}^{\infty} \operatorname{csch}^2\{\tfrac{1}{2}(2n+1)y\} = 4\sum_{n=1}^{\infty}\frac{n}{\sinh(ny)} \tag{14.15}$$

and

$$S^2 = 2\sum_{n=1}^{\infty}\frac{n\sin^2(n\theta)}{\sinh(ny)}.$$

Equality (14.15) is rather curious. In this connection, we record the following result found in Berndt's paper [3, Proposition 2.25]. If $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$, then

$$\alpha\sum_{n=1}^{\infty}\operatorname{csch}^2(\alpha n) + \beta\sum_{n=1}^{\infty}\operatorname{csch}^2(\beta n) = \frac{\alpha+\beta}{6} - 1.$$

Entry 15. *Let* φ *be defined by* (14.8). *Then*

(i) $$1 + 2\sum_{n=1}^{\infty}\frac{\cos(2n\theta)}{\cosh(ny)} = z\sqrt{1-x\sin^2\varphi},$$

(ii) $$\sum_{n=0}^{\infty}\frac{\cos(2n+1)\theta}{\cosh\{\tfrac{1}{2}(2n+1)y\}} = \tfrac{1}{2}z\sqrt{x}\cos\varphi,$$

(iii) $$\sum_{n=0}^{\infty}\frac{\sin(2n+1)\theta}{\sinh\{\tfrac{1}{2}(2n+1)y\}} = \tfrac{1}{2}z\sqrt{x}\sin\varphi,$$

(iv) $$\theta + \sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{n\cosh(ny)} = \varphi,$$

(v) $$\sum_{n=0}^{\infty}\frac{\sin(2n+1)\theta}{(2n+1)\cosh\{\tfrac{1}{2}(2n+1)y\}} = \tfrac{1}{2}\sin^{-1}(\sqrt{x}\sin\varphi),$$

and

(vi) $$\sum_{n=0}^{\infty}\frac{\cos(2n+1)\theta}{(2n+1)\sinh\{\tfrac{1}{2}(2n+1)y\}} = -\tfrac{1}{2}\operatorname{Log}\left(\frac{\sqrt{1-x\sin^2\varphi}-\sqrt{x}\cos\varphi}{\sqrt{1-x}}\right).$$

Proof. Parts (i)–(iii) are merely reiterations of (14.8) and (14.9).

To prove (iv), we integrate (i) and use (14.10). Accordingly,

$$\theta + \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n \cosh(ny)} = \int_0^{\theta} \left(1 + 2 \sum_{n=1}^{\infty} \frac{\cos(2n\theta)}{\cosh(ny)}\right) d\theta$$

$$= \int_0^{\theta} z\sqrt{1 - x \sin^2 \varphi}\, d\theta = \int_0^{\varphi} d\varphi = \varphi.$$

Similarly, to prove (v), we integrate (ii) and use (14.10). Thus,

$$\sum_{n=1}^{\infty} \frac{\sin(2n+1)\theta}{(2n+1)\cosh\{\frac{1}{2}(2n+1)y\}} = \int_0^{\theta} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\theta}{\cosh\{\frac{1}{2}(2n+1)y\}}\, d\theta$$

$$= \int_0^{\theta} \tfrac{1}{2} z\sqrt{x} \cos \varphi \, d\theta$$

$$= \frac{\sqrt{x}}{2} \int_0^{\varphi} \frac{\cos \varphi \, d\varphi}{\sqrt{1 - x \sin^2 \varphi}}$$

$$= \tfrac{1}{2} \sin^{-1} (\sqrt{x} \sin \varphi).$$

Lastly, we integrate (iii) and use (14.10) to establish (vi). Observe from (14.11) that $\theta = \pi/2$ if and only if $\varphi = \pi/2$. Hence,

$$\sum_{n=0}^{\infty} \frac{\cos(2n+1)\theta}{(2n+1)\sinh\{\frac{1}{2}(2n+1)y\}} = \int_{\theta}^{\pi/2} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\theta}{\sinh\{\frac{1}{2}(2n+1)y\}}\, d\theta$$

$$= \int_{\theta}^{\pi/2} \tfrac{1}{2} z\sqrt{x} \sin \varphi \, d\theta$$

$$= \frac{\sqrt{x}}{2} \int_{\varphi}^{\pi/2} \frac{\sin \varphi}{\sqrt{1 - x \sin^2 \varphi}}\, d\varphi$$

$$= -\tfrac{1}{2} \operatorname{Log}\left(\frac{\sqrt{1 - x \sin^2 \varphi} - \sqrt{x} \cos \varphi}{\sqrt{1 - x}}\right).$$

Formula (iv) is due to Jacobi [1], [2, p. 158]. Ramanujan (p. 222) omits the minus sign on the right side of (vi).

Entry 16 (First Part). *Suppose that θ and φ are related as in* (14.8). *If we replace θ by $\frac{1}{2}\pi - \theta$ in any formula involving θ, then we have the following table for converting certain functions of φ:*

Old Formula	New Formula
$\cot \varphi$	$\sqrt{1 - x} \tan \varphi$
$\sin \varphi$	$\dfrac{\cos \varphi}{\sqrt{1 - x \sin^2 \varphi}}$
$\cos \varphi$	$\dfrac{\sqrt{1 - x} \sin \varphi}{\sqrt{1 - x \sin^2 \varphi}}$
$\sqrt{1 - x \sin^2 \varphi}$	$\dfrac{\sqrt{1 - x}}{\sqrt{1 - x \sin^2 \varphi}}$

PROOF. From (14.1) and (14.8),

$$\frac{Kk}{\pi}\operatorname{sn}\left(\frac{2K\theta}{\pi}\right) = \tfrac{1}{2}z\sqrt{x}\sin\varphi.$$

Replacing θ by $\frac{1}{2}\pi - \theta$, using the identities (Whittaker and Watson [1, p. 500])

$$\operatorname{sn}(u+K) = \operatorname{cd} u = \frac{\operatorname{cn} u}{\operatorname{dn} u},$$

and employing (14.8) and (14.9), we find that

$$\frac{Kk}{\pi}\operatorname{sn}\left(\frac{2K}{\pi}\left(\frac{\pi}{2}-\theta\right)\right) = \frac{Kk}{\pi}\frac{\operatorname{cn}(-2K\theta/\pi)}{\operatorname{dn}(-2K\theta/\pi)} = \frac{Kk}{\pi}\frac{\operatorname{cn}(z\theta)}{\operatorname{dn}(z\theta)}$$
$$= \tfrac{1}{2}z\sqrt{x}\frac{\cos\varphi}{\sqrt{1-x\sin^2\varphi}}.$$

Hence, the second entry in the table follows.

By (14.1) and (14.8),

$$\frac{Kk}{\pi}\operatorname{cn}\left(\frac{2K\theta}{\pi}\right) = \tfrac{1}{2}z\sqrt{x}\cos\varphi.$$

Replacing θ by $\frac{1}{2}\pi - \theta$, using the identities (Whittaker and Watson [1, p. 500])

$$\operatorname{cn}(u+K) = -k'\operatorname{sd} u = -k'\frac{\operatorname{sn} u}{\operatorname{dn} u},$$

and employing (14.8) and (14.9) once again, we see that

$$\frac{Kk}{\pi}\operatorname{cn}\left(K - \frac{2K\theta}{\pi}\right) = \frac{Kkk'}{\pi}\frac{\operatorname{sn}(z\theta)}{\operatorname{dn}(z\theta)} = \tfrac{1}{2}z\sqrt{x(1-x)}\frac{\sin\varphi}{\sqrt{1-x\sin^2\varphi}}.$$

Hence, the third line of our table has been verified.

The first line of the table is now an immediate consequence of the second and third lines, and the fourth line follows readily from the second line.

Entry 16 (Second Part). *With θ and φ related by* (14.8),

(i) $$\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\theta}{\sinh\{\frac{1}{2}(2n+1)y\}} = \tfrac{1}{2}z\sqrt{x}\frac{\cos\varphi}{\sqrt{1-x\sin^2\varphi}},$$

(ii) $$\sum_{n=0}^{\infty}\frac{(-1)^n\sin(2n+1)\theta}{\cosh\{\frac{1}{2}(2n+1)y\}} = \tfrac{1}{2}z\sqrt{x(1-x)}\frac{\sin\varphi}{\sqrt{1-x\sin^2\varphi}},$$

(iii) $$\csc\theta + 4\sum_{n=0}^{\infty}\frac{\sin(2n+1)\theta}{e^{(2n+1)y}-1} = z\csc\varphi,$$

(iv) $$\sec\theta + 4\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\theta}{e^{(2n+1)y}-1} = z\sec\varphi\sqrt{1-x\sin^2\varphi},$$

and

$$\text{(v)}\ \operatorname{Log}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+4\sum_{n=0}^{\infty}\frac{(-1)^n\sin(2n+1)\theta}{(2n+1)(e^{(2n+1)y}-1)}=\operatorname{Log}\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right).$$

Proof of (i). Replace θ by $\frac{1}{2}\pi-\theta$ in Entry 15(iii) and employ the first part of Entry 16 to deduce the desired result.

Proof of (ii). Replace θ by $\frac{1}{2}\pi-\theta$ in Entry 15(ii) and use the table in the first part of Entry 16.

Proofs of (iii), (iv). We employ the Fourier expansion (Whittaker and Watson [1, p. 512])

$$\frac{\operatorname{dn}(z\theta)}{\operatorname{cn}(z\theta)}=\operatorname{dc}(z\theta)=\frac{\sec\theta}{z}+\frac{4}{z}\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\theta}{e^{(2n+1)y}-1},\qquad |\operatorname{Im}\theta|<\tfrac{1}{2}y.$$

Utilizing (14.1), (14.8), and (14.9), we find that

$$\sec\theta+4\sum_{n=0}^{\infty}\frac{(-1)^n\cos(2n+1)\theta}{e^{(2n+1)y}-1}=\frac{z\sqrt{1-x\sin^2\varphi}}{\cos\varphi},$$

which is (iv). Replacing θ by $\frac{1}{2}\pi-\theta$ and using the table from above, we complete the proof of (iii).

Proof of (v). Integrating Entry 16(iv) over $[0,\theta]$ and noting that

$$\frac{d}{d\theta}\operatorname{Log}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\sec\theta,$$

we find that

$$\operatorname{Log}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+4\sum_{n=0}^{\infty}\frac{(-1)^n\sin(2n+1)\theta}{(2n+1)(e^{(2n+1)y}-1)}$$
$$=\int_0^{\theta}z\sec\varphi\sqrt{1-x\sin^2\varphi}\,d\theta=\int_0^{\varphi}\sec\varphi\,d\varphi=\operatorname{Log}\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right),$$

by (14.10).

Entries 16(iii)–(v) are somewhat mysterious in that Ramanujan had not recorded the Fourier series of dc u, which we used in our proofs, or the Fourier series of ns u, which could have similarly been employed.

Entry 17. *Let θ and φ be related by* (14.8). *Define L, as in Section* 9 *of Chapter* 15, *by*

$$L(e^{-2y})=1-24\sum_{n=1}^{\infty}\frac{n}{e^{2ny}-1}.$$

Then

(i) $$\frac{\cos\theta}{\sin^3\theta} - 8\sum_{n=1}^{\infty}\frac{n^2\sin(2n\theta)}{e^{2ny}-1} = z^3\frac{\cos\varphi}{\sin^3\varphi}\sqrt{1-x\sin^2\varphi},$$

(ii) $$\frac{1}{\sin^2\theta} - 8\sum_{n=1}^{\infty}\frac{n\cos(2n\theta)}{e^{2ny}-1} = \frac{z^2}{\sin^2\varphi} - \tfrac{1}{3}z^2(1+x) + \tfrac{1}{3}L(e^{-2y}),$$

(iii) $$\cot\theta + 4\sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{e^{2ny}-1}$$
$$= z\cot\varphi\sqrt{1-x\sin^2\varphi} + z\int_0^{\varphi}\sqrt{1-x\sin^2\varphi}\,d\varphi$$
$$-\frac{2\theta z}{\pi}\int_0^{\pi/2}\sqrt{1-x\sin^2\varphi}\,d\varphi,$$

and

(iv) $$\sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{\sinh(ny)} = \frac{1}{2}z\int_0^{\varphi}\sqrt{1-x\sin^2\varphi}\,d\varphi - \frac{\theta z}{\pi}\int_0^{\pi/2}\sqrt{1-x\sin^2\varphi}\,d\varphi.$$

We are not sure how Ramanujan deduced these formulas, of which (ii) and (iv), according to Whittaker and Watson [1, p. 520], are due to Jacobi. In our proofs below, we rely on formulas from the theory of elliptic functions not recorded by Ramanujan but which are all found in Whittaker and Watson's book [1].

PROOF OF (ii). From Whittaker and Watson's treatise [1, p. 535, Exercise 57],

$$z^2\,\mathrm{ns}^2(z\theta) = \csc^2\theta + \frac{2z}{\pi}\left(\frac{\pi z}{2} - E\right) - 8\sum_{n=1}^{\infty}\frac{n\cos(2n\theta)}{e^{2ny}-1}, \tag{17.1}$$

where

$$E = \int_0^{\pi/2}\sqrt{1-x\sin^2\varphi}\,d\varphi, \tag{17.2}$$

the complete elliptic integral of the second kind. Using (14.1), (14.8), and the definition of ns u, we may rewrite (17.1) in the equivalent form

$$\csc^2\theta - 8\sum_{n=1}^{\infty}\frac{n\cos(2n\theta)}{e^{2ny}-1} = \frac{z^2}{\sin^2\varphi} - z^2 + \frac{2zE}{\pi}. \tag{17.3}$$

It therefore remains to show that

$$L(e^{-2y}) = \frac{6zE}{\pi} - 2z^2 + xz^2. \tag{17.4}$$

Now (Whittaker and Watson [1, p. 521]),

$$\frac{dK}{dk} = \frac{E}{kk'^2} - \frac{K}{k}.$$

If we convert this equality into Ramanujan's notation and solve for E, we find that

$$E = \pi x(1-x)\frac{dz}{dx} + \tfrac{1}{2}\pi z(1-x).$$

Putting this expression for E into (17.4), we find that it suffices to prove that

$$L(e^{-2y}) = 6x(1-x)z\frac{dz}{dx} + z^2(1-2x).$$

But this equality is precisely Entry 9(iv) in Chapter 17, and so Entry 17(ii) is established.

Proof of (i). Differentiating (ii) with respect to θ, we obtain the equality

$$-2\frac{\cos\theta}{\sin^3\theta} + 16\sum_{n=1}^{\infty}\frac{n^2\sin(2n\theta)}{e^{2ny}-1} = -2z^2\frac{\cos\varphi}{\sin^3\varphi}\frac{d\varphi}{d\theta}.$$

But by (14.10), $d\varphi/d\theta = z\sqrt{1-x\sin^2\varphi}$, and so (i) is immediate.

Proof of (iii). We first show that the derivatives of the left and right sides of (iii) are equal. Differentiating (iii) with respect to θ and using (14.10), we deduce, after some simplification, that

$$-\csc^2\theta + 8\sum_{n=1}^{\infty}\frac{n\cos(2n\theta)}{e^{2ny}-1} = z^2 - z^2\csc^2\varphi - \frac{2z}{\pi}E,$$

where E is defined by (17.2). But this equality is precisely (17.3), which we have seen is another form of (ii). Thus, it remains to show only that (iii) is valid for just one particular value of θ. Recalling that $\theta = \pi/2$ if and only if $\varphi = \pi/2$, we readily see that (iii) holds when $\theta = \pi/2$.

Proof of (iv). We first write (iv) in a more traditional form

$$\sum_{n=1}^{\infty}\frac{\sin(2n\theta)}{\sinh(ny)} = \frac{KE(\varphi)}{\pi} - \frac{2\theta KE}{\pi^2}, \tag{17.5}$$

where

$$E(\varphi) = \int_0^{\varphi}\sqrt{1-x\sin^2\varphi}\,d\varphi,$$

the incomplete elliptic integral of the second kind. We now see from Whittaker and Watson's text [1, pp. 518, 520] that (17.5) may easily be translated into a result of Jacobi, and so appealing to Jacobi, we complete the proof.

In Section 18, Ramanujan considers equations of the form

$$\Omega(x, e^{-y}, z, \theta, \varphi) = 0. \tag{18.1}$$

He transforms certain parameters and determines the effect of these changes on the remaining variables. The procedures he establishes are therefore analogous to the processes of duplication, dimidiation, and change of sign described in Section 13 of Chapter 17.

Entry 18(i). *If θ is replaced by $\frac{1}{2}\theta$ and y by $\frac{1}{2}y$ in* (18.1), *then*

$$\Omega\left(\frac{4\sqrt{x}}{(1+\sqrt{x})^2}, e^{-y/2}, (1+\sqrt{x})z, \tfrac{1}{2}\theta, \tfrac{1}{2}(\varphi + \sin^{-1}(\sqrt{x}\sin\varphi))\right) = 0.$$

Proof. Changing y to $\frac{1}{2}y$ yields the process of dimidiation described in Section 13 of Chapter 17. Thus, we only need to examine the effect on φ.

Consider Entry 15(iv). Replacing y by $\frac{1}{2}y$ and θ by $\frac{1}{2}\theta$, we find that the left side of Entry 15(iv) becomes

$$\tfrac{1}{2}\theta + \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n\cosh(\frac{1}{2}ny)}. \tag{18.2}$$

On the other hand, by Entries 15(iv) and (v),

$$\begin{aligned} &\tfrac{1}{2}(\varphi + \sin^{-1}(\sqrt{x}\sin\varphi)) \\ &= \tfrac{1}{2}\theta + \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{2n\cosh(\frac{1}{2}2ny)} + \sum_{n=0}^{\infty} \frac{\sin(2n+1)\theta}{(2n+1)\cosh\{\frac{1}{2}(2n+1)y\}} \\ &= \tfrac{1}{2}\theta + \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n\cosh(\frac{1}{2}ny)}. \end{aligned} \tag{18.3}$$

It follows from (18.2), (18.3), and Entry (iv) that φ is transformed into $\frac{1}{2}(\varphi + \sin^{-1}(\sqrt{x}\sin\varphi))$, as desired.

Entry 18(ii). *If θ is replaced by $\frac{1}{2}\pi - \theta$ and e^{-y} by $-e^{-y}$, then*

$$\Omega\left(\frac{x}{x-1}, -e^{-y}, z\sqrt{1-x}, \tfrac{1}{2}\pi - \theta, \tfrac{1}{2}\pi - \varphi\right) = 0.$$

Proof. Changing e^{-y} to $-e^{-y}$ means that we are "obtaining a formula by a change of sign," which is discussed in Section 13 of Chapter 17. Thus, we need only examine the effect on φ.

If we replace θ by $\frac{1}{2}\pi - \theta$ and e^{-y} by $-e^{-y}$ in Entry 15(iv), we find that the left side is transformed into

$$\begin{aligned} &\tfrac{1}{2}\pi - \theta + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(2n\theta)}{(-1)^n\cosh(ny)} \\ &= \tfrac{1}{2}\pi - \theta - \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{\cosh(ny)} = \tfrac{1}{2}\pi - \varphi, \end{aligned}$$

by Entry 15(iv) once again. Thus, φ is converted into $\frac{1}{2}\pi - \varphi$, and this completes the proof.

Entry 18(iii). *If* e^{-y} *is replaced by* $-e^{-y}$, *then*

$$\Omega\left(\frac{x}{x-1}, -e^{-y}, z\sqrt{1-x}, \theta, \cot^{-1}\left(\frac{\cot\varphi}{\sqrt{1-x}}\right)\right) = 0.$$

Proof. As in the proof of Entry 18(ii), we need only determine the transformation on φ.

From Whittaker and Watson's text [1, p. 512], (14.1), and (14.8),

$$\frac{1}{z}\left(\cot\theta - 4\sum_{n=1}^{\infty}\frac{e^{-2ny}\sin(2n\theta)}{1+e^{-2ny}}\right)$$

$$= \operatorname{cs}(z\theta) = \frac{\operatorname{cn}(z\theta)}{\operatorname{sn}(z\theta)} = \cot\varphi. \tag{18.4}$$

Replacing e^{-y} by $-e^{-y}$, we find that the left side of (18.4) is transformed into

$$\frac{1}{z\sqrt{1-x}}\left(\cot\theta - 4\sum_{n=1}^{\infty}\frac{e^{-2ny}\sin(2n\theta)}{1+e^{-2ny}}\right) = \frac{\cot\varphi}{\sqrt{1-x}},$$

by (18.4) again. Hence, $\cot\varphi$ is converted into $(\cot\varphi)/\sqrt{1-x}$, and this completes the proof.

Ramanujan (p. 223) incorrectly asserted that $\cot\varphi$ is changed into $\sqrt{1-x}\cot\varphi$ above.

Entry 18(iv). *Let*

$$z' = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-x) \quad \textit{and} \quad y' = \pi\frac{z}{z'}.$$

(Thus, y' and z' arise when x is replaced by $1-x$ or k^2 is replaced by k'^2.) If θ is replaced by $i\theta z/z'$ and y by y', then

$$\Omega\left(1-x, e^{-y'}, z', i\theta z/z', i\operatorname{Log}\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\right) = 0.$$

Furthermore, $\sin\varphi$ *is converted to* $i\tan\varphi$ *and* $\cos\varphi$ *to* $\sec\varphi$.

Proof. It is clear that it suffices to examine the effect on φ.

By (14.1) and (14.8), $\operatorname{sn}(z\theta) = \sin\varphi$. Applying the indicated transformations, we find that $\operatorname{sn}(z\theta)$ is converted into (Whittaker and Watson [1, pp. 505, 494])

$$\operatorname{sn}(i\theta z) = i\operatorname{sc}(\theta z') = i\frac{\operatorname{sn}(\theta z')}{\operatorname{cn}(\theta z')} = i\tan\varphi;$$

that is, $\sin\varphi$ is transformed into $i\tan\varphi$.

By (14.1) and (14.8), $\operatorname{cn}(z\theta) = \cos\varphi$. Applying the given transformations, we see that $\operatorname{cn}(z\theta)$ is transformed into (Whittaker and Watson [1, pp. 505, 494])

$$\operatorname{cn}(i\theta z) = \frac{1}{\operatorname{cn}(\theta z')} = \frac{1}{\cos \varphi};$$

that is, $\cos \varphi$ is sent into $\sec \varphi$.

Lastly, φ is transformed into

$$\sin^{-1}(i \tan \varphi) = i \operatorname{Log}\left(\frac{\cos \varphi}{1 - \sin \varphi}\right) = i \operatorname{Log} \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right),$$

after an elementary calculation. This completes the proof.

Entry 19(i). *Let an ellipse of eccentricity e be given by* $x = a \cos \varphi$, $y = b \sin \varphi$, $0 \le \varphi \le 2\pi$. *Let* $P = (a \cos \varphi, b \sin \varphi)$ *and* $A = (a, 0)$. *Then* $L(AP)$, *the length of the arc* AP, *is given by*

$$L(AP) = a \int_0^{\varphi} \sqrt{1 - e^2 \cos^2 \varphi}\, d\varphi.$$

Of course, this formula for $L(AP)$ follows from elementary calculus just as in Section 3. This formula for arc length is apparently due to Legendre [1, p. 617].

Entry 19(ii). *Let a hyperbola of eccentricity e be given by* $x = a \sec \varphi$, $y = b \tan \varphi$, $0 \le \varphi \le 2\pi$. *Let* $P = (a \sec \varphi, b \tan \varphi)$ *and* $A = (a, 0)$. *Then the arc length* $L(AP)$ *is given by*

$$L(AP) = a \tan \varphi \sqrt{e^2 - \cos^2 \varphi} - a \int_0^{\varphi} \sqrt{e^2 - \cos^2 \varphi}\, d\varphi + \frac{b^2}{a} \int_0^{\varphi} \frac{d\varphi}{\sqrt{e^2 - \cos^2 \varphi}}.$$

Proof. Using the standard formula for arc length, we find, after some simplification, that

$$L(AP) = a \int_0^{\varphi} \sec^2 \varphi \sqrt{e^2 - \cos^2 \varphi}\, d\varphi,$$

where $e = (1/a)\sqrt{a^2 + b^2}$. Integrating by parts, we arrive at

$$\begin{aligned} L(AP) &= a \tan \varphi \sqrt{e^2 - \cos^2 \varphi} - a \int_0^{\varphi} \frac{\sin^2 \varphi}{\sqrt{e^2 - \cos^2 \varphi}}\, d\varphi \\ &= a \tan \varphi \sqrt{e^2 - \cos^2 \varphi} - a \int_0^{\varphi} \sqrt{e^2 - \cos^2 \varphi}\, d\varphi \\ &\quad + a \int_0^{\varphi} \left(\sqrt{e^2 - \cos^2 \varphi} - \frac{1 - \cos^2 \varphi}{\sqrt{e^2 - \cos^2 \varphi}}\right) d\varphi. \end{aligned}$$

Upon simplifying the last integral above, we complete the proof.

Entry 19(ii) is due to Legendre [1, p. 652], although it is closely connected with Landen's [2] earlier work in 1775 on the expression for a hyperbolic arc in terms of the difference between two elliptic arcs.

Entry 19(iii). *Let the perimeter L of the ellipse $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$, be given by*

$$L = \pi(a+b)(1 + 4\sin^2 \tfrac{1}{2}\theta), \qquad 0 \le \theta \le \pi, \tag{19.1}$$

where

$$\sin\theta = \sqrt{x}\sin\varphi \quad \text{and} \quad x = \left(\frac{a-b}{a+b}\right)^2. \tag{19.2}$$

Then when $e = 1$, $\varphi = 30°18'6''$, and as e tends to 0, φ tends monotonically to $30°$.

Our statement of Entry 19(iii) is somewhat stronger than Ramanujan's, who says that φ "very rapidly diminishes to 30° when e becomes 0."

PROOF. We first show that $\varphi \ge \pi/6$. From (3.2), (3.6), (19.1), and (19.2),

$$3 - 2\sqrt{1 - x\sin^2\varphi} = 1 + 4\sin^2 \tfrac{1}{2}\theta = \sum_{n=0}^{\infty} \alpha_n x^n, \qquad |x| < 1. \tag{19.3}$$

From (19.3), (3.7), and Theorem 1 in Section 3, it follows that

$$3 - \sqrt{4 - x} \le 3 - 2\sqrt{1 - x\sin^2\varphi}.$$

Solving this inequality, we find that $\sin^2\varphi \ge 1/4$, or $\varphi \ge \pi/6$.

Second, we calculate φ when $e = 1$. Thus, $x = 1$ and $\theta = \varphi$. Therefore, from (19.1) and (3.2),

$$1 + 4\sin^2 \tfrac{1}{2}\varphi = {}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; 1) = \frac{4}{\pi}, \tag{19.4}$$

by Gauss' theorem (Bailey [4, p. 2]). Thus,

$$\sin^2 \tfrac{1}{2}\varphi = \frac{1}{\pi} - \frac{1}{4} = 0.0683098861.$$

It follows that $\varphi = 30°18'6''$.

Third, we calculate φ when $e = 0$. From (19.2) and (19.3),

$$\lim_{x\to 0} \sin^2\varphi = \lim_{x\to 0} \frac{\sin^2\theta}{x} = \lim_{x\to 0} \frac{4\sin^2 \frac{1}{2}\theta}{x}$$

$$= \lim_{x\to 0} \frac{\sum_{n=1}^{\infty} \alpha_n x^n}{x} = \alpha_1 = \frac{1}{4}.$$

Thus, φ tends to $\pi/6$ as e tends to 0.

Fourth, we show that φ is a monotonically increasing function of x. From (19.3),

$$4(1 - x\sin^2\varphi) = \left(2 - \sum_{n=1}^{\infty} \alpha_n x^n\right)^2,$$

or

$$x\sin^2\varphi = \sum_{n=1}^{\infty} \alpha_n x^n - \frac{1}{4}\left(\sum_{n=1}^{\infty} \alpha_n x^n\right)^2.$$

In order to show that φ is increasing, it suffices to show that if we write the right side above as a power series in x, then all of the coefficients are nonnegative. Putting

$$\left(\sum_{n=1}^{\infty} \alpha_n x^n\right)^2 = \sum_{n=2}^{\infty} \lambda_n x^n, \tag{19.5}$$

we observe that it suffices to show that

$$\lambda_n \le 4\alpha_n, \qquad n \ge 2. \tag{19.6}$$

We first prove that

$$\frac{1}{\alpha_r} + \frac{1}{\alpha_{n-r}} \ge \frac{1}{\alpha_{r+1}} + \frac{1}{\alpha_{n-r-1}}, \qquad 1 \le r \le \tfrac{1}{2}(n-1). \tag{19.7}$$

Using the definition of α_n in (3.7), we find that

$$\frac{\dfrac{1}{\alpha_{r+1}} - \dfrac{1}{\alpha_r}}{\dfrac{1}{\alpha_{n-r}} - \dfrac{1}{\alpha_{n-r-1}}} = \frac{(n-r)^2(2r+\frac{1}{2})\alpha_{n-r}}{(r-\frac{1}{2})^2(2n-2r-\frac{3}{2})\alpha_r}$$

$$\le \frac{(n-r)^2\alpha_{n-r}}{(r-\frac{1}{2})^2\alpha_r}$$

$$= \frac{(r+\frac{1}{2})^2(r+\frac{3}{2})^2\cdots(n-r-\frac{3}{2})^2}{(r+1)^2(r+2)^2\cdots(n-r-1)^2} \le 1.$$

Thus, (19.7) has been established.

Upon successive applications of (19.7),

$$\frac{\alpha_1\alpha_{n-1}}{\alpha_1+\alpha_{n-1}} \le \frac{\alpha_2\alpha_{n-2}}{\alpha_2+\alpha_{n-2}} \le \frac{\alpha_3\alpha_{n-3}}{\alpha_3+\alpha_{n-3}} \le \cdots \le \omega_n,$$

where

$$\omega_n = \begin{cases} \frac{1}{2}\alpha_m, & \text{if } n = 2m, \\ \dfrac{\alpha_m\alpha_{m+1}}{\alpha_m+\alpha_{m+1}}, & \text{if } n = 2m+1. \end{cases} \tag{19.8}$$

Hence, by (19.5),

$$\lambda_n = \sum_{j=1}^{n-1} \alpha_j \alpha_{n-j} \le 2\omega_n \sum_{j=1}^{n-1} \alpha_j$$
$$\le 2\omega_n \sum_{j=1}^{\infty} \alpha_j = 2\omega_n\left(\frac{4}{\pi} - 1\right), \tag{19.9}$$

by (19.4).

Next, we determine those values of n for which

$$\frac{\omega_{n+2}}{\omega_n} \le \frac{\alpha_{n+2}}{\alpha_n}. \tag{19.10}$$

First, let $n = 2m$. Thus, by (19.8), we investigate when

$$\frac{\alpha_{m+1}}{\alpha_m} \le \frac{\alpha_{2m+2}}{\alpha_{2m}}.$$

This inequality is equivalent to

$$\frac{(m - \frac{1}{2})^2}{(m + 1)^2} \le \frac{(2m + \frac{1}{2})^2(2m - \frac{1}{2})^2}{(2m + 2)^2(2m + 1)^2}.$$

It is easily verified that this inequality holds for each positive integer m, and hence (19.10) is valid for every even integer n.

Second, let $n = 2m + 1$. Thus, from (19.8), we wish to determine when

$$\frac{\alpha_{m+2}(\alpha_m + \alpha_{m+1})}{\alpha_m(\alpha_{m+1} + \alpha_{m+2})} \le \frac{\alpha_{2m+3}}{\alpha_{2m+1}}.$$

After some simplification, this inequality is found to be equivalent to the inequality

$$\frac{(m + \frac{1}{2})^2(2m^2 + m + \frac{5}{4})}{2m^2 + 5m + \frac{17}{4}} \le \frac{(2m + \frac{3}{2})^2(2m + \frac{1}{2})^2}{4(2m + 3)^2}.$$

After some additional manipulation and computation, we find that this inequality is valid for $m \ge 3$ but not for $m = 0, 1, 2$.

In conclusion, (19.10) is true for $n = 2, 4, 6, 7, 8, 9, \ldots$. It follows from (19.9) that, if n is even,

$$\lambda_n \le 2\alpha_n \frac{\omega_4}{\alpha_4}\left(\frac{4}{\pi} - 1\right) = 2\frac{128}{25}\left(\frac{4}{\pi} - 1\right) < 4\alpha_n, \qquad n \ge 4,$$

and if n is odd,

$$\lambda_n \le 2\alpha_n \frac{\omega_7}{\alpha_7}\left(\frac{4}{\pi} - 1\right) = 2\frac{409600}{96921}\left(\frac{4}{\pi} - 1\right) < 4\alpha_n, \qquad n \ge 7.$$

Now, $\lambda_2 = \alpha_1^2 = \frac{1}{16} = 4\alpha_2$, $\lambda_3 = 2\alpha_1\alpha_2 = \frac{1}{128} = 2\alpha_3$, and $\lambda_5 = 2(\alpha_1\alpha_4 + \alpha_2\alpha_3) = 29/2^{15} = \frac{58}{49}\alpha_5$. Hence, (19.6) has indeed been established, and this completes the proof.

Entry 19(iv). *Consider the same ellipse as in Entry* 19(iii), *but now set*

$$L = \pi(a+b)\left(1 + \frac{\sin^2\theta}{2+\cos^2\frac{1}{2}\theta}\right), \qquad 0 \le \theta \le \pi, \tag{19.11}$$

where

$$\sin\theta = \sqrt{x}\sin\varphi \quad \textit{and} \quad x = \left(\frac{a-b}{a+b}\right)^2.$$

Then when $e = 1$, $\varphi = 60°4'55''$, *and when* $e = 0$, $\varphi = 60°$.

PROOF. First, we prove that if $e = 1$, then $\varphi = 60°4'55''$. When $e = 1$, it follows that $x = 1$ and $\theta = \varphi$. Thus, from (3.2), (19.11), and (19.4),

$$1 + \frac{\sin^2\varphi}{2+\cos^2\frac{1}{2}\varphi} = 1 + \frac{\sin^2\theta}{2+\cos^2\frac{1}{2}\theta} = {}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; 1) = \frac{4}{\pi}.$$

It can be verified numerically that $\varphi = 60°4'55''$ is the solution of this equation.

Second, from (19.11), (3.2), and (3.6),

$$\lim_{x\to 0}\sin^2\varphi = \lim_{x\to 0}\frac{\sin^2\theta}{x} = 3\lim_{x\to 0}\frac{\sin^2\theta}{x(2+\cos^2\frac{1}{2}\theta)}$$

$$= 3\lim_{x\to 0}\frac{\sum_{n=1}^{\infty}\alpha_n x^n}{x} = 3\alpha_1 = \frac{3}{4}.$$

Hence, $\varphi = \pi/3$.

Although we have satisfactorily proved Ramanujan's assertions in Entry 19(iv), our result is weaker than Entry 19(iii). As we showed in Theorem 2 of Section 3,

$$1 + \frac{3x}{10+\sqrt{4-3x}} \le \sum_{n=0}^{\infty}\alpha_n x^n = 1 + \frac{2x\sin^2\varphi}{5+\sqrt{1-x\sin^2\varphi}}. \tag{19.12}$$

It easily follows from (19.12) that

$$\frac{3}{5+\sqrt{1-\frac{3}{4}x}} \le \frac{4\sin^2\varphi}{5+\sqrt{1+x\sin^2\varphi}}. \tag{19.13}$$

Now the right side is an increasing function of $\sin^2\varphi$. When $\sin^2\varphi = \frac{3}{4}$, we have equality in (19.13). Thus, (19.13) implies that $\sin^2\varphi \ge \frac{3}{4}$; that is, $\varphi \ge \pi/3$.

It seems quite likely that as e decreases from 1 to 0, φ decreases *monotonically* from $60°4'55''$ to $60°$. However, the calculations seemingly needed to prove this conjecture appear to be rather laborious.

Villarino [1] has strengthened Entries 19(iii) and 19(iv) by developing power series expansions for φ in terms of the eccentricity e.

Villarino has offered a very credible explanation for Ramanujan's representation (19.11). (A similar argument can be made for (19.1).) Setting $t = x = (a-b)^2/(a+b)^2$ in (3.5) and (3.6) and noting that $\sin^2(\pi/3) = \frac{3}{4}$, we see that

$$
\begin{aligned}
&{}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; x) \\
&= 1 + \cfrac{\frac{1}{3}x\sin^2(\pi/3)}{1 + \frac{1}{3}\left\{\frac{-\frac{1}{4}x\sin^2(\pi/3)}{1} + \frac{-\frac{1}{4}x\sin^2(\pi/3)}{1} + \frac{-\frac{1}{4}x\sin^2(\pi/3)}{1} + \frac{-\frac{11}{36}x\sin^2(\pi/3)}{1} + \cdots\right\}}.
\end{aligned}
$$

Now replace the kth numerator of the continued fraction by $-\frac{1}{4}x\sin^2(\pi/3 + \alpha_k)$. Thus, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, but $\alpha_4 \neq 0$. Next, replace $\pi/3 + \alpha_k$ by φ for each k, $1 \le k < \infty$, and also replace $\frac{1}{3}x\sin^2(\pi/3)$ by $\frac{1}{3}x\sin^2\varphi$. Lastly, set $\sin\theta = \sqrt{x}\sin\varphi$. Then, formally,

$$
\begin{aligned}
{}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; x) &= 1 + \cfrac{\frac{1}{3}\sin^2\theta}{1 + \frac{1}{3}\left\{\frac{-\frac{1}{4}\sin^2\theta}{1} + \frac{-\frac{1}{4}\sin^2\theta}{1} + \cdots\right\}} \\
&= 1 + \frac{\sin^2\theta}{3 + \frac{1}{2}(-1 + \cos\theta)} \\
&= 1 + \frac{\sin^2\theta}{2 + \cos^2(\theta/2)}.
\end{aligned}
$$

Hence, by (3.2), we have established a heuristic derivation of (19.11).

Corollary (i). *Let the perimeter L of an ellipse be given by*

$$L = \pi(a+b)\frac{\tan\theta}{\theta}, \qquad 0 \le \theta < \pi/2,$$

where

$$\tan\theta = \sqrt{x}\cos\varphi \quad \textit{and} \quad x = \left(\frac{a-b}{a+b}\right)^2. \tag{19.14}$$

Then as e increases from 0 to 1, φ decreases from $\pi/6$ to 0. Furthermore, φ is approximately given by

$$\frac{2\sqrt{ab}}{a+b}\left\{30^\circ + 6^\circ 18' 49'' \frac{(\sqrt{a} - \sqrt{b})^2}{a+b} - 1^\circ 10' 55'' \left(\frac{a-b}{a+b}\right)^2\right\}.$$

PROOF. We first examine the case $e = 0$. Then $x = 0$ and $\theta = 0$. From (3.2) and (3.6),

$$\frac{\tan\theta}{\theta} = 1 + \tfrac{1}{3}\theta^2 + \tfrac{2}{15}\theta^4 + \cdots = {}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; x) = \sum_{n=0}^{\infty} \alpha_n x^n, \tag{19.15}$$

for $|\theta| < \pi/2$ and $|x| \le 1$. Thus, from (19.14) and (19.15),

$$\lim_{x\to 0}\cos^2\varphi = \lim_{x\to 0}\frac{\tan^2\theta}{x} = \lim_{x\to 0}\frac{\theta^2}{x}\frac{\tan^2\theta}{\theta^2}$$

$$= \lim_{x\to 0}\frac{3\sum_{n=1}^{\infty}\alpha_n x^n\left(\sum_{n=0}^{\infty}\alpha_n x^n\right)^2}{x} = 3\alpha_1 = \frac{3}{4}.$$

Hence, $\varphi = \pi/6$.

We next determine φ when $e = 1$. Thus, $x = 1$ and $\tan\theta = \cos\varphi$. From (19.15) and (19.4),

$$\left.\frac{\tan\theta}{\theta}\right|_{x=1} = \sum_{n=0}^{\infty}\alpha_n = \frac{4}{\pi}. \qquad (19.16)$$

Hence, $\theta = \pi/4$ and $\varphi = 0$.

It appears to be extremely difficult to show that as x goes from 0 to 1, φ *monotonically* decreases from $\pi/6$ to 0. However, we can show that $0 \le \varphi \le \pi/6$. As a first step toward this end, we show that $g(x) := (\tan^{-1} x)/x$ is a monotonically decreasing function of x for $x \ge 0$. Now,

$$g'(x) = \frac{1}{x(1+x^2)} - \frac{\tan^{-1} x}{x^2}$$

$$= \frac{1}{x(1+x^2)} - \frac{1}{x^2}\int_0^x \frac{du}{1+u^2}$$

$$= \frac{1}{x^2}\int_0^x \left(\frac{1}{1+x^2} - \frac{1}{1+u^2}\right) du < 0.$$

It follows that g is monotonically decreasing for $x \ge 0$, as claimed.

Suppose that we can show that, for $0 \le x \le 1$,

$$\frac{\frac{1}{2}\sqrt{3x}}{\tan^{-1}(\frac{1}{2}\sqrt{3x})} \le \sum_{n=0}^{\infty}\alpha_n x^n = \frac{\tan\theta}{\theta} \le \frac{\sqrt{x}}{\tan^{-1}\sqrt{x}}. \qquad (19.17)$$

Since $x/\tan^{-1}x$ is increasing, it follows that

$$\tfrac{1}{2}\sqrt{3x} \le \tan\theta \le \sqrt{x},$$

or

$$\tfrac{1}{2}\sqrt{3} \le \cos\varphi \le 1;$$

that is, $0 \le \varphi \le \pi/6$. Thus, it remains to establish (19.17).

In Chapter 12, Section 18, Corollary 1 (Part II [9, p. 133]), Ramanujan records the continued fraction

$$\tan^{-1} x = \frac{x}{1+}\;\frac{x^2}{3+}\;\frac{(2x)^2}{5+}\;\frac{(3x)^2}{7+\cdots},$$

where here we take $x \ge 0$. (For a proof, see Perron's book [1, p. 155].) It follows that

$$\frac{\tan^{-1} x}{x} > \frac{1}{1 + x^2/3},$$

or

$$\frac{x}{\tan^{-1} x} < 1 + \tfrac{1}{3}x^2.$$

Hence, for $0 \le x \le 1$,

$$\frac{\frac{1}{2}\sqrt{3x}}{\tan^{-1}(\frac{1}{2}\sqrt{3x})} < 1 + \tfrac{1}{4}x < \sum_{n=0}^{\infty} \alpha_n x^n,$$

which establishes the first inequality of (19.17).

To prove the second inequality in (19.17), first define

$$G(x) := \tan^{-1} \sqrt{x} - \frac{\sqrt{x}}{F(x)},$$

where $F(x) = {}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; x)$. We want to show that $G(x) \le 0$, $0 \le x \le 1$. An elementary calculation gives

$$\begin{aligned} 2\sqrt{x}(1 + x)F^2(x)G'(x) &= F^2(x) - (1 + x)F(x) + 2x(1 + x)F'(x) \\ &= -\tfrac{1}{4}x + \sum_{n=2}^{\infty} \mu_n x^n, \end{aligned} \tag{19.18}$$

where $\mu_n > 0$, $2 \le n < \infty$. Equality (19.18) shows that $G'(x) < 0$ for $x > 0$ and x sufficiently small. It is also clear from (19.18) that $G'(x) = 0$ at most once on $[0, 1]$. But since $G(0) = 0$ and $G(1) = 0$, by (19.4) (or (19.16)), it follows that $G'(x) = 0$ exactly once on $(0, 1)$. Hence, $G(x) \le 0$ for $0 \le x \le 1$. Thus, the proof of (19.17) is complete.

Lastly, we establish Ramanujan's unusual approximation for φ. We observe that

$$\sqrt{1 - x} = \frac{2\sqrt{ab}}{a + b}$$

and

$$\sqrt{1 - x} - (1 - x) = \frac{2\sqrt{ab}}{(a + b)^2}(\sqrt{a} - \sqrt{b})^2.$$

Thus, Ramanujan is attempting to find an approximation to φ of the form

$$\sqrt{1 - x}(A + B\{1 - \sqrt{1 - x}\} + Cx), \tag{19.19}$$

which will be a good approximation both when x is close to 0 and when x is near 1. Our task is then to determine A, B, and C.

First, we find an approximation for φ in a neighborhood of $x = 0$. From (19.15), we seek an approximation for θ^2 in the form

$$\theta^2 = 3\alpha_1 x + \rho x^2 + \cdots,$$

where ρ may be determined by the equation

$$\alpha_1 x + \alpha_2 x^2 + \cdots = \tfrac{1}{3}(3\alpha_1 x + \rho x^2 + \cdots) + \tfrac{2}{15}(3\alpha_1 x + \rho x^2 + \cdots)^2 + \cdots.$$

Equating coefficients of x^2 on both sides, we find that

$$\alpha_2 = \tfrac{1}{3}\rho + \tfrac{6}{5}\alpha_1^2.$$

Solving for ρ, we find that $\rho = -\frac{57}{320}$. Hence,

$$\cos^2\varphi = \frac{\tan^2\theta\,\theta^2}{\theta^2\quad x} = (1 + \tfrac{1}{2}x + \cdots)(\tfrac{3}{4} - \tfrac{57}{320}x + \cdots)$$
$$= \tfrac{3}{4} + \tfrac{63}{320}x + \cdots.$$

Now write

$$\varphi = \frac{\pi}{6} + \beta x + \cdots.$$

Then

$$\begin{aligned}\tfrac{3}{4} + \tfrac{63}{320}x + \cdots &= \cos^2(\pi/6 + \beta x + \cdots)\\ &= \tfrac{1}{2}(1 + \cos(\pi/3 + 2\beta x + \cdots))\\ &= \tfrac{1}{2}(1 + \tfrac{1}{2}\cos(2\beta x + \cdots) - (\sqrt{3}/2)\sin(2\beta x + \cdots))\\ &= \tfrac{1}{2}(\tfrac{3}{2} + 2\beta^2x^2 + \cdots - \sqrt{3}\,\beta x + \cdots).\end{aligned}$$

Equating coefficients of x on both sides, we deduce that $\beta = -21\sqrt{3}/160$. Hence, in a neighborhood of $x = 0$, we have the following approximation for φ:

$$\varphi = \frac{\pi}{6} - \frac{21\sqrt{3}}{160}x + O(x^2). \tag{19.20}$$

Next, we want to approximate φ in a neighborhood of $x = 1$. As seen earlier in the proof, θ is then in a neighborhood of $\pi/4$. A straightforward calculation yields

$$\frac{\tan\theta}{\theta} = \frac{4}{\pi} + \frac{8(\pi - 2)}{\pi^2}\left(\theta - \frac{\pi}{4}\right) + \cdots, \qquad \left|\theta - \frac{\pi}{4}\right| < \frac{\pi}{4}. \tag{19.21}$$

Now ${}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; x)$ is not analytic at $x = 1$. However, for x in a neighborhood of 1, we can deduce that (Erdélyi [1, p. 110, Eq. (12)])

$$\frac{\tan\theta}{\theta} = {}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; x) = \frac{4}{\pi} + \frac{1}{\pi}(x - 1) + o(x - 1). \tag{19.22}$$

Comparing (19.21) and (19.22), we deduce the approximation

$$\theta = \frac{\pi}{4} + \frac{\pi}{8(\pi - 2)}(x - 1) + o(x - 1), \tag{19.23}$$

as x tends to 1.

Lastly, we need the elementary expansion

$$\frac{1}{\sqrt{x}} = 1 - \tfrac{1}{2}(x-1) + \cdots, \qquad |x-1| < 1. \tag{19.24}$$

Thus, from (19.22)–(19.24), we conclude that, as x tends to 1,

$$\begin{aligned}\cos\varphi &= \frac{\tan\theta}{\theta}\frac{\theta}{\sqrt{x}} \\ &= \left(\frac{4}{\pi} + \frac{1}{\pi}(x-1) + o(x-1)\right)\left(\frac{\pi}{4} + \frac{\pi}{8(\pi-2)}(x-1) + o(x-1)\right) \\ &\quad \times (1 - \tfrac{1}{2}(x-1) + o(x-1)) \\ &= 1 + \frac{4-\pi}{4(\pi-2)}(x-1) + o(x-1).\end{aligned}$$

Since $\cos\varphi = 1 - \frac{1}{2}\varphi^2 + \cdots$, we find that

$$\varphi^2 = \frac{4-\pi}{2(\pi-2)}(1-x) + o(1-x),$$

or

$$\varphi = \sqrt{\frac{4-\pi}{2\pi-4}}\sqrt{1-x} + o(\sqrt{1-x}), \tag{19.25}$$

as x tends to $1-$.

Our last task is then to use (19.20) and (19.25) in (19.19) to calculate A, B, and C. When x tends to 0, (19.19) tends to A. Thus, $A = \pi/6$ by (19.20). Next, examine $(\varphi - \pi/6)/x$ as x tends to 0. From (19.19) and (19.20), we find that

$$-\tfrac{1}{2}A + \tfrac{1}{2}B + C = -\frac{21\sqrt{3}}{160}. \tag{19.26}$$

Now check $\varphi/\sqrt{1-x}$ as x tends to $1-$. From (19.19) and (19.25), we see that

$$A + B + C = \sqrt{\frac{4-\pi}{2\pi-4}}. \tag{19.27}$$

Using the value $A = \pi/6$ and solving (19.26) and (19.27) simultaneously, we conclude that

$$B = 2\sqrt{\frac{4-\pi}{2\pi-4}} + \frac{21\sqrt{3}}{80} - \frac{\pi}{2} = 0.1101935$$

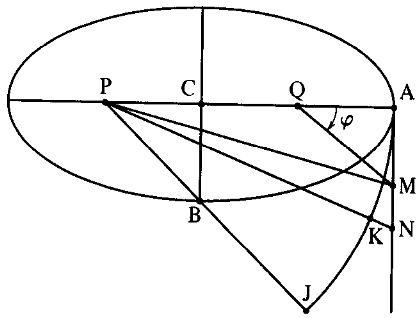
and

$$C = \frac{\pi}{3} - \sqrt{\frac{4-\pi}{2\pi-4}} - \frac{21\sqrt{3}}{80} = -0.0206291.$$

Converting A, B, and C to the sexagesimal system and substituting in (19.19), we finish the proof.

Before stating Corollary (ii), we describe a geometrical diagram given by Ramanujan. Consider an ellipse $x = a \cos u$, $y = b \sin u$, $0 \le u \le 2\pi$, with C as the center, $A = (a, 0)$, and $B = (0, b)$. Let AN be perpendicular to AC. With P and Q on the negative and positive x-axes, respectively, let $CP = CB = CQ$. Choose a point M on AN such that $\angle MPN = \frac{1}{2}\angle APM$. Furthermore, put $\varphi = \angle MQA$. Consider now a circle centered at P with radius PA. Suppose that this circle intersects PN at K. Let J denote that point on the circle obtained from the radius through B.

We abuse notation below in that, for example, PN may denote either the line segment with end points P and N or the length of this line segment. However, no confusion should arise.



Corollary (ii). *Let $L(UV)$ denote the length of an arc UV of an ellipse or circle. Suppose that PKN is rotated until*

$$\frac{L(AJ)}{L(AK)} = \frac{L(AB)}{AN}. \tag{19.28}$$

Then φ is approximately equal to

$$30° - x(1 - x)(11'22'' + 32'42''x), \tag{19.29}$$

where, as before, $x = ((a - b)/(a + b))^2$.

Ramanujan states this corollary in a somewhat retrorse manner, because (19.28) is given as part of his conclusion. He also made a calculational error and so incorrectly asserted that φ is approximately equal to

$$30° + x(1 - x)(5°19.4' - 6°3.5'x). \tag{19.30}$$

Ramanujan [7] also posed Corollary (ii) as a problem in the *Journal of the Indian Mathematical Society*. Villarino [1] has established a stronger version of Corollary (ii) in the spirit of Entries 19(iii) and 19(iv).

PROOF. We first show that $\varphi = \pi/6$ when $x = 0$ or 1. Observe that

$$\tan \varphi = \frac{AM}{a-b},$$

and so

$$\sqrt{x} \tan \varphi = \frac{AM}{a+b}.$$

Hence,

$$\tfrac{3}{2} \tan^{-1} (\sqrt{x} \tan \varphi) = \tfrac{3}{2} \tan^{-1} \left(\frac{AM}{a+b}\right) = \tfrac{3}{2} \angle APM = \angle APN, \qquad (19.31)$$

by Ramanujan's construction. Thus, if L denotes the perimeter of the ellipse, by (19.31) and (19.28),

$$\begin{aligned} \tan(\tfrac{3}{2} \tan^{-1} (\sqrt{x} \tan \varphi)) = \tan \angle APN &= \frac{AN}{a+b} = \frac{L(AB)L(AK)}{(a+b)L(AJ)} \\ &= \frac{L}{4(a+b)} \frac{L(AK)}{L(AJ)} \\ &= \frac{\pi}{4} \frac{L(AK)}{L(AJ)} {}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; x), \end{aligned} \qquad (19.32)$$

by (3.2).

We now show that

$$\angle APN = \frac{\pi}{4} \frac{L(AK)}{L(AJ)}. \qquad (19.33)$$

Since CPB is an isosceles right triangle, $\angle CPB = \pi/4$. Thus,

$$(\angle APJ)(a+b) = \frac{\pi}{4}(a+b) = L(AJ).$$

Also,

$$(\angle APN)(a+b) = L(AK).$$

Combining the last two equalities, we deduce (19.33).

Putting (19.33) in (19.32) and employing (19.31), we arrive at

$$\frac{\tan(\tfrac{3}{2} \tan^{-1} (\sqrt{x} \tan \varphi))}{\tfrac{3}{2} \tan^{-1} (\sqrt{x} \tan \varphi)} = {}_2F_1(-\tfrac{1}{2}, -\tfrac{1}{2}; 1; x). \qquad (19.34)$$

When $x = 1$, this equality reduces to

$$\frac{\tan(\tfrac{3}{2}\varphi)}{\tfrac{3}{2}\varphi} = \frac{4}{\pi},$$

by (19.4). Hence, $\varphi = \pi/6$.

For x sufficiently small,

$$\begin{aligned}\frac{\tan(\frac{3}{2}\tan^{-1}(\sqrt{x}\tan\varphi))}{\frac{3}{2}\tan^{-1}(\sqrt{x}\tan\varphi)} &= 1 + \tfrac{1}{3}(\tfrac{3}{2}\tan^{-1}(\sqrt{x}\tan\varphi))^2 + \tfrac{2}{15}(\tfrac{3}{2}\tan^{-1}(\sqrt{x}\tan\varphi))^4 + \cdots \\ &= 1 + \tfrac{3}{4}(\sqrt{x}\tan\varphi - \tfrac{1}{3}(\sqrt{x}\tan\varphi)^3 + \cdots)^2 \\ &\quad + \tfrac{27}{40}(\sqrt{x}\tan\varphi + \cdots)^4 + \cdots \\ &= 1 + \tfrac{3}{4}x\tan^2\varphi + \tfrac{7}{40}x^2\tan^4\varphi + \cdots. \end{aligned} \tag{19.35}$$

In order to obtain a first approximation for φ, we can ignore the fact that φ is a function of x. Thus, combining (19.34) and (19.35) and equating coefficients of x, we find that

$$\tfrac{3}{4}\tan^2\varphi = \tfrac{1}{4}.$$

Hence, we can conclude that φ tends to $\pi/6$ as x tends to 0.

Our procedure now is similar to that in the proof of Corollary (i). We find expansions for φ in neighborhoods of $x = 0$ and $x = 1$ and combine them together.

Thus, first we write

$$\varphi = \frac{\pi}{6} + \rho x + \cdots.$$

From (19.34) and (19.35),

$$\begin{aligned}\frac{1}{4}x + \frac{1}{64}x^2 + \cdots &= \frac{3}{4}x\tan^2\left(\frac{\pi}{6} + \rho x + \cdots\right) + \frac{7}{40}x^2\tan^4\left(\frac{\pi}{6} + \rho x + \cdots\right) + \cdots \\ &= \frac{3}{4}x\left(\frac{1}{3} + \frac{8\rho}{3\sqrt{3}}x + \cdots\right) + \frac{7}{40}x^2\left(\frac{1}{9} + \cdots\right) + \cdots.\end{aligned}$$

Equating coefficients of x^2 on both sides, we deduce that $\rho = -11\sqrt{3}/5760$. Thus, in a neighborhood of the origin,

$$\varphi = \frac{\pi}{6} - \frac{11\sqrt{3}}{5760}x + O(x^2). \tag{19.36}$$

Next, write

$$\varphi = \frac{\pi}{6} + \beta(x-1) + \cdots,$$

in a neighborhood of $x = 1$. From (19.22), (19.34), and a very laborious calculation, we find that

$$\begin{aligned}\frac{4}{\pi} + \frac{1}{\pi}(x-1) + o(x-1) &= \frac{\tan(\frac{3}{2}\tan^{-1}(\sqrt{x}\tan(\pi/6 + \beta(x-1) + \cdots)))}{\frac{3}{2}\tan^{-1}(\sqrt{x}\tan(\pi/6 + \beta(x-1) + \cdots))} \\ &= \frac{4}{\pi} + \frac{9}{\pi}\left\{\frac{1}{\sqrt{3}}\left(\frac{1}{2} - \frac{1}{\pi}\right) + \frac{4\beta}{3}\left(1 - \frac{2}{\pi}\right)\right\}(x-1) + \cdots.\end{aligned}$$

Equating coefficients of $(x-1)$ on both sides and solving for β, we deduce that

$$\beta = \frac{\pi}{12\pi - 24} - \frac{\sqrt{3}}{8}.$$

Thus, in neighborhood of $x = 1$,

$$\varphi = \frac{\pi}{6} + \left(\frac{\pi}{12\pi - 24} - \frac{\sqrt{3}}{8}\right)(x-1) + o(x-1). \tag{19.37}$$

We now want to combine the two approximations (19.36) and (19.37) into an estimate of the sort

$$\varphi \approx \frac{\pi}{6} - x(1-x)(D + Ex). \tag{19.38}$$

Examining $(\varphi - \pi/6)/x$ as x tends to 0, we deduce from (19.36) and (19.38) that

$$D = \frac{11\sqrt{3}}{5760} = 0.0033077.$$

Examining $(\varphi - \pi/6)/(x-1)$ as x tends to $1-$, we see from (19.37) and (19.38) that

$$D + E = \frac{\pi}{12\pi - 24} - \frac{\sqrt{3}}{8}.$$

Thus,

$$E = \frac{\pi}{12\pi - 24} - \frac{\sqrt{3}}{8} - \frac{11\sqrt{3}}{5760} = 0.0095141.$$

Putting these values of D and E in (19.38) and converting the numbers into the sexagesimal system, we complete the proof.

In a note following Corollary (ii), Ramanujan indicates a third value (besides $x = 0$ and $x = 1$) at which $\varphi = \pi/6$. He also records the values for which his version (19.30) of (19.29) achieves either a local maximum or minimum. In fact, from (19.29), we see that on $0 \le x \le 1$, the value $\pi/6$ is achieved only at the end points. An elementary calculation shows that (19.29) has a local minimum $29°52'33''$. This value is obtained at $x = 0.6213949$ or $e = 0.9929672$, since $e = 2x^{1/4}/(1 + \sqrt{x})$.

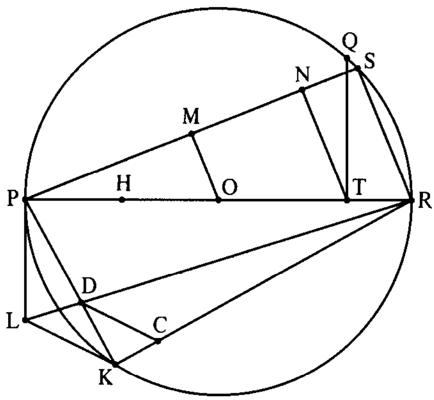
In Entry 20(i), Ramanujan attempts to "square the circle." His approximation in doing so is highly accurate and was given in his paper [1], [10, p. 22]. Since Ramanujan gave few details in [1], and since his argument is not very long, we give it below.

Entry 20(i). *Let O be the center and PR a diameter of a circle. Bisect OP at H and trisect OR at T; more precisely, $OT = 2TR$. Let TQ be perpendicular to OR, where Q is a point on the circle. Let $RS = TQ$, where S lies on the arc QR. Let OM and TN be parallel to RS, where M and N lie on PS. Let $PK = PM$, where K lies on the circle but on the opposite side of PR from Q and S. Draw*

PL perpendicular to OP and of length equal to MN, with L on the same side of PR as K. Let RC = RH, where C lies on RK. Draw CD parallel to KL, where D lies on RL. Then

$$RD^2 = \tfrac{1}{4}PR^2 \cdot \frac{355}{113}.$$

(We have again abused notation in that, for example, RC denotes the line segment with end points R and C as well as the length of this line segment.)



In fact, Ramanujan says that the area of the circle is (approximately) equal to RD^2. Thus, he is approximating π by

$$\frac{355}{113} = 3.14159292,$$

which differs from π by about 0.00000027. According to notes left by G. N. Watson, this approximation to π was discovered by the father of Adrian Metius. It might be recalled here that Hardy related that Ramanujan [3, p. xxxi] had "quite a small library of books by circle-squarers and other cranks."

PROOF. For brevity, set $d = PR$. Thus, $PT = \frac{5}{6}d$ and $TR = \frac{1}{6}d$. Therefore,

$$(\tfrac{5}{6}d)^2 + QT^2 = PQ^2 \quad \text{and} \quad (\tfrac{1}{6}d)^2 + QT^2 = QR^2.$$

Adding these equations and solving for QT^2, we find that

$$QT^2 = \tfrac{5}{36}d^2.$$

Also,

$$PS^2 = d^2 - RS^2 = d^2 - QT^2 = \tfrac{31}{36}d^2 \tag{20.1}$$

and

$$PK^2 = PM^2 = (\tfrac{1}{2}d)^2 - OM^2 = (\tfrac{1}{2}d)^2 - (\tfrac{1}{2}RS)^2 = \tfrac{31}{144}d^2.$$

Thus,

$$PL^2 = MN^2 = (\tfrac{1}{3}PS)^2 = \tfrac{31}{324}d^2$$

and

$$RK^2 = PR^2 - PK^2 = \tfrac{113}{144}d^2.$$

So,

$$RL^2 = PR^2 + PL^2 = \tfrac{355}{324}d^2.$$

Finally,

$$\frac{RD}{PR} = \frac{3RD}{4RC} = \frac{3RL}{4RK} = \frac{3}{4}\sqrt{\frac{355}{324}\frac{144}{113}} = \frac{1}{2}\sqrt{\frac{355}{113}}.$$

The desired result now follows.

In a note following Entry 20(i), Ramanujan remarks that "*RD* is $\frac{1}{100}$th of an inch greater than the true length if the given square is 14 square miles in area." Indeed, for a circle of area $\frac{1}{4}\pi d^2 = 14 \times 63360^2$ (in square inches),

$$\begin{aligned} RD - \tfrac{1}{2}d\sqrt{\pi} &= \tfrac{1}{2}d\sqrt{\frac{355}{113}} - \tfrac{1}{2}d\sqrt{\pi} \\ &= 63360\sqrt{14}\left(\sqrt{\frac{355}{113\pi}} - 1\right) \\ &= 0.0100653026, \end{aligned}$$

which justifies Ramanujan's claim.

The two mean proportionals between a and b are the two values x and y defined by

$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}.$$

Corollary (i). *Inscribe an equilateral triangle of side ℓ in the circle of Entry* 20(i). *Let m denote the first of two mean proportionals between ℓ and PS. Then $m/(\frac{1}{2}d\sqrt{\pi})$ differs from unity by approximately* 1/30,000.

PROOF. First, from elementary geometry, $\ell = \frac{1}{2}\sqrt{3}d$. From (20.1), $PS = \sqrt{\frac{31}{36}}d$. Thus, solving

$$\frac{\frac{1}{2}\sqrt{3}d}{m} = \frac{m}{y} = \frac{y}{\sqrt{\frac{31}{36}}d}$$

for m, we find that $m = \frac{1}{2}d(31)^{1/6}$. Now,

$$\frac{m}{\frac{1}{2}d\sqrt{\pi}} = \frac{(31)^{1/6}}{\sqrt{\pi}} \approx 1 - \frac{1}{29630},$$

which justifies the stated approximation.

We have rephrased and clarified Ramanujan's version of Corollary (i): "One of the two mean proportionals between a side of an equilateral triangle inscribed in the circle and the length PS is only less by 30000th part of it than the true length." In summary, Ramanujan has taken $(31)^{1/3}$ as an approximation to π. In fact,

$$(31)^{1/3} = 3.1413807,$$

which differs from π by about 0.00021.

Corollary (ii). *If we approximate π by $(97\frac{1}{2} - \frac{1}{11})^{1/4}$ in the expression $\frac{1}{2}d\sqrt{\pi}$, then if a circle of one million square miles is taken, the error made is approximately* 1/100th *of an inch.*

PROOF. Let $\frac{1}{4}\pi d^2 = 10^6 \times 63360^2$ square inches. Then

$$\begin{aligned}\tfrac{1}{2}d\sqrt{\pi} &- \tfrac{1}{2}d(97\tfrac{1}{2} - \tfrac{1}{11})^{1/8}\\ &= 10^3 \times 63360\left(1 - \frac{(97\frac{1}{2} - \frac{1}{11})^{1/8}}{\sqrt{\pi}}\right) = 0.0101561291,\end{aligned}$$

which justifies the given claim.

Recall that the approximation $(97\frac{1}{2} - \frac{1}{11})^{1/4}$ to π is given in Section 3. Our version of Corollary (ii) clarifies Ramanujan's original statement (p. 225).

The appearance below of Entries 20(ii) and 20(iii) is enigmatic indeed; there does not seem to be any connection between these entries and any other result in Chapter 18.

Entry 20(ii) is due to Euler [2], [4, pp. 428–458] and was rediscovered a century later by Hoppe [1]. Because the result is not widely known and a short proof can be given, we provide a proof of Entry 20(ii).

Entry 20(ii). *Parametric solutions of the equation*

$$A^3 + B^3 = C^2$$

are given by

$$A = 3n^3 + 6n^2 - n,$$

$$B = -3n^3 + 6n^2 + n,$$

and

$$C = 6n^2(3n^2 + 1),$$

where n is arbitrary.

PROOF. Assume that

$$A + B = 12n^2. \tag{20.2}$$

Thus, factoring $A^3 + B^3$, we find that

$$A^2 - AB + B^2 = \frac{C^2}{12n^2},$$

where we may assume that $n \neq 0$. Thus,

$$\begin{aligned}(A - B)^2 &= \tfrac{4}{3}(A^2 - AB + B^2) - \tfrac{1}{3}(A^2 + 2AB + B^2)\\ &= \frac{C^2}{9n^2} - 48n^4. \qquad (20.3)\end{aligned}$$

We next assume that C can be written in the form $C = n^2(\alpha n^2 + \beta)$ for some pair of integers α, β. Furthermore, we would like to write

$$(A - B)^2 = \tfrac{1}{9}n^2(\alpha n^2 - \beta)^2.$$

From (20.3), we see that if we choose $\alpha = 18$ and $\beta = 6$, both of these requirements can be met. Thus, we obtain the proposed formula for C, and we find that

$$A - B = 6n^3 - 2n. \qquad (20.4)$$

Solving (20.2) and (20.4) simultaneously, we derive the proffered parametric equations for A and B.

Entry 20(iii). *Parametric solutions of the equation*

$$A^3 + B^3 + C^3 = D^3$$

are given by

$$A = m^7 - 3(p + 1)m^4 + (3p^2 + 6p + 2)m,$$

$$B = 2m^6 - 3(2p + 1)m^3 + (3p^2 + 3p + 1),$$

$$C = m^6 - (3p^2 + 3p + 1),$$

and

$$D = m^7 - 3pm^4 + (3p^2 - 1)m,$$

where m and p denote arbitrary numbers.

This classical diophantine equation was perhaps first seriously discussed by Viète [1] in 1591. Euler [2], [4, pp. 428–458] completely solved the problem by finding the most general rational solution; Ramanujan's solution is less general. A general solution may be found in Hardy and Wright's text [1, pp. 199–201]. Now, in fact, sometime later, Ramanujan did find the most general solution and recorded it in his third notebook. See Berndt's book [11, Chap. 23, Entry 50] for details. A general solution in rational *integers* is not known. Many papers have been written on this venerable diophantine equation, and one should consult Dickson's book [1, pp. 550–561] for further references. In particular, there exist many ways in which to formulate solutions. Thus, we briefly indicate how Ramanujan's solution can be gotten.

PROOF. Some algebraic manipulation shows that the given equation may be put in the form

$$(B + C)\{3(B - C)^2 + (B + C)^2\} = (D - A)\{3(D + A)^2 + (D - A)^2\}. \quad (20.5)$$

Assume that

$$\frac{B + C}{D - A} = m^2. \quad (20.6)$$

(The most general solution, in fact, is obtained by putting $(B + C)/(D - A) = m^2 + 3k^2$.) After some additional manipulation, (20.5) can then be written

$$3\{m^2(B - C)^2 - (D + A)^2\} = (1 - m^6)(D - A)^2.$$

Now define n by

$$m(B - C) - (D + A) = nm(1 - m^6). \quad (20.7)$$

Thus,

$$3n\{m(B - C) + (D + A)\} = \frac{(D - A)^2}{m}. \quad (20.8)$$

From these last two equations, we see that multiplying A, B, C, and D by the same constant has the effect of multiplying n by the same constant, and conversely. Thus, without loss of generality, we can set $n = 1$. Solving (20.7) and (20.8) simultaneously for $B - C$ and $D + A$, we deduce that

$$B - C = \tfrac{1}{2}(1 - m^6) + \frac{1}{6m^2}(D - A)^2 \quad (20.9)$$

and

$$D + A = -\tfrac{1}{2}m(1 - m^6) + \frac{1}{6m}(D - A)^2. \quad (20.10)$$

Lastly, we solve for A, B, C, and D in terms of m and $D - A$. To avoid fractions, we introduce another parameter p defined by

$$D - A = 3m(m^3 - 1 - 2p). \quad (20.11)$$

The remainder of the proof is quite straightforward. Equalities (20.6) and (20.9) yield the proposed values of B and C; equalities (20.10) and (20.11) give the stated formulas for A and D.

Entry 20(iii) was discussed by Watson in his survey paper on the notebooks [5].

Ramanujan offers several examples to illustrate Entries 20(ii) and 20(iii). For each example, we append the values of the parameters needed to produce the example.

$$(11\tfrac{1}{2})^3 + (\tfrac{1}{2})^3 = 39^2 \quad (n = 2)$$

$$(3 - \tfrac{1}{105})^3 + (\tfrac{1}{105})^3 = (5\tfrac{6}{35})^2 \quad (n = \tfrac{15}{7})$$

$$(3\tfrac{1}{7})^3 - (\tfrac{1}{7})^3 = (5\tfrac{4}{7})^2 \quad (n = \tfrac{1}{7})$$

$$(3\tfrac{1}{104})^3 - (\tfrac{1}{104})^3 = (5\tfrac{23}{104})^2 \quad (n = \tfrac{13}{6})$$

$$11^3 + 37^3 = 228^2 \quad (n = \tfrac{1}{4})$$

$$71^3 - 23^3 = 588^2 \quad (n = 4)$$

$$3^3 + 4^3 + 5^3 = 6^3 \quad (m = p = 2)$$

$$1^3 + 12^3 = 9^3 + 10^3 \quad (m = 2, p = 3)$$

$$1^3 + 75^3 = (70\tfrac{1}{2})^3 + (41\tfrac{1}{2})^3 \quad (m = 3, p = 11)$$

$$3^3 + 509^3 + 34^6 = 1188^3 \quad (m = 4, p = \tfrac{53}{2})$$

$$18^3 + 19^3 + 21^3 = 28^3 \quad (m = 2, p = 1)$$

$$7^3 + 14^3 + 17^3 = 20^3 \quad (m = 2, p = -1)$$

$$19^3 + 60^3 + 69^3 = 82^3 \quad (m = 2, p = -2)$$

$$15^3 + 82^3 + 89^3 = 108^3 \quad (m = 2, p = -3)$$

$$3^3 + 36^3 + 37^3 = 46^3 \quad (m = 2, p = -4)$$

$$1^3 + 135^3 + 138^3 = 172^3 \quad (m = 2, p = -5)$$

$$23^3 + 134^3 = 95^3 + 116^3 \quad (m = 2, p = -10)$$

$$133^3 + 174^3 = 45^3 + 196^3 \quad (m = 2, p = -13)$$

$$1^3 + 6^3 + 8^3 = 9^3.$$

Observe that the example $1^3 + 6^3 + 8^3 = 9^3$ does not fall under the purview of Entry 20(iii). Thus, evidently, Ramanujan was aware that he had not found the most general solution of $A^3 + B^3 + C^3 = D^3$. The example $1^3 + 12^3 = 9^3 + 10^3$ was immortalized by Hardy, who, when writing about his recently deceased friend, recalled "I remember once going to see him when he was lying ill at Putney. I had ridden in taxi-cab no. 1729, and remarked that the number $(7 \cdot 13 \cdot 19)$ seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways'" (Ramanujan [10, p. xxxv]).

Hardy then asked Ramanujan if he knew the corresponding result for fourth powers. After thinking a moment, he replied that he did not know the answer and supposed that the first number is very large. In fact, the smallest solution

$$133^4 + 134^4 = 158^4 + 59^4 = 635{,}318{,}657$$

had been found by Euler [2], [4, pp. 428–458].

Actually, the example $1^3 + 12^3 = 9^3 + 10^3$ was found much earlier by Frenicle in 1657. Frenicle and J. Wallis each found several additional examples for two equal sums of two cubes. A bitter argument between Frenicle and Wallis ensued with each accusing the other of using trivial methods. A description of this feud may be found in Dickson's book [1, p. 552]. For more complete details, see Fermat's *Oeuvres* [1, Lettre X. Vicomte Brouncker à John Wallis, pp. 419–420; Lettre XVI. John Wallis à Kenelm Digby, pp. 427–457].

In 1898, C. Moreau [1] found the ten solutions of $A^3 + B^3 = C^3 + D^3$, where the sum is less than 100,000. After 1729, the next largest sum is $4104 = 2^3 + 16^3 = 9^3 + 15^3$.

The example $1^3 + 12^3 = 9^3 + 10^3$ can also be found on a fragment in the publication of Ramanujan's "lost notebook" [11, p. 341].

Entry 21. *Let x be any complex number, except that x cannot be a pole of the functions given on the left sides below. Then*

(i)
$$\frac{\pi}{3\sqrt{3}x^2}\frac{\cosh(\pi\sqrt{3}x) + 2\cos(\pi x)}{\cosh(\pi\sqrt{3}x) - \cos(\pi x)} = \frac{1}{2\sqrt{3}\pi x^4} + \sum_{n=1}^{\infty}\frac{n}{x^4 + n^2x^2 + n^4} + 2\sum_{n=1}^{\infty}\frac{(-1)^n n}{(x^4 + n^2x^2 + n^4)(e^{n\pi\sqrt{3}} - (-1)^n)},$$

(ii)
$$\frac{\pi}{3\sqrt{3}x^2}\frac{\cosh(\pi\sqrt{3}x) + 2\cos(\pi x) + 6\cosh(\pi x/\sqrt{3})}{\cosh(\pi\sqrt{3}x) - \cos(\pi x)} = \frac{\sqrt{3}}{2\pi x^4} + \sum_{n=1}^{\infty}\frac{n}{x^4 + n^2x^2 + n^4} + 2\sum_{n=1}^{\infty}\frac{(-1)^n n}{(x^4 + n^2x^2 + n^4)(e^{n\pi\sqrt{3}} - (-1)^n)} + 2\sum_{n=1}^{\infty}\frac{(-1)^n n\rho(n)}{x^4 + n^2x^2 + n^4},$$

where

$$\rho(n) = \begin{cases} \dfrac{\cosh(\frac{1}{2}\pi n/\sqrt{3})}{\sinh(\frac{1}{2}\pi n\sqrt{3})}, & \textit{if } n \textit{ is even}, \\[2ex] \dfrac{\sinh(\frac{1}{2}\pi n/\sqrt{3})}{\cosh(\frac{1}{2}\pi n\sqrt{3})}, & \textit{if } n \textit{ is odd}, \end{cases}$$

(iii)
$$\frac{2\pi}{\sqrt{3}x^2}\frac{1}{e^{2\pi\sqrt{3}x} - 2e^{\pi\sqrt{3}x}\cos(\pi x) + 1} = \frac{1}{2\sqrt{3}\pi x^4} - \frac{1}{2x^3} + \frac{2\pi}{3\sqrt{3}x^2} + 2\sum_{n=1}^{\infty}\frac{(-1)^n n}{(x^4 + n^2x^2 + n^4)(e^{n\pi\sqrt{3}} - (-1)^n)} - \frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{x^2 + nx + n^2},$$

and

(iv)
$$\frac{2\pi}{\sqrt{3}x^2}\frac{1}{e^{2\pi\sqrt{3}x}-2e^{\pi\sqrt{3}x}\cos(3\pi x)+1}$$
$$=\frac{1}{6\sqrt{3}\pi x^4}-\frac{1}{6x^3}+\frac{\pi}{3\sqrt{3}x^2}$$
$$+6\sum_{n=1}^{\infty}\frac{(-1)^n n}{(9x^4-3n^2x^2+n^4)(e^{n\pi\sqrt{3}}-(-1)^n)}$$
$$-\frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{3x^2+3nx+n^2}.$$

Ramanujan's formulation of Entry 21(ii) in the notebooks (p. 226) does not appear to be correct.

PROOF OF (i). Let

$$f(x):=\frac{\pi}{3\sqrt{3}x^2}\frac{\cosh(\pi\sqrt{3}x)+2\cos(\pi x)}{\cosh(\pi\sqrt{3}x)-\cos(\pi x)}. \tag{21.1}$$

We shall expand f into partial fractions.

We note that f has a quadruple pole at $x=0$ and simple poles at $x=n\exp(\pm\pi i/3)$, $1\le|n|<\infty$. After a moderate amount of calculation and considerable simplification, we find that

$$R_{n\exp(-\pi i/3)}=\begin{cases}\dfrac{ie^{\pi i/3}}{2\sqrt{3}n^2}\coth(\tfrac{1}{2}\pi n\sqrt{3}), & \text{if } n \text{ is even},\\[2ex] \dfrac{ie^{\pi i/3}}{2\sqrt{3}n^2}\tanh(\tfrac{1}{2}\pi n\sqrt{3}), & \text{if } n \text{ is odd},\end{cases}$$

and

$$R_{n\exp(\pi i/3)}=\begin{cases}-\dfrac{ie^{-\pi i/3}}{2\sqrt{3}n^2}\coth(\tfrac{1}{2}\pi n\sqrt{3}), & \text{if } n \text{ is even},\\[2ex] -\dfrac{ie^{-\pi i/3}}{2\sqrt{3}n^2}\tanh(\tfrac{1}{2}\pi n\sqrt{3}), & \text{if } n \text{ is odd},\end{cases}$$

where $1\le|n|<\infty$.

Consider now the contributions of the four poles $\pm n\exp(\pm\pi i/3)$ to the partial fraction decomposition of f. For n even, these contributions total

$$\frac{n}{x^4+n^2x^2+n^4}\left(1+\frac{2}{e^{n\pi\sqrt{3}}-1}\right), \tag{21.2}$$

while for n odd, they sum to

$$\frac{n}{x^4+n^2x^2+n^4}\left(1-\frac{2}{e^{n\pi\sqrt{3}}+1}\right), \tag{21.3}$$

where $1 \leq n < \infty$. We also find, after a straightforward calculation, that

$$f(x) = \frac{1}{2\sqrt{3}\pi x^4} + O(1) \tag{21.4}$$

in a neighborhood of $x = 0$. Summing (21.2) and (21.3) over all even n and odd n, respectively, $1 \leq n < \infty$, and using (21.4), we deduce the partial fraction expansion claimed in Entry 21(i).

PROOF OF (ii). Let

$$h(x) := f(x) + g(x),$$

where f is defined by (21.1) and

$$g(x) := \frac{2\pi}{3\sqrt{3}x^2} \frac{\cosh(\pi x/\sqrt{3})}{\cosh(\pi\sqrt{3}x) - \cos(\pi x)}.$$

We shall determine the partial fraction decomposition of g and combine it with that of f from Entry 21(i) to obtain the partial fraction expansion of h claimed in Entry 21(ii).

Observe that the poles of g are the same as those for f, with the same orders. By routine calculations, for the poles of g,

$$R_{n\exp(-\pi i/3)} = \begin{cases} \dfrac{ie^{\pi i/3}\cosh(\frac{1}{2}\pi n/\sqrt{3})}{\sqrt{3}n^2 \sinh(\frac{1}{2}\pi n\sqrt{3})}, & \text{if } n \text{ is even,} \\ -\dfrac{ie^{\pi i/3}\sinh(\frac{1}{2}\pi n/\sqrt{3})}{\sqrt{3}n^2 \cosh(\frac{1}{2}\pi n\sqrt{3})}, & \text{if } n \text{ is odd,} \end{cases}$$

and

$$R_{n\exp(\pi i/3)} = \begin{cases} -\dfrac{ie^{-\pi i/3}\cosh(\frac{1}{2}\pi n/\sqrt{3})}{\sqrt{3}n^2 \sinh(\frac{1}{2}\pi n\sqrt{3})}, & \text{if } n \text{ is even,} \\ \dfrac{ie^{-\pi i/3}\sinh(\frac{1}{2}\pi n/\sqrt{3})}{\sqrt{3}n^2 \cosh(\frac{1}{2}\pi n\sqrt{3})}, & \text{if } n \text{ is odd,} \end{cases}$$

where $1 \leq |n| < \infty$.

The four poles $\pm n\exp(\pm \pi i/3)$ contribute

$$\frac{2n\cosh(\frac{1}{2}\pi n/\sqrt{3})}{\sinh(\frac{1}{2}\pi n\sqrt{3})(x^4 + n^2x^2 + n^4)}, \quad \text{if } n \text{ is even,} \tag{21.5}$$

and

$$-\frac{2n\sinh(\frac{1}{2}\pi n/\sqrt{3})}{\cosh(\frac{1}{2}\pi n\sqrt{3})(x^4 + n^2x^2 + n^4)}, \quad \text{if } n \text{ is odd,} \tag{21.6}$$

to the partial fraction expansion of g. Finally, in a neighborhood of $x = 0$,

$$g(x) = \frac{1}{\pi\sqrt{3}x^4} + O(1). \tag{21.7}$$

From (21.5)–(21.7), we may determine the partial fraction decomposition of g. Combining this with the expansion of f from Entry 21(i), we deduce the desired expansion for h.

PROOF OF (iii). Let

$$f(x) := \frac{2\pi}{\sqrt{3}x^2} \frac{1}{e^{2\pi\sqrt{3}x} - 2e^{\pi\sqrt{3}x}\cos(\pi x) + 1},$$

which we wish to expand in partial fractions. Observe that f has a quadruple pole at the origin and simple poles at $x = \pm n\exp(\pm\pi i/3)$, $1 \le n < \infty$. Straightforward, but not quick, calculations show that

$$R_{n\exp(-\pi i/3)} = \begin{cases} \dfrac{ie^{\pi i/3}e^{-\pi n\sqrt{3}/2}}{2\sqrt{3}n^2\sinh(\frac{1}{2}\pi n\sqrt{3})}, & \text{if } n \text{ is even,} \\ -\dfrac{ie^{\pi i/3}e^{-\pi n\sqrt{3}/2}}{2\sqrt{3}n^2\cosh(\frac{1}{2}\pi n\sqrt{3})}, & \text{if } n \text{ is odd,} \end{cases}$$

and

$$R_{n\exp(\pi i/3)} = \begin{cases} -\dfrac{ie^{-\pi i/3}e^{-\pi n\sqrt{3}/2}}{2\sqrt{3}n^2\sinh(\frac{1}{2}\pi n\sqrt{3})}, & \text{if } n \text{ is even,} \\ \dfrac{ie^{-\pi i/3}e^{-\pi n\sqrt{3}/2}}{2\sqrt{3}n^2\cosh(\frac{1}{2}\pi n\sqrt{3})}, & \text{if } n \text{ is odd,} \end{cases}$$

where $1 \le |n| < \infty$.

Adding the principal parts for the four poles $x = \pm n\exp(\pm\pi i/3)$, we obtain the expressions

$$\frac{x^3\sinh(\frac{1}{2}\pi n\sqrt{3}) + n^3\cosh(\frac{1}{2}\pi n\sqrt{3})}{n^2\sinh(\frac{1}{2}\pi n\sqrt{3})(x^4 + n^2x^2 + n^4)}, \quad \text{if } n \text{ is even,}$$

and

$$\frac{x^3\cosh(\frac{1}{2}\pi n\sqrt{3}) + n^3\sinh(\frac{1}{2}\pi n\sqrt{3})}{n^2\cosh(\frac{1}{2}\pi n\sqrt{3})(x^4 + n^2x^2 + n^4)}, \quad \text{if } n \text{ is odd,}$$

where $1 \le n < \infty$. Lastly, the principal part about $x = 0$ is found to be

$$\frac{1}{2\sqrt{3}\pi x^4} - \frac{1}{2x^3} + \frac{2\pi}{3\sqrt{3}x^2} - \frac{\pi^2}{6x}.$$

Hence,

$$\frac{2\pi}{\sqrt{3}x^2} \frac{1}{e^{2\pi\sqrt{3}x} - 2e^{\pi\sqrt{3}x}\cos(\pi x) + 1}$$
$$= \frac{1}{2\sqrt{3}\pi x^4} - \frac{1}{2x^3} + \frac{2\pi}{3\sqrt{3}x^2}$$

$$-\frac{\pi^2}{6x}+x^3\sum_{n=1}^{\infty}\frac{1}{n^2(x^4+n^2x^2+n^4)}+\sum_{n=1}^{\infty}\frac{n}{x^4+n^2x^2+n^4}$$
$$+2\sum_{n=1}^{\infty}\frac{(-1)^n n}{(x^4+n^2x^2+n^4)(e^{n\pi\sqrt{3}}-(-1)^n)}.$$

Replacing $\pi^2/6$ by $\sum_{n=1}^{\infty}n^{-2}$, we find, after a short calculation, that

$$-\frac{\pi^2}{6x}+x^3\sum_{n=1}^{\infty}\frac{1}{n^2(x^4+n^2x^2+n^4)}+\sum_{n=1}^{\infty}\frac{n}{x^4+n^2x^2+n^4}$$
$$=-\frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{x^2+nx+n^2}.$$

Substituting this into the penultimate equality, we complete the proof.

PROOF OF (iv). Let

$$f(x):=\frac{2\pi}{\sqrt{3}x^2}\frac{1}{e^{2\pi\sqrt{3}x}-2e^{\pi\sqrt{3}x}\cos(3\pi x)+1},$$

which we now expand in partial fractions. We see that f has a quadruple pole at $x=0$ and simple poles at $x=\pm in\exp(\pm\pi i/3)/\sqrt{3}$, $1\le n<\infty$. Routine calculations yield

$$R_{n\exp(\pi i/3)/i\sqrt{3}}=\begin{cases}-\dfrac{e^{-\pi i/3}e^{-\pi n\sqrt{3}/2}}{2n^2\sinh(\frac{1}{2}\pi n\sqrt{3})}, & \text{if } n \text{ is even,}\\[2ex] \dfrac{e^{-\pi i/3}e^{-\pi n\sqrt{3}/2}}{2n^2\cosh(\frac{1}{2}\pi n\sqrt{3})}, & \text{if } n \text{ is odd,}\end{cases}$$

and

$$R_{n\exp(-\pi i/3)/i\sqrt{3}}=\begin{cases}\dfrac{e^{\pi i/3}e^{\pi n\sqrt{3}/2}}{2n^2\sinh(\frac{1}{2}\pi n\sqrt{3})}, & \text{if } n \text{ is even,}\\[2ex] \dfrac{e^{\pi i/3}e^{\pi n\sqrt{3}/2}}{2n^2\cosh(\frac{1}{2}\pi n\sqrt{3})}, & \text{if } n \text{ is odd,}\end{cases}$$

where $1\le|n|<\infty$.

The contributions of the four poles $\pm in\exp(\pm\pi i/3)/\sqrt{3}$ to the partial fraction expansion of f total

$$\frac{x^3\sinh(\frac{1}{2}\pi n\sqrt{3})-\frac{2}{3}n^2x\sinh(\frac{1}{2}\pi n\sqrt{3})+\frac{1}{3}n^3\cosh(\frac{1}{2}\pi n\sqrt{3})}{n^2\sinh(\frac{1}{2}\pi n\sqrt{3})(x^4-\frac{1}{3}n^2x^2+\frac{1}{9}n^4)},$$

if n is even, and

$$\frac{x^3\cosh(\frac{1}{2}\pi n\sqrt{3})-\frac{2}{3}n^2x\cosh(\frac{1}{2}\pi n\sqrt{3})+\frac{1}{3}n^3\sinh(\frac{1}{2}\pi n\sqrt{3})}{n^2\cosh(\frac{1}{2}\pi n\sqrt{3})(x^4-\frac{1}{3}n^2x^2+\frac{1}{9}n^4)},$$

if n is odd. Lastly, the principal part of f about $x = 0$ is equal to

$$\frac{1}{6\sqrt{3}\pi x^4} - \frac{1}{6x^3} + \frac{\pi}{3\sqrt{3}x^2} - \frac{\pi^2}{6x}.$$

Summing all of the principal parts, we arrive at

$$\begin{aligned}
&\frac{2\pi}{\sqrt{3}x^2}\,\frac{1}{e^{2\pi\sqrt{3}x} - 2e^{\pi\sqrt{3}x}\cos(3\pi x) + 1} \\
&\quad = \frac{1}{6\sqrt{3}\pi x^4} - \frac{1}{6x^3} + \frac{\pi}{3\sqrt{3}x^2} \\
&\qquad - \frac{\pi^2}{6x} + 9\sum_{n=1}^{\infty} \frac{x^3 - \frac{2}{3}n^2 x}{n^2(9x^4 - 3n^2x^2 + n^4)} + 3\sum_{n=1}^{\infty} \frac{n}{9x^4 - 3n^2x^2 + n^4} \\
&\qquad + 6\sum_{n=1}^{\infty} \frac{(-1)^n n}{(9x^4 - 3n^2x^2 + n^4)(e^{n\pi\sqrt{3}} - (-1)^n)}.
\end{aligned}$$

Replacing $\pi^2/6$ by $\sum_{n=1}^{\infty} n^{-2}$, we find, after a simple calculation, that

$$\begin{aligned}
&-\frac{\pi^2}{6x} + 9\sum_{n=1}^{\infty} \frac{x^3 - \frac{2}{3}n^2 x}{n^2(9x^4 - 3n^2x^2 + n^4)} + 3\sum_{n=1}^{\infty} \frac{n}{9x^4 - 3n^2x^2 + n^4} \\
&\quad = -\frac{1}{x}\sum_{n=1}^{\infty} \frac{1}{3x^2 + 3nx + n^2}.
\end{aligned}$$

Putting this in the penultimate equality, we deduce the desired partial fraction decomposition.

Example.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{81 + 9n^2 + n^4} = \frac{1}{324\sqrt{3}\pi} + \frac{11}{756} + \frac{\pi}{27\sqrt{3}} + \frac{\pi}{18\sqrt{3}(1 + \cosh(3\sqrt{3}\pi))}.$$

Our version of this example differs from that of Ramanujan (p. 226) in two respects. He has 25/756 instead of 11/756 and $-\pi/(54\sqrt{3})$ rather than $\pi/(27\sqrt{3})$. If in (21.8) below, there appeared 1/108 instead of $-1/108$, then we would obtain 25/756. We have no explanation for Ramanujan's other numerical error.

Proof. Putting $x = 3$ in Entry 21(iii) and rearranging terms, we find that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{(81 + 9n^2 + n^4)(e^{n\pi\sqrt{3}} - (-1)^n)} \\
&\quad = \frac{1}{324\sqrt{3}\pi} - \frac{1}{108} + \frac{\pi}{27\sqrt{3}} - \frac{1}{6}\sum_{n=1}^{\infty} \frac{1}{9 + 3n + n^2} \\
&\qquad - \frac{\pi e^{-3\sqrt{3}\pi}}{18\sqrt{3}(1 + \cosh(3\sqrt{3}\pi))}. \qquad (21.8)
\end{aligned}$$

To evaluate the series on the right side of (21.8), we employ the formula (Hansen [1, p. 105])

$$\sum_{n=0}^{\infty} \frac{1}{(nx+y)^2 - z^2} = \frac{1}{2xz}\left\{\psi\left(\frac{y+z}{x}\right) - \psi\left(\frac{y-z}{x}\right)\right\}, \tag{21.9}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$. Letting $x = 1$, $y = \frac{3}{2}$, and $z = \frac{1}{2}(3\sqrt{3})i$, we find that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2+3n+9} &= \frac{1}{3\sqrt{3}i}\left\{\psi\left(\frac{3}{2} + \frac{3\sqrt{3}i}{2}\right) - \psi\left(\frac{3}{2} - \frac{3\sqrt{3}i}{2}\right)\right\} \\ &= \frac{1}{3\sqrt{3}i}\left\{\frac{2}{1+3\sqrt{3}i} + \psi\left(\frac{1}{2} + \frac{3\sqrt{3}i}{2}\right)\right. \\ &\qquad \left. - \frac{2}{1-3\sqrt{3}i} - \psi\left(\frac{1}{2} - \frac{3\sqrt{3}i}{2}\right)\right\}.\end{aligned}$$

Since (Gradshteyn and Ryzhik [1, p. 945])

$$\psi(\tfrac{1}{2}+z) - \psi(\tfrac{1}{2}-z) = \pi \tan(\pi z), \tag{21.10}$$

we conclude that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2+3n+9} &= -\frac{1}{7} + \frac{\pi}{3\sqrt{3}} \tanh(\tfrac{1}{2}(3\sqrt{3})\pi) \\ &= -\frac{1}{7} + \frac{\pi}{3\sqrt{3}} \frac{\sinh(3\sqrt{3}\pi)}{1+\cosh(3\sqrt{3}\pi)}.\end{aligned}$$

Substituting this evaluation in (21.8) and simplifying, we complete the proof.

Ramanujan concludes Section 21 with a note claiming that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+nx+y}$$

can be evaluated exactly if x is an integer and y is arbitrary. This assertion is, indeed, correct, for (21.9) can first be used to write the sum in terms of ψ-functions. Using (21.10), the recursion formula

$$\psi(z+1) = \psi(z) + \frac{1}{z},$$

and (Gradshteyn and Ryzhik [1, p. 945])

$$\psi(1-z) = \psi(z) + \pi \cot(\pi z),$$

we can reduce this evaluation to elementary functions.

Entry 22. *If* $0 < x < 1$ *and* $n > 0$, *then*

(i) $$\int_0^{\infty} \exp\left(-n\int_0^{\varphi} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}\right) d\varphi = \frac{1}{n+}\,\frac{x}{n+}\,\frac{4}{n+}\,\frac{9x}{n+}\,\frac{16}{n+}\cdots,$$

(ii)
$$\int_0^\infty \exp\left(-n\int_0^\varphi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}\right)\frac{\cos\varphi}{\sqrt{1-x\sin^2\varphi}}\,d\varphi$$
$$=\frac{1}{n+}\frac{1}{n+}\frac{4x}{n}\,\frac{9}{+n+}\frac{16x}{n}+\cdots,$$

and

(iii)
$$\int_0^\infty \exp\left(-n\int_0^\varphi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}\right)\frac{\cos\varphi}{1-x\sin^2\varphi}\,d\varphi$$
$$=\frac{1}{n+}\frac{1-x}{n}\,\frac{4x}{-n+}\frac{9(1-x)}{n}\,\frac{16x}{-n}+\cdots.$$

PROOF OF (i). Let
$$u := \int_0^\varphi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}. \tag{22.1}$$

Then $\sin\varphi = \text{sn}\,u$ and $d\varphi/du = \text{dn}\,u$. Hence,
$$\int_0^\infty \exp\left(-n\int_0^\varphi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}\right)d\varphi = \int_0^\infty e^{-nu}\,\text{dn}\,u\,du.$$

However, Stieltjes [1], [2] and Rogers [3] have shown that
$$\int_0^\infty e^{-nu}\,\text{dn}\,u\,du = \frac{1/n}{1}\,\frac{x/n^2}{+\;1}\,\frac{2^2/n^2}{+\;1}\,\frac{3^2x/n^2}{+\;1}\,\frac{4^2/n^2}{+\;1}+\cdots,$$

which is easily seen to be equivalent to Ramanujan's formula.

PROOF OF (ii). Let u be given by (22.1). Then $d\varphi/du = \text{dn}\,u = \sqrt{1-x\,\text{sn}^2 u} = \sqrt{1-x\sin^2\varphi}$ and $\cos\varphi = \text{cn}\,u$. Hence,
$$\int_0^\infty \exp\left(-n\int_0^\varphi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}\right)\frac{\cos\varphi}{\sqrt{1-x\sin^2\varphi}}\,d\varphi = \int_0^\infty e^{-nu}\,\text{cn}\,u\,du.$$

But by a result of Stieltjes [1], [2] and Rogers [3] (see also (13.3)),
$$\int_0^\infty e^{-nu}\,\text{cn}\,u\,du = \frac{1}{n+}\frac{1^2}{n+}\frac{2^2x}{n}\,\frac{3^2}{+n+}\frac{4^2x}{n}+\cdots, \tag{22.2}$$

and the proof is complete.

PROOF OF (iii). Using the same substitutions as in the two proofs above, we find that
$$\int_0^\infty \exp\left(-n\int_0^\varphi \frac{d\theta}{\sqrt{1-x\sin^2\theta}}\right)\frac{\cos\varphi}{1-x\sin^2\varphi}\,d\varphi = \int_0^\infty e^{-nu}\frac{\text{cn}\,u}{\text{dn}\,u}\,du.$$

In the notation of Jacobi, sin coam $u = \text{cn}\,u/\text{dn}\,u$, and he [1], [2, p. 147] has shown that sin coam $u = \mathrm{cn}(k'u, ik/k')$. Hence, the integral on the right side above may be written

$$\int_0^\infty e^{-nu} \operatorname{cn}\left(k'u, \frac{ik}{k'}\right) du = \frac{1}{k'} \int_0^\infty e^{-nu/k'} \operatorname{cn}\left(u, \frac{ik}{k'}\right) du.$$

We may now employ (22.2) with k replaced by ik/k' to find that

$$\frac{1}{k'} \int_0^\infty e^{-nu/k'} \operatorname{cn}\left(u, \frac{ik}{k'}\right) du$$
$$= \frac{1}{k'}\left(\frac{1}{n/k'+}\,\frac{1^2}{n/k'+}\,\frac{2^2(ik/k')^2}{n/k'+}\,\frac{3^2}{n/k'+}\,\frac{4^2(ik/k')^2}{n/k'+}\cdots\right)$$
$$= \frac{1}{n+}\,\frac{1^2(1-x)}{n-}\,\frac{2^2x}{n+}\,\frac{3^2(1-x)}{n-}\,\frac{4^2x}{n+}\cdots.$$

In order to ensure that this continued fraction converges, we appeal to a theorem in Perron's text [1, p. 53, Satz 2.16]. The proof is now finished.

Entry 23. *Let x and y be complex with* $\operatorname{Re}(x \pm iy) > 0$. *Then*

(i)
$$\sqrt{2}\left\{\frac{1}{2} + \sum_{n=1}^\infty \exp\left(-\frac{n^2\pi x}{x^2+y^2}\right)\cos\left(\frac{n^2\pi y}{x^2+y^2}\right)\right\}$$
$$= (\sqrt{x^2+y^2}+x)^{1/2}\left\{\frac{1}{2} + \sum_{n=1}^\infty e^{-n^2\pi x}\cos(n^2\pi y)\right\}$$
$$+ (\sqrt{x^2+y^2}-x)^{1/2}\sum_{n=1}^\infty e^{-n^2\pi x}\sin(n^2\pi y)$$

and

(ii)
$$\sqrt{2}\sum_{n=1}^\infty \exp\left(-\frac{n^2\pi x}{x^2+y^2}\right)\sin\left(\frac{n^2\pi y}{x^2+y^2}\right)$$
$$= (\sqrt{x^2+y^2}-x)^{1/2}\left\{\frac{1}{2} + \sum_{n=1}^\infty e^{-n^2\pi x}\cos(n^2\pi y)\right\}$$
$$- (\sqrt{x^2+y^2}+x)^{1/2}\sum_{n=1}^\infty e^{-n^2\pi x}\sin(n^2\pi y).$$

PROOF. As we shall see, both (i) and (ii) follow from the inversion formula

$$\sqrt{\alpha}\left\{\frac{1}{2} + \sum_{n=1}^\infty e^{-\alpha^2 n^2}\right\} = \sqrt{\beta}\left\{\frac{1}{2} + \sum_{n=1}^\infty e^{-\beta^2 n^2}\right\}, \tag{23.1}$$

where $\alpha\beta = \pi$ and $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$. This formula has been given three times in the second notebook: as a corollary to Entry 7 in Chapter 14, as Entry 27(i) in Chapter 16, and as a corollary in Section 6 of Chapter 17.

Let $\alpha^2 = \pi(x + iy)$, and so $\beta^2 = \pi(x - iy)/(x^2 + y^2)$. Then (23.1) yields

$$(x + iy)^{1/4}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} e^{-n^2\pi x}(\cos(n^2\pi y) - i\sin(n^2\pi y))\right\}$$
$$= \left(\frac{x - iy}{x^2 + y^2}\right)^{1/4}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} \exp\left(-\frac{n^2\pi x}{x^2 + y^2}\right)\right.$$
$$\left.\times\left(\cos\left(\frac{n^2\pi y}{x^2 + y^2}\right) + i\sin\left(\frac{n^2\pi y}{x^2 + y^2}\right)\right)\right\}, \tag{23.2}$$

where principal values are taken. Letting $\alpha^2 = \pi(x - iy)$, so that $\beta^2 = \pi(x + iy)/(x^2 + y^2)$, in (23.1), we find that

$$(x - iy)^{1/4}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} e^{-n^2\pi x}(\cos(n^2\pi y) + i\sin(n^2\pi y))\right\}$$
$$= \left(\frac{x + iy}{x^2 + y^2}\right)^{1/4}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} \exp\left(-\frac{n^2\pi x}{x^2 + y^2}\right)\left(\cos\left(\frac{n^2\pi y}{x^2 + y^2}\right)\right.\right.$$
$$\left.\left. - i\sin\left(\frac{n^2\pi y}{x^2 + y^2}\right)\right)\right\}. \tag{23.3}$$

Elementary calculations give

$$\left(\frac{x + iy}{x - iy}(x^2 + y^2)\right)^{1/4} + \left(\frac{x - iy}{x + iy}(x^2 + y^2)\right)^{1/4} = \sqrt{2}(\sqrt{x^2 + y^2} + x)^{1/2}$$

and

$$\left(\frac{x + iy}{x - iy}(x^2 + y^2)\right)^{1/4} - \left(\frac{x - iy}{x + iy}(x^2 + y^2)\right)^{1/4} = i\sqrt{2}(\sqrt{x^2 + y^2} - x)^{1/2}.$$

Thus, adding (23.2) and (23.3), we obtain part (i), and subtracting (23.3) from (23.2), we deduce (ii).

Corollary. *If* $\operatorname{Re}(x) > 0$, *then*

$$\frac{1}{2} + \sum_{n=1}^{\infty} e^{-n^2\pi x}\cos(n^2\pi\sqrt{1 - x^2}) = \frac{\sqrt{2} + \sqrt{1 + x}}{\sqrt{1 - x}}\sum_{n=1}^{\infty} e^{-n^2\pi x}\sin(n^2\pi\sqrt{1 - x^2}).$$

PROOF. Putting $y = \sqrt{1 - x^2}$ in Entry 23(ii) and simplifying, we deduce the proposed formula.

Examples. *Recall from Entry* 22 *of Chapter* 16 *that the classical theta-function* φ *is defined by*

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad |q| < 1.$$

Then

$$\varphi(e^{-\pi}) = \sqrt{5\sqrt{5} - 10}\,\varphi(e^{-5\pi}) \tag{23.4}$$

and

$$(\sqrt{5} + \sqrt{3})\varphi(e^{-\pi\sqrt{5/3}}) = (3 + \sqrt{3})\varphi(e^{-3\pi\sqrt{5}}). \tag{23.5}$$

PROOF. If we set $\alpha^2 = 5\pi$, and hence $\beta^2 = \pi/5$, in (23.1), we find that

$$\sqrt{5}\varphi(e^{-5\pi}) = \varphi(e^{-\pi/5}). \tag{23.6}$$

Next, let $x = 1$ and $y = 2$ in Entry 23(i) to deduce that

$$\begin{aligned}
&(\sqrt{5} + 1)^{1/2}\varphi(e^{-\pi})\\
&= 2\sqrt{2}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} e^{-n^2\pi/5} \cos\frac{2\pi n^2}{5}\right\}\\
&= 2\sqrt{2}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} e^{-(5n)^2\pi/5} + \cos\frac{2\pi}{5} \sum_{\substack{n=1\\ n\not\equiv 0 \pmod 5}}^{\infty} e^{-n^2\pi/5}\right\}\\
&= 2\sqrt{2}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} e^{-5n^2\pi} - \cos\frac{2\pi}{5}\sum_{n=1}^{\infty} e^{-5n^2\pi} + \cos\frac{2\pi}{5}\sum_{n=1}^{\infty} e^{-n^2\pi/5}\right\}\\
&= \sqrt{2}\left\{\left(1 - \cos\frac{2\pi}{5}\right)\varphi(e^{-5\pi}) + \cos\frac{2\pi}{5}\varphi(e^{-\pi/5})\right\}\\
&= \sqrt{2}\left\{\frac{5 - \sqrt{5}}{4}\varphi(e^{-5\pi}) + \frac{\sqrt{5} - 1}{4}\sqrt{5}\varphi(e^{-5\pi})\right\},
\end{aligned}$$

by (23.6). Upon simplification, we may readily deduce (23.4).

To prove (23.5), we set $x = \sqrt{5/3}$ in the corollary to deduce that

$$\begin{aligned}
&\frac{1}{2} + \sum_{n=1}^{\infty} e^{-n^2\pi\sqrt{5/3}} \cos\frac{2\pi n^2}{3}\\
&\qquad = \frac{\sqrt{2} + \sqrt{1 + \sqrt{5/3}}}{\sqrt{1 - \sqrt{5/3}}} \sum_{n=1}^{\infty} e^{-n^2\pi\sqrt{5/3}} \sin\frac{2\pi n^2}{3}.
\end{aligned}$$

Since

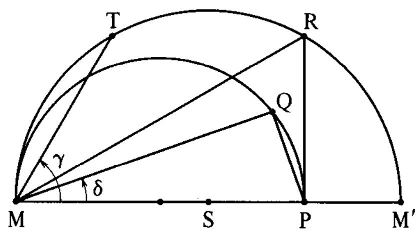
$$\frac{\sqrt{2} + \sqrt{1 + \sqrt{5/3}}}{\sqrt{1 - \sqrt{5/3}}} = \frac{2\sqrt{3} + \sqrt{5} + 1}{\sqrt{5} - 1},$$

the penultimate equality reduces to

$$\begin{aligned}
&\left(1 - \cos\frac{2\pi}{3}\right)\varphi(e^{-3\pi\sqrt{5}}) + \cos\frac{2\pi}{3}\varphi(e^{-\pi\sqrt{5/3}})\\
&\qquad = \frac{2\sqrt{3} + \sqrt{5} + 1}{\sqrt{5} - 1} \sin\frac{2\pi}{3}(\varphi(e^{-\pi\sqrt{5/3}}) - \varphi(e^{-3\pi\sqrt{5}})).
\end{aligned}$$

Multiplying both sides by $\sqrt{5} - 1$ and simplifying, we obtain (23.5).

Ramanujan commences the last section of Chapter 18 with a geometrical construction (Entry 24(i)). Let $\gamma = \angle TMM'$ be any angle, such that $0 < \gamma < \pi$, where MM' denotes the diameter of a semicircle cutting the bisector of γ at R. Let RP be perpendicular to MM' with $P \in MM'$. Suppose that MP is the diameter of another semicircle. Let Q be a point on this semicircle such that $PQ = PM'$. Let δ denote the angle QMP. Lastly, let S denote the midpoint of MM'. (As in Sections 19 and 20, we abuse notation by using XY to denote the line segment from X to Y as well as its length.)



Ramanujan now makes three claims.

Proposition 1. *"If RP divides MM' in medial section, then MQ coincides with MR."*

The words "medial section" indicate the golden mean. Thus, Ramanujan asserts that if

$$\frac{MP}{PM'} = \frac{\sqrt{5}+1}{2}, \tag{24.1}$$

then MQ and MR are coincident.

Proof. Transcribing Ramanujan's conclusion, we are required to show that $\cos\delta = \cos\frac{1}{2}\gamma$, or that

$$\frac{MQ}{MP} = \frac{MP}{MR}. \tag{24.2}$$

From similar triangles,

$$\frac{MR}{MM'} = \frac{MP}{MR}. \tag{24.3}$$

Hence, (24.2) is equivalent to

$$\begin{aligned} MP^2 &= MQ \cdot MR \\ &= \sqrt{MP^2 - QP^2}\,\sqrt{MP \cdot MM'} \\ &= \sqrt{MP^2 - PM'^2}\,\sqrt{MP \cdot MM'}, \end{aligned}$$

or

$$MP^4 = (MP^2 - PM'^2)MP(MP + PM'),$$

or

$$MP^2 - MP \cdot PM' - PM'^2 = 0. \tag{24.4}$$

In summary, we have shown that the conclusion of Proposition 1 is equivalent to (24.4). But solving (24.4) for MP, we immediately obtain (24.1).

Proposition 2. *If t_1 and t_2 denote the times it takes for a pendulum to oscillate through angles 4γ and 4δ, respectively, then*

$$t_1 = \frac{MM'}{MP} t_2 =: mt_2. \tag{24.5}$$

PROOF. From Hancock's book [1, p. 91],

$$t_1 = 2\sqrt{\frac{MM'}{g}} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \alpha \sin^2 \varphi}},$$
$$= \pi \sqrt{\frac{MM'}{g}} \, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha),$$

where g is the acceleration due to gravity, the length of the pendulum is MM', and $\alpha = \sin^2 \gamma$. Likewise,

$$t_2 = \pi \sqrt{\frac{MM'}{g}} \, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \beta),$$

where $\beta = \sin^2 \delta$. Hence,

$$\frac{t_1}{t_2} = \frac{{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha)}{{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \beta)}. \tag{24.6}$$

Since $QP = PM'$, a brief calculation shows that

$$\beta = \sin^2 \delta = (m - 1)^2, \tag{24.7}$$

where m is defined in (24.5). Second, by (24.3),

$$\alpha = \sin^2 \gamma = 4 \sin^2 \tfrac{1}{2}\gamma \cos^2 \tfrac{1}{2}\gamma = 4\left(\frac{RP}{MR}\right)^2 \left(\frac{MP}{MR}\right)^2$$
$$= 4\left(\frac{RP}{MM'}\right)^2 = 4\left(\frac{\sqrt{MR^2 - MP^2}}{MM'}\right)^2$$
$$= 4\left(\frac{MP \cdot MM' - MP^2}{MM'^2}\right) = 4\left(\frac{1}{m} - \frac{1}{m^2}\right). \tag{24.8}$$

Thus, using (24.7) and (24.8) in (24.6), we find that

$$\frac{t_1}{t_2} = \frac{{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; 4(m - 1)/m^2)}{{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; (m - 1)^2)}. \tag{24.9}$$

We now apply Landen's transformation (Erdélyi [1, p. 111, formula (5)])

$$ {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \frac{4z}{(1+z)^2}\right) = (1+z)\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; z^2) \tag{24.10} $$

with $z = m - 1$. We see immediately that (24.9) reduces to $t_1/t_2 = m$, and the proof is complete.

From the definition (24.5) of m, we observe that $m > 1$. Moreover, this fact and (24.9) imply that $m < 2$.

Proposition 3. *If m is defined by* (24.5), *then*

$$ \cos \gamma = \frac{2PS}{mMP}. $$

The factor MP was inadvertently omitted by Ramanujan (p. 228).

PROOF. Using (24.3) and (24.5), we observe that

$$ \cos \gamma = 2\left(\frac{MP}{MR}\right)^2 - 1 = 2\frac{MP}{MM'} - 1 $$
$$ = \frac{2PS}{MM'} = \frac{2PS}{mMP}. $$

In a note ending subsection (i) of Section 24, Ramanujan asserts that β is of the second degree in α. Indeed, from (24.8),

$$ m = \frac{2 - 2\sqrt{1-\alpha}}{\alpha}, \tag{24.11} $$

and so, by (24.7),

$$ \beta = \left(\frac{2 - 2\sqrt{1-\alpha} - a}{\alpha}\right)^2 = \frac{(1-\sqrt{1-\alpha})^4}{\alpha^2} = \frac{\alpha^2}{(1+\sqrt{1-\alpha})^4}. \tag{24.12} $$

However, more appropriately, as we shall see below, β is of degree 2 because the relation (24.12) is a modular equation of degree 2.

Before proceeding further, we precisely define a modular equation of degree (order) n. Let K, K', L, and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', ℓ, and ℓ', respectively. Suppose that the relation

$$ \frac{L'}{L} = n\frac{K'}{K} $$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and ℓ which is induced by the equality above.

Transcribing this definition into our notation and the terminology of hypergeometric functions, we conclude that a modular equation of degree n is an equation relating α and β that is induced by

$$n\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}=\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}. \tag{24.13}$$

Entry 24(ii). *Let*

$$m=\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}, \tag{24.14}$$

as in Section 24(i). *We call m the* multiplier. *Then modular equations of the second degree are given by*

$$m\sqrt{1-\alpha}+\sqrt{\beta}=1 \tag{24.15}$$

and

$$m^2\sqrt{1-\alpha}+\beta=1. \tag{24.16}$$

Furthermore,

$$\tfrac{1}{2}m^2=\frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}}=\frac{1+\beta}{1+(1-\alpha)}. \tag{24.17}$$

Proof. First, we show that the equalities (24.15)–(24.17) are valid. Then we demonstrate that these equalities, indeed, are modular equations of the second degree.

Both (24.15) and (24.16) are easily verified by substituting (24.7) and (24.8) into the left sides of (24.15) and (24.16). Likewise, the equalities of (24.17) are similarly verified.

In order to show that (24.15)–(24.17) are modular equations of the second degree, by (24.13) and (24.14), we need to show that

$$\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}=\frac{m}{2}.$$

By (24.7), (24.8), and (24.10) with $z=(2-m)/m$,

$$\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}=\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;((2-m)/m)^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;m(2-m))}=\frac{1}{1+(2-m)/m}=\frac{m}{2},$$

which completes the proof.

Entry 24(iii). *Modular equations of degree* 4 *are given by*

$$\sqrt{m}(1-\alpha)^{1/4}+\beta^{1/4}=1 \tag{24.18}$$

and

$$m(1-\alpha)^{1/4} + \sqrt{\beta} = 1. \tag{24.19}$$

Furthermore, the multiplier m is given by

$$\tfrac{1}{2}m = \frac{1+\beta^{1/4}}{1+(1-\alpha)^{1/4}} = \frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}}. \tag{24.20}$$

PROOF. It is clear from (24.13) that a modular equation of degree 2^r, $r \geq 2$, can be obtained by iterating a modular equation of degree 2^{r-1}.

The equality (24.12) may be written in the form

$$\beta = \left(\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2. \tag{24.21}$$

Iterating to obtain a modular equation of degree 4, we find that

$$\beta = \left(\frac{1-\left(1-\left(\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2\right)^{1/2}}{1+\left(1-\left(\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2\right)^{1/2}}\right)^2.$$

After a considerable amount of simplification, we deduce that

$$\sqrt{\beta} = \left(\frac{1-(1-\alpha)^{1/4}}{1+(1-\alpha)^{1/4}}\right)^2. \tag{24.22}$$

We thus obtain the following curious algorithm to derive a modular equation of the fourth degree from a modular equation of the second degree: replace β by $\sqrt{\beta}$ and $1-\alpha$ by $\sqrt{1-\alpha}$.

We now find an expression for the multiplier m in modular equations of degree 4. Using (24.14) and (24.11) and then iterating with the aid of (24.21), we find that

$$\begin{aligned} {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha) &= \frac{2-2\sqrt{1-\alpha}}{\alpha}\, {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \beta) \\ &= \frac{2-2\sqrt{1-\alpha}}{\alpha} \frac{2-2\left(1-\left(\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2\right)^{1/2}}{\left(\frac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2} \\ &\quad \times {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \left(\frac{1-\sqrt{1-\beta}}{1+\sqrt{1-\beta}}\right)^2\right). \end{aligned}$$

Thus, the new multiplier m is equal to

$$m = \frac{2-2\sqrt{1-\alpha}}{\alpha} \frac{2-2\left(1-\left(\dfrac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2\right)^{1/2}}{\left(\dfrac{1-\sqrt{1-\alpha}}{1+\sqrt{1-\alpha}}\right)^2}$$

$$= \frac{4}{\{1+(1-\alpha)^{1/4}\}^2}, \tag{24.23}$$

after considerable simplification.

If we now substitute (24.22) and (24.23) into the left sides of (24.18) and (24.19) and all expressions of (24.20), we readily verify each identity.

Observe that (24.18)–(24.20) may be obtained formally from (24.15)–(24.17), respectively, by replacing m by $\sqrt{m}$, $1-\alpha$ by $\sqrt{1-\alpha}$, and β by $\sqrt{\beta}$.

There appear to be some errors in Ramanujan's modular equations in Sections 24(iv) and 24(v). In Entry 24(iv), Ramanujan claims that

$$\sqrt{m}(1-\alpha)^{1/8} + \beta^{1/4} = 1 \tag{24.24}$$

is a modular equation of degree 8 and that

$$\tfrac{1}{2}\sqrt{m} = \frac{1+\beta^{1/4}}{1+(1-\alpha)^{1/4}}$$

is a modular equation of degree 16. Equations of the 8th and 16th degrees can be obtained by one and two further iterations, respectively, of (24.22) and (24.23). Our calculations indicate that modular equations of degrees 8 and 16 are not nearly as elegant as those claimed by Ramanujan. Because the equations are not attractive and no new ideas are involved, it does not seem worthwhile to pursue these details here.

Entry 24(v). *If we replace α by $1-\beta$, β by $1-\alpha$, and m by n/m, where n is the degree of the modular equation, we obtain a modular equation of the same degree.*

We call this process of obtaining a modular equation the method of reciprocation. Alternatively, we say that the latter equation is the reciprocal of the former. In the theory of modular forms, this modus operandi is called Fricke involution.

PROOF. Making the proffered substitutions in the definition of m given by (24.14), we find that

$$\frac{n}{m} = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}.$$

But by (24.14), this may be rewritten

$$n\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}=\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)};$$

that is, we obtain the defining relation (24.13) for a modular equation. This completes the proof.

Ramanujan now offers several examples to illustrate his algorithm. Thus, making the prescribed substitutions in (24.15), we obtain

$$\frac{2}{m}\sqrt{\beta}+\sqrt{1-\alpha}=1, \tag{24.25}$$

a modular equation of the second degree. Next, solve (24.15) for m and substitute its value in (24.25) to obtain the second degree modular equation

$$(1-\sqrt{1-\alpha})(1-\sqrt{\beta})=2\sqrt{\beta(1-\alpha)}.$$

Using Entry 24(v) in conjunction with (24.18), we derive the modular equation of degree 4,

$$\frac{2}{\sqrt{m}}\beta^{1/4}+(1-\alpha)^{1/4}=1.$$

This equation and (24.23) lead at once to another fourth degree equation

$$(1-(1-\alpha)^{1/4})(1-\beta^{1/4})=2(\beta(1-\alpha))^{1/4}.$$

If (24.24) were correct, Entry 24(v) would immediately yield the modular equation of degree 8,

$$\frac{2\sqrt{2}}{\sqrt{m}}\beta^{1/8}+(1-\alpha)^{1/4}=1. \tag{24.26}$$

The purported equalities (24.24) and (24.26) taken together would then give

$$(1-(1-\alpha)^{1/4})(1-\beta^{1/4})=2\sqrt{2}(\beta(1-\alpha))^{1/8}.$$

Entry 24(vi). *Consider again* (24.13) *and* (24.14). *Then*

$$n\frac{d\alpha}{d\beta}=\frac{\alpha(1-\alpha)}{\beta(1-\beta)}m^2. \tag{24.27}$$

Proof. Using Entry 9(i) of Chapter 17, we differentiate both sides of (24.13) with respect to β to find that

$$\frac{n}{\alpha(1-\alpha)\,{}_2F_1^2(\frac{1}{2},\frac{1}{2};1;\alpha)}\frac{d\alpha}{d\beta}=\frac{1}{\beta(1-\beta)\,{}_2F_1^2(\frac{1}{2},\frac{1}{2};1;\beta)}.$$

Noting the definition (24.14) of m, we see that the proposed formula follows immediately.

Entry 24(vi) is due originally to Jacobi [1], [2, p. 130]. See also Cayley's book [1, pp. 201, 216–217].

Ramanujan appears to remark that if we can find $d\alpha/d\beta$ by differentiating a modular equation (presumably a modular equation independent of m), then we can determine m from (24.27).

Section 24(vii) consists of the following statement: "Equations in terms of ψ functions can be transformed to those of φ functions and vice versa while those of f and χ functions remain unchanged. E.g. the identity

$$\frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)} = 1 + \left(\frac{\psi^4(q)}{q\psi^4(q^3)} - 1\right)^{1/3} \tag{24.28}$$

becomes

$$\frac{\varphi(q^{1/3})}{\varphi(q^3)} = 1 + \left(\frac{\varphi^4(q)}{\varphi^4(q^3)} - 1\right)^{1/3}." \tag{24.29}$$

We are uncertain about Ramanujan's intention in this claim. The functions ψ and φ are related, and we shall show that (24.28) and (24.29) are readily equivalent, so perhaps this explains part of the statement.

We first show that (24.28) implies (24.29). From the definition of ψ, we find that $\vartheta_2(0, \tau/2) = 2q^{1/8}\psi(q)$, where $q = e^{\pi i\tau}$. Thus, (24.28) is equivalent to

$$\frac{\vartheta_2(0, \tau/6)}{\vartheta_2(0, 3\tau/2)} = 1 + \left(\frac{\vartheta_2^4(0, \tau/2)}{\vartheta_2^4(0, 3\tau/2)} - 1\right)^{1/3}.$$

Replacing τ by $2\tau/(3\tau + 1)$, we transcribe the formula above into

$$\frac{\vartheta_2\left(0, \dfrac{\tau/3}{3\tau + 1}\right)}{\vartheta_2\left(0, \dfrac{3\tau}{3\tau + 1}\right)} = 1 + \left(\frac{\vartheta_2^4\left(0, \dfrac{\tau}{3\tau + 1}\right)}{\vartheta_2^4\left(0, \dfrac{3\tau}{3\tau + 1}\right)} - 1\right)^{1/3}. \tag{24.30}$$

By the transformation formulas for theta-functions (Rademacher [1, p. 182]),

$$\vartheta_2\left(0, \frac{\tau}{3\tau + 1}\right) = \sqrt{3\tau + 1}\,\vartheta_3(0, \tau),$$

$$\vartheta_2\left(0, \frac{3\tau}{3\tau + 1}\right) = \sqrt{3\tau + 1}\,\vartheta_3(0, 3\tau),$$

and

$$\vartheta_2\left(0, \frac{\tau/3}{3\tau + 1}\right) = \sqrt{3\tau + 1}\,\vartheta_3(0, \tau/3).$$

Using these equalities in (24.30) and putting the resulting equality in a slightly different form, we deduce that

$$\frac{\vartheta_3^4(0, \tau)}{\vartheta_3^4(0, 3\tau)} - 1 = \left(\frac{\vartheta_3(0, \tau/3)}{\vartheta_3(0, 3\tau)} - 1\right)^3. \tag{24.31}$$

Since $\vartheta_3(0, \tau) = \varphi(q)$, (24.31) reduces to (24.29).

Ramanujan restates (24.28) and (24.29) in Section 1 of Chapter 20, and so we defer proofs until then.

CHAPTER 19

Modular Equations of Degrees 3, 5, and 7 and Associated Theta-Function Identities

In several ways, this is a remarkable chapter. Not only are the results enormously interesting and often difficult to prove, but many questions arise in regard to Ramanujan's methods of proof. Undoubtedly, many of the proofs given here are quite unlike those found by Ramanujan. He evidently possessed methods that we have been unable to discern. No hints whatsoever of his methods are provided by Ramanujan.

As the chapter's title indicates, Ramanujan herein studies modular equations, primarily of degrees 3, 5, and 7. For each particular degree, Ramanujan appears to first derive a series of interesting identities involving theta-functions of appropriate arguments. These are then used to establish an astonishing battery of modular equations of that degree (order). We have not always been able to follow this process, and so, at times, we have had to reverse this procedure and employ modular equations to prove theta-function identities. We emphasize, however, that no circular reasoning is involved in our presentation. Frequently, we prove modular equations in an order different from that given by Ramanujan. It could be that, in arranging his numerous modular equations, Ramanujan gave priority to those he felt were more important and/or more elegant.

The theory of modular equations began with the work of Legendre and Jacobi. Informative source about modular equations are the books by Cayley [1] and Enneper [1]. The latter book also provides much of the history of the subject. Ramanujan's development of modular equations is vastly more substantial, however, than that of his predecessors. Most of the modular equations given in this chapter are not found elsewhere in the literature. Not only are the results new, but Ramanujan's methods are apparently original as well. Ramanujan published but one paper [2] in which modular equations are discussed, but because modular equations, per se, were not the raison d'etre for this paper, Ramanujan's methods in this theory are not disclosed.

Chapters 16 and 17 are crucial for the development of this chapter. Many basic properties of theta-functions found in Chapter 16 are repeatedly used here in proving theta-function identities. Many of the formulas in Chapter 17 are employed to establish modular equations and also to produce theta-function relations from modular equations.

It is always assumed that $|q| < 1$. Following Ramanujan, we frequently do not use a compact summation notation because the laws of formation of the signs and exponents are more easily ascertained by explicitly displaying the first several terms.

Entry 1.

(i) *Let* φ *and* ψ *be defined as in Entry* 22 *of Chapter* 16. *Then*

$$q^{1/8}\frac{\psi(q)}{\varphi(q)} = \frac{q^{1/8}}{1} \; \frac{q}{+\,1+q} \; \frac{q^2}{+\,1+q^2} \; \frac{q^3}{+\,1+q^3} + \cdots.$$

(ii) *Recall that* $f(a, b)$ *is defined by* (18.1) *in Chapter* 16. *Let*

$$v = q^{1/2}\frac{f(-q, -q^7)}{f(-q^3, -q^5)}.$$

Then

$$v = \frac{q^{1/2}}{1+q} \; \frac{q^2}{+\,1+q^3} \; \frac{q^4}{+\,1+q^5} \; \frac{q^6}{+\,1+q^7} + \cdots, \tag{1.1}$$

$$\frac{1}{v} - v = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)}, \tag{1.2}$$

and

$$\frac{1}{v} + v = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}. \tag{1.3}$$

Proof of (i). We employ Entry 12 of Chapter 16. Replace a^2, b^2, and q in Entry 12 by 0, $-q^{1/2}$, and $q^{1/2}$, respectively. This gives

$$\frac{(-q^2; q^2)_\infty}{(-q; q^2)_\infty} = \frac{1}{1} \; \frac{q}{+\,1+q} \; \frac{q^2}{+\,1+q^2} \; \frac{q^3}{+\,1+q^3} + \cdots.$$

Part (i) now follows immediately from Entry 22, Chapter 16.

K. G. Ramanathan [6] has also proved Entry 1(i).

Proof of (ii). Applying Entry 30(ii) in Chapter 16 with $a = iq^{1/2}$ and $b = -iq^{3/2}$, we find that

$$\psi(iq^{1/2}) + \psi(-iq^{1/2}) = f(iq^{1/2}, -iq^{3/2}) + f(-iq^{1/2}, iq^{3/2}) = 2f(-q^3, -q^5).$$

Similarly, by Entry 30(iii) in Chapter 16,

$$\psi(iq^{1/2}) - \psi(-iq^{1/2}) = f(iq^{1/2}, -iq^{3/2}) - f(-iq^{1/2}, iq^{3/2}) = 2iq^{1/2}f(-q, -q^7).$$

Thus,

$$iv = \frac{\psi(iq^{1/2}) - \psi(-iq^{1/2})}{\psi(iq^{1/2}) + \psi(-iq^{1/2})} = \frac{(-iq^{1/2}; -q)_\infty - (iq^{1/2}; -q)_\infty}{(-iq^{1/2}; -q)_\infty + (iq^{1/2}; -q)_\infty},$$

by Entry 22(ii) in Chapter 16. Now apply Entry 11 of Chapter 16 with a, b, and q replaced by $iq^{1/2}$, 0, and $-q$, respectively. Thus, (1.1) follows at once.

Next, by Entry 30(i) in Chapter 16, with $a = -q$ and $b = -q^3$,

$$f(-q, -q^7)f(-q^3, -q^5) = f(-q, -q^3)\psi(q^4) = \psi(-q)\psi(q^4).$$

Thus, by Example (iv) in Section 31 of Chapter 16,

$$v = q^{1/2}\frac{f^2(-q, -q^7)}{\psi(-q)\psi(q^4)} = \frac{\varphi(q) - \varphi(q^2)}{2q^{1/2}\psi(q^4)}$$

and

$$\frac{1}{v} = \frac{f^2(-q^3, -q^5)}{q^{1/2}\psi(-q)\psi(q^4)} = \frac{\varphi(q) + \varphi(q^2)}{2q^{1/2}\psi(q^4)}.$$

The truth of (1.2) and (1.3) are now evident.

Ramanathan [4] has independently given the same proof of (1.1).

Entry 2. *Recall that $f(-q)$ is defined in Entry 22 of Chapter 16. Then*

(i)
$$f(-q, -q^4)f^3(-q^{15}) = f(-q^5)f(-q^6, -q^9)f(-q, -q^{14})f(-q^4, -q^{11}),$$

$$f(-q^2, -q^3)f^3(-q^{15}) = f(-q^5)f(-q^3, -q^{12})f(-q^2, -q^{13})f(-q^7, -q^8),$$

(ii)
$$f(-q, -q^6)f^3(-q^{21}) = f(-q^7)f(-q^6, -q^{15})f(-q, -q^{20})f(-q^8, -q^{12}),$$

$$f(-q^2, -q^5)f^3(-q^{21}) = f(-q^7)f(-q^9, -q^{12})f(-q^2, -q^{19})f(-q^5, -q^{16}),$$

and

$$f(-q^3, -q^4)f^3(-q^{21}) = f(-q^7)f(-q^3, -q^{18})f(-q^4, -q^{17})f(-q^{10}, -q^{11}).$$

Proof. Each of these five equalities is proved in precisely the same fashion by expanding each side into an infinite product via Entry 19 (the Jacobi triple product identity) and Entry 22(iii), both in Chapter 16. We give the details for only the proof of the first part of (i).

By the two aforementioned theorems,

$$f(-q, -q^4)f^3(-q^{15}) = (q; q^5)_\infty(q^4; q^5)_\infty(q^5; q^5)_\infty(q^{15}; q^{15})^3_\infty$$

and

$$\begin{aligned} &f(-q^5)f(-q^6, -q^9)f(-q, -q^{14})f(-q^4, -q^{11}) \\ &\quad = (q^5; q^5)_\infty(q^6; q^{15})_\infty(q^9; q^{15})_\infty(q; q^{15})_\infty(q^{14}; q^{15})_\infty(q^4; q^{15})_\infty \\ &\qquad \times (q^{11}; q^{15})_\infty(q^{15}; q^{15})^3_\infty \\ &\quad = (q^5; q^5)_\infty(q; q^5)_\infty(q^4; q^5)_\infty(q^{15}; q^{15})^3_\infty, \end{aligned}$$

and the proof of the first part of (i) is complete.

In fact, Ramanujan appends the words "and so on" at the end of Entry 2. Thus, the next formula in this series would be for $f(-q, -q^8)f^3(-q^{27})$.

Entry 3. We have

(i) $$q\psi(q^2)\psi(q^6) = \frac{q}{1-q^2} - \frac{q^5}{1-q^{10}} + \frac{q^7}{1-q^{14}} - \frac{q^{11}}{1-q^{22}} + \cdots,$$

(ii) $$\varphi(q)\varphi(q^3) = 1 + 2\left(\frac{q}{1-q} - \frac{q^2}{1+q^2} + \frac{q^4}{1+q^4} - \frac{q^5}{1-q^5} + \frac{q^7}{1-q^7} - \cdots\right),$$

(iii) $$q\psi^2(q)\psi^2(q^3) = \frac{q}{1-q^2} + \frac{2q^2}{1-q^4} + \frac{4q^4}{1-q^8} + \frac{5q^5}{1-q^{10}} + \cdots,$$

and

(iv) $$\varphi^2(q)\varphi^2(q^3) = 1 + 4\left(\frac{q}{1-q} + \frac{4q^4}{1-q^4} + \frac{5q^5}{1-q^5} + \frac{7q^7}{1-q^7} + \cdots\right),$$

where in the last sum on the right side, the summation is over all values of n which are neither a multiple of 3 nor an odd multiple of 2.

Proof of (i). First employ (8.5) in Chapter 17 with $a = q$ and $b = q^5$. Then use Entries 19, 22(iii), (iv), 25(iv), and 24(iii) in Chapter 16. Accordingly, we find that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{6n+1}}{1-q^{12n+2}} - \sum_{n=0}^{\infty} \frac{q^{6n+5}}{1-q^{12n+10}} &= q\frac{f(-q^4, -q^8)}{f(-q^2, -q^{10})}\varphi(q^6)\psi(q^{12}) \\ &= q\frac{f(-q^4)\varphi(q^6)\psi(q^{12})}{(q^2; q^{12})_\infty(q^{10}; q^{12})_\infty(q^{12}; q^{12})_\infty} \end{aligned}$$

$$= q\frac{f(-q^4)(q^6;q^{12})_\infty\psi^2(q^6)}{(q^2;q^4)_\infty(q^{12};q^{12})_\infty}$$

$$= q\frac{f(-q^4)\chi(-q^6)\psi^2(q^6)}{\chi(-q^2)f(-q^{12})}$$

$$= q\psi(q^2)\psi(q^6).$$

PROOF OF (ii). Entry 3(ii) is the same as Entry 8(iv) in Chapter 17, and a proof was given there.

PROOF OF (iii). In part (i), replace q^2 by q, expand the summands into geometric series, and sum by columns, combining terms in alternate rows. Hence,

$$\sqrt{q}\,\psi(q)\psi(q^3) = \frac{q^{1/2}}{1-q} - \frac{q^{5/2}}{1-q^5} + \frac{q^{7/2}}{1-q^7} - \frac{q^{11/2}}{1-q^{11}} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{q^{n+1/2} - q^{5n+5/2}}{1-q^{6n+3}}$$

$$= -\sum_{n=0}^{\infty} \frac{q^{3n+3/2}(q^{2n+1} - q^{-2n-1})}{1-q^{6n+3}}$$

$$= -2i\sum_{n=0}^{\infty} \frac{q^{3n+3/2}\sin(2n+1)\theta}{1-q^{6n+3}},$$

where we have put $q = e^{i\theta}$. Hence,

$$q\psi^2(q)\psi^2(q^3) = -4\left(\sum_{n=0}^{\infty} \frac{q^{3n+3/2}\sin(2n+1)\theta}{1-q^{6n+3}}\right)^2. \tag{3.1}$$

Employing (13.10) and (14.12) in Chapter 18 and letting

$$T := \frac{1}{4}\sum_{n=-\infty}^{\infty} \operatorname{csch}^2((n+\tfrac{1}{2})y),$$

where now $e^{-y} = q^3$, we deduce that

$$S^2 = 4\left(\sum_{n=0}^{\infty} \frac{q^{3n+3/2}\sin(2n+1)\theta}{1-q^{6n+3}}\right)^2$$

$$= T - 2\sum_{n=1}^{\infty} \frac{n\cos(2n\theta)}{q^{-3n}-q^{3n}}$$

$$= T + 4\sum_{n=1}^{\infty} \frac{nq^{3n}\sin^2(n\theta)}{1-q^{6n}} - 2\sum_{n=1}^{\infty} \frac{n}{q^{-3n}-q^{3n}}$$

$$= T + 4\sum_{n=1}^{\infty} \frac{nq^{3n}\sin^2(n\theta)}{1-q^{6n}} - \sum_{n=1}^{\infty} \frac{n}{\sinh(ny)}$$

$$= 4\sum_{n=1}^{\infty} \frac{nq^{3n}\sin^2(n\theta)}{1-q^{6n}}, \tag{3.2}$$

by (14.15) in Chapter 18. Using (3.2) in (3.1), we find that

$$\begin{aligned}
q\psi^2(q)\psi^2(q^3) &= -4\sum_{n=1}^{\infty}\frac{nq^{3n}\sin^2(n\theta)}{1-q^{6n}} \\
&= \sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{6n}}(q^{2n}-2+q^{-2n}) \\
&= \sum_{n=1}^{\infty}\frac{n(q^n+q^{3n}+q^{5n})}{1-q^{6n}} - 3\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{6n}} \\
&= \sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}} - 3\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{6n}},
\end{aligned}$$

which immediately yields equality (iii).

PROOF OF (iv). Writing (ii) as a double series and transforming it by columns, while collecting terms in alternate rows, we first arrive at

$$\varphi(q)\varphi(q^3) = 1 + 2\left(\frac{q-q^2}{1-q^3} + \frac{q^2+q^4}{1+q^6} + \frac{q^3-q^6}{1-q^9} + \frac{q^4+q^8}{1+q^{12}} + \cdots\right). \tag{3.3}$$

Next, we employ the fundamental identity (14.3) of Chapter 18 along with (3.2) above. Thus,

$$\begin{aligned}
\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{\cosh(ny)}\right)^2 &= \tfrac{1}{4}\varphi^4(e^{-y}) - \left(\sum_{n=0}^{\infty}\frac{\sin\{\frac{1}{2}(2n+1)\theta\}}{\sinh\{\frac{1}{2}(2n+1)y\}}\right)^2 \\
&= \tfrac{1}{4}\varphi^4(e^{-y}) - 2\sum_{n=1}^{\infty}\frac{n\sin^2(\frac{1}{2}n\theta)}{\sinh(ny)},
\end{aligned} \tag{3.4}$$

since $e^{-y} = q^3$. Using (3.3), letting $e^{-y} = iq^{3/2}$ and $e^{i\theta} = iq^{1/2}$, and employing (3.4), we deduce that

$$\begin{aligned}
\tfrac{1}{4}\varphi^2(q)\varphi^2(q^3) &= \left(\frac{1}{2} + \frac{q^{-1/2}-q^{1/2}}{q^{-3/2}-q^{3/2}} + \frac{q^{-1}+q}{q^{-3}+q^3} + \frac{q^{-3/2}-q^{3/2}}{q^{-9/2}-q^{9/2}}\right. \\
&\qquad \left. + \frac{q^{-2}+q^2}{q^{-6}+q^6} + \cdots\right)^2 \\
&= \left(\frac{1}{2} + \frac{iq^{1/2}-iq^{-1/2}}{-iq^{-3/2}+iq^{3/2}} + \frac{-q-q^{-1}}{-q^{-3}-q^3} + \frac{-iq^{3/2}+iq^{-3/2}}{iq^{-9/2}-iq^{9/2}}\right. \\
&\qquad \left. + \frac{q^2+q^{-2}}{q^{-6}+q^6} + \cdots\right)^2 \\
&= \left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{\cosh(ny)}\right)^2 \\
&= \tfrac{1}{4}\varphi^4(e^{-y}) - 2\sum_{n=1}^{\infty}\frac{n\sin^2(\frac{1}{2}n\theta)}{\sinh(ny)}
\end{aligned}$$

$$= \tfrac{1}{4}\varphi^4(iq^{3/2}) + \sum_{n=1}^{\infty} \frac{n\{(iq^{1/2})^n - 2 + (-iq^{-1/2})^n\}}{(-iq^{-3/2})^n - (iq^{3/2})^n}$$

$$= \tfrac{1}{4}\varphi^4(iq^{3/2}) + \sum_{n=1}^{\infty} \frac{n\{(-1)^n q^{2n} - 2i^n q^{3n/2} + q^n\}}{1-(-1)^n q^{3n}}.$$

By Entry 8(ii) in Chapter 17,

$$\varphi^4(-q) = 1 + 8\sum_{n=1}^{\infty} \frac{(-1)^n nq^n}{1+q^n} = 1 + 8\sum_{n=1}^{\infty} \frac{(-1)^n n(q^n - q^{2n})}{1-q^{2n}}.$$

Thus, using this equality above, we see that

$$\varphi^2(q)\varphi^2(q^3) = 1 + 8\sum_{n=1}^{\infty} \frac{(-1)^n n\{(-iq^{3/2})^n - (-iq^{3/2})^{2n}\}}{1-(-iq^{3/2})^{2n}}$$

$$+ 4\sum_{n=1}^{\infty} \frac{n\{(-1)^n q^{2n} - 2i^n q^{3n/2} + q^n\}}{1-(-q)^{3n}}$$

$$= 1 + 4\sum_{n=1}^{\infty} \frac{n\{q^n + (-1)^n q^{2n} - 2q^{3n}\}}{1-(-q)^{3n}}$$

$$= 1 + 4\sum_{n=1}^{\infty} \frac{nq^n}{1-(-q)^n} - 12\sum_{n=1}^{\infty} \frac{nq^{3n}}{1-(-q)^{3n}},$$

where we have used the factorization $1 - x^3 = (1-x)(1+x+x^2)$. After canceling those terms involving powers of q^3, we transform the remaining odd powers via the formula

$$\frac{q}{1+q} = \frac{q}{1-q} - \frac{2q^2}{1-q^2}.$$

Noting that we have now also canceled those terms involving odd multiples of 2, we observe that the proof is complete.

Entry 4. *We have*

(i) $$q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3) = \frac{q}{1-q^2} - \frac{2^2q^2}{1-q^4} + \frac{4^2q^4}{1-q^8} - \frac{5^2q^5}{1-q^{10}} + \cdots,$$

(ii) $$9\varphi(q)\varphi^5(q^3) - \varphi^5(q)\varphi(q^3) = 8\left(1 + \frac{q}{1+q} - \frac{2^2q^2}{1-q^2} + \frac{4^2q^4}{1-q^4} - \frac{5^2q^5}{1+q^5} - \cdots\right),$$

(iii) $$\frac{\psi^3(q)}{\psi(q^3)} = 1 + 3\left(\frac{q}{1-q} - \frac{q^5}{1-q^5} + \frac{q^7}{1-q^7} - \frac{q^{11}}{1-q^{11}} + \cdots\right),$$

and

(iv) $$\frac{\varphi^3(q)}{\varphi(q^3)} = 1 + 6\left(\frac{q}{1-q} + \frac{q^2}{1+q^2} - \frac{q^4}{1+q^4} - \frac{q^5}{1-q^5} + \cdots\right).$$

PROOF OF (i). In (3.2), replace q^3 by q and differentiate both sides with respect to θ. Upon putting $\theta = \pi/3$, we find that

$$2\left(\sum_{n=0}^{\infty} \frac{q^{n+1/2}\sin\{(2n+1)\pi/3\}}{1-q^{2n+1}}\right)\left(\sum_{n=0}^{\infty} \frac{(2n+1)q^{n+1/2}\cos\{(2n+1)\pi/3\}}{1-q^{2n+1}}\right)$$
$$= \sum_{n=1}^{\infty} \frac{n^2 q^n \sin(2n\pi/3)}{1-q^{2n}}. \tag{4.1}$$

By Entry 3(i),

$$\sum_{n=0}^{\infty} \frac{q^{n+1/2}\sin\{(2n+1)\pi/3\}}{1-q^{2n+1}} = \frac{\sqrt{3}}{2} q^{1/2}\psi(q)\psi(q^3).$$

By Example (iii), Section 17, Chapter 17,

$$\sum_{n=0}^{\infty} \frac{(2n+1)q^{n+1/2}\cos\{(2n+1)\pi/3\}}{1-q^{2n+1}}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(2n+1)q^{n+1/2}}{1-q^{2n+1}} - \frac{3}{2}\sum_{n=0}^{\infty} \frac{(6n+3)q^{3n+3/2}}{1-q^{6n+3}}$$
$$= \tfrac{1}{2}(q^{1/2}\psi^4(q) - 9q^{3/2}\psi^4(q^3)).$$

Using these last two results in (4.1), we obtain at once the equality

$$q\psi^5(q)\psi(q^3) - 9q^2\psi(q)\psi^5(q^3) = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty} \frac{n^2 q^n \sin(2n\pi/3)}{1-q^{2n}},$$

which reduces at once to (i).

PROOF OF (ii). Recall the Fourier series (Whittaker and Watson [1, pp. 512, 535])

$$\left(\frac{2K}{\pi}\operatorname{cs} u\right)^2 = \left(\cot\theta - 4\sum_{n=1}^{\infty} \frac{q^n \sin(2n\theta)}{1+q^n}\right)^2$$

and

$$\left(\frac{2K}{\pi}\operatorname{ns} u\right)^2 = \csc^2\theta + \frac{4K(K-E)}{\pi^2} - 8\sum_{n=1}^{\infty} \frac{nq^n\cos(2n\theta)}{1-q^n}.$$

Here $u = 2K\theta/\pi$, $q = e^{-2y}$, θ is real, and K and E are the complete elliptic integrals of the first and second kinds, respectively. Since $\operatorname{ns}^2 u - \operatorname{cs}^2 u = 1$ (Whittaker and Watson [1, p. 493]),

$$\left(\cot\theta - 4\sum_{n=1}^{\infty} \frac{q^n\sin(2n\theta)}{1+q^n}\right)^2 \quad \text{and} \quad \csc^2\theta - 8\sum_{n=1}^{\infty} \frac{nq^n\cos(2n\theta)}{1-q^n}$$

differ by a constant. Differentiating with respect to θ and then letting $\theta = \pi/6$, we arrive at

$$\left(\cot(\pi/6) - 4\sum_{n=1}^{\infty}\frac{q^n\sin(n\pi/3)}{1+q^n}\right)\left(\csc^2(\pi/6) + 8\sum_{n=1}^{\infty}\frac{nq^n\cos(n\pi/3)}{1+q^n}\right)$$
$$= \cot(\pi/6)\csc^2(\pi/6) - 8\sum_{n=1}^{\infty}\frac{n^2q^n\sin(n\pi/3)}{1-q^n}. \tag{4.2}$$

By Entry 3(ii),

$$\cot(\pi/6) - 4\sum_{n=1}^{\infty}\frac{q^n\sin(n\pi/3)}{1+q^n} = \sqrt{3}\,\varphi(-q)\varphi(-q^3),$$

while, by Entry 8(ii) of Chapter 17,

$$\csc^2(\pi/6) + 8\sum_{n=1}^{\infty}\frac{nq^n\cos(n\pi/3)}{1+q^n} = 4 + 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^n}{1+q^n} - 12\sum_{n=1}^{\infty}\frac{(-1)^{n-1}3nq^{3n}}{1+q^{3n}}$$
$$= \tfrac{1}{2}\{9\varphi^4(-q^3) - \varphi^4(-q)\}.$$

Using these two results in (4.2), we deduce that

$$9\varphi(-q)\varphi^5(-q^3) - \varphi^5(-q)\varphi(-q^3)$$
$$= 8\left(1 - \frac{q}{1-q} - \frac{2^2q^2}{1-q^2} + \frac{4^2q^4}{1-q^4} + \frac{5^2q^5}{1-q^5} - \cdots\right).$$

Replacing q by $-q$, we finish the proof.

PROOF OF (iii). By Entry 8(x) of Chapter 17,

$$\psi(q^2)f^2(-q) = \sum_{n=-\infty}^{\infty}(3n+1)q^{3n^2+2n}$$
$$= \frac{d}{dz}\sum_{n=-\infty}^{\infty} q^{3n^2+2n}z^{3n+1}\Big|_{z=1}$$
$$= \frac{d}{dz}\{zf(q^5z^3, q/z^3)\}\Big|_{z=1}$$
$$= f(q,q^5)\frac{d}{dz}(\operatorname{Log}\{zf(q^5z^3, q/z^3)\})\Big|_{z=1}$$
$$= f(q,q^5)\frac{d}{dz}(\operatorname{Log}\{z(-q^5z^3;q^6)_\infty(-q/z^3;q^6)_\infty(q^6;q^6)_\infty\})\Big|_{z=1}$$
$$= f(q,q^5)\left(1 - 3\sum_{n=0}^{\infty}\left\{\frac{q^{6n+1}}{1+q^{6n+1}} - \frac{q^{6n+5}}{1-q^{6n+5}}\right\}\right), \tag{4.3}$$

where we have employed the Jacobi triple product identity. Now use Entries 19, 22(ii), (iv), 25(iv), and 24(i)–(iii) in Chapter 16 to deduce that

$$\frac{\psi(q^2)f^2(q)}{f(-q,-q^5)} = \frac{\psi(q^2)f^2(q)}{(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty} = \frac{\psi(q^2)f^2(q)(q^3;q^6)_\infty}{(q;q^2)_\infty(q^6;q^6)_\infty}$$
$$= \frac{\psi^2(q)f^2(q)}{\varphi(q)\chi(-q)\psi(q^3)} = \frac{\psi^2(q)\varphi(q)\psi(-q)}{\psi(q^3)\chi(-q)f(q)}$$
$$= \frac{\psi^2(q)\chi(q)\psi(-q)}{\psi(q^3)\chi(-q)} = \frac{\psi^3(q)}{\psi(q^3)}.$$

Replacing q by $-q$ in (4.3) and then using the result above, we finish the proof.

PROOF OF (iv). By Entry 8(ix), Chapter 17,

$$\varphi^2(-q)f(-q) = \sum_{n=-\infty}^{\infty} (6n+1)q^{(3n^2+n)/2}$$
$$= \frac{d}{dz}\sum_{n=-\infty}^{\infty} q^{(3n^2+n)/2}z^{6n+1}\Big|_{z=1}$$
$$= \frac{d}{dz}\{zf(q/z^6, q^2z^6)\}\Big|_{z=1}$$
$$= f(q,q^2)\frac{d}{dz}(\mathrm{Log}\{zf(q/z^6,q^2z^6)\})\Big|_{z=1}$$
$$= f(q,q^2)\left(1 - 6\sum_{n=0}^{\infty}\left\{\frac{q^{3n+1}}{1+q^{3n+1}} - \frac{q^{3n+2}}{1+q^{3n+2}}\right\}\right).$$

Replacing q by $-q$, we are led to examine $\varphi^2(q)f(q)/f(-q,-q^2)$. By Entries 19 and 22(iii) and (22.4) in Chapter 16,

$$\frac{\varphi^2(q)f(q)}{f(-q,-q^2)} = \frac{\varphi^2(q)(-q;-q)_\infty}{(q;-q^3)_\infty(-q^2;-q^3)_\infty(-q^3;-q^3)_\infty}$$
$$= \frac{\varphi^2(q)(-q;-q)_\infty}{(q;-q)_\infty}\frac{(q;-q)_\infty}{(q;-q^3)_\infty(-q^2;-q^3)_\infty}\frac{1}{(-q^3;-q^3)_\infty}$$
$$= \frac{\varphi^3(q)(q^3;-q^3)_\infty}{(-q^3;-q^3)_\infty} = \frac{\varphi^3(q)}{\varphi(q^3)},$$

from which the truth of (iv) is evident.

Recall that a modular equation of degree n is defined in Section 24 of Chapter 18 and in the Introduction. In Section 5, Ramanujan offers several modular equations of degree 3, and so we now summarize some of the notation that is used in this and succeeding sections. Let

$$z_1 = {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\alpha) \quad \text{and} \quad z_n = {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\beta),$$

where n is the degree of the modular equation. The expressions $\sqrt{\alpha}$ and $\sqrt{\beta}$ are the two moduli, and we say β is of the nth order (degree) in α. Recall, from

(6.4) in Chapter 17 and the definition of a modular equation (24.13) in Chapter 18, that

$$\varphi(q) = \varphi(e^{-y}) = z_1^{1/2}$$

and

$$\varphi(q^n) = \varphi(e^{-ny}) = z_n^{1/2}.$$

The multiplier m is defined by the equation

$$z_1 = mz_n.$$

It seems that Ramanujan derived his modular equations of degree 3 by taking formulas relating $\varphi(q), \psi(q), \ldots, \varphi(q^3), \psi(q^3), \ldots$ and transcribing them via formulas in Sections 10–12 in Chapter 17.

Entry 5. *The following are modular equations, formulas for multipliers, and identities for degree* 3:

(i) $$\left(\frac{\alpha^3}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} = 1 = \left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^3}{\alpha}\right)^{1/8};$$

(ii) $$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1;$$

(iii) $$m = 1 + 2\left(\frac{\beta^3}{\alpha}\right)^{1/8}; \qquad \frac{3}{m} = 1 + 2\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8};$$

(iv) $$m^2\left\{\left(\frac{\alpha^3}{\beta}\right)^{1/8} - \alpha\right\} = \left(\frac{\alpha^3}{\beta}\right)^{1/8} - \beta;$$

(v) $$m = \frac{1 - 2\left(\dfrac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}}{1 - 2(\alpha\beta)^{1/4}} = \left\{1 + 4\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}\right\}^{1/2};$$

$$\frac{3}{m} = \frac{2\left(\dfrac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8} - 1}{1 - 2(\alpha\beta)^{1/4}} = \left\{1 + 4\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}\right\}^{1/2};$$

(vi) *if* $\alpha = p\left(\dfrac{2+p}{1+2p}\right)^3$, *then* $\beta = p^3\left(\dfrac{2+p}{1+2p}\right)$,

$$1 - \alpha = (1+p)\left(\frac{1-p}{1+2p}\right)^3, \quad \textit{and} \quad 1 - \beta = (1+p)^3\left(\frac{1-p}{1+2p}\right);$$

(vii) $$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2};$$

$$\frac{9}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2};$$

(viii) $$(\alpha\beta^5)^{1/8} + \{(1-\alpha)(1-\beta)^5\}^{1/8} = 1 - \left(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\right)^{1/8}$$

$$= (\alpha^5\beta)^{1/8} + \{(1-\alpha)^5(1-\beta)\}^{1/8}$$

$$= \{\tfrac{1}{2}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})\}^{1/2};$$

(ix) $$\{\alpha(1-\beta)\}^{1/2} + \{\beta(1-\alpha)\}^{1/2} = 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$$

$$= m^2\{\alpha(1-\alpha)\}^{1/2} + \{\beta(1-\beta)\}^{1/2}$$

$$= \frac{9}{m^2}\{\beta(1-\beta)\}^{1/2} + \{\alpha(1-\alpha)\}^{1/2};$$

(x) $$m(1-\alpha)^{1/2} + (1-\beta)^{1/2} = \frac{3}{m}(1-\beta)^{1/2} - (1-\alpha)^{1/2}$$

$$= 2\{(1-\alpha)(1-\beta)\}^{1/8};$$

$$m\alpha^{1/2} - \beta^{1/2} = \frac{3}{m}\beta^{1/2} + \alpha^{1/2} = 2(\alpha\beta)^{1/8};$$

(xi) $$m - \frac{3}{m} = 2((\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4});$$

$$m + \frac{3}{m} = 4(\tfrac{1}{2}\{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}\})^{1/2};$$

(xii) *if* $P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ *and* $Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}$,

then
$$Q + \frac{1}{Q} + 2\sqrt{2}\left(P - \frac{1}{P}\right) = 0;$$

(xiii) *if* $P = (\alpha\beta)^{1/8}$ *and* $Q = (\beta/\alpha)^{1/4}$,

then
$$Q - \frac{1}{Q} = 2\left(P - \frac{1}{P}\right);$$

(xiv) *if* $\alpha = \sin^2(\mu + \nu)$ *and* $\beta = \sin^2(\mu - \nu)$,

then
$$\sin(2\mu) = 2\sin\nu;$$

(xv) *if* α *is an appropriately chosen root of the quadratic equation*

$$\alpha(1-\alpha) = q\left(\frac{2-q}{1+4q}\right)^3,$$

then

$$\beta(1-\beta) = q^3\left(\frac{2-q}{1+4q}\right).$$

We have written Entry 5(xv) in a more complete form than that given by Ramanujan (p. 231). The appropriate root α is given in (5.13) below.

PROOF OF (i). It is evident from Entries 4(iii) and (iv) that

$$2\frac{\psi^3(q)}{\psi(q^3)} = \frac{\varphi^3(q)}{\varphi(q^3)} + \frac{\varphi^3(-q^2)}{\varphi^3(-q^6)}.$$

We now transcribe this equality by means of Entries 10(i), (iii) and 11(i) in Chapter 17 and find that

$$2\frac{2^{-3/2}z_1^{3/2}(\alpha e^y)^{3/8}}{2^{-1/2}z_3^{1/2}(\beta e^{3y})^{1/8}} = \frac{z_1^{3/2}}{z_3^{1/2}} + \frac{z_1^{3/2}(1-\alpha)^{3/8}}{z_3^{1/2}(1-\beta)}.$$

Upon simplification, the first part of (i) is obtained.

The second equality of (i) is obtained from the first by employing Entry 24(v) of Chapter 18.

PROOF OF (ii). It is easily verified from Entries 3(i), (ii) that

$$4q\psi(q^2)\psi(q^6) = \varphi(q)\varphi(q^3) - \varphi(-q)\varphi(-q^3).$$

(This formula can also be readily deduced from (36.2) of Chapter 16.) Converting this formula with the aid of Entries 10(i), (ii) and 11(iii) in Chapter 17, we find that

$$(z_1z_3)^{1/2}e^{-y}(\alpha e^y)^{1/4}(\beta e^{3y})^{1/4} = (z_1z_3)^{1/2} - (z_1z_3)^{1/2}(1-\alpha)^{1/4}(1-\beta)^{1/4},$$

from which (ii) is obvious.

This form of the modular equation is due to Legendre [2, p. 229] and can be found in Cayley's book [1, p. 196] as well as Jacobi's *Fundamenta Nova* [1, p. 68], [2, p. 124].

PROOF OF (iii). From Entries 4(iv) and 3(ii), it is readily shown that

$$\frac{\varphi^3(q)}{\varphi(q^3)} + 2\frac{\varphi^3(-q^2)}{\varphi(-q^6)} = 3\varphi(q)\varphi(q^3).$$

Using Entries 10(i), (iii) in Chapter 17, we transcribe this equality and find that

$$\frac{z_1^{3/2}}{z_3^{1/2}} + 2\frac{z_1^{3/2}(1-\alpha)^{3/8}}{z_3^{1/2}(1-\beta)^{1/8}} = 3z_1^{1/2}z_3^{1/2}.$$

Canceling $z_1^{3/2}/z_3^{1/2}$, we derive the second equality of (iii).

The first equality of (iii) is simply the reciprocal of the second (in the sense of Entry 24(v) in Chapter 18).

PROOF OF (iv). From parts (i) and (iii),

$$\left(\frac{\beta^3}{\alpha}\right)^{1/8} = \frac{m-1}{2}, \qquad \left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} = \frac{m+1}{2},$$
$$\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} = \frac{3-m}{2m}, \quad \text{and} \quad \left(\frac{\alpha^3}{\beta}\right)^{1/8} = \frac{3+m}{2m}. \tag{5.1}$$

Taking the product of the cube of the first and the fourth, and then the product of the cube of the fourth and the first equalities of (5.1), we find that, respectively,

$$\beta = \frac{(m-1)^3(3+m)}{16m} \quad \text{and} \quad \alpha = \frac{(m-1)(3+m)^3}{16m^3}. \tag{5.2}$$

Hence, by (5.1) and (5.2),

$$m^2\alpha - \beta = (m^2-1)\frac{3+m}{2m} = (m^2-1)\left(\frac{\alpha^3}{\beta}\right)^{1/8},$$

from which (iv) is obtainable immediately.

Proof of (v). In (5.1), multiply the first and the second, the third and fourth, and the first and fourth equalities to deduce that, respectively,

$$\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8} = \frac{m^2-1}{4}, \qquad \left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8} = \frac{9-m^2}{4m^2},$$
$$\text{and} \quad (\alpha\beta)^{1/4} = \frac{m^2+2m-3}{4m}. \tag{5.3}$$

Solving the first equality for m and the second for $3/m$, we obtain two of the desired equalities.

The remaining two equalities of (v) are readily verified by substituting from (5.3).

Proof of (vi). Our procedure is logically somewhat different from that of Ramanujan. Define p by the equation

$$m = 1 + 2p. \tag{5.4}$$

Then the required formulas for α and β follow immediately from (5.2.).

Next, in (5.1), multiply the cube of the second equation by the third. Then take the cube of the third equality times the second. We then deduce, respectively, that

$$1-\beta = \frac{(m+1)^3(3-m)}{16m} \quad \text{and} \quad 1-\alpha = \frac{(m+1)(3-m)^3}{16m^3}. \tag{5.5}$$

Using (5.4), we deduce the desired formulas for $1-\beta$ and $1-\alpha$.

It might be noted that the first and third equalities of (5.1) immediately imply that $p > 0$ and $p < 1$, respectively.

For $0 \le p \le 1$, observe that

$$\frac{d\alpha}{dp} = \frac{2(1-p)^2(2+p)^2}{(1+2p)^4} \ge 0$$

and

$$\frac{d\beta}{dp} = \frac{6p^2(1+p)^2}{(1+2p)^2} \ge 0.$$

There is consequently a one-to-one correspondence between α and p and also between β and p when p lies between 0 and 1.

The parametric equations for α and β in (vi) were actually first discovered by Legendre [2, p. 223] and rediscovered by Jacobi [1, p. 25], [2, p. 76].

PROOF OF (vii). The proofs of (vii)–(xi) depend on (5.2) and (5.5). Thus, we first deduce that, respectively,

$$\left(\frac{\beta}{\alpha}\right)^{1/2} = \frac{m(m-1)}{3+m} \quad \text{and} \quad \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} = \frac{m(m+1)}{3-m}.$$

The first formula of (vii) now follows from a straightforward calculation, while the second follows from reciprocation.

PROOF OF (viii). From (5.2) and (5.5), respectively,

$$(\alpha\beta^5)^{1/8} = \frac{(m-1)^2(3+m)}{8m} \quad \text{and} \quad \{(1-\alpha)(1-\beta)^5\}^{1/8} = \frac{(m+1)^2(3-m)}{8m}.$$

Hence,

$$\begin{aligned}(\alpha\beta^5)^{1/8} + \{(1-\alpha)(1-\beta^5)\}^{1/8} &= \frac{3+m^2}{4m} \\ &= 1 - \frac{(3-m)(m-1)}{4m} \\ &= 1 - \left(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\right)^{1/8}, \qquad (5.6)\end{aligned}$$

by (5.2) and (5.5). This proves the first equality in (viii).

Taking the reciprocal of this equality, we find that

$$\begin{aligned}(\alpha^5\beta)^{1/8} + \{(1-\alpha)^5(1-\beta)\}^{1/8} &= 1 - \left(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\right)^{1/8} \\ &= \frac{3+m^2}{4m}, \qquad (5.7)\end{aligned}$$

by (5.6). On the other hand, from (5.2) and (5.5),

$$\begin{aligned}&1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} \\ &\quad = 1 + \frac{(m-1)^2(3+m)^2}{16m^2} + \frac{(m+1)^2(3-m)^2}{16m^2} = \frac{(m^2+3)^2}{8m^2}. \qquad (5.8)\end{aligned}$$

The truth of the second equality in (viii) is now manifest from (5.7) and (5.8).

PROOF OF (ix). The equality of the first and third expressions in (ix) is an immediate consequence of (vii), and so is the equality of the first and fourth expressions.

From (5.2) and (5.5),

$$\begin{aligned}&\{\alpha(1-\beta)\}^{1/2} + \{\beta(1-\alpha)\}^{1/2} \\ &\quad = \frac{(m+1)(m+3)\{(m^2-1)(9-m^2)\}^{1/2}}{16m^2}\end{aligned}$$

$$+\frac{(m-1)(3-m)\{(m^2-1)(9-m^2)\}^{1/2}}{16m^2}$$

$$=\frac{\{(m^2-1)(9-m^2)\}^{1/2}}{2m}=2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}, \qquad (5.9)$$

and so the proof of (ix) is complete.

PROOF OF (x). From (5.2) and (5.5), it is easily found that each of the first three expressions in (x) is equal to

$$\left\{\frac{1}{m}(m+1)(3-m)\right\}^{1/2}.$$

Likewise, from (5.2) and (5.5), we readily see that the latter three expressions of (x) are each equal to

$$\left\{\frac{1}{m}(m-1)(3+m)\right\}^{1/2}.$$

It is possible that this set of formulas was suggested by the simplicity of the expression for $m^2\alpha-\beta$, given in the proof of (iv); for this indicates the likely existence of a simple expression for the factor $m\sqrt{\alpha}-\sqrt{\beta}$.

PROOF OF (xi). From (5.2) and (5.5),

$$(\alpha\beta)^{1/4}-\{(1-\alpha)(1-\beta)\}^{1/4}=\frac{(m-1)(3+m)}{4m}-\frac{(m+1)(3-m)}{4m}$$

$$=\frac{m^2-3}{2m},$$

from which the first equality of (xi) is apparent.

The second part of (xi) follows immediately from (5.8).

PROOF OF (xii). From (5.2) and (5.5),

$$P^2=\frac{(m^2-1)(9-m^2)}{8m^2} \quad \text{and} \quad Q^2=\frac{(m^2-1)m^2}{9-m^2}.$$

Thus,

$$PQ=\frac{m^2-1}{\sqrt{8}} \quad \text{and} \quad \frac{P}{Q}=\frac{9-m^2}{m^2\sqrt{8}}.$$

The elimination of m from the latter pair of equalities yields

$$\frac{P}{Q}=\frac{\sqrt{8}-PQ}{\sqrt{8}\,PQ+1}.$$

Rearranging this equality, we easily deduce the result claimed in (xii).

PROOF OF (xiii). From (5.2) and (5.5),

$$P^2 = \frac{(m-1)(3+m)}{4m} \quad \text{and} \quad Q^2 = \frac{m(m-1)}{3+m}.$$

It follows that

$$PQ = \frac{m-1}{2} \quad \text{and} \quad \frac{P}{Q} = \frac{3+m}{2m}.$$

Eliminating m from this pair of equalities, we find that

$$\frac{P}{Q} = \frac{2+PQ}{2PQ+1},$$

which, upon rearrangement, yields the desired result.

PROOF OF (xiv). We assume that $\mu + \nu$ and $\mu - \nu$ are positive acute angles, and so 2ν is also an acute angle. Since it is clear that $\alpha > \beta$, it also follows that ν is positive.

Using the given values of α and β, we find that the first equality of (ix) can be written in the form

$$\sin(2\mu) = \{4\sin(2\mu+2\nu)\sin(2\mu-2\nu)\}^{1/4}.$$

Hence,

$$\sin^4(2\mu) = 4\sin^2(2\mu) - 4\sin^2(2\nu);$$

that is,

$$(2 - \sin^2(2\mu))^2 = 4\cos^2(2\nu).$$

Thus,

$$\sin^2(2\mu) = 2(1 - \cos(2\nu)),$$

and (xiv) follows at once, since ν is a positive, acute angle.

We observe that

$$\{(1-\alpha)(1-\beta)\}^{1/2} - (\alpha\beta)^{1/2} = \cos(2\mu).$$

On the other hand, by (5.2) and (5.5),

$$\{(1-\alpha)(1-\beta)\}^{1/2} - (\alpha\beta)^{1/2} = \frac{3-m^2}{2m}.$$

Thus, we deduce that

$$\cos(2\mu) = \frac{3-m^2}{2m} \tag{5.10}$$

We also shall later need an expression for $\cos\nu$. By (xiv) and (5.9), we find that

$$\cos^2\nu = 1 - \sin^2\nu = 1 - \tfrac{1}{4}\sin^2(2\mu)$$
$$= 1 - \tfrac{1}{4}(\{\alpha(1-\beta)\}^{1/2} + \{\beta(1-\alpha)\}^{1/2})^2$$

$$= 1 - \frac{(m^2 - 1)(9 - m^2)}{16m^2}$$

$$= \frac{(m^2 + 3)^2}{16m^2};$$

that is,

$$\cos v = \frac{m^2 + 3}{4m}. \tag{5.11}$$

PROOF OF (xv). Recall that p is defined by (5.4) and that, after the proof of (vi), we showed that $0 < p < 1$. Define q by

$$q = p + p^2, \tag{5.12}$$

so that $0 < q < 2$. We are then given that

$$\alpha(1 - \alpha) = p(1 + p)\frac{(1 - p)^3(2 + p)^3}{(1 + 2p)^6}.$$

Solving this quadratic equation, we find that either

$$\alpha = p\left(\frac{2 + p}{1 + 2p}\right)^3 = \left\{\frac{-1 + (4q + 1)^{1/2}}{2}\right\}\left\{\frac{3 + (4q + 1)^{1/2}}{2(4q + 1)^{1/2}}\right\}^3 \tag{5.13}$$

or

$$\alpha = (1 + p)\left(\frac{1 - p}{1 + 2p}\right)^3 = \left\{\frac{1 + (4q + 1)^{1/2}}{2}\right\}\left\{\frac{3 - (4q + 1)^{1/2}}{2(4q + 1)^{1/2}}\right\}^3. \tag{5.14}$$

Suppose that α is given by (5.13). Then from (vi) it follows that

$$\beta = p^3\left(\frac{2 + p}{1 + 2p}\right) \quad \text{and} \quad 1 - \beta = (1 + p)^3\left(\frac{1 - p}{1 + 2p}\right).$$

Hence,

$$\beta(1 - \beta) = q^3\left(\frac{2 - q}{1 + 4q}\right),$$

as desired.

Suppose that α is given by (5.14). If $\beta = p^3(2 + p)/(1 + 2p)$, then by the one-to-one correspondence established after the proof of (vi), $\alpha = p(2 + p)^3/(1 + 2p)^3$, which is a contradiction. Suppose that $\beta = (1 + p)^3(1 - p)/(1 + 2p)$. Then $1 - \alpha = p(2 + p)^3/(1 + 2p)^3$ and $1 - \beta = p^3(2 + p)/(1 + 2p)$. It follows that

$$\left(\frac{\alpha^3}{\beta}\right)^{1/8} - \left(\frac{(1 - \alpha)^3}{1 - \beta}\right)^{1/8} = \frac{1 - p}{1 + 2p} - \frac{2 + p}{1 + 2p} = -1.$$

However, this contradicts (i). Since the two values specified for β are the only two values that satisfy the equation $\beta(1-\beta) = q^3(2-q)/(1+4q)$, we must conclude that when α is given by (5.14), the value of $\beta(1-\beta)$ is not the one required in (xv). Hence, the appropriate root α in the statement of Entry 5(xv) is that specified by (5.13).

Suppose that the root given by (5.13) is the smaller root, that is,

$$p\left(\frac{2+p}{1+2p}\right)^3 < (1+p)\left(\frac{1-p}{1+2p}\right)^3.$$

This reduces to

$$2q^2 + 10q - 1 < 0.$$

Hence, $p(2+p)^3/(1+2p)^3$ is the smaller root when $q < \frac{1}{2}(3\sqrt{3}-5)$ and the greater root when $q > \frac{1}{2}(3\sqrt{3}-5)$.

Entry 6.

(i) *Let p be defined by* (5.4). *Then*

$${}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; p\left(\frac{2+p}{1+2p}\right)^3\right) = (1+2p)\,{}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; p^3\left(\frac{2+p}{1+2p}\right)\right).$$

(ii) *Let q be defined by* (5.12), *where p is defined by* (5.4). *Then if* $q < \frac{1}{2}(3\sqrt{3}-5)$,

$${}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{4}; 1; 4q\left(\frac{2-q}{1+4q}\right)^3\right) = (1+4q)^{1/2}\,{}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{4}; 1; 4q^3\left(\frac{2-q}{1+4q}\right)\right).$$

(iii) *If* $\tan\frac{1}{2}(A+B) = (1+p)\tan A$, *then*

$$(1+2p)\int_0^A \frac{d\varphi}{\left\{1-p^3\left(\frac{2+p}{1+2p}\right)\sin^2\varphi\right\}^{1/2}} = \int_0^B \frac{d\varphi}{\left\{1-p\left(\frac{2+p}{1+2p}\right)^3\sin^2\varphi\right\}^{1/2}}.$$

(iv) *If* $\tan\frac{1}{2}(A-B) = ((1-p)/(1+2p))\tan B$, *then*

$$(1+2p)\int_0^A \frac{d\varphi}{\left\{1-p^3\left(\frac{2+p}{1+2p}\right)\sin^2\varphi\right\}^{1/2}}$$

$$= 3\int_0^B \frac{d\varphi}{\left\{1-p\left(\frac{2+p}{1+2p}\right)^3\sin^2\varphi\right\}^{1/2}}.$$

(v) *If*

$$\tan\tfrac{1}{2}(C+B) = \frac{2\tan B + 2(1-x)\tan^3 B}{1-(1-x)\tan^4 B},$$

then

$$\int_0^C \frac{d\varphi}{\{1 - x\sin^2\varphi\}^{1/2}} = 3\int_0^B \frac{d\varphi}{\{1 - x\sin^2\varphi\}^{1/2}}.$$

It is tacitly assumed that $0 \le A, B, C \le \pi/2$.

PROOF OF (i). Part (i) is simply a version of the formula (5.1), that is,

$${}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha) = m\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \beta),$$

when α and β are given by the parametric equations of Entry 5(vi).

PROOF OF (ii). By Entry 33(ii) in Chapter 11 (Part II [9, p. 94]),

$$\begin{aligned} {}_2F_1(\tfrac{1}{4}, \tfrac{1}{4}; 1; 4\alpha(1-\alpha)) &= {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1}{2}(1 - \{1 - 4\alpha(1-\alpha)\}^{1/2})) \\ &= {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha) \\ &= m\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \beta) \\ &= (1 + 4q)^{1/2}\,{}_2F_1(\tfrac{1}{4}, \tfrac{1}{4}; 1; 4\beta(1-\beta)), \end{aligned}$$

where, to obtain the last equality, we merely repeat the prior steps, but in reverse order. The equality in (ii) now formally follows from Entry 5(xv).

Observe that in taking the square root above, it was assumed that $\alpha < \frac{1}{2}$, that is, $p(2+p)^3/(1+2p)^3 < \frac{1}{2}$. It is easily seen that the latter statement is equivalent to the inequality $q < \frac{1}{2}(3\sqrt{3} - 5)$, and so the proof of (ii) is complete. (With respect to the restriction on q, recall the remarks made at the conclusion of the proof of Entry 5(xv).)

PROOF OF (iii). This formula is the general transformation of the third order. There is no evidence as to how Ramanujan obtained it. Thus, rather than derive a proof *ab initio*, we merely content ourselves with demonstrating how it can be derived from the form of the general third-order transformation discovered by Jacobi [1], [2, p. 76]; namely,

$$(1+2p)\int_0^A \frac{d\varphi}{\left\{1 - p^3\left(\dfrac{2+p}{1+2p}\right)\sin^2\varphi\right\}^{1/2}} = \int_0^B \frac{d\varphi}{\left\{1 - p\left(\dfrac{2+p}{1+2p}\right)^3\sin^2\varphi\right\}^{1/2}}, \tag{6.1}$$

when

$$\sin B = \frac{(1+2p)\sin A + p^2\sin^3 A}{1 + p(2+p)\sin^2 A}.$$

Solving this quadratic equation for p, we find that

$$p = \frac{1 - \sin A\sin B \pm \cos A\cos B}{\sin A(\sin B - \sin A)}. \tag{6.2}$$

Since A and B vanish simultaneously, it is clear that the ambiguous sign above must be replaced by a minus sign. So,

$$p = \frac{1-\cos(A-B)}{\sin A(\sin B - \sin A)} = \frac{\sin\frac{1}{2}(B-A)}{\sin A\cos\frac{1}{2}(B+A)}$$
$$= \frac{\cos A\sin\frac{1}{2}(B+A)}{\sin A\cos\frac{1}{2}(B+A)} - 1.$$

In other words, $\tan\frac{1}{2}(A+B) = (1+p)\tan A$, and this establishes (iii).

Proof of (iv). In Jacobi's result (6.1), replace p by $-(2+p)/(1+2p)$, A by $-B$, and B by A. We then deduce that

$$\frac{3}{1+2p}\int_0^B \frac{d\varphi}{\left\{1 - p\left(\frac{2+p}{1+2p}\right)^3 \sin^2\varphi\right\}^{1/2}} = \int_0^A \frac{d\varphi}{\left\{1 - p^3\left(\frac{2+p}{1+2p}\right)\sin^2\varphi\right\}^{1/2}},$$

when

$$\sin A = \frac{3(1+2p)\sin B - (2+p)^2\sin^3 B}{(1+2p)^2 - 3p(2+p)\sin^2 B}.$$

Solving for $(2+p)/(1+2p)$, or employing (6.2) with the designated substitutions for p, A, and B, we find that

$$\frac{2+p}{1+2p} = \frac{1-\cos(A+B)}{\sin B(\sin A + \sin B)}$$
$$= \frac{\sin\frac{1}{2}(A+B)}{\sin B\cos\frac{1}{2}(A-B)} = \frac{\cos B\sin\frac{1}{2}(A-B)}{\sin B\cos\frac{1}{2}(A-B)} + 1;$$

that is

$$\tan\tfrac{1}{2}(A-B) = \left(\frac{2+p}{1+2p} - 1\right)\tan B = \frac{1-p}{1+2p}\tan B.$$

Therefore, the proof of (iv) is completed.

Proof of (v). By replacing B with C in (iii) and comparing (iii) and (iv), we deduce that

$$\int_0^C \frac{d\varphi}{\{1 - x\sin^2\varphi\}^{1/2}} = 3\int_0^B \frac{d\varphi}{\{1 - x\sin^2\varphi\}^{1/2}},$$

where $x = p(2+p)^3/(1+2p)^3$, and B and C are connected by the relation that is obtained by eliminating A from the equations

$$\tan\tfrac{1}{2}(A+C) = (1+p)\tan A \quad \text{and} \quad \tan\tfrac{1}{2}(A-B) = \frac{1-p}{1+2p}\tan B. \tag{6.3}$$

From the addition formula for $\tan u$ and the latter equality of (6.3), it follows that

$$\tan \tfrac{1}{2}(A + B) = \tan\{\tfrac{1}{2}(A - B) + B\} = \frac{(2 + p)\tan B}{(1 + 2p) - (1 - p)\tan^2 B}.$$

Hence,

$$\begin{aligned} \tan A &= \tan\{\tfrac{1}{2}(A - B) + \tfrac{1}{2}(A + B)\} \\ &= \frac{\dfrac{1 - p}{1 + 2p}\tan B + \dfrac{(2 + p)\tan B}{(1 + 2p) - (1 - p)\tan^2 B}}{1 - \dfrac{(1 - p)(2 + p)\tan^2 B}{(1 + 2p)^2 - (1 - p)(1 + 2p)\tan^2 B}} \\ &= \frac{3(1 + 2p)\tan B - (1 - p)^2 \tan^3 B}{(1 + 2p)^2 - 3(1 - p^2)\tan^2 B}. \end{aligned} \tag{6.4}$$

Using both equalities of (6.3) and then (6.4), we find that

$$\begin{aligned} \tan \tfrac{1}{2}(C + B) &= \tan\{\tfrac{1}{2}(A + C) - \tfrac{1}{2}(A - B)\} \\ &= \frac{(1 + p)(1 + 2p)\tan A - (1 - p)\tan B}{1 + 2p + (1 - p^2)\tan A \tan B} \\ &= \frac{2 \tan B + 2(1 - x)\tan^3 B}{1 - (1 - x)\tan^4 B}, \end{aligned}$$

after a somewhat lengthy computation. This finishes the proof of (v).

Although a triplication formula of this type is due to Jacobi [1, p. 29], [2, p. 80], this form of the triplication formula with the relatively simple expression connecting B and C is due to Ramanujan. The simpler relations between A and B in (iii) and (iv) evidently made it possible for Ramanujan to discover his elegant rendition of the triplication formula, whereas the analysis needed from Jacobi's relations would, indeed, be more formidable and less discernible. Jacobi's work is recapitulated in Cayley's treatise [1, pp. 201–202].

Entries 7(i), (ii). *Recall the definition of z in (6.2) of Chapter 17. Let $x = p(2 + p)^3/(1 + 2p)^3$. Let B denote an acute angle. If $\cos B = (1 - p)/(2 + p)$, then*

(i) $$\int_0^B \frac{d\varphi}{\{1 - x \sin^2 \varphi\}^{1/2}} = \frac{\pi}{3} z.$$

If $\sin B = (1 + 2p)/(2 + p)$, *then*

(ii) $$\int_0^B \frac{d\varphi}{\{1 - x \sin^2 \varphi\}^{1/2}} = \frac{\pi}{6} z.$$

Proof. We show that these two formulas are consequences of Entry 6(v) and the associated conditions (6.3). From (6.3), we see that as B increases from 0 to $\pi/2$ to π, $\frac{1}{2}(A - B)$ does the same. Thus, A increases from 0 to $3\pi/2$ to 3π. Also, if A increases from 0 to $\pi/2$ to π, $\frac{1}{2}(A + C)$ does likewise. So, C increases from 0 to $\pi/2$ to π. Thus, when C is equal to π, so is A, and, furthermore, B is a positive acute angle.

Using Entry 6(v) and recalling (6.9) in Chapter 17, we thus deduce that

$$\int_0^B \frac{d\varphi}{\{1 - x\sin^2\varphi\}^{1/2}} = \frac{\pi}{3}z,$$

when B is the positive acute angle that satisfies the equation

$$\tan\tfrac{1}{2}(\pi - B) = \frac{1-p}{1+2p}\tan B.$$

Using the identity

$$\tan\tfrac{1}{2}(\pi - B) = \cot\tfrac{1}{2}B = \frac{1+\cos B}{\sin B},$$

we eventually find that $\cos B$ satisfies the quadratic equation

$$(2+p)\cos^2 B + (1+2p)\cos B + (p-1) = 0.$$

Solving this equation for $\cos B$ and taking the proper root, we find that $\cos B = (1-p)/(2+p)$, as desired.

Also, when C is equal to $\pi/2$, so is A. Thus, by the same type of argument as that above,

$$\int_0^B \frac{d\varphi}{\{1 - x\sin^2\varphi\}^{1/2}} = \frac{\pi}{6}z,$$

where B is the positive acute angle that satisfies the equation

$$\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}B) = \frac{1-p}{1+2p}\tan B.$$

From elementary trigonometry, this equation may be put in the form

$$\frac{\sin B - 1 + \cos B}{\sin B + 1 - \cos B} = \frac{1-p}{1+2p}\tan B.$$

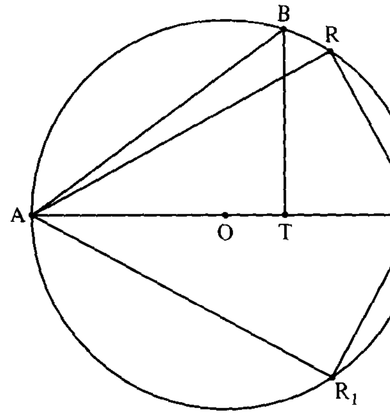
It is easily checked that $\sin B = (1+2p)/(2+p)$ is the solution to this equation.

In the classical notation of elliptic functions, formulas (i) and (ii) assume the respective forms

$$\operatorname{cn}\left(\frac{2K}{3}\right) = \frac{1-p}{2+p} \quad \text{and} \quad \operatorname{sn}\left(\frac{K}{3}\right) = \frac{1+2p}{2+p},$$

where $k^2 = x = p(2+p)^3/(1+2p)^3$. Other formulas of this nature are due to Forsyth [1], Glaisher [3], and Burnside [1].

Entry 7(iii). *Let AP denote the diameter of a circle $\mathscr{C}$ with center O. Let TB be perpendicular to AP, with $B \in \mathscr{C}$ and $T \in AP$. Draw chords PR and PR_1 equal in length to TB with R nearer to B. Form AB, AR,* and *AR_1. Then a pendulum oscillating through $\angle 4BAR_1$ takes $(AR - OT)/AO$ or $3AO/(AR + OT)$ times the time required to oscillate through $\angle 4BAR$.*



Proof. As we shall see, Entry 7(iii) can be derived from Entry 5(xiv).

Let $OP = a$ and $\angle PAB = \mu$. Then $AB = 2a \cos \mu$ and

$$BT = AB \sin \mu = 2a \sin \mu \cos \mu.$$

Thus,

$$\sin \angle PAR = \frac{RP}{2a} = \frac{BT}{2a} = \sin \mu \cos \mu = \sin \nu,$$

when ν is defined by the equation $\sin(2\mu) = 2 \sin \nu$. By the converse of Entry 5(xiv),

$$\alpha = \sin^2(\mu + \nu) = \sin^2 \angle BAR_1$$

and

$$\beta = \sin^2(\mu - \nu) = \sin^2 \angle BAR.$$

Let t_1 and t_2 denote the respective periodic times that it takes for a pendulum of length ℓ to oscillate through the angles $4BAR_1$ and $4BAR$. Then (Hancock [1, p. 91])

$$t_1 = \sqrt{\frac{\ell}{g}} \int_0^{2\pi} \frac{d\varphi}{\{1 - \alpha \sin^2 \varphi\}^{1/2}} \quad \text{and} \quad t_2 = \sqrt{\frac{\ell}{g}} \int_0^{2\pi} \frac{d\varphi}{\{1 - \beta \sin^2 \varphi\}^{1/2}}.$$

By Entry 6(i), $t_1/t_2 = m$.

Now observe that $\angle BOT = 2\mu$. Thus,

$$OT = a \cos(2\mu) = a \frac{3 - m^2}{2m}, \tag{7.1}$$

by (5.10). Also,

$$AR = 2a\cos\nu = 2a\frac{m^2+3}{4m},$$

by (5.11). Substituting these values for OT and AR into $(AR - OT)/AO$ and $3AO/(AR + OT)$, we find quite easily that each reduces to m, and so the proof is complete.

Corollary. *Suppose that T coincides with O. Then* $\angle BAR = \pi/12$, $\angle BAR_1 = 5\pi/12$, *and*

$$\frac{AR - OT}{AO} = \frac{3AO}{AR + OT} = \sqrt{3}.$$

Furthermore, a pendulum oscillating through 300° *takes* $\sqrt{3}$ *times the time required to oscillate through* 60°.

Proof. The hypotheses immediately imply that $\mu = \pi/4$, $\nu = \pi/6$, and $m = \sqrt{3}$, by (7.1). So, $\angle BAR = \mu - \nu = \pi/12$ and $\angle BAR_1 = \mu + \nu = 5\pi/12$. The assertion about the pendulum also follows at once from Entry 7(iii).

According to notes left by G. N. Watson, this special case concerning the angles 60° and 300° is due to Legendre several years before the discovery of the general cubic transformation.

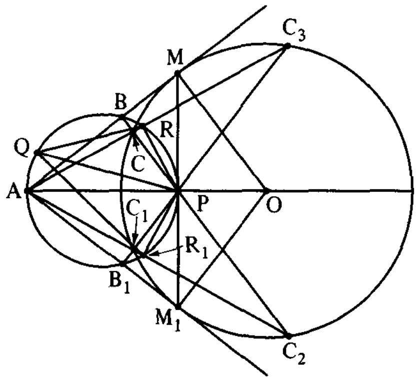
The following geometrical description is an extensive elaboration of that in Entry 7(iii). For convenience, we have divided Entry 7(iv) into three parts.

Entry 7(iv) (First Part). *Let AP denote a diameter and PQ a chord of a circle* $\mathscr{C}$. *Let B denote the midpoint of the arc PQ. Draw AB and PB. Let* B_1 *be the mirror image of B in AP and construct* AB_1 *and* PB_1. *Let R be a point on* $\mathscr{C}$ *such that* $PR = \frac{1}{2}PQ$ *and so that R is on the same side of AP as B and Q. Let* R_1 *be the image of R in AP. Form PR,* PR_1, *QR, and* QR_1. *Draw AR and* AR_1, *cutting PB and* PB_1 *at C and* C_1, *respectively. Construct a line perpendicular to AP at P. Let the extensions of AB and* AB_1 *meet this line at M and* M_1, *respectively. Extend BP and* AR_1 *to their point of intersection* C_2, *and extend* B_1P *and AR to their point of intersection* C_3.

Then a circle $\mathscr{C}'$ *will pass through M, C,* C_1, M_1, C_2, *and* C_3, *and this circle will be orthogonal to the circle* $\mathscr{C}$. *Furthermore,* $\mathscr{C}'$ *will be tangent to the straight lines AB and* AB_1 *at M and* M_1, *respectively. Let O denote the center of* $\mathscr{C}'$. *Form OM and* OM_1.

The circle $\mathscr{C}'$ *also passes through the intersections of the circles with centers A and P and radii AB and PR, respectively. The distances of any point on the circumference of* $\mathscr{C}'$ *from A and P bear a constant ratio. Lastly,*

$$QR \cdot QR_1 = 3RP^2.$$



Proof. Our procedure is logically somewhat different from that of Ramanujan.

Let the radius of $\mathscr{C}$ be denoted by a. Let $\angle PAQ = 2\mu$, where Q is any point on $\mathscr{C}$. Since the arcs PB and BQ are equal, it follows that $\angle PAB = \mu$. Also, $AB = 2a\cos\mu$ and

$$\sin(2\mu) = \frac{QP}{2a}. \tag{7.2}$$

Let $\nu = \angle PAR$. Then

$$\sin\nu = \frac{RP}{2a} = \frac{QP}{4a}. \tag{7.3}$$

Hence, (7.2) and (7.3) imply that

$$\sin(2\mu) = 2\sin\nu. \tag{7.4}$$

We now reorder the steps in Ramanujan's line of reasoning. Draw a second circle $\mathscr{C}'$ that is tangent to AB and AB_1 at M and M_1, respectively. We show that $\mathscr{C}'$ passes through C, C_1, C_2, and C_3.

Since, by construction, A is the pole of MM_1 with respect to $\mathscr{C}'$, A and P must be inverse points with respect to $\mathscr{C}'$ (Coxeter [1, p. 78]). Consequently, the circle $\mathscr{C}$ with AP as its diameter is orthogonal to $\mathscr{C}'$ (Coxeter [1, p. 80]), and $\mathscr{C}'$ is the locus of points whose distances from A and P are in the constant ratio (Court [1, p, 173])

$$\frac{AM}{PM} = \frac{1}{\sin\mu}. \tag{7.5}$$

To prove first that C lies on $\mathscr{C}'$, observe that, by the law of sines,

$$\frac{AC}{PC} = \frac{\sin\angle APC}{\sin\angle PAC} = \frac{\sin\angle APB}{\sin\angle PAR} = \frac{\cos\mu}{\sin\nu} = \frac{2\cos\mu}{\sin(2\mu)} = \frac{1}{\sin\mu},$$

by (7.4). Thus, by (7.5), C lies on $\mathscr{C}'$.

Examining next C_3, we have, by the law of sines,

$$\frac{AC_3}{PC_3} = \frac{\sin \angle APC_3}{\sin \angle PAC_3} = \frac{\sin \angle APB_1}{\sin \angle PAR} = \frac{\cos \mu}{\sin \nu} = \frac{1}{\sin \mu},$$

as above, and so, by (7.5), C_3 lies on $\mathscr{C}'$.

Since C_1 and C_2 are the images of C and C_3, respectively, in AP, C_1 and C_2 must also lie on $\mathscr{C}'$. Thus, since C, C_1, C_2, and C_3 lie on $\mathscr{C}'$, our definition of $\mathscr{C}'$ is reconciled with that of Ramanujan.

Next, draw the circles with centers A and P and radii $r_1 = AB$ and $r_2 = PR$, respectively. If X is either of their points of intersection, then

$$\frac{AX}{PX} = \frac{r_1}{r_2}.$$

Thus, by (7.5), X lies on $\mathscr{C}'$ provided that $r_2 = r_1 \sin \mu$. Since $r_1 = AB = 2a \cos \mu$ and $r_2 = PR = 2a \sin \nu$, we see that this condition is indeed met.

By the law of sines, elementary geometry, (7.4), (5.10), and (5.11),

$$\frac{QR}{RP} = \frac{\sin \angle QPR}{\sin \angle RQP} = \frac{\sin(2\mu - \nu)}{\sin \nu} = \frac{\sin(2\mu)\cos \nu - \cos(2\mu)\sin \nu}{\sin \nu}$$

$$= 2 \cos \nu - \cos(2\mu) = m. \tag{7.6}$$

Similar considerations show that

$$\frac{QR_1}{RP} = \frac{\sin \angle QPR_1}{\sin \angle R_1QP} = \frac{\sin(2\mu + \nu)}{\sin \nu} = 2 \cos \nu + \cos(2\mu) = \frac{3}{m}. \tag{7.7}$$

Thus, (7.6) and (7.7) imply that $QR \cdot QR_1 = 3RP^2$, as desired.

Most of the content of Entry 7(iv) (first part) was submitted by Ramanujan as a problem to the *Journal of the Indian Mathematical Society* [5], [10, p. 331]. A solution to this problem was never published. However, more recently, the first part of Entry 7(iv) was the basis of a Ramanujan Centenary Prize Competition [1] held in *Mathematics Today*, an Indian journal aimed at students of mathematics in high schools and colleges. A total of 24 solutions were received, and three were published.

Entry 7(iv) (Second Part). *A pendulum oscillating through the angle* $4BAR_1$ *takes* QR/RP *or* $3R_1P/R_1Q$ *times the time required to oscillate through the angle* $4BAR$.

Proof. By the last statement of Entry 7(iv) (first part) and (7.6), the two ratios QR/RP and $3R_1P/R_1Q$ are each equal to m. The given result now follows from the proof of Entry 7(iii), wherein it was shown that the ratio of the two respective designated times is equal to m.

Entry 7(iv) (Third Part). *With the notation of the first part of Entry* 7(iv) *and Entry* 5(xiv),

$$\alpha^{1/2} = \frac{BC_2}{AC_2}, \quad \beta^{1/2} = \frac{BC}{AC}, \quad (1-\alpha)^{1/2} = \frac{AB}{AC_2}, \quad (1-\beta)^{1/2} = \frac{AB}{AC}, \tag{7.8}$$

$$(\alpha\beta)^{1/8} = \left(\frac{BC \cdot BC_2}{AC_1 \cdot AC_3}\right)^{1/4} = \left(\frac{BM}{AM}\right)^{1/2} = \frac{BP}{AP}, \tag{7.9}$$

$$\{(1-\alpha)(1-\beta)\}^{1/8} = \left(\frac{AB}{AM}\right)^{1/2} = \frac{AB}{AP}, \tag{7.10}$$

$$m = \frac{QR}{RP}, \qquad \frac{3}{m} = \frac{QR_1}{R_1P}, \tag{7.11}$$

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = \frac{BM}{AM} + \frac{AB}{AM} = 1, \tag{7.12}$$

$$\begin{aligned}\left(\frac{\alpha^3}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} &= \frac{\alpha^{1/2}}{(\alpha\beta)^{1/8}} - \frac{(1-\alpha)^{1/2}}{\{(1-\alpha)(1-\beta)\}^{1/8}} \\ &= \frac{BC_2}{BP}\frac{AP}{AC_2} - \frac{AP}{AC_2} = \frac{PC_2}{BP}\frac{AP}{AC_2} \\ &= \frac{PC_2}{AC_2}\frac{AM}{PM} = 1, \end{aligned} \tag{7.13}$$

and

$$\begin{aligned}\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^3}{\alpha}\right)^{1/8} &= \frac{(1-\beta)^{1/2}}{\{(1-\alpha)(1-\beta)\}^{1/8}} - \frac{\beta^{1/2}}{(\alpha\beta)^{1/8}} \\ &= \frac{AP}{AC} - \frac{BC}{AC}\frac{AP}{BP} = \frac{AP}{AC}\frac{CP}{BP} = \frac{CP}{AC}\frac{AM}{PM} = 1. \end{aligned} \tag{7.14}$$

Proof. Since $\angle BAC_2 = \mu + \nu$ and $\angle BAC = \mu - \nu$, it follows that

$$\frac{BC_2}{AC_2} = \sin(\mu+\nu) = \alpha^{1/2}, \qquad \frac{BC}{AC} = \sin(\mu-\nu) = \beta^{1/2},$$

$$\frac{AB}{AC_2} = \cos(\mu+\nu) = (1-\alpha)^{1/2}, \quad \text{and} \quad \frac{AB}{AC} = \cos(\mu-\nu) = (1-\beta)^{1/2}.$$

Thus, (7.8) is established.

From (7.8),

$$(\alpha\beta)^{1/8} = \left(\frac{BC_2}{AC_2}\frac{BC}{AC}\right)^{1/4} = \left(\frac{BC_2}{AC_3}\frac{BC}{AC_1}\right)^{1/4}.$$

By (7.4),

$$\begin{aligned}(\alpha\beta)^{1/8} &= \{\sin(\mu+\nu)\sin(\mu-\nu)\}^{1/4} \\ &= \{\sin^2\mu\cos^2\nu - \sin^2\nu\cos^2\mu\}^{1/4} \\ &= \{\sin^2\mu - \tfrac{1}{4}\sin^2(2\mu)\}^{1/4} \\ &= \sin\mu = \frac{BP}{AP}. \end{aligned} \tag{7.15}$$

From similar triangles,

$$\frac{BP}{AP}=\frac{MP}{AM} \quad\text{and}\quad \frac{BP}{AP}=\frac{BM}{MP}. \tag{7.16}$$

We thus find at once that

$$\left(\frac{BP}{AP}\right)^2=\frac{BM}{AM}.$$

Thus, all the equalities of (7.9) are established.

Next, by (7.4),

$$\begin{aligned}\{(1-\alpha)(1-\beta)\}^{1/8} &= \{\cos(\mu+\nu)\cos(\mu-\nu)\}^{1/4}\\ &= \{\cos^2\mu\cos^2\nu-\sin^2\mu\sin^2\nu\}^{1/4}\\ &= \cos\mu=\frac{AB}{AP}.\end{aligned} \tag{7.17}$$

From similar triangles,

$$\frac{AB}{AP}=\frac{AP}{AM},$$

and so

$$\left(\frac{AB}{AP}\right)^2=\frac{AB}{AM}.$$

Thus, (7.10) has been proved.

The equalities of (7.11) have already been established in (7.6) and (7.7).

Equality (7.12) is a trivial consequence of (7.9) and (7.10).

Next, by (7.15), (7.17), and Entry 5(xiv) or (7.4),

$$\begin{aligned}\left(\frac{\alpha^3}{\beta}\right)^{1/8}-\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} &= \frac{\alpha^{1/2}}{(\alpha\beta)^{1/8}}-\frac{(1-\alpha)^{1/2}}{\{(1-\alpha)(1-\beta)\}^{1/8}}\\ &= \frac{\sin(\mu+\nu)}{\sin\mu}-\frac{\cos(\mu+\nu)}{\cos\mu}\\ &= \frac{\sin\nu}{\sin\mu\cos\mu}=\frac{\sin(2\mu)}{2\sin\mu\cos\mu}=1.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{\sin(\mu+\nu)}{\sin\mu}-\frac{\cos(\mu+\nu)}{\cos\mu} &= \frac{BC_2}{AC_2}\frac{AP}{BP}-\frac{AB}{AC_2}\frac{AP}{AB}\\ &= \frac{BC_2}{BP}\frac{AP}{AC_2}-\frac{AP}{AC_2}=\frac{BC_2-BP}{BP}\frac{AP}{AC_2}\\ &= \frac{PC_2}{BP}\frac{AP}{AC_2}=\frac{PC_2}{AC_2}\frac{AM}{MP},\end{aligned}$$

by (7.16). Thus, all equalities of (7.13) have been established.

Lastly, by (7.15), (7.17), and (7.4),

$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^3}{\alpha}\right)^{1/8} = \frac{(1-\beta)^{1/2}}{\{(1-\alpha)(1-\beta)\}^{1/8}} - \frac{\beta^{1/2}}{(\alpha\beta)^{1/8}}$$

$$= \frac{\cos(\mu-\nu)}{\cos\mu} - \frac{\sin(\mu-\nu)}{\sin\mu} = \frac{\sin\nu}{\sin\mu\cos\mu} = 1.$$

On the other hand,

$$\frac{\cos(\mu-\nu)}{\cos\mu} - \frac{\sin(\mu-\nu)}{\sin\mu} = \frac{AB}{AC}\frac{AP}{AB} - \frac{BC}{AC}\frac{AP}{BP}$$

$$= \frac{AP}{AC} - \frac{BC}{AC}\frac{AP}{BP} = \frac{AP}{AC}\frac{BP-BC}{BP} = \frac{AP}{AC}\frac{CP}{BP}$$

$$= \frac{CP}{AC}\frac{AM}{PM},$$

by (7.16). This completes the proof of (7.14) and all of Entry 7 as well.

This concludes, for this chapter, Ramanujan's study of modular equations of degree 3, with the concomitant theory of theta-functions and associated geometry. In Section 8, we begin the corresponding theory for degree 5.

Entry 8. *We have*

(i) $$q\psi^3(q)\psi(q^5) - 5q^2\psi(q)\psi^3(q^5) = \frac{q}{1-q^2} - \frac{2q^2}{1-q^4} - \frac{3q^3}{1-q^6} + \frac{4q^4}{1-q^8} + \frac{6q^6}{1-q^{12}} - \cdots,$$

(ii) $$5\varphi(q)\varphi^3(q^5) - \varphi^3(q)\varphi(q^5) = 4\left(1 + \frac{q}{1+q} - \frac{2q^2}{1-q^2} - \frac{3q^3}{1+q^3} + \frac{4q^4}{1-q^4} + \frac{6q^6}{1-q^6} - \cdots\right),$$

(iii) $$25\varphi(q)\varphi^3(q^5) - \frac{\varphi^5(q)}{\varphi(q^5)} = 24 + 40\left(\frac{q}{1+q} - \frac{3q^3}{1+q^3} - \frac{7q^7}{1+q^7} + \frac{9q^9}{1+q^9} + \cdots\right),$$

and

(iv) $$\frac{\psi^5(q)}{\psi(q^5)} - 25q^2\psi(q)\psi^3(q^5) = 1 + 5\left(\frac{q}{1+q} - \frac{2q^2}{1+q^2} - \frac{3q^3}{1+q^3} + \frac{4q^4}{1+q^4} + \frac{6q^6}{1+q^6} - \cdots\right).$$

The proofs of Entries 8(i), (ii) are rather difficult. Some results from Section 13 are employed in our proofs. However, no circular reasoning is involved, because the results from Section 8 are not subsequently utilized, except in Entries 9(i), (ii), (v), and (vi), which are not used in Section 13.

PROOF OF (i). From (3.4),

$$\left(\sum_{n=0}^{\infty} q^{n+1/2}\frac{\sin(n+\frac{1}{2})\theta}{1-q^{2n+1}}\right)^2 = \sum_{n=1}^{\infty}\frac{nq^n\sin^2(\frac{1}{2}n\theta)}{1-q^{2n}}.$$

Let $\theta = 3\pi/5$ and $\theta = \pi/5$ in turn and then subtract the two equalities to deduce that

$$\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}\left\{\sin^2\left(\frac{3n\pi}{10}\right)-\sin^2\left(\frac{n\pi}{10}\right)\right\}$$
$$=\left(\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\left\{\sin\frac{(6n+3)\pi}{10}+\sin\frac{(2n+1)\pi}{10}\right\}\right)$$
$$\times\left(\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\left\{\sin\frac{(6n+3)\pi}{10}-\sin\frac{(2n+1)\pi}{10}\right\}\right). \tag{8.1}$$

The cycles of values for the three expressions

$$\frac{4}{\sqrt{5}}\left(\sin^2\left(\frac{3n\pi}{10}\right)-\sin^2\left(\frac{n\pi}{10}\right)\right), \qquad \frac{2}{\sqrt{5}}\left(\sin\frac{(6n+3)\pi}{10}+\sin\frac{(2n+1)\pi}{10}\right),$$

and

$$2\left(\sin\frac{(6n+3)\pi}{10}-\sin\frac{(2n+1)\pi}{10}\right)$$

are, respectively,

$$\begin{gathered}1,1,-1,-1,0,-1,-1,1,1,0;\\ 1,1,0,1,1,-1,-1,0,-1,-1;\\ 1,-1,-4,-1,1,-1,1,4,1,-1.\end{gathered} \tag{8.2}$$

Thus, (8.1) is equal to

$$\frac{q}{1-q^2}+\frac{2q^2}{1-q^4}-\frac{3q^3}{1-q^6}-\frac{4q^4}{1-q^8}-\frac{6q^6}{1-q^{12}}-\frac{7q^7}{1-q^{14}}+\frac{8q^8}{1-q^{16}}$$
$$+\frac{9q^9}{1-q^{18}}+\cdots=\left(\frac{q^{1/2}}{1-q}+\frac{q^{3/2}}{1-q^3}+\frac{q^{7/2}}{1-q^7}+\frac{q^{9/2}}{1-q^9}-\frac{q^{11/2}}{1-q^{11}}\right.$$
$$\left.-\frac{q^{13/2}}{1-q^{13}}-\frac{q^{17/2}}{1-q^{17}}-\frac{q^{19/2}}{1-q^{19}}+\cdots\right)$$
$$\times\left(\frac{q^{1/2}}{1-q}-\frac{q^{3/2}}{1-q^3}-\frac{4q^{5/2}}{1-q^5}-\frac{q^{7/2}}{1-q^7}\right.$$
$$+\frac{q^{9/2}}{1-q^9}-\frac{q^{11/2}}{1-q^{11}}+\frac{q^{13/2}}{1-q^{13}}+\frac{4q^{15/2}}{1-q^{15}}$$
$$\left.+\frac{q^{17/2}}{1-q^{17}}-\frac{q^{19/2}}{1-q^{19}}+\cdots\right). \tag{8.3}$$

Let S_1 denote the first series on the right side of (8.3). Expanding the summands into geometric series and then summing by columns, we find that

$$S_1 = \sum_{n=0}^{\infty} \frac{q^{n+1/2} + q^{3n+3/2} + q^{7n+7/2} + q^{9n+9/2}}{1 + q^{10n+5}}.$$

We now apply (8.5) in Chapter 17 with $a = q^{1/2}$ and $b = -q^{9/2}$ and then with $a = q^{3/2}$ and $b = -q^{7/2}$. It follows at once that

$$\begin{aligned} S_1 &= q^{1/2} \frac{f(q^4, q^6)}{f(-q, -q^9)} \varphi(-q^5)\psi(q^{10}) + q^{3/2} \frac{f(q^2, q^8)}{f(-q^3, -q^7)} \varphi(-q^5)\psi(q^{10}) \\ &= \{q^{1/2} f(q^4, q^6) f(-q^3, -q^7) + q^{3/2} f(q^2, q^8) f(-q, -q^9)\} \\ &\quad \times \frac{\varphi(-q^5)\psi(q^{10})}{f(-q, -q^9) f(-q^3, -q^7)}. \end{aligned} \tag{8.4}$$

Applying Entries 29(i), (ii) of Chapter 16 with $a = q$, $b = -q^4$, $c = -q^2$, and $d = q^3$, we find that

$$f(q^4, q^6) f(-q^3, -q^7) = \tfrac{1}{2}\{f(q, -q^4) f(-q^2, q^3) + f(-q, q^4) f(q^2, -q^3)\} \tag{8.5}$$

and

$$q f(q^2, q^8) f(-q, -q^9) = \tfrac{1}{2}\{f(q, -q^4) f(-q^2, q^3) - f(-q, q^4) f(q^2, -q^3)\}. \tag{8.6}$$

Adding (8.5) and (8.6), we see that

$$f(q^4, q^6) f(-q^3, -q^7) + q f(q^2, q^8) f(-q, -q^9) = f(q, -q^4) f(-q^2, q^3).$$

Substituting this into (8.4), we find that

$$\begin{aligned} S_1 &= \frac{q^{1/2} f(q, -q^4) f(-q^2, q^3) \varphi(-q^5)\psi(q^{10})}{f(-q, -q^9) f(-q^3, -q^7)} \\ &= \frac{q^{1/2} f(q) f(q^5) \varphi(-q^5)\psi(q^{10})}{f(-q, -q^9) f(-q^3, -q^7)}, \end{aligned} \tag{8.7}$$

by the corollary to Entry 28 in Chapter 16. By the Jacobi triple product identity (Entry 19, Chapter 16),

$$\begin{aligned} f(-q, -q^9) f(-q^3, -q^7) &= (q; q^{10})_\infty (q^3; q^{10})_\infty (q^7; q^{10})_\infty (q^9; q^{10})_\infty (q^{10}; q^{10})_\infty^2 \\ &= \frac{(q; q^2)_\infty (q^{10}; q^{10})_\infty^2}{(q^5; q^{10})_\infty} = \frac{\chi(-q) f^2(-q^{10})}{\chi(-q^5)}, \end{aligned}$$

by Entry 22 in Chapter 16. Thus, (8.7) can be written

$$S_1 = \frac{q^{1/2} f(q) f(q^5) \chi(-q^5) \varphi(-q^5)\psi(q^{10})}{\chi(-q) f^2(-q^{10})}. \tag{8.8}$$

Letting S_2 denote the latter series on the right side of (8.3), we rewrite it in the form

$$S_2 = \frac{q^{1/2}}{1-q} - \frac{q^{3/2}}{1-q^3} + \frac{q^{5/2}}{1-q^5} - \frac{q^{7/2}}{1-q^7} + \frac{q^{9/2}}{1-q^9} - \frac{q^{11/2}}{1-q^{11}} + \frac{q^{13/2}}{1-q^{13}}$$
$$- \frac{q^{15/2}}{1-q^{15}} + \frac{q^{17/2}}{1-q^{17}} - \frac{q^{19/2}}{1-q^{19}} + \cdots - 5\left(\frac{q^{5/2}}{1-q^5} - \frac{q^{15/2}}{1-q^{15}} + \cdots\right)$$
$$= q^{1/2}\psi^2(q^2) - 5q^{5/2}\psi^2(q^{10}), \tag{8.9}$$

by (8.6) in Chapter 17.

Now put (8.8) and (8.9) in (8.3) and change the sign of q. Employing Entries 10(i), 11(iii), 12(ii), (iii), (v) in Chapter 17, we find that

$$\frac{q}{1-q^2} - \frac{2q^2}{1-q^4} - \frac{3q^3}{1-q^6} + \frac{4q^4}{1-q^8} + \frac{6q^6}{1-q^{12}} - \frac{7q^7}{1-q^{14}} - \frac{8q^8}{1-q^{16}}$$
$$+ \frac{9q^9}{1-q^{18}} + \cdots$$
$$= \frac{qf(-q)f(-q^5)\chi(q^5)\varphi(q^5)\psi(q^{10})}{\chi(q)f^2(-q^{10})}\{\psi^2(q^2) - 5q^2\psi^2(q^{10})\}$$
$$= \frac{z_1^{1/2}z_5^{3/2}}{2^{8/3}}(\alpha^3\beta)^{1/8}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}\left(m - 5\left(\frac{\beta}{\alpha}\right)^{1/2}\right)$$
$$= \frac{z_1^{1/2}z_5^{3/2}}{2^{8/3}}(\alpha^3\beta)^{1/8}2^{-4/3}\left(\frac{5}{m} - 1\right)\left(m - 5\left(\frac{2m-\rho}{5-m}\right)^2\right)$$
$$= \frac{z_1^{1/2}z_5^{3/2}}{4}(\alpha^3\beta)^{1/8}\frac{5\rho - 5m - m^2}{5-m}$$
$$= \frac{z_1^{1/2}z_5^{3/2}}{4}(\alpha^3\beta)^{1/8}\left(m - 5\left(\frac{\beta}{\alpha}\right)^{1/4}\right), \tag{8.10}$$

by Entry 13(iv) and (13.13), where $\rho = (m^3 - 2m^2 + 5m)^{1/2}$. On the other hand, by Entry 11(i) in Chapter 17, we easily deduce that

$$q\psi^3(q)\psi(q^5) - 5q^2\psi(q)\psi^3(q^5) = \frac{z_1^{1/2}z_5^{3/2}}{4}(\alpha^3\beta)^{1/8}\left(m - 5\left(\frac{\beta}{\alpha}\right)^{1/4}\right). \tag{8.11}$$

Combining (8.10) and (8.11), we at last complete the proof of Entry 8(i).

PROOF OF (ii). The proof of (ii) is not unlike that of (i). Examining Entries 16 (second part) (iii) and 17(ii) in Chapter 18, we observe that the difference of

$$\left(\csc\theta + 4\sum_{n=0}^{\infty}\frac{q^{2n+1}\sin(2n+1)\theta}{1-q^{2n+1}}\right)^2$$

and

$$\csc^2\theta - 8\sum_{n=1}^{\infty}\frac{nq^{2n}\cos(2n\theta)}{1-q^{2n}}$$

is independent of θ. To that end, letting $\theta = \pi/10$ and $\theta = 3\pi/10$ in turn and subtracting the two formulas, we deduce that

$$\csc^2\left(\frac{\pi}{10}\right) - \csc^2\left(\frac{3\pi}{10}\right) - 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\left\{\cos\left(\frac{n\pi}{5}\right) - \cos\left(\frac{3n\pi}{5}\right)\right\}$$
$$= \left(\csc\left(\frac{\pi}{10}\right) + \csc\left(\frac{3\pi}{10}\right) + 4\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{2n+1}}\left\{\sin\frac{(2n+1)\pi}{10}\right.\right.$$
$$\left.\left. + \sin\frac{(6n+3)\pi}{10}\right\}\right)\left(\csc\left(\frac{\pi}{10}\right) - \csc\left(\frac{3\pi}{10}\right) + 4\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{2n+1}}\right.$$
$$\left.\times\left\{\sin\frac{(2n+1)\pi}{10} - \sin\frac{(6n+3)\pi}{10}\right\}\right).$$

Now,

$$\frac{2}{\sqrt{5}}\left(\cos\left(\frac{n\pi}{5}\right) - \cos\left(\frac{3n\pi}{5}\right)\right), \qquad \frac{2}{\sqrt{5}}\left(\sin\frac{(2n+1)\pi}{10} + \sin\frac{(6n+3)\pi}{10}\right),$$

and

$$-2\left(\sin\frac{(2n+1)\pi}{10} - \sin\frac{(6n+3)\pi}{10}\right)$$

repeat in cycles of length 10, respectively, according to (8.2). Consequently,

$$1 - \frac{q^2}{1-q^2} - \frac{2q^4}{1-q^4} + \frac{3q^6}{1-q^6} + \frac{4q^8}{1-q^8} + \frac{6q^{12}}{1-q^{12}} + \frac{7q^{14}}{1-q^{14}} - \frac{9q^{16}}{1-q^{16}}$$
$$- \frac{11q^{18}}{1-q^{18}} - \cdots$$
$$= \left\{1 + \frac{q}{1-q} + \frac{q^3}{1-q^3} + \frac{q^7}{1-q^7} + \frac{q^9}{1-q^9}\right.$$
$$\left. - \frac{q^{11}}{1-q^{11}} - \frac{q^{13}}{1-q^{13}} - \frac{q^{17}}{1-q^{17}} - \frac{q^{19}}{1-q^{19}} + \cdots\right\}$$
$$\times\left\{1 - \frac{q}{1-q} + \frac{q^3}{1-q^3} + \frac{4q^5}{1-q^5} + \frac{q^7}{1-q^7} - \frac{q^9}{1-q^9} + \frac{q^{11}}{1-q^{11}}\right.$$
$$\left. - \frac{q^{13}}{1-q^{13}} - \frac{4q^{15}}{1-q^{15}} - \frac{q^{17}}{1-q^{17}} + \frac{q^{19}}{1-q^{19}} - \cdots\right\}. \tag{8.12}$$

Letting S_1 denote the former series on the right side of (8.12), we transform it just as in the previous proof. We then apply, from Chapter 16, the corollary to Entry 33(iii) twice, Entry 29(i), Entry 19, and Entry 22. Accordingly,

$$S_1 = 1 + \sum_{n=1}^{\infty}\frac{q^n + q^{3n} + q^{7n} + q^{9n}}{1+q^{10}}$$
$$= \frac{1}{2}\left\{\frac{f(q, q^9)}{f(-q, -q^9)} + \frac{f(q^3, q^7)}{f(-q^3, -q^7)}\right\}\varphi^2(-q^{10})$$

$$= \frac{f(-q^4, -q^{16}) f(-q^8, -q^{12})}{f(-q, -q^9) f(-q^3, -q^7)} \varphi^2(-q^{10})$$

$$= \frac{(q^4; q^{20})_\infty (q^8; q^{20})_\infty (q^{12}; q^{20})_\infty (q^{16}; q^{20})_\infty (q^{20}; q^{20})_\infty^2}{(q; q^{10})_\infty (q^3; q^{10})_\infty (q^7; q^{10})_\infty (q^9; q^{10})_\infty (q^{10}; q^{10})_\infty^2} \varphi^2(-q^{10})$$

$$= \frac{(q^4; q^4)_\infty (q^{20}; q^{20})_\infty (q^5; q^{10})_\infty}{(q; q^2)_\infty (q^{10}; q^{10})_\infty^2} \varphi^2(-q^{10})$$

$$= \frac{f(-q^4) f(-q^{20}) \chi(-q^5)}{\chi(-q) f^2(-q^{10})} \varphi^2(-q^{10}). \tag{8.13}$$

Letting S_2 denote the latter series on the right side of (8.12), we rewrite it in the form

$$S_2 = 1 - \frac{q}{1-q} + \frac{q^3}{1-q^3} - \frac{q^5}{1-q^5} + \frac{q^7}{1-q^7} - \frac{q^9}{1-q^9} + \frac{q^{11}}{1-q^{11}} - \frac{q^{13}}{1-q^{13}}$$
$$+ \frac{q^{15}}{1-q^{15}} - \frac{q^{17}}{1-q^{17}} + \frac{q^{19}}{1-q^{19}} - \cdots + 5\left(\frac{q^5}{1-q^5} - \frac{q^{15}}{1-q^{15}} + \cdots\right)$$
$$= \tfrac{1}{4}\{5\varphi^2(q^5) - \varphi^2(q)\}, \tag{8.14}$$

by Entry 8(i) in Chapter 17.

Denoting the left side of (8.12) by $\frac{1}{4}S$ and putting (8.13) and (8.14) in (8.12), we deduce that

$$S = \frac{f(-q^4) f(-q^{20}) \chi(-q^5)}{\chi(-q) f^2(-q^{10})} \varphi^2(-q^{10}) \{5\varphi^2(q^5) - \varphi^2(q)\}.$$

Invoking Entries 12(iii), (iv), and (vi) in Chapter 17 and Entry 13(iv) below, we find that

$$S = \frac{\varphi(q)\varphi(q^5)\varphi^2(-q^{10})}{4^{1/3}} \left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24} \frac{5-m}{(1-\alpha)^{1/4}}$$
$$= \frac{(5-m)^2 \varphi(q)\varphi(q^5)\varphi^2(-q^{10})}{4m(1-\alpha)^{1/4}}$$
$$= \left(5\frac{\rho - 3m + 5}{4m} - m\frac{5\rho - m^2 - 5m}{4m^2}\right) \frac{\varphi(q)\varphi(q^5)\varphi^2(-q^{10})}{(1-\alpha)^{1/4}}$$
$$= \left(5\{(1-\alpha)^3(1-\beta)\}^{1/8} - m\left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8}\right) \frac{\varphi(q)\varphi(q^5)\varphi^2(-q^{10})}{(1-\alpha)^{1/4}}$$
$$= 5\{\varphi(q)\varphi(-q)\varphi(q^5)\varphi(-q^5)\}^{1/2}\varphi^2(-q^{10}) - \left(\frac{\varphi^3(q)\varphi^3(-q)}{\varphi(q^5)\varphi(-q^5)}\right)^{1/2} \varphi^2(-q^{10})$$
$$= 5\varphi(-q^2)\varphi^3(-q^{10}) - \varphi^3(-q^2)\varphi(-q^{10}),$$

where we have used (13.10) and (13.5) below, Entries 10(i) and (ii) in Chapter 17, and Entry 25(iii) in Chapter 16. Replacing q^2 by $-q$, we complete the proof.

PROOF OF (iii). For brevity, in the proofs of (iii) and (iv), we put

$$S_1(q) = \frac{q}{1-q^2} - \frac{2q^2}{1-q^4} - \frac{3q^3}{1-q^6} + \frac{4q^4}{1-q^8} + \frac{6q^6}{1-q^{12}} - \cdots,$$

$$S_2(q) = \frac{q}{1+q} - \frac{2q^2}{1-q^2} - \frac{3q^3}{1+q^3} + \frac{4q^4}{1-q^4} + \frac{6q^6}{1-q^6} - \cdots,$$

$$S_3(q) = \frac{q}{1+q} - \frac{3q^3}{1+q^3} - \frac{7q^7}{1+q^7} + \frac{9q^9}{1+q^9} + \frac{11q^{11}}{1+q^{11}} - \cdots,$$

and

$$S_4(q) = \frac{q}{1+q} - \frac{2q^2}{1+q^2} - \frac{3q^3}{1+q^3} + \frac{4q^4}{1+q^4} + \frac{6q^6}{1+q^6} - \cdots.$$

First,

$$\begin{aligned} S_1(q) - S_1(-q) &= \frac{2q}{1-q^2} - \frac{6q^3}{1-q^6} - \frac{14q^7}{1-q^{14}} + \frac{18q^9}{1-q^{18}} + \frac{22q^{11}}{1-q^{22}} - \cdots \\ &= \frac{q}{1-q} - \frac{3q^3}{1-q^3} - \frac{7q^7}{1-q^7} + \frac{9q^9}{1-q^9} + \frac{11q^{11}}{1-q^{11}} - \cdots \\ &\quad + \frac{q}{1+q} - \frac{3q^3}{1+q^3} - \frac{7q^7}{1+q^7} + \frac{9q^9}{1+q^9} + \frac{11q^{11}}{1+q^{11}} - \cdots. \end{aligned} \tag{8.15}$$

Second,

$$\begin{aligned} &S_2(-q) + 2S_2(-q^2) \\ &= -\frac{q}{1-q} - \frac{2q^2}{1-q^2} + \frac{3q^3}{1-q^3} + \frac{4q^4}{1-q^4} + \frac{6q^6}{1-q^6} - \cdots \\ &\quad - \frac{2q^2}{1-q^2} - \frac{4q^4}{1-q^4} + \frac{6q^6}{1-q^6} + \frac{8q^8}{1-q^8} + \frac{12q^{12}}{1-q^{12}} + \cdots \\ &= -\frac{q}{1-q} + \frac{3q^3}{1-q^3} + \frac{7q^7}{1-q^7} - \frac{9q^9}{1-q^9} - \frac{11q^{11}}{1-q^{11}} + \cdots \\ &\quad - \frac{4q^2}{1-q^2} + \frac{12q^6}{1-q^6} + \frac{28q^{14}}{1-q^{14}} - \frac{36q^{18}}{1-q^{18}} - \frac{44q^{22}}{1-q^{22}} + \cdots \\ &= -3\left(\frac{q}{1-q} - \frac{3q^3}{1-q^3} - \frac{7q^7}{1-q^7} + \frac{9q^9}{1-q^9} + \frac{11q^{11}}{1-q^{11}} - \cdots\right) \\ &\quad +2\left(\frac{q}{1+q} - \frac{3q^3}{1+q^3} - \frac{7q^7}{1+q^7} + \frac{9q^9}{1+q^9} + \frac{11q^{11}}{1+q^{11}} - \cdots\right). \end{aligned} \tag{8.16}$$

Combining (8.15) and (8.16), we see that

$$5S_3(q) = S_2(-q) + 2S_2(-q^2) + 3\{S_1(q) - S_1(-q)\}.$$

Hence, by parts (i) and (ii), Entries 10(ii), (iii) and 11(i), (ii) in Chapter 17, and

(13.10) and (13.11) below,

$$
\begin{aligned}
&24 + 40S_3(q)\\
&\quad = 10\varphi(-q)\varphi^3(-q^5) - 2\varphi^3(-q)\varphi(-q^5) + 20\varphi(-q^2)\varphi^3(-q^{10})\\
&\qquad - 4\varphi^3(-q^2)\varphi(-q^{10}) + 24\{q\psi^3(q)\psi(q^5) + q\psi^3(-q)\psi(-q^5)\\
&\qquad - 5q^2\psi(q)\psi^3(q^5) + 5q^2\psi(-q)\psi^3(-q^5)\}\\
&\quad = \varphi(q)\varphi^3(q^5)(10\{(1-\alpha)(1-\beta)^3\}^{1/4} - 2m\{(1-\alpha)^3(1-\beta)\}^{1/4}\\
&\qquad + 20\{(1-\alpha)(1-\beta)^3\}^{1/8} - 4m\{(1-\alpha)^3(1-\beta)\}^{1/8} + 6m(\alpha^3\beta)^{1/8}\\
&\qquad + 6m\{\alpha^3(1-\alpha)^3\beta(1-\beta)\}^{1/8} - 30(\alpha\beta^3)^{1/8}\\
&\qquad + 30\{\alpha(1-\alpha)\beta^3(1-\beta)^3\}^{1/8})\\
&\quad = \varphi(q)\varphi^3(q^5)\Bigg(\frac{5(\rho - m^2 + 3m)^2}{8m^2} - \frac{(\rho - 3m + 5)^2}{8m} + \frac{5(\rho - m^2 + 3m)}{m}\\
&\qquad - (\rho - 3m + 5) + \frac{3(\rho + 3m - 5)}{2} + \frac{3(m^3 - 11m^2 + 35m - 25)}{8m}\\
&\qquad - \frac{15(\rho + m^2 - 3m)}{2m} + \frac{15(-m^3 + 7m^2 - 11m + 5)}{8m}\Bigg)\\
&\quad = \varphi(q)\varphi^3(q^5)(25 - m^2)\\
&\quad = 25\varphi(q)\varphi^3(q^5) - \frac{\varphi^5(q)}{\varphi(q^5)},
\end{aligned}
$$

which is formula (iii).

PROOF OF (iv). Expanding the summands of $S_1(q)$ into partial fractions, we see that

$$
\begin{aligned}
2S_1(q) &= \frac{q}{1-q} - \frac{2q^2}{1-q^2} - \frac{3q^3}{1-q^3} + \frac{4q^4}{1-q^4} + \frac{6q^6}{1-q^6} - \cdots\\
&\quad + \frac{q}{1+q} - \frac{2q^2}{1+q^2} - \frac{3q^3}{1+q^3} + \frac{4q^4}{1+q^4} + \frac{6q^6}{1+q^6} - \cdots\\
&= 2\left(\frac{q}{1-q} - \frac{3q^3}{1-q^3} - \frac{7q^7}{1-q^7} + \frac{9q^9}{1-q^9} + \frac{11q^{11}}{1-q^{11}} - \cdots\right)\\
&\quad - \left(\frac{q}{1-q} + \frac{2q^2}{1-q^2} - \frac{3q^3}{1-q^3} - \frac{4q^4}{1-q^4} - \frac{6q^6}{1-q^6} - \cdots\right) + S_4(q).
\end{aligned}
$$

Hence,

$$S_4(q) = 2S_1(q) + 2S_3(-q) - S_2(-q).$$

Applying parts (i)–(iii), Entry 11(ii) in Chapter 17, and (13.12) and (13.11) below, we arrive at

$$\begin{aligned}
4 &+ 20S_4(-q) \\
&= 40\{-q\psi^3(-q)\psi(-q^5) - 5q^2\psi(-q)\psi^3(-q^5)\} - \frac{\varphi^5(q)}{\varphi(q^5)} \\
&\quad + 5\varphi^3(q)\varphi(q^5) \\
&= 40\{-q\psi^3(-q)\psi(-q^5) - 5q^2\psi(-q)\psi^3(-q^5)\} \\
&\quad + \varphi(q)\varphi^3(q^5)(5m - m^2) \\
&= q^2\psi(-q)\psi^3(-q^5)\left(-\frac{40\psi^2(-q)}{q\psi^2(-q^5)} - 200 + \frac{\varphi(q)\varphi^3(q^5)(5m - m^2)}{q^2\psi(-q)\psi^3(-q^5)}\right) \\
&= q^2\psi(-q)\psi^3(-q^5)\left(-40m\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}\right. \\
&\quad \left. - 200 + \frac{4(5m - m^2)}{\{\alpha(1-\alpha)\beta^3(1-\beta)^3\}^{1/8}}\right) \\
&= q^2\psi(-q)\psi^3(-q^5)\left(-40m\frac{4m^2 - \rho^2}{m^2(m-1)^2} - 200 + \frac{64(5m - m^2)m^2}{\rho^2 - (m^2 - 3m)^2}\right) \\
&= q^2\psi(-q)\psi^3(-q^5)\left(\frac{40(m-5)}{m-1} - 200 + \frac{64m^2}{(m-1)^2}\right) \\
&= q^2\psi(-q)\psi^3(-q^5)\left(\frac{4(5-m)^2}{(m-1)^2} - 100\right). \qquad (8.17)
\end{aligned}$$

But by Entry 11(ii) in Chapter 17 and (13.12) below,

$$\frac{\psi^4(-q)}{q^2\psi^4(-q^5)} = m^2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} = \frac{(4m^2 - \rho^2)^2}{m^2(m-1)^4} = \frac{(m-5)^2}{(m-1)^2}.$$

Utilizing this in (8.17), we find that

$$\begin{aligned}
4 + 20S_4(-q) &= q^2\psi(-q)\psi^3(-q^5)\left(\frac{4\psi^4(-q)}{q^2\psi^4(-q^5)} - 100\right) \\
&= 4\frac{\psi^5(-q)}{\psi(-q^5)} - 100q^2\psi(-q)\psi^3(-q^5).
\end{aligned}$$

Replacing q by $-q$, we finish the proof.

Entry 9. *The following identities are valid:*

(i)
$$\frac{f^5(-q)}{f(-q^5)} = 1 - 5\left(\frac{q}{1+q} - \frac{3q^3}{1+q^3} + \frac{4q^4}{1+q^4} - \frac{7q^7}{1+q^7} + \frac{9q^9}{1+q^9} + \frac{11q^{11}}{1+q^{11}} - \frac{12q^{12}}{1+q^{12}} - \cdots\right),$$

where the powers of q are not multiples of 5 but are otherwise all the odd multiples of 2^{2k}, $k \geq 0$, and where the signs of the terms are $+$, $-$, $-$, $+$, *according as the power of q is congruent to* 1, 2, 3, 4 (mod 5), *respectively*;

(ii) $$4q\frac{f^5(q^5)}{f(q)}+\frac{\varphi^5(q^5)}{\varphi(q)}=\varphi(q)\varphi^3(q^5);$$

(iii) $$\varphi^2(q)-\varphi^2(q^5)=4q\chi(q)f(-q^5)f(-q^{20});$$

(iv) $$\{\varphi(q^5)+2q^{1/5}f(q^3,q^7)\}^2+\{\varphi(q^5)+2q^{4/5}f(q,q^9)\}^2$$
$$=\varphi^2(q^{1/5})-2\varphi^2(q)+3\varphi^2(q^5);$$

(v) $$1-\frac{f^5(-q)}{f(-q^5)}=5q\frac{d}{dq}\operatorname{Log}\frac{f(-q^2,-q^3)}{f(-q,-q^4)};$$

(vi) $$\frac{\psi^5(q)}{\psi(q^5)}-25q^2\psi(q)\psi^3(q^5)=1-5q\frac{d}{dq}\operatorname{Log}\frac{f(q^2,q^3)}{f(q,q^4)};$$

(vii) $$f(q,q^4)f(q^2,q^3)=\frac{\varphi(-q^5)f(-q^5)}{\chi(-q)};$$

$$f(-q,-q^4)f(-q^2,-q^3)=f(-q)f(-q^5);$$

and

$$f(q,q^9)f(q^3,q^7)=\chi(q)f(-q^5)f(-q^{20}).$$

The most fundamental result in Entry 9 appears to be (iii), and so we prove it first.

PROOF OF (iii). Employing Entry 8(i) of Chapter 17 and summing by columns, we first see that

$$\varphi^2(q)-\varphi^2(q^5)=4\left(\frac{q}{1-q}-\frac{q^3}{1-q^3}-\frac{q^7}{1-q^7}+\frac{q^9}{1-q^9}-\frac{q^{11}}{1-q^{11}}\right.$$
$$\left.+\frac{q^{13}}{1-q^{13}}+\frac{q^{17}}{1-q^{17}}-\frac{q^{19}}{1-q^{19}}+\cdots\right)$$
$$=4\left(\frac{q+q^9}{1+q^{10}}+\frac{q^2+q^{18}}{1+q^{20}}+\frac{q^3+q^{27}}{1+q^{30}}+\cdots\right)$$
$$-4\left(\frac{q^3+q^7}{1+q^{10}}+\frac{q^6+q^{14}}{1+q^{20}}+\frac{q^9+q^{21}}{1+q^{30}}+\cdots\right).$$

Now apply the corollary to Entry 33(iii) in Chapter 16 with $a=q$ and $b=q^9$ and then with $a=q^3$ and $b=q^7$. We next employ Entry 29(ii) of Chapter 16 with $a=q$, $b=q^9$, $c=-q^3$, and $d=-q^7$. Accordingly,

$$\varphi^2(q)-\varphi^2(q^5)=2\varphi^2(-q^{10})\left(\frac{f(q,q^9)}{f(-q,-q^9)}-\frac{f(q^3,q^7)}{f(-q^3,-q^7)}\right)$$
$$=4q\varphi^2(-q^{10})\frac{f(-q^2,-q^{18})f(-q^6,-q^{14})}{f(-q,-q^9)f(-q^3,-q^7)}$$
$$=4qf(q,q^9)f(q^3,q^7), \tag{9.1}$$

by two applications of Entry 30(iv) in Chapter 16.

By Entries 19 (Jacobi triple product identity) and 22 of Chapter 16, and also (22.2) of the same chapter,

$$\begin{aligned} f(q, q^9)f(q^3, q^7) &= (-q; q^{10})_\infty(-q^3; q^{10})_\infty(-q^7; q^{10})_\infty(-q^9; q^{10})_\infty(q^{10}; q^{10})_\infty^2 \\ &= \frac{\chi(q)(q^{10}; q^{10})_\infty^2}{(-q^5; q^{10})_\infty} \\ &= \chi(q)(q^{10}; q^{10})_\infty^2(q^5; q^{10})_\infty(-q^{10}; q^{10})_\infty \\ &= \chi(q)(q^{20}; q^{20})_\infty(q^5; q^5)_\infty \\ &= \chi(q)f(-q^{20})f(-q^5). \end{aligned} \tag{9.2}$$

Substituting (9.2) into (9.1), we complete the proof.

Proof of (ii). By Entry 24(iii) in Chapter 16 and Entries 12(i), (ii), and (iv) in Chapter 17,

$$\begin{aligned} 4q\frac{f^5(q^5)}{f(q)} = 4q\chi(q)\frac{f^5(q^5)}{\varphi(q)} &= \frac{4q\chi(q)z_5^{5/2}}{\varphi(q)}\left(\frac{\beta^5(1-\beta)^5}{2^{20}q^{25}}\right)^{1/24} \\ &= \frac{4q\chi(q)z_5^{3/2}}{\varphi(q)}f(-q^5)f(-q^{20}) \\ &= \frac{z_5^{3/2}\{\varphi^2(q)-\varphi^2(q^5)\}}{\varphi(q)}, \end{aligned}$$

by part (iii), and so (ii) is now immediate.

Proof of (i). Part (ii) may be rewritten in the form

$$\begin{aligned} m - 1 = \frac{\varphi^2(q)}{\varphi^2(q^5)} - 1 &= 4q\frac{\varphi(q)}{f(q)}\frac{f^5(q^5)}{\varphi^5(q^5)} \\ &= 4q\chi(q)\chi^{-5}(q^5) \\ &= 2^{4/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}, \end{aligned}$$

by Entry 24(iii) in Chapter 16 and Entry 12(v) in Chapter 17. The reciprocal of this modular equation, in the sense of Entry 24(v) of Chapter 18, is

$$\frac{5}{m} - 1 = 2^{4/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}.$$

Transcribing this via Entry 12(v) in Chapter 17, we find that

$$5\frac{\varphi^2(q^5)}{\varphi^2(q)} - 1 = 4\frac{\chi(q^5)}{\chi^5(q)} = 4\frac{f^5(q)}{\varphi^5(q)}\frac{\varphi(q^5)}{f(q^5)},$$

by Entry 24(iii) in Chapter 16. It follows that

$$4\frac{f^5(q)}{f(q^5)} = 5\varphi^3(q)\varphi(q^5) - \frac{\varphi^5(q)}{\varphi(q^5)},$$

which is complementary to (ii).

Hence, by Entries 8(ii), (iii),

$$4\frac{f^5(q)}{f(q^5)} = 25\varphi(q)\varphi^3(q^5) - \frac{\varphi^5(q)}{\varphi(q^5)} - 5\{5\varphi(q)\varphi^3(q^5) - \varphi^3(q)\varphi(q^5)\}$$

$$= 4 + 40\left(\frac{q}{1+q} - \frac{3q^3}{1+q^3} - \frac{7q^7}{1+q^7} + \frac{9q^9}{1+q^9} + \cdots\right)$$

$$-20\left(\frac{q}{1+q} - \frac{2q^2}{1-q^2} - \frac{3q^3}{1+q^3} + \frac{4q^4}{1-q^4} + \frac{6q^6}{1-q^6} - \cdots\right)$$

$$= 4 + 20\left(\frac{q}{1+q} + \frac{2q^2}{1-q^2} - \frac{3q^3}{1+q^3} - \frac{4q^4}{1-q^4} - \frac{6q^6}{1-q^6}\right.$$

$$\left. - \frac{7q^7}{1+q^7} + \frac{8q^8}{1-q^8} + \frac{9q^9}{1+q^9} + \cdots\right).$$

For each value of n which is an odd multiple of 2^{2k}, for some $k \geq 0$, we employ the trivial identity

$$\frac{q^n}{1 \pm q^n} = \frac{q^n}{1 \mp q^n} \mp \frac{2q^{2n}}{1 - q^{2n}}.$$

Upon simplification, we find that

$$\frac{f^5(q)}{f(q^5)} = 1 + 5\left(\frac{q}{1-q} - \frac{3q^3}{1-q^3} - \frac{4q^4}{1+q^4} - \frac{7q^7}{1-q^7} + \frac{9q^9}{1-q^9}\right.$$

$$\left. + \frac{11q^{11}}{1-q^{11}} + \frac{12q^{12}}{1+q^{12}} - \cdots\right),$$

where each of the indices is an odd multiple of 2^{2k}, $k \geq 0$, and the signs of the terms are $+, +, -, -, -, -, +, +$ according as the power of q is congruent to 1, 2, 3, 4, 6, 7, 8, 9 (mod 10). Replacing q by $-q$, we complete the proof.

PROOF OF (iv). By Corollary (i) of Entry 31, Chapter 16,

$$\varphi^2(q^{1/5}) = \{\varphi(q^5) + 2q^{1/5}f(q^3, q^7) + 2q^{4/5}f(q, q^9)\}^2$$

$$= \{\varphi(q^5) + 2q^{1/5}f(q^3, q^7)\}^2 + \{\varphi(q^5) + 2q^{4/5}f(q, q^9)\}^2$$

$$- \varphi^2(q^5) + 8qf(q^3, q^7)f(q, q^9).$$

If we now employ (9.1), (iv) follows at once.

PROOF OF (v). From the Jacobi triple product identity, observe that

$$q\frac{d}{dq}\operatorname{Log}\frac{f(-q^2, -q^3)}{f(-q, -q^4)} = q\frac{d}{dq}\operatorname{Log}\left(\frac{(q^2; q^5)_\infty(q^3; q^5)_\infty}{(q; q^5)_\infty(q^4; q^5)_\infty}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}} + \sum_{n=0}^{\infty}\frac{(5n+4)q^{5n+4}}{1-q^{5n+4}}$$

$$- \sum_{n=0}^{\infty}\frac{(5n+2)q^{5n+2}}{1-q^{5n+2}} - \sum_{n=0}^{\infty}\frac{(5n+3)q^{5n+3}}{1-q^{5n+3}}.$$

We now use the elementary identity

$$\frac{2mq^{2m}}{1-q^{2m}}=\frac{mq^m}{1-q^m}-\frac{mq^m}{1+q^m}$$

on each even indexed term above when m is an odd multiple of 2^{2k}, $k \geq 0$. Observe that all expressions of the type $nq^n/(1-q^n)$ cancel, and so we are left with only expressions of the form $\pm nq^n/(1+q^n)$. Furthermore, note that we obtain a plus sign when $n \equiv 1, 4 \pmod 5$ and a minus sign when $n \equiv 2, 3 \pmod 5$. Hence,

$$q\frac{d}{dq}\operatorname{Log}\frac{f(-q^2,-q^3)}{f(-q,-q^4)}=\frac{q}{1+q}-\frac{3q^3}{1+q^3}+\frac{4q^4}{1+q^4}-\frac{7q^7}{1+q^7}+\frac{9q^9}{1+q^9}+\cdots,$$

where the powers of q are odd multiples of 2^{2k}, $k \geq 0$. The truth of (v) is now manifest from (i).

PROOF OF (vi). Proceeding as we did at the start of the previous proof, we find that

$$\begin{aligned}5q\frac{d}{dq}\operatorname{Log}\frac{f(q^2,q^3)}{f(q,q^4)} &= 5q\frac{d}{dq}\operatorname{Log}\left(\frac{(-q^2;q^5)_\infty(-q^3;q^5)_\infty}{(-q;q^5)_\infty(-q^4;q^5)_\infty}\right)\\ &= 5\left(\sum_{n=0}^{\infty}\frac{(5n+2)q^{5n+2}}{1+q^{5n+2}}+\sum_{n=0}^{\infty}\frac{(5n+3)q^{5n+3}}{1+q^{5n+3}}\right.\\ &\quad\left.-\sum_{n=0}^{\infty}\frac{(5n+1)q^{5n+1}}{1+q^{5n+1}}-\sum_{n=0}^{\infty}\frac{(5n+4)q^{5n+4}}{1+q^{5n+4}}\right)\\ &= 1+25q^2\psi(q)\psi^3(q^5)-\frac{\psi^5(q)}{\psi(q^5)},\end{aligned}$$

by Entry 8(iv), and the proof is complete.

PROOF OF (vii). By the Jacobi triple product identity,

$$\begin{aligned}f(q,q^4)f(q^2,q^3) &= (-q;q^5)_\infty(-q^2;q^5)_\infty(-q^3;q^5)_\infty(-q^4;q^5)_\infty f^2(-q^5)\\ &= \frac{(-q;q)_\infty}{(-q^5;q^5)_\infty}f^2(-q^5)=\frac{(q^5;q^{10})_\infty}{(q;q^2)_\infty}f^2(-q^5)\\ &= \frac{\chi(-q^5)}{\chi(-q)}f^2(-q^5)=\frac{\varphi(-q^5)f(-q^5)}{\chi(-q)},\end{aligned}$$

by (22.3) and Entry 24(iii), both in Chapter 16.

The second identity in (vii) is found in the corollary of Entry 28 in Chapter 16.

Lastly, the proof of the third equality in (vii) is given in (9.2.).

There are, in fact, several proofs of Entry 9(i) in the literature. The first is due to Darling [1], who employs a heavy dosage of the theory of theta-functions. Mordell [1] shortly thereafter gave a shorter proof based on a certain Hauptmodul in the theory of modular functions. Bailey's [2] first proof of Entry 9(i) depends on a certain formula for a well-poised basic bilateral hypergeometric series, while his [3] second proof rests on the theory of the Weierstrass $\mathscr{P}$-function. A more recent proof of Raghavan [1] depends on the theory of modular forms.

Now, in fact, Entry 9(i) also appears in a manuscript of Ramanujan [11] on the partition function $p(n)$. The formula is mentioned as a companion to a formula for $f^5(q^5)/f(q)$, which leads to a rapid proof of Ramanujan's famous congruence $p(5n+4) \equiv 0 \pmod 5$, $n \geq 0$. For a further elaboration of this fact, see the papers mentioned above. Another proof by Ramanujan for this congruence is discussed by Hardy [3, Chap. 6].

An application of Entry 9(iii) to lattice sums has been given by I. J. Zucker [3].

Ramanathan [1] has utilized Entry 9(v), which has also been proved by Bailey [3], in providing proofs of some formulas from Ramanujan's "lost notebook" [11].

Entry 10. *We have*

(i) $$\psi(q^{1/5}) - q^{3/5}\psi(q^5) = f(q^2, q^3) + q^{1/5}f(q, q^4),$$

(ii) $$\varphi(q^{1/5}) - \varphi(q^5) = 2q^{1/5}f(q^3, q^7) + 2q^{4/5}f(q, q^9),$$

(iii) $$f(-q)\{f(-q^{1/5}) + q^{1/5}f(-q^5)\} = f^2(-q^2, -q^3) - q^{2/5}f^2(-q, -q^4),$$

(iv) $$\varphi^2(q) - \varphi^2(q^5) = 4qf(q, q^9)f(q^3, q^7),$$

(v) $$\psi^2(q) - q\psi^2(q^5) = f(q, q^4)f(q^2, q^3),$$

(vi) $$f^5(q^2, q^3) + qf^5(q, q^4) = \left(\frac{\psi^2(q)}{\psi(q^5)} - q\psi(q^5)\right)\{\psi^4(q) - 4q\psi^2(q)\psi^2(q^5) + 11q^2\psi^4(q^5)\},$$

(vii) $$32qf^5(q^3, q^7) + 32q^4f^5(q, q^9) = \left(\frac{\varphi^2(q)}{\varphi(q^5)} - \varphi(q^5)\right)\{\varphi^4(q) - 4\varphi^2(q)\varphi^2(q^5) + 11\varphi^4(q^5)\},$$

and

(viii) $$f^{10}(-q^2, -q^3) - q^2f^{10}(-q, -q^4) = \frac{f^{11}(-q)}{f(-q^5)} + 11qf^5(-q)f^5(-q^5).$$

Proofs of (i)–(iv). To prove (i), merely replace q by $q^{1/5}$ in the third equality of Corollary (ii), Section 31, Chapter 16. Likewise, (ii) follows from the second equality of Corollary (i), Section 31, Chapter 16.

Part (iii) is simply a repetition of Entry 38(iv) of Chapter 16.

Part (iv) follows immediately from Entries 9(iii) and (vii).

PROOF OF (v). The proof of (v) is more difficult and is similar to that of Entry 9(iii), which is obviously an analogue of Entry 10(v), a fact made even more transparent by Entry 10(iv). To avoid fractions as much as possible, we shall work with $\psi^2(q^2) - q^2\psi^2(q^{10})$.

Employing Example (iv) in Section 17 of Chapter 17, expanding the summands into geometric series, and summing by columns, we find that

$$\psi^2(q^2) - q^2\psi^2(q^{10}) = \sum_{n=0}^{\infty} \frac{q^n}{1+q^{2n+1}} - \sum_{n=0}^{\infty} \frac{q^{5n+2}}{1+q^{10n+5}}$$

$$= \frac{1+q^4}{1-q^5} - \frac{q+q^{13}}{1-q^{15}} + \frac{q^2+q^{22}}{1-q^{25}} - \cdots$$

$$+ \frac{q+q^3}{1-q^5} - \frac{q^4+q^{10}}{1-q^{15}} + \frac{q^7+q^{17}}{1-q^{25}} - \cdots.$$

We now turn to (8.5) in Chapter 17 and apply it twice, with $a, b = iq^{1/2}, -iq^{9/2}$ and $a, b = iq^{3/2}, -iq^{7/2}$. Adding the two results, we see that the equality above is equal to

$$\psi^2(q^2) - q^2\psi^2(q^{10}) = \varphi(q^5)\psi(q^{10})\left(\frac{f(q^4, q^6)}{f(q, q^9)} + q\frac{f(q^2, q^8)}{f(q^3, q^7)}\right)$$

$$= \frac{\varphi(q^5)\psi(q^{10})\{f(q^4, q^6)f(q^3, q^7) + qf(q^2, q^8)f(q, q^9)\}}{f(q, q^9)f(q^3, q^7)}.$$

Next, add the two formulas in Entries 29(i), (ii) of Chapter 16 with $a = q$, $b = q^4$, $c = q^2$, and $d = q^3$. Obtaining a formula for the expression in curly brackets above, we deduce that

$$\psi^2(q^2) - q^2\psi^2(q^{10}) = \frac{\varphi(q^5)\psi(q^{10})f(q, q^4)f(q^2, q^3)}{f(q, q^9)f(q^3, q^7)}$$

$$= \frac{\varphi(q^5)\psi(q^{10})\varphi(-q^5)}{\chi(-q)\chi(q)f(-q^{20})}$$

$$= \frac{\varphi^2(-q^{10})\psi(q^{10})}{\chi(-q^2)f(-q^{20})}$$

$$= \frac{\varphi(-q^{10})f(-q^{10})}{\chi(-q^2)}$$

$$= f(q^2, q^8)f(q^4, q^6),$$

where we have successively applied Entry 9(vii), Entry 25(iii) in Chapter 16, Entry 24(iii) in Chapter 16, and Entry 9(vii) once again. Replacing q^2 by q, we obtain the required result.

For another approach to (v) via modular forms, see a paper of Raghavan and Rangachari [1].

PROOF OF (vi). Formulas (vi) and (vii) are the first of a type which is rather numerous.

Let ζ denote an arbitrary fifth root of unity and replace $q^{1/5}$ by $q^{1/5}\zeta$ in Entry 10(i). Hence,

$$\prod_{\zeta} \{\psi(q^{1/5}\zeta)\} = \prod_{\zeta} \{f(q^2, q^3) + q^{1/5}\zeta f(q, q^4) + q^{3/5}\zeta^3\psi(q^5)\}, \quad (10.1)$$

where each product is over all fifth roots of unity. From the product representation of $\psi(q)$ given in Entry 22(ii) of Chapter 16, we see that

$$\prod_{\zeta} \{\psi(q^{1/5}\zeta)\} = \frac{(q^2; q^2)_\infty^5}{(q; q^2)_\infty^5} \prod_{\substack{n=1 \\ n \not\equiv 0 \pmod 5}}^{\infty} \frac{1 - q^{2n}}{1 - q^{2n-1}} = \frac{\psi^6(q)}{\psi(q^5)}.$$

Thus, on multiplying out the product on the right side of (10.1) and using part (v) and the equality above, we find that

$$\begin{aligned}\frac{\psi^6(q)}{\psi(q^5)} &= q^3\psi^5(q^5) + f^5(q^2, q^3) + qf^5(q, q^4) \\ &\quad - 5q^2\psi^3(q^5)f(q, q^4)f(q^2, q^3) + 5q\psi(q^5)f^2(q, q^4)f^2(q^2, q^3) \\ &= q^3\psi^5(q^5) + f^5(q^2, q^3) + qf^5(q, q^4) \\ &\quad - 5q^2\psi^3(q^5)\{\psi^2(q) - q\psi^2(q^5)\} + 5q\psi(q^5)\{\psi^2(q) - q\psi^2(q^5)\}^2 \\ &= 11q^3\psi^5(q^5) - 15q^2\psi^2(q)\psi^3(q^5) + 5q\psi^4(q)\psi(q^5) \\ &\quad + f^5(q^2, q^3) + qf^5(q, q^4),\end{aligned}$$

which, upon factorization, yields the result we sought.

Proof of (vii). Formula (vii) is obtained in the same manner as (vi). Let ζ again denote an arbitrary fifth root of unity. Employing (ii), we find that

$$\prod_{\zeta} \{\varphi(q^{1/5}\zeta)\} = \prod_{\zeta} \{\varphi(q^5) + 2q^{1/5}\zeta f(q^3, q^7) + 2q^{4/5}\zeta^4 f(q, q^9)\}.$$

Multiplying out and using the same argument as above, we arrive at

$$\begin{aligned}\frac{\varphi^6(q)}{\varphi(q^5)} &= \varphi^5(q^5) + 32qf^5(q^3, q^7) + 32q^4f^5(q, q^9) \\ &\quad - 20q\varphi^3(q^5)f(q^3, q^7)f(q, q^9) + 80q^2\varphi(q^5)f^2(q^3, q^7)f^2(q, q^9).\end{aligned}$$

Thus, by (iv),

$$\begin{aligned}32qf^5(q^3, q^7) + 32q^4f^5(q, q^9) &= \frac{\varphi^6(q)}{\varphi(q^5)} - \varphi^5(q^5) + 5\varphi^3(q^5)\{\varphi^2(q) - \varphi^2(q^5)\} \\ &\quad - 5\varphi(q^5)\{\varphi^2(q) - \varphi^2(q^5)\}^2,\end{aligned}$$

which, upon simplification and factorization, yields the proposed result.

Proof of (viii). Proceeding in the same fashion as above, we find from (iii) that

$$\begin{aligned}&f^5(-q)\prod_{\zeta} \{f(-q^{1/5}\zeta)\} \\ &= \prod_{\zeta} \{f^2(-q^2, -q^3) - q^{2/5}\zeta^2 f^2(-q, -q^4) - q^{1/5}\zeta f(-q)f(-q^5)\}.\end{aligned}$$

Upon expanding the products, we see that

$$\frac{f^{11}(-q)}{f(-q^5)} = f^{10}(-q^2, -q^3) - q^2 f^{10}(-q, -q^4) - qf^5(-q)f^5(-q^5)$$
$$- 5qf^3(-q)f^3(-q^5)f^2(-q, -q^4)f^2(-q^2, -q^3)$$
$$- 5qf(-q)f(-q^5)f^4(-q, -q^4)f^4(-q^2, -q^3).$$

Substituting the expression for $f(-q, -q^4)f(-q^2, -q^3)$ given in Entry 9(vii) and simplifying, we complete the proof.

See Ramanathan's paper [4] for another proof of Entry 10(viii).

Entry 11.

(i) *There exist positive functions* μ *and* ν *such that*

$$\varphi(q^{1/5}) = \varphi(q^5) + \mu^{1/5} + \nu^{1/5}, \tag{11.1}$$

where

$$\mu + \nu = \frac{\varphi^2(q) - \varphi^2(q^5)}{\varphi(q^5)}\{\varphi^4(q) - 4\varphi^2(q)\varphi^2(q^5) + 11\varphi^4(q^5)\},$$

$$\mu - \nu = \frac{\varphi^2(q) - \varphi^2(q^5)}{\varphi(q^5)}\{5\varphi^2(q^5) - \varphi^2(q)\}\{\varphi^4(q) - 2\varphi^2(q)\varphi^2(q^5)$$
$$+ 5\varphi^4(q^5)\}^{1/2},$$

and

$$(\mu\nu)^{1/5} = \varphi^2(q) - \varphi^2(q^5).$$

(ii) *There are positive functions* μ *and* ν *satisfying the equations*

$$q^{1/40}\psi(q^{1/5}) = q^{5/8}\psi(q^5) + \mu^{1/5} + \nu^{1/5}, \tag{11.2}$$

where

$$\mu + \nu = q^{1/8}\frac{\psi^2(q) - q\psi^2(q^5)}{\psi(q^5)}\{\psi^4(q) - 4q\psi^2(q)\psi^2(q^5) + 11q^2\psi^4(q^5)\},$$

$$\mu - \nu = q^{1/8}\frac{\psi^2(q) - q\psi^2(q^5)}{\psi(q^5)}\{\psi^2(q) - 5q\psi^2(q^5)\}$$
$$\times \{\psi^4(q) - 2q\psi^2(q)\psi^2(q^5) + 5q^2\psi^4(q^5)\}^{1/2},$$

and

$$(\mu\nu)^{1/5} = q^{1/4}\{\psi^2(q) - q\psi^2(q^5)\}.$$

(iii) *If*

$$2\mu = 11 + \frac{f^6(-q)}{qf^6(-q^5)} \quad \textit{and} \quad 2\nu = 1 + \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)},$$

then

$$\{(\mu^2+1)^{1/2}-\mu\}^{1/5}=(\nu^2+1)^{1/2}-\nu=\frac{q^{1/5}}{1}\,\frac{q}{+1}\,\frac{q^2}{+1}\,\frac{q^3}{+1}\,\frac{q^4}{+1}+\cdots$$

$$=q^{1/5}\frac{f(-q,-q^4)}{f(-q^2,-q^3)}.$$

(iv) *For certain positive functions μ and ν,*

$$\frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}=(5+\mu^{1/5}-\nu^{1/5})^{1/3}, \tag{11.3}$$

where

$$(\mu\nu)^{1/5}=25+3\frac{f^6(-q)}{qf^6(-q^5)}$$

and

$$\nu-\mu=5^5\cdot 11+75^2\frac{f^6(-q)}{qf^6(-q^5)}+15^2\frac{f^{12}(-q)}{q^2f^{12}(-q^5)}-\frac{f^{18}(-q)}{q^3f^{18}(-q^5)}.$$

The functions μ and ν have different identities in different parts. In fact, explicit identifications will be made in the proofs. In the sequel, when we speak of a "positive function" μ (or ν), we mean that μ (or ν) is positive for sufficiently small positive values of q.

Proof of (i). A consideration of Entry 10(ii) immediately shows that (11.1) is valid with

$$\mu^{1/5}=2q^{1/5}f(q^3,q^7)\quad\text{and}\quad \nu^{1/5}=2q^{4/5}f(q,q^9). \tag{11.4}$$

The formulas for $\mu+\nu$ and $(\mu\nu)^{1/5}$ then follow at once from Entries 10(vii), (iv), respectively.

It remains to prove the formula for $\mu-\nu$. Observe that

$$\begin{aligned}(\mu-\nu)^2&=(\mu+\nu)^2-4\mu\nu\\&=\left(\frac{\varphi^2(q)-\varphi^2(q^5)}{\varphi(q^5)}\right)^2\{\varphi^4(q)-4\varphi^2(q)\varphi^2(q^5)+11\varphi^4(q^5)\}^2\\&\quad-4\{\varphi^2(q)-\varphi^2(q^5)\}^5\\&=\left(\frac{\varphi^2(q)-\varphi^2(q^5)}{\varphi(q^5)}\right)^2\{\varphi^8(q)-12\varphi^6(q)\varphi^2(q^5)+50\varphi^4(q)\varphi^4(q^5)\\&\quad-100\varphi^2(q)\varphi^6(q^5)+125\varphi^8(q^5)\}.\end{aligned}$$

Upon factoring the expression in curly brackets on the far right side above and then taking the square root of both sides, we complete the proof.

Proof of (ii). From Entry 10(i), we can at once deduce (11.2) with

$$\mu^{1/5}=q^{1/40}f(q^2,q^3)\quad\text{and}\quad \nu^{1/5}=q^{9/40}f(q,q^4).$$

The formulas for $\mu + v$ and $(\mu v)^{1/5}$ now follow from Entries 10(vi), (v), respectively.

It remains to prove the displayed formula for $\mu - v$. Now,

$$\begin{aligned}(\mu - v)^2 &= (\mu + v)^2 - 4\mu v \\ &= q^{1/4}\left(\frac{\psi^2(q) - q\psi^2(q^5)}{\psi(q^5)}\right)^2 \{\psi^4(q) - 4q\psi^2(q)\psi^2(q^5) \\ &\quad + 11q^2\psi^4(q^5)\}^2 - 4q^{5/4}\{\psi^2(q) - q\psi^2(q^5)\}^5.\end{aligned}$$

The remainder of the calculations are identical with those of the previous proof, with $\varphi(q)$ and $\varphi(q^5)$ being replaced by $\psi(q)$ and $q^{1/2}\psi(q^5)$, respectively. Some care must be exercised in the determination of the proper square root.

Proof of (iii). By the celebrated Rogers–Ramanujan continued fraction, Entry 38(iii) of Chapter 16,

$$\frac{q^{1/5}}{1}\,\frac{q}{+1}\,\frac{q^2}{+1}\,\frac{q^3}{+1}+\cdots = q^{1/5}\frac{f(-q,-q^4)}{f(-q^2,-q^3)}.$$

Let each of these equal expressions be denoted by J. Thus,

$$\begin{aligned}\frac{1}{J} - J &= \frac{f^2(-q^2,-q^3) - q^{2/5}f^2(-q,-q^4)}{q^{1/5}f(-q,-q^4)f(-q^2,-q^3)} \\ &= \frac{f(-q)\{f(-q^{1/5}) + q^{1/5}f(-q^5)\}}{q^{1/5}f(-q)f(-q^5)} \\ &= \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} + 1 = 2v,\end{aligned} \tag{11.5}$$

by applications of Entries 10(iii) and 9(vii). Also, by Entries 10(viii) and 9(vii),

$$\begin{aligned}\frac{1}{J^5} - J^5 &= \frac{f^{10}(-q^2,-q^3) - q^2f^{10}(-q,-q^4)}{qf^5(-q,-q^4)f^5(-q^2,-q^3)} \\ &= \frac{f^{11}(-q) + 11qf^5(-q)f^6(-q^5)}{qf^5(-q)f^6(-q^5)} \\ &= \frac{f^6(-q)}{qf^6(-q^5)} + 11 = 2\mu.\end{aligned} \tag{11.6}$$

Solving each of (11.5) and (11.6) for J, we deduce that

$$(v^2+1)^{1/2} - v = J = \{(\mu^2+1)^{1/2} - \mu\}^{1/5},$$

and the proof is complete.

Before proceeding with the proof of part (iv), we derive parametric representations that will subsequently be useful.

From Entry 11(iii) and the binomial theorem,

$$\begin{aligned}(\mu^2+1)^{1/2} \pm \mu &= \{(v^2+1)^{1/2} \pm v\}^5 \\ &= (16v^4 + 12v^2 + 1)(v^2+1)^{1/2} \pm (16v^5 + 20v^3 + 5v).\end{aligned}$$

Hence, upon subtraction of these two equalities,

$$2\mu = 32v^5 + 40v^3 + 10v = (1+w)^5 + 5(1+w)^3 + 5(1+w),$$

where w is defined by $v = \frac{1}{2}(1+w)$; that is, from the definition of v,

$$w = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}. \tag{11.7}$$

Furthermore, by the definition of μ and the formula for μ above,

$$\frac{f^6(-q)}{qf^6(-q^5)} = w^5 + 5w^4 + 15w^3 + 25w^2 + 25w. \tag{11.8}$$

Thus, from (11.7) and (11.8),

$$\frac{q^{1/5}f^6(-q)}{f^6(-q^{1/5})} = \frac{f^6(-q)}{qf^6(-q^5)}\frac{q^{6/5}f^6(-q^5)}{f^6(-q^{1/5})}$$

$$= \frac{w^5 + 5w^4 + 15w^3 + 25w^2 + 25w}{w^6}. \tag{11.9}$$

Proof of (iv). Let J be as in the previous proof. Now, from (11.5),

$$\frac{f^3(-q^{1/5})}{q^{3/5}f^3(-q^5)} = \left(\frac{1}{J} - 1 - J\right)^3 = 5 + \frac{1-3J^5}{J^3} - \frac{3+J^5}{J^2}.$$

In light of (11.3), we are motivated to define

$$\mu^{1/5} = \frac{1-3J^5}{J^3} \quad \text{and} \quad v^{1/5} = \frac{3+J^5}{J^2}. \tag{11.10}$$

We then need to verify the proffered formulas for $(\mu v)^{1/5}$ and $v - \mu$.

First, by (11.6),

$$(\mu v)^{1/5} = 3(J^{-5} - J^5) - 8 = 3\left(\frac{f^6(-q)}{qf^6(-q^5)} + 11\right) - 8$$

$$= 25 + 3\frac{f^6(-q)}{qf^6(-q^5)}.$$

Second,

$$v - \mu = \left(\frac{3+J^5}{J^2}\right)^5 - \left(\frac{1-3J^5}{J^3}\right)^5$$

$$= (J^{15} - J^{-15}) + 258(J^{10} + J^{-10}) - 315(J^5 - J^{-5}) + 540$$

$$= 1056 + 312(J^{-5} - J^5) + 258(J^{-5} - J^5)^2 - (J^{-5} - J^5)^3$$

$$= 34375 + 5625\frac{f^6(-q)}{qf^6(-q^5)} + 225\frac{f^{12}(-q)}{q^2f^{12}(-q^5)} - \frac{f^{18}(-q)}{q^3f^{18}(-q^5)},$$

upon the use of (11.6). The proof of (iv) is now complete.

In Entries 12(i)–(iv), we have used the variable Q instead of q in order to present the proofs more clearly. As in Entry 11, the functions μ and ν change from formula to formula. J. M. and P. B. Borwein [6] have employed Entry 12(iii) in devising a quintic algorithm for calculating π.

Entry 12.

(i) *There are positive functions* μ *and* ν *such that*

$$1 + 5Q\frac{f(-Q^{25})}{f(-Q)} = \mu^{1/5} - \nu^{1/5}, \tag{12.1}$$

where

$$\mu\nu = 1 \tag{12.2}$$

and

$$\mu - \nu = 11 + 125Q\frac{f^6(-Q^5)}{f^6(-Q)}. \tag{12.3}$$

(ii) *For certain positive functions* μ *and* ν,

$$Q\frac{f(-Q^{25})}{f(-Q)} = \left(\frac{1 + \mu^{1/5} - \nu^{1/5}}{25}\right)^{1/3}, \tag{12.4}$$

where

$$(\mu\nu)^{1/5} = 1 + 15Q\frac{f^6(-Q^5)}{f^6(-Q)}$$

and

$$\nu - \mu = 11 + 15^2Q\frac{f^6(-Q^5)}{f^6(-Q)} + 5\cdot 15^2Q^2\frac{f^{12}(-Q^5)}{f^{12}(-Q)} - 25^2Q^3\frac{f^{18}(-Q^5)}{f^{18}(-Q)}.$$

(iii) *There exist positive functions* μ *and* ν *such that*

$$5\frac{\varphi(Q^{25})}{\varphi(Q)} = 1 + \mu^{1/5} + \nu^{1/5}, \tag{12.5}$$

where

$$(\mu\nu)^{1/5} = 5\frac{\varphi^2(Q^5)}{\varphi^2(Q)} - 1$$

and

$$\mu + \nu = \left(5\frac{\varphi^2(Q^5)}{\varphi^2(Q)} - 1\right)\left(11 - 20\frac{\varphi^2(Q^5)}{\varphi^2(Q)} + 25\frac{\varphi^4(Q^5)}{\varphi^4(Q)}\right).$$

(iv) *For certain positive functions* μ *and* ν,

$$5Q^3\frac{\psi(Q^{25})}{\psi(Q)} = 1 - \mu^{1/5} + \nu^{1/5},$$

where

$$(\mu\nu)^{1/5} = 1 - 5Q\frac{\psi^2(Q^5)}{\psi^2(Q)}$$

and

$$\mu - \nu = \left(1 - 5Q\frac{\psi^2(Q^5)}{\psi^2(Q)}\right)\left(11 - 20Q\frac{\psi^2(Q^5)}{\psi^2(Q)} + 25Q^2\frac{\psi^4(Q^5)}{\psi^4(Q)}\right).$$

(v) $$\frac{f(-q^{1/5})}{f(-q^5)} = \frac{f(-q^2, -q^3)}{f(-q, -q^4)} - q^{1/5} - q^{2/5}\frac{f(-q, -q^4)}{f(-q^2, -q^3)}.$$

(vi) $$\frac{\varphi(-q^{2/5})\varphi(-q^{10})}{\varphi^2(-q^2)} + q^{2/5}\left(\frac{\psi(q^{1/5})\psi(q^5)}{\psi^2(q)} + \frac{\psi(-q^{1/5})\psi(-q^5)}{\psi^2(-q)}\right) = 1.$$

PROOF OF (i). Given q, define Q by the equation

$$5\,\mathrm{Log}(1/Q)\mathrm{Log}(1/q) = 4\pi^2.$$

Letting $\alpha = \frac{1}{2}\,\mathrm{Log}(1/q)$ and $\beta = \frac{1}{2}\,\mathrm{Log}(1/Q^5)$ in Entry 27(iii) of Chapter 16 and noting that $\alpha\beta = \pi^2$, from above, we find that

$$q^{1/24}\,\mathrm{Log}^{1/4}(1/q)f(-q) = Q^{5/24}\,\mathrm{Log}^{1/4}(1/Q^5)f(-Q^5). \qquad (12.6)$$

Replacing q and Q by $q^{1/5}$ and Q^5, respectively, and then by q^5 and $Q^{1/5}$, respectively, we deduce that

$$q^{1/120}\,\mathrm{Log}^{1/4}(1/q)f(-q^{1/5}) = \sqrt{5}\,Q^{25/24}\,\mathrm{Log}^{1/4}(1/Q^5)f(-Q^{25}) \quad (12.7)$$

and

$$q^{5/24}\,\mathrm{Log}^{1/4}(1/q^5)f(-q^5) = Q^{1/24}\,\mathrm{Log}^{1/4}(1/Q)f(-Q). \qquad (12.8)$$

Dividing (12.6) by (12.8) and then (12.7) by (12.8), we deduce that, respectively,

$$\frac{f(-q)}{q^{1/6}f(-q^5)} = \sqrt{5}\,Q^{1/6}\frac{f(-Q^5)}{f(-Q)} \qquad (12.9)$$

and

$$\frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} = 5Q\frac{f(-Q^{25})}{f(-Q)}. \qquad (12.10)$$

Now suppose that we can show, for certain positive functions μ and ν, that

$$1 + \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} = \mu^{1/5} - \nu^{1/5}, \qquad (12.11)$$

where

$$\mu\nu = 1 \qquad (12.12)$$

and

$$\mu - \nu = \frac{f^6(-q)}{qf^6(-q^5)} + 11. \qquad (12.13)$$

Then, from (12.10),

$$1+\frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}=1+5Q\frac{f(-Q^{25})}{f(-Q)},$$

and from (12.9),

$$\frac{f^6(-q)}{qf^6(-q^5)}=125Q\frac{f^6(-Q^5)}{f^6(-Q)}.$$

Thus, we see that (12.11)–(12.13) translate into (12.1)–(12.3), respectively, and so it remains to prove (12.11)–(12.13).

From Entry 10(iii),

$$1+\frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}=\frac{f^2(-q^2,-q^3)}{q^{1/5}f(-q)f(-q^5)}-\frac{q^{1/5}f^2(-q,-q^4)}{f(-q)f(-q^5)}.$$

This dictates to us the choices

$$\mu=\frac{f^{10}(-q^2,-q^3)}{qf^5(-q)f^5(-q^5)} \quad \text{and} \quad \nu=\frac{qf^{10}(-q,-q^4)}{f^5(-q)f^5(-q^5)},$$

and so (12.11) is established. Second, formula (12.12) is an immediate consequence of the second part of Entry 9(vii). Lastly, divide both sides of Entry 10(viii) by $qf^5(-q)f^5(-q^5)$, and we arrive at (12.13) at once to complete the proof.

PROOF OF (ii). From (12.10) and Entry 11(iv),

$$Q\frac{f(-Q^{25})}{f(-Q)}=\frac{f(-q^{1/5})}{5q^{1/5}f(-q^5)}=\tfrac{1}{5}(5+\mu^{*1/5}-\nu^{*1/5})^{1/3}$$
$$=\left(\frac{1+\frac{1}{5}\mu^{*1/5}-\frac{1}{5}\nu^{*1/5}}{25}\right)^{1/3},$$

where we have replaced μ and ν in Entry 11(iv) by μ^* and ν^*, respectively. Thus, μ^* and ν^* are defined in (11.10). Therefore, by (12.4), we are compelled to define μ and ν in Entry 12(ii) by $\mu^{1/5}=\frac{1}{5}\mu^{*1/5}$ and $\nu^{1/5}=\frac{1}{5}\nu^{*1/5}$. This establishes (12.4).

Now by Entry 11(iv) and (12.9),

$$(\mu\nu)^{1/5}=\tfrac{1}{25}(\mu^*\nu^*)^{1/5}=1+\frac{3f^6(-q)}{25qf^6(-q^5)}=1+15Q\frac{f^6(-Q^5)}{f^6(-Q)}$$

and

$$\nu-\mu=5^{-5}(\nu^*-\mu^*)$$
$$=5^{-5}\Bigg(5^5\cdot 11+75^2\cdot 5^3Q\frac{f^6(-Q^5)}{f^6(-Q)}$$
$$+15^2\cdot 5^6Q^2\frac{f^{12}(-Q^5)}{f^{12}(-Q)}-5^9Q^3\frac{f^{18}(-Q^5)}{f^{18}(-Q)}\Bigg),$$

which finishes the proof of (ii).

PROOF OF (iii). In Entry 27(i) of Chapter 16, let $\alpha^2 = \mathrm{Log}(1/q)$ and $\beta^2 = \mathrm{Log}(1/Q^5)$. Thus,

$$\mathrm{Log}^{1/4}(1/q)\varphi(q) = \mathrm{Log}^{1/4}(1/Q^5)\varphi(Q^5), \tag{12.14}$$

where

$$5\,\mathrm{Log}(1/q)\mathrm{Log}(1/Q) = \pi^2.$$

Replacing q and Q by $q^{1/5}$ and Q^5, respectively, and then by q^5 and $Q^{1/5}$, respectively, we derive

$$\mathrm{Log}^{1/4}(1/q)\varphi(q^{1/5}) = \sqrt{5}\,\mathrm{Log}^{1/4}(1/Q^5)\varphi(Q^{25}) \tag{12.15}$$

and

$$\mathrm{Log}^{1/4}(1/q^5)\varphi(q^5) = \mathrm{Log}^{1/4}(1/Q)\varphi(Q), \tag{12.16}$$

respectively. Thus, (12.14) and (12.16) yield

$$\frac{\varphi(q)}{\varphi(q^5)} = \sqrt{5}\frac{\varphi(Q^5)}{\varphi(Q)}, \tag{12.17}$$

while (12.15) and (12.16) imply

$$\frac{\varphi(q^{1/5})}{\varphi(q^5)} = 5\frac{\varphi(Q^{25})}{\varphi(Q)}. \tag{12.18}$$

By (12.18) and Entry 11(i),

$$5\frac{\varphi(Q^{25})}{\varphi(Q)} = \frac{\varphi(q^{1/5})}{\varphi(q^5)} = 1 + \frac{\mu^{*1/5}}{\varphi(q^5)} + \frac{\nu^{*1/5}}{\varphi(q^5)},$$

where we have replaced μ and ν in Entry 11(i) by μ^* and ν^*, respectively. Thus, (12.5) is established if we define

$$\mu^{1/5} = \frac{\mu^{*1/5}}{\varphi(q^5)} \quad \text{and} \quad \nu^{1/5} = \frac{\nu^{*1/5}}{\varphi(q^5)}.$$

By Entry 11(i) and (12.17),

$$(\mu\nu)^{1/5} = \frac{(\mu^*\nu^*)^{1/5}}{\varphi^2(q^5)} = \frac{\varphi^2(q)}{\varphi^2(q^5)} - 1 = 5\frac{\varphi^2(Q^5)}{\varphi^2(Q)} - 1$$

and

$$\mu + \nu = \frac{\mu^* + \nu^*}{\varphi^5(q^5)} = \frac{\varphi^2(q) - \varphi^2(q^5)}{\varphi^6(q^5)}\{\varphi^4(q) - 4\varphi^2(q)\varphi^2(q^5) + 11\varphi^4(q^5)\}$$
$$= \left(\frac{\varphi^2(q)}{\varphi^2(q^5)} - 1\right)\left(\frac{\varphi^4(q)}{\varphi^4(q^5)} - 4\frac{\varphi^2(q)}{\varphi^2(q^5)} + 11\right).$$

Upon employing (12.17), we complete the proof.

PROOF OF (iv). We proceed as we did in Section 24(vii) of Chapter 18. Thus, we transcribe the formulas involving ψ into formulas involving ϑ_2. We then use classical transformation formulas for theta-functions, which apparently are not found in the notebooks. The transformed formulas are now given in terms of φ. Appealing to (iii), we then complete the proof.

Put $Q = \exp(\pi i\tau)$, where $\text{Im}(\tau) > 0$. In the notation of Whittaker and Watson [1], $\vartheta_2(0, \tau/2) = Q^{1/8}\psi(Q)$. Thus, transcribing Entry 12(iv), we want to show that

$$5\frac{\vartheta_2(0, 25/\tau)}{\vartheta_2(0, \tau/2)} = 1 - \mu^{1/5} + \nu^{1/5}, \tag{12.19}$$

where

$$(\mu\nu)^{1/5} = 1 - 5\frac{\vartheta_2^2(0, 5\tau/2)}{\vartheta_2^2(0,\tau/2)} \tag{12.20}$$

and

$$\mu - \nu = \left(1 - 5\frac{\vartheta_2^2(0, 5\tau/2)}{\vartheta_2^2(0, \tau/2)}\right)\left(11 - 20\frac{\vartheta_2^2(0, 5\tau/2)}{\vartheta_2^2(0, \tau/2)} + 25\frac{\vartheta_2^4(0, 5\tau/2)}{\vartheta_2^4(0, \tau/2)}\right). \tag{12.21}$$

We further replace τ by $2\tau/(25\tau + 1)$. Now, from Rademacher's book [1, p. 182], we may readily deduce that

$$\vartheta_2\left(0, \frac{25\tau}{25\tau + 1}\right) = (25\tau + 1)^{1/2}\vartheta_3(0, 25\tau),$$

$$\vartheta_2\left(0, \frac{\tau}{25\tau + 1}\right) = (25\tau + 1)^{1/2}\vartheta_3(0, \tau),$$

and

$$\vartheta_2\left(0, \frac{5\tau}{25\tau + 1}\right) = (25\tau + 1)^{1/2}\vartheta_3(0, 5\tau).$$

Hence, replacing τ by $2\tau/(25\tau + 1)$ in (12.19)–(12.21), employing the three equalities above, and using the fact that $\vartheta_3(0, \tau) = \varphi(Q)$, we find that (12.19)–(12.21) are transformed into the equations

$$5\frac{\varphi(Q^{25})}{\varphi(Q)} = 1 - \mu^{1/5} + \nu^{1/5}, \tag{12.22}$$

$$(\mu\nu)^{1/5} = 1 - 5\frac{\varphi^2(Q^5)}{\varphi^2(Q)}, \tag{12.23}$$

and

$$\mu - \nu = \left(1 - 5\frac{\varphi^2(Q^5)}{\varphi^2(Q)}\right)\left(11 - 20\frac{\varphi^2(Q^5)}{\varphi^2(Q)} + 25\frac{\varphi^4(Q^5)}{\varphi^4(Q)}\right). \tag{12.24}$$

We now apply Entry 12(iii), but with μ replaced by $-\mu$. Then (12.22)–(12.24) follow immediately. Examining (11.4), which gives rise to the values of μ and ν in Entry 12(iii), we see that ν is always positive for real q but that μ takes on both positive and negative values for real values of q. However, the positivity of v and the formulas for $(\mu v)^{1/5}$ and $\mu - v$ in Entry 12(iv) clearly imply that $\mu > 0$ for Q sufficiently small and positive.

PROOF OF (v). By Entries 10(iii) and 9(vii),

$$\frac{f(-q^2, -q^3)}{f(-q, -q^4)} - q^{2/5}\frac{f(-q, -q^4)}{f(-q^2, -q^3)} = \frac{f(-q)\{f(-q^{1/5}) + q^{1/5}f(-q^5)\}}{f(-q^2, -q^3)f(-q, -q^4)}$$
$$= \frac{f(-q)\{f(-q^{1/5}) + q^{1/5}f(-q^5)\}}{f(-q)f(-q^5)},$$

from which (v) is apparent.

In fact, Entry 12(v) is a special instance of a more general theorem which has been established independently by Ramanathan [8] and Evans [1]. Since Entry 17(v) in this chapter and Entries 6(iii), 8(i), and 12(i) in Chapter 20 are also particular case of this theorem, we state and prove it here.

Theorem 12.1. *Let n be a natural number with $n \equiv \pm 1 \pmod 6$. If $n = 6g + 1$, where $g \geq 1$, then*

$$\frac{f(-q^{1/n})}{f(-q^n)} = (-1)^g q^{(n^2-1)/(24n)} + \sum_{k=1}^{(n-1)/2} (-1)^{k+g} q^{(k-g)(3k-3g-1)/(2n)} \frac{f(-q^{2k}, -q^{n-2k})}{f(-q^k, -q^{n-k})}, \tag{12.25}$$

while if $n = 6g - 1$, where $g \geq 1$, then

$$\frac{f(-q^{1/n})}{f(-q^n)} = (-1)^g q^{(n^2-1)/(24n)} + \sum_{k=1}^{(n-1)/2} (-1)^{k+g} q^{(k-g)(3k-3g+1)/(2n)} \frac{f(-q^{2k}, -q^{n-2k})}{f(-q^k, -q^{n-k})}. \tag{12.26}$$

Before proving Theorem 12.1, we note that setting $n = 5$ in (12.26) immediately yields Entry 12(v).

PROOF. Let $U_k = a^{k(k+1)/2}b^{k(k-1)/2}$ and $V_k = a^{k(k-1)/2}b^{k(k+1)/2}$. From (31.2) in Chapter 16, if n is odd,

$$f(U_1, V_1) = f(U_n, V_n) + \sum_{k=1}^{(n-1)/2} U_k f\left(\frac{U_{n+k}}{U_k}, \frac{V_{n-k}}{U_k}\right) + \sum_{k=1}^{(n-1)/2} V_k f\left(\frac{V_{n+k}}{V_k}, \frac{U_{n-k}}{V_k}\right).$$

Putting $U_1 = a = -q^{1/n}$ and $V_1 = b = -q^{2/n}$, we find that

$$f(-q^{1/n}) = f(-q^{(3n-1)/2}, -q^{(3n+1)/2}) + \sum_{k=1}^{(n-1)/2} (-1)^k q^{k(3k-1)/(2n)} f(-q^{(3n+6k-1)/2}, -q^{(3n-6k+1)/2}) + \sum_{k=1}^{(n-1)/2} (-1)^k q^{k(3k+1)/(2n)} f(-q^{(3n+6k+1)/2}, -q^{(3n-6k-1)/2}). \tag{12.27}$$

We now assume that $n \equiv \pm 1 \pmod 6$. Suppose first that $n = 6g + 1, g \geq 1$. The gth term in the second sum on the right side of (12.27) equals

$$(-1)^g q^{(n^2-1)/(24n)} f(-q^n).$$

We partition the remaining terms in the two sums on the right side of (12.27) into six subintervals. Note that the first term on the right side of (12.27) is equal to the term when $k = 0$ in the second sum on the right side of (12.27). Accordingly, we find that

$$\begin{aligned} &f(-q^{1/n}) - (-1)^g q^{(n^2-1)/(24n)} f(-q^n) \\ &= \left(\sum_{k=1}^{g} + \sum_{k=g+1}^{2g} + \sum_{k=2g+1}^{3g}\right)(-1)^k q^{k(3k-1)/(2n)} f(-q^{(3n+6k-1)/2}, -q^{(3n-6k+1)/2}) \\ &\quad + \left(\sum_{k=0}^{g-1} + \sum_{k=g+1}^{2g} + \sum_{k=2g+1}^{3g}\right)(-1)^k q^{k(3k+1)/(2n)} f(-q^{(3n+6k+1)/2}, -q^{(3n-6k-1)/2}). \end{aligned} \tag{12.28}$$

We now combine the first and sixth sums above. Replacing k by $2g + k$ in the latter sum, we find that the sum of the kth terms equals

$$\begin{aligned} &(-1)^k q^{k(3k-1)/(2n)} \{ f(-q^{n+3k+3g}, -q^{2n-3k-3g}) \\ &\qquad + q^{k+g} f(-q^{n-3k-3g}, -q^{2n+3k+3g}) \} \\ &\quad = (-1)^k q^{k(3k-1)/(2n)} f(-q^n) \frac{f(-q^{2k+2g}, -q^{n-2k-2g})}{f(-q^{k+g}, -q^{n-k-g})}, \qquad 1 \leq k \leq g, \end{aligned}$$

by an application of the quintuple product identity (38.8) in Chapter 16 with $x = q^{k+g}$ and $\lambda = q^{n-3k-3g}$. Replacing k by $k - g$, we conclude that the sum of the first and sixth sums on the right side of (12.28) equals

$$(-1)^g f(-q^n) \sum_{k=g+1}^{2g} (-1)^k q^{(k-g)(3k-3g-1)/(2n)} \frac{f(-q^{2k}, -q^{n-2k})}{f(-q^k, -q^{n-k})}. \tag{12.29}$$

Next, we combine the fourth and fifth sums together on the right side of (12.28). Replace k by $-k$ and k by $2g + k$, respectively, in these two sums. Then by identically the same argument as above, these sums equal

$$(-1)^g f(-q^n) \sum_{k=1}^{g} (-1)^k q^{(k-g)(3k-3g-1)/(2n)} \frac{f(-q^{2k}, -q^{n-2k})}{f(-q^k, -q^{n-k})}. \tag{12.30}$$

Third, we combine the second and third sums on the right side of (12.28). In the third sum, we replace k by $4g + 1 - k$. Upon simplification, we find that the sum of the kth terms equals

$$\begin{aligned} &(-1)^k q^{k(3k-1)/(2n)} \{ f(-q^{(3n+6k-1)/2}, -q^{(3n-6k+1)/2}) \\ &\qquad - q^{4g+1-2k} f(-q^{21g+4-3k}, -q^{-3g-1+3k}) \}, \qquad g + 1 \leq k \leq 2g. \end{aligned}$$

Applying the quintuple product identity, (38.2) of Chapter 16, with $B =$

$q^{(4g+1-2k)/2}$ and q replaced by $-q^{n/2}$, we deduce that the expression above equals

$$(-1)^k q^{k(3k-1)/(2n)} f(-q^n) \frac{f(-q^{n-2k-2g}, -q^{2k+2g})}{f(-q^{k+g}, -q^{n-k-g})}, \qquad g+1 \le k \le 2g.$$

Replacing k by $k-g$, we conclude that the sum of the second and third sums on the right side of (12.28) equals

$$(-1)^g f(-q^n) \sum_{k=2g+1}^{3g} (-1)^k q^{(k-g)(3k-3g-1)/(2n)} \frac{f(-q^{n-2k}, -q^{2k})}{f(-q^k, -q^{n-k})}. \tag{12.31}$$

Finally, substituting (12.29)–(12.31) in (12.28), we readily deduce (12.25).

Now suppose that $n = 6g-1$. The gth term in the first sum on the right side of (12.27) equals

$$(-1)^g q^{(n^2-1)/(24n)} f(-q^n).$$

The remainder of the proof of (12.26) is parallel to the proof above with the roles of the two sums on the right side of (12.27) reversed. This concludes our proof.

Evans' [1] version of Theorem 12.1 is more general because it holds for all odd n. His proof is different from that of Ramanathan and will be given in Chapter 20.

The proof of (vi) that we give here is very difficult. Ramanujan must have had an easier proof. Before proving (vi), however, we establish a "rational" version of (vi), namely,

$$\frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)} + q\left(\frac{\psi^2(q^5)}{\psi^2(q)} - \frac{\psi^2(-q^5)}{\psi^2(-q)}\right) = 1, \tag{12.32}$$

which will be used later in this chapter. We first prove a needed lemma.

Lemma. $f(-q)f(q) = f(-q^2)\varphi(-q^2)$.

PROOF. By Entry 22 of Chapter 16,

$$\begin{aligned} f(-q)f(q) &= (q;q)_\infty(-q;q)_\infty = (q;q)_\infty(-q;q^2)_\infty(q^2;q^2)_\infty \\ &= (q;q^2)_\infty(q^2;q^2)_\infty^2(-q;q^2)_\infty = (q^2;q^2)_\infty^2(q^2;q^4)_\infty \\ &= f(-q^2)\varphi(-q^2). \end{aligned}$$

PROOF OF (12.32). By Entries 10(v) and 9(vii) in this chapter, Entry 25(iii) in Chapter 16, and the lemma above,

$$\begin{aligned} &1 - q\left(\frac{\psi^2(q^5)}{\psi^2(q)} - \frac{\psi^2(-q^5)}{\psi^2(-q)}\right) \\ &\quad = \left(1 - q\frac{\psi^2(q^5)}{\psi^2(q)}\right)\left(1 + q\frac{\psi^2(-q^5)}{\psi^2(-q)}\right) + q^2\frac{\psi^2(q^5)\psi^2(-q^5)}{\psi^2(q)\psi^2(-q)} \end{aligned}$$

$$= \frac{\varphi(-q^5)f(-q^5)\varphi(q^5)f(q^5)}{\chi(-q)\chi(q)\psi^2(q)\psi^2(-q)} + q^2\frac{\psi^2(q^5)\psi^2(-q^5)}{\psi^2(q)\psi^2(-q)}$$

$$= \frac{\varphi^2(-q^{10})f(-q^5)f(q^5)}{\chi(-q^2)\psi^2(q^2)\varphi^2(-q^2)} + q^2\frac{\psi^2(q^{10})\varphi^2(-q^{10})}{\psi^2(q^2)\varphi^2(-q^2)}$$

$$= \frac{\varphi^3(-q^{10})f(-q^{10})}{\chi(-q^2)\psi^2(q^2)\varphi^2(-q^2)} + q^2\frac{\psi^2(q^{10})\varphi^2(-q^{10})}{\psi^2(q^2)\varphi^2(-q^2)}$$

$$= \frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)\psi^2(q^2)}\left(\frac{\varphi(-q^{10})f(-q^{10})}{\chi(-q^2)} + q^2\psi^2(q^{10})\right)$$

$$= \frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)\psi^2(q^2)}(f(q^2, q^8)f(q^4, q^6) + q^2\psi^2(q^{10}))$$

$$= \frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)\psi^2(q^2)}(\psi^2(q^2) - q^2\psi^2(q^{10}) + q^2\psi^2(q^{10}))$$

$$= \frac{\varphi^2(-q^{10})}{\varphi^2(-q^2)},$$

which completes the proof of (12.32).

PROOF OF (vi). Unless otherwise stated, all references in this proof are to results in Chapter 16.

Using the third equality of Corollary (ii) in Section 31, Entry 25(iv), Entry 30(i), and Entry 25(iii), we deduce that

$$\Psi := q^{2/5}\left(\frac{\psi(q^{1/5})\psi(q^5)}{\psi^2(q)} + \frac{\psi(-q^{1/5})\psi(-q^5)}{\psi^2(-q)}\right)$$

$$= \frac{q\psi^2(q^5) + q^{2/5}\psi(q^5)f(q^2, q^3) + q^{3/5}\psi(q^5)f(q, q^4)}{\psi^2(q)}$$

$$- \frac{q\psi^2(-q^5) - q^{2/5}\psi(-q^5)f(q^2, -q^3) + q^{3/5}\psi(-q^5)f(-q, q^4)}{\psi^2(-q)}$$

$$= \frac{q\psi(q^{10})\varphi(q^5) + q^{2/5}f(q^2, q^8)f(q^3, q^7) + q^{3/5}f(q, q^9)f(q^4, q^6)}{\psi(q^2)\varphi(q)}$$

$$- \frac{q\psi(q^{10})\varphi(-q^5) - q^{2/5}f(q^2, q^8)f(-q^3, -q^7) + q^{3/5}f(-q, -q^9)f(q^4, q^6)}{\psi(q^2)\varphi(-q)}$$

$$= \frac{1}{\psi(q^2)\varphi^2(-q^2)}(q\psi(q^{10})\{\varphi(q^5)\varphi(-q) - \varphi(-q^5)\varphi(q)\}$$
$$+ q^{2/5}f(q^2, q^8)\{f(q^3, q^7)f(-q, -q) + f(-q^3, -q^7)f(q, q)\}$$
$$+ q^{3/5}f(q^4, q^6)\{f(q, q^9)f(-q, -q) - f(-q, -q^9)f(q, q)\}). \tag{12.33}$$

To simplify this expression, we first apply (36.2) with $A = 1, B = -1, \mu = 3,$

and $\nu = 2$. Using also Entries 18(iii), (iv), we find that

$$\tfrac{1}{2}\{\varphi(q^5)\varphi(-q) - \varphi(-q^5)\varphi(q)\}$$
$$= q^5 f(-q^{40}, -q^{20}) f(-q^{16}, -q^{-4}) - q^{21} f(-q^{60}, -1) f(-q^{24}, -q^{-12})$$
$$+ q^{49} f(-q^{80}, -q^{-20}) f(-q^{32}, -q^{-20})$$
$$= -2q f(-q^4, -q^8) f(-q^{20}, -q^{40}).$$

Second, we apply (36.1) with $A = q^2$, $B = -1$, $\mu = 3$, and $\nu = 2$. Again using Entries 18(iii), (iv), we arrive at

$$f(q^3, q^7) f(-q, -q) + f(-q^3, -q^7) f(q, q)$$
$$= 2\{f(-q^{32}, -q^{28}) f(-q^8, -q^4) - q^8 f(-q^{52}, -q^8) f(-q^{16}, -q^{-4})$$
$$+ q^{28} f(-q^{72}, -q^{-12}) f(-q^{24}, -q^{-12})\}$$
$$= 2f(-q^8, -q^4)\{f(-q^{32}, -q^{28}) + q^4 f(-q^{52}, -q^8)\}.$$

Third, utilize (36.2) again with $A = q^4$, $B = -1$, $\mu = 3$, and $\nu = 2$ to realize that

$$f(q, q^9) f(-q, -q) - f(-q, -q^9) f(q, q)$$
$$= 2q^4\{q^5 f(-q^{44}, -q^{16}) f(-q^{20}, -q^{-8})$$
$$- q^{25} f(-q^{64}, -q^{-4}) f(-q^{28}, -q^{-16})$$
$$+ q^{57} f(-q^{84}, -q^{-24}) f(-q^{36}, -q^{-24})\}$$
$$= 2q f(-q^4, -q^8)\{-f(-q^{44}, -q^{16}) + q^4 f(-q^4, -q^{56})\},$$

by Entries 18(iii), (iv) again. Using the last three calculations in (12.33) above, we deduce that

$$\Psi = \frac{f(-q^4)}{\psi(q^2)\varphi^2(-q^2)}(-4q^2\psi(q^{10})f(-q^{20})$$
$$+ 2q^{2/5} f(q^2, q^8)\{f(-q^{28}, -q^{32}) + q^4 f(-q^8, -q^{52})\}$$
$$- 2q^{8/5} f(q^4, q^6)\{f(-q^{16}, -q^{44}) - q^4 f(-q^4, -q^{56})\}). \qquad (12.34)$$

We now simplify this expression by making several substitutions. First, by Entry 24(iii),

$$f(-q^4) = \psi(q^2)\chi(-q^2). \qquad (12.35)$$

Second, applying Entry 25(iii) and Entry 24(iii) twice, we see that

$$\psi(q^{10})f(-q^{20}) = \frac{\psi(q^{20})\varphi(-q^{20})f(-q^{20})}{\psi(-q^{10})}$$
$$= \frac{\psi(q^{20})\varphi(-q^{20})f(q^{10})}{f(-q^{20})} = f(-q^{40})f(q^{10}). \qquad (12.36)$$

Third, in (38.6), replace q by $-q^{10}$ and let $B = 1/q^2$. We accordingly deduce that

$$f(-q^4, -q^{36})f(q^8, q^{42}) = f(-q^{40}, -q^{80})\{f(-q^{16}, -q^{44}) - q^4 f(-q^4, -q^{56})\}. \tag{12.37}$$

Similarly, replacing q by $-q^{10}$ and putting $B = q^6$, we find that

$$f(-q^{52}, -q^{-12})f(q^4, q^{16}) = f(-q^{40}, -q^{80})\{f(-q^{68}, -q^{-8}) - q^{-12}f(-q^{32}, -q^{28})\}.$$

By Entry 18(iv),

$$f(-q^{-12}, -q^{52}) = -q^{-12}f(-q^{28}, -q^{12})$$

and

$$f(-q^{-8}, -q^{68}) = -q^{-8}f(-q^{52}, -q^8).$$

Using these two equalities in the foregoing equality, we find that

$$f(-q^{28}, -q^{12})f(q^4, q^{16}) = f(-q^{40}, -q^{80})\{f(-q^{28}, -q^{32}) + q^4 f(-q^8, -q^{52})\}. \tag{12.38}$$

Substituting (12.35)–(12.38) into (12.34), we arrive at

$$\Psi = \frac{\chi(-q^2)}{\varphi^2(-q^2)}\left(-4q^2 f(q^{10})f(-q^{40}) + \frac{2q^{2/5}f(q^2, q^8)f(-q^{12}, -q^{28})f(q^4, q^{16})}{f(-q^{40})} - \frac{2q^{8/5}f(q^4, q^6)f(-q^4, -q^{36})f(q^8, q^{12})}{f(-q^{40})}\right). \tag{12.39}$$

By the Jacobi triple product identity, Entry 22(i), and (22.4),

$$\begin{aligned}\frac{f(-q^6, -q^{14})f(q^6, q^{14})}{\varphi(-q^{20})} &= \frac{(q^{12}; q^{40})_\infty(q^{28}; q^{40})_\infty(q^{20}; q^{20})_\infty^2}{(q^{20}; q^{20})_\infty(q^{20}; q^{40})_\infty}\\ &= (q^{12}; q^{40})_\infty(q^{28}; q^{40})_\infty(q^{20}; q^{20})_\infty(-q^{20}; q^{20})_\infty\\ &= (q^{12}; q^{40})_\infty(q^{28}; q^{40})_\infty(q^{40}; q^{40})_\infty\\ &= f(-q^{12}, -q^{28}).\end{aligned}$$

Similarly,

$$\frac{f(-q^2, -q^{18})f(q^2, q^{18})}{\varphi(-q^{20})} = f(-q^4, -q^{36}).$$

Thus, (12.39) above may be written in the form

$$\Psi = \frac{\chi(-q^2)}{\varphi^2(-q^2)}\left(-4q^2 f(q^{10})f(-q^{40}) + \frac{2q^{2/5}f(q^2, q^8)f(-q^6, -q^{14})f(q^6, q^{14})f(q^4, q^{16})}{f(-q^{40})\varphi(-q^{20})}\right.$$

$$\left. - \frac{2q^{8/5}f(q^4,q^6)f(-q^2,-q^{18})f(q^2,q^{18})f(q^8,q^{12})}{f(-q^{40})\varphi(-q^{20})}\right)$$

$$= \frac{\chi(-q^2)}{\varphi^2(-q^2)}\left(-4q^2f(q^{10})f(-q^{40})\right.$$

$$+ \frac{2q^{2/5}f(q^2,q^8)f(-q^6,-q^{14})f(q^4,q^6)\psi(q^{10})}{f^2(-q^{20})}$$

$$\left. - \frac{2q^{8/5}f(q^4,q^6)f(-q^2,-q^{18})f(q^2,q^8)\psi(q^{10})}{f^2(-q^{20})}\right),$$

where we have employed Entry 24(iii) and Entry 30(i) twice, first with $a = q^6$ and $b = q^4$ and second with $a = q^2$ and $b = q^8$.

Invoking next Entry 24(iii) of Chapter 16 and Entries 9(vii), 9(iii), and 10(ii) of this chapter, we deduce that

$$\Psi = \frac{\chi(-q^2)}{\varphi^2(-q^2)}\left(-4q^2f(q^{10})f(-q^{40})\right.$$

$$\left. + \frac{\varphi(-q^{10})f(-q^{10})\psi(q^{10})}{\chi(-q^2)f^2(-q^{20})}\{2q^{2/5}f(-q^6,-q^{14}) - 2q^{8/5}f(-q^2,-q^{18})\}\right)$$

$$= \frac{1}{\varphi^2(-q^2)}\left(-4q^2\chi(-q^2)f(q^{10})f(-q^{40})\right.$$

$$\left. + \varphi(-q^{10})\{2q^{2/5}f(-q^6,-q^{14}) - 2q^{8/5}f(-q^2,-q^{18})\}\right)$$

$$= \frac{1}{\varphi^2(-q^2)}(\varphi^2(-q^2) - \varphi^2(-q^{10}) + \varphi(-q^{10})\{\varphi(-q^{10}) - \varphi(-q^{2/5})\})$$

$$= 1 - \frac{\varphi(-q^{10})\varphi(-q^{2/5})}{\varphi^2(-q^2)},$$

which, at last, is the proposed formula.

We come now to a panoply of fifth-order modular equations, some of which we have already utilized in Section 8.

Entry 13. *The following are modular equations and formulas for multipliers for degree 5:*

(i) $(\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1;$

(ii) $$\left(\frac{\alpha^5}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8} = 1 + 2^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24};$$

(iii) $$\left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^5}{\alpha}\right)^{1/8} = 1 + 2^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24};$$

(iv) $m = 1 + 2^{4/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}$ *and* $\frac{5}{m} = 1 + 2^{4/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}$;

(v) $$m = \frac{1 + \left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8}}{1 + \{(1-\alpha)^3(1-\beta)\}^{1/8}} = \frac{1 - \left(\frac{\beta^5}{\alpha}\right)^{1/8}}{1 - (\alpha^3\beta)^{1/8}};$$

(vi) $$\frac{5}{m} = \frac{1 + \left(\frac{\alpha^5}{\beta}\right)^{1/8}}{1 + (\alpha\beta^3)^{1/8}} = \frac{1 - \left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8}}{1 - \{(1-\alpha)(1-\beta)^3\}^{1/8}};$$

(vii) $$(\alpha\beta^3)^{1/8} + \{(1-\alpha)(1-\beta)^3\}^{1/8} = 1 - 2^{1/3}\left(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right)^{1/24}$$
$$= (\alpha^3\beta)^{1/8} + \{(1-\alpha)^3(1-\beta)\}^{1/8}$$
$$= \left(\frac{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right)^{1/2};$$

(viii) *if a and b are arbitrary complex numbers, then*
$$m = \frac{a + 2^{4/3}(a-b)\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24} + 4^{1/3}b\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}}{a - b\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}}$$
$$= \frac{1 - 2^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24} - 4^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}}{(1 - 3\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} + \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/3})^{1/2}};$$

(ix) $$1 + 4^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12} = \tfrac{1}{2}m(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2});$$
$$1 + 4^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/12} = \frac{5}{2m}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2});$$

(x) $$\{\alpha(1-\beta)\}^{1/4} + \{\beta(1-\alpha)\}^{1/4} = 4^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}$$
$$= m\{\alpha(1-\alpha)\}^{1/4} + \{\beta(1-\beta)\}^{1/4}$$
$$= \{\alpha(1-\alpha)\}^{1/4} + \frac{5}{m}\{\beta(1-\beta)\}^{1/4};$$

(xi) $$\left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8} + \left(\frac{\beta^5}{\alpha}\right)^{1/8} = m\left(\frac{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right)^{1/2};$$
$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} + \left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8} = \frac{5}{m}\left(\frac{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right)^{1/2};$$

(xii) $$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4};$$

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4};$$

(xiii)
$$m - \frac{5}{m} = \frac{4((\alpha\beta)^{1/2} - \{(1-\alpha)(1-\beta)\}^{1/2})}{\left(\dfrac{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right)^{1/2}};$$

$$m + \frac{5}{m} = 2(2 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2});$$

(xiv) *if* $P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/2}$ *and* $Q = (\beta(1-\beta)/\alpha(1-\alpha))^{1/8}$, *then*

$$Q + \frac{1}{Q} + 2\left(P - \frac{1}{P}\right) = 0;$$

(xv) *if* $P = (\alpha\beta)^{1/4}$ *and* $Q = (\beta/\alpha)^{1/8}$, *then*

$$\left(Q - \frac{1}{Q}\right)^3 + 8\left(Q - \frac{1}{Q}\right) = 4\left(P - \frac{1}{P}\right).$$

The formulas (i)–(iii) do not appear to be translations of any simple combinations of formulas given in Sections 8–12. It is likely that Ramanujan's method of attack proceeded along the following lines. He first discovered (iv)–(vi), and then when he had deduced (i)–(iii) from (iv)–(vi), he decided to give (i)–(iii) priority in placement because of their elegance and simplicity. This conjecture is supported by an apparently similar rearrangement in Section 15.

PROOF OF (iv). Transcribing Entry 9(iii) via Entries 12(ii), (iv), and (v) in Chapter 17, we obtain the formula

$$z_1 - z_5 = \frac{4 \cdot 2^{1/6}}{\{\alpha(1-\alpha)\}^{1/24}} \frac{z_5^{1/2}}{2^{1/6}}(1-\beta)^{1/6}\beta^{1/24} \frac{z_5^{1/2}}{4^{1/3}}(1-\beta)^{1/24}\beta^{1/6}.$$

Dividing both sides by z_5 and simplifying, we deduce the first formula of (iv).

The second formula is the reciprocal of the first, in the language of Entry 24(v) of Chapter 18.

PROOF OF (v). Replacing q by $-q$ in Entry 9(iii), we may deduce the formula

$$\frac{\varphi^2(-q) - \varphi^2(-q^5)}{\varphi^2(q) - \varphi^2(q^5)} = -\frac{\chi(-q)f(q^5)}{\chi(q)f(-q^5)}.$$

Utilizing Entries 10(i), (ii) and 12(i), (ii), (v), (vi) of Chapter 17, we translate the formula above into

$$\frac{z_1(1-\alpha)^{1/2} - z_5(1-\beta)^{1/2}}{z_1 - z_5} = -\frac{2^{1/6}(1-\alpha)^{1/12}\alpha^{-1/24}z_5^{1/2}2^{-1/6}\{\beta(1-\beta)\}^{1/24}}{2^{1/6}\{\alpha(1-\alpha)\}^{-1/24}z_5^{1/2}2^{-1/6}(1-\beta)^{1/6}\beta^{1/24}},$$

which reduces to

$$\frac{m(1-\alpha)^{1/2}-(1-\beta)^{1/2}}{m-1}=-\left(\frac{1-\alpha}{1-\beta}\right)^{1/8}.$$

On solving for m, we deduce the first part of (v).

From Entries 9(iii), (vii) and 10(v),

$$\frac{\psi^2(q^2)-q^2\psi^2(q^{10})}{\varphi^2(q)-\varphi^2(q^5)}=\frac{\varphi(-q^{10})f(-q^{10})}{4q\chi(-q^2)\chi(q)f(-q^5)f(-q^{20})}.$$

By Entries 11(iii), 10(iii), and 12(ii), (iii), (iv), (v), (vii) in Chapter 17, the formula above transforms into the equality

$$\frac{m\sqrt{\alpha}-\sqrt{\beta}}{m-1}=\left(\frac{\alpha}{\beta}\right)^{1/8},$$

after simplification. Solving for m, we obtain the second formula of (v).

PROOF OF (vi). The formulas of (vi) are simply the reciprocals of the respective formulas of (v).

Before proving the remainder of the formulas in Entry 13, we derive parametric representations for m and various radical expressions in α and β. Put

$$u=\left(\frac{\alpha^5}{\beta}\right)^{1/24} \quad\text{and}\quad v=\left(\frac{\beta^5}{\alpha}\right)^{1/24}.$$

Then, from (v) and (vi), respectively,

$$m=\frac{1-v^3}{1-u^2v} \quad\text{and}\quad \frac{5}{m}=\frac{1+u^3}{1+uv^2}. \tag{13.1}$$

To eliminate v, we rewrite the first equation as

$$m-1=v(mu^2-v^2),$$

square this, and then substitute the value

$$v^2=\frac{m(1+u^3)-5}{5u},$$

given by the second equation of (13.1). Accordingly, we obtain the cubic equation in u^3:

$$125u^3(m-1)^2=(mu^3+m-5)(4mu^3+5-m)^2.$$

It is easily checked that $u^3=-1$ is one root of this equation. Upon dividing out the extraneous factor u^3+1 and performing some tedious algebra, we arrive at the quadratic equation in u^3

$$16m^3u^6-(8m^3+40m^2)u^3+m^3-15m^2+75m-125=0.$$

The roots of this polynomial are

$$u^3 = \frac{m^2 + 5m \pm 5\rho}{4m^2}, \tag{13.2}$$

where

$$\rho = (m^3 - 2m^2 + 5m)^{1/2}. \tag{13.3}$$

If q tends to 0, then m tends to 1 and v approaches 0. Thus, from (13.1), u^3 tends to 4. Thus, we are forced to take the plus sign above in the determination of u^3; that is,

$$u^3 = \frac{m^2 + 5m + 5\rho}{4m^2}.$$

We next want to determine v^3. This is most easily accomplished by first realizing that if u, v, and m are replaced by $-v$, $-u$, and $5/m$, respectively, in the equations (13.1), they are invariant. Thus, using (13.2), we find, after a brief calculation, that

$$v^3 = \frac{-m - 1 \pm \rho}{4}.$$

Since v tends to 0 as m tends to 1, we must take the plus sign above, and so

$$v^3 = \frac{\rho - m - 1}{4}.$$

We now summarize, in terms of α and β, the formulas that we have derived, namely,

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} = \frac{5\rho + m^2 + 5m}{4m^2} \quad \text{and} \quad \left(\frac{\beta^5}{\alpha}\right)^{1/8} = \frac{\rho - m - 1}{4}. \tag{13.4}$$

Their reciprocals are, respectively,

$$\left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8} = \frac{\rho + m + 1}{4} \quad \text{and} \quad \left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8} = \frac{5\rho - m^2 - 5m}{4m^2}. \tag{13.5}$$

We are now in a position to prove (i)–(iii).

Proofs of (ii), (iii). By (13.4) and (13.5),

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8} = \frac{m^2 + 5m}{2m^2} = \frac{1}{2} + \frac{5}{2m}.$$

Using the formula for $5/m$ from (iv), we complete the proof.

Formula (iii) is the reciprocal of (ii). Alternatively, (13.4) and (13.5) can be employed once again.

It might be noted here, that by Entries 10(i), (iii), 11(i), and 12(i) in Chapter 17, Entry 13(ii) is equivalent to the formula

$$\frac{\varphi^5(q)}{\varphi(q^5)}+\frac{\varphi^5(-q^2)}{\varphi(-q^{10})}+2\frac{f^5(q)}{f(q^5)}=4\frac{\psi^5(q)}{\psi(q^5)},$$

for which no direct proof has been constructed.

PROOF OF (i). Multiplying the equalities of (13.4) and those of (13.5), we deduce that

$$\begin{aligned}(\alpha\beta)^{1/2}&+\{(1-\alpha)(1-\beta)\}^{1/2}\\ &=\frac{10\rho^2-2m^3-12m^2-10m}{16m^2}=\frac{m^2-4m+5}{2m}\\ &=1-\frac{1}{2}(m-1)\left(\frac{5}{m}-1\right)=1-2^{5/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6},\end{aligned}\qquad(13.6)$$

by part (iv). This completes the proof of (i).

It might be observed that, by Entries 10(i), (ii), 11(iii), and 12(iii) in Chapter 17, (i) is a translation of

$$\begin{aligned}\varphi^2(q)\varphi^2(q^5)&-\varphi^2(-q)\varphi^2(-q^5)-16q^3\psi^2(q^2)\psi^2(q^{10})\\ &=8qf^2(-q^2)f^2(-q^{10}),\end{aligned}$$

of which a direct proof has not been given.

We now derive some parametric equations for further radicals that will be useful in the proofs of the remainder of the formulas of Entry 13.

From (13.6),

$$\left(\frac{1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right)^{1/2}=\left(\frac{m^2-2m+5}{4m}\right)^{1/2}=\frac{\rho}{2m},\qquad(13.7)$$

by (13.3). Furthermore, from (13.6),

$$\begin{aligned}\{16\alpha\beta&(1-\alpha)(1-\beta)\}^{1/6}\\ &=\frac{1}{4}(m-1)\left(\frac{5}{m}-1\right)=\frac{(m-1)-\frac{1}{4}(m-1)^2}{m}.\end{aligned}\qquad(13.8)$$

Now substitute for $m-1$ from Entry 13(iv) and deduce that

$$m=\frac{2^{4/3}\left(\dfrac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}-2^{2/3}\left(\dfrac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}}{\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}}.\qquad(13.9)$$

Next, by combining (13.4) and (13.5) with Entries 13(v), (vi), we may easily derive the formulas

$$(\alpha^3\beta)^{1/8}=\frac{\rho+3m-5}{4m},\qquad\{(1-\alpha)^3(1-\beta)\}^{1/8}=\frac{\rho-3m+5}{4m},\qquad(13.10)$$

$$(\alpha\beta^3)^{1/8} = \frac{\rho + m^2 - 3m}{4m}, \quad \text{and} \quad \{(1-\alpha)(1-\beta)^3\}^{1/8} = \frac{\rho - m^2 + 3m}{4m}. \tag{13.11}$$

Hence, by division,

$$\left(\frac{\alpha}{\beta}\right)^{1/4} = \frac{2m + \rho}{m(m-1)} \quad \text{and} \quad \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} = \frac{2m - \rho}{m(m-1)}, \tag{13.12}$$

and by inversion,

$$\left(\frac{\beta}{\alpha}\right)^{1/4} = \frac{2m - \rho}{5 - m} \quad \text{and} \quad \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} = \frac{2m + \rho}{5 - m}. \tag{13.13}$$

Multiplication of (13.10) and (13.11) yields

$$(\alpha\beta)^{1/2} = \frac{4m^3 - 16m^2 + 20m + \rho(m^2 - 5)}{16m^2} \tag{13.14}$$

and

$$\{(1-\alpha)(1-\beta)\}^{1/2} = \frac{4m^3 - 16m^2 + 20m - \rho(m^2 - 5)}{16m^2}. \tag{13.15}$$

From (13.4) and (13.5),

$$\begin{aligned}
\left(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right)^{1/8} &= \frac{1}{16m^2}\{6m^3 - 4m^2 + 30m - \rho(m^2 + 10m + 5)\} \\
&= \frac{1}{16m^3}\{6\rho^2 m + 8m^3 - \rho(\rho^2 + 12m^2)\} \\
&= \frac{(2m - \rho)^3}{16m^3}.
\end{aligned}$$

Therefore,

$$\frac{\rho}{2m} = 1 - 2^{1/3}\left(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right)^{1/24}. \tag{13.16}$$

PROOF OF (vii). If we properly combine (13.10), (13.11), (13.7), and (13.16), we deduce all of the equalities of (vii).

PROOF OF (viii). Multiply the first equality of (iv) by a and multiply the numerator and denominator of the right side of (13.9) by b. The first formula for m in (viii) is now easily verified.

To prove the second, first observe that

$$\begin{aligned}
\left(\frac{5 - m^2}{4m}\right)^2 = \left(\frac{5 + m^2}{4m}\right)^2 - \frac{5}{4} &= \left(\frac{6m - m^2 - 5}{4m} - \frac{3}{2}\right)^2 - \frac{5}{4} \\
&= 1 - 3\left(\frac{6m - m^2 - 5}{4m}\right) + \left(\frac{6m - m^2 - 5}{4m}\right)^2.
\end{aligned}$$

Therefore, by (13.8) and the equality above,

$$\begin{aligned} m(1 - 3\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} &+ \{16\alpha\beta(1-\alpha)(1-\beta)^{1/3}\}^{1/2} \\ &= \tfrac{1}{4}(5 - m^2) = 1 - \tfrac{1}{2}(m-1) - \tfrac{1}{4}(m-1)^2 \\ &= 1 - 2^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24} - 2^{2/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12}, \end{aligned}$$

by Entry 13(iv), and so we obtain the second equality of (viii).

Proof of (ix). Again, from Entry 13(iv),

$$2^{2/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12} = \tfrac{1}{4}(m-1)^2 = \frac{m^3 - 2m^2 + 5m}{4m} - 1 = \frac{\rho^2}{4m} - 1.$$

The desired result now follows from (13.7).

The second formula follows from the first by reciprocation.

Proof of (x). Rewrite (13.16) in the form

$$\left(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right)^{1/24} = \frac{2m-\rho}{2^{4/3}m}.$$

Using the process of reciprocation, we derive the companion formula

$$\left(\frac{\alpha^5(1-\beta)^5}{\beta(1-\alpha)}\right)^{1/24} = \frac{2m+\rho}{2^{4/3}m}.$$

Upon adding the last two equalities, we readily derive the first part of (x).

From (iv) and (13.8),

$$\begin{aligned} m\{\alpha(1-\alpha)\}^{1/4} &+ \{\beta(1-\beta)\}^{1/4} \\ &= \frac{1}{4}\left\{m\left(\frac{5}{m} - 1\right)^{5/4}(m-1)^{1/4} + (m-1)^{5/4}\left(\frac{5}{m} - 1\right)^{1/4}\right\} \\ &= \left\{\left(\frac{5}{m} - 1\right)(m-1)\right\}^{1/4} \\ &= 4^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}, \end{aligned}$$

which is the second equality of (x).

The last equality follows from the second equality, reciprocation, and the invariance of $\alpha\beta(1-\alpha)(1-\beta)$.

Proof of (xi). The first formula follows immediately from (13.4), (13.5), and (13.7), while the second is the reciprocal of the first.

Proof of (xii). The first equality follows from (12.32), the rational version of Entry 12(vi), by the use of Entries 10(iii) and 11(i), (ii) in Chapter 17. An alternative proof can easily be constructed with the aid of (13.13). The second part is simply the reciprocal of the first.

PROOF OF (xiii). The former formula is apparent from (13.14), (13.15), and (13.7). The latter formula is obtained from squaring (13.7), substituting for ρ^2 via (13.3), and then rearranging the terms.

PROOF OF (xiv). Let P and Q be as defined in (xiv). Then from (13.8) and (13.13), respectively,

$$4P^2 = (m-1)\left(\frac{5}{m}-1\right) \quad \text{and} \quad Q^2 = \frac{m-1}{\dfrac{5}{m}-1}.$$

Hence,

$$2PQ = m-1 \quad \text{and} \quad \frac{2P}{Q} = \frac{5}{m}-1,$$

from whence it follows that

$$(1+2PQ)\left(1+\frac{2P}{Q}\right) = 5.$$

The desired result now follows by rearranging the terms.

PROOF OF (xv). With P and Q as defined in (xv), we write parts of (v) and (vi) in the forms

$$m = \frac{1-PQ^3}{1-P/Q} \quad \text{and} \quad \frac{5}{m} = \frac{1+P/Q^3}{1+PQ},$$

respectively. It is now obvious that

$$(1-PQ^3)\left(1+\frac{P}{Q^3}\right) = 5(1+PQ)\left(1-\frac{P}{Q}\right),$$

whence (xv) follows upon rearrangement.

Entry 14.

(i) *Let β be of the fifth degree in α. If $\alpha = \sin^2(\mu+\nu)$ and $\beta = \sin^2(\mu-\nu)$, then*

$$\sin(2\mu) = \sin\nu(1+\cos^2\nu).$$

(ii) *If p is defined by*

$$m = 1+2p, \tag{14.1}$$

then for $0 < p < \frac{1}{2}(5\sqrt{5}-11)$,

$$4\alpha(1-\alpha) = p\left(\frac{2-p}{1+2p}\right)^5$$

and

$$4\beta(1-\beta) = p^5\left(\frac{2-p}{1+2p}\right).$$

(iii) *If p is defined by* (14.1) *and* $0 < p < 2$, *then*

$$1 - 2\alpha = \frac{1 - 11p - p^2}{(1 + 2p)^2}\left(\frac{1 + p^2}{1 + 2p}\right)^{1/2}$$

and

$$1 - 2\beta = (1 + p - p^2)\left(\frac{1 + p^2}{1 + 2p}\right)^{1/2}.$$

(iv) *If α and β are given by the equalities immediately above, then*

$${}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha) = (1 + 2p)\,{}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; \beta).$$

(v) *If* $0 < p < \frac{1}{2}(5\sqrt{5} - 11)$, *then*

$${}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{4}; 1; p\left(\frac{2 - p}{1 + 2p}\right)^5\right) = (1 + 2p)\,{}_2F_1\left(\tfrac{1}{4}, \tfrac{1}{4}; 1; p^5\left(\frac{2 - p}{1 + 2p}\right)\right).$$

Observe that (i) is an analogue of Entry 5(xiv). Formulas (ii) and (v) are more accurate versions than those stated by Ramanujan.

PROOF OF (i). Substituting in Entry 13(i), we find that

$$\sin(\mu + \nu)\sin(\mu - \nu) + \cos(\mu + \nu)\cos(\mu - \nu) + 2\{4 \sin(\mu + \nu)\sin(\mu - \nu)\cos(\mu + \nu)\cos(\mu - \nu)\}^{1/3} = 1,$$

or

$$\cos(2\nu) + 2\{\sin 2(\mu + \nu)\sin 2(\mu - \nu)\}^{1/3} = 1,$$

or

$$\sin^2(2\mu) - \sin^2(2\nu) = \tfrac{1}{8}(1 - \cos(2\nu))^3,$$

or

$$\sin^2(2\mu) = \tfrac{1}{8}(1 - \cos(2\nu))(3 + \cos(2\nu))^2 = \sin^2 \nu(1 + \cos^2 \nu)^2,$$

and so (i) is established.

PROOF OF (ii). By Entry 13(iv),

$$4\alpha(1 - \alpha) = \frac{1}{64}\left(\frac{5}{m} - 1\right)^5(m - 1). \tag{14.2}$$

Using (14.1), we find that the foregoing equality takes the shape

$$4\alpha(1 - \alpha) = p\left(\frac{2 - p}{1 + 2p}\right)^5 =: f(p). \tag{14.3}$$

Using elementary calculus, we can easily show that f increases as p goes from 0 to $\frac{1}{2}(5\sqrt{5} - 11)$, and then decreases back to 0 as p varies from $\frac{1}{2}(5\sqrt{5} - 11)$

to 2. Hence, for each value of α between 0 and 1, there are two values of p such that (14.3) is satisfied. Now, as α tends to 0, m approaches 1. By (14.1), p then tends to 0. Thus, the appropriate value of p is that with $0 < p < \frac{1}{2}(5\sqrt{5} - 11)$.

By Entry 13(iv) and (14.1), for p as above,

$$4\beta(1-\beta) = \tfrac{1}{64}(m-1)^5\left(\frac{5}{m} - 1\right) = p^5\left(\frac{2-p}{1+2p}\right), \tag{14.4}$$

and this completes the proof.

PROOF OF (iii). For brevity, set

$$a = p\left(\frac{2-p}{1+2p}\right)^5.$$

Then solving for α from part (ii), we find that

$$\alpha = \tfrac{1}{2}(1 \pm \sqrt{1-a}).$$

If $p = 0$, then $m = 1$ and $\alpha = 0$. Thus, we must take the minus sign above. Hence,

$$1 - 2\alpha = \sqrt{1-a} = \left(1 - p\left(\frac{2-p}{1+2p}\right)^5\right)^{1/2} = \frac{1 - 11p - p^2}{(1+2p)^2}\left(\frac{1+p^2}{1+2p}\right)^{1/2}.$$

Denoting the far right side by $g(p)$ and employing elementary calculus, we find that g decreases monotonically from 1 to -1 as p goes from 0 to 2. Thus, for each value of α, $0 < \alpha < 1$, there exists just one value of p, $0 < p < 2$, such that $1 - 2\alpha = g(p)$. Clearly, this representation for α is valid for $0 < p < 2$.

The proposed formula for $1 - 2\beta$ follows in the same fashion. Thus,

$$1 - 2\beta = \left(1 - p^5\left(\frac{2-p}{1+2p}\right)\right)^{1/2} = (1 + p - p^2)\left(\frac{1+p^2}{1+2p}\right)^{1/2}.$$

PROOF OF (iv). By (14.1) and part (iii), we observe that (iv) is simply a version of the equality $z_1 = mz_5$.

PROOF OF (v). By (ii), the proposed formula reads

$${}_2F_1(\tfrac{1}{4}, \tfrac{1}{4}; 1; 4\alpha(1-\alpha)) = {}_2F_1(\tfrac{1}{4}, \tfrac{1}{4}; 1; 4\beta(1-\beta)).$$

The proof of this is exactly the same as that for Entry 6(ii).

In Section 15, we assume that β is of the fifth degree in α and γ is of the fifth degree in β, so that γ is of the 25th degree in α. Let m denote the multiplier connecting α and β, and let m' be the multiplier associated with β and γ. Put (see (13.3))

$$\rho = (m^3 - 2m^2 + 5m)^{1/2} \quad \text{and} \quad \rho' = (m'^3 - 2m'^2 + 5m')^{1/2}. \tag{15.1}$$

Ramanujan's formulation of Entry 15 is in terms of hypergeometric series; for simplicity, we employ the notations m and m'.

Entry 15. *If* α, β, *and* γ *are as defined above, then*

(i) $$\left(\frac{\gamma}{\alpha}\right)^{1/8}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}-\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8}-2\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/12}=(mm')^{1/2},$$

(ii) $$\left(\frac{\alpha}{\gamma}\right)^{1/8}+\left(\frac{1-\alpha}{1-\gamma}\right)^{1/8}-\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/8}-2\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/12}=\frac{5}{(mm')^{1/2}},$$

(iii) $$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/8}+\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/8}=\left(\frac{m'}{m}\right)^{1/2},$$

(iv) $$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4}+\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4}+\left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}$$
$$-2\left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8}\left\{1+\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/8}+\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/8}\right\}$$
$$=5\frac{m}{m'},$$

and

(v) $$\frac{1+4^{1/3}\left(\dfrac{\beta^{10}(1-\beta)^{10}}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/24}}{1+4^{1/3}\left(\dfrac{\alpha^5\gamma^5(1-\alpha)^5(1-\gamma)^5}{\beta^2(1-\beta)^2}\right)^{1/24}}=\frac{m}{m'}.$$

It seems likely that, in arranging these formulas, Ramanujan gave (i) and (ii) priority over (iii) because they involve α and γ only and not the intermediate modulus $\sqrt{\beta}$. As we shall see, (i) and (ii) arise from (iii).

PROOF OF (iii). Replacing q by q^5 in Entry 12(vi), we find that

$$\frac{\varphi(-q^2)\varphi(-q^{50})}{\varphi^2(-q^{10})}+q^2\left(\frac{\psi(q)\psi(q^{25})}{\psi^2(q^5)}+\frac{\psi(-q)\psi(-q^{25})}{\psi^2(-q^5)}\right)=1.$$

Employing Entries 10(iii) and 11(i), (ii) in Chapter 17, we translate the formula above into

$$\frac{z_1^{1/2}(1-\alpha)^{1/8}z_{25}^{1/2}(1-\gamma)^{1/8}}{z_5(1-\beta)^{1/4}}+\frac{z_1^{1/2}\alpha^{1/8}z_{25}^{1/2}\gamma^{1/8}}{z_5\beta^{1/4}}$$
$$+\frac{z_1^{1/2}\{\alpha(1-\alpha)\}^{1/8}z_{25}^{1/2}\{\gamma(1-\gamma)\}^{1/8}}{z_5\{\beta(1-\beta)\}^{1/4}}=1.$$

Multiplying both sides by $z_5/(z_1z_{25})^{1/2}=(m'/m)^{1/2}$, we finish the proof.

PROOF OF (i). We first derive, from (iii), two equalities connecting m, m', ρ, and ρ' by using the trivial equalities

$$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/8}=\left(\frac{\alpha}{\beta^5}\right)^{1/8}(\beta^3\gamma)^{1/8}=(\alpha\beta^3)^{1/8}\left(\frac{\gamma}{\beta^5}\right)^{1/8}$$

and a corresponding set for $\{(1-\alpha)(1-\gamma)/(1-\beta)^2\}^{1/8}$. First, by (13.4) and (13.10),

$$\left(\frac{\alpha}{\beta^5}\right)^{1/8}(\beta^3\gamma)^{1/8}=\frac{\rho'+3m'-5}{m'(\rho-m-1)}.$$

Second, by (13.5) and (13.10),

$$\left(\frac{1-\alpha}{(1-\beta)^5}\right)^{1/8}\{(1-\beta)^3(1-\gamma)\}^{1/8}=\frac{\rho'-3m'+5}{m'(\rho+m+1)}.$$

Upon multiplication, we obtain the equality

$$\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/8}=\frac{(m'-1)(5-m')^2}{m'^2(m-1)^3},$$

where, of course, (15.1) was utilized. Thus, Entry 15(iii) now assumes the form

$$\frac{\rho'+3m'-5}{m'(\rho-m-1)}+\frac{\rho'-3m'+5}{m'(\rho+m+1)}+\frac{(m'-1)(5-m')^2}{m'^2(m-1)^3}=\left(\frac{m'}{m}\right)^{1/2}.$$

Combining terms together, we obtain our first new form of Entry 15(iii), namely,

$$\frac{2m\rho\rho'+2m'(m+1)(3m'-5)+(m'-1)(5-m')^2}{m'^2(m-1)^3}=\left(\frac{m'}{m}\right)^{1/2}. \qquad (15.2)$$

Similarly, by (13.11) and (13.4),

$$(\alpha\beta^3)^{1/8}\left(\frac{\gamma}{\beta^5}\right)^{1/8}=\frac{m'^2(\rho+m^2-3m)}{m(5\rho'+m'^2+5m')}.$$

By (13.11) and (13.5),

$$\{(1-\alpha)(1-\beta)^3\}^{1/8}\left(\frac{1-\gamma}{(1-\beta)^5}\right)^{1/8}=\frac{m'^2(\rho-m^2+3m)}{m(5\rho'-m'^2-5m')}.$$

Upon multiplication of the two equalities above, we deduce that

$$\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/8}=\frac{m'^3(m-1)^2(5-m)}{m(5-m')^3}.$$

Hence, Entry 15(iii) may be written in the form

$$\frac{m'^2(\rho+m^2-3m)}{m(5\rho'+m'^2+5m')}+\frac{m'^2(\rho-m^2+3m)}{m(5\rho'-m'^2-5m')}+\frac{m'^3(m-1)^2(5-m)}{m(5-m')^3}=\left(\frac{m'}{m}\right)^{1/2},$$

which simplifies to

$$\frac{10m'\rho\rho'-2mm'^2(m-3)(m'+5)+m'^3(m-1)^2(5-m)}{m(5-m')^3}=\left(\frac{m'}{m}\right)^{1/2}, \qquad (15.3)$$

the second desired new form of Entry 15(iii).

We remark that (15.2) and (15.3) make it evident that $\rho\rho'$ and $(m'/m)^{1/2}$ are expressible as rational functions of m and m'; this fact will be useful as a guide in proving (iv). In fact, in the proof of (i), only (15.3) is used, but the foregoing work should provide some rationale for the seemingly unmotivated proof of (iv).

By (13.4) and (13.5),

$$\begin{aligned}\left(\frac{\gamma}{\alpha}\right)^{1/8}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}&=\left(\frac{\gamma}{\beta^5}\right)^{1/8}\left(\frac{\beta^5}{\alpha}\right)^{1/8}+\left(\frac{1-\gamma}{(1-\beta)^5}\right)^{1/8}\left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8}\\&=\frac{m'^2(\rho-m-1)}{5\rho'+m'^2+5m'}+\frac{m'^2(\rho+m+1)}{5\rho'-m'^2-5m'}\\&=\frac{10m'\rho\rho'+2m'^2(m+1)(m'+5)}{(5-m')^3}.\end{aligned}$$

Hence, by (15.3),

$$\begin{aligned}&\left(\frac{\gamma}{\alpha}\right)^{1/8}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}-(mm')^{1/2}\\&=\frac{1}{(5-m')^3}\{2m'^2(m+1)(m'+5)+2mm'^2(m-3)(m'+5)\\&\quad-m'^3(m-1)^2(5-m)\}\\&=\frac{(m-1)^2}{(5-m')^3}\{2m'^2(m'+5)-m'^3(5-m)\}\\&=\frac{(m-1)^2}{(5-m')^3}\{m'^3(m-1)-2(m'^3-5m'^2)\}\\&=\frac{(m-1)^3}{\left(\frac{5}{m'}-1\right)^3}+2\frac{(m-1)^2}{\left(\frac{5}{m'}-1\right)^2}\\&=\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8}+2\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/12},\end{aligned}\tag{15.4}$$

by Entry 13(iv). Hence, formula (i) is established.

PROOF OF (ii). Formula (ii) is simply the reciprocal of (i).

It might be remarked here that, with the help of Entries 10(iii), 11(i), (ii), and 12(iii) in Chapter 17, it can easily be shown that Entry 15(ii) is a translation of the formula

$$\frac{\psi(q)}{\psi(q^{25})}-\frac{\psi(-q)}{\psi(-q^{25})}=5q^3+2q\frac{f(-q^2)}{f(-q^{50})}-q^3\frac{\varphi(-q^2)}{\varphi(-q^{50})},$$

for which no direct proof has been constructed.

Formula (iv) appears to be more recondite than the preceding three formulas, and it is not obvious how it can be deduced from them in any simple manner. Undoubtedly, Ramanujan had some ingenious method of obtaining it, for it is inconceivable that it could have been discovered by the process of verification given below.

PROOF OF (iv). We first express m and m' as simple functions of a parameter t. By Entry 13(iv),

$$(m-1)^5\left(\frac{5}{m}-1\right) = 2^8\beta(1-\beta) = (m'-1)\left(\frac{5}{m'}-1\right)^5. \tag{15.5}$$

Now put

$$t = \frac{m-1}{\dfrac{5}{m'}-1}, \tag{15.6}$$

so t approaches 0 as m tends to 1. It is also evident from (15.5) that

$$t^5 = \frac{m'-1}{\dfrac{5}{m}-1}. \tag{15.7}$$

Moreover, from Entry 13(iv) again,

$$t = \left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/24}. \tag{15.8}$$

When we substitute in (15.6) the value of m' obtained from (15.7), we obtain the quadratic equation in m,

$$m^2(t^5-1) + m\{(t-1)(t^5-1) - 5(t^5-t)\} - 5t^5(t-1) = 0.$$

Solving this quadratic equation, we find that

$$\begin{aligned} 2m(t^5-1) = {} & -(t-1)(t^5-1) + 5(t^5-t) \\ & \pm(\{(t-1)(t^5-1) - 5(t^5-t)\}^2 + 20t^5(t-1)(t^5-1))^{1/2}. \end{aligned} \tag{15.9}$$

In order to simplify the radical, we write

$$\begin{aligned} &\{(t-1)(t^5-1) - 5(t^5-t)\}^2 + 20t^5(t-1)(t^5-1) \\ &\quad = \{(t-1)(t^5-1) + 5(t^5+t)\}^2 - 100t^6 \\ &\quad = \{(t-1)(t^5-1) + 5(t^5+t) - 10t^3\}\{(t-1)(t^5-1) + 5(t^5+t) + 10t^3\} \\ &\quad = \{(t-1)(t^5-1) + 5t(t^2-1)^2\}\{(t-1)(t^5-1) + 5t(t^2+1)^2\} \\ &\quad = (t-1)^2(t^2+3t+1)^2\{t^6+4t^5+10t^3+4t+1\}. \end{aligned}$$

Substituting this in (15.9) and letting t tend to 0 in order to determine the proper sign on the radical, we conclude that

$$m = \frac{A + BR}{2(t^5 - 1)}, \tag{15.10}$$

where

$$\left.\begin{aligned} A &= -(t-1)(t^5-1) + 5(t^5 - t), \\ B &= (t-1)(t^2+3t+1), \\ R &= (t^6 + 4t^5 + 10t^3 + 4t + 1)^{1/2}, \\ B^2R^2 - A^2 &= 20t^5(t-1)(t^5-1). \end{aligned}\right\} \tag{15.11}$$

We now determine mm' in terms of t. From (5.6), $m' = 5t/(m + t - 1)$. Also, from (15.10) and (15.11),

$$m = \frac{B^2R^2 - A^2}{2(t^5-1)(BR - A)} = \frac{10t^5(t-1)}{BR - A}.$$

Hence,

$$\begin{aligned} mm' &= \frac{5t}{1 + \dfrac{t-1}{m}} = \frac{5t}{1 + \dfrac{BR - A}{10t^5}} = \frac{50t^6}{BR - (A - 10t^5)} \\ &= \frac{50t^6(BR + A - 10t^5)}{B^2R^2 - A^2 + 20At^5 - 100t^6} \\ &= \frac{5t(BR + A - 10t^5)}{2(t-1)(t^5-1) + 2A - 10t^5} \\ &= \tfrac{1}{2}(10t^5 - A - BR), \end{aligned}$$

by (15.11). Thus, by (15.11) again,

$$4mm' = 2t^6 + 8t^5 + 8t + 2 - 2BR$$

and

$$2\sqrt{mm'} = R - (t-1)(t^2 + 3t + 1).$$

From this, (15.10), and (15.11), it follows that

$$m(t^5 - 1) = 5(t^5 - t^3) + B(mm')^{1/2}. \tag{15.12}$$

Combining (15.12) and (15.7), we find that $(mm')^{1/2}$ satisfies the quadratic equation

$$mm' + B(mm')^{1/2} - 5t^3 = 0. \tag{15.13}$$

After these preliminary calculations, we are now in a position to establish (iv). Multiplying (iii) by $2(\beta^2(1-\beta)^2/\{\alpha\gamma(1-\alpha)(1-\gamma)\})^{1/4}$ and using the proposed formula (iv), we see that we are required to prove that

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4}+\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4}+\left(1-2\left(\frac{m'}{m}\right)^{1/2}\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}=5\frac{m}{m'},$$

which can be rewritten in the form

$$L:=1-\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4}-\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4}+5\frac{m}{m'}\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4}$$
$$=2\left(1-\left(\frac{m'}{m}\right)^{1/2}\right). \tag{15.14}$$

We now prove (15.14) by using several previously derived formulas to reduce the left side. Employing the following formulas in turn: (13.12) and (13.13), (13.10) and (13.11), Entry 13(iv), (15.4) and (15.6), (15.12), (15.13), (15.11), and (15.11) and (15.12), we find that

$$L=1-\frac{(2m+\rho)(2m'-\rho')}{m(m-1)(5-m')}-\frac{(2m-\rho)(2m'+\rho')}{m(m-1)(5-m')}+\frac{5(m'-1)(5-m)}{(m-1)(5-m')}$$
$$=\frac{2\rho\rho'+2m(m-3)(5-3m')}{m(m-1)(5-m')}$$
$$=\frac{(\rho+m^2-3m)(\rho'-3m'+5)+(\rho-m^2+3m)(\rho'+3m'-5)}{m(m-1)(5-m')}$$
$$=\frac{16m'}{(m-1)(5-m')}((\alpha\beta^3)^{1/8}\{(1-\beta)^3(1-\gamma)\}^{1/8}$$
$$+\{(1-\alpha)(1-\beta)^3\}^{1/8}(\beta^3\gamma)^{1/8})$$
$$=\frac{16m'\{\alpha(1-\alpha)\beta^3(1-\beta)^3\}^{1/8}}{(m-1)(5-m')}\left\{\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}+\left(\frac{\gamma}{\alpha}\right)^{1/8}\right\}$$
$$=\frac{m'(5-m)(m-1)}{m(5-m')}\left\{\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}+\left(\frac{\gamma}{\alpha}\right)^{1/8}\right\}$$
$$=\frac{t(5-m)}{m}(t^3+2t^2+(mm')^{1/2})$$
$$=\frac{t}{m(t^5-1)}\{5(t^3-1)-B(mm')^{1/2}\}(t^3+2t^2+(mm')^{1/2})$$
$$=\frac{t}{m(t^5-1)}(5(t^3-1)(t^3+2t^2)+\{5(t^3-1)-B(t^3+2t^2)\}(mm')^{1/2}$$
$$-5t^3B+B^2(mm'))$$
$$=\frac{t}{m(t^5-1)}\{10(t^4-t^2)-(2t^4-2t^2-4t+4)(mm')^{1/2}\}$$
$$=\frac{2}{m}\{m-(mm')^{1/2}\},$$

and this completes the proof of (15.14) and hence of (iv).

PROOF OF (v). By Entry 13(iv), the left side of (v) is equal to

$$\frac{1 + \frac{1}{4}(m-1)\left(\frac{5}{m'} - 1\right)}{1 + \frac{1}{4}(m'-1)\left(\frac{5}{m} - 1\right)} = \frac{m}{m'} \frac{4m' + (m-1)(5-m')}{4m + (m'-1)(5-m)} = \frac{m}{m'},$$

which completes the proof.

Entry 16(i). *Let β be of the fifth degree in α. If*

$$\int_0^A \frac{d\varphi}{(1-\alpha\sin^2\varphi)^{1/2}} = m\int_0^B \frac{d\varphi}{(1-\beta\sin^2\varphi)^{1/2}},$$

for some pair A, B, with $0 \le A, B \le \pi/2$, then

$$\tan\{\tfrac{1}{2}(A-B)\} = \frac{p\tan B}{1 + \frac{1}{2}(1 + p + \{(1+2p)(1+p^2)\}^{1/2})\tan^2 B},$$

where $m = 2p + 1$, as in (14.1).

PROOF. Entry 16(i) gives the general transformation of the fifth order, which is due to Jacobi [1, pp. 26–28], [2, pp. 77–79]. As in the corresponding analysis of Entry 6(iii), it seems sufficient to derive Ramanujan's formulation from that given by Jacobi. Converting Jacobi's result into our notation, we see that

$$\sin A = \frac{a\sin B + a'\sin^3 B + a''\sin^5 B}{1 + b'\sin^2 B + b''\sin^4 B}, \tag{16.1}$$

where

$$\begin{aligned} 1 + b'\sin^2 B + b''\sin^4 B &\pm (a\sin B + a'\sin^3 B + a''\sin^5 B) \\ &= (1 \pm \sin B)(1 \pm p\sin B + q\sin^2 B)^2, \end{aligned} \tag{16.2}$$

where q is the positive root of the equation

$$2q(q + p + 1) = p^3;$$

that is,

$$2q + p + 1 = \{(1+2p)(1+p^2)\}^{1/2}. \tag{16.3}$$

Moreover, $a = m$, $a' = 2q + p^2 + 2pq$, $a'' = q^2$, $b' = 2p + 2q + p^2$, and $b'' = 2pq + q^2$.

We first prove that

$$\tan(\tfrac{1}{4}\pi - \tfrac{1}{2}A) = \tan(\tfrac{1}{4}\pi - \tfrac{1}{2}B)\frac{1 - p\sin B + q\sin^2 B}{1 + p\sin B + q\sin^2 B}. \tag{16.4}$$

By the "half-angle" formula, $\tan(\frac{1}{4}\pi - \frac{1}{2}A) = \cos A/(1 + \sin A)$. Thus, (16.4) can be written in the form

$$\frac{\cos A}{1+\sin A}=\frac{\cos B}{1+\sin B}\frac{1-p\sin B+q\sin^2 B}{1+p\sin B+q\sin^2 B}.$$

Squaring both sides and simplifying, we determine that it now suffices to show that

$$\frac{1-\sin A}{1+\sin A}=\frac{1-\sin B}{1+\sin B}\frac{(1-p\sin B+q\sin^2 B)^2}{(1+p\sin B+q\sin^2 B)^2}. \tag{16.5}$$

Now substitute (16.1) for $\sin A$ on the left side of (16.5). Simplifying while employing (16.2), we immediately ascertain the truth of (16.5), and hence of (16.4).

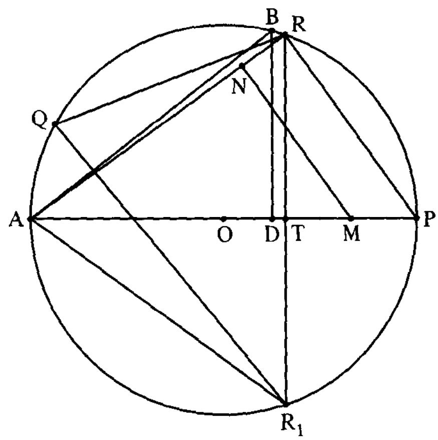
Hence, by (16.4),

$$\begin{aligned}
&\tan\{\tfrac{1}{2}(A-B)\}\\
&=\frac{\tan(\frac{1}{2}A-\frac{1}{4}\pi)-\tan(\frac{1}{2}B-\frac{1}{4}\pi)}{1+\tan(\frac{1}{2}A-\frac{1}{4}\pi)\tan(\frac{1}{2}B-\frac{1}{4}\pi)}\\
&=\frac{-\tan(\frac{1}{4}\pi-\frac{1}{2}B)\dfrac{1-p\sin B+q\sin^2 B}{1+p\sin B+q\sin^2 B}+\tan(\frac{1}{4}\pi-\frac{1}{2}B)}{1+\tan^2(\frac{1}{4}\pi-\frac{1}{2}B)\dfrac{1-p\sin B+q\sin^2 B}{1+p\sin B+q\sin^2 B}}\\
&=\frac{2p\sin B\tan(\frac{1}{4}\pi-\frac{1}{2}B)}{1+p\sin B+q\sin^2 B+\tan^2(\frac{1}{4}\pi-\frac{1}{2}B)(1-p\sin B+q\sin^2 B)}\\
&=\frac{2p\sin B\sin(\frac{1}{4}\pi-\frac{1}{2}B)\cos(\frac{1}{4}\pi-\frac{1}{2}B)}{\cos^2(\frac{1}{4}\pi-\frac{1}{2}B)(1+p\sin B+q\sin^2 B)+\sin^2(\frac{1}{4}\pi-\frac{1}{2}B)(1-p\sin B+q\sin^2 B)}\\
&=\frac{p\sin B\cos B}{1+q\sin^2 B+p\sin^2 B}\\
&=\frac{p\tan B}{1+(q+p+1)\tan^2 B},
\end{aligned}$$

and, because of (16.3), this is Ramanujan's formula.

Ramanujan begins Section 16(ii) with the following geometrical construction. Let O be the center and AP a diameter of a circle $\mathscr{C}$ of radius a. Let G denote the point of medial section on AP; that is, $AG/GP=\frac{1}{2}(1+\sqrt{5})$. Let T be any point between G and P. Draw a perpendicular line segment RR_1 to AP at T, where $R, R_1\in\mathscr{C}$. Form PR, AR, and AR_1. Letting M denote the midpoint of TP, draw MN parallel to PR, with N on AR. With $B\in\mathscr{C}$ on the same side of AP as R, draw BD perpendicular to AP such that $BD=MN$. Let $Q\in\mathscr{C}$ be such that the arcs BQ and BP are equal. Form AB, QR, and QR_1.

It appears, at first glance, that the introduction of the point of medial section G is irrelevant. However, it can be shown that the condition that T be to the right of G is necessary to ensure that $MN\le a$, which, in turn, is necessary for the construction of BD. In fact, if $T=G$, $MN=a$.



Entry 16(ii). *Let t_1 and t_2 denote the times required for a pendulum of length ℓ to oscillate through the angles $4BAR_1$ and $4BAR$, respectively. Then $t_1 = mt_2$, where m is the multiplier of degree 5. Furthermore,*

$$1 + m = \frac{2QR}{RT}, \tag{16.6}$$

$$1 + \frac{5}{m} = \frac{2QR_1}{RT}, \tag{16.7}$$

and

$$\frac{5}{m} - m = 8\frac{OD}{AR}. \tag{16.8}$$

Proof. Let $v = \angle PAR$. Then $AR = 2a \cos v$. Since also $\cos v = AT/AR$, we see that $AT = 2a \cos^2 v$. Now,

$$AT = AO + OM - TM = a + OM - (a - OM) = 2OM.$$

Thus,

$$AM = a + OM = a + \tfrac{1}{2}AT = a(1 + \cos^2 v)$$

and

$$MN = AM \sin v = a(1 + \cos^2 v)\sin v. \tag{16.9}$$

Now define μ to be the angle between 0 and $\pi/4$ determined by the equation

$$\sin(2\mu) = (1 + \cos^2 v)\sin v. \tag{16.10}$$

Then, by (16.9), (16.10), and the fact $BD = MN$,

$$\sin(2\mu) = \frac{BD}{a} = 2\frac{BP}{2a}\frac{BD}{BP} = 2 \sin \angle PAB \cos \angle PAB = \sin 2\angle PAB.$$

Hence, $\mu = \angle PAB$. By the converse of Entry 14(i), we conclude that

$$\alpha = \sin^2(\mu + \nu) = \sin^2 \angle BAR_1$$

and

$$\beta = \sin^2(\mu - \nu) = \sin^2 \angle BAR.$$

Furthermore, β is of the fifth degree in α.

The equality $t_1 = mt_2$ now follows just as in the proof of Entry 7(iii), but now, of course, Entry 16(i) is utilized in place of Entry 6(i).

By Entry 13(xiii),

$$\begin{aligned}\frac{5}{m} - m &= \frac{4\{\cos(\mu+\nu)\cos(\mu-\nu) - \sin(\mu+\nu)\sin(\mu-\nu)\}}{\left(\dfrac{1 + \sin(\mu+\nu)\sin(\mu-\nu) + \cos(\mu+\nu)\cos(\mu-\nu)}{2}\right)^{1/2}} \\ &= \frac{4\cos(2\mu)}{\left(\dfrac{1+\cos(2\nu)}{2}\right)^{1/2}} = \frac{4\cos(2\mu)}{\cos\nu} \\ &= 4\frac{OD/a}{AR/(2a)} = 8\frac{OD}{AR},\end{aligned} \tag{16.11}$$

which is (16.8).

From Entry 13(xiii) again and (16.10),

$$\frac{5}{m} + m + 2 = 6 + 2\cos(2\nu) = 4(1 + \cos^2\nu) = \frac{4\sin(2\mu)}{\sin\nu}. \tag{16.12}$$

Subtracting (16.11) from (16.12), we find that

$$m + 1 = 2\left(\frac{\sin(2\mu)}{\sin\nu} - \frac{\cos(2\mu)}{\cos\nu}\right) = \frac{4\sin(2\mu - \nu)}{\sin(2\nu)}. \tag{16.13}$$

Since $\angle BOP = 2\mu$, $\angle ROP = 2\nu$, and arc QB = arc BP, it is not difficult to see that $\angle OQR = \frac{1}{2}\pi + \nu - 2\mu$. Thus, by the law of cosines,

$$\sin(2\mu - \nu) = \cos(\tfrac{1}{2}\pi + \nu - 2\mu) = \frac{QR^2 + a^2 - a^2}{2aQR} = \frac{QR}{2a}.$$

Hence, from (16.13),

$$m + 1 = 4\frac{QR/(2a)}{RT/a} = 2\frac{QR}{RT}, \tag{16.14}$$

which establishes (16.6).

Adding (16.11) and (16.12), we find that

$$\frac{5}{m} + 1 = 2\left(\frac{\cos(2\mu)}{\cos\nu} + \frac{\sin(2\mu)}{\sin\nu}\right) = \frac{4\sin(2\mu + \nu)}{\sin(2\nu)}. \tag{16.15}$$

Since $\angle QOR_1 = 4\mu + 2\nu$, we find that $\angle OQR_1 = \frac{1}{2}\pi - 2\mu - \nu$. Thus, by the law of cosines,

$$\sin(2\mu + \nu) = \cos(\tfrac{1}{2}\pi - 2\mu - \nu) = \frac{QR_1^2 + a^2 - a^2}{2aQR_1} = \frac{QR_1}{2a}.$$

Hence, by (16.15),

$$\frac{5}{m} + 1 = 4\frac{QR_1/(2a)}{RT/a} = 2\frac{QR_1}{RT},$$

which is (16.7).

Example (i). *If* $AP = 1$, *then*

$$TP = \{16\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/6},$$

$$DT = (\alpha\beta)^{1/2},$$

$$OD + OT = \{(1 - \alpha)(1 - \beta)\}^{1/2},$$

and

$$(\alpha\beta)^{1/2} + \{(1 - \alpha)(1 - \beta)\}^{1/2} + 2\{16\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/6} = 1.$$

The last equality above is a trivial consequence of the first three, since $AP = 1$. Thus, Ramanujan has found an interesting geometric derivation of the fifth-order modular equation Entry 13(i).

In our proof below, we proceed without the assumption $AP = 1$, which yields only a trivial simplification.

PROOF. First, by (16.10),

$$\begin{aligned} TP &= RP \sin \nu = 2a \sin^2 \nu \\ &= 2a\{\sin^2 \nu(1 + \cos^2 \nu)^2 - 4 \sin^2 \nu \cos^2 \nu\}^{1/3} \\ &= 2a\{\sin^2(2\mu) - \sin^2(2\nu)\}^{1/3} \\ &= 2a\{\sin(2\mu + 2\nu)\sin(2\mu - 2\nu)\}^{1/3} \\ &= 2a\{4\alpha^{1/2}(1 - \alpha)^{1/2}\beta^{1/2}(1 - \beta)^{1/2}\}^{1/3} \\ &= 2a\{16\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/6}. \end{aligned}$$

Second,

$$\begin{aligned} DT = OT - OD &= a \cos(2\nu) - a \cos(2\mu) \\ &= 2a \sin(\mu + \nu)\sin(\mu - \nu) = 2a(\alpha\beta)^{1/2}. \end{aligned}$$

Third,

$$\begin{aligned} OD + OT &= a \cos(2\mu) + a \cos(2\nu) \\ &= 2a \cos(\mu + \nu)\cos(\mu - \nu) = 2a\{(1 - \alpha)(1 - \beta)\}^{1/2}, \end{aligned}$$

which completes the proof of Example (i).

Example (ii). *Let $Q = A$. Then $D = O$, $m = \sqrt{5}$, and T divides AP in medial section, that is, $T = G$.*

Proof. Since the arcs BP and BQ are equal, it is clear that $D = O$. Thus, from (16.11), $5/m - m = 0$, that is, $m = \sqrt{5}$.

Lastly, from similar triangles,

$$\frac{AT}{TP} = \frac{AT}{RT^2/AT} = \frac{AR^2 - RT^2}{RT^2}.$$

From (16.14), with $m = \sqrt{5}$ and $Q = A$, we conclude that

$$\frac{AT}{TP} = \left(\frac{\sqrt{5}+1}{2}\right)^2 - 1 = \frac{\sqrt{5}+1}{2};$$

that is, T divides AP in medial section.

Our formulation of Example (ii) is somewhat different from that of Ramanujan (p. 238).

The last three sections of Chapter 19 are devoted to modular equations of degree 7 and associated theta-series identities.

Entry 17.

(i)

$$q\psi(q)\psi(q^7) = \frac{q}{1-q} - \frac{q^3}{1-q^3} - \frac{q^5}{1-q^5} + \frac{q^9}{1-q^9} + \frac{q^{11}}{1-q^{11}} - \frac{q^{13}}{1-q^{13}} + \cdots,$$

where the cycle of coefficients is of length 14.

(ii)

$$\begin{aligned}\varphi(q)\varphi(q^7) = 1 + 2\biggl(&\frac{q}{1-q} - \frac{q^2}{1-q^2} - \frac{q^3}{1-q^3} + \frac{q^4}{1-q^4} - \frac{q^5}{1-q^5} \\ &+ \frac{q^6}{1-q^6} + \frac{q^8}{1-q^8} + \frac{q^9}{1-q^9} + \frac{q^{10}}{1-q^{10}} \\ &+ \frac{q^{11}}{1-q^{11}} - \frac{q^{12}}{1-q^{12}} - \frac{q^{13}}{1-q^{13}} + \frac{q^{15}}{1-q^{15}} \\ &+ \frac{q^{16}}{1-q^{16}} - \frac{q^{17}}{1-q^{17}} - \frac{q^{18}}{1-q^{18}} - \frac{q^{19}}{1-q^{19}} \\ &- \frac{q^{20}}{1-q^{20}} - \frac{q^{22}}{1-q^{22}} + \frac{q^{23}}{1-q^{23}} - \frac{q^{24}}{1-q^{24}} \\ &+ \frac{q^{25}}{1-q^{25}} + \frac{q^{26}}{1-q^{26}} - \frac{q^{27}}{1-q^{27}} + \cdots\biggr),\end{aligned}$$

where the cycle of coefficients is of length 28.

(iii) $\varphi(q^{1/7}) - \varphi(q^7) = 2q^{1/7}f(q^5, q^9) + 2q^{4/7}f(q^3, q^{11}) + 2q^{9/7}f(q, q^{13}).$

(iv) $\psi(q^{1/7}) - q^{6/7}\psi(q^7) = f(q^3, q^4) + q^{1/7}f(q^2, q^5) + q^{3/7}f(q, q^6).$

(v)

$$\frac{f(-q^{1/7})}{f(-q^7)} = \frac{f(-q^2, -q^5)}{f(-q, -q^6)} - q^{1/7}\frac{f(-q^3, -q^4)}{f(-q^2, -q^5)} - q^{2/7} + q^{5/7}\frac{f(-q, -q^6)}{f(-q^3, -q^4)}.$$

The first two formulas are of extreme interest, since they appear to indicate that Ramanujan was acquainted with a theorem equivalent to the addition theorem for elliptic integrals of the second kind. Although it would appear to be very difficult to prove (i) without this addition theorem, it is apparently not found in the notebooks.

PROOF OF (i). In the notation of the notebooks, the addition theorem assumes the form

$$\frac{qf(-a, -q^2/a)f(-b, -q^2/b)f(-ab, -q^2/ab)}{abf(-aq, -q/a)f(-bq, -q/b)f(-abq, -q/ab)}$$
$$= \frac{\varphi(-q)}{f^3(-q^2)}\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\left(\frac{1}{a^nb^n} - \frac{1}{a^n} - \frac{1}{b^n} + a^n + b^n - a^nb^n\right). \quad (17.1)$$

A direct proof of this theorem has been constructed by Glaisher [1] who wrote the left side of (17.1) as a product of three Laurent series and multiplied them together. It does not seem worthwhile to give the details here.

To deduce (i) from (17.1), replace q, a, and b by q^7, q^2, and q^4, respectively. Thus,

$$\sum_{n=1}^{\infty}\frac{q^n - q^{3n} - q^{5n} + q^{9n} + q^{11n} - q^{13n}}{1 - q^{14n}}$$
$$= q\frac{f^3(-q^{14})}{\varphi(-q^7)}\frac{f(-q^2, -q^{12})f(-q^4, -q^{10})f(-q^6, -q^8)}{f(-q^5, -q^9)f(-q^3, -q^{11})f(-q, -q^{13})}$$
$$= q\frac{f^3(-q^{14})f(-q^2)\chi(-q^7)}{\varphi(-q^7)f(-q^{14})\chi(-q)}$$
$$= q\psi(q)\psi(q^7).$$

In the analysis above, we expanded six of the theta-functions by means of the Jacobi triple product identity (Entry 19 of Chapter 16), simplified, and then employed Entry 24(iii) in Chapter 16 to obtain the final form.

Now take the summands in the series on the left side and expand them into geometric series. After inverting the order of summation in the resulting double series, we obtain the series displayed on the right side of (i) to complete the proof.

We remark that, in more classical notation, (17.1) is equivalent to the identity

$$E(u) + E(v) - E(u+v) = k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u+v),$$

where $E(u)$ denotes the incomplete elliptic integral of the second kind.

PROOF OF (ii). In order to prove (ii), we appeal to a modular equation of the seventh order given in Entry 19(i). Multiply both sides of it by $\{(1-\alpha)(1-\beta)\}^{1/8}$ to put it in the form

$$\{(1-\alpha)(1-\beta)\}^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/4} = \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}.$$

Transforming this equality by Entries 10(ii), (iii) and 11(ii) in Chapter 17, we obtain the theta-function identity

$$\varphi(-q)\varphi(-q^7) - \varphi(-q^2)\varphi(-q^{14}) = -2q\psi(-q)\psi(-q^7).$$

Replace q by q^{2^n} and sum both sides on n, $0 \le n < \infty$. Using part (i), we arrive at

$$\begin{aligned}\varphi(-q)\varphi(-q^7) - 1 &= -2\sum_{n=0}^{\infty} q^{2^n}\psi(-q^{2^n})\psi(-q^{2^n7})\\ &= -2\left(\frac{q}{1+q} + \frac{q^2}{1+q^2} - \frac{q^3}{1+q^3} + \frac{q^4}{1+q^4} - \frac{q^5}{1+q^5}\right.\\ &\quad - \frac{q^6}{1+q^6} + \frac{q^8}{1+q^8} + \frac{q^9}{1+q^9} - \frac{q^{10}}{1+q^{10}} + \frac{q^{11}}{1+q^{11}}\\ &\quad \left. - \frac{q^{12}}{1+q^{12}} - \frac{q^{13}}{1+q^{13}} + \frac{q^{15}}{1+q^{15}} + \cdots\right),\end{aligned}$$

where the cycle of coefficients is of length 7. Replacing q by $-q$, we deduce that

$$\begin{aligned}\varphi(q)\varphi(q^7) = 1 + 2&\left(\frac{q}{1-q} - \frac{q^2}{1+q^2} - \frac{q^3}{1-q^3} - \frac{q^4}{1+q^4} - \frac{q^5}{1-q^5}\right.\\ &+ \frac{q^6}{1+q^6} - \frac{q^8}{1+q^8} + \frac{q^9}{1-q^9} + \frac{q^{10}}{1+q^{10}} + \frac{q^{11}}{1-q^{11}} + \frac{q^{12}}{1+q^{12}}\\ &\left. - \frac{q^{13}}{1-q^{13}} + \frac{q^{15}}{1-q^{15}} - \cdots\right),\end{aligned}$$

where the cycle of coefficients is now of length 14. For each even value of n above, write

$$\frac{q^n}{1+q^n} = \frac{q^n}{1-q^n} - \frac{2q^{2n}}{1-q^{2n}}.$$

We now readily see that the right side of (ii) is obtained, with a cycle of coefficients of length 28. This completes the proof.

PROOF OF (iii). In Entry 31 of Chapter 16, put $a = b = q$ and $n = 7$. Then

$$\varphi(q) = \varphi(q^{49}) + 2qf(q^{35}, q^{63}) + 2q^4 f(q^{21}, q^{77}) + 2q^9 f(q^7, q^{91}).$$

Replacing q by $q^{1/7}$, we complete the proof.

PROOF OF (iv). In Entry 31 of Chapter 16, let $a = 1$, $b = q$, and $n = 7$. Using Entry 18(ii) in Chapter 16 and the definition of ψ, we find that

$$\psi(q) = f(q^{21}, q^{28}) + qf(q^{14}, q^{35}) + q^3 f(q^7, q^{42}) + q^6\psi(q^{49}).$$

Replacing q by $q^{1/7}$, we complete the proof.

PROOF OF (v). Set $n = 7$ in (12.25).

Entry 18(ii) below serves as the basis for a septic algorithm for calculating π that has been devised by J. M. and P. B. Borwein [6].

Entry 18.

(i) *There are positive functions u, v, and w such that*

$$1 + \frac{f(-q^{1/7})}{q^{2/7}f(-q^7)} = u^{1/7} - v^{1/7} + w^{1/7}, \tag{18.1}$$

where

$$u - v + w = 57 + 14\frac{f^4(-q)}{qf^4(-q^7)} + \frac{f^8(-q)}{q^2 f^8(-q^7)}, \tag{18.2}$$

$$uv - uw + vw = 289 + 126\frac{f^4(-q)}{qf^4(-q^7)} + 19\frac{f^8(-q)}{q^2 f^8(-q^7)} + \frac{f^{12}(-q)}{q^3 f^{12}(-q^7)}, \tag{18.3}$$

and

$$uvw = 1. \tag{18.4}$$

(ii) *With u, v, and w as above,*

$$1 + 7q^2\frac{f(-q^{49})}{f(-q)} = u^{1/7} - v^{1/7} + w^{1/7},$$

where

$$u - v + w = 57 + 2\cdot 7^3 q\frac{f^4(-q^7)}{f^4(-q)} + 7^4 q^2\frac{f^8(-q^7)}{f^8(-q)}$$

and

$$uv - uw + vw = 289 + 18\cdot 7^3 q\frac{f^4(-q^7)}{f^4(-q)} + 19\cdot 7^4 q^2\frac{f^8(-q^7)}{f^8(-q)} + 7^6 q^3\frac{f^{12}(-q^7)}{f^{12}(-q)}.$$

(iii) $$f(q, q^6)f(q^2, q^5)f(q^3, q^4) = \frac{f^2(-q^7)}{\chi(-q)}\varphi(-q^7).$$

(iv) $$f(-q, -q^6)f(-q^2, -q^5)f(-q^3, -q^4) = f(-q)f^2(-q^7).$$

(v) $$f(q, q^{13})f(q^3, q^{11})f(q^5, q^9) = \chi(q)\psi(-q^7)f^2(-q^{14}).$$

(vi) *If*

$$\mu = \frac{f^4(-q)}{qf^4(-q^7)} \quad \textit{and} \quad v = \frac{f(-q^{1/7})}{q^{2/7}f(-q^7)},$$

then

$$2\mu = 7(v^3 + 5v^2 + 7v) + (v^2 + 7v + 7)\{4v^3 + 21v^2 + 28v\}^{1/2}.$$

PROOFS OF (i), (vi). Our primary aim here is to establish (i), but, along the way, we also prove (vi).

In view of Entry 17(v), define α, β, γ, v, u, v, and w by

$$\alpha = u^{1/7} = \frac{f(-q^2, -q^5)}{q^{2/7}f(-q, -q^6)}, \qquad -\beta = v^{1/7} = \frac{f(-q^3, -q^4)}{q^{1/7}f(-q^2, -q^5)},$$

$$\gamma = w^{1/7} = q^{3/7}\frac{f(-q, -q^6)}{f(-q^3, -q^4)}, \quad \text{and} \quad v = \frac{f(-q^{1/7})}{q^{2/7}f(-q^7)}.$$

Thus, from Entry 17(v), we deduce immediately that $v + 1 = \alpha + \beta + \gamma$, which is (18.1). It is also clear that (18.4) is trivially satisfied. It remains to establish (18.2) and (18.3).

Let the cubic equation satisfied by α, β, and γ be given by

$$z^3 - pz^2 + sz - r = 0.$$

Thus, $p = v + 1$, by (18.1). It also follows from (18.4) that $r = -1$.

For brevity, we let $J_0, J_1, \ldots, J_6$ denote power series in q. From Entry 24(ii) in Chapter 16, it is easy to see that v^3 has the form

$$v^3 = J_0 + q^{-6/7}J_1 + q^{-5/7}J_2 + q^{-3/7}J_4. \tag{18.5}$$

But on the other hand,

$$\begin{aligned} v^3 = (\alpha + \beta + \gamma - 1)^3 = J_0 + q^{-6/7}J_1 + q^{-5/7}J_2 + q^{-4/7}J_3 \\ + q^{-3/7}J_4 + q^{-2/7}J_5 + q^{-1/7}J_6, \end{aligned} \tag{18.6}$$

where

$$J_0 = -1 + 6\alpha\beta\gamma = -7,$$

$$\left.\begin{aligned} q^{-4/7}J_3 &= 3(\gamma - \alpha^2 + \alpha\beta^2), \\ q^{-2/7}J_5 &= 3(\alpha - \beta^2 + \beta\gamma^2), \\ q^{-1/7}J_6 &= 3(\beta - \gamma^2 + \gamma\alpha^2). \end{aligned}\right\} \tag{18.7}$$

By comparing (18.5) and (18.6), we find, from (18.7), that

$$\left.\begin{aligned} \gamma - \alpha^2 + \alpha\beta^2 &= 0, \\ \alpha - \beta^2 + \beta\gamma^2 &= 0, \\ \beta - \gamma^2 + \gamma\alpha^2 &= 0. \end{aligned}\right\} \tag{18.8}$$

That these three formulas are equivalent to a single relation may be seen by multiplying them by γ, α, and β, respectively, and observing that, since $\alpha\beta\gamma = -1$, they merely undergo a cyclical interchange. To obtain this relation in a symmetrical form, multiply them by α, β, and γ, respectively, and add them to get the equation

$$\sum \beta\gamma - \sum \alpha^3 - \sum \beta^2\gamma^2 = 0, \tag{18.9}$$

where each sum is over all cyclical interchanges of the summands. Easy calculations show that

$$\sum \alpha^3 = p^3 - 3ps + 3r \tag{18.10}$$

and

$$\sum \beta^2\gamma^2 = s^2 - 2pr. \tag{18.11}$$

Thus, (18.9) becomes

$$\begin{aligned} 0 &= s - (p^3 - 3ps + 3r) + s^2 - 2pr \\ &= s^2 + (3p + 1)s - p^3 - 3r - 2pr \\ &= s^2 + (3v + 4)s - v^3 - 3v^2 - v + 4, \end{aligned} \tag{18.12}$$

since $p = v + 1$. Solving (18.12) for s, we find that

$$2s = -(3v + 4) - (4v^3 + 21v^2 + 28v)^{1/2}. \tag{18.13}$$

To see why we chose the minus sign on the radical, observe that, from the definitions of α, β, γ, v, and s,

$$v \sim q^{-2/7} \quad \text{and} \quad s \sim -q^{-3/7},$$

as q tends to 0.

Next, replace $q^{1/7}$ by $\zeta q^{1/7}$, where ζ is any seventh root of unity. Then, since $v = \alpha + \beta + \gamma - 1$,

$$\frac{f(-\zeta q^{1/7})}{\zeta^2 q^{2/7} f(-q^7)} = \zeta^5\alpha + \zeta^6\beta + \zeta^3\gamma - 1.$$

Taking the product of each side over all seventh roots of unity and using an argument analogous to that used in the proof of Entry 10(vi), we deduce that

$$\frac{f^8(-q)}{q^2 f^8(-q^7)} = \prod_\zeta \frac{f(-\zeta q^{1/7})}{\zeta^2 q^{2/7} f(-q^7)} = \prod_\zeta (\zeta^5\alpha + \zeta^6\beta + \zeta^3\gamma - 1). \tag{18.14}$$

We now face the challenging task of calculating the product on the right side above. For brevity, in the sequel, we put $s_n = \alpha^n + \beta^n + \gamma^n$.

First, replacing ζ by $1/\zeta$, we determine the product

$$(\zeta^5\alpha + \zeta^6\beta + \zeta^3\gamma - 1)(\zeta^2\alpha + \zeta\beta + \zeta^4\gamma - 1)$$
$$= 1 + s_2 - (\alpha - \gamma\alpha)(\zeta^2 + \zeta^5) - (\beta - \alpha\beta)(\zeta + \zeta^6) - (\gamma - \beta\gamma)(\zeta^3 + \zeta^4).$$

There will be two additional products of this type. Multiplying these three equalities and the equality $\alpha + \beta + \gamma - 1 = p - 1$ together, we deduce, from (18.14), that

$$\begin{aligned}\frac{f^8(-q)}{q^2 f^8(-q^7)} = (p-1)\{&(1+s_2)^3 + (1+s_2)^2\sum(\alpha - \gamma\alpha)\\ &- 2(1+s_2)\sum(\alpha-\gamma\alpha)^2 + 3(1+s_2)\sum(\beta-\alpha\beta)(\gamma-\beta\gamma)\\ &- \sum(\alpha-\gamma\alpha)^3 + 4\sum(\beta-\alpha\beta)^2(\gamma-\beta\gamma)\\ &- 3\sum(\beta-\alpha\beta)(\gamma-\beta\gamma)^2 + (\alpha-\gamma\alpha)(\beta-\alpha\beta)(\gamma-\beta\gamma)\}, \quad (18.15)\end{aligned}$$

where the summations are extended over all cyclical (not symmetrical) interchanges of $\alpha - \gamma\alpha$, $\beta - \alpha\beta$, and $\gamma - \beta\gamma$. We now are presented with the task of evaluating all terms on the right side of (18.15).

First, since

$$s_2 = p^2 - 2s, \quad (18.16)$$

$$\begin{aligned}(1+s_2)^3 &= 1 + 3(p^2 - 2s) + 3(p^4 - 4p^2 s + 4s^2)\\ &\quad + p^6 - 6p^4 s + 12p^2 s^2 - 8s^3\\ &= -8s^3 + 12(p^2+1)s^2 - 6(p^2+1)^2 s + (p^2+1)^3. \quad (18.17)\end{aligned}$$

Second, by (18.16) and the definitions of p and s,

$$\begin{aligned}&(1+s_2)^2\sum(\alpha-\gamma\alpha)\\ &\quad = (1 + 2(p^2 - 2s) + (p^2 - 2s)^2)(p - s)\\ &\quad = -4s^3 + 4(p^2 + p + 1)s^2 - (p^4 + 4p^3 + 2p^2 + 4p + 1)s\\ &\qquad + (p^5 + 2p^3 + p). \quad (18.18)\end{aligned}$$

Next, by (18.8) and (18.16),

$$\sum\alpha^2\gamma = \sum\alpha^2 - \sum\alpha = p^2 - 2s - p. \quad (18.19)$$

Thus, by (18.19), (18.16), and (18.11),

$$\begin{aligned}&-2(1+s_2)\sum(\alpha-\gamma\alpha)^2\\ &\quad = -2(1 + p^2 - 2s)\sum(\alpha^2 - 2\alpha^2\gamma + \alpha^2\gamma^2)\\ &\quad = -2(1 + p^2 - 2s)(p^2 - 2s - 2(p^2 - 2s - p) + s^2 + 2p)\\ &\quad = 4s^3 + (6 - 2p^2)s^2 - (8p^2 - 16p + 4)s\\ &\qquad + (2p^4 - 8p^3 + 2p^2 - 8p). \quad (18.20)\end{aligned}$$

With the help of (18.19), it can readily be verified that

$$\sum \beta^2\gamma = ps - 3r - p(p-1) + 2s. \tag{18.21}$$

Thus, by (18.16) and (18.21),

$$\begin{aligned} 3(1 + s_2)&\sum(\beta - \alpha\beta)(\gamma - \beta\gamma) \\ &= 3(1 + s_2)\sum(\beta\gamma - \alpha\beta\gamma - \beta^2\gamma + \alpha\beta^2\gamma) \\ &= 3(1 + p^2 - 2s)(s - 3r - ps + 3r + p^2 - p - 2s + pr) \\ &= 6(p+1)s^2 - (3p^3 + 9p^2 - 9p + 3)s + 3(p^2+1)(p^2 - 2p). \end{aligned} \tag{18.22}$$

By (18.19),

$$\begin{aligned} \sum \alpha^3\gamma &= (p\alpha^2 - s\alpha + r)\gamma + (p\beta^2 - s\beta + r)\alpha + (p\gamma^2 - s\gamma + r)\beta \\ &= p\sum \alpha^2\gamma - s^2 + rp \\ &= p(p(p-1) - 2s) - s^2 + rp \\ &= -s^2 - 2ps + p^3 - p^2 - p. \end{aligned} \tag{18.23}$$

By (18.11), (18.21), and (18.16),

$$\begin{aligned} \sum \alpha^3\gamma^2 &= (p\alpha^2 - s\alpha + r)\gamma^2 + (p\beta^2 - s\beta + r)\alpha^2 + (p\gamma^2 - s\gamma + r)\beta^2 \\ &= p\sum \alpha^2\gamma^2 - s\sum \alpha\gamma^2 + r\sum \alpha^2 \\ &= p(s^2 - 2pr) - s(ps - 3r - p^2 + p + 2s) + r(p^2 - 2s) \\ &= -2s^2 + (p^2 - p - 1)s + p^2. \end{aligned} \tag{18.24}$$

By (18.10), (18.23), and (18.24),

$$\begin{aligned} \sum \alpha^3\gamma^3 &= (p\gamma^2 - s\gamma + r)\alpha^3 + (p\alpha^2 - s\alpha + r)\beta^3 + (p\beta^2 - s\beta + r)\gamma^3 \\ &= p\sum \alpha^3\gamma^2 - s\sum \alpha^3\gamma + r\sum \alpha^3 \\ &= p(-2s^2 + (p^2 - p - 1)s + p^2) \\ &\quad - s(-s^2 - 2ps + p^3 - p^2 - p) + r(p^3 - 3ps + 3r) \\ &= s^3 + 3ps + 3. \end{aligned} \tag{18.25}$$

Hence, by (18.10) and (18.23)–(18.25),

$$\begin{aligned} -\sum(\alpha - \gamma\alpha)^3 &= -\sum \alpha^3 + 3\sum \alpha^3\gamma - 3\sum \alpha^3\gamma^2 + \sum \alpha^3\gamma^3 \\ &= -(p^3 - 3ps - 3) + 3(-s^2 - 2ps + p^3 - p^2 - p) \\ &\quad - 3(-2s^2 + (p^2 - p - 1)s + p^2) + s^3 + 3ps + 3 \\ &= s^3 + 3s^2 - (3p^2 - 3p - 3)s + (2p^3 - 6p^2 - 3p + 6). \end{aligned} \tag{18.26}$$

Next, by (18.21),

$$\begin{aligned} \sum \beta^3\gamma &= \sum \gamma(p\beta^2 - s\beta + r) \\ &= p(ps - 3r - p(p-1) + 2s) - s^2 + pr. \end{aligned} \tag{18.27}$$

Thus, by (18.21), (18.27), (18.16), and (18.19),

$$\begin{aligned}
&4\sum(\beta-\alpha\beta)^2(\gamma-\beta\gamma)\\
&\quad=4\sum(\beta^2\gamma-\beta^3\gamma-2\alpha\beta^2\gamma+2\alpha\beta^3\gamma+\alpha^2\beta^2\gamma-\alpha^2\beta^3\gamma)\\
&\quad=4\sum(\beta^2\gamma-\beta^3\gamma+2\beta-2\beta^2-\alpha\beta+\alpha\beta^2)\\
&\quad=4\{ps-3r-p(p-1)+2s-p(ps-3r-p(p-1)+2s)+s^2\\
&\qquad-pr+2p-2(p^2-2s)-s+p^2-2s-p\}\\
&\quad=4\{s^2-(p^2+p-3)s+(p^3-3p^2+3)\}. \qquad (18.28)
\end{aligned}$$

Now, by (18.11), (18.19), and (18.16),

$$\begin{aligned}
\sum\beta^3\gamma^2&=\sum(p\beta^2-s\beta+r)\gamma^2\\
&=p\sum\alpha^2\beta^2-s\sum\beta\gamma^2-\sum\alpha^2\\
&=p(s^2-2pr)-s(p^2-2s-p)-(p^2-2s)\\
&=s^2(p+2)-s(p^2-p-2)+p^2. \qquad (18.29)
\end{aligned}$$

Hence, by (18.19), (18.11), (18.29), and (18.21),

$$\begin{aligned}
&-3\sum(\beta-\alpha\beta)(\gamma-\beta\gamma)^2\\
&\quad=-3\sum(\beta\gamma^2-\alpha\beta\gamma^2-2\beta^2\gamma^2+2\alpha\beta^2\gamma^2+\beta^3\gamma^2-\alpha\beta^3\gamma^2)\\
&\quad=-3\sum(\beta\gamma^2+\gamma-2\beta^2\gamma^2-2\beta\gamma+\beta^3\gamma^2+\beta^2\gamma)\\
&\quad=-3(p^2-2s-p+p-2(s^2-2pr)-2s+(p+2)s^2\\
&\qquad-s(p^2-p-2)+p^2+ps-3r-p(p-1)+2s)\\
&\quad=-3(ps^2+(2p-p^2)s+p^2-3p+3). \qquad (18.30)
\end{aligned}$$

Lastly, a direct calculation gives

$$(\alpha-\gamma\alpha)(\beta-\alpha\beta)(\gamma-\beta\gamma)=p-s-2. \qquad (18.31)$$

Substituting (18.17), (18.18), (18.20), (18.22), (18.26), (18.28), (18.30), and (18.31) into (18.15), we at last derive the formula

$$\begin{aligned}
\frac{f^8(-q)}{q^2f^8(-q^7)}&=(p-1)\{-7s^3+(14p^2+7p+35)s^2\\
&\quad-(7p^4+7p^3+35p^2-14p)s+p^6+p^5\\
&\quad+8p^4-6p^3-13p^2-6p+8\}\\
&=-7(p-1)s^3+(14p^3-7p^2+28p-35)s^2\\
&\quad-(7p^5+28p^3-49p^2+14p)s\\
&\quad+p^7+7p^5-14p^4-7p^3+7p^2+14p-8. \qquad (18.32)
\end{aligned}$$

Since $p=v+1$, (18.32) may be written in the form

$$\frac{f^8(-q)}{q^2f^8(-q^7)} = -7vs^3 + 7v(2v^2+5v+8)s^2$$
$$- 7v(v^4+5v^3+14v^2+15v+5)s$$
$$+ v^7 + 7v^6 + 28v^5 + 56v^4 + 42v^3 - 7v^2 - 7v.$$

Using (18.12) to reduce the right side to a linear function of s and then employing (18.13), we find that

$$\frac{2f^8(-q)}{q^2f^8(-q^7)} = 2\{-7v(v^4+12v^3+49v^2+84v+49)s$$
$$+ v^7 + 21v^6 + 126v^5 + 322v^4 + 294v^3 - 147v^2 - 343v\}$$
$$= 2v^7 + 63v^6 + 532v^5 + 2009v^4 + 3724v^3 + 3087v^2 + 686v$$
$$+ 7v(v^4+12v^3+49v^2+84v+49)(4v^3+21v^2+28v)^{1/2}.$$

Taking the appropriate square root of both sides, we deduce that

$$2\mu = 2\frac{f^4(-q)}{qf^4(-q^7)} = 7(v^3+5v^2+7v) + (v^2+7v+7)(4v^3+21v^2+28v)^{1/2}, \tag{18.33}$$

which is formula (vi).

We next calculate s_7. To do this, we employ a general formula for s_n, which may be found in Littlewood's book [1, p. 83]. Omitting the rather tedious algebraic details, we deduce that

$$s_7 = \begin{vmatrix} p & 1 & 0 & 0 & 0 & 0 & 0 \\ 2s & p & 1 & 0 & 0 & 0 & 0 \\ 3r & s & p & 1 & 0 & 0 & 0 \\ 0 & r & s & p & 1 & 0 & 0 \\ 0 & 0 & r & s & p & 1 & 0 \\ 0 & 0 & 0 & r & s & p & 1 \\ 0 & 0 & 0 & 0 & r & s & p \end{vmatrix}$$

$$= -7ps^3 + (7r + 14p^3)s^2 - (7p^5 + 21p^2r)s + p^7 + 7p^4r + 7pr^2. \tag{18.34}$$

Hence, from (18.32) and (18.34),

$$\frac{f^8(-q)}{q^2f^8(-q^7)} - s_7 = 7s^3 + (-7p^2 + 28p - 28)s^2$$
$$+ (-28p^3 + 28p^2 - 14p)s$$
$$+ 7p^5 - 7p^4 - 7p^3 + 7p^2 + 7p - 8.$$

By (18.2), it remains to show that

$$2\frac{f^4(-q)}{qf^4(-q^7)} = -s^3 + (p^2 - 4p + 4)s^2 + (4p^3 - 4p^2 + 2p)s$$
$$- p^5 + p^4 + p^3 - p^2 - p - 7. \tag{18.35}$$

With the use of (18.12) and (18.13), it is a straightforward, but laborious, task to reduce the right side of (18.35) to the right side of (18.33). This concludes the proof of (18.2).

Examining (18.3), we are led to calculate $\sum \beta^7\gamma^7$. To do this, we can use (18.34). Thus, in (18.34), suppose that α, β, and γ are replaced by $\alpha\beta$, $\beta\gamma$, and $\gamma\alpha$, respectively. Then r, p, and s are replaced by r^2, s, and rp, respectively. Hence,

$$\sum \beta^7\gamma^7 = s^7 + 7ps^5 + 7s^4 + 14p^2s^3 + 21ps^2 + 7(p^3 + 1)s + 7p^2. \tag{18.36}$$

By (18.3), we want to show that the right side of (18.36) is equal to

$$-289 - 126\mu - 19\mu^2 - \mu^3. \tag{18.37}$$

To do this by hand would be a superhuman feat. Therefore, we employ MACSYMA. In (18.36), we set $p = v + 1$ and substitute the value of s given by (18.13). In (18.37), we substitute the value of μ given by (18.33). Both (18.37) and the right side of (18.36) then reduce to

$$-\tfrac{1}{2}\{21v^{10} + 595v^9 + 6468v^8 + 37229v^7 + 127421v^6$$
$$+ 270445v^5 + 355103v^4 + 275723v^3 + 113484v^2 + 19208v + 578$$
$$+ (v^9 + 63v^8 + 910v^7 + 5929v^6 + 21007v^5 + 43099v^4 + 51107v^3$$
$$+ 32907v^2 + 9800v + 882)(4v^3 + 21v^2 + 28v)^{1/2}\}.$$

Of course, the proof that we have given is quite unsatisfactory, because it is a verification which could not have been achieved without knowledge of the result. Ramanujan obviously possessed a more natural, transparent, and ingenious proof. A proof of (18.2) via the theory of modular forms can be found near the end of the introductory material in Chapter 20.

Compared to our proofs of (i) and (vi), the proofs of the remaining four parts are almost trivial.

PROOF OF (ii). Applying Entry 27(iii) of Chapter 16 with $\alpha = \frac{1}{4}\operatorname{Log}(1/Q^7)$ and $\beta = \frac{1}{2}\operatorname{Log}(1/q)$, where q and Q are chosen so that

$$7\operatorname{Log}(1/Q)\operatorname{Log}(1/q) = 4\pi^2,$$

we find that

$$q^{1/24}\operatorname{Log}^{1/4}(1/q)f(-q) = Q^{7/24}\operatorname{Log}^{1/4}(1/Q^7)f(-Q^7). \tag{18.38}$$

Replacing q by $q^{1/7}$ and Q by Q^7, we find that

$$q^{1/168}\,\mathrm{Log}^{1/4}(1/q)f(-q^{1/7})=\sqrt{7}\,Q^{49/24}\,\mathrm{Log}^{1/4}(1/Q^7)f(-Q^{49}). \tag{18.39}$$

Lastly, reversing the roles of q and Q in (18.38), we deduce that

$$q^{7/24}\,\mathrm{Log}^{1/4}(1/q^7)f(-q^7)=Q^{1/24}\,\mathrm{Log}^{1/4}(1/Q)f(-Q). \tag{18.40}$$

Dividing (18.38) by (18.40) and then (18.39) by (18.40), we deduce that, respectively,

$$\frac{f(-q)}{q^{1/4}f(-q^7)}=\sqrt{7}\frac{Q^{1/4}f(-Q^7)}{f(-Q)} \tag{18.41}$$

and

$$\frac{f(-q^{1/7})}{q^{2/7}f(-q^7)}=7\frac{Q^2f(-Q^{49})}{f(-Q)}. \tag{18.42}$$

Thus, by (18.42) and part (i),

$$1+7Q^2\frac{f(-Q^{49})}{f(-Q)}=u^{1/7}-v^{1/7}+w^{1/7}.$$

Furthermore, by (18.41) and part (i),

$$u-v+w=57+14\cdot 7^2Q\frac{f^4(-Q^7)}{f^4(-Q)}+7^4Q^2\frac{f^8(-Q^7)}{f^8(-Q)}.$$

The formula for $uv-uw+vw$ is proved in the same way.

Proof of (iii). By Entry 19, (22.4), and Entry 22, all in Chapter 16,

$$\begin{aligned}
&f(q,q^6)f(q^2,q^5)f(q^3,q^4)\\
&\quad=(-q;q^7)_\infty(-q^2;q^7)_\infty(-q^3;q^7)_\infty(-q^4;q^7)_\infty(-q^5;q^7)_\infty\\
&\qquad\times(-q^6;q^7)_\infty(-q^7;q^7)_\infty^3\\
&\quad=\frac{(-q;q)_\infty(q^7;q^7)_\infty^3}{(-q^7;q^7)_\infty}=\frac{(q^7;q^7)_\infty^2}{(q;q^2)_\infty}\frac{(q^7;q^7)_\infty}{(-q^7;q^7)_\infty}=\frac{f^2(-q^7)}{\chi(-q)}\varphi(-q^7),
\end{aligned}$$

which completes the proof.

Proof of (iv). The proof employs Entries 19 and 22 of Chapter 16 and is even easier than the proof of (iii) above.

Proof of (v). With the aid of the Jacobi triple product identity, it readily transpires that

$$\begin{aligned}
f(q,q^{13})f(q^3,q^{11})f(q^5,q^9)&=\frac{(-q;q^2)_\infty(q^{14};q^{14})_\infty^3}{(-q^7;q^{14})_\infty}\\
&=\chi(q)\psi(-q^7)f^2(-q^{14}),
\end{aligned}$$

by Entry 22 in Chapter 16.

Chapter 19 ends with a battery of modular equations of degree 7.

Entry 19. *If β is of the seventh degree in α, and m is the multiplier for degree 7, then*

(i)
$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1,$$
$$(\tfrac{1}{2}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}))^{1/2} = 1 - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8},$$

(ii)
$$m = \frac{1 - 4\left(\dfrac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}}{\{(1-\alpha)(1-\beta)\}^{1/8} - (\alpha\beta)^{1/8}},$$
$$\frac{7}{m} = \frac{1 - 4\left(\dfrac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}}{(\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}},$$

(iii)
$$\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^7}{\alpha}\right)^{1/8} = m(\tfrac{1}{2}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}))^{1/2},$$
$$\left(\frac{\alpha^7}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^7}{1-\beta}\right)^{1/8} = \frac{7}{m}(\tfrac{1}{2}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}))^{1/2},$$

(iv)
$$\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} - 1 = (\alpha\beta)^{1/8}\left\{\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^7}{\alpha}\right)^{1/8}\right\},$$
$$\left(\frac{\alpha^7}{\beta}\right)^{1/8} - 1 = \{(1-\alpha)(1-\beta)\}^{1/8}\left\{\left(\frac{\alpha^7}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^7}{1-\beta}\right)^{1/8}\right\},$$

(v)
$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} - 8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3},$$
$$\frac{49}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} - 8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3},$$

(vi)
$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4} + \left(\frac{\beta^3}{\alpha}\right)^{1/4} - \left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/4} = m^2\left(\frac{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right),$$
$$\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/4} + \left(\frac{\alpha^3}{\beta}\right)^{1/4} - \left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/4} = \frac{49}{m^2}\left(\frac{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right),$$

(vii)
$$\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} + \left(\frac{\beta^7}{\alpha}\right)^{1/8} + 2\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24} = \frac{3+m^2}{4},$$
$$\left(\frac{(1-\alpha)^7}{1-\beta}\right)^{1/8} + \left(\frac{\alpha^7}{\beta}\right)^{1/8} + 2\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24} = \frac{3}{4} + \frac{49}{4m^2},$$

and

(viii)

$$m - \frac{7}{m} = 2\left((\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}\right)\left(2 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right).$$

(ix) *If*

$$P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \quad \text{and} \quad Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6},$$

then

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left(P + \frac{1}{P}\right).$$

(x) *If*

$$P = (\alpha\beta)^{1/2} \quad \text{and} \quad Q = (\beta/\alpha)^{1/2},$$

then

$$P + \frac{1}{P} = Q + \frac{1}{Q} + (P^{1/8} - P^{-1/8})^8.$$

(xi) *If* $\alpha = \sin^2(\mu + \nu)$ *and* $\beta = \sin^2(\mu - \nu)$, *then*

$$\cos(2\mu) = (2\cos\nu - 1)(4\cos\nu - 3)^{1/2}.$$

The seventh-order modular equation given in the first equality of (i) is due to Guetzlaff [1] in 1834. Fiedler [1] in 1835 and Schröter [1], [2], [4] in 1854 also proved this modular equation. More complicated modular equations of degree 7 have been discovered by Schläfli [1], Klein [1], Sohncke [1], [2], and Russell [1].

Proof of (i). Let $\mu = 4$ and $\nu = 3$ in (36.8) of Chapter 16. This yields the equality

$$\psi(q)\psi(q^7) = \varphi(q^{28})\psi(q^8) + q\psi(q^{14})\psi(q^2) + q^6\psi(q^{56})\varphi(q^4), \tag{19.1}$$

where we have used the equality

$$q^4 f(q^{12}, q^{-4}) = f(q^4, q^4) = \varphi(q^4),$$

deducible from Entry 18(iv) in Chapter 16. Transforming (19.1) by means of Entries 10(v) and 11(i), (iii), (v) in Chapter 17, we deduce that

$$\begin{aligned}\tfrac{1}{2}(\alpha\beta)^{1/8} &= \tfrac{1}{8}\{1 - (1-\alpha)^{1/4}\}\{1 + (1-\beta)^{1/4}\} + \tfrac{1}{4}(\alpha\beta)^{1/4} \\ &\quad + \tfrac{1}{8}\{1 + (1-\alpha)^{1/4}\}\{1 - (1-\beta)^{1/4}\}.\end{aligned}$$

Simplifying, transposing, and taking the square root of each side, we arrive at the first equality of (i).

Second, square the first formula of (i) and transpose. Then square the result and transpose again. This gives, in succession,

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1 - 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$$

and

$$\begin{aligned} 1 &+ (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2} \\ &= 2 - 4\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} + 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}. \end{aligned}$$

Dividing both sides by 2 and then taking the square root of both sides, we deduce the second equality of (i).

In order to derive further modular equations, we need to obtain expressions for α, β, and various radicals in α and β in terms of a positive parameter t, which we define by

$$\alpha\beta = t^8. \tag{19.2}$$

Hence, from part (i),

$$(1-\alpha)(1-\beta) = (1-t)^8. \tag{19.3}$$

It now follows that α and β are roots of the quadratic equation

$$x^2 - \{1 + t^8 - (1-t)^8\}x + t^8 = 0. \tag{19.4}$$

Now,

$$\begin{aligned} (1 &+ t^8 - (1-t)^8)^2 - 4t^8 \\ &= \{1 + t^8 - (1-t)^8 + 2t^4\}\{1 + t^8 - (1-t)^8 - 2t^4\} \\ &= \{(1+t^4)^2 - (1-t)^8\}\{(1-t^4)^2 - (1-t)^8\} \\ &= \{(1+t^4) - (1-t)^4\}\{(1+t^4) + (1-t)^4\}\{(1-t^4) \\ &\quad - (1-t)^4\}\{(1-t^4) + (1-t)^4\} \\ &= 16t^2(2t^2 - 3t + 2)(1 - t + t^2)^2(2 - t + t^2)(1-t)^2(1 - t + 2t^2). \end{aligned}$$

Hence, solving (19.4), we obtain the roots

$$\begin{aligned} \alpha, \beta &= \tfrac{1}{2}(1 + t^8) - \tfrac{1}{2}(1-t)^8 \pm 2t(1-t)(1-t+t^2) \\ &\quad \times \{(2t^2 - 3t + 2)(2 - t + t^2)(1 - t + 2t^2)\}^{1/2}. \end{aligned} \tag{19.5}$$

Clearly, from the definition of a modular equation, α is the larger root. For brevity, we then write

$$\alpha = A + BR \quad \text{and} \quad \beta = A - BR,$$

where

$$A = \tfrac{1}{2}(1 + t^8) - \tfrac{1}{2}(1-t)^8,$$

$$B = 2t(1-t)(1-t+t^2),$$

and

$$R = \{(2 - 3t + 2t^2)(2 - t + t^2)(1 - t + 2t^2)\}^{1/2}. \tag{19.6}$$

From (19.4),

$$(\alpha \pm t^4)^2 = \{(1 \pm t^4)^2 - (1 - t)^8\}\alpha,$$

and so by the same factorization process as used above,

$$\alpha + t^4 = 2(1 - t + t^2)\{\alpha t(2 - 3t + 2t^2)\}^{1/2}$$

and

$$\alpha - t^4 = 2(1 - t)\{\alpha t(1 - t + 2t^2)(2 - t + t^2)\}^{1/2}, \tag{19.7}$$

for clearly, by (19.2), $\alpha > t^4$. Consequently, by addition,

$$\sqrt{\alpha} = (1 - t + t^2)\{t(2 - 3t + 2t^2)\}^{1/2} + (1 - t)\{t(1 - t + 2t^2)(2 - t + t^2)\}^{1/2}. \tag{19.8}$$

If α is replaced by β, the only change in the preceding argument is that the sign of the second radical must be changed, since $\beta < t^4$. Hence,

$$\sqrt{\beta} = (1 - t + t^2)\{t(2 - 3t + 2t^2)\}^{1/2} - (1 - t)\{t(1 - t + 2t^2)(2 - t + t^2)\}^{1/2}. \tag{19.9}$$

It is quite obvious from (19.2) and (19.3) that expressions for $\sqrt{1 - \alpha}$ and $\sqrt{1 - \beta}$ can be obtained by replacing t by $1 - t$ and choosing the appropriate signs of the radicals. Therefore,

$$\sqrt{1 - \alpha} = (1 - t + t^2)\{(1 - t)(1 - t + 2t^2)\}^{1/2} - t\{(1 - t)(2 - 3t + 2t^2)(2 - t + t^2)\}^{1/2} \tag{19.10}$$

and

$$\sqrt{1 - \beta} = (1 - t + t^2)\{(1 - t)(1 - t + 2t^2)\}^{1/2} + t\{(1 - t)(2 - 3t + 2t^2)(2 - t + t^2)\}^{1/2}. \tag{19.11}$$

To calculate $\sqrt{\alpha(1 - \alpha)}$ in its simplest form, it is perhaps wise to let $t = \frac{1}{2}(1 - u)$. Thus, replacing t by $1 - t$ has the effect of changing the sign of u. Under this change of variable, (19.8) and (19.10) take the shapes, respectively,

$$8\sqrt{\alpha} = (3 + u^2)\{(1 - u)(2 + u + u^2)\}^{1/2} + (1 + u)\{(1 - u)(2 - u + u^2)(7 + u^2)\}^{1/2}$$

and

$$8\sqrt{1 - \alpha} = (3 + u^2)\{(1 + u)(2 - u + u^2)\}^{1/2} - (1 - u)\{(1 + u)(2 + u + u^2)(7 + u^2)\}^{1/2}.$$

Hence,

$$64\left(\frac{\alpha(1-\alpha)}{1-u^2}\right)^{1/2} = \{(3+u^2)^2 - (1-u^2)(7+u^2)\}\{4+3u^2+u^4\}^{1/2}$$
$$+ (3+u^2)(7+u^2)^{1/2}\{(1+u)(2-u+u^2)$$
$$- (1-u)(2+u+u^2)\}$$
$$= (2+12u^2+2u^4)\{4+3u^2+u^4)^{1/2}$$
$$+ 2u(3+4u^2+u^4)(7+u^2)^{1/2}$$
$$= \tfrac{1}{2}\{(4+3u^2+u^4)^{1/2} + u(7+u^2)^{1/2}\}^3.$$

Therefore,

$$\left(\frac{\alpha(1-\alpha)}{t(1-t)}\right)^{1/6} = \tfrac{1}{4}\{(4+3u^2+u^4)^{1/2} + u(7+u^2)^{1/2}\}. \qquad (19.12)$$

Similarly, from (19.9) and (19.11),

$$\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/6} = \tfrac{1}{4}\{(4+3u^2+u^4)^{1/2} - u(7+u^2)^{1/2}\}. \qquad (19.13)$$

On squaring (19.12), we deduce that

$$\left(\frac{\alpha(1-\alpha)}{t(1-t)}\right)^{1/3} = \tfrac{1}{8}\{2+5u^2+u^4+u(28+25u^2+10u^4+u^6)^{1/2}\}$$
$$= \tfrac{1}{2}\{2-7t+11t^2-8t^3+4t^4+(1-2t)R\}, \qquad (19.14)$$

where R is defined by (19.6). It is seen from (19.13) that an analogous analysis yields

$$\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3} = \tfrac{1}{8}\{2+5u^2+u^4-u(28+25u^2+10u^4+u^6)^{1/2}\}$$
$$= \tfrac{1}{2}\{2-7t+11t^2-8t^3+4t^4-(1-2t)R\}. \qquad (19.15)$$

Hence, by (19.14) and (19.15),

$$\left(\frac{\alpha(1-\alpha)}{t(1-t)}\right)^{1/3} \quad \text{and} \quad \left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3}$$

are the roots of the quadratic equation

$$x^2 - \tfrac{1}{4}(2+5u^2+u^4)x + \tfrac{1}{16}(1-u^2)^2 = 0, \qquad (19.16)$$

or

$$x^2 - (2-7t+11t^2-8t^3+4t^4)x + t^2(1-t)^2 = 0.$$

In the proofs of Entries 13(iv), (v), we derived formulas for the multiplier m from Entry 9(iii). In the absence of any formula analogous to Entry 9(iii), we must proceed differently here. Thus, in order to find a parametric representation for m, it seems necessary to use Entry 24(vi) of Chapter 18 with

$n = 7$, namely,

$$m^2 = 7\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\frac{d\alpha}{d\beta}. \tag{19.17}$$

By differentiating (19.4) with respect to t, we find that

$$(2\alpha - \{1 + t^8 - (1-t)^8\})\frac{d\alpha}{dt} = 8(\{t^7 + (1-t)^7\}\alpha - t^7),$$

with a similar equation involving β. Hence, from (19.17) and (19.2),

$$\begin{aligned} m^2 &= -7\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\frac{\{t^7 + (1-t)^7\}\alpha - t^7}{\{t^7 + (1-t)^7\}\beta - t^7} \\ &= -7\frac{1-\beta}{1-\alpha}\frac{t^8 + t(1-t)^7 - \beta}{t^8 + t(1-t)^7 - \alpha} \\ &= -7\frac{t(1-t)^7 - \beta\{(1-t)^8 + t(1-t)^7\}}{t(1-t)^7 - \alpha\{(1-t)^8 + t(1-t)^7\}}, \end{aligned}$$

where in the last step we multiplied out the numerator and denominator and then substituted for β^2 and α^2 by (19.4). Hence, upon cancellation and the use of (19.2) and (19.4),

$$\begin{aligned} m^2 &= -7\frac{t-\beta}{t-\alpha} \\ &= -7\frac{(t-\beta)^2}{t^2 - (\alpha+\beta)t + \alpha\beta} \\ &= -7\frac{(t-\beta)^2}{t^2 - \{1 + t^8 - (1-t)^8\}t + t^8} \\ &= \frac{(t-\beta)^2}{t^2(1-t)^2(1-t+t^2)^2}. \end{aligned} \tag{19.18}$$

Consequently,

$$m = \frac{t-\beta}{t(1-t)(1-t+t^2)}. \tag{19.19}$$

We have, indeed, chosen the proper square root, because from the first equality in (19.18) and the fact that $\alpha > \beta$, it follows that $\beta < t$.

We are now in a position to easily prove (ii)–(iv).

PROOF OF (ii). By (19.5), (19.6), and (19.19),

$$\begin{aligned} m &= \frac{t - \frac{1}{2}(1+t^8) + \frac{1}{2}(1-t)^8 + 2t(1-t)(1-t+t^2)R}{t(1-t)(1-t+t^2)} \\ &= -3 + 8t - 6t^2 + 4t^3 + 2R. \end{aligned} \tag{19.20}$$

It follows that

$$(1-2t)m = -3 + 14t - 22t^2 + 16t^3 - 8t^4 + 2(1-2t)R.$$

Hence, from (19.15),

$$(1-2t)m = um = 1 - 4\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3}, \tag{19.21}$$

and from (19.2) and (19.3),

$$m = \frac{1 - 4\left(\dfrac{\beta^8(1-\beta)^8}{t^8(1-t)^8}\right)^{1/24}}{(1-t)-t} = \frac{1 - 4\left(\dfrac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}}{\{(1-\alpha)(1-\beta)\}^{1/8} - (\alpha\beta)^{1/8}},$$

and this is the first formula of (ii).

The second formula of (ii) follows from the first by the process of reciprocation.

Proof of (iii). Observe that, by (19.2), (19.3), and (19.19),

$$\begin{aligned}\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^7}{\alpha}\right)^{1/8} &= \frac{1-\beta}{1-t} - \frac{\beta}{t} = \frac{t-\beta}{t(1-t)} \\ &= m\{1 - t(1-t)\} \\ &= m(1 - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}).\end{aligned}$$

Employing part (i), we readily deduce the first formula of (iii).

The second formula of (iii) is simply the reciprocal of the first.

Proof of (iv). The first formula in (iv) is achieved by substituting for t and $1-t$ from (19.2) and (19.3) in the obvious identity

$$\frac{1-\beta}{1-t} - 1 = t\left(\frac{1-\beta}{1-t} - \frac{\beta}{t}\right).$$

The second part of (iv) is the reciprocal of the first.

Proof of (v). Let

$$T := (t^4 - \beta)\{(1-\beta) - (1-t)^4\}.$$

First, by (19.3), (19.10), and (19.11),

$$\begin{aligned}1 - \beta - (1-t)^4 &= \sqrt{1-\beta}(\sqrt{1-\beta} - \sqrt{1-\alpha}) \\ &= \sqrt{1-\beta}\,2t\{(1-t)(2-3t+2t^2)(2-t+t^2)\}^{1/2}.\end{aligned}$$

Using an analogue of (19.7) and the identity above, we deduce that

$$\begin{aligned}T = {}& 2(1-t)\{\beta t(1-t+2t^2)(2-t+t^2)\}^{1/2} \\ &\times 2t\{(1-\beta)(1-t)(2-3t+2t^2)(2-t+t^2)\}^{1/2}\end{aligned}$$

$$= 4t^2(1-t)^2(2-t+t^2)^2\{(1-t+2t^2)(2-3t+2t^2)\}^{1/2}\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/2}$$

$$= \tfrac{1}{32}(1-u^2)^2(7+u^2)(4+3u^2+u^4)^{1/2}\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/2}$$

$$= \tfrac{1}{16}(1-u^2)^2(7+u^2)\left\{\left(\frac{\alpha(1-\alpha)}{t(1-t)}\right)^{1/6}+\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/6}\right\}\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/2},$$

where, as above, $u = 1-2t$ and where we utilized (19.12) and (19.13). Rearranging terms, we arrive at

$$T = \tfrac{1}{64}(1-u^2)^2(7+u^2)\left\{(1-u^2)\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3}+4\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{2/3}\right\}$$

$$= \tfrac{1}{64}(1-u^2)^2\left\{32\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{2/3}+(7+u^2)(1-u^2)\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3}\right.$$
$$\left.-4(1-u^2)\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{2/3}\right\}$$

$$= \frac{(1-u^2)^2}{2}\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{2/3}+\frac{(1-u^2)^3}{64u^2}\left\{(2+5u^2+u^4)\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3}\right.$$
$$\left.-2(1-u^2)\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3}-4u^2\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{2/3}\right\}$$

$$= \frac{(1-u^2)^2}{2}\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{2/3}+\frac{(1-u^2)^3}{64u^2}\left\{\tfrac{1}{4}(1-u^2)^2\right.$$
$$\left.-2(1-u^2)\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3}+4(1-u^2)\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{2/3}\right\},$$

by (19.16). Now, from (19.21),

$$-2\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3}+4\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{2/3}=\tfrac{1}{4}m^2u^2-\tfrac{1}{4}.$$

Hence,

$$T = \frac{(1-u^2)^2}{2}\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{2/3}+\frac{(1-u^2)^3}{64u^2}\{\tfrac{1}{4}(1-u^2)^2-\tfrac{1}{4}(1-u^2)$$
$$+\tfrac{1}{4}m^2u^2(1-u^2)\}$$

$$= \frac{(1-u^2)^2}{2}\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{2/3}+\frac{(1-u^2)^4}{256}(m^2-1)$$

$$= 8t^2(1-t)^2\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{2/3}+t^4(1-t)^4(m^2-1).$$

Solving for m^2, we deduce that

$$m^2 = 1 + \left(1 - \frac{\beta}{t^4}\right)\left(\frac{1-\beta}{(1-t)^4} - 1\right) - \frac{8}{t^2(1-t)^2}\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{2/3}$$
$$= \frac{\beta}{t^4} + \frac{1-\beta}{(1-t)^4} - \frac{\beta(1-\beta)}{t^4(1-t)^4} - 8\left(\frac{\beta(1-\beta)}{t^4(1-t)^4}\right)^{2/3}$$
$$= \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} - 8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3},$$

which is precisely the first formula of (v).

The second formula of (v) is the reciprocal of the first.

We remark here that Entries 10(iii), 11(i), (ii), and 12(iii) in Chapter 17 can be employed to convert the former equality of (v) into the theta-function identity

$$\frac{\varphi^4(-q^{14})}{\varphi^4(-q^2)} + q^3\left(\frac{\psi^4(q^7)}{\psi^4(q)} - \frac{\psi^4(-q^7)}{\psi^4(-q)}\right) - 8q^2\frac{f^4(-q^{14})}{f^4(-q^2)} = 1.$$

No direct proof of this fascinating identity has ever been constructed.

Proof of (vi). By (19.18),

$$(1 - t + t^2)^2 m^2 = \frac{(t-\beta)^2}{t^2(1-t)^2} = \frac{1-\beta}{(1-t)^2} + \frac{\beta}{t^2} - \frac{\beta(1-\beta)}{t^2(1-t)^2}.$$

Hence, by (19.2), (19.3), and part (i),

$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4} + \left(\frac{\beta^3}{\alpha}\right)^{1/4} - \left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/4}$$
$$= m^2\{1 - t(1-t)\}^2$$
$$= m^2(\tfrac{1}{2}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}))^{1/2},$$

which is the former formula of (vi). The second is the reciprocal of the first.

Proof of (vii). From (19.20),

$$m = -(1-2t)(3 - 2t + 2t^2) + 2R,$$

and so

$$m^2 = (3 - 8t + 6t^2 - 4t^3)^2 + 4R^2 - 4R(1-2t) - 8R(1-2t)(1-t+t^2).$$

Using (19.5) and (19.15), we deduce from the foregoing equality that

$$m^2 - 8\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3} - \frac{4(1-2t)\beta}{t(1-t)}$$

$$= (3 - 8t + 6t^2 - 4t^3)^2 + 4R^2 - 4(2 - 7t + 11t^2 - 8t^3 + 4t^4)$$

$$- \frac{2 - 4t}{t(1-t)}\{1 + t^8 - (1-t)^8\}$$

$$= \frac{1 + 3t}{1 - t},$$

after a rather tedious calculation. Hence, from the equality above, (19.2), and (19.3),

$$m^2 + 3 = 8\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3} + \frac{4(1-2t)\beta + 4t}{t(1-t)}$$

$$= 8\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3} + \frac{4\beta}{t} + \frac{4(1-\beta)}{1-t}$$

$$= 8\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24} + 4\left(\frac{\beta^7}{\alpha}\right)^{1/8} + 4\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8},$$

which yields the first formula of (vii). The second is the reciprocal of the first.

PROOF OF (viii). From the analysis leading to (19.20), it is clear that similar reasoning yields the companion formula

$$-\frac{7}{m} = -3 + 8t - 6t^2 + 4t^3 - 2R. \tag{19.22}$$

Adding this formula to (19.20), we find that

$$m - \frac{7}{m} = -6 + 16t - 12t^2 + 8t^3$$

$$= 2\{t - (1-t)\}\{2 + t^2 + (1-t)^2\}. \tag{19.23}$$

Formula (viii) now follows at once upon the use of (19.2) and (19.3).

In the notebooks (p. 240), Entry 19(viii) contains two misprints.

PROOF OF (ix). By (19.2) and (19.3), $P = t(1-t)\sqrt{2}$, and by (19.16),

$$Q + \frac{1}{Q} = \left(\frac{\beta(1-\beta)}{t^4(1-t)^4}\right)^{1/3} + \left(\frac{\alpha(1-\alpha)}{t^4(1-t)^4}\right)^{1/3}$$

$$= \frac{2 + 5(1-2t)^2 + (1-2t)^4}{4t(1-t)}$$

$$= \frac{2 - 7t(1-t) + 4t^2(1-t)^2}{t(1-t)}$$

$$= \frac{2\sqrt{2}}{P} - 7 + 2\sqrt{2}P,$$

which completes the proof of (ix).

Ramanathan [10] has given a different proof of (ix).

PROOF OF (x). From part (i),

$$(1 - P^{1/4})^8 = (1 - (\alpha\beta)^{1/8})^8 = (1 - \alpha)(1 - \beta) = \left(1 - \frac{P}{Q}\right)(1 - PQ).$$

Dividing both sides by P, we find that

$$(P^{-1/8} - P^{1/8})^8 = \left(\frac{1}{P} - \frac{1}{Q}\right)(1 - PQ),$$

which immediately yields the desired result.

PROOF OF (xi). From the second formula of part (i),

$$\begin{aligned} 1 - \{\sin(\mu + \nu)\sin(\mu - \nu)\cos(\mu + \nu)\cos(\mu - \nu)\}^{1/4} \\ = (\tfrac{1}{2}(1 + \sin(\mu + \nu)\sin(\mu - \nu) + \cos(\mu + \nu)\cos(\mu - \nu)))^{1/2} \\ = (\tfrac{1}{2}(1 + \cos(2\nu)))^{1/2} = \cos \nu, \end{aligned}$$

and so

$$\sin(2\mu + 2\nu)\sin(2\mu - 2\nu) = 4(1 - \cos \nu)^4;$$

that is,

$$\cos^2(2\nu) - \cos^2(2\mu) = 4(1 - \cos \nu)^4.$$

Hence,

$$\begin{aligned} \cos^2(2\mu) &= (2 \cos^2 \nu - 1)^2 - 4(1 - \cos \nu)^4 \\ &= 16 \cos^3 \nu - 28 \cos^2 \nu + 16 \cos \nu - 3 \\ &= (2 \cos \nu - 1)^2(4 \cos \nu - 3), \end{aligned}$$

and so the proof of (xi) is complete.

Employing the theory of modular forms, Raghavan [1], [2] and Raghavan and Rangachari [1] have proved several results in this chapter.

CHAPTER 20

Modular Equations of Higher and Composite Degrees

In this chapter, we continue to examine Ramanujan's discoveries about modular equations. In the previous chapter, modular equations of degrees 3, 5, and 7 were derived. Modular equations of degrees 11, 13, 17, 19, 23, 31, 47, and 71 are established in this chapter. Also, modular equations of composite degree, or "mixed" modular equations, are studied. Most of the equations of the latter type involve four distinct moduli, and so we begin by defining such a modular equation. Let K, K', L_1, L_1', L_2, L_2', L_3, and L_3' denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, and $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2, and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1 \frac{K'}{K} = \frac{L_1'}{L_1}, \quad n_2 \frac{K'}{K} = \frac{L_2'}{L_2}, \quad \text{and} \quad n_3 \frac{K'}{K} = \frac{L_3'}{L_3} \tag{0.1}$$

hold. Then a "mixed" modular equation is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$, and $\sqrt{\delta}$ that is induced by (0.1). In such an instance, we say that β, γ, and δ are of degrees n_1, n_2, and n_3, respectively. Recalling that $z_r = \varphi^2(q^r)$, we define the multipliers m and m' by

$$m = z_1/z_{n_1} \quad \text{and} \quad m' = z_{n_2}/z_{n_3}.$$

Amazingly, in this chapter, Ramanujan derives modular equations for *twenty* distinct sets $\{n_1, n_2, n_3\}$ of degrees, namely,

3, 5, 15;	5, 27, 135;
3, 7, 21;	7, 9, 63;
3, 9, 27;	7, 17, 119;
3, 11, 33;	7, 25, 175;
3, 13, 39;	9, 15, 135;
3, 21, 63;	9, 23, 207;

3, 29, 87;	11, 13, 143;
5, 7, 35;	11, 21, 231;
5, 11, 55;	13, 19, 247;
5, 19, 95;	15, 17, 255.

Hardy [1, p. 220] recorded, without proof, two modular equations for the triple $\{3, 5, 15\}$ of degrees. Otherwise, none of Ramanujan's work on "mixed" modular equations had been published until Berndt, Biagioli, and Purtilo [1]–[3] published proofs of a small portion of Ramanujan's modular equations in Chapters 19 and 20.

In Chapter 19, we employed the theory of theta-functions and elementary, but often complicated and tedious, algebra to prove Ramanujan's modular equations. For many of the modular equations of this chapter, we have been unable to establish them by these techniques. Instead, we have had to invoke the theory of modular forms. In some ways, this approach is the best of the three methods of attack. The first two methods become rapidly more difficult as the degree of the modular equation increases, while the complexity of the approach through modular forms increases only slightly as the degree increases. Because a modular equation is always equivalent to an identity among theta-functions of several arguments, the theory of modular forms provides the theoretical backdrop explaining the raison d'etre for such identities. The primary disadvantage of this method, as well as most elementary algebraic approaches, is that the modular equation must be known in advance. Thus, the proofs are perhaps more aptly called verifications.

In order to avoid a lengthy diversion later in the sequel, it seems advisable, at this juncture, to present the theory of modular forms that will be necessary to establish many of Ramanujan's modular equations. General references are Rankin's book [2] and Petersson's notes [1].

Let $\mathscr{H}$ denote the upper half-plane, that is, $\mathscr{H} = \{\tau\colon \operatorname{Im}(\tau) > 0\}$. Put $q = e^{\pi i\tau}$, where $\tau \in \mathscr{H}$. For each $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2^+(\mathbb{R})$, the set of real 2×2 matrices with positive determinant, the bilinear transformation $M(\tau)$ is defined by

$$M\tau := M(\tau) := \frac{a\tau + b}{c\tau + d}.$$

It is easy to see that composition of bilinear transformations is compatible with matrix multiplication; that is, $M(S\tau) = (MS)\tau$, for any $M, S \in M_2^+(\mathbb{R})$. For $M \in M_2^+(\mathbb{R})$, it is well known that $M(\tau)$ maps $\mathscr{H}$ onto $\mathscr{H}$ and $\mathbb{R} \cup \{\infty\}$ onto itself.

For each real number r, we define the stroke operator of weight r by

$$(f|_{(r)}M)(\tau) = (\det M)^{r/2}(M:\tau)^{-r}f(M\tau),$$

where $(M:\tau) = (c\tau + d)$ and the power is determined by taking $-\pi \le \arg(M:\tau) < \pi$. We usually suppress the index and write the left side above as $f|M$. This operator satisfies the equality

$$f|MS = \sigma(M, S)f|M|S, \tag{0.2}$$

which Knopp [1, p. 52] calls the consistency condition, where

$$\sigma(M, S) = e^{2\pi i r w(M,S)},$$

and where

$$2\pi w(M, S) = \arg(M : S\tau) + \arg(S : \tau) - \arg(MS : \tau).$$

The value of $w(M, S)$ is either 1, 0, or -1 and is independent of τ. It is not difficult to see that if either M or S has the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, with a and d positive, then $w(M, S) = 0$. Hence,

$$f|MS = f|M|S,$$

a fact that we shall use without comment many times in the sequel.

We shall usually write $f|n$ and $f|\frac{m}{n}$ as abbreviations for $f|\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $f|\left(\begin{smallmatrix} m & 0 \\ 0 & n \end{smallmatrix}\right)$, respectively. Similarly, $f|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}$ is an abbreviation for $f|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

If $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and n is a positive integer, we define

$$^{(n)}A = \begin{pmatrix} a & nb \\ c/n & d \end{pmatrix} \quad \text{and} \quad {}_{(n)}A = \begin{pmatrix} na & b \\ c & d/n \end{pmatrix}, \tag{0.3}$$

which have the properties

$$f|^{(n)}A|n = f|n|A = f|_{(n)}A\Big|\frac{1}{n}. \tag{0.4}$$

The modular group $\Gamma(1)$ is defined by

$$\Gamma(1) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2^+(\mathbb{Z}): ad - bc = 1\right\}.$$

We shall be concerned with certain subgroups of the modular group, namely,

$$\Gamma(2) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1): \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod } 2)\right\},$$

$$\Gamma_\theta = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1): a + c \equiv b + d (\text{mod } 2)\right\},$$

$$\Gamma^0(n) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1): b \equiv 0 (\text{mod } n)\right\},$$

and

$$\Gamma_0(n) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1): c \equiv 0 (\text{mod } n)\right\},$$

where n is a positive integer. The index of $\Gamma(2)$ in $\Gamma(1)$ is 6. The subgroups $\Gamma_0(2)$, $\Gamma^0(2)$, and Γ_θ form a conjugacy class in $\Gamma(1)$, and their intersection is $\Gamma(2)$. Each of them has index 3 in $\Gamma(1)$ and index 2 over $\Gamma(2)$. Also,

$$(\Gamma(1): \Gamma_0(n)) = n \prod_{p|n} \left(1 + \frac{1}{p}\right),$$

where the product is over all primes p dividing n. See the books of Rankin [2, pp. 26, 29] or Schoeneberg [1, Chap. 4, §§3.2, 4.2] for proofs of these facts.

We always denote by Γ a subgroup of $\Gamma(1)$ with finite index. Such a group acts on $\mathscr{H} \cup Q \cup \{\infty\}$ by the transformation $V(\tau)$, for $V \in \Gamma$, and this induces an equivalence relation; the equivalence classes are called orbits. We call $\mathscr{F} \subseteq \mathscr{H} \cup Q \cup \{\infty\}$ a fundamental set for Γ if it contains one element of each equivalence class, and $\mathscr{F} \cap (Q \cup \{\infty\})$, which is always a finite set, is called a complete set of inequivalent cusps.

A function $f: \mathscr{H} \to \mathscr{C}$ is a modular form if there is a subgroup $\Gamma \subseteq \Gamma(1)$ of finite index, a real number r, and a function $v: \Gamma \to \{z \in \mathscr{C}: |z| = 1\}$ such that the following three conditions hold:

(i) f is analytic on $\mathscr{H}$.
(ii) $f|V = v(V)f$, for all $V \in \Gamma$.
(iii) Let $A \in \Gamma(1)$ and $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Define

$$N = \min\{k > 0: \pm A^{-1}U^kA \in \Gamma\}$$

and

$$f_A = \sigma(A, A^{-1})f|A^{-1}.$$

Then there exist an integer m_0, complex numbers b_m with $m \geq m_0$, and a real number κ with $0 \leq \kappa < 1$, such that f_A has the expansion

$$f_A(\tau) = \sum_{m=m_0}^{\infty} b_m e^{2\pi i\tau(m+\kappa)/N},$$

in some half-plane $\{\tau: \operatorname{Im} \tau > h \geq 0\}$.

The *weight* of f is r and the *multiplier system* for f is v. The set of all modular forms on Γ of weight r and multiplier system v is denoted by $\{\Gamma, r, v\}$. The positive integer $N = N(\Gamma; \zeta)$ is called the *width* of Γ at the cusp $\zeta = A^{-1}\infty$. The cusp parameter $\kappa = \kappa(\Gamma; \zeta)$ is defined by

$$e^{2\pi i\kappa} = v(A^{-1}U^NA), \qquad 0 \leq \kappa < 1.$$

If $b_{m_0} \neq 0$, then we write

$$\operatorname{Ord}_\Gamma(f; \zeta) = m_0 + \kappa,$$

which is called the *order* of f at ζ *with respect to* Γ. We also write

$$\operatorname{ord}(f; \zeta) = \frac{m_0 + \kappa}{N},$$

which is called the *invariant order* of f at ζ. For each $z \in \mathscr{H}$, $\operatorname{ord}(f; z)$ denotes the order of f at z, as an analytic function of z. The order of f with respect to Γ is defined by

$$\operatorname{Ord}_\Gamma(f; z) = \frac{1}{\ell}\operatorname{ord}(f; z),$$

where $\ell \in \{1, 2, 3\}$ is the order of z as a fixed point of Γ.

We are now in a position to state the valence formula (see Rankin's book [2, Theorem 4.1.4, p. 98]), which is the most important fact for us as we employ modular forms to establish theta-function identities. If $f \in \{\Gamma, r, v\}$ and $\mathscr{F}$ is any fundamental set for Γ, then, provided that f is not constant,

$$\sum_{z \in \mathscr{F}} \operatorname{Ord}_\Gamma(f; z) = r\rho_\Gamma, \tag{0.5}$$

where

$$\rho_\Gamma = \tfrac{1}{12}(\Gamma(1) \colon \Gamma). \tag{0.6}$$

If $f \in \{\Gamma_1, r_1, v_1\}$ and $g \in \{\Gamma_2, r_2, v_2\}$, then $fg \in \{\Gamma_1 \cap \Gamma_2, r_1 + r_2, v_1 v_2\}$ and $f/g \in \{\Gamma_1 \cap \Gamma_2, r_1 - r_2, v_1/v_2\}$. Observe that $f + g$ is a modular form only if $r_1 = r_2$, in which case it is a form on the subgroup $\Gamma \subseteq \Gamma_1 \cap \Gamma_2$ determined by $v_1 = v_2$.

If $f \in \{\Gamma, r, v\}$ and $M \in M_2^+(\mathbb{Z})$, then by (0.2),

$$f|M|V = \frac{\sigma(MVM^{-1}, M)}{\sigma(M, V)} f|MVM^{-1}|M.$$

Thus, $f|M$ is a modular form on $M^{-1}\Gamma M$ with multiplier system $v|M$ defined by

$$(v|M)(V) = \frac{\sigma(MVM^{-1}, M)}{\sigma(M, V)} v(MVM^{-1}). \tag{0.7}$$

Let $M = \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $MVM^{-1} = {}^{(n)}V$, in the notation (0.3). If $f \in \{\Gamma, r, v\}$, then $f(n\tau) = n^{-r/2} f|n$ has multiplier system $v|n(V) = v({}^{(n)}V)$, by (0.7), since $\sigma(M, *) = 1$. Thus, $f(\tau)$ and $f(n\tau)$ are modular forms on $\Gamma \cap M^{-1}\Gamma M$. In particular, if $\Gamma = \Gamma(1)$, then $\Gamma(1) \cap M^{-1}\Gamma(1)M = \Gamma_0(n)$, and $f(\tau)$ and $f(n\tau)$ are modular forms on $\Gamma_0(n)$.

We shall need to determine the multiplier systems for the various theta-functions that appear in the identities to be proved. To determine these multiplier systems, it will be convenient to introduce some notation. Let $\operatorname{neg}(c) = 1$ or 0 according as $c < 0$ or not, respectively. We define a pair of symbols closely related to the Legendre–Jacobi symbol $\left(\frac{d}{c}\right)$. Let $(c, d) = 1$ with c odd. Define

$$\left(\frac{d}{c}\right)^* = \left(\frac{d}{|c|}\right), \tag{0.8}$$

and for $d \neq 0$,

$$\left(\frac{d}{c}\right)_* = (-1)^{\operatorname{neg}(c)\operatorname{neg}(d)}\left(\frac{d}{|c|}\right). \tag{0.9}$$

Finally, define

$$\left(\frac{0}{-1}\right)_* = -1. \tag{0.10}$$

These symbols possess properties analogous to those of the Legendre–Jacobi symbol, and we summarize those that are needed in the sequel. If c and d are odd, with $(c, d) = 1$, then

$$\left(\frac{c}{d}\right)_* = (-1)^{(c-1)(d-1)/4}\left(\frac{d}{c}\right)^*. \tag{0.11}$$

Also,

$$\left(\frac{c_1 c_2}{d}\right)^* = \left(\frac{c_1}{d}\right)^*\left(\frac{c_2}{d}\right)^*$$

and

$$\left(\frac{c}{d_1 d_2}\right)^* = \left(\frac{c}{d_1}\right)^*\left(\frac{c}{d_2}\right)^*.$$

Similar properties hold for $(\frac{c}{d})_*$.

Recall that the Dedekind eta-function $\eta(\tau)$, $\tau \in \mathscr{H}$, is defined by $\eta(\tau) = e^{\pi i\tau/12}(q^2; q^2)_\infty$, where $q = e^{\pi i\tau}$. (See Section 22 of Chapter 16.) Now define

$$\begin{gathered} f_0(\tau) = \eta\left(\frac{\tau}{2}\right), \qquad f_1(\tau) = e^{-\pi i/24}\eta\left(\frac{\tau+1}{2}\right), \qquad f_2(\tau) = \eta(2\tau), \\ g_0(\tau) = \frac{\eta^2\left(\frac{\tau}{2}\right)}{\eta(\tau)}, \qquad g_1(\tau) = \frac{\eta^2\left(\frac{\tau+1}{2}\right)}{\eta(\tau+1)}, \qquad g_2(\tau) = \frac{\eta^2(2\tau)}{\eta(\tau)}, \\ h_0(\tau) = \frac{\eta^2(\tau)}{\eta\left(\frac{\tau}{2}\right)}, \qquad h_1(\tau) = e^{-\pi i/8}\frac{\eta^2(\tau+1)}{\eta\left(\frac{\tau+1}{2}\right)}, \qquad h_2(\tau) = \frac{\eta^2(\tau)}{\eta(2\tau)}. \end{gathered} \tag{0.12}$$

Note that if τ is replaced by $\tau \pm 1$, then q is replaced by $-q$, and if 2τ is substituted for τ, q is supplanted by q^2. Thus, from Entry 22 of Chapter 16, we obtain the relations

$$\begin{gathered} q^{1/24}f(-q) = f_0(\tau), \qquad q^{1/24}f(q) = f_1(\tau), \qquad q^{1/12}f(-q^2) = \eta(\tau), \\ \varphi(-q) = g_0(\tau), \qquad \varphi(q) = g_1(\tau), \qquad q^{1/4}\psi(q^2) = g_2(\tau), \\ q^{1/8}\psi(q) = h_0(\tau), \qquad q^{1/8}\psi(-q) = h_1(\tau), \qquad \varphi(-q^2) = h_2(\tau). \end{gathered} \tag{0.13}$$

We shall need to determine the multiplier systems for f_j, g_j, and h_j, $0 \le j \le 2$. From the definitions of these functions, it is clear that we should employ the multiplier system of $\eta(\tau)$. From either the books of Knopp [1, p. 51] or Rademacher [1, p. 163], $\eta(\tau)$ is a modular form of weight $\frac{1}{2}$ on the full modular group $\Gamma(1)$ with multiplier system v_η given by

$$v_\eta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{d}{c}\right)^* e^{2\pi i\{-3c - bd(c^2-1) + c(a+d)\}/24}, & \text{if } c \text{ is odd}, \\ \left(\frac{c}{d}\right)_* e^{2\pi i\{3(d-1) - ac(d^2-1) + d(b-c)\}/24}, & \text{if } d \text{ is odd}. \end{cases} \tag{0.14}$$

(Recall that the definitions of $(\frac{c}{d})^*$ and $(\frac{c}{d})_*$ are given by (0.8)–(0.10).) Setting

$M_0 = \left(\begin{smallmatrix}1 & 0\\ 0 & 2\end{smallmatrix}\right)$, $M_1 = \left(\begin{smallmatrix}1 & 1\\ 0 & 2\end{smallmatrix}\right)$, and $M_2 = \left(\begin{smallmatrix}2 & 0\\ 0 & 1\end{smallmatrix}\right)$, we may readily verify that f_j, g_j, and h_j are modular forms of weight $\frac{1}{2}$ on $\Gamma_j = \Gamma(1) \cap M_j^{-1}\Gamma(1)M_j$, $0 \le j \le 2$. Note that $\Gamma_0 = \Gamma^0(2)$, $\Gamma_1 = \Gamma_\theta$, and $\Gamma_2 = \Gamma_0(2)$. As observed earlier, $\Gamma_0 \cap \Gamma_1 \cap \Gamma_2 = \Gamma(2)$. Thus, all nine functions are modular forms of weight $\frac{1}{2}$ on $\Gamma(2)$. For $A = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma(2)$, their multiplier systems are given by

$$v_{f_0}(A) = \left(\frac{2c}{d}\right)_* e^{\pi i(d-1)/4} e^{2\pi i\{ac(d^2-1)+(b-4c)d/2\}/24}, \tag{0.15}$$

$$v_{f_1}(A) = \left(\frac{c}{d}\right)_* e^{\pi i(d-1)/4} e^{2\pi i\{ac(d^2-1)+(b-c)d/2\}/24}, \tag{0.16}$$

$$v_{f_2}(A) = \left(\frac{2c}{d}\right)_* e^{\pi i(d-1)/4} e^{2\pi i\{ac(d^2-1)+(4b-c)d/2\}/24}, \tag{0.17}$$

$$v_{g_0}(A) = \left(\frac{c}{d}\right)_* e^{\pi i(d-1)/4} e^{-\pi icd/4}, \tag{0.18}$$

$$v_{g_1}(A) = \left(\frac{c}{d}\right)_* e^{\pi i(d-1)/4}, \tag{0.19}$$

$$v_{g_2}(A) = \left(\frac{c}{d}\right)_* e^{\pi i(d-1)/4} e^{\pi ibd/4}, \tag{0.20}$$

$$v_{h_0}(A) = \left(\frac{2c}{d}\right)_* e^{\pi i(d-1)/4} e^{\pi ibd/8}, \tag{0.21}$$

$$v_{h_1}(A) = \left(\frac{c}{d}\right)_* e^{\pi i(d-1)/4} e^{\pi i(b-c)d/8}, \tag{0.22}$$

and

$$v_{h_2}(A) = \left(\frac{2c}{d}\right)_* e^{\pi i(d-1)/4} e^{-\pi icd/8}. \tag{0.23}$$

To prove these formulas, we first note that

$$\eta|2|A = \eta|^{(2)}A|2 \quad \text{and} \quad \eta\left|\frac{1}{2}\right|A = \eta\left|\begin{pmatrix}a & b/2\\ 2c & d\end{pmatrix}\right|\frac{1}{2}.$$

Thus, (0.15) and (0.17) follow upon observing that $3(d^2-1)$ is a multiple of 24. From the identity

$$(q;q)_\infty(q^4;q^4)_\infty(-q;-q)_\infty = (q^2;q^2)_\infty^3,$$

we deduce that

$$\eta\left(\frac{\tau}{2}\right)\eta(2\tau)\eta\left(\frac{\tau+1}{2}\right) = e^{\pi i/24}\eta^3(\tau).$$

Thus,

$$v_{f_1} = \frac{v_\eta^3}{v_{f_0}v_{f_2}},$$

and (0.16) easily follows. Since $g_0(\tau) = f_0^2(\tau)/\eta(\tau)$, $h_0(\tau) = \eta^2(\tau)/f_0(\tau)$, and so on, the multiplier systems for g_j and h_j, $0 \le j \le 2$, can readily be derived from those of f_j and η. Hence, (0.18)–(0.23) readily follow.

Each of the nine multiplier systems may be written in the form $v_F = \xi_0\xi_1\xi_2$, where

$$\xi_0 = \left(\frac{c}{d}\right)_* \quad \text{or} \quad \left(\frac{2c}{d}\right)_*,$$

$$\xi_1 = e^{\pi i(d-1)/4},$$

and

$$\xi_2 = e^{2\pi i\Phi(a,b,c,d)/48},$$

where Φ is a polynomial in a, b, c, and d with integral coefficients.

If $F(\tau)$ is any of the functions f_j, g_j, or h_j, $0 \le j \le 2$, then, when n is odd, $F(n\tau)$ is a modular form on

$$\Gamma(2) \cap \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma(2) \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} = \Gamma(2) \cap \Gamma_0(n),$$

as seen after (0.7). Since $(2, n) = 1$, the index is multiplicative (Rankin [2, Theorems 1.4.2, 1.4.3 ff]), and so

$$(\Gamma(1)\colon \Gamma(2) \cap \Gamma_0(n)) = 6n \prod_{p|n} \left(1 + \frac{1}{p}\right). \tag{0.24}$$

Also, from discourse after (0.7), the multiplier system of $F(n\tau)$ is $(v_F|n)(A) = v_F({}^{(n)}A)$. If $(6, n) = 1$, then

$$\frac{c}{n} \equiv nc \pmod{48},$$

and since every (nontrivial) term of Φ has exactly one factor of b or c, it follows that

$$(v_F|n)(A) = \left(\frac{n}{d}\right)_* \xi_0\xi_1\xi_2^n. \tag{0.25}$$

By (0.25), the multiplier system of $F(n\tau)/F(\tau)$ is equal to

$$\frac{(v_F|n)(A)}{v_F(A)} = \left(\frac{n}{d}\right)_* \xi_2^{n-1}, \tag{0.26}$$

and the multiplier system for $F(\tau)F(n\tau)$ equals

$$v_F(A)(v_F|n)(A) = \left(\frac{n}{d}\right)_* \xi_1^2\xi_2^{n+1}. \tag{0.27}$$

We now offer a few remarks on orders and poles. The following lemma allows us to calculate the orders at the cusps of a transformed function $f|M$. The conditions permit ∞ as a cusp, in the form $r/s = 1/0$.

Lemma 0.1. *Suppose r and s are relatively prime integers, $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2^+(\mathbb{Z})$, $m = ad - bc$, and $g = (ar + bs, cr + ds)$. If f is a modular form, then*

$$\operatorname{ord}\left(f|M; \frac{r}{s}\right) = \frac{g^2}{m}\operatorname{ord}\left(f; M\left(\frac{r}{s}\right)\right).$$

Proof. If $A \in \Gamma(1)$, then from the definition of *order*,

$$\operatorname{ord}(f; A^{-1}\infty) = \operatorname{ord}(f|A^{-1}; \infty).$$

If $M = \left(\begin{smallmatrix} g & h \\ 0 & k \end{smallmatrix}\right)$, then $M\tau$ takes $e^{2\pi i\tau}$ into $e^{2\pi ih/k}e^{2\pi ig\tau/k}$. Thus,

$$\operatorname{ord}(f|M; \infty) = \frac{g}{k}\operatorname{ord}(f; \infty). \tag{0.28}$$

More generally, if $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we choose $A = \left(\begin{smallmatrix} \alpha & \beta \\ -s & r \end{smallmatrix}\right) \in \Gamma(1)$ so that $A^{-1}\infty = r/s$. Then

$$\operatorname{ord}\left(f|M; \frac{r}{s}\right) = \operatorname{ord}(f|MA^{-1}; \infty). \tag{0.29}$$

Observe that the first column of MA^{-1} is $\left(\begin{smallmatrix} ar+bs \\ cr+ds \end{smallmatrix}\right)$. Thus, there exists $B \in \Gamma(1)$ so that, for some h,

$$B^{-1}MA^{-1} = \begin{pmatrix} g & h \\ 0 & m/g \end{pmatrix} =: M_1.$$

Hence, by (0.28),

$$\begin{aligned}
\operatorname{ord}(f|MA^{-1}; \infty) &= \operatorname{ord}(f|BM_1; \infty) = \operatorname{ord}(f|B|M_1; \infty) \\
&= \frac{g^2}{m}\operatorname{ord}(f|B; \infty) = \frac{g^2}{m}\operatorname{ord}(f; B\infty) \\
&= \frac{g^2}{m}\operatorname{ord}(f; BM_1\infty) = \frac{g^2}{m}\operatorname{ord}(f; MA^{-1}\infty) \\
&= \frac{g^2}{m}\operatorname{ord}\left(f; M\left(\frac{r}{s}\right)\right),
\end{aligned}$$

which, by (0.29), completes the proof.

Applying Lemma 0.1 to $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and recalling (e.g., see Rankin's book [2, Theorem 4.1.2(i)]) that $\operatorname{ord}(\eta; \zeta) = \frac{1}{24}$ for each $\zeta \in Q \cup \{\infty\}$, we obtain the following expressions for $\operatorname{ord}(F; r/s)$:

Table 1

F	$48\operatorname{ord}(F; r/s)$	F	$24\operatorname{ord}(F; r/s)$	F	$48\operatorname{ord}(F; r/s)$
f_0	$(r, 2)^2$	g_0	$(r, 2)^2 - 1$	h_0	$4 - (r, 2)^2$
f_1	$(r + s, 2)^2$	g_1	$(r + s, 2)^2 - 1$	h_1	$4 - (r + s, 2)^2$
f_2	$(s, 2)^2$	g_2	$(s, 2)^2 - 1$	h_2	$4 - (s, 2)^2$

Thus, for each of the nine functions, $\operatorname{ord}(F; r/s) \geq 0$ for every $r/s \in Q \cup \{\infty\}$.

From the definition of η, it follows that $\eta(\tau)$ has no zeros or poles on $\mathscr{H}$. This and the conclusion above show that if $F = f_j$, g_j, or h_j, $0 \le j \le 2$, and $M \in M_2^+(\mathbb{Z})$,

$$\operatorname{Ord}_\Gamma(F|M; z) \ge 0, \qquad z \in \mathscr{H} \cup Q \cup \{\infty\},$$

for any group Γ on which $F|M$ is a modular form.

Suppose now that F is a modular form of weight r without poles on $\Gamma = \Gamma(2) \cap \Gamma_0(n)$, for some odd integer $n \ge 3$. If the coefficients of $q^0, q^1, \ldots, q^\mu$ in F are equal to zero, then, since the width of Γ at ∞ is 2, it follows that $\operatorname{Ord}_\Gamma(F; \infty) \ge \mu + 1$. Suppose further that $\mu + 1 > r\rho_\Gamma$. Then

$$\sum_{z \in \mathscr{F}} \operatorname{Ord}_\Gamma(F; z) \ge \operatorname{Ord}_\Gamma(F; \infty) \ge \mu + 1 > r\rho_\Gamma. \tag{0.30}$$

We conclude that $F \equiv 0$, for otherwise we have a contradiction to the valence formula (0.5). In our applications, generally F will be a linear combination of products of functions f_j, g_j, and h_j, $0 \le j \le 2$.

We now obtain what is known as the reciprocal relation by applying a stroke operator and show how this helps in reducing the amount of computation needed in proving a modular equation.

Let n be odd. Choose $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ such that $d \equiv 0 \pmod{n}$, and set $M = B\left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In the present situation, we assume $B \in \Gamma(2)$, and, to aid in computation, we require that $b \equiv c \equiv 0 \pmod{16}$, $\left(\frac{c}{n}\right) = 1$, $d \equiv 1 \pmod{8}$, and $ac \equiv cd \equiv bd = 0 \pmod{3}$.

For each modular form F of weight $\frac{1}{2}$ on $\Gamma(2)$ with multiplier system v_F,

$$F(\tau)|M = v_F(B)F|n = n^{1/4}v_F(B)F(n\tau) \tag{0.31}$$

and

$$\begin{aligned} F(n\tau)|M = n^{-1/4}F|n|M &= n^{-1/4}F\left|\begin{matrix} an & b \\ c & d/n \end{matrix}\right|\begin{matrix} n & 0 \\ 0 & n \end{matrix} \\ &= n^{-1/4}v_F\begin{pmatrix} an & b \\ c & d/n \end{pmatrix}F(\tau). \end{aligned} \tag{0.32}$$

If $F = \eta$, f_j, g_j, or h_j, $0 \le j \le 2$, then the conditions that we imposed on B imply that $v_F(B) = \left(\frac{c}{d}\right)_*$ and also that

$$v_F\begin{pmatrix} an & b \\ c & d/n \end{pmatrix} = \left(\frac{c_0}{d/n}\right)_* e^{\pi i(d/n-1)/4} = \left(\frac{c_0}{n}\right)_* \left(\frac{c}{d}\right)_* e^{\pi i(n-1)/4},$$

where $c_0 = c$ or $2c$ and where we have used the facts that $\left(\frac{2}{d}\right) = 1$ and $d/n \equiv n \pmod{8}$. Hence, by (0.31) and (0.32),

$$F(\tau)|M = n^{1/4}\left(\frac{c}{d}\right)_* F(n\tau), \quad \text{if } F = \eta, f_j, g_j, h_j, 0 \le j \le 2, \tag{0.33}$$

and

$$F(n\tau)|M = \begin{cases} n^{-1/4}\left(\dfrac{c}{d}\right)_* e^{\pi i(n-1)/4}F(\tau), & \text{if } F = \eta, f_1, g_0, g_1, g_2, h_1, \\ n^{-1/4}\left(\dfrac{c}{d}\right)_*\left(\dfrac{2}{n}\right)_* e^{\pi i(n-1)/4}F(\tau), & \text{if } F = f_0, f_2, h_0, h_2, \end{cases} \tag{0.34}$$

since $(\frac{c}{n}) = 1$. Combining these, we deduce that

$$F(\tau)F(n\tau)|M = \begin{cases} e^{\pi i(n-1)/4}F(\tau)F(n\tau), & \text{if } F = \eta, f_1, g_0, g_1, g_2, h_1, \\ \left(\dfrac{2}{n}\right)e^{\pi i(n-1)/4}F(\tau)F(n\tau), & \text{if } F = f_0, f_2, h_0, h_2, \end{cases} \tag{0.35}$$

and

$$\frac{F(n\tau)}{F(\tau)}\bigg| M = \begin{cases} n^{-1/2}e^{\pi i(n-1)/4}\dfrac{F(\tau)}{F(n\tau)}, & \text{if } F = \eta, f_1, g_0, g_1, g_2, h_1, \\ n^{-1/2}\left(\dfrac{2}{n}\right)e^{\pi i(n-1)/4}\dfrac{F(\tau)}{F(n\tau)}, & \text{if } F = f_0, f_2, h_0, h_2. \end{cases} \tag{0.36}$$

For $n = 11$, 19, we shall also be considering $g_j(2\tau)$ and $g_j(2n\tau)$, $j = 1, 2$, in the sequel. First, by (0.33),

$$\begin{aligned} (g_j|2(\tau))|M &= g_j\begin{vmatrix} a & 2b \\ c/2 & d \end{vmatrix} n \bigg| 2 \\ &= n^{1/4}\left(\frac{c/2}{d}\right)_* g_j|2(n\tau). \end{aligned}$$

Second, by (0.34),

$$\begin{aligned} (g_j|2(n\tau))|M &= n^{-1/4}g_j|2|n|B|n \\ &= n^{-1/4}g_j\bigg|2\begin{vmatrix} na & b \\ c & d/n \end{vmatrix}\begin{matrix} 1 \\ \overline{n} \end{matrix}\bigg| n \\ &= n^{-1/4}g_j\begin{vmatrix} na & 2b \\ c/2 & d/n \end{vmatrix} 2 \\ &= n^{-1/4}\left(\frac{c/2}{d/n}\right)_* e^{\pi i(n-1)/4}g_j(\tau)|2. \end{aligned}$$

Hence, if $n = 11$ or 19 and $j = 1$ or 2,

$$(g_j|2(\tau))\cdot g_j|2(n\tau))|M = -ig_j|2(\tau)g_j|2(n\tau). \tag{0.37}$$

Applying this operator M to an equality involving modular forms yields a new equality that is either valid or invalid together with the original. This new equality can also be obtained from the theory of modular equations and is known as the reciprocal relation. (See Section 24(*v*) of Chapter 18.)

Lemma 0.2. *Let F and M be as above, and let* $\Gamma = \Gamma(2) \cap \Gamma_0(n)$. *Then*

(i) $\dfrac{a}{c} = M(\infty)$ *is a cusp that is not equivalent to* ∞ *modulo* Γ;

(ii) $\operatorname{Ord}_\Gamma(F|M;\infty) = \operatorname{Ord}_\Gamma\left(F;\dfrac{a}{c}\right)$.

PROOF OF (i). If $A \in \Gamma$ and $A\infty = r/s$, then $n|s$. Since $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$ and $n|d$, then $n \nmid c$. Hence, $r/s \neq a/c$, and the proof of (i) is complete.

PROOF OF (ii). Recall that $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. If $k \in \mathbb{Z}^+$,

$$BU^kB^{-1} = \begin{pmatrix} 1 - ack & ka^2 \\ -c^2k & 1 + ack \end{pmatrix} \in \Gamma$$

if and only if $2n|k$. Thus, $N(\Gamma; a/c) = 2n$. Since $M = \left(\begin{smallmatrix} na & b \\ nc & d \end{smallmatrix}\right)$, an application of Lemma 0.1 with $r/s = 1/0$ and $g = n$ yields

$$\begin{aligned}\operatorname{Ord}_\Gamma(F|M;\infty) &= N(\Gamma;\infty)\operatorname{ord}(F|M;\infty)\\ &= 2\frac{n^2}{n}\operatorname{ord}\left(F;\frac{a}{c}\right)\\ &= N\left(\Gamma;\frac{a}{c}\right)\operatorname{ord}\left(F;\frac{a}{c}\right)\\ &= \operatorname{Ord}_\Gamma\left(F;\frac{a}{c}\right).\end{aligned}$$

This completes the proof of (ii).

As before, suppose that F is a modular form of weight r without poles on $\Gamma = \Gamma(2) \cap \Gamma_0(n)$, where n is an odd integer, $n \geq 3$. If $\operatorname{Ord}_\Gamma(F; a/c), \operatorname{Ord}_\Gamma(F;\infty) \geq \mu + 1 > \frac{1}{2}r\rho_\Gamma$, then

$$\begin{aligned}\sum_{z \in \mathscr{F}} \operatorname{Ord}_\Gamma(F;z) &\geq \operatorname{Ord}_\Gamma(F;\infty) + \operatorname{Ord}_\Gamma\left(F;\frac{a}{c}\right)\\ &\geq 2\mu + 2 > r\rho_\Gamma. \end{aligned} \tag{0.38}$$

By the valence formula (0.5), this is a contradiction unless $F \equiv 0$. In conclusion, by Lemma 0.2 and (0.38), we only need to show that the coefficients of $q^0, q^1, \ldots, q^\mu$ are equal to 0 in both F and $F|M$ to conclude that $F \equiv 0$.

In summary, the number of terms that must be computed without using the reciprocal relation is generally equal to the total number of terms that must be computed using the reciprocal relation. But in the latter approach, half are in one identity and half are in the reciprocal identity. This produces a considerable savings of time, as lower order terms are much easier to compute than higher order terms. Moreover, if the modular equation, or theta-function equivalent, is self-reciprocal, then the number of terms to be computed is actually halved.

Another approach, based on the theory of modular forms, to proving certain identities in Chapters 19 and 20 has been devised by R. J. Evans [1]. His elegant methods are more analytical and less computational than the methods described in the preceding pages. Instead of working on the subgroup $\Gamma_0(n)$, Evans employs $\Gamma^0(n)$. Evans' ideas are especially valuable in proving identities in which the quotients

$$G_m(z) := G_{m,p}(z) := (-1)^m q^{m(3m-p)/(2p^2)} \frac{f(-q^{2m/p}, -q^{1-2m/p})}{f(-q^{m/p}, -q^{1-m/p})} \tag{0.39}$$

appear, where m is a positive integer, p is an odd positive integer (usually a prime), and $q = e^{2\pi i z}$.

Let $z \in \mathscr{H}$ and $\gamma \in \mathscr{C}$. The fundamental function employed by Evans is the classical theta-function

$$\begin{aligned} \vartheta_1(\gamma, z) &= \sum_{n=-\infty}^{\infty} \exp(\pi i z(n + \tfrac{1}{2})^2 + 2\pi i(n + \tfrac{1}{2})(\gamma - \tfrac{1}{2})) \\ &= -ie^{\pi i(\gamma + z/4)}(e^{2\pi i\gamma}q)_\infty(e^{-2\pi i\gamma})_\infty(q)_\infty, \end{aligned} \tag{0.40}$$

by the Jacobi triple product identity.

So that we may eventually relate ϑ_1 to modular forms in z of arbitrary weight, following H. M. Stark [1, Eq. (10)], we set, for $u, v \in \mathscr{C}$ and $z \in \mathscr{H}$,

$$\varphi(u, v; z) = e^{\pi i u(uz+v)} \frac{\vartheta_1(uz + v, z)}{\eta(z)}.$$

The function φ is analytic in each variable, for $z \in \mathscr{H}$ and $u, v \in \mathscr{C}$. Furthermore, put

$$F(u, v; z) := \eta(z)\varphi(u, v; z), \tag{0.41}$$

and, when $v = 0$,

$$F(u; z) := F(u, 0; z) = \eta(z)\varphi(u, 0; z). \tag{0.42}$$

Combining (0.40), (0.41), and (0.42), we find that

$$\begin{aligned} F(u, v; z) &= e^{\pi i u(uz+v)}\vartheta_1(uz + v, z) \\ &= -i \sum_{n=-\infty}^{\infty} (-1)^n \exp(\pi i z(n + u + \tfrac{1}{2})^2 + \pi i v(2n + u + 1)) \\ &= -ie^{\pi i\{z(u+1/2)^2 + v(u+1)\}}(e^{2\pi i v}q^{u+1})_\infty(e^{-2\pi i v}q^{-u})_\infty(q)_\infty. \end{aligned} \tag{0.43}$$

In particular, for $v = 0$,

$$\begin{aligned} F(u; z) = e^{\pi i u^2 z}\vartheta_1(uz, z) &= -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+u+1/2)^2/2} \\ &= -iq^{(u+1/2)^2/2}(q^{u+1})_\infty(q^{-u})_\infty(q)_\infty. \end{aligned} \tag{0.44}$$

From the series in (0.43), it is easily seen that, for arbitrary integers r and s,

$$F(u+r, v+s; z) = (-e^{\pi i u})^s(-e^{-\pi i v})^r(-1)^{rs}F(u, v; z) \tag{0.45}$$

and that

$$F(-u, -v; z) = -F(u, v; z).$$

In particular, when $v = 0$,

$$F(u+1; z) = -F(u; z) \tag{0.46}$$

and

$$F(-u; z) = -F(u; z). \tag{0.47}$$

By (0.44), for fixed $z \in \mathscr{H}$, the zeros of $F(u; z)$ are the points u in the lattice $\mathbb{Z} + \mathbb{Z}z^{-1}$, and these zeros are simple. Thus, $F(2u; z)/F(u; z)$ is an entire function of u. The following lemma shows, in fact, that $F(2u; z)/F(u; z)$ is a linear combination of $F(\frac{1}{3} + u; 3z)$ and $F(\frac{1}{3} - u; 3z)$.

Lemma 0.3. *For $z \in \mathscr{H}$ and $u \in \mathscr{C}$,*

$$i\eta(z)\frac{F(2u; z)}{F(u; z)} = F(\tfrac{1}{3} + u; 3z) + F(\tfrac{1}{3} - u; 3z). \tag{0.48}$$

PROOF. Replace all the functions in (0.48) by their triple products, found in (0.44) above and Entry 22(iii) of Chapter 16. Setting $a = q^u$ and simplifying, we find that

$$\begin{aligned}(-aq; q)_\infty(-a^{-1}; q)_\infty(a^2q; q^2)_\infty(a^{-2}q; q^2)_\infty(q; q)_\infty \\ = a^{-1}(a^3q; q^3)_\infty(a^{-3}q^2; q^3)_\infty(q^3; q^3)_\infty \\ + (a^3q^2; q^3)_\infty(a^{-3}q; q^3)_\infty(q^3; q^3)_\infty.\end{aligned} \tag{0.49}$$

However, (0.49) is just a version of the quintuple product identity. To see this, apply the Jacobi triple product identity to each theta-function on the left side of (38.9) in Chapter 16 and then replace z by $-a$ in (38.9). This completes the proof.

We focus on the quotients $F(2u; z)/F(u; z)$ when u is a rational number m/p, with p odd. Thus, if m and p are integers with p odd and $p > 1$, define

$$G(m; z) := (-1)^m \frac{F(2m/p; z)}{F(m/p; z)}. \tag{0.50}$$

By (0.46), for fixed p and z, $G(m; z)$ depends only on the residue class of m (mod p), since p is odd. By (0.47),

$$G(m; z) = G(-m; z) = G(p-m; z). \tag{0.51}$$

By the product representation of $F(u; z)$ in (0.44), $G(0; z) = 2$, and so

$$G(m; z) = 2, \quad \text{if } p|m. \tag{0.52}$$

By the Jacobi triple product identity, (0.44), and (0.47),

$$f(-q^u, -q^{1-u}) = -iq^{-(u-1/2)^2/2}F(u; z).$$

Thus,

$$\frac{F(2u; z)}{F(u; z)} = q^{u(3u-1)/2}\frac{f(-q^{2u}, -q^{1-2u})}{f(-q^u, -q^{1-u})},$$

and so

$$G(m; pz) = (-1)^m q^{m(3m-p)/(2p)}\frac{f(-q^{2m}, -q^{p-2m})}{f(-q^m, -q^{p-m})}. \tag{0.53}$$

This shows that

$$G_m(z) = G(m; z),$$

where $G_m(z)$ is given by (0.39) and $G(m; z)$ is given by (0.50).

We need one additional fact before we prove some theorems about $G_m(z)$. From (0.41), (0.14), and Stark's work [1, Eq. (17)],

$$F(u, v; Vz) = v_\eta^3(V)\sqrt{cz + d}\, F(u_V, v_V; z), \tag{0.54}$$

where $u, v \in \mathscr{C}$, $z \in \mathscr{H}$, $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$, and (u_V, v_V) is a row vector defined by

$$(u_V, v_V) = (u, v)V = (au + cv, bu + dv). \tag{0.55}$$

Theorem 0.4. *Let p be an odd integer exceeding* 1, *and let ε_r, β_r $(1 \le r \le s)$ be nonzero integers with*

$$\varepsilon_1\beta_1^2 + \cdots + \varepsilon_s\beta_s^2 \equiv 0 \pmod{p}. \tag{0.56}$$

Then

$$g(z) := \sum_m \prod_r G(m\beta_r; z)^{\varepsilon_r} \in \{\Gamma^0(p), 0, 1\}, \tag{0.57}$$

where the sum is over all m (mod p), *and the product is over all r with* $1 \le r \le s$. *Moreover, $g(z)$ has no poles on $\mathscr{H}$ or at the cusp* 0.

Proof. Let $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^0(p)$. We first prove that g satisfies the transformation formula

$$g(Vz) = g(z), \qquad z \in \mathscr{H}, \tag{0.58}$$

of a modular function. By (0.42), (0.50), and (0.57),

$$g(Vz) = \sum_m \prod_r \left\{(-1)^{m\beta_r}\frac{F(2m\beta_r/p, 0; Vz)}{F(m\beta_r/p, 0; Vz)}\right\}^{\varepsilon_r}. \tag{0.59}$$

By (0.52), the expression within braces in (0.59) is to be interpreted as 2 when $p|m\beta_r$. Applying the transformation (0.54), with the notation (0.55), we find that

$$g(Vz) = \sum_m \prod_r \left\{(-1)^{m\beta_r} \frac{F(2m\beta_r a/p, 2m\beta_r b/p; z)}{F(m\beta_r a/p, m\beta_r b/p; z)}\right\}^{\varepsilon_r}. \tag{0.60}$$

Since $p|b$, $m\beta_r b/p$ is an integer. Thus, by (0.60) and (0.45), with $r = 0 = v$,

$$g(Vz) = \sum_m \prod_r \left\{E_r(m)(-1)^{m\beta_r a} \frac{F(2m\beta_r a/p, 0; z)}{F(m\beta_r a/p, 0; z)}\right\}^{\varepsilon_r}, \tag{0.61}$$

where

$$E_r(m) = (-1)^{m\beta_r(a+1+b/p)} e^{3\pi iabm^2\beta_r^2/p^2}. \tag{0.62}$$

Using (0.50), we may rewrite (0.61) as

$$g(Vz) = \sum_m \prod_r \{E_r(m)G(m\beta_r a; z)\}^{\varepsilon_r}. \tag{0.63}$$

Now, by (0.62),

$$\prod_r E_r(m)^{\varepsilon_r} = \exp\left(\pi im(a + 1 + b/p) \sum_r \varepsilon_r\beta_r + \frac{3\pi iabm^2}{p^2} \sum_r \varepsilon_r\beta_r^2\right). \tag{0.64}$$

The sums $\sum_r \varepsilon_r\beta_r$ and $\sum_r \varepsilon_r\beta_r^2$ clearly have the same parity, and the latter sum is a multiple of p by (0.56). Thus, if a is odd, the right side of (0.64) equals 1. If a is even, then b is odd because $ad - bc = 1$, and so again the right side of (0.64) equals 1. Thus, (0.63) reduces to the equality

$$g(Vz) = \sum_m \prod_r G(m\beta_r a; z)^{\varepsilon_r}. \tag{0.65}$$

Since $ad - bc = 1$ and $p|b$, $(a, p) = 1$. Thus, am runs through a complete residue system (mod p) when m does. Thus, (0.58) follows from (0.65).

If $m/p \in \mathbb{Z}$, then $G(m; z) = 2$ for all $z \in \mathscr{H}$, by (0.52). If $m/p \notin \mathbb{Z}$, then since m/p is not half of an integer, both $F(2m/p; z)$ and $F(m/p; z)$ are analytic functions of z on $\mathscr{H}$ that never vanish on $\mathscr{H}$ by (0.44). It follows from the definition of g in (0.57), that $g(z)$ is analytic on $\mathscr{H}$. It remains to show that $g(z)$ is meromorphic at every cusp $L\infty$ ($L \in \Gamma(1)$) with no pole at the cusp 0.

By (0.50), (0.54), and (0.55), for each integer m and $L = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma(1)$,

$$G(m; Lz) = (-1)^m \frac{F(2m\alpha/p, 2m\beta/p; z)}{F(m\alpha/p, m\beta/p; z)}, \tag{0.66}$$

where, by (0.52), the right side of (0.66) is to be interpreted as 2 if $p|m$. By (0.43) and (0.66), for each pair of integers m, ε, $G(m; Lz)^\varepsilon$ has a Fourier expansion of the form

$$G(m; Lz)^\varepsilon = \sum_{n=N}^{\infty} a_k e^{2\pi izn/p}, \qquad a_n \in \mathscr{C},$$

where N is finite. Thus, $g(Lz)$ also has a Fourier expansion of this form and therefore is meromorphic at every cusp. This completes the proof that $g(z) \in \{\Gamma^0(p), 0, 1\}$. It remains to show that $g(z)$ has no pole at the cusp 0.

To prove this, we show that, for each integer m, $G_m(-1/z)$ has a Fourier expansion of the form

$$G_m(-1/z) = \sum_{n=0}^{\infty} c_n q^n, \tag{0.67}$$

where

$$c_0 = 2(-1)^m \cos(\pi m/p) \neq 0. \tag{0.68}$$

If $p|m$, then (0.67) and (0.68) are true because $G_m(-1/z) = 2$, by (0.52). Suppose that $p \nmid m$. Then by (0.66),

$$G_m(-1/z) = (-1)^m \frac{F(0, -2m/p; z)}{F(0, -m/p; z)}. \tag{0.69}$$

Now, from (0.43),

$$\begin{aligned} q^{-1/8}F(0, -m/p; z) &= -i \sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi i(2n+1)m/p} q^{(n^2+n)/2} \\ &= -2 \sum_{n=0}^{\infty} (-1)^n \sin\left(\frac{\pi(2n+1)m}{p}\right) q^{(n^2+n)/2}. \end{aligned} \tag{0.70}$$

From elementary trigonometry, we see that (0.67) and (0.68) follow from (0.69) and (0.70).

Corollary 0.5. *Let p be a prime such that $p > 5$ and $p \equiv 1$ (mod 4). Let R denote the set of quadratic residues (mod p) between 1 and $p/2$. Then*

$$h(z) := \prod_{\beta \in R} G_\beta(z) + (-1)^{(p^2-1)/8} \prod_{\beta \in R} G_\beta^{-1}(z) \tag{0.71}$$

is in $\{\Gamma^0(p), 0, 1\}$ and has no poles on $\mathscr{H}$ or at the cusp 0.

Proof. In Theorem 0.4, let $s = (p-1)/4$, let $\varepsilon_r = 1$ $(1 \le r \le s)$, and let $\beta_1, \ldots, \beta_s$ denote the members of R. (Note that $\{\pm\beta_r: 1 \le r \le s\}$ is a complete set of quadratic residues (mod p).) Set $B = \beta_1^2 + \cdots + \beta_s^2$. If g is a primitive root (mod p),

$$2\sum_{r \in R} \beta_r^2 + 2\sum_{r \in R} (g\beta_r)^2 \equiv \sum_{m=1}^{p-1} m^2 \equiv 0 \pmod{p},$$

and therefore $B(1+g^2) \equiv 0 \pmod{p}$. Since $p > 5$, $1 + g^2 \not\equiv 0 \pmod{p}$, and so $B \equiv 0 \pmod{p}$. Thus, (0.56) holds. For $g(z)$, as defined by (0.57),

$$g(z) = \sum_m \prod_{\beta \in R} G(m\beta; z) = 2^{(p-1)/4} + \sum_{\substack{m \\ (\frac{m}{p})=1}} \prod_{\beta \in R} G_\beta(z) + \sum_{\substack{m \\ (\frac{m}{p})=-1}} \prod_{\beta \in N} G_\beta(z),$$

by (0.52), where N is the set of s quadratic nonresidues (mod p) between 0 and $p/2$. Therefore,

$$g(z) = 2^{(p-1)/4} + \tfrac{1}{2}(p-1)\left(\prod_{\beta \in R} G_\beta(z) + \sum_{\beta \in N} G_\beta(z)\right). \tag{0.72}$$

By (0.50),

$$\prod_{\beta\in R} G_\beta(z) \prod_{\beta\in N} G_\beta(z) = \prod_{m=1}^{(p-1)/2} G_m(z) = (-1)^{(p^2-1)/8} \prod_{m=1}^{(p-1)/2} \frac{F(2m/p; z)}{F(m/p; z)}$$
$$= (-1)^{(p^2-1)/8},$$

since $F(u; z) = F(1-u; z)$ by (0.46) and (0.47). Using this calculation in (0.72), we deduce that

$$g(z) = 2^{(p-1)/4} + \tfrac{1}{2}(p-1)h(z),$$

where $h(z)$ is defined by (0.71). The result now follows from Theorem 0.4.

Corollary 0.6. *Let p be a prime* $\equiv 1 \pmod 4$, *and let β be a primitive fourth root of unity* (mod p). *Then for $\varepsilon = 1$ or -1,*

$$k_\varepsilon(z) := \sum_{m(\text{mod } p)} G^\varepsilon(m; z) G^\varepsilon(\beta m; z)$$

is in $\{\Gamma^0(p), 0, 1\}$ and has no poles on $\mathscr{H}$ or at the cusp 0.

PROOF. Apply Theorem 0.4 with $s = 2$, $\varepsilon_1 = \varepsilon_2 = \varepsilon$, $\beta_1 = 1$, and $\beta_2 = \beta$.

Corollary 0.7. *Let p be odd and greater than* 1. *Then*

$$g(z) := \sum_{m(\text{mod } p)} G_m^p(z)$$

is in $\{\Gamma^0(p), 0, 1\}$ and has no poles on $\mathscr{H}$ or at the cusp 0.

PROOF. This follows from Theorem 0.4 with $s = 1$, $\varepsilon_1 = p$, and $\beta_1 = 1$.

If p is an odd prime, Theorem 0.4 provides a method for proving identities of the type

$$g(z) = E(z), \tag{0.73}$$

where $g(z)$ is given by (0.57) and $E(z)$ is a relatively simple function in $\{\Gamma^0(p), 0, 1\}$. The idea is to construct a function $E(z) \in \{\Gamma^0(p), 0, 1\}$ with no poles, except possibly at the cusp ∞, such that $g(z) - E(z)$ has a zero at ∞. Then since 0 and ∞ are the only inequivalent cusps (mod $\Gamma^0(p)$) when p is prime (Schoeneberg [1, pp. 87–88]), it follows from Theorem 0.4 that $g(z) - E(z)$ has no poles at all. But the only entire modular functions in $\{\Gamma^0(p), 0, 1\}$ are constants (Rankin [2, p. 108]), and so $g(z) - E(z)$ is a constant, which, of course, must be zero. Thus, (0.73) follows.

To examine $g(z)$ at ∞, we need the Fourier expansion of $g(z)$. Thus, we must determine the Fourier expansion of $G_m(z)$. This can be obtained by utilizing Lemma 0.3 in conjunction with (0.44) and Entry 22(iii) of Chapter 16. Thus, by Lemma 0.3, which we saw was equivalent to the quintuple product identity, and (0.44),

$$\eta(z)(-1)^m G(m; z) = \sum_{n=-\infty}^{\infty} (-1)^n (q^{3(n+m/p-1/6)^2/2} + q^{3(n-m/p-1/6)^2/2}). \tag{0.74}$$

Since $G(m; z) = G(p - m; z)$, by (0.51), we may assume that $1 \le m \le (p-1)/2$. Isolating the terms in (0.74) with $n = 0, \pm 1$, we find that

$$\eta(z)(-1)^m G(m; z) = q^{1/24} q^{(3m^2 - mp)/(2p^2)} \{1 + q^{m/p} - q^{(p-2m)/p} - q^{(p+3m)/p} - q^{(2p-3m)/p} + O(q^2)\}. \tag{0.75}$$

By Entry 22(iii) in Chapter 16,

$$\eta(z) = q^{1/24}(1 - q - q^2 + O(q^5)). \tag{0.76}$$

Thus, by (0.75) and (0.76), for $1 \le m \le (p-1)/2$,

$$\begin{aligned}(-1)^m G(m; z) &= q^{(3m^2 - mp)/(2p^2)}\{1 + q^{m/p} - q^{(p-2m)/p} - q^{(p+3m)/p} \\ &\qquad - q^{(2p-3m)/p} + q + q^{(p+m)/p} - q^{(2p-2m)/p} - q^{(3p-3m)/p} + O(q^2)\} \\ &= q^{(3m^2 - mp)/(2p^2)}\{1 + q^{m/p} - q^{(p-2m)/p} + O(q^{2/p})\}. \end{aligned} \tag{0.77}$$

To first illustrate the usefulness of the theory developed above, we give a second proof of (18.2) in Entry 18(i) of Chapter 19. Recall that our first proof was laboriously tedious. We reformulate (18.2) in terms of $G_m(z)$ and $\eta(z)$ before proving it.

Theorem 0.8 ((18.2) of Chapter 19). *If $p = 7$, then*

$$G_1^7(z) + G_2^7(z) + G_3^7(z) = -57 - 14\left(\frac{\eta(z/7)}{\eta(z)}\right)^4 - \left(\frac{\eta(z/7)}{\eta(z)}\right)^8. \tag{0.78}$$

PROOF. With

$$g(z) := \sum_{m=0}^{6} G_m^7(z),$$

we see that (0.78) is equivalent to

$$g(z) = 14 - 28\left(\frac{\eta(z/7)}{\eta(z)}\right)^4 - 2\left(\frac{\eta(z/7)}{\eta(z)}\right)^8. \tag{0.79}$$

By Corollary 0.7, the left side of (0.79) belongs to $\{\Gamma^0(7), 0, 1\}$ and has no poles on $\mathscr{H}$ or at the cusp 0.

Now, for $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma^0(p)$ and $p \equiv 1 \pmod 6$,

$$\begin{aligned}\eta^4(Vz/p) &= v_\eta^4\begin{pmatrix} a & b/p \\ pc & d\end{pmatrix}(cz+d)^2\eta^4(z/p) \\ &= v_\eta^4(V)(cz+d)^2\eta^4(z/p).\end{aligned}$$

Thus, for $p \equiv 1 \pmod 6$,

$$\eta^4(z/p)/\eta^4(z) \in \{\Gamma^0(p), 0, 1\}.$$

In particular, for $p = 7$, if $E(z)$ denotes the right side of (0.79), then $E(z)$ is in $\{\Gamma^0(7), 0, 1\}$ and has no poles except at ∞.

Since both sides of (0.79) are in $\{\Gamma^0(7), 0, 1\}$, by the procedure described above, (0.79) will follow if $g(z) - E(z)$ has a zero at ∞. By (0.76),

$$E(z) = -2q^{-2/7} - 12q^{-1/7} + 86 + O(q^{1/7}). \tag{0.80}$$

By (0.77),

$$G_1^7(z) = -q^{-2/7}\{1 + q^{1/7} + O(q^{5/7})\}^7, \tag{0.81}$$

$$G_2^7(z) = q^{-1/7}\{1 + O(q^{2/7})\}^7, \tag{0.82}$$

and

$$G_3^7(z) = O(q^{3/7}). \tag{0.83}$$

Thus, using (0.81)–(0.83), we find that

$$g(z) = -2q^{-2/7} - 12q^{-1/7} + 86 + O(q^{1/7}). \tag{0.84}$$

Hence, by (0.80) and (0.84), $g(z) - E(z)$ has a zero at ∞, and the proof is complete.

We close this long introduction to Chapter 20 by proving a generalization of Theorem 12.1 in Section 12 of Chapter 19.

Theorem 0.9. *For each odd integer $p > 1$,*

$$\sum_{m \pmod p} G(m; z) = 2\left(\frac{3}{p}\right)\frac{\eta(z/p^2)}{\eta(z)}, \tag{0.85}$$

where $(\frac{3}{p})$ denotes the Legendre symbol.

Proof. By (0.74),

$$\begin{aligned}
\eta(z)\sum_m G(m; z) &= \sum_m \sum_{n=-\infty}^{\infty} (-1)^{m+n}(q^{3(n+m/p+1/6)^2/2} + q^{3(n-m/p+1/6)^2/2}) \\
&= 2\sum_{j=-\infty}^{\infty} (-1)^j q^{3(j/p+1/6)^2/2} \\
&= 2\sum_{j=-\infty}^{\infty} (-1)^j q^{3(j+p/6)^2/(2p^2)}.
\end{aligned}$$

However, from Entry 22(iii) of Chapter 16,

$$\eta(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{3(n+1/6)^2/2}. \tag{0.86}$$

It therefore remains to show that

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{3(j+p/6)^2/2} = \left(\frac{3}{p}\right)\sum_{n=-\infty}^{\infty} (-1)^n q^{3(n+1/6)^2/2}. \tag{0.87}$$

This is easily verified in the cases $p \equiv \pm 1 \pmod{12}$ and $p \equiv \pm 5 \pmod{12}$, where $\left(\frac{3}{p}\right) = 1$ and -1, respectively. Suppose that $3|p$, so that $\left(\frac{3}{p}\right) = 0$. Then the left side of (0.87) equals

$$S := \pm \sum_{j=-\infty}^{\infty} (-1)^j q^{3(j+1/2)^2/2}.$$

Replacing j by $-j-1$, we see that $S = -S$, that is, $S = 0$. Hence, (0.87) holds, and the proof is complete.

Recall that the functions $\chi(q)$, $\psi(q)$, $\varphi(q)$, and $f(-q)$ are defined in Entry 22 of Chapter 16.

Entry 1. *Define v by $v := q^{1/3}\chi(-q)/\chi^3(-q^3)$. The following equalities are then valid:*

(i)
$$v = \frac{q^{1/3}}{1} \, \frac{}{+} \, \frac{q+q^2}{1} \, \frac{}{+} \, \frac{q^2+q^4}{1} \, \frac{}{+} \, \frac{q^3+q^6}{1} + \cdots,$$

$$1 + \frac{1}{v} = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)},$$

and

$$1 + \frac{1}{v^3} = \frac{\psi^4(q)}{q\psi^4(q^3)};$$

(ii)
$$1 + \frac{\psi(-q^{1/3})}{q^{1/3}\psi(-q^3)} = \left(1 + \frac{\psi^4(-q)}{q\psi^4(-q^3)}\right)^{1/3},$$

$$2v = 1 - \frac{\varphi(-q^{1/3})}{\varphi(-q^3)},$$

and

$$1 + 3q\frac{\psi(-q^9)}{\psi(-q)} = \left(1 + 9q\frac{\psi^4(-q^3)}{\psi^4(-q)}\right)^{1/3};$$

(iii)
$$\frac{\varphi(q^{1/3})}{\varphi(q^3)} = 1 + \left(\frac{\varphi^4(q)}{\varphi^4(q^3)} - 1\right)^{1/3},$$

$$3\frac{\varphi(q^9)}{\varphi(q)} = 1 + \left(\frac{9\varphi^4(q^3)}{\varphi^4(q)} - 1\right)^{1/3},$$

$$\cos 40^\circ + \cos 80^\circ = \cos 20^\circ,$$

and

$$\frac{1}{\cos 40^\circ} + \frac{1}{\cos 80^\circ} = \frac{1}{\cos 20^\circ} + 6;$$

(iv)
$$3 + \frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)} = \left(27 + \frac{f^{12}(-q)}{qf^{12}(-q^3)}\right)^{1/3} = \frac{1}{v} + 4v^2$$

and

$$1+9q\frac{f^3(-q^9)}{f^3(-q)}=\left(1+27q\frac{f^{12}(-q^3)}{f^{12}(-q)}\right)^{1/3};$$

(v) $$f^3(-q^{1/3})+3q^{1/3}f^3(-q^3)=f(-q)\left\{1+6\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}}-\frac{q^{3n+2}}{1-q^{3n+2}}\right)\right\}.$$

The introduction of the continued fraction in (i) appears adventitious, for it seems to have no intrinsic connection with the remainder of the results in Entry 1. Entry 1(i) was communicated by Ramanujan [10, p. xxviii] in his second letter to Hardy.

The first equalities of (ii) and (iii) are, respectively, (24.28) and (24.29) in Chapter 18. A. J. Biagioli [2] and J. M. and P. B. Borwein [2, pp. 142–144] have given proofs of these interesting identities. The proofs provided below are in the spirit of much of the material of the present book.

We are unable to relate the trigonometric equalities in (iii) to any other material in Entry 1.

Proof of (i). By (22.4), Entry 24(iii), Entry 19, and Corollary (ii) of Entry 31, all in Chapter 16,

$$\frac{1}{v}=\frac{(-q;q)_\infty\varphi(-q^3)}{q^{1/3}\psi(q^3)}=\frac{(-q^3;q^3)_\infty f(q,q^2)\varphi(-q^3)}{q^{1/3}\psi(q^3)(q^3;q^3)_\infty}$$
$$=\frac{f(q,q^2)}{q^{1/3}\psi(q^3)}=\frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}-1. \tag{1.1}$$

Thus, the second equality of (i) is apparent.

In the result just proved, replace $q^{1/3}$ by $\omega q^{1/3}$ and $\omega^2q^{1/3}$ in turn, where ω is a primitive cube root of unity. Note that v is replaced by ωv and $\omega^2 v$, respectively. Multiplying these three equalities together, we find that

$$1+\frac{1}{v^3}=\frac{\psi(q^{1/3})\psi(\omega q^{1/3})\psi(\omega^2q^{1/3})}{q\psi^3(q^3)}. \tag{1.2}$$

Now using the product representation for ψ given in Entry 22(ii) of Chapter 16, we find that (1.2) becomes

$$1+\frac{1}{v^3}=\frac{(q^2;q^2)_\infty^3}{q(q;q^2)_\infty^3}\frac{(q^2;q^2)_\infty(q^3;q^6)_\infty}{(q;q^2)_\infty(q^6;q^6)_\infty}\frac{1}{\psi^3(q^3)}=\frac{\psi^4(q)}{q\psi^4(q^3)}.$$

This establishes the third part of (i).

Lastly, Watson [4] has shown that the continued fraction of (i) is equal to $q^{1/3}\psi(q^3)/f(q,q^2)$. By (1.1), the truth of the first equality of (i) is evinced.

This continued fraction also appears in the third notebook [9, p. 373] in the form

$$\frac{f(-q,-q^5)}{f(-q^3,-q^3)}=\frac{1}{1+}\frac{q+q^2}{1}\frac{q^2+q^4}{1}+\cdots, \tag{1.3}$$

which, by the Jacobi triple product identity, is readily seen to be equivalent to the formulation in (i). Proofs of (1.3) have also been found by A. Selberg [1, p. 19], [2, p. 17], Gordon [2], Andrews [1], and Hirschhorn [6]. See also Ramanathan's papers [1], [4]. The convergence of (1.3) on $|q| = 1$ has been thoroughly examined by L.-C. Zhang [1]. In fact, Ramanujan has recorded a considerable generalization of (1.3) in his "lost notebook" [11]. For proofs and discussion of this more general theorem, see papers by Andrews [10], Hirschhorn [3], and Ramanathan [4].

PROOF OF (ii). The last two equalities of part (i) yield two expressions for v, and when these are equated and the sign of q is changed, we obtain the first equality of (ii).

From (1.1) above, (22.3) of Chapter 16, and Example (v) and Corollary (i), both in Section 31 of Chapter 16, we find that

$$2v = \frac{2q^{1/3}\chi(-q)\psi(q^3)}{\varphi(-q^3)} = \frac{2q^{1/3}f(-q,-q^5)}{\varphi(-q^3)} = 1 - \frac{\varphi(-q^{1/3})}{\varphi(-q^3)},$$

and so the second part of (ii) is established.

The third equality of (ii) follows from the first equality of (ii) by elementary algebra as follows: replace q by q^3, cube both sides, cancel 1 on each side, multiply both sides by $9q^4\psi^4(-q^9)/\psi^4(-q)$, add 1 to both sides, and finally take the cube root of each side.

PROOF OF (iii). In Section 24 (vii) of Chapter 18, we showed the equivalence of the first equalities of (ii) and (iii).

Alternatively, a proof dependent on the second equality of (ii) can be given. In the aforementioned equality, replace $q^{1/3}$ by $\omega q^{1/3}$ and $\omega^2 q^{1/3}$ in turn, where ω is a primitive cube root of unity. Multiplying the three equalities together, we deduce that

$$1 - 8v^3 = \frac{\varphi(-q^{1/3})\varphi(-\omega q^{1/3})\varphi(-\omega^2 q^{1/3})}{q^3(-q^3)} = \frac{\varphi^4(-q)}{\varphi^4(-q^3)},$$

where the last equality is readily ascertained from the product representation of φ given in Entry 22(i) or (22.4) of Chapter 16. Hence,

$$-2v = \left(\frac{\varphi^4(-q)}{\varphi^4(-q^3)} - 1\right)^{1/3}.$$

Equating this expression for $-2v$ with that from part (ii) and changing the sign of q, we deduce the first equality of (iii).

The second equality of (iii) follows from the first by very elementary algebra, completely analogous to that outlined in the proof of (ii).

To prove the first trigonometric equality, we employ the elementary identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

By taking $\theta = 20°$, $40°$, and $80°$ in turn, we see that $-\cos 20°$, $\cos 40°$, and cos 80° are the three roots of

$$4x^3 - 3x + \tfrac{1}{2} = 0.$$

The first trigonometric equality is now evident.

Replacing x by $1/x$ above, we deduce that the reciprocals of $-\cos 20°$, $\cos 40°$, and $\cos 80°$ are the roots of

$$x^3 - 6x^2 + 8 = 0,$$

and the second trigonometric equality follows at once.

PROOF OF (iv). To obtain one part of (iv), first observe that, by Entry 24(ii) in Chapter 16 and parts (i) and (ii) above,

$$\frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)} = \frac{\varphi^2(-q^{1/3})\psi(q^{1/3})}{q^{1/3}\varphi^2(-q^3)\psi(q^3)} = (1-2v)^2\left(1+\frac{1}{v}\right) = 4v^2 + \frac{1}{v} - 3,$$

which gives part of (iv).

Now replace $q^{1/3}$ by $\omega q^{1/3}$ and $\omega^2 q^{1/3}$ in turn, where ω is a primitive cube root of unity. Observe that v is replaced by ωv and $\omega^2 v$, respectively. Multiplying these three equalities together and using the product representation for f found in Entry 22(iii) of Chapter 16, we deduce that

$$\frac{f^{12}(-q)}{qf^{12}(-q^3)} = \left(4v^2 + \frac{1}{v} - 3\right)\left(4v^2 + \frac{1}{v} - 3\omega\right)\left(4v^2 + \frac{1}{v} - 3\omega^2\right)$$
$$= \left(4v^2 + \frac{1}{v}\right)^3 - 27.$$

This easily yields the second equality of (iv).

The last equality of (iv) follows from elementary algebra. With q replaced by q^3, cube the first equality of (iv). Cancel 27 and multiply both sides by $27q^4f^{12}(-q^9)/f^{12}(-q)$. Add 1 to both sides and take the cube root of each side to complete the proof.

PROOF OF (v). We employ the series representation for $f^3(-q^{1/3})$ given in Entry 24(ii) of Chapter 16 and separate the terms into three subsets according to the residuacity of the index modulo 3. Hence,

$$f^3(-q^{1/3}) = \sum_{n=1}^{\infty} (-1)^n(6n-3)q^{(3n-1)(3n-2)/6} + \sum_{n=1}^{\infty} (-1)^{n-1}(6n-1)q^{n(3n-1)/2}$$
$$+ \sum_{n=0}^{\infty} (-1)^n(6n+1)q^{n(3n+1)/2}$$
$$= -3q^{1/3}f^3(-q^3) + f(-q) + 6\sum_{n=1}^{\infty} (-1)^n nq^{n(3n+1)/2}$$
$$+ 6\sum_{n=1}^{\infty} (-1)^{n-1} nq^{n(3n-1)/2},$$

where we have utilized once again Entry 24(ii) as well as the series representation for $f(-q)$ given in Entry 22(iii) of Chapter 16. Therefore,

$$\begin{aligned}
&f^3(-q^{1/3}) + 3q^{1/3}f^3(-q^3) - f(-q)\\
&= 6\sum_{n=-\infty}^{\infty} (-1)^n nq^{n(3n+1)/2}\\
&= 6\left(\frac{\partial}{\partial\alpha}\sum_{n=-\infty}^{\infty} (-\alpha)^n q^{n(3n+1)/2}\right)_{\alpha=1}\\
&= 6\left(\frac{\partial}{\partial\alpha} f(-q/\alpha, -\alpha q^2)\right)_{\alpha=1}\\
&= 6\left(f(-q/\alpha, -\alpha q^2)\frac{\partial}{\partial\alpha}\operatorname{Log} f(-q/\alpha, -\alpha q^2)\right)_{\alpha=1}\\
&= 6f(-q)\left(\frac{\partial}{\partial\alpha}\{\operatorname{Log}(q/\alpha; q^3)_\infty + \operatorname{Log}(\alpha q^2; q^3)_\infty\}\right)_{\alpha=1}\\
&= 6f(-q)\left(\sum_{n=0}^{\infty}\frac{\alpha^{-2}q^{3n+1}}{1-\alpha^{-1}q^{3n+1}} - \sum_{n=0}^{\infty}\frac{q^{3n+2}}{1-\alpha q^{3n+2}}\right)_{\alpha=1}\\
&= 6f(-q)\sum_{n=0}^{\infty}\left(\frac{q^{3n+1}}{1-q^{3n+1}} - \frac{q^{3n+2}}{1-q^{3n+2}}\right),
\end{aligned}$$

where we have employed the Jacobi triple product identity. Identity (v) is now obvious.

Entry 2. *We have*

(i) $\varphi(q)\varphi(q^9) - \varphi^2(q^3) = 2q\varphi(-q^2)\psi(q^9)\chi(q^3)$,

(ii) $\psi(q) - 3q\psi(q^9) = \dfrac{\varphi(-q)}{\chi(-q^3)}$,

(iii) $\varphi(q)\varphi(q^9) + \varphi^2(q^3) = 2\psi(q)\varphi(-q^{18})\chi(q^3)$,

(iv) $\psi(q^{1/9}) - q^{1/9}\psi(q) = f(q^4, q^5) + q^{1/3}f(q^2, q^7) + q^{2/3}f(q, q^8)$,

(v) $f(-q^{1/3}) = f(-q^4, -q^5) - q^{1/3}f(-q^2, -q^7) - q^{2/3}f(-q, -q^8)$,

(vi) $f(-q, -q^8)f(-q^2, -q^7)f(-q^4, -q^5) = \dfrac{f(-q)f^3(-q^9)}{f(-q^3)}$,

(vii) $\dfrac{f(-q^4, -q^5)}{f(-q^2, -q^7)} + q\dfrac{f(-q, -q^8)}{f(-q^4, -q^5)} = \dfrac{f(-q^2, -q^7)}{f(-q, -q^8)}$,

(viii) $\dfrac{f(-q^4, -q^5)}{f(-q, -q^8)} + q\dfrac{f(-q^2, -q^7)}{f(-q^4, -q^5)} = q\dfrac{f(-q, -q^8)}{f(-q^2, -q^7)} + \dfrac{f^4(-q^3)}{f(-q)f^3(-q^9)}$,

and

(ix) $\varphi(q^{1/9}) - \varphi(q) = 2q^{1/9}f(q^7, q^{11}) + 2q^{4/9}f(q^5, q^{13}) + 2q^{16/9}f(q, q^{17})$.

PROOF OF (i). By Corollary (i) and Example (v), both in Section 31 of Chapter 16,

$$\begin{aligned}\{\varphi(q^{1/3})-\varphi(q^3)\}\varphi(q^3)&=2q^{1/3}f(q,q^5)\varphi(q^3)\\&=2q^{1/3}\psi(-q^3)\chi(q)\varphi(q^3)\\&=2q^{1/3}\psi(q^3)\varphi(-q^6)\chi(q),\end{aligned}\tag{2.1}$$

where we have used the equality $\psi(-q)\varphi(q)=\psi(q)\varphi(-q^2)$, which is easily deducible from Entries 25(iii), (iv) in Chapter 16. Using again the same two facts from Section 31 of Chapter 16, we arrive at

$$\begin{aligned}&\{\varphi(q^{1/3})-\varphi(q^3)\}\varphi(q^3)\\&=2q^{1/3}\psi(q^3)\{\varphi(-q^{2/3})+2q^{2/3}f(-q^2,-q^{10})\}\chi(q)\\&=2q^{1/3}\psi(q^3)\varphi(-q^{2/3})\chi(q)+4q\psi(q^3)\psi(q^6)\chi(q)\chi(-q^2).\end{aligned}\tag{2.2}$$

Next, by Entries 11(i), (iii) and 12(v), (vii) in Chapter 17, if β is of the third degree in α,

$$\frac{4q\psi(q^3)\psi(q^6)\chi(q)\chi(-q^2)}{\varphi^2(q^3)}=2\left(\frac{\beta^3}{\alpha}\right)^{1/8}=\frac{\varphi^2(q)}{\varphi^2(q^3)}-1,$$

by Entry 5(iii) in Chapter 19. Substituting in (2.2), we deduce that

$$\varphi(q^{1/3})\varphi(q^3)=2q^{1/3}\psi(q^3)\varphi(-q^{2/3})\chi(q)+\varphi^2(q).$$

Replacing q by q^3, we complete the proof of (i).

PROOF OF (ii). Employing Entries 1(i), (ii) above and Entry 24(iii) of Chapter 16, we find that

$$\begin{aligned}\psi(q^{1/3})-3q^{1/3}\psi(q^3)&=\frac{q^{1/3}\psi(q^3)}{v}(1-2v)=\frac{q^{1/3}\psi(q^3)\varphi(-q^{1/3})}{v\varphi(-q^3)}\\&=\frac{q^{1/3}\varphi(-q^{1/3})}{v\chi^3(-q^3)}=\frac{\varphi(-q^{1/3})}{\chi(-q)},\end{aligned}$$

where we lastly used the definition of v. Replacing q by q^3, we complete the proof of (ii).

PROOF OF (iii). By Entry 19, (22.3), and Entry 24(iii), all in Chapter 16,

$$f(q,q^2)=\frac{(-q;q)_\infty(q^3;q^3)_\infty}{(-q^3;q^3)_\infty}=\frac{\chi(-q^3)f(-q^3)}{\chi(-q)}=\frac{\varphi(-q^3)}{\chi(-q)}.\tag{2.3}$$

Hence, by (2.1), Corollary (ii) of Section 31 in Chapter 16, and (2.3),

$$\begin{aligned}\varphi(q^{1/3})\varphi(q^3)-\varphi^2(q^3)&=2q^{1/3}\psi(q^3)\varphi(-q^6)\chi(q)\\&=2\{\psi(q^{1/3})-f(q,q^2)\}\varphi(-q^6)\chi(q)\\&=2\psi(q^{1/3})\varphi(-q^6)\chi(q)-2\frac{\varphi(-q^3)\varphi(-q^6)\chi(q)}{\chi(-q)}.\end{aligned}\tag{2.4}$$

Next, by Entries 10(i)–(iii) and 12(v), (vi) in Chapter 17 and Entries 5(i), (iii) in Chapter 19,

$$2\frac{\varphi(-q^3)\varphi(-q^6)\chi(q)}{\varphi^2(q^3)\chi(-q)} = 2\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8}$$
$$= 2 + 2\left(\frac{\beta^3}{\alpha}\right)^{1/8} = 1 + \frac{\varphi^2(q)}{\varphi^2(q^3)}.$$

Substituting this in (2.4), we deduce that

$$\varphi(q^{1/3})\varphi(q^3) = 2\psi(q^{1/3})\varphi(-q^6)\chi(q) - \varphi^2(q).$$

Replacing q by q^3 completes the proof.

PROOF OF (iv). We apply Entry 31 in Chapter 16 with $a = 1$, $b = q$, and $n = 9$. Using also Entry 18(ii) in Chapter 16, we find that

$$\psi(q) = f(q^{36}, q^{45}) + qf(q^{27}, q^{54}) + q^3f(q^{18}, q^{63}) + q^6f(q^9, q^{72}) + q^{10}\psi(q^{81}).$$

Replacing q by $q^{1/9}$ and then utilizing Corollary (ii) in Section 31 of Chapter 16, we complete the proof of (iv).

PROOF OF (v). In Entry 31 of Chapter 16, set $a = -q$, $b = -q^2$, and $n = 3$. Hence,

$$f(-q) = f(-q^{12}, -q^{15}) - qf(-q^6, -q^{21}) - q^2f(-q^3, -q^{24}).$$

Replacing q by $q^{1/3}$, we complete the proof.

PROOF OF (vi). Set $n = 4$ in (28.1) of Chapter 16.

PROOFS OF (vii), (viii). With $A = f(-q^4, -q^5)$, $B = f(-q^2, -q^7)$, and $C = f(-q, -q^8)$, we write (v) in the form

$$f(-q^{1/3}) = A - Bq^{1/3} - Cq^{2/3},$$

which when cubed yields

$$f^3(-q^{1/3}) = A^3 - B^3q - C^3q^2 + 6ABCq - 3(A^2B + B^2Cq - AC^2q)q^{1/3}$$
$$- 3(A^2C - AB^2 + BC^2q)q^{2/3}.$$

Comparing this with Entry 1(v), we deduce that

$$A^2C - AB^2 + BC^2q = 0$$

and

$$A^2B + B^2Cq - AC^2q = f^3(-q^3),$$

which, with the use of (vi), immediately yield (vii) and (viii), respectively.

Alternatively, (vii) is a corollary of Theorem 0.9.

PROOF OF (ix). Apply Entry 31 of Chapter 16 with $a = b = q$ and $n = 9$. Hence,

$$\varphi(q) = \varphi(q^{81}) + 2qf(q^{63}, q^{99}) + 2q^4f(q^{45}, q^{117})$$
$$+ 2q^9f(q^{27}, q^{135}) + 2q^{16}f(q^9, q^{153}).$$

Replacing q by $q^{1/9}$ and using Corollary (i) in Section 31 of Chapter 16, we complete the proof.

Using Entries 2(v), (vi), (viii) and Theorem 0.9, Evans [1, Theorem 7.2] has established the following beautiful identity. For $p = 9$,

$$G_1(z)G_2^2(z) + G_2(z)G_4^2(z) + G_4(z)G_1^2(z) = 6 - \frac{\eta(z/3)\{3\eta^3(z/3) + \eta^3(z/27)\}}{\eta(z/9)\eta^3(z)}.$$

Entry 3. *Let β and γ be of the third and ninth degrees, respectively, with respect to α. Let $m = z_1/z_3$ and $m' = z_3/z_9$. Then the following modular equations are valid:*

(i) $$1 + 4^{1/3}\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/24} = \frac{3}{\sqrt{mm'}},$$

(ii) $$1 + 4^{1/3}\left(\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}\right)^{1/24} = \sqrt{mm'},$$

(iii) $$1 - 2^{4/3}\left(\frac{\alpha^3\gamma^3(1-\alpha)^3(1-\gamma)^3}{\beta^2(1-\beta)^2}\right)^{1/24} = \frac{m'}{m},$$

(iv) $$1 - 4^{1/3}\left(\frac{\gamma^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/24} = \sqrt{\frac{m'}{m}} = 4^{1/3}\left(\frac{\alpha^3(1-\gamma)^3}{\beta(1-\beta)}\right)^{1/24} - 1,$$

(v) $$(\alpha\gamma)^{1/2} + \{(1-\alpha)(1-\gamma)\}^{1/2} + 2\{4\beta(1-\beta)\}^{1/3} = 1 + 8\{\beta(1-\beta)\}^{1/4}\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{1/8},$$

(vi) $$\{\alpha(1-\gamma)\}^{1/8} + \{\gamma(1-\alpha)\}^{1/8} = 2^{1/3}\{\beta(1-\beta)\}^{1/24},$$

(vii) $$\frac{\left(\dfrac{(1-\beta)^3}{1-\alpha}\right)^{1/4} - 1}{1 - \{(1-\alpha)(1-\beta)\}^{1/4}} = m = \frac{1 - \left(\dfrac{\beta^3}{\alpha}\right)^{1/4}}{1 - (\alpha\beta)^{1/4}},$$

(viii) $$1 + \left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8} = m(\tfrac{1}{2}\{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}\})^{1/2},$$

(ix) $$1 + \left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8} = \frac{3}{m}(\tfrac{1}{2}\{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}\})^{1/2},$$

(x) $$\left(\frac{\gamma}{\alpha}\right)^{1/8} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8} - \left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8} = \sqrt{mm'},$$

(xi) $$\left(\frac{\alpha}{\gamma}\right)^{1/8} + \left(\frac{1-\alpha}{1-\gamma}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/8} = \frac{3}{\sqrt{mm'}},$$

(xii) $$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4} - \left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} = -3\frac{m}{m'},$$

(xiii) $$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4}-\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4}=\frac{m'}{m},$$

(xiv) $$\frac{2^{1/3}\{\beta(1-\beta)\}^{1/24}}{\{\alpha(1-\gamma)\}^{1/8}-\{\gamma(1-\alpha)\}^{1/8}}=\sqrt{\frac{m}{m'}},$$

(xv) $$(\alpha^{1/4}-\gamma^{1/4})^4+\{(1-\gamma)^{1/4}-(1-\alpha)^{1/4}\}^4=(\{\alpha(1-\gamma)\}^{1/4}-\{\gamma(1-\alpha)\}^{1/4})^4,$$

and

(xvi) $$1=(\alpha\gamma)^{1/2}+\{(1-\alpha)(1-\gamma)\}^{1/2}+2\{4\beta(1-\beta)\}^{1/3}\frac{m'}{m}.$$

It is not clear why (vii)–(ix) are placed here, because they are third degree modular equations. For several other modular equations of degree 3, see Section 5 of Chapter 19.

PROOFS OF (i)–(iii). In order to prove these formulas, we first need to express $\alpha(1-\alpha)$, $\beta(1-\beta)$, and $\gamma(1-\gamma)$ as rational functions of a parameter t.

Let q be defined by (5.12) in Chapter 19. Thus, by (5.4) and Entry 5(xv) in Chapter 19,

$$m=\sqrt{1+4q}, \tag{3.1}$$

$$\alpha(1-\alpha)=q\left(\frac{2-q}{1+4q}\right)^3, \quad \text{and} \quad \beta(1-\beta)=q^3\left(\frac{2-q}{1+4q}\right). \tag{3.2}$$

Analogously, let q' connect β and γ. Then

$$m'=\sqrt{1+4q'}, \tag{3.3}$$

$$\beta(1-\beta)=q'\left(\frac{2-q'}{1+4q'}\right)^3, \quad \text{and} \quad \gamma(1-\gamma)=q'^3\left(\frac{2-q'}{1+4q'}\right). \tag{3.4}$$

Trivially, (3.2) and (3.4) indicate that

$$q^3\left(\frac{2-q}{1+4q}\right)=q'\left(\frac{2-q'}{1+4q'}\right)^3. \tag{3.5}$$

Regarding this as an equation to determine q as a function of q', we see that one solution is $q=(2-q')/(1+4q')$. However, because $q=O(\alpha)$ and $q'=O(\beta)=O(\alpha^3)$ as α tends to 0, this solution must be irrelevant. Dividing out this root in the quartic equation (3.5), we find that the relevant solution is a root of

$$q^3-\frac{9q'}{1+4q'}q^2-\frac{9q'(2-q')}{(1+4q')^2}q-\frac{q'(2-q')^2}{(1+4q')^2}=0.$$

Setting $q=x+3q'/(1+4q')$, we transform this equation into

$$x^3-\frac{18q'(q'+1)}{(1+4q')^2}x-\frac{2q'(2q'^3+6q'^2+33q'+2)}{(1+4q')^3}=0.$$

This cubic equation may be solved by Tartaglia's method (Hall and Knight [1, p. 480]). Omitting the details, we find that the roots of the original cubic equation are

$$q = \frac{3q' + 3\omega(2q'^2)^{1/3} + \omega^2(1 + q')(4q')^{1/3}}{1 + 4q'}, \tag{3.6}$$

where ω is any cube root of unity. Of course, the relevant root is the real one. Moreover, q is a rational function of $(q')^{1/3}$.

Accordingly, if we set $q' = 2t^3$, then $\alpha(1-\alpha)$, $\beta(1-\beta)$, and $\gamma(1-\gamma)$ are expressible as rational functions of t. Thus, from (3.6) and (3.1)–(3.4), we deduce that

$$q = \frac{2t(1 + t + t^2)}{1 - 2t + 4t^2},$$

$$\alpha(1-\alpha) = 16t\left(\frac{1-t}{1+2t}\right)^8 \frac{1-t^3}{1+8t^3}, \tag{3.7}$$

$$\beta(1-\beta) = 16t^3\left(\frac{1-t^3}{1+8t^3}\right)^3, \tag{3.8}$$

$$\gamma(1-\gamma) = 16t^9\left(\frac{1-t^3}{1+8t^3}\right), \tag{3.9}$$

$$m^2 = \frac{(1+2t)^4}{1+8t^3}, \tag{3.10}$$

and

$$m'^2 = 1 + 8t^3. \tag{3.11}$$

Moreover, from (3.7)–(3.9),

$$\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)} = 256\left(\frac{1-t}{1+2t}\right)^{24} \tag{3.12}$$

and

$$\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)} = 256t^{24}. \tag{3.13}$$

It is now a simple matter to establish (i)–(iii). By (3.10)–(3.12),

$$1 + 4^{1/3}\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/24} = 1 + 2\frac{1-t}{1+2t} = \frac{3}{1+2t} = \frac{3}{\sqrt{mm'}},$$

which is (i). By (3.10), (3.11), and (3.13),

$$1 + 4^{1/3}\left(\frac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}\right)^{1/24} = 1 + 2t = \sqrt{mm'},$$

which establishes (ii). By (3.10)–(3.13),

$$1-2^{4/3}\left(\frac{\alpha^3\gamma^3(1-\alpha)^3(1-\gamma)^3}{\beta^2(1-\beta)^2}\right)^{1/24}=\frac{1-2t+4t^2}{1+2t}=\frac{1+8t^3}{(1+2t)^2}=\frac{m'}{m},$$

and so (iii) is proved.

J. M. and P. B. Borwein [2, pp. 142–144] have also given a proof of (i).

Proof of (iv). Applying Entries 10(i), (iii), 11(i), and 12(v) in Chapter 17 to Entry 2(i), we readily find that it translates into the first equality of (iv).

Similarly, the same entries in Chapter 17 can be invoked to translate Entry 2(iii) into the second equality of (iv).

Proof of (v). By (5.2), (5.3), and (5.5) in Chapter 19,

$$\begin{aligned}(\alpha\gamma)^{1/2}&+\{(1-\alpha)(1-\gamma)\}^{1/2}\\
&=(\alpha\beta)^{1/2}\left(\frac{\gamma}{\beta}\right)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}\left(\frac{1-\gamma}{1-\beta}\right)^{1/2}\\
&=\frac{(m+3)^2(m-1)^2}{16m^2}\frac{m'(m'-1)}{3+m'}+\frac{(m+1)^2(3-m)^2}{16m^2}\frac{m'(m'+1)}{3-m'}\\
&=\frac{m'}{16m^2(9-m'^2)}\{(m+3)^2(m-1)^2(m'-1)(3-m')\\
&\qquad+(m+1)^2(3-m)^2(m'+1)(3+m')\}\\
&=\frac{m'}{2m^2(9-m'^2)}\{m'(m^4-2m^2+9)-m(m^2-3)(m'^2+3)\}.\end{aligned}$$

We now substitute (3.10) and (3.11) into the foregoing equality, and, after considerable simplification, we deduce that

$$\begin{aligned}(\alpha\gamma)^{1/2}&+\{(1-\alpha)(1-\gamma)\}^{1/2}\\
&=\frac{1-4t+4t^2+8t^4}{(1+2t)^2}=1-\frac{8t(1-t^3)}{(1+2t)^2}\\
&=1-\frac{8t(1-t^3)}{1+8t^3}+\frac{32t^2(1-t)(1-t^3)}{(1+2t)(1+8t^3)}\\
&=1-2\{4\beta(1-\beta)\}^{1/3}+8\{\beta(1-\beta)\}^{1/4}\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{1/8}, \qquad (3.14)\end{aligned}$$

where we have employed (3.7)–(3.9). Thus, (v) is evident.

Proof of (vi). Using (5.1) in Chapter 19, then (3.10) and (3.11), and then finally (3.8), we find that

$$\left(\frac{\alpha(1-\gamma)}{\beta^3(1-\beta)^3}\right)^{1/8}+\left(\frac{\gamma(1-\alpha)}{\beta^3(1-\beta)^3}\right)^{1/8}=\frac{4m'}{(m-1)(3-m')}+\frac{4m'}{(m+1)(3+m')}$$

$$= \frac{8m'(m' + 3m)}{(m^2 - 1)(9 - m'^2)}$$
$$= \frac{1 + 8t^3}{2t(1 - t^3)}$$
$$= \frac{1}{\{\frac{1}{2}\beta(1 - \beta)\}^{1/3}}, \tag{3.15}$$

from which (vi) follows.

PROOF OF (vii). By (5.1) in Chapter 19,

$$\frac{\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/4} - 1}{1 - \{(1-\alpha)(1-\beta)\}^{1/4}} = \frac{\left(\frac{m+1}{2}\right)^2 - 1}{1 - \frac{(m+1)(3-m)}{4m}} = m.$$

The proof of the second equality is similar.

PROOF OF (viii). By (5.1) in Chapter 19,

$$1 + \left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8} = \frac{m^2 + 3}{4},$$

while

$$m(\tfrac{1}{2}\{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}\})^{1/2}$$
$$= \frac{m}{\sqrt{2}}\left(1 + \frac{(3+m)^2(m-1)^2}{16m^2} + \frac{(m+1)^2(3-m)^2}{16m^2}\right)^{1/2} = \frac{m^2+3}{4}.$$

Hence, (viii) follows.

PROOF OF (ix). This formula is simply the reciprocal of (viii), in the sense of Entry 24(v) of Chapter 18.

PROOF OF (x). By (vi) and (3.7)–(3.11),

$$\left(\frac{\gamma}{\alpha}\right)^{1/8} + \left(\frac{1-\gamma}{1-\alpha}\right)^{1/8} = \left(\frac{256\beta(1-\beta)}{\alpha^3(1-\alpha)^3}\right)^{1/24}$$
$$= \frac{1+2t}{1-t} = \frac{t(1+2t)}{1-t} + (1+2t)$$
$$= \left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8} + \sqrt{mm'},$$

from which (x) is apparent.

PROOF OF (xi). This is the reciprocal of (x).

PROOF OF (xii). Employing once again (5.1) of Chapter 19 and (3.7)–(3.11), we find that

$$\left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} + \left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4}$$

$$= \left(\frac{\beta^3}{\alpha}\right)^{1/4} (\beta\gamma)^{-1/4} + \left(\frac{(1-\beta)^3}{(1-\alpha)}\right)^{1/4} \{(1-\beta)(1-\gamma)\}^{-1/4}$$

$$= \frac{m'(m-1)^2}{(m'-1)(3+m')} + \frac{m'(m+1)^2}{(m'+1)(3-m')}$$

$$= \frac{4m'^2(m^2+1) - 4mm'(3-m'^2)}{(m'^2-1)(9-m'^2)}$$

$$= \frac{1+2t}{t(1-t)}$$

$$= \frac{(1+2t)(1+t+t^2)}{t(1-t)(1-2t+4t^2)} - \frac{3(1+2t)^2}{1+8t^3}$$

$$= \left(\frac{\beta^2(1-\beta)^2}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} - 3\frac{m}{m'}, \tag{3.16}$$

which completes the proof of (xii).

Proof of (xiii). By (3.16) and (3.7)–(3.11),

$$\left(\frac{\alpha\gamma}{\beta^2}\right)^{1/4} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)^2}\right)^{1/4} - \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4}$$

$$= \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta^2(1-\beta)^2}\right)^{1/4} \left\{\left(\frac{(1-\beta)^2}{(1-\alpha)(1-\gamma)}\right)^{1/4} + \left(\frac{\beta^2}{\alpha\gamma}\right)^{1/4} - 1\right\}$$

$$= \frac{t(1-t)(1-2t+4t^2)}{(1+2t)(1+t+t^2)} \left\{\frac{1+2t}{t(1-t)} - 1\right\}$$

$$= \frac{1-2t+4t^2}{1+2t} = \frac{m'}{m}.$$

Proof of (xiv). As in (3.15),

$$\left(\frac{\alpha(1-\gamma)}{\beta^3(1-\beta)^3}\right)^{1/8} - \left(\frac{\gamma(1-\alpha)}{\beta^3(1-\beta)^3}\right)^{1/8} = \frac{4m'}{(m-1)(3-m')} - \frac{4m'}{(m+1)(3+m')}$$

$$= \frac{8m'(mm'+3)}{(m^2-1)(9-m'^2)}$$

$$= \frac{m'(1-2t+4t^2)}{2t(1-t^3)}$$

$$= \frac{1+8t^3}{2t(1-t^3)}\sqrt{\frac{m'}{m}}$$

$$= \frac{1}{\{\frac{1}{2}\beta(1-\beta)\}^{1/3}}\sqrt{\frac{m'}{m}},$$

where we have utilized (3.10), (3.11), and lastly (3.8). Inverting the last equality and dividing both sides by $\{\frac{1}{2}\beta(1-\beta)\}^{1/3}$, we complete the proof.

PROOF OF (xv). By (5.3) in Chapter 19,

$$(\alpha\beta)^{1/4}-(\beta\gamma)^{1/4}=\frac{(m-1)(3+m)}{4m}-\frac{(m'-1)(3+m')}{4m'}=\frac{(m-m')(mm'+3)}{4mm'};$$

by (5.1) in Chapter 19,

$$\{(1-\beta)(1-\gamma)\}^{1/4}-\{(1-\alpha)(1-\beta)\}^{1/4}=\frac{(3-m')(m'+1)}{4m'}-\frac{(3-m)(m+1)}{4m}$$
$$=\frac{(m-m')(mm'+3)}{4mm'};$$

and by foregoing expressions,

$$(\alpha\beta)^{1/4}\{(1-\beta)(1-\gamma)\}^{1/4}-(\beta\gamma)^{1/4}\{(1-\alpha)(1-\beta)\}^{1/4}$$
$$=\frac{(m-m')(mm'+3)}{4mm'}.$$

From these formulas it is evident that

$$(1-\beta)\{(\alpha\beta)^{1/4}-(\beta\gamma)^{1/4}\}^4+\beta(\{(1-\beta)(1-\gamma)\}^{1/4}-\{(1-\alpha)(1-\beta)\}^{1/4})^4$$
$$=((\alpha\beta)^{1/4}\{(1-\beta)(1-\gamma)\}^{1/4}-(\beta\gamma)^{1/4}\{(1-\alpha)(1-\beta)\}^{1/4})^4.$$

Dividing both sides by $\beta(1-\beta)$, we obtain (xv).

PROOF OF (xvi). By (3.14), (3.10), (3.11), and (3.8),

$$(\alpha\gamma)^{1/2}+\{(1-\alpha)(1-\gamma)\}^{1/2}=1-\frac{8t(1-t^3)}{(1+2t)^2}$$
$$=1-\frac{8t(1-t^3)}{1+8t^3}\frac{m'}{m}$$
$$=1-2\{4\beta(1-\beta)\}^{1/3}\frac{m'}{m}, \qquad (3.17)$$

which establishes (xvi).

Entry 4. *We have*

(i) $$\frac{\varphi(-q^{18})}{\varphi(-q^2)}+q\left(\frac{\psi(q^9)}{\psi(q)}-\frac{\psi(-q^9)}{\psi(-q)}\right)=1,$$

(ii) $$\frac{\varphi(-q^2)}{\varphi(-q^{18})}+\frac{1}{q}\left(\frac{\psi(q)}{\psi(q^9)}-\frac{\psi(-q)}{\psi(-q^9)}\right)=3,$$

(iii) $$\frac{\varphi(-q^2)\varphi(-q^{54})}{\varphi(-q^6)\varphi(-q^{18})}+q^2\left(\frac{\psi(q)\psi(q^{27})}{\psi(q^3)\psi(q^9)}+\frac{\psi(-q)\psi(-q^{27})}{\psi(-q^3)\psi(-q^9)}\right)=1,$$

and

(iv) $\varphi(q)\varphi(q^{27}) - \varphi(-q)\varphi(-q^{27}) = 4qf(-q^6)f(-q^{18}) + 4q^7\psi(q^2)\psi(q^{54})$.

PROOFS OF (i), (ii). If we transcribe the proposed equalities via Entries 10(iii) and 11(i), (ii) in Chapter 17, we obtain Entries 3(x), (xi), respectively, and thus (i) and (ii) are established.

PROOF OF (iii). Employing, in turn, Corollary (i), (ii) in Section 31 of Chapter 16, Entry 4(i), Example (v) in Section 31 of Chapter 16, (2.3), and Entry 24(iii) in Chapter 16, we find that

$$\begin{aligned}
&\frac{\varphi(-q^2)\varphi(-q^{54})}{\varphi(-q^6)\varphi(-q^{18})} + q^2\left(\frac{\psi(q)\psi(q^{27})}{\psi(q^3)\psi(q^9)} + \frac{\psi(-q)\psi(-q^{27})}{\psi(-q^3)\psi(-q^9)}\right) - 1 \\
&= \frac{\{\varphi(-q^{18}) - 2q^2f(-q^6, -q^{30})\}\varphi(-q^{54})}{\varphi(-q^6)\varphi(-q^{18})} \\
&\quad + q^2\left(\frac{\{f(q^3,q^6)+q\psi(q^9)\}\psi(q^{27})}{\psi(q^3)\psi(q^9)} + \frac{\{f(-q^3,-q^6)-q\psi(-q^9)\}\psi(-q^{27})}{\psi(-q^3)\psi(-q^9)}\right) - 1 \\
&= -\frac{2q^2f(-q^6,-q^{30})\varphi(-q^{54})}{\varphi(-q^6)\varphi(-q^{18})} + \frac{q^2f(q^3,q^6)\psi(q^{27})}{\psi(q^3)\psi(q^9)} + \frac{q^2f(-q^3,-q^6)\psi(-q^{27})}{\psi(-q^3)\psi(-q^{27})} \\
&= -q^2\left(\frac{2\psi(q^{18})\chi(-q^6)\varphi(-q^{54})}{\varphi(-q^6)\varphi(-q^{18})} - \frac{\varphi(-q^9)\psi(q^{27})}{\psi(q^3)\psi(q^9)\chi(-q^3)} - \frac{\varphi(q^9)\psi(-q^{27})}{\psi(-q^3)\psi(-q^9)\chi(q^3)}\right) \\
&= -\frac{q^2}{f(-q^6)}\left(\frac{2\psi(q^{18})\varphi(-q^{54})}{\varphi(-q^{18})} - \frac{\varphi(-q^9)\psi(q^{27})}{\psi(q^9)} - \frac{\varphi(q^9)\psi(-q^{27})}{\psi(-q^9)}\right). \qquad (4.1)
\end{aligned}$$

Now by Entries 10(i)–(iii) and 11(i)–(iii) in Chapter 17,

$$\begin{aligned}
&\frac{2\psi(q^2)\varphi(-q^6)}{\varphi(-q^2)} - \frac{\varphi(-q)\psi(q^3)}{\psi(q)} - \frac{\varphi(q)\psi(-q^3)}{\psi(-q)} \\
&= \sqrt{z_3}\left(\frac{\beta(1-\beta)}{q^2\alpha(1-\alpha)}\right)^{1/8}\left\{\left(\frac{\alpha^3}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} - 1\right\} = 0,
\end{aligned}$$

by Entry 5(i) of Chapter 19. Hence, the far right side of (4.1) is also equal to 0, and so the proof of (iii) is complete.

PROOF OF (iv). We employ in turn the following results from Chapter 16: Corollary (i) in Section 31; (36.2) with $A = B = 1$, $\mu = 2$, $\nu = 1$, and q replaced by q^9; (36.1) with $A = 1$, $B = q^6$, $\mu = 2$, $\nu = 1$, and q replaced by q^9; Entry 18(iv) three times with $n = 1$; Entry 31 with $a = \pm q$, $b = \pm q^2$, and $n = 2$; (36.2) with $A = q^9$, $B = q^{-3}$, $\mu = 2$, $\nu = 1$, and q replaced by q^9; Entry 18(iv) three times with $n = 1$; and Corollary (ii) in Section 31. Hence,

$$\begin{aligned}
&\varphi(q)\varphi(q^{27}) - \varphi(-q)\varphi(-q^{27}) \\
&\quad = \varphi(q^9)\varphi(q^{27}) - \varphi(-q^9)\varphi(-q^{27}) + 2q\{f(q^3, q^{15})\varphi(q^{27}) \\
&\quad\quad + f(-q^3, -q^{15})\varphi(-q^{27})\}
\end{aligned}$$

$$
\begin{aligned}
&= 2\{q^{27}f(q^{54}, q^{162})f(q^{90}, q^{-18}) + q^{117}f(q^{270}, q^{-54})f(q^{126}, q^{-54})\} \\
&\quad + 4q\{f(q^{90}, q^{126})f(q^{30}, q^{42}) + q^{24}f(q^{18}, q^{198})f(q^{6}, q^{66})\} \\
&= 4q^{9}f(q^{54}, q^{162})f(q^{18}, q^{54}) \\
&\quad + 2q(\{f(q^{90}, q^{126}) + q^{18}f(q^{18}, q^{198})\}\{f(q^{30}, q^{42}) + q^{6}f(q^{6}, q^{66})\} \\
&\quad + \{f(q^{90}, q^{126}) - q^{18}f(q^{18}, q^{198})\}\{f(q^{30}, q^{42}) - q^{6}f(q^{6}, q^{66})\}) \\
&= 4q^{9}\psi(q^{54})\psi(q^{18}) + 2q\{f(q^{18}, q^{36})f(q^{6}, q^{12}) \\
&\quad + f(-q^{18}, -q^{36})f(-q^{6}, -q^{12})\} \\
&= 4q^{9}f(q^{54}, q^{162})f(q^{18}, q^{54}) + 4qf(-q^{18}, -q^{36})f(-q^{6}, -q^{12}) \\
&\quad + 2q\{f(q^{18}, q^{36})f(q^{6}, q^{12}) - f(-q^{18}, -q^{36})f(-q^{6}, -q^{12})\} \\
&= 4q^{9}\psi(q^{54})\psi(q^{18}) + 4qf(-q^{18})f(-q^{6}) \\
&\quad + 4q^{10}\{q^{27}f(q^{162}, q^{54})f(q^{102}, q^{-30}) + q^{123}f(q^{270}, q^{-54})f(q^{138}, q^{-66})\} \\
&= 4q^{9}\psi(q^{54})\psi(q^{18}) + 4qf(-q^{18})f(-q^{6}) \\
&\quad + 4q^{7}f(q^{54}, q^{162})\{f(q^{30}, q^{42}) + q^{6}f(q^{6}, q^{66})\} \\
&= 4q^{9}\psi(q^{54})\psi(q^{18}) + 4qf(-q^{18})f(-q^{6}) + 4q^{7}\psi(q^{54})f(q^{6}, q^{12}) \\
&= 4qf(-q^{6})f(-q^{18}) + 4q^{7}\psi(q^{54})\psi(q^{2}).
\end{aligned}
$$

This completes the proof of Entry 4(iv).

Entry 5. *Let α, β, γ, and δ be of the first, third, ninth, and twenty-seventh degrees, respectively. Let m be the multiplier connecting α and β, and let m'' denote the multiplier associated with γ and δ. Then*

(i) $$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} + \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} = \sqrt{\frac{m''}{m}},$$

(ii) $$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/4} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/4} + \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4}$$
$$-2\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}\left\{1+\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8}+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8}\right\} = -3\frac{m}{m''},$$

(iii) $$\frac{1-(\alpha\delta)^{1/4}-\{(1-\alpha)(1-\delta)\}^{1/4}}{2\{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/12}} = \sqrt{\frac{m''}{m}},$$

and

(iv) $$\frac{\{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/24} + \{\alpha\delta(1-\alpha)(1-\delta)\}^{1/8}}{\{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/24} + \{\beta\gamma(1-\beta)(1-\gamma)\}^{1/8}} = \sqrt{\frac{m''}{m}}.$$

PROOF OF (i). If we utilize Entries 10(iii) and 11(i), (ii) of Chapter 17 in Entry 4(iii), we obtain the desired result immediately.

PROOF OF (iii). Employing Entries 10(i), (ii), 11(iii), and 12(iii) of Chapter 17 in Entry 4(iv), we obtain (iii) with no difficulty.

PROOF OF (ii). Since δ has degree 3 over γ, we proceed as in Section 3 and (see (3.11)) define a parameter u by

$$m'' = 1 + 8u^3. \tag{5.1}$$

Thus, we deduce analogues of (3.7)–(3.9), namely,

$$\beta(1-\beta) = 16u\left(\frac{1-u}{1+2u}\right)^8 \frac{1-u^3}{1+8u^3}, \tag{5.2}$$

$$\gamma(1-\gamma) = 16u^3\left(\frac{1-u^3}{1+8u^3}\right)^3, \tag{5.3}$$

and

$$\delta(1-\delta) = 16u^9\left(\frac{1-u^3}{1+8u^3}\right). \tag{5.4}$$

Let m' be as in Section 3. It follows from (3.10) and (3.11) that

$$1 + 8t^3 = m'^2 = \frac{(1+2u)^4}{1+8u^3}, \tag{5.5}$$

from which we readily deduce that

$$t^3(1-2u+4u^2) = u(1+u+u^2). \tag{5.6}$$

From (3.7), (3.9), (5.2), and (5.4),

$$\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\frac{\delta(1-\delta)}{\beta(1-\beta)}\right)^{1/8} = \frac{u(1+2u)(1-t)}{t(1+2t)(1-u)}, \tag{5.7}$$

while from (3.10), (3.11), (5.1), and (5.5),

$$\sqrt{\frac{m''}{m}} = \sqrt{\frac{m'}{m}\frac{m''}{m'}} = \frac{1+2u}{1+2t}. \tag{5.8}$$

Thus, from Entry 5(i), (5.7), and (5.8),

$$\begin{aligned}\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} &= \frac{1+2u}{1+2t} - \frac{u(1+2u)(1-t)}{t(1+2t)(1-u)} \\ &= \frac{(1+2u)(t-u)}{t(1+2t)(1-u)}.\end{aligned} \tag{5.9}$$

Next, by (5.7)–(5.9),

$$\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/4} + \left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/4} + 1 + 3\frac{m}{m''}\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/4}$$

$$= \frac{(1+2u)^2(t-u)^2}{t^2(1+2t)^2(1-u)^2} - 2\frac{u(1+2u)(1-t)}{t(1+2t)(1-u)} + 1 + 3\left(\frac{1+2t}{1+2u}\right)^2 \frac{u^2(1+2u)^2(1-t)^2}{t^2(1+2t)^2(1-u)^2}$$

$$= \frac{1}{t^2(1+2t)^2(1-u)^2}(2t^2(1-u)^2(1+2t)(1+2u)$$
$$+ 4(t-u)\{t^3(1-2u+4u^2)-(u+u^2+u^3)\})$$
$$= \frac{2(1+2u)}{1+2t} = 2\sqrt{\frac{m''}{m}}, \tag{5.10}$$

where we have omitted a heavy amount of tedious elementary algebra to obtain the penultimate line, where (5.6) was used in the penultimate step, and where (5.8) was employed at the end. Multiplying the extremal sides of (5.10) by $(\beta\gamma(1-\beta)(1-\gamma)/\alpha\delta(1-\alpha)(1-\delta))^{1/4}$, we arrive at

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/4} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/4} + \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4}$$
$$- 2\sqrt{\frac{m''}{m}}\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4} = -3\frac{m}{m''}.$$

Lastly, if we substitute the formula for $\sqrt{m''/m}$, given in (i), into the left side above, we obtain (ii).

Proof of (iv). Using (3.7)–(3.9) and (5.2)–(5.4), we observe that

$$\frac{\{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/24} + \{16\alpha\delta(1-\alpha)(1-\delta)\}^{1/8}}{\{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/24} + \{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/8}}$$
$$= \frac{1 + \left(256\dfrac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\dfrac{\delta^3(1-\delta)^3}{\gamma(1-\gamma)}\right)^{1/24}}{1 + \left(256\dfrac{\beta^3(1-\beta)^3}{\gamma(1-\gamma)}\dfrac{\gamma^3(1-\gamma)^3}{\beta(1-\beta)}\right)^{1/24}}$$
$$= \frac{1 + 2\dfrac{1-t}{1+2t}u}{1 + 2\dfrac{1-u}{1+2u}t} = \frac{1+2u}{1+2t} = \sqrt{\frac{m''}{m}},$$

where we have used (5.8). Thus, the truth of (iv) is manifest.

Entry 6. *We have*

(i) $$\psi(q^{1/11}) - q^{15/11}\psi(q^{11}) = f(q^5, q^6) + q^{1/11}f(q^4, q^7) + q^{3/11}f(q^3, q^8) + q^{6/11}f(q^2, q^9) + q^{10/11}f(q, q^{10}),$$

(ii) $$\varphi(q^{1/11}) - \varphi(q^{11}) = 2q^{1/11}f(q^9, q^{13}) + 2q^{4/11}f(q^7, q^{15}) + 2q^{9/11}f(q^5, q^{17}) + 2q^{16/11}f(q^3, q^{19}) + 2q^{25/11}f(q, q^{21}),$$

and

(iii) $$\frac{f(-q^{1/11})}{f(-q^{11})} = \frac{f(-q^4, -q^7)}{f(-q^2, -q^9)} - q^{1/11}\frac{f(-q^2, -q^9)}{f(-q, -q^{10})} - q^{2/11}\frac{f(-q^5, -q^6)}{f(-q^3, -q^8)} + q^{5/11} + q^{7/11}\frac{f(-q^3, -q^8)}{f(-q^4, -q^7)} - q^{15/11}\frac{f(-q, -q^{10})}{f(-q^5, -q^6)}.$$

PROOF OF (i). In Entry 31 of Chapter 16, set $a = 1$, $b = q$, and $n = 11$. Using Entry 18(ii) of Chapter 16, we obtain the equality

$$\psi(q) = f(q^{55}, q^{66}) + qf(q^{44}, q^{77}) + q^3 f(q^{33}, q^{88}) + q^6 f(q^{22}, q^{99}) + q^{10} f(q^{11}, q^{110}) + q^{15}\psi(q^{121}).$$

If we replace q by $q^{1/11}$, we complete the proof.

PROOF OF (ii). Putting $a = b = q$ and $n = 11$ in Entry 31 of Chapter 16, we find that

$$\varphi(q) = \varphi(q^{121}) + 2qf(q^{99}, q^{143}) + 2q^4 f(q^{77}, q^{165}) + 2q^9 f(q^{55}, q^{187}) + 2q^{16} f(q^{33}, q^{209}) + 2q^{25} f(q^{11}, q^{231}).$$

Replacing q by $q^{1/11}$, we achieve the desired result.

PROOF OF (iii). Set $n = 11$ in (12.26) of Chapter 19.

Entry 7. *The following are modular equations of degree* 11:

(i) $$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = 1,$$

(ii) $$m - \frac{11}{m} = 2((\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4}) \times (4 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}),$$

(iii) $$m + \frac{11}{m} = 2\sqrt{2}(2 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}) \times (1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})^{1/2},$$

(iv) $$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^3}{\alpha}\right)^{1/8} - \left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8} = \frac{m}{\sqrt{2}}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})^{1/2},$$

(v) $$\left(\frac{\alpha^3}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} - \left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8} = \frac{11}{m\sqrt{2}}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})^{1/2},$$

(vi) $$\frac{1}{m}\left\{1+2^{10/3}\left(\frac{\beta^{11}(1-\beta)^{11}}{\alpha(1-\alpha)}\right)^{1/24}\right\}-\frac{m}{11}\left\{1+2^{10/3}\left(\frac{\alpha^{11}(1-\alpha)^{11}}{\beta(1-\beta)}\right)^{1/24}\right\}$$
$$= 2((\alpha\beta)^{1/2} - \{(1-\alpha)(1-\beta)\}^{1/2}),$$

and

(vii) $$\frac{1}{m}\left\{1+2^{10/3}\left(\frac{\beta^{11}(1-\beta)^{11}}{\alpha(1-\alpha)}\right)^{1/24}\right\}+\frac{m}{11}\left\{1+2^{10/3}\left(\frac{\alpha^{11}(1-\alpha)^{11}}{\beta(1-\beta)}\right)^{1/24}\right\}$$
$$= 2\sqrt{2}((\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4})$$
$$\times (1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})^{1/2}.$$

These modular equations are followed by two further modular equations of degree 11, designated by (viii) and (ix). However, Ramanujan (p. 244) has crossed them out.

The first modular equation of degree 11 to appear in the literature was established by Sohncke [1], [2]. The modular equation (i) is due, in 1858, to Schröter [3], [4], who earlier [1], [2] had established slightly more complicated modular equations of degree 11. The remaining six modular equations in Entry 7 are due to Ramanujan. More complex modular equations of degree 11 were discovered by Schläfli [1], Fiedler [1], Fricke [1], and Russell [1], [2].

Because of its extensive use throughout the sequel, we record here the equality

$$f(a, b) = f(a^3b, ab^3) + af\left(\frac{b}{a}, \frac{a}{b}a^4b^4\right), \tag{7.1}$$

which can be deduced either by adding Entries 30(ii), (iii) or by applying Entry 31 with $n = 2$, where the cited entries are in Chapter 16.

PROOF OF (i). Setting $A = B = 1$, $\mu = 6$, and $\nu = 5$ in (36.2) of Chapter 16, we find that

$$\varphi(Q)\varphi(q) - \varphi(-Q)\varphi(-q)$$
$$= 2\sum_{n=0}^{5} q^{11(2n+1)+12n^2} f(Q^{10-4n}, Q^{14+4n}) f(q^{-10-20n}, q^{34+20n}), \tag{7.2}$$

where $Q = q^{11}$. Next, in (36.10) of Chapter 16, replace q by q^2 and set $\mu = 6$ and $\nu = 5$. Thus,

$$\psi(Q^2)\psi(q^2) = \sum_{n=0}^{2} q^{12n(n+1)} f(Q^{10-4n}, Q^{14+4n}) f(q^{2-20n}, q^{22+20n})$$
$$= \frac{1}{2}\sum_{n=0}^{5} q^{12n(n+1)} f(Q^{10-4n}, Q^{14+4n}) f(q^{2-20n}, q^{22+20n}). \tag{7.3}$$

The last equality can be demonstrated by showing that the terms of index n and $5 - n$, $0 \le n \le 2$, are equal. In order to do this, we apply Entry 18(iv) of

Chapter 16 twice to deduce that

$$f(Q^{-10+4n}, Q^{34-4n}) = Q^{-10+4n} f(Q^{10-4n}, Q^{14+4n})$$

and

$$f(q^{-98+20n}, q^{122-20n}) = q^{15(-98+20n)+10(122-20n)} f(q^{2-20n}, q^{22+20n}).$$

Multiplying (7.3) by $4q^3$ and subtracting the result from (7.2), we deduce that

$$\begin{aligned}&\varphi(Q)\varphi(q) - \varphi(-Q)\varphi(-q) - 4q^3\psi(Q^2)\psi(q^2)\\ &\quad = 2\sum_{n=0}^{5} q^{12n^2+22n+11} f(Q^{10-4n}, Q^{14+4n})\{f(q^{-10-20n}, q^{34+20n})\\ &\qquad - q^{-10n-8} f(q^{2-20n}, q^{22+20n})\}.\end{aligned}$$

We next apply (7.1) with $a = -q^{-8-10n}$ and $b = -q^{14+10n}$. The expression within curly brackets above is thus found to equal $f(-q^{-8-10n}, -q^{14+10n})$. Hence,

$$\begin{aligned}&\varphi(Q)\varphi(q) - \varphi(-Q)\varphi(-q) - 4q^3\psi(Q^2)\psi(q^2)\\ &\quad = 2\sum_{n=0}^{5} q^{12n^2+22n+11} f(Q^{10-4n}, Q^{14+4n}) f(-q^{-8-10n}, -q^{14+10n}). \qquad (7.4)\end{aligned}$$

Replacing n by $n + 3$ in the last three summands above, we find, after an application of Entry 18(iv) in Chapter 16, that

$$\begin{aligned}&q^{12(n+3)^2+22(n+3)+11} f(Q^{10-4(n+3)}, Q^{14+4(n+3)}) f(-q^{-8-10(n+3)}, -q^{14+10(n+3)})\\ &\quad = q^{12n^2+94n+185} f(Q^{-2-4n}, Q^{26+4n})(-q^{-38-10n})^{15}(-q^{44+10n})^{10}\\ &\qquad \times f(-q^{-8-10n}, -q^{14+10n})\\ &\quad = -q^{12n^2+22n+11} Q^{4+2n} f(Q^{-2-4n}, Q^{26+4n}) f(-q^{-8-10n}, -q^{14+10n}).\end{aligned}$$

Thus, the right side of (7.4) may be rewritten in the form

$$\begin{aligned}&2\sum_{n=0}^{2} q^{12n^2+22n+11}\{f(Q^{10-4n}, Q^{14+4n}) - Q^{4+2n} f(Q^{-2-4n}, Q^{26+4n})\}\\ &\qquad \times f(-q^{-8-10n}, -q^{14+10n}).\end{aligned}$$

Applying (7.1) with $a = -Q^{4+2n}$ and $b = -Q^{2-2n}$, we find that the expression in curly brackets above equals $f(-Q^{4+2n}, -Q^{2-2n})$. Thus, from (7.4) and the observations just made, we deduce that

$$\begin{aligned}&\varphi(Q)\varphi(q) - \varphi(-Q)\varphi(-q) - 4q^3\psi(Q^2)\psi(q^2)\\ &\quad = 2\sum_{n=0}^{2} q^{12n^2+22n+11} f(-Q^{2-2n}, -Q^{4+2n}) f(-q^{-8-10n}, -q^{14+10n})\\ &\quad = 2q^{11}\{f(-Q^2) f(-q^{-8}, -q^{14}) + q^{92} f(-Q^{-2}, -Q^8) f(-q^{-28}, -q^{34})\}\\ &\quad = 4q f(-Q^2) f(-q^2), \qquad (7.5)\end{aligned}$$

where, to obtain the penultimate line, we used the fact that $f(-1, -Q^6) = 0$ (Entry 18(iii) of Chapter 16), and, to obtain the last line, we employed Entry 18(iv) of Chapter 16 three times.

Lastly, we utilize Entries 10(i), (ii), 11(iii), and 12(iii) in Chapter 17 to transcribe (7.5) into the proposed modular equation.

The theta-function identity (7.5) has also been proved by Kondo and Tasaka [1, Eq. (T24)].

Before proceeding with the remaining proofs, which utilize the theory of modular forms, we observe that (v) is the reciprocal of (iv). Thus, it remains to prove (ii), (iii), (iv), (vi), and (vii). Transcribing these modular equations via Entries 10(i)–(iv), 11(i)–(iv), and 12(i) of Chapter 17, we deduce that, respectively,

$$\varphi^4(q)-11\varphi^4(q^{11}) = 2(4q^3\psi(q^2)\psi(q^{22}) - \varphi(-q)\varphi(-q^{11})) \times (4\varphi(q)\varphi(q^{11})+4q^3\psi(q^2)\psi(q^{22})+\varphi(-q)\varphi(-q^{11})), \tag{7.6}$$

$$\varphi^4(q)+11\varphi^4(q^{11}) = 4(\varphi(q^2)\varphi(q^{22})+4q^6\psi(q^4)\psi(q^{44})) \times (2\varphi(q)\varphi(q^{11}) + 4q^3\psi(q^2)\psi(q^{22}) + \varphi(-q)\varphi(-q^{11})), \tag{7.7}$$

$$\varphi(q^2)\varphi(q^{22})+4q^6\psi(q^4)\psi(q^{44}) = \frac{\varphi^3(-q^{22})}{\varphi(-q^2)} - 2q^4\frac{\psi^3(q^{11})}{\psi(q)} - 2q^4\frac{\psi^3(-q^{11})}{\psi(-q)}, \tag{7.8}$$

$$2(16q^6\psi^2(q^2)\psi^2(q^{22})-\varphi^2(-q)\varphi^2(-q^{11})) = \varphi^4(q^{11})\left(1+32q^5\frac{f^{11}(q^{11})\varphi(q)}{f(q)\varphi^{11}(q^{11})}\right) - \frac{\varphi^4(q)}{11}\left(1+32\frac{f^{11}(q)\varphi(q^{11})}{f(q^{11})\varphi^{11}(q)}\right), \tag{7.9}$$

and

$$4(4q^3\psi(q^2)\psi(q^{22}) + \varphi(-q)\varphi(-q^{11}))(\varphi(q^2)\varphi(q^{22}) + 4q^6\psi(q^4)\psi(q^{44})) = \varphi^4(q^{11})\left(1 + 32q^5\frac{f^{11}(q^{11})\varphi(q)}{f(q)\varphi^{11}(q^{11})}\right) + \frac{\varphi^4(q)}{11}\left(1 + 32\frac{f^{11}(q)\varphi(q^{11})}{f(q^{11})\varphi^{11}(q)}\right). \tag{7.10}$$

Next, we convert (7.6)–(7.10) into equalities relating the modular forms f_1, g_0, g_1, g_2, h_0, h_1, and h_2, defined by (0.12). Thus, we deduce from (0.13) that

$$g_1^4(\tau)-11g_1^4(11\tau) = 2(4g_2(\tau)g_2(11\tau) - g_0(\tau)g_0(11\tau)) \times (4g_1(\tau)g_1(11\tau)+4g_2(\tau)g_2(11\tau)+g_0(\tau)g_0(11\tau)), \tag{7.11}$$

$$g_1^4(\tau)+11g_1^4(11\tau) = 4(g_1(2\tau)g_1(22\tau) + 4g_2(2\tau)g_2(22\tau)) \times (2g_1(\tau)g_1(11\tau) + 4g_2(\tau)g_2(11\tau) + g_0(\tau)g_0(11\tau)), \tag{7.12}$$

$$g_1(2\tau)g_1(22\tau) + 4g_2(2\tau)g_2(22\tau) = \frac{h_2^3(11\tau)}{h_2(\tau)} - 2\frac{h_0^3(11\tau)}{h_0(\tau)} - 2\frac{h_1^3(11\tau)}{h_1(\tau)}, \qquad (7.13)$$

$$2(16g_2^2(\tau)g_2^2(11\tau) - g_0^2(\tau)g_0^2(11\tau))$$
$$= g_1^4(11\tau)\left(1+32\frac{f_1^{11}(11\tau)g_1(\tau)}{f_1(\tau)g_1^{11}(11\tau)}\right) - \frac{g_1^4(\tau)}{11}\left(1+32\frac{f_1^{11}(\tau)g_1(11\tau)}{f_1(11\tau)g_1^{11}(\tau)}\right), \qquad (7.14)$$

and

$$4(4g_2(\tau)g_2(11\tau) + g_0(\tau)g_0(11\tau))(g_1(2\tau)g_1(22\tau) + 4g_2(2\tau)g_2(22\tau))$$
$$= g_1^4(11\tau)\left(1+32\frac{f_1^{11}(11\tau)g_1(\tau)}{f_1(\tau)g_1^{11}(11\tau)}\right) + \frac{g_1^4(\tau)}{11}\left(1+32\frac{f_1^{11}(\tau)g_1(11\tau)}{f_1(11\tau)g_1^{11}(\tau)}\right). \qquad (7.15)$$

We next determine the multiplier system of each term in (7.11)–(7.15). From the theory at the beginning of this chapter, $f_1(\tau)$, $f_1(11\tau)$, $g_j(\tau)$, $g_j(11\tau)$, $h_j(\tau)$, and $h_j(11\tau)$, $0 \le j \le 2$, are modular forms on the subgroup $\Gamma(2) \cap \Gamma_0(11)$. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2) \cap \Gamma_0(11)$. From (0.16), (0.19), and (0.25), we find that $g_1^4(\tau)$, $g_1^4(11\tau)$, $f_1^{11}(11\tau)g_1(\tau)/\{f_1(\tau)g_1^{11}(11\tau)\}$, and $f_1^{11}(\tau)g_1(11\tau)/\{f_1(11\tau)g_1^{11}(\tau)\}$ each have a multiplier system identically equal to 1. Also, $g_j(\tau)g_j(11\tau)$, $0 \le j \le 2$, has a multiplier system equal to $(\frac{d}{11})$, by (0.18)–(0.20), (0.27), and (0.11), since b and c are even, where $(\frac{u}{v})$ denotes the Legendre–Jacobi symbol. By (0.21)–(0.23), (0.25), and (0.11), for $0 \le j \le 2$, $h_j^3(11\tau)/h_j(\tau)$ has a multiplier system equal to

$$\left(\frac{11}{d}\right)_*^3 \xi_1^2 \xi_2^{3\cdot 11-1} = \left(\frac{11}{d}\right)_* (-1)^{(d-1)/2} = \left(\frac{d}{11}\right). \qquad (7.16)$$

It remains to find the multiplier system for $g_1(2\tau)g_1(22\tau) + 4g_2(2\tau)g_2(22\tau)$.

Lemma 7.1. *The function $g_1(2\tau)g_1(22\tau) + 4g_2(\tau)g_2(22\tau)$ is a modular form on $\Gamma(2) \cap \Gamma_0(11)$ with multiplier system $(\frac{d}{11})$.*

By multiplying out, if necessary, in (7.11)–(7.15), we now see that each side is a modular form on $\Gamma(2) \cap \Gamma_0(11)$. Furthermore, the modular forms on each side of each equality sign have the same multiplier system. Except for (7.13), each of these multiplier systems is identically equal to 1; the modular forms on each side of (7.13) have a multiplier system equal to $(\frac{d}{11})$.

PROOF OF LEMMA 7.1. Note that $g_j(2\tau)$, $j = 1, 2$, is a modular form on $\Gamma' = \Gamma(2) \cap \Gamma_0(4)$. Since

$$\Gamma(2) = \Gamma' \cup \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \Gamma',$$

it follows that $(\Gamma(2): \Gamma') = 2$. Now $g_j(22\tau)$, $j = 1, 2$, is a modular form on $\Gamma' \cap \Gamma_0(11)$. The index of $\Gamma' \cap \Gamma_0(11)$ in $\Gamma(2) \cap \Gamma_0(11)$ equals 2, for by a lemma in Schoeneberg's book [1, p. 74], it can be shown that

$$\Gamma(2)\cap\Gamma_0(11) = (\Gamma'\cap\Gamma_0(11))\cup A(\Gamma'\cap\Gamma_0(11)),$$

where $A = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, with $a\equiv d\equiv b+1\equiv 1 \pmod 2$, $c\equiv 2 \pmod 4$, and $c\equiv 0 \pmod{11}$. (The indices of these two subgroups will not be explicitly used in the sequel.)

Suppose first that $A = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma(1)$, with a, c, and d odd. Then

$$g_1|A = \frac{\sqrt{2}\left(\eta\left|\begin{matrix}1 & 1\\ 0 & 2\end{matrix}\right| A\right)^2}{\eta\left|\begin{matrix}1 & 1\\ 0 & 1\end{matrix}\right| A} = \frac{\sqrt{2}\left(\eta\left|\begin{matrix}\frac{a+c}{2} & b+d\\ c & 2d\end{matrix}\right| 2\right)^2}{\eta\left|\begin{matrix}a+c & b+d\\ c & d\end{matrix}\right|}$$

$$= \frac{v_\eta\begin{pmatrix}\frac{a+c}{2} & b+d\\ c & 2d\end{pmatrix}^2}{v_\eta\begin{pmatrix}a+c & b+d\\ c & d\end{pmatrix}}2g_2$$

$$=: v_g'(A)2g_2,$$

where, by (0.14), with c odd,

$$v_g'(A) = \left(\frac{d}{c}\right)^* e^{\pi i c(d-1)/4}. \tag{7.17}$$

Similarly,

$$g_2|A = \frac{\left(\eta\left|\begin{matrix}2a & b-a\\ c & \frac{d-c}{2}\end{matrix}\right|\begin{matrix}1 & 1\\ 0 & 2\end{matrix}\right)^2}{\sqrt{2}\eta\left|\begin{matrix}a & b-a\\ c & d-c\end{matrix}\right|\begin{matrix}1 & 1\\ 0 & 1\end{matrix}}$$

$$= \frac{v_\eta\begin{pmatrix}2a & b-a\\ c & \frac{d-c}{2}\end{pmatrix}^2}{v_\eta\begin{pmatrix}a & b-a\\ c & d-c\end{pmatrix}}\tfrac{1}{2}g_1$$

$$= \left(\frac{d}{c}\right)^* e^{\pi i c(a-1)/4}\tfrac{1}{2}g_1, \tag{7.18}$$

where again we have employed (0.14).

Now let $A\in\Gamma(2)\cap\Gamma_0(11)$ with $c\equiv 2 \pmod 4$. Then, by (7.17),

$$\begin{aligned}(g_1|2\cdot g_1|22)|A &= v_g'({}^{(2)}A)v_g'({}^{(22)}A)4g_2|2\cdot g_2|22\\ &= \left(\frac{d}{c/2}\right)^*\left(\frac{d}{c/22}\right)^* e^{\pi i(c/2+c/22)(d-1)/4}4g_2|2\cdot g_2|22\\ &= \left(\frac{d}{11}\right)4g_2|2\cdot g_2|22, \qquad (7.19)\end{aligned}$$

by (0.8). Similarly, by (7.18),

$$(4g_2|2\cdot g_2|22)|A = \left(\frac{d}{11}\right)g_1|2\cdot g_1|22. \qquad (7.20)$$

Let $j = 1$ or 2 and $A \in \Gamma(2) \cap \Gamma_0(11)$ with $c \equiv 0 \pmod 4$. Then $c/2$ is even. By (0.4), (0.19), (0.20), (0.25), and (0.11), it follows that

$$\begin{aligned}(g_j|2\cdot g_j|22)|A &= (g_j\cdot g_j|11)|^{(2)}A|2\\ &= \left(\frac{d}{11}\right)(g_j\cdot g_j|11)|2\\ &= \left(\frac{d}{11}\right)g_j|2\cdot g_j|22. \qquad (7.21)\end{aligned}$$

Equalities (7.19)–(7.21) imply the truth of Lemma 7.1.

We now are in a position to prove (7.11)–(7.13). Clearing denominators (if necessary) and collecting terms on one side, we can write each proposed equality in the form

$$F := F_1 + \cdots + F_m = 0,$$

where F is a modular form of weight r on the group $\Gamma(2) \cap \Gamma_0(11)$. From (0.6), (0.24), and (0.30), if we can show that the coefficients of $q^0, q^1, \ldots, q^\mu$ in F are equal to zero, where $\mu + 1 > 6r$, then, in each case, (7.11)–(7.13) are established. For $0 \le j \le 2$, g_j and h_j each have weight $\frac{1}{2}$. Thus, we obtain the following table:

	r	μ
(7.11)	2	12
(7.12)	2	12
(7.13)	5/2	15.

By employing the computer algebra system MACSYMA, we have, indeed, verified that all of the required coefficients are equal to 0. Thus, the proofs of (7.11)–(7.13) are completed.

In principle, the same procedure can be used to verify (7.14) and (7.15). However, in each case, $r = 10$, and so $\mu = 60$. Since the amount of computation is considerably greater, we show how to decrease the value of μ by deriving more information about the orders at the cusps.

Let $\Gamma := \Gamma(2) \cap \Gamma_0(11)$, and recall that $N(\Gamma; \zeta)$ denotes the width of Γ at the cusp $\zeta \in Q \cup \{\frac{1}{0}\}$, where $\frac{1}{0}$ denotes the point at ∞.

Lemma 7.2. *If r and s are coprime integers, then*

$$N\left(\Gamma; \frac{r}{s}\right) = 2\frac{11}{(11, s)}.$$

Proof. If we choose $B \in \Gamma(1)$ so that

$$B\frac{r}{s} = \frac{1}{0},$$

we find that

$$P := B^{-1}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} B = \begin{pmatrix} 1 - krs & kr^2 \\ -ks^2 & 1 + krs \end{pmatrix}.$$

Thus, $P \in \Gamma$ if and only if $2|k$ and $11|ks^2$. The smallest positive integer k with these properties is $k = 2 \cdot 11/(11, s)$.

Lemma 7.3. *If we set*

$$\zeta_1 = \frac{1}{0}, \quad \zeta_2 = \frac{2}{11}, \quad \zeta_3 = \frac{1}{11}, \quad \zeta_4 = \frac{1}{2}, \quad \zeta_5 = \frac{0}{1}, \quad \text{and} \quad \zeta_6 = \frac{1}{1},$$

then

(i) *$\zeta_1, \ldots, \zeta_6$ is a complete set of inequivalent cusps for Γ, and*
(ii) *if r_1, r_2, s_1, and s_2 are integers such that $(r_1, s_1) = (r_2, s_2) = 1$, then r_1/s_1 and r_2/s_2 are equivalent cusps modulo Γ if and only if*

$$r_1 \equiv r_2 \text{ and } s_1 \equiv s_2 \pmod 2 \quad \text{and} \quad (11, s_1) = (11, s_2). \tag{7.22}$$

Proof. If r_1/s_1 and r_2/s_2 are equivalent cusps modulo Γ, we can choose $B \in \Gamma$ so that

$$B\frac{r_1}{s_1} = \frac{r_2}{s_2}.$$

Then the conditions (7.22) follow, which shows that they are necessary.

Using (7.22), we easily check that no two of $\zeta_1, \ldots, \zeta_6$ are equivalent cusps modulo Γ. Then an application of Lemma 7.2 shows that

$$\sum_{i=1}^{6} N(\Gamma; \zeta_i) = 72 = (\Gamma(1) \colon \Gamma).$$

By a theorem in Rankin's book [2, Eq. (2.4.10)], this shows that $\zeta_1, \ldots, \zeta_6$ is a *complete* set of inequivalent cusps for Γ.

Now suppose that the conditions (7.22) hold. Choose i and j so that $r_1/s_1 \sim \zeta_i$ and $r_2/s_2 \sim \zeta_j$ modulo Γ. But then by (7.22) and the definitions of $\zeta_1, \ldots, \zeta_6$, it follows that $i = j$. Thus, $r_1/s_1 \sim r_2/s_2$, and the proof is complete.

Recall that for a cusp ζ and a modular form f, the order of f with respect to Γ at ζ, $\mathrm{Ord}_\Gamma(f;\zeta)$, and the invariant order of f at ζ, $\mathrm{ord}(f;\zeta)$, are related by the equality

$$\mathrm{Ord}_\Gamma(f;\zeta) = N(\Gamma;\zeta)\mathrm{ord}(f;\zeta), \tag{7.23}$$

where, as above, $N(\Gamma;\zeta)$ is the width of Γ at ζ. We also recall from Lemma 0.1 that if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an integral matrix with determinant $m = ad - bc > 0$, and if $(r, s) = 1$, then

$$\mathrm{ord}\left(f|M;\frac{r}{s}\right) = \frac{g^2}{m}\mathrm{ord}\left(f;M\frac{r}{s}\right), \tag{7.24}$$

where $g = (ar + bs, cr + ds)$.

Lemma 7.4. *Let* $\Gamma = \Gamma(2) \cap \Gamma_0(11)$ *and* $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}$, *where* $\delta \equiv 0 \pmod{11}$ *and* $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(2)$. *Let f denote any modular form on Γ, and let ζ denote any cusp of Γ. Then*

$$\mathrm{Ord}_\Gamma(f|M;\zeta) = \mathrm{Ord}_\Gamma(f;M\zeta).$$

PROOF. Set

$$M\frac{r}{s} = \frac{11\alpha r + \beta s}{11\gamma r + \delta s} = \frac{r'}{s'},$$

where $(r', s') = 1$. Then, by (7.23), (7.24), and Lemma 7.2, if $\zeta = r/s$,

$$\begin{aligned}\mathrm{Ord}_\Gamma(f|M;r/s) &= N\left(\Gamma;\frac{r}{s}\right)\frac{g^2}{m}\mathrm{Ord}\left(f;M\frac{r}{s}\right)\\ &= \frac{N\left(\Gamma;\frac{r}{s}\right)\frac{g^2}{m}}{N\left(\Gamma;\frac{r'}{s'}\right)}\mathrm{Ord}_\Gamma\left(f;M\frac{r}{s}\right)\\ &= \frac{(11,s')(11\alpha r + \beta s, 11\gamma r + \delta s)^2}{11(11,s)}\mathrm{Ord}_\Gamma\left(f;M\frac{r}{s}\right).\end{aligned}$$

It thus suffices to show that, for each cusp r/s,

$$\frac{(11,s')(11\alpha r + \beta s, 11\gamma r + \delta s)^2}{11(11,s)} = 1. \tag{7.25}$$

By examining each of the inequivalent cusps in Lemma 7.3, we may easily verify that (7.25) holds in each case. This completes the proof.

Lemma 7.5. *Let M and Γ be as given in Lemma 7.4, and let $\zeta_1, \ldots, \zeta_6$ be as defined in Lemma 7.3. Then*

$$M\zeta_i \sim \zeta_{i+3}, \qquad 1 \le i \le 3,$$

where, of course, the symbol $\sim$ denotes equivalence.

PROOF. Lemma 7.5 follows by direct computation with the aid of Lemma 7.3.

We are now in a position to find improved values of μ in order to prove (7.14) and (7.15).

By (0.33), (0.34), (0.35), and (0.37), we find that both (7.14) and (7.15) are self-reciprocal. In particular, if we set

$$F_1(\tau) := \frac{f_1^{11}(11\tau)g_1(\tau)}{f_1(\tau)g_1^{11}(11\tau)} \quad \text{and} \quad F_2(\tau) := \frac{f_1^{11}(\tau)g_1(11\tau)}{f_1(11\tau)g_1^{11}(\tau)},$$

then $F_2 = F_1|M$. We use Table 1 in the introduction of this chapter to calculate $\mathrm{Ord}_\Gamma(F_1;\zeta)$ and $\mathrm{Ord}_\Gamma(F_2;\zeta)$, when $\zeta = \zeta_1, \zeta_2, \zeta_3$. We next employ Lemmas 7.4 and 7.5 to calculate $\mathrm{Ord}_\Gamma(F_1;\zeta)$ and $\mathrm{Ord}_\Gamma(F_2;\zeta)$, when $\zeta = \zeta_4, \zeta_5, \zeta_6$. Each of the remaining terms in (7.14) and (7.15) does not have any poles, and so we just use 0 as a lower bound for the order at each cusp of each such term. Now write each of the proposed identities (7.14) and (7.15) in the form $F(\tau) = 0$. Suppose that the coefficients of $q^0, q^1, \ldots, q^\mu$ in $F(\tau)$ are each equal to 0. We now summarize our calculations in the following table:

ζ	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
$\mathrm{Ord}_\Gamma(F_1;\zeta)$	5	5	-10	0	0	0
$\mathrm{Ord}_\Gamma(F_2;\zeta)$	0	0	0	5	5	-10
Lower bound for $\mathrm{Ord}_\Gamma(F;\zeta)$	$\mu+1$	0	-10	$\mu+1$	0	-10

By (0.30), we may conclude that $F(\tau) \equiv 0$, provided that

$$2(\mu+1) - 20 > \tfrac{1}{12} r(\Gamma(1):\Gamma) = 12,$$

where we have used (0.6), (0.24), and the fact that the weight r equals 2. Thus, $\mu \geq 16$.

Using the computer algebra system MACSYMA, we have, indeed, verified that the coefficients of $q^0, q^1, \ldots, q^{16}$ are each equal to 0 for each of (7.14) and (7.15). This finally completes the proofs of Ramanujan's modular equations of degree 11.

Entry 8(i). *Define*

$$\mu_1 = \frac{f(-q^4,-q^9)}{q^{7/13}f(-q^2,-q^{11})}, \qquad \mu_2 = \frac{f(-q^6,-q^7)}{q^{6/13}f(-q^3,-q^{10})},$$

$$\mu_3 = \frac{f(-q^2,-q^{11})}{q^{5/13}f(-q,-q^{12})}, \qquad \mu_4 = \frac{f(-q^5,-q^8)}{q^{2/13}f(-q^4,-q^9)},$$

$$\mu_5 = \frac{q^{5/13}f(-q^3,-q^{10})}{f(-q^5,-q^8)}, \quad \textit{and} \quad \mu_6 = \frac{q^{15/13}f(-q,-q^{12})}{f(-q^6,-q^7)}.$$

Then

$$\frac{f(-q^{1/13})}{q^{7/13}f(-q^{13})} = \mu_1 - \mu_2 - \mu_3 + \mu_4 + 1 - \mu_5 + \mu_6, \tag{8.1}$$

$$1 + \frac{f^2(-q)}{qf^2(-q^{13})} = \mu_1\mu_2 - \mu_3\mu_5 - \mu_4\mu_6, \tag{8.2}$$

$$-4 - \frac{f^2(-q)}{qf^2(-q^{13})} = \frac{1}{\mu_1\mu_2} - \frac{1}{\mu_3\mu_5} - \frac{1}{\mu_4\mu_6}, \tag{8.3}$$

$$3 + \frac{f^2(-q)}{qf^2(-q^{13})} = \mu_2\mu_3\mu_4 - \mu_1\mu_5\mu_6, \tag{8.4}$$

and

$$1 = \mu_1\mu_2\mu_3\mu_4\mu_5\mu_6. \tag{8.5}$$

PROOF OF (8.1). Put $n = 13$ in (12.25) of Chapter 19.

PROOF OF (8.2). Using (0.39), (0.51), and (0.52), we first translate (8.2) into the equivalent form

$$\sum_{m(\mathrm{mod}\,13)} G(m; z)G(5m; z) = -4\frac{\eta^2(z/13)}{\eta^2(z)}. \tag{8.6}$$

This, in turn, is a special case of the following theorem of R. J. Evans [1, Theorem 6.2].

Theorem 8.1. *For each prime* $p \equiv 1 \pmod 4$,

$$\sum_{m(\mathrm{mod}\,p)} G(m; z)G(m\beta; z) = 2a_p\frac{\eta^2(z/p)}{\eta^2(z)}, \tag{8.7}$$

where β *is any primitive fourth root of unity* (mod p), *and where*

$$a_p = \sum_{\substack{m,n \in Z \\ (6m-1)^2+(6n-1)^2=2p}} (-1)^{m+n}. \tag{8.8}$$

To see that (8.6) follows from Theorem 8.1, let $p = 13$ and $\beta = 5$ and observe that $a_{13} = -2$.

PROOF OF THEOREM 8.1. By a general theorem on Hecke operators (Rankin [2, pp. 289–290, Theorem 9.2.1]), the space of cusp forms $\{\Gamma(12), 1, 1\}_0$ is invariant under the Hecke operator T_p defined for $f \in \{\Gamma(12), 1, 1\}_0$ by

$$f(z)|T_p := f(pz) + \frac{1}{p}\sum_{n=0}^{p-1} f\left(\frac{z+12n}{p}\right).$$

Evans [1, Lemma 3.1] has shown that the dimension of $\{\Gamma(12), 1, 1\}_0$ is 1. Moreover, since $\eta(z)$ is a modular form of weight $\frac{1}{2}$ on $\Gamma(1)$ with multiplier system given by (0.14), we easily see that $\eta^2(z) \in \{\Gamma(12), 1, 1\}_0$. It follows that for some complex number α_p,

$$\eta^2(pz) + \frac{1}{p}\sum_{n=0}^{p-1} \eta^2\left(\frac{z+12n}{p}\right) = \alpha_p \eta^2(z). \tag{8.9}$$

Since by (0.76), $\eta^2(z) = q^{1/12}(1 - 2q + \cdots)$, a comparison of the coefficients of $q^{1/12}$ in (8.9) shows that α_p is the coefficient of $q^{p/12}$ in the Fourier expansion of $\eta^2(z)$. Squaring the Fourier series of $\eta(z)$ given in (0.86), we see that α_p equals the expression for a_p given by (8.8).

For a modular form $h(z)$ with a Fourier expansion of the form

$$h(z) = \sum_{k \in \mathbb{Z}} b_k q^{k/(12p)}, \qquad b_k \in \mathscr{C}, \tag{8.10}$$

define

$$I(h) = \sum_{p|k} b_k q^{k/(12p)} = \frac{1}{p}\sum_{n=0}^{p-1} h(z+12n).$$

Thus, $I(h)$ is the sum of those terms in (8.10) that are *integral* powers of $q^{1/12}$. Therefore, (8.9) can be written in the form

$$\eta^2(pz) + I(\eta^2(z/p)) = a_p \eta^2(z). \tag{8.11}$$

Squaring both sides of (0.86), we arrive at

$$\eta^2(z/p) = \tfrac{1}{4} \sum_{m,n (\mathrm{mod}\, p)} \{\eta(pz)G_m(pz)\}\{\eta(pz)G_n(pz)\}. \tag{8.12}$$

By (0.74),

$$\eta(pz)G_m(pz) = (-1)^m q^{1/24} \sum_{k=-\infty}^{\infty} (-1)^k (q^{(pk+m)(3pk+3m-p)/(2p)}$$
$$+ q^{(pk-m)(3pk-3m-p)/(2p)}).$$

Thus, either all or none of the terms in the Fourier expansion of the product $\{\eta(pz)G_m(pz)\}\{\eta(pz)G_n(pz)\}$ will contain integral powers of $q^{1/12}$ according as $m^2 + n^2$ is divisible by p or not. Now $m^2 + n^2$ is divisible by p if and only if $n \equiv \pm m\beta \pmod p$, for β as defined above, and then there are two such values for each nonzero $m \pmod p$. Thus, by (8.12), (0.51), and (0.52),

$$I(\eta^2(z/p)) = -\eta^2(pz) + \tfrac{1}{2} \sum_{m (\mathrm{mod}\, p)} \{\eta(pz)G(m; pz)\}\{\eta(pz)G(m\beta; pz)\}. \tag{8.13}$$

In conclusion, (8.11) and (8.13) yield

$$\sum_{m (\mathrm{mod}\, p)} G(m; pz)G(m\beta; pz) = 2(\eta^2(pz) + I(\eta^2(z/p)))/\eta^2(pz)$$
$$= 2a_p \eta^2(z)/\eta^2(pz),$$

and (8.7) follows.

PROOFS OF (8.3), (8.4). Transcribing (8.3) and (8.4) by means of (0.39), (0.51), and (0.52), we find that, respectively,

$$G_1^{-1}(z)G_5^{-1}(z) + G_2^{-1}(z)G_3^{-1}(z) + G_4^{-1}(z)G_6^{-1}(z) = 4 + \frac{\eta^2(z/p)}{\eta^2(z)} \tag{8.14}$$

and

$$G_1(z)G_3(z)G_4(z) - G_1^{-1}(z)G_3^{-1}(z)G_4^{-1}(z) = 3 + \frac{\eta^2(z/p)}{\eta^2(z)}, \tag{8.15}$$

where $p = 13$. By Corollaries 0.6 and 0.5, respectively, the left sides of (8.14) and (8.15) are in $\{\Gamma^0(13), 0, 1\}$ and have no poles on $\mathscr{H}$ or at the cusp 0.

We now follow the procedure outlined at the end of the introductory material in this chapter. By (0.77), for $p = 13$,

$$G_1(z) = -q^{-5/p^2}\{1 + q^{1/p} + O(q^{2/p})\}, \tag{8.16}$$

$$G_2(z) = q^{-7/p^2}\{1 + O(q^{2/p})\}, \tag{8.17}$$

$$G_3(z) = -q^{-6/p^2}\{1 + O(q^{2/p})\}, \tag{8.18}$$

$$G_4(z) = q^{-2/p^2}\{1 + O(q^{2/p})\}, \tag{8.19}$$

$$G_5(z) = -q^{5/p^2}\{1 + O(q^{2/p})\}, \tag{8.20}$$

and

$$G_6(z) = q^{15/p^2}\{1 - q^{1/p} + O(q^{2/p})\}. \tag{8.21}$$

For $p \equiv 1 \pmod{12}$ and $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(p)$, by (0.14),

$$\eta^2(Vz/p) = v^2\begin{pmatrix} a & b/p \\ pc & d \end{pmatrix}(cz + d)\eta^2(z/p) = v^2(V)(cz + d)\eta^2(z/p).$$

Hence, for $p \equiv 1 \pmod{12}$, $\eta^2(z/p)/\eta^2(z) \in \{\Gamma^0(p), 0, 1\}$. Thus, both sides of (8.14) and (8.15) belong to $\{\Gamma^0(13), 0, 1\}$ and have no poles except at ∞. By (0.76),

$$\eta(z/p)/\eta(z) = q^{(1-p)/(24p)}\{1 - q^{1/p} + O(q^{2/p})\}.$$

Thus, for $p = 13$,

$$\eta^2(z/p)/\eta^2(z) = q^{-1/p} - 2 + O(q^{1/p}). \tag{8.22}$$

Finally, from (8.16)–(8.21) and (8.22), both sides of (8.14) equal

$$q^{-1/p} + 2 + O(q^{1/p}),$$

while both sides of (8.15) equal

$$q^{-1/p} + 1 + O(q^{1/p}).$$

This completes the proof of (8.14) and (8.15).

Proof of (8.5). The desired result is an immediate consequence of the definitions of $\mu_1, \ldots, \mu_6$.

Evans [1, Theorem 7.1] has proved another beautiful identity in the spirit of (8.2)–(8.4). For $t = q^{1/13}$,

$$\frac{1}{(t^2)_\infty(t^3)_\infty(t^{10})_\infty(t^{11})_\infty}+\frac{t}{(t^4)_\infty(t^6)_\infty(t^7)_\infty(t^9)_\infty}=\frac{1}{(t)_\infty(t^5)_\infty(t^8)_\infty(t^{12})_\infty},$$

which is equivalent to, for $p = 13$,

$$G_1^{-1}(z)G_5^{-1}(z) + G_4(z)G_6(z) = 1.$$

Entry 8(ii). *We have*

$$\begin{aligned} &f(-q, -q^{12})f(-q^2, -q^{11})f(-q^3, -q^{10})f(-q^4, -q^9)f(-q^5, -q^8) \\ &\qquad \times f(-q^6, -q^7) \\ &\quad = f(-q)f^5(-q^{13}). \end{aligned}$$

PROOF. This identity is just the special case $n = 6$ of (28.1) in Chapter 16.

Entries 8(iii), (iv). *If β is of the* 13th *degree,*

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} - 4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6} \tag{8.23}$$

and

$$\frac{13}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} - 4\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6}. \tag{8.24}$$

PROOF. The modular equation (8.24) is simply the reciprocal of (8.23), in the sense of Entry 24(v) of Chapter 18, and so it suffices to prove (8.23).

By Entries 10(i), (ii), 11(i), (ii), and 12(i) in Chapter 17, (8.23) is equivalent to the theta-function identity

$$\begin{aligned} &q^3\frac{\psi(q^{26})\varphi(q^{13})}{\psi(q^2)\varphi(q)} + \frac{\varphi(-q^{13})\varphi(q^{13})}{\varphi(-q)\varphi(q)} - q^3\frac{\psi(q^{26})\varphi(-q^{13})}{\psi(q^2)\varphi(-q)} \\ &\qquad - 4q^2\frac{f^4(q^{13})\varphi^2(q)}{f^4(q)\varphi^2(q^{13})} = 1. \end{aligned} \tag{8.25}$$

Employing (0.13), we translate this identity into an identity involving modular forms,

$$\begin{aligned} &\frac{g_2(13\tau)g_1(13\tau)}{g_2(\tau)g_1(\tau)} + \frac{g_0(13\tau)g_1(13\tau)}{g_0(\tau)g_1(\tau)} - \frac{g_2(13\tau)g_0(13\tau)}{g_2(\tau)g_0(\tau)} \\ &\qquad - 4\frac{f_1^4(13\tau)g_1^2(\tau)}{f_1^4(\tau)g_1^2(13\tau)} = 1. \end{aligned} \tag{8.26}$$

Since b and c are even, it follows from (0.26), (0.16), and (0.18)–(0.20) that the multiplier system of each term of (8.26) is trivial; that is, $v(A) = 1$ for each $A \in \Gamma(2) \cap \Gamma_0(13)$. Clearing denominators in (8.26) and collecting terms on one side, we can write the transformed equation in the form

$$F := F_1 + \cdots + F_5 = 0,$$

where F is a modular form of weight $\frac{9}{2}$ on $\Gamma = \Gamma(2) \cap \Gamma_0(13)$. By (0.6) and (0.24), $\rho_\Gamma = 7$. From (0.30), we then see that $\mu = 31$.

To decrease computation, it seems advisable to use the reciprocal relation. Applying (0.36) in (8.26) and then converting the new equality back to an equality involving q-series, we find that

$$\frac{\psi(q^2)\varphi(q)}{q^3\psi(q^{26})\varphi(q^{13})} + \frac{\varphi(-q)\varphi(q)}{\varphi(-q^{13})\varphi(q^{13})} - \frac{\psi(q^2)\varphi(-q)}{q^3\psi(q^{26})\varphi(-q^{13})} - 4\frac{f^4(q)\varphi^2(q^{13})}{q^2 f^4(q^{13})\varphi^2(q)} = 13. \tag{8.27}$$

(Of course, we can also obtain (8.27) directly from (8.24) by using Entries 10(i), (ii), 11(i), (ii), and 12(i) in Chapter 17.) Thus, by (0.38), $\mu = 15$. Clearing denominators in both (8.25) and (8.27), transforming all terms to one side of the equation in each case, and employing the computer algebra system MACSYMA, we have indeed verified that the coefficients of $q^0, q^1, \ldots, q^{15}$ are equal to 0 for each of the two proposed identities. This then completes the proof.

Modular equations of degree 13 have been developed by Sohncke [1], [2], Schläfli [1], Klein [1], and Russell [2], but all of these modular equations are considerably more complicated than those of Ramanujan established above.

Entry 9. *We have*

(i) $$\psi(q^3)\psi(q^5) - \psi(-q^3)\psi(-q^5) = 2q^3\psi(q^2)\psi(q^{30}),$$

(ii) $$\varphi(-q^6)\varphi(-q^{10}) + 2q\psi(q^3)\psi(q^5) = \varphi(q)\varphi(q^{15}),$$

(iii) $$\varphi(-q^2)\varphi(-q^{30}) + 2q^2\psi(q)\psi(q^{15}) = \varphi(q^3)\varphi(q^5),$$

(iv) $$\psi(q)\psi(q^{15}) + \psi(-q)\psi(-q^{15}) = 2\psi(q^6)\psi(q^{10}),$$

(v) $$\varphi(q)\varphi(q^{15}) - \varphi(q^3)\varphi(q^5) = 2qf(-q^2)f(-q^{30})\chi(q^3)\chi(q^5),$$

(vi) $$\varphi(q)\varphi(q^{15}) + \varphi(q^3)\varphi(q^5) = 2f(-q^6)f(-q^{10})\chi(q)\chi(q^{15}),$$

and

(vii) $$\{\psi(q^3)\psi(q^5) - q\psi(q)\psi(q^{15})\}\varphi(-q^3)\varphi(-q^5) = \{\psi(q^3)\psi(q^5) + q\psi(q)\psi(q^{15})\}\varphi(-q)\varphi(-q^{15}) = f(-q)f(-q^3)f(-q^5)f(-q^{15}).$$

PROOF OF (i). In (36.8) of Chapter 16, let $\mu = 4$ and $\nu = 1$ to obtain the identity

$$\psi(q^3)\psi(q^5) = \psi(q^8)\varphi(q^{60}) + q^3\psi(q^2)\psi(q^{30}) + q^{14}\varphi(q^4)\psi(q^{120}).$$

Now replace q by $-q$ and subtract the result from the equality above. This yields (i) at once.

Proof of (ii). As in Entry 11 below, let α, β, γ, and δ be of the first, third, fifth, and fifteenth degrees. As in previous work, we set $\varphi(q^n) = \sqrt{z_n}$. Then, translating Entry 9(i) via Entries 11(i)–(iii) in Chapter 17, we find that

$$(\beta\gamma)^{1/8} - \{\beta\gamma(1-\beta)(1-\gamma)\}^{1/8} = (\alpha\delta)^{1/4}\left(\frac{z_1 z_{15}}{z_3 z_5}\right)^{1/2}. \tag{9.1}$$

Inverting the roles of α and δ and also of β and γ, we derive the reciprocal modular equation

$$\{(1-\gamma)(1-\beta)\}^{1/8} - \{(1-\gamma)(1-\beta)\gamma\beta\}^{1/8} = \{(1-\delta)(1-\alpha)\}^{1/4}\left(\frac{z_{15} z_1}{z_5 z_3}\right)^{1/2}. \tag{9.2}$$

By Entries 10(ii), (iii) and 11(ii) in Chapter 17, the translation of this equality is the identity

$$\varphi(-q^6)\varphi(-q^{10}) - 2q\psi(-q^3)\psi(-q^5) = \varphi(-q)\varphi(-q^{15}).$$

Changing the sign of q gives (ii).

Proof of (iv). In (36.10) of Chapter 16, replace q by q^2 and set $\mu = 4$ and $\nu = 1$. We then apply Entries 30(ii), (iii) in Chapter 16 twice apiece. Thus,

$$\begin{aligned}2\psi(q^6)\psi(q^{10}) &= 2f(q^6, q^{10})f(q^{90}, q^{150}) + 2q^{16}f(q^2, q^{14})f(q^{30}, q^{210}) \\ &= \tfrac{1}{2}\{f(q, q^3) + f(-q, -q^3)\}\{f(q^{15}, q^{45}) + f(-q^{15}, -q^{45})\} \\ &\quad + \tfrac{1}{2}\{f(q, q^3) - f(-q, -q^3)\}\{f(q^{15}, q^{45}) - f(-q^{15}, -q^{45})\} \\ &= \psi(q)\psi(q^{15}) + \psi(-q)\psi(-q^{15}).\end{aligned}$$

Thus, (iv) has been established.

Proof of (iii). Translating Entry 9(iv) via Entries 11(i)–(iii) in Chapter 17, we deduce that

$$(\alpha\delta)^{1/8} + \{\alpha\delta(1-\alpha)(1-\delta)\}^{1/8} = (\beta\gamma)^{1/4}\left(\frac{z_3 z_5}{z_1 z_{15}}\right)^{1/2}.$$

The reciprocal of this formula is

$$\{(1-\alpha)(1-\delta)\}^{1/8} + \{\alpha\delta(1-\alpha)(1-\delta)\}^{1/8} = \{(1-\beta)(1-\gamma)\}^{1/4}\left(\frac{z_3 z_5}{z_1 z_{15}}\right)^{1/2}.$$

Employing Entries 10(ii), (iii) and Entry 11(ii) in Chapter 17, we find that the translation of this formula is the identity

$$\varphi(-q^2)\varphi(-q^{30}) + 2q^2\psi(-q)\psi(-q^{15}) = \varphi(-q^3)\varphi(-q^5).$$

Replacing q by $-q$, we deduce (iii).

We postpone the proofs of Entries 9(v)–(vii) until Section 11 where it will be convenient to use the theory developed there.

Entry 10. *We have*

(i) $$f(-q^7,-q^8)+qf(-q^2,-q^{13})=\frac{f(-q^2,-q^3)}{f(-q,-q^4)}f(-q^5),$$

(ii) $$f(-q^4,-q^{11})-qf(-q,-q^{14})=\frac{f(-q,-q^4)}{f(-q^2,-q^3)}f(-q^5),$$

(iii) $$f(-q^7,-q^8)-qf(-q^2,-q^{13})=f(-q^{2/3},-q)+q^{2/3}f(-q^3,-q^{12}),$$

(iv) $$\{f(-q^4,-q^{11})+qf(-q,-q^{14})\}q^{1/3}=f(-q^6,-q^9)-f(-q^{1/3},-q^{4/3}),$$

(v) $$q\psi(q^3)\psi(q^5)+q^2\psi(q)\psi(q^{15})=\frac{q}{1-q}-\frac{q^7}{1-q^7}-\frac{q^{11}}{1-q^{11}}-\frac{q^{13}}{1-q^{13}}+\frac{q^{17}}{1-q^{17}}+\frac{q^{19}}{1-q^{19}}+\frac{q^{23}}{1-q^{23}}-\frac{q^{29}}{1-q^{29}}+\cdots,$$

and

(vi) $$\varphi(q^3)\varphi(q^5)+\varphi(q)\varphi(q^{15})=2\left(1+\frac{q}{1-q}-\frac{q^2}{1-q^2}+\frac{q^4}{1-q^4}-\frac{q^7}{1-q^7}+\frac{q^8}{1-q^8}-\frac{q^{11}}{1-q^{11}}-\frac{q^{13}}{1-q^{13}}+\frac{q^{14}}{1-q^{14}}+\frac{q^{16}}{1-q^{16}}+\frac{q^{17}}{1-q^{17}}+\frac{q^{19}}{1-q^{19}}+\frac{q^{22}}{1-q^{22}}+\frac{q^{23}}{1-q^{23}}+\frac{q^{26}}{1-q^{26}}-\frac{q^{28}}{1-q^{28}}-\frac{q^{29}}{1-q^{29}}+\frac{q^{31}}{1-q^{31}}+\frac{q^{32}}{1-q^{32}}-\frac{q^{34}}{1-q^{34}}-\frac{q^{37}}{1-q^{37}}-\frac{q^{38}}{1-q^{38}}-\frac{q^{41}}{1-q^{41}}-\frac{q^{43}}{1-q^{43}}-\frac{q^{44}}{1-q^{44}}-\frac{q^{46}}{1-q^{46}}+\frac{q^{47}}{1-q^{47}}+\frac{q^{49}}{1-q^{49}}-\frac{q^{52}}{1-q^{52}}+\frac{q^{53}}{1-q^{53}}-\frac{q^{56}}{1-q^{56}}+\frac{q^{58}}{1-q^{58}}-\frac{q^{59}}{1-q^{59}}+\cdots\right).$$

In (v), the cycle of coefficients is of length 30, while in (vi), the cycle of coefficients is of length 60. These rules of formation were not made explicit by

Ramanujan (p. 245), since he recorded only six and four terms, respectively, in the two series.

PROOFS OF (i), (ii). For the derivation of (i), first use the identity $f(-q^{-2}, -q^{17}) = -q^{-2}f(-q^{13}, -q^2)$, which is deducible from Entry 18(iv) in Chapter 16. In the quintuple product identity (38.2) of Chapter 16, replace q by $q^{5/2}$ and let $B = -q^{3/2}, -q^{1/2}$, in turn. Formulas (i) and (ii), respectively, now follow.

PROOFS OF (iii), (iv). In Entry 31 of Chapter 16, set $n = 3$, $a = -q^{2/3}$, and $b = -q$ to achieve (iii), and let $n = 3$, $a = -q^{1/3}$, and $b = -q^{4/3}$ to obtain (iv).

The proofs of (v) and (vi) are considerably more difficult.

PROOF OF (v). Let S denote the series on the right side of (v). Taking the summands of S, expanding them into geometric series, and then inverting the order of summation, we find that

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{q^n - q^{7n} - q^{11n} - q^{13n} + q^{17n} + q^{19n} + q^{23n} - q^{29n}}{1 - q^{30n}} \\
&= \sum_{n=1}^{\infty} \frac{q^n - q^{5n} - q^{11n} + q^{19n} + q^{25n} - q^{29n}}{1 - q^{30n}} \\
&\quad + \sum_{n=1}^{\infty} \frac{q^{5n} - q^{7n} - q^{13n} + q^{17n} + q^{23n} - q^{25n}}{1 - q^{30n}}.
\end{aligned}$$

Applying next the addition theorem, (17.1) of Chapter 19, with q replaced by q^{15}, $a = q^4, q^2$, and $b = q^{10}, q^8$, respectively, we deduce that

$$\begin{aligned}
S &= \frac{f^3(-q^{30})}{\varphi(-q^{15})}\left(\frac{qf(-q^4, -q^{26})f(-q^{10}, -q^{20})f(-q^{14}, -q^{16})}{f(-q^{19}, -q^{11})f(-q^{25}, -q^5)f(-q^{29}, -q)}\right. \\
&\quad \left. + \frac{q^5 f(-q^2, -q^{28})f(-q^8, -q^{22})f(-q^{10}, -q^{20})}{f(-q^{17}, -q^{13})f(-q^{23}, -q^7)f(-q^{25}, -q^5)}\right) \\
&= \frac{f^3(-q^{30})f(-q^{10})}{\varphi(-q^{15})f(-q^{25}, -q^5)}\left(\frac{qf(-q^4, -q^{26})f(-q^{14}, -q^{16})f(-q^{24}, -q^6)}{f(-q, -q^{29})f(-q^{11}, -q^{19})f(-q^{21}, -q^9)}\right. \\
&\quad \times \frac{f(-q^9, -q^{21})}{f(-q^6, -q^{24})} \\
&\quad \left. + \frac{q^5 f(-q^2, -q^{28})f(-q^{12}, -q^{18})f(-q^{22}, -q^8)}{f(-q^7, -q^{23})f(-q^{17}, -q^{13})f(-q^{27}, -q^3)} \frac{f(-q^3, -q^{27})}{f(-q^{12}, -q^{18})}\right) \\
&= \frac{f^3(-q^{30})f(-q^{10})}{\varphi(-q^{15})\psi(q^{15})\chi(-q^5)}\left(\frac{qf(-q^4, -q^6)f(-q^9, -q^{21})}{f(-q, -q^9)f(-q^6, -q^{24})}\right. \\
&\quad \left. + \frac{q^5 f(-q^2, -q^8)f(-q^3, -q^{27})}{f(-q^3, -q^7)f(-q^{12}, -q^{18})}\right),
\end{aligned}$$

where we have employed the Jacobi triple product identity and Example (v) in Section 31 of Chapter 16. Putting everything under a common denominator and then using the Jacobi triple product identity to transform the denominator, we arrive at

$$S = \frac{qf^3(-q^{30})f(-q^{10})\chi(-q^5)}{\varphi(-q^{15})\psi(q^{15})\chi(-q^5)\chi(-q)f^2(-q^{10})f(-q^6)f(-q^{30})}$$
$$\times (f(-q^3, -q^7)f(-q^4, -q^6)f(-q^9, -q^{21})f(-q^{12}, -q^{18})$$
$$+ q^4f(-q, -q^9)f(-q^2, -q^8)f(-q^3, -q^{27})f(-q^6, -q^{24}))$$
$$= \frac{q}{2\chi(-q^{15})f(-q^{10})f(-q^6)\chi(-q)}$$
$$\times (\{f(-q^3, -q^7)f(-q^4, -q^6) + qf(-q, -q^9)f(-q^2, -q^8)\}$$
$$\times \{f(-q^9, -q^{21})f(-q^{12}, -q^{18}) + q^3f(-q^3, -q^{27})f(-q^6, -q^{24})\}$$
$$+ \{f(-q^3, -q^7)f(-q^4, -q^6) - qf(-q, -q^9)f(-q^2, -q^8)\}$$
$$\times \{f(-q^9, -q^{21})f(-q^{12}, -q^{18}) - q^3f(-q^3, -q^{27})f(-q^6, -q^{24})\}), \tag{10.1}$$

where we have used two of the equalities in Entry 24(iii) of Chapter 16.

Next, applying Entries 29(i), (ii) of Chapter 16, we find that, respectively,

$$f(q, q^4)f(-q^2, -q^3) + f(-q, -q^4)f(q^2, q^3) = 2f(-q^3, -q^7)f(-q^4, -q^6)$$

and

$$f(q, q^4)f(-q^2, -q^3) - f(-q, -q^4)f(q^2, q^3) = 2qf(-q^2, -q^8)f(-q, -q^9).$$

The results gotten by adding these two equalities and then by subtracting them are used in (10.1) to give us

$$S = \frac{q}{2\chi(-q^{15})f(-q^{10})f(-q^6)\chi(-q)}$$
$$\times (f(q, q^4)f(-q^2, -q^3)f(q^3, q^{12})f(-q^6, -q^9)$$
$$+ f(-q, -q^4)f(q^2, q^3)f(-q^3, -q^{12})f(q^6, q^9))$$
$$= \frac{qf^2(-q^5)}{2f^6(-q^{15})\chi(-q^{15})f(-q^{10})f(-q^6)\chi(-q)}$$
$$\times (f(q, q^{14})f(q^6, q^9)f(q^{11}, q^4)f(-q^2, -q^{13})f(-q^7, -q^8)f(-q^{12}, -q^3)$$
$$\times f(q^3, q^{12})f(-q^6, -q^9)$$
$$+ f(-q, -q^{14})f(-q^6, -q^9)f(-q^{11}, -q^4)f(q^2, q^{13})f(q^7, q^8)f(q^{12}, q^3)$$
$$\times f(-q^3, -q^{12})f(q^6, q^9))$$

$$= \frac{qf^2(-q^5)f(q^3,q^{12})f(q^6,q^9)f(-q^3,-q^{12})f(-q^6,-q^9)}{2f^6(-q^{15})\chi(-q^{15})f(-q^{10})f(-q^6)\chi(-q)}$$
$$\times (f(q,q^{14})f(q^4,q^{11})f(-q^2,-q^{13})f(-q^7,-q^8)$$
$$+f(-q,-q^{14})f(-q^4,-q^{11})f(q^2,q^{13})f(q^7,q^8)),$$

where in the penultimate equality we utilized the Jacobi triple product identity several times. Employing the Jacobi triple product identity as well as Entry 18(iv) in Chapter 16, using the corollary of Section 30 in Chapter 16, utilizing the Jacobi triple product identity three more times, and invoking results in Entries 24(ii), (iii) of Chapter 16, we deduce that

$$S = \frac{q^2f^2(-q^5)\varphi^2(-q^{15})f(-q^6)f(-q^{30})}{2f^6(-q^{15})\chi(-q^{15})f(-q^{10})f(-q^6)\chi(-q)}$$
$$\times (f(q^{-1},q^{16})f(-q^2,-q^{13})f(q^4,q^{11})f(-q^7,-q^8)$$
$$-f(-q^{-1},-q^{16})f(q^2,q^{13})f(-q^4,-q^{11})f(q^7,q^8))$$
$$= \frac{qf^2(-q^5)\varphi^2(-q^{15})f(-q^{30})}{f^6(-q^{15})\chi(-q^{15})f(-q^{10})\chi(-q)}f(-q^3,-q^{12})f(-q^9,-q^6)$$
$$\times f(q^5,q^{10})\psi(q^{15})$$
$$= \frac{qf^2(-q^5)\varphi^2(-q^{15})f(-q^{30})f(-q^3)f^3(-q^{15})f(-q^{10})\psi(q^{15})}{f^6(-q^{15})\chi(-q^{15})f(-q^{10})\chi(-q)f(-q^5)f(-q^{30})}$$
$$= \frac{qf(-q^5)f(-q^3)\varphi^2(-q^{15})\psi(q^{15})}{f^3(-q^{15})\chi(-q^{15})\chi(-q)}$$
$$= \frac{qf(-q)f(-q^3)f(-q^5)}{\varphi(-q)}\frac{\psi(q^{15})\varphi^2(-q^{15})}{f^3(-q^{15})\chi(-q^{15})}$$
$$= \frac{qf(-q)f(-q^3)f(-q^5)f(-q^{15})}{\varphi(-q)\varphi(-q^{15})}.$$

Invoking Entry 9(vii), we complete the proof.

PROOF OF (vi). Combining Entries 9(ii), (iii) and 10(v), we find that

$$\{\varphi(-q^3)\varphi(-q^5)+\varphi(-q)\varphi(-q^{15})\}-\{\varphi(-q^6)\varphi(-q^{10})+\varphi(-q^2)\varphi(-q^{30})\}$$
$$= -2q\psi(-q^3)\psi(-q^5) + 2q^2\psi(-q)\psi(-q^{15})$$
$$= -2\left(\frac{q}{1+q} - \frac{q^7}{1+q^7} - \frac{q^{11}}{1+q^{11}} - \frac{q^{13}}{1+q^{13}} + \frac{q^{17}}{1+q^{17}} + \frac{q^{19}}{1+q^{19}}\right.$$
$$\left. + \frac{q^{23}}{1+q^{23}} - \frac{q^{29}}{1+q^{29}} + \cdots\right).$$

Replacing q by q^{2^n} and summing on n, $0 \le n < \infty$, we deduce that

$$\varphi(-q^3)\varphi(-q^5)+\varphi(-q)\varphi(-q^{15})-2$$
$$=-2\left(\frac{q}{1+q}+\frac{q^2}{1+q^2}+\frac{q^4}{1+q^4}-\frac{q^7}{1+q^7}+\frac{q^8}{1+q^8}-\frac{q^{11}}{1+q^{11}}\right.$$
$$\left.-\frac{q^{13}}{1+q^{13}}-\frac{q^{14}}{1+q^{14}}+\cdots\right),$$

where the cycle of coefficients is of length 15. If we change the sign of q, we find that

$$\varphi(q^3)\varphi(q^5)+\varphi(q)\varphi(q^{15})$$
$$=2\left(1+\frac{q}{1-q}-\frac{q^2}{1+q^2}-\frac{q^4}{1+q^4}-\frac{q^7}{1-q^7}-\frac{q^8}{1+q^8}-\frac{q^{11}}{1-q^{11}}\right.$$
$$-\frac{q^{13}}{1-q^{13}}+\frac{q^{14}}{1+q^{14}}-\frac{q^{16}}{1+q^{16}}+\frac{q^{17}}{1-q^{17}}+\frac{q^{19}}{1-q^{19}}+\frac{q^{22}}{1+q^{22}}$$
$$\left.+\frac{q^{23}}{1-q^{23}}+\frac{q^{26}}{1+q^{26}}+\frac{q^{28}}{1+q^{28}}-\frac{q^{29}}{1-q^{29}}+\cdots\right),$$

where now the cycle of coefficients is of length 30. For each even value of n, use the trivial identity

$$\frac{q^n}{1+q^n}=\frac{q^n}{1-q^n}-\frac{2q^{2n}}{1-q^{2n}}$$

in the foregoing series. We then obtain the proffered identity, with a cycle of coefficients of length 60.

Entry 11. *Let α, β, γ, and δ be of the first, third, fifth, and fifteenth degrees, respectively. Let m denote the multiplier connecting α and β, and let m' be the multiplier relating γ and δ. Then,*

(i) $$(\alpha\delta)^{1/8}+\{(1-\alpha)(1-\delta)\}^{1/8}=\sqrt{\frac{m'}{m}},$$

(ii) $$(\beta\gamma)^{1/8}+\{(1-\beta)(1-\gamma)\}^{1/8}=\sqrt{\frac{m}{m'}}$$
$$=\frac{(\beta\gamma)^{1/8}-\{\beta\gamma(1-\beta)(1-\gamma)\}^{1/8}}{(\alpha\delta)^{1/4}}$$
$$=\frac{\{(1-\beta)(1-\gamma)\}^{1/8}-\{\beta\gamma(1-\beta)(1-\gamma)\}^{1/8}}{\{(1-\alpha)(1-\delta)\}^{1/4}},$$

(iii) $$(\alpha\delta)^{1/8}-\{(1-\alpha)(1-\delta)\}^{1/8}=(\beta\gamma)^{1/8}-\{(1-\beta)(1-\gamma)\}^{1/8},$$

(iv) $$1+(\beta\gamma)^{1/8}+\{(1-\beta)(1-\gamma)\}^{1/8}=4^{1/3}\left(\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24},$$

(v) $1-(\alpha\delta)^{1/8}-\{(1-\alpha)(1-\delta)\}^{1/8}=4^{1/3}\left(\dfrac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/24}$,

(vi) $$(\alpha\delta)^{1/16}(\{(1+\sqrt{\alpha})(1+\sqrt{\delta})\}^{1/4}+\{(1-\sqrt{\alpha})(1-\sqrt{\delta})\}^{1/4}) + \{(1-\alpha)(1-\delta)\}^{1/16}(\{(1+\sqrt{1-\alpha})(1+\sqrt{1-\delta})\}^{1/4} + \{(1-\sqrt{1-\alpha})(1-\sqrt{1-\delta})\}^{1/4})=\sqrt{2},$$

(vii) $$(\beta\gamma)^{1/16}(\{(1+\sqrt{\beta})(1+\sqrt{\gamma})\}^{1/4}-\{(1-\sqrt{\beta})(1-\sqrt{\gamma})\}^{1/4}) + \{(1-\beta)(1-\gamma)\}^{1/16}(\{(1+\sqrt{1-\beta})(1+\sqrt{1-\gamma})\}^{1/4} - \{(1-\sqrt{1-\beta})(1-\sqrt{1-\gamma})\}^{1/4})=\sqrt{2},$$

(viii) $$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8}-\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}=\sqrt{\frac{m'}{m}},$$

(ix) $$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8}+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8}-\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}=-\sqrt{\frac{m}{m'}},$$

(x) $$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4}-\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} -4\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/6}=mm',$$

(xi) $$\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/4}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/4}-\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/4} -4\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/6}=\frac{9}{mm'},$$

(xii) $$\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/4}+\left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/4}+\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/4} -2\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/8}\left\{1+\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/8}+\left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/8}\right\} =\frac{z_1z_3}{z_5z_{15}},$$

(xiii) $$\left(\frac{\alpha\beta}{\gamma\delta}\right)^{1/4}+\left(\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}\right)^{1/4}+\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{1/4} -2\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{1/8}\left\{1+\left(\frac{\alpha\beta}{\gamma\delta}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}\right)^{1/8}\right\} =25\frac{z_5z_{15}}{z_1z_3},$$

and

(xiv)
$$(\alpha\beta\gamma\delta)^{1/8} + \{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8} + 2^{1/3}\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/24} = 1.$$

(xv) *If*
$$P = \{256\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/48}$$
and
$$Q = \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/16},$$
then
$$Q + \frac{1}{Q} = \sqrt{2}\left(P + \frac{1}{P}\right).$$

If we multiply (i) by the first equality in (ii), we obtain a modular equation sent by Ramanujan [10, p. xxix] in his second letter to Hardy. Entry 11(xiv) also appears in the same letter [10, p. xxix]. Both (iii) and (vi) were recorded by Hardy [3, p. 220] in his brief description of some of Ramanujan's work on modular equations. (In his statement of (vi), Hardy made two sign errors.)

If we multiply (viii) and (ix) together, we obtain a modular equation established by Weber [1]. Weber [1] also established (xiv).

Proof of (i). If we translate Entry 9(iii) via Entries 10(i), (iii) and 11(i) in Chapter 17, we obtain Entry 11(i) at once.

Proof of (ii). Transcribing Entry 9(ii) by means of Entries 10(i), (iii) and 11(i) in Chapter 17, we obtain the first part of Entry 11(ii) immediately. The second and third equalities of (ii) are (9.1) and (9.2), respectively.

Proof of (iii). Let
$$A = (\alpha\delta)^{1/8}, \qquad A' = \{(1-\alpha)(1-\delta)\}^{1/8}, \qquad B = (\beta\gamma)^{1/8},$$
$$B' = \{(1-\beta)(1-\gamma)\}^{1/8}, \quad \text{and} \quad M = \left(\frac{z_1 z_{15}}{z_3 z_5}\right)^{1/2} = \left(\frac{m}{m'}\right)^{1/2},$$
where the notation of Section 9 is used. Then in this abbreviated notation, Entries 11(i) and (ii) yield the equalities
$$\frac{1}{A+A'} = B + B' = \frac{B - BB'}{A^2} = \frac{B' - BB'}{A'^2} = M. \tag{11.1}$$

The last three expressions yield
$$B - BB' - (B' - BB') = MA^2 - MA'^2,$$
or
$$\frac{B - B'}{(A - A')(A + A')} = M.$$

Using the extremal parts of (11.1), we conclude that

$$B - B' = A - A', \tag{11.2}$$

which is (iii).

Proofs of (iv), (v). According to (11.1) and (11.2), we may set

$$\begin{aligned} A &= \tfrac{1}{2}(M^{-1} - \rho), & A' &= \tfrac{1}{2}(M^{-1} + \rho), \\ B &= \tfrac{1}{2}(M - \rho), \quad \text{and} & B' &= \tfrac{1}{2}(M + \rho), \end{aligned} \tag{11.3}$$

where ρ is positive when α is small. Taking the equality $B - BB' = AM^2$ from (11.1), substituting for B, B', and A from (11.3), and solving for ρ^2, we deduce that

$$\rho^2 = \frac{1 + M - M^2}{M}. \tag{11.4}$$

Thus, from (11.4) and (11.1),

$$\begin{aligned} 4\left(\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} &= 4\frac{B^2B'^2}{AA'} = \frac{(M^2-\rho^2)^2}{M^{-2}-\rho^2} = (M+1)^3 \\ &= ((\beta\gamma)^{1/8} + \{(1-\beta)(1-\gamma)\}^{1/8} + 1)^3. \end{aligned} \tag{11.5}$$

Taking the cube root of both sides, we deduce (iv).

Similarly, from (11.4) and (11.1),

$$\begin{aligned} 4\left(\frac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} &= 4\frac{A^2A'^2}{BB'} = \frac{(M^{-2}-\rho^2)^2}{M^2-\rho^2} = \left(1 - \frac{1}{M}\right)^3 \\ &= (1 - (\alpha\delta)^{1/8} - \{(1-\alpha)(1-\delta)\}^{1/8})^3. \end{aligned} \tag{11.6}$$

Part (v) is now apparent.

Proof of (vi). First, by (11.3) and (11.4),

$$\begin{aligned} &(\{(1+\sqrt{\alpha})(1+\sqrt{\delta})\}^{1/2} + \{(1-\sqrt{\alpha})(1-\sqrt{\delta})\}^{1/2})^2 \\ &= 2(1 + \sqrt{\alpha\delta} + \sqrt{(1-\alpha)(1-\delta)}) \\ &= 2(1 + A^4 + A'^4) \\ &= 2(1 + \tfrac{1}{16}(M^{-1}-\rho)^4 + \tfrac{1}{16}(M^{-1}+\rho)^4) \\ &= \frac{(M^3 - M^2 + 3M + 1)^2}{4M^4}, \end{aligned}$$

after some algebraic manipulation and simplification.

Hence, from the last calculation, (11.3), and (11.4),

$$\begin{aligned} &(\alpha\delta)^{1/8}(\{(1+\sqrt{\alpha})(1+\sqrt{\delta})\}^{1/4} + \{(1-\sqrt{\alpha})(1-\sqrt{\delta})\}^{1/4})^2 \\ &= (\alpha\delta)^{1/8}\left(\frac{M^3 - M^2 + 3M + 1}{2M^2} + 2A'^2\right) \end{aligned}$$

$$= (\alpha\delta)^{1/8}\left(\frac{M^3 - M^2 + 3M + 1}{2M^2} + \frac{1 + 2M\rho + M + M^2 - M^3}{2M^2}\right)$$

$$= \frac{1}{2M}(1 - \rho M)\frac{1 + M\rho + 2M}{M^2}$$

$$= \frac{(M - \rho)^2}{2M^2}.$$

Thus, we have shown that the first expression on the left side of (vi) is equal to $(M - \rho)/(M\sqrt{2})$.

Suppose now that we repeat the analysis above, but with α and δ replaced by $1 - \alpha$ and $1 - \delta$, respectively. The calculations are seen to be exactly the same, except that ρ is replaced by $-\rho$. Hence, the second expression on the left side of (vi) is equal to $(M + \rho)/(M\sqrt{2})$. The truth of (vi) is now apparent.

Proof of (vii). The proof is analogous to that above. First, by (11.3) and (11.4),

$$(\{(1 + \sqrt{\beta})(1 + \sqrt{\gamma})\}^{1/2} + \{(1 - \sqrt{\beta})(1 - \sqrt{\gamma})\}^{1/2})^2$$

$$= 2(1 + \sqrt{\beta\gamma} + \sqrt{(1 - \beta)(1 - \gamma)})$$

$$= 2(1 + \tfrac{1}{16}(M - \rho)^4 + \tfrac{1}{16}(M + \rho)^4)$$

$$= \frac{(1 + M + 3M^2 - M^3)^2}{4M^2}.$$

Thus,

$$(\beta\gamma)^{1/8}(\{(1 + \sqrt{\beta})(1 + \sqrt{\gamma})\}^{1/4} - \{(1 - \sqrt{\beta})(1 - \sqrt{\gamma})\}^{1/4})^2$$

$$= (\beta\gamma)^{1/8}\left(\frac{1 + M + 3M^2 - M^3}{2M} - 2B'^2\right)$$

$$= (\beta\gamma)^{1/8}\left(\frac{1 + M + 3M^2 - M^3}{2M} - \frac{M^3 + 2M^2\rho + 1 + M - M^2}{2M}\right)$$

$$= \frac{(M - \rho)(2M^2 - M^3 - M^2\rho)}{2M}$$

$$= \tfrac{1}{2}(1 - M\rho)^2.$$

Hence, the first expression on the left side of (vii) is equal to $(1 - M\rho)/\sqrt{2}$.

We now repeat the procedure above but with β and γ replaced by $1 - \beta$ and $1 - \gamma$, respectively. As in the proof of (vi), we see that the calculations are the same except that ρ is replaced by $-\rho$. Hence, the second expression on the left side of (vii) is found to equal $(1 + M\rho)/\sqrt{2}$. The truth of (vii) is now manifest.

Proof of (viii). Observe that, by (11.3),

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8}-\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}$$
$$=\frac{M^{-1}-\rho}{M-\rho}+\frac{M^{-1}+\rho}{M+\rho}-\frac{M^{-2}-\rho^2}{M^2-\rho^2}$$
$$=\frac{2-2\rho^2}{M^2-\rho^2}-\frac{M^{-2}-\rho^2}{M^2-\rho^2}=\frac{1}{M},$$

upon the use of (11.4). Thus, (viii) is established.

Proof of (ix). The proof is analogous to that of (viii). Thus, by (11.3) and (11.4),

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8}+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8}-\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}$$
$$=\frac{M-\rho}{M^{-1}-\rho}+\frac{M+\rho}{M^{-1}+\rho}-\frac{M^2-\rho^2}{M^{-2}-\rho^2}=\frac{2-M^2-\rho^2}{M^{-2}-\rho^2}=-M,$$

which verifies (ix).

The proofs of (x)–(xiii) are somewhat difficult. It seems necessary to express α, β, γ, and δ as algebraic functions of a single parameter. To that end, we set

$$t=\left(\frac{z_3z_5}{z_1z_{15}}\right)^{1/2}=\frac{1}{M},$$

or

$$m'=mt^2. \tag{11.7}$$

Thus, from (ii) and (iv),

$$4^{1/3}\left(\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24}=1+\frac{1}{t}, \tag{11.8}$$

and from (i) and (v),

$$4^{1/3}\left(\frac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/24}=1-t. \tag{11.9}$$

Since β and δ are of the third degree in α and γ, respectively, it follows from (5.2) and (5.5) of Chapter 19 that

$$\begin{aligned}
\alpha&=\frac{(m-1)(3+m)^3}{16m^3}, & \beta&=\frac{(m-1)^3(3+m)}{16m},\\
\gamma&=\frac{(m'-1)(3+m')^3}{16m'^3}, & \delta&=\frac{(m'-1)^3(3+m')}{16m'},\\
1-\alpha&=\frac{(m+1)(3-m)^3}{16m^3}, & 1-\beta&=\frac{(m+1)^3(3-m)}{16m},\\
1-\gamma&=\frac{(m'+1)(3-m')^3}{16m'^3}, & 1-\delta&=\frac{(m'+1)^3(3-m')}{16m'}.
\end{aligned} \tag{11.10}$$

Put

$$\mu = \frac{z_1}{z_5} \quad \text{and} \quad \mu' = \frac{z_3}{z_{15}}$$

and note that, from (11.7),

$$\mu' = \mu t^2. \tag{11.11}$$

Since γ and δ are of the fifth degree in α and β, respectively, it follows from (14.2) and (14.4) of Chapter 19 that

$$\begin{aligned} \alpha(1-\alpha) &= \frac{(\mu-1)(5-\mu)^5}{2^8\mu^5}, \qquad & \beta(1-\beta) &= \frac{(\mu'-1)(5-\mu')^5}{2^8\mu'^5}, \\ \gamma(1-\gamma) &= \frac{(\mu-1)^5(5-\mu)}{2^8\mu}, \qquad & \delta(1-\delta) &= \frac{(\mu'-1)^5(5-\mu')}{2^8\mu'}. \end{aligned} \tag{11.12}$$

Substituting these values in (11.8) and (11.9), we find that, respectively,

$$\left(\frac{(m^2-1)^5(9m'^{-2}-1)^5}{(m'^2-1)(9m^{-2}-1)}\right)^{1/24} = 1 + \frac{1}{t} = \left(\frac{(\mu-1)^3(5\mu'^{-1}-1)^3}{(\mu'-1)(5\mu^{-1}-1)}\right)^{1/8}$$

and

$$\left(\frac{(m'^2-1)^5(9m^{-2}-1)^5}{(m^2-1)(9m'^{-2}-1)}\right)^{1/24} = 1 - t = \left(\frac{(\mu'-1)^3(5\mu^{-1}-1)^3}{(\mu-1)(5\mu'^{-1}-1)}\right)^{1/8}.$$

If follows that

$$\left(1+\frac{1}{t}\right)^5(1-t) = (m^2-1)(9m'^{-2}-1), \tag{11.13}$$

$$\left(1+\frac{1}{t}\right)(1-t)^5 = (m'^2-1)(9m^{-2}-1), \tag{11.14}$$

$$\left(1+\frac{1}{t}\right)^3(1-t) = (\mu-1)(5\mu'^{-1}-1), \tag{11.15}$$

and

$$\left(1+\frac{1}{t}\right)(1-t)^3 = (\mu'-1)(5\mu^{-1}-1). \tag{11.16}$$

From either (11.13) or (11.14) and (11.7), it readily follows that

$$m^2 + \frac{9}{m^2t^4} = \frac{t^6+5t^5+5t^4-5t^2+5t-1}{t^5}. \tag{11.17}$$

Also, from either (11.15) or (11.16) and (11.11), it follows easily that

$$\mu + \frac{5}{\mu t^2} = \frac{t^4+3t^3+3t-1}{t^3}. \tag{11.18}$$

Solving (11.17) for m^2 and (11.18) for μ, we find after a considerable amount of elementary algebra that

$$2t^5m^2 = t^6 + 5t^5 + 5t^4 - 5t^2 + 5t - 1 - 4t^2(t^2 + 2t - 1)RS \quad (11.19)$$

and

$$2t^3\mu = t^4 + 3t^3 + 3t - 1 - 4t^2RS, \quad (11.20)$$

where

$$4t^2R^2 = t^4 + t^3 + 2t^2 - t + 1 \quad (11.21)$$

and

$$4t^2S^2 = t^4 + 5t^3 + 2t^2 - 5t + 1 = (t^2 + 4t - 1)(t^2 + t - 1). \quad (11.22)$$

The negative signs on the two radicals were chosen in order to be consistent with the fact that t tends to 1 as m and m' tend to 1. Since α, β, γ, and δ can be expressed in terms of m and m', it follows from (11.7) and (11.19) that α, β, γ, and δ can be expressed in terms of a single parameter t.

PROOF OF (x). By (11.12), (11.15), and (11.16),

$$\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} = \left(\frac{(\mu'-1)^6(5\mu'^{-1}-1)^6}{(\mu-1)^6(5\mu^{-1}-1)^6}\right)^{1/4}$$
$$= \frac{(\mu'-1)^3(5\mu'^{-1}-1)^3}{(1+t^{-1})^6(1-t)^6} = \frac{(\mu'-1)^3(5\mu'^{-1}-1)^3}{(t^{-1}-t)^6}.$$

On the other hand, by (11.10), (11.13), and (11.14),

$$\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} = \left(\frac{(m^2-1)^2(m'^2-1)^2}{(9m^{-2}-1)^2(9m'^{-2}-1)^2}\right)^{1/4} = \frac{(m^2-1)(m'^2-1)}{(t^{-1}-t)^3}. \quad (11.23)$$

Equating the right sides above, taking cube roots, and using (11.11), (11.18), (11.20), (11.21), and (11.22), we find that

$$\{(m^2-1)(m'^2-1)\}^{1/3} = \frac{1}{t^{-1}-t}\left(6 - \mu' - \frac{5}{\mu'}\right)$$
$$= \frac{t^4 + 3t^3 + 2t^2 - 3t + 1 - 4t^2RS}{2t^2}$$
$$= (R-S)^2. \quad (11.24)$$

Thus, by (11.23),

$$\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4} = \frac{(R-S)^6}{(t^{-1}-t)^3}. \quad (11.25)$$

Next, by (11.10), (11.23), (11.13), (11.14), (11.7), and (11.17),

$$\left\{\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4}\right\}^2$$
$$=\frac{(m-1)(m'-1)}{(3m^{-1}+1)(3m'^{-1}+1)}+\frac{(m+1)(m'+1)}{(3m^{-1}-1)(3m'^{-1}-1)}+\frac{2(m^2-1)(m'^2-1)}{(t^{-1}-t)^3}$$
$$=\frac{20+2mm'+18/(mm')+6(m+m')^2/(mm')}{(9m^{-2}-1)(9m'^{-2}-1)}+\frac{2(m^2-1)(m'^2-1)}{(t^{-1}-t)^3}$$
$$=\frac{2(m^2-1)(m'^2-1)}{(t^{-1}-t)^6}\left(10+mm'+\frac{9}{mm'}+\frac{3(m+m')^2}{mm'}+(t^{-1}-t)^3\right)$$
$$=\frac{2(m^2-1)(m'^2-1)}{(t^{-1}-t)^6}\left(10+\frac{t^6+5t^5+5t^4-5t^2+5t-1}{t^3}\right.$$
$$\left.+\frac{3(1+t^2)^2}{t^2}+\frac{(1-t^2)^3}{t^3}\right)$$
$$=\frac{16(m^2-1)(m'^2-1)(t^4+t^3+2t^2-t+1)}{(t^{-1}-t)^6t^2}.$$

So, taking square roots and utilizing (11.24) and (11.21), we deduce that

$$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4}=\frac{8R(R-S)^3}{(t^{-1}-t)^3}. \tag{11.26}$$

Because of the pervasiveness of $t^{-1}-t$, it will be convenient to introduce a new parameter

$$u:=t^{-1}-t. \tag{11.27}$$

Thus, by (11.19), (11.21), and (11.22), respectively,

$$2m^2t^2=10-8u+5u^2-u^3-4RS(2-u), \tag{11.28}$$

$$4R^2=u^2-u+4, \tag{11.29}$$

and

$$4S^2=u^2-5u+4. \tag{11.30}$$

We are now prepared for the final calculations necessary to complete the proof of (x). Employing (11.26), (11.25), (11.28), (11.29), and (11.30), we deduce that

$$u^3\left\{\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4}-\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}-mm'\right\}$$
$$=8R(R-S)^3-(R-S)^6-u^3m^2t^2$$
$$=8R^4+24R^2S^2-(R^2+S^2)(R^4+14R^2S^2+S^4)$$
$$-\tfrac{1}{2}u^3(10-8u+5u^2-u^3)$$
$$-RS\{24R^2+8S^2-6R^4-20R^2S^2-6S^4-2u^3(2-u)\}$$

$$\begin{aligned}
&= \tfrac{1}{2}(u^2-u+4)^2 + \tfrac{3}{2}(u^2-u+4)(u^2-5u+4) \\
&\quad - \tfrac{1}{2}u^3(10-8u+5u^2-u^3) \\
&\quad - \tfrac{1}{32}(u^2-3u+4)\{(u^2-u+4)^2 + 14(u^2-u+4)(u^2-5u+4) \\
&\quad + (u^2-5u+4)^2\} - RS\{6(u^2-u+4) \\
&\quad + 2(u^2-5u+4) - \tfrac{3}{8}(u^2-u+4)^2 - \tfrac{5}{4}(u^2-u+4)(u^2-5u+4) \\
&\quad - \tfrac{3}{8}(u^2-5u+4)^2 - 2u^3(2-u)\} \\
&= 2u^5 - 12u^4 + 30u^3 - 48u^2 + 32u - 8RS(u^3-3u^2+4u) \\
&= 4u(R^4+6R^2S^2+S^4) - 16uRS(R^2+S^2) \\
&= 4u(R-S)^4 \\
&= 4u^3\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/6},
\end{aligned}$$

by (11.25) again. Hence, at last, (x) is established.

Proof of (xi). Formula (xi) is the reciprocal of (x).

Proof of (xii). First, by (11.10), (11.13), and (11.14),

$$\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/4} = \frac{(m'^2-1)(9m'^{-2}-1)}{(m^2-1)(9m^{-2}-1)} = \frac{(t^{-1}-t)^6}{(m^2-1)^2(9m^{-2}-1)^2}. \tag{11.31}$$

Second, by (11.10), (11.12), (11.15), and (11.16),

$$\begin{aligned}
\{(m^2-1)(9m^{-2}-1)\}^4 &= 2^{16}\alpha(1-\alpha)\beta(1-\beta) \\
&= (\mu-1)(\mu'-1)(5\mu^{-1}-1)^5(5\mu'^{-1}-1)^5 \\
&= (t^{-1}-t)^4(5\mu^{-1}-1)^4(5\mu'^{-1}-1)^4;
\end{aligned}$$

that is

$$(m^2-1)(9m^{-2}-1) = (t^{-1}-t)(5\mu^{-1}-1)(5\mu'^{-1}-1). \tag{11.32}$$

We want to express $(m^2-1)(9m^{-2}-1)$ entirely in terms of t. By (11.17), (11.19), (11.21), and (11.22),

$$\begin{aligned}
(m^2-1)(9m^{-2}-1) &= \frac{(1-t^2)}{2t^5}(t^8+5t^7+6t^6+5t^5+2t^4-5t^3 \\
&\quad + 6t^2-5t+1+4t^2(t^2+1)(t^2+2t-1)RS) \\
&= (t^{-1}-t)\{(2-t^{-1}+t)R+(t^{-1}+t)S\}^2.
\end{aligned} \tag{11.33}$$

We have omitted some rather tedious, but straightforward, algebraic computations. We now claim that

$$(2 - t^{-1} + t)R + (t^{-1} + t)S = \frac{(t^{-1} - t)^2}{(2 - t^{-1} + t)R - (t^{-1} + t)S}. \quad (11.34)$$

To verify this equality, cross-multiply and express everything in terms of t via (11.21) and (11.22). Putting (11.34) in (11.33), we see that

$$(m^2 - 1)(9m^{-2} - 1) = (t^{-1} - t)^5\{(2 - t^{-1} + t)R - (t^{-1} + t)S\}^{-2}. \quad (11.35)$$

Hence, by (11.10), (11.31), and (11.17),

$$\begin{aligned}\left\{\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/8} + \left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/8}\right\}^2 &= \frac{(m'-1)(3m'^{-1}+1)}{(m-1)(3m^{-1}+1)} + \frac{(m'+1)(3m'^{-1}-1)}{(m+1)(3m^{-1}-1)} + \frac{2(t^{-1}-t)^3}{(m^2-1)(9m^{-2}-1)} \\ &= \frac{1}{(m^2-1)(9m^{-2}-1)}\left(8 - 2\left(\frac{3}{m} - m\right)\left(\frac{3}{m'} - m'\right) + 2\frac{(1-t^2)^3}{t^3}\right) \\ &= \frac{1}{(m^2-1)(9m^{-2}-1)}\left(8 - 2\left(m^2t^2 + \frac{9}{m^2t^2}\right) + \frac{6(1+t^4)}{t^2} + \frac{2(1-t^2)^3}{t^3}\right) \\ &= \frac{4(1-t^2)(1-t+2t^2+t^3+t^4)}{(m^2-1)(9m^{-2}-1)t^3} \\ &= \frac{16(1-t^2)R^2}{t(m^2-1)(9m^{-2}-1)}. \end{aligned} \quad (11.36)$$

Thus, by (11.36), (11.35), (11.27), and (11.29),

$$\begin{aligned}\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/8} + \left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/8} + 1 &= \frac{4R\{(2-t^{-1}+t)R - (t^{-1}+t)S\}}{(t^{-1}+t)^2} + 1 \\ &= u^{-2}\{4R^2(2 - t^{-1} + t) \\ &\quad - 4RS(t^{-1}+t) + u^2\} \\ &= u^{-2}\{8 - 6u + 4u^2 - u^3 \\ &\quad - 4RS(t^{-1}+t)\}. \end{aligned} \quad (11.37)$$

Using our calculations in (11.36), as well as (11.32), (11.31), (11.17), (11.27), and (11.35), we find that

$$\begin{aligned}&\left\{\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/4} + \left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/4} + \left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/4} - \frac{z_1z_3}{z_5z_{15}}\right\}(m^2-1) \\ &\quad \times (9m^{-2}-1) \\ &= 8 - 2\left(m^2t^2 + \frac{9}{m^2t^2}\right) + \frac{6(1+t^4)}{t^2} + \frac{(t^{-1}+t)^6}{(m^2-1)(9m^{-2}-1)} \\ &\quad - (t^{-1}-t)(5-\mu)(5-\mu')\end{aligned}$$

$$= \frac{2 - 4t + 10t^2 + 8t^3 - 10t^4 - 4t^5 - 2t^6}{t^3}$$

$$+ \frac{(t^{-1} + t)^6}{(m^2 - 1)(9m^{-2} - 1)} - u(5 - \mu)(5 - \mu')$$

$$= u(2u^2 - 4u + 16) + u\{(2 - u)R - (t^{-1} + t)S\}^2$$

$$- u(25 - 5\mu(1 + t^2) + \mu^2 t^2). \tag{11.38}$$

The expressions in μ must be converted to terms in u. By (11.18) and (11.20),

$$5(1 + t^2)\mu - t^2\mu^2$$

$$= \frac{5(1 + t^2)}{2t^3}(t^4 + 3t^3 + 3t - 1) - \frac{(t^4 + 3t^3 + 3t - 1)^2}{2t^4} - 5$$

$$+ \frac{2}{t^2}(t^4 + 3t^3 + 3t - 1 - 5t(1 + t^2))RS$$

$$= \tfrac{1}{2}(u^2 + 4)(u + 2)(3 - u) - 5 - 2(2 + u)(t + t^{-1})RS.$$

Using this in (11.38) and utilizing (11.29) and (11.30), we find that

$$\left\{\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/4} + \left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/4} + \left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/4} - \frac{z_1 z_3}{z_5 z_{15}}\right\}$$

$$\times (m^2 - 1)(9m^{-2} - 1)$$

$$= u\{2u^2 - 4u + 16 + \{(2 - u)R - (t^{-1} + t)S\}^2 - 20$$

$$+ \tfrac{1}{2}(u^2 + 4)(u + 2)(3 - u) - 2(2 + u)(t^{-1} + t)RS\}$$

$$= u\{16 - 12u + 8u^2 - 2u^3 - 8(t^{-1} + t)RS\}. \tag{11.39}$$

By comparing (11.37) and (11.39), we conclude that

$$\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/4} + \left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/4} + \left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/4} - \frac{z_1 z_3}{z_5 z_{15}}$$

$$= \frac{2u^3}{(m^2 - 1)(9m^{-2} - 1)}\left\{1 + \left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/8} + \left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/8}\right\}$$

$$= 2\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/8}\left\{1 + \left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/8} + \left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/8}\right\},$$

by (11.31). This completes the proof of (xii).

PROOF OF (xiii). This formula is the reciprocal of (xii).

After the proofs of (x) and (xii), the proofs of (xiv) and (xv) are comparatively simple.

PROOF OF (xiv). By (11.3) and (11.4),

$$4M(\alpha\beta\gamma\delta)^{1/8} = 4MAB = M(M^{-1} - \rho)(M - \rho) = 1 + 2m - M^2 - \rho(M^2 + 1).$$

Similarly,

$$4M\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8} = 1 + 2M - M^2 + \rho(M^2 + 1).$$

Thus, by (11.4),

$$\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8}$$
$$= \frac{1}{16M^2}\{1 + 2M - M^2 - \rho(M^2+1)\}\{1 + 2M - M^2 + \rho(M^2+1),$$
$$= \frac{1}{16M^2}\{(1 + 2M - M^2)^2 - \frac{1}{M}(1 + M - M^2)(M^2+1)^2\}$$
$$= \frac{1}{16M^3}(M^2 - 1)^3.$$

It is now clear that by combining these last three calculations, we achieve (xiv).

Proof of (xv). From (11.6),

$$PQ = \left(\frac{256\alpha^4\delta^4(1-\alpha)^4(1-\delta)^4}{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}\right)^{1/48} = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{M}\right), \qquad (11.40)$$

and from (11.5),

$$\frac{P}{Q} = \left(\frac{256\beta^4\gamma^4(1-\beta)^4(1-\gamma)^4}{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}\right)^{1/48} = \frac{1}{\sqrt{2}}(M+1). \qquad (11.41)$$

Hence, upon multiplication,

$$P^2 = \frac{1}{2}\left(M - \frac{1}{M}\right),$$

Adding (11.40) and (11.41) and using the formula above, we deduce that

$$P\left(Q + \frac{1}{Q}\right) = \frac{1}{\sqrt{2}}\left(M + 2 - \frac{1}{M}\right) = \frac{1}{\sqrt{2}}(2P^2 + 2).$$

Dividing both sides by P, we complete the proof.

With all the groundwork developed in Section 11, it will now be a relatively easy task to prove Entries 9(v)–(vii), which we left unproved in Section 9.

Proof of Entry 9(v). By Entries 12(iii), (v) in Chapter 17, as well as (11.40) above,

$$2qf(-q^2)f(-q^{30})\chi(q^3)\chi(q^5) = 2^{2/3}\sqrt{z_1 z_{15}}\left(\frac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/24}$$
$$= \sqrt{z_1 z_{15}}\left(1 - \frac{1}{M}\right)$$

$$= \sqrt{z_1 z_{15}} - \sqrt{z_3 z_5}$$
$$= \varphi(q)\varphi(q^{15}) - \varphi(q^3)\varphi(q^5),$$

which completes the proof.

PROOF OF ENTRY 9(vi). Again, by Entries 12(iii), (v) in Chapter 17, as well as (11.41) above,

$$\begin{aligned}2f(-q^6)f(-q^{10})\chi(q)\chi(q^{15}) &= 2^{2/3}\sqrt{z_3 z_5}\left(\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24}\\ &= \sqrt{z_3 z_5}(M+1)\\ &= \sqrt{z_1 z_{15}} + \sqrt{z_3 z_5}\\ &= \varphi(q)\varphi(q^{15}) + \varphi(q^3)\varphi(q^5).\end{aligned}$$

PROOF OF ENTRY 9(vii). Employing Entries 11(i), 10(ii), and 12(ii) of Chapter 17, (11.3), (11.40), (11.41), and (11.4), we find that

$$\begin{aligned}&\{q\psi(q^3)\psi(q^5) - q^2\psi(q)\psi(q^{15})\}\varphi(-q^3)\varphi(-q^5)\\ &= \tfrac{1}{2}\sqrt{z_1 z_3 z_5 z_{15}}\left\{\frac{1}{M}(\beta\gamma)^{1/8} - (\alpha\delta)^{1/8}\right\}(1-\beta)^{1/4}(1-\gamma)^{1/4}\\ &= \frac{2^{-5/3}\sqrt{z_1 z_3 z_5 z_{15}}\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/6}(\alpha\beta\gamma\delta)^{1/24}}{\{(1-\alpha)(1-\delta)\}^{1/4}}\\ &\quad\times\left\{\frac{1}{M}\{(1-\alpha)(1-\delta)\}^{1/8}2^{2/3}\left(\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24}\right.\\ &\quad\left. - \{(1-\beta)(1-\gamma)\}^{1/8}2^{2/3}\left(\frac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/24}\right\}\\ &= \frac{qf(-q)f(-q^3)f(-q^5)f(-q^{15})}{2\{(1-\alpha)(1-\delta)\}^{1/4}}\\ &\quad\times\left\{\frac{(M^{-1}+\rho)}{2M}(M+1) - \frac{(M+\rho)}{2}\left(1-\frac{1}{m}\right)\right\}\\ &= \frac{qf(-q)f(-q^3)f(-q^5)f(-q^{15})}{(M^{-1}+\rho)^2 M^2}\\ &\quad\times\{(1+\rho M)(M+1) - M(M+\rho)(M-1)\}\\ &= qf(-q)f(-q^3)f(-q^5)f(-q^{15}).\end{aligned}$$

This concludes the proof of part of (vii).

The second equality of (vii) is established by completely analogous reasoning. Thus,

$$\begin{aligned}\{q\psi(q^3)\psi(q^5) + q^2\psi(q)\psi(q^{15})\}\varphi(-q)\varphi(-q^{15}) &= \tfrac{1}{2}\sqrt{z_1z_3z_5z_{15}}\{(\beta\gamma)^{1/8} + M(\alpha\delta)^{1/8}\}(1-\alpha)^{1/4}(1-\delta)^{1/4} \\ &= \frac{qf(-q)f(-q^3)f(-q^5)f(-q^{15})}{2\{(1-\beta)(1-\gamma)\}^{1/4}} \\ &\quad \times \left\{\frac{(M^{-1}+\rho)(M+1)}{2} + \frac{M(M+\rho)}{2}\left(1 - \frac{1}{M}\right)\right\} \\ &= qf(-q)f(-q^3)f(-q^5)f(-q^{15}),\end{aligned}$$

which completes the proof of Entry 9(vii).

Entry 12.

(i) *Let*

$$\mu_1 = \frac{f(-q^6,-q^{11})}{q^{12/17}f(-q^3,-q^{14})}, \qquad \mu_2 = \frac{f(-q^4,-q^{13})}{q^{11/17}f(-q^2,-q^{15})},$$

$$\mu_3 = \frac{f(-q^8,-q^9)}{q^{10/17}f(-q^4,-q^{13})}, \qquad \mu_4 = \frac{f(-q^2,-q^{15})}{q^{7/17}f(-q,-q^{16})},$$

$$\mu_5 = \frac{f(-q^7,-q^{10})}{q^{5/17}f(-q^5,-q^{12})}, \qquad \mu_6 = \frac{q^{3/17}f(-q^5,-q^{12})}{f(-q^6,-q^{11})},$$

$$\mu_7 = \frac{q^{14/17}f(-q^3,-q^{14})}{f(-q^7,-q^{10})}, \quad \text{and} \quad \mu_8 = \frac{q^{28/17}f(-q,-q^{16})}{f(-q^8,-q^9)}.$$

Then

$$\frac{f(-q^{1/17})}{q^{12/17}f(-q^{17})} = \mu_1 - \mu_2 - \mu_3 + \mu_4 + \mu_5 - 1 - \mu_6 + \mu_7 - \mu_8, \tag{12.1}$$

$$\mu_1\mu_5\mu_6\mu_7 = \mu_2\mu_8\mu_3\mu_4 = 1, \tag{12.2}$$

and

$$\mu_1\mu_5 + \mu_2\mu_8 - \mu_3\mu_4 - \mu_6\mu_7 = -1. \tag{12.3}$$

(ii) $f(-q)f^7(-q^{17})$

$$\begin{aligned}&= f(-q,-q^{16})f(-q^2,-q^{15})f(-q^3,-q^{14})f(-q^4,-q^{13}) \\ &\quad \times f(-q^5,-q^{12})f(-q^6,-q^{11})f(-q^7,-q^{10})f(-q^8,-q^9).\end{aligned}$$

The following are modular equations of degree 17:

(iii)
$$\begin{aligned}m &= \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \\ &\quad - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}\left\{1 + \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8}\right\}\end{aligned}$$

and

(iv)
$$\frac{17}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} + \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} - 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}\left\{1 + \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8}\right\}.$$

Proof of (i). We first observe that (12.1) follows from (12.26) of Chapter 19 with $n = 17$.

The equalities of (12.2) follow immediately from the definitions of $\mu_1, \ldots, \mu_8$.

A proof of (12.3) depending on Entry 22(iii) in Chapter 16 could be constructed. However, it seems easier to use the identity

$$f^2(-q) = \sum_{\substack{m,n=-\infty \\ n \geq 2|m|}}^{\infty} (-1)^{m+n} q^{n(n+1)/2 - m(3m-1)/2}, \tag{12.4}$$

first proved by L. J. Rogers [1] in 1894, rediscovered by Hecke [1], [2, pp. 418–427] in 1925, and more recently proved by Andrews [16], [17], Bressoud [3], and Kac and Peterson [1].

First, square both sides of (12.1) and equate rational parts on each side to deduce that

$$\mathscr{R}\left(q^{-24/17}\frac{f^2(-q^{1/17})}{f^2(-q^{17})}\right) = 2(\mu_1\mu_5 + \mu_2\mu_8 - \mu_3\mu_4 - \mu_6\mu_7) + 1,$$

where $\mathscr{R}(\cdot)$ denotes the rational part of $(\cdot)$. From (12.3), it thus suffices to show that

$$\mathscr{R}\left(q^{-24/17}\frac{f^2(-q^{1/17})}{f^2(-q^{17})}\right) = -1. \tag{12.5}$$

Now, by (12.4),

$$q^{-24/17}f^2(-q^{1/17}) = \sum_{\substack{m,n=-\infty \\ n \geq 2|m|}}^{\infty} (-1)^{m+n} q^{n(n+1)/34 - m(3m-1)/34 - 24/17}.$$

We thus want to isolate those terms where

$$n(n+1)/2 - m(3m-1)/2 \equiv 7 \pmod{17}. \tag{12.6}$$

Examining complete residue systems of m and n (mod 17), we see that the only solutions of (12.6) are

$$n \equiv 8 \pmod{17} \quad \text{and} \quad m \equiv 3 \pmod{17}.$$

Thus, setting $m = 3 + 17k$ and $n = 8 + 17\ell$, we discover that

$$\begin{aligned}
&\mathscr{R}(q^{-24/17}f^2(-q^{1/17}))\\
&= -\sum_{\substack{k,\ell=-\infty\\ \ell\geq 2|k|}}^{\infty} (-1)^{k+\ell} q^{\{(8+17\ell)^2+(8+17\ell)\}/34-\{3(3+17k)^2-(3+17k)\}/34-24/17}\\
&= -\sum_{\substack{k,\ell=-\infty\\ \ell\geq 2|k|}}^{\infty} (-1)^{k+\ell} q^{17\ell(\ell+1)/2-17k(3k+1)/2}\\
&= -f^2(-q^{17}),
\end{aligned}$$

by (12.4). Hence, (12.5) is established, and the proof of (12.3) is complete.

We offer a second proof of (12.3) that depends on Theorem 8.1. By (0.53), (0.51), and (0.52), (12.3) is equivalent to the identity

$$\sum_{m \pmod{17}} G(m;z)G(4m;z) = 0. \tag{12.7}$$

However, (12.7) follows immediately from (8.7) for $p = 17$, since $a_{17} = 0$.

Proof of (ii). This identity is merely the special case $n = 8$ of (28.1) of Chapter 16.

Proofs of (iii), (iv). First, observe that (iv) is the reciprocal of (iii).

Using Entries 10(i)–(iii) and 11(i)–(iii) of Chapter 17, we find that (iii) and (iv) are equivalent to, respectively,

$$\begin{aligned}
1 &= q^4\frac{\psi(q^{34})\varphi(q^{17})}{\psi(q^2)\varphi(q)} + \frac{\varphi(-q^{17})\varphi(q^{17})}{\varphi(-q)\varphi(q)} + q^4\frac{\psi(q^{34})\varphi(-q^{17})}{\psi(q^2)\varphi(-q)}\\
&\quad - 2q^2\frac{\psi(-q^{17})\varphi(q^{17})}{\psi(-q)\varphi(q)} - 2q^4\frac{\psi(q^{34})\varphi(-q^{34})}{\psi(q^2)\varphi(-q^2)} - 2q^2\frac{\psi(q^{17})\varphi(-q^{17})}{\psi(q)\varphi(-q)}
\end{aligned}\tag{12.8}$$

and

$$\begin{aligned}
17 &= \frac{\psi(q^2)\varphi(q)}{q^4\psi(q^{34})\varphi(q^{17})} + \frac{\varphi(-q)\varphi(q)}{\varphi(-q^{17})\varphi(q^{17})} + \frac{\psi(q^2)\varphi(-q)}{q^4\psi(q^{34})\varphi(-q^{17})}\\
&\quad - 2\frac{\psi(-q)\varphi(q)}{q^2\psi(-q^{17})\varphi(q^{17})} - 2\frac{\psi(q^2)\varphi(-q^2)}{q^4\psi(q^{34})\varphi(-q^{34})} - 2\frac{\psi(q)\varphi(-q)}{q^2\psi(q^{17})\varphi(-q^{17})}.
\end{aligned}\tag{12.9}$$

Using (0.13), we transform (12.8) and (12.9) into equalities involving modular forms. Thus, respectively,

$$\begin{aligned}
1 &= \frac{g_2(17\tau)g_1(17\tau)}{g_2(\tau)g_1(\tau)} + \frac{g_0(17\tau)g_1(17\tau)}{g_0(\tau)g_1(\tau)} + \frac{g_2(17\tau)g_0(17\tau)}{g_2(\tau)g_0(\tau)} - 2\frac{h_1(17\tau)g_1(17\tau)}{h_1(\tau)g_1(\tau)}\\
&\quad - 2\frac{h_2(17\tau)g_2(17\tau)}{h_2(\tau)g_2(\tau)} - 2\frac{h_0(17\tau)g_0(17\tau)}{h_0(\tau)g_0(\tau)}
\end{aligned}\tag{12.10}$$

and

$$17 = \frac{g_2(\tau)g_1(\tau)}{g_2(17\tau)g_1(17\tau)} + \frac{g_0(\tau)g_1(\tau)}{g_0(17\tau)g_1(17\tau)} + \frac{g_2(\tau)g_0(\tau)}{g_2(17\tau)g_0(17\tau)} - 2\frac{h_1(\tau)g_1(\tau)}{h_1(17\tau)g_1(17\tau)}$$
$$- 2\frac{h_2(\tau)g_2(\tau)}{h_2(17\tau)g_2(17\tau)} - 2\frac{h_0(\tau)g_0(\tau)}{h_0(17\tau)g_0(17\tau)}. \tag{12.11}$$

Since b and c are even, we see from (0.18)–(0.23) and (0.26) that the multiplier system of each term in (12.10) and (12.11) is trivial; that is, $v(A) \equiv 1$ for each $A \in \Gamma(2) \cap \Gamma_0(17)$.

By (0.6) and (0.24), $\rho_\Gamma = 9$ for $\Gamma = \Gamma(2) \cap \Gamma_0(17)$. By clearing denominators in (12.10) and (12.11), we may write each of the proposed identities in the form $F := F_1 + \cdots + F_7 = 0$, where F_j, $1 \le j \le 7$, is a modular form of weight 3 on Γ. Thus, by (0.38), it suffices to show that the coefficients of $q^0, q^1, \ldots, q^{13}$ are equal to 0 for the functions F arising from (12.10) and (12.11). Using the computer algebra system MACSYMA, we have, indeed, shown that all of these coefficients are equal to 0. Thus, (12.10) and (12.11) are established, and the proofs of (iii) and (iv) are complete.

Modular equations of degree 17 have previously been established by Sohncke [1], [2], Schläfli [1], Russell [1], [2], and Greenhill [2], but none of these has the simplicity of the two modular equations of Ramanujan that we have just proved.

At the end of Section 12, Ramanujan remarks:

> N.B, Thus we see that $\varphi(x^{1/n})$, $\psi(x^{1/n})$ or $f(-x^{1/n})$ n being any prime number can be expressed as the sum of $\frac{1}{2}(n-1)$, nth roots of several functions and $\varphi(x^n)$, $\psi(x^n)$ or $f(-x^n)$. In finding the values of these functions, quadratics only appear in the case of the 5th, 17th, 257th, etc. degrees, and cubics in case of the 7th, 13th, 19th, 37th, 73rd, 97th, 109th, 163rd, 193rd etc. degrees not as cube roots but as $\sin(\frac{1}{3}\sin^{-1}\theta)$ and quintics in case of the 11th, 41st, 101st etc. degrees. $f^3(-x^{1/n})$ can also be similarly expressed.

We are unable to provide a proper interpretation for most of this statement. As we have seen in past entries, for example, Entries 12(i) and 18(i) in Chapter 19 and Entries 6(i), 8(i), and 12(i) in Chapter 20, by employing Entry 31 in Chapter 16, we may express $\varphi(q^{1/n})$, $\psi(q^{1/n})$, and $f(-q^{1/n})$ in terms of $\varphi(q^n)$, $\psi(q^n)$, and $f(-q^n)$, respectively. Moreover, in each case, there appear $\frac{1}{2}(n-1)$ additional expressions that involve nth roots of q. The words "these functions" of Ramanujan evidently refer to what he calls "nth roots of several functions." We cannot identify the "quadratics," "cubics," and "quintics" to which Ramanujan refers, nor do we know what θ denotes. It does not seem possible to find the values of "these functions" by purely algebraical means, which is what is seemingly indicated; Ramanujan evidently has something else in mind. In the last sentence, Ramanujan possibly is employing the identity $f^3(-q) = \varphi^2(-q)\psi(q)$, which is part of Entry 24(ii) of Chapter 16.

Entry 13. *If β, γ, and δ are of degrees* 3, 7, *and* 21, *respectively*, $m = z_1/z_3$, *and* $m' = z_7/z_{21}$, *then*

(i) $$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/4}+\left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/4}-\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/4}$$
$$+4\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/6}=\frac{m}{m'},$$

(ii) $$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/4}+\left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/4}-\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/4}$$
$$+4\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/6}=\frac{m'}{m},$$

(iii) $$\left(\frac{\gamma\delta}{\alpha\beta}\right)^{1/8}+\left(\frac{(1-\gamma)(1-\delta)}{(1-\alpha)(1-\beta)}\right)^{1/8}-\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/8}$$
$$-2\left(\frac{\gamma\delta(1-\gamma)(1-\delta)}{\alpha\beta(1-\alpha)(1-\beta)}\right)^{1/12}=\left(\frac{z_1z_3}{z_7z_{21}}\right)^{1/2},$$

(iv) $$\left(\frac{\alpha\beta}{\gamma\delta}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\beta)}{(1-\gamma)(1-\delta)}\right)^{1/8}-\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{1/8}$$
$$-2\left(\frac{\alpha\beta(1-\alpha)(1-\beta)}{\gamma\delta(1-\gamma)(1-\delta)}\right)^{1/12}=7\left(\frac{z_7z_{21}}{z_1z_3}\right)^{1/2},$$

(v) $$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/4}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/4}+\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/4}$$
$$-2\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8}\left\{1+\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8}+\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8}\right\}=mm',$$

and

(vi) $$\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/4}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/4}+\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/4}$$
$$-2\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/8}\left\{1+\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8}+\left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/8}\right\}=\frac{9}{mm'}.$$

Observe that (ii), (iv), and (vi) are the reciprocals, respectively, of (i), (iii), and (v).

PROOFS. Our proofs utilize the theory of modular forms.

Using Entries 10(i)–(iii), 11(i)–(iii), and 12(iii) in Chapter 17, we convert (i), (iii), and (v) above into the equivalent theta-function identities

$$\frac{\psi(q^6)\psi(q^{14})\varphi(q^3)\varphi(q^7)}{q^3\psi(q^2)\psi(q^{42})\varphi(q)\varphi(q^{21})}+\frac{\varphi(q^3)\varphi(q^7)\varphi(-q^3)\varphi(-q^7)}{\varphi(q)\varphi(q^{21})\varphi(-q)\varphi(-q^{21})}$$
$$-\frac{\psi(q^6)\psi(q^{14})\varphi(-q^3)\varphi(-q^7)}{q^3\psi(q^2)\psi(q^{42})\varphi(-q)\varphi(-q^{21})}+4\frac{f^2(-q^6)f^2(-q^{14})}{q^2f^2(-q^2)f^2(-q^{42})}=1, \quad (13.1)$$

$$\frac{q^3\psi(q^7)\psi(q^{21})}{\psi(q)\psi(q^3)}+\frac{\varphi(-q^{14})\varphi(-q^{42})}{\varphi(-q^2)\varphi(-q^6)}-\frac{q^3\psi(-q^7)\psi(-q^{21})}{\psi(-q)\psi(-q^3)}$$
$$-2\frac{q^2f(-q^{14})f(-q^{42})}{f(-q^2)f(-q^6)}=1, \tag{13.2}$$

and

$$\frac{q^4\psi(q^6)\psi(q^{42})\varphi(q^3)\varphi(q^{21})}{\psi(q^2)\psi(q^{14})\varphi(q)\varphi(q^7)}+\frac{\varphi(q^3)\varphi(q^{21})\varphi(-q^3)\varphi(-q^{21})}{\varphi(q)\varphi(q^7)\varphi(-q)\varphi(-q^7)}$$
$$+\frac{q^4\psi(q^6)\psi(q^{42})\varphi(-q^3)\varphi(-q^{21})}{\psi(q^2)\psi(q^{14})\varphi(-q)\varphi(-q^7)}-2\frac{q^2\psi(-q^3)\psi(-q^{21})\varphi(q^3)\varphi(q^{21})}{\psi(-q)\psi(-q^7)\varphi(q)\varphi(q^7)}$$
$$-2\frac{q^4\psi(q^6)\psi(q^{42})\varphi(-q^6)\varphi(-q^{42})}{\psi(q^2)\psi(q^{14})\varphi(-q^2)\varphi(-q^{14})}-2\frac{q^2\psi(q^3)\psi(q^{21})\varphi(-q^3)\varphi(-q^{21})}{\psi(q)\psi(q^7)\varphi(-q)\varphi(-q^7)}=1, \tag{13.3}$$

respectively. Because the theta-function identities equivalent to (ii), (iv), and (vi) are similar to (13.1)–(13.3), respectively, we do not record them. Next, we transcribe, via (0.13), the identities (13.1)–(13.3) into the modular form identities

$$\frac{g_1(3\tau)g_1(7\tau)g_2(3\tau)g_2(7\tau)}{g_1(\tau)g_1(21\tau)g_2(\tau)g_2(21\tau)}+\frac{g_0(3\tau)g_0(7\tau)g_1(3\tau)g_1(7\tau)}{g_0(\tau)g_0(21\tau)g_1(\tau)g_1(21\tau)}$$
$$-\frac{g_0(3\tau)g_0(7\tau)g_2(3\tau)g_2(7\tau)}{g_0(\tau)g_0(21\tau)g_2(\tau)g_2(21\tau)}+4\frac{\eta^2(3\tau)\eta^2(7\tau)}{\eta^2(\tau)\eta^2(21\tau)}=1, \tag{13.4}$$

$$\frac{h_0(7\tau)h_0(21\tau)}{h_0(\tau)h_0(3\tau)}+\frac{h_2(7\tau)h_2(21\tau)}{h_2(\tau)h_2(3\tau)}-\frac{h_1(7\tau)h_1(21\tau)}{h_1(\tau)h_1(3\tau)}-2\frac{\eta(7\tau)\eta(21\tau)}{\eta(\tau)\eta(3\tau)}=1, \tag{13.5}$$

and

$$\frac{g_1(3\tau)g_1(21\tau)g_2(3\tau)g_2(21\tau)}{g_1(\tau)g_1(7\tau)g_2(\tau)g_2(7\tau)}+\frac{g_0(3\tau)g_0(21\tau)g_1(3\tau)g_1(21\tau)}{g_0(\tau)g_0(7\tau)g_1(\tau)g_1(7\tau)}$$
$$+\frac{g_0(3\tau)g_0(21\tau)g_2(3\tau)g_2(21\tau)}{g_0(\tau)g_0(7\tau)g_2(\tau)g_2(7\tau)}-2\frac{g_1(3\tau)g_1(21\tau)h_1(3\tau)h_1(21\tau)}{g_1(\tau)g_1(7\tau)h_1(\tau)h_1(7\tau)}$$
$$-2\frac{g_2(3\tau)g_2(21\tau)h_2(3\tau)h_2(21\tau)}{g_2(\tau)g_2(7\tau)h_2(\tau)h_2(7\tau)}-2\frac{g_0(3\tau)g_0(21\tau)h_0(3\tau)h_0(21\tau)}{g_0(\tau)g_0(7\tau)h_0(\tau)h_0(7\tau)}=1, \tag{13.6}$$

respectively.

In the introduction to this chapter, we showed that if $F = g_j$, h_j, or η, $0 \le j \le 2$, then F has a multiplier system v_F of the form

$$v_F\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{c_0}{d}\right)_* \xi_1\xi_2. \tag{13.7}$$

Furthermore, if n is a positive odd integer, then $F(n\tau)$ is a modular form on

$\Gamma(2) \cap \Gamma_0(n)$ with a multiplier system $v_F|n$ that has the values

$$v_F|n\begin{pmatrix} a & b \\ b & d \end{pmatrix} = \left(\frac{n}{d}\right)_* \left(\frac{c_0}{d}\right)_* \xi_1 \xi_2^n, \tag{13.8}$$

provided that $(n, 6) = 1$. Here, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(2) \cap \Gamma_0(n)$ and $c_0 = c$ if $F = \eta, g_0, g_1, g_2, h_1$, while $c_0 = 2c$ if $F = h_0, h_2$. If n is odd and $3|n$, (13.8) is still valid for $F = g_j$ or h_j, $0 \le j \le 2$, but is not valid when $F = \eta$. Moreover, from (0.14) and (0.18)–(0.23) we note that on $\Gamma(2)$, $\xi_2^4 = 1$ for $F = g_j$ and $\xi_2^8 = 1$ for $F = h_j$, where $0 \le j \le 2$. Using (13.7), (13.8), and the foregoing observations, we may easily verify that each expression in (13.4)–(13.6) has a multiplier system that is identically equal to 1, with the possible exceptions of the two expressions involving eta-functions. Now observe that, in the instance at hand, $3 \nmid d$, so that $d^2 - 1 \equiv 0 \pmod{24}$. Thus, from (0.14), the multiplier system of $\eta(n\tau)$ is equal to

$$v_\eta|n\begin{pmatrix} a & b \\ c & d \end{pmatrix} = v_\eta\begin{pmatrix} a & nb \\ c/n & d \end{pmatrix} = \left(\frac{c/n}{d}\right)_* \xi_1 e^{\pi i d(nb - c/n)/12}. \tag{13.9}$$

Using (13.9) for $n = 1, 3, 7$, and 21, we readily verify that the last expressions on the left sides of (13.4) and (13.5) each have a trivial multiplier system. In conclusion, for each proposed identity, (13.4)–(13.6), each expression has a trivial multiplier system.

If we were now to use (0.30) or (0.38), we would find that the amount of computation is fairly high because the weights r are relatively large. Thus, as in Section 7, we derive additional information about the orders at the cusps in order to decrease the amount of computation. Because similar knowledge is needed in Sections 18 and 19, we proceed more generally than is necessary here. Furthermore, we discuss the function f_1, which appears in Sections 18 and 19 but which is irrelevant in this section.

Let $n = pq$, where p and q are odd primes, let $\Gamma = \Gamma(2) \cap \Gamma_0(n)$, and let $N = N(\Gamma; \zeta)$ denote the width of Γ at the cusp $\zeta \in Q \cup \{\frac{1}{0}\}$, where $\frac{1}{0}$ denotes the point at ∞.

Lemma 13.1. *If r and s are coprime integers, then*

$$N\left(\Gamma; \frac{r}{s}\right) = 2\frac{n}{(n, s)}.$$

PROOF. The proof is exactly like that of Lemma 7.2; just replace 11 by n throughout the proof.

Lemma 13.2. *If we set*

$$\zeta_1 = \frac{1}{0}, \quad \zeta_2 = \frac{2}{n}, \quad \zeta_3 = \frac{1}{n}, \quad \zeta_4 = \frac{1}{2q}, \quad \zeta_5 = \frac{2}{q}, \quad \zeta_6 = \frac{1}{q},$$

$$\zeta_7 = \frac{1}{2p}, \quad \zeta_8 = \frac{2}{p}, \quad \zeta_9 = \frac{1}{p}, \quad \zeta_{10} = \frac{1}{2}, \quad \zeta_{11} = \frac{2}{1}, \quad \textit{and} \quad \zeta_{12} = \frac{1}{1},$$

then

(i) $\zeta_1, \ldots, \zeta_{12}$ *is a complete set of inequivalent cusps for* Γ, *and*
(ii) *if* r_1, r_2, s_1, *and* s_2 *are integers such that* $(r_1, s_1) = (r_2, s_2) = 1$, *then* r_1/s_1 *and* r_2/s_2 *are equivalent cusps modulo* Γ *if and only if*

$$r_1 \equiv r_2 \quad \textit{and} \quad s_1 \equiv s_2 \pmod{2} \quad \textit{and} \quad (n, s_1) = (n, s_2). \tag{13.10}$$

PROOF. If r_1/s_1 and r_2/s_2 are equivalent cusps modulo Γ, we can choose $B \in \Gamma$ so that

$$B\left(\frac{r_1}{s_1}\right) = \frac{r_2}{s_2}.$$

Then the conditions (13.10) follow, which shows that they are necessary.

Using (13.10), we easily check that no two of $\zeta_1, \ldots, \zeta_{12}$ are equivalent cusps modulo Γ. Then an application of Lemma 13.1 shows that

$$\sum_{i=1}^{12} N(\Gamma; \zeta_i) = 6(p+1)(q+1) = (\Gamma(1):\Gamma),$$

which, by Rankin's book [2, Eq. (2.4.10)], shows that $\zeta_1, \ldots, \zeta_{12}$ is a complete set of inequivalent cusps for Γ.

Now suppose that the conditions (13.10) hold. Choose i and j so that $r_1/s_1 \sim \zeta_i$ and $r_2/s_2 \sim \zeta_j$ modulo Γ. But then by (13.10) and the definitions of $\zeta_1, \ldots, \zeta_{12}$, it follows that $i = j$. Thus, $r_1/s_1 \sim r_2/s_2$, and the proof is complete.

Recall that for a cusp ζ and a modular form F, the order of F with respect to Γ at ζ, $\operatorname{Ord}_\Gamma(F; \zeta)$, and the invariant order of F at ζ, $\operatorname{ord}(f; \zeta)$, are related by the equality

$$\operatorname{Ord}_\Gamma(F; \zeta) = N(\Gamma; \zeta)\operatorname{ord}(F; \zeta), \tag{13.11}$$

where, as above, $N(\Gamma; \zeta)$ is the width of Γ at ζ. We also recall from Lemma 0.1 that if $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is an integral matrix with determinant $m = ad - bc > 0$, and if $(r, s) = 1$, then

$$\operatorname{ord}(F|M; r/s) = \frac{g^2}{m}\operatorname{ord}\left(F; M\frac{r}{s}\right), \tag{13.12}$$

where $g = (ar + bs, cr + ds)$.

Now let d be one of the factors p, q, or pq of $n = pq$ and choose $B = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma(2)$ such that

$$\delta \equiv 0 \pmod{d} \quad \text{and} \quad \gamma \equiv 0 \pmod{e},$$

where $e = n/d$. We call

$$M_d := B\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$$

a *Fricke involution* of level d for $\Gamma = \Gamma(2) \cap \Gamma_0(n)$. For completeness, we also define $M_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

The following lemma is very important for later computations.

Lemma 13.3. *Let $r/s \in Q \cup \{\infty\}$ and let F be any modular form on Γ. Let $M = M_d$, as above. Then*

$$\mathrm{Ord}_\Gamma\left(F|M; \frac{r}{s}\right) = \mathrm{Ord}_\Gamma\left(F; M\frac{r}{s}\right).$$

In our proof of Lemma 13.3, we use the following simple, general lemma.

Lemma 13.4. *Let M and A be 2×2 and 2×1 integral matrices, respectively, and let $G = MA$. Let m, a, and g denote the greatest common divisors of the entries of M, A, and G, respectively, and let $d = \det(M)$. Then*

$$ma|g \quad \text{and} \quad mg|da.$$

Proof. The first conclusion is clear. The second claim follows from the first and the equality

$$dM^{-1}G = dA.$$

Proof of Lemma 13.3. We employ the elementary properties $((a, b), c) = (a, b, c)$ and $(a + b, b) = (a, b)$.

Let $(r, s) = 1$, let $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$, and set

$$\begin{pmatrix} r' \\ s' \end{pmatrix} = M\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} d\alpha r + \beta s \\ d\gamma r + \delta s \end{pmatrix}.$$

With $g = (r', s')$, the width of Γ at r'/s' is, by Lemma 13.1,

$$N\left(\Gamma; \frac{r'}{s'}\right) = N\left(\Gamma; \frac{r'/g}{s'/g}\right) = 2\frac{n}{(n, s'/g)} = 2\frac{ng}{(ng, s')} = 2\frac{n(r', s')}{(nr', s')}. \tag{13.13}$$

By Lemma 13.4, $g|d$, since $(r, s) = 1$. Thus, since $d|\delta$ and $(d, \beta) = 1$,

$$g = (d, g) = (d, r', s') = (d, d\alpha r + \beta s) = (d, \beta s) = (d, s). \tag{13.14}$$

We now apply Lemma 13.4 to

$$G = \begin{pmatrix} nr' \\ s' \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} M \begin{pmatrix} r \\ s \end{pmatrix}.$$

We see that m, a, g, and d of Lemma 13.4 are replaced by d, 1, $g' := (nr', s')$, and nd, respectively. We thus conclude that $g'|n$. Hence,

$$\begin{aligned} g' = (n, g') &= (n, nd\alpha r + n\beta s, d\gamma r + \delta s) \\ &= (n, d\gamma r + \delta s) \\ &= d\left(e, \gamma r + \frac{\delta}{d}s\right) \\ &= d(e, s), \end{aligned} \tag{13.15}$$

since $e|\gamma$ and $(e, \delta/d) = 1$. Finally, from (13.13)–(13.15),

$$N\left(\Gamma; M\frac{r}{s}\right) = 2\frac{n(d, s)}{d(e, s)}. \tag{13.16}$$

Thus, by Lemma 13.1, (13.11), (13.12), (13.14), and (13.16),

$$\begin{aligned}\mathrm{Ord}_\Gamma(F|M; r/s) &= 2\frac{n}{(n, s)}\frac{g^2}{d}\mathrm{ord}\left(F; \mathrm{M}\frac{\mathrm{r}}{\mathrm{s}}\right)\\ &= 2\frac{n(d, s)^2}{(d, s)(e, s)d}\mathrm{ord}\left(F; M\frac{r}{s}\right)\\ &= N\left(\Gamma; M\frac{r}{s}\right)\mathrm{ord}\left(F; M\frac{r}{s}\right)\\ &= \mathrm{Ord}_\Gamma\left(F; M\frac{r}{s}\right),\end{aligned}$$

which completes the proof.

Lemma 13.5. *The Fricke involution M_d interchanges the cusps $\zeta_1, \zeta_2, \dots, \zeta_{12}$ of $\Gamma = \Gamma(2) \cap \Gamma_0(n)$ pairwise. More precisely, we can group the indices into pairs $\{i, j\}$ such that $M_d\zeta_i \sim \zeta_j$ and $M_d\zeta_j \sim \zeta_i$ modulo Γ, where j is determined by*

$$j = \begin{cases} i+9, & \text{for } i = 1, 2, 3, \quad \text{when } d = pq,\\ i+3, & \text{for } i = 4, 5, 6, \quad \text{when } d = pq,\\ i+6, & \text{for } i = 1, 2, \dots, 6, \quad \text{when } d = q,\\ i+3, & \text{for } i = 1, 2, 3, 7, 8, 9, \quad \text{when } d = p.\end{cases}$$

Proof. The result follows from Lemma 13.2 and the definition of M_d.

We conclude that if we know the orders of F at ζ_1, ζ_2, and ζ_3, then we can use Lemmas 13.3 and 13.5 to determine the orders of F at $\zeta_1, \zeta_2, \dots, \zeta_{12}$.

In order to uniformly examine $F(\tau)$, $F(p\tau)$, $F(q\tau)$, and $F(pq\tau)$, where $F = \eta$, f_1, g_j, or h_j, $0 \le j \le 2$, we consider the effect of M_d on a modular form $F(\ell m\tau)$, where $\ell|d$ and $m|e$. Then also $\ell|\delta$ and $m|\gamma$, and so

$$\begin{aligned}F(\ell m\tau)|M_d &= (\ell m)^{-1/4}F\left|\begin{matrix}\ell\alpha & m\beta\\ \gamma/m & \delta/\ell\end{matrix}\right|\frac{md}{\ell}\\ &= (\ell m)^{-1/4}v_F(\tbinom{m}{\ell}B)(md/\ell)^{1/4}F\left(\frac{d}{\ell}m\tau\right)\\ &= (d/\ell^2)^{1/4}v_F(\tbinom{m}{\ell}B)F\left(\frac{d}{\ell}m\tau\right).\end{aligned} \tag{13.17}$$

In order to simply determine the multiplier system v_F above, we impose further restrictions on the matrix B. We require that $B \equiv \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) \pmod{2^4 3e}$, in addition to our previous requirements that $\gamma \equiv 0 \pmod{e}$ and $\delta \equiv 0 \pmod{d}$. An examination of the multiplier systems (0.14), (0.16), and (0.18)–(0.23) for η, f_1, g_j, and h_j, $0 \le j \le 2$, now shows that

$$\Phi(\ell\alpha, m\beta, \gamma/m, \delta/\ell) \equiv 0 \pmod{48},$$

in each case. Furthermore, since $\delta \equiv 1 \pmod 8$ and ℓ is odd, $\delta/\ell \equiv \ell \pmod 8$. Hence, by (13.7),

$$v_F\begin{pmatrix} \ell\alpha & m\beta \\ \gamma/m & \delta/\ell \end{pmatrix} = \left(\frac{\gamma_0/m}{\delta/\ell}\right)_* e^{\pi i(\ell-1)/4}, \tag{13.18}$$

where

$$\gamma_0 = \begin{cases} \gamma, & \text{if } F = \eta, f_1, g_0, g_1, g_2, h_1, \\ 2\gamma, & \text{if } F = h_0, h_2. \end{cases}$$

Employing (13.18) in (13.17), we conclude that

$$F(\ell m\tau)|M_d$$
$$= \begin{cases} (d/\ell^2)^{1/4}\left(\dfrac{m}{\ell}\right)\left(\dfrac{\gamma}{\delta}\right)_* e^{\pi i(\ell-1)/4} F\left(\dfrac{d}{\ell} m\tau\right), & \text{if } F = \eta, f_1, g_0, g_1, g_2, h_1, \\ (d/\ell^2)^{1/4}\left(\dfrac{2m}{\ell}\right)\left(\dfrac{\gamma}{\delta}\right)_* e^{\pi i(\ell-1)/4} F\left(\dfrac{d}{\ell} m\tau\right), & \text{if } F = h_0, h_2. \end{cases}$$

Now apply M_d to the proposed identities (13.4)–(13.6). First, M_3 and M_7 transform (13.4) into the reciprocal identity, that is, the identity relating modular forms that corresponds to the reciprocal modular equation (ii). On the other hand, M_{21} (and M_1) preserves (13.4). Second, M_7 and M_{21} transform (13.5) into its reciprocal, and M_1 and M_3 preserve (13.5). Formula (13.6) is preserved by M_3 and M_7 but is transformed into its reciprocal by M_{21}.

If we clear denominators in (13.4)–(13.6) and collect terms on one side, we obtain, in each case, an equation $F = 0$, where F is a modular form of weight $r = \frac{1}{2}v$ on $\Gamma(2) \cap \Gamma_0(n)$, and where v is the number of factors in each term of F. Since η, g_j, and h_j, $0 \le j \le 2$, have no poles, neither does F.

Let F^* denote the modular form corresponding to F that arises from the reciprocal relation, and suppose we now calculate the coefficients of $q^0, q^1, \ldots, q^\mu$ in both F and F^* and find that these coefficients are equal to 0. Then by Lemmas 13.3 and 13.5,

$$\operatorname{Ord}_\Gamma(F; \zeta_i) \ge \mu + 1,$$

for $i = 1, 4, 7, 10$. By the valence formula (0.30), (0.6), and (0.24), we may conclude that $F \equiv 0$, provided that

$$4(\mu + 1) > \tfrac{1}{12} r(\Gamma(1) : \Gamma) = \frac{v}{4}(p + 1)(q + 1),$$

or

$$\mu + 1 > \frac{\nu(p+1)(q+1)}{16}. \tag{13.19}$$

In the present applications, $p = 3$ and $q = 7$; that is, $(p+1)(q+1)/16 = 2$. The following table records the minimum value of μ necessary to effect a proof of the identity, at least according to the reasoning we have used.

	(13.4)	(13.5)	(13.6)
ν	10	8	12
μ	20	16	24

Using the computer algebra system MACSYMA, we have calculated the coefficients of F and F^* through q^μ and have verified that all, indeed, are equal to zero. This then completes the proof of (13.4)–(13.6) and therefore of Entry 13 as well.

Entry 14. *Let β, γ, and δ be of the third, eleventh, and thirty-third degrees, respectively. Then*

(i)
$$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8} + \left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8} - \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8} - 2\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/12} = \sqrt{mm'}$$

and

(ii)
$$\left(\frac{\alpha\gamma}{\beta\delta}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\gamma)}{(1-\beta)(1-\delta)}\right)^{1/8} - \left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/8} - 2\left(\frac{\alpha\gamma(1-\alpha)(1-\gamma)}{\beta\delta(1-\beta)(1-\delta)}\right)^{1/12} = \frac{3}{\sqrt{mm'}},$$

where m and m' are the multipliers associated with the pairs α, β and γ, δ, respectively.

PROOF OF (i). Applying (36.10) with $\mu = 6$ and $\nu = 5$ and utilizing Entry 18(iv), both in Chapter 16, we find that

$$\begin{aligned}
2q\psi(q)\psi(q^{11}) &= 2qf(q^{55}, q^{77})f(q, q^{11}) + 2q^{13}f(q^{33}, q^{99})f(q^{-9}, q^{21}) \\
&\quad + 2q^{37}f(q^{11}, q^{121})f(q^{-19}, q^{31}) \\
&= 2qf(q, q^{11})f(q^{55}, q^{77}) + 2q^4f(q^3, q^9)f(q^{33}, q^{99}) \\
&\quad + 2q^{11}f(q^5, q^7)f(q^{11}, q^{121}) \\
&= \tfrac{1}{2}\{f(q, q^2) - f(-q, -q^2)\}\{f(q^{11}, q^{22}) + f(-q^{11}, -q^{22})\}
\end{aligned}$$

$$+\tfrac{1}{2}\{f(q,q^2)+f(-q,-q^2)\}\{f(q^{11},q^{22})-f(-q^{11},-q^{22})\}$$
$$+2q^4\psi(q^3)\psi(q^{33})$$
$$=f(q,q^2)f(q^{11},q^{22})-f(-q,-q^2)f(-q^{11},-q^{22})$$
$$+2q^4\psi(q^3)\psi(q^{33}), \tag{14.1}$$

where we have used Entries 30(ii), (iii) in Chapter 16. Now by Entry 19, (22.3), and Entry 22, all in Chapter 16, we find that

$$f(q,q^2)=(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty$$
$$=\frac{(-q;q)_\infty(q^3;q^3)_\infty}{(-q^3;q^3)_\infty}=\frac{(q^3;q^6)_\infty(q^3;q^3)_\infty}{(q;q^2)_\infty}=\frac{\varphi(-q^3)}{\chi(-q)}. \tag{14.2}$$

Using (14.2) in (14.1), we deduce that

$$2q\psi(q)\psi(q^{11})=\frac{\varphi(-q^3)\varphi(-q^{33})}{\chi(-q)\chi(-q^{11})}-f(-q)f(-q^{11})+2q^4\psi(q^3)\psi(q^{33}). \tag{14.3}$$

Translating (14.3) by Entries 10(ii), 11(i), and 12(ii), (vi) in Chapter 17, we arrive at

$$\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8}+\frac{\{(1-\beta)(1-\delta)\}^{1/4}}{2^{1/3}\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{1/12}}$$
$$=\left(\frac{z_1z_{11}}{z_3z_{33}}\right)^{1/2}\left\{1+2^{-1/3}\left(\frac{(1-\alpha)^2(1-\gamma)^2}{\alpha\gamma}\right)^{1/12}\right\}. \tag{14.4}$$

Next, replace q by $-q$ in (14.3) and transcribe the resulting identity via Entries 10(i), 11(ii), and 12(i), (v) in Chapter 17 to find that

$$\left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8}+2^{-1/3}\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{-1/12}$$
$$=\left(\frac{z_1z_{11}}{z_3z_{33}}\right)^{1/2}(-1+2^{-1/3}\{\alpha\gamma(1-\alpha)(1-\gamma)\}). \tag{14.5}$$

Now we apply (36.3) with $\mu=6$ and $\nu=5$ and Enry 18(iv) seven times, both in Chapter 16, to deduce that

$$\tfrac{1}{2}\{\varphi(q)\varphi(q^{11})+\varphi(-q)\varphi(-q^{11})\}$$
$$=f(q^{132},q^{132})f(q^{12},q^{12})+q^{12}f(q^{176},q^{88})f(q^{32},q^{-8})$$
$$+q^{48}f(q^{220},q^{44})f(q^{52},q^{-28})+q^{108}f(q^{264},1)f(q^{72},q^{-48})$$
$$+q^{192}f(q^{308},q^{-44})f(q^{92},q^{-68})+q^{300}f(q^{352},q^{-88})f(q^{112},q^{-88})$$
$$=\varphi(q^{12})\varphi(q^{132})+2q^4f(q^8,q^{16})f(q^{88},q^{176})$$
$$+2q^{16}f(q^4,q^{20})f(q^{44},q^{220})+4q^{36}\psi(q^{24})\psi(q^{264}). \tag{14.6}$$

But by Entries 25(i), (ii) in Chapter 16,

$$\begin{aligned}\tfrac{1}{2}\{\varphi(q)\varphi(q^{11}) + \varphi(-q)\varphi(-q^{11})\}\\ &= \tfrac{1}{4}\{\varphi(q) + \varphi(-q)\}\{\varphi(q^{11}) + \varphi(-q^{11})\}\\ &\quad + \tfrac{1}{4}\{\varphi(q) - \varphi(-q)\}\{\varphi(q^{11}) - \varphi(-q^{11})\}\\ &= \varphi(q^4)\varphi(q^{44}) + 4q^{12}\psi(q^8)\psi(q^{88}). \end{aligned}\tag{14.7}$$

Substitute (14.7) in (14.6) and replace q^4 by $-q^2$ to deduce that

$$\begin{aligned}\varphi(-q^2)\varphi(-q^{22}) - 4q^6\psi(q^4)\psi(q^{44})\\ &= \varphi(-q^6)\varphi(-q^{66}) - 2q^2f(q^4, q^8)f(q^{44}, q^{88})\\ &\quad + 2q^8f(-q^2, -q^{10})f(-q^{22}, -q^{110}) - 4q^{18}\psi(q^{12})\psi(q^{132}).\end{aligned}$$

Now replace q by q^4 in (14.3) to obtain the equality

$$\begin{aligned}2q^4\psi(q^4)\psi(q^{44}) = \frac{\varphi(-q^{12})\varphi(-q^{132})}{\chi(-q^4)\chi(-q^{44})} - f(-q^4)f(-q^{44})\\ + 2q^{16}\psi(q^{12})\psi(q^{132}).\end{aligned}$$

Combining the last two equalities, using (14.2), and invoking Example (v) of Section 31 in Chapter 16, we deduce that

$$\begin{aligned}\varphi(-q^2)\varphi(-q^{22}) &= \varphi(-q^6)\varphi(-q^{66}) + 2q^8f(-q^2, -q^{10})f(-q^{22}, -q^{110})\\ &\quad - 2q^2f(-q^4)f(-q^{44})\\ &= \varphi(-q^6)\varphi(-q^{66}) + 2q^8\psi(q^6)\chi(-q^2)\psi(q^{66})\chi(-q^{22})\\ &\quad - 2q^2f(-q^4)f(-q^{44}).\end{aligned}$$

Translating this via Entries 10(iii), 11(iii), and 12(iv), (vii) in Chapter 17, we find that

$$\begin{aligned}\left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8} + \frac{(\beta\delta)^{1/4}}{2^{1/3}\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{1/12}}\\ = \left(\frac{z_1z_{11}}{z_3z_{33}}\right)^{1/2}\left\{1 + 2^{-1/3}\left(\frac{\alpha^2\gamma^2}{(1-\alpha)(1-\gamma)}\right)^{1/12}\right\}.\end{aligned}\tag{14.8}$$

Finally, we add (14.4) and (14.8) and subtract (14.5) to deduce that

$$\begin{aligned}\left(\frac{\beta\delta}{\alpha\gamma}\right)^{1/8} + \left(\frac{(1-\beta)(1-\delta)}{(1-\alpha)(1-\gamma)}\right)^{1/8} - \left(\frac{\beta\delta(1-\beta)(1-\delta)}{\alpha\gamma(1-\alpha)(1-\gamma)}\right)^{1/8}\\ + 2^{-1/3}\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{-1/12}(\{(1-\beta)(1-\delta)\}^{1/4} + (\beta\delta)^{1/4} - 1)\\ = \left(\frac{z_1z_{11}}{z_3z_{33}}\right)^{1/2}\{3 + 2^{-1/3}\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{-1/12}(\{(1-\alpha)(1-\gamma)\}^{1/4}\\ + (\alpha\gamma)^{1/4} - 1)\}.\end{aligned}\tag{14.9}$$

By Entry 7(i), the last expression in parentheses on the left side is equal to $-2^{4/3}\{\beta\delta(1-\beta)(1-\delta)\}^{1/12}$, while the last expression in parentheses on the right side is equal to $-2^{4/3}\{\alpha\gamma(1-\alpha)(1-\gamma)\}^{1/12}$. Hence, (14.9) reduces to Entry 14(i), and the proof is complete.

PROOF OF (ii). Formula (ii) is simply the reciprocal of (i).

Entry 15. *If β is of the twenty-third degree, then*

$$\text{(i)}\qquad (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} = 1,$$

$$\text{(ii)}\qquad 1 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2^{4/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} = \{2(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})\}^{1/2},$$

and

$$\text{(iii)}\quad m - \frac{23}{m} = 2((\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8})(11 - 13\cdot 4^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24} + 18\cdot 2^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} - 14\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} + 2^{5/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}).$$

Ramanujan's formulation of Entry 15(iii) (p. 249) appears to be incorrect, and we have corrected it here.

Entry 15(i) is due to Schröter [3], [4] who does not give his proof. A more complicated modular equation of degree 23 is proved in his thesis [1] and is also found in his paper [2]. Since the proof of (i) is decidedly nontrivial and since there seems to be no proof in print, we provide one here.

Formula (ii) follows from (i) by elementary algebra, and so we begin with this deduction.

PROOF OF (ii). Put

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1 - t \quad \text{and} \quad \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} = u.$$

Then (i) is equivalent to the equality $t^3 = 4u$. Now,

$$\begin{aligned}
&\{2(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})\}^{1/2} \\
&\quad = \{2(1 + (1-t)^4 - 4u(1-t)^2 + 2u^2)\}^{1/2} \\
&\quad = 2\{1 - 2t + 3t^2 - \tfrac{5}{2}t^3 + \tfrac{3}{2}t^4 - \tfrac{1}{2}t^5 + \tfrac{1}{16}t^6\}^{1/2} \\
&\quad = 2 - 2t + 2t^2 - \tfrac{1}{2}t^3 \\
&\quad = 1 + (1-t)^2 - 2u + (4u)^{2/3} \\
&\quad = 1 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} + 2^{4/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12},
\end{aligned}$$

which completes the proof.

PROOF OF (i). We derive two modular equations of degree 23 and then combine them to deduce (i).

In (36.11) of Chapter 16, set $\mu = 12$, $\omega = 1$, and $Q = q^{23}$ to deduce that

$$\tfrac{1}{2}\{f(AQ, Q/A)f(Bq, q/B) + f(-AQ, -Q/A)f(-Bq, -q/B)\} = \sum_{n=0}^{11} q^{4n^2}B^{-2n} f\left(\frac{AQ^{24+4n}}{B^{23}}, \frac{B^{23}Q^{24-4n}}{A}\right) f\left(ABq^{24-4n}, \frac{q^{24+4n}}{AB}\right). \tag{15.1}$$

Apply (15.1) with $A = B = 1$; $A = -B = i$; $A = Q$, $B = q$; and $A = iQ$, $B = -iq$, in turn, to deduce that

$$\tfrac{1}{2}\{\varphi(Q)\varphi(q) + \varphi(-Q)\varphi(-q)\} = \sum_{n=0}^{11} q^{4n^2} f(Q^{24+4n}, Q^{24-4n}) f(q^{24-4n}, q^{24+4n}), \tag{15.2}$$

$$\tfrac{1}{2}\{f(iQ, -iQ)f(-iq, iq) + f(-iQ, iQ)f(iq, -iq)\} = \sum_{n=0}^{11} (-1)^n q^{4n^2} f(Q^{24+4n}, Q^{24-4n}) f(q^{24-4n}, q^{24+4n}), \tag{15.3}$$

$$\tfrac{1}{2}\{f(Q^2, 1)f(q^2, 1) + f(-Q^2, -1)f(-q^2, -1)\} = \sum_{n=0}^{11} q^{4n^2-2n} f(Q^{24+4n}, Q^{24-4n}) f(q^{48-4n}, q^{4n}), \tag{15.4}$$

and

$$\tfrac{1}{2}\{f(iQ^2, -i)f(-iq^2, i) + f(-iQ^2, i)f(iq^2, -i)\} = \sum_{n=0}^{11} (-1)^n q^{4n^2-2n} f(Q^{24+4n}, Q^{24-4n}) f(q^{48-4n}, q^{4n}). \tag{15.5}$$

We now use Entries 18(ii), (iii) of Chapter 16 in (15.4). Also, easy exercises show that

$$f(iq, -iq) = \varphi(-q^4) \quad \text{and} \quad f(iq^2, -i) = (1 - i)\psi(-q^2).$$

Multiply both (15.4) and (15.5) by q^6 to get (15.4)′ and (15.5)′, say, respectively. Then add (15.2) and (15.5)′ and subtract (15.3) and (15.4)′ from them. Using the aforementioned simplifications, we deduce that

$$\begin{aligned} S &:= \tfrac{1}{2}\{\varphi(Q)\varphi(q) + \varphi(-Q)\varphi(-q)\} - \varphi(-Q^4)\varphi(-q^4) - 2q^6\psi(Q^2)\psi(q^2) \\ &\quad + 2q^6\psi(-Q^2)\psi(-q^2) \\ &= 2\sum_{n=1}^{6} q^{4(2n-1)^2} f(Q^{20+8n}, Q^{28-8n}) f(q^{28-8n}, q^{20+8n}) \\ &\quad - 2\sum_{n=1}^{6} q^{4(2n-1)^2-2(2n-1)+6} f(Q^{20+8n}, Q^{28-8n}) f(q^{52-8n}, q^{8n-4}) \end{aligned}$$

$$= 2 \sum_{n=1}^{6} q^{4(2n-1)^2} f(Q^{20+8n}, Q^{28-8n}) \{ f(q^{28-8n}, q^{20+8n})$$
$$- q^{8-4n} f(q^{52-8n}, q^{8n-4}) \}$$
$$= 2 \sum_{n=1}^{6} q^{4(2n-1)^2} f(Q^{20+8n}, Q^{28-8n}) f(-q^{4+4n}, -q^{8-4n}), \tag{15.6}$$

where we have applied (7.1) with $a = -q^{8-4n}$ and $b = -q^{4+4n}$. Combining the terms with indices n and $7 - n$, $1 \le n \le 3$, and employing Entries 18(iii), (iv) in Chapter 16 and (7.1) once again, we find that, from (15.6),

$$S = 4 \sum_{n=1}^{3} q^{4(2n-1)^2} f(Q^{20+8n}, Q^{28-8n}) f(-q^{4+4n}, -q^{8-4n})$$
$$= 4q^4 f(Q^{28}, Q^{20}) f(-q^8, -q^4) + 4q^{100} f(Q^{44}, Q^4) f(-q^{16}, -q^{-4})$$
$$= 4q^4 \{ f(Q^{28}, Q^{20}) - Q^4 f(Q^{44}, Q^4) \} f(-q^4, -q^8)$$
$$= 4q^4 f(-Q^4, -Q^8) f(-q^4, -q^8)$$
$$= 4q^4 f(-Q^4) f(-q^4).$$

Employing (14.7) with q^{11} replaced by Q, we find that the foregoing equality becomes

$$\varphi(Q^4)\varphi(q^4) + 4q^{24}\psi(Q^8)\psi(q^8) - \varphi(-Q^4)\varphi(-q^4) - 2q^6\psi(Q^2)\psi(q^2)$$
$$+ 2q^6\psi(-Q^2)\psi(-q^2) = 4q^4 f(-Q^4) f(-q^4).$$

Next, replace q^2 by q above, and, consequently, Q^2 by Q. Translate the resulting equality by means of Entries 10(iii), (iv), 11(i), (ii), (iv), and 12(iii) in Chapter 17 to obtain the modular equation

$$\tfrac{1}{2}\{(1 + \sqrt{1-\alpha})^{1/2}(1 + \sqrt{1-\beta})^{1/2} + (1 - \sqrt{1-\alpha})^{1/2}(1 - \sqrt{1-\beta})^{1/2}\}$$
$$- \{(1-\alpha)(1-\beta)\}^{1/8} - (\alpha\beta)^{1/8} + \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$$
$$= 2^{4/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12};$$

that is,

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$$
$$+ 2^{4/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}$$
$$= \{\tfrac{1}{2}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})\}^{1/2}. \tag{15.7}$$

This is the first of the two modular equations that we have sought.

To obtain the second, we let $\mu = 16$, $\omega = 3$, and $Q = q^{23}$ in (36.14) of Chapter 16. Hence,

$$\varphi(Q)\varphi(q) + \varphi(-Q)\varphi(-q) + 4q^6\psi(Q^2)\psi(q^2)$$
$$= 2 \sum_{n=0}^{15} q^{4n^2} f(Q^{32+4n}, Q^{32-4n}) f(q^{8-6n}, q^{8+6n})$$

$$= 2\varphi(Q^{32})\varphi(q^8) + 2q^{256}f(Q^{64}, 1)f(q^{-40}, q^{56})$$
$$+ 4\sum_{n=1}^{7} q^{4n^2}f(Q^{32+4n}, Q^{32-4n})f(q^{8-6n}, q^{8+6n}),$$

where we have combined the terms with indices n and $16 - n$, $1 \le n \le 7$. Combining the terms with indices n and $8 - n$, $1 \le n \le 3$, and utilizing Entries 18(ii), (iv) in Chapter 16, we find that

$$\begin{aligned}
&\varphi(Q)\varphi(q) + \varphi(-Q)\varphi(-q) + 4q^6\psi(Q^2)\psi(q^2)\\
&\quad = 2\{\varphi(Q^{32}) + 2Q^8\psi(Q^{64})\}\varphi(q^8) + 8q^{48}\psi(Q^{16})\psi(q^{16})\\
&\qquad + 4q^4\{f(Q^{28}, Q^{36}) + Q^6f(Q^4, Q^{60})\}f(q^2, q^{14})\\
&\qquad + 4q^{12}\{f(Q^{24}, Q^{40}) + Q^4f(Q^8, Q^{56})\}f(q^4, q^{12})\\
&\qquad + 4q^{26}\{f(Q^{20}, Q^{44}) + Q^2f(Q^{12}, Q^{52})\}f(q^6, q^{10})\\
&\quad = 2\varphi(Q^8)\varphi(q^8) + 8q^{48}\psi(Q^{16})\psi(q^{16}) + 4q^4f(Q^6, Q^{10})f(q^2, q^{14})\\
&\qquad + 4q^{12}\psi(Q^4)\psi(q^4) + 4q^{26}f(Q^2, Q^{14})f(q^6, q^{10})\\
&\quad = \{\varphi(Q^8) + 2Q^2\psi(Q^{16})\}\{\varphi(q^8) + 2q^2\psi(q^{16})\}\\
&\qquad + \{\varphi(Q^8) - 2Q^2\psi(Q^{16})\}\{\varphi(q^8) - 2q^2\psi(q^{16})\}\\
&\qquad + 2q^3\{f(Q^6, Q^{10}) + Qf(Q^2, Q^{14})\}\{f(q^6, q^{10}) + qf(q^2, q^{14})\}\\
&\qquad - 2q^3\{f(Q^6, Q^{10}) - Qf(Q^2, Q^{14})\}\{f(q^6, q^{10}) - qf(q^2, q^{14})\}\\
&\qquad + 4q^{12}\psi(Q^4)\psi(q^4),
\end{aligned} \tag{15.8}$$

where in the penultimate equality we employed (7.1) three times as well as the equality

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8), \tag{15.9}$$

which is a ready consequence of Entries 25(i), (ii) in Chapter 16. Now use (15.9) four times and (7.1) also four times to reduce (15.8) to the identity

$$\begin{aligned}
&\varphi(Q)\varphi(q) + \varphi(-Q)\varphi(-q) + 4q^6\psi(Q^2)\psi(q^2)\\
&\quad = \varphi(Q^2)\varphi(q^2) + \varphi(-Q^2)\varphi(-q^2) + 2q^3\psi(Q)\psi(q) - 2q^3\psi(-Q)\psi(-q)\\
&\qquad + 4q^{12}\psi(Q^4)\psi(q^4).
\end{aligned}$$

Lastly, we translate this identity with the aid of Entries 10(i)–(iv) and 11(i)–(iv) in Chapter 17 to deduce that

$$\begin{aligned}
&1 + \{(1-\alpha)(1-\beta)\}^{1/4} + (\alpha\beta)^{1/4}\\
&\quad = \tfrac{1}{2}(1 + \sqrt{1-\alpha})^{1/2}(1 + \sqrt{1-\beta})^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/8} + (\alpha\beta)^{1/8}\\
&\qquad - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} + \tfrac{1}{2}(1 - \sqrt{1-\alpha})^{1/2}(1 - \sqrt{1-\beta})^{1/2},
\end{aligned}$$

which can be recast in the form

$$1 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} - (\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}$$
$$+ \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$$
$$= \{\tfrac{1}{2}(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2})\}^{1/2}, \tag{15.10}$$

which is the second of the modular equations that we sought.

Equating the left sides of (15.7) and (15.10), we deduce that

$$\{1 - (\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}\}^2 = 2^{4/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}.$$

For small α, the expression within the outside curly brackets on the left side is approximately equal to $\frac{1}{8}\alpha > 0$. Thus, in taking the square root of both sides, we arrive at

$$1 - (\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8} = 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24},$$

which is formula (i).

Observe that, by adding (15.7) and (15.10), we obtain (ii).

PROOF OF (iii). We employ the theory of modular forms. We first translate (i) by means of Entries 10(i), (ii), 11(i), (ii), and 12(i), (iii) in Chapter 17 to find that

$$\varphi^4(q) - 23\varphi^4(q^{23})$$
$$= 2(2q^3\psi(q)\psi(q^{23}) - \varphi(-q^2)\varphi(-q^{46}))\Bigg(11\varphi(q)\varphi(q^{23}) - 26qf(q)f(q^{23})$$
$$+ 36q^2f(-q^2)f(-q^{46}) - 28q^3\psi(-q)\psi(-q^{23})$$
$$+ 8q^4\frac{f^2(-q^2)f^2(-q^{46})}{\varphi(q)\varphi(q^{23})}\Bigg).$$

Using (0.13), we write this proposed identity as an identity involving modular forms,

$$g_1^4(\tau) - 23g_1^4(23\tau)$$
$$= (4h_0(\tau)h_0(23\tau) - 2h_2(\tau)h_2(23\tau))\Bigg(11g_1(\tau)g_1(23\tau) - 26f_1(\tau)f_1(23\tau)$$
$$+ 36\eta(\tau)\eta(23\tau) - 28h_1(\tau)h_1(23\tau) + 8\frac{\eta^2(\tau)\eta^2(23\tau)}{g_1(\tau)g_1(23\tau)}\Bigg). \tag{15.11}$$

Using (0.14), (0.16), (0.19), and (0.21)–(0.23), we may easily verify that each expression in (15.11) has a multiplier system identically equal to 1.

Letting $\Gamma = \Gamma(2) \cap \Gamma_0(23)$, we see from (0.6) and (0.24) that $\rho_\Gamma = 12$. If we clear fractions in (15.11) and rewrite (15.11) in the form $F := F_1 + F_2 + \cdots + F_{12} = 0$, we observe that the weight r of each form equals 3. Thus, by (0.30), in order to prove (15.11), we need to show that the coefficients of q^j, $0 \le j \le 36$, in F are equal to 0. However, if M is as in (0.31)–(0.37), we readily find that $F|M = -F$. Thus, by (0.38), we need only show that the coefficients of q^j,

$0 \le j \le 18$, in F are equal to 0. Using MACSYMA, we have, indeed, verified this, and so the proof of (15.11), and hence of (iii), is complete.

Entry 16. *If* β *is of degree* 19, *then*

(i)
$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^3}{\alpha}\right)^{1/8} + \left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/8}$$
$$- 2\left(\frac{\beta^3(1-\beta)^3}{\alpha(1-\alpha)}\right)^{1/16}\left\{\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} - 1 - \left(\frac{\beta^3}{\alpha}\right)^{1/8}\right\}^{1/2}$$
$$= m\{\tfrac{1}{2}(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)})\}^{1/2}$$

and

(ii)
$$\left(\frac{\alpha^3}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8} + \left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/8}$$
$$- 2\left(\frac{\alpha^3(1-\alpha)^3}{\beta(1-\beta)}\right)^{1/16}\left\{\left(\frac{\alpha^3}{\beta}\right)^{1/8} - 1 - \left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8}\right\}^{1/2}$$
$$= \frac{19}{m}\{\tfrac{1}{2}(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)})\}^{1/2}.$$

Observe that (ii) is the reciprocal of (i), and so it suffices to establish (i). However, we use the theory of modular forms and apply (0.38). Thus, in fact, (i) and (ii) will be established simultaneously.

Previously proven modular equations of degree 19 by Sohncke [2], Schläfli [1], Fiedler [1], Weber [1], Russell [1], and Fricke [1] do not have the simplicity of Ramanujan's modular equations in Entry 16.

Proof of (i). Utilizing Entries 10(i), (iii), (iv) and 11(i), (ii), (iv) in Chapter 17, we rewrite (i) in the form

$$\varphi(q^2)\varphi(q^{38}) + 4q^{10}\psi(q^4)\psi(q^{76})$$
$$= \frac{\varphi^3(-q^{38})}{\varphi(-q^2)} - 2q^7\frac{\psi^3(q^{19})}{\psi(q)} + 2q^7\frac{\psi^3(-q^{19})}{\psi(-q)}$$
$$- 2\left(2q^7\frac{\psi^3(-q^{19})}{\psi(-q)}\left\{\frac{\varphi^3(-q^{38})}{\varphi(-q^2)} - \frac{\varphi^3(q^{19})}{\varphi(q)} - 2q^7\frac{\psi^3(q^{19})}{\psi(q)}\right\}\right)^{1/2}.$$

Written in terms of modular forms, by (0.13), the foregoing equality becomes

$$\left\{g_1(2\tau)g_1(38\tau) + 4g_2(2\tau)g_2(38\tau)\right.$$
$$\left. - \frac{h_2^3(19\tau)}{h_2(\tau)} + 2\frac{h_0^3(19\tau)}{h_0(\tau)} - 2\frac{h_1^3(19\tau)}{h_1(\tau)}\right\}^2$$
$$= 8\frac{h_1^3(19\tau)}{h_1(\tau)}\left\{\frac{h_2^3(19\tau)}{h_2(\tau)} - \frac{g_1^3(19\tau)}{g_1(\tau)} - 2\frac{h_0^3(19\tau)}{h_0(\tau)}\right\}. \tag{16.1}$$

We now must examine the multiplier systems for each term in (16.1). By identically the same proof, with 11 replaced by 19, we deduce as in Lemma 7.1 that $g_1(2\tau)g_1(38\tau) + 4g_2(2\tau)g_2(38\tau)$ is a modular form on $\Gamma = \Gamma(2) \cap \Gamma_0(19)$ with the multiplier system $(\frac{d}{19})$. If $F = g_1$, h_0, h_1, or h_2, then by the same reasoning that gave (7.16), we find that $F^3(19\tau)/F(\tau)$ has a multiplier system equal to $(\frac{d}{19})$. Thus, upon multiplying out in (16.1), we find that each term has a multiplier system identically equal to 1.

Multiplying out and clearing denominators in (16.1), we may rewrite (16.1) in the form $F := F_1 + F_2 + \cdots + F_{19} = 0$. The weight r of each modular form F_j, $1 \le j \le 19$, is equal to $\frac{11}{2}$. Also, by (0.6) and (0.24), $\rho_\Gamma = 10$. By (0.38), if we verify that the coefficients of q^j, $0 \le j \le 27$, in both F and $F|M$ are equal to 0, then the proof of (16.1), and hence of both (i) and (ii), is complete. Using the computer algebra system MACSYMA, we have, indeed, found that all of the required coefficients are equal to 0.

Entry 17. *We have*

(i) $\varphi(q)\varphi(q^{35}) = \varphi(-q)\varphi(-q^{35}) + 4qf(-q^{10})f(-q^{14}) + 4q^9\psi(q^2)\psi(q^{70})$,

(ii) $\varphi(q^5)\varphi(q^7) = \varphi(-q^5)\varphi(-q^7) + 4q^3\psi(q^{10})\psi(q^{14}) - 4q^3f(-q^2)f(-q^{70})$,

(iii) $\varphi(q)\varphi(q^{135}) = \varphi(-q^{10})\varphi(-q^{54}) + 2qf(q^9)f(q^{15}) + 2q^4\psi(q^5)\psi(q^{27})$,

and

(iv) $\varphi(q^5)\varphi(q^{27}) = \varphi(-q^2)\varphi(-q^{270}) + 2q^{17}\psi(q)\psi(q^{135}) + 2q^2f(q^3)f(q^{45})$.

PROOF OF (ii). First, putting $A = B = 1$, $\mu = 6$, and $\nu = 1$ in (36.2) of Chapter 16, we find that

$$\begin{aligned}\tfrac{1}{2}\{\varphi(q^5)\varphi(q^7) &- \varphi(-q^5)\varphi(-q^7)\} \\ &= \sum_{n=0}^{5} q^{12n^2+14n+7} f(q^{35(14+4n)}, q^{35(10-4n)}) f(q^{26+4n}, q^{-2-4n}) \\ &= 2\sum_{n=0}^{2} q^{12n^2+14n+7} f(q^{35(14+4n)}, q^{35(10-4n)}) f(q^{26+4n}, q^{-2-4n}),\end{aligned}$$

where we have employed Entry 18(iv) of Chapter 16 to show that the terms with index n and $5-n$, $0 \le n \le 2$, are equal. Second, setting $\mu = 6$ and $\nu = 1$ and replacing q by q^2 in (36.10) of Chapter 16, we arrive at

$$\begin{aligned}2q^3&\psi(q^{10})\psi(q^{14}) \\ &= 2\sum_{n=0}^{2} q^{12n^2+12n+3} f(q^{35(14+4n)}, q^{35(10-4n)}) f(q^{14+4n}, q^{10-4n}).\end{aligned}$$

Subtracting the latter equality from the former, employing Entry 18(iv) in Chapter 16, and noting the cancellation of the terms with index 1, we see that

$$\begin{aligned}
&\tfrac{1}{2}\{\varphi(q^5)\varphi(q^7) - \varphi(-q^5)\varphi(-q^7)\} - 2q^3\psi(q^{10})\psi(q^{14}) \\
&\quad = 2f(q^{490}, q^{350})\{q^5 f(q^2, q^{22}) - q^3 f(q^{10}, q^{14})\} \\
&\qquad + 2f(q^{770}, q^{70})\{q^{73} f(q^{10}, q^{14}) - q^{75} f(q^2, q^{22})\} \\
&\quad = -2q^3\{f(q^{10}, q^{14}) - q^2 f(q^2, q^{22})\}\{f(q^{350}, q^{490}) - q^{70} f(q^{70}, q^{770})\} \\
&\quad = -2q^3 f(-q^2, -q^4) f(-q^{70}, -q^{140}) \\
&\quad = -2q^3 f(-q^2) f(-q^{70}),
\end{aligned}$$

where we have applied (7.1) twice. This concludes the proof of (ii).

PROOF OF (i). In (ii), replace q by $q^{1/5}$ to obtain the equality

$$\varphi(q)\varphi(q^{7/5}) = \varphi(-q)\varphi(-q^{7/5}) + 4q^{3/5}\psi(q^2)\psi(q^{14/5}) - 4q^{3/5} f(-q^{2/5}) f(-q^{14}).$$

Employing Entries 10(i)–(iii) in Chapter 19, we deduce that

$$\begin{aligned}
&\varphi(q)\{\varphi(q^{35}) + 2q^{7/5} f(q^{21}, q^{49}) + 2q^{28/5} f(q^7, q^{63})\} \\
&\quad = \varphi(-q)\{\varphi(-q^{35}) - 2q^{7/5} f(-q^{21}, -q^{49}) + 2q^{28/5} f(-q^7, -q^{63})\} \\
&\qquad + 4q^{3/5}\psi(q^2)\{q^{42/5}\psi(q^{70}) + f(q^{28}, q^{42}) + q^{14/5} f(q^{14}, q^{56})\} \\
&\qquad - 4q^{3/5}\{-q^{2/5} f(-q^2) f(-q^{10}) + f^2(-q^4, -q^6) \\
&\qquad - q^{4/5} f^2(-q^2, -q^8)\}\frac{f(-q^{14})}{f(-q^2)}.
\end{aligned}$$

If we now equate rational parts on both sides, we obtain (i).

PROOF OF (iii). The proof of (iii) is very troublesome, indeed. First, replace q by $-q$ in (iii) to find that

$$\begin{aligned}
&\varphi(-q^{10})\varphi(-q^{54}) - 2q f(-q^9) f(-q^{15}) + 2q^4\psi(-q^5)\psi(-q^{27}) \\
&\quad = \varphi(-q)\varphi(-q^{135}).
\end{aligned}$$

We now translate this into a modular equation via Entries 10(ii), (iii), 11(ii), and 12(ii) in Chapter 17. With the subscript on each parameter below corresponding to its degree, we find that

$$\begin{aligned}
&(z_5 z_{27})^{1/2}\{(1-\alpha_5)(1-\alpha_{27})\}^{1/8} \\
&\quad - (z_9 z_{15})^{1/2} 2^{2/3}(\alpha_9\alpha_{15})^{1/24}\{(1-\alpha_9)(1-\alpha_{15})\}^{1/6} \\
&\quad + (z_5 z_{27})^{1/2}\{\alpha_5\alpha_{27}(1-\alpha_5)(1-\alpha_{27})\}^{1/8} \\
&= (z_1 z_{135})^{1/2}\{(1-\alpha_1)(1-\alpha_{135})\}^{1/4}.
\end{aligned}$$

Note that we have obtained a modular equation involving six moduli! Now replace each modulus α_n by its complementary modulus $1 - \alpha_{135/n}$ to obtain the reciprocal formula

$$\begin{aligned}
&(z_{27} z_5)^{1/2}(\alpha_5\alpha_{27})^{1/8} - (z_{15} z_9)^{1/2} 2^{2/3}(\alpha_9\alpha_{15})^{1/6}\{(1-\alpha_9)(1-\alpha_{15})\}^{1/24} \\
&\quad + (z_{27} z_5)^{1/2}\{\alpha_5\alpha_{27}(1-\alpha_5)(1-\alpha_{27})\}^{1/8} = (z_{135} z_1)^{1/2}(\alpha_1\alpha_{135})^{1/4}.
\end{aligned}$$

Now this formula can be translated into the theta-function identity

$$2q^4\psi(q^5)\psi(q^{27}) + 2q^4\psi(-q^5)\psi(-q^{27}) - 4q^4f(-q^{36})f(-q^{60}) = 4q^{34}\psi(q^2)\psi(q^{270}), \tag{17.1}$$

via Entries 11(i)–(iii) and 12(iv) in Chapter 17. Since (17.1) and Entry 17(iii) have been shown to be equivalent, it suffices to prove (17.1).

In the remainder of the proof, all references, unless otherwise stated, will be to results found in Chapter 16.

In (36.8), set $\mu = 16$, $\nu = 11$, and $Q = q^{135}$ to deduce that

$$q^4\psi(q^5)\psi(q^{27}) = q^4\varphi(Q^{16})\psi(q^{32}) + q^{940}\psi(Q^{32})f(q^{176}, q^{-144}) + \sum_{n=1}^{7} q^{16n^2-11n+4}f(Q^{16+2n}, Q^{16-2n})f(q^{22n}, q^{32-22n}).$$

Replacing q by $-q$ and adding the result to the foregoing equality, we find that

$$\begin{aligned} q^4\psi(q^5)\psi(q^{27}) &+ q^4\psi(-q^5)\psi(-q^{27}) \\ &= 2q^4\varphi(Q^{16})\psi(q^{32}) + 2q^{940}\psi(Q^{32})f(q^{176}, q^{-144}) \\ &\quad + 2\sum_{n=1}^{3} q^{64n^2-22n+4}f(Q^{16+4n}, Q^{16-4n})f(q^{44n}, q^{32-44n}) \\ &= 2q^4\varphi(Q^{16})\psi(q^{32}) + 2q^{540}\psi(Q^{32})\varphi(q^{16}) \\ &\quad + 2q^{34}f(Q^{12}, Q^{20})f(q^{12}, q^{20}) + 2q^{136}f(Q^8, Q^{24})f(q^8, q^{24}) \\ &\quad + 2q^{306}f(Q^4, Q^{28})f(q^4, q^{28}), \end{aligned}$$

where we have made several applications of Entry 18(iv). Rewriting the right side with the aid of Entries 25(i), (ii) and 30(ii), (iii), we find that

$$\begin{aligned} q^4\psi(q^5)\psi(q^{27}) &+ q^4\psi(-q^5)\psi(-q^{27}) \\ &= \tfrac{1}{4}\{\varphi(Q^4) + \varphi(-Q^4)\}\{\varphi(q^4) - \varphi(-q^4)\} \\ &\quad + \tfrac{1}{4}\{\varphi(q^4) + \varphi(-q^4)\}\{\varphi(Q^4) - \varphi(-Q^4)\} + 2q^{136}\psi(Q^8)\psi(q^8) \\ &\quad + \tfrac{1}{2}q^{34}\{f(Q^2, Q^6) + f(-Q^2, -Q^6)\}\{f(q^2, q^6) + f(-q^2, -q^6)\} \\ &\quad + \tfrac{1}{2}q^{34}\{f(Q^2, Q^6) - f(-Q^2, -Q^6)\}\{f(q^2, q^6) - f(-q^2, -q^6)\} \\ &= \tfrac{1}{2}\varphi(Q^4)\varphi(q^4) - \tfrac{1}{2}\varphi(-Q^4)\varphi(-q^4) + 2q^{136}\psi(Q^8)\psi(q^8) \\ &\quad + q^{34}\psi(Q^2)\psi(q^2) + q^{34}\psi(-Q^2)\psi(-q^2) \\ &= \tfrac{1}{8}\{\varphi(Q) + \varphi(-Q)\}\{\varphi(q) + \varphi(-q)\} \\ &\quad + \tfrac{1}{8}\{\varphi(Q) - \varphi(-Q)\}\{\varphi(q) - \varphi(-q)\} \\ &\quad - \tfrac{1}{2}\varphi(-Q^4)\varphi(-q^4) + q^{34}\psi(Q^2)\psi(q^2) + q^{34}\psi(-Q^2)\psi(-q^2) \\ &= \tfrac{1}{4}\varphi(Q)\varphi(q) + \tfrac{1}{4}\varphi(-Q)\varphi(-q) - \tfrac{1}{2}\varphi(-Q^4)\varphi(-q^4) \\ &\quad + q^{34}\psi(Q^2)\psi(q^2) + q^{34}\psi(-Q^2)\psi(-q^2). \end{aligned} \tag{17.2}$$

Next, we employ (36.1) with q replaced by q^{18} and $\mu = 4$, $\nu = 1$, $A = q^{30}$, $B = q^{18}$, and $Q = q^{135}$. Accordingly,

$$\begin{aligned}
&\tfrac{1}{2}\{f(q^{36}, q^{72})f(q^{60}, q^{120}) + f(-q^{36}, -q^{72})f(-q^{60}, -q^{120})\}\\
&\quad = \sum_{n=0}^{3} q^{144n^2+12n} f(Q^{16+8n}, Q^{16-8n}) f(q^{192+72n}, q^{96-72n})\\
&\quad = \varphi(Q^{16})f(q^{96}, q^{192}) + q^{156}\psi(Q^8)f(q^{24}, q^{264})\\
&\qquad + 2q^{552}\psi(Q^{32})f(q^{48}, q^{240}) + q^{1212}f(Q^{-8}, Q^{40})f(q^{120}, q^{168})\\
&\quad = \tfrac{1}{4}\{\varphi(Q^4) + \varphi(-Q^4)\}\{f(q^{12}, q^{60}) + f(-q^{12}, -q^{60})\}\\
&\qquad + \tfrac{1}{4}\{\varphi(Q^4) - \varphi(-Q^4)\}\{f(q^{12}, q^{60}) - f(-q^{12}, -q^{60})\}\\
&\qquad + q^{132}\psi(Q^8)\{f(q^{120}, q^{168}) + q^{24}f(q^{24}, q^{264})\},
\end{aligned}$$

where we have utilized Entry 18(iv) several times and employed Entries 25(i), (ii) and 30(ii), (iii) as well. Employing once again these four entries as well as (7.1) in this chapter, we deduce that

$$\begin{aligned}
&2q^4\{f(q^{36}, q^{72})f(q^{60}, q^{120}) + f(-q^{36}, -q^{72})f(-q^{60}, -q^{120})\}\\
&\quad = 2q^4\varphi(Q^4)f(q^{12}, q^{60}) + 2q^4\varphi(-Q^4)f(-q^{12}, -q^{60})\\
&\qquad + 4q^{136}\psi(Q^8)f(q^{24}, q^{48})\\
&\quad = \tfrac{1}{2}q\{\varphi(Q) + \varphi(-Q)\}\{f(q^3, q^{15}) - f(-q^3, -q^{15})\}\\
&\qquad + 2q^4\varphi(-Q^4)f(-q^{12}, -q^{60}) + \tfrac{1}{2}q\{\varphi(Q) - \varphi(-Q)\}\{f(q^3, q^{15})\\
&\qquad + f(-q^3, -q^{15})\}\\
&\quad = q\varphi(Q)f(q^3, q^{15}) - q\varphi(-Q)f(-q^3, -q^{15})\\
&\qquad + 2q^4\varphi(-Q^4)f(-q^{12}, -q^{60})\\
&\quad = \tfrac{1}{2}\varphi(Q)\{\varphi(q) - \varphi(q^9)\} + \tfrac{1}{2}\varphi(-Q)\{\varphi(-q) - \varphi(-q^9)\}\\
&\qquad - \varphi(-Q^4)\{\varphi(-q^4) - \varphi(-q^{36})\},
\end{aligned} \tag{17.3}$$

where we have used Corollary (i) in Section 31 three times.

Third, we invoke (36.2) with q replaced by q^{18}, $\mu = 4$, $\nu = 1$, $A = q^{30}$, $B = q^{-18}$, and $Q = q^{135}$. Then in the second equality below, we apply Entry 18(iv) several times. In the third, Entries 30(ii), (iii) are employed. In the fourth, we make four applications of (7.1) of the present chapter with $a = \pm q^6$, $b = \pm q^{66}$ and $a = \pm q^{30}$, $b = \pm q^{42}$. Two further applications of (7.1), with $a = \pm q^6$, $b = q^{12}$, are made in the penultimate equality. Lastly, Corollary (ii) in Section 31 is utilized. Thus,

$$\begin{aligned}
&\tfrac{1}{2}\{f(q^{36}, q^{72})f(q^{60}, q^{120}) - f(-q^{36}, -q^{72})f(-q^{60}, -q^{120})\}\\
&\quad = \sum_{n=0}^{3} q^{144n^2+192n+120} f(Q^{20+8n}, Q^{12-8n}) f(q^{372+72n}, q^{-84-72n})
\end{aligned}$$

$$\begin{aligned}
&= q^{36}f(Q^{12},Q^{20})f(q^{84},q^{204}) + q^{300}f(Q^{4},Q^{28})f(q^{132},q^{156})\\
&\quad + q^{312}f(Q^{4},Q^{28})f(q^{60},q^{228}) + q^{60}f(Q^{12},Q^{20})f(q^{12},q^{276})\\
&= \tfrac{1}{2}f(Q^2,Q^6)\{q^{36}f(q^{84},q^{204}) + q^{30}f(q^{132},q^{156})\\
&\quad + q^{42}f(q^{60},q^{228}) + q^{60}f(q^{12},q^{276})\}\\
&\quad + \tfrac{1}{2}f(-Q^2,-Q^6)\{q^{36}f(q^{84},q^{204}) - q^{30}f(q^{132},q^{156})\\
&\quad - q^{42}f(q^{60},q^{228}) + q^{60}f(q^{12},q^{276})\}\\
&= \tfrac{1}{2}\psi(Q^2)\{q^{36}f(q^{6},q^{66}) + q^{30}f(q^{30},q^{42})\}\\
&\quad + \tfrac{1}{2}\psi(-Q^2)\{q^{36}f(-q^{6},-q^{66}) - q^{30}f(-q^{30},-q^{42})\}\\
&= \tfrac{1}{2}q^{30}\psi(Q^2)f(q^6,q^{12}) - \tfrac{1}{2}q^{30}\psi(-Q^2)f(-q^6,q^{12})\\
&= \tfrac{1}{2}q^{30}\psi(Q^2)\{\psi(q^2) - q^2\psi(q^{18})\}\\
&\quad - \tfrac{1}{2}q^{30}\psi(-Q^2)\{\psi(-q^2) + q^2\psi(-q^{18})\}.
\end{aligned}$$

Rearranging slightly, in summary, we have shown that

$$\begin{aligned}
&2q^4\{f(q^{36},q^{72})f(q^{60},q^{120}) - f(-q^{36},-q^{72})f(-q^{60},-q^{120})\}\\
&\quad = 2q^{34}\{\psi(q^2)\psi(Q^2) - \psi(-q^2)\psi(-Q^2)\}\\
&\quad\quad - 2q^{36}\{\psi(q^{18})\psi(Q^2) + \psi(-q^{18})\psi(-Q^2)\}. \qquad (17.4)
\end{aligned}$$

Subtracting (17.4) from (17.3), we deduce that

$$\begin{aligned}
&4q^4f(-q^{36},-q^{72})f(-q^{60},-q^{120})\\
&\quad = \tfrac{1}{2}\{\varphi(q)\varphi(Q) + \varphi(-q)\varphi(-Q) - 2\varphi(-q^4)\varphi(-Q^4)\\
&\quad\quad - 4q^{34}\psi(q^2)\psi(Q^2) + 4q^{34}\psi(-q^2)\psi(-Q^2)\}\\
&\quad\quad - \tfrac{1}{2}\{\varphi(q^9)\varphi(Q) + \varphi(-q^9)\varphi(-Q) - 2\varphi(-q^{36})\varphi(-Q^4)\\
&\quad\quad - 4q^{36}\psi(q^{18})\psi(Q^2) - 4q^{36}\psi(-q^{18})\psi(-Q^2)\}. \qquad (17.5)
\end{aligned}$$

Quite remarkably, the latter expression within curly brackets vanishes, as we demonstrate in the next paragraph.

Let $\mu = 8$ and $\nu = 7$ in (36.3). Combine the terms with indices n and $8 - n$, $1 \le n \le 3$, with the aid of Entry 18(iv). Then apply Entries 25(i), (ii), 18(iv) again, and 30(ii), (iii). Accordingly,

$$\begin{aligned}
&\tfrac{1}{2}\{\varphi(q^{15})\varphi(q) + \varphi(-q^{15})\varphi(-q)\}\\
&\quad = \sum_{n=0}^{7} q^{16n^2}f(q^{240+60n},q^{240-60n})f(q^{16+28n},q^{16-28n})\\
&\quad = \varphi(q^{240})\varphi(q^{16}) + 4q^{64}\psi(q^{480})\psi(q^{32})\\
&\quad\quad + 2\sum_{n=1}^{3} q^{16n^2}f(q^{240-60n},q^{240+60n})f(q^{16-28n},q^{16+28n})
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{4}\{\varphi(q^{60}) + \varphi(-q^{60})\}\{\varphi(q^4) + \varphi(-q^4)\} \\
&\quad + \tfrac{1}{4}\{\varphi(q^{60}) - \varphi(-q^{60})\}\{\varphi(q^4) - \varphi(-q^4)\} \\
&\quad + 2q^4 f(q^{180}, q^{300}) f(q^{12}, q^{20}) + 2q^{16}\psi(q^{120})\psi(q^8) \\
&\quad + 2q^{36} f(q^{60}, q^{420}) f(q^4, q^{28}) \\
&= \tfrac{1}{2}\varphi(q^{60})\varphi(q^4) + \tfrac{1}{2}\varphi(-q^{60})\varphi(-q^4) + 2q^{16}\psi(q^{120})\psi(q^8) \\
&\quad + \tfrac{1}{2}q^4\{f(q^{30}, q^{90}) + f(-q^{30}, -q^{90})\}\{f(q^2, q^6) + f(-q^2, -q^6)\} \\
&\quad + \tfrac{1}{2}q^4\{f(q^{30}, q^{90}) - f(-q^{30}, -q^{90})\}\{f(q^2, q^6) - f(-q^2, -q^6)\} \\
&= \tfrac{1}{8}\{\varphi(q^{15}) + \varphi(-q^{15})\}\{\varphi(q) + \varphi(-q)\} \\
&\quad + \tfrac{1}{8}\{\varphi(q^{15}) - \varphi(-q^{15})\}\{\varphi(q) - \varphi(-q)\} \\
&\quad + \tfrac{1}{2}\varphi(-q^{60})\varphi(-q^4) + q^4\psi(q^{30})\psi(q^2) + q^4\psi(-q^{30})\psi(-q^2) \\
&= \tfrac{1}{4}\varphi(q^{15})\varphi(q) + \tfrac{1}{4}\varphi(-q^{15})\varphi(-q) + \tfrac{1}{2}\varphi(-q^{60})\varphi(-q^4) \\
&\quad + q^4\psi(q^{30})\psi(q^2) + q^4\psi(-q^{30})\psi(-q^2).
\end{aligned}
$$

If we replace q by q^9 in the identity above, we see that we have, indeed, shown that the latter expression in curly brackets on the right side of (17.5) is equal to 0. Thus, (17.5) reduces to the equality

$$
\begin{aligned}
4q^4 f(-q^{36}) f(-q^{60}) &= \tfrac{1}{2}\{\varphi(q)\varphi(Q) + \varphi(-q)\varphi(-Q)\} - \varphi(-q^4)\varphi(-Q^4) \\
&\quad - 2q^{34}\psi(q^2)\psi(Q^2) + 2q^{34}\psi(-q^2)\psi(-Q^2). \qquad (17.6)
\end{aligned}
$$

If we now multiply (17.2) by 2 and then subtract (17.6) from the resulting equality, we obtain (17.1), which is what we sought to prove.

Proof of (iv). Replacing q by $q^{1/5}$ in (iii), we find that

$$
\begin{aligned}
\varphi(q^{1/5})\varphi(q^{27}) &= \varphi(-q^2)\varphi(-q^{54/5}) + 2q^{1/5} f(q^{9/5}) f(q^3) \\
&\quad + 2q^{4/5}\psi(q)\psi(q^{27/5}).
\end{aligned}
$$

Applying Entries 10(i)–(iii) of Chapter 19, we find that the previous formula becomes

$$
\begin{aligned}
&\varphi(q^{27})\{\varphi(q^5) + 2q^{1/5} f(q^3, q^7) + 2q^{4/5} f(q, q^9)\} \\
&\quad = \varphi(-q^2)\{\varphi(-q^{270}) - 2q^{54/5} f(-q^{162}, -q^{378}) \\
&\qquad + 2q^{216/5} f(-q^{54}, -q^{486})\} \\
&\qquad + 2q^{1/5}\{q^{9/5} f(q^9) f(q^{45}) + f^2(-q^{18}, q^{27}) - q^{18/5} f^2(q^9, -q^{36})\}\frac{f(q^3)}{f(q^9)} \\
&\qquad + 2q^{4/5}\psi(q)\{q^{81/5}\psi(q^{135}) + f(q^{54}, q^{81}) + q^{27/5} f(q^{27}, q^{104})\}.
\end{aligned}
$$

Equating rational parts on both sides, we complete the proof.

Entry 18. *Let β, γ, and δ have degrees 5, 7, and 35, respectively. Let m and m' denote the multipliers connecting α, β and γ, δ, respectively. Then*

$$\text{(i)}\quad (\alpha\delta)^{1/4} + \{(1-\alpha)(1-\delta)\}^{1/4} + 2^{4/3}\{\alpha\delta(1-\alpha)(1-\delta)\}^{1/12} + (\beta\gamma)^{1/4} + \{(1-\beta)(1-\gamma)\}^{1/4} + 2^{4/3}\{\beta\gamma(1-\beta)(1-\gamma)\}^{1/12} = 1 + \{1 + 2^{4/3}\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/24}\}^2,$$

$$\text{(ii)}\quad \{(\alpha\delta)^{1/4} + \{(1-\alpha)(1-\delta)\}^{1/4} + 2^{4/3}\{\alpha\delta(1-\alpha)(1-\delta)\}^{1/12}\} \times \{(\beta\gamma)^{1/4} + \{(1-\beta)(1-\gamma)\}^{1/4} + 2^{4/3}\{\beta\gamma(1-\beta)(1-\gamma)\}^{1/12}\} = 1 - 2^{7/3}\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/24} \times ((\alpha\beta\gamma\delta)^{1/8} + \{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8}),$$

$$\text{(iii)}\quad (\alpha\delta)^{1/4} + \{(1-\alpha)(1-\delta)\}^{1/4} + 2^{4/3}\{\beta\gamma(1-\beta)(1-\gamma)\}^{1/12}\left(\frac{m'}{m}\right)^{1/2} = 1,$$

$$\text{(iv)}\quad (\beta\gamma)^{1/4} + \{(1-\beta)(1-\gamma)\}^{1/4} - 2^{4/3}\{\alpha\delta(1-\alpha)(1-\delta)\}^{1/12}\left(\frac{m}{m'}\right)^{1/2} = 1,$$

$$\text{(v)}\quad \frac{\{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/24} - \{16\alpha\delta(1-\alpha)(1-\delta)\}^{1/8}}{\{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/24} + \{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/8}} = \left(\frac{m'}{m}\right)^{1/2} = \frac{\{16\alpha\delta(1-\alpha)(1-\delta)\}^{1/8} + \{16\alpha\delta(1-\alpha)(1-\delta)\}^{1/24}}{\{16\beta\gamma(1-\beta)(1-\gamma)\}^{1/8} - \{16\alpha\delta(1-\alpha)(1-\delta)\}^{1/24}},$$

$$\text{(vi)}\quad \left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8} + 2\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/12} = \left(\frac{m'}{m}\right)^{1/2},$$

and

$$\text{(vii)}\quad \left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8} + 2\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12} = -\left(\frac{m}{m'}\right)^{1/2}.$$

PROOFS OF (iii), (iv). If we transcribe (iii) and (iv) via Entries 10(i), (ii), 11(iii), and 12(iii) in Chapter 17, we obtain Entries 17(i) and 17(ii), respectively. Hence, the proofs are complete.

Proofs of the remaining modular equations are accomplished by means of the theory of modular forms.

PROOFS OF (i), (ii), (v)–(vii). Using Entries 10(i)–(iii), 11(i)–(iii), and 12(i), (iii) in

Chapter 17, we translate the six modular equations to be yet proved into the proposed theta-function identities

$$4\frac{q^9\psi(q^2)\psi(q^{70})}{\varphi(q)\varphi(q^{35})}+\frac{\varphi(-q)\varphi(-q^{35})}{\varphi(q)\varphi(q^{35})}+4\frac{q^3f(-q^2)f(-q^{70})}{\varphi(q)\varphi(q^{35})}$$
$$+4\frac{q^3\psi(q^{10})\psi(q^{14})}{\varphi(q^5)\varphi(q^7)}+\frac{\varphi(-q^5)\varphi(-q^7)}{\varphi(q^5)\varphi(q^7)}+4\frac{qf(-q^{10})f(-q^{14})}{\varphi(q^5)\varphi(q^7)}$$
$$=1+\left(1+4\frac{q^2f(q)f(q^5)f(q^7)f(q^{35})}{\varphi(q)\varphi(q^5)\varphi(q^7)\varphi(q^{35})}\right)^2, \tag{18.1}$$

$$(4q^9\psi(q^2)\psi(q^{70})+\varphi(-q)\varphi(-q^{35})+4q^3f(-q^2)f(-q^{70}))$$
$$\times(4q^3\psi(q^{10})\psi(q^{14})+\varphi(-q^5)\varphi(-q^7)+4qf(-q^{10})f(-q^{14}))$$
$$=\varphi(q)\varphi(q^5)\varphi(q^7)\varphi(q^{35})-8\frac{q^2f(q)f(q^5)f(q^7)f(q^{35})}{\varphi(q)\varphi(q^5)\varphi(q^7)\varphi(q^{35})}$$
$$\times(4q^6\psi(q)\psi(q^5)\psi(q^7)\psi(q^{35})+\varphi(-q^2)\varphi(-q^{10})\varphi(-q^{14})\varphi(-q^{70})), \tag{18.2}$$

$$\varphi(q^5)\varphi(q^7)(2q\psi(-q^5)\psi(-q^7)+2q^4\psi(-q)\psi(-q^{35})+f(q^5)f(q^7))$$
$$=\varphi(q)\varphi(q^{35})f(q^5)f(q^7), \tag{18.3}$$

$$\varphi(q)\varphi(q^{35})(2\psi(-q^5)\psi(-q^7)-2q^3\psi(-q)\psi(-q^{35})-f(q)f(q^{35}))$$
$$=\varphi(q^5)\varphi(q^7)f(q)f(q^{35}), \tag{18.4}$$

$$\frac{q^3\psi(q)\psi(q^{35})}{\psi(q^5)\psi(q^7)}+\frac{\varphi(-q^2)\varphi(-q^{70})}{\varphi(-q^{10})\varphi(-q^{14})}-\frac{q^3\psi(-q)\psi(-q^{35})}{\psi(-q^5)\psi(-q^7)}$$
$$+2\frac{q^2f(-q^2)f(-q^{70})}{f(-q^{10})f(-q^{14})}=1, \tag{18.5}$$

and

$$\frac{\psi(q^5)\psi(q^7)}{q^3\psi(q)\psi(q^{35})}+\frac{\varphi(-q^{10})\varphi(-q^{14})}{\varphi(-q^2)\varphi(-q^{70})}-\frac{\psi(-q^5)\psi(-q^7)}{q^3\psi(-q)\psi(-q^{35})}$$
$$+2\frac{f(-q^{10})f(-q^{14})}{q^2f(-q^2)f(-q^{70})}=-1. \tag{18.6}$$

Next, we employ (0.13) to rewrite (18.1)–(18.6), respectively, as

$$4\frac{g_2(\tau)g_2(35\tau)}{g_1(\tau)g_1(35\tau)}+\frac{g_0(\tau)g_0(35\tau)}{g_1(\tau)g_1(35\tau)}+4\frac{\eta(\tau)\eta(35\tau)}{g_1(\tau)g_1(35\tau)}$$
$$+4\frac{g_2(5\tau)g_2(7\tau)}{g_1(5\tau)g_1(7\tau)}+\frac{g_0(5\tau)g_0(7\tau)}{g_1(5\tau)g_1(7\tau)}+4\frac{\eta(5\tau)\eta(7\tau)}{g_1(5\tau)g_1(7\tau)}$$
$$=1+\left(1+4\frac{f_1(\tau)f_1(5\tau)f_1(7\tau)f_1(35\tau)}{g_1(\tau)g_1(5\tau)g_1(7\tau)g_1(35\tau)}\right)^2, \tag{18.7}$$

$$(4g_2(\tau)g_2(35\tau) + g_0(\tau)g_0(35\tau) + 4\eta(\tau)\eta(35\tau))$$
$$\times (4g_2(5\tau)g_2(7\tau) + g_0(5\tau)g_0(7\tau) + 4\eta(5\tau)\eta(7\tau))$$
$$= g_1(\tau)g_1(5\tau)g_1(7\tau)g_1(35\tau) - 8\frac{f_1(\tau)f_1(5\tau)f_1(7\tau)f_1(35\tau)}{g_1(\tau)g_1(5\tau)g_1(7\tau)g_1(35\tau)}$$
$$\times (4h_0(\tau)h_0(5\tau)h_0(7\tau)h_0(35\tau) + h_2(\tau)h_2(5\tau)h_2(7\tau)h_2(35\tau)), \quad (18.8)$$

$$g_1(5\tau)g_1(7\tau)(2h_1(5\tau)h_1(7\tau) + 2h_1(\tau)h_1(35\tau) + f_1(5\tau)f_1(7\tau))$$
$$= g_1(\tau)g_1(35\tau)f_1(5\tau)f_1(7\tau), \quad (18.9)$$

$$g_1(\tau)g_1(35\tau)(2h_1(5\tau)h_1(7\tau) - 2h_1(\tau)h_1(35\tau) - f_1(\tau)f_1(35\tau))$$
$$= g_1(5\tau)g_1(7\tau)f_1(\tau)f_1(35\tau), \quad (18.10)$$

$$\frac{h_0(\tau)h_0(35\tau)}{h_0(5\tau)h_0(7\tau)} + \frac{h_2(\tau)h_2(35\tau)}{h_2(5\tau)h_2(7\tau)} - \frac{h_1(\tau)h_1(35\tau)}{h_1(5\tau)h_1(7\tau)} + 2\frac{\eta(\tau)\eta(35\tau)}{\eta(5\tau)\eta(7\tau)} = 1, \quad (18.11)$$

and

$$\frac{h_0(5\tau)h_0(7\tau)}{h_0(\tau)h_0(35\tau)} + \frac{h_2(5\tau)h_2(7\tau)}{h_2(\tau)h_2(35\tau)} - \frac{h_1(5\tau)h_1(7\tau)}{h_1(\tau)h_1(35\tau)} + 2\frac{\eta(5\tau)\eta(7\tau)}{\eta(\tau)\eta(35\tau)} = -1. \quad (18.12)$$

We now apply the theory developed in Section 13. By employing the multiplier systems (0.14) and (0.18)–(0.23) in (13.7) and (13.8), we may easily verify that each expression in (18.7), (18.8), (18.11), and (18.12) has a multiplier system identically equal to 1, while each expression in (18.9) and (18.10) has a multiplier system equal to $\exp\{\pi i(b - c)d/2\}$. In conclusion, for each proposed identity, the terms have identical multiplier systems.

We now apply the operator M_d to the proposed identities (18.7)–(18.12). In all cases, (18.7) and (18.8) are invariant under the Fricke involutions. Each of (18.9)–(18.12) is transformed into its reciprocal by M_5 and M_7 but is left invariant under M_{35}.

Proceed now as in Section 13. Here $p = 5$ and $q = 7$. Thus, $(p + 1)(q + 1)/16 = 3$. Using (13.19), we obtain the following table of values for μ and ν.

	(18.7)	(18.8)	(18.9)	(18.10)	(18.11)	(18.12)
ν	8	8	4	4	8	8
μ	24	24	12	12	24	24

Using the computer algebra system MACSYMA, we have calculated the coefficients of F and F^* (in the notation of Section 13) through q^μ and have verified that, in each of the six cases, all, indeed, are equal to 0. This completes the proofs of (18.7)–(18.12) and hence also of Entry 18.

If we multiply (vi) and (vii) together, we obtain a modular equation first discovered by Weber [1].

Entry 19.

(i) $\varphi(q)\varphi(q^{63}) - \varphi(q^7)\varphi(q^9) = 2qf(q^3)f(q^{21})$.

(ii) $\psi(q^7)\psi(q^9) - q^6\psi(q)\psi(q^{63}) = f(-q^6)f(-q^{42})$.

(iii) *Let* β, γ, *and* δ *have degrees* 3, 13, *and* 39; *or* 5, 11, *and* 55; *or* 7, 9, *and* 63, *respectively, in each case. Then*

$$\frac{1 + \{(1-\alpha)(1-\delta)\}^{1/4} + (\alpha\delta)^{1/4}}{1 + \{(1-\beta)(1-\gamma)\}^{1/4} + (\beta\gamma)^{1/4}} = \frac{\{(1-\alpha)(1-\delta)\}^{1/8} - (\alpha\delta)^{1/8}}{\{(1-\beta)(1-\gamma)\}^{1/8} - (\beta\gamma)^{1/8}}$$

$$= \frac{(\alpha\delta)^{1/8} \pm \{\alpha\delta(1-\alpha)(1-\delta)\}^{1/8}}{(\beta\gamma)^{1/8} - \{\beta\gamma(1-\beta)(1-\gamma)\}^{1/8}}$$

$$= \sqrt{\frac{m'}{m}},$$

where m is the multiplier associated with α *and* β, *and* m' *is that associated with* γ *and* δ. *In the third expression, the plus sign is to be taken in the first two cases, and the minus sign is chosen in the third instance.*

(iv) *If* β, γ, *and* δ *are of degrees* 3, 13, *and* 39 *or of degrees* 5, 7, *and* 35, *respectively, in each case, then*

$$\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}$$
$$+ 2\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/12} = \sqrt{\frac{m'}{m}}$$

and

$$\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}$$
$$+ 2\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12} = \pm\sqrt{\frac{m}{m'}},$$

where the plus sign is taken in the first case, and the minus sign is assumed in the second. The multipliers m and m' *have the same meaning as in* (iii).

In fact, in the case of degrees 5, 7, and 35, the two formulas of (iv) are identical to Entries 18(vi), (vii), respectively.

The first and third equalities in (iii) appeared in Ramanujan's [10, pp. xxix, 353] second letter to Hardy.

When β, γ, and δ have degrees 3, 13, and 39, respectively, multiplying the two modular equations of (iv) yields a modular equation first found by Weber [1].

It will be simplest to first prove (ii) and then deduce (i) from it.

PROOF OF (ii). Unless otherwise stated, all references in this proof are to results in Chapter 16.

First, we employ Corollary (ii) of Section 31 and Entry 18(ii). With the use of Entry 18(iii), we next apply (36.1) with $\mu = 4$, $\nu = 3$, q replaced by $q^{9/2}$, and, in turn, $A = q^{21/2}, q^{63/2}$ and $B = q^{-9/2}, q^{-3/2}$. Hence,

$$\begin{aligned}
&\psi(q^7)\psi(q^9) - q^6\psi(q)\psi(q^{63}) \\
&\quad = \{f(q^{21}, q^{42}) + q^7\psi(q^{63})\}\psi(q^9) - q^6\{f(q^3, q^6) + q\psi(q^9)\}\psi(q^{63}) \\
&\quad = \tfrac{1}{2}f(q^{21}, q^{42})f(1, q^9) - \tfrac{1}{2}f(1, q^{63})f(q^3, q^6) \\
&\quad = \sum_{n=0}^{3} q^{36n^2+15n}f(q^{42(7+3n)}, q^{42(5-3n)})f(q^{42+54n}, q^{30-54n}) \\
&\qquad - \sum_{n=0}^{3} q^{36n^2+33n+6}f(q^{42(7+3n)}, q^{42(5-3n)})f(q^{66+54n}, q^{6-54n}) \\
&\quad = \sum_{n=0}^{3} q^{36n^2+15n}f(q^{42(7+3n)}, q^{42(5-3n)})\{f(q^{42+54n}, q^{30-54n}) \\
&\qquad - q^{18n+6}f(q^{66+54n}, q^{6-54n})\}.
\end{aligned} \tag{19.1}$$

If we now apply (7.1) of this chapter with $a = -q^6$ and $b = -q^{12}$, we easily deduce that

$$f(-q^6, -q^{12}) = f(q^{30}, q^{42}) - q^6f(q^6, q^{66}). \tag{19.2}$$

Furthermore, applying Entry 18(iv) twice with $n = 2$, we find that

$$f(-q^6, -q^{12}) = q^{132}f(q^{-102}, q^{174}) - q^{90}f(q^{-78}, q^{150}). \tag{19.3}$$

Using again Entry 18(iv) with $n = 2$, we see that

$$f(q^{-24}, q^{96}) - q^{24}f(q^{-48}, q^{120}) = 0, \tag{19.4}$$

while employing Entry 18(iv) with $n = 5$, we arrive at

$$f(q^{-132}, q^{204}) - q^{60}f(q^{-156}, q^{228}) = 0. \tag{19.5}$$

Using equalities (19.2)–(19.5) to simplify the terms with indices 0, 2, 1, and 3, respectively, we find that (19.1) reduces to the equality

$$\begin{aligned}
&\psi(q^7)\psi(q^9) - q^6\psi(q)\psi(q^{63}) \\
&\quad = f(q^{210}, q^{294})f(-q^6, -q^{12}) - q^{84}f(q^{-42}, q^{546})f(-q^6, -q^{12}) \\
&\quad = f(-q^6)\{f(q^{210}, q^{294}) - q^{42}f(q^{42}, q^{462})\} \\
&\quad = f(-q^6)f(-q^{42}),
\end{aligned}$$

where we have applied Entry 18(iv) and then utilized (19.2) above with q replaced by q^7. This concludes the proof of (ii).

Proof of (i). Let α_j have degree j, and let $z_j = \varphi^2(q^j)$, as usual. Translating Entry 19(ii) via Entries 11(i) and 12(iii) in Chapter 17, we find that

$$\begin{aligned}
&(z_7z_9)^{1/2}(\alpha_7\alpha_9)^{1/8} - (z_1z_{63})^{1/2}(\alpha_1\alpha_{63})^{1/8} \\
&\quad = 2^{1/3}(z_3z_{21})^{1/2}\{\alpha_3\alpha_{21}(1 - \alpha_3)(1 - \alpha_{21})\}^{1/12}.
\end{aligned}$$

Replacing α_j by its "complement" $1 - \alpha_{63/j}$, we deduce that

$$(z_9 z_7)^{1/2}\{(1-\alpha_9)(1-\alpha_7)\}^{1/8} - (z_{63}z_1)^{1/2}\{(1-\alpha_{63})(1-\alpha_1)\}^{1/8}$$
$$= 2^{1/3}(z_{21}z_3)^{1/2}\{(1-\alpha_{21})(1-\alpha_3)\alpha_{21}\alpha_3\}^{1/12}.$$

Translating this formula with the aid of Entries 10(iii) and 12(iii) in Chapter 17, we deduce that

$$\varphi(-q^{14})\varphi(-q^{18}) - \varphi(-q^2)\varphi(-q^{126}) = 2q^2 f(-q^6)f(-q^{42}).$$

If we replace q^2 by $-q$, we obtain (i) at once.

Proof of (iii). All references in this proof are to Chapter 16, unless otherwise stated.

Apply (36.6) with $\mu = 8$ and put $Q = q^{64-\nu^2}$. Then

$$S := \tfrac{1}{2}\{\varphi(q^{8+\nu})\varphi(q^{8-\nu}) + \varphi(-q^{8+\nu})\varphi(-q^{8-\nu})\} + 2q^4\psi(q^{16+2\nu})\psi(q^{16-2\nu})$$
$$= \sum_{n=0}^{7} q^{16n^2} f(Q^{16+4n}, Q^{16-4n}) f(q^{4+2\nu n}, q^{4-2\nu n}). \tag{19.6}$$

By two applications of Entry 18(iv), with $n = 1, 2\nu$, we find that

$$q^{16(8-n)^2} f(Q^{16+4(8-n)}, Q^{16-4(8-n)}) f(q^{4+2\nu(8-n)}, q^{4-2\nu(8-n)})$$
$$= q^{16(8-n)^2} Q^{-16+4n} q^{(4-2\nu(8-n))\nu(2\nu+1)+(4+2\nu(8-n))\nu(2\nu-1)}$$
$$\times f(Q^{16-4n}, Q^{16+4n}) f(q^{4-2\nu n}, q^{4+2\nu n})$$
$$= q^{16n^2} f(Q^{16-4n}, Q^{16+4n}) f(q^{4-2\nu n}, q^{4+2\nu n}).$$

Using this equality in (19.6) for $1 \le n \le 3$, we find that

$$S = \varphi(Q^{16})\varphi(q^4) + 2q^{256}\psi(Q^{32}) f(q^{4+8\nu}, q^{4-8\nu})$$
$$+ 2\sum_{n=1}^{3} q^{16n^2} f(Q^{16+4n}, Q^{16-4n}) f(q^{4-2\nu n}, q^{4+2\nu n}). \tag{19.7}$$

If ν is an odd integer, we apply Entry 18(iv) with $n = (\nu-1)/2$ and twice with $n = \nu$ to deduce that, respectively,

$$f(q^{4+4\nu}, q^{4-4\nu}) = q^{1-\nu^2} f(1, q^8),$$
$$f(q^{4+6\nu}, q^{4-6\nu}) = q^{-2\nu^2} f(q^{4+2\nu}, q^{4-2\nu}),$$

and

$$f(q^{4+8\nu}, q^{4-8\nu}) = q^{-4\nu^2}\varphi(q^4).$$

Employing these results in (19.7), we find that

$$S = \varphi(Q^{16})\varphi(q^4) + 2Q^4\psi(Q^{32})\varphi(q^4) + 4qQ\psi(Q^8)\psi(q^8)$$
$$+ 2q^{16}\{f(Q^{12}, Q^{20}) + Q^2 f(Q^4, Q^{28})\} f(q^{4+2\nu}, q^{4-2\nu})$$
$$= \varphi(Q^4)\varphi(q^4) + 4qQ\psi(Q^8)\psi(q^8) + 2q^{16} f(Q^2, Q^6) f(q^{4+2\nu}, q^{4-2\nu}). \tag{19.8}$$

In the last equality, we applied Entries 25(i), (ii) and (7.1) in this chapter with $a = Q^2$ and $b = Q^6$. Now in the three cases of part (iii), $v = 5$, 3, and 1, respectively. By using Entry 18(iv) and Entries 25(i), (ii) once again, we see that we can write (19.8), in all cases, in the form

$$S = \tfrac{1}{2}\{\varphi(Q)\varphi(q) + \varphi(-Q)\varphi(-q)\} + 2(Qq)^{1/4}\psi(Q^2)\psi(q^2).$$

In summary, for $v = 5$, 3, and 1,

$$\varphi(q^{8+v})\varphi(q^{8-v}) + \varphi(-q^{8+v})\varphi(-q^{8-v}) + 4q^4\psi(q^{16+2v})\psi(q^{16-2v})$$
$$= \varphi(Q)\varphi(q) + \varphi(-Q)\varphi(-q) + 4(Qq)^{1/4}\psi(Q^2)\psi(q^2).$$

Translating this formula by means of Entries 10(i), (ii) and 11(iii) in Chapter 17, we deduce that

$$(z_{8+v}z_{8-v})^{1/2}(1 + \{(1-\beta)(1-\gamma)\}^{1/4} + (\beta\gamma)^{1/4})$$
$$= (z_{64-v^2}z_1)^{1/2}(1 + \{(1-\alpha)(1-\delta)\}^{1/4} + (\alpha\delta)^{1/4}).$$

This establishes the first part of (iii).

Next, the three parts of the corollary in Section 37 may be collectively written in the form

$$q^2\{\psi(q^{8+v})\psi(q^{8-v}) - \psi(-q^{8+v})\psi(-q^{8-v})\}$$
$$= (Qq)^{1/8}\{\psi(Q)\psi(q) \pm \psi(-Q)\psi(-q)\},$$

where $v = 5$, 3, and 1, and where the plus sign is chosen in the first two cases and the minus sign is assumed in the last case. Translating this formula via Entries 11(i), (ii) in Chapter 17, we find that

$$(z_{8+v}z_{8-v})^{1/2}((\beta\gamma)^{1/8} - \{\beta\gamma(1-\beta)(1-\gamma)\}^{1/8})$$
$$= (z_{64-v^2}z_1)^{1/2}((\alpha\delta)^{1/8} \pm \{\alpha\delta(1-\alpha)(1-\delta)\}^{1/8}), \qquad (19.9)$$

which establishes the third equality of (iii).

The reciprocal of the preceding modular equation is obtained by replacing α, β, γ, and δ by $1-\delta$, $1-\gamma$, $1-\beta$, and $1-\alpha$, respectively. Accordingly,

$$(z_{8-v}z_{8+v})^{1/2}(\{(1-\gamma)(1-\beta)\}^{1/8} - \{(1-\gamma)(1-\beta)\gamma\beta\}^{1/8})$$
$$= (z_1z_{64-v^2})^{1/2}(\{(1-\delta)(1-\alpha)\}^{1/8} \pm \{(1-\delta)(1-\alpha)\delta\alpha\}^{1/8}). \quad (19.10)$$

For brevity, set $A = (\alpha\delta)^{1/8}$, $A' = \{(1-\alpha)(1-\delta)\}^{1/8}$, $B = (\beta\gamma)^{1/8}$, and $B' = \{(1-\beta)(1-\gamma)\}^{1/8}$. Then combining (19.9) and (19.10), we deduce that

$$\frac{A \pm AA'}{B - BB'} = \frac{A' \pm AA'}{B' - BB'}.$$

Consulting the statement of (iii), we see that it suffices to prove that

$$\frac{A \pm AA'}{B - BB'} = \frac{A' - A}{B' - B}.$$

By cross-multiplication, it is easily seen that the last two equalities are equivalent. Thus, the proof of (iii) is complete.

Proof of (iv). As we have already seen, we need only prove the case when β, γ, and δ are of degrees 3, 13, and 39, respectively. As in the proofs of Entries 18(vi), (vii), we employ the theory of modular forms. Since the modular equations to be established are exactly of the same shapes as those of Entries 18(vi), (vii), the proofs are almost identical. Thus, we forego almost all of the details. In particular, we do not record the relevant theta-function and modular form identities.

In the instance at hand, $p = 3$ and $q = 13$, and so $(p + 1)(q + 1)/16 = \frac{7}{2}$. Also, in each case, $v = 8$ as before. Thus, $\mu = 28$ in each case. With the same notation as in Sections 13 and 18, we need to show that the coefficients of q^0, $q^1, \ldots, q^{28}$ are equal to 0 for both F and F^*. Using MACSYMA, we have, indeed, done this, and so the proof of (iv) is complete.

Entry 20.

(i) *Let β, γ, and δ have one of the following sequences of degrees:*

$$\begin{array}{c} 3, 21, \ 63; \\ 5, 19, \ 95; \\ 11, 13, 143; \\ 7, 17, 119; \\ 9, 15, 135; \end{array}$$

respectively. Then

$$\begin{aligned} &\{\tfrac{1}{2}(1 + (\alpha\delta)^{1/2} + \{(1 - \alpha)(1 - \delta)\}^{1/2})\}^{1/2} \\ &\quad = (\alpha\delta)^{1/8} + \{(1 - \alpha)(1 - \delta)\}^{1/8} \pm \{\alpha\delta(1 - \alpha)(1 - \delta)\}^{1/8} \\ &\quad\quad + 2^{4/3}\{\beta\gamma(1 - \beta)(1 - \gamma)\}^{1/2}\sqrt{\frac{m'}{m}}, \end{aligned}$$

where the plus sign is taken in the first three cases and the minus sign is chosen in the latter two cases. Here m is the multiplier associated with α and β, and m' is that attached to γ and δ.

(ii) *Let β, γ, and δ have one of the following sequences of degrees:*

$$\begin{array}{c} 5, 19, \ 95; \\ 7, 17, 119; \\ 11, 13, 143; \end{array}$$

respectively. Then

$$\begin{aligned} &\{\tfrac{1}{2}(1 + (\beta\gamma)^{1/2} + \{(1 - \beta)(1 - \gamma)\}^{1/2})\}^{1/2} \\ &\quad = (\beta\gamma)^{1/8} + \{(1 - \beta)(1 - \gamma)\}^{1/8} - \{\beta\gamma(1 - \beta)(1 - \gamma)\}^{1/8} \\ &\quad\quad \pm 2^{4/3}\{\alpha\delta(1 - \alpha)(1 - \delta)\}^{1/12}\sqrt{\frac{m}{m'}}, \end{aligned}$$

where the minus sign is chosen in the first two cases and the plus sign is assumed in the last case. The multipliers m and m′ are as in part (i).

We first prove the three formulas of (ii). Then the second, fourth, and third formulas in (i) can be deduced immediately. Lastly, the first and fifth formulas of (i) are established. Because 3 and 21 as well as 9 and 15 have a factor in common, a somewhat different argument is needed to establish these two formulas.

PROOF OF (ii). Unless otherwise stated, all references in this proof are to Chapter 16.

Let ν denote one of the integers 7, 5, and 1 and put $Q = q^{144-\nu^2}$. First, in (36.8), set $\mu = 12$ to deduce that

$$\psi(q^{12+\nu})\psi(q^{12-\nu}) = \varphi(Q^{12})\psi(q^{24}) + q^{432-6\nu}\psi(Q^{24})f(q^{12\nu}, q^{24-12\nu}) + \sum_{n=1}^{5} q^{12n^2-\nu n}f(Q^{12+2n}, Q^{12-2n})f(q^{2\nu n}, q^{24-2\nu n}).$$

Replace q by $-q$ and subtract the result from the equality above to arrive at

$$\psi(q^{12+\nu})\psi(q^{12-\nu}) - \psi(-q^{12+\nu})\psi(-q^{12-\nu}) = 2\sum_{n=0}^{2} q^{12(2n+1)^2-\nu(2n+1)}f(Q^{14+4n}, Q^{10-4n})f(q^{4\nu n+2\nu}, q^{24-4\nu n-2\nu}). \quad (20.1)$$

Second, in (36.2), let $\mu = 12$, set $A = B = 1$, and replace q by q^2. Accordingly,

$$\tfrac{1}{2}\{\varphi(q^{24+2\nu})\varphi(q^{24-2\nu}) - \varphi(-q^{24+2\nu})\varphi(-q^{24-2\nu})\} = \sum_{n=0}^{11} q^{(4n+2)(12+\nu)+48n^2}f(Q^{52+8n}, Q^{44-8n})f(q^{96+4\nu+8\nu n}, q^{-4\nu-8\nu n}). \quad (20.2)$$

Third, in (36.10), let $\mu = 12$ and replace q by q^4 to deduce that

$$\psi(q^{48+4\nu})\psi(q^{48-4\nu}) = \sum_{n=0}^{5} q^{48n(n+1)}f(Q^{52+8n}, Q^{44-8n})f(q^{48+4\nu+8\nu n}, q^{48-4\nu-8\nu n}) = \frac{1}{2}\sum_{n=0}^{11} q^{48n(n+1)}f(Q^{52+8n}, Q^{44-8n})f(q^{48+4\nu+8\nu n}, q^{48-4\nu-8\nu n}). \quad (20.3)$$

We have extended the sum to $0 \le n \le 11$ by using the identity

$$q^{48(11-n)(12-n)}f(Q^{52+8(11-n)}, Q^{44-8(11-n)})f(q^{48+4\nu+8\nu(11-n)}, q^{48-4\nu-8\nu(11-n)}) = q^{48n(n+1)}f(Q^{52+8n}, Q^{44-8n})f(q^{48+4\nu+8\nu n}, q^{48-4\nu-8\nu n}),$$

which is established by two applications of Entry 18(iv), with $n = 1, \nu$ there.

Combining (20.2) and (20.3), we find that

$$\tfrac{1}{2}\{\varphi(q^{24+2\nu})\varphi(q^{24-2\nu}) - \varphi(-q^{24+2\nu})\varphi(-q^{24-2\nu})\} + 2q^{12}\psi(q^{48+4\nu})\psi(q^{48-4\nu})$$
$$= \sum_{n=0}^{11} q^{48n(n+1)+12} f(Q^{52+8n}, Q^{44-8n})\{q^{12+2\nu+4\nu n} f(q^{96+4\nu+8\nu n}, q^{-4\nu-8\nu n})$$
$$+ f(q^{48+4\nu+8\nu n}, q^{48-4\nu-8\nu n})\}$$
$$= \sum_{n=0}^{11} q^{48n(n+1)+12} f(Q^{52+8n}, Q^{44-8n}) f(q^{12+2\nu+4\nu n}, q^{12-2\nu-4\nu n})$$
$$= \sum_{n=0}^{5} q^{48n(n+1)+12}\{f(Q^{52+8n}, Q^{44-8n})$$
$$+ Q^{14+4n} f(Q^{100+8n}, Q^{-4-8n})\} f(q^{12+2\nu+4\nu n}, q^{12-2\nu-4\nu n})$$
$$= \sum_{n=0}^{5} q^{48n(n+1)+12} f(Q^{14+4n}, Q^{10-4n}) f(q^{12+2\nu+4\nu n}, q^{12-2\nu-4\nu n})$$
$$= 2\sum_{n=0}^{2} q^{48n(n+1)+12} f(Q^{14+4n}, Q^{10-4n}) f(q^{12+2\nu+4\nu n}, q^{12-2\nu-4\nu n}), \qquad (20.4)$$

where we have applied (7.1) of this chapter with $a = q^{12+2\nu+4\nu n}$ and $b = q^{12-2\nu-4\nu n}$, utilized Entry 18(iv) with $n = \nu$ to combine the terms with indices n and $n+6$, $0 \le n \le 5$, invoked (7.1) once more but with $a = Q^{14+4n}$ and $b = Q^{10-4n}$, and lastly utilized Entry 18(iv) again with $n = \nu$ to show that the terms with indices n and $5-n$, $0 \le n \le 2$, are equal.

Combining (20.1) and (20.4), applying (7.1) of this chapter with $a = -q^{3-\nu-2\nu n}$ and $b = -q^{3+\nu+2\nu n}$, invoking Entries 18(iv), (iii), and employing (7.1) once again but with $a = -Q^2$ and $b = -Q^4$, we find that

$$\tfrac{1}{2}\{\varphi(q^{24+2\nu})\varphi(q^{24-2\nu}) - \varphi(-q^{24+2\nu})\varphi(-q^{24-2\nu})\}$$
$$+ 2q^{12}\psi(q^{48+4\nu})\psi(q^{48-4\nu}) - q^3\psi(q^{12+\nu})\psi(q^{12-\nu})$$
$$+ q^3\psi(-q^{12+\nu})\psi(-q^{12-\nu})$$
$$= 2\sum_{n=0}^{2} q^{48n(n+1)+12} f(Q^{14+4n}, Q^{10-4n})\{f(q^{12+2\nu+4\nu n}, q^{12-2\nu-4\nu n})$$
$$- q^{3-\nu-2\nu n} f(q^{4\nu n+2\nu}, q^{24-4\nu n-2\nu})\}$$
$$= 2\sum_{n=0}^{2} q^{48n(n+1)+12} f(Q^{14+4n}, Q^{10-4n}) f(-q^{3-\nu-2\nu n}, -q^{3+\nu+2\nu n})$$
$$= 2q^{12} f(Q^{10}, Q^{14}) f(-q^{3-\nu}, -q^{3+\nu})$$
$$+ 2q^{300} f(Q^2, Q^{22}) f(-q^{3-5\nu}, -q^{3+5\nu})$$
$$= 2q^{12}\{f(Q^{10}, Q^{14}) - Q^2 f(Q^2, Q^{22})\} f(-q^{3-\nu}, -q^{3+\nu})$$
$$= 2q^{12} f(-Q^2, -Q^4) f(-q^{3-\nu}, -q^{3+\nu}).$$

When $\nu = 7$, 5, and 1, the last expression has the values

$$-2q^8 f(-Q^2) f(-q^2), \quad -2q^{10} f(-Q^2) f(-q^2), \quad \text{and} \quad 2q^{12} f(-Q^2) f(-q^2),$$

respectively, by Entry 18(iv). In summary, we have shown that

$$\begin{aligned}\varphi(q^{24+2\nu})\varphi(q^{24-2\nu}) &+ 4q^{12}\psi(q^{48+4\nu})\psi(q^{48-4\nu})\\ &= 2q^3\psi(q^{12+\nu})\psi(q^{12-\nu}) + \varphi(-q^{24+2\nu})\varphi(-q^{24-2\nu})\\ &\quad - 2q^3\psi(-q^{12+\nu})\psi(-q^{12-\nu}) \pm 4(qQ)^{1/12}f(-Q^2)f(-q^2), \qquad (20.5)\end{aligned}$$

where the minus sign is taken in the first two cases and the plus sign is chosen in the last case.

We now translate this equality via Entries 10(iii), (iv), 11(i), (ii), (iv), and 12(iii) in Chapter 17. If we furthermore use the identity

$$\begin{aligned}\tfrac{1}{2}(1 + \sqrt{1-\beta})^{1/2}(1 + \sqrt{1-\gamma})^{1/2} &+ \tfrac{1}{2}(1 - \sqrt{1-\beta})^{1/2}(1 - \sqrt{1-\gamma})^{1/2}\\ &= \{\tfrac{1}{2}(1 + \sqrt{\beta\gamma} + \sqrt{(1-\beta)(1-\gamma)})\}^{1/2}, \qquad (20.6)\end{aligned}$$

which is easily established by squaring both sides, we complete the proof.

PROOF OF (i). In (20.5), replace q by $q^{1/(12-\nu)}$ to deduce that

$$\begin{aligned}\varphi(q^{(24+2\nu)/(12-\nu)})\varphi(q^2) &+ 4q^{12/(12-\nu)}\psi(q^{(48+4\nu)/(12-\nu)})\psi(q^4)\\ &= 2q^{3/(12-\nu)}\psi(q^{(12+\nu)/(12-\nu)})\psi(q) + \varphi(-q^{(24+2\nu)/(12-\nu)})\varphi(-q^2)\\ &\quad - 2q^{3/(12-\nu)}\psi(-q^{(12+\nu)/(12-\nu)})\psi(-q)\\ &\quad \pm 4q^{(145-\nu^2)/(12(12-\nu))}f(-q^{2(12+\nu)})f(-q^{2/(12-\nu)}).\end{aligned}$$

We now equate the rational parts on both sides. In order to do this, we must employ Entries 10(i)–(iii) in Chapter 19 for the case $\nu = 7$, Entries 17(iii)–(v) in Chapter 19 for the case $\nu = 5$, and Entry 6 in Chapter 20 in the case $\nu = 1$. The details are somewhat tedious as each case must be examined separately. However, the details are straightforward, as in the similar proof of Entry 17(iv), and we eventually find that

$$\begin{aligned}\varphi(Q^2)\varphi(q^2) &+ 4(Qq)^{1/2}\psi(Q^4)\psi(q^4)\\ &= 2(Qq)^{1/8}\psi(Q)\psi(q) + \varphi(-Q^2)\varphi(-q^2) \pm 2(Qq)^{1/8}\psi(-Q)\psi(-q)\\ &\quad + 4q^2f(-q^{24+2\nu})f(-q^{24-2\nu}),\end{aligned}$$

where the plus sign is correct when ν is equal to 7 or 1, and the minus sign is chosen when $\nu = 5$. Employing Entries 10(iii), (iv), 11(i), (ii), (iv), and 12(iii) in Chapter 17, we see that the foregoing equality transcribes into the modular equation

$$\begin{aligned}\tfrac{1}{2}(1 + \sqrt{1-\alpha})^{1/2}(1 + \sqrt{1-\delta})^{1/2} &+ \tfrac{1}{2}(1 - \sqrt{1-\alpha})^{1/2}(1 - \sqrt{1-\delta})^{1/2}\\ &= (\alpha\delta)^{1/8} + \{(1-\alpha)(1-\delta)\}^{1/8} \pm \{\alpha\delta(1-\alpha)(1-\delta)\}^{1/8}\\ &\quad + 2^{4/3}\{\beta\gamma(1-\beta)(1-\gamma)\}^{1/12}\sqrt{\frac{m'}{m}}.\end{aligned}$$

If we use (20.6), we find that the equality above yields the required identity.

We now establish part (i) in case 1. In (36.8) of Chapter 16, put $\mu = 8$ and $\nu = 1$. Then apply Entries 25(i), (ii) and Corollary (ii) in Section 31 of Chapter

16. Accordingly,

$$\begin{aligned}
&4q^2\psi(q^7)\psi(q^9)\\
&\quad= 4q^2\varphi(q^{504})\psi(q^{16}) + 4q^{126}\psi(q^{1008})\varphi(q^8)\\
&\qquad + 4\sum_{n=1}^{3} q^{8n^2-n+2}f(q^{126(4+n)}, q^{126(4-n)})f(q^{2n}, q^{16-2n})\\
&\quad= \tfrac{1}{2}\{\varphi(q^{126}) + \varphi(-q^{126})\}\{\varphi(q^2) - \varphi(-q^2)\}\\
&\qquad + \tfrac{1}{2}\{\varphi(q^{126}) - \varphi(-q^{126})\}\{\varphi(q^2) + \varphi(-q^2)\}\\
&\qquad + 4q^9f(q^{630}, q^{378})f(q^2, q^{14}) + 4q^{71}f(q^{882}, q^{126})f(q^6, q^{10})\\
&\qquad + 4q^{32}\psi(q^{252})\psi(q^4)\\
&\quad= \varphi(q^{126})\varphi(q^2) - \varphi(-q^{126})\varphi(-q^2) + 4q^{32}\psi(q^{252})\psi(q^4)\\
&\qquad + q^8\{\psi(q^{63}) + \psi(-q^{63})\}\{\psi(q) - \psi(-q)\}\\
&\qquad + q^8\{\psi(q^{63}) - \psi(-q^{63})\}\{\psi(q) + \psi(-q)\}\\
&\quad= \varphi(q^{126})\varphi(q^2) - \varphi(-q^{126})\varphi(-q^2) + 4q^{32}\psi(q^{252})\psi(q^4)\\
&\qquad + 2q^8\psi(q)\psi(q^{63}) - 2q^8\psi(-q)\psi(-q^{63}).
\end{aligned}$$

Hence, by Entry 19(ii) of this chapter,

$$\begin{aligned}
&4q^8\psi(q)\psi(q^{63}) + 4q^2f(-q^6)f(-q^{42})\\
&\quad= \varphi(q^{126})\varphi(q^2) - \varphi(-q^{126})\varphi(-q^2) + 4q^{32}\psi(q^{252})\psi(q^4)\\
&\qquad + 2q^8\psi(q)\psi(q^{63}) - 2q^8\psi(-q)\psi(-q^{63}).
\end{aligned}$$

Transcribing this equality via Entries 10(iii), (iv), 11(i), (ii), (iv), and 12(iii) in Chapter 17, we complete the proof of (i) in case 1.

Lastly, we establish (i) in the fifth case. Rewriting (17.6) and using Entries 25(i), (ii) in Chapter 16, we find that

$$\begin{aligned}
&2q^{34}\psi(q^2)\psi(q^{270}) + \varphi(-q^4)\varphi(-q^{540}) - 2q^{34}\psi(-q^2)\psi(-q^{270})\\
&\qquad + 4q^4f(-q^{36})f(-q^{60})\\
&\quad= \tfrac{1}{2}\{\varphi(q)\varphi(q^{135}) + \varphi(-q)\varphi(-q^{135})\}\\
&\quad= \tfrac{1}{4}\{\varphi(q) + \varphi(-q)\}\{\varphi(q^{135}) + \varphi(-q^{135})\}\\
&\qquad + \tfrac{1}{4}\{\varphi(q) - \varphi(-q)\}\{\varphi(q^{135}) - \varphi(-q^{135})\}\\
&\quad= \varphi(q^4)\varphi(q^{540}) + 4q^{136}\psi(q^8)\psi(q^{1080}).
\end{aligned}$$

Replacing q by $\sqrt{q}$, we find that

$$\begin{aligned}
&\varphi(q^2)\varphi(q^{270}) + 4q^{68}\psi(q^4)\psi(q^{540})\\
&\quad= 2q^{17}\psi(q)\psi(q^{135}) + \varphi(-q^2)\varphi(-q^{270}) - 2q^{17}\psi(-q)\psi(-q^{135})\\
&\qquad + 4q^2f(-q^{18})f(-q^{30}).
\end{aligned}$$

Transcribing this equality by the same entries in Chapter 17 as in the previous four cases, we complete the proof.

Entry 21.

(i) *Let* α *and* β *has degrees* 1, 7; 3, 5; *or* 1, 15, *respectively. Then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} \pm \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$$
$$= \{\tfrac{1}{2}(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)})\}^{1/2},$$

where the minus sign is chosen in the first two cases and the plus sign is selected in the last case.

(ii) *Let* β, γ, *and* δ *have one of the following sequences of degrees:*

3, 13, 39;
5, 11, 55;
7, 9, 63.

Let m *and* m' *denote the multipliers associated with the pairs* α, β *and* γ, δ, *respectively. Then*

$$\{(1-\alpha)(1-\delta)\}^{1/8} + ((\beta\gamma)^{1/8} + \{\beta\gamma(1-\beta)(1-\gamma)\}^{1/8})\sqrt{\frac{m'}{m}}$$
$$= \{\tfrac{1}{2}(1 + \sqrt{\alpha\delta} + \sqrt{(1-\alpha)(1-\delta)})\}^{1/2}$$

and

$$\{(1-\beta)(1-\gamma)\}^{1/8} + ((\alpha\delta)^{1/8} \pm \{\alpha\delta(1-\alpha)(1-\delta)\}^{1/8})\sqrt{\frac{m}{m'}}$$
$$= \{\tfrac{1}{2}(1 + \sqrt{\beta\gamma} + \sqrt{(1-\beta)(1-\gamma)})\}^{1/2},$$

where the minus sign is correct in the first two cases and the plus sign is correct in the last case.

Russell [2, p. 388] has derived a modular equation of degree 15 similar to, but more complicated than, Ramanujan's modular equation in Entry 21(i).

All references in the proofs of (i) and (ii) are to Chapter 16 unless otherwise stated.

Proof of (i). We first apply (36.12) and (36.13) when $\mu = 8$ and $\omega = 3$ and subtract the results. Second, we employ (7.1) of this chapter with $a = -q^{4-6n}$ and $b = -q^{4+6n}$. Third, we apply Entry 18(iv) to the term of index 4 to find that

$$f(-q^{20}, -q^{28}) = -q^{-36}\varphi(-q^4).$$

By using Entry 18(iv), we next show that the terms with indices n and $8-n$, $1 \le n \le 3$, are equal. Then we use Entries 25(i), (ii) and further simplify the terms by using Entries 18(iii), (iv). Lastly, we invoke (7.1) again but with

$a = -q^{14}$ and $b = -q^{42}$. Accordingly,

$$\begin{aligned}
&\tfrac{1}{2}\{\varphi(q^7)\varphi(q) + \varphi(-q^7)\varphi(-q)\} - 2q^2\psi(q^{14})\psi(q^2) \\
&\quad = \sum_{n=0}^{7} q^{4n^2} f(q^{112+28n}, q^{112-28n}) \\
&\qquad \times \{f(q^{16-12n}, q^{16+12n}) - q^{4-6n} f(q^{32-12n}, q^{12n})\} \\
&\quad = \sum_{n=0}^{7} q^{4n^2} f(q^{112+28n}, q^{112-28n}) f(-q^{4-6n}, -q^{4+6n}) \\
&\quad = \varphi(q^{112})\varphi(-q^4) - 2q^{28}\psi(q^{224})\varphi(-q^4) \\
&\qquad + 2\sum_{n=1}^{3} q^{4n^2} f(q^{112+28n}, q^{112-28n}) f(-q^{4-6n}, -q^{4+6n}) \\
&\quad = \tfrac{1}{2}\{\varphi(q^{28}) + \varphi(-q^{28})\}\varphi(-q^4) - \tfrac{1}{2}\{\varphi(q^{28}) - \varphi(-q^{28})\}\varphi(-q^4) \\
&\qquad - 2q^2 f(q^{140}, q^{84}) f(-q^2, -q^6) + 2q^{16} f(q^{196}, q^{28}) f(-q^2, -q^6) \\
&\quad = \varphi(-q^{28})\varphi(-q^4) - 2q^2\psi(-q^{14})\psi(-q^2).
\end{aligned}$$

Replacing q by $\sqrt{q}$, we find that

$$\begin{aligned}
&\tfrac{1}{2}\{\varphi(q^{7/2})\varphi(q^{1/2}) + \varphi(-q^{7/2})\varphi(-q^{1/2})\} \\
&\qquad = 2q\psi(q^7)\psi(q) + \varphi(-q^{14})\varphi(-q^2) - 2q\psi(-q^7)\psi(-q).
\end{aligned}$$

Transcribing this identity by means of Entries 10(iii), (vi), (vii) and 11(i), (ii) in Chapter 17, we arrive at

$$\begin{aligned}
&\tfrac{1}{2}(1 + \sqrt{\alpha})^{1/2}(1 + \sqrt{\beta})^{1/2} + \tfrac{1}{2}(1 - \sqrt{\alpha})^{1/2}(1 - \sqrt{\beta})^{1/2} \\
&\qquad = (\alpha\beta)^{1/8} + \{(1 - \alpha)(1 - \beta)\}^{1/8} - \{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/8}.
\end{aligned}$$

Using an obvious analogue of (20.6) above, we conclude the proof of (i) in the first case.

To prove (i) in the third case, we again first employ (36.12) and (36.13), but now with $\mu = 8$ and $\omega = 1$. Subtract the results and employ (7.1) in this chapter with $a = -q^{4-2n}$ and $b = -q^{4+2n}$. We then apply Entry 18(iv) to simplify the term with index 4 and to show that the terms with index n and $8 - n$, $1 \le n \le 3$, are equal. Next apply Entries 25(i), (ii) and 18(iii), (iv) to further simplify the terms. Lastly, we appeal to (7.1) again but with $a = -q^{30}$ and $b = -q^{90}$. Thus,

$$\begin{aligned}
&\tfrac{1}{2}\{\varphi(q^{15})\varphi(q) + \varphi(-q^{15})\varphi(-q)\} - 2q^4\psi(q^{30})\psi(q^2) \\
&\quad = \sum_{n=0}^{7} q^{4n^2} f(q^{240+60n}, q^{240-60n})\{f(q^{16-4n}, q^{16+4n}) - q^{4-2n} f(q^{32-4n}, q^{4n})\} \\
&\quad = \sum_{n=0}^{7} q^{4n^2} f(q^{240+60n}, q^{240-60n}) f(-q^{4-2n}, -q^{4+2n}) \\
&\quad = \varphi(q^{240})\varphi(-q^4) - 2q^{60}\psi(q^{480})\varphi(-q^4)
\end{aligned}$$

$$+ 2\sum_{n=1}^{3} q^{4n^2} f(q^{240+60n}, q^{240-60n}) f(-q^{4-2n}, -q^{4+2n})$$
$$= \varphi(-q^{60})\varphi(-q^4) + 2q^4 f(q^{300}, q^{180}) f(-q^2, -q^6)$$
$$- 2q^{34} f(q^{420}, q^{60}) f(-q^2, -q^6)$$
$$= \varphi(-q^{60})\varphi(-q^4) + 2q^4\psi(-q^{30})\psi(-q^2). \tag{21.1}$$

Replacing q by $\sqrt{q}$, we find that

$$\tfrac{1}{2}\{\varphi(q^{15/2})\varphi(q^{1/2}) + \varphi(-q^{15/2})\varphi(-q^{1/2})\}$$
$$= 2q^2\psi(q^{15})\psi(q) + \varphi(-q^{30})\varphi(-q^2) + 2q^2\psi(-q^{15})\psi(-q).$$

Employing Entries 10(iii), (vi), (vii) and 11(i), (ii) in Chapter 17, we easily transcribe the equality above into the desired modular equation in case 3. The details are completely analogous to those in the first case.

In order to establish the desired modular equation in the second case, we first replace q by $q^{1/3}$ in (21.1) to arrive at

$$\tfrac{1}{2}\{\varphi(q^5)\varphi(q^{1/3}) + \varphi(-q^5)\varphi(-q^{1/3})\}$$
$$= 2q^{4/3}\psi(q^{10})\psi(q^{2/3}) + \varphi(-q^{20})\varphi(-q^{4/3}) + 2q^{4/3}\psi(-q^{10})\psi(-q^{2/3}).$$

Using Corollaries (i), (ii) in Section 31, we equate rational parts on both sides above and deduce that

$$\tfrac{1}{2}\{\varphi(q^5)\varphi(q^3) + \varphi(-q^5)\varphi(-q^3)\}$$
$$= 2q^2\psi(q^{10})\psi(q^6) + \varphi(-q^{20})\varphi(-q^{12}) - 2q^2\psi(-q^{10})\psi(-q^6).$$

When q is replaced by $\sqrt{q}$, the foregoing equality becomes

$$\tfrac{1}{2}\{\varphi(q^{5/2})\varphi(q^{3/2}) + \varphi(-q^{5/2})\varphi(-q^{3/2})\}$$
$$= 2q\psi(q^5)\psi(q^3) + \varphi(-q^{10})\varphi(-q^6) - 2q\psi(-q^5)\psi(-q^3).$$

Translating this equality by the same set of results from Chapter 17 that we used above, we complete the proof of (i) in the second case.

PROOF OF (ii). Apply (36.3) and (36.4) with $\mu = 8$ and $\nu = 5$, 3, or 1. Set $Q = q^{64-\nu^2}$. The theorems that we use below are precisely the same that we used in the proof of part (i), and so we proceed without further comment. Hence,

$$\tfrac{1}{2}\{\varphi(q^{8+\nu})\varphi(q^{8-\nu}) + \varphi(-q^{8+\nu})\varphi(-q^{8-\nu})\} - 2q^4\psi(q^{16+2\nu})\psi(q^{16-2\nu})$$
$$= \sum_{n=0}^{7} q^{16n^2} f(Q^{16+4n}, Q^{16-4n})\{f(q^{16+4\nu n}, q^{16-4\nu n})$$
$$- q^{2\nu n+4} f(q^{32+4\nu n}, q^{-4\nu n})\}$$
$$= \sum_{n=0}^{7} q^{16n^2} f(Q^{16+4n}, Q^{16-4n}) f(-q^{4+2\nu n}, -q^{4-2\nu n})$$

$$= \varphi(Q^{16})\varphi(-q^4) + 2q^{256}\psi(Q^{32})f(-q^{4+8\nu}, -q^{4-8\nu})$$
$$+ 2\sum_{n=1}^{3} q^{16n^2} f(Q^{16+4n}, Q^{16-4n}) f(-q^{4+2\nu n}, -q^{4-2\nu n})$$
$$= \varphi(Q^{16})\varphi(-q^4) - 2Q^4\psi(Q^{32})\varphi(-q^4) + 2q^{16}f(Q^{20}, Q^{12})f(-q^{4+2\nu}, -q^{4-2\nu})$$
$$- 2q^{16}Q^2 f(Q^{28}, Q^4) f(-q^{4+2\nu}, -q^{4-2\nu})$$
$$= \varphi(-Q^4)\varphi(-q^4) + 2q^{16}\psi(-Q^2)f(-q^{4+2\nu}, -q^{4-2\nu})$$
$$= \varphi(-Q^4)\varphi(-q^4) \pm 2(Qq)^{1/4}\psi(-Q^2)\psi(-q^2), \tag{21.2}$$

where the plus sign is taken when $\nu = 1$ and the minus sign is chosen when $\nu = 3$ or 5. (We emphasize that we used Entry 18(iv) several times above.)

Replacing q by $\sqrt{q}$, we deduce that

$$\tfrac{1}{2}\{\varphi(q^{(8+\nu)/2})\varphi(q^{(8-\nu)/2}) + \varphi(-q^{(8+\nu)/2})\varphi(-q^{(8-\nu)/2})\}$$
$$- 2q^2\psi(q^{8+\nu})\psi(q^{8-\nu})$$
$$= \varphi(-Q^2)\varphi(-q^2) \pm 2(Qq)^{1/8}\psi(-Q)\psi(-q).$$

Employing Entries 10(iii), (vi), (vii) and 11(i), (ii) in Chapter 17 and an obvious analogue of (20.6) above, we readily find that

$$\{\tfrac{1}{2}(1 + \sqrt{\beta\gamma} + \sqrt{(1-\beta)(1-\gamma)})\}^{1/2} - (\beta\gamma)^{1/8}$$
$$= (\{(1-\alpha)(1-\delta)\}^{1/8} \pm \{\alpha\delta(1-\alpha)(1-\delta)\}^{1/8})\left(\frac{z_1 z_{64-\nu^2}}{z_{8-\nu}z_{8+\nu}}\right)^{1/2}.$$

Replacing each modulus by its complementary modulus, that is, taking the reciprocal of this modular equation, we obtain the second part of (ii).

To prove the first part of (ii), return to (21.2) and replace q by $q^{1/(8-\nu)}$ to find that

$$\tfrac{1}{2}\{\varphi(q^{(8+\nu)/(8-\nu)})\varphi(q) + \varphi(-q^{(8+\nu)/(8-\nu)})\varphi(-q)\} - 2q^{4/(8-\nu)}\psi(q^{2(8+\nu)/(8-\nu)})\psi(q^2)$$
$$= \varphi(-q^{4(8+\nu)})\varphi(-q^{4/(8-\nu)}) \pm 2(q^{8+\nu+1/(8-\nu)})^{1/4}\psi(-q^{2(8+\nu)})\psi(-q^{2/(8-\nu)}),$$

where the plus sign is correct when $\nu = 1$ and the minus sign is correct when $\nu = 3$ or 5. We now equate rational parts on both sides. In the case $\nu = 5$, we use Corollaries (i), (ii) in Section 31. For $\nu = 3$, we appeal to Entries 10(i), (ii) of Chapter 19, and for $\nu = 1$, we employ Entries 17(iii), (iv) of Chapter 19. Omitting the straightforward details, we conclude that, in all three cases,

$$\tfrac{1}{2}\{\varphi(Q)\varphi(q) + \varphi(-Q)\varphi(-q)\} - 2(Qq)^{1/4}\psi(Q^2)\psi(q^2)$$
$$= \varphi(-q^{32+4\nu})\varphi(-q^{32-4\nu}) + 2q^4\psi(-q^{16+2\nu})\psi(-q^{16-2\nu}).$$

Replacing q by $\sqrt{q}$, we deduce that

$$\tfrac{1}{2}\{\varphi(Q^{1/2})\varphi(q^{1/2}) + \varphi(-Q^{1/2})\varphi(-q^{1/2})\} - 2(Qq)^{1/8}\psi(Q)\psi(q)$$
$$= \varphi(-q^{16+2\nu})\varphi(-q^{16-2\nu}) + 2q^2\psi(-q^{8+\nu})\psi(-q^{8-\nu}).$$

By Entries 10(iii), (vi), (vii) and 11(i), (ii) in Chapter 17, the translation of this is the modular equation

$$\tfrac{1}{2}(1+\sqrt{\alpha})^{1/2}(1+\sqrt{\delta})^{1/2}+\tfrac{1}{2}(1-\sqrt{\alpha})^{1/2}(1-\sqrt{\delta})^{1/2}$$
$$=(\alpha\delta)^{1/8}+(\{(1-\beta)(1-\gamma)\}^{1/8}+\{\beta\gamma(1-\beta)(1-\gamma)\}^{1/8})\left(\frac{z_{8-v}z_{8+v}}{z_1 z_{64-v^2}}\right)^{1/2}.$$

Using an analogue of (20.6) and replacing each modulus by its complementary modulus, we obtain the first part of (ii).

Entry 22. *Each of the following modular equations is of degree* 31.

(i) *Let*

$$\Omega(\alpha,\beta)=(\alpha\beta)^{1/32}(\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\}^{1/8}\{1+(\alpha\beta)^{1/4}$$
$$+\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\}^{1/2}\}^{1/2}$$
$$+\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\}^{1/8}\{1+(\alpha\beta)^{1/4}$$
$$+\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\}^{1/2}\}^{1/2}).$$

Then

$$\Omega(\alpha,\beta)+\Omega(1-\beta,1-\alpha)=8^{1/4}.$$

(ii) $$1+(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}$$
$$-2((\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}+\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8})$$
$$=2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/16}(1+(\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8})^{1/2}.$$

(iii) $$1+(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}-(\tfrac{1}{2}\{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\})^{1/2}$$
$$=(\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}+\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}.$$

The statement of (i) in the second notebook (p. 252) is somewhat obscure. The first and third of these modular equations of the thirty-first degree are new. Entry 22(ii) is due to Russell [1]. See also Greenhill's book [1, p. 327]. The only other modular equation of degree 31 which is comparable to Ramanujan's in simplicity is due to Schröter [1], [2], [3], who showed that

$$(\alpha\beta)^{1/16}\left(\left(\frac{1+\sqrt{\alpha}}{2}\right)^{1/4}\left(\frac{1+\sqrt{\beta}}{2}\right)^{1/4}+\left(\frac{1-\sqrt{\alpha}}{2}\right)^{1/4}\left(\frac{1-\sqrt{\beta}}{2}\right)^{1/4}\right)-(\alpha\beta)^{1/8}$$
$$=\{(1-\alpha)(1-\beta)\}^{1/16}\left(\left(\frac{1+\sqrt{1-\alpha}}{2}\right)^{1/4}\left(\frac{1+\sqrt{1-\beta}}{2}\right)^{1/4}\right.$$
$$\left.+\left(\frac{1-\sqrt{1-\alpha}}{2}\right)^{1/4}\left(\frac{1-\sqrt{1-\beta}}{2}\right)^{1/4}\right)-\{(1-\alpha)(1-\beta)\}^{1/8}.$$

This can be proved with the aid of (36.8) in Chapter 16, but we do not give any details. Schröter [2] further remarks that "une autre forme de cette équation modulaire, plus analogue aux formes précédents, mais plus compliqueés." Of course, it is pure speculation to conjecture that Schröter had in mind one of the three modular equations given above.

Considerably more complicated modular equations of degree 31 were found by Weber [1], Berry [1], and Hanna [1].

It will be convenient to prove Ramanujan's modular equations in the reverse order in which they are stated.

All references in the proof of Entry 22 are to Chapter 16, unless stated otherwise.

PROOF OF (iii). Put $Q = q^{31}$. We first apply (36.14) with $\mu = 16$ and $\omega = 1$. The resulting term of index 8 is

$$q^{256}f(q^{-8}, q^{24})f(1, Q^{64}) = 2q^{248}\varphi(q^8)\psi(Q^{64}),$$

by Entries 18(ii), (iv). For $1 \le n \le 7$, we apply Entry 18(iv) to show that the terms of index n and $16 - n$ are equal. After combining terms with the aid of Entry 18(iv), we employ Entries 25(i), (ii) and (7.1) of this chapter three times. Lastly, we invoke Corollary (ii) in Section 31. Accordingly,

$$\begin{aligned}
&\tfrac{1}{2}\{\varphi(q)\varphi(Q) + \varphi(-q)\varphi(-Q)\} + 2q^8\psi(q^2)\psi(Q^2)\\
&= \sum_{n=0}^{15} q^{4n^2}f(q^{8-2n}, q^{8+2n})f(Q^{32-4n}, Q^{32+4n})\\
&= \varphi(q^8)\varphi(Q^{32}) + 2q^{248}\varphi(q^8)\psi(Q^{64})\\
&\quad + 2\sum_{n=1}^{7} q^{4n^2}f(q^{8-2n}, q^{8+2n})f(Q^{32-4n}, Q^{32+4n})\\
&= \tfrac{1}{2}\varphi(q^8)\{\varphi(Q^8) + \varphi(-Q^8) + \varphi(Q^8) - \varphi(-Q^8)\} + 4q^{64}\psi(q^{16})\psi(Q^{16})\\
&\quad + 2q^4f(q^6, q^{10})\{f(Q^{28}, Q^{36}) + Q^6f(Q^4, Q^{60})\}\\
&\quad + 2q^{16}f(q^4, q^{12})\{f(Q^{24}, Q^{40}) + Q^4f(Q^8, Q^{56})\}\\
&\quad + 2q^{36}f(q^2, q^{14})\{f(Q^{20}, Q^{44}) + Q^2f(Q^{12}, Q^{52})\}\\
&= \varphi(q^8)\varphi(Q^8) + 4q^{64}\psi(q^{16})\psi(Q^{16}) + 2q^4f(q^6, q^{10})f(Q^6, Q^{10})\\
&\quad + 2q^{16}f(q^4, q^{12})f(Q^4, Q^{12}) + 2q^{36}f(q^2, q^{14})f(Q^2, Q^{14})\\
&= \tfrac{1}{4}\{\varphi(q^2) + \varphi(-q^2)\}\{\varphi(Q^2) + \varphi(-Q^2)\}\\
&\quad + \tfrac{1}{4}\{\varphi(q^2) - \varphi(-q^2)\}\{\varphi(Q^2) - \varphi(-Q^2)\}\\
&\quad + 2q^{16}\psi(q^4)\psi(Q^4) + \tfrac{1}{2}q^4\{\psi(q) + \psi(-q)\}\{\psi(Q) + \psi(-Q)\}\\
&\quad + \tfrac{1}{2}q^4\{\psi(q) - \psi(-q)\}\{\psi(Q) - \psi(-Q)\}\\
&= \tfrac{1}{2}\{\varphi(q^2)\varphi(Q^2) + \varphi(-q^2)\varphi(-Q^2)\} + 2q^{16}\psi(q^4)\psi(Q^4)\\
&\quad + q^4\{\psi(q)\psi(Q) + \psi(-q)\psi(-Q)\}.
\end{aligned}$$

Using Entries 10(i)–(iv) and 11(i)–(iv) in Chapter 17 to translate the foregoing formula, we find that

$$1 + \{(1-\alpha)(1-\beta)\}^{1/4} + (\alpha\beta)^{1/4} = \tfrac{1}{2}(1+\sqrt{1-\alpha})^{1/2}(1+\sqrt{1-\beta})^{1/2} + \tfrac{1}{2}(1-\sqrt{1-\alpha})^{1/2}(1-\sqrt{1-\beta})^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/8} + (\alpha\beta)^{1/8} + \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}.$$

Using (20.6) to simplify, we complete the proof of (iii).

PROOF OF (ii). Let

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = t \quad \text{and} \quad \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} = u.$$

Then (iii) assumes the form

$$1 - t + t^2 - 3u = \{\tfrac{1}{2}(1 + t^4 - 4ut^2 + 2u^2)\}^{1/2}.$$

Squaring both sides, we arrive at

$$(1-t)^4 - 4u(3 - 3t + 2t^2) + 16u^2 = 0,$$

or

$$\{(1-t)^2 - 4u\}^2 = 4u(1+t). \tag{22.1}$$

Since $\beta = O(\alpha^{31})$ as α tends to 0, we find that

$$1 - t - 2\sqrt{u} \sim 1 - (1 - \tfrac{1}{8}\alpha) + O(\alpha^2) \sim \tfrac{1}{8}\alpha,$$

as α tends to 0. Thus, when α is small and positive, $(1-t)^2 > 4u$. Hence, taking the square root in (22.1), we find that

$$(1-t)^2 - 4u = 2\sqrt{u(1+t)}.$$

Rephrasing this equality in terms of α and β, we deduce that

$$1 - 2((\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8}) + ((\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8})^2 = 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/16}(1 + (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8})^{1/2},$$

which readily is seen to be equivalent to (ii).

PROOF OF (i). The proof of (i) is a bit more difficult than those for (ii) and (iii).

As above, we set $Q = q^{31}$. First, we apply (36.3) with $\mu = 16$ and $\nu = 15$. For the term of index 8, we apply Entry 18(iv), with $n = 7$ there, and Entry 18(ii) to deduce that

$$q^{2048}f(q^{512}, q^{-448}) = 2q^{256}\psi(q^{64}).$$

We then show, with the aid of Entry 18(iv), that the terms with index n and $16 - n$, $1 \le n \le 7$, are equal. Employ next Entries 25(i), (ii). We now simplify the terms somewhat by using Entry 18(iv) to show that

$$q^{32n^2}f(q^{32+60n}, q^{32-60n}) = q^{4n^2}f(q^{32+4n}, q^{32-4n}).$$

Next, we combine the terms of index n with those of index $8 - n$, $1 \le n \le 3$, and then make several applications of Entries 30(ii), (iii). Accordingly,

$$
\begin{aligned}
&\tfrac{1}{2}\{\varphi(q)\varphi(Q)+\varphi(-q)\varphi(-Q)\}\\
&=\sum_{n=0}^{15} q^{32n^2} f(q^{32+60n}, q^{32-60n}) f(Q^{32+4n}, Q^{32-4n})\\
&=\varphi(q^{32})\varphi(Q^{32})+4q^{256}\psi(q^{64})\psi(Q^{64})\\
&\quad+2\sum_{n=1}^{7} q^{32n^2} f(q^{32+60n}, q^{32-60n}) f(Q^{32+4n}, Q^{32-4n})\\
&=\tfrac{1}{2}\{\varphi(q^8)\varphi(Q^8)+\varphi(-q^8)\varphi(-Q^8)\}\\
&\quad+2\sum_{n=1}^{7} q^{4n^2} f(q^{32+4n}, q^{32-4n}) f(Q^{32+4n}, Q^{32-4n})\\
&=\tfrac{1}{2}\{\varphi(q^8)\varphi(Q^8)+\varphi(-q^8)\varphi(-Q^8)\}+2q^{64}\psi(q^{16})\psi(Q^{16})\\
&\quad+2\sum_{n=1}^{3} q^{4n^2}\{f(q^{32+4n}, q^{32-4n}) f(Q^{32+4n}, Q^{32-4n})\\
&\quad+q^{64(4-n)} f(q^{64-n}, q^{4n}) f(Q^{64-4n}, Q^{4n})\}\\
&=\tfrac{1}{2}\{\varphi(q^8)\varphi(Q^8)+\varphi(-q^8)\varphi(-Q^8)\}+2q^{64}\psi(q^{16})\psi(Q^{16})\\
&\quad+\frac{1}{2}\sum_{n=1}^{3} q^{4n^2}(\{f(q^{2(4-n)}, q^{2(4+n)})+f(-q^{2(4-n)}, -q^{2(4+n)})\}\\
&\quad\times\{f(Q^{2(4-n)}, Q^{2(4+n)})+f(-Q^{2(4-n)}, -Q^{2(4+n)})\}\\
&\quad+\{f(q^{2(4-n)}, q^{2(4+n)})-f(-q^{2(4-n)}, -q^{2(4+n)})\}\\
&\quad\times\{f(Q^{2(4-n)}, Q^{2(4+n)})-f(-Q^{2(4-n)}, -Q^{2(4+n)})\})\\
&=\tfrac{1}{2}\{\varphi(q^8)\varphi(Q^8)+\varphi(-q^8)\varphi(-Q^8)\}+2q^{64}\psi(q^{16})\psi(Q^{16})\\
&\quad+q^4\{f(q^6, q^{10})f(Q^6, Q^{10})+f(-q^6, -q^{10})f(-Q^6, -Q^{10})\}\\
&\quad+q^{16}\{\psi(q^4)\psi(Q^4)+\psi(-q^4)\psi(-Q^4)\}\\
&\quad+q^{36}\{f(q^2, q^{14})f(Q^2, Q^{14})+f(-q^2, -q^{14})f(-Q^2, -Q^{14})\}\\
&=\tfrac{1}{2}\{\varphi(q^8)\varphi(Q^8)+\varphi(-q^8)\varphi(-Q^8)\}+2q^{64}\psi(q^{16})\psi(Q^{16})\\
&\quad+q^{16}\{\psi(q^4)\psi(Q^4)+\psi(-q^4)\psi(-Q^4)\}\\
&\quad+\tfrac{1}{4}q^4\{f(q, q^3)+f(-q, -q^3)\}\{f(Q, Q^3)+f(-Q, -Q^3)\}\\
&\quad+\tfrac{1}{4}q^4\{f(iq, -iq^3)+f(-iq, iq^3)\}\{f(-iQ, iQ^3)+f(iQ, -iQ^3)\}\\
&\quad+\tfrac{1}{4}q^4\{f(q, q^3)-f(-q, -q^3)\}\{f(Q, Q^3)-f(-Q, -Q^3)\}\\
&\quad+\tfrac{1}{4}q^4\{f(iq, -iq^3)-f(-iq, iq^3)\}\{f(-iQ, iQ^3)-f(iQ, -iQ^3)\}\\
&=\tfrac{1}{2}\{\varphi(q^8)\varphi(Q^8)+\varphi(-q^8)\varphi(-Q^8)\}+2q^{64}\psi(q^{16})\psi(Q^{16})\\
&\quad+q^{16}\{\psi(q^4)\psi(Q^4)+\psi(-q^4)\psi(-Q^4)\}\\
&\quad+\tfrac{1}{2}q^4\{f(q, q^3)f(Q, Q^3)+f(-q, -q^3)f(-Q, -Q^3)\\
&\quad+f(iq, -iq^3)f(-iQ, iQ^3)+f(-iq, iq^3)f(iQ, -iQ^3)\}.
\end{aligned}
$$

Replacing q by $q^{1/4}$, we find that

$$\begin{aligned}
&q\{\psi(q^{1/4})\psi(Q^{1/4}) + \psi(-q^{1/4})\psi(-Q^{1/4}) \\
&\qquad + \psi(iq^{1/4})\psi(-iQ^{1/4}) + \psi(-iq^{1/4})\psi(iQ^{1/4})\} \\
&\quad = \varphi(q^{1/4})\varphi(Q^{1/4}) + \varphi(-q^{1/4})\varphi(-Q^{1/4}) - \varphi(q^2)\varphi(Q^2) - \varphi(-q^2)\varphi(-Q^2) \\
&\quad - 4q^{16}\psi(q^4)\psi(Q^4) - 2q^4\{\psi(q)\psi(Q) + \psi(-q)\psi(-Q)\}.
\end{aligned} \tag{22.2}$$

By Entries 25(iv), (i), (ii),

$$\begin{aligned}
\psi^2(iq^{1/4}) &= \varphi(iq^{1/4})\psi(-q^{1/2}) \\
&= (\tfrac{1}{2}\{\varphi(iq^{1/4}) + \varphi(-iq^{1/4})\} \\
&\quad + \tfrac{1}{2}\{\varphi(iq^{1/4}) - \varphi(-iq^{1/4})\})\psi(-q^{1/2}) \\
&= \{\varphi(q) + 2iq^{1/4}\psi(q^2)\}\psi(-q^{1/2}).
\end{aligned}$$

Using this in (22.2), we find that the left side of (22.2) equals

$$\begin{aligned}
&q\{\psi(q^{1/4})\psi(Q^{1/4}) + \psi(-q^{1/4})\psi(-Q^{1/4}) \\
&\qquad + \{\psi(-q^{1/2})\psi(-Q^{1/2})\}^{1/2}(\{\varphi(q) + 2iq^{1/4}\psi(q^2)\}^{1/2} \\
&\qquad \times \{\varphi(Q) - 2iQ^{1/4}\psi(Q^2)\}^{1/2} \\
&\qquad + \{\varphi(q) - 2iq^{1/4}\psi(q^2)\}^{1/2}\{\varphi(Q) + 2iQ^{1/4}\psi(Q^2)\}^{1/2})\} \\
&\quad = (z_1z_{31})^{1/2}(\alpha\beta)^{1/32}\left(\frac{1+\sqrt{\alpha}}{2}\frac{1+\sqrt{\beta}}{2}\right)^{1/8}\{(1+\alpha^{1/4})^{1/2}(1+\beta^{1/4})^{1/2} \\
&\qquad + (1-\alpha^{1/4})^{1/2}(1-\beta^{1/4})^{1/2}\} \\
&\qquad + (z_1z_{31})^{1/2}(\alpha\beta)^{1/32}\left(\frac{1-\sqrt{\alpha}}{2}\frac{1-\sqrt{\beta}}{2}\right)^{1/8} \\
&\qquad \times \{(1+i\alpha^{1/4})^{1/2}(1-i\beta^{1/4})^{1/2} + (1-i\alpha^{1/4})^{1/2}(1+i\beta^{1/4})^{1/2}\} \\
&\quad = (z_1z_{31})^{1/2}(\alpha\beta)^{1/32}\left(\frac{1+\sqrt{\alpha}}{2}\frac{1+\sqrt{\beta}}{2}\right)^{1/8} \\
&\qquad \times \{2 + 2(\alpha\beta)^{1/4} + 2\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\}^{1/2}\}^{1/2} \\
&\qquad + (z_1z_{31})^{1/2}(\alpha\beta)^{1/32}\left(\frac{1-\sqrt{\alpha}}{2}\frac{1-\sqrt{\beta}}{2}\right)^{1/8} \\
&\qquad \times \{2 + 2(\alpha\beta)^{1/4} + 2\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\}^{1/2}\}^{1/2} \\
&\quad = 2^{1/4}(z_1z_{31})^{1/2}\Omega(\alpha, \beta),
\end{aligned} \tag{22.3}$$

where we have employed Entries 11(iii), (vii)–(ix) in Chapter 17 and analogues of (20.6) above.

Using (22.3) and employing Entries 10(iii), (iv), (viii), (ix) and 11(i), (ii), (iv) of Chapter 17 in (22.2), we deduce the modular equation

$$\begin{aligned}2^{1/4}\Omega(\alpha, \beta) &= (1 + \alpha^{1/4})(1 + \beta^{1/4}) + (1 - \alpha^{1/4})(1 - \beta^{1/4})\\&\quad - \tfrac{1}{2}(1 + \sqrt{1-\alpha})^{1/2}(1 + \sqrt{1-\beta})^{1/2} - \{(1-\alpha)(1-\beta)\}^{1/8}\\&\quad - \tfrac{1}{2}(1 - \sqrt{1-\alpha})^{1/2}(1 - \sqrt{1-\beta})^{1/2} - (\alpha\beta)^{1/8}\\&\quad - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\\&= 2 + 2(\alpha\beta)^{1/4} - (\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}\\&\quad - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} - (\tfrac{1}{2}\{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\})^{1/2}.\end{aligned}$$

Next, take the reciprocal of the modular equation above and add it to the original equation to deduce that

$$\begin{aligned}&2^{1/4}\Omega(\alpha, \beta) + 2^{1/4}\Omega(1-\beta, 1-\alpha)\\&\quad = 4 + 2(\alpha\beta)^{1/4} + 2\{(1-\alpha)(1-\beta)\}^{1/4} - 2(\alpha\beta)^{1/8} - 2\{(1-\alpha)(1-\beta)\}^{1/8}\\&\qquad - 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} - 2(\tfrac{1}{2}\{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\})^{1/2}\\&\quad = 2,\end{aligned}$$

by (iii). Thus, (i) follows immediately.

Entry 23.

(i) *If β is of degree* 47, *then*

$$\begin{aligned}2(\tfrac{1}{2}\{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\})^{1/2} &= 1 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\\&\quad + 4^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}(1 + (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8}).\end{aligned}$$

(ii) *If β is of degree* 71, *then*

$$\begin{aligned}&1 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} - (\tfrac{1}{2}\{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\})^{1/2}\\&\quad = (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\\&\qquad + 4^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}(1 - (\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}).\end{aligned}$$

These two modular equations are the climax of Ramanujan's modular equations involving two moduli only.

The first modular equation of degree 47 was offered without proof and with two sign errors by Hurwitz in a paper by Klein [2]. Russell [1] corrected and proved the result shortly thereafter. More complicated modular equations of degree 47 were established by Fiedler [1] and Hanna [1]. Fiedler [1] also constructed a modular equation of degree 71. Simpler forms of Fiedler's equation were obtained soon thereafter by Weber [1] and Russell [2].

Before embarking on a proof of Entry 23, we show that Russell's modular equations can easily be derived from those of Ramanujan.

Set

$$P = 1 + (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} \quad \text{and} \quad R = \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}.$$

Then Russell's modular equation of degree 47 assumes the form

$$(P-2)^2 - P(4R)^{1/3} - 2(4R)^{2/3} - 4R = 0, \tag{23.1}$$

while Ramanujan's equation takes the shape

$$2(\tfrac{1}{2}\{(P-1)^4 + 1 - 4R(P-1)^2 + 2R^2\})^{1/2} = (P-1)^2 + 1 + P(4R)^{1/3} - 2R.$$

Squaring Ramanujan's equation and rearranging the terms, we derive the equality

$$P\{P-(4R)^{1/3}\}\{(P-2)^2 - P(4R)^{1/3} - 2(4R)^{2/3} - 4R\} = 0.$$

By examining each of the first two factors above as α tends to 0, we see that they cannot vanish identically. Thus, the third factor must vanish; that is, Russell's equation (23.1) holds.

In the case of degree 71, set

$$P = (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} - 1 \quad \text{and} \quad R = -\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}.$$

Then Russell's equation takes the form

$$P^3 - 4(4R)^{1/3}(P^2 + P + 1) + 2P(4R)^{2/3} - 4R - (4R)^{4/3} = 0, \tag{23.2}$$

while Ramanujan's equation of degree 71 assumes the shape

$$P^2 + P + 1 - P(4R)^{1/3} + R$$
$$= (\tfrac{1}{2}\{(P+1)^4 + 1 + 4R(P+1)^2 + 2R^2\})^{1/2}.$$

Squaring and rearranging Ramanujan's equation, we arrive at

$$\tfrac{1}{2}P\{P^3 - 4(4R)^{1/3}(P^2 + P + 1) + 2P(4R)^{2/3} - 4R - (4R)^{4/3}\} = 0. \tag{23.3}$$

Now as α tends to 0,

$$P \sim O(\alpha^9) + 1 - \tfrac{1}{8}\alpha + \cdots - 1.$$

Thus, P cannot identically vanish. Hence, the second factor in (23.3) is identically equal to 0; that is, (23.2) is valid.

It is interesting to note that, by Entry 19(i) in Chapter 19, P does vanish identically in the case that β is of degree 7.

All references in the proofs of (i) and (ii) are from Chapter 16, unless indicated otherwise.

Proof of (i). Our proof rests on two representations for 47, namely,

$$47 = 3\cdot 2^4 - 1^2 = 3\cdot 2^5 - 7^2.$$

In (36.14), let $\mu = 48$ and $\omega = 7$ and set $Q = q^{47}$. After combining the terms with index n and $n + 24$, $0 \le n \le 23$, by an application of Entry 18(iv), and using (7.1) in this chapter, we deduce that

$$\begin{aligned}
&\tfrac{1}{2}\{\varphi(Q)\varphi(q)+\varphi(-Q)\varphi(-q)\}+2q^{12}\psi(Q^2)\psi(q^2)\\
&\quad=\sum_{n=0}^{47} q^{4n^2}f(Q^{96+4n},Q^{96-4n})f(q^{24+14n},q^{24-14n})\\
&\quad=\sum_{n=0}^{23} q^{4n^2}\{f(Q^{96+4n},Q^{96-4n})+Q^{24+2n}f(Q^{192+4n},Q^{-4n})\}\\
&\qquad\times f(q^{24+14n},q^{24-14n})\\
&\quad=\sum_{n=0}^{23} q^{4n^2}f(Q^{24+2n},Q^{24-2n})f(q^{24+14n},q^{24-14n}). \qquad (23.4)
\end{aligned}$$

Second, in (36.12), set $\mu=24$, $\omega=1$, and $Q=q^{47}$. Replacing q by $\sqrt{q}$, we find that

$$\begin{aligned}
&\tfrac{1}{2}\{\varphi(Q^{1/2})\varphi(q^{1/2})+\varphi(-Q^{1/2})\varphi(-q^{1/2})\}\\
&\quad=\sum_{n=0}^{23} q^{2n^2}f(Q^{24+2n},Q^{24-2n})f(q^{24+2n},q^{24-2n}).
\end{aligned}$$

Subtracting (23.4) from the last equality, we deduce that

$$\begin{aligned}
S&:=\tfrac{1}{2}\{\varphi(Q^{1/2})\varphi(q^{1/2})+\varphi(-Q^{1/2})\varphi(-q^{1/2})\}\\
&\quad-\tfrac{1}{2}\{\varphi(Q)\varphi(q)+\varphi(-Q)\varphi(-q)\}-2q^{12}\psi(Q^2)\psi(q^2)\\
&=\sum_{n=1}^{23} q^{2n^2}f(Q^{24+2n},Q^{24-2n})\{f(q^{24+2n},q^{24-2n})-q^{2n^2}f(q^{24+14n},q^{24-14n})\}\\
&=2\sum_{n=1}^{11} q^{2n^2}f(Q^{24+2n},Q^{24-2n})\{f(q^{24+2n},q^{24-2n})\\
&\quad-q^{2n^2}f(q^{24+14n},q^{24-14n})\}. \qquad (23.5)
\end{aligned}$$

In the analysis above, we used the fact that the term with index 12 vanishes and that the terms of index n and $24-n$, $1\le n\le 11$, are equal. These deductions are easily made with the help of Entry 18(iv). Next, by repeated applications of Entry 18(iv), the terms with indices $n=3, 4, 6, 8, 9$ likewise vanish. Further transforming via Entry 18(iv), we find that

$$\begin{aligned}
S&=2q^2f(Q^{22},Q^{26})\{f(q^{22},q^{26})-q^2f(q^{10},q^{38})\}\\
&\quad+2q^8f(Q^{20},Q^{28})\{f(q^{20},q^{28})-q^4f(q^4,q^{44})\}\\
&\quad+2q^{50}f(Q^{14},Q^{34})\{f(q^{14},q^{34})-q^4f(q^2,q^{46})\}\\
&\quad+2q^{98}f(Q^{10},Q^{38})\{f(q^{10},q^{38})-q^{-2}f(q^{22},q^{26})\}\\
&\quad+2q^{200}f(Q^4,Q^{44})\{f(q^4,q^{44})-q^{-4}f(q^{20},q^{28})\}\\
&\quad+2q^{242}f(Q^2,Q^{46})\{f(q^2,q^{46})-q^{-4}f(q^{14},q^{34})\}\\
&=2q^2\{f(q^{22},q^{26})-q^2f(q^{10},q^{38})\}\{f(Q^{22},Q^{26})-Q^2f(Q^{10},Q^{38})\}\\
&\quad+2q^{50}\{f(q^{14},q^{34})-q^4f(q^2,q^{46})\}\{f(Q^{14},Q^{34})-Q^4f(Q^2,Q^{46})\}\\
&\quad+2q^8\{f(q^{20},q^{28})-q^4f(q^4,q^{44})\}\{f(Q^{20},Q^{28})-Q^4f(Q^4,Q^{44})\}.
\end{aligned}$$

By Entries 30(ii), (iii) and (7.1) in this chapter,

$$\begin{aligned}
&2\{f(q^{22}, q^{26}) - q^2 f(q^{10}, q^{38})\} \\
&\quad = f(q^5, q^7) + f(-q^5, -q^7) - qf(q, q^{11}) + qf(-q, -q^{11}) \\
&\quad = f(-q, -q^2) + f(q, -q^2) \\
&\quad = f(-q) + f(q)
\end{aligned} \tag{23.6}$$

and

$$\begin{aligned}
&2q\{f(q^{14}, q^{34}) - q^4 f(q^2, q^{46})\} \\
&\quad = qf(q, q^{11}) + qf(-q, -q^{11}) - f(q^5, q^7) + f(-q^5, -q^7) \\
&\quad = f(q) - f(-q).
\end{aligned} \tag{23.7}$$

Consequently,

$$\begin{aligned}
S &= \tfrac{1}{2}q^2\{f(-q) + f(q)\}\{f(-Q) + f(Q)\} \\
&\quad + \tfrac{1}{2}q^2\{f(q) - f(-q)\}\{f(Q) - f(-Q)\} + 2q^8 f(-q^4) f(-Q^4) \\
&= q^2 f(Q) f(q) + q^2 f(-Q) f(-q) + 2q^8 f(-Q^4) f(-q^4).
\end{aligned}$$

Referring back to (23.5) for the definition of S and transcribing the equality above via Entries 10(i), (ii), (vi), (vii), 11(iii), and 12(i), (ii), (iv) in Chapter 17, we find that

$$\begin{aligned}
&\tfrac{1}{2}\{(1 + \sqrt{\alpha})^{1/2}(1 + \sqrt{\beta})^{1/2} + (1 - \sqrt{\alpha})^{1/2}(1 - \sqrt{\beta})^{1/2}\} \\
&\qquad - \tfrac{1}{2}(1 + \{(1 - \alpha)(1 - \beta)\}^{1/4}) - \tfrac{1}{2}(\alpha\beta)^{1/4} \\
&\quad = 2^{-1/3}\{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/24} + 2^{-1/3}\{(1 - \alpha)(1 - \beta)\}^{1/6}(\alpha\beta)^{1/24} \\
&\qquad + 2^{-1/3}\{(1 - \alpha)(1 - \beta)\}^{1/24}(\alpha\beta)^{1/6}.
\end{aligned}$$

Simplifying by an obvious analogue of (20.6) above and rearranging terms, we complete the proof.

PROOF OF (ii). Our proof depends on two representations for 71, namely,

$$71 = 3 \cdot 2^5 - 5^2 = 3 \cdot 2^6 - 11^2.$$

In (36.14), let $\mu = 48$, $\omega = 5$, and $Q = q^{71}$. Combine the terms with indices n and $n + 24$, $0 \le n \le 23$, with the aid of Entry 18(iv). Then using (7.1) in this chapter, we deduce that

$$\begin{aligned}
&\tfrac{1}{2}\{\varphi(Q)\varphi(q) + \varphi(-Q)\varphi(-q)\} + 2q^{18}\psi(Q^2)\psi(q^2) \\
&= \sum_{n=0}^{47} q^{4n^2} f(Q^{96+4n}, Q^{96-4n}) f(q^{24+10n}, q^{24-10n}) \\
&= \sum_{n=0}^{23} q^{4n^2}\{f(Q^{96+4n}, Q^{96-4n}) + Q^{24+2n} f(Q^{192+4n}, Q^{-4n})\} f(q^{24+10n}, q^{24-10n}) \\
&= \sum_{n=0}^{23} q^{4n^2} f(Q^{24+2n}, Q^{24-2n}) f(q^{24+10n}, q^{24-10n}).
\end{aligned} \tag{23.8}$$

Apply (36.14) once again, but now with $\mu = 96$, $\omega = 11$, and q replaced by q^2. Proceeding as above, we find that

$$\begin{aligned}
&\tfrac{1}{2}\{\varphi(Q^2)\varphi(q^2) + \varphi(-Q^2)\varphi(-q^2)\} + 2q^{36}\psi(Q^4)\psi(q^4) \\
&\quad = \sum_{n=0}^{95} q^{8n^2} f(Q^{384+8n}, Q^{384-8n}) f(q^{96+44n}, q^{96-44n}) \\
&\quad = \sum_{n=0}^{47} q^{8n^2}\{f(Q^{384+8n}, Q^{384-8n}) + Q^{96+4n} f(Q^{768+8n}, Q^{-8n})\} \\
&\qquad \times f(q^{96+44n}, q^{96-44n}) \\
&\quad = \sum_{n=0}^{47} q^{8n^2} f(Q^{96+4n}, Q^{96-4n}) f(q^{96+44n}, q^{96-44n}).
\end{aligned} \tag{23.9}$$

Third, let $\mu = 96$ and $\omega = 11$ in (36.13) to infer that

$$2q^{18}\psi(Q^2)\psi(q^2) = \sum_{n=0}^{95} q^{4n^2-22n+48} f(Q^{192+4n}, Q^{192-4n}) f(q^{384-44n}, q^{44n}).$$

Replace q^2 by q and $-q$, in turn. Add the resulting two equalities to find that

$$\begin{aligned}
&q^9\psi(Q)\psi(q) - q^9\psi(-Q)\psi(-q) \\
&\quad = \sum_{n=0}^{47} q^{8n^2-22n+24} f(Q^{96+4n}, Q^{96-4n}) f(q^{192-44n}, q^{44n}).
\end{aligned} \tag{23.10}$$

Adding (23.9) and (23.10), employing (7.1) of this chapter, and combining the terms with indices n and $n + 24$, $0 \le n \le 23$, we deduce that

$$\begin{aligned}
&\tfrac{1}{2}\{\varphi(Q^2)\varphi(q^2) + \varphi(-Q^2)\varphi(-q^2)\} + 2q^{36}\psi(Q^4)\psi(q^4) \\
&\qquad + q^9\psi(Q)\psi(q) - q^9\psi(-Q)\psi(-q) \\
&\quad = \sum_{n=0}^{47} q^{8n^2} f(Q^{96+4n}, Q^{96-4n})\{f(q^{96+44n}, q^{96-44n}) \\
&\qquad + q^{24-22n} f(q^{192-44n}, q^{44n})\} \\
&\quad = \sum_{n=0}^{47} q^{8n^2} f(Q^{96+4n}, Q^{96-4n}) f(q^{24+22n}, q^{24-22n}) \\
&\quad = \sum_{n=0}^{23} q^{8n^2}\{f(Q^{96+4n}, Q^{96-4n}) + Q^{24+2n} f(Q^{192+4n}, Q^{-4n})\} \\
&\qquad \times f(q^{24+22n}, q^{24-22n}) \\
&\quad = \sum_{n=0}^{23} q^{8n^2} f(Q^{24+2n}, Q^{24-2n}) f(q^{24+22n}, q^{24-22n}),
\end{aligned}$$

by a calculation made in (23.4) above.

By subtracting the last result from (23.8), we see that

$$\begin{aligned}
S &:= \tfrac{1}{2}\{\varphi(Q)\varphi(q) + \varphi(-Q)\varphi(-q)\} + 2q^{18}\psi(Q^2)\psi(q^2) \\
&\quad - \tfrac{1}{2}\{\varphi(Q^2)\varphi(q^2) + \varphi(-Q^2)\varphi(-q^2)\} - 2q^{36}\psi(Q^4)\psi(q^4) \\
&\quad - q^9\psi(Q)\psi(q) + q^9\psi(-Q)\psi(-q)
\end{aligned}$$

$$= \sum_{n=1}^{23} q^{4n^2} f(Q^{24+2n}, Q^{24-2n}) \{ f(q^{24+10n}, q^{24-10n}) - q^{4n^2} f(q^{24+22n}, q^{24-22n}) \}$$

$$= 2 \sum_{n=1}^{11} q^{4n^2} f(Q^{24+2n}, Q^{24-2n}) \{ f(q^{24+10n}, q^{24-10n}) - q^{4n^2} f(q^{24+22n}, q^{24-22n}) \},$$

where we have used Entry 18(iv) to show that the term with $n = 12$ vanishes and that the terms with indices n and $24 - n$, $1 \le n \le 11$, are equal. With several applications of Entry 18(iv), it is easy to show that the terms with indices $n = 3, 4, 6, 8, 9$ vanish. Thus, by further applications of Entry 18(iv) and the same calculations that we made in the proof of (i), we deduce that

$$\begin{aligned} S = {} & 2q^4 f(Q^{22}, Q^{26}) \{ f(q^{14}, q^{34}) - q^4 f(q^2, q^{46}) \} \\ & + 2q^{16} f(Q^{20}, Q^{28}) \{ f(q^4, q^{44}) - q^{-4} f(q^{20}, q^{28}) \} \\ & + 2q^{74} f(Q^{14}, Q^{34}) \{ f(q^{22}, q^{26}) - q^2 f(q^{10}, q^{38}) \} \\ & + 2q^{146} f(Q^{10}, Q^{38}) \{ q^4 f(q^2, q^{46}) - f(q^{14}, q^{34}) \} \\ & + 2q^{296} f(Q^4, Q^{44}) \{ f(q^{20}, q^{28}) - q^4 f(q^4, q^{44}) \} \\ & + 2q^{358} f(Q^2, Q^{46}) \{ q^2 f(q^{10}, q^{38}) - f(q^{22}, q^{26}) \} \\ = {} & 2q^4 \{ f(Q^{22}, Q^{26}) - Q^2 f(Q^{10}, Q^{38}) \} \{ f(q^{14}, q^{34}) - q^4 f(q^2, q^{46}) \} \\ & + 2q^{74} \{ f(Q^{14}, Q^{34}) - Q^4 f(Q^2, Q^{46}) \} \{ f(q^{22}, q^{26}) - q^2 f(q^{10}, q^{38}) \} \\ & - 2q^{12} \{ f(Q^{20}, Q^{28}) - Q^4 f(Q^4, Q^{44}) \} \{ f(q^{20}, q^{28}) - q^4 f(q^4, q^{44}) \} \\ = {} & \tfrac{1}{2} q^3 \{ f(-Q) + f(Q) \} \{ f(q) - f(-q) \} \\ & + \tfrac{1}{2} q^3 \{ f(Q) - f(-Q) \} \{ f(-q) + f(q) \} - 2q^{12} f(-q^4) f(-Q^4) \\ = {} & q^3 f(Q) f(q) - q^3 f(-Q) f(-q) - 2q^{12} f(-q^4) f(-Q^4), \end{aligned}$$

by (23.6) and (23.7).

Finally, we employ Entries 10(i)–(iv), 11(i)–(iv), and 12(i), (ii), (iv) in Chapter 17 to transcribe the equality above. Using (20.6), we immediately obtain (ii) to complete the proof.

Entry 24. *Let β, γ, and δ have one of the following sequences of degrees:*

3, 29, 87;

5, 27, 135;

11, 21, 231;

13, 19, 247;

$$7, 25, 175;$$
$$9, 23, 207;$$
$$15, 17, 255.$$

Let m and m' denote the multipliers associated with the pairs α, β and γ, δ, respectively. Then

(i)
$$\left(\tfrac{1}{2}\{1+\sqrt{\beta\gamma}+\sqrt{(1-\beta)(1-\gamma)}\}\right)^{1/2} + (\beta\gamma)^{1/8} + \{(1-\beta)(1-\gamma)\}^{1/8} + \{\beta\gamma(1-\beta)(1-\gamma)\}^{1/8} = (1+(\alpha\delta)^{1/4}+\{(1-\alpha)(1-\delta)\}^{1/4})\sqrt{\frac{m}{m'}}$$

and

(ii)
$$\left(\tfrac{1}{2}\{1+\sqrt{\alpha\delta}+\sqrt{(1-\alpha)(1-\delta)}\}\right)^{1/2} + (\alpha\delta)^{1/8} + \{(1-\alpha)(1-\delta)\}^{1/8} \pm \{\alpha\delta(1-\alpha)(1-\delta)\}^{1/8} = (1+(\beta\gamma)^{1/4}+\{(1-\beta)(1-\gamma)\}^{1/4})\sqrt{\frac{m'}{m}},$$

where the minus sign is chosen in the first four cases and the plus sign is assumed in the last three cases.

A phrase about the appropriate signs is absent in the notebooks (p. 252).

Entry 24(i) can be found in Ramanujan's [10, p. 353] second letter to Hardy.

All references in the proofs of (i) and (ii) are to Chapter 16, unless otherwise stated.

Proof of (i). First, we invoke (36.6) with $\mu = 16$, ν taking the values 13, 11, 5, 3, 9, 7, and 1, respectively, q replaced by q^2, and $Q = q^{256-\nu^2}$. Thus,

$$\varphi(q^{32+2\nu})\varphi(q^{32-2\nu}) + \varphi(-q^{32+2\nu})\varphi(-q^{32-2\nu}) + 4q^{16}\psi(q^{64+4\nu})\psi(q^{64-4\nu}) = 2\sum_{n=0}^{15} q^{64n^2} f(Q^{64+8n}, Q^{64-8n}) f(q^{16+4\nu n}, q^{16-4\nu n}).$$

Second, let $\mu = 16$ in (36.4) with q replaced by $\sqrt{q}$. This yields

$$2q^4\psi(q^{16+\nu})\psi(q^{16-\nu}) = \sum_{n=0}^{15} q^{16n^2+\nu n+4} f(Q^{16+2n}, Q^{16-2n}) f(q^{32+2\nu n}, q^{-2\nu n}).$$

Replace q by $-q$ and add the result to the preceding two equalities. Then combine the terms with indices n and $8+n$, $0 \le n \le 7$, by making use of Entry 18(iv). Lastly, we apply (7.1) of this chapter twice. Accordingly, we find that

$$\begin{aligned} S := {} & \varphi(q^{32+2\nu})\varphi(q^{32-2\nu}) + \varphi(-q^{32+2\nu})\varphi(-q^{32-2\nu}) \\ & + 4q^{16}\psi(q^{64+4\nu})\psi(q^{64-4\nu}) + 2q^4\psi(q^{16+\nu})\psi(q^{16-\nu}) \\ & + 2q^4\psi(-q^{16+\nu})\psi(-q^{16-\nu}) \end{aligned}$$

$$= 2\sum_{n=0}^{7} q^{64n^2}\{f(Q^{64+8n}, Q^{64-8n})$$
$$+ Q^{16+4n}f(Q^{128+8n}, Q^{-8n})\}f(q^{16+4\nu n}, q^{16-4\nu n})$$
$$+ 2\sum_{n=0}^{7} q^{64n^2+2\nu n+4}f(Q^{16+4n}, Q^{16-4n})f(q^{32+4\nu n}, q^{-4\nu n})$$
$$+ 2\sum_{n=0}^{7} q^{64n^2}f(Q^{16+4n}, Q^{16-4n})\{f(q^{16+4\nu n}, q^{16-4\nu n})$$
$$+ q^{4+2\nu n}f(q^{32+4\nu n}, q^{-4\nu n})\}$$
$$= 2\sum_{n=0}^{7} q^{64n^2}f(Q^{16+4n}, Q^{16-4n})f(q^{4+2\nu n}, q^{4-2\nu n}).$$

In the next step, we combine the terms with indices n and $n + 4, 0 \le n \le 3$, with the aid of Entry 18(iv). We then apply (7.1) again. The terms of indices 1 and 3 are now found to be equal, by Entry 18(iv). Thus,

$$S = 2\sum_{n=0}^{3} q^{64n^2}\{f(Q^{16+4n}, Q^{16-4n}) + Q^{4+2n}f(Q^{32+4n}, Q^{-4n})\}f(q^{4+2\nu n}, q^{4-2\nu n})$$
$$= 2\sum_{n=0}^{3} q^{64n^2}f(Q^{4+2n}, Q^{4-2n})f(q^{4+2\nu n}, q^{4-2\nu n})$$
$$= 2\varphi(Q^4)\varphi(q^4) + 4q^{64}\psi(Q^2)f(q^{4+2\nu}, q^{4-2\nu}) + 4q^{256}\psi(Q^8)f(q^{4+4\nu}, q^{4-4\nu}).$$

Now we apply Entry 18(iv) with $n = (\nu \pm 1)/4$, according as $\nu \equiv \mp 1 \pmod 4$, to discover that

$$q^{64}f(q^{4+2\nu}, q^{4-2\nu}) = q^{64+(1-\nu^2)/4}f(q^2, q^6) = (Qq)^{1/4}\psi(q^2).$$

We also apply Entry 18(iv) with $n = (\nu - 1)/2$ to deduce that

$$f(q^{4+4\nu}, q^{4-4\nu}) = 2q^{1-\nu^2}\psi(q^8).$$

Employing lastly Entries 25(i), (ii), we find that

$$S = 2\varphi(Q^4)\varphi(q^4) + 4(Qq)^{1/4}\psi(Q^2)\psi(q^2) + 8Qq\psi(Q^8)\psi(q^8)$$
$$= \tfrac{1}{2}\{\varphi(Q) + \varphi(-Q)\}\{\varphi(q) + \varphi(-q)\}$$
$$+ \tfrac{1}{2}\{\varphi(Q) - \varphi(-Q)\}\{\varphi(q) - \varphi(-q)\} + 4(Qq)^{1/4}\psi(Q^2)\psi(q^2)$$
$$= \varphi(Q)\varphi(q) + \varphi(-Q)\varphi(-q) + 4(Qq)^{1/4}\psi(Q^2)\psi(q^2). \tag{24.1}$$

Finally, employing Entries 10(i)–(iv) and 11(i)–(iv) in Chapter 17, we translate the equality above into the sought modular equation and so complete the proof of (i).

PROOF OF (ii). In the extremal parts of (24.1), we replace q by $q^{1/(16-\nu)}$ to find that

$$\varphi(q^{(32+2\nu)/(16-\nu)})\varphi(q^2) + \varphi(-q^{(32+2\nu)/(16-\nu)})\varphi(-q^2)$$
$$+ 4q^{16/(16-\nu)}\psi(q^{(64+4\nu)/(16-\nu)})\psi(q^4) + 2q^{4/(16-\nu)}\psi(q^{(16+\nu)/(16-\nu)})\psi(q)$$

$$+ 2q^{4/(16-\nu)}\psi(-q^{(16+\nu)/(16-\nu)})\psi(-q)$$
$$= \varphi(q^{16+\nu})\varphi(q^{1/(16-\nu)}) + \varphi(-q^{16+\nu})\varphi(-q^{1/(16-\nu)})$$
$$+ 4q^{(257-\nu^2)/(4(16-\nu))}\psi(q^{32+2\nu})\psi(q^{2/(16-\nu)}).$$

We now equate rational parts on both sides to deduce, except when $\nu = 7$, that

$$\varphi(Q^2)\varphi(q^2) + \varphi(-Q^2)\varphi(-q^2) + 4(Qq)^{1/2}\psi(Q^4)\psi(q^4)$$
$$+ 2(Qq)^{1/8}\psi(Q)\psi(q) \pm 2(Qq)^{1/8}\psi(-Q)\psi(-q)$$
$$= \varphi(q^{16+\nu})\varphi(q^{16-\nu}) + \varphi(-q^{16+\nu})\varphi(-q^{16-\nu}) + 4q^8\psi(q^{32+2\nu})\psi(q^{32-2\nu}), \tag{24.2}$$

where the plus sign is chosen when $\nu = 9$ or 1 and the minus sign is taken when $\nu = 13, 11, 5$, or 3. The details in demonstrating the validity of (24.2) are rather tedious, and we shall be content with merely indicating the requisite steps. Each of the six cases must be examined separately. Corollaries (i), (ii) of Section 31 in Chapter 16 are used when $\nu = 13$. Entries 10(i), (ii) of Chapter 19 are employed when $\nu = 11$. For $\nu = 9$, the rational parts are obtained by using Entries 17(iii), (iv) in Chapter 19. When $\nu = 5$, utilize Entries 6(i), (ii) of the present chapter. For $\nu = 1, 3$, Ramanujan has not explicitly recorded the appropriate formulas, but they are very easily obtained from Entry 31 of Chapter 16 in the same manner as the aforementioned results were derived. We emphasize that when $\nu = 7$, (24.2) is not obtained, because when rational parts are equated, additional terms arise. Note that $16 - 7 = 3^2$.

The translation of (24.2) into Entry 24(ii) uses precisely the same formulas from Chapter 17 that were employed in the proof of part (i).

There remains the proof of (ii) in the case $\nu = 7$. In this instance, the requisite formula to be established is

$$\varphi(Q^2)\varphi(q^2) + 4(Qq)^{1/2}\psi(Q^4)\psi(q^4) + 2(Qq)^{1/8}\psi(Q)\psi(q)$$
$$+ \varphi(-Q^2)\varphi(-q^2) + 2(Qq)^{1/8}\psi(-Q)\psi(-q)$$
$$= \varphi(q^{23})\varphi(q^9) + 4q^8\psi(q^{46})\psi(q^{18}) + \varphi(-q^{23})\varphi(-q^9), \tag{24.3}$$

where $Q = q^{207}$. The translation of (24.3) into (ii) is exactly the same as above, and so the proof will be completed on establishing (24.3).

We apply (36.6) with $\mu = 16$ and $\nu = 7$. Using Entry 18(iv), we first combine the terms with indices n and $8 + n$, $0 \le n \le 7$. Simplify the resulting sum with the aid of (7.1) in this chapter. After separating the terms with $n = 0$, 4 and simplifying with the aid of Entries 18(ii), (iv), we use Entry 18(iv) to show that the terms with indices n and $8 - n$, $1 \le n \le 3$, are equal. Next, we employ Entries 25(i), (ii), and lastly we invoke Corollary (ii) of Section 31. Accordingly, we deduce that

$$\tfrac{1}{2}\{\varphi(q^{23})\varphi(q^9) + \varphi(-q^{23})\varphi(-q^9)\} + 2q^8\psi(q^{46})\psi(q^{18})$$
$$= \sum_{n=0}^{15} q^{32n^2} f(Q^{32+4n}, Q^{32-4n}) f(q^{8+14n}, q^{8-14n})$$

$$= \sum_{n=0}^{7} q^{32n^2}\{f(Q^{32+4n}, Q^{32-4n}) + Q^{8+2n}f(Q^{64+4n}, Q^{-4n})\}f(q^{8+14n}, q^{8-14n})$$

$$= \sum_{n=0}^{7} q^{32n^2}f(Q^{8+2n}, Q^{8-2n})f(q^{8+14n}, q^{8-14n})$$

$$= \varphi(Q^8)\varphi(q^8) + 4Q^2q^2\psi(Q^{16})\psi(q^{16})$$
$$+ 2\sum_{n=1}^{3} q^{32n^2}f(Q^{8+2n}, Q^{8-2n})f(q^{8+14n}, q^{8-14n})$$

$$= \tfrac{1}{4}\{\varphi(Q^2) + \varphi(-Q^2)\}\{\varphi(q^2) + \varphi(-q^2)\}$$
$$+ \tfrac{1}{4}\{\varphi(Q^2) - \varphi(-Q^2)\}\{\varphi(q^2) - \varphi(-q^2)\} + 2q^{26}f(Q^6, Q^{10})f(q^6, q^{10})$$
$$+ 2q^{104}\psi(Q^4)\psi(q^4) + 2q^{234}f(Q^2, Q^{14})f(q^2, q^{14})$$

$$= \tfrac{1}{2}\varphi(Q^2)\varphi(q^2) + \tfrac{1}{2}\varphi(-Q^2)\varphi(-q^2) + 2q^{104}\psi(Q^4)\psi(q^4)$$
$$+ q^{26}(\{f(Q^6, Q^{10}) + Qf(Q^2, Q^{14})\}\{f(q^6, q^{10}) + qf(q^2, q^{14})\}$$
$$+ \{f(Q^6, Q^{10}) - Qf(Q^2, Q^{14})\}\{f(q^6, q^{10}) - qf(q^2, q^{14})\})$$

$$= \tfrac{1}{2}\varphi(Q^2)\varphi(q^2) + \tfrac{1}{2}\varphi(-Q^2)\varphi(-q^2) + 2q^{104}\psi(Q^4)\psi(q^4)$$
$$+ q^{26}\{\psi(Q)\psi(q) + \psi(-Q)\psi(-q)\}.$$

Thus, the proof of (24.3) and, consequently, the proof of (ii) in the sixth case are complete.

This concludes a truly fascinating chapter!

CHAPTER 21

Eisenstein Series

Chapter 21 concludes the organized portion of Ramanujan's second notebook; after Chapter 21, there are 100 pages of unorganized material. Chapter 21 constitutes only four pages and thus is the shortest chapter in the second notebook. Almost all of the previous chapters are twelve pages in length.

The focus of this chapter is similar to those of the immediately preceding chapters. However, whereas in Chapters 19 and 20, the goal was to establish identities involving theta-functions, here our task is to prove equalities relating a certain linear combination of Eisenstein series with theta-functions. From the viewpoint of modular forms, just as in Chapter 20, both the Eisenstein series and theta-functions are forms on $\Gamma(2) \cap \Gamma_0(n)$ for some odd integer $n \geq 3$.

The key to establishing Ramanujan's formulas is apparently (2.3) below. This formula is not explicitly stated by Ramanujan, but we conjecture that it is this formula to which Ramanujan makes allusion in Entry 2(v). Unfortunately, we have not always been successful in applying this formula or the related formula (5.3). Thus, for seven of the results in this chapter, we have had to rely on the theory of modular forms that was developed in Chapter 20. As in the last chapter, the theory of modular forms provides the best means of explaining why these identities exist. However, again as before, it is necessary to know the identity in advance, and so the proofs are more properly called verifications.

As in previous chapters, we employ the notation introduced in Chapter 17, especially in Section 6.

We shall precisely quote Ramanujan (p. 253) for Entry 1.

Entry 1.

(i)
$$1 - \frac{3}{y} - 24 \sum_{n=1}^{\infty} \frac{n}{e^{2ny} - 1} \tag{1.1}$$

is a complete series which when divided by z^2 *can be expressed by radicals precisely in the same manner as the series*

$$1 + 240 \sum_{n=1}^{\infty} \frac{n^3}{e^{2ny} - 1} \tag{1.2}$$

and the series

$$1 - 504 \sum_{n=1}^{\infty} \frac{n^5}{e^{2ny} - 1} \tag{1.3}$$

when divided by z^4 *and* z^6*, respectively.*

(ii) $$1 - 24 \sum_{n=0}^{\infty} \frac{2n+1}{e^{(2n+1)y} + 1} = z^2(1 - 2x).$$

(iii) $$1 - 240 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{e^{ny} - (-1)^n} = z^4\{1 - 16x(1 - x)\}.$$

(iv) $$1 + 504 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^5}{e^{ny} - (-1)^n} = z^6(1 - 2x)\{1 + 32x(1 - x)\}.$$

The Eisenstein series (1.1)–(1.3) were introduced by Ramanujan in Section 9 of Chapter 15 and were denoted by $L - 3/y$, M, and N, respectively. In Ramanujan's celebrated paper [6], [10, pp. 136–162], the series L, M, and N are designated by P, Q, and R, respectively.

The definition of "complete" is given rather vaguely by Ramanujan in Section 10 of Chapter 15 (Part II [9, p. 320]).

In fact, (i) is not quite accurately stated by Ramanujan, since the condition that y^2/π^2 be rational should be added. With this additional stipulation, (i) was established by Ramanujan in his paper [2], [10, pp. 32, 33]. The reader should consult [2] and the Borweins' book [2, Chap. 5] to learn how Ramanujan used such results to derive excellent approximations to π.

It might be noted that, in general,

$$S := \frac{1}{z^2}\left(1 - \frac{3}{y} - 24 \sum_{n=1}^{\infty} \frac{n}{e^{2ny} - 1}\right) = \frac{3}{2}\left(\frac{E}{K} - \frac{E'}{K'}\right) + k^2 - \tfrac{1}{2}, \tag{1.4}$$

where, as usual, K and K' denote the complete elliptic integrals of the first kind attached to the moduli k and k', respectively, while E and E' are the complete elliptic integrals of the second kind associated with k and k', respectively.

To prove (1.4), recall from Entry 2 of Chapter 18 that

$$S = \frac{3}{z}\,{}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}; 1; x) - 2 + x - \frac{3}{yz^2}.$$

Since $x = k^2$, $y = \pi K'/K$, and $z = 2K/\pi$, we rewrite this last equality in the form

$$S = \frac{3\pi}{2K}\,{}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) - 2 + k^2 - \frac{3\pi}{4KK'}.$$

From (3.7) in Chapter 18, ${}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) = 2E/\pi$. Using also Legendre's relation (Whittaker and Watson [1, p. 520]), we find that

$$S = \frac{3E}{K} - 2 + k^2 - \frac{3}{2}\left(\frac{E}{K} + \frac{E'}{K'} - 1\right),$$

which yields (1.4) at once. (A recent, somewhat simpler proof of Legendre's relation has been given by Almkvist and Berndt [1].)

With regard to (1.2) and (1.3), we recall from Entries 13(i), (ii) of Chapter 17 that

$$\frac{1}{z^4}\left(1 + 240 \sum_{n=1}^{\infty} \frac{n^3}{e^{2ny} - 1}\right) = 1 - x + x^2$$

and

$$\frac{1}{z^6}\left(1 - 504 \sum_{n=1}^{\infty} \frac{n^5}{e^{2ny} - 1}\right) = (1 + x)(1 - \tfrac{1}{2}x)(1 - 2x),$$

respectively.

Proof of (ii). By Entries 13(viii), (ix) in Chapter 17,

$$\begin{aligned} 1 - 24 \sum_{n=0}^{\infty} \frac{2n+1}{e^{(2n+1)y} + 1} &= \left(2 + 24 \sum_{n=1}^{\infty} \frac{2n}{e^{2ny} + 1}\right) - \left(1 + 24 \sum_{n=1}^{\infty} \frac{n}{e^{ny} + 1}\right) \\ &= 2z^2(1 - \tfrac{1}{2}x) - z^2(1 + x) \\ &= z^2(1 - 2x). \end{aligned}$$

Proof of (iii). We use the procedure of "obtaining a formula by change of sign," described in Section 13 of Chapter 17. Thus, in Entry 13(iii) of Chapter 17, replace x by $-x/(1 - x)$, which induces the replacements of e^{-y} by $-e^{-y}$ and z by $z\sqrt{1 - x}$. Hence,

$$1 + 240 \sum_{n=1}^{\infty} \frac{(-1)^n n^3 e^{-ny}}{1 - (-1)^n e^{-ny}} = z^4(1 - x)^2\left(1 - \frac{14x}{1 - x} + \frac{x^2}{(1 - x)^2}\right),$$

which upon simplification yields the proposed formula.

Proof of (iv). We employ the "change of sign" process to Entry 13(iv) of Chapter 17 to find that

$$\begin{aligned} 1 - 504 &\sum_{n=1}^{\infty} \frac{(-1)^n n^5 e^{-ny}}{1 - (-1)^n e^{-ny}} \\ &= z^6(1 - x)^3\left(1 - \frac{x}{1 - x}\right)\left(1 + \frac{34x}{1 - x} + \frac{x^2}{(1 - x)^2}\right). \end{aligned}$$

After simplification, we obtain the desired result.

Entry 2.

(i) $$12q\frac{\varphi'(q)}{\varphi(q)} = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} - \left(1 - 24 \sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1 + q^{2n+1}}\right).$$

(ii)
$$24q\frac{\dfrac{d}{dq}\{q^{1/8}\psi(q)\}}{q^{1/8}\psi(q)} = 4\left(1 - 24\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{4n}}\right) - \left(1 - 24\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1+q^{2n+1}}\right).$$

(iii)
$$24q\frac{\dfrac{d}{dq}\{q^{1/24}f(-q)\}}{q^{1/24}f(-q)} = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

(iv)
$$24q\frac{\dfrac{d}{dq}\{q^{1/24}/\chi(q)\}}{q^{1/24}/\chi(q)} = 1 - 24\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1+q^{2n+1}}.$$

(v) *By differentiating the equation for m once or the equation for α, β twice we can calculate the value of the first series.*

Parts (i)–(iv) are quite easy to prove. However, the meaning of (v), for which we have quoted Ramanujan (p. 253) exactly, is rather opaque. Perhaps Ramanujan is referring to a more precise version of Entry 1(i), or to a certain formula, (2.3) below, which will be needed to prove many of the formulas in the remainder of the chapter.

Proof of (iii). By Entry 23(iii) in Chapter 16,

$$\operatorname{Log}\{q^{1/24}f(-q)\} = \tfrac{1}{24}\operatorname{Log} q - \sum_{k=1}^{\infty}\frac{q^k}{k(1-q^k)}.$$

Differentiating both sides with respect to q, expanding the summands, and inverting the order of summation, we find that

$$\begin{aligned}24q\frac{\dfrac{d}{dq}\{q^{1/24}f(-q)\}}{q^{1/24}f(-q)} &= 1 - 24\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)^2}\\ &= 1 - 24\sum_{k=1}^{\infty}q^k\sum_{n=1}^{\infty}nq^{k(n-1)}\\ &= 1 - 24\sum_{n=1}^{\infty}n\sum_{k=1}^{\infty}q^{nk}\\ &= 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},\end{aligned}$$

which completes the proof.

Proof of (iv). From Entry 23(iv) in Chapter 16,

$$\operatorname{Log}\{q^{1/24}/\chi(q)\} = \tfrac{1}{24}\operatorname{Log} q + \sum_{k=1}^{\infty}\frac{(-1)^k q^k}{k(1-q^{2k})}.$$

Proceeding in exactly the same manner as in the proof above, we deduce that

$$\begin{aligned}
24q\frac{\dfrac{d}{dq}\{q^{1/24}/\chi(q)\}}{q^{1/24}/\chi(q)} &= 1 + 24\sum_{k=1}^{\infty}(-1)^k\frac{q^k+q^{3k}}{(1-q^{2k})^2}\\
&= 1 + 24\sum_{k=1}^{\infty}(-1)^k(q^k+q^{3k})\sum_{n=1}^{\infty}nq^{2k(n-1)}\\
&= 1 + 24\sum_{n=1}^{\infty}n\sum_{k=1}^{\infty}(-1)^k(q^{k(2n-1)}+q^{k(2n+1)})\\
&= 1 - 24\sum_{n=1}^{\infty}n\left(\frac{q^{2n-1}}{1+q^{2n-1}}+\frac{q^{2n+1}}{1+q^{2n+1}}\right)\\
&= 1 - 24\sum_{n=0}^{\infty}\left(\frac{(n+1)q^{2n+1}}{1+q^{2n+1}}+\frac{nq^{2n+1}}{1+q^{2n+1}}\right)\\
&= 1 - 24\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1+q^{2n+1}}.
\end{aligned}$$

Proof of (i). From Entry 22(i) and (22.2) in Chapter 16, we can easily see that $\varphi(q) = f(-q^2)\chi^2(q)$. Thus,

$$\operatorname{Log}\varphi(q) = \operatorname{Log}\{q^{1/12}f(-q^2)\} - 2\operatorname{Log}\{q^{1/24}/\chi(q)\}.$$

Differentiating and employing parts (iii) and (iv) above, we find that

$$12q\frac{\varphi'(q)}{\varphi(q)} = 1 - 24\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}} - \left(1 - 24\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1+q^{2n+1}}\right).$$

Proof of (ii). Differentiating Entry 23(ii) of Chapter 16 and proceeding as in the proofs of (iii) and (iv), we find that

$$\begin{aligned}
&24q\frac{\dfrac{d}{dq}\{q^{1/8}\psi(q)\}}{q^{1/8}\psi(q)}\\
&= 3 + 24\sum_{k=1}^{\infty}\frac{q^k}{(1+q^k)^2}\\
&= 3 + 24\sum_{n=1}^{\infty}\frac{(-1)^{n-1}nq^n}{1-q^n}\\
&= 3 + 24\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1-q^{2n+1}} - 24\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}\\
&= 3 + 24\sum_{n=0}^{\infty}\left(\frac{(2n+1)q^{2n+1}}{1+q^{2n+1}}+\frac{(4n+2)q^{4n+2}}{1-q^{4n+2}}\right) - 24\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}\\
&= 3 + 24\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1+q^{2n+1}} - 24\sum_{n=1}^{\infty}\frac{4nq^{4n}}{1-q^{4n}},
\end{aligned}$$

from which (ii) follows.

PROOFS OF TWO INTERPRETATIONS OF (v). We first prove a formula on which much of the remainder of this chapter is based.

Let n denote any positive integer, and put $u = q^2 = e^{-2y}$. By using Entry 2(iii) above, we find that

$$\frac{d}{dy}\operatorname{Log}\left(\frac{e^{-ny}f^{12}(-e^{-2ny})}{e^{-y}f^{12}(-e^{-2y})}\right)$$
$$= -2u\frac{d}{du}\operatorname{Log}\left(\frac{u^{n/2}f^{12}(-u^n)}{u^{1/2}f^{12}(-u)}\right)$$
$$= -24u\frac{d}{du}\operatorname{Log}\left(\frac{u^{n/24}f(-u^n)}{u^{1/24}f(-u)}\right)$$
$$= 24u\frac{\frac{d}{du}\{u^{1/24}f(-u)\}}{u^{1/24}f(-u)} - 24u\frac{\frac{d}{du}\{u^{n/24}f(-u^n)\}}{u^{n/24}f(-u^n)}$$
$$= 1 - 24\sum_{k=1}^{\infty}\frac{ku^k}{1-u^k} - n\left(1 - 24\sum_{k=1}^{\infty}\frac{ku^{nk}}{1-u^{nk}}\right). \tag{2.1}$$

Now let β have degree n over α, and let m denote the multiplier associated with α and β. By Entries 12(iii) and 9(i) in Chapter 17,

$$\frac{d}{dy}\operatorname{Log}\left(\frac{e^{-ny}f^{12}(-e^{-2ny})}{e^{-y}f^{12}(-e^{-2y})}\right)$$
$$= \frac{d}{dy}\operatorname{Log}\left(\frac{z_n^6\beta(1-\beta)}{z_1^6\alpha(1-\alpha)}\right)$$
$$= -\alpha(1-\alpha)z_1^2\frac{d}{d\alpha}\operatorname{Log}\left(\frac{\beta(1-\beta)}{m^6\alpha(1-\alpha)}\right). \tag{2.2}$$

Thus, combining (2.1) and (2.2), we derive the important formula

$$1 - 24\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - n\left(1 - 24\sum_{k=1}^{\infty}\frac{kq^{2nk}}{1-q^{2nk}}\right)$$
$$= -\alpha(1-\alpha)z_1^2\frac{d}{d\alpha}\operatorname{Log}\left(\frac{\beta(1-\beta)}{m^6\alpha(1-\alpha)}\right). \tag{2.3}$$

To derive our second possible interpretation of (v), we utilize Entry 27(iii) of Chapter 16 in the form

$$e^{-y/12}y^{1/4}f(-e^{-2y}) = e^{-y'/12}y'^{1/4}f(-e^{-2y'}), \tag{2.4}$$

where $y, y' > 0$ with $yy' = \pi^2$. Set $q = e^{-y}$ and $q' = e^{-y'}$. Logarithmically differentiating (2.4) with respect to y and using Entry 2(iii) above, we see that

$$\frac{1}{4y} - \frac{1}{12}\left(1 - 24\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}\right) = -\left\{\frac{1}{4y'} - \frac{1}{12}\left(1 - 24\sum_{k=1}^{\infty}\frac{kq'^{2k}}{1-q'^{2k}}\right)\right\}\frac{\pi^2}{y^2},$$

or

$$1-24\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}=\frac{6}{y}-\frac{y'}{y}\left(1-24\sum_{k=1}^{\infty}\frac{kq'^{2k}}{1-q'^{2k}}\right). \tag{2.5}$$

Now set $y=\pi/\sqrt{n}$, where n is a natural number. Thus, $y'=\pi\sqrt{n}$. Combining (2.3) and (2.5), we conclude that

$$\frac{6\sqrt{n}}{\pi}-n\left(1-24\sum_{k=1}^{\infty}\frac{k}{e^{2\pi k\sqrt{n}}-1}\right)$$
$$=n\left(1-24\sum_{k=1}^{\infty}\frac{k}{e^{2\pi k\sqrt{n}}-1}\right)-\alpha(1-\alpha)z_1^2\frac{d}{d\alpha}\operatorname{Log}\left(\frac{\beta(1-\beta)}{m^6\alpha(1-\alpha)}\right)\Bigg|_{y=\pi/\sqrt{n}},$$

or

$$n\left(1-24\sum_{k=1}^{\infty}\frac{k}{e^{2\pi k\sqrt{n}}-1}\right)-\frac{3\sqrt{n}}{\pi}=\tfrac{1}{2}\alpha(1-\alpha)z_1^2\frac{d}{d\alpha}\operatorname{Log}\left(\frac{\beta(1-\beta)}{m^6\alpha(1-\alpha)}\right)\Bigg|_{y=\pi/\sqrt{n}}.$$

Entry 3.

(i)
$$1+12\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-36\sum_{k=1}^{\infty}\frac{kq^{3k}}{1-q^{3k}}$$
$$=\left\{1+6\left(\frac{q}{1-q}-\frac{q^2}{1-q^2}+\frac{q^4}{1-q^4}-\frac{q^5}{1-q^5}+\cdots\right)\right\}^2$$
$$=\left\{1+24\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}+216\sum_{k=1}^{\infty}\frac{k^3q^{3k}}{1-q^{3k}}\right\}^{1/2}$$
$$=\left\{\frac{\psi^4(q)+3q\psi^4(q^3)}{\psi(q)\psi(q^3)}\right\}^2$$
$$=\left\{\frac{f^{12}(-q)+27qf^{12}(-q^3)}{f^3(-q)f^3(-q^3)}\right\}^{2/3}$$

(ii)
$$1+12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-36\sum_{k=1}^{\infty}\frac{kq^{6k}}{1-q^{6k}}$$
$$=\left\{\frac{\varphi^4(q)+3\varphi^4(q^3)}{4\varphi(q)\varphi(q^3)}\right\}^2$$
$$=\varphi^2(q)\varphi^2(q^3)-4q\psi^2(-q)\psi^2(-q^3).$$

(iii) *Let β have degree 3 with respect to α. Then*

$$1+12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-36\sum_{k=1}^{\infty}\frac{kq^{6k}}{1-q^{6k}}$$
$$=\tfrac{1}{2}\varphi^2(q)\varphi^2(q^3)\{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\}.$$

Ramanujan actually expresses (iii) in the variable y, where $q = e^{-y}$. It will be convenient to prove (ii) and (iii) before (i).

PROOF OF (ii). Putting $n = 3$ in (2.3), we find that

$$S := 1 + 12 \sum_{k=1}^{\infty} \frac{kq^{2k}}{1 - q^{2k}} - 36 \sum_{k=1}^{\infty} \frac{kq^{6k}}{1 - q^{6k}}$$

$$= \tfrac{1}{2}\alpha(1 - \alpha)z_1^2 \frac{d}{d\alpha} \operatorname{Log}\left(\frac{\beta(1 - \beta)}{m^6\alpha(1 - \alpha)}\right). \tag{3.1}$$

Now from (5.2) and (5.5) of Chapter 19,

$$\alpha(1 - \alpha) = \frac{(m^2 - 1)(9 - m^2)^3}{256m^6}, \qquad \frac{\beta(1 - \beta)}{\alpha(1 - \alpha)} = \frac{m^4(m^2 - 1)^2}{(9 - m^2)^2}, \tag{3.2}$$

and

$$\frac{dm}{d\alpha} = \frac{16m^4}{(9 - m^2)^2}.$$

Using these equalities and the chain rule in (3.1), we arrive at

$$S = \frac{(m^2 - 1)(9 - m^2)}{16m^2} z_1^2 \frac{d}{dm} \operatorname{Log}\left(\frac{m^2 - 1}{m(9 - m^2)}\right)$$

$$= \frac{z_1^2}{16m^3}(m^2 + 3)^2$$

$$= \frac{\varphi^6(q^3)}{16\varphi^2(q)}\left(\frac{\varphi^4(q)}{\varphi^4(q^3)} + 3\right)^2, \tag{3.3}$$

which establishes the first equality in (ii).

From the second equality in (3.3) and (3.2),

$$S = z_1 z_3\left(1 - \frac{(9 - m^2)(m^2 - 1)}{16m^2}\right)$$

$$= z_1 z_3(1 - \{\alpha\beta(1 - \alpha)(1 - \beta)\}^{1/4})$$

$$= \varphi^2(q)\varphi^2(q^3) - 4q\psi^2(-q)\psi^2(-q^3),$$

by Entry 11(ii) of Chapter 17. Thus, the second equality of (ii) has been proved.

PROOF OF (iii). From (3.3) and from (5.8) of Chapter 19,

$$S = \frac{z_1 z_3}{16m^2}(m^2 + 3)^2 = \tfrac{1}{2}z_1 z_3\{1 + \sqrt{\alpha\beta} + \sqrt{(1 - \alpha)(1 - \beta)}\},$$

from which the truth of (iii) is manifest.

PROOF OF (i). Once again from (3.3), and from (5.1) of Chapter 19,

$$S = z_1 z_3 \left(\frac{(3+m)^2 + 3(m-1)^2}{16m} \right)^2$$
$$= z_1 z_3 \left\{ \frac{m}{4} \left(\frac{\alpha^3}{\beta} \right)^{1/4} + \frac{3}{4m} \left(\frac{\beta^3}{\alpha} \right)^{1/4} \right\}^2$$
$$= \left\{ \frac{\psi^3(q^2)}{\psi(q^6)} + 3q^2 \frac{\psi^3(q^6)}{\psi(q^2)} \right\}^2, \tag{3.4}$$

by Entry 11(iii) of Chapter 17. If we now replace q^2 by q, we obtain the equality between the first and fourth expressions in (i).

By Entry 11(i) of Chapter 17, (5.2) and (5.3) of Chapter 19, and Entry 11(iii) of Chapter 17,

$$\frac{\psi^4(q) + 3q\psi^4(q^3)}{\psi(q)\psi(q^3)} = \tfrac{1}{2}\sqrt{z_1 z_3} \left\{ m \left(\frac{\alpha^3}{\beta} \right)^{1/8} + \frac{3}{m} \left(\frac{\beta^3}{\alpha} \right)^{1/8} \right\}$$
$$= \frac{\sqrt{z_1 z_3}}{2} \left\{ \frac{m+3}{2} + \frac{3(m-1)}{2m} \right\}$$
$$= \frac{\sqrt{z_1 z_3}}{4m} (m^2 + 6m - 3)$$
$$= \sqrt{z_1 z_3} \left(1 + \frac{m^2 + 2m - 3}{4m} \right)$$
$$= \sqrt{z_1 z_3} (1 + (\alpha\beta)^{1/4})$$
$$= \varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6). \tag{3.5}$$

We next invoke Entries 3(i), (ii) of Chapter 19. For each summand of even index, we use one of the two equalities,

$$\frac{q^n}{1-q^{2n}} = \frac{q^n}{1-q^n} - \frac{q^{2n}}{1-q^{2n}} \quad \text{and} \quad \frac{q^{2n}}{1+q^{2n}} = \frac{q^{2n}}{1-q^{2n}} - \frac{2q^{4n}}{1-q^{4n}}.$$

We thus deduce that

$$\varphi(q)\varphi(q^3) + 4q\psi(q^2)\psi(q^6)$$
$$= 1 + 6\left(\frac{q}{1-q} - \frac{q^2}{1-q^2} + \frac{q^4}{1-q^4} - \frac{q^5}{1-q^5} + \cdots \right). \tag{3.6}$$

Combining (3.5) and (3.6), we establish the equality between the second and fourth expressions of part (i).

Next, by Entry 13(i) in Chapter 17 and (3.2) above,

$$1 + 24 \sum_{k=1}^{\infty} \frac{k^3 q^{2k}}{1-q^{2k}} + 216 \sum_{k=1}^{\infty} \frac{k^3 q^{6k}}{1-q^{6k}}$$
$$= \tfrac{1}{10}\{z_1^4(1 - \alpha + \alpha^2) + 9z_3^4(1 - \beta + \beta^2)\}$$

$$= \tfrac{1}{10}z_1^2z_3^2\left(m^2 + \frac{9}{m^2} - m^2\alpha(1-\alpha) - \frac{9}{m^2}\beta(1-\beta)\right)$$

$$= \tfrac{1}{10}z_1^2z_3^2\left(m^2 + \frac{9}{m^2} - \frac{(m^2-1)(9-m^2)^3}{256m^4} - \frac{9(m^2-1)^3(9-m^2)}{256m^4}\right)$$

$$= \tfrac{1}{256}z_1^2z_3^2(m^4 + 12m^2 + 54 + 108m^{-2} + 81m^{-4})$$

$$= \frac{z_1^2z_3^2}{256m^4}(m^2+3)^4$$

$$= \left\{\frac{\psi^3(q^2)}{\psi(q^6)} + 3q^2\frac{\psi^3(q^6)}{\psi(q^2)}\right\}^4,$$

by the same calculation as in (3.4). Replacing q^2 by q, we establish the equality between the third and fourth expressions in (i).

By Entries 24(i), (iii) in Chapter 16,

$$f(-q) = \frac{f(q)\chi(-q)}{\chi(q)} = f(-q^2)\chi(-q) = \psi(q)\chi^2(-q). \tag{3.7}$$

Using Entries 1(i), (iv) of Chapter 20, the definition of v given in Entry 1 of Chapter 20, and (3.7), we find that

$$\frac{\psi^4(q) + 3q\psi^4(q^3)}{\psi(q)\psi(q^3)} = \frac{q\psi^3(q^3)}{\psi(q)}\left(4 + \frac{1}{v^3}\right)$$

$$= \frac{q\psi^3(q^3)}{v^2\psi(q)}\left(27 + \frac{f^{12}(-q)}{qf^{12}(-q^3)}\right)^{1/3}$$

$$= \frac{\psi^3(q^3)\chi^6(-q^3)}{\psi(q)\chi^2(-q)f^4(-q^3)}\{f^{12}(-q) + 27qf^{12}(-q^3)\}^{1/3}$$

$$= \left\{\frac{f^{12}(-q) + 27qf^{12}(-q^3)}{f^3(-q)f^3(-q^3)}\right\}^{1/3}.$$

Thus, we have shown that the fourth and fifth expressions in (i) are equal. In conclusion, the equality of all five expressions of (i) has been established.

Entry 4.

(i) $$1 + 6\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} - 30\sum_{k=1}^{\infty}\frac{kq^{5k}}{1-q^{5k}}$$

$$= \frac{\{f^{12}(-q) + 22qf^6(-q)f^6(-q^5) + 125q^2f^{12}(-q^5)\}^{1/2}}{f(-q)f(-q^5)}$$

$$= \frac{\psi^4(q) + 2q\psi^2(q)\psi^2(q^5) + 5q^2\psi^4(q^5)}{\psi(q)\psi(q^5)}$$

$$\times \{\psi^4(q) - 2q\psi^2(q)\psi^2(q^5) + 5q^2\psi^4(q^5)\}^{1/2}.$$

(ii) $1+6\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-30\sum_{k=1}^{\infty}\frac{kq^{10k}}{1-q^{10k}}$

$$=\{\varphi^2(q)\varphi^2(q^5)-2qf^2(-q^2)f^2(-q^{10})\}\left(1-\frac{4q}{\chi^4(q)\chi^4(q^5)}\right)^{1/2}.$$

(iii) *Let β have degree 5 over α. Then*

$$\begin{aligned}1+6\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}&-30\sum_{k=1}^{\infty}\frac{kq^{10k}}{1-q^{10k}}\\&=\tfrac{1}{4}\varphi^2(q)\varphi^2(q^5)\{3+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\}\\&\quad\times\{\tfrac{1}{2}(1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)})\}^{1/2}\\&=\varphi^2(q)\varphi^2(q^5)(\tfrac{1}{2}\{1+\alpha\beta+(1-\alpha)(1-\beta)\}\\&\quad-\tfrac{3}{4}\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/3})^{1/2}.\end{aligned}$$

Ramanujan has stated (iii) in terms of y, where $q=e^{-y}$.

PROOF OF (i). Setting $n=5$ in (2.3), we find that

$$\begin{aligned}S:=1+6\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}&-30\sum_{k=1}^{\infty}\frac{kq^{10k}}{1-q^{10k}}\\&=\tfrac{1}{4}\alpha(1-\alpha)z_1^2\frac{d}{d\alpha}\operatorname{Log}\left(\frac{\beta(1-\beta)}{m^6\alpha(1-\alpha)}\right).\end{aligned}\tag{4.1}$$

Now take (14.2) of Chapter 19 and differentiate both sides with respect to m. After simplification, we find that

$$\frac{d\alpha}{dm}=\frac{\alpha(1-\alpha)}{1-2\alpha}\left(\frac{25-20m-m^2}{m(m-1)(5-m)}\right).$$

Furthermore, from (14.2) and (14.4) of Chapter 19,

$$\frac{\beta(1-\beta)}{\alpha(1-\alpha)}=\frac{(m-1)^4m^4}{(5-m)^4}.\tag{4.2}$$

Lastly, in Entry 14(ii) of Chapter 19, make the substitution $p=(m-1)/2$, from (14.1), simplify, and use the definition of ρ given in (13.3) of the same chapter. Accordingly, we find that

$$1-2\alpha=\frac{(25-20m-m^2)\rho}{8m^3}.$$

Employing these last three equalities in (4.1), we deduce that

$$\begin{aligned}S&=\tfrac{1}{4}(1-2\alpha)z_1^2\frac{m(m-1)(5-m)}{25-20m-m^2}\frac{d}{dm}\operatorname{Log}\left(\frac{(m-1)^4}{m^2(5-m)^4}\right)\\&=\frac{\rho(m-1)(5-m)z_1^2}{32m^2}\frac{d}{dm}\operatorname{Log}\left(\frac{(m-1)^4}{m^2(5-m)^4}\right)\end{aligned}$$

$$= \frac{\rho(m^2 + 2m + 5)z_1^2}{16m^3}$$

$$= \frac{z_1^2}{16m^3}(m^7 + 2m^6 + 11m^5 + 12m^4 + 55m^3 + 50m^2 + 125m)^{1/2}. \quad (4.3)$$

By Entry 12(iii) in Chapter 17 and Entry 13(iv) in Chapter 19,

$$\frac{f^5(-q^2)}{f(-q^{10})} = \frac{z_1^{5/2}}{2^{4/3}z_5^{1/2}}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/12} = \frac{z_1^2\sqrt{m}}{16}\left(\frac{5}{m} - 1\right)^2.$$

Also, by Entry 12(iii) in Chapter 17 and (4.2),

$$q^2\frac{f^6(-q^{10})}{f^6(-q^2)} = \frac{1}{m^3}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} = \frac{(m-1)^2}{m(5-m)^2}.$$

Employing these last two results in (4.3), we find that

$$S = \frac{f^5(-q^2)}{f(-q^{10})}\left(\frac{m^6 + 2m^5 + 11m^4 + 12m^3 + 55m^2 + 50m + 125}{m^2(5-m)^4}\right)^{1/2}$$

$$= \frac{f^5(-q^2)}{f(-q^{10})}\left(\frac{m^2(5-m)^4 + 22(m-1)^2m(5-m)^2 + 125(m-1)^4}{m^2(5-m)^4}\right)^{1/2}$$

$$= \frac{f^5(-q^2)}{f(-q^{10})}\left(1 + 22q^2\frac{f^6(-q^{10})}{f^6(-q^2)} + 125q^4\frac{f^{12}(-q^{10})}{f^{12}(-q^2)}\right)^{1/2}, \quad (4.4)$$

from which, upon the replacement of q^2 by q, the first equality of (i) is apparent.

We now prove the second equality of (i). First, from (4.3) and the equality $m = \varphi^2(q)/\varphi^2(q^5)$,

$$S = \frac{\rho(m^2 + 2m + 5)z_1^2}{16m^3}$$

$$= \frac{\varphi^4(q) + 2\varphi^2(q)\varphi^2(q^5) + 5\varphi^4(q^5)}{16\varphi(q)\varphi(q^5)}\{\varphi^4(q) - 2\varphi^2(q)\varphi^2(q^5) + 5\varphi^4(q^5)\}^{1/2}.$$

Observe that (4.4) is invariant under a change of sign of q. Thus, by (4.4), the first part of (i), and the foregoing equality,

$$\frac{\{f^{12}(-q^2) + 22q^2f^6(-q^2)f^6(-q^{10}) + 125q^4f^{12}(-q^{10})\}^{1/2}}{\varphi^2(q)\varphi^2(q^5)f(-q^2)f(-q^{10})}$$

$$= \frac{\varphi^4(-q) + 2\varphi^2(-q)\varphi^2(-q^5) + 5\varphi^4(-q^5)}{16\varphi^2(q)\varphi^2(q^5)\varphi(-q)\varphi(-q^5)}$$

$$\times \{\varphi^4(-q) - 2\varphi^2(-q)\varphi^2(-q^5) + 5\varphi^4(-q^5)\}^{1/2}. \quad (4.5)$$

Converting (4.5) into a modular equation via Entries 10(i), (ii) and 12(iii) in Chapter 17, we find that

$$\frac{\{m^3\alpha(1-\alpha) + 22\sqrt{\alpha\beta(1-\alpha)(1-\beta)} + 125m^{-3}\beta(1-\beta)\}^{1/2}}{2^{4/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}}$$

$$= \frac{m(1-\alpha) + 2\sqrt{(1-\alpha)(1-\beta)} + 5m^{-1}(1-\beta)}{16\{(1-\alpha)(1-\beta)\}^{1/4}}$$
$$\times \{m(1-\alpha) - 2\sqrt{(1-\alpha)(1-\beta)} + 5m^{-1}(1-\beta)\}^{1/2}. \qquad (4.6)$$

The left side of (4.6) is self-reciprocal and so is equal to the reciprocal of the right side of (4.6). Hence, by (4.5) and (4.6),

$$\frac{\{f^{12}(-q^2) + 22q^2f^6(-q^2)f^6(-q^{10}) + 125q^4f^{12}(-q^{10})\}^{1/2}}{\varphi^2(q)\varphi^2(q^5)f(-q^2)f(-q^{10})}$$
$$= \frac{5m^{-1}\beta + 2\sqrt{\alpha\beta} + m\alpha}{16(\alpha\beta)^{1/4}}\{5m^{-1}\beta - 2\sqrt{\alpha\beta} + m\alpha\}^{1/2}$$
$$= \frac{\psi^4(q^2) + 2q^2\psi^2(q^2)\psi^2(q^{10}) + 5q^4\psi^4(q^{10})}{\varphi^2(q)\varphi^2(q^5)\psi(q^2)\psi(q^{10})}$$
$$\times \{\psi^4(q^2) - 2q^2\psi^2(q^2)\psi^2(q^{10}) + 5q^4\psi^4(q^{10})\}^{1/2},$$

by Entry 11(iii) in Chapter 17. Replacing q^2 by q, we complete the task of showing that the second and third expressions in (i) are equal.

Proof of (ii). By Entry 12(iii) of Chapter 17 and (13.8) of Chapter 19,

$$1 - 2q\frac{f^2(-q)f^2(-q^{10})}{\varphi^2(q)\varphi^2(q^5)} = 1 - 2^{-1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}$$
$$= 1 - \frac{(m-1)(5-m)}{8m}$$
$$= \frac{m^2 + 2m + 5}{8m}. \qquad (4.7)$$

Similarly, by Entry 12(v) in Chapter 17 and (13.8) in Chapter 19,

$$1 - \frac{4q}{\chi^4(q)\chi^4(q^5)} = 1 - 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}$$
$$= 1 - \frac{(m-1)(5-m)}{4m}$$
$$= \frac{m^3 - 2m^2 + 5m}{4m^2} = \frac{\rho^2}{4m^2}, \qquad (4.8)$$

by (13.3) of Chapter 19. Hence, combining (4.7) and (4.8), we conclude that

$$\left(1 - 2q\frac{f^2(-q)f^2(-q^{10})}{\varphi^2(q)\varphi^2(q^5)}\right)\left(1 - \frac{4q}{\chi^4(q)\chi^4(q^5)}\right)^{1/2} = \frac{\rho(m^2 + 2m + 5)}{16m^2}.$$

Appealing to (4.3), we finish the proof of (ii).

Proof of (iii). By (13.7) in Chapter 19,

$$\{\tfrac{1}{2}(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)})\}^{1/2} = \frac{\rho}{2m}. \qquad (4.9)$$

Thus,

$$3 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} = 2 + \{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\}$$
$$= 2 + \frac{\rho^2}{2m^2}$$
$$= \frac{m^2 + 2m + 5}{2m}. \tag{4.10}$$

Hence, by (4.9) and (4.10),

$$\tfrac{1}{4}\{3 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\}\{\tfrac{1}{2}(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)})\}^{1/2}$$
$$= \frac{\rho(m^2 + 2m + 5)}{16m^2}.$$

By (4.3), we complete the proof of the first equality of (iii).

By (13.7), (13.8), and (13.3) of Chapter 19,

$$\tfrac{1}{2}\{1 + \alpha\beta + (1-\alpha)(1-\beta)\} - \tfrac{3}{4}\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/3}$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\left(\frac{\rho^2}{2m^2} - 1\right)^2 - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/2} - \tfrac{3}{4}\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/3}$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\left(\frac{\rho^2}{2m^2} - 1\right)^2 + \frac{(m^2 - 6m + 5)^3}{256m^3} - \frac{3(m^2 - 6m + 5)^2}{64m^2}$$
$$= \frac{1}{256m^3}(m^6 + 2m^5 + 11m^4 + 12m^3 + 55m^2 + 50m + 125)$$
$$= \frac{\rho^2}{256m^4}(m^2 + 2m + 5)^2.$$

(It is quite clear that we have omitted a heavy dosage of tedious algebra.) Taking the square root of both sides and using (4.3) again, we complete the proof of the second equality of (iii).

Entry 5.

(i) $$1 + 4\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k} - 28\sum_{k=1}^{\infty} \frac{kq^{7k}}{1-q^{7k}}$$
$$= \left\{1 + 2\left(\frac{q}{1-q} + \frac{q^2}{1-q^2} - \frac{q^3}{1-q^3} + \frac{q^4}{1-q^4} - \frac{q^5}{1-q^5} - \frac{q^6}{1-q^6} + \frac{q^8}{1-q^8} + \cdots\right)\right\}^2$$
$$= \left\{\frac{f^8(-q) + 13qf^4(-q)f^4(-q^7) + 49q^2f^8(-q^7)}{f(-q)f(-q^7)}\right\}^{2/3}.$$

In the middle expression, the cycle of coefficients has length 7.

(ii)
$$1+4\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-28\sum_{k=1}^{\infty}\frac{kq^{14k}}{1-q^{14k}}$$
$$=\{\varphi(q)\varphi(q^7)-2q\psi(-q)\psi(-q^7)\}^2.$$

(iii) *Let β have degree 7 over α. Then*
$$1+4\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-28\sum_{k=1}^{\infty}\frac{kq^{14k}}{1-q^{14k}}$$
$$=\tfrac{1}{2}\varphi^2(q)\varphi^2(q^7)\{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\}.$$

In our proof of Entry 5, all references of the form (19.-) arise from Chapter 19.

Proof of (i). We begin by replacing q by $-q$ on the left side of (i). Thus, we first derive an analogue of (2.3) wherein q^2 has been replaced by $-q$.

With $q=e^{-y}$ and $u=q^n$, where n is a positive integer,
$$2\frac{d}{dy}\operatorname{Log}\left(\frac{e^{-ny/2}f^{12}(e^{-ny})}{e^{-y/2}f^{12}(e^{-y})}\right)$$
$$=-24q\frac{d}{dq}\operatorname{Log}\left(\frac{q^{n/24}f(q^n)}{q^{1/24}f(q)}\right)$$
$$=24q\frac{\dfrac{d}{dq}(q^{1/24}f(q))}{q^{1/24}f(q)}-24nu\frac{\dfrac{d}{du}(u^{1/24}f(u))}{u^{1/24}f(u)}$$
$$=1-24\sum_{k=1}^{\infty}\frac{k(-q)^k}{1-(-q)^k}-n\left(1-24\sum_{k=1}^{\infty}\frac{k(-q^n)^k}{1-(-q^n)^k}\right),\tag{5.1}$$
by the same argument that we used in (2.1).

On the other hand, by Entries 12(i) and 9(i) in Chapter 17,
$$2\frac{d}{dy}\operatorname{Log}\left(\frac{e^{-ny/2}f^{12}(e^{-ny})}{e^{-y/2}f^{12}(e^{-y})}\right)=2\frac{d}{dy}\operatorname{Log}\left(\frac{z_n^6}{z_1^6}\left\{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right\}^{1/2}\right)$$
$$=-\alpha(1-\alpha)z_1^2\frac{d}{d\alpha}\operatorname{Log}\left(\frac{\beta(1-\beta)}{m^{12}\alpha(1-\alpha)}\right).\tag{5.2}$$

Thus, combining (5.1) and (5.2), we conclude that
$$1-24\sum_{k=1}^{\infty}\frac{k(-q)^k}{1-(-q)^k}-n\left(1-24\sum_{k=1}^{\infty}\frac{k(-q^n)^k}{1-(-q^n)^k}\right)$$
$$=-\alpha(1-\alpha)z_1^2\frac{d}{d\alpha}\operatorname{Log}\left(\frac{\beta(1-\beta)}{m^{12}\alpha(1-\alpha)}\right).\tag{5.3}$$

Now, setting $n=7$ in (5.3), we find that
$$S:=1+4\sum_{k=1}^{\infty}\frac{k(-q)^k}{1-(-q)^k}-28\sum_{k=1}^{\infty}\frac{k(-q^7)^k}{1-(-q^7)^k}$$

$$= \tfrac{1}{6}\alpha(1-\alpha)z_1^2 \frac{d}{d\alpha} \operatorname{Log}\left(\frac{\beta(1-\beta)}{m^{12}\alpha(1-\alpha)}\right)$$

$$= \tfrac{1}{6}\alpha(1-\alpha)z_1^2\left(\frac{1-2\beta}{\beta(1-\beta)}\frac{d\beta}{d\alpha} - \frac{1-2\alpha}{\alpha(1-\alpha)} - 12\frac{d}{d\alpha}\operatorname{Log} m\right)$$

$$= \tfrac{7}{6}(1-2\beta)z_7^2 - \tfrac{1}{6}(1-2\alpha)z_1^2 - 2\alpha(1-\alpha)z_1^2\frac{d}{d\alpha}\operatorname{Log} m, \tag{5.4}$$

where (19.17) has been utilized.

Let $t > 0$ be defined by (19.2). Observe that, for any positive integer n,

$$\left(m + \frac{n}{m}\right)^{-1}\frac{d}{dt}\left(m - \frac{n}{m}\right) = \left(m + \frac{n}{m}\right)^{-1}\left(1 + \frac{n}{m^2}\right)\frac{dm}{dt} = \frac{1}{m}\frac{dm}{dt}. \tag{5.5}$$

Also, in general, if β has degree n and F is any differentiable function, then, from (19.17), it is easily shown that

$$\alpha(1-\alpha)z_1^2\frac{dF}{d\alpha} = n\beta(1-\beta)z_n^2\frac{dF}{d\beta}. \tag{5.6}$$

Thus, from (5.4)–(5.6),

$$S = \tfrac{1}{6}z_1 z_7\left(\frac{7}{m}(1-2\beta) - m(1-2\alpha) - 12m\alpha(1-\alpha)\frac{dt}{d\alpha}\frac{d}{dt}\operatorname{Log} m\right)$$

$$= \tfrac{1}{6}z_1 z_7\left(\frac{7}{m}(1-2\beta) - m(1-2\alpha)\right.$$

$$\left. - 12\left\{7\alpha\beta(1-\alpha)(1-\beta)\frac{dt}{d\alpha}\frac{dt}{d\beta}\right\}^{1/2}\left(m + \frac{7}{m}\right)^{-1}\frac{d}{dt}\left(m - \frac{7}{m}\right)\right). \tag{5.7}$$

By (19.23),

$$m - \frac{7}{m} = -6 + 16t - 12t^2 + 8t^3, \tag{5.8}$$

and so

$$\frac{d}{dt}\left(m - \frac{7}{m}\right) = 16 - 24t + 24t^2. \tag{5.9}$$

From (19.20) and (19.22), respectively,

$$m = -3 + 8t - 6t^2 + 4t^3 + 2R$$

and

$$\frac{7}{m} = 3 - 8t + 6t^2 - 4t^3 + 2R,$$

where R is defined by (19.6). Thus,

$$m + \frac{7}{m} = 4R. \tag{5.10}$$

Hence, from (5.7), (5.9), and (5.10),

$$S = \tfrac{1}{6}z_1 z_7\Bigg(\left(m + \frac{7}{m}\right)(\alpha - \beta) + \left(m - \frac{7}{m}\right)(\alpha + \beta - 1) - 24\left\{7\alpha\beta(1-\alpha)(1-\beta)\frac{dt}{d\alpha}\frac{dt}{d\beta}\right\}^{1/2}\frac{2 - 3t + 3t^2}{R}\Bigg). \tag{5.11}$$

Our next goal is to obtain a suitable expression for $(dt/d\alpha)\,(dt/d\beta)$. From (19.2),

$$\beta + \alpha\frac{d\beta}{d\alpha} = 8t^7\frac{dt}{d\alpha} \quad \text{and} \quad \alpha + \beta\frac{d\alpha}{d\beta} = 8t^7\frac{dt}{d\beta}.$$

Therefore, from these equalities, (19.17), and (19.18),

$$\begin{aligned}
\frac{dt}{d\alpha}\frac{dt}{d\beta} &= \frac{1}{64t^{14}}\left(\beta + \alpha\frac{d\beta}{d\alpha}\right)\left(\alpha + \beta\frac{d\alpha}{d\beta}\right)\\
&= \frac{t^2}{64\alpha^2\beta^2}\left(\beta + \frac{7\beta(1-\beta)}{m^2(1-\alpha)}\right)\left(\alpha + \frac{m^2\alpha(1-\alpha)}{7(1-\beta)}\right)\\
&= \frac{t^2(14m^2\alpha\beta(1-\alpha)(1-\beta) + m^4\alpha\beta(1-\alpha)^2 + 49\alpha\beta(1-\beta)^2)}{448\alpha^2\beta^2(1-\alpha)(1-\beta)m^2}\\
&= -\frac{t^2(-2(t-\alpha)(t-\beta)(1-\alpha)(1-\beta) + (t-\beta)^2(1-\alpha)^2 + (t-\alpha)^2(1-\beta)^2)}{64\alpha\beta(1-\alpha)(1-\beta)(t-\alpha)(t-\beta)}\\
&= -\frac{t^2\{(t-\beta)(1-\alpha) - (t-\alpha)(1-\beta)\}^2}{64\alpha\beta(1-\alpha)(1-\beta)(t-\alpha)(t-\beta)}\\
&= \frac{t^2(\alpha-\beta)^2(1-t)^2}{64\alpha\beta(1-\alpha)(1-\beta)(\alpha-t)(t-\beta)}.
\end{aligned}$$

As we observed after (19.19), $\alpha > t$ and $\beta < t$. Thus, taking the square root on each side above, we deduce that

$$\left(\frac{dt}{d\alpha}\frac{dt}{d\beta}\right)^{1/2} = \frac{t(\alpha-\beta)(1-t)}{8\{\alpha\beta(1-\alpha)(1-\beta)(\alpha-t)(t-\beta)\}^{1/2}}.$$

Hence, from (5.8), (5.10), and (5.11),

$$S = \tfrac{1}{6}z_1 z_7\Bigg(4R(\alpha-\beta) + (-6 + 16t - 12t^2 + 8t^3)(\alpha + \beta - 1) - 3t(1-t)(\alpha-\beta)\left(\frac{7}{(\alpha-t)(t-\beta)}\right)^{1/2}\frac{2 - 3t + 3t^2}{R}\Bigg). \tag{5.12}$$

From (19.18) and (19.19),

$$(\alpha - t)(t - \beta) = \frac{7}{m^2}(t-\beta)^2 = 7t^2(1-t)^2(1 - t + t^2)^2.$$

Hence,

$$\left(\frac{7}{(\alpha-t)(t-\beta)}\right)^{1/2}=\frac{1}{t(1-t)(1-t+t^2)}.$$

By (19.5), $\alpha-\beta=2BR$ and $\alpha+\beta=2A$. Substituting these expressions and then the values of A, B, and R into (5.12), we discern that

$$\begin{aligned}S&=\tfrac{1}{6}z_1z_7\left(8BR^2+(-6+16t-12t^2+8t^3)(2A-1)-\frac{6B(2-3t+3t^2)}{1-t+t^2}\right)\\&=\tfrac{1}{6}z_1z_7\Big(16t(1-t)(1-t+t^2)(2-3t+2t^2)(2-t+t^2)(1-t+2t^2)\\&\qquad+(-6+16t-12t^2+8t^3)\{(1+t^8)-(1-t)^8-1\}\\&\qquad-12t(1-t)(2-3t+3t^2)\Big)\\&=z_1z_7(1-2t+2t^2)^2\\&=z_1z_7\{(1-t)^2+t^2\}^2,\end{aligned}\tag{5.13}$$

where the algebra was effected by a computer algebra package. Utilizing (19.2), (19.3), and Entries 10(ii) and 11(iii) in Chapter 17, we deduce that

$$\begin{aligned}S&=z_1z_7(\{(1-\alpha)(1-\beta)\}^{1/4}+(\alpha\beta)^{1/4})^2\\&=\{\varphi(-q)\varphi(-q^7)+4q^2\psi(q^2)\psi(q^{14})\}^2.\end{aligned}\tag{5.14}$$

Lastly, we use Entries 17(i), (ii) in Chapter 19 to find that

$$\begin{aligned}&\varphi(q)\varphi(q^7)+4q^2\psi(q^2)\psi(q^{14})\\&\quad=1+2\left(\frac{q}{1-q}+\frac{q^2}{1-q^2}-\frac{q^3}{1-q^3}+\frac{q^4}{1-q^4}-\frac{q^5}{1-q^5}-\frac{q^6}{1-q^6}+\cdots\right),\end{aligned}\tag{5.15}$$

where the cycle of coefficients is of length 7. Changing the sign of q in (5.14) and combining the result with (5.15), we complete the proof of the first equality in part (i).

In order to establish the second equality of (i), we first observe, by Entry 12(i) in Chapter 17, (19.2), and (19.3), that

$$\left\{\frac{2q}{t(1-t)}\right\}^{2/3}f^2(q)f^2(q^7)=z_1z_7.$$

It follows that

$$\begin{aligned}&z_1z_7(1-2t+2t^2)^2\\&\quad=(2q)^{2/3}f^2(q)f^2(q^7)\left(\frac{1-6t+18t^2-32t^3+36t^4-24t^5+8t^6}{t(1-t)}\right)^{2/3}.\end{aligned}\tag{5.16}$$

On the other hand, by Entry 12(i) in Chapter 17, (19.2), (19.3), (19.14), (19.15), (19.6), (5.8), and (5.10),

$$
\begin{aligned}
&\frac{f^4(q)}{qf^4(q^7)} + 49\frac{qf^4(q^7)}{f^4(q)} \\
&= m^2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6} + \frac{49}{m^2}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6} \\
&= \frac{1}{t(1-t)}\left\{m^2\left(\frac{\alpha(1-\alpha)}{t(1-t)}\right)^{1/3} + \frac{49}{m^2}\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/3}\right\} \\
&= \frac{1}{2t(1-t)}\left\{(2-7t+11t^2-8t^3+4t^4)\left(m^2+\frac{49}{m^2}\right)\right. \\
&\qquad \left. + (1-2t)R\left(m^2-\frac{49}{m^2}\right)\right\} \\
&= \frac{1}{2t(1-t)}\left\{\left(m-\frac{7}{m}\right)\left((2-7t+11t^2-8t^3+4t^4)\left(m-\frac{7}{m}\right)\right.\right. \\
&\qquad \left.\left. + (1-2t)R\left(m+\frac{7}{m}\right)\right) + 14(2-7t+11t^2-8t^3+4t^4)\right\} \\
&= \frac{1}{t(1-t)}\left\{2(-3+8t-6t^2+4t^3)\Big((2-7t+11t^2-8t^3+4t^4)\right. \\
&\qquad \times(-3+8t-6t^2+4t^3) \\
&\qquad + 2(1-2t)(2-3t+2t^2)(2-t+t^2)(1-t+2t^2)\Big) \\
&\qquad \left. + 7(2-7t+11t^2-8t^3+4t^4)\right\} \\
&= \frac{1}{t(1-t)}\left\{2(-3+8t-6t^2+4t^3)(2-3t-3t^2+2t^3)\right. \\
&\qquad \left. + 7(2-7t+11t^2-8t^3+4t^4)\right\} \\
&= \frac{1}{t(1-t)}(2+t+23t^2-64t^3+72t^4-48t^5+16t^6). \qquad (5.17)
\end{aligned}
$$

Combining (5.16) and (5.17), we deduce that

$$
\begin{aligned}
z_1z_7(1-2t+2t^2)^2 &= q^{2/3}f^2(q)f^2(q^7)\left(\frac{f^4(q)}{qf^4(q^7)} + 49\frac{qf^4(q^7)}{f^4(q)} - 13\right)^{2/3} \\
&= \left(\frac{f^8(q)+49q^2f^8(q^7)-13qf^4(q)f^4(q^7)}{f(q)f(q^7)}\right)^{2/3}.
\end{aligned}
$$

Using the equality above in (5.13) and changing the sign of q, we complete the proof of the second part of (i).

PROOFS OF (ii), (iii). Replacing q by $-q^2$, we showed in (5.14) that

$$1+4\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-28\sum_{k=1}^{\infty}\frac{kq^{14k}}{1-q^{14k}}$$
$$=\{\varphi(q^2)\varphi(q^{14})+4q^4\psi(q^4)\psi(q^{28})\}^2. \tag{5.18}$$

By Entries 10(iv) and 11(iv) in Chapter 17, (20.6) in Chapter 20, (19.2), and (19.3),

$$\begin{aligned}
&\varphi(q^2)\varphi(q^{14})+4q^4\psi(q^4)\psi(q^{28})\\
&=\tfrac{1}{2}\sqrt{z_1z_7}(\{1+\sqrt{1-\alpha}\}^{1/2}\{1+\sqrt{1-\beta}\}^{1/2}\\
&\quad+\{1-\sqrt{1-\alpha}\}^{1/2}\{1-\sqrt{1-\beta}\}^{1/2})\\
&=\sqrt{z_1z_7}(\tfrac{1}{2}\{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\})^{1/2}\\
&=\sqrt{z_1z_7}(\tfrac{1}{2}\{1+t^4+(1-t)^4\})^{1/2}\\
&=\sqrt{z_1z_7}\{1-t(1-t)\}\\
&=\sqrt{z_1z_7}(1-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8})\\
&=\varphi(q)\varphi(q^7)-2q\psi(-q)\psi(-q^7),
\end{aligned} \tag{5.19}$$

where we have invoked Entries 10(i) and 11(ii) of Chapter 17. Substituting the far right side of (5.19) into (5.18), we complete the proof of (ii). Furthermore, substituting the second expression on the right side of (5.19) into (5.18), we deduce (iii).

Entry 6.

(i) *If β has degree 3, then*

$$1+12\sum_{k=1}^{\infty}\frac{k(-q)^k}{1-(-q)^k}-36\sum_{k=1}^{\infty}\frac{k(-q)^{3k}}{1-(-q)^{3k}}$$
$$=\varphi^2(q)\varphi^2(q^3)((\alpha\beta)^{1/4}-\{(1-\alpha)(1-\beta)\}^{1/4})^2.$$

(ii) *If β has degree 5, then*

$$1+6\sum_{k=1}^{\infty}\frac{k(-q)^k}{1-(-q)^k}-30\sum_{k=1}^{\infty}\frac{k(-q)^{5k}}{1-(-q)^{5k}}$$
$$=\varphi^2(q)\varphi^2(q^5)(\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)})$$
$$\times(\tfrac{1}{2}\{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}\})^{1/2}.$$

(iii) *If β has degree 7, then*

$$1+4\sum_{k=1}^{\infty}\frac{k(-q)^k}{1-(-q)^k}-28\sum_{k=1}^{\infty}\frac{k(-q)^{7k}}{1-(-q)^{7k}}$$
$$=\varphi^2(q)\varphi^2(q^7)((\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4})^2.$$

PROOF OF (i). By Entry 3(i), the series on the left side of Entry 6(i) is equal to

$$\Psi := \left\{\frac{\psi^4(-q) - 3q\psi^4(-q^3)}{\psi(-q)\psi(-q^3)}\right\}^2.$$

Using Entry 11(ii) of Chapter 17 and (5.2) and (5.5) from Chapter 19, we deduce that

$$\begin{aligned}\Psi &= \left\{\frac{z_1^2\{\alpha(1-\alpha)\}^{1/2} - 3z_3^2\{\beta(1-\beta)\}^{1/2}}{2(z_1z_3)^{1/2}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}}\right\}^2 \\ &= \left\{\frac{z_1^2(9-m^2) - 3z_3^2(m^2-1)m^2}{8m^2(z_1z_3)^{1/2}}\right\}^2 \\ &= \left\{\frac{(9-m^2) - 3(m^2-1)}{8m}\sqrt{z_1z_3}\right\}^2 \\ &= z_1z_3\left\{\frac{(m+1)(3-m)}{4m} - \frac{(m-1)(3+m)}{4m}\right\}^2 \\ &= z_1z_3(\{(1-\alpha)(1-\beta)\}^{1/4} - (\alpha\beta)^{1/4})^2. \end{aligned} \tag{6.1}$$

Hence, the truth of (i) is made manifest.

PROOF OF (ii). By Entry 4(i), the series on the left side of Entry 6(ii) is equal to

$$\Psi := \frac{\psi^4(-q) - 2q\psi^2(-q)\psi^2(-q^5) + 5q^2\psi^4(-q^5)}{\psi(-q)\psi(-q^5)} \times \{\psi^4(-q) + 2q\psi^2(-q)\psi^2(-q^5) + 5q^2\psi^4(-q^5)\}^{1/2}.$$

By Entry 11(ii) in Chapter 17 and (14.2) and (14.4) in Chapter 19,

$$\begin{aligned}\Psi &= \frac{\psi^5(-q^5)q^3}{z_5\psi(-q)}\left(\frac{2m^2\{\alpha(1-\alpha)\}^{1/2}}{\{\beta(1-\beta)\}^{3/4}} - \frac{4m\{\alpha(1-\alpha)\}^{1/4}}{\{\beta(1-\beta)\}^{1/2}} + \frac{10}{\{\beta(1-\beta)\}^{1/4}}\right) \\ &\quad \times (\tfrac{1}{4}z_1^2\{\alpha(1-\alpha)\}^{1/2} + \tfrac{1}{2}z_1z_5\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4} + \tfrac{5}{4}z_5^2\{\beta(1-\beta)\}^{1/2})^{1/2} \\ &= \frac{\psi^5(-q^5)q^3}{\psi(-q)}\left(\frac{m^2\left(\frac{5}{m}-1\right)^{7/4}}{(m-1)^{13/4}} - \frac{2m\left(\frac{5}{m}-1\right)^{3/4}}{(m-1)^{9/4}} + \frac{5}{\left(\frac{5}{m}-1\right)^{1/4}(m-1)^{5/4}}\right) \\ &\quad \times \left(m^2\left(\frac{5}{m}-1\right)^{5/2}(m-1)^{1/2} + 2m\left(\frac{5}{m}-1\right)^{3/2}(m-1)^{3/2}\right. \\ &\quad \left. + 5(m-1)^{5/2}\left(\frac{5}{m}-1\right)^{1/2}\right)^{1/2}\end{aligned}$$

$$= \frac{\psi^5(-q^5)q^3}{\psi(-q)(m-1)^3}\left((5-m)^2 - 2(5-m)(m-1) + 5(m-1)^2\right)$$
$$\times\ ((5-m)^2 + 2(5-m)(m-1) + 5(m-1)^2)^{1/2}$$
$$= \frac{\psi^5(-q^5)q^3}{\psi(-q)(m-1)^3}(40 - 32m + 8m^2)(20 - 8m + 4m^2)^{1/2}. \tag{6.2}$$

From Entry 11(ii) in Chapter 17 and (13.3)–(13.5) in Chapter 19,

$$\frac{16q^3\psi^5(-q^5)}{\psi(-q)(m-1)^3} = \frac{4z_5^{5/2}}{z_1^{1/2}(m-1)^3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/8}$$
$$= \frac{z_5^{5/2}(\rho^2 - (m+1)^2)}{4z_1^{1/2}(m-1)^3} = \frac{z_5^{5/2}}{4z_1^{1/2}} = \frac{z_1 z_5}{4m^{3/2}}.$$

Using the calculation above in (6.2) and then employing (13.6) and (13.7) in Chapter 19, we discern that

$$\Psi = z_1 z_5 \frac{m^2 - 4m + 5}{2m}\left(\frac{m^2 - 2m + 5}{4m}\right)^{1/2}$$
$$= z_1 z_5(\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)})(\tfrac{1}{2}\{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\})^{1/2},$$

which completes the proof of (ii).

Proof of (iii). Observe that (5.14) is precisely Entry 6(iii), and so the proof has already been accomplished.

Entry 7.

(i) $$1 + 3\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} - 27\sum_{k=1}^{\infty}\frac{kq^{9k}}{1-q^{9k}}$$
$$= \frac{f^6(-q^3)}{f^2(-q)f^2(-q^9)}\{f^6(-q) + 9qf^3(-q)f^3(-q^9) + 27q^2f^6(-q^9)\}^{1/3}$$
$$= \left\{\frac{\psi^4(q^3) + 3q\psi^2(q)\psi^2(q^9)}{\psi(q)\psi(q^9)}\right\}^2 \frac{\psi^2(q^3)}{\psi(q)\psi(q^9)}.$$

(ii) $$1 + 3\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 27\sum_{k=1}^{\infty}\frac{kq^{18k}}{1-q^{18k}}$$
$$= \left\{\frac{\varphi^4(q^3) + 3\varphi^2(q)\varphi^2(q^9)}{4}\right\}^2 \frac{\varphi^2(q^3)}{\varphi^3(q)\varphi^3(q^9)}.$$

(iii) $$1 + \sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} - 25\sum_{k=1}^{\infty}\frac{kq^{25k}}{1-q^{25k}}$$
$$= \frac{f^5(-q^5)}{f(-q)f(-q^{25})}\{f^2(-q) + 2qf(-q)f(-q^{25}) + 5q^2f^2(-q^{25})\}^{1/2}.$$

Throughout the proofs of (i) and (ii), we use the notation of Section 3 in Chapter 20.

PROOF OF (i). It will again be judicious to make a change of sign. By (6.1),

$$\begin{aligned}
S &:= 1 + 3\sum_{k=1}^{\infty}\frac{k(-q)^k}{1-(-q)^k} - 27\sum_{k=1}^{\infty}\frac{k(-q)^{9k}}{1-(-q)^{9k}} \\
&= \frac{1}{4}\left(1 + 12\sum_{k=1}^{\infty}\frac{k(-q)^k}{1-(-q)^k} - 36\sum_{k=1}^{\infty}\frac{k(-q)^{3k}}{1-(-q)^{3k}}\right) \\
&\quad + \frac{3}{4}\left(1 + 12\sum_{k=1}^{\infty}\frac{k(-q)^{3k}}{1-(-q)^{3k}} - 36\sum_{k=1}^{\infty}\frac{k(-q)^{9k}}{1-(-q)^{9k}}\right) \\
&= \tfrac{1}{4}z_1z_3\left(\frac{(m+1)(3-m)}{4m} - \frac{(m-1)(3+m)}{4m}\right)^2 \\
&\quad + \tfrac{3}{4}z_3z_9\left(\frac{(m'+1)(3-m')}{4m'} - \frac{(m'-1)(3+m')}{4m'}\right)^2 \\
&= \tfrac{1}{4}z_1z_3\left(\frac{3-m^2}{2m}\right)^2 + \tfrac{3}{4}z_3z_9\left(\frac{3-m'^2}{2m'}\right)^2.
\end{aligned} \tag{7.1}$$

Our next task is to express these last expressions in terms of t. By (3.10) and (3.11) of Chapter 20 and (7.1),

$$\begin{aligned}
S &= \tfrac{1}{4}mm'z_3z_9\left(\frac{3-m^2}{2m}\right)^2 + \tfrac{3}{4}z_3z_9\left(\frac{3-m'^2}{2m'}\right)^2 \\
&= \frac{z_3z_9}{16}\left(\frac{\{3(1+8t^3)-(1+2t)^4\}^2}{(1+2t)^2(1+8t^3)} + \frac{3\{3-(1+8t^3)\}^2}{1+8t^3}\right) \\
&= \frac{z_3z_9}{16(1+8t^3)}(\{3(1-2t+4t^2)-(1+2t)^3\}^2 + 12(1-4t^3)^2) \\
&= \frac{z_3z_9}{4(1+8t^3)}(\{1-6t-4t^3\}^2 + 3(1-4t^3)^2) \\
&= \frac{z_3z_9}{1+8t^3}(1-3t+9t^2-8t^3+12t^4+16t^6) \\
&= \frac{z_3z_9}{1+8t^3}(1+t+t^2)(1-2t+4t^2)^2 \\
&= \frac{z_3z_9}{1+2t}(1+t+t^2)(1-2t+4t^2).
\end{aligned} \tag{7.2}$$

Next, we attempt to identify this last expression with the middle expression of (i). By Entry 12(i) of Chapter 17 and (3.7)–(3.11) of Chapter 20,

$$\frac{q^{1/3}f^6(q^3)}{f(q)f(q^9)} = \frac{z_3^3\{\beta(1-\beta)\}^{1/4}}{(z_1z_9)^{1/2}2^{2/3}\{\alpha(1-\alpha)\gamma(1-\gamma)\}^{1/24}}$$

$$= \frac{z_3 z_9 m'^{3/2} t^{1/3}(1-t^3)^{2/3}(1+2t)^{1/3}}{m^{1/2}(1-t)^{1/3}(1+8t^3)^{2/3}}$$

$$= \frac{z_3 z_9 (1+8t^3) t^{1/3}(1-t^3)^{2/3}(1+2t)^{1/3}}{(1+2t)(1-t)^{1/3}(1+8t^3)^{2/3}}$$

$$= \frac{z_3 z_9 t^{1/3}(1-2t+4t^2)^{1/3}(1-t)^{1/3}(1+t+t^2)^{2/3}}{(1+2t)^{1/3}}.$$

Hence, we may rewrite (7.2) in the form

$$S = \frac{q^{1/3} f^6(q^3)}{f(q)f(q^9)} \left(\frac{(1+t+t^2)(1-2t+4t^2)^2}{t(1+2t)^2(1-t)} \right)^{1/3}$$

$$= \frac{q^{1/3} f^6(q^3)}{f(q)f(q^9)} \left(\frac{(1+2t)^2(1-t)}{t} - 9 + 27 \frac{t}{(1+2t)^2(1-t)} \right)^{1/3}, \tag{7.3}$$

where the last equality is verified by straightforward algebra.

It now suffices to prove that

$$\frac{f^3(q)}{qf^3(q^9)} = \frac{(1+2t)^2(1-t)}{t}, \tag{7.4}$$

for upon substituting (7.4) into (7.3) and changing the sign of q, we then obtain the first equality of part (i).

Employing Entry 12(i) in Chapter 17 and (3.7), (3.9), (3.10), and (3.11) in Chapter 20, we find that

$$\frac{f^3(q)}{qf^3(q^9)} = \left(\frac{z_1}{z_9} \right)^{3/2} \left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)} \right)^{1/8}$$

$$= (mm')^{3/2} \frac{(1-t)}{t(1+2t)} = \frac{(1+2t)^2(1-t)}{t},$$

as desired.

To establish the second part of (ii), we first observe that by Entry 11(ii) in Chapter 17 and (3.7)–(3.11) in Chapter 20,

$$q^2\psi^2(-q^3)\psi(-q)\psi(-q^9)$$

$$= \tfrac{1}{4} z_3 \sqrt{z_1 z_9} \{\beta(1-\beta)\}^{1/4} \{\alpha(1-\alpha)\gamma(1-\gamma)\}^{1/8}$$

$$= z_3 z_9 \frac{(1-t)(1-t^3)t^2}{1+8t^3}.$$

Thus, by (7.2), we see that

$$S = q^2\psi^2(-q^3)\psi(-q)\psi(-q^9) \left(\frac{1-2t+4t^2}{t(1-t)} \right)^2$$

$$= q^2\psi^2(-q^3)\psi(-q)\psi(-q^9) \left(\frac{1+t+t^2}{t(1-t)} - 3 \right)^2. \tag{7.5}$$

By Entry 11(ii) in Chapter 17 and (3.7)–(3.11) of Chapter 20,

$$\frac{\psi^4(-q^3)}{q\psi^2(-q)\psi^2(-q^9)} = \frac{m'\{\beta(1-\beta)\}^{1/2}}{m\{\alpha(1-\alpha)\gamma(1-\gamma)\}^{1/4}} = \frac{1+t+t^2}{t(1-t)}. \tag{7.6}$$

Substituting (7.6) into (7.5) and changing the sign of q, we establish the second part of (i).

PROOF OF (ii). By part (i), the series on the left side of (ii) is equal to

$$T := \frac{q^{2/3}f^6(-q^6)}{f(-q^2)f(-q^{18})}\left(\frac{f^3(-q^2)}{q^2f^3(-q^{18})} + 9 + 27\frac{q^2f^3(-q^{18})}{f^3(-q^2)}\right).$$

Utilizing Entry 12(iii) in Chapter 17 and (3.7)–(3.11) in Chapter 20, we deduce that

$$\begin{aligned}
T &= \frac{z_3^3\{\beta(1-\beta)\}^{1/2}}{2^{4/3}(z_1z_9)^{1/2}\{\alpha(1-\alpha)\gamma(1-\gamma)\}^{1/12}} \\
&\quad \times \left\{\left(\frac{z_1}{z_9}\right)^{3/2}\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/4} + 9 + 27\left(\frac{z_9}{z_1}\right)^{3/2}\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/4}\right\}^{1/3} \\
&= \frac{z_3^3t^{2/3}(1-t^3)^{4/3}(1+2t)^{2/3}}{(z_1z_9)^{1/2}(1-t)^{2/3}(1+8t^3)^{4/3}} \\
&\quad \times \left\{\frac{(1+2t)(1-t)^2}{t^2} + 9 + 27\frac{t^2}{(1+2t)(1-t)^2}\right\}^{1/3} \\
&= z_3(z_1z_9)^{1/2}\frac{m'}{m}\frac{(1+t+t^2)^{4/3}(1+3t^2+4t^3+9t^4+6t^5+4t^6)^{1/3}}{(1+8t^3)(1-2t+4t^2)^{1/3}} \\
&= z_3(z_1z_9)^{1/2}\frac{1+8t^3}{(1+2t)^2}\frac{(1+t+t^2)^2}{1+8t^3} \\
&= z_3(z_1z_9)^{1/2}\left(\frac{1+t+t^2}{1+2t}\right)^2 \\
&= z_3(z_1z_9)^{1/2}\left(\frac{3}{4} + \frac{1+8t^3}{4(1+2t)^2}\right)^2 \\
&= z_3(z_1z_9)^{1/2}\left(\frac{3}{4} + \frac{m'}{4m}\right)^2 \\
&= \varphi^2(q^3)\varphi(q)\varphi(q^9)\left(\frac{3}{4} + \frac{\varphi^4(q^3)}{4\varphi^2(q)\varphi^2(q^9)}\right)^2,
\end{aligned} \tag{7.7}$$

from which (ii) now readily follows.

PROOF OF (iii). We employ the notation

$$w = \frac{f(-q)}{qf(-q^{25})}, \tag{7.8}$$

which is a modification of (11.7) in Chapter 19. Returning to Entry 4(i) and utilizing (11.8) of Chapter 19, we deduce that

$$1+6\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-30\sum_{k=1}^{\infty}\frac{kq^{5k}}{1-q^{5k}}$$

$$=\frac{f^5(-q)}{f(-q^5)}\left(1+22q\frac{f^6(-q^5)}{f^6(-q)}+125q^2\frac{f^{12}(-q^5)}{f^{12}(-q)}\right)^{1/2}$$

$$=\frac{f^5(-q)}{f(-q^5)}\left(1+\frac{22}{w^6}\frac{f^6(-q^5)}{q^5f^6(-q^{25})}+\frac{125}{w^{12}}\frac{f^{12}(-q^5)}{q^{10}f^{12}(-q^{25})}\right)^{1/2}$$

$$=\frac{f^5(-q)}{f(-q^5)}\left(1+\frac{22}{w^6}(w^5+5w^4+15w^3+25w^2+25w)\right.$$
$$\left.+\frac{125}{w^{12}}(w^5+5w^4+15w^3+25w^2+25w)^2\right)^{1/2}$$

$$=\frac{f^5(-q)}{f(-q^5)w^5}(w^{10}+22w^9+235w^8+1580w^7+7425w^6+25550w^5$$
$$+65625w^4+125000w^3+171875w^2+156250w+78125)^{1/2}$$

$$=\frac{q^5f^5(-q^{25})}{f(-q^5)}(w^4+10w^3+45w^2+100w+125)(w^2+2w+5)^{1/2}.$$

Again using Entry 4(i) and proceeding in the same manner as above, we discover that

$$1+6\sum_{k=1}^{\infty}\frac{kq^{5k}}{1-q^{5k}}-30\sum_{k=1}^{\infty}\frac{kq^{25k}}{1-q^{25k}}$$

$$=\frac{q^5f^5(-q^{25})}{f(-q^5)}\left(\frac{f^{12}(-q^5)}{q^{10}f^{12}(-q^{25})}+22\frac{f^6(-q^5)}{q^5f^6(-q^{25})}+125\right)^{1/2}$$

$$=\frac{q^5f^5(-q^{25})}{f(-q^5)}((w^5+5w^4+15w^3+25w^2+25w)^2$$
$$+22(w^5+5w^4+15w^3+25w^2+25w)+125)^{1/2}$$

$$=\frac{q^5f^5(-q^{25})}{f(-q^5)}(w^{10}+10w^9+55w^8+200w^7+525w^6+1022w^5$$
$$+1485w^4+1580w^3+1175w^2+550w+125)^{1/2}$$

$$=\frac{q^5f^5(-q^{25})}{f(-q^5)}(w^4+4w^3+9w^2+10w+5)(w^2+2w+5)^{1/2}.$$

Multiplying (7.9) by $\frac{1}{6}$ and (7.10) by $\frac{5}{6}$, adding the two resulting equalities, and using (11.8) of Chapter 19 and (7.8), we conclude that

$$1+\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-25\sum_{k=1}^{\infty}\frac{kq^{25k}}{1-q^{25k}}$$

$$=\frac{q^5f^5(-q^{25})}{f(-q^5)}\left(\tfrac{1}{6}(w^4+10w^3+45w^2+100w+125)\right.$$

$$\left.+\tfrac{5}{6}(w^4+4w^3+9w^2+10w+5)\right)(w^2+2w+5)^{1/2}$$

$$=\frac{q^5f^5(-q^{25})}{f(-q^5)}(w^4+5w^3+15w^2+25w+25)(w^2+2w+5)^{1/2}$$

$$=\frac{qf^5(-q^5)}{f(-q)}(w^2+2w+5)^{1/2}$$

$$=\frac{qf^5(-q^5)}{f(-q)}\left(\frac{f^2(-q)}{q^2f^2(-q^{25})}+2\frac{f(-q)}{qf(-q^{25})}+5\right)^{1/2},$$

from which the truth of Entry 7(iii) is evident.

Entry 8.

(i) $$5+12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-132\sum_{k=1}^{\infty}\frac{kq^{22k}}{1-q^{22k}}$$

$$=5\varphi^2(q)\varphi^2(q^{11})-20qf^2(q)f^2(q^{11})$$

$$+32q^2f^2(-q^2)f^2(-q^{22})-20q^3\psi^2(-q)\psi^2(-q^{11}).$$

(ii) *Let β have degree 11 over α. Then*

$$5+12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-132\sum_{k=1}^{\infty}\frac{kq^{22k}}{1-q^{22k}}$$

$$=\varphi^2(q)\varphi^2(q^{11})(2+2(\alpha\beta)^{1/2}+2\{(1-\alpha)(1-\beta)\}^{1/2}+(\alpha\beta)^{1/4}$$

$$+\{(1-\alpha)(1-\beta)\}^{1/4}-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}).$$

(iii) *If β has degree 19, then*

$$3+4\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-76\sum_{k=1}^{\infty}\frac{kq^{38k}}{1-q^{38k}}$$

$$=\varphi^2(q)\varphi^2(q^{19})(1+(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}+(\alpha\beta)^{1/4}$$

$$+\{(1-\alpha)(1-\beta)\}^{1/4}-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}).$$

We are unable to prove either (i) or (iii) using (2.3) or (5.3). Thus, the proofs of (i) and (iii) will be deferred until Section 11 where the theory of modular forms will be invoked to establish several results that we cannot otherwise prove. Part (ii) follows from (i), and so this proof will be the only one given in this section.

PROOF OF (ii). By Entries 11(ii) and 12(i), (iii) in Chapter 17 and Entry 7(i) in Chapter 20,

$$
\begin{aligned}
&5\varphi^2(q)\varphi^2(q^{11}) - 20qf^2(q)f^2(q^{11}) \\
&\qquad + 32q^2f^2(-q^2)f^2(-q^{22}) - 20q^3\psi^2(-q)\psi^2(-q^{11}) \\
&\quad = z_1z_{11}(5 - 20\cdot 2^{-2/3}\{\alpha(1-\alpha)\beta(1-\beta)\}^{1/12} \\
&\qquad\qquad + 32\cdot 2^{-4/3}\{\alpha(1-\alpha)\beta(1-\beta)\}^{1/6} - 5\{\alpha(1-\alpha)\beta(1-\beta)\}^{1/4}) \\
&\quad = z_1z_{11}(5 - 5(1 - (\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4}) \\
&\qquad\qquad + 2(1 - (\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4})^2 \\
&\qquad\qquad - 5\{\alpha(1-\alpha)\beta(1-\beta)\}^{1/4}) \\
&\quad = z_1z_{11}(2 + 2(\alpha\beta)^{1/2} + 2\{(1-\alpha)(1-\beta)\}^{1/2} + (\alpha\beta)^{1/4} \\
&\qquad\qquad + \{(1-\alpha)(1-\beta)\}^{1/4} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}).
\end{aligned}
$$

Hence, (i) implies the truth of (ii).

Entry 9.

(i)
$$
\begin{aligned}
&11 + 12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 276\sum_{k=1}^{\infty}\frac{kq^{46k}}{1-q^{46k}} \\
&\quad = \varphi^2(q)\varphi^2(q^{23})\{\tfrac{11}{2}(1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}) \\
&\qquad - 10\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} \\
&\qquad - 8\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}(1 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4})\}.
\end{aligned}
$$

(ii) *If β is of degree* 15, *then*

$$
\begin{aligned}
&7 + 12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 180\sum_{k=1}^{\infty}\frac{kq^{30k}}{1-q^{30k}} \\
&\quad = \tfrac{1}{2}\varphi^2(q)\varphi^2(q^{15})(\{1 + (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8}\}^4 \\
&\qquad - 1 - \sqrt{\alpha\beta} - \sqrt{(1-\alpha)(1-\beta)}).
\end{aligned}
$$

(iii) *If β is of degree* 31, *then*

$$
\begin{aligned}
&5 + 4\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 124\sum_{k=1}^{\infty}\frac{kq^{62k}}{1-q^{62k}} \\
&\quad = \varphi^2(q)\varphi^2(q^{31})(\tfrac{1}{2}\{1 + \sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}\} \\
&\qquad + \{1 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\}^2 \\
&\qquad - 2\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}\{1 + (\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8}\}).
\end{aligned}
$$

We defer the proofs for Entry 9 until Section 11, where we employ the theory of modular forms.

Entry 10.

(i) *Let β have degree 5 with respect to α. Then*

$$1+6\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-30\sum_{k=1}^{\infty}\frac{kq^{10k}}{1-q^{10k}}$$
$$=\varphi^2(q)\varphi^2(q^5)\left(\tfrac{1}{2}\{1+\alpha\beta+(1-\alpha)(1-\beta)\}\right.$$
$$\left.-\tfrac{3}{16}\{1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\}^2\right)^{1/2}.$$

(ii) *If β has degree 9, then*

$$1+3\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-27\sum_{k=1}^{\infty}\frac{kq^{18k}}{1-q^{18k}}$$
$$=\varphi^2(q)\varphi^2(q^9)\left(\tfrac{1}{2}\{1+\alpha\beta+(1-\alpha)(1-\beta)\}\right.$$
$$-\tfrac{9}{32}\{1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\}^2$$
$$\left.+\frac{3}{2}\frac{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/2}}{1-(\alpha\beta)^{1/2}-\{(1-\alpha)(1-\beta)\}^{1/2}}\right)^{1/2}.$$

(iii) *If β is of degree 17, then*

$$2+3\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}}-51\sum_{k=1}^{\infty}\frac{kq^{34k}}{1-q^{34k}}$$
$$=\varphi^2(q)\varphi^2(q^{17})\left(2\{1+\alpha\beta+(1-\alpha)(1-\beta)\}\right.$$
$$-\tfrac{21}{16}\{1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\}^2$$
$$-\tfrac{51}{32}\{1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\}\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}$$
$$\left.-3\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/3}\right)^{1/2}.$$

Proof of (i). Comparing Entry 10(i) with Entry 4(iii), we see that it suffices to prove that

$$\{1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\}^2=4\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/3}.$$

But this equality follows immediately from Entry 13(i) of Chapter 19, and so the proof is complete.

Proof of (ii). Employing the notation of Section 7, (7.7), and (3.10) and (3.11) of Chapter 20, we deduce that the left side of (ii) is equal to

$$z_1z_9\left(\frac{m'}{m}\right)^{1/2}\left(\frac{1+t+t^2}{1+2t}\right)^2=z_1z_9\frac{(1+8t^3)^{1/2}}{1+2t}\left(\frac{1+t+t^2}{1+2t}\right)^2$$
$$=z_1z_9\frac{(1-2t+4t^2)^{1/2}(1+t+t^2)^2}{(1+2t)^{5/2}}. \tag{10.1}$$

Now by (3.17), (3.7), and (3.9) in Chapter 20,

$$\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)} = 1 - \frac{8t(1-t^3)}{(1+2t)^2} \tag{10.2}$$

and

$$\sqrt{\alpha\beta(1-\alpha)(1-\beta)} = 16t^5\left(\frac{1-t}{1+2t}\right)^4 \frac{1-t^3}{1+8t^3}. \tag{10.3}$$

Employing (10.2) and (10.3) and performing a very laborious calculation, we find that

$$\begin{aligned}
&\frac{(1-2t+4t^2)(1+t+t^2)^4}{(1+2t)^5} - \tfrac{1}{2}\{1+\alpha\beta+(1-\alpha)(1-\beta)\}\\
&= \frac{(1-2t+4t^2)(1+t+t^2)^4}{(1+2t)^5} - \tfrac{1}{2}\{1+(\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)})^2\\
&\quad - 2\sqrt{\alpha\beta(1-\alpha)(1-\beta)}\}\\
&= \frac{(1-2t+4t^2)(1+t+t^2)^4}{(1+2t)^5} - 1 + \frac{8t(1-t^3)}{(1+2t)^2} - \frac{32t^2(1-t^3)^2}{(1+2t)^4}\\
&\quad + \frac{16t^5(1-t)^4(1-t^3)}{(1+2t)^4(1+8t^3)}\\
&= \frac{-18t^2+36t^3-69t^4+30t^5-78t^6+168t^7-39t^8+42t^9-72t^{10}}{(1+2t)^3(1+8t^3)}.
\end{aligned} \tag{10.4}$$

Next, from (10.2)–(10.4),

$$\begin{aligned}
&\frac{(1-2t+4t^2)(1+t+t^2)^4}{(1+2t)^5} - \tfrac{1}{2}\{1+\alpha\beta+(1-\alpha)(1-\beta)\}\\
&\quad + \tfrac{9}{32}\{1-\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)}\}^2\\
&= \frac{3t^4(1-2t-2t^2+8t^3-7t^4+2t^5)}{(1+2t)^3(1+8t^3)}\\
&= \frac{3t^4(1-t)^4}{(1+2t)^2(1+8t^3)}\\
&= \frac{3}{2}\left(16t^5\left(\frac{1-t}{1+2t}\right)^4\frac{1-t^3}{1+8t^3}\right)\left(\frac{(1+2t)^2}{8t(1-t^3)}\right)\\
&= \frac{3}{2}\frac{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/2}}{1-(\alpha\beta)^{1/2}-\{(1-\alpha)(1-\beta)\}^{1/2}}.
\end{aligned}$$

By combining (10.1) with the result just obtained, we obtain (ii) to complete the proof.

The proof of (iii) will be deferred until Section 11.

Entry 11. *If β has degree 35 over α, then*

$$17 + 12 \sum_{k=1}^{\infty} \frac{kq^{2k}}{1-q^{2k}} - 420 \sum_{k=1}^{\infty} \frac{kq^{70k}}{1-q^{70k}}$$
$$= \varphi^2(q)\varphi^2(q^{35})\left(\frac{(1-(\alpha\beta)^{1/4} - \{(1-\alpha)(1-\beta)\}^{1/4})^3}{2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}}\right.$$
$$\left. + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} - \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/4}\right).$$

In fact, Entry 11 is listed as 11(i) in the second notebook (p. 256), but no further result is stated in this section.

There remain now seven formulas in Chapter 21 which we have not yet proved but which we now establish via the theory of modular forms. Our first task is to identify the series on the left sides of these formulas as modular forms.

Theorem. *Let*

$$E_2(\tau) = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^{2k}}{1-q^{2k}},$$

where $q = e^{\pi i \tau}$ and $\tau \in \mathscr{H}$. Define

$$F_n(\tau) = E_2(\tau) - nE_2(n\tau).$$

Then F_n is a modular form on $\Gamma_0(n)$ of weight 2 and trivial multiplier system.

Proof. Set $E_2^*(\tau) = E_2(\tau) + 3/(\pi y)$, where $y = \text{Im}(\tau)$. It is well known (e.g., see the treatises of Rankin [2, pp. 194–195] or Schoeneberg [1, p. 68]) that if $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$, then $E_2^*(\tau)$ satisfies the transformation formula

$$E_2^*(V\tau) = (c\tau + d)^2 E_2^*(\tau) - \frac{6ic}{\pi}(c\tau + d). \tag{11.1}$$

Observe that $F_n(\tau) = E_2^*(\tau) - nE_2^*(n\tau)$. So, by (11.1) and (0.4) of Chapter 20, if $V \in \Gamma_0(n)$,

$$F_n(V\tau) = (c\tau + d)^2 E_2^*(\tau) - \frac{6ic}{\pi}(c\tau + d)$$
$$- n\left(\left(\frac{c}{n}n\tau + d\right)^2 E_2^*(n\tau) - \frac{6ic}{\pi n}\left(\frac{c}{n}n\tau + d\right)\right)$$
$$= (c\tau + d)^2 F_n(\tau).$$

The desired conclusion now follows.

PROOFS OF ENTRIES 8(i), (iii), 9(i)–(iii), 10(iii), 11. We first translate those six entries written in terms of α and β into proposed identities involving theta-functions. Thus, using Entries 10(i)–(iii), 11(i)–(iii), and 12(iii) in Chapter 17, we find that Entries 8(iii), 9(i)–(iii), 10(iii), and 11 can be written as the respective theta-function relations

$$\begin{aligned}3 + 4\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 76\sum_{k=1}^{\infty}\frac{kq^{38k}}{1-q^{38k}}\\ = \varphi^2(q)\varphi^2(q^{19}) + 16q^{10}\psi^2(q^2)\psi^2(q^{38}) + \varphi^2(-q)\varphi^2(-q^{19})\\ + 4q^5\psi(q^2)\psi(q^{38})\varphi(q)\varphi(q^{19}) + \varphi(q)\varphi(q^{19})\varphi(-q)\varphi(-q^{19})\\ - 4q^5\psi(q^2)\psi(q^{38})\varphi(-q)\varphi(-q^{19}),\end{aligned} \tag{11.2}$$

$$\begin{aligned}11 + 12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 276\sum_{k=1}^{\infty}\frac{kq^{46k}}{1-q^{46k}}\\ = \tfrac{11}{2}(\varphi^2(q)\varphi^2(q^{23}) + 16q^{12}\psi^2(q^2)\psi^2(q^{46}) + \varphi^2(-q)\varphi^2(-q^{23}))\\ -16q^2f(-q^2)f(-q^{46})(\varphi(q)\varphi(q^{23})+4q^6\psi(q^2)\psi(q^{46})+\varphi(-q)\varphi(-q^{23}))\\ - 40q^4f^2(-q^2)f^2(-q^{46}),\end{aligned} \tag{11.3}$$

$$\begin{aligned}7 + 12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 180\sum_{k=1}^{\infty}\frac{kq^{30k}}{1-q^{30k}}\\ = \frac{1}{2\varphi^2(q)\varphi^2(q^{15})}(\varphi(q)\varphi(q^{15}) + 2q^2\psi(q)\psi(q^{15}) + \varphi(-q^2)\varphi(-q^{30}))^4\\ - \tfrac{1}{2}(\varphi^2(q)\varphi^2(q^{15}) + 16q^8\psi^2(q^2)\psi^2(q^{30}) + \varphi^2(-q)\varphi^2(-q^{15})),\end{aligned} \tag{11.4}$$

$$\begin{aligned}5 + 4\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 124\sum_{k=1}^{\infty}\frac{kq^{62k}}{1-q^{62k}}\\ = \tfrac{1}{2}(\varphi^2(q)\varphi^2(q^{31}) + 16q^{16}\psi^2(q^2)\psi^2(q^{62}) + \varphi^2(-q)\varphi^2(-q^{31}))\\ + (\varphi(q)\varphi(q^{31}) + 4q^8\psi(q^2)\psi(q^{62}) + \varphi(-q)\varphi(-q^{31}))^2\\ - 4q^4\psi(-q)\psi(-q^{31})(\varphi(q)\varphi(q^{31}) + 2q^4\psi(q)\psi(q^{31})\\ + \varphi(-q^2)\varphi(-q^{62})),\end{aligned} \tag{11.5}$$

$$\begin{aligned}\left(2 + 3\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 51\sum_{k=1}^{\infty}\frac{kq^{34k}}{1-q^{34k}}\right)^2\\ = 2(\varphi^4(q)\varphi^4(q^{17}) + 256q^{18}\psi^4(q^2)\psi^4(q^{34}) + \varphi^4(-q)\varphi^4(-q^{17}))\\ - \tfrac{21}{16}(\varphi^2(q)\varphi^2(q^{17}) - 16q^9\psi^2(q^2)\psi^2(q^{34}) - \varphi^2(-q)\varphi^2(-q^{17}))^2\\ - \tfrac{51}{8}q^3f^2(-q^2)f^2(-q^{34})(\varphi^2(q)\varphi^2(q^{17}) - 16q^9\psi^2(q^2)\psi^2(q^{34})\\ - \varphi^2(-q)\varphi^2(-q^{17})) - 48q^6f^4(-q^2)f^4(-q^{34}),\end{aligned} \tag{11.6}$$

and

$$17 + 12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 420\sum_{k=1}^{\infty}\frac{kq^{70k}}{1-q^{70k}}$$
$$= \frac{1}{4q^3 f(-q^2)f(-q^{70})}(\varphi(q)\varphi(q^{35}) - 4q^9\psi(q^2)\psi(q^{70}) - \varphi(-q)\varphi(-q^{35}))^3$$
$$+ 4q^9\psi(q^2)\psi(q^{70})\varphi(q)\varphi(q^{35}) + \varphi(q)\varphi(q^{35})\varphi(-q)\varphi(-q^{35})$$
$$- 4q^9\psi^2(-q)\psi^2(-q^{35}). \tag{11.7}$$

Next, we rewrite Entry 8(i) and (11.2)–(11.7) as proposed identities relating modular forms. Thus, by (0.13) in Chapter 20, we find that, respectively,

$$5 + 12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 132\sum_{k=1}^{\infty}\frac{kq^{22k}}{1-q^{22k}}$$
$$= 5g_1^2(\tau)g_1^2(11\tau) - 20f_1^2(\tau)f_1^2(11\tau) + 32\eta^2(\tau)\eta^2(11\tau) - 20h_1^2(\tau)h_1^2(11\tau), \tag{11.8}$$

$$3 + 4\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 76\sum_{k=1}^{\infty}\frac{kq^{38k}}{1-q^{38k}}$$
$$= g_1^2(\tau)g_1^2(19\tau) + 16g_2^2(\tau)g_2^2(19\tau) + g_0^2(\tau)g_0^2(19\tau)$$
$$+ 4g_2(\tau)g_2(19\tau)g_1(\tau)g_1(19\tau) + g_1(\tau)g_1(19\tau)g_0(\tau)g_0(19\tau)$$
$$- 4g_2(\tau)g_2(19\tau)g_0(\tau)g_0(19\tau), \tag{11.9}$$

$$11 + 12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 276\sum_{k=1}^{\infty}\frac{kq^{46k}}{1-q^{46k}}$$
$$= \tfrac{11}{2}(g_1^2(\tau)g_1^2(23\tau) + 16g_2^2(\tau)g_2^2(23\tau) + g_0^2(\tau)g_0^2(23\tau))$$
$$- 16\eta(\tau)\eta(23\tau)(g_1(\tau)g_1(23\tau) + 4g_2(\tau)g_2(23\tau) + g_0(\tau)g_0(23\tau))$$
$$- 40\eta^2(\tau)\eta^2(23\tau), \tag{11.10}$$

$$7 + 12\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 180\sum_{k=1}^{\infty}\frac{kq^{30k}}{1-q^{30k}}$$
$$= \frac{1}{2g_1^2(\tau)g_1^2(15\tau)}(g_1(\tau)g_1(15\tau) + 2h_0(\tau)h_0(15\tau) + h_2(\tau)h_2(15\tau))^4$$
$$- \tfrac{1}{2}(g_1^2(\tau)g_1^2(15\tau) + 16g_2^2(\tau)g_2^2(15\tau) + g_0^2(\tau)g_0^2(15\tau)), \tag{11.11}$$

$$5 + 4\sum_{k=1}^{\infty}\frac{kq^{2k}}{1-q^{2k}} - 124\sum_{k=1}^{\infty}\frac{kq^{62k}}{1-q^{62k}}$$
$$= \tfrac{1}{2}(g_1^2(\tau)g_1^2(31\tau) + 16g_2^2(\tau)g_2^2(31\tau) + g_0^2(\tau)g_0^2(31\tau))$$
$$+ (g_1(\tau)g_1(31\tau) + 4g_2(\tau)g_2(31\tau) + g_0(\tau)g_0(31\tau))^2$$
$$- 4h_1(\tau)h_1(31\tau)(g_1(\tau)g_1(31\tau) + 2h_0(\tau)h_0(31\tau) + h_2(\tau)h_2(31\tau)), \tag{11.12}$$

$$\left(2 + 3\sum_{k=1}^{\infty} \frac{kq^{2k}}{1-q^{2k}} - 51\sum_{k=1}^{\infty} \frac{kq^{34k}}{1-q^{34k}}\right)^2$$
$$= 2(g_1^4(\tau)g_1^4(17\tau) + 256g_2^4(\tau)g_2^4(17\tau) + g_0^4(\tau)g_0^4(17\tau))$$
$$- \tfrac{21}{16}(g_1^2(\tau)g_1^2(17\tau) - 16g_2^2(\tau)g_2^2(17\tau) - g_0^2(\tau)g_0^2(17\tau))^2$$
$$- \tfrac{51}{8}\eta^2(\tau)\eta^2(17\tau)(g_1^2(\tau)g_1^2(17\tau) - 16g_2^2(\tau)g_2^2(17\tau) - g_0^2(\tau)g_0^2(17\tau))$$
$$- 48\eta^4(\tau)\eta^4(17\tau), \tag{11.13}$$

and

$$17 + 12\sum_{k=1}^{\infty} \frac{kq^{2k}}{1-q^{2k}} - 420\sum_{k=1}^{\infty} \frac{kq^{70k}}{1-q^{70k}}$$
$$= \frac{1}{4\eta(\tau)\eta(35\tau)}(g_1(\tau)g_1(35\tau) - 4g_2(\tau)g_2(35\tau) - g_0(\tau)g_0(35\tau))^3$$
$$+ 4g_2(\tau)g_2(35\tau)g_1(\tau)g_1(35\tau) + g_1(\tau)g_1(35\tau)g_0(\tau)g_0(35\tau)$$
$$- 4h_1^2(\tau)h_1^2(35\tau). \tag{11.14}$$

We next demonstrate that the multiplier system for each term on each side of (11.8)–(11.14) is trivial, for transformations belonging to $\Gamma = \Gamma(2) \cap \Gamma_0(n)$, where $n = 11, 19, 23, 15, 31, 17$, and 35, respectively.

First, by the theorem proved above, the multiplier system of each of the seven left sides is trivial. Using (0.14), (0.16), and (0.18)–(0.23) in Chapter 20, we easily check that each term on the right sides in (11.8)–(11.14) has a trivial multiplier system. We note that (0.27) in Chapter 20 facilitates the computations. Furthermore, for (11.11), we need to use the remarks made after (13.8) in Chapter 20, because here $3|n$.

After clearing denominators in (11.11) and (11.14), we write each of the proposed modular form identities (11.8)–(11.14) in the form

$$F := F_1 + F_2 + \cdots + F_m = 0.$$

The following table indicates the weight r of each modular form F, the value ρ_Γ calculated from (0.6) and (0.24) in Chapter 20, and the number μ determined from (0.30) in Chapter 20.

	r	ρ_Γ	μ
(11.8)	2	6	12
(11.9)	2	10	20
(11.10)	2	12	24
(11.11)	4	12	48
(11.12)	2	16	32
(11.13)	4	9	36
(11.14)	3	24	72

By the theory surrounding (0.30) in Chapter 20, if we can show that the coefficients of $q^0, q^1, \ldots, q^\mu$ for F are equal to 0, then $F \equiv 0$. With the help of the computer algebra system MACSYMA, we have, indeed, verified that the required coefficients are equal to 0. Hence, the truths of (11.8)–(11.14) have been established, and so the proofs are complete.

After the statement of Entry 11, Ramanujan draws a short horizontal bar and offers below it three equalities relating φ at certain arguments. Although the material is unrelated to the subject matter of Chapter 21, we provide proofs here because the results appear on the last page (p. 256) of Chapter 21.

Final Entry. *If the principal branch of each root is taken, then*

$$\frac{\varphi(q)-\varphi(-q)}{\varphi(q)+\varphi(-q)} = \left(\frac{\varphi^2(q^2)-\varphi^2(-q^2)}{\varphi^2(q^2)+\varphi^2(-q^2)}\right)^{1/2} = \left(\frac{\varphi^4(q^4)-\varphi^4(-q^4)}{\varphi^4(q^4)}\right)^{1/4} \tag{11.15}$$

and

$$\{\varphi(q)+i\varphi(-q)\}^{1/2} = \left(\frac{\varphi(q)+\varphi(q^2)\sqrt{2}}{2}\right)^{1/2} + \left(\frac{\varphi(q)-\varphi(q^2)\sqrt{2}}{2}\right)^{1/2}. \tag{11.16}$$

PROOF. Squaring and employing Entries 10(i)–(iv) in Chapter 17, we find that the first proposed identity of (11.15) is equivalent to the equality

$$\left(\frac{1-(1-x)^{1/4}}{1+(1-x)^{1/4}}\right)^2 = \frac{\frac{1}{2}(1+\sqrt{1-x})-(1-x)^{1/4}}{\frac{1}{2}(1+\sqrt{1-x})+(1-x)^{1/4}}.$$

This equality is obvious.

From (10.1) and Entries 10(iii), (iv), all in Chapter 17, we find that

$$\varphi^4(-q^4) = \varphi^2(q^2)\varphi^2(-q^2) = \tfrac{1}{2}z^2(1+\sqrt{1-x})(1-x)^{1/4}. \tag{11.17}$$

Taking the fourth power of the extremal sides of (11.15), using Entries 10(i), (ii), (v) in Chapter 17, and utilizing (11.17) above, we deduce the equivalent identity

$$\left(\frac{1-(1-x)^{1/4}}{1+(1-x)^{1/4}}\right)^4 = \frac{\frac{1}{16}(1+(1-x)^{1/4})^4 - \frac{1}{2}(1+\sqrt{1-x})(1-x)^{1/4}}{\frac{1}{16}(1+(1-x)^{1/4})^4}.$$

A modest calculation verifies the truth of the latter identity.

Squaring both sides of (11.16), we find that it is equivalent to the formula

$$\varphi(q)+i\varphi(-q) = \varphi(q) + \{\varphi^2(q) - 2\varphi^2(q^2)\}^{1/2}.$$

By Entries 10(i), (ii), (iv) in Chapter 17, the last identity is equivalent to the equality

$$1 + i(1-x)^{1/4} = 1 + \{1-(1+\sqrt{1-x})\}^{1/2},$$

which is obvious.

Some of the results in this chapter were independently derived by S. Ghosh in her doctoral dissertation [1].

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$$1+\frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})}\cdot x+\frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)(1-q^{2})(1-q^{\gamma})(1-q^{\gamma+1})}\cdot x^{2}+\cdots,$$

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