Trees and expectations

TSILB*

Version 0.91, 10 March 2003

Notation for probability trees

Elementary probability theory is hampered by the lack of a good linear notation for probability trees. Here we propose the $\rightarrow$ notation, based on representing a probability distribution with outcomes $x_1, \ldots, x_n$ and associated probabilities $p_1, \ldots, p_n$ as

$$\{p_1 \rightarrow x_1, \ldots, p_n \rightarrow x_n\}.$$

Definition. A *probability tree* is defined recursively as either an outcome (assumed distinguishable as such), or an expression of the form

$$\{p_1 \rightarrow T_1, \ldots, p_n \rightarrow T_n\}$$

where $T_1, \ldots, T_n$ are probability trees, and $p_1, \ldots, p_n$ are non-negative real numbers adding to 1.

Example: I bet a dollar on the toss of a fair coin:

$$\{\frac{1}{2} \rightarrow 1, \frac{1}{2} \rightarrow -1\}.$$

Example: I am playing a best-of-3 coin-tossing tournament. The prize is a, and I have already won one toss:

$$\{\frac{1}{2} \rightarrow a, \frac{1}{2} \rightarrow \{\frac{1}{2} \rightarrow a, \frac{1}{2} \rightarrow 0\}\}.$$

*This Space Intentionally Left Blank. Contributors include: Peter Doyle.

This is assuming that we stop if I win the second toss. If we really play best-of-3, meaning we play out all 3 tosses no matter what, then the payoff is

$$\{\frac{1}{2} \to \{\frac{1}{2} \to a, \frac{1}{2} \to a\}, \frac{1}{2} \to \{\frac{1}{2} \to a, \frac{1}{2} \to 0\}\}.$$

These two situations are equivalent, and we want some transformation rules that will allow us to convert between the two representations.

Transformation rules

The most basic transformation rules are those that allow us to permute comma-separated terms, e.g.

$$\{p_1 \to x_1, p_2 \to x_2\} \longleftrightarrow \{p_2 \to x_2, p_1 \to x_1\}.$$

We won't bother to spell out these rules formally.

In the following rules, $T, T_1, \ldots$ are metavariables representing probability trees.

Certainty:

$$\{1 \to T\} \longleftrightarrow T.$$

Pruning:

$$\{0 \to p_1, p_2 \to T_2, \ldots\} \longleftrightarrow \{p_2 \to T_2, \ldots\}.$$

Identical outcomes:

$$\{p_1 \to T, p_2 \to T, p_3 \to T_3, \ldots\} \longleftrightarrow \{(p_1 + p_2) \to T, p_3 \to T_3, \ldots\}$$

Conditioning:

$$p \to \{q_1 \to T_1, \ldots, q_n \to T_n\} \longleftrightarrow \{pq_1 \to T_1, \ldots, pq_n \to T_n\}.$$

These rules are sufficient to allow us to transform any gamble into *standard form*

$$\{p_1 \to x_1, \ldots, p_n \to x_n\},$$

where the outcomes $x_1, \ldots x_n$ are all distinct, and the probabilities $p_1, \ldots, p_n$ are all positive (and ≤ 1, since their sum is necessarily 1).

The expected value of a gamble

A *gamble* is a probability tree whose outcomes are all real numbers, representing payments. A gamble is *rational* if all branch probabilities are rational numbers. We call the gamble x, where $x \in \mathrm{R}$, a *constant gamble*, or an *ungamble*.

The *expected value* of a gamble is defined recursively as follows:

$$\mathrm{Val}(x) = x, \quad x \in \mathrm{R};$$

$$\mathrm{Val}(\{p_1 \to T_1, \ldots, p_n \to T_n\}) = p_1 \mathrm{Val}(T_1) + \ldots + p_n Val(T_n).$$

We wish to interpret $\mathrm{Val}(T)$ as the fair market value of the gamble T. To justify this, we are going to introduce one additional transformation rule, which changes a gamble only to the extent of incorporating a manifestly fair side bet.

Side bet:

$$\{p \to a, p \to b, p_3 \to T_3, \ldots, p_n \to T_n\} \longleftrightarrow \{p \to a-c, p \to b+c, p_3 \to T_3, \ldots, p_n \to T_n\}.$$

Like the previously introduced transformation rules, this rule preserves $\mathrm{Val}(T)$. With its help we may transform any rational gamble T to (or from) an ungamble, which will necessarily be the constant gamble $\mathrm{Val}(T)$, the 'gamble' that always pays $\mathrm{Val}(T)$. Indeed, we can first transform T to normal form, explode it into a normal form gamble where all probabilities are equal, use side bets to make all outcomes equal, and finally consolidate to an ungamble. We illustrate the procedure for Huygens's example of a gamble where you have two chance to win 8, and three chances to win 13.

$$\begin{aligned}
& \{\tfrac{2}{5} \to 8, \tfrac{3}{5} \to 13\} \\
\longleftrightarrow \quad & \{\tfrac{1}{5} \to 8, \tfrac{1}{5} \to 8, \tfrac{1}{5} \to 13, \tfrac{1}{5} \to 13, \tfrac{1}{5} \to 13\} \\
\longleftrightarrow \quad & \{\tfrac{1}{5} \to 8+8, \tfrac{1}{5} \to 0, \tfrac{1}{5} \to 13, \tfrac{1}{5} \to 13, \tfrac{1}{5} \to 13\} \\
\longleftrightarrow \quad & \{\tfrac{1}{5} \to 8+8+13, \tfrac{1}{5} \to 0, \tfrac{1}{5} \to 0, \tfrac{1}{5} \to 13, \tfrac{1}{5} \to 13\} \\
\longleftrightarrow \quad & \{\tfrac{1}{5} \to 8+8+13+13, \tfrac{1}{5} \to 0, \tfrac{1}{5} \to 0, \tfrac{1}{5} \to 0, \tfrac{1}{5} \to 13\}
\end{aligned}$$

$$
\begin{aligned}
&\longleftrightarrow && \{\frac{1}{5} \to 8+8+13+13+13, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 0\} \\
&= && \{\frac{1}{5} \to 55, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 0\} \\
&\longleftrightarrow && \{\frac{1}{5} \to 44, \frac{1}{5} \to 11, \frac{1}{5} \to 0, \frac{1}{5} \to 0, \frac{1}{5} \to 0\} \\
&\longleftrightarrow && \{\frac{1}{5} \to 33, \frac{1}{5} \to 11, \frac{1}{5} \to 11, \frac{1}{5} \to 0, \frac{1}{5} \to 0\} \\
&\longleftrightarrow && \{\frac{1}{5} \to 22, \frac{1}{5} \to 11, \frac{1}{5} \to 11, \frac{1}{5} \to 11, \frac{1}{5} \to 0\} \\
&\longleftrightarrow && \{\frac{1}{5} \to 11, \frac{1}{5} \to 11, \frac{1}{5} \to 11, \frac{1}{5} \to 11, \frac{1}{5} \to 11\} \\
&\longleftrightarrow && \{1 \to 11\} \\
&\longleftrightarrow && 11
\end{aligned}
$$

This is not the most efficient method of reducing this gamble to an ungamble. What we did was to first convert the gamble to a *lottery*

$$\{1/n \to nc, 1/n \to 0, \ldots, 1/n \to 0\},$$

and then convert the lottery to an ungamble. This was done in honor of Huygens, who in addition to side bets allowed an additional transformation:

Lottery:

$$c \longleftrightarrow \{1/n \to nc, 1/n \to 0, \ldots, 1/n \to 0\}$$

In fact this transformation is redundant: As we have seen, it can be accomplished by means of side bets.

Sources

This work is based on earlier works copyright 2003 Peter G. Doyle.