GROUP DISCUSSION QUESTIONS

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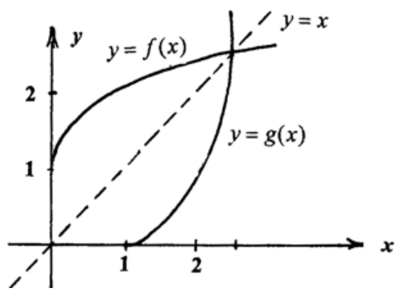
1. a. $(g \circ f)(x) = g(f(x)) = [f(x) - 1]^2 = [(\sqrt{x} + 1) - 1]^2 = (\sqrt{x})^2 = x.$

   $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} + 1 = \sqrt{(x-1)^2} + 1 = (x-1) + 1 = x.$

   b. The graphs follow.



From the figure, we see that the graph of one is the mirror reflection of the other if we place a mirror along the line $y = x$.

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1. a. As x approaches 0 (from either direction), $h(x)$ oscillates more and more rapidly between –1 and 1 and therefore cannot approach a specific number. But this says $\lim_{x\to 0} h(x)$ does not exist.

   b. The function f fails to have a limit at $x = 0$ because $f(x)$ approaches 1 from the right but –1 from the left. The function g fails to have a limit at $x = 0$ because $g(x)$ is unbounded on either side of $x = 0$. The function h here does not approach any number from either the right or the left and has no limit at 0 as explained earlier.

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1. a. $\lim_{x\to\infty} f(x)$ does not exist because no matter how large x is, $f(x)$ takes on values between –1 and 1. In other words, $f(x)$ does not approach a definite number as x approaches infinity. Similarly, $\lim_{x\to -\infty} f(x)$ fails to exist.

b. The function of Example 10 fails to have a limit at infinity (minus infinity) because $f(x)$ increases (decreases) without bound or x approaches infinity (minus infinity). On the other hand, the function whose graph is depicted here, though bounded (its values lie between –1 and 1), does not approach any specific number as x increases (decreases) without bound and this is the reason it fails to have a limit at infinity or minus infinity.

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1. The average rate of change of a function f is measured over an interval. Thus, the average rate of change of f over the interval $[a, b]$ is the number $\dfrac{f(b)-f(a)}{b-a}$. On the other hand, the instantaneous rate of change of a function measures the rate of change of the function at a point. As we have seen, this quantity can be found by taking the limit of an appropriate difference quotient. Specifically, the instantaneous rate of change of f at $x = a$ is $\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$.

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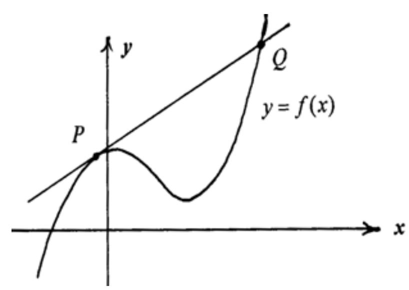
1. Yes. Refer to the following figure. Here the line tangent to the graph of f at P also intersects the graph at the point Q lying on the graph of f.



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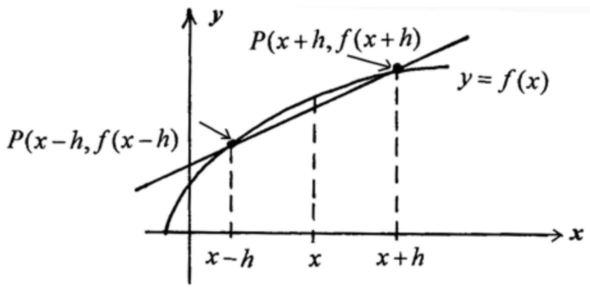
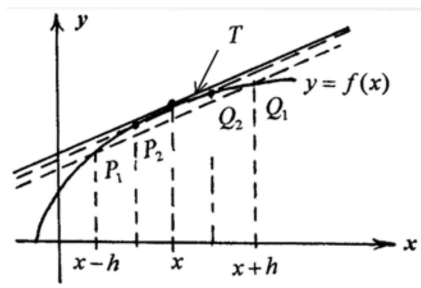
1. a. The quotient gives the slope of the secant line passing through $P(x-h, f(x-h))$ and $Q(x + h, f(x + h))$ [see figure].



It also gives the average rate of change of *f* over the interval [*x* - *h*, *x* + *h*].
b. The limit gives the slope of the tangent line to the graph of *f* at the point $(x, f(x))$ [see figure]. It also gives the (instantaneous) rate of change of *f* at the point $(x, f(x))$.
As *h* gets smaller and smaller, the secant lines approach the tangent line *T*.



c. The observation in part (b) suggests that this definition makes sense. We can also justify this observation as follows:

From the definition of $f'(x)$, we have $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Replacing *h* by –*h* gives

$$f'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}.$$

So $$2f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{f(x) - f(x-h)}{h} \right]$$

or $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

in agreement with the result of Example 3.

d. **Step 1** Compute $f(x+h)$ and $f(x-h)$. z

Step 2 Form the difference $f(x+h) - f(x-h)$.

Step 3 Form the quotient $\dfrac{f(x+h)-f(x-h)}{2h}$.

Step 4 Compute $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x-h)}{2h}$.

For the function $f(x)=x^2$, we have

Step 1 $f(x+h)=(x+h)^2=x^2+2xh+h^2$

$f(x-h)=(x-h)^2=x^2-2xh+h^2$

Step 2 $f(x+h)-f(x-h)=(x^2+2xh+h^2)-(x^2-2xh+h^2)=4xh.$

Step 3 $\dfrac{f(x+h)-f(x)}{2h}=\dfrac{4xh}{2h}=2x.$

Step 4 $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{2h}=\lim_{h\to 0}2x=2x$

in agreement with the result of Example 3.

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1. No. The slope of the tangent line to the graph of f at $(a, f(a))$ is defined by $f'(a)=\lim_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ and since the limit must be unique (see the definition of a limit), there is only one number $f'(a)$ giving the slope of the tangent line. Furthermore, since there can only be one straight line with a given slope, $f'(a)$, passing through a given point, $(a,\ f(a))$, our conclusion follows.

EXPLORING WITH TECHNOLOGY QUESTIONS

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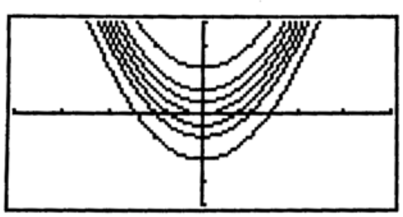
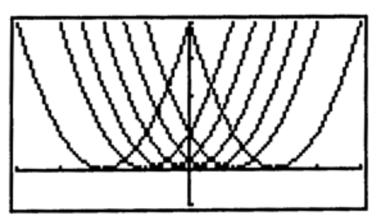
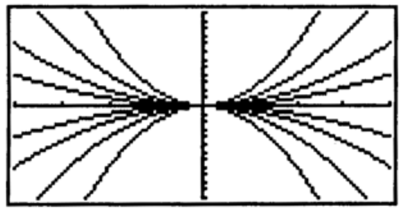
1. a.



b.



c.



d. The graph of $f(x)+c$ is obtained by translating the graph of *f* along the *y*–axis by *c* units. The graph of $f(x+c)$ is obtained by translating the graph of *f* along the *x*–axis by *c* units. Finally, the graph of *cf* is obtained from that of *f* by expanding" ($c > 1$) or "contracting" ($0 < c < 1$) that of *f*. If $c < 0$, the graph of *cf* is obtained from that of *f* by reflecting it with respect to the *x*–axis as well as "expanding" it or "contracting" it.

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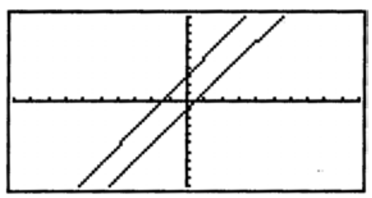
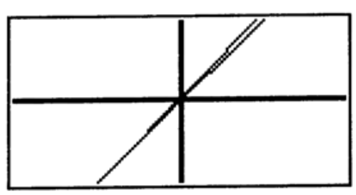
1. a.



The lines seem to be parallel to each other and they appear to not intersect.
b.



They appear to intersect. But finding the point of intersection using **TRACE** and **ZOOM** with any degree of accuracy seems to be an impossible task. Using the "intersection" function of the graphing utility yields the point of intersection (–40,–81) immediately.

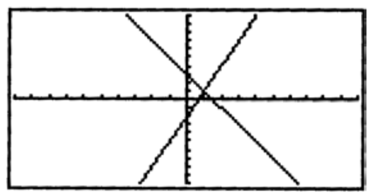
c. Substituting the first equation into the second gives

$$2x - 1 = 2.1x + 3$$
$$-4 = 0.1x,$$

or $x = -40$. The corresponding y-value is -81.

d. The **Trace** and **Zoom** technique is not effective. The "intersection" function gives the desired result immediately. The algebraic method also yields the answer with little effort and without the use of a graphing utility.

2. a. Plotting the straight lines L_1 and L_2 and using **TRACE** and **ZOOM** repeatedly, you will see that the iterations approach the answer (1,1). Using the "intersection" function of the graphing utility gives the result $x = 1$ and $y = 1$,



immediately.

b. Substituting the first equation into the second yields

$$3x - 2 = -2x + 3; \quad 5x = 5$$

and $x = 1$. Substituting this value of x into either equation gives $y = 1$.

c. The iterations obtained using **TRACE** and **ZOOM** converge to the solution (1,1) with a little effort. The use of the "intersection" function is clearly superior to the first method. The algebraic method also yields the desired result accurately.

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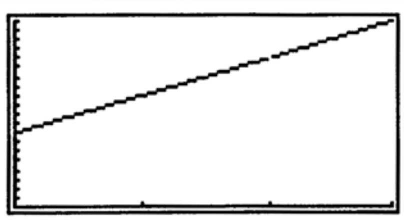
1. The graph of g follows.



2. Using **ZOOM** and **TRACE** repeatedly, we find that $g(x)$ approaches 16 as x approaches 2.
3. If we try to use the "evaluation function" of the graphing utility to find $g(2)$ it will fail. It fails because $x = 2$ is not in the domain of g.

4. The results obtained here confirm those obtained in the preceding example.

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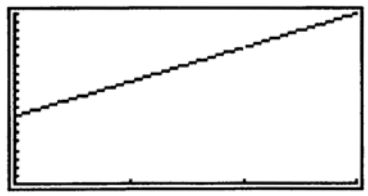
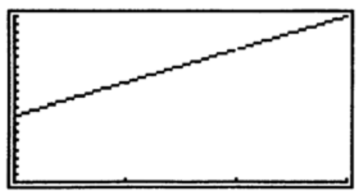
1. The graph of f follows.



Using **TRACE**, we find $\lim_{x \to 2} \frac{4(x^2-4)}{x-2} = 16.$

2. The graph of g is shown below.



Using **TRACE**, we find $\lim_{x \to 2} 4(x+2) = 16.$ When $x = 2$, $y = 16$. The function $f(x) = 4(x+2)$ is defined at $x = 2$ and so $f(2) = 16$ is defined.

3. No.
4. As we saw in Example 5, the function f is not defined at $x = 2$ but g is defined there.

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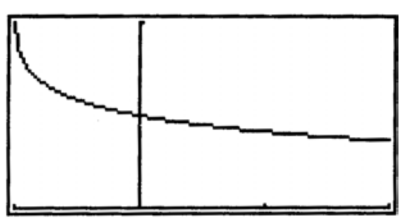
1. The graph of g follows.



Using **TRACE** and **ZOOM**, we see that $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = 0.5$.

2. The graph of f is the same as that of g except that the domain of f includes $x = 0$ (this will not be evident by just looking at the graphs!) Using the "evaluation" function to obtain the value of y, we obtain $y = 0.5$ when $x = 0$. This is to be expected since $x = 0$ lies in the domain of g.
3. As mentioned in part (2), the graphs are indistinguishable even though $x = 0$ is in the domain of g but not in the domain of f.
4. The functions f and g are the same everywhere except at $x = 0$ and so

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$$ as obtained in Example 6.

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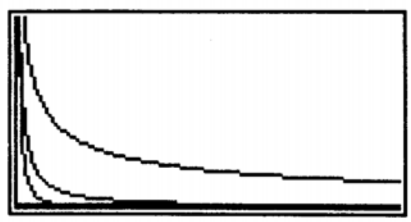
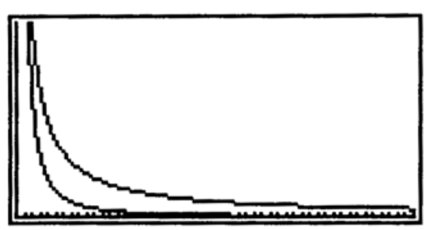
1. The graphs of the given functions follow.



The results suggest that $\frac{1}{x^n}$ goes to zero (as x increases) with increasing rapidity as n gets larger. This is as predicted by Theorem 2.

2. The graphs of the given functions follow



The results suggest that $\frac{1}{x^n}$ goes to zero (as x decreases) with increasing rapidity as n gets larger. This is as predicted by Theorem 2.

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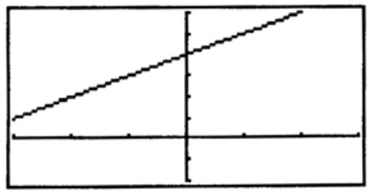
1. The graph of g follows



2. Using **ZOOM–IN** repeatedly, we find $\lim_{x\to 0} g(x) = 4$.

3. That the limit found in (2) is $f'(2)$ is a consequence of the definition of a derivative.

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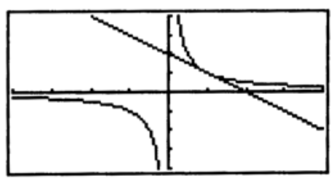
1. The graph of f and its tangent line at (1,1) are shown in the figure that follows.



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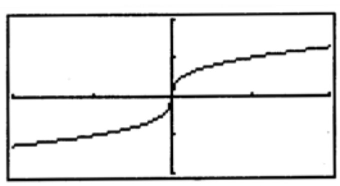
1. The graph of f follows.



2. The graphing utility will indicate an error was made when you try to draw the tangent line to the graph of f at (0,0). This happens because the slope of f is not defined when $x = 0$.