Kemeny's constant

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Sources

This work is derived in part from Grinstead and Snell's *Introduction to Probability, Second Edition.* Thanks to the the far-sightedness of the authors, and of their publisher the American Mathematical Society, this splendid work has now been made freely redistributable.

Two exercises on Kemeny's constant

Grinstead and Snell gave the following two problems as exercises:

Exercise 11.5.19 *Show that, for an ergodic Markov chain (see Theorem such-and-such),*

$$\sum_j m_{ij} w_j = \sum_j z_{jj} - 1 = K.$$

The second expression above shows that the number K is independent of i. The number K is called Kemeny's constant. *A prize was offered to the first person to give an intuitively plausible reason for the above sum to be independent of i. (See also Exercise 11.5.24.)*

Exercise 11.5.24 *Peter Doyle (private communication) has suggested the following interpretation for Kemeny's constant. We are given an ergodic chain and do not know the starting state. However, we would like to start watching it at a time when it can be considered to be in equilibrium (i.e., as*

*if we had started with the fixed vector w or as if we had waited a long time).
However, we don't know the starting state and we don't want to wait a long
time. Peter says to choose a state according to the fixed vector w. That is,
choose state j with probability w_j using a spinner, for example. Then wait
until the time T that this state occurs for the first time. We consider T as
our starting time and observe the chain from this time on. Of course the
probability that we start in state j is w_j, so we are starting in equilibrium.
Kemeny's constant is the expected value of T, and it is independent of the
way in which the chain was started. Should Peter have been given the prize?*

These two exercises have given readers some trouble, and with good reason. It is high time to dispel some of the mystery.

Notation

We quickly recall the notation used by Grinstead and Snell. We're dealing
with a regular Markov chain with transition matrix P. The row vector w is
the fixed vector of P:

$$wP = w.$$

The matrix $W = \lim_{n \to \infty} P^n$ has all rows equal to w.

M is the mean first passage matrix. The off-diagonal entries m_{ij} tell the
mean number of steps to get to j, starting from i; the diagonal entries M_{ii}
are zero. It is easy to see that M satisfies the fundamental relation

$$(I - P)M = C - D.$$

Here C is a matrix of all 1's; D is a matrix with all entries 0 except the
diagonal entries $d_{ii} = r_i$ telling the mean time starting from i to *return* to
i. These mean recurrence times r_i are obviously (and easily proven to be)
related to the equilibrium probabilities w_i:

$$r_i = \frac{1}{w_i}.$$

The matrix Z is the so-called *fundamental matrix* $Z = (I - P + W)^{-1}$.
Its basic properties are

$$(I - P)Z = Z(I - P) = I - W;$$

$$WZ = W;$$

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$$ZC = C.$$

According to Grinstead and Snell's Theorem such-and-such, we can compute the mean first passage matrix M from Z:

$$m_{ij} = \frac{z_{jj} - z_{ij}}{w_j},$$

or equivalently,

$$m_{ij} w_j = z_{jj} - z_{ij}.$$

Grinstead and Snell prove this relation with a blizzard of formulas. It is possible to give a human-readable derivation of this formula, by developing some of the notions of potential theory for Markov chains. We won't bother with this, because we will find that we can get along quite well without this formula, and indeed without Z altogether.

Problem 19

Summing the relation

$$m_{ij} w_j = z_{jj} - z_{ij}$$

yields

$$
\begin{aligned}
& \sum_j m_{ij} w_j \\
= & \sum_j z_{jj} - z_{ij} \\
= & \sum_j z_{jj} - \sum_j z_{ij} \\
= & \sum_j z_{jj} - 1 \\
= & K,
\end{aligned}
$$

where we have used the fact that the rows of Z sum to 1 (expressed above in the equation $ZC = C$.

This solves the problem as stated, and this is clearly the solution Grinstead and Snell had in mind. However, as they indicate, this derivation leaves open the question of why the quantity $\sum_j m_{ij} w_j$ should be independent of i. They say that 'A prize was offered to the first person to give an intuitively plausible reason for the above sum to be independent of i', and refer us to Problem 24.

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Problem 24

Let's let $K_i = \sum_j m_{ij} w_j$. Problem 24 proposes to interpret K_i as what we could call the 'expected time to equilibrium': We pick a state j at random according to the equilibrium probability distribution w, and ask for the expected time to get to j from the starting state i. Some attempt is made to show why we might be interested in the expected time to equilbrium, and we are then asked whether Peter Doyle, who supposedly proposed this interpretation of the quantity $\sum_j m_{ij} w_j$, should have been given the prize described in Problem 19.

Looking back at Problem 19, we see that a prize was offered *not* for an intuitive interpretation for Kemeny's constant, but for an intuitively plausible reason for K_i to be constant. So far we've seen nothing approaching an argument for why the expected time to equilibrium should be independent of the starting state i, so it's hard to see why Peter should have been given the prize.

On the other hand, we may note that this question does not ask whether Peter should have been given the prize on the basis of this interpretation, but just whether he should have been given the prize. Did Peter offer an intuitively plausible reason for K_i to be constant, somehow related to its being the expected time to equilibrium? How can we decide this question on the basis of the information we've been given? Grinstead and Snell are not in the habit of leaving us high and dry, so the most likely explanation is that there is a simple and intuitively plausible reason for the constancy of K_i, immediately related to the interpretation as the time to equilibrium.

So let's just think about this for a second. We're starting from i, and heading for the random state j. 'A journey of a thousand miles begins with a single step.' This old saw is the basis for the formula

$$(I - P)M = C - D$$

discussed above. (Indeed, it is the basis for all of potential theory, the general mathematical theory to which the computation of hitting times naturally belongs.)

Now after taking one step from i, the expected time to equilibrium will be $\sum_k P_{ik} K_k$, so the expected time K_i to equilibrium starting at i would seem to be $1 + \sum_k P_{ik} K_k$: That's 1 for the first step, plus $\sum_k P_{ik} K_k$ to get to equilibrium from wherever we are after the first step. However, *maybe we were already in equilibrium when we started!* If the randomly selected target

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state j happened to coincide with the starting state i, then that first step was a mistake. The probability of making this mistake is w_i, and if we make it, the expected extra time it will cost is the mean recurrence time $r_i = \frac{1}{w_i}$. The expected cost attributable to the possibility of abandoning equilibrium is thus $w_i r_i = w_i \frac{1}{w_i} = 1$, which just cancels the benefit of taking one step of our journey:

$$K_i = 1 + \sum_k P_{ik} K_k - w_i r_i = \sum_k P_{ik} K_k.$$

This equation says that the quantities K_i have the *averaging property*: The value at any state is the same as the average value at its neighbors, where the average is taken according to the matrix P. This means that the numbers K_i must all be the same, by a fundamental principle of potential theory called the *maximum principle*. The proof of the maximum principle involves looking at a state i where the quantity K_i takes on its maximum value, and noting that this maximum value must be attained also at any other state k for which $P_{ik} > 0$. Eventually, you find (for an ergodic chain), that the maximum must be attained everywhere, i.e., the quantity is constant (independent of state). Actually, we don't really need to drag in the maximum principle here, because if we write the quantities K_i as a column vector v, the equation above says that $Pv = v$, and we're supposed to know already that the only column vectors fixed by P are constant vectors. (Of course the proof of this was via the maximum principle, in spirit if not in name!)

To recapitulate, the expected time to equilbrium has the averaging property, because moving to a neighbor take you one step closer to equilibrium—except if you were already in equilibrium, which happens with probability w_i and costs you $r_i = \frac{1}{w_i}$. Since the expected time to equilibrium has the averaging property, it is constant.

This argument makes the constancy of K_i not just intuitively plausible, but intuitively obvious. Furthermore, we were led immediately to this argument from the interpretation of K_i as the expected time to equilibrium, as a consequence of the 'thousand-miles/single-step' principle. Doubtless Peter Doyle proposed the 'time to equilibrium' interpretation in conjunction with this argument, and thus should have been awarded the prize. Turning to the answer key for confirmation, we find the terse phrase, 'He got it!'

Yikes! You mean the prize for explaining Kemeny's constant has already been paid out? Sad, but true. If you look carefully at Problem 19, you will observe that it states that 'a prize *was* offered' (emphasis added), rather than 'a prize *is* offered'....

Final words

Let's close with a few more observations about Kemeny's constant.

First, let us observe that there is a simple computational proof of the constancy of the expected time to equilibrium, which parallels the intuitive argument above. This proof uses nothing beyond the fundamental equation

$$(I - P)M = C - D,$$

which as noted above is the result of applying the 'thousand-miles/single-step' principle to the mean first passage matris M. The quantities K_i we've been discussing are just the entries of the column vector Mw^t, where w^t is the transpose of the row vector w. Multiplying the fundamental equation above by w^t gives

$$(I - P)Mw^t = Cw^t - Dw^t.$$

But $Cw^t = \mathbf{1}$, because w is a probability vector, and $Dw^t = \mathbf{1}$, because $r_i = \frac{1}{w_i}$, so

$$(I - P)Mw^t = \mathbf{1} - \mathbf{1} = \mathbf{0}.$$

But the only vectors v for which $(I - P)v = \mathbf{0}$ are constant vectors, q.e.d. Note how all the same considerations of our intuitive argument are showing up in these more formal manipulations.

Another important point to consider is that while the matrix Z is used to compute the value of the constant K, it plays no role in our proof that the expected time to equilibrium is constant. Of course once we have shown that the time to equilibrium is constant, we may well ask what its value is. To answer this, we'll need to use the matrix Z, or some close relative. The approach taken in Problem 19 leaves much to be desired, because it depends on the formula of Theorem such-and-such, whose derivation by Grinstead and Snell appears more 'alphabetical' than 'mathematical'. As noted above, a more humane approach is possible, based on developing the basic notions of potential theory, but we're not going to undertake that here.

One last thing. When Laurie Snell mailed Peter Doyle the prize for Kemeny's constant, he first made the mistake of trying to send a $50 bill by mail. That first letter never arrived. You have probably heard before that you should not send cash through the mail. Let this be a further lesson!