Instructor's Manual
Solutions to Exercises

Chapter 0

1. Using MATLAB:

a. 2*x*cos(2/x) + 2*sin(2/x)

b.
$$-4\frac{y}{(y^2-x)^2} + 8\frac{(x^2+y)y^2}{(y^2-x)^3} - 2\frac{x^2+y}{(y^2-x)^2}$$

c. -exp(-x)*cos(b*x^2) - 2*exp(-x)*sin(b*x^2)*b*x

d. $1/y^2$

2. From the TI92:

a. 2cos(2/x)x + 2sin(2/x)

b.
$$\frac{2(y^3 + 3y^2x^2 + 3yx + x^3)}{(y^2 - x)^3}$$

c. $-e^{-x}\cos(bx^2) - 2bxe^{-x}\sin(bx^2)$

d. $1/y^2$

3. Because the tick marks are spaced apart by 0.2, reading the zero is not more accurate than estimating the minimum from Fig. 0.2. Using Fig 0.2 is preferred because it avoids having to find the derivative.

4. On the TI92, there are no tick marks on the x-axis to use to estimate the zero, but the correct zero can be found through F5, 2. It is similar on the HP48G.

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* An asterisk by the exercise number indicates the solution is in <u>Answers to Selected Exercises</u>.

5. This is a 3-D geometry problem. While it is possible to write expressions that give L as a function of angle c, then solve dL/dc = 0, there is better alternative. Project the ladder against ground level; Figure 0.1b then represents this except L1 and L2 are now the lengths of the projection. We observe that the maximum length of the tipped ladder corresponds to the maximum length of the projection. Hence the optimum angle is the value that maximizes L1 + L2: c = 0.4677 radians as before.

We then compute the maximum length of the tipped ladder as the hypotenuse of a right triangle with sides equal to 33.42 and 6 ft: 33.95 ft (about 6.4 in. longer).

6* L = 181.557

7. Answer is system dependent.

8. Answer is language dependent.

9* There is an endless loop at TOL = 1E-8. Stop with "BREAK".

10. Add, at end of program:

```
L = 9 / SIN(.9946 - X3) + 7 / SIN(X3)
L1 = 9 / SIN(.9946 - X1) + 7 / SIN(X1)
L2 = 9 / SIN(.9946 - X2) + 7 / SIN(X2)
PRINT "THE LENGTH OF THE LADDER IS ";
PRINT USING "##.#### "; L; " +/- ";
PRINT USING ".#### "; ABS((L1 - L2) / 2)
```

11.

a. From [-3,-4]: -3.221472, 19 iterations, TOL = 1E-6
   From [-6,-7]: -6.279436, 19 iterations, TOL = 1E-6

b* From [0,1]: 0.453398, 19 iterations, TOL = 1E-6

c. From [1,2]: 1.850615, 19 iterations, TOL = 1E-6
   From [3,4]: 3.584627, 19 iterations, TOL = 1E-6

12. Endless loop at TOL = 1D-17.

13. Using the TI92,

  a. From the graph: -6.279436, -3.22147

  b. From F2, 1 (solve): 0.4533976

  c. From the graph: 1.8506156, 3.5846277

  Results from the HP48G are the same.

14a. .12345678E4.

  b. -.1020304E-2.

  c. .1234567890E10.

  d. .1E-8.

15a. 655,361 (includes zero).

  b. $.EFFF*16^5 = 983024_{10}$.

  c. $-.EFFF*16^5 = -983024_{10}$.

  d. $.1000*16^{-4} = .95367E-6$

  e. $-.1000*16^{-4} = -.95367E-6$.

16*a. 180,001 (includes zero).

  b. .9999E5.

  c. -.9999E5.

  d. .1000E-4.

  e. -.1000E-4.

17. Answer is system dependent.

18*

	Chopped	Rounded
a.	.123E2	.123E2
b.	-.319E-1	-.320E-1
c.	.122E2	.123E2
d.	-.288E3	-.289E3
e.	.130E3	.130E3
f.	-.156E5	-.156E5
g.	.123E-6	.123E-6

19. Answer is system dependent.

20. Answer is system dependent.

21. Answer is system and language dependent.

22.

	Chopped		Rounded	
	abs err	rel err	abs err	rel err
a.	.0234	.00190	----- Same -----	
b.	-.00126	.0395	----- Same -----	
c.	.0766	.00624	-.0234	-.00191
d.	-.9	.00312	.1	-.000346
e.	.284	.00218	----- Same -----	
f.	-20.4	.00131	----- Same -----	
g.	.9E-10	.123E-6	----- Same -----	

23. For the TI92: adding 1E-13 > 1.0, but adding 1E-14 gives 1.0. (The result must be brought to the entry line to see this.) For the HP48G, adding 1E-11 > 1.0, but adding 1E-12 gives 1.0. Both results agree with the stated internal precision of the calculators.

24* Exact value = -1.297387.
Chopped, 3 digits gives -1.31,
abs, rel err = .0126, -.00972.
Rounded, 3 digits gives -1.30,
abs, rel err = .00261, -.00201.

25. Chopped, 3 digits gives -1.31,
abs, rel err = .0126, -.00972.
Rounded, 3 digits gives -1.30,
abs, rel err = .00261, -.00201.

26. Chopped, 3 digits gives -1.32,
 abs, rel err = .0226, -.0174.
 Rounded, 3 digits gives -1.30,
 abs, rel err = .00261, -.00201.

27. Using QBASIC, single precision:

True	---------- Amount added -----------		
sum	0.001	0.0001	0.00001
0.1	0.1	0.1000011	0.0999915
0.2	0.2000002	0.2000028	0.1999783
0.3	0.2999997	0.2999970	0.3000396
0.4	0.3999984	0.3999838	0.4001754
0.5	0.4999971	0.4999706	0.5003111
0.6	0.5999959	0.5999872	0.6004469
0.7	0.6999946	0.7000038	0.7005827
0.8	0.7999933	0.8000204	0.8007185
0.9	0.8999920	0.9000370	0.9008543
1.0	0.9999907	1.000054	1.000990
abs err	0.93E-5	-0.54E-4	-0.99E-4

28* The series converges because, for very large N, 1/N evaluates as zero.

29. The answer is system dependent.

30. If f(x) is discontinuous, it may change sign but have no root within [a,b].

31* The answer depends on the spacing of the roots. If evenly spaced, we obtain the middle one. Closely spaced roots act like a multiple root. If f(x) > 0 beyond x = b, the root found tends to be larger than the middle one.

32. There are many examples. A typical one:

$$f(x) = 1 + x/2 - x^2.$$

Starting with [1,2], the fifth iterate has a smaller error than the sixth. Also the eleventh has a smaller error than the twelfth.

33. Answer is system dependent.

34. Since successive computations depend on the previous one, parallel processors cannot help (except they may speed the evaluations of the function).

35* Parallel processing is applicable when iterate n+1 does not require the knowledge of iterate n.

Chapter 1

1. After eight iterations, x_3 = 1.05078 (error 0.000735). The actual error is always less than the bound. The error does not always decrease: the error after five iterations is larger than after four.

2. Intervals are [-∞,0.6], and [0.4,∞]. Starting with [0.5,1], we get x3 = 0.615625. Error bound = 0.00781; actual error = 0.00156.

3* A graph indicates a root near -1.5. Beginning from [-2, -1], root is -1.491644 in 19 iterations, tol = 1E-6.

4.

	Root	Start from	Iter.	Rel acc
a.	0.328125	[0.3,0.5]	6	0.15%
b.	1.390625	[1, 1.5]	5	0.34%
c.	0.446875	[0.3,0.5]	6	0.39%
d.	6.723294	[6.5,6.9]	3	*

* First iterate = 6.70, rel error = 0.34%, but next has error of 1.14%. From iterate #3 on, error always < 0.5%

5. Roots at 1.222032, 1.649883, 1.774114, 1.833272.

6.

	Root	Start from	Iterations
a.	1.292696	[1,1.5]	3
b.	0.6180399	[0,0.9]	3
c.	0.244525	[0,0.5]	3
d.	f(x) has no real root		

7.

	Root	Start from	Iterations
a.	1.292696	[1,1.5]	6
b.	0.6180399	[0,0.9]	14
c.	0.244525	[0,0.5]	7
d.	f(x) has no real root		

8. The plots intersect near x = 4.5, y about 56. Using *regula falsi* from [4, 5], x = 4.53786 after 8 iterations, tol = 1E-5. The secant method, from [4, 5] gets the same value in 5 iterations, tol = 1E-5. Substituting this value into either equation gets y = 55.7978.

9. Bisection is slower because it doesn't recognize when an iterate is near the root. Linear interpolation (Regula Falsi) can get "hung up" near one end of the interval because it must always bracket the root.

10. Program.

11. Newton, starting with x = 1, gets 0.494193 in 4 iterations; this has a relative error of 0.030%. The number of correct digits is:

Iter #:	1	2	3	4	5
No. digits:	0	<1	1+	3	6

Bisection, starting from [0,1] gives results with these relative errors:

Iter #:	8	9	10	11	12	14
% rel error:	0.414	0.019	0.178	0.079	0.030	0.005

12.

	Start value for Newton's	Number of iterations required: Newton	Bisection	Reg. Falsi	Secant
a.	1	2	5	6	3
b.	0	3	6	14	3
c.	0	2	8	7	3

d. (No real root).

13* Let $f(x) = x^2 - N = 0$, so $f'(x) = 2x$. Then:

$$x_1 = x_0 - \frac{x_0^2 - N}{2\,x_0} = \frac{x_0^2 + N}{2\,x_0} = \frac{x_0 + N/x_0}{2}.$$

14. For $N^{1/3}$: $x_1 = \dfrac{2x_0 + N/x_0^2}{3}$.

For $N^{1/4}$: $x_1 = \dfrac{3x_0 + N/x_0^3}{4}$.

15. If N = A*B, let $x_0 = A$, $N/x_0 = B$, then $x_1 = \dfrac{A + B}{2}$.

$$x_2 = \frac{(A + B)/2 + 2N/(A + B)}{2} = \frac{A + B}{4} + \frac{N}{A + B}.$$

16. It is easiest to show this by an experiment. For N = 3:

A	B	A/B	|Actual error|	|Expression|
1.5	2.0	0.75	9.200E-5	5.206E-5
1.6	1.875	0.8533	8.552E-6	4.903E-6
1.7	1.763	0.9633	3.109E-8	1.521E-8
1.8	1.667	1.080	4.457E-7	2.735E-7
1.9	1.579	1.203	1.582E-5	9.066E-6
2.1	1.429	1.470	2.945E-4	1.639E-4
2.5	1.2	2.083	3.760E-3	1.905E-3
3.0	1.0	3.000	1.795E-2	7.812E-3

The error expression is conservative in this example.

17. From f(r) = f(x) + (x - r)f'(x), solve for r: r = x + f(x)/ f'(x). If more terms are included, we can get the error term).

18* From x_0 = 0.9 or 1.1, converge in 3 iterations with x-tol = 1.E-6. From x_0 = -0.9 or -1.1, it takes 18 iterations with x-tol = 1.E-6.

19. The secant method, from [-0.9, -1.1], gets the root within 0.0004 in 9 iterations while Newton, from [-0.9], takes 13 iterations to achieve this accuracy. Secant works well here because the function is nearly straight near x = -1.

20* f'(x) = 0 at x = -1.0 and x = 0.5.

21a. f'(x) = 0 at x = -0.888912, -3.931893, -5.49672, and other negative values.
  b. f'(x) = 0 at x = 0.793700
  c. f'(x) = 0 at x = 0.0
  d. f'(x) = 0 at x = -0.301220, x = 0.435335

22a. ±1.41421i in 4 iterations, starting with x = i.
  b. -0.5437, 0.7718 ± 1.115i in 3 iterations, starting with 1 + i.
  c. 1, -0.4450, 1.2470, -1.80193 (there are no complex roots).
  d. 0.4314 ± 0.9786i in 3 iterations, starting with 0.5 + i.

23a. Errors are: 0.000533, 0.000001, 0.000000, starting from [0.8,1.0,1.2].
  b. Errors are: 0.000768, 0.000002, 0.000000, starting from [6.0,6.2,6.4].
  Another root is at x = 1.173745.
  c. Errors are: 0.000446, 0.000001, 0.000000, starting from [3.5,3.7,3.9].
  d. Errors are: 0.000687, 0.000005 0.000000, starting from [4.1,4.3,4.5].

24* For parts (a), (b), and (c), Muller's method does get the root closest to zero starting with [-0.5,0,0.5]. In part (d), [-0.5,0,0.5] fails but [-1,0,1] works. There are cases where this technique does not find the smallest root, such as for f(x) = (x + 0.3)(x - 0.2)(x - 0.3), or when there is a root near to -0.5 or +0.5 in addition to a smaller root.

25a. First root = -0.618034; then, after deflating, -1, 1.618034.

b. [-0.5,0,0.5] doesn't work to start (parabola doesn't cut axis). [1,2,3] gives 1.648844; deflation fails because other roots are complex.

c. First root = 0.2395827; then, after deflating, 1.4896300, -1.590191, and 1.8019873.

26* If the parabola doesn't cut the x-axis, an attempt is made to get the square root of a negative number. Try different starting values.

27a. [2,2.0001,2.0002] fails but [1.5,1.50001,1.50002] is OK.

b. [4,4.00001,4.00002] fails but [4,4.0001,4.0002] is OK.

c. Close spaced values near zero give a negative root.

d. Close spaced values near 2 fail.

28. Same answers as in Exercise 22.

29. $\sqrt{(e^x/3)}$ converges in 16 iterations to 0.91001 from x = 0.
$-\sqrt{(e^x/3)}$ converges in 9 iterations to -0.45896 from x = 0.
$\ln(3x^2)$ converges in 17 iterations to 3.7331 from x = 3.

30* Converges to 0.618033 in 12 iterations. Acceleration gives 0.618034 after six iterations.

31. After 18 iterations, result is 0.61803399. With acceleration, this value reached after 6 iterations.

32. Converges for all starting values but to the positive root.

33* Starting from x = 1.2:
(1) $((6+4x-4x^2)/2)^{1/3}$, 26 iterations (6 if accel).
(2) $((6+4x-2x^3)/4)^{1/2}$, 23 iterations (6 if accel).
(3) $((6+4x)/(2x+4))^{1/2}$, 4 iterations (3 if accel).

34. None of the functions of Exercise 33 work to get roots near -2.3 or

-0.9. However:

$(2x^3+4x^2-6)/4$ will get the root at -1.0;

$(6-4x^2)/(2x-4)$ will get the root at -2.30177.

35. $P(x) = (x + 1)(x - 1.4)(x^2 + 5x + 10)$.

36* $P(x) = (x + 4.56155)(x + 0.438447)(2x^2 - 3x + 7)$.
(Roots of quadratic: $0.75 \pm 1.71391i$).

37. Program.

38.

Deflate from	Roots obtained	Avg. Error
x = 1.44504*.99	-0.246980, 2.80194	0.3808E-5
x = 2.80193*.99	1.44504, -0.246980	0.4048E-5
x = -.246979*.99	2.80194, 1.44504	0.1784E-5

The effect is surprisingly small.

39*

Change in coefficient	Max change in any root
2.00 -> 2.02	0.73%
7.00 -> 7.07	0.56%
4.00 -> 4.04	0.71%
29.00 -> 29.29	0.80%
14.00 -> 14.14	0.89%

40. Unless the sequential program is written specifically to take advantage of the zero coefficients, parallel processors will speed up the computation just as much.

41. Continued synthetic division does not give the higher derivatives directly, but the remainders divided by n! give $P^{(n)}(a)$.

42. Quadratic factors are: $(x^2+2.2x-3.7)$, $(x^2-1.3x+3.2)$.
Roots are -3.315852, 1.115852, $0.65 \pm 1.666583i$.

43* $(x^2 - 1.5x + 4.3)(x^2 - 4.2x + 16.1)$;
(Roots: $0.75 \pm 1.93326i$, $2.1 \pm 3.4191i$).

44. Roots: -1, 1.50122, -2.55061 ± 2.0378i

45* $(x^2 - 1.5x + 3.5)(2x^2 + 10x + 4)$;
(Roots: 0.75 ± 1.7139i, -4.5616, -0.43845).

46. Six iterations required from [R,S] = [0,0] with tolerance on ΔR, ΔS = E-6. Modulus of roots = 1.

47. Steps in the algorithm:

    Get degree of polynomial, the coefficients, and starting values
      for R and S from user.
    Set up b and c arrays according to equations in Section 1.8.
    Repeat:
      Repeat:
        Compute partial derivatives, ∂b/∂r, ∂b/∂s,
        Compute ΔR, ΔS, reset R, S,
      Until Δr, ΔS < tolerance;
      Print a factor.
      Reduce polynomial,
    Until degree = 2 or 1.
    Print last factor.

48. Program.

49 a* After convergence, q's give roots: 1.8012, -1.2462, 0.4450.

b* There are two real roots: -0.6475 and -3.5526 and a quadratic factor:

$x^2 - 2.1x + 3.1$.

c. After 100 iterations, quadratic factors are:

$x^2 - 1.7183x + 4.5942$ and $x^2 + 2.7183s + 3.9667$.

The true factors: $x^2 - 1.7x + 4.6$ and $x^2 + 2.7x + 4.1$

50. QD fails, gives division by zero even when roots are perturbed. QD always fails if all coefficients are of the same magnitude!

51. Steps in the algorithm:

Get degree of polynomial, coefficients from user.
If any $a_i = 0$, perturb a's until no a = 0.
Compute initial row of q's and e's.
Repeat:
  Compute new rows of q's and e's
Until values stabilize.
Have user read real roots from last row of q's; have user get quadratic factors (and complex roots) from last two rows of q's.

52. Graeffe's method gives magnitude of real roots easily (0.64742, 3.5526 here), but converges more slowly to magnitude of the complex pair (1.7606 here).

53* From -1, get -0.64742; from -4, get -3.5526. As described, Laguerre's method does not get complex roots.

54. Bisection cannot get a double roots because the function does not change sign at the root. This is true for any roots of even multiplicity.

55. The secant method can get both roots if starting values are fairly close to the root. From (0,4), immediately get root at x = 3 because the secant line crosses the x-axis at that point.

56. Newton's method "flies off into space" from x = 2. From x = 2.9, there is linear convergence to the root; errors decrease in the ratio of 2/3.

57* With TOL = 1E-6, Bairstow's method gives complex roots but with very small imaginary parts. Real parts are 1, 2.99377; last real root is -3.012453. This suggests that propagated errors are significant in this example.

58. Starting values that are symmetrical about x = 1 gives the root at

x = 1 immediately. Starting values far from x = 1 fail (such as -2,-1,0). Starting values near x = 3 converge but convergence is slow unless they are
symmetrical about x = 3. If starting values are all greater than x = 3, the method fails.

59. The convergence is quadratic for both roots.
Starting from x = 1.8, with k = 2, errors are 0.2, 0.024, 0.004194.
Starting from x = 3.3, with k = 3, errors are 0.3, 0.024, 0.000188.

60* Starting from x = 1.2, errors are 0.2, 0.03636, 0.00103.
Starting from x = 3.3, errors are 0.3, 0.0224, 0.000172.
Quadratic convergence is seen in both cases.

61a. Errors: 0.09, 0.0405, 0.0162, 0.00503, 0.00084, 0.0000325, 0.0000000509. Conclusion: At start, convergence is faster than linear but not quadratic; as root is approached closely, it becomes quadratic.
  b. Same conclusion as in part (a).
  c. Iterates "fly off to infinity."
  d. Same as part (c).

62. Root = 2.618014. Errors: 0.38199, 0.11450, 0.01398, 0.0002401, 0.00007236; convergence is quadratic.

63. Because slope of curve is -1.2735; at x = 3, slope is -23.05.

64* Slope at x = 2.05 is -0.7886; converges to root at 0.
Slope at x = 2.00 is -0.3561; converges to root at 9.41756.

65a. Regula falsi from [2,3] gives root = 2.618014, 19 iterations.
  b. Secant from [2,3] gets this root in 9 iterations.
  c. Muller from [2, 2.5, 3] gets it in 4 iterations.

66. One possibility: f(x) has a small jump discontinuity just to right of the root.

67a. Starting from x = 1.1: errors are 0.1, 0.04737, 0.02311, 0.01142, 0.005677, 0.002830; linear convergence.

b. Starting from x = 6.5: errors are 0.2168, 0.1080, 0.05394, 0.02696, 0.01348, 0.006740; linear convergence.

c* Starting from x = 0.1, errors are 0.11, 0.06719, 0.04503, 0.03013, 0.02014, 0.01345; linear convergence.

68a. Using the derivative function, starting from x = 1.5, errors are 0.5, 0.125, 0.00781, 0.000305; quadratic convergence.

Using the k-factor, starting from x = 1.5, errors are 0.5, 0.1, 0.004762; quadratic convergence.

b. Using the derivative function, starting from x = 6.5, errors are 0.2168, 0.001695, 0.000000108; quadratic convergence.

Using the k-factor, starting from x = 6.5, errors are 0.2168, -0.0008534, 0.000000107; quadratic convergence.

c. Using the derivative function, starting from x = 0.1, errors are 0.1, -0.00149, -0.0000003747; quadratic convergence.

Using the k-factor, starting from x = 0.1, errors are 0.1, 0.001559, 0.0000004044; quadratic convergence.

69. Starting from x = 0.5, errors are 0.06351, 0.0005122,0.000000298; faster than quadratic.

70. Using x = fzero('cos(x)-x*sin(x)',1): 0.8603

71* The soLve command does not find the roots, but using NEWTON as listed in Section 10.1 of the DERIVE manual on programming finds a root at 0.86033 after three iterations from x = 1. (From x = -1, a root at -0.86033 is found in three iterations.)

72. Both MATLAB and DERIVE find -0.61803 and 1.61803.

73. This M-file (named secant.m) does it:

```
function rtn = secant(fx,xa,xb,n)
% does the secant method n times
x=xa; fa=eval(fx);
x=xb; fb=eval(fx)
if abs(fa)>abs(fb)
  xc=xa; xa=xb; xb=xc;
  x=xa: fa=eval(fx); x=xb; fb=eval(fx);
end  % of the if
for i=1:n
  xc=xa-fa*(xa-xb)/(fa-fb); x=xc; fc=eval(fx);
  X = [i,xa,xb,xc,fc];
  disp(X)
  xa=xb; x=xa;fa=evl(fx);
  xb=xc; x=xb; fb=eval(fx);
end  % of the for loop
```

74. Modify the program as follows:

(1) In line 1: "bisec" becomes "regfls"

(2) In line 7: "xc=(xa+xb)/2) becomes "xc=xa-fa*(xa-xb)/(fa-fb)"

(3) Save the file as "regfls.m"

75. In this, the first line is a declarative. Lines 2 and 3 are auxillary functions that are used in the last line. The procedure is invoked by first defining f(x), then authoring SECNT(a,b,n) (where a and b are the starting values and n is the number of iterations to be done), and then approXimating.

```
     F(x) :=
     vc:= v SUB 1 - F(v SUB 1)*(v SUB 1 - v SUB 2)/(F(v SUB 1) - F(v
SUB2))
     SC(v) := IF ABS(F(v SUB 1)*(F(vc) < ABS (F(v SUB 2)*F(vc)),
              [v SUB 1, vc], [vc, vSUB 2])
     SECNT(a,b,n) := ITERATES(SC(v),v,[a,b],n)
```

76. In this, the first two lines are declaratives. Lines 3 and 4 are auxillary functions that are used in the last line. Invoke in the same manner as in Exercise 75.

```
     F(x) :=
     v := []
     vc:= v SUB 1 - F(v SUB 1)*(v SUB 1 - v SUB 2)/(F(v SUB 1) - F(v
SUB2))
     RF(v) := IF (F(vSUB 1)*(F(vc) < 0, [v SUB 1,vc], [vc,v SUB 2])
     REGFL(a,b,n) := ITERATES (RF(v), v, [a,b], n)
```

To employ the ITERATE function, just replace "ITERATES" with that word. Only the final iterate is then displayed.

77. These results are obtained with either calculator:

a. $\pm\sqrt{(2)}$

b. 1.46557, $-0.23278\pm0.79266i$

c. -1.80194, -0.445042, 1.0, 1.246980

d. Has no real or complex roots

78* From the graphs, using a command to get zeros:

a. 1.1462718

b. 6.1353472, 1.1737446

c. 3.7333079, -0.4589623, 0.91000757

d. 4.3026887 (and many others)

79. The same answers are found either from the graph or from the equation:
   a. 0.328625
   b. -0.474627, 1.39534
   c. ±0.44865
   d. -1.23709, 8.72329

80. (S - 1.4812)(S + 0.8111)(S + 2.1701).

81. y = -0.028997.

82. A = 0.1176 radians (6.74°).

83. 5.12E-3

84. 4.7576.

85a. T_1 = 6.0096E-6, f = 6848.9, duty cycle = 4.12%.
  b. R_B = 15531 (and also 20629).
  c. For f = 5000 and duty cycle of 10%, T_1 = 2E-5, T_2 = 1.8E-4.

86. 1.5707, 4.7123, 7.7252 (and negative roots, too).

87. Maximum at x = 0.95991 (found by a search program).

88. Zeros at ±0.2386, ±0.6612, ±0.9325.

89a. Zeros at 2.2942, 0.41579, 6.2899.
  b. Zeros at 1.7457, 0.32255, 9.3949, 4.5367.

90. Zeros are ±0.26433, ±0.70711, ±0.96593.

91. The sphere sinks more than halfway, h/r = 1.1341.

Chapter 2

1a.

$$3A = \begin{vmatrix} 9 & 0 & 6 & -12 \\ 12 & -9 & 3 & 6 \\ 15 & 3 & -3 & 6 \end{vmatrix};\ 2A+3B = \begin{vmatrix} 30 & 36 & 12 & -12 \\ 8 & 2 & 6 & 16 \\ 8 & 6 & -10 & 28 \end{vmatrix};\ 2x-3y = \begin{vmatrix} 4 \\ -18 \\ -6 \\ -16 \end{vmatrix}$$

b*

$$A-B = \begin{vmatrix} -3 & -9 & 0 & -3 \\ 4 & -5 & 0 & -1 \\ 3 & 0 & 1 & -4 \end{vmatrix};\ Ax = \begin{vmatrix} 2 \\ 19 \\ 9 \end{vmatrix};\ By = \begin{vmatrix} 34 \\ 28 \\ 36 \end{vmatrix}$$

c*

$$x^T y = |-6|;\ xy^T = \begin{vmatrix} 0 & 8 & 4 & 12 \\ 0 & -12 & -6 & -18 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & -6 \end{vmatrix}$$

d.

$$B^T = \begin{vmatrix} 6 & 0 & 2 \\ 9 & 2 & 1 \\ 2 & 1 & -2 \\ -1 & 3 & 6 \end{vmatrix}$$

2a*

$$BA = \begin{vmatrix} -18 & 7 & 9 \\ -15 & -8 & -1 \\ 8 & 11 & 26 \end{vmatrix};\ B^3 = \begin{vmatrix} -203 & 45 & 190 \\ -40 & -28 & 55 \\ -150 & 45 & -58 \end{vmatrix};\ AA^T = \begin{vmatrix} 14 & -3 & -1 \\ -3 & 13 & 4 \\ -1 & 4 & 14 \end{vmatrix}.$$

b. det(A) = -47; det(B) = -113.

c.

$$\begin{vmatrix} -3 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 2 & 3 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & 2 & 1 \end{vmatrix} + \begin{vmatrix} -4 & 1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix}.$$

3a. Both products = I_3.

b. True.

c. p $AC = \begin{vmatrix} -3 & 2 & 6 \\ 9 & 5 & 6 \\ 4 & 3 & 5 \end{vmatrix}$; $CA = \begin{vmatrix} 5 & -2 & 4 \\ 8 & -6 & 9 \\ 13 & -1 & 8 \end{vmatrix}$;

$BC = \begin{vmatrix} -1 & 8 & 4 \\ 3 & -9 & -6 \\ 4 & -13 & -7 \end{vmatrix}$; $CB = \begin{vmatrix} 3 & 6 & -10 \\ -4 & -6 & 13 \\ 3 & 9 & -14 \end{vmatrix}$.

d. $A = \begin{vmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$.

4a. $P(A) = x^2 - 6x - 7$; $P(B) = -x^3 + 8x^2 + 7x - 110$.

b. eig(A) = -1, 7; eig(B) = 5, 6.42442, -3.42442.

5.
$$\begin{aligned} 2x_1 + 4x_2 - x_3 - 2x_4 &= 10 \\ 4x_1 + 2x_3 + x_4 &= 7 \\ x_1 + 3x_2 - 2x_3 &= 3 \\ 3x_1 + 2x_2 + 5x_4 &= 2 \end{aligned}$$

6*
$$\begin{vmatrix} 2 & -6 & 1 \\ -5 & 1 & -2 \\ 1 & 2 & 7 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 11 \\ -12 \\ 20 \end{vmatrix}$$

7a. $x_3 = 2$; $x_2 = (-10 + 6)/4 = -1$; $x_1 = (-11 - 2 - 3)/2 = -8$.

b. $x_3 = 2$; $x_2 = (3 + 6)/3 = 3$; $x_1 = (7 - 4 + 3)/2 = 3$.

8. x = (1, 2, 2, -1)

9. x = (1, -1, 3)

10. Using elementary row operations (without pivoting) gives:

$$\begin{vmatrix} 1 & 1 & -2 & 3 \\ 4 & -2 & 1 & 5 \\ 3 & -1 & 3 & 8 \end{vmatrix} \Longrightarrow \begin{vmatrix} 1 & 1 & -2 & 3 \\ 0 & -6 & 9 & -7 \\ 0 & 0 & 18 & 22 \end{vmatrix}.$$

Using back-substitution, we find: $z = 11/9$, $y = 3$, $x = 22/9$.

11* Elementary row operations reduce the augmented matrix to:

$$\begin{vmatrix} 3 & 2 & -1 & -4 & 10 \\ 1 & -1 & 3 & -1 & -4 \\ 2 & 1 & -3 & 0 & 16 \\ 0 & -1 & 8 & -5 & 3 \end{vmatrix} \Longrightarrow \begin{vmatrix} 3 & 2 & -1 & -4 & 10 \\ 0 & 5 & -10 & -1 & 22 \\ 0 & 0 & -45 & 39 & 162 \\ 0 & 0 & 0 & 0 & 435 \end{vmatrix}.$$

The last row leads to the contradiction: $0 = 435$.

12. Elementary row operations reduce the augmented matrix to:

$$\begin{vmatrix} 3 & 2 & -1 & -4 & 2 \\ 1 & -1 & 3 & -1 & 3 \\ 2 & 1 & -3 & 0 & 1 \\ 0 & -1 & 8 & -5 & 3 \end{vmatrix} \Longrightarrow \begin{vmatrix} 3 & 2 & -1 & -4 & 2 \\ 0 & 5 & -10 & -1 & -7 \\ 0 & 0 & -45 & 39 & -12 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Then, for any x_4, the solutions are: $x_3 = (4 + 13x_4)/15$, $x_2 = (-7 + x_4 + 10x_3)/5$, $x_1 = (2 + 4x_4 + x_3 - 2x_2)/3$.

13* $R_1 + R_2 - 2R_3 = R_4$ (The R's are rows of the coefficient matrix).

14a. $x = (1, 2, 2, -1)$

b. $\det(a) = 75$

c.

$$LU = \begin{vmatrix} 2 & 2 & -0.5 & -1.0 \\ 4 & -8 & -0.5 & -0.625 \\ 1 & 1 & -1.0 & -1.625 \\ 3 & -4 & -0.5 & 4.688 \end{vmatrix}$$

(U has ones on its diagonal).

15* a. From back substitution: $x_3 = 11/9$; $x_2 = 3$; $x_1 = 22/9$.

b. $\det(A) = -18$.

c.

$$L = \begin{vmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/4 & 1/3 & 1 \end{vmatrix}, \quad U = \begin{vmatrix} 4 & -2 & 1 \\ 0 & 3/2 & -9/4 \\ 0 & 0 & 3 \end{vmatrix}.$$

16* a. x = (1.30, -1.35, -0.275).

b. x = (1.45, -1.59, -0.276).

c. Calculated right-hand sides are:
(0.02, 1.02, -0.21) and (0.04, 1.03, -0.54).

17. The final augmented matrices are:

$$(X5): \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{vmatrix}; \quad (X6): \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{vmatrix}; \quad (X10): \begin{vmatrix} 1 & 0 & 0 & 11/9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 22/9 \end{vmatrix}.$$

18. Augmenting A with all three b's, then doing Gaussian elimination, gives, ready for back substitution:

$$\begin{vmatrix} 4 & 2 & 1 & -3 & 4 & 9 & 4 \\ 0 & -5/2 & 5/4 & 25/4 & 5 & 5/4 & -10 \\ 0 & 0 & 5 & 8 & 13 & 5 & -3 \\ 0 & 0 & 0 & 53/10 & 53/10 & 0 & -53/10 \end{vmatrix}$$

The solutions are (1, 1, 1, 1), (2, 0, 1, 0), (-1, 2, 1, -1).

19a. In col 1: n-1 rows, 1 div + n mult per row = (n-1)(n+1),
col 2: n-2 rows, 1 div + n-1 mult per row = (n-2)(n),
. . . . .
col n-1: 1 row, 1 div + 2 mult = (1)(3).
Summing over i: SUM[(i)(i+2)] = SUM[i^2+2i] (for i = 1 .. (n-1)). We now need only use given formulas (n-1 replaces n) to get
(n-1)(2n-2+1)(n-1+1)/6 + 2*(n-1)(n+1-1)/2 which equals
n(n-1)(2n-1)/6 + n(n-1).

b. The development parallels part (a).

c. In general, the number of multiplications/divisions for the Gauss-Jordan method is $O(n^3/2)$ versus $O(n^3/3)$ for Gaussian elimination.

20* a. Let A = B + Ci, z = x + yi, and b = p + qi; then Az = b can be written as (B+Ci)(x+yi) = p + qi, so we solve:

$$\begin{aligned} Bx - Cy &= p \\ Cx + By &= q \end{aligned} \qquad \begin{vmatrix} B & -C \\ C & B \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} \quad \begin{vmatrix} p \\ q \end{vmatrix}$$

b. $2n^2+2n$ versus $4n^2+2n$.

21. a. Using the answer in Exercise 20:

$$B = \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix}, \quad C = \begin{vmatrix} 1 & 2 \\ -3 & 1 \end{vmatrix},$$

so that $A = \begin{vmatrix} 3 & 1 & -1 & -2 \\ 0 & 2 & 3 & -1 \\ 1 & 2 & 3 & 1 \\ -3 & 1 & 0 & 2 \end{vmatrix}$, with a right-hand side of $\begin{vmatrix} 6 \\ 1 \\ 2 \\ -1 \end{vmatrix}$.

b* x = (1, 2), y = (-1, 0), z = (1-i, 2).

22* Here is the matrix in compact form after pivoting, the b' vector from forward substitution, and the solution vector:

$$\begin{vmatrix} 4.000 & -0.50 & 0.25 \\ 1.000 & 1.50 & -1.50 \\ 3.000 & 0.50 & 3.00 \end{vmatrix}, \quad \begin{vmatrix} 1.250 \\ 1.167 \\ 1.222 \end{vmatrix}, \quad \begin{vmatrix} 2.444 \\ 3.000 \\ 1.222 \end{vmatrix}.$$

23. The answer is the same as for Exercises 8, 14a, and 17.

24. Use double precision in forming the sums that compute L(i,j) and U(j,i), also in the back substitution.

25. For both parts, because A = LU, Ax = LUx = b. Let Ux = y: solve Ly = b by forward substitution to give $y = L^{-1}b = b'$. Then solve Ux = b' by backsubstitution.

a.
$$LU = \begin{vmatrix} 4 & 2 & 1 & -3 \\ 1/4 & 3/2 & -5/4 & 3/4 \\ 3/4 & -5/3 & -5/6 & 15/2 \\ 0 & 4/3 & -34/5 & 53 \end{vmatrix}, \quad b' = (4, 1, 20/3, 53), \quad x = (1, 1, 1, 1).$$

(other solutions match Exercise 18)

b.
$$LU = \begin{vmatrix} 4 & 0.5 & 0.25 & -0.75 \\ 1 & 1.5 & -0.833 & 0.5 \\ 3 & -2.5 & -0.833 & -9 \\ 0 & 2 & 5.667 & 5.3 \end{vmatrix}, \quad b' = (1, 2/3, -8, 1), \quad x = (1, 1, 1, 1).$$

26. a. x = (46.154, 84.615, 92.308, 84.615, 46.154).

b. During reduction: 3(n-1) multiply/divides, 2(n-1) subtracts;
During back substitution 2(n-1)+1 m/d, 3(n-1) subtracts;
Total: 5(n-1) multiply/divides, 3(n-1) subtracts.

27. One way to determine this is to get the determinants.

a. det(A) = 0, singular.
b. det(A) = 21, nonsingular.
c. det(A) = 0, singular.

28. det(A) = 7(x - y + 2). Whenever (x - y) = -2, the matrix is singular (such as (1,3), (2,4)) and whenever (x - y) ≠ 2, it is nonsingular (such as (1,2), (2,5)).

29. a. Rows as vectors are dependent: 2*R1 + 3*R2 - 1*R3 = 0

b* Columns as vectors are dependent: 13*C1 + 12*C2 - 5*C3 = 0.

30a. A solution exists, x = -1, y = -1, z = 3, that satisfies all four equations.

b* There is NO solution; the first three equations gives us a unique solution: (1.5, -0.5, -1.5), but substituting this into the fourth equation does not produce the correct result.

c. Matrix A is singular; 2*R1 - 2R2 = R3. There is no solution because this not true for the rhs.

d. Matrix A is singular; 2*R1 - 2R2 = R3. There is an infinity of solutions since this relationship is also true for the rhs.

31a. det(H) = 1.65E-5. (A zero determinant means singular.)

b. (1.11, 0.228, 1.95, 0.797).

c. (0.988, 1.42, -0.428, 2.10).

32. The value of the determinant is 232.

33. The value of the determinant is -723.

34.

$$A^{-1} = 1/75 \begin{vmatrix} -26 & 33 & 46 & -17 \\ 44 & -27 & -49 & 23 \\ 53 & -24 & -88 & 26 \\ -2 & -9 & -8 & 16 \end{vmatrix}, \quad A^{-1}b = (1, 2, 2, -1).$$

35. x = (-2, 1, 0, 3).

36. Trying to get the inverse involves a division by zero.

37a. Det(H) = 1.653E-7.

b*

$$H^{-1} = \begin{vmatrix} 16 & -120 & 240 & -140 \\ -120 & 1200 & -2700 & 1680 \\ 240 & -2700 & 6480 & -4200 \\ -140 & 1680 & -4200 & 2800 \end{vmatrix}$$

38* Gauss Elimination: 25 mult/div; 11 add/subtracts.
Gauss-Jordan: 29 mult/div; 15 add/subtracts.
The system here is too small to illustrate the true difference between the methods.

39. a. 1-norm = 17.45, 2-norm = 10.912, ∞-norm = 10.0
b. 1-norm = 19, 2-norm = 9.9499, ∞-norm = 7.0
c. 1-norm = 22, f-norm = 18.841, ∞-norm = 23
d. 1-norm = 12, f-norm = 10.344, ∞-norm = 11

40* 25/12, which is the sum of the elements of the first row.

41. 13,620, the sum of the elements of the third row of the inverse.

42a* (1592.61, -631.911, -493.62).
b. (-118, 47.1, 37.0) with pivoting; even with rounding there is a large difference.
c. e = (1710, -697, -530), 2-norm is 1914.
d. Yes, there is a small element (about 0.020) on the diagonal after reduction; also, arithmetic precision makes a large difference.

43a. (0.15094, 0.145246, -0.165916).
b. (0.153, 0.144, -0.166) with pivoting.
c. e = (-0.00206, 0.00125, 0.000084), 2-norm is 0.0309.
d. No, there is no small diagonal element after reduction; also the arithmetic precision makes less difference.

44* x = (119.53, -47.14, -36.84). This is further evidence of ill-condition, in that small changes in the coefficients make a large change in the solution vector.

45. r = (-1.463, 0.434, -1.563) and (0.00149, 0.00247, -0.008811).

46. Cond(A) = 55,228 for Exercise 42,
Cond(A) = 16.05 for Exercise 43.

47.

	Exercise 42	Exercise 43
norm(r)	2.1844	0.009271
norm(A)	1914	15
norm (A^{-1})	3682	1.0702
Left side	0.1456	0.000618
Right side	8042	0.009921
norm(e)	1914	0.002411

In both cases, norm(e) falls between.

48* Lefthand side of Eq. (2.30) = 3.95E-5
Righthand side of Eq. (2.30) = 120,640
Central part = 1.07, falls between the two.

49. If 14 digits precison is used, the norms of both r and e are essentially zero.

50. xbar = (-118, 47.1, 37.0), r = (-1.463, 0.434, -1.5633),
ebar = (1710, -679, -529) using 6-digit precision,
improved x = (1592, -632, -492) which much better.
(If one gets e with only 3-digit precision, there is very little improvement.)

51* x = (0.153, 0.144,-0.166), r = (0.00149, 0.00247, -0.00881),
e = (-0.00271, 0.00126, 0.00103) using 3-digit precision, improved
x = (0.15029, 0.14526, -0.16497) a much better result even though only 3-digits were used. (With 6-digits, there is 6-digit accuracy in the final result.)

52. After interchanging rows 1 and 2, and starting with (0, 0, 0), Jacobi gets (1, -1, 3) accurate to 5 digits in 11 iterations; Gauss-Seidel gets this in 5 iterations.

53. Converges to (46.1539, 84.6154, 92.3077, 84.6154, 46.1538) in 9 iterations.

54* Both methods diverge; after 10 iterations, Jacobi gets

(-76.76, -76.76), Guass-Seidel gets(201,551.9, 604,659.8).

55* After interchanging rows 2 and 3, and starting with (0, 0, 0):

a. In ten iterations, the Jacobi method produces the answer: (-2.00000, 1.00000, -3.00000).

b. The Gauss-Seidel method produces the same result as in part (a), but does it in just five iterations.

56. x = (1, -1, 2).

57. x = (-2, 1, 3).

58. The answers are the same as for Exercise 26a.

59* The two solutions are (0.72595, 0.50295) and (-1.6701, 0.34513).

60. (x,y,z) = (2.49137, 0.242745, 1.65351).

61. (x,y) = (1.64303,-2.34978) and (-2.07929, -3.16174).

62. Using the analytical partials, there is convergence after just three iterations to the answer:

x = 0.90223, y = 1.10035, z = 0.95013

63. In both exercises, the answers are the same but convergence is slower. To match to four significant digits, it takes 8 iterations versus 4 in Exercise 59 and 5 versus 3 in Exercise 62.

64a. Multiply $P_{1,3}$*A where $P_{1,3}$ is the order-4 identity matrix with the first and third rows interchanged.

b* Multiply $P_{1,4}$*A where $P_{1,4}$ is the identity matrix with the first and fourth rows interchanged.

c* Multiply A*$P_{1,2}$ where $P_{1,2}$ has columns 1 and 2 interchanged.

d* Do: $P_{2,4}$*A*$P_{2,4}$

65. If $P^{-1} = P$, P^2 should equal I. This is true.

66. This is confirmed when the matrices are multiplied.

67. This is confirmed when the matrices are multiplied.

68. Both systems give the same answers as in Exercise 10. In either, rref, solve, or linsolve can be used.

69. Operations on arrays are like operations on scalars in MATLAB. Maple requires: "with (linalg):", then operations are as for scalars.

70. With MATLAB, just perform the operations.

71. The same answers are obtained as in Exercise 10.

72. Answers are the same as for Exercise 4.

73. a. If a = hilb(4) and bt = right-hand sides as a column vector, x = b\bt gives the answer: (1, 1, 1, 1).
   b. cond(hilb(4)) gives 1.5514E4 for the condition number.
   c. cond(hilb(10)) gives 1.6025E13.

74. After doing: with(linalg)
   a. det(hilb(4)) gives 604800.
   b. inverse(hilb(4)) gives the same matrix as in Exercise 37b.

75. The answers duplicate those of Exercise 59.

76. Using fsolve without specifying a range for (x,y) gives the leftmost intersection: (-2.07930, -3.16174). If the range is {x = 0..2,y = -3..0}, the intersection at (1.64304, -2.34978) is obtained.

77. Making upper-triangular (see Exercise 19):

Multiply/divides = $(n - 1)(2n - 1)(n)/6 + (n - 1)(n)$
$= (2n^3 + 3n^2 - 5n)/6$.

Add/subtracts = $(n - 1)(2n - 2 + 1)(n - 1 + 1)/6 + (n - 1)(n)/2$
$= (n^3 - n)/3$.

Back substitution:

Multiply/divides = $(n - 1)(n)/2) + n = (n^2 - n)/2$.

For subtracts, the same: $(n^2 - n)/2)$.

78. Making diagonal: $(n^3 + 2n^2 - 3n)/2$ multiply/divides,
$(n^3 - n)/2$ add/subtracts.

"Back substitution", n divides. This is $O(n^3/2)$ while Gaussian elimination is only $O(n^3/3)$.

79. Work on matrix A (N x N) augmented with the N x N identity matrix, and use a Gauss-Jordan method, taking advantage of the zeros in the identity matrix. Assign numbers (i,j), i = 1 TO n, j = 1 TO n to the n^2 processors.

```
For i = 1 to N  ' counts rows
 {ON PROCESSOR (j,k-i)}:
   A(j,k) = A(j,k) - A(j,i) / A(i,i) * A(i,k) FOR
      j = 1 TO N (j ¹ i), k = i + 1 TO i + N
(Now divide by diagonals)
     {ON PROCESSOR (i,j)}
   A(i,j+N) = A(i,j+N)/A(i,i), FOR
      i = 1 TO N, j = 1 TO N.
```

80. When doing row i, all elements to the left of the diagonal will become zero; we do not have to specifically calculate them. So we reassign one of the processors from this set, say PROCESSOR(i,i-1) to replace PROCESSOR(i,n+1). The n^2 processors are adequate to perform the back substitution phase.

81. Number the processors: PROCESSOR(i,j), i = 1 TO n, j = 1 TO n and assign PROCESSOR(i,i) to the corresponding variable. We can perform each of the computations in the next iteration simultaneously according to the assignment statement in the algorithm for Jacobi iteration:

```
new_x[i] = new_x[i] - A[i,j]*old_x[j].
```

82. The transformed vector is (1.965, 0.664, -2.672).

83a. Cond. no. = 9.870E7 using Euclidean norms.

b. The fifth component changes most but the system is so ill-conditioned that the specific values are uncertain.

85. The system is overdetermined. Using the data for peaks 2, 3, 4, 5, and 6 gives p = {2.17, 0.002, 6.611, 8.323, 4.348} and these values are reasonably consistent with the sum and the value for peak 1.

86.

P_1: x	f	P_2: x	f	P_3: x	f
0.24	-1035	0.80	51	0.22	-707
-0.59	732	-0.78	1964	-0.45	500
0.12	-152	0.33	94	0.08	464
-0.56	-531	-0.80	-739	-0.53	-562
-0.32	-232	-0.73	-260	-0.45	72
0.10	469	0.60	1261	0.07	437
0.25	616	0.67	630	0.20	464
-0.42	268	-0.84	1036	-0.53	500
0.29	-378	0.84	1465	0.29	-707

87. Correct values are near (425, 351, 346, 167).

Chapter 3

1.

$$\frac{(x - 0.5)(x - 3.1)}{(-2.3 - 0.5)(-2.3 - 3.1)}(2.1) + \frac{(x + 2.3)(x - 3.1)}{(0.5 + 2.3)(0.5 - 3.1)}(-1.3)$$

$$+ \frac{(x + 2.3)(x - 0.5)}{(3.1 + 2.3)(3.1 - 0.5)}(4.2)$$

2* $P_3(x) = 0.08333x^3 - 1.125x^2 + 4.41667x - 1.375$

3. $P_2(x) = -0.043596x^2 + 0.623429x - 0.379325$.

x:	1	2	3	4	5	6	7
P(x):	0.2005	0.6932	1.0986	1.4169	1.6479	1.7918	1.8485
error:	0.2005	0	0	0.0306	0.0385	0	0.0971

x:	8	9	10
P(x):	1.8183	1.7007	1.4959
error:	0.2610	0.4965	0.8066

Conclusion: interpolation is good, extrapolation is poor.

4. Estimate = 1.22183, actual error = -4.31E-4.
Error bounds: -3.333E-4, -4.499E-4.

5. Estimate = 1.4894, actual error = 0.0024.
Error bounds: 0.00200, 0.00298.

6.

3.0	20.7180	75.4040	46.5355	53.7704	56.0694
5.0	130.0900	-40.0700	10.3609	39.9768	
7.0	470.4100	312.9461	247.2881		
-2.0	-1.9817	94.0861			
-3.0	-17.9930				

$P_2(4) = 46.5355$, $P_3(4) = 53.7704$, $P_4(4) = 56.0694$

7. Table for x = 0.2:

0.1	1.1052	1.2276	1.2218
0.3	1.3499	1.2333	
0.0	1.0000		

$P_2(0.2)$ reproduces the result of Exercise 4.

8. Table for x = 0.4:

0.1	1.1052	1.4723	1.4894
0.3	1.3499	1.4665	
0.0	1.0000		

$P_2(0.4)$ reproduces result of Exercise 5, linear interpolation gives 1.4723.

9. From Neville table: $P_1(0.2) = 1.2276$, $P_2(0.2) = 1.22183$.
Lagrange interpolation gives the same results.

10. (Neglecting the cost of rearranging the data pairs.)
If there are (n+1) data pairs,
steps in sequential processing = (n)(n+1)/2;
steps in parallel processing = n.
Some values:

n	2	3	4	5	10	20
Sequential	3	6	10	15	55	210
Parallel	2	3	4	5	10	20
Ratio	0.667	0.500	0.400	0.333	0.182	0.095

11.

0.50	-1.1518	-2.6494	1.0955	1.0286	0.0036
-0.20	0.7028	-2.4303	0.6841	1.0267	
0.70	-1.4845	-2.2251	0.8894		
0.10	-0.1494	-2.8477			
0.00	0.1353				

12. The polynomial is identical to that of Exercise 2.

13. Estimate = 1.22183, identical to that of Exercise 4.

14. a. -0.3587
    b. -0.2851
    c. -0.28940
    d. -0.28938
    e. -0.28938
    f. Because each polynomial is different

15* a. At x = 0.0, 0.1, 0.5 or -0.2
    b. At x = -0.2, 0.0, 0.1
    c. At x = 0.1, 0.5, 0.7

16. Estimate = 1.4894, identical to that of Exercise 5.

17. $P_3(0.2) = -0.42672$, error estimate = 0.00002

18* Bounds: -3.333E-4, -4.499E-4; actual error (-4.31E-4) falls between.

19.

i	x	f	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1	1.200	0.1823	0.0408	-0.0015	-0.0001	0.0004	-0.0007
2	1.250	0.2231	0.0393	-0.0016	0.0003	-0.0003	0.0004
3	1.300	0.2624	0.0377	-0.0013	0.0000	0.0004	
4	1.350	0.3001	0.0364	-0.0013	0.0001		
5	1.400	0.3365	0.0351	-0.0012			
6	1.450	0.3716	0.0339				
7	1.500	0.4055					

20. Sixth degree but third degree will almost fit because the third differences are nearly constant.

21. Third differences are constant at 0.096. $a_0 n! h^3 = 0.096$.

22* $\Delta^3 f_0 = 0.3365 - 3(0.3001) + 3(0.2624) - 0.2231$
$= 0.0003$ at $x_0 = 1.25$; agrees with value in the table.

23. $f[x_0,x_1] = 0.7860$,
$f[x_0,x_1,x_2] = -0.3200$.
$f[x_0,x_1,x_2,x_3] = 0.4000$.

From Exercise 22: $f_0^{[3]} = \dfrac{\Delta^3 f_0}{3!h^3} = \dfrac{0.0003}{(6)(0.05^3)} = 0.4000$.

24. Estimate = 0.3148, estimate of error = 1.7E-5.

25. Estimate = -0.2305, estimate of error = 0.01458.
A larger error because we extrapolate outside the table.

26* $P_2(0.203) = 0.78024$, estimate of error = 1.32E-3.
$P_3(0.203) = 0.78156$, estimate of error = 7.2-5.

27. $P_2(0.612) = 0.66867$, error = -6.99E-3.
$P_3(0.612) = 0.66168$. No "next term" from $x_0 = 0.375$.

28. $P_2(0.612) = 0.72023$, error = -6.370E-2.
$P_3(0.612) = 0.65654$, error = 4.778E-3.

29. $P_3(0.54) = 0.166$.

30* Fourth degree because the fourth differences are constant.

31. Each divided difference is the corresponding ordinary difference divided by ($h^n n!$), where n is the order of the differences.

32. From data rounded to 3 places, the third differences become:
-0.001, 0.003, 0.003.
From data chopped to three places, they are:
0.003, 0.000, 0.003.
Compare to the original:
0.001, 0.002, 0.002.

33. $P_2(x) = 1 - x^2$, maximum error = 0.9375.

$P_3(x) \equiv 0$, maximum error = 1.0.

$P_4(x) = 4x^4 - 5x^2 + 1$, maximum error = 0.703.

$P_5(x) = 0.65104x^4 - 0.88542x^2 + 0.23438$, max error = 0.7656.

34*
$$\begin{vmatrix} 1.220 & 0.490 & 0.000 & 0.000 & -1.326 \\ 0.490 & 1.240 & 0.130 & 0.000 & -1.715 \\ 0.000 & 0.130 & 0.620 & 0.180 & -0.829 \\ 0.000 & 0.000 & 0.180 & 2.440 & -1.865 \end{vmatrix}$$

35.

x_i :	0.15	0.27	0.76	0.89	1.07	2.11
S-values:	0.000	-0.678	-1.018	-0.922	-0.696	0.000
a:	-0.9413	-0.1159	0.1240	0.2087	0.1136	
b:	0	-0.3389	-0.5092	-0.4609	-0.3482	
c:	1.0919	1.0512	0.6356	0.5095	0.3639	
d:	0.1680	0.2974	0.7175	0.7918	0.8698	

The above are coefficients of $a(x - x_i)^3 + b(x - x_i)^2 + c(x - x_i) + d$ in each interval. Interpolating with these polynomials:

x:	0.33	0.92	2.05
Interpolate:	0.3592	0.8067	0.9971
True value:	0.3593	0.8067	0.9963
Error:	0.0001	0	-0.0008

36.

End condition	x: 0.33	0.92	2.05	Maximum error
2	0.3605	0.8067	0.9961	-0.0012
3	0.3588	0.8066	1.0041	-0.0078
4	0.3589	0.8067	0.9953	0.0010

(End condition 1 gives best accuracy.)

37. Maximum error = 0.607 at $x = \pm 0.25$. Compare to maximum error of $P_4(x) = 0.703$. Note: evenly spaced points are not the best choice.

38. With end condition 3: maximum error = 0.625 at $x = \pm 0.25$.
With end condition 4: maximum error = 0.656 at $x = \pm 0.25$.
Note: evenly spaced points are not the best choice.

39* Maximum error = 0.5938 at $x = \pm 0.25$ with end slopes = 0.
Note: evenly spaced points are not the best choice.

40. n-2 equations are the same; one more equation is $S_0 - S_n = 0$.
The final equation, based on equal slopes at the end:

$$-2h_0S_0 + h_0S_1 - 4h_{n-1}S_{n-2} + 3h_{n-1}S_{n-1} = 6(f[x_{n-2},x_{n-1}] - f[x_0,x_1])$$

41. Some representative values:

Time:	0.05	0.1	0.45	0.75	0.9	0.95
Data:	0.280	0.253	0.133	0.511	0.386	0.341
Interpolate:	0.278	0.252	0.133	0.511	0.385	0.343

(These are for end condition 1).

42. Multiply the matrices and compare terms.

43*

$$(u^4, u^3, u^2, u, 1) \begin{vmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} (p_0, p_1, p_3, p_4)^T$$

44. Each p (for both Bezier and B-spline curves) is of the form $\Sigma a_i p_i$ and each $a_i \le 0$. On multiplying out and collecting terms we find, for each curve, $\Sigma a_i = 1$.

45* $dx/du = -(3/6)(1 - u)^2 x_{i-1} + (1/6)(6u^2 - 6u)x_i + (1/6)(-9u^2 + 6u + 3)x_{i+1} + (3/6)(u^2)x_{i+2}$;

at $u = 0$, $dx/du = (3/6)(x_{i+1} - x_{i-1})$.

The expression for dy/du is similar, so

$$\frac{dy}{dx} = \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}} = \text{slope between points adjacent to } p_i.$$

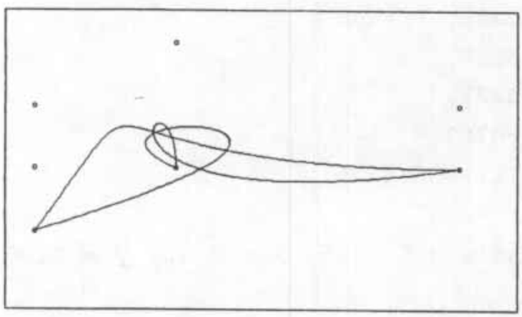
46. For both Bezier and B-spline curves, changing a single point changes the curve only within the intervals where that point enters the equations. Its influence is localized in contrast to a cubic spline where changing any one point affects the entire curve.

47. A quadratic B-spline will have these conditions at the joints:

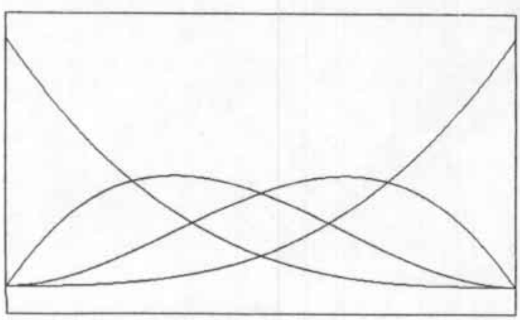
$B_i(1) = B_{i+1}(0)$,

$B_i'(1) = B_{i+1}'(0)$.

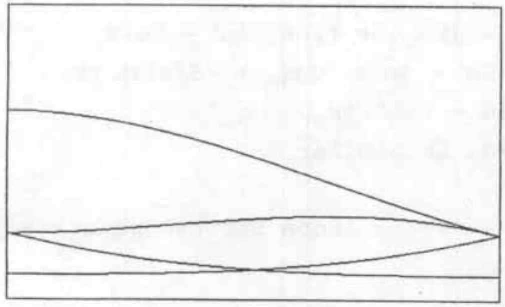
48.



49.



50.



51. Value = 1.841 (as before).

52* f(1.6,0.33) = 1.833.

53. f(1.62,0.2) = 1.1287,
f(1.62,0.4) = 2.3167,
f(1.62,0.3) = 1.6927;
from these, f(1.62,0.31) = 1.7491.

54* f(1.1,0.71) = 0.70725,
f(3.0,0.71) = 5.26137,
f(3.7,0.71) = 8.00277,
f(5.2,0.71) = 15.80769;
from these, f(3.32,0.71) = 6.4435.

55. When u = 0.8736 and v = 0.9325, x = 3.70, y = 0.60,
f(3.70,0.60) = 8.8534.

56. When u = 0.6124 and v = 0.6327, x = 3.70, y = 0.60,
f(3.70,0.60) = 9.3853.

57.

True value of R	From curve by eye	From curve by least squares
765	771.75	771.798
826	814.45	813.216
873	875.50	875.345
942	956.20	950.714
1032	1034.95	1027.101
Sum(deviations)^2	419.56	315.05
Maximum deviation	14.20	12.78

58. From the normal equations:

$$a = \frac{N\Sigma xy - \Sigma x\Sigma y}{N\Sigma x^2 - (\Sigma x)^2}, \quad b = \frac{\Sigma x^2\Sigma y - \Sigma x\Sigma y}{N\Sigma x^2 - (\Sigma x)^2},$$

Using these in $y = ax + b$, and substituting $y = \Sigma y/N$, $x = \Sigma x/N$, we get

$$\Sigma y/N = a(\Sigma x/N) + b.$$

59* $y = 2.908x + 2.02533$.

60. $x = 0.34207y - 0.674347$, or $y = 2.923x + 1.9714$.

61* Normal equations:

$$\begin{matrix} N & \Sigma x & \Sigma y & \Sigma z \\ \Sigma x & \Sigma x^2 & \Sigma xy & \Sigma xz \\ \Sigma y & \Sigma xy & \Sigma y^2 & \Sigma yz \end{matrix}$$

$z = 2.85297x - 1.91454y + 1.03987$.

62. From two points: $y = 0.5x + 2$.

a. $y = 0.5x + 2.333$,

b. $y = 0.5x + 1.333$,

c. $y = 0.5385x + 2$

d. $y = 0.6154x + 1.0769$.

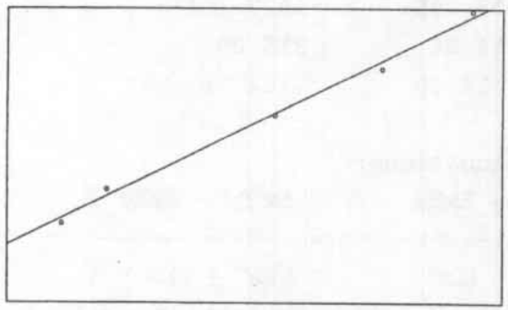
63. Multipliying shows $A*A^T$ = coefficient matrix, and $A*y$ gives $(\Sigma y_i, \Sigma x_i y_i, \Sigma x_i^2 y_i, \ldots)$.

64. $A*A^T$ = coefficient matrix, which is symmetric.

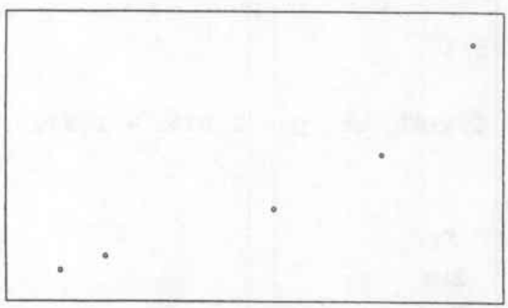
Proof of positive definite:

Consider $x(AA^T)x^T = (xA)(A^Tx^T) = v*v^T$ where $v = xA$. For any vector, v, $v*v^T = \Sigma v_i^2 \geq 0$ and zero only if v = zero vector; hence AA^T is positive definite.

65. Plot of ln(S) versus T:



66. Plot of S versus T:



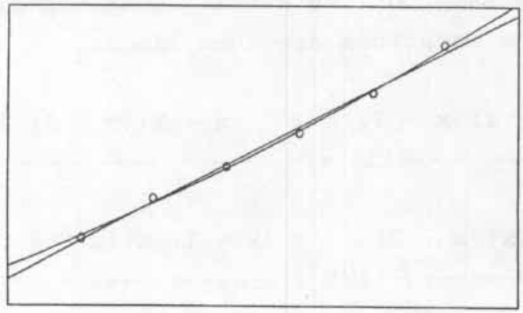
67* $\ln(S) = 0.009602T + 0.18396$.

68. $\ln(F) = 3.4083 + 0.49101*\ln(P)$, or $F = 30.214\ P^{0.49101}$.

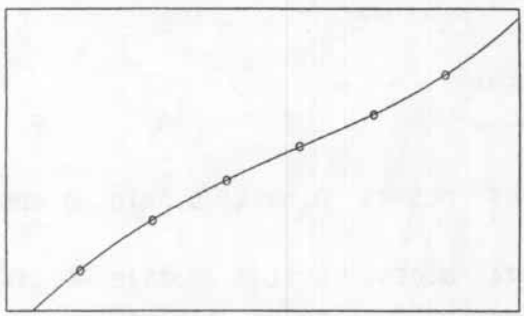
69. $F = -0.01341P^2 + 3.5836P + 62.149$.

70. Linear: $y = 2.908x + 2.0253$, $S(dev)^2 = 0.7832$, variance = 0.1958. Quadratic: $0.1036x^2 + 2.1830x + 2.9920$, $S(dev)^2 = 0.3829$, variance = 0.1276.

71. $-8.408E-5*x^5 - 3.499E-3*x^4 + 0.1366x^3 - 1.0041x^2 + 5.1903x + 0.7208$.
Plot of least squares line and quadratic:



Plot of interpolating polynomial:



72.

	Maximum slope	Minimum slope
Linear:	2.908	2.908
Quadratic:	3.4262	2.3902
Fifth degree:	4.3258	2.4414

(These are slopes within [1,6]).

73*

Degree:	2	3	4	5
Variance:	533.3	85.47	86.73	64.84

The cubic polynomial is preferred.

74. Using points 1, 3, 5, ..., third degree is preferred:

Degree:	2	3	4	5
Variance:	544.1	21.97	24.56	21.76

Using points 2, 4, 6, ..., fifth degree is preferred:

Degree:	2	3	4	5	6
Variance:	629.3	123.6	99.45	66.36	79.63

75. The method is the same but the normal equations are now nonlinear. If C is specified, the equations are then linear.

76.

$$P_3(x) = \frac{x(x-1)(x-2)}{6}(11) + \frac{(x+1)(x-1)(x-2)}{2}(-7)$$

$$+ \frac{(x+1)(x)(x-2)}{2}(7) + \frac{(x+1)(x)(x-1)}{6}(-5)$$

$$= ((x-2)x+1)x-7.$$

(There are many others).

77. magnitude of errors:

x:	3	4	6	7	8	9	10
Actual error:	0.2005	0.0306	0.0385	0.0971	0.2610	0.4964	0.8066
Lower bound:	0.0026	0.0374	0.0556	0.1166	0.2346	0.3440	0.4480

The upper bounds are so large as to be unhelpful.

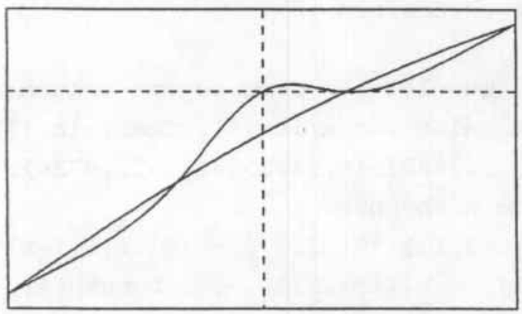
78*

Degree	P(0.1)	Error
2	0.99	-0.39
3	0	0.60
4	0.9504	-0.3504
5	0.2256	0.3744

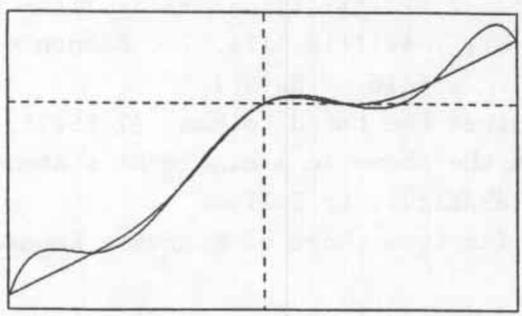
Cannot find bounds because f'(x) is discontinuous.

79. Beginning value: 0.2, $P_2(x) = 0.2427x^2 + 0.2179x - 0.02054$,
$P_2(0.5) = 0.14908$, Error = 0.0025, bounds: 0.0015, 0.0040.
Beginning value: 0.3, $P_2(x) = 0.0047x^2 + 0.4321x - 0.06337$.
$P_2(0.5) = 0.15384$, Error = -0.0022, bounds: 0.0008, 0.0042.
Beginning value: 0.6, $P_2(x) = -0.1339x^2 + 0.6401x - 0.1383$,
$P_2(0.5) = 0.14830$, Error = 0.0033, bounds: 0.0012, 0.0066.

80. $P_2(x) = 2x - x^2$.
$P_3(x) = -0.3703x^2 + 2x - 0.6297$.
$P_4(x) = 3.562x^4 - 4.562x^3 + 2x$.
$P_5(x) = 2.049x^4 - 2.829x^2 - 0.2198$
$P_6(x) = -10.397x^6 + 17.222x^4 -7.824x^2 + 2x$
(The odd degree polynomials miss the point at x = 0).
Plot of $P_3(x)$ and $f(x)$:



Plot of $P_6(x)$ and $f(x)$:



82. It is easiest to compute one column at a time:

If the original data (in [1]) is

```
{{.5,1.0025},{-2,1.3940},{.7,1.0084},{.1,1.1221},{0,1.1884}}
```

This request:

```
Table [(%1[[i+1,2]] - %1[[i,2]])/(%1[[i+1,1]]-%1[[i,1]]),
        {i,Length[%1]-1}]
```

produces the first column of differences as [2]:

```
{0.559286, -0.428444, -0.1895, -0.663}
```

and a similar request:

```
Table [(%2[[i+1]] - %2[[i]])/(%1[[i+2,1]]-%1[[i,1]]),
        {i,Length[%2]-1}]
```

produces the second column as [3]: {0.654206, 0.796481, 0.676429}

We get the third differences with:

```
Table [(%3[[i+1]] - %3[[i]])/(%1[[i+3,1]]-%1[[i,1]]),
        {i,Length[%3]-1}], giving {-0.355688, -0.600265]
```

We can continue as far as desired.

83. As in Exercise 82, we elect to compute one column at a time.

We begin by defining a value for x in [1]. Then, in [2], we give the data:

```
{{2,3.4899},{5,21.7889},{6,31.3585},{-1,.8726},{-2,3.4899}}
```

We get the first column with this:

```
Table [((x - %[[i+1,1]])*%[[i,2]] + (%[[i,1]]-x)*%[[i+1,2]])
        / (%[[i,1]] - %[[i+1,1]]), {i, Length[%] - 1}]
```

and Mathematica gives [2]: {9.5889, 2.6497, 18.2931, -9.5971}

The second column comes from the request:

```
Table [((x - %2[[i+2,1]])*%[[i]] + (%2[[i,1]] - x)*%[[i+1]])
        / (%2[[i,1]] - %2[[i+2,1]]), {i, Length[%] - 1}]
```

which produces {7.8541, 7.86416, 7.83426}

An analogous request gives the third column: {7.85075, 7.85561}

It is possible to nest the above in a single DO statement in Mathematica but the logic is then difficult to follow.

Note: these result differ from those of Exercise 6 because we used more accurate f(x) values.

84.
$$(-1 + x)\left(-\left(\frac{16}{3}\right) + \left(\frac{13}{15} - \frac{119(-3 + x)}{120}\right)(2+x)\right)$$

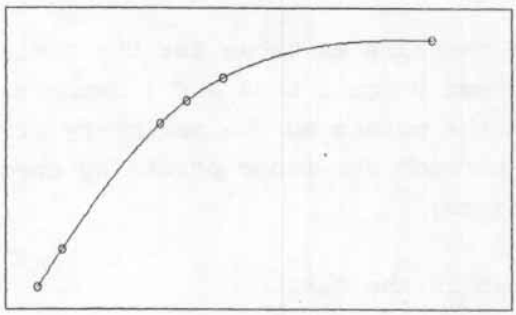
$$= -\left(\frac{47}{20}\right) + \frac{59x}{120} + \frac{57x^2}{20} - \frac{119x^3}{120}$$

85* 1.8407.

86. 1.03987 + 2.85297x - 1.91454y.

87. ln(S) = 0.183932 + 0.0096028T.

88. Plot of spline curve:



89. The agreement cannot be better than three decimal places, not three digits, because the accuracy of the original data is only that good. Linear interpolation agrees with the formula except at T = 650 and T = 750. Even a quartic does not agree at these points. This is because the formula itself does not give three decimal agreement with the given value for T = 700; the formula gives 0.0705, the table shows 0.067.

90. Dosage at 2.5 is 3.27. Both quadratic and cubic polynomials give this result.

91.

N:	0	1	2	3	4	5
D:	0.93	3.12	3.05	2.75	2.43	2.09

These values are from a quadratic through the three nearest points. Values at N = 0, 4, and 5 are uncertain.

93. u(0.7,1.2) = 10.68.
u(1.6,2.4) = 0.
u(0.65,0.82) = 9.40.
Quadratics in both directions were used.

94. Using a cyclic cubic spline with an assumed value of 7.92 at phase = 120, we get

Phase:	-100	-60	-20	20	60	100
Estimate:	8.23	9.79	11.56	10.58	8.74	7.93
Error:	0.14	-0.39	-0.17	0.26	-0.21	-0.04

95. The M-matrices are the same as those for the Bezier curves given in Section 3.6. Setting u and v equal to 0 and 1 demonstrates that the surface passes through the points on the periphery of the patch. The surface can be forced through the inner points by specifying duplicates a sufficient number of times.

96. The curves are shown in the text.

Chapter 4

1. $T_{11}(x) = 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x$
   $T_{12}(x) = 2048x^{12} - 6144x^{10} + 6912x^8 - 3584x^6 + 840x^4 - 72x^2 + 1$

2. Substitute cos(θ) for x; the integrand becomes -cos(nθ)*cos(mθ). The integrations then are easy.
   a. For (0,1): 0
   b. For (1,1): π/2
   c. For (1,2): 0

3. Maximum magnitudes on [-2,2]:
   $T_1(x) = 2$ (linear, symmetric about origin).
   $T_2(x) = 7$ (parabola, symmetric about y = 0.
   $T_3(x) = 26$ (cubic, symmetric about origin).

4* The zeros are at 0, ±0.5877852526, ±0.951056162.
Analytically, these are 0, $\pm[(5/8)-\sqrt{(5/8)}]^{1/2}$, $\pm[(5/8)+\sqrt{(5/8)}]^{1/2}$.

5. Write cos(6x) as cos(3x + 3x) = cos(3x)*cos(3x) - sin(3x)*sin(3x)
   $= 2\cos^2(3x) - 1$
   $= 2[4\cos^3(x) - 3\cos(x)]^2 - 1$
   $= 32\cos^6(x) - 48\cos^4(x) + 18\cos^2(x) - 1$.

6. e^x is approximated by
   $1.0000434 + 0.9973958x + 0.4992188x^2 + 0.1770833x^3 + 0.04375x^4$.
The maximum error is 0.00079051 at x = 1.

7. The maximum error for the fourth degree Taylor series is 0.0099485 at x = 1; for the fourth degree economized polynomial, 0.0007905 at x = 1.

8* The ninth degree Maclaurin series is very accurate near x = 0 but the error increases very rapidly near x = ±1 to 0.04952. The third degree economized polynomial has a maximum error at x = ±0.4 of 0.0349.

9. The truncated Chebyshev series is $Q(x) = 0.99985x - 0.16650x^3$. Comparing to the truncated Maclaurin series $P(x) = x - x^3/6$, we get these typical values:

x:	0	0.2	0.6	1.0
Q(x):	0	0.1982638	0.563946	0.833350
P(x):	0	0.1986667	0.564000	0.8333333
Exact:	0	0.1986693	0.564625	0.8414710

The maximum error in P(x) is 0.008138 at x = ±1 while the maximum error in Q(x) is 0.008121 at x = ±1.0.

10. The Chebyshev series of degree two is

$$0.99748T_0(x) + 0.10038T_1(x) - 0.002532T_2(x)$$
$$= 1.000001 + 0.10038x - 0.005064x^2.$$

Maximum errors: Chebyshev series = -0.000139 at x = -1, truncated Maclaurin series = -0.000573 at x = -1. The Chebyshev series has a smaller error by a factor of 4.1.

11* Let $y = \dfrac{2x - b - a}{b - a}$. When x = a, y = -1, when x = b, y = +1.

12* a. In trying to get R3,3, the matrix for the b's is singular.

$$R_{4,2} = \frac{15x^2 - 3x^4}{15 + 2x^2}$$

b. $R_{3,3} = 1$.
This is a very poor approximation except near x = 0.

c.
$$R_{3,3} = \frac{x^3 + 12x^2 + 60x + 120}{-x^3 + 12x^2 - 60x + 120}.$$

13. Maximum errors:

a. For $R_{4,2}$ is 0.002191 (Taylor series: -0.003038).

b. For $R_{3,3}$ is 0.4597 (Taylor series: 0.040302).

c. For $R_{3,3}$ is -0.000028 (Taylor series: 0.000226).

14a.

$$1 - \cfrac{4}{x + 3 + \cfrac{1}{x - 1}} .$$

b.

$$2x + 3 + \cfrac{48}{4x - 9 - \cfrac{19}{4x + 5}} .$$

c.

$$2x + 3 + \cfrac{4}{x + 5 + \cfrac{6}{x + 7 + \cfrac{8}{x + 9}}} .$$

15*

a.

$$R_{3,3} = -3x/2 - \cfrac{15}{4 + 30/x} .$$

b. $R_{3,3}$ is already a "continued fraction!"

(Exercise 15 continued)

c.

$$R_{3,3} = -1 - \cfrac{24}{x - 12 + \cfrac{50}{x + 10/x}}.$$

16a. Next term: -6.36E-3; actual error : -6.213E-3.

b. Next term: zero; actual error: -0.4597.

c. Next term: -1.7E-5; actual error: -2.8E-5.

17.

$$R_{4,2} = \frac{0.851632 - 0.14202T_2 + 0.001162T_4}{1 + 0.00518619T_2}.$$

18. The expression is not minimax. If it were, the error curve would have nine equal max/min on [0,1].

19a. Periodic, period = 2π.

b. Not periodic.

c. Periodic, period = 2π.

d. Periodic, period = π.

20. The plot is a series of "tent" functions going from f(x) = 0 at 0, $\pm 2\pi$, ... to f(x) = 1 at $\pm\pi$, $\pm 3\pi$, ...

21.

$$f(x) \approx 2 + \frac{12}{\pi^2} \Sigma \frac{\cos(n\pi x)}{n^2} + \frac{1}{\pi^3} \Sigma (12 - 8n^2\pi^2)\sin(npx),$$

with the sums from n = 1 to N, N being the number of terms.

22*

$$f(x) \approx \frac{1}{3} + \frac{4}{\pi^2} \Sigma \frac{\cos(n\pi x)}{n^2} - \frac{4}{\pi^2} \Sigma \frac{\sin(n\pi x)}{n}$$

with the sums from n = 1 to N, N being the number of terms.

23. Add the series for Exercises 21 and 22.

24. No. This is true only for f(x) or g(x) equal to a constant.

25a. Reflect about the y-axis.
b. Reflect about the origin.

26. Let y = x + 1, then reflect f(y), finally, let x = y - 1.

27* a. $f(x) \approx A_n \cos(n\pi x/2)$, n = 0, 1, 2, ..., where

n:	0	1	2	3	4
A_n:	0.09760	-0.25074	-0.30642	-0.094459	-0.062950

b. $f(x) \approx B_n \sin(n\pi x/2)$, n = 1, 2, 3, ..., where

n:	1	2	3	4	5
B_n:	0.19903	-0.15982	-0.17348	-0.13388	-0.094460

28a. $f(x) \approx A_n \cos(n\pi x/4)$, n = 0, 1, 2, ..., where

n:	0	1	2	3	4
A_n:	0.071659	0.079405	0.102211	0.068481	0.020304

b. $f(x) \approx B_n \sin(n\pi x/4)$, n = 1, 2, 3, ..., where

n:	1	2	3	4
B_n:	0.0004739	0.031651	0.082033	0.072761

29* a. Maxima are 0.12 at x = 0 and 0.17 at x = ±1.0;
minima are -0.1056 at x = ±0.6892;
$T_4/8$ has maxima of 0.125 and minima of -0.125.
b. Maxima are 0.126 at x = 0 and 0.086 at x = ±1.0;
minima are -0.1444 at x = ±0.7211;
$T_4/8$ has maxima of 0.125 and minima of -0.125.

30. Chebyshev polynomials have all their maxima/minima equal 1 in magnitude in [-1,1]. All Legendre polynomials have maxima/minima equal to 1 at x = -1 or x = +1 but their intermediate maxima.minima are less than 1 in magnitude.

31. Using $x = \cos(\theta)$, $dx = \sin(\theta)d\theta$, $\sqrt{(1-x2)} = \sqrt{(1-\cos^2(\theta))} = \sqrt{(\sin^2(\theta))} = \sin(\theta)$, $T_n(x) = \cos(n\theta)$, $T_m(x) = \cos(m\theta)$. The integral in Eq. (4.3) becomes

$$\int_{-\pi}^{0} \cos(n\theta)\cos(m\theta)d\theta$$

because, at $x = -1$, $\theta = -\pi$ and at $x = 1$, $\theta = 0$.
If $n = m = 0$, the integrand equals 1; integration gives π.
If $n = m \neq 0$, the integrand equals $\cos^2(n\theta)$; integration gives $\theta/2 + (\sin(n\theta)\cos(n\theta))/(2n)$. This evaluated between $[-\pi,0]$ equals $\pi/2$.
If $n \neq m$, the integrand equals $\cos(n\theta)\cos(m\theta)$ and $\cos(n\theta)\cos(m\theta) = \cos((n-m)\theta)/2 + \cos((n+m)\theta)/2$. Integration gives $\sin(n-m)\theta/(2(n-m)) + \sin(n+m)\theta/(2(n+m))$ to be evaluated between $\theta = -\pi$ and $\theta = 0$. Both terms are zero because $\sin(n\theta) = 0$ for any integer value for n.

32. $\sin(nx/\pi)$ is orthogonal over [-1,1]. The graph of $\sin(5\pi x/2)$ $[n = 5\pi^2/2]$ has six maxima/minima each equal to +1 or -1 but these do not occur at the same x-values as those for $T_4(x)$, except at $x = -1$, $x = 0$, and at $x = +1$.

33. $\cos(nx/\pi)$ is orthogonal over [-1,1]. The graph of $\cos(5\pi x/2)$ $[n = 5\pi^2/2]$ has five maxima/minima each equal to +1 or -1 but these do not occur at the same x-values as those for $T_4(x)$; at $x = -1$ or $x = +1$, $\cos(5\pi x/2)$ equals zero.

34. The Maple command is: orthopoly[T](n,x); the results are precisely the same as Eqs. (4.1).

35. The *Mathematica* command is: LegendreP[n,x]. The results are:
$L_1 = x$; $L_2 = (-1 + 3x^2)/2$; $L_3 = (-3x + 5x^3)/2$;
$L_4 = (3 - 30x^2 + 35x^4)/8$; $L_5 = (15x - 70x^3 + 63x^5)/8$;
$L_6 = (-5 + 105x^2 - 315x^4 - 231x^6)/16$

36. Using the file Tch.m from Section 4.6, we do:

```
f = 'xxxxx'  % define the function
ts = symsub(taylor(f,7),'O(x^7)')
cs = symop(Tch(6),'/','2^5','*','C6')  %C6 = coeff of x^6 in ts
es = symsub(ts,cs)
vpa(collect(es),5)
```

which gives

a. $1.0026 - x - 0.54609x^2 + 0.83333x^3 - 2.0840e\text{-}3x^4 - 0.175x^5$

b. $0.15927 - x + 5.7635x^2 + 9.7029x^3 - 12.778x^4 - 30.833x^5$

c. The Taylor series is just x for x > 0, -x for x < 0.

37. Using the results of Exercise 36, we get

a. $1.0026 - 0.94531x - 0.54609x^2 + 0.61458x^3 - 2.0840e\text{-}3x^4$

b. $0.15927 + 8.6353x + 5.7635x^2 - 28.838x^3 - 12.778x^4$

c. The Taylor series is just x for x > 0, -x for x < 0.

38*

x	Errors, part (a) TS(6)	Econ(4)	Errors, part(b) TS(6)	Econ(4)
-1.0	0.01414	0.00258	11.7154	10.6291
-0.5	0.00010	0.00591	0.2114	0.0886
0	0	0.00046	0	0.8407
0.5	0.00007	0.00502	0.2899	1.5144
1.0	0.00724	0.00341	30.1945	27.4266

Observe that the maximum error is less for Econ(4) than for TS(6) in both parts. In Part (b), the errors of Econ(4) are actually less than for Econ(5).

39. Coefficients are:

i	Part(a) a_i	b_i	Part(b) a_i	b_i	Part(c) a_i	b_i
0	-1.4053		2.0891		1	
1	3.8106	-0.1716	-2.3545	-1.8056	-0.8106	0
2	-3.2175	2.8648	1.2995	1.4054	0	0
3	1.1356	-2.6714	-1.0312	-1.0087	-0.0901	0

40. With so few terms, the graphs in parts (a) and (b) do not match well to the graphs of the functions. However, in part (c), the graphs do match well. In parts (a) and (b), both series get the average values of the function at $x = \pi$ and at $x = -\pi$.

41. Let $x \log_2(e) = c + f$ where c is an integer and f is a fraction such that $0 \le f < 1$. Then we have

$$2^{x\log(e)} = 2^{c + f}, \quad \text{or } e^x = 2^c * 2^f.$$

On digital computers, 2^c is simple shift of the binary point (an adjustment to the exponent part of the value). We then are left with the evaluation over the interval [0,ln(2)] = [0,0.69315], but we want zero to be the center so we change the variable by subtracting ln(2)/2 to get an interval defined as [-ln(2)/2,ln(2)/2]. (Of course we need to reverse the process when the Padé approximation is employed. [See Ralston, (1965)].

42,43. Make sure that students do not bother the system personnel of your computer center in researching these exercises. The best way to avoid that is to have the necessary technical manuals available in the library.

44. The size of the "ear" is about 9% above the square wave regardless of the size of n.

46. A good starting place for a literature search on the use of Lanczos factors is Hamming, 1973.

Chapter 5

1. f'(0.242) = 1.9754 - 3.9088(0.032 + 0.012) = 1.8034
(true value = 1.7946).

2. Error from next term = -0.0074, actual error = -0.0088.

3. The recomputed table is

0.15	0.1761	2.4350	-5.7500	15.6253
0.21	0.3222	1.9750	-3.8750	8.1060
0.23	0.3617	1.7425	-2.9833	6.7359
0.27	0.4314	1.4740	-2.1750	
0.32	0.5051	1.3000		
0.35	0.5441			

f'(0.242) = 1.9750 - 3.8750(0.032 + 0.012) = 1.8045. The error is -0.0099. Truncation causes a greater error than does rounding.

4* Using the same quadratic as in Exercise 1:

x	Computed f'(x)	Exact value	Error
0.21	2.0536	2.0681	0.0145
0.22	1.9754	1.9741	-0.0013
0.23	1.8972	1.8882	-0.0090
0.24	1.8190	1.8096	-0.0095
0.25	1.7409	1.7372	-0.0037
0.26	1.6627	1.6704	-0.0077
0.27	1.5845	1.6085	0.0240

The least error is at x = 0.22 because this is best centered among the data used for the polynomial.

5.

x	Computed f'(x)	Exact value	Error
0.21	2.0525	2.0681	0.0156
0.22	1.9750	1.9741	-0.0009
0.23	1.8975	1.8882	-0.0093
0.24	1.8200	1.8096	-0.0104
0.25	1.7425	1.7372	-0.0053
0.26	1.6650	1.6704	-0.0054
0.27	1.5875	1.6085	0.0210

The average of the magnitudes of the errors is essentially the same as in Exercise 4.

6.

x	Lower bound	Upper bound
0.21	0.0088	0.0187
0.23	-0.0058	-0.0125
0.27	0.0177	0.0375

(These bracket the actual errors.)

7.

x	Next term	Actual error
0.21	0.0105	0.0145
0.22	-0.0009	-0.0013
0.23	-0.0070	-0.0090
0.24	-0.0079	-0.0095
0.25	-0.0035	-0.0037
0.26	0.0061	-0.0077
0.27	0.0210	0.0240

8.

x	Next term	Actual error
0.21	0.0097	0.0156
0.22	-0.0008	-0.0009
0.23	-0.0065	-0.0093
0.24	-0.0073	-0.0104
0.25	-0.0032	-0.0053
0.26	0.0057	-0.0054
0.27	0.0195	0.0210

9. i = 0: 2.4355 - 0.124(5.7505) = 1.7224, error = 0.0722.
i = 2: 1.7409 + 0.016(2.9464) = 1.7880, error = 0.0066.
i = 3: 1.4757 - 0.106(2.2307) = 1.7122, error = 0.0824.
(At i = 1, error is -0.0088).

10.

Degree	1	2	3	4	5	Exact
Value	1.7409	1.8034	1.7960	1.7839	1.8217	1.7946
Error	0.0537	-0.0088	-0.0014	0.0107	-0.0271	--

Least error from $P_3(x)$.

11* a. 1.2502.
b. 1.0843.
c. 1.2935.

12. a. Error = 0.0004.
b. Error = -0.0048.
c. Error = -0.0010, bounds: 4.8E-5, 1.0E-4.

13.

	Next term	Actual error
a.	0.00003	0.00044
b.	-0.00050	-0.00485
c.	-0.00000	-0.0009

14* 1.2905, actual error = 0.0020, bounds: 0.0017, 0.0021.

15. 1.2745, actual error = 0.0180, next term = 0.0215, bounds: 0.0160, 0.0215.

16. 1.2960, actual error = -0.0035, next term = -0.0030, bounds: -0.0003, -0.0005.

17. The recomputed table is only very slightly different, even up to the fourth differences. Repeating the exercises produces insignificnt differences.

18.

h	Central difference Value	Central difference Error	Forward difference Value	Forward difference Error
0.1	-2.5433	0.0220	-2.8054	0.2840
0.01	-2.5216	2.3E-4	-2.5476	0.0262
0.001	-2.5214	1E-5	-2.5240	0.0026

19.

h	Central difference Value	Central difference Error	Forward difference Value	Forward difference Error
0.1	-2.540	0.019	-2.800	0.279
0.01	-2.500	-0.021	-2.500	-0.021
0.001	-2.500	-0.021	-2.000	-0.521

20. 3rd deriv. = $(1/h^3)(\Delta^3 - (3/2)\Delta^4 + (7/4)\Delta^5 - (15/8)\Delta^6 + \ldots]f_0$
4th deriv. = $(1/h^4)(\Delta^4 - 2\Delta^5 + (17/6)\Delta^6 - (7/2)\Delta^7 + \ldots]f_0$

21. From just the first term and using double precision:
$f^{(3)}(0.3) = 12.219$, true value = 10.650.
$f^{(4)}(0.3) = 22.936$, true value = 19.560.

22* One term: -0.404148, error = -0.0007082, est. = -0.007048
Two terms: -0.411988, error = -0.0000758, est. = 0
Three terms: The same as with two terms with single precision.

23. The best formula will use function values between x_{-2} and x_2:

$$f^{(4)}(x)_n = \frac{f_{-2} - 4f_{-1} + 6f_0 - 4f_1 + f_2}{h^4} + O(h^2).$$

A symmetrical formula for $f^{(3)}(x_0)$ will also use these same function values.

24. Double precision arithmetic is required. Answer with h = 0.05 is -0.0366428, error = 0.000504. With h = 0.025, answer is -0.0362647, error = 0.000126. The ratio of the errors is 4.00, confirming $O(h^2)$.

25. The equation for $f'(x_0)$ is confirmed with its error term.

26* Forward: $(1/h)(f_1 - f_0) - (h/2)f''(x)$.
Backward: $(1/h)(f_0 - f_{-1}) - (h/2)f''(x)$.

27. Central: $(1/h^2)(f_1 - 2f_0 + f_{-1}) + (h^2/12)f^{(4)}(x)$.
Forward: $(1/h^2)(f_2 - 2f_1 + f_0) - hf^{(4)}(x)$.

28. With h = 0.2: 1.2060.
With h = 0.4: 1.20175.
Extrapolated: 1.20742; (exact = 1.20720). Cannot extrapolate further, we need f(0.1), f(1.7).

29. With step size h: $f_0' = (f_1 - f_{-1})/(2h) + Ch^2$.
With step size 2h: $f_0' = (f_2 - f_{-1-2})/(2*2h) + C(2h^2)$.
Then 4*(first equation) - (second equation) gives

$$f_0' = \frac{1}{h}\left(\frac{f_1 - f_{-1}}{2} - \frac{f_2 - 2f_1 + 2f_{-1} - f_{-2}}{12}\right).$$

which can be reduced to the formula.

30. With unevenly spaced data, extrapolation is virtually impossible. In any event, the derivative based on three unevenly spaced points will be only of O(h), where h is the average separation.

31* Using double precision, the Richardson table is

0.157021273		
0.157217754	0.157283248	
0.157266897	0.157283278	0.157283280

Exact = 0.157283; the estimate agrees to six places.

32. Using double precision, the Richardson table is

0.474222326		
0.474518839	0.474617676	
0.474592990	0.474617707	0.474617709

Exact = 0.474617; the estimate agrees to six places.

33. Extrapolation formula: more accurate + (more - less).
A Richardson table (using double precision,) requires seven stages (to h = 7.8125E-4) to get repeated values the same to five places: (0.157275728 and 0.157281820).

34. Program.

35. Value = 0.28135, exact = 0.275294, error = -0.00606, bounds: -0.1117, 0.03312.

36. Value = 0.269609, exact = 0.275294, error = 0.00569, bounds: 0.00023, 0.01376.

37. Value = 0.272742, exact = 0.275294, error = 0.00255, bounds: 0.00105, 0.04643.

38* Using undetermined coefficients:

$n = 4$: $(h/45)(14f_0 + 64f_1 + 24f_2 + 64f_3 + 14f_4) + O(h^7)$.

$n = 5$: $(h/288)(95f_0 + 375f_1 + 250f_2 + 250f_3 + 375f_4 + 95f_5) + O(h^7)$.

39. $n = 1$: $hf^{[0]} + [(x_1 - x_0)/2]f^{[1]} - hx_1f^{[1]} = (h/2)(f_0 + f_1)$.

$n = 2$: $2hf^{[0]} + [(x_2^2 - x_0^2)/2 - 2hx_0]f^{[1]} + [(x_2^3 - x_0^3)/3 - x_0(x_2^2 - x_0^2)/2 - x_1(x_2^2 - x_0^2)/2 + 2hx_0x_1]f^{[2]} = (h/3)(f_0 + 4f_1 + f_2)$.

For $n = 3$, operations are similar but messy.

40a. 1.7684.

b. 1.7728.

c. 1.7904.

41* Errors equal about $-0.147*h^2$.

a. -0.00143.

b. -0.0058.

c. -0.0234.

42. Second differences range from 0.242 to 1.469. These predict values for f"(x) from 6.075 to 36.725 (compare to exact values of 4.953 to 44.701). From the second differences, bounds will be -0.016, -0.098.

43. With 1600 intervals (h = 0.001), value is 23.914454, error = -1.4E-6.

44. 0.874705.

45* The Romberg table:

0.70833		
0.69702	0.69325	
0.69412	0.69315	0.69315

(Compare 0.69315 to exact value of 0.693147).

46. The Romberg table:

h = 0.1:	1.76845	1.76697	1.76697
h = 0.2:	1.77286	1.76699	
h = 0.4:	1.79047		

(Compare 1.76697 to exact value of 1.76697).

47.

h	Value	Extrapolations	
0.25	0.340088	0.341358	0.341294
0.50	0.336275	0.341550	
1.0	0.320450		

48. h = 0.1: 1.76693.
h = 0.2: 1.76693.
h = 0.4: 1.76720.

49a. Error = 4.3E-5, bounds: -6.85E-7, -13.8E-7.
b. Error = 4.3E-5, bounds: -1.10E-5, -2.21E-5.
c. Error = -2.3E-4, bounds: -1.76E-4, -3.54E-4.

50* Using Simpson's 1/3 rule, h = 0.125: 1.718284, exact value = 1.71828182, error = -2.30E-6.

51. For n an even integer, let T_h, T_{2h}, be trapezoidal rule integrals with step sizes h and 2h. It is easy to show that

$T_h - T_{2h} = (h/2)(-f_0 + 2f_1 - 2f_2 + \ldots - f_n)$, from which

$T_h + (1/3)(T_h - T_{2h}) = (h/3)(f_0 + 4f_1 + 2f_3 + 4f_3 + \ldots + f_n)$

which is Simpson's 1/3 rule.

52. h = 0.5: 0.946146.
h = 0.25: 0.946087, extrapolation: 0.9460831.
Analytical: 0.9460831.

53. With 12 intervals, integral = 1.718283, error = -1.0E-6.

54.

Range for 3/8 rule	Integral	Error
[3.0,4.5]	10.228808	-2.0E-4 (best)
[4.0,5.5]	10.228860	-2.6E-4
[5.0,6.5]	10.228857	-2.5E-4

55* Let $P_3(x) = a + bx + cx^2 + dx^3$. By change of variable, the integration can be from -h to h with midpoint at x = 0. The quadratic that fits at three evenly spaced points is $a + (b + dh^2)x + cx^2$. The integral of this and of $P_3(x)$ are both $= 2ah + 2ch^3/3$.

56. $3h^2f_0 + (9/2)h^3\Delta f_0 + 9h^3(2h-1)/4\Delta^2 f_0 + 3h^3(3h-2)^2/8\Delta^3 f_0 + \ldots$

57. When limits are from s = -1 to s = 0, we must divide $h\Delta f_0$ by

$$\Delta + (1/2)\Delta^2 - (1/6)\Delta^3 + (1/12)\Delta^4 - (1/20)\Delta^5 + \ldots$$

which gives $h[1 + (1/2)\Delta - (10/24)\Delta^2 + (9/24)\Delta^3 - (73/360)\Delta^4 + \ldots]f_0$.

58. Integral $= h[2\Delta + (1/3)\Delta^2 - (1/3)\Delta^3 + (29/90)\Delta^4 - \ldots]f_0$.

59* We want the coefficients of: Integral $= af_0 + bf_1$. The limits can be [0,h]. Using f(x) = 1, then f(x) = x, we get two equations: a + b = h, $bh = h^2/2$, from which a = h/2, b = h/2.

60. We want the coefficients of: Integral = $af_0 + bf_1 + cf_2$. The limits can be [-h,h]. Using f(x) = 1, then f(x) = x, and f(x) = x^2, we get three equations: a + b + c = 2h, -a + c = 0, a + c = 2h/3, from which a = h/3, b = 4h/3, c = h/3.

61. We want the coefficients of: $f'(x) = af_{-1} + bf_1$. Using f(x) = 1, then f(x) = x, we get two equations: a + b = 0, -ah + bh = 1, from which a = -1/2h, b = 1/2h.

62* Value = 1.718281 which is accurate to six decimal places. Gauss quadrature requires three function evaluations, Simpson's 1/3 rule requires eight.

63. Value = 0.9460831. Gauss three-point quadrature gives 0.9460832. We get this same value with 12 intervals (h = 1/12) with Simpson's 1/3 rule.

64* Correct value is -0.700943. Even five terms in the Gaussian formula is not enough. Simpson's 1/3 rule attains five digits of accuracy with 400 intervals. The result from an extrapolated Simpson's rule gets this in seven levels, using 128 intervals.

65. The error of Guassian quadrature = $1/(4^n n!) f^{[2n]}(x)$ -- see Atkinson, (1978). Polynomial error bounds are usually smaller. For two-term Gauss (n = 2), comparable error term [$P_3(x)$] is about 1/6 as large. For three terms (n = 3), $P_5(x)$ has an error term about 1/2 as large.

66. The values are readily confirmed.

67. The values are confirmed.

68. If computed without adaptive integration, value = 4.00001, if extrapolated from computations with 64 and 128 intervals. Using adaptive integration, value = 4.00001 requiring 45 function evaluations. Adaptive Simpson's rule gets this from 17 evaluations.

69* With TOL set at 0.4, the result is 3.657243; this differs from the exact answer by 0.003% and requires 9 function evaluations. With TOL set at 0.5, the accuracy criterion is not met.

70. Break the interval into subintervals: [0,1], [1,π/2].

71. Program.

72. The same result is obtained.

73a*

		1	2	2	1
Δx	Δy	4	8	8	4
--	--	2	4	4	2
3	2	4	8	8	4
		1	2	2	1

b.

		1	4	2	4	1
Δx	Δy	4	16	8	16	4
--	--	2	8	4	8	2
3	3	4	16	8	16	4
		1	4	2	4	1

c.

		1	3	3	2	3	3	1
		3	9	9	6	9	9	3
$3\Delta x$	$3\Delta y$	3	9	9	6	9	9	3
---	---	2	6	6	4	6	6	2
8	8	3	9	9	3	9	9	3
		3	9	9	3	9	9	3
		1	3	3	2	3	3	1

d* for a: Any number in the y-direction, even number in the x-direction.
for b: Even number in both direction.
for c: Divisible by 3 in both directions.

74. Analytical value = -1/6. This is confirmed by Simpson's rule.

75. Top plane: 1 4 1

	4	16	4
	1	4	1
Middle plane:	4	16	4
	16	64	16
	4	16	4
Bottom plane:	1	4	1
	4	16	4
	1	4	1

The final sum is to be multiplied by $(\Delta x/3)(\Delta y/3)(\Delta z/3)$.

76a. 0.408064.
  b. 0.408065.
  c. 0.408088.
Analytical value = 0.408064.

77. h = 0.2: 0.408058.
    h = 0.1: 0.408063.
Extrapolated: 0.408065.

78* Analytical value = 2/3.

Δx	Δy	Integral	Error	Error/h^2
0.5	0.5	0.75	-0.0833	-0.3333
0.25	0.5	0.7185	-0.05208	-0.3333*
0.5	0.25	0.7185	-0.05208	-0.3333*
0.25	0.25	0.6875	-0.0208	-0.3333
0.125	0.125	0.6719	-0.0052	-0.3333

(*using the average of the squares of the h-values.)

79. a. 0.27704 (error = 0.00404), analytical = 0.281081.
    b. 0.28118 (error = -0.00010)

80. Answers are the same as for Exercise 79.

81. Integrating with x constant and using 16 y-intervals, then varying x from -1 to 1 with $\Delta x = 0.125$, the integral is 1.29205. Exact answer = 1.29199; error is -6E-5.

82. Procedure:

(1) Locate the Gauss points on the y-axis between[0,1].

(2) For each of these y-values, locate the Gauss points for x between x = 0 and the x-value on the circle.

(3) For each y-value in (1), compute the weighted sum of function values at each of the Gauss points in (2); divide the sum by 2.

(4) Compute the weighted sum of the sums in (3); divide this by 2.

Results: (a) 0.28108 (b) Same result Analytical value = 0.28108

83. (2.755,4.397), (4.545,4.397), (2.755,6.302), (4.545,6.302).

84.

		End condition:			Exact	Central diff.
	x	1	3	4	value	(h = 0.1)
	1.5	-0.0841	-0.0823	-0.0819	-0.0816	-0.0817
f'(x):	2.0	-0.0627	-0.0630	-0.0632	-0.0625	-0.0625
	2.5	-0.0489	-0.0497	-0.0494	-0.0494	-0.0494
	1.5	0.0596	0.0467	0.0440	0.0466	0.0466
f"(x):	2.0	0.0257	0.0307	0.0310	0.0313	0.0313
	2.5	0.0296	0.0227	0.0240	0.0219	0.0220

85. As indicated by the answers to Exercise 84, the plots are very close to the plots of the analytical values.

86. With polynomials formed from f(x) at values with $\Delta x = 0.25$. For the cubic, values from x_{-1} to x_2 were used.

	x	Degree 3	Degree 4	Exact value
	1.5	-0.0814	-0.0819	-0.0816
f'(x):	2.0	-0.0633	-0.0633	-0.0625
	2.5	-0.0470	-0.0475	-0.0494
	1.5	0.0484	0.0611*	0.0466
f"(x):	2.0	0.0398	0.0412*	0.0313
	2.5	0.0290	0.0225	0.0219

* These values distorted from round-off.

87*

x:	1.5	2.0	2.5
f'(x):	-0.0844	-0.0619	-0.0512
f"(x):	0.0653	0.0245	0.0272

88* Value = 1.29919; Simpson's rule: 1.30160; exact: 1.30176.

89.

End condition:	2	3	4
Value:	1.30177	1.30160	1.30030

90. Best agreement with exact value from end condition 2: 1.30177.

91. a.

	By Trapezoid rule		Analytical	
N	A	B	A	B
0	4.0100		4.0000	
1	1.2259	-2.1385	1.2156	-2.1595
2	0.3142	-1.1827	0.3040	-1.2249
3	0.1456	-0.7708	0.1351	-0.8245
4	0.0868	-0.5458	0.0760	-0.6306

(Exercise 91 continued)

b. By Trapezoid rule / Analytical

N	A (Trapezoid)	B (Trapezoid)	A (Analytical)	B (Analytical)
0	2.6800		2.6667	
1	-2.5255	-1.6345	-2.5465	-1.6211
2	-0.4189	1.2311	-0.4053	1.2732
3	0.7850	0.1941	0.8488	0.1801
4	0.1158	-0.5506	0.1013	-0.6366

c. By Trapezoid rule / Analytical

N	A (Trapezoid)	B (Trapezoid)	A (Analytical)	B (Analytical)
0	0.0889		0.0890	
1	-0.1332	0.0527	-0.1229	0.0524
2	0.0495	-0.0355	0.0508	-0.0363
3	0.0001	0.0337	0.0002	0.0361
4	-0.0192	-0.0115	-0.0220	-0.0130

92* Results with Simpson's rule. (See Exercise 91 for analytical values.)

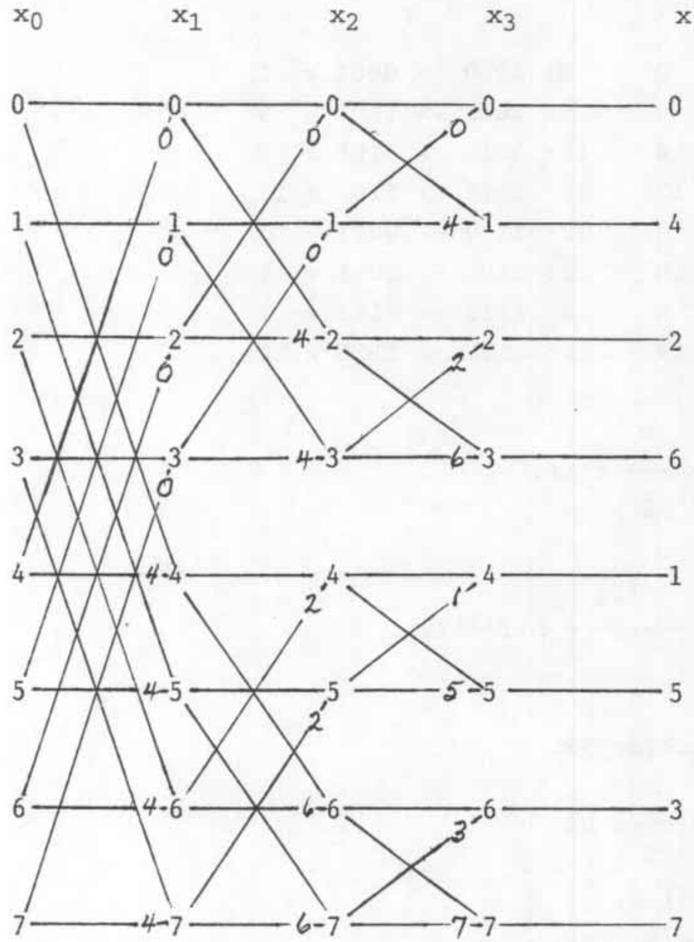
	part (a)		part (b)		part (c)	
N	A	B	A	B	A	B
0	4.0000		2.6667		0.0890	
1	1.2156	-2.1595	-2.5466	-1.6209	-0.1339	0.0524
2	0.3031	-1.2259	-0.4041	1.2744	0.0509	-0.0364
3	0.1330	-0.8384	0.8530	0.1773	0.0002	0.0363
4	0.0716	-0.6409	0.0954	-0.6474	-0.0225	-0.0133

93. a. About 2100 panels.
    b. About 1600 panels.
    c. About 1100 panels.

94. a. About 160 panels.
    b. About 140 panels.
    c. About 60 panels.

95. Multiply the matrices, add exponents of $(W^i)(W^j)$, write $W^0 = 1$, write W^n as $W^{n \bmod 4}$, then unscramble the rows.

96.



97* After stage

After stage																
1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	2	1	3	0	0	0	0	0	0	0	0	0	0	0	0
3	0	4	2	6	1	5	3	7	0	0	0	0	0	0	0	0
4	0	8	4	12	2	10	6	14	1	9	5	13	3	11	7	15

98.

0: 0000 -> 0000 = 0	8: 1000 -> 0001 = 1
1: 0001 -> 1000 = 8	9: 1001 -> 1001 = 9
2: 0010 -> 0100 = 4	10: 1010 -> 0101 = 5
3: 0011 -> 1100 = 12	11: 1011 -> 1101 = 13
4: 0100 -> 0010 = 2	12: 1100 -> 0011 = 3
5: 0101 -> 1010 = 10	13: 1101 -> 1011 = 11
6: 0110 -> 0110 = 6	14: 1110 -> 0111 = 7
7: 0111 -> 1110 = 14	15: 1111 -> 1111 = 15

99.

$$f'(x_0) = \frac{f_0 - f_{-1}}{h} + \frac{h}{2} f''(x).$$

$$f''(x)_0 = \frac{f_0 - 2f_{-1} + f_{-2}}{h^2} + h\, f^{(3)}(x).$$

100. Same results as Exercise 99.

101. Same results as Exercise 99.

102. $n = 4$: $-(8/945)h^7 f^{(6)}(x)$.
$n = 5$: $-(275/12096)h^7 f^{(6)}(x)$.

103. Integrate $s(s - 1)(s - 2) \ldots (s - n)ds$ from 0 to n. This will be zero when n is even. This is apparent when a plot of the integrand is studied -- every loop above the x-axis has a matching partner that goes below the x-axis.

104* Suppose $f(x) = e^x$. To get the local error, integrate over two panels:

Limits	h	Integral	Error	Error/h^5
[1,2]	0.5	4.67235	-1.575E-3	-0.0504
[1,1.5]	0.25	1.76344	-3.802E-5	-0.0389
[1,1.25]	0.125	0.77206	-9.938E-7	-0.0326
[1,1.125]	0.0625	0.36194	-2.884E-8	-0.0302

To get the global error, integrate between limits of [1,2] with varying h:

h	Integral	Error	Error/h^4
0.5	4.67235	-1.575E-3	-0.0252
0.25	4.67088	-1.008E-4	-0.0258
0.125	4.67078	-6.388E-6	-0.0259
0.0625	4.67078	-6.662E-7	-0.0259

105. Demonstrations confirm the statement.

106. Derivative at x = 0.5; single precision:

h	f'(x)	Error	f"(x)	Error
0.10000	0.495574445	0.000524402	0.959482014	0.001546979
0.01000	0.496093184	0.000005662	0.961124957	-0.000095963
0.00100	0.496089488	0.000009358	0.968575597	-0.007546604
0.00010	0.496134222	-0.000035375	2.980232716	-2.019203663
0.00001	0.496581256	-0.000482410	74.505821228	-73.544792175

Double precision:

0.10000	0.495574397	0.000524436	0.959480062	0.001548921
0.01000	0.496093649	0.000005185	0.961013529	0.000015455
0.00100	0.496098782	0.000000052	0.961028829	0.000000155
0.00010	0.496098833	0.000000001	0.961028982	0.000000002
0.00001	0.496098834	0.000000000	0.961029144	-0.000000160

107. Derivative at x = 0.5; single precision:

h	f'(x)	Error	f"(x)	Error
0.10000	0.543548524	-0.047449678	0.917188048	0.043840945
0.01000	0.500898838	-0.004799992	0.957772195	0.003256798
0.00100	0.496573776	-0.000474930	0.961125016	-0.000096023
0.00010	0.496283233	-0.000184387	-2.980232716	3.941261768
0.00001	0.496953785	-0.000854939	0.000000000	0.961028993

Double precision:

0.10000	0.543548401	-0.047449567	0.917191131	0.043837853
0.01000	0.500898717	-0.004799883	0.957808611	0.003220373
0.00100	0.496579296	-0.000480463	0.960716863	0.000312121
0.00010	0.496146885	-0.000048051	0.960997871	0.000031113
0.00001	0.496103639	-0.000004805	0.961025397	0.000003587

108. Single precision:

h	Integral	Error
0.10000	0.010994081	-0.000332289
0.01000	0.010665112	-0.000003320
0.00100	0.010661796	-0.000000004
0.00010	0.010669500	-0.000007708
0.00001	0.010658802	0.000002990

Double precision:

0.10000	0.010994081	-0.000332289
0.01000	0.010665114	-0.000003322
0.00100	0.010661825	-0.000000033
0.00010	0.010661792	0.000000000
0.00001	0.010668985	-0.000007193

The optimum step size is the same as for differentiation.

109. Single precision:

h	Integral	Error
0.10000	0.01066171	0.00000008
0.01000	0.01066179	-0.00000000
0.00100	0.01066180	-0.00000000
0.00010	0.01066180	-0.00000001
0.00001	0.01066179	0.00000000

Double precision:

h	Integral	Error
0.10000	0.01066171	0.00000008
0.01000	0.01066179	-0.00000000
0.00100	0.01066179	-0.00000000
0.00010	0.01066179	-0.00000000
0.00001	0.01066179	-0.00000000

The optimum step size is larger than for Exercise 108.

110. Procedure: (1) Define f(x) as symbolic expression;
(2) Issue command: diff(f,'x')

Answers:

```
Exercise 2: 1/x/log(1)
Exercise 11: 1 + 1/3*cos(x)
Exercise 18: exp(x)/(x-2) = exp(x)/(x-2)^2
Exercise 31: sin(1/2*x)*cos(1/2*x)
```

111. Procedure: (1) Define f(x) as a symbolic expression;
(2) Issue command: int(f,'x',a,b) where a,b are limits.

The analytical answers are complicated. Numerical values are:

Exercise 35: 0.2753
Exercise 41: 1.7670
Exercise 50: 1.7183
Exercise 52: Si(1) (This is the sine integral; value is 0.9406.)

112. Procedure: Define f = '(x - x^2/2 + x^3/3 - x^4/4)/h', then use symmul(f,f). The results agree with those of Exercise 16 but the terms are not in the same order.

113. The MATLAB command: legendre(n,'x') does not give a symbolic expression but: legendre (n,x), where x has a value, gives the numerical value of $L_n(x)$ as the first element of a vector.

114. The first differences of log(Δt) will be constant when the values are linear. For the given data, a fairly straight line occurs between 9.5 and 13 min, and a second straight line is from 15 min to the end of the data. It is not clear which line segment should be considered indicative of the completion of the reaction; perhaps there two reactions occurring. The numerical method is more quantitative and less subjective but is more influenced by errors in the data that would be smoothed out by the graphical procedure.

115. Assuming that the reaction ceases at 15 min (see comments in Exercise 114), the integral is 166.4, Simpson's 1/3 rule would be preferred (adjusted if necessary for an uneven number of panels). Gaussian quadrature is not appropriate for tabulated data.

116. Using Simpson's 1/3 rule, the integral is 6.1245.

117. The effect of precision of the data is most noticeable on S for the second point for end conditions 1 and 3 and on S for the first point for end condition 4.

Values for S at the second point:

End condition:	1	3	4
3 digits	0.9664	0.7605	0.6640
4 digits	0.9709	0.7641	0.6664
5 digits	0.9714	0.7644	0.6666
6 digits	0.9714	0.7644	0.6666

For end condition 3, values of S at the first point are

1.125, 1.1333, 1.1332, and 1.1325.

Chapter 6

1a. $y(x) = 1 + x + 3/2\ x^2 + 5/6\ x^3 + 7/12\ x^4 + 17/60 x^5 + O(x^6)$

$y(0.1) = 1.11589$, $y(0.5) = 2.02448$

b. $y(x) = 1 + x + x^2 + 1/3\ x^3 + 1/12\ x^4 + 1/60\ x^5 + O(x^6)$

$y(0.1) = 1.11034$, $y(0.5) = 1.79740$

c. $y(x) = 1 + 1/2\ x^2 - 3/8\ x^4 + O(x^6)$

$y(0.1) = 1.00496$, $y(0.5) = 1.10156$

d. $y(x) = 1 + 2x/5 + 3x^2/50 + x^3/500 + 3x^4/10000 + x^5/250000$

2. For $y(1) = 0$: $y(x) \equiv 0$.

For $y(0) = 1$, $y(x) = 1 + 1/3\ x^3 + 1/9\ x^6 + 1/27\ x^9 + \ldots$

The analytical solution is $y = 3/(3 - x^2)$

For x =	0.1	0.2	0.3	0.4
Series:	1.00033	1.00267	1.00908	1.02179
Analytic:	1.00033	1.00267	1.00908	1.02180

3. $y(x) = 1 + x + x^3/6 + x^4/12 + x^6/180 + x^7/504$

x =	0.2	0.4	0.6
y =	1.20147	1.41283	1.64704

4. $x(t) = 1 - 3t^2/10 + 3t^4/250 - 3t^6/6250$

5. a* With h = 0.01, y(0.1) = 1.11418. Error is 0.00171. To reduce error to 0.00005 (34-fold), one must reduce h 34-fold, to about 0.00029.

b. Error = 0.001096, reduce 22-fold, to about 0.00034.

c. Error = 0.000489, reduce 10-fold, to about 0.001.

d. Error = 0.00006, almost good enough, reduce to about 0.008.

6. y(1) = 1.38556 with h = 0.1, y(1) = 1.35504 with h = 0.2.
Extrapolating, we get

$y(1) = 1.38556 + (1/(1))(1.38556-1.35504) = 1.41608$ (versus 1.41421).

7. With modified Euler, all results are already accurate to four decimals.
   a* y(0.1) = 1.11587. The simple Euler method would require about 340 steps and 340 function evaluations compared to 4 steps and 8 function evaluations here.
   b. y(0.1) = 1.110319, 4 steps here versus 220.
   c. y(0.1) = 1.004963, 4 steps here versus 100.
   d. y(0.1) = 1.040600, 4 steps here versus 12.

8. y(2) = 6.15633 with h = 0.1; y(2) = 6.51879 with h = 0.05. Extrapolating (error = $O(h^2)$) gives 6.63961 (versus analytical of 6.703888). Estimate of error is 0.12082, actual error is 0.18509 when h = 0.05.

9*

x:	0.1	0.2	0.3	0.4	0.5
y:	2.2150	2.4630	2.7473	3.0715	3.4394

10. Equation is $dv/dt = 32.2 - cv^{3/2}$, v(0) = 0. At 80 mi/hr (117.333 ft/sec) dv/dt = 0, giving c = 0.025335.

t:	0.2	0.4	0.6	0.8	1.0	1.2
v:	6.3986	12.6822	18.7997	24.7198	30.4209	35.8883

t:	1.4	1.6	1.8	2.0
v:	41.1127	46.0888	50.8149	55.2919

11. y(0.1) = 1.11589, which is correct to 5 decimals. With the simple Euler formula, about 3400 steps would be required (3400 function evaluations). With the modified Euler method, about 16 steps would be required
(32 function evaluations) while with the Runge-Kutta method, only 4 evaluations are needed.

12.

h:	0.2	0.1	0.05
y(0.2):	6.61982	6.69432	6.70305

13.

x:	0.2	0.4	0.6
y:	2.09327	2.17549	2.24927

14. Interpolating linearly between v(6.0) and v(6.5), v = 105.60 ft/sec at t = 6.36 sec. Distance traveled is about 435 ft.

15. Using h = 0.3, 90% of the terminal velocity is reached in 6.305 sec. At t = 6.0, the two results agree to 6 digits, so that level of accuracy is assured. Accuracy is improved about 8-fold with h = 0.3 because $(0.5/0.3)^4 = 7.72$

16. Using h = 0.1 for each result:
For Exercise 11: y(0.1) = 1.11589465
For Exercise 12: y(2.0) = 6.705276
For Exercise 13: y(0.6) = 2.249272

17. Using h = 0.1 for each result:
For Exercise 1: y(0.1) = 1.115895, y(0.5) = 2.027337
For Exercise 6: y(1.0) = 1.414214
For Exercise 9: y(0.5) = 3.443299

18* The exact answer is: $y(x) = -5e^{-x} + 2x^2 - 4x + 4$

x	y(RKF)	Analytical
0.00	-1.000000	-1.000000
0.10	-0.904187	-0.904187
0.20	-0.813654	-0.813654
0.30	-0.724091	-0.724091
	. . . . .	
1.80	2.453505	2.453506
1.90	2.872157	2.872157
2.00	3.323325	3.323324

19. From Exercise 10 we have: $dv/dt = 32.2 - 0.025178\, v^{3/2}$.
With RKF, v(2.0) = 55.32416

20* a. -0.28326.
b. -0.28387.
c. -0.28396.

21.

t:	0.8	1.0	1.2
y:	2.0146	2.2822	2.5207
analytical:	2.0145	2.2817	2.5199

22. Exact results are obtained because dy/dt is a quadratic.

23. y(1.2) = 2.5199 versus 2.5199 (analytical).

24. y(4) = 4.1149 (predicted), y(4) = 4.2229 (corrected). The error estimate is -0.0031; the corrected value should be correct to three digits, but the actual error is -0.0998. The original data must be correct to at least 3-digits.

25.

By RKF:	x:	0.2	0.4	0.6
	y:	1.06268	1.24601	1.51691
By Milne:	x:	0.8	1.0	
	y:	1.74687	1.95374	

26*

x:	0.8	1.0	1.2	1.6	2.0
y:	2.3163	2.3780	2.4350	2.5380	2.6294
est. error:	0.0003	<5E-5	0	-5E-5	-<2E-5

(h was increased to 0.4 at x = 1.2).

27.

x:	0.8	1.0	1.2
y:	2.0145	2.2817	2.5199

(These match the analytical results.)

28.

x:	0.8	1.0
y:	1.74687	1.95374

(y(0.8) is more accurate than by Milne, y(1.0) is less accurate).

29* Using Runge-Kutta:

x:	0	0.2	0.4	0.6
y:	0	0.0004	0.0064	0.0324

Using Adams-Moulton:

x:	0.8	0.9	1.0	1.1	1.2	1.25	1.3	1.35	1.4
y:	0.1025	0.1644	0.2513	0.3704	0.5321	0.6340	0.7544	0.8990	1.0772

(The step size was was halved after x = 0.8 and again after x = 1.2).

30. One relatively easy technique is the method of undetermined coefficients.

31a. $f_y = \sin(x)$ so h(max) = (24/9)/1 = 2.67.

b. With h = 0.267, D cannot exceed 10E-N for N-decimal place accuracy.

c. For D = 14.2E-N, h cannot exceed (1/14.2)h(max) = 0.188h(max).

32a. $f_y = 2y = 0.30$ near (1.0,0.15), so h(max) = (24/9)/0.3 = 8.89.

b. With h = 0.889, D cannot exceed (24/9)/(0.3*0.889)E-N = 10E-N.

c. For D = 14.2E-N, h cannot exceed (1/14.2)h(max) = 0.626h(max).

33a. h(max) = 3/1 = 3.

b. With h = 0.3, D must be less than 3/0.3/1 = 10E-N.

c. For D = 29E-N, h must be less than (1/29)h(max) = 0.103.

34. The derivation parallels that for Adams-Moulton with 24/9 replaced by 3 as shown by Eq. (6.8) compared with Eq. (6.16).

35. The development parallels that in Section 6.8 with the factor 9/24 replaced by 1/2 as shown by comparing Eq. (6.4) with Eq. (6.18). There is no accuracy criterion because the predictor and corrector formulas are the same and we do not have two different error terms to compare.

37* Let y' = z so that y" = z'. Then we have

$$y' = z, \quad y(0) = 0$$

$$EIz' = M(1 + z^2)^{3/2}, \quad z(0) = 0$$

38. Let $y_1' = y_3$, $y_2' = y_4$. Then we have

$$y_3' = (-k_1y_1 - k_2y_1 + k_2y_2)/m_1, \quad y_3(0) = B$$
$$y_4' = (k_2y_1 - k_2y_2)/m_2, \quad y_4(0) = D$$
$$y_1' = y_3, \quad y_1(0) = A$$
$$y_2' = y_4, \quad y_2(0) = C$$

39*

t:	0	0.2	0.4	0.6
y:	1	0.982	0.934	0.865
x:	0	0.022	0.093	0.221

40.

t:	0.8	1.0
y:	0.780	0.699
x:	0.407	0.660

41.

t:	0.8	1.0
y:	0.786	0.714
x:	0.411	0.672

42* Starting with a Taylor series with terms through x^6:

x:	0	0.1	0.2	0.3
y:	1	0.8950	0.7802	0.6561
y':	-1	-1.0995	-1.1956	-1.2847

With Adams-Moulton:

x:	0.4	0.5	0.6
y:	0.5236	0.3840	0.2389
y':	-1.3629	-1.4263	-1.4715

43.

t	y	y'	y"
0.2	0.2003	1.0053	0.0793
0.4	0.4042	1.0418	0.3095
0.6	0.6210	1.1380	0.6733

44.

t	y	y'	y"
0.8	0.8651	1.3186	1.1530
1.0	1.1555	1.6058	1.7374

At t = 1.0, $y_C - y_P$ = 2.1E-4 giving an estimated error of 1E-5. However, the value at t = 1.0 using RK4 with h = 0.1 gives y = 1.15558. The difference is probably due to round-off of the previous values.

45. Using Δt = 0.0625:

t	x	x'	y	y'
0.0	0.4	0.0	0.0	2.0
0.5	0.3044	-0.3180	0.4314	1.6550
1.0	0.1191	-0.4040	1.6370	1.1637
1.5	-0.4095	-0.0866	2.0939	0.6640
2.0	-0.3277	-0.3516	2.3412	0.0419
		. . . . .		
5.0	0.0409	0.8056	-1.0651	-1.4210
5.5	0.4277	0.7247	-1.6518	-0.9285
6.0	0.7540	0.5027	-2.0650	-0.2784

The motion is not purely periodic but the maxima in x and y reoccur about ever 8.3 time units.

46* Using RKF to start the solution:

t:	0	0.1	0.2	0.3
x:	0	0.0717	0.1998	0.2028
x(anal):	0	0.0717	0.1999	0.2026

Values by Eq. (6.28):

t:	0.4	0.5	0.6	0.7	0.8
x:	-0.0196	-0.3532	-0.5676	-0.4489	0.0898
x(anal):	-0.0233	-0.3784	-0.5977	-0.4419	0.0932

47. The results depart greatly from the analytical.

48. a. L > 2
b. L > 1
c. L > 20

49. a. $|x^2 - y_1^2 - x^2 + y_2^2| = |y_1^2 - y_2^2| = |y_1 - y_2||y_1 + y_2|$, so that $L > |y_1 + y_2| \le 2$ on the unit square.

b. Does not satisfy the Lipschitz condition since $|f(x, y_1) - f(x, y_2)| = x^2/|y_1 y_2||y_1 - y_2|$ is unbounded at $y = 0$.

c. $|xt_1 - xt_2| = |x||t_1 - t_2|$ so that $L > \max|x| = 5$.

50. Whenever $x \neq 1$.

51* Examine $f(x,y) = x|y|$ on the unit square. In general, consider the integral of a bounded function with a finite number of discontinuities.

52* Parts (b) and (c) are stable; parts (a) and (d) are unstable.

53. Using Eq. (6.42) with K = 1, M = 2, we get this table.

x	Eq. (6.24)	Actual error
0	0	0
0.02	0.00404	0.000403
0.04	0.000816	0.000822
0.06	0.001237	0.001257
0.08	0.001666	0.001710
0.10	0.002103	0.002180

54*

x	y	f	$1 + hf_y$	$h^2y''/2$	Est. error	Actual error
1.0	1.000	1.000	1.200	0.015	0	0
1.1	1.100	1.331	1.242	0.022	0.019	0.017
1.2	1.233	1.825	1.296	0.035	0.052	0.049
1.3	1.416	2.605	1.368	0.058	0.118	0.111
1.4	1.676	3.933	1.469	0.106	0.256	0.247
1.5	2.069	6.423	1.621	0.221	0.578	0.597
1.6	2.712	11.765	1.868	0.547	1.464	1.833

55. Using s = 1+hK, the equation for e_n in Section 6.11 can be written as this approximate inequality:

$$e_n \le (h^2/2)(1 + S + S^2 + \ldots + S^{n-1})y''(x)$$
$$\le [(h^2/2)(S^n-1)/(S-1)]y''(x).$$

Now, using M as the bound for abs(y"(x)) and noting that $S^n = (1+hK)^n < e^{nhK}$,

$$e_n \le [(h^2/2)((1+hK)^n-1)/(hK)]M$$
$$= [(hM)/(2K)](e^{nhK}-1)$$

and $nhK - (x_n-x_0)K$.

56. a. Since Δx is contant, the differences can be written in terms of ordinary differences.
   b. Same as for (a).
   c. By change of variable: $x = x_n + ht$. Then $(x - x_{n-3})$ becomes $h(t+3)$, $(x - x_{n-2})$ becomes $h(t+2)$, and ds becomes h dt.

57. MATLAB commands are:
   a. dsolve ('Dy = x^2+x*y','y(1)=2') resulting in a complicated expression involving both exponentials and ERF.
   b. dsolve('Dy = sin(t)', x(0)=1') giving ans = -cos(t) + 2.
   c. dsolve('Dy = 2 - x - y') gives 3 - x + exp(-x) + C1.

58. a. The plot resembles that of cubic polynomial; there is one real zero at about x = -2.06, a maximum near (-1.2, 1.2) and a minimum near (0,0.96).
   b. The plot is a cosine curve: maxima at y = 3, minima at y = 1.

59*

```
          C exp(x^2/2)
     ---------------------
      exp(x) - C exp(x^2/2)
```

60. The commands to Maple are of this form:

```
dsolve({deq, y(x0) = y0}, y(x), series);
```

a. $y(x) = 2 + 3(x-1) + 7/2(x-1)^2 + 5/2(x-1)^3 + 3/2(x-1)^4 + 4/5(x-1)^5$
   $y(2) = 16.3$

b. $x(t) = -1 + t^2/2 - t^4/24 + O(t^6)$
   $x(2) = 2.3333$

c. $y(x) = -3 + 3(x-2) - 2(x-2)^2 + 2/3(-2)^3 - 1/6(x-2)^4 + 1/30(x-2)^5$
   $y(2) = -3$

61. All of these match the analytical values.

a. $y(2) = 13.962$

b. $x(2) = 2.416$

c. $y(2) = 3$

62* $y(x) = 1 - x + 3/2x^2 - 7/6x^3 + 19/24x^4 - 9/24x^5$
   $y(2) = -3.6667$ (which is far from correct!)

63. $y(2) = 1.9313$. (If the Taylor series of Exercise 62 is carried to x^{20}, $y(2) = 1.93134$ from it)

64. The plot slopes downward from (1, 0.743) to (2.-3.667), crossing the x-axis at $x = 1.529$

65. The plots are the same as Fig. 6.12

66. Take M(x) constant, so the simplified equation is

$$y'' = M(x)/(EI) = C.$$

The analytical solution at $x = 1$ ($y(1) = C/2$) is to be compared to the numerical solution of the nonlinear equation in Exercise 37. One finds that, at $C = 0.198$, $y(1) = 0.0999$ which is 1% different from $C/2 = 0.0990$.

67 - 70 are programs.

71. Some representative values:

t (sec):	0	0.01	0.02	0.04	0.06	0.08	0.10
I (amp):	0	1.366	0.401	-0.279	0.133	-0.049	0.014
q/C (V):	0	9.6	19.3	14.3	14.8	15.2	14.9

(I(max) is about 1.37 amp at about t = 0.009 sec.)

72. Representative values with h = 0.002:

t (sec):	0.01	0.02	0.04	0.06	0.08	0.10
I (amp):	0.428	-0.397	0.962	0.514	-0.497	-0.851
q/C (V):	5.99	-1.402	-1.563	2.248	1.799	-0.611

A plot of q/C versus t appears very much like a sine curve for t in [0.083,0.10].

74. Set up as four first-order equations by eliminating one second derivative from each equation. Then, if $i_1 = w$, $i_1' = x$, $i_2' = y$, $i_2' = z$, we get

$$w' = x,$$
$$y' = z,$$
$$x' = (e_2' + 2e_1' + 67.3z + 150000y - 91x)/0.0055,$$
$$z' = (e_1' + 6e_2' + 213x + 550000w - 113.2z - 200000y)/0.0055.$$

76. After stabilizing, the flux resembles a sine curve of amplitude about 7.5E-4 but the maxima and minima themselves oscillate. It is about 180 degrees out of phase with the exciting voltage. With the parameter values given, the values of phi never exceed 1.5E-3 in magnitude so neglecting the $(\text{phi})^3$ term makes no difference up to six significant figures.

78. The simplest way to handle the varying "constant" is by incorporating a look-up table in the program. If we use a subroutine that interpolates from the table using a cubic polynomial with the x-values centered, and using
h = 0.2, RK4 gives:

T:	0.0	0.2	0.4 ...	5.4	5.6	5.8	...	7.0
N:	100	115	133 ...	5519	5651	5633	...	3844

Observe that N reaches a maximum at about T = 5.6

Chapter 7

1. Rate of heat leaving is $-[k + k'dx][A + a'dx][du/dx + u''dx]$; equating rates in and out, canceling like terms, and dropping the $(dx)^2$ term results in Eq. (7.3).

2* The temperatures are linear within each portion. The gradient from x = 0 to x = X is proportional to A/k_1; from x = X to x = L, it is proportional to A/k_2. From these, the temperature at the junction is

$$U = 100k_2x/[k_1(L - X) + k_2X].$$

3. Take $u_{out} = u_{in} + (du_{in}/dx)dx$. Substitute $a + bu + cu^2$ for the k's, expand, cancel common terms, and drop terms in $(dx)^2$; the result:

$$(a + bu + cu^2)(d^2u/dx^2) + (b + 2cu)(du/dx)^2 = Qp/A.$$

4. In addition to the substitution in Exercise 3, take $A_{out} = A_{in} + m dx$. After expanding, canceling common terms, and dropping terms in $(dx)^2$ and $(dx)^3$, we get:

$$(a + bu + cu^2)(mx + n)(d^2u/dx^2) + (b + 2cu)(mx + n)(du/dx)^2$$
$$+ (a + bu + cu^2)(m)(du/dx) = Qp.$$

5* With modified Euler method: y'(1) = 5.48408.
With Runge-Kutta-Fehlberg: y'(1) = 5.50012.
With Runge-Kutta : y'(1) = 5.49872.
Analytical (exact): y'(1) = 5.50000.

6. Runge-Kutta-Fehlberg method was used, with h = 0.25.

x	Computed from y'(1) = 5	Computed from y'(1) = 6	Interpolated (y'(1) = 5.50012)
1.0	1.5000	1.5000	1.5000
1.25	3.1421	3.4204	3.2813
1.50	5.7008	6.2991	6.0000
1.75	9.3790	10.3083	9.8438
2.00	14.3871	15.6126	15.0000

7* Truncation errors cause the modified Euler results to be inexact. With h = 0.01, we match to 6 digits for y(2) when y'(1) = 5.5.

8. The same y values are obtained.

9. Runge-Kutta-Fehlberg was used with h = 0.2.

Initial slopes	Interpolated slope	y(1)
0.5, 1.0	0.91383	3.0009
0.91383, 0.913	0.9111393	3.0000

It is more difficult to get the correct result because the problem is nonlinear (but this one is not strongly nonlinear.)

10. With Runge-Kutta-Fehlberg method and h = 0.05, y'(1) = 0.910804 gives the correct results. Reducing h does not change this.

11. Using Runge-Kutta fourth order method and h = -0.2, y'(1) = -0.5106, gives y(0) = 0. Intermediate results:

t:	1.8	1.6	1.4	1.2	1.0	0.8	0.6	0.4	0.2
y:	-0.9109	-0.8891	-0.9648	-1.1147	-1.2642	-1.3104	-1.1677	-0.8282	-0.3843

12* a.

θ	y	%error
0	0	0
π/4	0.77015	0.625
π/2	1.42153	0.518
3π/4	1.85370	0.321
π	2	0

b. With h = π/5, largest error is 0.404%.

c. Shooting has a maximum error <0.5% with h = π/2.

13. The results with 64 intervals match those from RK4 (h = 0.25) to four digits. These also match to RKF (h = 0.25) which is more accurate.

14. Extrapolated results:

t	0.2	0.4	0.6	0.8
x	0.520865	0.062106	-0.355993	-0.715310

These agree to 5 digits with results from RKF with h = 0.1.

15* It requires 32 intervals, h = 0.03125.

16. Using four intervals:

t:	-1.0	-0.5	0.0	0.5	1.0
y:	2.000	2.338	2.522	2.717	3.000
RKF:	2.000	2.367	2.598	2.804	3.000

17. a.

t	y	error
0.00	-2.0398	0.0398
0.25	-1.8825	0.0348
0.50	-1.4349	0.0207
0.75	-0.7661	0.0007
1.00	0.0210	-0.0210

b. With 8 intervals, the largest error is 0.0097 at t = 0, 0.48%.

c. RKF with four intervals matches the analytical to 5 digits.

18. The solution obtained is the trivial solution, y ≡ 0.

19*

x:	0	π/8	π/4	3π/8	π/2
y:	1.5000	1.5828	1.4215	1.0410	0.5000
Anal:	1.5000	1.5772	1.4142	1.0360	0.5000

20. Computer program.

21. Index the nodes in [0, 1] as i = 1, 2, 3, 4, 5, 6. Add fictitious points at x_0 and x_7. We can write six equations, one at each interior point. The boundary conditions add two more. The augmented matrix is:

-1.000	0.400	1.000	0.000	0.000	-1.000	0.400	1.000	1.6000
1.000	-2.000	1.000	0.000	0.000	0.000	0.000	0.000	0.0000
0.000	1.020	-1.998	0.980	0.000	0.000	0.000	0.000	0.0003
0.000	0.000	1.040	-1.994	0.960	0.000	0.000	0.000	0.0026
0.000	0.000	0.000	1.060	-1.986	0.940	0.000	0.000	0.0086
0.000	0.000	0.000	0.000	1.080	-1.974	0.920	0.000	0.0205
0.000	0.000	0.000	0.000	0.000	1.100	-1.960	0.900	0.0400
1.000	0.400	-1.000	0.000	0.000	1.000	0.400	-1.000	1.2000

12. The augmented matrix (with two rows from the boundary conditions):

10.000	-12.000	6.000	-1.000	0.000	0.000	0.0195
2.040	0.000	-2.040	1.000	0.000	0.000	0.0239
-1.000	2.040	0.000	-2.040	1.000	0.000	0.0292
0.000	-1.000	2.040	0.000	-2.040	1.000	0.0356
0.000	0.000	0.000	0.000	1.000	0.000	1.0000
0.000	0.000	0.000	-1.000	0.000	1.000	0.0000

The solution is (0, 0.417, 0.706, 0.884, 0.978, 1).
By Runge-Kutta: (0, 0.3552, 0.6363, 0.8389, 0.9602, 1).

23. If we replace the second derivative with a central difference approximation, the typical equations is

$$y_{i-1} + (4h^2 y_i/\sin^2(x) - 2)y_i + y_{i+1} = 2.$$

A fictitious node must be added at the left of x = 1; for this:

$$y_F = y_1 - 2h(0.9093).$$

24. a. The analytical solution is $y = A\cosh(kx) + B\sinh(kx)$. $y(0) = 0$ implies $A = 0$ and $y(1) = 0$ implies $B = 0$ (since $k \neq 0$) so $y \equiv 0$.

(Exercise 24 continued)

b. The set of equations is $(A + kI)y = 0$. If $Z = 2 + 0.04k^2$, we evaluate this determinant:

$$\det\begin{vmatrix} Z & -1 & 0 & 0 \\ -1 & Z & -1 & 0 \\ 0 & -1 & Z & -1 \\ 0 & 0 & -1 & Z \end{vmatrix} = Z^4 - 3Z^2 + 1.$$

Solving for Z and substituting $Z = 2 + 0.04k^2$ gives only complex values for k.

c. The shooting method finds $y \equiv 0$.

25* The exact answer is 2.46166.

a. (h = 1/2): k = 2.0000,
b. (h = 1/3): k = 2.25895,
c. (h = 1/4): k = 2.34774,
d. Extrapolated: k = 2.46366.

26. h = 1/4 gives k = ±5.37981; with h = 1/5, k = ±5.44068.

27. Analytical solution: $C\, e^{3x/2} \sin(px)$. Typical values (for C = 1):

x:	0	0.25	0.50	0.75	1
y:	0	1.02883	2.11700	2.17804	0

28. We cannot use the exact eigenvalue. With h = 1/4, the computed value of the second eigenvalue is k = 10.8111, $k^2 = 116.88$. When the values for k and k^2 are substituted into the three equations, the system is redundant; we can chose any value for one of the unknowns. Taking $x_2 = 1$, we find that $x_1 = 0.0809$ and $x_3 = 0.1149$. The eigenvector is then any nonzero multiple of (0.0809, 1, 0.1149).

29* We cannot get the second eigenvalue with h = 1/2.
With h = 1/3: 3.59125,
with h = 1/4: 4.00000,
with h = 1/5: 4.19885.

30a. 9.3166; vector (0.1583, 1).
 b. 8; vector (0.5, 1).
 c. 3.6056; vector (1, 0.5352) and -3.6056; vector (-0.5352, 1).
 d. 7.2702; vector (1, 0.6351, 0.0768).
 e. 4.8845; vector (1, 0.6601, 0.8547).

31a. Intervals that contain the eigenvalues:
 Matrix A: [-8, -2], [-11, -7], [4, 10].
 Matrix B: Circles: Center at -4 + 2i, radius = 6; center at 7 + i, radius = 4; center at 4 - i, radius = 3.
 b. Neither is singular but Gerschgorin's theorems cannot tell this.

32a. Eigenvalue = 0.
 b. One eigenvalue = 0, the other is -1.73206, vector (-0.5774, 1).
 c. One eigenvalue = 0, others are
 -1.31101 (vector (-0.4545, 0, 1),
 -2.62202 (vector (0.4545, -0.9535, 1).

33. Each eigenvalue is the reciprocal; the vectors are the same.
 a. 0.10723; vector (0.1583, 1).
 b. 0.125; vector (0.5, 1).
 c. 0.277350; vector (1, 0.5352) and -0.277350; vector (-0.5352, 1).
 d. 0.137548; vector (1, 0.6351, 0.0768).
 e. 0.204729; vector (1, 0.6601, 0.8547).

34* Characteristic polynomial is $-w^3 - 7w^2 + 58w + 319$ whose roots are -4.6241, 7.2024, -9.5783. From the inverse matrix, the characteristic polynomial is $(-319w^3 - 58w^2 + 7w + 1)/319$, whose roots are the reciprocals.

35. -9.5782, -4.6241, 7.2017.

36. For a_{21}: $\begin{vmatrix} -5/d & 1/d & 0 \\ -1/d & -5/d & 0 \\ 0 & 0 & 1 \end{vmatrix}$ where $d = \sqrt{(26)}$.

(Exercsie 36 continued)

For a_{31}:

$$\begin{vmatrix} -5/d & 0 & 2/d \\ 0 & 1 & 0 \\ -2/d & 0 & -5/d \end{vmatrix} \quad \text{where } d = \sqrt{(29)}.$$

For a_{32}:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & -9/d & -1/d \\ 0 & 1/d & -9/d \end{vmatrix} \quad \text{where } d = \sqrt{(82)}.$$

37. After 102 rotations, A is

$$\begin{vmatrix} -9.5783 & -0.3816 & 0.9160 \\ 0.00007 & 7.2024 & 1.0076 \\ -0.00007 & 0.00008 & -4.6241 \end{vmatrix}$$

The diagonal elements match the eigenvalues of Exercise 34.

38. The upper Hessenberg matrix:

$$\begin{vmatrix} 3 & 21 & -2.375 & 7 \\ 1 & 0 & 0.625 & -1 \\ 0 & 8 & -2.125 & 5 \\ 0 & 0 & -5.8281 & 7.125 \end{vmatrix}$$

39. With rows and columns 2 and 3 interchanged, the upper Hessenberg matrix is:

$$\begin{vmatrix} 3 & 5.25 & 30.5 & 7 \\ 4 & 2 & 6.5 & 1 \\ 0 & -0.5 & -4.125 & -1.25 \\ 0 & 0 & 23.3125 & 7.125 \end{vmatrix}$$

40* Upper Hessenberg matrix is

$$\begin{vmatrix} -5 & -1.7889 & -1.416 \\ -2.2361 & 3 & -7 \\ 0 & -7 & -5 \end{vmatrix}$$

After 6 rotations, the eigenvalues are 7.2024, -9.5783, -4.6241. (Without getting the upper Hessenberg matrix, 102 rotations were required.)

41. We use the subscript notation for partial derivatives. When the thickness (t) is variable, the rate of flow out is

$-k^*(t + t_x dx)^*dy^*(u_x + u_{xx}dx) - k^*(t + t_y dy)^*dx^*(u_y + u_{yy}dy) + Q\,dx\,dy.$
Equating to the rate of flow in and canceling terms gives Eq. (7.9).

42. We use the subscript notation for partial derivatives. When both t and k vary, the rate of flow out is

$$-(k + k_x dx)^*(t + t_x dx)^*dy^*(u_x + u_{xx}dx)$$
$$- (k + k_y dy)^*(t + t_y dy)^*dx^*(u_y + u_{yy}dy) + Q\,dx\,dy.$$

Equating to the rate of flow in and canceling terms gives Eq. (7.10).

43. We use the subscript notation for partial derivatives and $\nabla^2 u$ for the Laplacian.

$$kt\nabla^2 u + (kt_x + tk_x)u_x + (kt_y + tk_y)u_y + (kt_z + tk_z)u_z = Q.$$

44. Substitute $a + bu + cu^2$ for k in the development. After canceling common terms and dropping terms in $(dx)^2$, this is added to the net flow:

$$(b + 2cu)[u_x^2 + u_y^2].$$

45. $u_y = (u_{i,j+1} - u_{i,j-1})/(2h)$

$$u_{yx} = \frac{(u_{i+1,j+1} - u_{i+1,j-1})/(2h) - (u_{i-1,j+1} - u_{i-1,j-1})/(2h)}{2h}$$

$$= \frac{u_{i+1,j+1} + u_{i-1,j-1} - u_{i+1,j-1} - u_{i-1,j+1}}{4h^2}$$

which is the same as the given operator.

46*

$$\frac{1}{12h^2}\begin{array}{ccccc} & & -1 & & \\ & & 16 & & \\ -1 & 16 & -60 & 16 & -1 \\ & & 16 & & \\ & & -1 & & \end{array} \; u_{i,j}$$

47. If we look at the grid tilted 45°, we see five points with a spacing of $\sqrt{(2)}*h$. Laplace's equation for this five point star is:

$$\nabla^2 u_{i,j} = \begin{array}{ccc} 1 & 0 & 1 \\ 0 & -4 & 1 \\ 1 & 0 & 1 \end{array} \; u_{i,j}/(2h^2) = 0.$$

If we use a weighted average of this operator (weight = 1/3) and the standard operator (weight = 2/3), we get the nine point operator of Equation (7.13).

48* The gradient is 100/L where L is the width of the plate. Let h = L/n. Nodes are at $x_i = i*h$, for i = 0 .. n, measured from the left end. (For points on the insulated boundaries, add fictitious points with the same gradient.) Then $u_i = 100 + ih(100/L)$. This gives $u_{i-1} + u_{i+1} = 2u_i$ and $u_{i-2} + u_{i+2} = 2u_i$. From these we have

a.

$$\begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array} \; u_{i,j}/h^2 = \frac{2u_i - 2u_i}{h^2} = 0.$$

$$\begin{array}{ccc} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{array} \; u_{i,j}/(6h^2) = \frac{12u_i - 12u_i}{6h^2} = 0.$$

b.

$$\begin{array}{ccccc} & & -1 & & \\ & & 16 & & \\ -1 & 16 & -60 & 16 & -1 \\ & & 16 & & \\ & & -1 & & \end{array} \; u_{i,j}/(12h^2) = \frac{-2u_i + 32u_i - 30u_i}{12h^2} = 0.$$

49. Interior temperatures:

58.53	70.87	70.87	58.53
43.24	54.08	54.08	43.24
40.35	48.14	48.14	40.35

50* Interior temperatures:

64.21	105.20	146.65	186.41
61.63	89.94	114.99	134.00
52.39	77.93	89.38	84.59

51. Temperatures at interior nodes:

63.45	104.32	145.95	186.09
60.81	88.53	113.79	133.58
51.87	76.42	88.05	84.52

52*

$$3u_{xx} + 2u_{yy} = \frac{3(u_L - 2u_0 + u_R)}{h^2} + \frac{2(u_A - 2u_0 + u_B)}{h^2}$$

$$= \begin{matrix} & 2 & \\ 3 & -10 & 3 \\ & 2 & \end{matrix} \quad u_{i,j}/h^2 = 0.$$

53. Temperatures at interior nodes:

	89.35		
47.39	57.39	61.49	52.87
32.18	31.32	35.70	

54. The temperatures are the same as for Exercise 50. A tolerance value of 0.00001 was used. With initial values all equal to zero, 31 Iterations were needed. With initial all equal to 300, 32 iterations were needed. With initial values all equal to 93.89 (the average of the boundary temperatures), 27 iterations were needed. The final values are not exactly the same for these three cases.

55. With w_{opt} = 1.293, we converge (TOL = 0.00001) to the same values as in Exercise 54 after 17 iterations. This started with all interior nodes at zero. Liebmann's method took 31 iterations.

56* a. With h = 2/3, f = 0.444 at each point.

b. With h = 1/3. there are 25 interior points. The values are symmetrical about the center point. Values in upper left quadrant:

0.2115	0.3120	0.3419
0.3120	0.4722	0.5214
0.3419	0.5214	0.5769

57. There is symmetry about the center point. Values in first octant:

		-1.794
	-3.134	-2.337
-2.859	-3.119	-2.357
-2.099	-2.814	-2.223
-1.909	-2.690	-2.159

58. Values at interior nodes:

-0.087	-0.166	-0.226	-0.251	-0.202
-0.120	-0.226	-0.304	-0.327	-0.251
-0.116	-0.217	-0.288	-0.304	-0.226
-0.088	-0.165	-0.217	-0.226	-0.166
-0.047	-0.088	-0.116	-0.120	-0.087

There is symmetry about the line y = x.

59. Starting with all values equal to zero, and w = 1.35, we converge in 12 iterations to the same values as in Exercise 56. The predicted value for w_{opt} is 1.333. It takes 14 iterations with w = 1.34 or w = 1.36.

60* Iterations required with varying values of w (TOL = 0.00001):

w:	1.30	1.32	1.34	1.35	1.36	1.40	1.50
Iterations:	21	19	16	15	17	18	21

Equation (7.15) does not apply because the region is not a rectangle.

61. Values at interior points, laid out as in the figure:

	93.40		
82.13	73.62	67.56	54.39
54.91	51.37	42.23	
36.13	34.72		

62. There is no unique solution; if u(x,y) is a solution, so is u(x,y) + C where C is any constant.

63.

0.431	0.557	1.010	2.056	4.317	9.163	19.668	43.212
0.609	0.787	1.426	2.897	6.050	12.668	26.296	53.180
0.431	0.557	1.010	2.056	4.317	9.163	19.668	43.212

64.

9.501	13.092	14.906	16.502	18.944	23.729	33.851	56.198
4.910	7.961	10.031	12.158	15.545	22.122	35.478	62.088
2.180	3.810	5.099	6.554	8.958	13.733	23.852	46.198

65* The same answers as in Exercise 50 are obtained after 27 iterations. Exercise 54 required 27 iterations; in Exercise 55, only 18 were needed.

66. a. With ρ = 1, converge to exact answer on second iteration.
    b. Optimum value of ρ is 1.72, but converges (TOL = 0.001) in nine iterations, giving results that match those of Exercise 56.

67. There are 6 "layers" of nodes; each layer has 6*6 = 36 nodes; the total number of nodes is 6*36 = 216 so there are 216 equations. There are 3 sets of these, one or each direction (x, y and z). Even though each system is tridiagonal, getting a convergent solution is not done quickly.

68. In addition to the "layers" of nodes as described for Exercise 67, the surface where there is a temperature gradient must also be included as an additional layer. There are then three sets with 8*36 = 288 equations in each set.

69*

```
                         1/(hA(hA+hB))

                   -(hL + hR)        (hA + hB)
  2*  1/(hL(hL+hR)) ------------ - ------------   1/(hR(hL+hR))
                   hL*hR(hL+hR)   hA*hB(hA+hB)

                         1/(hB(hA+hB))
```

70. Values in the first octant (other values are symmetrical):

a. Uneven star:

	9.259	
37.421	18.517	0.402
56.325	26.987	2.183
60.893	30.922	8.822

b. Distorted boundary:

	12.531	
42.837	25.077	10.156
60.597	34.778	15.545
64.772	37.843	17.246

71. Because of radial symmetry, all nodes equally distant from the center have the same values. This means that the problem can be solved in one dimension. Using h = 0.5, there are 8 nodes at x-values within [2,5.5]:

x:	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
f:	2.0	3.5	4.5	5.0	5.0	4.5	3.5	2.0

72. With nodes on a uniform grid, 1.5 cm between each, and using an uneven start near the circumference, values in the first octant are:

5.535	3.425
5.394	6.323
5.220	5.592

There is eight-fold symmetry.

73. There are 28 nodes in the quarter circle within [0,4] and there is two-fold symmetry. The coefficient matrix will then be 28 × 28. The first seven elements on the diagonal of the matrix are:

-12.186, -4.546, -3.132, -2.363, -2.408, -2.283, -2.208,

and this is repeated three times for the 28 elements.

74. Equations for 49 nodes can be set up in the quarter circle within [0,4] (taking advantage of symmetry).

75. All have unique solutions even though the condition is violated at:

a. x = -1.

b. x = 1.

c. x = 0.

76* Augmented matrices:

a.
$$\left|\begin{array}{cccc} -2.125 & 1.375 & 0 & 0.4375 \\ 0.750 & -2.250 & 1.250 & -0.2500 \\ 0 & 0.875 & -2.375 & -1.3125 \end{array}\right| \text{, det} = -6.5820.$$

b.
$$\left|\begin{array}{cccc} -1.625 & 0.875 & 0 & 1.4375 \\ 1.250 & -1.750 & 0.750 & 0.2500 \\ 0 & 1.375 & -1.875 & -0.3125 \end{array}\right| \text{, det} = -3.0352.$$

c.
$$\left|\begin{array}{cccc} -1.750 & 0.750 & 0 & 1.125 \\ 1.125 & -2.000 & 0.750 & 0 \\ 0 & 1.250 & -2.250 & 0.625 \end{array}\right| \text{, det} = -4.3359.$$

77.

y'(-1)	y(1)
0.0	2.65933
0.5	2.85196
0.88424	2.99067

Passing a parabola through these points suggests y'(-1) = 0.91094. Using this value gives y(1) = 3.000047.

78. Using y'(-1) values of 0.5, 0.88424, 0.91094, a parabola suggests y'(-1) = 0.910081 which is less accurate that linear interpolation after three trials which gets y(1) = 3.000000 from y'(-1) = 0.9108056.

79. The value of $A^m v$ does not converge to $c_1\lambda_1{}^m x_1{}^m$ but to a combination involving the two largest eigenvalues and their vectors.

80* a. 19.333

b. 6.001

81. Multiplying gives

$$T^2 = \begin{vmatrix} L^2 & 0 \\ 2Lx & L^2 \end{vmatrix}, \quad T^3 = \begin{vmatrix} L^3 & 0 \\ 3L^2x & L^3 \end{vmatrix}, \ldots$$

so, if $|L| < 1$, $L^n \to 0$ and $T^n \to$ zero matrix.

82 a. Multiply out.

b. Use mathematical induction.

c. Use mathematical induction.

83* Largest eigenvalue with

	w_{opt}	$w = w_{opt}$	$w = 1$	w_{opt} from Eq. (7.15)
a.	1.01612	0.01524	0.0625	1.01613
b.	1.20377	0.2038	0.5625	1.21 (see note)
c.	?	0.0486	0.125	1.0334

Note: In part (b), Equation (7.15) does not apply. Value obtained by trials.

84. For Jacobi, $w_{opt} = 0.932$; maximum change after nine iterations is 3.45E-5. For Gauss-Seidel, $w_{opt} = 0.967$; maximum change after nine iterations is 1.4E-5.

85. For Jacobi, modulus of largest eigenvalue with w = 0.932 is 0.2689, versus 0.2874 with w = 1. For Gauss-Seidel, largest eigenvalue with w = 0.967 is 0.0951, versus 0.1372 with w = 1.

86. For both parts (a) and (b), the values at interior nodes are:

1.25	3.75
1.25	3.75

The principle is confirmed.

87. Take the origin at the lower left corner with nodes:
#1 at (h,h), #2 at (2h,h), #3 at (h,2h) and #4 at (2h,2h).

a. Equation for u_1 is $u_1 = u_2 + u_3 - h^2 * f(x,y)$; the others are similar.

b.

h	f(x,y)	u_1	u_2	u_3	u_4
1	0	1.25	3.75	1.25	3.75
2	0	1.25	3.75	1.25	3.75
1	1	101.25	103.75	101.25	103.75
2	1	401.25	403.75	401.25	403.75
1	x	126.25	178.75	126.25	178.75
2	y	1001.25	1003.75	1401.25	1403.75
1	xy	159.6	220.4	217.9	312.1

88. MATLAB's ode23 (and ode45 as well) sets the step size automatically and this is not under user control so $\Delta t \neq 0.2$. With the default tolerance of 1.E-3, the results at t = 3.0 differ from Table 7.1 in the fourth decimal place. With tolerance = 1.E-5, the results at t = 3.0 do match the table. (There is a way to make $\Delta t = 0.2$ without modifying the M-file. Challenge the sutdents to find it.)

89. The final results at t = 3.0 match those in Table 7.1.

90. The results at t = 3.0 match those in Table 7.2.

91. After defining matrices: M = coefficients, B = right-hand sides,
   a. x = inv(M)*B gives (0.5520, -0.4244, -0.9644).
   b. vpa(linsolve(M,B,4) gives the same values.
   c. x = rref([M B]) gives the same values in the fourth column.

92* Create matrix B as symbolic: B = sym('[5,3,2;-2,6,3;3,2,4]')
   Then: vpa(eigensys(B),5) gives 3.6431, 3.8080, 8.0489

93. After creating the modified matrix (B1) as symbolic:
   vpa(eigensys((B1),5) gives 3.4520 ± 0.13468i, 8.3961

94.

Eigenvalue	Eigenvector
3.6431	(-0.3306, 0.6225, -0.7094)
8.0489	(0.7021, 0.2759, 0.6565)
3.3080	(0.2516, -0.6310, 0.7339)

95. charpoly(B) = $-w^3 - 15w^2 + 68w + 104$

96. Let $M = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$, $A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$,

then $MAM^{-1} = 1/2 \begin{vmatrix} 5 & 1 & -1 \\ 2 & 4 & -2 \\ 1 & -1 & 3 \end{vmatrix}$.

Both have the same eigenvalues but the eigenvectors are different.

97. Start the solution from the right-hand end, stepping x backward. Using units of lb and in, I = 10.6667 in^4 if the 4 in dimension is vertical and 2.6667 if it is horizontal. Since the problem is linear, only two trials are needed. With RK4:
   a. (4 in dimension vertical): y(L) = 3.4665, at midpoint, y = 2.19867.
   b. (4 in dimension horizontal): y(L) = 9.12485, at midpoint, y = 5.5077.

98a. Using h = 12 in (the same as in Exercise 97), y(120) = 3.50138 in.
   With h = 6, y(120) = 3.47520; extrapolating gives y(120) = 3.4665 in.

b. Using h = 12 in, y(120) = 9.22189 in. With h = 6, y(120) = 9.14912; extrapolating gives y(120) = 9.11430 in.

99. The distance from the end of the beam to the wall is 0.37 in less than 120 in under the loads (4 in edge horizontal). Using the exact equation and allowing for the shorter distance, y(120) = 9.0546 in.

100. The equation is linear so two trials by the shooting method are sufficient. With y'(1) = -779.06, we get these results (RK4 with h = 0.1):

r:	1.0	1.2	1.4	1.6	1.8	2.0
T:	540.0	403.2	287.6	187.4	99.0	20.0

101. Set y'(2) equal to 0.83(T(2) - 20) and solve with a negative step from r = 2 to r = 1. With t(2) = 486.34, we get T(1) = 540, and these intermediate values:

r:	1.0	1.2	1.4	1.6	1.8	2.0
T:	540.0	525.9	514.0	503.6	494.5	486.3

102. This is not a characteristic problem because it is not linear. (This is easy to see if we approximate the differential equation by finite differences.) If we attempt to linearize by moving the nonlinear terms to the right-hand side, it then is no longer homogeneous.

103. For $F(x) = 4x(\pi - x)/\pi^2$ (a parabolic curve) with $h = \pi/8$, a y-dimension of 20h is adequate (rather than infinity). The agreement with the analytical solution depends on the size of h.

Chapter 8

1. a. Hyperbolic
   b. Parabolic
   c. When k, m, and a are nonzero scalars, it is hyperbolic if k and a are of the same sign. When they have opposite signs, it is parabolic if $|4ka| = m^2$, elliptic if $|4ka| > m^2$, hyperbolic if $|4ka| < m^2$.
   d. Parabolic. This is an eigenvalue problem.

2* The discriminant is $4(1-x^2) + 4(1+y)(1-y)$. When set to zero, this describes a hyperbola whose center is at (1,0) and whose vertices are at (1,1) and (1,-1). The equation is parabolic at points on this curve. Above the upper branch and below the lower branch, it is elliptic. Between the two branches, it is hyperbolic.

3. The discriminant is $4x^4(y-1)$. The equation is parabolic on the y-axis and the line y = 1. It is hyperbolic above the line y = 1 (but not for x = 0). It is elliptic below the line y = 1 (but not for x = 0).

4. For t measured in seconds, units of the other parameters are

$$k: \frac{\text{BTU/sec}}{\text{ft}^2\ (^\circ\text{F/ft})},\quad c: \frac{\text{BTU/lb}}{^\circ\text{F}},\quad r: \frac{\text{lb}}{\text{ft}^3}.$$

5. $(ku_x)_x + Q(x) = c(x)\rho(x)u_t$.

6* Using k = 2.156 BTU/(hr*in*°F)
   a. -29.53 °F/in.
   b. -75.59 °F/in.
   c. -34.91 °F/in.

7. $u_{tt} = Tgu_{xx}/(W(x) + W_x/2)$.

8. With $r = 0.5$:

x:	0	0.25	0.50	0.75	1.00	
u:	0	17.34	32.04	41.86	45.31	(symmetrical to right of x = 1.0).
anal:	0	17.72	32.74	42.78	46.30	

9* With $r = 1$:

x:	0	0.25	0.50	0.75	1.00	
u:	0	17.85	32.98	43.09	46.64	(symmetrical to right of x = 1.0).

10. With $r = 1$:

	x:	0	0.25	0.50	0.75	1.00
$\theta = 2/3$	u:	0	18.19	33.61	43.92	47.53
$\theta = 0.878$	u:	0	18.61	34.38	44.92	48.62
$\theta = 1$	u:	0	18.82	34.82	45.49	49.24

For this problem, Crank-Nicolson is more accurate; if $\theta = 0.435$, there is even less error.

11. With units of BTU, lb, in, sec, °F, $k = 0.00517$, $c = 0.0919$, $\rho = 0.322$. With $\Delta x = 1$ in., $\Delta t = 2.862$ sec. Using $r = 0.5$, at $t = 28.62$:

x:	0	1	2	3	4	5	6	7	8
u:	100	85.94	73.44	60.94	50.00	39.06	26.56	14.06	0

12. At $t = 28.62$ sec ($\Delta t = 0.7155$), and with $r = 0.5$:

x:	1	3	6
u, $\Delta x = 0.5$:	85.70	60.70	27.54
u, $\Delta x = 1.0$:	85.94	60.94	26.56

13. At $t = 28.62$ sec,

x:	1	3	6	No. steps	Calc/ step	Δt	r	Δx
Exercise 13:	85.70	60.70	27.54	20	7	1.43	0.25	1
Exercise 12:	85.70	60.70	27.54	40	14	0.72	0.5	0.5
Exercise 11:	85.94	60.94	26.56	10	7	2.86	0.5	1

14. The formula gives f = 444.03 cycles/sec. If the string is divided into seven equal segments, displacements repeat every 14 time steps. Since $\Delta t = 1.6086E-4$, computations show that $f = 1/(14\Delta t) = 444.03$.

15. a. $\Delta t = 3$ sec. Displacements versus time:

t	x = 0	6	12	18	24	30	36	42	48
0.00	0.00	-0.11	-0.19	-0.23	-0.25	-0.23	-0.19	-0.11	0.00
3.00	0.00	-0.09	-0.17	-0.22	-0.23	-0.22	-0.17	-0.09	0.00
6.00	0.00	-0.06	-0.13	-0.17	-0.19	-0.17	-0.13	-0.06	0.00
9.00	0.00	-0.03	-0.06	-0.09	-0.11	-0.09	-0.06	-0.03	0.00
12.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
15.00	0.00	0.03	0.06	0.09	0.11	0.09	0.06	0.03	0.00
18.00	0.00	0.06	0.13	0.17	0.19	0.17	0.13	0.06	0.00
			. . . . .						
45.00	0.00	-0.09	-0.17	-0.22	-0.23	-0.22	-0.17	-0.09	0.00
48.00	0.00	-0.11	-0.19	-0.23	-0.25	-0.23	-0.19	-0.11	0.00

b. $\Delta t = 3$ sec.

t	x = 0	6	12	18	24	30	36	42	48
0.00	0.00	1.00	2.00	0.00	-2.00	-4.00	-2.67	-1.33	0.00
3.00	0.00	1.00	0.50	0.00	-2.00	-2.33	-2.67	-1.33	0.00
6.00	0.00	-0.50	-1.00	-1.50	-0.33	-0.67	-1.00	-1.33	0.00
9.00	0.00	-2.00	-2.50	-1.33	-0.17	1.00	0.67	0.33	0.00
12.00	0.00	-2.00	-2.33	-1.17	0.00	1.17	2.33	2.00	0.00
15.00	0.00	-0.33	-0.67	-1.00	0.17	1.33	2.50	2.00	0.00
18.00	0.00	1.33	1.00	0.67	0.33	1.50	1.00	0.50	0.00
			. . . . .						
45.00	0.00	1.00	0.50	0.00	-2.00	-2.33	-2.67	-1.33	0.00
48.00	0.00	1.00	2.00	0.00	-2.00	4.00	-2.67	-1.33	0.00

(Exercise 15 continued)

c. Δt = 3 sec.

t	x = 0	6	12	18	24	30	36	42	48
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3.00	0.00	0.33	0.56	0.70	0.75	0.70	0.56	0.33	0.00
6.00	0.00	0.56	1.03	1.31	1.41	1.31	1.03	0.56	0.00
9.00	0.00	0.70	1.31	1.73	1.88	1.73	1.31	0.70	0.00
12.00	0.00	0.75	1.41	1.88	2.06	1.88	1.41	0.75	0.00
15.00	0.00	0.70	1.31	1.73	1.88	1.73	1.31	0.70	0.00
18.00	0.00	0.56	1.03	1.31	1.41	1.31	1.03	0.56	0.00
			. . . . .						
48.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
51.00	0.00	0.33	0.56	0.70	0.75	0.70	0.56	0.33	0.00

d. Δt = 3 sec.

t	x = 0	6	12	18	24	30	36	42	48
0.00	0.00	0.25	0.50	0.75	1.00	0.75	0.50	0.25	0.00
3.00	0.00	-0.50	-1.00	-1.50	-2.25	-1.50	-1.00	-0.50	0.00
6.00	0.00	-1.25	-2.50	-4.00	-4.00	-4.00	-2.50	-1.25	0.00
9.00	0.00	-2.00	-4.25	-5.00	-5.75	-5.00	-4.25	-2.00	0.00
12.00	0.00	-3.00	-4.50	-6.00	-6.00	-6.00	-4.50	-3.00	0.00
15.00	0.00	-2.50	-4.75	-5.50	-6.25	-5.50	-4.75	-2.50	0.00
			. . . . .						
45.00	0.00	1.00	2.00	3.00	3.75	3.00	2.00	1.00	0.00
48.00	0.00	0.25	0.50	0.75	1.00	0.75	0.50	0.25	0.00

16* The computed values agree with the analytical. There is symmetry about x = 0.5. some values:

	--------	x-value	----------	
t	0	1/8	1/4	3/8	1/2
0.0	0	0.383	0.707	0.924	1.000
0.1	0	0.354	0.653	0.854	0.924
0.2	0	0.271	0.500	0.633	0.707
0.8	0	-0.383	-0.707	-0.924	-1.000
0.9	0	-0.354	-0.653	-0.854	-0.924

17. Number the nodes from 0 to N and put y(N) = sin(πt/4), also, using a backward difference for the derivative, y(N-1) = y(N). If Δx = 0.2, then Δt = 0.2. Some representative values (they repeat after 32 time steps):

t	x = 0	0.2	0.4	0.6	0.8	1.0
0.00	0.000	0.000	0.000	0.000	0.000	0.000
0.20	0.000	0.000	0.000	0.000	0.156	0.156
0.40	0.000	0.000	0.000	0.156	0.309	0.309
0.60	0.000	0.000	0.156	0.309	0.454	0.454
0.80	0.000	0.156	0.309	0.454	0.588	0.588
1.00	0.000	0.309	0.454	0.588	0.707	0.707
1.20	0.000	0.298	0.588	0.707	0.809	0.809
1.40	0.000	0.279	0.551	0.809	0.891	0.891
1.60	0.000	0.253	0.500	0.735	0.951	0.951
1.80	0.000	0.221	0.437	0.642	0.988	0.988
		. . . . .				
7.60	0.000	-0.000	-0.000	-0.156	-0.309	-0.309
7.80	0.000	-0.000	-0.000	-0.000	-0.156	-0.156
8.00	0.000	-0.000	-0.000	-0.000	-0.000	-0.000
8.20	0.000	-0.000	-0.000	-0.000	0.156	0.156

18* For part (c), a full period is still 16 time steps (48 sec). Values change when Eq. (8.26) is used. Comparison of values:

t:	0	3	6	9	12	15	18
Eq. (8.26):	0	0.734	1.375	1.828	2.000	1.828	1.375
Eq. (8.19):	0	0.750	1.406	1.875	2.063	1.875	1.406

For part (d):

t:	0	3	6	9	12	15	18
Eq. (8.26):	1.000	-2.000	-4.000	-5.500	-6.000	-6.000	-5.000
Eq. (8.19):	1.000	-2.250	-4.000	-5.750	-6.000	-6.250	-5.000

19. For part (c), no difference because the initial velocities are a quadratic in x and Simpson's rule is exact for a quadratic. For part (d), Simpson's rule is exact except at the midpoint (because velocities are linear in x except at that point). At the midpoint, the exact integral of the velocity is -10.5; we get this value with 4, 8, 12, ... intervals within [18,30] but not with 2, 6, 10, ...:

Number of intervals:	2	4	6	8	10	12
Value of integral:	-11.0	-10.5	-10.555	-10.5	-10.52	-10.5

20. With $\Delta x = 0.3$, $\Delta t = 0.003344$ sec. After three time steps ($t = 0.01003$), $y(1.5) = 0.0067334$ ft = 0.0808 in. (same as analytical). Other values agree with the series solution.

21*

34.722	38.589	50.744	70.106	100.00
29.644	33.296	45.066	65.376	100.00
19.495	21.816	30.152	49.058	100.00
0.000	0.000	0.000	0.000	----

These values are within 3.5° of the steady-state values.

22. Let the faces that lose heat be the top face, the front face, and the left side. Looking at the cube from the front, we see four "layers" Of nodes where the temperatures vary with time. The top face is one of these. Each layer has 16 nodes so there are 64 equations but each can be solved explicitly. Fictitious nodes are assumed outside the surface nodes where heat is being lost and these have u-values that are related to the values at the surface node and the node immediately inside. The basic equation is:

$$u^{k+1} = r(u_L + u_R + u_A + u_B + u_{front} + u_{back})^k + (1-6r)u^k$$

The maximum value for r is 1/6. using $c = 0.226$, $\rho = 0.0975$, $k = 0.00291$ (c.g.s. units), and $\Delta x = 1$, Δt is 1.26 sec. It takes 12 steps to reach $t = 15.12$ sec.

23. There are still 64 equations. These are not tridiagonal but they are banded. After getting the LU equivalent, solving the system amounts to two multiplications of a matrix times a vector. Since r can be 1, only two time steps are needed to reach $t = 15.12$.

24. The answer is the same as that of Exercise 23.

25. There are three sets of equations with 64 in each set but these are tridiagonal. Getting the LU equivalent requires at most 3*63 = 189 multiplications/divisions and this needs to be done only once. Using the LU's to solve the equations for the next time step requires only 63*2 + 1 = 127 multiplications/divisions in each set after the right-hand sides have been updated. Since r can be 1, only two time steps are needed to reach t = 15.12 sec, but more accurate results are obtained after every third time step.

26. Values laid out in nodal positions:

t = .2041301

0.000	0.000	0.000	0.000	0.000
0.000	0.029	0.077	0.086	0.000
0.000	0.077	0.204	0.230	0.000
0.000	0.086	0.230	0.258	0.000
0.000	0.000	0.000	0.000	0.000

t = .4082603

0.000	0.000	0.000	0.000	0.000
0.000	0.077	0.159	0.153	0.000
0.000	0.159	0.306	0.274	0.000
0.000	0.153	0.274	0.230	0.000
0.000	0.000	0.000	0.000	0.000

t = .6123904

0.000	0.000	0.000	0.000	0.000
0.000	0.131	0.191	0.131	0.000
0.000	0.191	0.230	0.115	0.000
0.000	0.131	0.115	0.016	0.000
0.000	0.000	0.000	0.000	0.000

(Exercise 26 continued)

t = .8165205

0.000	0.000	0.000	0.000	0.000
0.000	0.115	0.086	0.000	0.000
0.000	0.086	0.000	-0.086	0.000
0.000	0.000	-0.086	-0.115	0.000
0.000	0.000	0.000	0.000	0.000

t = 1.020651

0.000	0.000	0.000	0.000	0.000
0.000	-0.045	-0.134	-0.131	0.000
0.000	-0.134	-0.230	-0.172	0.000
0.000	-0.131	-0.172	-0.102	0.000
0.000	0.000	0.000	0.000	0.000

27. Nodal displacements:

At t = 0

0.000	0.000	0.000	0.000	0.000
0.000	0.141	0.375	0.422	0.000
0.000	0.375	1.000	1.125	0.000
0.000	0.422	1.125	1.266	0.000
0.000	0.000	0.000	0.000	0.000

At t = .2041301

0.000	0.000	0.000	0.000	0.000
0.000	0.188	0.391	0.375	0.000
0.000	0.391	0.750	0.672	0.000
0.000	0.375	0.672	0.563	0.000
0.000	0.000	0.000	0.000	0.000

At t = .4082603

0.000	0.000	0.000	0.000	0.000
0.000	0.250	0.281	0.109	0.000
0.000	0.281	0.063	-0.281	0.000
0.000	0.109	-0.281	-0.594	0.000
0.000	0.000	0.000	0.000	0.000

(Exercise 27 continued)

At t = .6123904

0.000	0.000	0.000	0.000	0.000
0.000	0.094	-0.180	-0.375	0.000
0.000	-0.180	-0.750	-0.883	0.000
0.000	-0.375	-0.883	-0.844	0.000
0.000	0.000	0.000	0.000	0.000

There is no repetitive pattern.

28. We assumed that the initial displacements form a pyramid with flat faces whose peak is at (1,1). Using $\Delta x = 0.5$, Δt is 0.00544 sec. There appears to be no repetitive pattern.

The initial displacements:

0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.000	0.500	0.500	0.500	0.500	0.250	0.000
0.000	0.500	1.000	0.750	0.500	0.250	0.000
0.000	0.500	0.500	0.500	0.500	0.250	0.000
0.000	0.000	0.000	0.000	0.000	0.000	0.000

Some values for the node at (2,1):

Steps:	0	1	2	4	6	8	10	14
u(2,1):	0.500	0.500	0.250	-0.234	-0.625	0.313	0.897	-0.932

29. Δt becomes 0.00172 sec (reduced by a factor of $\sqrt{(10)}$).

30. The initial displacements are similar to those in Exercise 28 except the ridge from (1,1) to (3,2) is horizontal. There is no repetitive pattern.

The initial displacements:

0.000	0.167	0.333	0.500	0.667	0.833	1.000
0.000	0.500	0.667	0.833	1.000	0.875	0.750
0.000	0.500	1.000	0.875	0.750	0.625	0.500
0.000	0.500	0.500	0.500	0.500	0.375	0.250
0.000	0.000	0.000	0.000	0.000	0.000	0.000

(Exerxcise 30 continued)

Some values for the node at (2,1):

Steps:	0	1	2	4	6	8	10	14
u(2,1):	0.750	0.750	0.542	-0.070	-0.188	0.595	1.081	-0.443

31. After 22 time steps, a single error grows to become larger than the original error and then continues to grow by a factor of 1.0485 at each succeeding time step.

32. After 7 time steps, the maximum error has decreased to 0.1167 times the original error. This is larger than the factor in Table 8.9, but the maximum error continues to decrease by a factor of 0.8538 at each succeeding time step.

33. After 7 time steps, the maximum error has decreased to 0.219 times the original error. As time increases, the maximum error decreases by a factor of 0.875 for two time steps and this factor gets smaller as time progresses.

34* After 7 time steps, the maximum error has decreased to 0.234 times the original error. The maximum error at each succeeding time step is about 0.85 times the previous error.

35. The errors damp out very rapidly. After four time steps, the maximum error is less than 0.02% of the original error.

36.

N:	4	4	5	5
r:	0.5	0.6	0.5	0.6
Eigenvalue:	0.8090	-1.1708	0.8660	-1.2392

The statement is confirmed.

37. With N = 4, the largest eigenvalue with r = 1.0 is 0.679285; with r = 2.0, it is -0.566915. The statement is confirmed.

38.

N:	3	3	3	3
r:	0.5	1.0	2.0	3.0
Eigenvalue:	0.7735	0.6306	0.4605	0.3627
N:	4	4	4	4
r:	0.5	1.0	2.0	3.0
Eigenvalue:	0.8396	0.7236	0.5669	0.4660

39. a. The errors develop a complicated pattern and sometimes are larger than the original error but they ever are more than 1.5 times the original error.

b. The method is unstable when the ratio is 2. Errors grow rapidly; after 9 time steps the largest is 8.6E5 times the original error. Eventually they get so large as to cause overflow.

40. The table resembles Table 8.11 except the errors are reflected earlier.

41. When $r = Tg(\Delta t)^2 / [w(\Delta x)^2] = 1$, the equation becomes

$$y_i^{j+1} = [y_{i+1} + y_{i-1} - (1 - S)y_i]^j/(1 + S)$$

where $S = B\Delta t/2$. For the first time step, substitute $(y^1 - 2v_0\Delta t)$ for y^{-1}, giving this equation to initiate the computations:

$$y_i^1 = (y_{i+1} + y_{i-1})^0/2 + (1 - S)v_0(\Delta t).$$

With $\Delta x = 1$, Δt is 0.0114 sec. The values show typical damped behavior, the largest y-values at x = 3 occur at time steps 0, 10, 20, ... and each of these is 0.8925 times the previous:

t:	0	0.1138	0.2275	0.3413	0.4550
y(3):	3.0	2.677	2.390	2.133	1.904

42. Using the A.D.I. method with Δx = 1 in. = 2.54 cm and r = 0.2 so that Δt = 8.52 sec, the center point is above 2000° after only 11 time steps (in 93.7 sec). If this were a real-world problem, it would be pointless to solve as a three-dimensional problem.

43. One way to cope with the nonlinearity of radiant heat transfer is to convert to a boundary condition with heat flowing according to $hA(u_{surface} - 2350)$ and equate this to the rate of heat flow given by the radiation formula. In effect, we use a value of h that varies with the surface temperature. If a table of such values is computed, a program can use this table to evaluate h as time progresses. The variation of h with surface temperature is less than might be expected — from 45.5 at 500° to 116 at 2250°.

44. Using finite differences, with $\Delta x = L/4$, Δt is 5.50E-4 sec.

Some representative values:

	x/L =	0	0.25	0.50	0.75	1.00
(Time	0	0	0	0	0	0.700
steps)	1	0	0	0	0.350	0.700
	3	0	0.350	0.700	0.700	0.700
	6	0	0	0.350	0.700	0.700

45. Since there is radial symmetry, the derivative with respect to q vanishes and the equation reduces to one involving only r and t — a one-dimensional problem.

47. This is a lengthy and challenging project!

Chapter 9

1. Let $G(x,u,u') = (u')^2 - Qu^2 + 2Fu$, the integrand. The Euler-Lagrange condition is $G_u = d[G_{u'}]/dx$. Compute: $G_u = -2Qu + 2F$ and $G_{u'} = 2u'$, giving $d[G_{u'}]/dx = 2u''$. From the Euler-Lagrange condition, we have $-2Qu + 2F = 2u''$, which is the same as $u'' + Qu = F$.

2* Let $u(x) = C(x)(x - 1)$. The Rayleigh-Ritz integral gives

$2c/3 + 0 = -2(5/12)$, so $c = 5/4$. Some values:

x:	0	0.2	0.4	0.6	0.8	1.0
u:	0	-0.200	-0.300	-0.300	-0.200	0
anal:	0	-0.176	-0.288	-0.312	-0.242	0

3. $I_a = (4a + 2b - 5)/6 = 0$,
$I_b = (5a + 4b - 7)/15 = 0$.
Solving, we get $a = 1$, $b = 1/2$. which gives $u(x) = x^3/2 + x^2/2 - x$, matching the analytical solution.

4. $I_a = (4a - 2b - 5)/6 = 0$,
$I_b = (-10a + 8b + 11)/30 = 0$.
Solving, $a = 3/2$, $b = 1/2$, giving

$w(x) = x^3/2 + x^2/2 - x$, the analytical solution.

5. Change variable: $v = y - 2x - 1$ so that $v = 0$ at $x = 0$ and at $x = 1$. The equation becomes $v'' = 3x + 1$. $v(0) = 0$, $v(1) = 1$. Solution is

$u(x) = (x^3 + x^2 + 2x + 2)/2$.

6* $R(x) = y'' - 3x - 1$. If $u = cx(x - 1)$, $u'' = 2c$. Since there is only one constant, set $R = 0$ at $x = 1/2$. We then have $2c - 3(1/2) - 1 = 0$ giving $c = 5/4$. This is identical to the answer of Exercise 2.

7. If we set $R(x) = 0$ at $x = 1/4$, $1/3$, $2/3$, and $3/4$, in turn, we get $c = 7/8$, 1, $3/2$, $13/8$. None of these is as close to the analytical solution (which has $c = 5/4$) obtained with $R(x) = 0$ at $x = 1/2$.

8. The residual is $2a - 2b + (6b - 3)x - 1$. This equals zero for any value of x if $a = 1$, $b = 1/2$. This means that any pair of points in [0,1] gets the same answer as in Exercise 3.

9* Integral is $\int[x(x - 1)][2c - 3x - 1]\,dx$ between $x = 0$ and $x = 1$. This gives $c = 5/4$, identical to the answers of Exercises 2 and 6.

10. The two integrals evaluate to

$$5/12 - a/3 - b/6 = 0 \text{ and}$$

$$7/30 - a/6 - 2b/15 = 0.$$

From these, $a = 1$, $b = 1/2$, giving the analytical solution:

$$u(x) = x^3/2 + x^2/2 - x.$$

11a. $N_L = (0.45 - x)/0.12$, $N_R = (x - 0.33)/0.12$.

b. $(1/0.12)\int(0.45 - x)(u'' + u \sin(x) - x^2 - 2)\,dx$, limits [0.33,0.45], and $(1/0.12)\int(x - 0.33)(u'' + u \sin(x) - x^2 - 2)\,dx$, same limits.

c. Equations are

$$8.3181\ c_L - 8.3409\ c_R = -0.1291,$$

$$-8.3409\ c_L + 8.3181\ c_R = -0.1291.$$

d. $Q_{av} = \sin(0.39)$, F_{av} - 2.1521.

12a. For element between [0.21, 0.33]:

$N_L = (0.33 - x)/0.12$, $N_R = (x - 0.21)/0.12$.

For element between [0.45, 0.71]:

$N_L = (0.71 - x)/0.26$, $N_R = (x - 0.45)/0.26$.

b. Coefficients are

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

where $A = (1/0.12)\int(0.33 - x)\,R(x)\,dx$, limits [0.21, 0.33]

$B = (1/0.26)\int(0.71 - x)\,R(x)\,dx$, limits [0.45, 0.71]

$C = (1/0.12)\int(x - 0.21)\,R(x)\,dx$, limits [0.21, 0.33]

$D = (1/0.26)\int(x - 0.45)\,R(x)\,dx$, limits [0.45, 0.71]

$R(x) = u'' + u \sin(x) - x^2 - 2.$

(Exercise 12 continued)

c. $8.323c_L - 8.339c_R = -0.1244,$
$-8.339c_L + 8.323c_R = -0.1244.$
$3.799c_L - 3.870c_R = -0.3037,$
$-3.870c_L + 3.799c_R = -0.3037.$

d. $Q_{av} = 0.2667,\ F_{av} = 2.0729.$
$Q_{av} = 0.5480,\ F_{av} = 2.3364.$

13. Call the values at the nodes c_1, c_2, c_3, c_4. The system is

$$\begin{vmatrix} 8.3227 & -8.3387 & & \\ -8.3387 & 16.6408 & -8.3409 & \\ & -8.3409 & 12.1168 & -3.8699 \\ & & -3.8699 & 3.7987 \end{vmatrix} \begin{vmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{vmatrix} = \begin{vmatrix} -0.1244 \\ 0.2535 \\ 0.4328 \\ -0.3037 \end{vmatrix}.$$

14*

x:	1.0	1.2	1.5	1.75	2
u(x):	-1	-0.2307	0.9174	1.9197	3
anal:	-1	-0.2267	0.9167	1.9196	3

15.

x:	1.0	1.2	1.5	1.75	2
u(x):	-1.1281	-0.3425	0.8402	1.8791	3
anal:	-1	-0.2267	0.9167	1.9196	3

The solution should be identical to that of Exercise 14. The FE method is not very accurate with a derivative boundary when y(x) is steep near that boundary.

16. The errors range from 1.75E-4 to 1.36E-3. The average error is 41% as large as the average error in Exercise 14.

17. Multiply the matrices; the product is the identity matrix.

18a.

$$M^{-1} = \begin{vmatrix} 0.753 & -0.228 & 0.475 \\ 0.613 & -0.534 & -0.079 \\ -0.158 & 0.280 & -0.123 \end{vmatrix} \quad \{a\} = \begin{vmatrix} 2.534 \\ -8.170 \\ 3.945 \end{vmatrix},$$

N = (0.753 + 0.613x - 0.158y, -0.228 - 0.534x + 0.280y,
0.475 - 0.079x - 0.123y),
u(-1,0) = 10.704.

b.

$$M^{-1} = \begin{vmatrix} -0.333 & 0 & 1.333 \\ 0 & 0.022 & -0.022 \\ 0.033 & -0.011 & -0.022 \end{vmatrix} \quad \{a\} = \begin{vmatrix} 9.300 \\ -0.087 \\ 0.123 \end{vmatrix},$$

N = (-0.333 + 0.033y, 0.022x - 0.011y, 1.333 - 0.022x - 0.022y),
u(20,20) = 10.033.

c*

$$M^{-1} = \begin{vmatrix} -4.650 & 3.982 & 1.668 \\ 0 & -0.217 & 0.217 \\ 0.500 & -0.120 & -0.380 \end{vmatrix} \quad \{a\} = \begin{vmatrix} 405.16 \\ -32.17 \\ 9.30 \end{vmatrix},$$

N = (-4.650 + 0.500y, 3.982 - 0.217x - 0.120y,
1.668 + 0.217x - 0.380y),
u(10.6,9.6) = 153.44.

19. The sum of the elements in the top row of M^{-1} is NOT the area (it always equals 1). It is the sum of the elements in the top row of the matrix in Eq. (9.45) that equals twice the area.

a. Area = 5.71.
b. Area = 675.
c. Area = 4.60.

20* The augmented matrix is:

$$\begin{vmatrix} -974.54 & -488.12 & -488.72 & 1.738 \\ -488.12 & -975.41 & -487.85 & 1.738 \\ -488.72 & -487.85 & -974.81 & 1.738 \end{vmatrix}.$$

21. The is some ambiguity about what temperatures to assign to the corner nodes at the right end. One choice is to set these at 50°, the average of the temperatures on the adjacent edges. The alternative is to use a double node, with one of these paired nodes at 0°, the other at 100°. Results:

With nodes at the average temp.	2.007, 23.084. 60.076.
With a pair of nodes:	1.460, 16.788, 43.691.
Answers from Example 7.14:	1.289, 12.654, 53.177.

Neither choice gives close match but the second alternative is better.

22. The answers are the same as for Exercise 21.

23* The element equations are formed from:

$$c_{i,j} = 0.2825 \text{ if } i = j,\ 0.1412 \text{ if } i \neq j,$$

$$[K] = \frac{k*A}{c\rho}\begin{vmatrix} 0.489 & 0.089 & -0.573 \\ 0.089 & 0.196 & -0.285 \\ -0.573 & -0.285 & 0.857 \end{vmatrix}, \text{ (A is area, 1.695)}$$

$$b_i = 0.565\ F_{av}.$$

24. Equation (8.9) has no heat generation, so the element equations are

$$(1/\Delta t)[C]\{u\}^{m+1} = \{(1/\Delta t)[C] - (k/c\rho)[K]\}\{u\}^m + \{b\},$$

where

$$[C] = \frac{h_i}{2}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad [K] = \frac{k}{c\rho h_i}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix},$$

$$\{u\} = \begin{vmatrix} u_{i-1} \\ u_i \end{vmatrix} \quad \text{and } b = \begin{vmatrix} 0 \\ 0 \end{vmatrix}.$$

(When there are derivative end conditions, b is modified.)
These element equations are assembled in the usual way.

25. Number the nodes 0, 1, 2, ... and number the elements 1, 2, 3, ... starting from the left end. Consider element n with nodes n-1 and n. The element equation for node n comes from

$$\int N_n\, c_n\, dx = -\alpha\int N'_n\, N'_{n-1}\, c_{n-1}\, dx - \alpha\int N'_n\, N'_n\, c_n\, dx$$

and is $(h_n/2\Delta t)(c_n^{m+1} - c_n^{m}) = [(\alpha/h_n)\, c_{n-1} - (\alpha/h_n) c_n]^m$.
Element n+1 (that has nodes n and n+1) contributes another equation for node n:

$$(h_{n+1}/2\Delta t)(c_n^{m+1} - c_n^{m}) = [-(\alpha/h_{n+1}) c_n + (\alpha/h_{n+1}) c_{n+1}]^m.$$

If $h_n = h_{n+1} = h$, these assemble to give

$$(2h/2\Delta t)(c_n^{m+1} - c_n^{m}) = [(\alpha/h) c_{n-1} - 2(\alpha/h) c_n + (\alpha/h) c_{n+1}]^m.$$

Collecting terms gives an equation that matches that for the explicit method:

$$c_n^{m+1} = [r c_{n-1} + (1 - 2r) c_n + r c_{n+1}]^m.$$

When applied to solve the exercise, identical results are obtained as expected.

26, 27. Use a commercial FE program.

28. $(2 + r)\{c^{m+1}\} = (2 - r)\{c^m\} + 2r[K^{-1}]\{b\}$.

29. $(1 - r\theta)\{c^{m+1}\} = (1 - (1-\theta)\{c^m\} + r[K^{-1}]\{b\}$.

30. The element equations can be reduced to

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \{c\}^{m+1} = \begin{vmatrix} -2 & 8 \\ 8 & -2 \end{vmatrix} \{c\}^{m} - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \{c\}^{m-1}$$

After assembly, the equations form a tridiagonal matrix with all diagonal elements equal to 4 and all off-diagonal elements equal to 1. The right-hand sides are $4(y_{i-1} + y_{i+1})^m - 3(y_{i-1} + y_{i+1})^{m-1}$.

31. The element equations for $t = t_1$ are

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \{c\}^{1} = (1/2)\begin{vmatrix} -2 & 8 \\ 8 & -2 \end{vmatrix} \{c\}^{0}$$

32* This will always be true.

33, 34, 35, 36. Use a commercial FE program.

37, 38, 39, 40. Programs.