CHAPTER 9, REVIEW EXERCISES, page 651

1. $y = C_1e^{2x} + C_2e^{-3x}$; $y' = 2C_1e^{2x} - 3C_2e^{-3x}$; $y'' = 4C_1e^{2x} + 9C_2e^{-3x}$.
Substituting these values into the differential equation, we have
$$4C_1e^{2x} + 9C_2e^{-3x} + 2C_2e^{2x} - 3C_2e^{-3x} - 6(C_1e^{2x} + C_2e^{-3x}) = 0$$
$$[y'' + y' - 6y = 0]$$
and the differential equation is satisfied.

2. $y = 2e^{2x} + 3x - 2$. So $y' = 4e^{2x} + 3$ and $y'' = 8e^{2x}$. Therefore,
$y'' - y' - 2y = 8e^{2x} - (4e^{2x} + 3) - 2(2e^{2x} + 3x - 2) = -6x + 1$
which is equal to the right-hand side of the given differential equation. So *y* is a solution of the differential equation.

3. $y = Cx^{-4/3}$. So $y' = -\frac{4}{3}Cx^{-7/3}$. Substituting these values into the given differential equation, which can be written in the form $\frac{dy}{dx} = -\frac{4xy^3}{3x^2y^2} = -\frac{4y}{3x}$, we find
$$-\frac{4}{3}Cx^{-7/3} = -\frac{4(Cx^{-4/3})}{3x} = -\frac{4}{3}Cx^{-7/3}$$
we see that *y* is a solution of the differential equation.

4. $y = \frac{1}{x^2 - C}$; $y(0) = 1$ implies that $1 = -\frac{1}{C}$, or C = -1. Therefore, $y = \frac{1}{x^2 + 1}$.

5. $y = (9x + C)^{-1/3}$. So $y' = -\frac{1}{3}(9x + C)^{-4/3}(9) = -3(9x + C)^{-4/3}$. Substituting into the differential equation, we have
$$-3(9x + C)^{-4/3} = -3[(9x + C)^{-1/3}]^{-4} = -3(9x + C)^{-4/3}$$
$$[y' = -3y^4]$$
and it is satisfied. Therefore, *y* is indeed a solution. Next, using the side condition,
we find $y(0) = C^{-1/3} = \frac{1}{2}$, or $C = 8$.
Therefore, the required solution is $y = (9x + 8)^{-1/3}$.

6. We have $\dfrac{dy}{dx}=\dfrac{x^3+1}{y^2}$. Separating variables and integrating gives

$$\int y^2\,dy=\int (x^3+1)\,dx$$
$$\tfrac{1}{3}y^3=\tfrac{1}{4}x^4+x+C.$$

7. $\dfrac{dy}{4-y}=2\,dt$ implies that $-\ln|4-y|=2t+C_1$.

$4-y=Ce^{-2t}$, or $y=4-Ce^{-2t}$, where $C=e^{-C_1}$.

8. $\dfrac{dy}{dx}=\dfrac{y\ln x}{x}$. Separating variables and integrating, we obtain

$$\int \frac{dy}{y}=\int \frac{\ln x}{x}\,dx$$
$$\ln y=\tfrac{1}{2}(\ln x)^2+C_1$$
$$y=Ce^{(1/2)(\ln x)^2}.$$

9. We have $\dfrac{dy}{dx}=3x^2y+y^2=y^2(3x^2+1)$. Separating variables and integrating, we have

$$\int y^{-2}dy=\int (3x^2+1)\,dx$$
$$-\frac{1}{y}=x^3+x+C.$$

Using the side condition, we find $\tfrac{1}{2}=C$. So the solution is

$$y=-\frac{1}{x^3+x+\frac{1}{2}}=-\frac{2}{2x^3+2x+1}.$$

10. $\dfrac{dy}{1-y}=x^2dx$ implies that $-\ln|1-y|=\tfrac{1}{3}x^3+C_1$, or $1-y=Ce^{-(1/3)x^3}$.

Next, $y(0)=-2$ implies that 1 + 2 = C. Therefore, $y=1-3e^{-(1/3)x^3}$.

11. We have $\dfrac{dy}{dx}=-\tfrac{3}{2}x^2y$. Separating variables and integrating, we have

$$\int \frac{dy}{y} = \int -\frac{3}{2}x^2\,dx$$

$$\ln|y| = -\frac{1}{2}x^3 + C$$

Using the condition $y(0) = 3$, we have ln 3 = C. Therefore,

$$\ln|y| = -\frac{1}{2}x^3 + \ln 3 \quad \text{or} \quad y = e^{-(1/2)x^3+\ln 3} = 3e^{-x^3/2}.$$

12. $\frac{dy}{dx} = -\frac{4xy}{x^2+1}$. Separating variables and integrating, we obtain

$$\int \frac{dy}{y} = -\int \frac{4x}{x^2+1}dx$$

$$\ln y = -2\ln(x^2+1) + \ln C = -\ln(x^2+1)^2 + \ln C$$

$$\ln y(x^2+1)^2 = \ln C,\ y(x^2+1)^2 = C$$

$$y = \frac{C}{(x^2+1)^2}.$$

When x = 1 and y = 1, we find that $1 = \frac{C}{4}$, or $C = 4$. Therefore, the required function is $y = \frac{4}{(x^2+1)^2}$.

13. a. $x_0 = 0,\ b = 1$, and $n = 4$. Therefore, $h = 0.25$. So

$x_0 = 0,\ x_1 = 0.25,\ x_2 = 0.5,\ x_3 = 0.75,\ x_4 = 1.$

Also $F(x,y) = x + y^2$, $y(0) = 0$. Therefore,

$$\tilde{y}_0 = y_0 = 0$$

$$\tilde{y}_1 = \tilde{y}_0 + hF(x_0, \tilde{y}_0) = 0 + 0.25(0) = 0$$

$$\tilde{y}_2 = \tilde{y}_1 + hF(x_1, \tilde{y}_1) = 0 + 0.25(0.25 + 0) = 0.0625.$$

$$\tilde{y}_3 = \tilde{y}_2 + hF(x_2, \tilde{y}_2) = 0.0625 + 0.25(0.5 + 0.0625^2) = 0.1884766.$$

$$\tilde{y}_4 = \tilde{y}_3 + hF(x_3, \tilde{y}_3) = 0.1884766 + 0.25(0.75 + 0.884766^2) = 0.3848575.$$

Therefore, $y(1) = 0.3849$.

b. n = 6. Therefore, h = 0.1666667.

So $x_0 = 0,\ x_1 = 1.666667,\ x_2 = 0.3333334,\ x_3 = 0.5,\ x_4 = 0.6666667,$
$x_5 = 0.8333334$, and $x_6 = 1.$

So

$\widetilde{y}_0 = y_0 = 0$

$\widetilde{y}_1 = \widetilde{y}_0 + hF(x_0, \widetilde{y}_0) = 0 + 0.1666667(0+0) = 0$

$\widetilde{y}_2 = \widetilde{y}_1 + hF(x_1, \widetilde{y}_1) = 1 + 0.1666667[0.1666667 + 0^2] = 0.0277778.$

$\widetilde{y}_3 = \widetilde{y}_2 + hF(x_2, \widetilde{y}_2) = 0.0277778 + 0.1666667(0.3333334 + 0.0277778^2) = 0.0834619.$

$\widetilde{y}_4 = \widetilde{y}_3 + hF(x_3, \widetilde{y}_3) = 0.0834619 + 0.1666667(0.5 + 0.0834619^2)$
$\approx 0.1679562.$

$\widetilde{y}_5 = \widetilde{y}_4 + hF(x_4, \widetilde{y}_4) = 0.1679562 + 0.1666667(0.666667 + 0.1679562^2)$
$\approx 0.2837689.$

$\widetilde{y}_6 = \widetilde{y}_5 + hF(x_5, \widetilde{y}_5) = 0.2837689 + 0.1666667(0.8333334 + 0.2837689^2)$
$\approx 0.4360785.$

Therefore, $y(1) \approx 0.4361.$

14. a. Here $x_0 = 0, b = 1,$ and $n = 4$. So $h = 1/4 = 0.25$ and
$x_0 = 0,\ x_1 = 0.25,\ x_2 = 0.5,\ x_3 = 0.75,$ and $x_4 = 1.$

$F(x, y) = x^2 + 2y^2$ and $y(0) = 0.$ So

$\widetilde{y}_0 = y_0 = 0$

$\widetilde{y}_1 = \widetilde{y}_0 + hF(x_0, \widetilde{y}_0) = 0 + 0.25(0) = 0$

Proceeding in a similar manner, we obtain

$\widetilde{y}_2 = 0.015625,\ \widetilde{y}_3 = 0.078247,\ \widetilde{y}_4 = 0.221933.$

Therefore, $y(1) \approx 0.2219.$

b. Here $n = 6$ and so $h = 1/6 \approx 0.166667,$ and

$x_0 = 0,\ x_1 = 0.166667,\ x_2 = 0.333333,\ x_3 = 0.5,\ x_4 = 0.666667,\ \ x_5 = 0.833333,$ and

$x_6 = 1.$

So

$\widetilde{y}_0 = y_0 = 0$

$\widetilde{y}_1 = \widetilde{y}_0 + hF(x_0, \widetilde{y}_0) = 0 + 0.166667(0) = 0$

Proceeding in a similar manner, we obtain

$\widetilde{y}_2 = \widetilde{y}_1 + hF(x_1, \widetilde{y}_1) = 0.004630,\ \widetilde{y}_3 = 0.023156,\ \widetilde{y}_4 = 0.065001,\ \widetilde{y}_5 = 0.140483,$

and $y_6 = 0.262802.$

Therefore, $y(1) = 0.26280.$

15. a. Here $x_0 = 0, b = 1$, and $n = 4$. So h = 1/4 = 0.25. So
$x_1 = 0,\ x_1 = 0.25,\ x_2 = 0.5,\ x_3 = 0.75$, and $x_4 = 1.$
$F(x,y) = 1 + 2xy^2$ and $y(0) = 0.$ So
$\tilde{y}_0 = y_0 = 0$
$\tilde{y}_1 = \tilde{y}_0 + hF(x_0,\tilde{y}_0) = 0 + 0.25[1+0] = 0.25.$
$\tilde{y}_2 = \tilde{y}_1 + hF(x_1,\tilde{y}_1) = 0.25 + 0.25[1+2(0.05)(0.25)^2] = 0.507812.$
$\tilde{y}_3 = \tilde{y}_2 + hF(x_2,\tilde{y}_2) = 0.507813 + 0.25[1+2(0.5)(0.507813)^2] \approx 0.822281.$
$\tilde{y}_4 = \tilde{y}_3 + hF(x_3,\tilde{y}_3) = 0.822281 + 0.25[1+2(0.75)(0.822281)^2] \approx 1.32584.$
Therefore, $y(1) = 1.3258$.
b. $n = 6$. Therefore, $h = 1/6 = 0.1666667$.
So
$x_0 = 0,\ x_1 = 1.666667,\ x_2 = 0.3333334,\ x_3 = 0.5,\ x_4 = 0.6666667,\ x_5 = 0.8333334,$
and $x_6 = 1.$ So
$\tilde{y}_0 = y_0 = 0$
$\tilde{y}_1 = \tilde{y}_0 + hF(x_0,\tilde{y}_0) = 0 + 0.1666667(1+0) = 0.166667.$
$\tilde{y}_2 = \tilde{y}_1 + hF(x_1,\tilde{y}_1) = 1 + 0.1666667[1+2(0.166667)(0.166667)^2 = 0.334877.$
$\tilde{y}_3 = \tilde{y}_2 + hF(x_2,\tilde{y}_2) = 0.334877 + 0.1666667[1+2(0.333333)(0.334877)^2]$
$\approx 0.514003.$
$\tilde{y}_4 = \tilde{y}_3 + hF(x_3,\tilde{y}_3) = 0.514004 + 0.1666667[1+2(0.50000)(0.515005)^2]$
$\approx 0.724703.$
$\tilde{y}_5 = \tilde{y}_4 + hF(x_4,\tilde{y}_4) = 0.724704 + 0.1666667(1+2(0.666667)(0.724704)^2]$
$\approx 1.008081.$
$\tilde{y}_6 = \tilde{y}_5 + hF(x_5,\tilde{y}_5) = 1.008081 + 0.1666667[1+2(0.833333)(1.008081)^2]$
$\approx 1.457034.$

Therefore, $y(1) = 1.4570$.

16. a. $x_0 = 0,\ b = 1$, and $n = 4$; therefore, $h = 0.25.$
So $x_0 = 0,\ x_1 = 0.25,\ x_2 = 0.5,\ x_3 = 0.75$, and $x_4 = 1.$ Also,
$F(x,y) = e^x + y^2,\ y(0) = 0.$ Therefore,
$\tilde{y}_0 = y_0 = 0$
$\tilde{y}_1 = \tilde{y}_0 + hF(x_0,\tilde{y}_0) = 0 + 0.25(e^0 + 0) = 0.25.$
Proceeding in a similar manner, we obtain

$\widetilde{y}_2 = 0.5866314,\ \widetilde{y}_3 = 1.0848458,\ \widetilde{y}_4 = 1.9083184.$

Therefore, $y(1) \approx 1.9083.$

b. Here $n = 6$ and so $h = 1/6 \approx 0.1666667$ and

$x_0 = 0,\ x_1 = 0.166667,\ x_2 = 0.333333,\ x_3 = 0.5,\ x_4 = 0.666667,$

$x_5 = 0.833333,$ and $x_6 = 1.$ So

$\widetilde{y}_0 = y_0 = 0$

$\widetilde{y}_1 = \widetilde{y}_0 + hF(x_0, \widetilde{y}_0) = 0 + 0.166667(e^0 + 0^2) = 0.166667.$

Proceeding in a similar manner, we obtain

$\widetilde{y}_2 = 0.3681898,\ \widetilde{y}_3 = 0.6233859,\ \widetilde{y}_4 = 0.9629412,\ \widetilde{y}_5 = 0.14421064,$

and $\widetilde{y}_6 = 2.1722143.$

Therefore, $y(1) = 2.1722.$

17. Here $x_0 = 0,\ b = 1,$ so with $n = 5,$ and $h = 1/5 = 0.2$ Then

$x_0 = 0,\ x_1 = 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8,$ and $x_5 = 1.$

Also, $F(x, y) = 2xy$ and $y_0 = y(0) = 1.$

Therefore,

$\widetilde{y}_0 = y_0 = 1$

$\widetilde{y}_1 = \widetilde{y}_0 + hF(x_0, \widetilde{y}_0) = 1 + (0.2)(2)(0)(1) = 1$

$\widetilde{y} = \widetilde{y}_1 + hF(x_1, \widetilde{y}_1) = 1 + (0.2)(2)(0.2)(1) = 1.08.$

$\widetilde{y}_3 = \widetilde{y}_2 + hF(x_2, \widetilde{y}_2) = 1.08 + (0.2)(2)(0.4)(1.08) = 1.2528.$

$\widetilde{y}_4 = \widetilde{y}_3 + hF(x_3, \widetilde{y}_3) = 1.2528 + (0.2)(2)(0.6)(1.2528) = 1.553472.$

$\widetilde{y}_5 = \widetilde{y}_4 + hF(x_4, \widetilde{y}_4) = 1.553472 + (0.2)(2)(0.8)(1.553472) = 2.05058.$

Therefore, $y(1) = 2.0506.$

18. Here $x_0 = 0,\ b = 1,$ and $n = 5.$ So $h = 1/5 = 0.2$ and

$x_0 = 0,\ x_1 = 0.2,\ x_2 = 0.4,\ x_3 = 0.6,\ x_4 = 0.8,$ and $x_5 = 1.$

$F(x, y) = x^2 + y^2$ and $y(0) = y_0 = 1.$ Therefore,

$\widetilde{y}_0 = y_0 = 1$

$\widetilde{y}_1 = \widetilde{y}_0 + hF(x_0, \widetilde{y}_0) = 1 + (0.2)[0 + 1] = 1.2$

$\widetilde{y}_2 = \widetilde{y}_1 + hF(x_1, \widetilde{y}_1) = 1 + (0.2)[(0.2)^2 + (1.2)^2] = 1.496.$

$\widetilde{y}_3 = \widetilde{y}_2 + hF(x_2, \widetilde{y}_2) = 1.496 + 0.2[(0.4)^2 + (1.496)^2] \approx 1.975603.$

$\widetilde{y}_4 = \widetilde{y}_3 + hF(x_3, \widetilde{y}_3) = 1.975603 + 0.2[(0.6)^2 + (1.975603)^2] \approx 2.828204.$

$\widetilde{y}_5 = \widetilde{y}_4 + hF(x_4, \widetilde{y}_4) = 2.828204 + 0.2[(0.8)^2 + (2.828204)^2] \approx 4.555952.$

The solutions are summarized below:

x	0	0.2	0. 4	0.6	0.8	1
$\tilde{y}_n$	1	1.2	1.496	1.9756	2.8282	4.5560

19. $\dfrac{dS}{dT} = -kS;$ Thus, $S(t) = S_0 e^{-kt}$, $S(t) = 50{,}000e^{-kt}$.

$S(2) = 32{,}000 = 50{,}000e^{-2k}$, $e^{-2k} = \dfrac{32}{50} = 0.64.$

$-2k \ln e \approx \ln 0.64,$ or $k = 0.223144.$

a. Therefore, $S = 50{,}000e^{-0.223144t} = 50{,}000(0.8)^t$.

b. $S(5) = 50{,}000(0.8)^5 = 16{,}384,$ or \$16,384.

20. Separating variables and integrating, we have

$$\int \frac{dR}{R} = k\int \frac{dS}{S}.$$

$\ln R = k \ln S + \ln C$ (where C is a constant).

$= \ln CS^k$.

So $R = CS^k$.

21. Separating variables and integrating, we have

$$\int \frac{dA}{rA + P} = \int dt$$

$\dfrac{1}{r}\ln(rA + P) = t + C_1$

$\ln(rA + P) = rt + C_2 \quad (C_2 = C_1 r)$

$rA + P = Ce^{rt}$.

So $A = \dfrac{1}{r}(Ce^{rt} - P)$. Using the condition $A(0) = 0$, we have

$0 = \dfrac{1}{r}(C - P)$, or $C = P$.

So $A = \dfrac{P}{R}(e^{rt} - 1)$. The size of the fund after five years is

$A = \dfrac{50{,}000}{0.12}(e^{(0.12)(5)} - 1) \approx 342{,}549.50,$ or approximately \$342,549.50.

22. $\dfrac{dS}{dt} = k(D - S),\ S(0) = S_0.$ Separating variables, we obtain

$$\frac{dS}{D - S} = k\,dt.$$

Integrating, we find

$$-\ln|D - S| = kt + C_1$$

$$\ln|D - S| = -kt - C_1$$

$$D - S = Ce^{-kt}$$

$$S = D - Ce^{-kt}.$$

Next, $S(0) = S_0$ implies $S_0 = D - C$, or $C = D - S_0$. Therefore,

$$S = D - (D - S_0)e^{-kt}.$$

23. According to Newton's Law of cooling,

$$\frac{dT}{dt} = k(350 - T)$$

where T is the temperature of the roast. We also have the conditions $T(0) = 68$ and $T(2) = 118$. Separating the variables in the differential equation and integrating, we have

$$\frac{dT}{350 - T} = k\,dt$$

$$\ln|350 - T| = kt + C_1$$

$$350 - T = Ce^{kt},$$

or $$T = 350 - Ce^{kt}.$$

Using the condition $T(0) = 68$, we have

$$350 - C = 68, \text{ or } C = 282.$$

So $$T = 350 - 282e^{kt}.$$

Next, we use the condition $T(2) = 118$ to find

$$118 = 350 - 282e^{2k}$$

$$e^{2k} = 0.822695035.$$

So $$k = \tfrac{1}{2}\ln 0.822695035 \approx -0.097584.$$

Therefore, $T = 350 - 282e^{-0.097584t}$.

We want to find t when $T = 150$. So we solve

$$150 = 350 - 282e^{-0.097584t}$$
$$e^{-0.097584t} = 0.709219858$$
$$-0.097584t = \ln 0.709219858 = -0.343589704,$$
or $t = 3.52096,$ or approximately 3.5 hours.

So the roast would have been $150° F$ at approximately 7:30 P.M.

24. Separating variables and integrating, we have

$$\int \frac{dQ}{120-Q} = \int k\,dt$$
$$-\ln|120-Q| = kt + C_1$$
$$\ln|120-Q| = -kt - C_1$$
$$120 - Q = Ce^{-kt} \qquad (C = e^{-C_1})$$
$$Q = 120 - Ce^{-kt}.$$

Using the condition $Q(0) = 60$ gives

$$60 = 120 - C, \text{ or } C = 60.$$

So $Q = 120 - 60e^{-kt}.$

Next, $Q(10) = 90$ gives

$$90 = 120 - 60e^{-10k}$$
$$e^{-10k} = \tfrac{1}{2}$$
$$-10k = \ln\tfrac{1}{2}, \text{ or } k \approx 0.069315.$$

Therefore, $Q(t) = 120 - 60e^{-0.069315t}$ .Upon completion of the course, the student can take $Q(20) = 120 - 60e^{-0.069315(20)} = 105,$ or 105 words per minute.

25. $N = \dfrac{200}{1+49e^{-200kt}}.$ When $t = 2$, $N = 40$, and

$$40 = \frac{200}{1+49e^{-400k}}$$
$$40 + 1960e^{-400k} = 200$$
$$e^{-400k} = \frac{160}{1960}(0.0816327)$$
$$-400\ln k = \ln 0.0816327$$
$$k = 0.00626.$$

Therefore, $N(5) = \dfrac{200}{1 + 49e^{-200(5)(0.00626)}} \approx 183,$ or 183 families.

26. Let x denote the amount of salt in the tank at any time t. Then

$$\frac{dx}{dt} = (3)(4) - \frac{x}{40}(4), \text{ or } \frac{dx}{dt} = 12 - \frac{1}{10}x.$$

Also, $x(0) = 0$. To solve the differential equation, separate variables and integrate, obtaining

$$\int \frac{dx}{12 - \frac{1}{10}x} = \int dt.$$

$$-10\ln(12 - \tfrac{1}{10}x) = t + C_1$$

$$\ln(12 - \tfrac{1}{10}x) = -0.1t + C_2$$

$$12 - \tfrac{1}{10}x = C_3 e^{-0.1t}$$

or $$x = 120 - Ce^{-0.1t}.$$

$x(0) = 0$ implies 0 = 120 - C, or C = 120. Therefore,

$$x = 120(1 - e^{-0.1t}).$$

Letting t approach infinity, we see that in the long run, the amount of salt in the tank will be 120 lbs.