Isospectral hyperbolic surfaces have matching geodesics

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Abstract

We show that if two closed hyperbolic surfaces (not necessarily orientable or even connected) have the same Laplace spectrum, then for every length they have the same number of orientation-preserving geodesics and the same number of orientation-reversing geodesics. In the orientable case, this result dates back to 1959, and can be proved by a straight-forward application of the Selberg Trace Formula. The extension to the non-orientable case involves a not-so-straightforward application of the Trace Formula.

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1 Introduction

We say that two hyperbolic surfaces (assumed closed but not necessarily orientable or even connected) are *almost conjugate* if their closed geodesics match, in the sense that for every length l they have the same number of

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orientation-preserving geodesics and the same number of orientation-reversing geodesics.

More generally, we say that two hyperbolic d-manifolds (assumed closed, but not necessarily orientable or even connected) are almost conjugate if their geodesics match with respect to length and 'twist'. The twist of a geodesic (also called its 'holonomy') is measured by the conjugacy class in $O(d)$ of the action of parallel translation around the geodesic. To say that geodesics have matching length and twist amounts to saying that the corresponding deck transformations are conjugate under the action of the full isometry group of hyperbolic d-space.

While we haven't specifically required it in the definition, the matching of geodesics between almost conjugate hyperbolic manifolds, whether surfaces or manifolds of higher dimension, can and should be taken to respect the imprimitivity index of the geodesics as well as their length and twist.

Please note that here and throughout, by 'geodesics' we mean oriented closed geodesics. Because our geodesics carry a designated orientation, the number of geodesics of length l will always be even, with each unoriented geodesic being counted twice, once for each orientation. So when we say, for example, that the number of geodesics of length at most l is asymptotically $\frac{e^l}{l}$, we're talking about oriented geodesics; the asymptotic number of unoriented geodesics would be $\frac{e^l}{2l}$.

According to the Selberg Trace Formula, almost conjugate hyperbolic manifolds are *isospectral*: They have the same Laplace eigenvalues with the same multiplicity. (Cf. Randol's chapter in [3]; Gangolli [8]; Bérard-Bergery [1].) And using Sunada's method [13], it is easy to construct pairs of non-isometric hyperbolic manifolds that are almost conjugate and hence isospectral. Sunada's method is very flexible, and works already in dimension 2: For an exposition, see Buser [2]. The isospectral surfaces that Buser describes are all orientable, but with trivial modifications the constructions can be made to yield isospectral pairs of non-orientable surfaces. Since we don't know of any handy reference for examples of non-orientable surfaces, we'll elaborate on this in Section 4. (You'll see very shortly why we care.)

We've said that according to Selberg, almost conjugate manifolds are isospectral. What about the converse:

Question 1 *If two hyperbolic manifolds are isospectral, must they be almost conjugate?*

In the case of orientable surfaces, where there is no twisting to contend with, the answer is yes: This is 'Huber's Theorem' [9, 10, 11], dating back to 1959. Nowadays we recognize this as a straight-forward consequence of the Trace Formula.

The purpose of this paper is to prove that the answer is still yes for surfaces, even without the orientability assumption. As we will see, this is a not-so-straight-forward consequence of the Trace Formula.

Theorem 1 *If two hyperbolic surfaces (not necessarily orientable or even connected) are isospectral, then they are almost conjugate.*

Now it will be apparent why we care about the existence of isospectral pairs of non-orientable surfaces: If such pairs didn't exist, Theorem 1 wouldn't go beyond Huber's Theorem.

In higher dimensions Question 1 remains open, even in the case of connected orientable manifolds. The issues at stake in higher dimensions are well illustrated in the proof of Theorem 1—so you might be interested in this theorem even if you don't see why anyone would care about non-orientable surfaces.

To see that the possible existence of isospectral hyperbolic manifolds that are not almost conjugate is a question that must be taken seriously, we note that in dimension $d \geq 3$, there exist isospectral flat manifolds that are not almost conjugate. The best possible example of this is the 3-manifold pair 'Tetra and Didi' [5]. In the flat case, some care is needed in defining almost conjugacy, because while in a hyperbolic manifold geodesics come only in isolation, in a flat manifold geodesics come in parallel families of varying dimension. So in the flat case, matching geodesics between manifolds involves measuring, rather than just counting. But Tetra and Didi will fail to be almost conjugate according to any definition.

Note. For further insight into the possible existence of isospectral spaces that are not almost conjugate, it is natural to expand the class of spaces we're considering from manifolds to orbifolds. (Cf. Dryden [6], Dryden and Strohmaier [7].) Of course we need to extend the definition of 'almost conjugacy' appropriately. We don't propose to discuss orbifolds in detail here, but for the benefit of those familiar with orbifolds, we have appended some comments in Section 5 below. Briefly, what we find is this: Theorem 1 extends to rule out examples among hyperbolic 2-orbifolds. However, there are examples of isospectral flat 2-orbifolds (necessarily disconnected) that are not almost conjugate. And we still don't know what happens in the hyperbolic case in dimension ≥ 3.

2 Outline

Let M be a hyperbolic surface, and γ a geodesic of length l and imprimitivity index ν (the number of times γ runs around a primitive ancestor). Define the *weight* $\mathrm{wt}(\gamma)$ as follows:

$$\mathrm{wt}_M(\gamma) = \begin{cases} \frac{1}{\nu} & \text{if } \gamma \text{ is orientation-preserving;} \\ \\ \frac{1}{\nu}\tanh(l/2) & \text{if } \gamma \text{ is orientation-reversing.} \end{cases} \tag{1}$$

Define the *total weight function*

$$W_M(l) = \sum_{l(\gamma)=l} \mathrm{wt}(\gamma). \tag{2}$$

From the Selberg Trace Formula, we have

Proposition 1 *Let M and N be hyperbolic surfaces, possibly non-orientable or disconnected. M and N are isospectral if and only if $W_M = W_N$.*

Sketch of proof. The weight $\mathrm{wt}_M(\gamma)$ tells the spectral contribution of γ, measured in units of the contribution of a primitive orientation-preserving geodesic of length $l(\gamma)$. Geodesics of different lengths make distinguishable contributions to the spectrum, but the contributions from geodesics of any given length get pooled together. (For details, see Randol's chapter in Chavel [3], specifically page 294; cf. also Gangolli [8] and Bérard-Bergery [1].) ∎

To prove Theorem 1 above, it suffices to show

Theorem 2 *If M and N are hyperbolic surfaces and $W_M = W_N$, then M and N are almost conjugate.*

Observe that this is a purely geometrical statement: All reference to the Laplace spectrum has been laundered through the total weight function.

To prove Theorem 2, we will analyze how we might engineer agreement between W_M and W_N without having total agreement between the geodesics of M and N, and show that this is not possible without having infinitely many lengths l for which the number of geodesics of length exactly l is at least $C\frac{e^l}{l}$, for $C > 0$. This will contradict the following Proposition, which is a simple consequence of the so-called 'Prime Geodesic Theorem'.

Proposition 2 *For any compact hyperbolic surface, the number of geodesics of length exactly l is $o(\frac{e^l}{l})$.*

Proof. According to the Prime Geodesic Theorem (see [12]), for a connected hyperbolic surface (whether orientable or not) the number $F(l)$ of geodesics of length at most l is asymptotic to $\frac{e^l}{l}$. The number $f(l)$ of geodesics of length exactly l is given by the jump of F at l:

$$f(l) = \lim_{s \to l+} F(s) - \lim_{s \to l-} F(s). \tag{3}$$

But if F is any positive increasing function asymptotic to G, the jumps of F are $o(G)$. So $f(l) = o(\frac{e^l}{l})$. This establishes our claim for connected surfaces. The extension to the general case is immediate, because the $o(\frac{e^l}{l})$ estimate holds separately on each of the finitely many connected components. ∎

3 Proof of Theorem 2

Let $\alpha_M(l)$ denote the number of primitive orientation-preserving geodesics in M of length exactly l, and $\beta_M(l)$ the number of primitive orientation-reversing geodesics.

Fix two surfaces M and N with $W_M = W_N$, and set

$$a(l) = \alpha_M(l) - \alpha_N(l); \tag{4}$$

$$b(l) = \beta_N(l) - \beta_M(l). \tag{5}$$

Note how, in the second definition, M and N have traded places. The reason for this switch is so that a and b will tend to have the same sign (though they may sometimes have opposite signs).

Let

$$L = \{l : \, a(l) \neq 0 \text{ or } b(l) \neq 0\} \tag{6}$$

and

$$L_0 = \{l \in L : \, l \text{ is not a multiple of any other element of } L\}. \tag{7}$$

According to this definition, L is the set of lengths of geodesics where M and N exhibit different behavior, and L_0 consists of those lengths which are minimal with respect to the partial order where $l \preceq m$ means $m = kl$, $k \in \mathbf{N}^+$. Every element of L sits above some minimal element, i.e.

$$L \subseteq L_0\mathbf{N}^+. \tag{8}$$

Our job is to show that $L_0 = \emptyset$.

Lemma 1 $|L_0| < \infty$

Proof. Suppose $l \in L_0$. By assumption, $W_M(l) = W_N(l)$. Because l is minimal in L, any contributions by imprimitive geodesics to $W_M(l)$ are exactly matched by contributions to $W_N(l)$. This means that the contributions of primitive geodesics of length l must match:

$$a(l) = \tanh(l/2)b(l). \tag{9}$$

Assume for convenience that $a(l) > 0$, and hence $b(l) > a(l)$. Rewrite the equation above:

$$b(l) - a(l) = b(l)(1 - \tanh(l/2)); \tag{10}$$

$$b(l) = \frac{b(l) - a(l)}{1 - \tanh(l/2)}. \tag{11}$$

When l is large,

$$b(l) = \frac{b(l) - a(l)}{1 - \tanh(l/2)} \approx \frac{e^l}{2}(b(l) - a(l)) \geq \frac{e^l}{2}. \tag{12}$$

According to Proposition 2, the total number of geodesics of length exactly l is $o(\frac{e^l}{l})$. Here we have at least something on the order of $\frac{e^l}{2}$ geodesics of length l. This puts an upper bound on l, and thus forces $|L_0| < \infty$. ■

Let P_{odd} denote the set of odd primes.

Lemma 2 *For any $l \in L_0$, only a finite number of the odd prime multiples of l are also multiples of an element of L_0 differing from l. Specifically,*

$$|lP_{\text{odd}} \cap (L_0 - \{l\})\mathbf{N}^+| \leq |L_0| - 1. \tag{13}$$

Proof. If $l_1 \in L_0$, $l_1 \neq l$, then $|lP_{\text{odd}} \cap l_1\mathbf{N}^+| \leq 1$. (This is a simple fact about about divisibility: It has nothing special to do with lengths of geodesics!) ■

Now fix any $l \in L_0$, and let p be an odd prime that avoids the finite set for which $pl \in (L_0 - \{l\})\mathbf{N}^+$. Since l is minimal in L, as above we have

$$a(l) = \tanh(l/2)b(l). \tag{14}$$

As above, assume for convenience that $a(l) > 0$, and hence $b(l) > a(l)$.

By assumption, $W_M(l) = W_N(l)$. The only geodesics that are 'in play' at length pl are those of length l or pl: That was the whole point of the restriction we've placed on p. So

$$a(pl) + \frac{1}{p}a(l) = \tanh(pl/2)\left(b(pl) + \frac{1}{p}b(l)\right). \tag{15}$$

Let's rework this:

$$a(pl) - \tanh(pl/2)b(pl) = \frac{1}{p}(\tanh(pl/2)b(l) - a(l)); \tag{16}$$

$$a(pl) - b(pl) + b(pl)(1 - \tanh(pl/2)) = \frac{1}{p}(\tanh(pl/2)b(l) - a(l)); \tag{17}$$

$$b(pl) = \frac{\frac{1}{p}(\tanh(pl/2)b(l) - a(l)) + (b(pl) - a(pl))}{1 - \tanh(pl/2)}. \tag{18}$$

Look at the numerator here. For p large, $\frac{1}{p}(\tanh(pl/2)b(l) - a(l))$ is close to $\frac{1}{p}(b(l) - a(l))$, and $b(l) - a(l)$ is a positive integer. And $b(pl) - a(pl)$ is always an integer: Not necessarily a positive integer, just some integer. As soon as p is larger than $2(b(l) - a(l))$, $\frac{1}{p}(b(l) - a(l))$ will be a positive fraction smaller than $1/2$, and adding an integer to it can only increase its absolute value. This means that for p large, the smallest the numerator can be in absolute value is something like $\frac{1}{p}(b(l) - a(l))$, which is at least $\frac{1}{p}$.

Meanwhile, the denominator is $1 - \tanh(pl/2) \approx 2e^{-pl}$. So $b(pl)$ is bigger than something like $\frac{e^{pl}}{2p}$. This contradicts Proposition 2—unless L_0 is empty! So $L_0 = \emptyset$, and M and N are almost conjugate. ∎

4 Isospectral nonorientable surfaces

It is well known that there are many examples of isospectral closed hyperbolic surfaces. The first example goes back to Vigneras [14], who constructed arithmetic examples from quaternion algebras. More recent constructions have used Sunada's method. Sunada's method is very flexible, and can produce nonorientable examples as easily as orientable examples. But as we don't know of a reference for this, we briefly outline the procedure here. For necessary backgound, see Buser [2].

If you take any pair of Sunada isospectral closed hyperbolic surfaces without boundary, then they have a common quotient. Now just add what Conway calls a 'cross-handle' to this quotient, i.e., take the connected sum with a Klein bottle. Or more generally, take the connected sum with any closed non-orientable surface. Put the hyperbolic metric on this new quotient, and lift everything (the cross-handles and the metric) back up to the covers. The resulting surfaces are isospectral and nonorientable.

Another way of producing isospectral nonorientable pairs is to change some of the gluings in known orientable examples where isospectrality is proven using transplantation. To take a specific example, consider the surfaces described by Buser [2], Chapter 11, page 304. If you reinterpret Buser's glueing diagrams (Figures 11.5.1 and 11.5.2) so that the identifications on the β geodesics are by translation, you get a non-orientable isospectral pair. The β identifications now add four cross-handles, rather than four handles. The transplantation method proving isospectrality in the orientable case continues to work here as well.

5 Comments on orbifolds

Here, as promised above, are some brief comments about orbifolds.

There are three independent examples of isospectral flat (disconnected) 2-orbifolds that are not almost conjugate, one involving quotients of a square torus, and two involving quotients of a hexagonal torus. We describe them using Conway's orbifold notation [4].

A standard square torus has as 2- and 4-fold quotients a 2222 orbifold and a 244 orbifold. If we call the torus S_1 and the quotients S_2 and S_4, spectrally

$$S_1 + 2S_4 = 3S_2, \tag{19}$$

i.e., you can't hear the difference between a torus with two 244s, and a trio of 2222s.

A standard hexagonal torus H_1 has as 2-, 3-, and 6-fold quotients a 2222 orbifold H_2 (this is a regular tetrahedron); a 333 orbifold H_3; and a 236 orbifold H_6. Spectrally,

$$H_2 + H_6 = 2H_3 \tag{20}$$

and

$$H_1 + H_3 + H_6 = 3H_2. \tag{21}$$

From these relations we can derive, for example:

$$H_1 + 3H_3 = 4H_2; \tag{22}$$

$$H_1 + 4H_6 = 5H_3; \tag{23}$$

$$2H_1 + 3H_6 = 5H_2. \tag{24}$$

These examples arise from a careful analysis of the contributions of rotations of various orders to the spectrum via the Selberg Trace Formula. To explain just how this works would take us too far afield. However, it is possible to verify isospectrality in these examples in a direct and elementary way by using Fourier series to represent explicitly the eigenfunctions of the component orbifolds, and checking that eigenvalues match up.

Among hyperbolic 2-orbifolds, no such examples exist, whether connected or not. This is a corollary of Theorem 1, together with the observation that, in contrast to the flat case, in the hyperbolic case elliptic elements of differing order make distinguishable contributions to the spectrum. Again, to go further into detail would take us too far afield.

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