The match set of a random permutation has the FKG property

By Peter C. Fishburn, Peter G. Doyle and L. A. Shepp

AT&T Bell Laboratories

15 March 1987

Abstract

We prove a conjecture of Kumar Joag-Dev and Prem Goel that if $M = M(\sigma) = \{i : \sigma(i) = i\}$ is the (random) match set, or set of fixed points, of a random permutation σ of $1, 2, \ldots, n$ then $f(M)$ and $g(M)$ are correlated whenever f and g are increasing real-valued set functions on $2^{\{1,\ldots,n\}}$, i.e., $Ef(M)g(M) \geq Ef(M)Eg(M)$. No simple use of the FKG or Ahlswede-Daykin inequality seems to establish this, despite the fact that the FKG hypothesis is "almost" satisfied. Instead we reduce to the case where f and g take values in $\{0, 1\}$, and make a case-by-case argument: Depending on the specific form of f and g, we move the probability weights around so as to make them satisfy the FKG or Ahlswede-Daykin hypotheses, without disturbing the expectations Ef, Eg, Efg. This approach extends the methodology by which FKG-style correlation inequalities can be proved.

American Mathematical Society 1980 subject classifications. Primary 60B15, 60E15; secondary 06A10, 06D99.

Key words and phrases. Random permutations, fixed points, FKG inequality, Ahlswede-Daykin inequality, correlated functions.

Running head. Match sets of random permutations.

1 Introduction.

Pick a permutation of $\{1, 2, \ldots, 10\}$ at random, and look at its match set (set of fixed points). The probability that the match set contains at least one odd-numbered element is $1458120/3628800 = .40181\ldots$. The conditional probability that the match set contains at least one odd-numbered element, given that it contains at least one even-numbered element, is $622401/1458120 = .42685\ldots > .40181\ldots$. Thus knowing that the match set is big in the sense that it contains an even-numbered element makes it more likely that it is big in the sense that it contains an odd-numbered element.

More generally, we will show in this paper that for a random permutation of $\mathbf{n} = \{1, \ldots, n\}$, any two reasonable definitions of what it means for the match set to be big are positively correlated: Knowing that the match set is big in the first sense makes it more likely (or rather, no less likely) that it is big in the second sense. A probability distribution on $2^{\mathbf{n}}$ for which any two notions of bigness are positively correlated is said to have the FKG property (after Fortuin, Kasteleyn, and Ginibre, 1971). Thus our result says that the distribution of the match set of a random permutation of $\mathbf{n}$ has the FKG property. This result was originally conjectured by Joag-Dev (1985) and Prem Goel, and proven by Joag-Dev for $n \leq 6$. For other correlation inequalities of a similar kind, see Ahlswede and Daykin (1978), Shepp (1980, 1982), Fishburn (1984) and Hwang and Shepp (1987).

To formulate our result precisely, call a set $\mathcal{A} \subseteq 2^{\mathbf{n}}$ an *up-set* if $A \in \mathcal{A}$, $B \supseteq A$ implies $B \in \mathcal{A}$. Let σ be a random permutation uniformly distributed over all permutations of $\mathbf{n}$, and let $P(A)$ be the probability that $M(\sigma) \stackrel{\text{def}}{=} \{i : \sigma(i) = i\} = A$. As usual, for any set $\mathcal{A} \subseteq 2^{\mathbf{n}}$ let $P(\mathcal{A}) = \sum_{A \in \mathcal{A}} P(A)$.

Theorem. *For any pair $\mathcal{A}, \mathcal{B}$ of up-sets,*

$$P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A} \cap \mathcal{B}). \tag{1.1}$$

Note that as long as $\mathcal{A}$ is nonempty, (1.1) is equivalent to $P(\mathcal{B}) \leq P(\mathcal{B}|\mathcal{A})$.

Corollary. *For all real-valued nondecreasing functions f, g on $2^{\mathbf{n}}$,*

$$Ef(M)g(M) \geq Ef(M)Eg(M). \tag{1.2}$$

Here f is nondecreasing means $f(A) \leq f(B)$ if $A \subseteq B$.

Proof of the corollary. Assume without loss of generality (*wlog*) that $f(\emptyset) = g(\emptyset) = 0$, $f(\mathbf{n}) = g(\mathbf{n}) = 1$. Then f and g are finite positive linear combinations of nondecreasing functions from $2^{\mathbf{n}}$ into $\{0, 1\}$ (constructed sequentially up the lattice), and hence are convex combinations of indicator functions of up-sets. The corollary follows from applications of (1.1). ♣

As was noted by Joag-Dev, the theorem is almost a consequence of the FKG theorem (Fortuin, Kasteleyn and Ginibre, 1971):

FKG Theorem. *If μ is a probability measure on $2^{\mathbf{n}}$ satisfying*

$$\mu(A)\mu(B) \leq \mu(A \cup B)\mu(A \cap B) \text{ for all } A, B \subseteq \mathbf{n} \tag{1.3}$$

then (1.1) holds for any pair of up-sets $\mathcal{A}, \mathcal{B}$.

Unfortunately, if we set $\mu = P$, we find that the FKG hypothesis fails to hold when $|A \cup B| = n-1 > \max(|A|, |B|)$. The problem is that a permutation can't fix $n-1$ of the points without fixing the remaining point. However, the condition is satisfied for all other pairs A, B, so it seems as if the distribution is trying as hard as it possibly can to satisfy the FKG hypothesis.

For a measure μ for which $\mu(A)$ depends only on $|A|$, the FKG hypothesis is equivalent to the condition

$$\mu(\{1, \ldots, k\})^2 \leq \mu(\{1, \ldots, k-1\})\mu(\{1, \ldots, k+1\}) \text{ for } k = 2, \ldots, n-1. \tag{1.4}$$

If we set $\mu = P$, this equivalent condition is satisfied except for $k = n - 1$. Thus in a sense the FKG hypothesis only fails in one spot.

Now you might think that since the measure P comes so close to satisfying the FKG hypothesis, it should be possible to twiddle the problem into a form where the FKG theorem would apply. For example, while we have formulated the FKG theorem only for the lattice $2^{\mathbf{n}}$, it applies to an arbitrary distributive lattice. Maybe we could transfer our problem to another distributive lattice and apply FKG there, as was done in Shepp (1980). We tried this approach, but we couldn't make it work.

Another idea is to stick with the lattice $2^{\mathbf{n}}$, but to move the measure around. To see how this might work, define a new measure P^*,

$$P^*(A) = \begin{cases} P(\mathbf{n}) - \alpha, & A = \mathbf{n} \\ \alpha/n, & |A| = n-1 \\ P(A) & \text{otherwise.} \end{cases} \tag{1.5}$$

Because P assigns measure 0 to sets A with $|A| = n-1$, we may assume that $\mathcal{A}$ and $\mathcal{B}$ each contain all subsets of $\mathbf{n}$ of size $n-1$. But then

$$P^*(\mathcal{A}) = P(\mathcal{A}),\ P^*(\mathcal{B}) = P(\mathcal{B}),\ P^*(\mathcal{A}\cap\mathcal{B}) = P(\mathcal{A}\cap\mathcal{B}), \tag{1.6}$$

so if we could choose α so as to get P^* to satisfy (1.4), we would be all set. Unfortunately, this can't be done.

Now there is good news, and bad news. The good news is that an argument based on this reallocation idea can be made to work. The bad news is that the argument depends on a case-by-case analysis of a zillion different possibilities for the pair $\mathcal{A}, \mathcal{B}$. In each case, we move the weights around until we get things into a form where we can apply the FKG theorem (or the Ahlswede-Daykin theorem, which is a strengthened form of the FKG theorem).

As an indication that a case-by-case argument may be needed, consider the following example.

Example. *Let*

$$\mathcal{A} = \{1\}^+, \mathcal{B} = \{B : 1 \notin B, |B| = n-2\} \cup \{B : |B| \geq n-1\}. \tag{1.7}$$

Then $P(\mathcal{A}) = 1/n$, $P(\mathcal{B}) = \left[\binom{n-1}{n-2} + 1\right]/n! = 1/(n-1)!$ *and* $P(\mathcal{A} \cap \mathcal{B}) = P(\mathbf{n}) = 1/n!$ *so equality holds in (1.1).*

This example shows that there isn't much slack in the inequalities we are trying to prove, whereas inequalities proven by appealing directly to the FKG theorem tend to have some slack in them.

The method of moving mass around in a way that depends on the particular form of the up-sets provides a new methodology for proving correlation inequalities. Certainly this method is something of a cop-out. By using it to prove Joag-Dev's conjecture, we seem to be saying that the conjecture is true because the distribution of the match set of a random permutation nearly satisfies the FKG hypothesis. Perhaps this seems like too frivolous a reason. Perhaps you would prefer a short, slick proof based on some nice property of permutations. So would we. But even if it turns out that Joag-Dev's conjecture is true for some really good reason, there must be situations where the FKG property holds for no better reason than that the distributions involved nearly satisfy the FKG hypothesis. In such situations, the methodology we have developed here may be the only way to go.

2 Preliminaries.

We use the following notations and definitions, some of which were already introduced in §1.

- $\mathbf{n} = \{1, 2, \ldots, n\}$.
- $2^{\mathbf{n}}$ is the set of all subsets of $\mathbf{n}$.

- $\mathcal{A} \subseteq 2^{\mathbf{n}}$ is an up-set if $A \in \mathcal{A}$ and $B \supseteq A$ implies $B \in \mathcal{A}$.

- $\mathcal{A} \wedge \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

- $\mathcal{A} \vee \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$.

- N. B. If $\mathcal{A}$ and $\mathcal{B}$ are up-sets then $\mathcal{A} \vee \mathcal{B} = \mathcal{A} \cap \mathcal{B}$.

- A *base* of an up-set is a minimal set in the up-set.

- The empty up-set, denoted by $\emptyset$, has no base.

- If $A \in 2^{\mathbf{n}}$, $A^+ \stackrel{\text{def}}{=} \{B \in 2^{\mathbf{n}} : B \supseteq A\}$ has the single base A.

- N. B. $\emptyset^+ = 2^{\mathbf{n}}$; $\mathbf{n}^+ = \{\mathbf{n}\}$; if $\mathcal{A}$ is a non-empty up-set whose bases are $A_1, \ldots, A_n$ then $\mathcal{A} = \bigcup_i A_i^+$.

- The *match set* $M(\sigma)$ of a permutation σ is the set of fixed points of σ, i.e. $M(\sigma) = \{i : \sigma(i) = i\}$.

- The probability that a given $A \in 2^{\mathbf{n}}$ is the match set of a random permutation is $P(A) = T(A)/n!$, where $T(A) = |\{\sigma : M(\sigma) = A\}|$.

- For any $\mathcal{A} \subseteq 2^{\mathbf{n}}$ we define $P(\mathcal{A})$ by $P(\mathcal{A}) = \sum_{A \in \mathcal{A}} P(A)$ so $P(\mathcal{A})$ is the probability that the match set of a random permutation lies in $\mathcal{A}$.

Since $T(A)$ depends only on $|A|$, and since the classical formula for $T(A)$ is most simply written in terms of $n - |A|$, we define

$$T_i = T(A), \text{ where } |A| = n - i, \; 0 \le i \le n \tag{2.1}$$

i	T_i	$i!/e$
0	1	$.367\ldots$
1	0	$.367\ldots$
2	1	$.753\ldots$
3	2	$2.207\ldots$
4	9	$8.829\ldots$
5	44	$44.145\ldots$
6	265	$264.873\ldots$
7	1854	$1854.112\ldots$
8	14833	$14832.899\ldots$
9	133496	$133496.091\ldots$

Table 1: Values of T_i.

so that $T_0 = 1$, $T_1 = 0$, $\sum \binom{n}{i} T_i = n!$ and as is well known from an inclusion-exclusion argument (Feller, 1968),

(2.2)
$$T_i = i! \sum_{j=0}^{i} (-1)^j \frac{1}{j!}.$$

This leads to the important recurrence

(2.3)
$$T_{i+1} = (i+1)T_i + (-1)^{i+1}, i \geq 0.$$

The first few values of T_i are shown in Table 1. It is easy to show that

(2.4)
$$T_i^2 < T_{i-1}T_{i+1} \text{ for } i \geq 3,$$

and

(2.5)
$$T_i \sim i!/e, i \to \infty.$$

We remark that the asymptotic relation is remarkably accurate even for small i, as can be seen from Table 1.

3 Proof of the theorem.

Without loss of generality, we assume henceforth that $\mathcal{A}$ and $\mathcal{B}$ are up-sets, that each contains all A with $|A| = n-1$, and that neither one is $2^{\mathbf{n}}$. We will also assume that (1.1) is true for the first few n, say $n \leq 5$.

The proof divides into two cases:

Lemma 1. *(1.1) holds if* $\{i\} \in \mathcal{A} \cup \mathcal{B}$ *for some* $i \in \mathbf{n}$, *i.e., if there is a singleton set either in* $\mathcal{A}$ *or in* $\mathcal{B}$.

Lemma 2. *(1.1) holds if* $\min\{|A| : A \in \mathcal{A} \cup \mathcal{B}\} \geq 2$.

These lemmas are proved by somewhat different methods. The proof of Lemma 1 uses the FKG inequality and a matching argument in which each $B \in \mathcal{B} \setminus \mathcal{A}$ with $|B| \leq n-3$ is paired with $B \cup \{1\} \in \mathcal{A} \cap \mathcal{B}$ under the hypothesis that $\{1\} \in \mathcal{A}$. The proof of Lemma 2 is based on the Ahlswede-Daykin theorem. Both use the idea of redefining the measure.

The cores of our proofs of Lemmas 1 and 2 do not cover various special cases of small k and n; these special cases have to be considered separately. Certain of these cases are isolated in Lemmas 3 and 4, which we will prove before attacking Lemmas 1 and 2; the rest of the cases will be cleaned up afterwards.

Lemma 3. *(1.1) holds if* $\min\{|A| : A \in \mathcal{A} \cap \mathcal{B}\} = n-k$ *and* $2n \leq (k-1)!$.

Lemma 4. *Suppose* $\min\{|A| : A \in \mathcal{A} \cap \mathcal{B}\} = n-k$. *Then (1.1) holds if any of the following holds:*

(3.1) $$k = 3 \text{ and } n \leq 4; \quad k = 4 \text{ and } n \leq 9;$$

$$k = 5 \text{ and } n \le 25; \quad k = 6 \text{ and } n \le 101;$$

$$k = 7 \text{ and } n \le 532; \quad k = 8 \text{ and } n \le 2715.$$

The order of events is as follows: proof of Lemma 3; proof of Lemma 4; core proof of Lemma 1; core proof of Lemma 2; remnants of Lemma 1; remnants of Lemma 2.

4 Proof of Lemma 3.

We are to prove that (1.1) holds if $2n \le (k-1)!$ where k is defined by

$$n - k = \min\{|A| : A \in \mathcal{A} \cap \mathcal{B}\}. \tag{4.1}$$

We will use the FKG theorem but since $P(A) = 0$ for $|A| = n-1$, P does not satisfy the FKG hypothesis. We want to move some of the mass of P from sets $A \in \mathcal{A} \cap \mathcal{B}$ with $|A| \ne n-1$ to sets A with $|A| = n-1$ and get a new measure P^* that satisfies the FKG hypothesis:

$$P^*(A)P^*(B) \le P^*(A \cap B)P^*(A \cup B), \text{ for all } A, B \in 2^{\mathbf{n}}. \tag{4.2}$$

The FKG theorem will then yield

$$P(\mathcal{A})P(\mathcal{B}) = P^*(\mathcal{A})P^*(\mathcal{B}) \le P^*(\mathcal{A} \cap \mathcal{B}) = P(\mathcal{A} \cap \mathcal{B}). \tag{4.3}$$

The following simple lemma makes it easy to verify the FKG hypothesis for P^* when $P^*(A)$ is nearly a function of $|A|$:

Lemma 3A. *Suppose* $u : 2^{\mathbf{n}} \to \mathbf{R}^+$. *Let*

$$u_i = \min_{|A|=i} u(A); \ U_i = \max_{|A|=i} u(A). \tag{4.4}$$

If

$$U_i^2 \le u_{i-1}u_{i+1} \text{ for } i = 2, \ldots, n-1 \tag{4.5}$$

then

$$u(A)u(B) \le u(A\cup B)u(A\cap B) \text{ for all } A, B \in 2^{\mathbf{n}}. \tag{4.6}$$

Proof. Since $|A|+|B| = |A\cup B|+|A\cap B|$, the conclusion holds whenever $|A| = |B|$ and $|A\cup B| = |A|+1$. A simple induction argument with the observation that $u_i \le U_i$ completes the proof. ♣

Proof of Lemma 3. Pick a set $A_0 \in \mathcal{A}\cap\mathcal{B}$ with $|A_0| = n-k$. Set $P^*(A) = T^*(A)/n!$, where T^* is obtained from T by taking $1/2$ away from $T(\mathbf{n})$ and $(n-1)/2$ away from $T(A_0)$ and using the total of $n/2$ thus obtained to make $T(A) = 1/2$ for $|A| = n-1$. Thus

$$T^*(A) = \begin{cases} 1/2, & |A| \ge n-1 \\ T_k - (n-1)/2, & A = A_0 \\ T_{n-|A|} & \text{otherwise.} \end{cases} \tag{4.7}$$

From (2.4) we have

$$T_i^2 < T_{i-1}T_{i+1} \text{ for } i \ge 3, \tag{4.8}$$

and

$$U_2^{*2} = T_2^2 = 1 \le u_1^* u_3^* = \frac{1}{2}T_3 = 1, \tag{4.9}$$

so for (4.5) to hold it is only necessary to check that

$$T_{k-1}^2 \le T_{k-2}T^*(A_0);\ T_{k+1}^2 \le T^*(A_0)T_{k+2}. \tag{4.10}$$

Since the second inequality is more restrictive than the first when $k \ge 6$, these cases require

$$T^*(A_0) = T_k - \frac{n-1}{2} \ge \frac{T_{k+1}^2}{T_{k+2}}. \tag{4.11}$$

But this follows from the assumption that $2n \le (k-1)!$ when $k \ge 6$, as may easily be checked. This proves Lemma 3 for $k \ge 6$; the remaining cases $k \le 5$ are covered by Lemma 4. ♣

	i									
	0	1	2	3	4	5	6	7	8	9
T	1	0	1	2	9	44	265	1854	14833	133496
$T^{(1)}$	1/2	1/2	$[1/2, 1]$	2	$[8, 9]$	44	$[242, 265]$	1854	$[14204, 14833]$	133496
$T^{(2)}$	1/2	1/2	$[8/15, 1]$	2	$[7.5, 9]$	44	$\longrightarrow$ same as T			
$T^{(3)}$	1/2	1/2	$[1/2, 1]$	2	$[8.36, 9]$	44	$[231.74, 265]$	1854	$\longrightarrow$ same as T	

Table 2: Reallocations.

5 Proof of Lemma 4.

The idea here, as in the proof of Lemma 3, is to move the mass of the measure T around inside $\mathcal{A} \cap \mathcal{B}$ to get a new measure T^* satisfying the strong FKG hypothesis (4.5) of Lemma 3A. Table 2 illustrates three different strategies for doing this. As long as the values used fall within the intervals indicated on one of the three rows of the table, (4.5) will be satisfied. For instance, looking at the line of the table labelled $T^{(2)}$, we see that (4.5) will be satisfied as long as $T^*(\mathbf{n}) = \frac{1}{2}$; $T^*(A) = \frac{1}{2}, |A| = n-1$; $8/15 \leq T^*(A) \leq 1, |A| = n-2$; etc.

In order for one of these strategies to work in a particular case, it must be possible to reallocate the T weights so as to satisfy the constraints without changing the weights of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{A} \cap \mathcal{B}$. To insure that we do not change these weights, we will choose $A_0 \in \mathcal{A} \cap \mathcal{B}$ with $|A_0| = n-k$ and change only the weights of sets A for which $A \supseteq A_0$ or $|A| = n-1$. All of these strategies require us to increase $T(A) = 0$ to $T^*(A) = \frac{1}{2}$ for all $|A| = n-1$. This leaves us with a net increase of $\frac{n}{2}$, and the question is whether we can make up for this by decreases in the weights of the supersets $A \supseteq A_0$, $|A| \neq n-1$.

When $k = 5$, so that $T(A_0) = T_5$, we choose strategy $T^{(2)}$: A_0 has $\binom{5}{1} = 5$ supersets of size $n-4$, each of whose weight can be reduced by 1.5 (from 9 to 7.5); $\binom{5}{3} = 10$ supersets of size $n-2$, each of whose weight can be reduced by $1-\frac{8}{15}=\frac{7}{15}$; and 1 superset of size n, whose weight is reduced from 1 to $\frac{1}{2}$. Thus the total savings possible is

$$5(1.5)+10(\frac{7}{15})+\frac{1}{2}=12.66. \tag{5.1}$$

This is $\geq n/2$ as long as $n \leq 25$, so we conclude that (1.1) holds when $k=5$ and $n \leq 25$.

The other assertions of Lemma 4 are obtained by using strategy $T^{(1)}$ for $k=3,4,8$ and strategy $T^{(3)}$ for $k=6,7$. ♣

6 Core proof of Lemma 1.

We are to show that $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ whenever $\mathcal{A}$ or $\mathcal{B}$ has a singleton. Suppose *wlog* that $\{1\} \in \mathcal{A}$, so $\mathcal{A} \supseteq \{1\}^+$. We partition $\mathcal{B}$ into *disjoint* sets, one or more of which may be empty:

$$\mathcal{B}=\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_3^1+\mathcal{B}_r \tag{6.1}$$

where

$$\begin{aligned} \mathcal{B}_1 &= \{\mathbf{n}\} \\ \mathcal{B}_2 &= \{B \in \mathcal{B}\setminus\mathcal{A} : |B|=n-2\} \\ \mathcal{B}_3 &= \{B \in \mathcal{B}\setminus\mathcal{A} : |B|\leq n-3\} \\ \mathcal{B}_3^1 &= \{B\cup\{1\} : B \in \mathcal{B}_3\} \\ \mathcal{B}_r &= (\mathcal{A}\cap\mathcal{B})\setminus(\mathcal{B}_3^1\cup\{\mathbf{n}\}). \end{aligned} \tag{6.2}$$

We set $t=|\mathcal{B}_2|$, and note that $0 \leq t \leq n-1$.

Now rewrite (1.1) as

$$\begin{aligned} & P(\mathcal{A})\left[P(\mathbf{n})+P(\mathcal{B}_2)+P(\mathcal{B}_3)+P(\mathcal{B}_3^1)+P(\mathcal{B}_r)\right] \\ \leq\ & P(\mathbf{n})+P(\mathcal{B}_3^1)+P(\mathcal{B}_r). \end{aligned} \tag{6.3}$$

Since $P(\mathcal{A}) < 1$, $P(\mathcal{A})P(\mathcal{B}_r) \le P(\mathcal{B}_r)$. Hence to prove (1.1) it is enough to prove

$$P(\mathcal{A})(P(\mathbf{n}) + P(\mathcal{B}_2)) \le P(\mathbf{n}) \tag{6.4}$$

and

$$P(\mathcal{A})(P(\mathcal{B}_3) + P(\mathcal{B}_3^1)) \le P(\mathcal{B}_3^1). \tag{6.5}$$

We will show that the first inequality always holds. Then we will show that in any particular case, either the second inequality holds, or (1.1) can be proven by some other means.

Lemma 1A. *(6.4) holds.*

Proof. Since $P(\mathbf{n}) = P(B) = 1/n!$ for each $B \in \mathcal{B}_2$ we need to show

$$P(\mathcal{A}) \le \frac{1}{t+1}, \tag{6.6}$$

where $t = |\mathcal{B}_2|$, $0 \le t \le n-1$. This is clear for $t = 0$. For $t = n-1$, $P(\mathcal{A}) = P(\{1\}^+) = 1/n$ so equality holds. (This is the Example of §1.) For $1 \le t \le n-2$ assume *wlog* after permuting elements that

$$\mathcal{B}_2 = \{\mathbf{n} \setminus \{1,2\}, \mathbf{n} \setminus \{1,3\}, \ldots, \mathbf{n} \setminus \{1, t+1\}\}. \tag{6.7}$$

If $A \in \mathcal{A}$ and $1 \notin A$ then $2 \in A$ or else $\mathbf{n} \setminus \{1,2\} \in \mathcal{A}$ since $\mathcal{A}$ is an up-set, but $\mathbf{n} \setminus \{1,2\} \in \mathcal{B}_2$ and $\mathcal{A} \cap \mathcal{B}_2 = \emptyset$. Similarly if $A \in \mathcal{A}$ and $1 \notin A$ then $3 \in A$, $\ldots$, $t+1 \in A$. Thus

$$\mathcal{A} \subseteq \{1\}^+ \cup \{2, 3, \ldots, t+1\}^+ \tag{6.8}$$

and so if M is the match set,

$$\begin{aligned} P(\mathcal{A}) &\le P(1 \in M) + P(2 \in M, 3 \in M, \ldots, t+1 \in M;\ 1 \notin M) \\ &= \frac{1}{n} + \frac{1}{n}\frac{1}{n-1} \cdots \frac{1}{n-t+1}\ \frac{n-t-1}{n-t} \\ &= \frac{1}{n} + \frac{(n-t-1)!(n-t-1)}{n!}. \end{aligned} \tag{6.9}$$

The conclusion of the lemma holds if the right side is $\leq 1/(t+1)$, or transposing $1/n$ and canceling $n-t-1$, if

$$t+1 \leq (n-1)(n-2)\cdots(n-t), 1 \leq t \leq n-1. \tag{6.10}$$

This inequality holds since $t \leq n-2$ and $1 \leq n-2$. ♣

Remark. Equality holds in Lemma 1A if and only if $t = n-1$, i.e., $\mathcal{A} = \{i\}^+$ for some i and $\mathcal{B}_2 = \{$ all $(n-2)$-sets not containing $i\}$.

We now consider (6.5). We may assume $\mathcal{B}_3 \neq \emptyset$ (since otherwise both sides vanish) and define k' by

$$\min\{|B| : B \in \mathcal{B}_3\} = n - k' - 1,\ k' \geq 2. \tag{6.11}$$

For B that realize (6.11), $B \cup \{1\} \in \mathcal{A} \cap \mathcal{B}$ and $|B \cup \{1\}| = n - k'$. It follows that with k as in Lemma 3, $n-k \leq n-k'$, i.e.,

$$k' \leq k. \tag{6.12}$$

Clearly (6.5) holds if for every $B \in \mathcal{B}_3$,

$$P(\mathcal{A})\,[P(B) + P(B \cup \{1\})] \leq P(B \cup \{1\}) \tag{6.13}$$

since the correspondence $B \leftrightarrow B \cup \{1\}$ between $\mathcal{B}_3$ and $\mathcal{B}_3^1$ is 1-1 and onto. Thus (6.5) follows from (6.13), or transposing, from

$$P(\mathcal{A}) \leq \frac{P(B \cup \{1\})}{P(B) + P(B \cup \{1\})} = \frac{T_{i-1}}{T_i + T_{i-1}} \text{ when } |B| = n - i. \tag{6.14}$$

For $i \geq 3$, as in (2.4),

$$\frac{T_i}{T_{i+1}} \leq \frac{T_{i-1}}{T_i} \tag{6.15}$$

so that (6.14) holds if and only if it holds in the worst case when i is largest, $i = k'+1$, or (using (2.3))

$$P(\mathcal{A}) \leq \frac{T_{k'}}{T_{k'+1} + T_{k'}} = \frac{1}{k' + 2 + (-1)^{k'+1}/T_{k'}}. \tag{6.16}$$

Since there is a set $B_0 \in \mathcal{B}_3 \setminus \mathcal{A}$ by (6.11) with $|B_0| = n - k' - 1$ say

$$B_0 = \{k' + 2, \ldots, n\} \tag{6.17}$$

$\mathcal{A}$ must be a subset of $\{1\}^+ \cup \{2\}^+ \cup \cdots \cup \{k'+1\}^+$ for if $A \in \mathcal{A}$ and none of $1, 2, \ldots, k'+1$ are in A then B_0 would be a superset of A and B_0 would be in $\mathcal{A}$. Thus

$$P(\mathcal{A}) \leq \frac{k'+1}{n}. \tag{6.18}$$

If $n \leq (k'-1)!/2$ then by (6.12), $n \leq (k-1)!/2$ and then by Lemma 3, (1.1) holds. Thus we may assume

$$n > (k'-1)!/2. \tag{6.19}$$

But then the right side of (6.18) is less than the right side of (6.16) at least for $k' \geq 6$ since

$$(k'+1)(k'+2+\frac{(-1)^{k'+1}}{T_{k'}}) < (k'-1)!/2 \text{ for } k' \geq 6. \tag{6.20}$$

For $k' \geq 6$, we have proven that *either* (1.1) holds *or* (6.14) and hence (6.5) holds. But (6.5) implies (1.1) also since we have proven (6.4). Thus we have completed the proof of the following lemma.

Lemma 1B. *Suppose $\{1\} \in \mathcal{A}$ and either $\mathcal{B} \setminus \mathcal{A}$ has no set with fewer than $n-2$ elements ($\mathcal{B}_3 = \emptyset$) or the smallest set in $\mathcal{B} \setminus \mathcal{A}$ has $n-k'-1$ elements with $k' \geq 6$. Then (1.1) holds.*

It remains to show that (1.1) holds when $k' \leq 5$. This will be done in §8.

7 Core proof of Lemma 2.

We are to show that (1.1) holds if every $A \in \mathcal{A} \cup \mathcal{B}$ has $|A| \geq 2$. The proof will depend on the Ahlswede-Daykin generalization (Ahlswede and Daykin, 1978) of the FKG inequality, used in a way similar to the proof of Lemma 1. To state it we assume given nonnegative functions $\alpha, \beta, \gamma, \delta$ on $2^{\mathbf{n}}$ and for any $f \in \{\alpha, \beta, \gamma, \delta\}$ and any $\mathcal{C} \subseteq 2^{\mathbf{n}}$ define

$$f(\mathcal{C}) = \sum_{C \in \mathcal{C}} f(C). \tag{7.1}$$

Ahlswede-Daykin Theorem. *If for all $A, B \in 2^{\mathbf{n}}$*

$$\alpha(A)\beta(B) \le \gamma(A \cup B)\delta(A \cap B) \tag{7.2}$$

then for all $\mathcal{C}, \mathcal{D} \subseteq 2^{\mathbf{n}}$,

$$\alpha(\mathcal{C})\beta(\mathcal{D}) \le \gamma(\mathcal{C} \vee \mathcal{D})\delta(\mathcal{C} \wedge \mathcal{D}), \tag{7.3}$$

where $\mathcal{C} \vee \mathcal{D}$ and $\mathcal{C} \wedge \mathcal{D}$ are defined as in §2.

We may assume *wlog* that $\mathcal{A} \cap \mathcal{B}$ contains all sets with $n-1$ elements. For convenience we drop the normalization by $n!$ in $P(A) = T(A)/n!$ and work directly with $T(A)$.

Analogously to (4.7) in the proof of Lemma 3, we will redefine T on $2^{\mathbf{n}}$, this time so that (7.2) will hold. Thus define $T'(A)$ for $A \in 2^{\mathbf{n}}$ by

$$T'(A) = \begin{cases} 0, & A = \mathbf{n} \\ 1/n, & |A| = n-1 \\ T(A) & \text{otherwise.} \end{cases} \tag{7.4}$$

Thus T' removes the weight of 1 from $\mathbf{n}$ and redistributes it evenly over the $n-1$ element subsets to which T assigns weight zero. Otherwise $T' \equiv T$. We want to define $\alpha, \beta, \gamma, \delta$ so that (7.3) implies (1.1) and to that end set α and β as follows:

$$\alpha(A) = \begin{cases} 0, & A \notin \mathcal{A} \\ T'(A), & A \in \mathcal{A} \end{cases} ; \ \beta(B) = \begin{cases} 0 & B \notin \mathcal{B} \\ T'(B) & B \in \mathcal{B}. \end{cases} \tag{7.5}$$

Since each of $\mathcal{A}$ and $\mathcal{B}$ contains all $n-1$ element subsets,

$$\alpha(\mathcal{A}) = P(\mathcal{A})n!; \ \beta(\mathcal{B}) = P(\mathcal{B})n!. \tag{7.6}$$

Set γ as follows:

$$(7.7)\qquad \gamma(A) = \begin{cases} \frac{1}{2n}, & A = \mathbf{n} \\ 0, & A \notin \mathcal{A} \cap \mathcal{B} \\ T'(A) & \text{otherwise.} \end{cases}$$

Since the sum of $\gamma(A)$ over $(n-1)$-sets is 1 we have

$$(7.8)\qquad \gamma(\mathcal{A} \vee \mathcal{B}) = \gamma(\mathcal{A} \cap \mathcal{B}) = \frac{1}{2n} + n!P(\mathcal{A} \cap \mathcal{B}).$$

Since this is a wee bit greater than $n!P(\mathcal{A} \cap \mathcal{B})$ we must choose $\delta(\mathcal{A} \wedge \mathcal{B})$ somewhat less than $n!$ in order to make (7.3) agree with (1.1).

We choose δ constant on all sets of fixed cardinality,

$$(7.9)\qquad \delta(A) = \begin{cases} 0, & A = \mathbf{n} \\ 1/n, & |A| = n-1 \\ 1, & |A| = n-2 \\ nT_2, & |A| = n-3 \\ nT_3, & |A| = n-4 \\ \vdots & \\ nT_{n-2}, & |A| = 1 \\ 2nT_{n-2}, & A = \emptyset. \end{cases}$$

Consequently

$$(7.10)\qquad \begin{aligned} \delta(\mathcal{A} \wedge \mathcal{B}) \ &\leq \ Z_n \stackrel{\text{def}}{=} \sum_{i=0}^{n} \binom{n}{i} \delta_i \\ &= \ 2nT_{n-2} + n \sum_{i=1}^{n-3} \binom{n}{i} T_{n-i-1} + \binom{n}{2} + 1. \end{aligned}$$

With $\alpha, \beta, \gamma, \delta$ as above it will suffice to check that (7.2) holds and that (7.3) implies (1.1). We begin by checking that in all relevant cases, either (7.2) holds, or (1.1) can be verified by some other means.

	$\lvert D\rvert$	n	$n-1$	$n-2$	$n-3$	$n-4$	$\cdots$	3	2	1	0
$\lvert C\rvert$		0	$1/n$	1	n	$2n$		nT_{n-4}	nT_{n-3}	nT_{n-2}	$2nT_{n-2}$
n	$1/(2n)$	0	0	$1/n^2$	$1/n$	T_2		T_{n-5}	T_{n-4}	T_{n-3}	T_{n-2}
$n-1$	$1/n$	0	$1/n^2$	$1/n$	T_2	T_3		T_{n-4}	T_{n-3}	T_{n-2}	$2T_{n-2}$
$n-2$	T_2	0	0	T_2^2	T_2T_3	T_2T_4		T_2T_{n-3}	T_2T_{n-2}	T_3T_{n-2}	T_4T_{n-2}
$n-3$	T_3	0	0	0	T_3^2	T_3T_4		T_3T_{n-3}	T_3T_{n-2}	T_4T_{n-2}	T_5T_{n-2}
$\vdots$											
$n-k$	T_k						${T_k}^2$ at $\lvert D\rvert = n-k$	T_kT_{n-3}	T_kT_{n-2}	$T_{k+1}T_{n-2}$	$T_{k+2}T_{n-2}$

Table 3: Restrictions on γ and δ.

Let k be as in (4.1), the largest integer for which there is a set $A_0 \in \mathcal{A} \cap \mathcal{B}$ with $|A_0| = n-k$, so that A_0 is a smallest set in $\mathcal{A} \cap \mathcal{B}$. To check (7.2), we need only consider the case $A \in \mathcal{A}$, $B \in \mathcal{B}$ since otherwise both $\alpha(A)\beta(B)$ and $\gamma(A \cup B)\delta(A \cap B)$ will vanish. Let $C = A \cup B$ and $D = A \cap B$. The values $\alpha(A), \beta(B), \gamma(C), \delta(D)$ depend only on the sizes $|A|, |B|, |C|, |D|$. The entries in Table 3 show for given values of $|C|, |D|$ the biggest possible value of $\alpha(A)\beta(B)$ for any pair of sets A, B for which $A \cup B = C$, $A \cap B = D$. For example, the entry in row $|C| = n-2$, column $|D| = 3$ is the maximum of $\alpha(A)\beta(B)$ with $|A| + |B| = n - 2 + 3 = n + 1$, or $n - |A| + n - |B| = n - 1$, which is T_2T_{n-3} as indicated.

To make sure that $\alpha(A)\beta(B) \leq \gamma(C)\delta(D)$, we must check that each entry is $\leq$ the product of the the corresponding values of $\gamma(C), \delta(D)$, which are shown at the borders of the table. For row $|C| = n-1$, equality holds. For row $|C| = n$, equality holds at $|D| = n-4$ and $|D| = 0$, and the desired inequality holds elsewhere. For any other row the columns $|D| = 2, 3, \ldots$ are similar and require

$$nT_{n-j-1} \geq T_{n-j}, \text{ i.e., approx. } n > n-j, \tag{7.11}$$

which holds using (2.3). For the last two columns the entry in the lower right corner, which is the worst case, requires

$$2nT_k \geq T_{k+2}, \text{ i.e., approx. } 2n \geq (k+2)(k+1), \tag{7.12}$$

which holds for $k \geq 5$ except for small n which are covered by Lemma 4. Thus for example when $k = 5$ Lemma 4 allows us to assume that $n \geq 26$ and $2n \geq T_7/T_5 = 1854/44$ for $n \geq 26$.

We are now in a position to prove most of Lemma 2.

Lemma 2A. *Suppose $\mathcal{A} \cup \mathcal{B}$ contains no singleton. Then (1.1) holds if $k \geq 4$ or if $(k = 3, n \geq 8)$, $(k = 2, n \geq 7)$, or $(k = 1, n \geq 8)$.*

Proof. We consider Z_n defined in (7.10) first. For $n = 7$, $Z_7 = 4852$ compared to $7! = 5040$. Assume $k \geq 5$ or $k = 2$. Then (7.2) may be seen to hold as described previously and so (7.3) and (7.8) give

$$\begin{aligned} (7!)^2 P(\mathcal{A})P(\mathcal{B}) &= \alpha(\mathcal{A})\beta(\mathcal{B}) \\ &\leq \gamma(\mathcal{A}\cap\mathcal{B})\delta(\mathcal{A}\wedge\mathcal{B}) \\ &\leq \left[\frac{1}{14} + 7!P(\mathcal{A}\cap\mathcal{B})\right] Z_7. \end{aligned} \tag{7.13}$$

The last term in (7.13) is supposed to be below $(7!)^2 P(\mathcal{A}\cap\mathcal{B})$ and $P(\mathcal{A}\cap\mathcal{B}) \geq 2/7!$ whenever $k \geq 2$ in which case (7.13) is easily checked. Therefore (1.1) holds whenever $n = 7$ and $k = 2$ or $k \geq 5$.

Suppose next that $n = 8$ and either $k \geq 5$ or $k \in \{1, 2\}$. We compute $Z_8 = 36685$ with $8! = 40320$. By (7.3),

$$(8!)^2 P(\mathcal{A})P(\mathcal{B}) \leq [1/16 + 8!P(\mathcal{A}\cap\mathcal{B})]Z_8. \tag{7.14}$$

But $[1/16 + 8!P(\mathcal{A}\cap\mathcal{B})]Z_8 \leq (8!)^2 P(\mathcal{A}\cap\mathcal{B})$ if and only if $P(\mathcal{A}\cap\mathcal{B}) \geq (36685/58160)/8! \doteq$

(0.63)/8!, and this is valid since $P(\mathcal{A} \cap \mathcal{B}) \geq 1/8!$. Hence $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A} \cap \mathcal{B})$ whenever $n = 8$ and either $k \in \{1, 2\}$ or $k \geq 5$.

The same conclusion holds also for all $n \geq 9$ since $Z_n/n!$ is decreasing in n. In particular, reduction gives

$$(n+1)!\left[\frac{Z_n}{n!} - \frac{Z_{n+1}}{(n+1)!}\right] = (n+1)\left\{\begin{array}{l} [2T_{n-2} - 2(-1)^{n-1}] \\ + \sum_{i=1}^{n-3}\left[\binom{n}{i-1}T_{n-i-1} - \binom{n+1}{i}(-1)^{n-i}\right] \\ - \binom{n+1}{n-2} + \frac{n(n-2)}{2} + 1 \end{array}\right\} - 1. \tag{7.15}$$

Since all terms in square brackets are strictly positive and $2T_{n-2} > \binom{n+1}{n-2}$ when $n \geq 8$, it follows that $Z_n/n! > Z_{n+1}/(n+1)!$ for $n \geq 8$. Since $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A} \cap \mathcal{B})$ is known to be true for small values of n when $k \geq 5$ by Lemma 3, we have verified the following conclusions of Lemma 2A at this point:

$$P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A} \cap \mathcal{B}) \text{ if } k \geq 5 \text{ or } (k = 1, n \geq 8) \text{ or } (k = 2, n \geq 7)\ .$$

It remains to consider $k \in \{3, 4\}$. Suppose first that $k = 4$. To satisfy the (7.2) hypotheses at $k = 4$ we increase δ_0 from $2nT_{n-2}$ to $(T_{k+2}/T_k)T_{n-2} = (265/9)T_{n-2}$ when $n \leq 14$. Since the original δ_0 value suffices when $n \geq 15$, and since Lemma 4 says that $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A} \cap \mathcal{B})$ if $n \leq 9$, the preceding proofs suffice for $k = 4$ except when $10 \leq n \leq 14$. When $n = 10$, the increase in Z_{10} caused by the addition to δ_0 is not enough to invalidate the desired conclusion. In particular, (7.3) applied to δ_0 revised gives

$$(10!)^2 P(\mathcal{A})P(\mathcal{B}) \leq [1/20 + 10!P(\mathcal{A} \cap \mathcal{B})]\left[Z_{10} + \left(\frac{265}{9} - 20\right)T_8\right] \tag{7.16}$$

with $Z_{10} + (265/9 - 20)T_8 = 3192875.4$ and $10! = 3628800$, and this implies $P(\mathcal{A})P(\mathcal{B}) \leq$

$P(\mathcal{A}\cap\mathcal{B})$. The situation is even more favorable for $k=4$ at $n=11,\ldots,14$, so we obtain the desired result for all n at $k=4$.

Finally, suppose $k=3$. We increase the value of δ_0 from $2nT_{n-2}$ to $(T_{k+2}/T_2)T_{n-2} = 22T_{n-2}$ when $n\le 10$. By Lemma 4 and the preceding proof with Z_n, we know that the desired result holds for all n except $5\le n\le 9$. By a method similar to that for $k=4$ in the preceding paragraph, it is easily verified that $P(\mathcal{A})P(\mathcal{B})\le P(\mathcal{A}\cap\mathcal{B})$ when $n\in\{8,9\}$. However, that method does not give the desired conclusion for $n\le 7$, so we are left with the unresolved cases of $n\in\{6,7\}$ to consider further (with $n=5$ covered by earlier presumption). ♣

We are left only with the following remnant of Lemma 2 to prove.

Remnant 2. *Given no singleton in* $\mathcal{A}\cup\mathcal{B}$*, and letting* $\min\{|A| : A\in\mathcal{A}\cap\mathcal{B}\} = n-k$ *as before, if*

(a) $k=3$ *and* $n\in\{6,7\}$*, or*

(b) $k=2$ *and* $n=6$*, or*

(c) $k=1$ *and* $n\in\{6,7\}$*,*

then (1.1) holds.

The proof will be given in §9.

8 Remnants of Lemma 1.

Let k and k' be as in (4.1) and (6.11). Suppose $\{1\}\in\mathcal{A}$ and $k'\le 5$. Since (6.16) implies (1.1) as in §6, we must prove the following remnant of Lemma 1.

Remnant 1. *Given* $\{1\}\in\mathcal{A}$, $\mathcal{B}_2=\{B : |B|=n-2$ *and* $B\in\mathcal{B}\setminus\mathcal{A}\}$*, and letting* $\min\{|B| : B\in\mathcal{B}\setminus\mathcal{A}\} = n-k'-1$ *as before, if* $k'\le 5$ *then (6.16) or (1.1) holds.*

Proof. For $k' = 5$, (6.16) says

$$P(\mathcal{A}) \leq \frac{T_5}{T_5 + T_6} \doteq 0.1424. \tag{8.1}$$

If $t = |\mathcal{B}_2| \geq 1$, say with $\{3,4,\ldots,n\} \in \mathcal{B} \setminus \mathcal{A}$, then no subset of $\{3,4,\ldots,n\} \in \mathcal{A}$ and so

$$P(\mathcal{A}) \leq P(\{1\}^+ \cup \{2\}^+) < 2/n < 0.1424 \text{ for } n \geq 15. \tag{8.2}$$

Thus (8.1) holds for $n \geq 15$ and since $k' \leq k$ by (6.12) and (1.1) holds for $k \leq 5$, $n \leq 25$ by Lemma 4, (6.16) or (1.1) holds for $k' \leq 5$, for $t \geq 1$. The only other case is $t = 0$ and so every $n-2$ element superset of some particular $B_0 \in \mathcal{B} \setminus \mathcal{A}$ with $|B_0| = n - k' - 1 = n - 6$ must be in $\mathcal{A}$. Since there are $\binom{6}{2} = 15$ such supersets of B_0 the total savings from reallocation $T^{(2)}$ in Table 2 is at least $\frac{1}{2} + \binom{6}{2}\frac{7}{15} + 5(1.5) = 15$ so that if $n/2 \leq 15$ then (1.1) holds by the type of analysis used in the proof of Lemma 4. We thus may assume for $k' = 5$ that $t = 0, n \geq 31$. Suppose next that some $|A| = n - 4$ is *not* in $\mathcal{A}$. Then $P(\mathcal{A}) < 4/n$ as in (8.2). But $4/n < 0.1424$ for $n \geq 29$ so (8.1) and (6.16) holds unless all $|A| = n - 4$ are in $\mathcal{A}$. But if all A with $|A| = n - 4$ are in $\mathcal{A}$ then the total savings from reallocation $T^{(2)}$ accruing from $B_0 \in \mathcal{B} \setminus \mathcal{A}$ with $|B| = n - 6$ is at least $\frac{1}{2} + \frac{7}{15}\binom{6}{4} + (1.5)\binom{6}{2} = 30$ and so (1.1) holds if $n \leq 60$. Finally if $n \geq 61$ and some $|A| = n - 6$ is not in $\mathcal{A}$ (which we know to be true since $B_0 \in \mathcal{B} \setminus \mathcal{A}$ and $|B_0| = n - 6$), then $P(\mathcal{A}) < 6/n$ by the analog to (8.2). But $6/n < 0.1424$ for $n \geq 43$ and since $n \geq 61$ at this point, we conclude that (1.1) or (6.16) holds for $k' = 5$. This completes the cases $k' = 5$ of Remnant 1 and of course proves (1.1) for this case (since we have shown that either (6.16) or (1.1) holds and in §6 that (6.16) implies (1.1)). We now turn to $k' < 5$.

Suppose first that $t = |\mathcal{B}_2| \geq 2$ with $\{3,4,\ldots,n\}$ and $\{2,4,5,\ldots,n\}$ in $\mathcal{B} \setminus \mathcal{A}$ for definiteness. Then every set $A \in \mathcal{A}$ contains either 1 or both 2 and 3, so

$$P(\mathcal{A}) \leq P(\{1\}^+ \cup \{2,3\}^+) = \frac{1}{n} + \frac{n-3}{n(n-1)(n-2)} < \frac{1}{n-1}. \tag{8.3}$$

Since the right side of (6.16) is $1/3$ at $k' = 2$ and $1/(n-1) \le 1/3$ for $n \ge 4$, (6.16) holds for $k' = 2$, $n \ge 4$. For $k' = 3$, $T_3/(T_3+T_4) = 2/11$ and $1/(n-1) < 2/11$ for $n \ge 6$. Since we can check directly that (1.1) holds for $n \le 5$ this covers every case for $t \ge 2$, $k' < 4$. For $k' = 4$, $T_4/(T_4+T_5) = 9/53$ and $1/(n-1) < 9/53$ for $n \ge 7$. Lemma 4 settles $n \le 9$ and so for $t \ge 2$ and arbitrary k', (1.1) holds when $\{1\} \in \mathcal{A}$. For $t = 1$, say with $\{3, 4, \ldots, n\} \in \mathcal{B} \setminus \mathcal{A}$

$$P(\mathcal{A}) \le P(\{1\}^+ \cup \{2^+\}) = \frac{1}{n} + \frac{1}{n} - \frac{1}{n(n-1)} = \frac{2n-3}{n(n-1)}. \tag{8.4}$$

At $k' = 4$, $T_4/(T_4+T_5) = 9/53$ and so (6.16) and (1.1) holds for $n \ge 12$. Similarly at $k' = 3, n \ge 11$ or $k' = 2, n \ge 6$. We are done with Remnant 1 except for $|\mathcal{B}_2| = 1$; ($k' = 3$ and $n \le 10$) or ($k' = 4$ and $n \le 11$) and for $\mathcal{B}_2 = \emptyset, k' \in \{2, 3, 4\}$. We remind the reader that $\{1\} \in \mathcal{A}$ for Remnant 1.

Case 1. $|\mathcal{B}_2| = 1$, $k' = 4, n \le 11$. Since $k' = 4$, some $B_0 \in \mathcal{B} \setminus \mathcal{A}$ has $n-5$ elements, and $\mathcal{B} \setminus \mathcal{A}$ has no smaller set. Since $t = 1$ ($\mathcal{B} \setminus \mathcal{A}$ has only one set with $n-2$ elements) and B_0 has $\binom{5}{3} = 10$ supersets with $n-2$ elements each, nine of these must be in $\mathcal{A} \cap \mathcal{B}$. In addition, $\mathcal{A} \cap \mathcal{B}$ has a set with $n-4$ elements, $B_0 \cup \{1\}$. The savings from $T^{(1)}$ of Table 2 is therefore at least $1 + 9(1/2) + 1/2 = 6$ and since $n/2 \le 6$ for $n \le 12$ it follows as in the proof of Lemma 3 that (1.1) holds.

Case 2. $|\mathcal{B}_2| = 1$, $k' = 3, n \le 10$. Let $B = \{5, 6, \ldots, n\}$ be in $\mathcal{B} \setminus \mathcal{A}$ with no smaller set in $\mathcal{B} \setminus \mathcal{A}$. Since B has six supersets with $|A| = n-2$ and $t = 1$, five of these must be in $\mathcal{A} \cap \mathcal{B}$ so the savings by $T^{(1)}$ is at least $5 \cdot 1/2 + 1/2 = 3$ and so (1.1) holds for $n \le 6$. We may thus consider only $7 \le n \le 10$. We proceed on the basis of the number of sets in $\mathcal{B}$ that have $n-4$ elements. Let x denote this number. By assumption $x \ge 1$ since $k' = 3$.

Case 2.1. Suppose $x \ge 3$. These x have at least ten distinct $|A| = n-2$ supersets and, since $t = 1$, nine of these must be in $\mathcal{A} \cap \mathcal{B}$. The $T^{(1)}$ savings is at least $9 \cdot 1/2 + 1/2 = 5$ so the usual argument gives (1.1) for $n \le 10$.

Case 2.2. Suppose $x = 2$. These two have at least nine supersets with $|A| = n-2$, so the $T^{(1)}$ savings is at least $1/2 + 8(1/2) = 4.5$. Hence $P(\mathcal{A})P(\mathcal{B}) \le P(\mathcal{A}\cap\mathcal{B})$ by the usual argument if $n \le 9$. Moreover, if the two $\mathcal{B}$ sets with $n-4$ elements have fewer than $n-5$ elements in their intersection, they have more than nine supersets with $|A| = n-2$, in which case the argument implies the desired result if $n \le 10$. We can therefore assume that $n = 10$ and that the second set in $\mathcal{B}$ with $n-4$ elements is $B' = \{4,6,7,\ldots,10\}$. (If B' were to contain 1 then $B' \in \mathcal{A}\cap\mathcal{B}$, and we would obtain the desired result. We may therefore suppose that, in general, $B' \notin \mathcal{A}$.) Since the usual Lemma 3 analysis yields the desired result for $n = 10$ if there are more than nine $|A| = n-2$ in $\mathcal{B}$, we assume that only the nine $(n-2)$-element supersets of B' and $B = \{5,\ldots,10\}$ are in $\mathcal{B}$ at level $n-2$. The effect of this is to force $\mathcal{B} = B^+ \cup (B')^+$ with $P(\mathcal{B}) = [2(9)+7(2)+9(1)+1]/10! = 42/10!$. At the same time,

$$P(\mathcal{A}\cap\mathcal{B}) \ge [2(2)+8(1)+1]/10! = 13/10!, \tag{8.5}$$

so $P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B}) \ge 13/42$. However, since $P(\mathcal{A}) \le P(\{1\}^+ \cup \{2\}^+) < 1/5$ and $1/5 < 13/42$, we conclude that $P(\mathcal{A})P(\mathcal{B}) \le P(\mathcal{A}\cap\mathcal{B})$.

Case 2.3. Suppose $x = 1$. Then every set in $\mathcal{B}$ other than $B = \{5,\ldots,n\}$ has at least $n-3$ elements. Suppose

$$\begin{aligned} &\mathcal{B} \text{ has } z_1\ |B'| = n-3 \text{ with } 1 \in B' \\ &\mathcal{B} \text{ has } z_2\ |B'| = n-3 \text{ with } 1 \notin B' \\ &\mathcal{B} \text{ has } y\ |B'| = n-2. \end{aligned} \tag{8.6}$$

Then $P(\mathcal{B}) = (9 + 2z_1 + 2z_2 + y + 1)/n!$ and $P(\mathcal{A}\cap\mathcal{B}) \ge (2z_1 + \max\{z_2, y-1\} + 1)/n!$, so

$$\frac{P(\mathcal{A}\cap\mathcal{B})}{P(\mathcal{B})} \ge \frac{2z_1 + \max\{z_2, y-1\} + 1}{9 + 2z_1 + 2z_2 + y + 1} \tag{8.7}$$

with $z_1 \ge 1$, $z_2 \ge 3$ and $y \ge 5$. Since the right side of this inequality increases in z_1, the worst case for $P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B})$ has $z_1 = 1$, so at $z_1 = 1$ we get

$$\frac{P(\mathcal{A}\cap\mathcal{B})}{P(\mathcal{B})} \geq \frac{3+\max\{z_2, y-1\}}{12+2z_2+y}. \tag{8.8}$$

Because $z_2 \geq 3$ and $y \geq 5$, it is easily shown that the right side of the new inequality is at least 11/42. Hence $P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B}) \geq 11/42$. On the other hand,

$$P(\mathcal{A}) \leq P(\{1\}^+ \cup \{2\}^+) = (2n-3)/[n(n-1)], \tag{8.9}$$

which equals 11/42 at $n = 7$ and is smaller for larger n. Therefore $P(\mathcal{A}) \leq P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B})$. This completes the proof of Case 2 of Remnant 1.

Case 3. This case assumes that $t = 0$ (no $|A| = n-2$ is in $\mathcal{B}(\mathcal{A})$) and $k \in \{2, 3, 4\}$. We consider the possible k in turn. (For convenience, at this point we drop the prime on k.)

Case 3.1. $k = 2$. Assume for definiteness that $\{4,\ldots,n\} \in \mathcal{B}\setminus\mathcal{A}$. If $\mathcal{A}$ *omits* more than one $(n-3)$-element set, another such set either has the form $\{3,5,\ldots,n\}$ or $\{2,3,6,\ldots,n\}$. In the first case it is easily checked that $P(\mathcal{A}) \leq 2/n$ if $n \geq 4$, and in the second that $P(\mathcal{A}) \leq 2/n$ if $n \geq 6$. Since we assume the desired result for $n \leq 5$, we can presume that $P(\mathcal{A}) \leq 2/n$ when $\mathcal{A}$ omits more than one $|A| = n-3$. Then, since $2/n \leq T_2/(T_2+T_3) = 1/3$ if $6 \leq n$, (6.16) holds, and we conclude that $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$. Suppose then that only $\{4,\ldots,n\}$ is not in $\mathcal{A}$ at level $n-3$, so $P(\mathcal{A})$ can be near to $3/n$. Let $(c,d) =$ (number of $B \in \mathcal{B}$ with $|B| = n-3$ other than $\{4,\ldots,n\}$, number of $B \in \mathcal{B}$ with $|B| = n-2$ besides the supersets of $\{4,\ldots,n\}$). Then

$$\frac{P(\mathcal{A}\cap\mathcal{B})}{P(\mathcal{B})} = \frac{3+2c+d+1}{2+3+2c+d+1} = \frac{4+2c+d}{6+2c+d} \geq \frac{2}{3}, \tag{8.10}$$

and therefore $P(\mathcal{A}) \leq P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B})$ if $3/n \leq 2/3$ or $n \geq 5$.

Case 3.2. $k = 3$. Let $B = \{5,6,\ldots,n\} \in \mathcal{B}\setminus\mathcal{A}$. Since $t = 0$, every $(n-2)$-element superset of B is in $\mathcal{A}\cap\mathcal{B}$, and it follows from the usual analysis that $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ if $n \leq 7$. Assume henceforth for Case 3.2 that $n \geq 8$.

Assume further throughout this paragraph that B is the only set in $\mathcal{B}$ that has $n-4$ elements. If $\mathcal{A}$ omits none of the $|A| = n-3$, then $P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B}) \geq (2(4)+6+1+X)/(9+2(4)+6+1+X) \geq 15/24$ and, since $P(\mathcal{A}) \leq 4/n$, we get $P(\mathcal{A}) \leq P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B})$ if $4/n \leq 15/24$, or $n \geq 7$, so we're all right here. If $\mathcal{A}$ omits exactly one $|A| = n-3$, then $P(\mathcal{A}) < 3/n$ and $P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B}) \geq 13/24$, so $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ if $n \geq 6$. If $\mathcal{A}$ omits two $|A| = n-3$, then $P(\mathcal{A}) \leq 2/n$ when $n \geq 6$ (see Case 3.1 above) and $P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B}) \geq 11/24$, so $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ if $n \geq 5$. Next if $\mathcal{A}$ omits three sets with $|A| = n-3$, then $P(\mathcal{A}) \leq 2/n$ and $P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B}) \geq 9/24$, so the desired result holds if $2/n \leq 9/24$, or $n \geq 6$. Finally, if $\mathcal{A}$ omits four or more $|A| = n-3$, leaving more room at level $n-3$ for sets in $\mathcal{B}\setminus\mathcal{A}$, and if x denotes the number of $B' \in \mathcal{B}\setminus\mathcal{A}$ with $|B'| = n-3$ in addition to the three supersets of B that do not contain 1 at level $n-3$, then $P(\mathcal{A}) \leq 2/n$ and

$$\frac{P(\mathcal{A}\cap\mathcal{B})}{P(\mathcal{B})} \geq \frac{9+x}{24+2x} \geq \frac{9}{24}, \tag{8.11}$$

so again $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ if $n \geq 6$.

Suppose next that $\mathcal{B}$ has exactly one $(n-4)$-element set besides $B = \{5,\ldots,n\}$. The usual analysis gives $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ if $n \leq 10$, so assume henceforth that $n \geq 11$. If $\mathcal{A}$ omits no $|A| = n-3$, then $P(\mathcal{A}) \leq 4/n$ and $P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B}) \geq (2(7)+9+1)/(9(2)+2(7)+9+1) = 24/42$, so $P(\mathcal{A}) \leq P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B})$ if $n \geq 7$. If $\mathcal{A}$ omits one $|A| = n-3$, then $P(\mathcal{A}) \leq 3/n$ and $P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B}) \geq 22/42$, so the desired result holds if $n \geq 6$. If $\mathcal{A}$ omits two or more $|A| = n-3$, then $P(\mathcal{A}) \leq 2/n$ and $P(\mathcal{A}\cap\mathcal{B})/P(\mathcal{B}) \geq (14+X)/(42+2X) \geq 1/3$, and again $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ if $n \geq 6$.

If $\mathcal{B}$ has more than two sets with $n-4$ elements, similar analysis shows that $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ for all n.

Case 3.3. $k = 4$. Let $B = \{6,\ldots,n\} \in \mathcal{B}\setminus\mathcal{A}$. The savings for $T^{(1)}$ accruing from B is at least $1 + 10(1/2) + 1/2 = 13/2$, so $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ for all $n \leq 13$ by the usual analysis of Lemma 3. Assume henceforth that $n \geq 14$.

Let x denote the number of $|A| = n-3$ not in $\mathcal{A}$. Suppose $x \geq 2$. Then $P(\mathcal{A}) \leq 2/n$ (see Case 3.1 above), and since $2/n \leq T_4/(T_4+T_5) = 9/53$ for all $n \geq 12$, (6.16) holds and hence $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$.

Suppose next that $x = 1$. Then $P(\mathcal{A}) \leq 3/n$, and (6.16) holds if $3/n \leq 9/53$, or $n \geq 18$. Hence $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ if $n \geq 18$. If $\mathcal{B}$ has two or more $(n-5)$-element sets, then the savings for $T^{(1)}$ is at least $1 + 14(1/2) + 1/2 = 17/2$, so $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ by the Lemma 3 method if $n \leq 17$. Since this covers all n, assume henceforth that B is the only $(n-5)$-element set in $\mathcal{B}$. Then

$$(8.12) \qquad \frac{P(\mathcal{A}\cap\mathcal{B})}{P(\mathcal{B})} \geq \frac{9(1)+2(9)+10+1+2a+b}{44+9(5)+2(10)+10+1+9a+2b} \geq \frac{2}{9},$$

where $(a,b) =$ (number of $B' \in \mathcal{B}\setminus\mathcal{A}$ with $|B'| = n-4$ other than supersets of B, number of $B' \in \mathcal{B}\setminus\mathcal{A}$ with $|B'| = n-3$ other than supersets of B). Then $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ if $3/n \leq 2/9$, or $n \geq 14$. Since the lead paragraph of Case 3.3 covers all $n \leq 13$, the proof for $x = 1$ is complete.

Suppose finally that $x = 0$, so that all $|A| = n-3$ are in $\mathcal{A}$. Let y denote the number of $|A| = n-4$ *not* in $\mathcal{A}$. If $y = 0$ then $P(\mathcal{A}) \leq 5/n$, if $y = 1$ then $P(\mathcal{A}) \leq 4/n$, and if $y \geq 2$ then $P(\mathcal{A}) \leq 3/n$. We assume $n \geq 14$.

Suppose first for $x = 0$ that $\mathcal{B}\setminus\mathcal{A}$ has two or more $(n-5)$-element sets. Then the $T^{(1)}$ savings is at least $1/2 + 16(1/2) + 2 = 21/2$, so $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ by the Lemma 3 method if $n \leq 21$. If $y \geq 2$, then (6.16) holds if $3/n \leq 9/53$, or $n \geq 18$, so all n are covered in this case. If $y \leq 1$ then (6.16) holds for $5/n \leq 9/53$, or $n \geq 30$, but in this case the $T^{(1)}$ savings is at least $1/2 + 16(1/2) + 8(1) = 33/2$, so $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ by the Lemma 3 method for $n \leq 33$. Hence $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$ if $\mathcal{B}\setminus\mathcal{A}$ has two or more $(n-5)$-element sets.

Assume henceforth for $x = 0$ that B is the only set in $\mathcal{B} \setminus \mathcal{A}$ with $n-5$ elements and, with no loss of generality, assume also that $\mathcal{B}$ has no other set with fewer than $n-4$ elements. Suppose first that $y \leq 3$. Then $P(\mathcal{A}) < 5/n$ and

$$\text{(8.13)} \qquad \frac{P(\mathcal{A} \cap \mathcal{B})}{P(\mathcal{B})} \geq \frac{9(2) + 2(10) + 11 + z}{44 + 9(5) + 2(10) + 11 + z} \geq \frac{49 + z}{120 + z} \geq \frac{49}{120},$$

where z is the T weight from all sets in $\mathcal{A} \cap \mathcal{B}$ that are not supersets of B. We then have $P(\mathcal{A}) \leq P(\mathcal{A} \cap \mathcal{B})/P(\mathcal{B})$ if $5/n \leq 49/120$, or $n \geq 13$. Since all $n \leq 13$ are covered by the opening paragraph of Case 3.3, the desired result holds if $y \leq 3$. Because $P(\mathcal{A}) \leq 3/n$ if $y \geq 2$, a similar calculation shows that $P(\mathcal{A}) \leq P(\mathcal{A} \cap \mathcal{B})/P(\mathcal{B})$ if $y \leq 12$: at $y = 12$, $P(\mathcal{A} \cap \mathcal{B})/P(\mathcal{B}) \geq (9 + 20 + 11)/(120 + 7(9)) = 40/183$, and $3/n \leq 40/183$ if $14 \leq n$. The same conclusion fails to hold for $y > 12$ only if $\mathcal{B} \setminus \mathcal{A}$ has at least 12 $(n-4)$-element sets. But in this case $\mathcal{A} \cap \mathcal{B}$ has at least 15 $(n-2)$-element sets, the $T^{(1)}$ savings is at least $1/2 + 15(1/2) + 1 = 18/2$, and therefore the Lemma 3 method implies $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A} \cap \mathcal{B})$ if $n \leq 18$. On the other hand, (6.16) holds if $3/n \leq 9/53$, or $n \geq 18$, so all n are covered for $y > 12$.

This completes the proof of $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A} \cap \mathcal{B})$, i.e., (1.1) for all cases left open in Remnant 1. The proof of Lemma 1 is complete. ♣

9 Remnants of Lemma 2.

We consider parts (a), (b) and (c) of Remnant 2 in that order. It is assumed that no singleton is in $\mathcal{A} \cup \mathcal{B}$.

(a). $k = 3$ and $n \in \{6, 7\}$. Suppose first that $\mathcal{A} \cap \mathcal{B}$ has three or more $|A| = n - 3$. Then the savings for reallocation $T^{(1)}$ of Table 2 is at least $1/2 + (1/2)6 = 7/2$, and since $n/2 \leq 7/2$ for $n \leq 7$, the method of Lemma 3 implies $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A} \cap \mathcal{B})$ for $n \leq 7$.

Suppose next that exactly two $|A| = n-3$ are in $\mathcal{A} \cap \mathcal{B}$. Then the $T^{(1)}$ savings is at least $1/2 + (1/2)5 = 3$, which covers $n = 6$ by the method of Lemma 3. To avoid (1.1) at $n = 7$, the two $|A| = n-3$ in $\mathcal{A} \cap \mathcal{B}$ must have a three-element intersection, say $\{4,5,6,7\}$ and $\{3,5,6,7\}$ for these two, and only their five $(n-2)$-element supersets can be in $\mathcal{A} \cap \mathcal{B}$ for $|A| = n-2$. So assume that

$$\mathcal{A} \cap \mathcal{B} = \{3,5,6,7\}^+ \cup \{4,5,6,7\}^+, \tag{9.1}$$

and let $\mathcal{B}_0 = \mathcal{A} \cap \mathcal{B}$. We compute $P(\mathcal{A} \cap \mathcal{B}) = 10/7!$. If $\mathcal{B} = \mathcal{B}_0$, $\max P(\mathcal{A}) = 1331/7!$, which occurs when all $|A| \geq 2$ are in $\mathcal{A}$. In this case $\mathcal{B}$ cannot be increased from $\mathcal{B}_0$, and $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A} \cap \mathcal{B})$ since $(1331)(10) < (10)(5040)$. Starting at $\mathcal{B}_0$, we can expand it to get a larger $\mathcal{B}$, but any such expansion reduces the maximal allowable $\mathcal{A}$ substantially. One example is $\mathcal{B} = \{5,6,7\}^+$, but then $P(\mathcal{B}) = 24/7!$, which is much less than is needed to violate $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A} \cap \mathcal{B})$ even if $P(\mathcal{A})$ remains at $1331/7!$ (which is impossible: $\max P(\mathcal{A})$ at $\mathcal{B} = \{5,6,7\}^+$ is $528/7!$). In general, unless at least one of $P(\mathcal{A})$ and $P(\mathcal{B})$ exceeds $224/7!$, then it is not possible to violate $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A} \cap \mathcal{B})$. However, if $P(\mathcal{A}) > 224/7!$, then either $\mathcal{A}$ contains most of the three-element subsets of $\mathbf{7}$ or it contains several two-element subsets, and in both cases the restriction on $\mathcal{A} \cap \mathcal{B}$ forces $\mathcal{B}$ to be comparatively small. Further details are left to the reader.

Finally, suppose $\mathcal{A} \cap \mathcal{B}$ has exactly one $|A| = n-3$, say $\{1,2,3\}$ for $n = 6$ or $\{1,2,3,4\}$ for $n = 7$. Assume first that $n = 6$. Then, by the method of reallocation analysis of Lemma 3, $\mathcal{A} \cap \mathcal{B}$ can have at most one more $|A| = n-2$ besides the three produced as supersets of $\{1,2,3\}$, so that either $P(\mathcal{A} \cap \mathcal{B}) = 6/6!$ or $P(\mathcal{A} \cap \mathcal{B}) = 7/6!$. Then $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A} \cap \mathcal{B})$ can be violated only if the larger of $P(\mathcal{A})$ and $P(\mathcal{B})$ exceeds $65/6!$. Suppose $P(\mathcal{A}) \geq 66/6!$. Then $\mathcal{A}$ must contain two or more two-element sets and most of the three-element sets, and the restriction on $\mathcal{A} \cap \mathcal{B}$ will force $\mathcal{B}$ to contain little more than $\{1,2,3\}^+$. The extreme case of $\mathcal{B} = \{1,2,3\}^+$ has $\max P(\mathcal{A}) = 191/6!$. When $\mathcal{B} = \{1,2\}^+$ and $\mathcal{A} \cap \mathcal{B}$ contains $\{1,2,4,5\}$

along with $\{1,2,3\}^+, P(\mathcal{B}) = 24/6!$ and $\max P(\mathcal{A}) = 94/6!$, compared to $P(\mathcal{A}\cap\mathcal{B}) = 7/6!$, and $P(\mathcal{A})P(\mathcal{B})/P(\mathcal{A}\cap\mathcal{B}) = 0.448$. We omit further details.

Assume next that $n = 7$ and $\{1,2,3,4\}$ is the only four-element subset in $\mathcal{A}\cap\mathcal{B}$. The FKG reallocation analysis allows at most two more $|A| = n-2$ in $\mathcal{A}\cap\mathcal{B}$ besides the three in $\{1,2,3,4\}^+$ (else the desired result follows for $n = 7$). Hence $6/7! \leq P(\mathcal{A}\cap\mathcal{B}) \leq 8/7!$. To violate $P(\mathcal{A})P(\mathcal{B}) \leq P(\mathcal{A}\cap\mathcal{B})$, we require at least one of $P(\mathcal{A})$ and $P(\mathcal{B})$ to exceed $173/7!$ when $P(\mathcal{A}\cap\mathcal{B}) = 6/7!$, or $187/7!$ when $P(\mathcal{A}\cap\mathcal{B}) = 7/7!$, or $200/7!$ when $P(\mathcal{A}\cap\mathcal{B}) = 8/7!$. If $P(\mathcal{A}) \geq P(\mathcal{B})$, this forces $\mathcal{A}$ to contain several two-element sets or many of the three-element sets, and as before the restriction on $\mathcal{A}\cap\mathcal{B}$ then forces $\mathcal{B}$ to be comparatively small. Again, we omit the details.

(b). $k = 2$ and $n = 6$. In this case $\mathcal{A}\cap\mathcal{B}$ has one or more four-element sets and no three-element set. Suppose it contains s four-element sets with $1 \leq s \leq 4$ since the reallocation analysis of Lemma 3 applies if $s \geq 5$. Then $P(\mathcal{A}\cap\mathcal{B}) = (s+1)/6!$.

Consider $s = 1$ first, so $P(\mathcal{A}\cap\mathcal{B}) = 2/6!$. Let $\{1,2,3,4\}$ be in $\mathcal{A}\cap\mathcal{B}$. The only way to have two-sets in both $\mathcal{A}$ and $\mathcal{B}$ is to have exactly one $|A| = 2$ in each with empty intersection and $\{1,2,3,4\}$ as their union, say $\{1,2\} \in \mathcal{A}$ and $\{3,4\} \in \mathcal{B}$. The maximum $P(\mathcal{A})P(\mathcal{B})$ in this case is $768/(6!)^2$, compared to $P(\mathcal{A}\cap\mathcal{B}) = 1440/(6!)^2$. Further calculations show that this value of $P(\mathcal{A})P(\mathcal{B})$ cannot be exceeded.

Suppose next that $s = 2$, so $P(\mathcal{A}\cap\mathcal{B}) = 3/6!$. Then $\max P(\mathcal{A})P(\mathcal{B}) = 1104/(6!)^2$, obtained with one two-element set in $\mathcal{A}$ and two two-element sets in $\mathcal{B}$ (that have empty intersection with the one in $\mathcal{A}$). By comparison, $P(\mathcal{A}\cap\mathcal{B}) = 2160/(6!)^2$.

Cases for $s = 3$ and $s = 4$ are similar. For example, a comparatively large value of $P(\mathcal{A})P(\mathcal{B})$ for $s = 4$ is obtained with $\mathcal{A} = \{1,3\}^+ \cup \{1,5\}^+$ and $\mathcal{B} = \{2,4\}^+ \cup \{2,6\}^+ \cup \{3,4,5,6\}$. This gives $P(\mathcal{A})P(\mathcal{B}) = 1806/(6!)^2$, compared to $P(\mathcal{A}\cap\mathcal{B}) = 2880/(6!)^2$.

(c). $k = 1$ and $n \in \{6,7\}$. Here $P(\mathcal{A}\cap\mathcal{B}) = 1/n!$. For $n = 6$, $\max P(\mathcal{A})P(\mathcal{B}) = 432/(6!)^2$

with $\mathcal{A} = \{1,2\}^+$ and $\mathcal{B} = \{3,4,5\}^+ \cup \{3,4,6\}^+ \cup \{3,5,6\}^+ \cup \{4,5,6\}^+$. The best at $n = 7$ appears to be $P(\mathcal{A})P(\mathcal{B}) = 2640/(7!)^2$ with $\mathcal{A} = \{1,2\}^+$ and $\mathcal{B} = \bigcup_A A^+$ with each A a four-element subset of $\{3,4,5,6,7\}$.

This completes the proof of Lemma 2. ♣

References.

Ahlswede, R. and Daykin, D. E. (1978). An inequality for the weights of two families of sets, their unions and intersections. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 43* 183–185.

Feller,W. (1968). *An Introduction to Probability Theory and its Applications, Vol. I, 3rd ed.* Wiley, New York.

Fishburn, P. C. (1984). A correlational inequality for linear extensions of posets. *Order 1* 127–137.

Fortuin, C. M., Kasteleyn, P. W. and Ginibre, J. (1971). Correlation inequalities for some partially ordered sets. *Comm. Math. Phys. 22* 89–103.

Hwang, F. K. and Shepp, L. A. (1987) *IEEE Trans. Information Theory* (to appear).

Joag-Dev, K. (1985). Association of matchmakers. (Widely distributed preprint).

Shepp, L. A. (1980). The FKG inequality and some monotonicity properties of partial orders. *SIAM J. Alg. Disc. Meth. 1* 295–299.

Shepp, L. A. (1982). The XYZ conjecture and the FKG inequality. *Ann. Probab. 10* 824–827.

AT&T Bell Laboratories

Murray Hill, NJ 07974