On deciding whether a surface is parabolic or hyperbolic

Peter G. Doyle

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1 Introduction

In our book [3], Laurie Snell and I tell how a method from the classical theory of electricity called *Rayleigh's short-cut method* can be used to determine whether or not a person walking around at random on the vertices of a given infinite graph is certain to return to the starting point. In this paper, I will present an application of Rayleigh's method to the classical type problem for Riemann surfaces, which is really just the same random walk problem, only now instead of walking around on an infinite graph our walker is diffusing around on a surface. My aim will be to convince you that if you want to figure out whether or not a random walker gets lost, you should use Rayleigh's method.

The problem to which we will apply Rayleigh's method was raised by Milnor [5]. Milnor considered infinite surfaces that are rotationally symmetric about some point p, and showed that if the Gaussian curvature is negative enough then the surface is conformally hyperbolic, i.e., can be mapped conformally onto the unit disk, while if the curvature is just a little less negative then the surface is conformally parabolic, i.e., can be mapped conformally onto the whole plane. In probabilistic terms, this means that if the curvature is negative enough a particle diffusing around on the surface will eventually wander off and never come back, while if the curvature is just a little less negative the particle is bound to come back near where it started, no matter how far off it may have wandered.

Milnor goes on to suggest that his hyperbolicity and parabolicity criteria should remain valid even without the stringent condition of rotational symmetry, and tells how to extend the parabolicity criterion using a theorem of Ahlfors. Here I will finish the story by showing how to extend the hyperbolicity criterion also.

Note. The work described here formed part of the author's Ph.D. thesis [2] at Dartmouth College.

2 Milnor's criteria

In order to formulate Milnor's criteria in the absence of rotational symmetry, we need to make some assumptions about the surface. What we will assume is that the surface is a complete C^∞ Riemannian 2-manifold having a global geodesic polar coordinate system (r, θ) about some point p. (See Figure 1.) This means that geodesics emanating from p can be extended indefinitely without ever running into each other, and we can identify each point of the surface uniquely by telling how far it is from p, and the direction you have to go to get there. This condition will certainly be satisfied for simply-connected infinite surfaces of negative curvature, e.g., the graph of the function $z = xy$.

In terms of the (r, θ) coordinates the metric takes the form

$$ds = \sqrt{dr^2 + g(r, \theta)^2 d\theta^2}.$$

Thus

$$G(r) = \int_0^{2\pi} g(r, \theta) d\theta$$

is the length of the geodesic circle consisting of all points at distance r from p. The Gaussian curvature takes the form

$$K(r, \theta) = -\frac{\frac{\partial^2 g}{\partial r^2}(r, \theta)}{g(r, \theta)}.$$

Milnor wanted to relate the conformal type of the surface to the Gaussian curvature K. Specifically, he proposed the following two criteria:

P: If $K \geq -1/(r^2 \log r)$ for large r then the surface is parabolic.

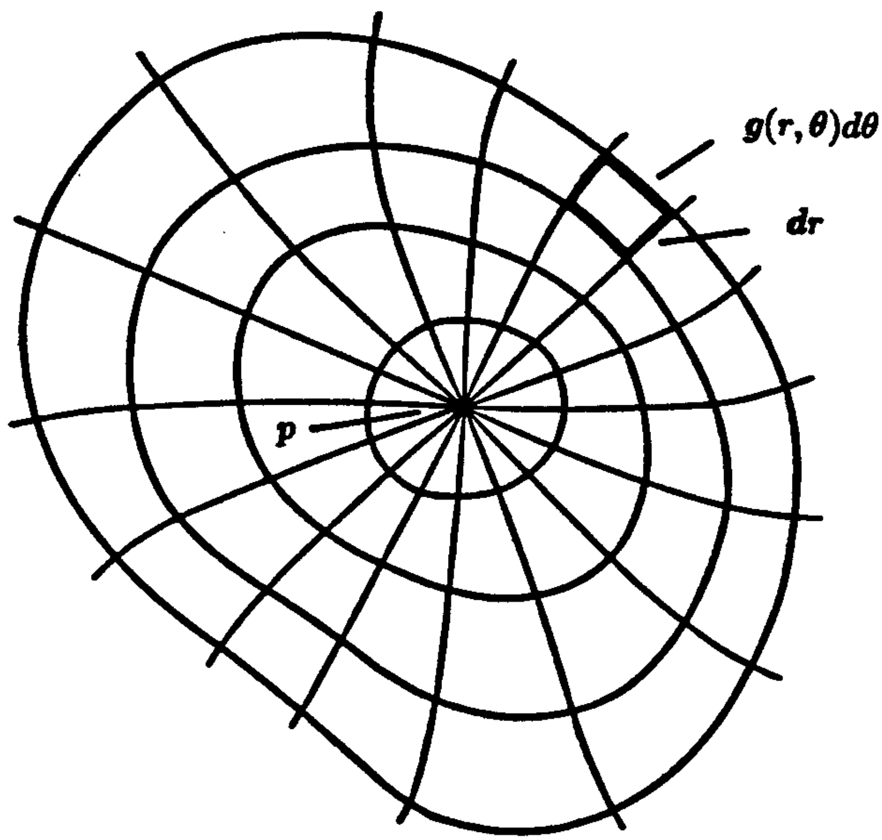

Figure 1: Geodesic polar coordinates.

H: If $K \le -(1+\epsilon)/(r^2 \log r)$ for large r, and if $G(r)$ is unbounded, then the surface is hyperbolic.

Milnor established these criteria for the case of a surface symmetric about p (that is, for a surface where $g(r,\theta)$ depends only on r) by constructing explicitly a conformal mapping of the surface onto a disk or the whole complex plane. He then asked for a proof in the more general case. Apparently Robert Osserman pointed out to him that by applying a method of Ahlfors we can conclude that the surface is parabolic whenever

$$\int_\alpha^\infty \frac{dr}{G(r)} = \infty.$$

The criterion **P** is a simple consequence of this parabolicity criterion. Hence what was missing was a proof of the criterion **H**.

3 Shorting

To someone familiar with Rayleigh's method, the expression appearing in Ahlfors's criterion has a particularly concrete significance. If we think of the surface as being made of an isotropic resistive material of "constant thickness," then the type problem can be interpreted as the problem of determining whether the resistance of the surface out to infinity is infinite (parabolic type) or finite (hyperbolic type). (See Figure 2.) The expression

$$\int_\alpha^\infty \frac{dr}{G(r)}$$

represents the resistance out to infinity of the electrical system obtained from the surface by "shorting the points of the surface together along the circles $r = \text{const.}$" Indeed, shorting along circles $r = \text{const}$ reduces the electrical system to an infinite number of infinitely thin rings hooked up in series, so we can compute the resistance simply by adding up the resistances of the rings; the resistance of each ring is proportional to its width dr, and inversely proportional to its length $G(r)$.

According to Rayleigh's shorting law, shorting always decreases resistance, so the resistance of the shorted system is $\le$ the resistance of the old

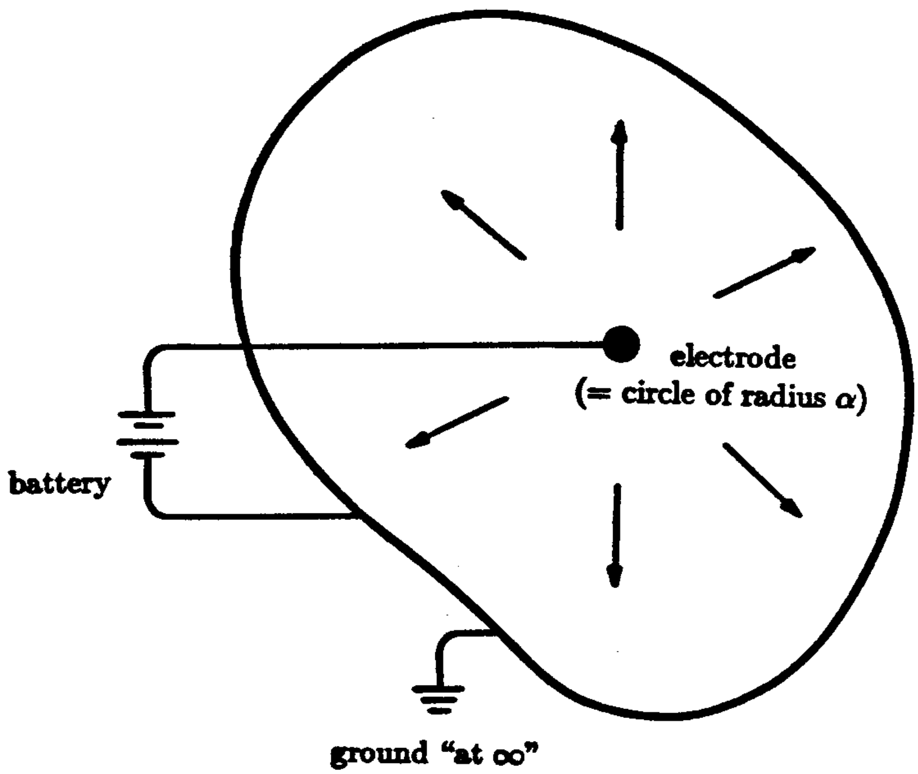

Figure 2: Measuring the resistance out to ∞.

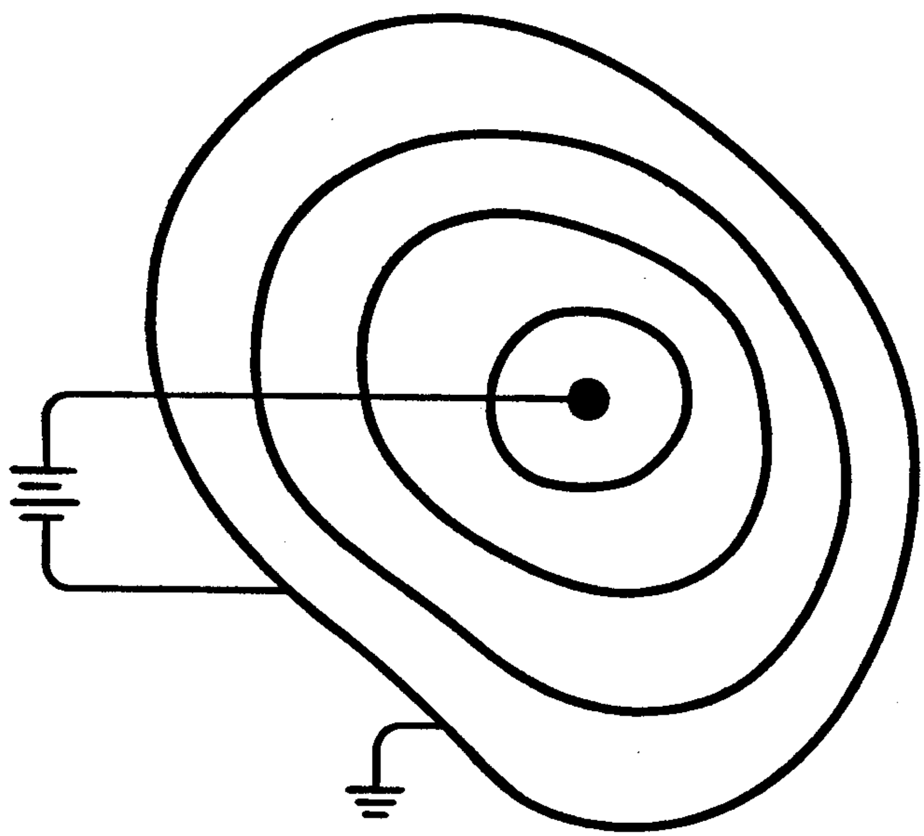

Figure 3: Shorting along circles $r = \text{const}$.

system. Thus we conclude that the surface is parabolic whenever

$$\int_\alpha^\infty \frac{dr}{G(r)} = \infty.$$

This is precisely Ahlfors's criterion. Thus from an electrical point of view this criterion is perfectly straight-forward and natural.

4 Cutting

To someone familiar with Rayleigh's method, the obvious next step is to get an upper bound for the resistance of the surface by making some cuts in it. In the case we are looking at, the most natural thing to do is to cut along the rays $\theta = \text{const}$, i.e., along the geodesics radiating from p. (See Figure 4.) The conductance (i.e., inverse resistance) of the resulting system out to infinity is

$$\int_0^{2\pi} \frac{d\theta}{\displaystyle\int_\alpha^\infty \frac{dr}{g(r,\theta)}}.$$

Indeed, cutting along rays $\theta = \text{const}$ reduces the electrical system to an infinite number of infinitely skinny strips hooked up in parallel, so we can compute the conductance simply by adding up the conductances of the strips. The resistance of a single strip is

$$\int_\alpha^\infty \frac{dr}{g(r,\theta)d\theta}.$$

(See Figure 5.) The conductance of a single strip is

$$\frac{d\theta}{\displaystyle\int_\alpha^\infty \frac{dr}{g(r,\theta)}},$$

and the conductance of the whole system is

$$\int_0^{2\pi} \frac{d\theta}{\displaystyle\int_\alpha^\infty \frac{dr}{g(r,\theta)}}.$$

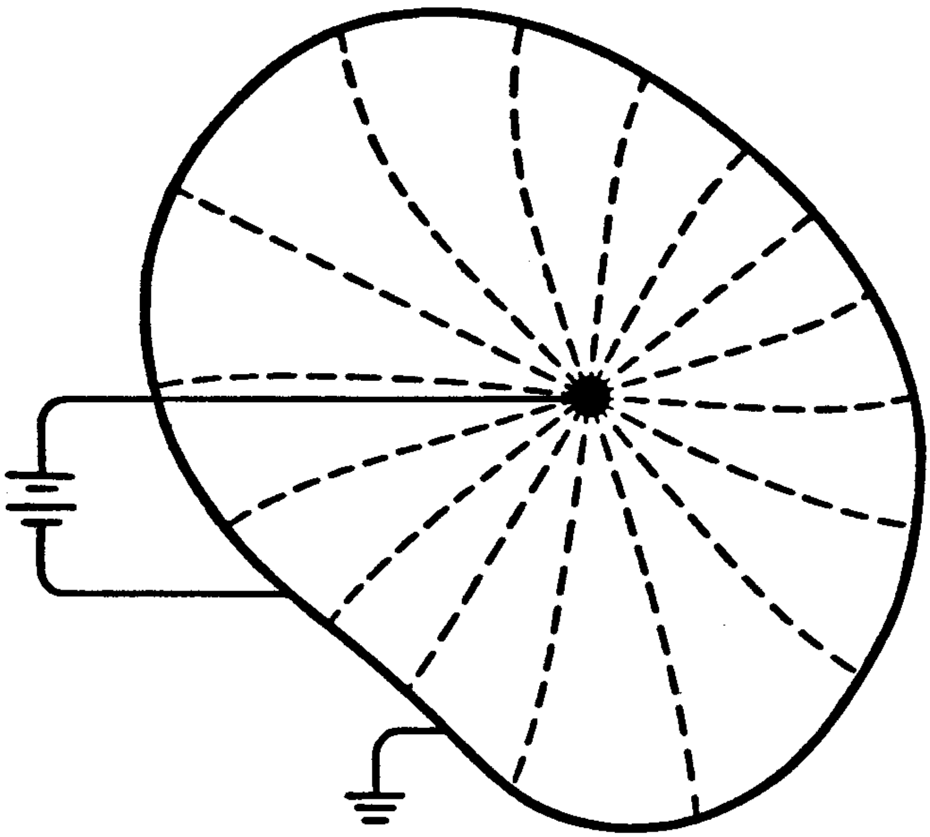

Figure 4: Cutting along rays $\theta = \mathrm{const.}$

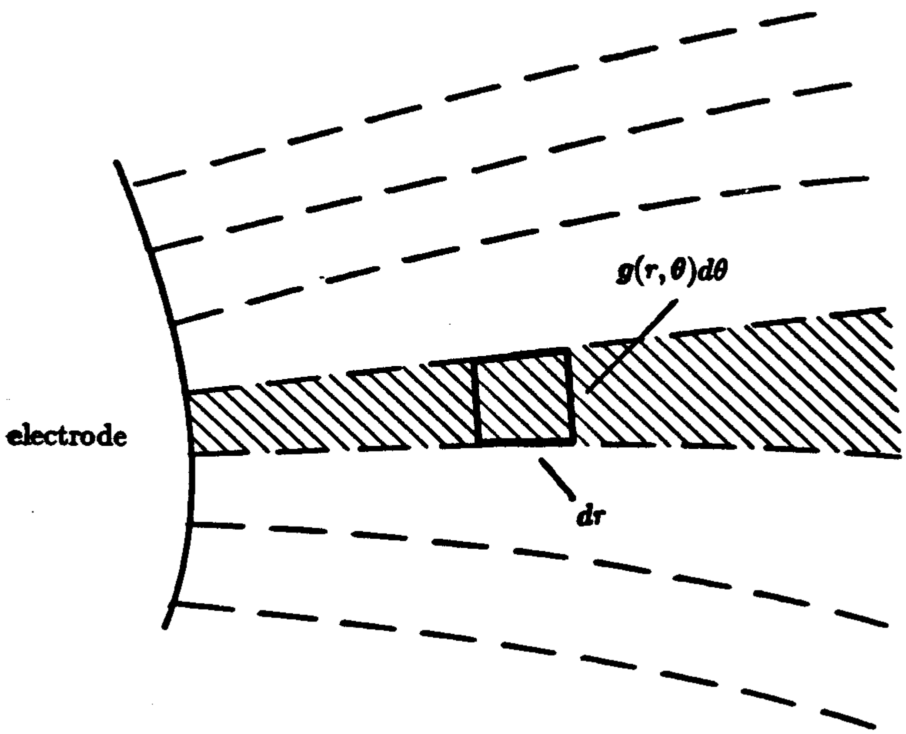

Figure 5: Determining the resistance of a strip.

According to Rayleigh's cutting law, cutting always reduces conductance, so the conductance of the cut system is $\leq$ the conductance of the original surface to infinity. Thus we conclude that the surface is hyperbolic whenever

$$\int_0^{2\pi} \frac{d\theta}{\displaystyle\int_\alpha^\infty \frac{dr}{g(r,\theta)}} > 0.$$

This hyperbolicity criterion obviously resembles Ahlfors's parabolicity criterion above, and just as **P** followed from the earlier criterion, so **H** follows from this one.

5 But can you make this rigorous?

When faced with a physicist's proof like the one I have just given, mathematicians invariably ask, "But can you make this rigorous?" This is fair enough, though still annoying. In this case we are in luck, because as Alfred Huber has pointed out to me, once you have used Rayleigh's method to discover the hyperbolicity criterion dual to Ahlfors's parabolicity criterion, you can use the time-honored method of extremal length to prove it. (If you want to know what this method is, see Ahlfors and Sario [1]; to see that it is really just Rayleigh's method in disguise, see Duffin [4].)

6 Conclusion

Rayleigh's method gives us a way of understanding and extending Milnor's investigations of the relationship between growth of the Gaussian curvature and the conformal type of a surface. This is just one example of how Rayleigh's method can be used to determine the conformal type of a surface. I hope this example will convince you of the beauty and power of Rayleigh's method, and inspire you to buy a copy of our book [3].

Acknowledgement. I would like to thank Alfred Huber for helpful correspondence about this problem.

References

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