CHAPTER 11 REVIEW EXERCISES, page 767

1. $f(x) = \dfrac{1}{x+2} = \dfrac{1}{(x+1)+2-1} = \dfrac{1}{1+(x+1)}$.

$f(x) = 1-(x+1)+(x+1)^2-(x+1)^3+(x+1)^4$.

2. $f(x) = e^{-x},\ f'(x) = -e^{-x},\ f''(x) = e^{-x},\ f'''(x) = -e^{-x},\ f^{(4)}(x) = e^{-x}$

$f(1) = e^{-1},\ f'(1) = -e^{-1},\ f''(1) = e^{-1},\ f'''(1) = -e^{-1},\ f^{(4)}(1) = e^{-1}$.

So

$$\begin{aligned} P_4(x) &= f(1) - f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 \\ &= e^{-1} - e^{-1}(x-1) + \frac{e^{-1}}{2!}(x-1)^2 - \frac{e^{-1}}{3!}(x-1)^3 + \frac{e^{-1}}{4!}(x-1)^4 \\ &= e^{-1}\left[1-(x-1)+\frac{(x-1)^2}{2!}-\frac{(x-1)^3}{3!}+\frac{(x-1)^4}{4!}\right]. \end{aligned}$$

3. Observe that $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$. Therefore,

$$\begin{aligned} \ln(1+x^2) &= x^2 - \tfrac{1}{2}(x^2)^2 + \tfrac{1}{3}(x^2)^3 - \tfrac{1}{4}(x^2)^4 \\ &= x^2 - \tfrac{1}{2}x^4. \end{aligned}$$

4. $$\begin{aligned} f(x) &= (1+x)^{-2} \\ &= 1 - 2x + \frac{-2(-2-1)}{2!}x^2 + \frac{-2(-2-1)(-2-2)}{3!}x^3 + \frac{(-2)(-3)(-4)(-5)}{4!}x^4 \end{aligned}$$

$f(x) = 1 - 2x + 3x^2 - 4x^3 + 5x^4$.

5. $f(x) = x^{1/3},\ f'(x) = \tfrac{1}{3}x^{-2/3},\ f''(x) = -\tfrac{2}{9}x^{-5/3}$. So

$f(8) = 2,\ f'(8) = \dfrac{1}{12},\ f''(8) = -\dfrac{1}{144}$. Therefore,

$$f(x) = f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

$$\sqrt[3]{7.8} = f(7.8) \approx 2 + \frac{1}{12}(-0.2) - \frac{1}{288}(-0.2)^2 \approx 1.9832.$$

6. $\int_0^1 e^{-x^2}\,dx = \int_0^1 \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}\right) dx = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 \Big|_0^1$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} \approx 0.74.$$

7. $f(x) = x^{1/3},\ f'(x) = \frac{1}{3}x^{-2/3},\ f''(x) = -\frac{2}{9}x^{-5/3},\ f'''(x) = \frac{10}{27}x^{-8/3}$

$f(27) = 3,\ f'(27) = \frac{1}{27},\ f'(27) = -\frac{2}{2187}.$

$$f(x) = f(27) + f'(27)(x-27) + \frac{f''(27)}{2!}(x-27)^2$$

$$= 3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2$$

$$\sqrt[3]{26.98} = f(26.98) = 3 + \frac{1}{27}(-0.02) - \frac{1}{2187}(-0.02)^2 \approx 2.9992591.$$

The error is less than $\frac{M}{3!}|x-27|^3$ where M is a bound for $f'''(x) = \frac{10}{27x^{7/3}}$ on [26.98, 26]. Observe that $f'''(x)$ is decreasing on the interval and so $|f'''(x)| \le \frac{10}{27(26.98)^{8/3}} < 0.00006.$ So the error is less than

$$\frac{0.0002}{6}(0.02)^3 < 8 \times 10^{-11}.$$

8. $f(x) = \frac{1}{1+x} = (1+x)^{-1},\ f'(x) = -(1+x)^{-2},\ f''(x) = 2(1+x)^{-3}$

$f'''(x) = -6(1+x)^{-4},\ f^{(4)}(x) = 24(1+x)^{-5}.$

Therefore, $f(0) = 1,\ f'(0) = -1,\ f''(0) = 2,\ f'''(0) = -6,\ f^{(4)}(0) = 24,$ and the third Taylor polynomial about $x = 0$ is $P(x) = 1 - x + x^2 - x^3$.

So $f(0.1) = 1 - 0.1 + (0.1)^2 - (0.1)^3 = 0.909.$ To find a bound for the error, we observe that $f^{(4)}(x)$ is decreasing on (0, 0.1) and so a bound for $f^{(4)}(x)$ on (0, 0.1) is $M = f^{(4)}(0) = 24.$ Therefore, by (3) a required error bound is

$$|R_3(x)| \le \frac{24}{4!}(0.1)^4 = 0.0001.$$

The exact value of $f(0.1)$ is $\dfrac{1}{1+0.1} = 0.9090909\ldots$.

9. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$. Therefore,

$$e^{-1} \approx 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \cdots \approx 0.367.$$

10. $$\int_0^{0.2} e^{-x^2/2} dx \approx \int_0^{0.2} \left(1 - \frac{x^2}{2} + \frac{x^4}{8}\right) dx = \left(x - \frac{1}{6}x^3 + \frac{1}{40}x^5\right)\Bigg|_0^{0.2}$$

$$= 0.2 - \frac{1}{6}(0.2)^3 + \frac{1}{40}(0.2)^5 \approx 0.199.$$

11. $\lim\limits_{n\to\infty} \dfrac{2n^2+1}{3n^2-1} = \lim\limits_{n\to\infty} \dfrac{2+\frac{1}{n^2}}{3-\frac{1}{n^2}} = \dfrac{2}{3}.$

12. The sequence diverges because if *n* is large and odd then a_n is close to 1 but if *n* is large and even, then a_n is close to -1.

13. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(1 - \dfrac{1}{2^n}\right) = \lim\limits_{n\to\infty} 1 - \lim\limits_{n\to\infty} \dfrac{1}{2^n} = 1 - 0 = 1.$

14. $\lim\limits_{n\to\infty} \dfrac{1+\sqrt{n}}{1-\sqrt{n}} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{\sqrt{n}}+1}{\frac{1}{\sqrt{n}}-1} = -1.$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{2}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = \tfrac{2}{3}\cdot 3 = 2.$

16. $\displaystyle\sum_{n=1}^{\infty} 2^{-n}3^{-n+1} = 3\sum_{n=1}^{\infty} \frac{1}{6^n} = \frac{1}{2}\left(1 + \frac{1}{6} + \left(\frac{1}{6}\right)^2 + \cdots\right) = \frac{1}{2}\left[\frac{1}{1-\frac{1}{6}}\right] = \frac{3}{5}.$

17. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}}\right)^2 + \cdots = \frac{\frac{1}{\sqrt{2}}}{1-\left(-\frac{1}{\sqrt{2}}\right)} = \frac{1}{\sqrt{2}}\cdot\frac{1}{1+\frac{1}{\sqrt{2}}} = \frac{1}{1+\sqrt{2}}.$

18. $\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^n = \frac{1}{e}\left(\frac{1}{1-\frac{1}{e}}\right) = \frac{1}{e}\cdot\frac{e}{e-1} = \frac{1}{e-1}.$

19. $$\begin{aligned} 1.424242... &= 1+\frac{42}{10^2}+\frac{42}{10^4}+\cdots \\ &= 1+\frac{42}{100}\left[1+\frac{1}{100}+\left(\frac{1}{100}\right)^2+\cdots\right] \\ &= 1+\frac{42}{100}\left(\frac{1}{1-\frac{1}{100}}\right) = 1+\frac{42}{100}\left(\frac{100}{99}\right) = 1+\frac{42}{99} = \frac{141}{99}. \end{aligned}$$

20. $$\begin{aligned} 3.142142142 &= 3+\frac{142}{10^3}+\frac{142}{10^6}+\cdots \\ &= 3+\frac{142}{1000}\left[1+\frac{1}{1000}+\left(\frac{1}{1000}\right)^2+\cdots\right] \\ &= 3+\frac{142}{1000}\left[\frac{1}{1-\frac{1}{1000}}\right] = 3+\frac{142}{1000}\left(\frac{1000}{999}\right) = \frac{3139}{999}. \end{aligned}$$

21. Let $a_n = \frac{n^2+1}{2n^2-1}$. Since $\lim_{n\to\infty}\frac{n^2+1}{2n^2-1} = \lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{2-\frac{1}{n^2}} = \frac{1}{2} \neq 0,$ the divergence test implies that the series $\sum_{n=1}^{\infty} a_n$ diverges.

22. Let $f(x) = \frac{x+1}{2x^2+4x}$. Then f is nonnegative and decreasing on $[1,\infty)$. Next,

$$\begin{aligned} \int_1^{\infty}\frac{x+1}{2x^2+4x}\,dx &= \lim_{b\to\infty}\int_1^{b}\frac{x+1}{2x^2+4x}\,dx \\ &= \lim_{b\to\infty}\left[\frac{1}{4}\ln(2x^2+4x)\Big|_1^b\right] \qquad \text{(Use the substitution } u = 2x^2+4x\text{)} \\ &= \lim_{b\to\infty}\left[\frac{1}{4}\ln(2b^2+4b)-\frac{1}{4}\ln b\right] = \infty \end{aligned}$$

So by the integral test, the series diverges.

23. $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^{1.1} = \sum_{n=1}^{\infty}\frac{1}{n^{1.1}}$ is a convergent p-series with $p = 1.1 > 1$.

24. Let $a_n = \frac{n^3}{n^5+2} \le \frac{n^3}{n^5} = \frac{1}{n^2} = b_n$. Since $\sum b_n$ is a convergent p-series. The comparison test shows that $\sum_{n=1}^{\infty} a_n$ is convergent.

25. $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\frac{1}{n^2+2}}{\frac{1}{(n+1)^2+2}} = \lim_{n\to\infty}\frac{(n+1)^2+2}{n^2+2} = \lim_{n\to\infty}\frac{n^2+2n+3}{n^2+2} = 1.$

The interval of convergence is (-1,1).

26. $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n+1}}} = \lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n}} = 1.$ The interval of convergence is (-1,1).

27. $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)(n+2)}{n(n+1)} = 1.$

So $R = 1$ and the interval of convergence is (0,2).

28. $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{\frac{e^n}{n^2}}{\frac{e^{n+1}}{(n+1)^2}} = \lim_{n\to\infty}\frac{(n+1)^2}{n^2 e} = \frac{1}{e}.$

The interval of convergence is $(2-\frac{1}{e}, 2+\frac{1}{e})$.

29. We have $\frac{1}{1+x} = 1 + x + x^2 + x^3 + \cdots \qquad (-1 < x < 1)$

Replacing x by $2x$ in the expression, we find

$$f(x) = \frac{1}{2x-1} = -\frac{1}{1-2x} = -[1 + 2x + (2x)^2 + (2x)^3 + \cdots]$$

$$= -1 - 2x - 4x^2 - 8x^3 - \cdots - 2^n x^n - \cdots. \qquad (-\tfrac{1}{2} < x < \tfrac{1}{2}).$$

30. $f(x) = e^{-x} = e^{-(x-1)-1} = e^{-1}e^{-(x-1)}$

$$= e^{-1}[1-(x-1)+\frac{1}{2!}(x-1)^2-\frac{1}{3!}(x-1)^3+\cdots].$$

Therefore, $f(x)=\frac{1}{e}-\frac{1}{e}(x-1)+\frac{1}{2!e}(x-1)^2-\cdots+\frac{(-1)^n}{n!e}(x-1)^n+\cdots;$

The interval of convergence is $(-\infty,\infty)$.

31. We know that

$$\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots+(-1)^{n+1}\frac{x^n}{n} \qquad (-1<x<1).$$

Replace x by $2x$, to obtain

$$f(x)=\ln(1+2x)=2x-2x^2+\frac{8}{3}x^3-\cdots+(-1)^{n+1}\frac{2^n x^n}{n}=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^n}{n}x^n.$$

The interval of convergence is $(-\frac{1}{2},\frac{1}{2}]$..

32. Replacing x by -2x in the following

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$$

we obtain $e^{-2x}=1-2x+2x^2+\cdots+\frac{(-2x)^n}{n!}+\cdots$. Therefore,

$$f(x)=x^2e^{-2x}=x^2-2x^3+2x^5-\cdots+\frac{(-1)^n x^{n+2}}{n!}+\cdots$$

The interval of convergence is $(-\infty,\infty)$.

33. $f(x)=x^3-12,\ f'(x)=3x^2$

Therefore, $x_{n+1}=x_n-\frac{x_n^3-12}{3x_n^2}=\frac{3x_n^3-x_n^3+12}{3x_n^2}=\frac{2x_n^3+12}{3x_n^2}$.

Using $x_0=2,$ we have $x_1=2.3333333,\ x_1=2.2902491,\ x_3=2.2894277,$ and the root is approximately 2.28943.

34. $f'(x)=3x^2+2x.$ The iteration is

$$x_{n+1}=x_n-\frac{x_n^3+x_n^2-1}{3x_n^2+2x_n}=\frac{2x_n^3+x_n^2+1}{3x_n^2+2x_n}.$$

With $x_0=0.5,$ we find

$x_1=0.857143,\ x_2=0.764137,\ x_1=0.754963,\ x_4=0.754878.$

So the root is approximately 0.7549.

35. We solve the equation $F(x) = 2x - e^{-x}$. $F'(x) = 2 + e^{-x}$. So the iteration is

$$x_{n+1} = x_n - \frac{2x_n - e^{-x_n}}{2 + e^{-x_n}} = \frac{(x_n + 1)e^{-x_n}}{2 + e^{-x_n}}.$$

Taking $x_0 = 0.5$, we find $x_1 = 0.349045$, $x_2 = 0.351733$, $x_3 = 0.351733$.
So the point of intersection is approximately (0.35173, 0.70346).

36. We solve the equation $f(t) = 27(t + 3e^{-t/3} - 3) = 24$ for t. Now,

$$t + 3e^{-t/3} - 3 = \frac{24}{27} = \frac{8}{9}$$

$$t + 3e^{-t/3} - \frac{35}{9} = 0$$

Let $g(t) = t + 3e^{-t/3} - \frac{35}{9}$ and use the Newton-Raphson Method to solve $g(t) = 0$.
$g'(t) = 1 - e^{-t/3}$. So the iteration is

$$t_{n+1} = t_n - \frac{t_n + e^{-t_n/3} - \frac{35}{9}}{1 - \frac{1}{3}e^{-t_n/3}} = \frac{t_n - t_n e^{-t_n/3} - t_n - 3e^{-t_n/3} + \frac{35}{9}}{1 - \frac{1}{3}e^{-t_n/3}} = \frac{\frac{35}{9} - (t_n + 3)e^{-t_n/3}}{1 - e^{-t_n/3}}.$$

Taking the initial guess $t_0 = 2$, we find (keeping 5 decimal places)

$t_1 = 2.71650$, $t_2 = 2.64829$, $t_3 = 2.64775$, $t_4 = 2.64775$

so it takes approximately 2.65 seconds for the suitcase to reach the bottom.

37. The amount required

$$A = 10{,}000[e^{-0.09} + e^{-0.09(2)} + \cdots] = \frac{10{,}000e^{-0.09}}{1 - e^{-0.09}} = 106{,}186.10,$$

or \$106,186.10.

38. a. The economic impact is

$$(0.92)(10) + (0.92)^2(10) + \cdots = (0.92)(10)(1 + 0.92 + 0.92^2 + \cdots)$$

$$= 9.2\left(\frac{1}{1 - 0.92}\right) = 115,$$

or \$115 billion.

b. The economic impact is

$$(0.9)(10) + (0.9)^2(10) + \cdots = (0.9)(10)(1 + 0.9 + 0.9^2 + \cdots) = 9\left(\frac{1}{1-0.9}\right) = 90,$$

or $90 billion.

39. We compute

$$P(63.5 \le x \le 65.5) = \frac{1}{2.5\sqrt{2\pi}}\int_{63.5}^{65.5} e^{-1/2[(x-64.5)/2.5]^2}.$$

Replacing x with $-\frac{1}{2}[(x-64.5)/2.5]^2$ in the expression

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3,$$

we obtain

$$e^{-1/2[(x-64.5)/2.5]^2} \approx 1 - \tfrac{1}{2}[(x-64.5)/2.5)^2 + \tfrac{1}{2!}\left\{-\tfrac{1}{2}\left[(x-64.5)/2.5\right]^2\right\}^2]$$

$$+ \tfrac{1}{3!}\left\{-\tfrac{1}{2}[(x-64.5)/2.5]^2\right\}^3$$

$$= 1 - \tfrac{1}{12.5}(x-64.5)^2 + \tfrac{1}{312.5}(x-64.5)^4 - \tfrac{1}{11718.75}(x-64.5)^6.$$

Therefore,

$P(63.5 \le x \le 65.5) \approx$

$$\frac{1}{2.5\sqrt{2\pi}}\int_{63.5}^{65.5}\left[1 - \frac{1}{12.5}(x-64.5)^2 + \frac{1}{312.5}(x-64.5)^4 - \frac{1}{11718.75}(x-64.5)^6\right]dx$$

$$= \frac{1}{2.5\sqrt{2\pi}}\left[x - \frac{1}{37.5}(x-64.5)^3 + \frac{1}{1607.5}(x-64.5)^5 - \frac{1}{82031.25}(x-64.5)^7\right]_{63.5}^{65.5}$$

$$= \frac{1}{2.5\sqrt{2\pi}}[(65.5 - 0.02667 + 0.00062 - 0.00001)$$

$$-(63.5 + 0.02667 - 0.00062 + 0.00001)]$$

≈ 0.3108, or approximately 31.08%.