Lower bounds for lowest eigenvalues

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UNDER CONSTRUCTION
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Introduction

In this abortive paper, we will discuss how to find lower bounds for lowest eigenvalues. The aim will be to get across the basic ideas, so we will discuss only the simplest cases, and we won't worry about matters like exactly how many derivatives we are demanding of our functions.

The work described here is joint work with Bob Brooks and Bob Kohn.

Caveat. I seem to recall that I was changing the notation around when I turned to other things, so there may be some inconsistencies. Also, you'll note the the story breaks off abruptly near the end.

Continuous version

Consider a nice bounded domain Ω in the Euclidean plane. A positive number λ is an eigenvalue of the *positive* Laplacian

$$\Delta = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

if there exists a function u (other than the zero function) such that

$$\Delta u = \lambda u$$

and

$$u|\partial\Omega = 0.$$

We label the eigenvalues $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$. The lowest eigenvalue λ_0 is simple, and the corresponding eigenfunction u_0 does not change sign in the interior of Ω. By taking $u_0 \geq 0$ and $\int {u_0}^2 = 1$, we get a unique 'lowest eigenfunction' u_0.

The lowest eigenvalue λ_0 is the minimum of the Rayleigh quotient:

$$\begin{aligned} \lambda_0 &= \min_{u|\partial\Omega=0} \frac{\int |\nabla u|^2}{\int u^2} \\ &= \min_{\substack{u|\partial\Omega=0 \\ \int u^2=1}} \int |\nabla u|^2. \end{aligned}$$

These minima are attained when $u = u_0$, and since $u_0 \geq 0$ we can, if we wish, add the additional constraint $u \geq 0$ to the classes of functions over which we take these minima.

Theorem.

$$\begin{aligned} \lambda_0 &= \sup(\lambda | \exists \mathbf{v}, \nabla \cdot \mathbf{v} - |\mathbf{v}|^2 \geq \lambda) \\ &= \sup_{\mathbf{v}} \inf(\nabla \cdot \mathbf{v} - |\mathbf{v}|^2). \end{aligned}$$

Proof. Suppose

$$\nabla \cdot \mathbf{v} - |\mathbf{v}|^2 \geq \lambda.$$

Then

$$\begin{aligned} \lambda \int u^2 &\leq \int u^2(\nabla \cdot \mathbf{v}) - u^2|\mathbf{v}|^2 \\ &= \int -2u\nabla u \cdot \mathbf{v} - u^2|\mathbf{v}|^2 \\ &= \int |\nabla u|^2 - |\nabla u + u\mathbf{v}|^2 \\ &\leq \int |\nabla u|^2. \end{aligned}$$

Thus

$$\lambda_0 \geq \sup(\lambda | \exists \mathbf{v}, \nabla \cdot \mathbf{v} - |\mathbf{v}|^2 \geq \lambda).$$

On the other hand, if we put

$$\mathbf{v}_0 = -\frac{\nabla u_0}{u_0}$$

then

$$\nabla \cdot \mathbf{v}_0 = \frac{\Delta u_0}{u_0} + \frac{|\nabla u_0|^2}{{u_0}^2}$$

and

$$\begin{aligned} \nabla \cdot \mathbf{v}_0 - |\mathbf{v}_0|^2 &= \frac{\Delta u_0}{u_0} \\ &= \lambda_0. \end{aligned}$$

$\blacksquare$

Corollary. Let

$$\gamma_0 = \sup(\gamma | \exists \mathbf{w}, \nabla \cdot \mathbf{w} \geq \gamma, |\mathbf{w}| \leq 1).$$

Then

$$\lambda_0 \geq \frac{{\gamma_0}^2}{4}.$$

Proof. Suppose

$$\nabla \cdot \mathbf{w} \geq \gamma, |\mathbf{w}| \leq 1.$$

Let

$$\mathbf{v} = \frac{\gamma}{2}\mathbf{w}.$$

Then

$$\begin{aligned} \nabla \cdot \mathbf{v} - |\mathbf{v}|^2 &\geq \frac{\gamma}{2} \cdot \gamma - \left(\frac{\gamma}{2}\right)^2 \\ &= \frac{\gamma^2}{4}. \end{aligned}$$

$\blacksquare$

Corollary (Cheeger's inequality). Let

$$\chi_0 = \min_{D \subseteq \Omega} \frac{\text{length}(\partial D)}{\text{area}(D)}.$$

Then

$$\lambda_0 \geq \frac{{\chi_0}^2}{4}.$$

Proof. By the max-flow min-cut theorem, $\gamma_0 = \chi_0$.

$\blacksquare$

Generalized continuous version

Now introduce conductivity tensor σ and capacity function ρ: In this setting the Laplacian Δ becomes

$$\Delta u = -\frac{1}{\rho} \nabla \cdot (\sigma \nabla u)$$

and the eigenvalue equation becomes

$$-\nabla \cdot (\sigma \nabla u) = \lambda \rho u.$$

Now

$$\begin{aligned}
\lambda_0 &= \min_{u|\partial\Omega=0} \frac{\int \nabla u \cdot (\sigma \nabla u)}{\int \rho u^2} \\
&= \min_{\substack{u|\partial\Omega=0 \\ \int \rho u^2 = 1}} \int \nabla u \cdot (\sigma \nabla u)
\end{aligned}$$

and our theorem becomes:

Theorem.

$$\begin{aligned}
\lambda_0 &= \sup(\lambda | \exists \mathbf{v}, \nabla \cdot \mathbf{v} - \mathbf{v} \cdot \sigma^{-1} \mathbf{v} \geq \lambda \rho) \\
&= \sup_{\mathbf{v}} \inf(\frac{1}{\rho}(\nabla \cdot \mathbf{v} - \mathbf{v} \cdot \sigma^{-1} \mathbf{v})).
\end{aligned}$$

$\blacksquare$

In this case, the choice of $\mathbf{v}$ that achieves the supremum is

$$\mathbf{v}_0 = -\frac{\sigma \nabla u_0}{u_0}.$$

Discrete version

Let $\Omega = \partial\Omega \cup \mathrm{int}\Omega$ be a graph with adjacency matrix $C(x, y)$ and valence function

$$D(x) = \sum_y C(x, y).$$

A positive number λ is an eigenvalue of the discrete Laplacian Δ, where

$$\Delta u(x) = \frac{1}{D(x)} \sum_y C(x, y)(u(x) - u(y))$$

if

$$\Delta u = \lambda u$$

and

$$u|\partial\Omega = 0.$$

We label the eigenvalues $0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$. The lowest eigenvalue λ_0 is simple, and the corresponding eigenfunction u_0 does not change sign in the interior of Ω. By taking $u_0 \geq 0$ and $\sum_x u_0(x)^2 = 1$, we get a unique 'lowest eigenfunction' u_0.

The lowest eigenvalue λ_0 is the minimum of the Rayleigh quotient:

$$\begin{aligned}
\lambda_0 &= \min_{u|\partial\Omega=0} \frac{\frac{1}{2}\sum_{x,y} C(x, y)(u(x) - u(y))^2}{\sum_x D(x)u(x)^2} \\
&= \min_{\substack{u|\partial\Omega=0 \\ \sum_x u(x)^2=1}} \frac{1}{2}\sum_{x,y} C(x, y)(u(x) - u(y))^2.
\end{aligned}$$

These minima are attained when $u = u_0$, and since $u_0 \geq 0$ we can, if we wish, add the additional constraint $u \geq 0$ to the classes of functions over which we take these minima.

Theorem.

$$\begin{aligned}
\lambda_0 &= \sup\{\lambda | \exists \eta : \mathrm{int}\Omega \times \Omega \to [0,\infty), \ {}_{\sum_{y\in\Omega} C(x,y)(1-\eta(x,y)) \geq \lambda D(x)}^{y\in \mathrm{int}\Omega \Rightarrow \eta(x,y)\eta(y,x)\geq 1} \ \} \\
&= \sup_{\eta\in H} \inf_{x\in \mathrm{int}\Omega} \frac{1}{D(x)} \sum_{y\in\Omega} C(x,y)(1-\eta(x,y)),
\end{aligned}$$

where

$$H = \{\eta, \mathrm{int}\Omega \times \Omega \to [0,\infty) | y \in \mathrm{int}\Omega \Rightarrow \eta(x,y)\eta(y,x) \geq 1\}.$$

Proof. Suppose $\eta \in H$ and for every $x \in \mathrm{int}\Omega$,

$$\sum_y C(x,y)(1-\eta(x,y)) \geq \lambda.$$

For any u with $u|\partial\Omega = 0$,

$$\begin{aligned}
\lambda \sum_x u(x)^2 &\leq \sum_x u(x)^2 \cdot \sum_y C(x,y)(1-\eta(x,y)) \\
&= \sum_{x,y} C(x,y)u(x)^2 - C(x,y)\eta(x,y)u(x)^2 \\
&= \sum_{x,y} C(x,y)\frac{1}{2}\left[u(x)^2 + u(y)^2 - \eta(x,y)u(x)^2 - \eta(y,x)u(y)^2\right] \\
&\leq \sum_{x,y} C(x,y)\frac{1}{2}\left[u(x)^2 + u(y)^2 - \eta(x,y)u(x)^2 - \frac{1}{\eta(x,y)}u(y)^2\right] \\
&= \frac{1}{2}\sum_{x,y} C(x,y)\left[u(x)^2 + u(y)^2 - \left(\sqrt{\eta(x,y)}u(x) - \frac{1}{\sqrt{\eta(x,y)}}u(y)\right)^2 - 2u(x)u(y)\right] \\
&= \frac{1}{2}\sum_{x,y} C(x,y)\left[(u(x)-u(y))^2 - \left(\sqrt{\eta(x,y)}u(x) - \frac{1}{\sqrt{\eta(x,y)}}u(y)\right)^2\right] \\
&\leq \frac{1}{2}\sum_{x,y} C(x,y)(u(x)-u(y))^2.
\end{aligned}$$

(This isn't quite right, of course, since we've been too cavalier about what happens when $y \in \partial\Omega$, but not to worry ...)

On the other hand, if we set

$$\eta_0(x,y) = \frac{u_0(y)}{u_0(x)}$$

we have

$$\begin{aligned}
& \sum_y C(x,y)(1-\eta_0(x,y)) \\
= & \sum_y C(x,y)\left(1-\frac{u_0(y)}{u_0(x)}\right) \\
= & \frac{\Delta u_0(x)}{u_0(x)} \\
= & \lambda_0.
\end{aligned}$$

∎

If we think of setting

$$v(x,y) = C(x,y)(1-\eta(x,y)),$$

we can get a form of this principle that is in a certain sense the discrete limit of the continuous variational principle:

Corollary.

$$\lambda_0 = \sup\{\lambda | \exists v : \mathrm{int}\Omega\times\Omega \to \mathbf{R}, v(x,y) \leq C(x,y), \ {}_{\substack{\sum_{y\in\Omega} v(x,y) \geq \lambda D(x) \\ -v(x,y)-v(y,x)+v(x,y)v(y,x)/C(x,y)\geq 0}} \}$$

Proof. Set

$$\eta(x,y) = 1 - \frac{v(x,y)}{C(x,y)}.$$

∎

More later

We've run out of steam, and we haven't even covered the discrete Cheeger inequality yet. . . .