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Chapter 3

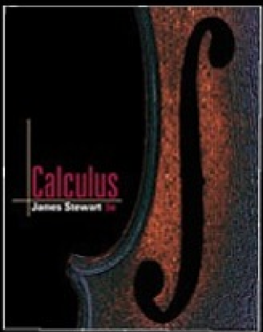
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



Calculus 5e
James Stewart
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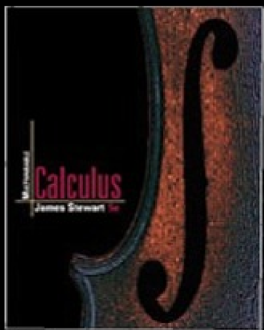

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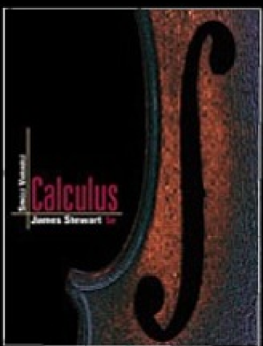

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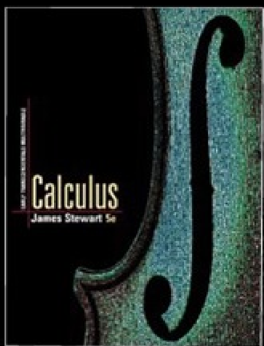

Calculus 5e - Multivariable
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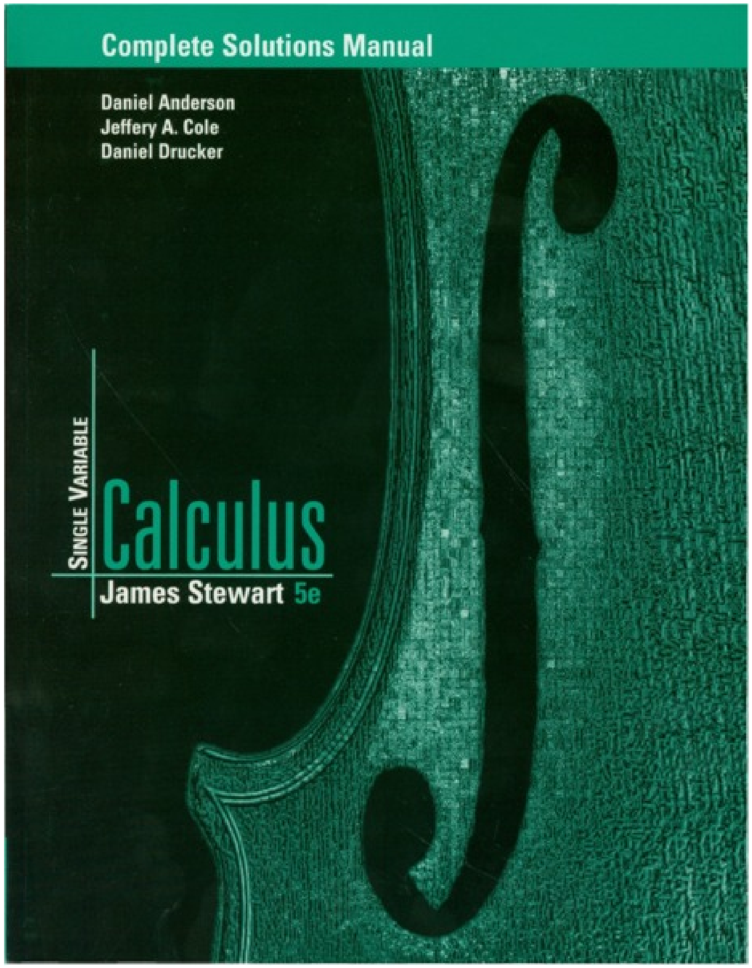
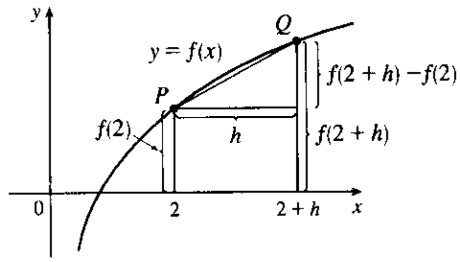


3 ☐ DERIVATIVES

3.1 Derivatives

1. 

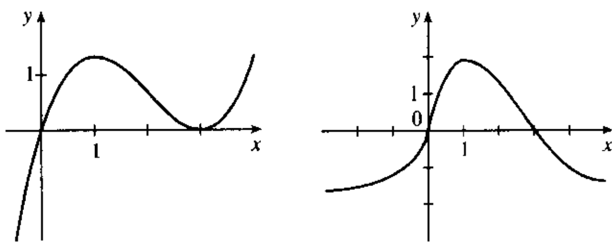
The line from $P(2, f(2))$ to $Q(2+h, f(2+h))$ is the line that has slope $\dfrac{f(2+h)-f(2)}{h}$.

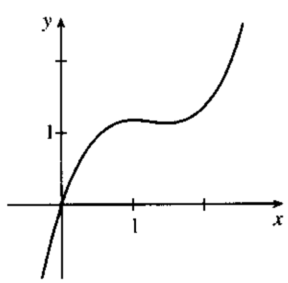
2. As h decreases, the line PQ becomes steeper, so its slope increases. So
$0 < \dfrac{f(4)-f(2)}{4-2} < \dfrac{f(3)-f(2)}{3-2} < \lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$. Thus, $0 < \frac{1}{2}\left[f(4)-f(2)\right] < f(3)-f(2) < f'(2)$.

3. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

4. Since $(4, 3)$ is on $y = f(x)$, $f(4) = 3$. The slope of the tangent line between $(0, 2)$ and $(4, 3)$ is $\frac{1}{4}$, so $f'(4) = \frac{1}{4}$.

5. We begin by drawing a curve through the origin at a slope of 3 to satisfy $f(0) = 0$ and $f'(0) = 3$. Since $f'(1) = 0$, we will round off our figure so that there is a horizontal tangent directly over $x = 1$. Lastly, we make sure that the curve has a slope of -1 as we pass over $x = 2$. Two of the many possibilities are shown.



6. 

7. Using Definition 2 with $f(x) = 3x^2 - 5x$ and the point $(2, 2)$, we have

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{\left[3(2+h)^2 - 5(2+h)\right] - 2}{h}$$

$$= \lim_{h\to 0} \frac{(12 + 12h + 3h^2 - 10 - 5h) - 2}{h} = \lim_{h\to 0} \frac{3h^2 + 7h}{h} = \lim_{h\to 0}(3h + 7) = 7.$$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = 7(x - 2)$ or $y = 7x - 12$.

8. Using Definition 2 with $g(x) = 1 - x^3$ and the point $(0, 1)$, we have

$$g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \to 0} \frac{\left[1 - (0+h)^3\right] - 1}{h} = \lim_{h \to 0} \frac{(1 - h^3) - 1}{h} = \lim_{h \to 0} (-h^2) = 0.$$

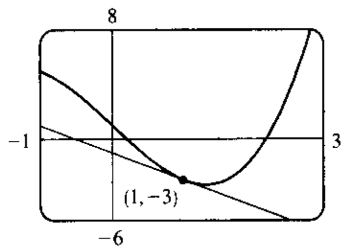
So an equation of the tangent line is $y - 1 = 0(x - 0)$ or $y = 1$.

9. (a) Using Definition 2 with $F(x) = x^3 - 5x + 1$ and the point $(1, -3)$, we have

$$\begin{aligned} F'(1) &= \lim_{h \to 0} \frac{F(1+h) - F(1)}{h} = \lim_{h \to 0} \frac{\left[(1+h)^3 - 5(1+h) + 1\right] - (-3)}{h} \\ &= \lim_{h \to 0} \frac{(1 + 3h + 3h^2 + h^3 - 5 - 5h + 1) + 3}{h} = \lim_{h \to 0} \frac{h^3 + 3h^2 - 2h}{h} \\ &= \lim_{h \to 0} \frac{h(h^2 + 3h - 2)}{h} = \lim_{h \to 0} (h^2 + 3h - 2) = -2 \end{aligned}$$

So an equation of the tangent line at $(1, -3)$ is $y - (-3) = -2(x - 1) \quad \Leftrightarrow \quad y = -2x - 1$.

(b)

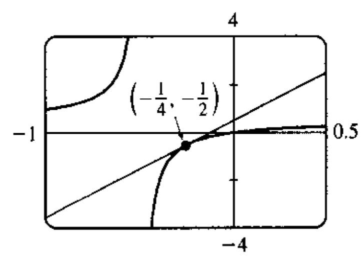


10. (a) $$G'(a) = \lim_{h \to 0} \frac{G(a+h) - G(a)}{h} = \lim_{h \to 0} \frac{\dfrac{a+h}{1 + 2(a+h)} - \dfrac{a}{1 + 2a}}{h}$$

$$= \lim_{h \to 0} \frac{a + 2a^2 + h + 2ah - a - 2a^2 - 2ah}{h(1 + 2a + 2h)(1 + 2a)} = \lim_{h \to 0} \frac{1}{(1 + 2a + 2h)(1 + 2a)} = (1 + 2a)^{-2}$$

So the slope of the tangent at the point $\left(-\frac{1}{4}, -\frac{1}{2}\right)$ is $m = \left[1 + 2\left(-\frac{1}{4}\right)\right]^{-2} = 4$, and thus an equation is $y + \frac{1}{2} = 4\left(x + \frac{1}{4}\right)$ or $y = 4x + \frac{1}{2}$.

(b)



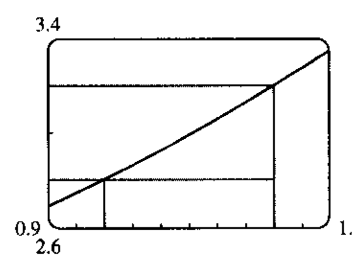
11. (a) $$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{3^{1+h} - 3^1}{h}.$$

So let $F(h) = \dfrac{3^{1+h} - 3}{h}$. We calculate:

h	$F(h)$	h	$F(h)$
0.1	3.484	−0.1	3.121
0.01	3.314	−0.01	3.278
0.001	3.298	−0.001	3.294
0.0001	3.296	−0.0001	3.296

We estimate that $f'(1) \approx 3.296$.

(b)



From the graph, we estimate that the slope of the tangent is about

$$\frac{3.2 - 2.8}{1.06 - 0.94} = \frac{0.4}{0.12} \approx 3.3.$$

12. (a) $g'\left(\frac{\pi}{4}\right) = \lim_{h\to 0} \dfrac{g\left(\frac{\pi}{4}+h\right) - g\left(\frac{\pi}{4}\right)}{h}$

$= \lim_{h\to 0} \dfrac{\tan\left(\frac{\pi}{4}+h\right) - \tan\left(\frac{\pi}{4}\right)}{h}$.

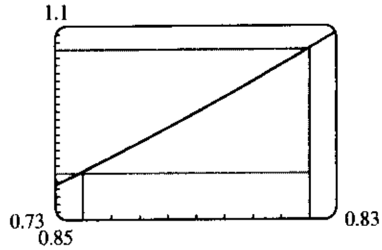
So let $G(h) = \dfrac{\tan\left(\frac{\pi}{4}+h\right) - 1}{h}$.

We calculate:

h	$G(h)$	h	$G(h)$
0.1	2.2305	−0.1	1.8237
0.01	2.0203	−0.01	1.9803
0.001	2.0020	−0.001	1.9980
0.0001	2.0002	−0.0001	1.9998

We estimate that $g'\left(\frac{\pi}{4}\right) = 2$.

(b)



From the graph, we estimate that the slope of the tangent is about $\dfrac{1.07 - 0.91}{0.82 - 0.74} = \dfrac{0.16}{0.08} = 2$.

13. Use Definition 2 with $f(x) = 3 - 2x + 4x^2$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\left[3 - 2(a+h) + 4(a+h)^2\right] - \left(3 - 2a + 4a^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(3 - 2a - 2h + 4a^2 + 8ah + 4h^2\right) - \left(3 - 2a + 4a^2\right)}{h}$$

$$= \lim_{h\to 0} \frac{-2h + 8ah + 4h^2}{h} = \lim_{h\to 0} \frac{h(-2 + 8a + 4h)}{h} = \lim_{h\to 0}(-2 + 8a + 4h) = -2 + 8a$$

14. $$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\left[(a+h)^4 - 5(a+h)\right] - \left(a^4 - 5a\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(a^4 + 4a^3h + 6a^2h^2 + 4ah^3 + h^4 - 5a - 5h\right) - \left(a^4 - 5a\right)}{h}$$

$$= \lim_{h\to 0} \frac{4a^3h + 6a^2h^2 + 4ah^3 + h^4 - 5h}{h} = \lim_{h\to 0} \frac{h\left(4a^3 + 6a^2h + 4ah^2 + h^3 - 5\right)}{h}$$

$$= \lim_{h\to 0} \left(4a^3 + 6a^2h + 4ah^2 + h^3 - 5\right) = 4a^3 - 5$$

15. Use Definition 2 with $f(t) = (2t+1)/(t+3)$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{2(a+h)+1}{(a+h)+3} - \dfrac{2a+1}{a+3}}{h}$$

$$= \lim_{h\to 0} \frac{(2a + 2h + 1)(a+3) - (2a+1)(a+h+3)}{h(a+h+3)(a+3)}$$

$$= \lim_{h\to 0} \frac{\left(2a^2 + 6a + 2ah + 6h + a + 3\right) - \left(2a^2 + 2ah + 6a + a + h + 3\right)}{h(a+h+3)(a+3)}$$

$$= \lim_{h\to 0} \frac{5h}{h(a+h+3)(a+3)} = \lim_{h\to 0} \frac{5}{(a+h+3)(a+3)} = \frac{5}{(a+3)^2}$$

16. $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \dfrac{\dfrac{(a+h)^2+1}{(a+h)-2} - \dfrac{a^2+1}{a-2}}{h}$

$$= \lim_{h\to 0} \frac{(a^2+2ah+h^2+1)(a-2) - (a^2+1)(a+h-2)}{h(a+h-2)(a-2)}$$

$$= \lim_{h\to 0} \frac{(a^3-2a^2+2a^2h-4ah+ah^2-2h^2+a-2) - (a^3+a^2h-2a^2+a+h-2)}{h(a+h-2)(a-2)}$$

$$= \lim_{h\to 0} \frac{a^2h-4ah+ah^2-2h^2-h}{h(a+h-2)(a-2)} = \lim_{h\to 0} \frac{h\left(a^2-4a+ah-2h-1\right)}{h(a+h-2)(a-2)}$$

$$= \lim_{h\to 0} \frac{a^2-4a+ah-2h-1}{(a+h-2)(a-2)} = \frac{a^2-4a-1}{(a-2)^2}$$

17. Use Definition 2 with $f(x) = 1/\sqrt{x+2}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{(a+h)+2}} - \dfrac{1}{\sqrt{a+2}}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{\sqrt{a+2}-\sqrt{a+h+2}}{\sqrt{a+h+2}\,\sqrt{a+2}}}{h} = \lim_{h\to 0} \left[\frac{\sqrt{a+2}-\sqrt{a+h+2}}{h\sqrt{a+h+2}\,\sqrt{a+2}} \cdot \frac{\sqrt{a+2}+\sqrt{a+h+2}}{\sqrt{a+2}+\sqrt{a+h+2}}\right]$$

$$= \lim_{h\to 0} \frac{(a+2)-(a+h+2)}{h\sqrt{a+h+2}\,\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)}$$

$$= \lim_{h\to 0} \frac{-h}{h\sqrt{a+h+2}\,\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)}$$

$$= \lim_{h\to 0} \frac{-1}{\sqrt{a+h+2}\,\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)}$$

$$= \frac{-1}{\left(\sqrt{a+2}\right)^2\left(2\sqrt{a+2}\right)} = -\frac{1}{2\,(a+2)^{3/2}}$$

18. $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \dfrac{\sqrt{3\,(a+h)+1} - \sqrt{3a+1}}{h}$

$$= \lim_{h\to 0} \frac{\left(\sqrt{3a+3h+1}-\sqrt{3a+1}\right)\left(\sqrt{3a+3h+1}+\sqrt{3a+1}\right)}{h\left(\sqrt{3a+3h+1}+\sqrt{3a+1}\right)}$$

$$= \lim_{h\to 0} \frac{(3a+3h+1)-(3a+1)}{h\left(\sqrt{3a+3h+1}+\sqrt{3a+1}\right)} = \lim_{h\to 0} \frac{3h}{h\left(\sqrt{3a+3h+1}+\sqrt{3a+1}\right)}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3a+3h+1}+\sqrt{3a+1}} = \frac{3}{2\sqrt{3a+1}}$$

Note that the answers to Exercises 19–24 are not unique.

19. By Definition 2, $\lim_{h\to 0} \dfrac{(1+h)^{10}-1}{h} = f'(1)$, where $f(x) = x^{10}$ and $a = 1$.

Or: By Definition 2, $\lim_{h\to 0} \dfrac{(1+h)^{10}-1}{h} = f'(0)$, where $f(x) = (1+x)^{10}$ and $a = 0$.

20. By Definition 2, $\lim\limits_{h \to 0} \dfrac{\sqrt[4]{16+h}-2}{h} = f'(16)$, where $f(x) = \sqrt[4]{x}$ and $a = 16$.

Or: By Definition 2, $\lim\limits_{h \to 0} \dfrac{\sqrt[4]{16+h}-2}{h} = f'(0)$, where $f(x) = \sqrt[4]{16+x}$ and $a = 0$.

21. By Equation 3, $\lim\limits_{x \to 5} \dfrac{2^x - 32}{x-5} = f'(5)$, where $f(x) = 2^x$ and $a = 5$.

22. By Equation 3, $\lim\limits_{x \to \pi/4} \dfrac{\tan x - 1}{x - \pi/4} = f'(\pi/4)$, where $f(x) = \tan x$ and $a = \pi/4$.

23. By Definition 2, $\lim\limits_{h \to 0} \dfrac{\cos(\pi+h)+1}{h} = f'(\pi)$, where $f(x) = \cos x$ and $a = \pi$.

Or: By Definition 2, $\lim\limits_{h \to 0} \dfrac{\cos(\pi+h)+1}{h} = f'(0)$, where $f(x) = \cos(\pi + x)$ and $a = 0$.

24. By Equation 3, $\lim\limits_{t \to 1} \dfrac{t^4+t-2}{t-1} = f'(1)$, where $f(t) = t^4 + t$ and $a = 1$.

25. $v(2) = f'(2) = \lim\limits_{h \to 0} \dfrac{f(2+h)-f(2)}{h} = \lim\limits_{h \to 0} \dfrac{\left[(2+h)^2 - 6(2+h) - 5\right] - \left[2^2 - 6(2) - 5\right]}{h}$

$= \lim\limits_{h \to 0} \dfrac{(4+4h+h^2-12-6h-5)-(-13)}{h} = \lim\limits_{h \to 0} \dfrac{h^2-2h}{h} = \lim\limits_{h \to 0}(h-2) = -2 \text{ m/s}$

26. $v(2) = f'(2) = \lim\limits_{h \to 0} \dfrac{f(2+h)-f(2)}{h}$

$= \lim\limits_{h \to 0} \dfrac{\left[2(2+h)^3 - (2+h) + 1\right] - \left[2(2)^3 - 2 + 1\right]}{h}$

$= \lim\limits_{h \to 0} \dfrac{(2h^3 + 12h^2 + 24h + 16 - 2 - h + 1) - 15}{h}$

$= \lim\limits_{h \to 0} \dfrac{2h^3 + 12h^2 + 23h}{h} = \lim\limits_{h \to 0} (2h^2 + 12h + 23) = 23 \text{ m/s}$

27. (a) $f'(x)$ is the rate of change of the production cost with respect to the number of ounces of gold produced. Its units are dollars per ounce.

(b) After 800 ounces of gold have been produced, the rate at which the production cost is increasing is \$17/ounce. So the cost of producing the 800th (or 801st) ounce is about \$17.

(c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of start-up costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

28. (a) $f'(5)$ is the rate of growth of the bacteria population when $t = 5$ hours. Its units are bacteria per hour.

(b) With unlimited space and nutrients, f' should increase as t increases; so $f'(5) < f'(10)$. If the supply of nutrients is limited, the growth rate slows down at some point in time, and the opposite may be true.

29. (a) $f'(v)$ is the rate at which the fuel consumption is changing with respect to the speed. Its units are $(\text{gal/h})/(\text{mi/h})$.

(b) The fuel consumption is decreasing by $0.05\ (\text{gal/h})/(\text{mi/h})$ as the car's speed reaches 20 mi/h. So if you increase your speed to 21 mi/h, you could expect to decrease your fuel consumption by about $0.05\ (\text{gal/h})/(\text{mi/h})$.

30. (a) $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per pound when the price is \$8 per pound. The units for $f'(8)$ are pounds/(dollars/pound).

(b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

31. $T'(10)$ is the rate at which the temperature is changing at 10:00 A.M. To estimate the value of $T'(10)$, we will average the difference quotients obtained using the times $t = 8$ and $t = 12$. Let

$A = \dfrac{T(8) - T(10)}{8 - 10} = \dfrac{72 - 81}{-2} = 4.5$ and $B = \dfrac{T(12) - T(10)}{12 - 10} = \dfrac{88 - 81}{2} = 3.5$. Then

$$T'(10) = \lim_{t \to 10} \frac{T(t) - T(10)}{t - 10} \approx \frac{A + B}{2} = \frac{4.5 + 3.5}{2} = 4°\text{F/h}.$$

32. For 1910: We will average the difference quotients obtained using the years 1900 and 1920.

Let $A = \dfrac{E(1900) - E(1910)}{1900 - 1910} = \dfrac{48.3 - 51.1}{-10} = 0.28$ and

$B = \dfrac{E(1920) - E(1910)}{1920 - 1910} = \dfrac{55.2 - 51.1}{10} = 0.41$. Then

$E'(1910) = \displaystyle\lim_{t \to 1910} \frac{E(t) - E(1910)}{t - 1910} \approx \frac{A + B}{2} = 0.345$. This means that life expectancy at birth was increasing at about 0.345 year/year in 1910.

For 1950: Using data for 1940 and 1960 in a similar fashion, we obtain $E'(1950) \approx [0.31 + 0.10]/2 = 0.205$. So life expectancy at birth was increasing at about 0.205 year/year in 1950.

33. (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are (mg/L)/°C.

(b) For $T = 16°$C, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So $S'(16) \approx \dfrac{6 - 14}{32 - 0} = -\dfrac{8}{32} = -0.25$ (mg/L)/°C. This means that as the temperature increases past 16°C, the oxygen solubility is decreasing at a rate of 0.25 (mg/L)/°C.

34. (a) $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are (cm/s)/°C.

(b) For $T = 15$ °C, it appears the tangent line to the curve goes through the points $(10, 25)$ and $(20, 32)$. So $S'(15) \approx \dfrac{32 - 25}{20 - 10} = 0.7$ (cm/s)/°C. This tells us that at $T = 15$ °C, the maximum sustainable speed of Coho salmon is changing at a rate of 0.7 (cm/s)/°C. In a similar fashion for $T = 25$ °C, we can use the points $(20, 35)$ and $(25, 25)$ to obtain $S'(25) \approx \dfrac{25 - 35}{25 - 20} = -2$ (cm/s)/°C. As it gets warmer than 20 °C, the maximum sustainable speed decreases rapidly.

35. Since $f(x) = x \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \displaystyle\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \to 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0. (Compare with Example 4 in Section 2.2.)

36. Since $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{h^2 \sin(1/h) - 0}{h} = \lim\limits_{h \to 0} h \sin(1/h)$. Since $-1 \leq \sin \dfrac{1}{h} \leq 1$, we have

$-|h| \leq |h| \sin \dfrac{1}{h} \leq |h| \quad \Rightarrow \quad -|h| \leq h \sin \dfrac{1}{h} \leq |h|$. Because $\lim\limits_{h \to 0} (-|h|) = 0$ and $\lim\limits_{h \to 0} |h| = 0$, we know that

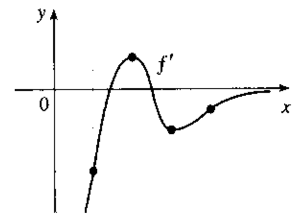
$\lim\limits_{h \to 0} \left(h \sin \dfrac{1}{h} \right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

3.2 The Derivative as a Function

1. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f, it appears that

(a) $f'(1) \approx -2$ (b) $f'(2) \approx 0.8$

(c) $f'(3) \approx -1$ (d) $f'(4) \approx -0.5$

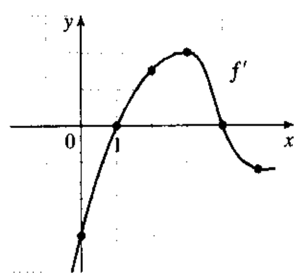


2. *Note:* Your answers may vary depending on your estimates. By estimating the slopes of tangent lines on the graph of f, it appears that

(a) $f'(0) \approx -3$ (b) $f'(1) \approx 0$

(c) $f'(2) \approx 1.5$ (d) $f'(3) \approx 2$

(e) $f'(4) \approx 0$ (f) $f'(5) \approx -1.2$



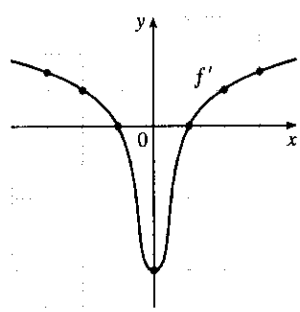
3. It appears that f is an odd function, so f' will be an even function—that is, $f'(-a) = f'(a)$.

(a) $f'(-3) \approx 1.5$ (b) $f'(-2) \approx 1$

(c) $f'(-1) \approx 0$ (d) $f'(0) \approx -4$

(e) $f'(1) \approx 0$ (f) $f'(2) \approx 1$

(g) $f'(3) \approx 1.5$



4. (a)$'$ = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

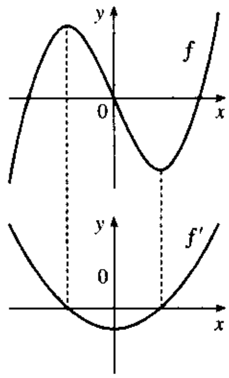
(b)$'$ = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)$'$ = I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

(d)$'$ = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 5–13: First plot x-intercepts on the graph of f' for any horizontal tangents on the graph of f. Look for any corners on the graph of f — there will be a discontinuity on the graph of f'. On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

5.



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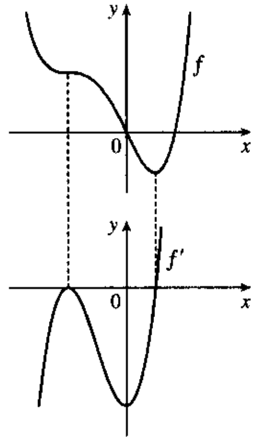

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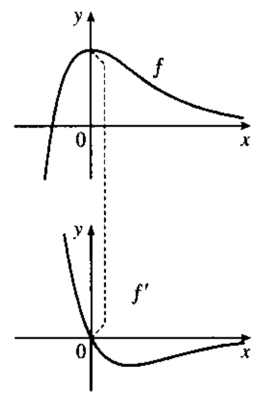

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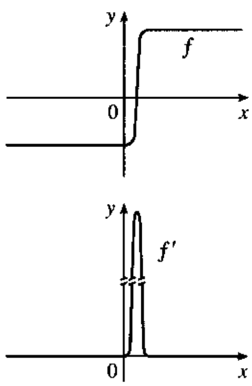

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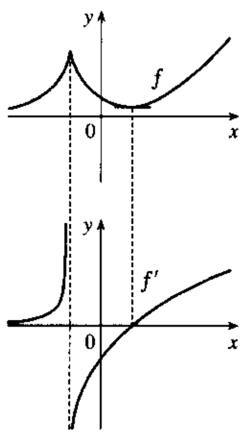

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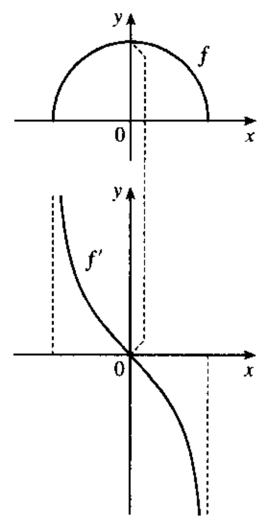

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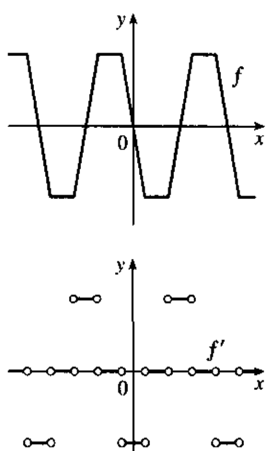

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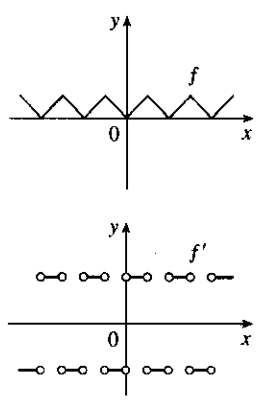

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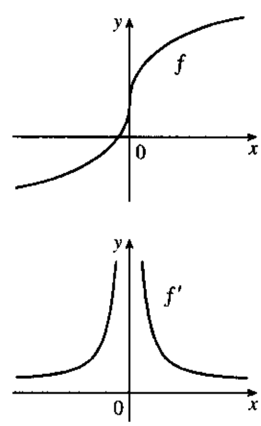
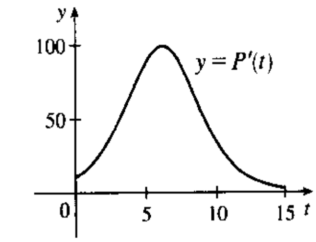
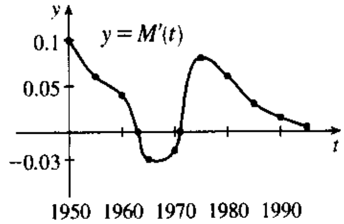


14. The slopes of the tangent lines on the graph of $y = P(t)$ are always positive, so the y-values of $y = P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t = 6$. Then the y-values of $y = P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.



15. It appears that there are horizontal tangents on the graph of M for $t = 1963$ and $t = 1971$. Thus, there are zeros for those values of t on the graph of M'. The derivative is negative for the years 1963 to 1971.

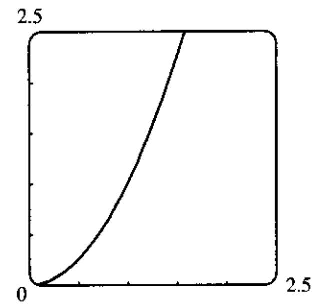


16. See Figure 1 in Section 3.5.

17. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.

(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

(c) It appears that $f'(x)$ is twice the value of x, so we guess that $f'(x) = 2x$.

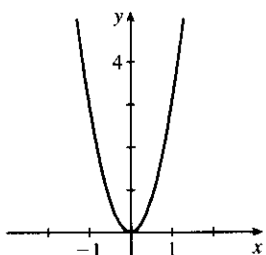


(d) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$
$$= \lim_{h\to 0} \frac{(x^2 + 2hx + h^2) - x^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h} = \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x+h) = 2x$$

18. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) \approx 0.75$, $f'(1) \approx 3$, $f'(2) \approx 12$, and $f'(3) \approx 27$.

(b) By symmetry, $f'(-x) = f'(x)$. So $f'\left(-\frac{1}{2}\right) \approx 0.75$, $f'(-1) \approx 3$, $f'(-2) \approx 12$, and $f'(-3) \approx 27$.

(c)



(d) Since $f'(0) = 0$, it appears that f' may have the form $f'(x) = ax^2$. Using $f'(1) = 3$, we have $a = 3$, so $f'(x) = 3x^2$.

(e) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0} \frac{h\left(3x^2 + 3xh + h^2\right)}{h} = \lim_{h\to 0} \left(3x^2 + 3xh + h^2\right) = 3x^2$$

19. $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{37 - 37}{h} = \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$
Domain of f = domain of f' = $\mathbb{R}$.

20. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[12+7(x+h)]-(12+7x)}{h}$

$= \lim_{h\to 0} \dfrac{12+7x+7h-12-7x}{h} = \lim_{h\to 0} \dfrac{7h}{h} = \lim_{h\to 0} 7 = 7$

Domain of f = domain of $f' = \mathbb{R}$.

21. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{[1-3(x+h)^2]-(1-3x^2)}{h}$

$= \lim_{h\to 0} \dfrac{[1-3(x^2+2xh+h^2)]-(1-3x^2)}{h} = \lim_{h\to 0} \dfrac{1-3x^2-6xh-3h^2-1+3x^2}{h}$

$= \lim_{h\to 0} \dfrac{-6xh-3h^2}{h} = \lim_{h\to 0} \dfrac{h(-6x-3h)}{h} = \lim_{h\to 0}(-6x-3h) = -6x$

Domain of f = domain of $f' = \mathbb{R}$.

22. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\left[5(x+h)^2+3(x+h)-2\right]-(5x^2+3x-2)}{h}$

$= \lim_{h\to 0} \dfrac{5x^2+10xh+5h^2+3x+3h-2-5x^2-3x+2}{h} = \lim_{h\to 0} \dfrac{10xh+5h^2+3h}{h}$

$= \lim_{h\to 0} \dfrac{h(10x+5h+3)}{h} = \lim_{h\to 0}(10x+5h+3) = 10x+3$

Domain of f = domain of $f' = \mathbb{R}$.

23. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\left[(x+h)^3-3(x+h)+5\right]-(x^3-3x+5)}{h}$

$= \lim_{h\to 0} \dfrac{(x^3+3x^2h+3xh^2+h^3-3x-3h+5)-(x^3-3x+5)}{h}$

$= \lim_{h\to 0} \dfrac{3x^2h+3xh^2+h^3-3h}{h} = \lim_{h\to 0} \dfrac{h(3x^2+3xh+h^2-3)}{h}$

$= \lim_{h\to 0} (3x^2+3xh+h^2-3) = 3x^2-3$

Domain of f = domain of $f' = \mathbb{R}$.

24. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\left(x+h+\sqrt{x+h}\right)-(x+\sqrt{x})}{h}$

$= \lim_{h\to 0} \left(\dfrac{h}{h} + \dfrac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) = \lim_{h\to 0} \left[1 + \dfrac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\right]$

$= \lim_{h\to 0} \left(1 + \dfrac{1}{\sqrt{x+h}+\sqrt{x}}\right) = 1 + \dfrac{1}{\sqrt{x}+\sqrt{x}} = 1 + \dfrac{1}{2\sqrt{x}}$

Domain of $f = [0,\infty)$, domain of $f' = (0,\infty)$.

25. $g'(x) = \lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h} \left[\dfrac{\sqrt{1+2(x+h)}+\sqrt{1+2x}}{\sqrt{1+2(x+h)}+\sqrt{1+2x}}\right]$

$= \lim_{h\to 0} \dfrac{(1+2x+2h)-(1+2x)}{h\left[\sqrt{1+2(x+h)}+\sqrt{1+2x}\right]} = \lim_{h\to 0} \dfrac{2}{\sqrt{1+2x+2h}+\sqrt{1+2x}} = \dfrac{2}{2\sqrt{1+2x}} = \dfrac{1}{\sqrt{1+2x}}$

Domain of $g = \left[-\frac{1}{2},\infty\right)$, domain of $g' = \left(-\frac{1}{2},\infty\right)$.

26. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{3+(x+h)}{1-3(x+h)} - \dfrac{3+x}{1-3x}}{h}$

$= \lim_{h\to 0} \dfrac{(3+x+h)(1-3x)-(3+x)(1-3x-3h)}{h(1-3x-3h)(1-3x)}$

$= \lim_{h\to 0} \dfrac{(3-9x+x-3x^2+h-3hx)-(3-9x-9h+x-3x^2-3hx)}{h(1-3x-3h)(1-3x)}$

$= \lim_{h\to 0} \dfrac{10h}{h(1-3x-3h)(1-3x)} = \lim_{h\to 0} \dfrac{10}{(1-3x-3h)(1-3x)} = \dfrac{10}{(1-3x)^2}$

Domain of f = domain of $f' = \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

27. $G'(t) = \lim_{h\to 0} \dfrac{G(t+h)-G(t)}{h} = \lim_{h\to 0} \dfrac{\dfrac{4(t+h)}{(t+h)+1} - \dfrac{4t}{t+1}}{h} = \lim_{h\to 0} \dfrac{\dfrac{4(t+h)(t+1)-4t(t+h+1)}{(t+h+1)(t+1)}}{h}$

$= \lim_{h\to 0} \dfrac{(4t^2+4ht+4t+4h)-(4t^2+4ht+4t)}{h(t+h+1)(t+1)}$

$= \lim_{h\to 0} \dfrac{4h}{h(t+h+1)(t+1)} = \lim_{h\to 0} \dfrac{4}{(t+h+1)(t+1)} = \dfrac{4}{(t+1)^2}$

Domain of G = domain of $G' = (-\infty, -1) \cup (-1, \infty)$.

28. $g'(x) = \lim_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} = \lim_{h\to 0} \dfrac{x^2-(x+h)^2}{h(x+h)^2x^2}$

$= \lim_{h\to 0} \dfrac{x^2-(x^2+2xh+h^2)}{h} = \lim_{h\to 0} \dfrac{-2xh-h^2}{h(x+h)^2x^2} = \lim_{h\to 0} \dfrac{-2x-h}{(x+h)^2x^2} = \dfrac{-2x}{x^4}$

$= -2x^{-3}$

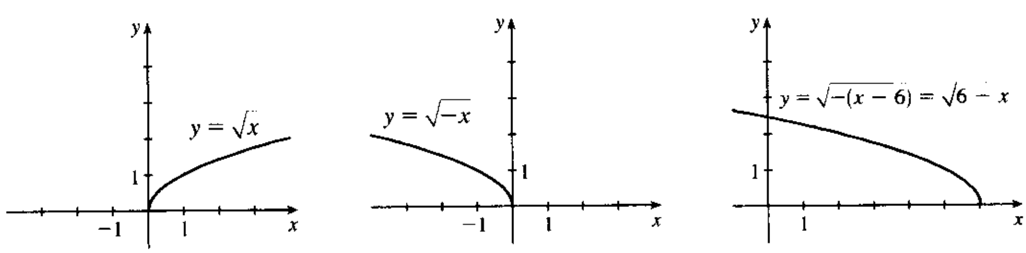
Domain of g = domain of $g' = \{x \mid x \neq 0\}$.

29. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^4-x^4}{h} = \lim_{h\to 0} \dfrac{(x^4+4x^3h+6x^2h^2+4xh^3+h^4)-x^4}{h}$

$= \lim_{h\to 0} \dfrac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim_{h\to 0} \left(4x^3+6x^2h+4xh^2+h^3\right) = 4x^3$

Domain of f = domain of $f' = \mathbb{R}$.

30. (a)



(b) Note that the third graph in part (a) has small negative values for its slope, f'; but as $x \to 6^-$, $f' \to -\infty$. See the graph in part (d).

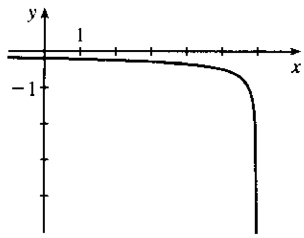
(c) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\sqrt{6-(x+h)}-\sqrt{6-x}}{h}\left[\frac{\sqrt{6-(x+h)}+\sqrt{6-x}}{\sqrt{6-(x+h)}+\sqrt{6-x}}\right]$$

$$=\lim_{h\to 0}\frac{[6-(x+h)]-(6-x)}{h\left[\sqrt{6-(x+h)}+\sqrt{6-x}\right]}=\lim_{h\to 0}\frac{-h}{h\left(\sqrt{6-x-h}+\sqrt{6-x}\right)}$$

$$=\lim_{h\to 0}\frac{-1}{\sqrt{6-x-h}+\sqrt{6-x}}=\frac{-1}{2\sqrt{6-x}}$$

Domain of $f=(-\infty,6]$, domain of $f'=(-\infty,6)$.

(d)

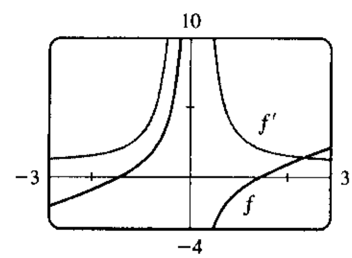


31. (a) $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\left[x+h-\left(\frac{2}{x+h}\right)\right]-\left[x-\left(\frac{2}{x}\right)\right]}{h}$$

$$=\lim_{h\to 0}\left[\frac{h-\frac{2}{(x+h)}+\frac{2}{x}}{h}\right]=\lim_{h\to 0}\left[1+\frac{-2x+2(x+h)}{hx(x+h)}\right]=\lim_{h\to 0}\left[1+\frac{2h}{hx(x+h)}\right]$$

$$=\lim_{h\to 0}\left[1+\frac{2}{x(x+h)}\right]=1+\frac{2}{x^2}$$

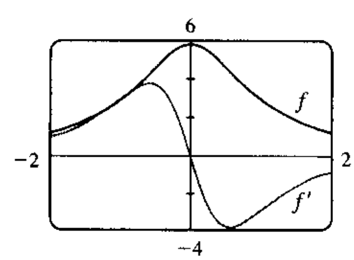
(b) Notice that when f has steep tangent lines, $f'(x)$ is very large. When f is flatter, $f'(x)$ is smaller.



32. (a) $$f'(t)=\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}=\lim_{h\to 0}\frac{\frac{6}{1+(t+h)^2}-\frac{6}{1+t^2}}{h}=\lim_{h\to 0}\frac{6+6t^2-6-6(t+h)^2}{h\left[1+(t+h)^2\right]\left(1+t^2\right)}$$

$$=\lim_{h\to 0}\frac{-12th-6h^2}{h\left[1+(t+h)^2\right]\left(1+t^2\right)}=\lim_{h\to 0}\frac{-12t-6h}{\left[1+(t+h)^2\right]\left(1+t^2\right)}=\frac{-12t}{(1+t^2)^2}$$

(b) Notice that f has a horizontal tangent when $t=0$. This corresponds to $f'(0)=0$. f' is positive when f is increasing and negative when f is decreasing.



33. (a) $U'(t)$ is the rate at which the unemployment rate is changing with respect to time. Its units are percent per year.

(b) To find $U'(t)$, we use $\lim\limits_{h\to 0} \dfrac{U(t+h)-U(t)}{h} \approx \dfrac{U(t+h)-U(t)}{h}$ for small values of h.

For 1991: $U'(1991) = \dfrac{U(1992)-U(1991)}{1992-1991} = \dfrac{7.5-6.8}{1} = 0.70$

For 1992: We estimate $U'(1992)$ by using $h=-1$ and $h=1$, and then average the two results to obtain a final estimate.

$h=-1 \;\Rightarrow\; U'(1992) \approx \dfrac{U(1991)-U(1992)}{1991-1992} = \dfrac{6.8-7.5}{-1} = 0.70;$

$h=1 \;\Rightarrow\; U'(1992) \approx \dfrac{U(1993)-U(1992)}{1993-1992} = \dfrac{6.9-7.5}{1} = -0.60.$

So we estimate that $U'(1992) \approx \frac{1}{2}[0.70+(-0.60)] = 0.05$.

t	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
$U'(t)$	0.70	0.05	−0.70	−0.65	−0.35	−0.35	−0.45	−0.35	−0.25	−0.20

34. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim\limits_{h\to 0} \dfrac{P(t+h)-P(t)}{h} \approx \dfrac{P(t+h)-P(t)}{h}$ for small values of h.

For 1950: $P'(1950) = \dfrac{P(1960)-P(1950)}{1960-1950} = \dfrac{35.7-31.1}{10} = 0.46$

For 1960: We estimate $P'(1960)$ by using $h=-10$ and $h=10$, and then average the two results to obtain a final estimate.

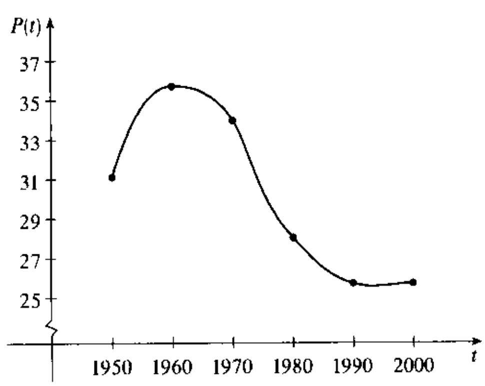
$h=-10 \;\Rightarrow\; P'(1960) \approx \dfrac{P(1950)-P(1960)}{1950-1960} = \dfrac{31.1-35.7}{-10} = 0.46$

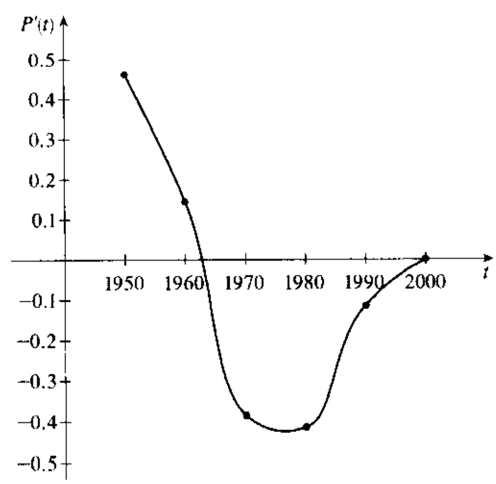
$h=10 \;\Rightarrow\; P'(1960) \approx \dfrac{P(1970)-P(1960)}{1970-1960} = \dfrac{34.0-35.7}{10} = -0.17$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46+(-0.17)] = 0.145$.

t	1950	1960	1970	1980	1990	2000
$P'(t)$	0.460	0.145	−0.385	−0.415	−0.115	0.000

(c)





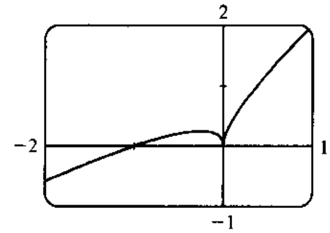
(d) We could get more accurate values for $P'(t)$ by obtaining data for the mid-decade years 1955, 1965, 1975, 1985, and 1995.

35. f is not differentiable at $x = -1$ or at $x = 11$ because the graph has vertical tangents at those points; at $x = 4$, because there is a discontinuity there; and at $x = 8$, because the graph has a corner there.

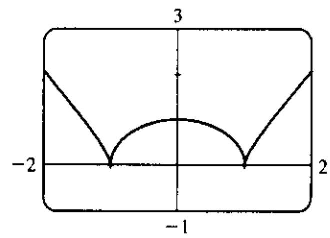
36. (a) g is discontinuous at $x = -2$ (a removable discontinuity), at $x = 0$ (g is not defined there), and at $x = 5$ (a jump discontinuity).

(b) g is not differentiable at the points mentioned in part (a) (by Theorem 4), nor is it differentiable at $x = -1$ (corner), $x = 2$ (vertical tangent), or $x = 4$ (vertical tangent).

37. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so $f(x) = x + \sqrt{|x|}$ is differentiable at $x = -1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = 0$.



38. As we zoom in toward $(0, 1)$, the curve appears more and more like a straight line, so f is differentiable at $x = 0$. But no matter how much we zoom in toward $(1, 0)$ or $(-1, 0)$, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = \pm 1$.



39. (a) Note that we have factored $x - a$ as the difference of two cubes in the third step.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

$$= \lim_{x \to a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{1}{3a^{2/3}} \text{ or } \tfrac{1}{3}a^{-2/3}$$

(b) $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt[3]{h} - 0}{h} = \lim\limits_{h \to 0} \dfrac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

(c) $\lim\limits_{x \to 0} |f'(x)| = \lim\limits_{x \to 0} \dfrac{1}{3x^{2/3}} = \infty$ and f is continuous at $x = 0$ (root function), so f has a vertical tangent at $x = 0$.

40. (a) $g'(0) = \lim\limits_{x \to 0} \dfrac{g(x) - g(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x^{2/3} - 0}{x} = \lim\limits_{x \to 0} \dfrac{1}{x^{1/3}}$, which does not exist.

(b) $$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} \frac{x^{2/3} - a^{2/3}}{x - a} = \lim_{x \to a} \frac{(x^{1/3} - a^{1/3})(x^{1/3} + a^{1/3})}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

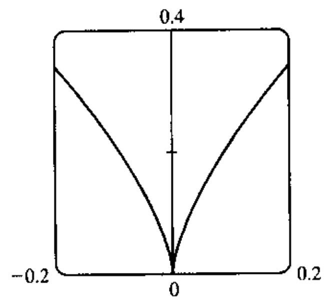
$$= \lim_{x \to a} \frac{x^{1/3} + a^{1/3}}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3a^{1/3}} \text{ or } \tfrac{2}{3}a^{-1/3}$$

(c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and

$\lim_{x\to 0} |g'(x)| = \lim_{x\to 0} \frac{2}{3\,|x|^{1/3}} = \infty$. This shows that

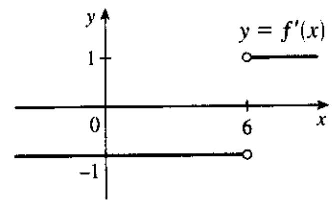
g has a vertical tangent line at $x = 0$.

(d)



41. $f(x) = |x - 6| = \begin{cases} -(x-6) & \text{if } x < 6 \\ x - 6 & \text{if } x \geq 6 \end{cases} = \begin{cases} 6 - x & \text{if } x < 6 \\ x - 6 & \text{if } x \geq 6 \end{cases}$

$$\lim_{x\to 6^+} \frac{f(x) - f(6)}{x - 6} = \lim_{x\to 6^+} \frac{|x-6| - 0}{x - 6} = \lim_{x\to 6^+} \frac{x-6}{x-6} = \lim_{x\to 6^+} 1 = 1.$$

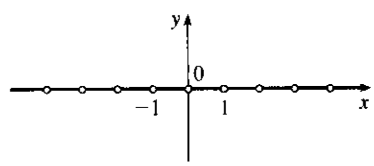


But $\lim_{x\to 6^-} \frac{f(x) - f(6)}{x - 6} = \lim_{x\to 6^-} \frac{|x-6| - 0}{x - 6} = \lim_{x\to 6^-} \frac{6-x}{x-6}$
$= \lim_{x\to 6^-} (-1) = -1$

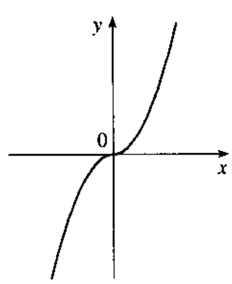
So $f'(6) = \lim_{x\to 6} \frac{f(x) - f(6)}{x - 6}$ does not exist. However, $f'(x) = \begin{cases} -1 & \text{if } x < 6 \\ 1 & \text{if } x > 6 \end{cases}$

Another way of writing the answer is $f'(x) = \frac{x - 6}{|x - 6|}$.

42. $f(x) = [\![x]\!]$ is not continuous at any integer n, so f is not differentiable at n by the contrapositive of Theorem 4. If a is not an integer, then f is constant on an open interval containing a, so $f'(a) = 0$. Thus, $f'(x) = 0$, x not an integer.



43. (a) $f(x) = x\,|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$



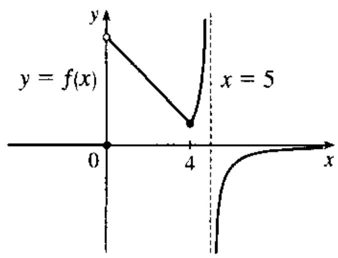
(b) Since $f(x) = x^2$ for $x \geq 0$, we have $f'(x) = 2x$ for $x > 0$. [See Exercise 3.2.17(d).] Similarly, since $f(x) = -x^2$ for $x < 0$, we have $f'(x) = -2x$ for $x < 0$. At $x = 0$, we have $f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x\,|x|}{x} = \lim_{x\to 0} |x| = 0$. So f is differentiable at 0. Thus, f is differentiable for all x.

(c) From part (b), we have $f'(x) = \left\{\begin{matrix} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{matrix}\right\} = 2\,|x|$.

44. (a) $f'_{-}(4) = \lim_{h\to 0^-} \dfrac{f(4+h)-f(4)}{h} = \lim_{h\to 0^-} \dfrac{5-(4+h)-1}{h} = \lim_{h\to 0^-} \dfrac{-h}{h} = -1$ and

$$f'_{+}(4) = \lim_{h\to 0^+} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0^+} \frac{\dfrac{1}{5-(4+h)}-1}{h} = \lim_{h\to 0^+} \frac{1-(1-h)}{h(1-h)} = \lim_{h\to 0^+} \frac{1}{1-h} = 1.$$

(b)



(c) $f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 5-x & \text{if } 0 < x < 4 \\ 1/(5-x) & \text{if } x \ge 4 \end{cases}$

These expressions show that f is continuous on the intervals $(-\infty, 0)$, $(0, 4)$, $(4, 5)$ and $(5, \infty)$. Since $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (5-x) = 5 \ne 0 = \lim_{x\to 0^-} f(x)$, $\lim_{x\to 0} f(x)$ does not exist, so f is discontinuous (and therefore not differentiable) at 0.

At 4 we have $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (5-x) = 1$ and $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \dfrac{1}{5-x} = 1$, so $\lim_{x\to 4} f(x) = 1 = f(4)$ and f is continuous at 4. Since $f(5)$ is not defined, f is discontinuous at 5.

(d) From (a), f is not differentiable at 4 since $f'_{-}(4) \ne f'_{+}(4)$, and from (c), f is not differentiable at 0 or 5.

45. (a) If f is even, then

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0} \frac{f[-(x-h)]-f(-x)}{h} = \lim_{h\to 0} \frac{f(x-h)-f(x)}{h}$$

$$= -\lim_{h\to 0} \frac{f(x-h)-f(x)}{-h} \quad [\text{let } \Delta x = -h] \quad = -\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = -f'(x)$$

Therefore, f' is odd.

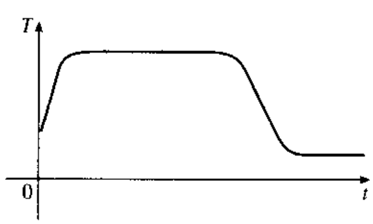
(b) If f is odd, then

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} = \lim_{h\to 0} \frac{f[-(x-h)]-f(-x)}{h} = \lim_{h\to 0} \frac{-f(x-h)+f(x)}{h}$$

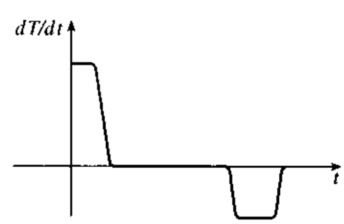
$$= \lim_{h\to 0} \frac{f(x-h)-f(x)}{-h} \quad [\text{let } \Delta x = -h] \quad = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$

Therefore, f' is even.

46. (a)

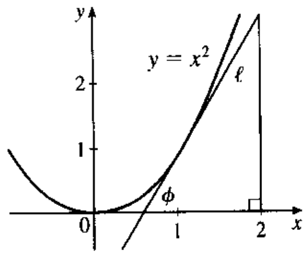


(c)



(b) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt = 0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.

47.



In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent angle ϕ. Then the slope of the tangent line ℓ is $m = \Delta y/\Delta x = \tan\phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 17) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \tan^{-1} 2 \approx 63°$.

3.3 Differentiation Formulas

1. $f(x) = 186.5$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

2. $f(x) = \sqrt{30}$ is a constant function, so its derivative is 0, that is, $f'(x) = 0$.

3. $f(x) = 5x - 1 \quad\Rightarrow\quad f'(x) = 5 - 0 = 5$

4. $F(x) = -4x^{10} \quad\Rightarrow\quad F'(x) = -4(10x^{10-1}) = -40x^9$

5. $f(x) = x^2 + 3x - 4 \quad\Rightarrow\quad f'(x) = 2x^{2-1} + 3 - 0 = 2x + 3$

6. $g(x) = 5x^8 - 2x^5 + 6 \quad\Rightarrow\quad g'(x) = 5(8x^{8-1}) - 2(5x^{5-1}) + 0 = 40x^7 - 10x^4$

7. $f(t) = \frac{1}{4}(t^4 + 8) \quad\Rightarrow\quad f'(t) = \frac{1}{4}(t^4 + 8)' = \frac{1}{4}(4t^{4-1} + 0) = t^3$

8. $f(t) = \frac{1}{2}t^6 - 3t^4 + t \quad\Rightarrow\quad f'(t) = \frac{1}{2}(6t^5) - 3(4t^3) + 1 = 3t^5 - 12t^3 + 1$

9. $V(r) = \frac{4}{3}\pi r^3 \quad\Rightarrow\quad V'(r) = \frac{4}{3}\pi(3r^2) = 4\pi r^2$

10. $R(t) = 5t^{-3/5} \quad\Rightarrow\quad R'(t) = 5\left[-\frac{3}{5}t^{(-3/5)-1}\right] = -3t^{-8/5}$

11. $Y(t) = 6t^{-9} \quad\Rightarrow\quad Y'(t) = 6(-9)t^{-10} = -54t^{-10}$

12. $R(x) = \dfrac{\sqrt{10}}{x^7} = \sqrt{10}\,x^{-7} \quad\Rightarrow\quad R'(x) = -7\sqrt{10}\,x^{-8} = -\dfrac{7\sqrt{10}}{x^8}$

13. $F(x) = (\frac{1}{2}x)^5 = \left(\frac{1}{2}\right)^5 x^5 = \frac{1}{32}x^5 \quad\Rightarrow\quad F'(x) = \frac{1}{32}(5x^4) = \frac{5}{32}x^4$

14. $f(t) = \sqrt{t} - \dfrac{1}{\sqrt{t}} = t^{1/2} - t^{-1/2} \quad\Rightarrow\quad f'(t) = \frac{1}{2}t^{-1/2} - \left(-\frac{1}{2}t^{-3/2}\right) = \dfrac{1}{2\sqrt{t}} + \dfrac{1}{2t\sqrt{t}}$

15. $y = x^{-2/5} \quad\Rightarrow\quad y' = -\frac{2}{5}x^{(-2/5)-1} = -\frac{2}{5}x^{-7/5} = -\dfrac{2}{5x^{7/5}}$

16. $y = \sqrt[3]{x} = x^{1/3} \quad\Rightarrow\quad y' = \frac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$

17. $y = 4\pi^2 \quad\Rightarrow\quad y' = 0$ since $4\pi^2$ is a constant.

18. $g(u) = \sqrt{2}\,u + \sqrt{3u} = \sqrt{2}\,u + \sqrt{3}\sqrt{u} \quad\Rightarrow\quad g'(u) = \sqrt{2}\,(1) + \sqrt{3}\left(\frac{1}{2}u^{-1/2}\right) = \sqrt{2} + \sqrt{3}/(2\sqrt{u})$

19. $v = t^2 - \dfrac{1}{\sqrt[4]{t^3}} = t^2 - t^{-3/4} \quad\Rightarrow\quad v' = 2t - \left(-\frac{3}{4}\right)t^{-7/4} = 2t + \dfrac{3}{4t^{7/4}} = 2t + \dfrac{3}{4t\,\sqrt[4]{t^3}}$

20. $u = \sqrt[3]{t^2} + 2\sqrt{t^3} = t^{2/3} + 2t^{3/2} \quad\Rightarrow\quad u' = \frac{2}{3}t^{-1/3} + 2\left(\frac{3}{2}\right)t^{1/2} = \dfrac{2}{3\sqrt[3]{t}} + 3\sqrt{t}$

21. Product Rule: $y = (x^2 + 1)(x^3 + 1) \quad\Rightarrow$

$$y' = (x^2 + 1)(3x^2) + (x^3 + 1)(2x) = 3x^4 + 3x^2 + 2x^4 + 2x = 5x^4 + 3x^2 + 2x.$$

Multiplying first: $y = (x^2 + 1)(x^3 + 1) = x^5 + x^3 + x^2 + 1 \quad\Rightarrow\quad y' = 5x^4 + 3x^2 + 2x$ (equivalent).

22. Quotient Rule: $F(x) = \dfrac{x - 3x\sqrt{x}}{\sqrt{x}} = \dfrac{x - 3x^{3/2}}{x^{1/2}} \quad\Rightarrow$

$$F'(x) = \frac{x^{1/2}\left(1 - \frac{9}{2}x^{1/2}\right) - \left(x - 3x^{3/2}\right)\left(\frac{1}{2}x^{-1/2}\right)}{\left(x^{1/2}\right)^2}$$

$$= \frac{x^{1/2} - \frac{9}{2}x - \frac{1}{2}x^{1/2} + \frac{3}{2}x}{x} = \frac{\frac{1}{2}x^{1/2} - 3x}{x} = \tfrac{1}{2}x^{-1/2} - 3$$

Simplifying first: $F(x) = \dfrac{x - 3x\sqrt{x}}{\sqrt{x}} = \sqrt{x} - 3x = x^{1/2} - 3x \quad\Rightarrow\quad F'(x) = \frac{1}{2}x^{-1/2} - 3$ (equivalent).

For this problem, simplifying first seems to be the better method.

The notations $\overset{\text{PR}}{\Rightarrow}$ and $\overset{\text{QR}}{\Rightarrow}$ indicate use of the Product and Quotient Rules, respectively.

23. $V(x) = (2x^3 + 3)(x^4 - 2x) \quad \overset{\text{PR}}{\Rightarrow}$

$V'(x) = (2x^3 + 3)(4x^3 - 2) + (x^4 - 2x)(6x^2) = (8x^6 + 8x^3 - 6) + (6x^6 - 12x^3) = 14x^6 - 4x^3 - 6$

24. $Y(u) = (u^{-2} + u^{-3})(u^5 - 2u^2) \quad \overset{\text{PR}}{\Rightarrow}$

$$Y'(u) = (u^{-2} + u^{-3})(5u^4 - 4u) + (u^5 - 2u^2)(-2u^{-3} - 3u^{-4})$$

$$= (5u^2 - 4u^{-1} + 5u - 4u^{-2}) + (-2u^2 - 3u + 4u^{-1} + 6u^{-2}) = 3u^2 + 2u + 2u^{-2}$$

25. $F(y) = \left(\dfrac{1}{y^2} - \dfrac{3}{y^4}\right)(y + 5y^3) = (y^{-2} - 3y^{-4})(y + 5y^3) \quad \overset{\text{PR}}{\Rightarrow}$

$$F'(y) = (y^{-2} - 3y^{-4})(1 + 15y^2) + (y + 5y^3)(-2y^{-3} + 12y^{-5})$$

$$= (y^{-2} + 15 - 3y^{-4} - 45y^{-2}) + (-2y^{-2} + 12y^{-4} - 10 + 60y^{-2})$$

$$= 5 + 14y^{-2} + 9y^{-4} \text{ or } 5 + 14/y^2 + 9/y^4$$

26. $y = \sqrt{x}\,(x - 1) = x^{3/2} - x^{1/2} \quad\Rightarrow\quad y' = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} = \frac{1}{2}x^{-1/2}(3x - 1)$ [factor out $\frac{1}{2}x^{-1/2}$] or $y' = \dfrac{3x - 1}{2\sqrt{x}}$.

27. $g(x) = \dfrac{3x - 1}{2x + 1} \quad \overset{\text{QR}}{\Rightarrow} \quad g'(x) = \dfrac{(2x + 1)(3) - (3x - 1)(2)}{(2x + 1)^2} = \dfrac{6x + 3 - 6x + 2}{(2x + 1)^2} = \dfrac{5}{(2x + 1)^2}$

28. $f(t) = \dfrac{2t}{4 + t^2} \quad \overset{\text{QR}}{\Rightarrow} \quad f'(t) = \dfrac{(4 + t^2)(2) - (2t)(2t)}{(4 + t^2)^2} = \dfrac{8 + 2t^2 - 4t^2}{(4 + t^2)^2} = \dfrac{8 - 2t^2}{(4 + t^2)^2}$

29. $y = \dfrac{t^2}{3t^2 - 2t + 1} \quad \overset{\text{QR}}{\Rightarrow}$

$$y' = \frac{(3t^2 - 2t + 1)(2t) - t^2(6t - 2)}{(3t^2 - 2t + 1)^2} = \frac{2t\left[3t^2 - 2t + 1 - t(3t - 1)\right]}{(3t^2 - 2t + 1)^2}$$

$$= \frac{2t(3t^2 - 2t + 1 - 3t^2 + t)}{(3t^2 - 2t + 1)^2} = \frac{2t(1 - t)}{(3t^2 - 2t + 1)^2}$$

30. $y = \dfrac{t^3 + t}{t^4 - 2} \quad \overset{\text{QR}}{\Rightarrow} \quad y' = \dfrac{(t^4 - 2)(3t^2 + 1) - (t^3 + t)(4t^3)}{(t^4 - 2)^2} = \dfrac{(3t^6 + t^4 - 6t^2 - 2) - (4t^6 + 4t^4)}{(t^4 - 2)^2}$

$$= \frac{-t^6 - 3t^4 - 6t^2 - 2}{(t^4 - 2)^2} = -\frac{t^6 + 3t^4 + 6t^2 + 2}{(t^4 - 2)^2}$$

31. $y = \dfrac{v^3 - 2v\sqrt{v}}{v} = v^2 - 2\sqrt{v} = v^2 - 2v^{1/2} \Rightarrow y' = 2v - 2\left(\frac{1}{2}\right)v^{-1/2} = 2v - v^{-1/2}$.

We can change the form of the answer as follows: $2v - v^{-1/2} = 2v - \dfrac{1}{\sqrt{v}} = \dfrac{2v\sqrt{v} - 1}{\sqrt{v}} = \dfrac{2v^{3/2} - 1}{\sqrt{v}}$

32. $y = \dfrac{\sqrt{x} - 1}{\sqrt{x} + 1} \Rightarrow$

$$y' = \frac{(\sqrt{x}+1)\left(\dfrac{1}{2\sqrt{x}}\right) - (\sqrt{x}-1)\left(\dfrac{1}{2\sqrt{x}}\right)}{(\sqrt{x}+1)^2} = \frac{\dfrac{1}{2} + \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2} + \dfrac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2} = \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}$$

33. $y = \dfrac{1}{x^4 + x^2 + 1} \Rightarrow y' = \dfrac{(x^4 + x^2 + 1)(0) - 1\,(4x^3 + 2x)}{(x^4 + x^2 + 1)^2} = -\dfrac{2x(2x^2 + 1)}{(x^4 + x^2 + 1)^2}$

34. $y = x^2 + x + x^{-1} + x^{-2} \Rightarrow y' = 2x + 1 - x^{-2} - 2x^{-3}$

35. $y = ax^2 + bx + c \Rightarrow y' = 2ax + b$

36. $y = A + \dfrac{B}{x} + \dfrac{C}{x^2} = A + Bx^{-1} + Cx^{-2} \Rightarrow y' = -Bx^{-2} - 2Cx^{-3} = -\dfrac{B}{x^2} - 2\dfrac{C}{x^3}$

37. $y = \dfrac{r^2}{1 + \sqrt{r}} \Rightarrow$

$$y' = \frac{(1+\sqrt{r})(2r) - r^2\left(\frac{1}{2}r^{-1/2}\right)}{(1+\sqrt{r})^2} = \frac{2r + 2r^{3/2} - \frac{1}{2}r^{3/2}}{(1+\sqrt{r})^2} = \frac{2r + \frac{3}{2}r^{3/2}}{(1+\sqrt{r})^2} = \frac{\frac{1}{2}r(4 + 3r^{1/2})}{(1+\sqrt{r})^2} = \frac{r(4 + 3\sqrt{r})}{2(1+\sqrt{r})^2}$$

38. $y = \dfrac{cx}{1 + cx} \Rightarrow y' = \dfrac{(1+cx)(c) - (cx)(c)}{(1+cx)^2} = \dfrac{c + c^2x - c^2x}{(1+cx)^2} = \dfrac{c}{(1+cx)^2}$

39. $y = \sqrt[3]{t}\,(t^2 + t + t^{-1}) = t^{1/3}(t^2 + t + t^{-1}) = t^{7/3} + t^{4/3} + t^{-2/3} \Rightarrow$
$y' = \frac{7}{3}t^{4/3} + \frac{4}{3}t^{1/3} - \frac{2}{3}t^{-5/3} = \frac{1}{3}t^{-5/3}(7t^{9/3} + 4t^{6/3} - 2) = (7t^3 + 4t^2 - 2)/(3t^{5/3})$

40. $y = \dfrac{u^6 - 2u^3 + 5}{u^2} = u^4 - 2u + 5u^{-2} \Rightarrow$
$y' = 4u^3 - 2 - 10u^{-3} = 2u^{-3}(2u^6 - u^3 - 5) = 2(2u^6 - u^3 - 5)/u^3$

41. $f(x) = \dfrac{x}{x + c/x} \Rightarrow$

$$f'(x) = \frac{(x + c/x)(1) - x(1 - c/x^2)}{\left(x + \dfrac{c}{x}\right)^2} = \frac{x + c/x - x + c/x}{\left(\dfrac{x^2 + c}{x}\right)^2} = \frac{2c/x}{\dfrac{(x^2+c)^2}{x^2}} \cdot \frac{x^2}{x^2} = \frac{2cx}{(x^2+c)^2}$$

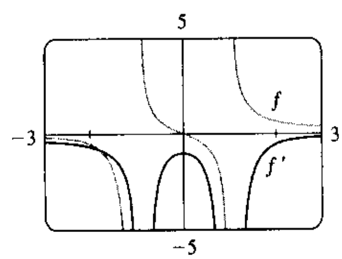
42. $f(x) = \dfrac{ax + b}{cx + d} \Rightarrow f'(x) = \dfrac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2} = \dfrac{acx + ad - acx - bc}{(cx+d)^2} = \dfrac{ad - bc}{(cx+d)^2}$

43. $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \Rightarrow$
$P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1$

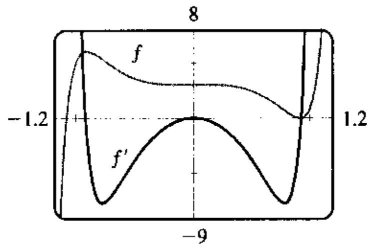
44. $f(x) = \dfrac{x}{x^2 - 1} \Rightarrow$

$$f'(x) = \frac{(x^2-1)1 - x(2x)}{(x^2-1)^2} = \frac{-x^2 - 1}{(x^2-1)^2} = -\frac{x^2+1}{(x^2-1)^2}$$

Notice that the slopes of all tangents to f are negative and $f'(x) < 0$ always.

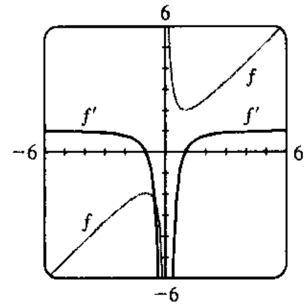


45. $f(x) = 3x^{15} - 5x^3 + 3 \Rightarrow$
$f'(x) = 45x^{14} - 15x^2$.



Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

46. $f(x) = x + 1/x = x + x^{-1} \Rightarrow$
$f'(x) = 1 - x^{-2} = 1 - 1/x^2$.



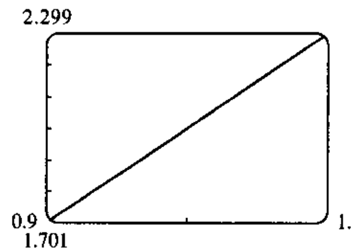
Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

47. To graphically estimate the value of $f'(1)$ for $f(x) = 3x^2 - x^3$, we'll graph f in the viewing rectangle $[1 - 0.1, 1 + 0.1]$ by $[f(0.9), f(1.1)]$, as shown in the figure. [When assigning values to the window variables, it is convenient to use $Y_1(0.9)$ for Y_{min} and $Y_1(1.1)$ for Y_{max}.] If we have sufficiently zoomed in on the graph of f, we should obtain a graph that looks like a diagonal line; if not, graph again with $1 - 0.01$ and $1 + 0.01$, etc.

Estimated value:

$$f'(1) \approx \frac{2.299 - 1.701}{1.1 - 0.9} = \frac{0.589}{0.2} = 2.99.$$

Exact value: $f(x) = 3x^2 - x^3 \Rightarrow f'(x) = 6x - 3x^2$, so $f'(1) = 6 - 3 = 3$.

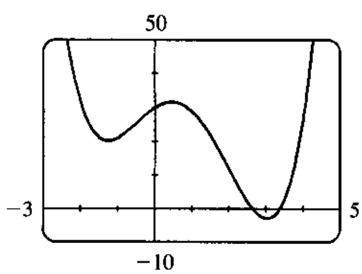


48. See the previous exercise. Since f is a decreasing function, assign $Y_1(3.9)$ to Y_{max} and $Y_1(4.1)$ to Y_{min}.

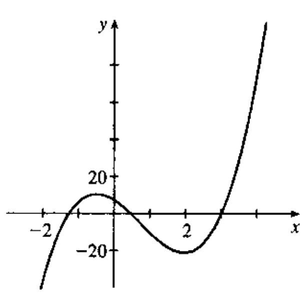
Estimated value: $f'(4) \approx \dfrac{0.49386 - 0.50637}{4.1 - 3.9} = \dfrac{-0.01251}{0.2} = -0.06255$.

Exact value: $f(x) = x^{-1/2} \Rightarrow f'(x) = -\frac{1}{2}x^{-3/2}$, so $f'(4) = -\frac{1}{2}(4^{-3/2}) = -\frac{1}{2}(\frac{1}{8}) = -\frac{1}{16} = -0.0625$.

49. (a)

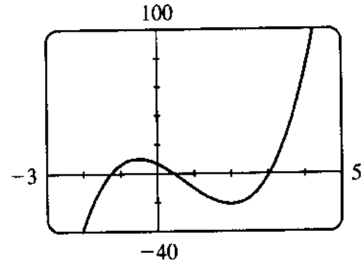


(b)

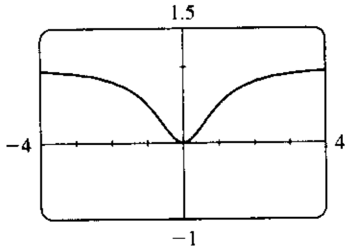


From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3). The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞).

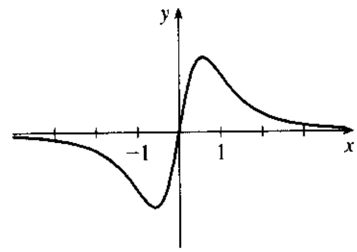
(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \Rightarrow$
$f'(x) = 4x^3 - 9x^2 - 12x + 7$



50. (a)



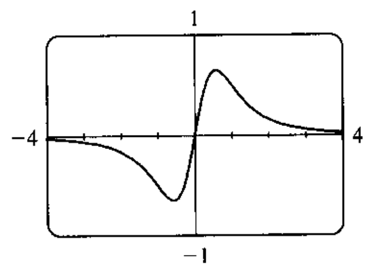
(b)



From the graph in part (a), it appears that g' is zero at $x = 0$. The slopes are negative (so g' is negative) on $(-\infty, 0)$. The slopes are positive (so g' is positive) on $(0, \infty)$.

(c) $g(x) = \dfrac{x^2}{x^2+1} \Rightarrow$

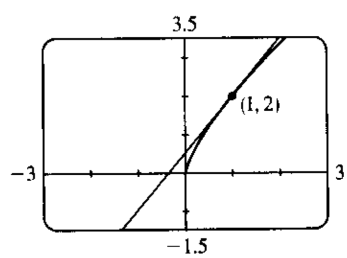
$$g'(x) = \frac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$$



51. $y = \dfrac{2x}{x+1} \Rightarrow y' = \dfrac{(x+1)(2) - (2x)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$. At $(1,1)$, $y' = \frac{1}{2}$, and an equation of the tangent line is $y - 1 = \frac{1}{2}(x-1)$, or $y = \frac{1}{2}x + \frac{1}{2}$.

52. $y = \dfrac{\sqrt{x}}{x+1} \Rightarrow y' = \dfrac{(x+1)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}\,(1)}{(x+1)^2} = \dfrac{(x+1)-(2x)}{2\sqrt{x}\,(x+1)^2} = \dfrac{1-x}{2\sqrt{x}\,(x+1)^2}$. At $(4, 0.4)$, $y' = \frac{-3}{100} = -0.03$, and an equation of the tangent line is $y - 0.4 = -0.03(x-4)$, or $y = -0.03x + 0.52$.

53. $y = f(x) = x + \sqrt{x} \Rightarrow f'(x) = 1 + \frac{1}{2}x^{-1/2}$. So the slope of the tangent line at $(1,2)$ is $f'(1) = 1 + \frac{1}{2}(1) = \frac{3}{2}$ and its equation is $y - 2 = \frac{3}{2}(x-1)$ or $y = \frac{3}{2}x + \frac{1}{2}$.

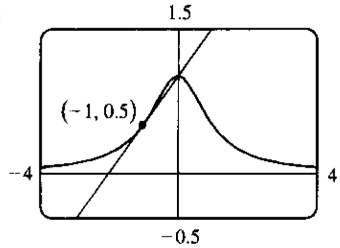


54. $y = (1+2x)^2 = 1 + 4x + 4x^2 \Rightarrow y' = 4 + 8x$. At $(1,9)$, $y' = 12$ and an equation of the tangent line is $y - 9 = 12(x-1)$ or $y = 12x - 3$.

55. (a) $y = f(x) = \dfrac{1}{1+x^2} \quad\Rightarrow$

$$f'(x) = \frac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}.$$ So the slope of the tangent line at the point $\left(-1, \frac{1}{2}\right)$ is $f'(-1) = \dfrac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x+1)$ or $y = \frac{1}{2}x + 1$.

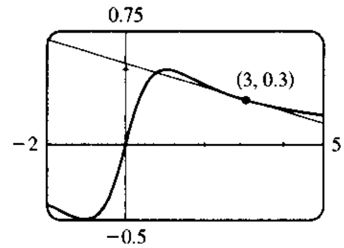
(b)



56. (a) $y = f(x) = \dfrac{x}{1+x^2} \quad\Rightarrow$

$$f'(x) = \frac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}.$$ So the slope of the tangent line at the point $(3, 0.3)$ is $f'(3) = \frac{-8}{100}$ and its equation is $y - 0.3 = -0.08(x-3)$ or $y = -0.08x + 0.54$.

(b)



57. We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

(b) $\left(\dfrac{f}{g}\right)'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{(-3)(6) - (1)(2)}{(-3)^2} = -\dfrac{20}{9}$

(c) $\left(\dfrac{g}{f}\right)'(5) = \dfrac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \dfrac{(1)(2) - (-3)(6)}{(1)^2} = 20$

58. We are given that $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$, and $g'(3) = 5$.

(a) $(f+g)'(3) = f'(3) + g'(3) = -6 + 5 = -1$

(b) $(fg)'(3) = f(3)g'(3) + g(3)f'(3) = (4)(5) + (2)(-6) = 20 - 12 = 8$

(c) $\left(\dfrac{f}{g}\right)'(3) = \dfrac{g(3)f'(3) - f(3)g'(3)}{[g(3)]^2} = \dfrac{(2)(-6) - (4)(5)}{(2)^2} = \dfrac{-32}{4} = -8$

(d) $$\left(\frac{f}{f-g}\right)'(3) = \frac{[f(3) - g(3)]\,f'(3) - f(3)\,[f'(3) - g'(3)]}{[f(3) - g(3)]^2} = \frac{(4-2)(-6) - 4(-6-5)}{(4-2)^2} = \frac{-12+44}{2^2} = 8$$

59. $f(x) = \sqrt{x}\, g(x) \quad\Rightarrow\quad f'(x) = \sqrt{x}\, g'(x) + g(x)\cdot \frac{1}{2}x^{-1/2}$, so $f'(4) = \sqrt{4}\, g'(4) + g(4)\cdot \frac{1}{2\sqrt{4}} = 2\cdot 7 + 8 \cdot \frac{1}{4} = 16$.

60. $$\frac{d}{dx}\left[\frac{h(x)}{x}\right] = \frac{xh'(x) - h(x)\cdot 1}{x^2} \quad\Rightarrow\quad \frac{d}{dx}\left[\frac{h(x)}{x}\right]_{x=2} = \frac{2h'(2) - h(2)}{2^2} = \frac{2(-3) - (4)}{4} = \frac{-10}{4} = -2.5$$

61. (a) From the graphs of f and g, we obtain the following values: $f(1) = 2$ since the point $(1, 2)$ is on the graph of f; $g(1) = 1$ since the point $(1, 1)$ is on the graph of g; $f'(1) = 2$ since the slope of the line segment between $(0, 0)$ and $(2, 4)$ is $\dfrac{4-0}{2-0} = 2$; $g'(1) = -1$ since the slope of the line segment between $(-2, 4)$ and $(2, 0)$ is $\dfrac{0-4}{2-(-2)} = -1$. Now $u(x) = f(x)g(x)$, so $u'(1) = f(1)g'(1) + g(1)\,f'(1) = 2\cdot(-1) + 1\cdot 2 = 0$.

(b) $v(x) = f(x)/g(x)$, so $v'(5) = \dfrac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \dfrac{2\left(-\frac{1}{3}\right) - 3\cdot\frac{2}{3}}{2^2} = \dfrac{-\frac{8}{3}}{4} = -\dfrac{2}{3}$

62. (a) $P(x) = F(x)G(x)$, so $P'(2) = F(2)G'(2) + G(2)F'(2) = 3 \cdot \frac{2}{4} + 2 \cdot 0 = \frac{3}{2}$.

(b) $Q(x) = F(x)/G(x)$, so $Q'(7) = \dfrac{G(7)F'(7) - F(7)G'(7)}{[G(7)]^2} = \dfrac{1 \cdot \frac{1}{4} - 5 \cdot \left(-\frac{2}{3}\right)}{1^2} = \dfrac{1}{4} + \dfrac{10}{3} = \dfrac{43}{12}$

63. (a) $y = xg(x) \quad \Rightarrow \quad y' = xg'(x) + g(x) \cdot 1 = xg'(x) + g(x)$

(b) $y = \dfrac{x}{g(x)} \quad \Rightarrow \quad y' = \dfrac{g(x) \cdot 1 - xg'(x)}{[g(x)]^2} = \dfrac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y = \dfrac{g(x)}{x} \quad \Rightarrow \quad y' = \dfrac{xg'(x) - g(x) \cdot 1}{(x)^2} = \dfrac{xg'(x) - g(x)}{x^2}$

64. (a) $y = x^2 f(x) \quad \Rightarrow \quad y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \dfrac{f(x)}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \dfrac{xf'(x) - 2f(x)}{x^3}$

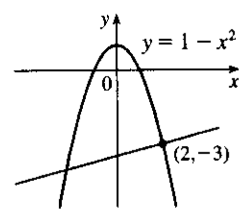
(c) $y = \dfrac{x^2}{f(x)} \quad \Rightarrow \quad y' = \dfrac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \dfrac{1 + xf(x)}{\sqrt{x}} \quad \Rightarrow$

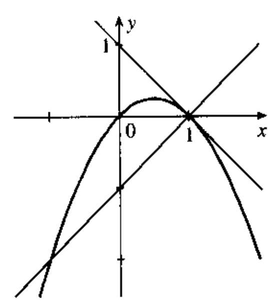
$$y' = \frac{\sqrt{x}\,[xf'(x) + f(x)] - [1 + xf(x)]\,\dfrac{1}{2\sqrt{x}}}{\left(\sqrt{x}\,\right)^2}$$

$$= \frac{x^{3/2} f'(x) + x^{1/2} f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2} f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2 f'(x) - 1}{2x^{3/2}}$$

65. $y = f(x) = 1 - x^2 \quad \Rightarrow \quad f'(x) = -2x$, so the tangent line at $(2, -3)$ has slope $f'(2) = -4$. The normal line has slope $-\frac{1}{-4} = \frac{1}{4}$ and equation $y + 3 = \frac{1}{4}(x - 2)$ or $y = \frac{1}{4}x - \frac{7}{2}$.

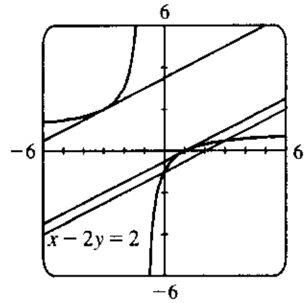


66. $y = f(x) = x - x^2 \quad \Rightarrow \quad f'(x) = 1 - 2x$. So $f'(1) = -1$, and the slope of the normal line is the negative reciprocal of that of the tangent line, that is, $-1/(-1) = 1$. So the equation of the normal line at $(1, 0)$ is $y - 0 = 1(x - 1)$ $\Leftrightarrow \quad y = x - 1$. Substituting this into the equation of the parabola, we obtain $x - 1 = x - x^2 \quad \Leftrightarrow \quad x = \pm 1$. The solution $x = -1$ is the one we require. Substituting $x = -1$ into the equation of the parabola to find the y-coordinate, we have $y = -2$. So the point of intersection is $(-1, -2)$, as shown in the sketch.



67. $y = x^3 - x^2 - x + 1$ has a horizontal tangent when $y' = 3x^2 - 2x - 1 = 0$. $(3x + 1)(x - 1) = 0 \quad \Leftrightarrow$ $x = 1$ or $-\frac{1}{3}$. Therefore, the points are $(1, 0)$ and $\left(-\frac{1}{3}, \frac{32}{27}\right)$.

68. $y = \dfrac{x-1}{x+1} \quad \Rightarrow \quad y' = \dfrac{(x+1)(1)-(x-1)(1)}{(x+1)^2} = \dfrac{2}{(x+1)^2}$. If the tangent intersects the curve when $x = a$, then its slope is $2/(a+1)^2$. But if the tangent is parallel to $x - 2y = 2$, that is, $y = \frac{1}{2}x - 1$, then its slope is $\frac{1}{2}$. Thus, $\dfrac{2}{(a+1)^2} = \dfrac{1}{2} \quad \Rightarrow$ $(a+1)^2 = 4 \quad \Rightarrow \quad a+1 = \pm 2 \quad \Rightarrow \quad a = 1$ or -3. When $a = 1$, $y = 0$ and the equation of the tangent is $y - 0 = \frac{1}{2}(x-1)$ or $y = \frac{1}{2}x - \frac{1}{2}$. When $a = -3$, $y = 2$ and the equation of the tangent is $y - 2 = \frac{1}{2}(x+3)$ or $y = \frac{1}{2}x + \frac{7}{2}$.



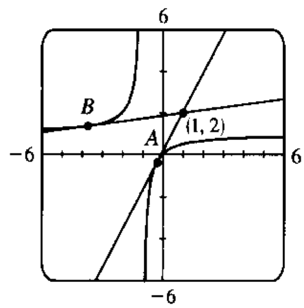
69. If $y = f(x) = \dfrac{x}{x+1}$, then $f'(x) = \dfrac{(x+1)(1) - x(1)}{(x+1)^2} = \dfrac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is $y - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(x-a)$. This line passes through $(1,2)$ when $2 - \dfrac{a}{a+1} = \dfrac{1}{(a+1)^2}(1-a) \quad \Leftrightarrow$ $2(a+1)^2 - a(a+1) = 1 - a \quad \Leftrightarrow \quad 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \quad \Leftrightarrow \quad a^2 + 4a + 1 = 0$.

The quadratic formula gives the roots of this equation as $a = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$, so there are two such tangent lines. Since

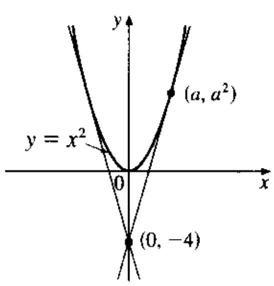
$$f\left(-2 \pm \sqrt{3}\right) = \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1} = \frac{-2 \pm \sqrt{3}}{-1 \pm \sqrt{3}} \cdot \frac{-1 \mp \sqrt{3}}{-1 \mp \sqrt{3}}$$

$$= \frac{2 \pm 2\sqrt{3} \mp \sqrt{3} - 3}{1 - 3} = \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \mp \sqrt{3}}{2},$$

the lines touch the curve at $A\left(-2 + \sqrt{3}, \frac{1-\sqrt{3}}{2}\right) \approx (-0.27, -0.37)$ and $B\left(-2 - \sqrt{3}, \frac{1+\sqrt{3}}{2}\right) \approx (-3.73, 1.37)$.



70.



Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y - (-4) = 2a(x - 0) \quad \Leftrightarrow \quad y = 2ax - 4$. Since (a, a^2) also lies on the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

71. $y = 6x^3 + 5x - 3 \quad \Rightarrow \quad m = y' = 18x^2 + 5$, but $x^2 \geq 0$ for all x, so $m \geq 5$ for all x.

72. If $y = x^2 + x$, then $y' = 2x + 1$. If the point at which a tangent meets the parabola is $(a, a^2 + a)$, then the slope of the tangent is $2a + 1$. But since it passes through $(2, -3)$, the slope must also be $\dfrac{\Delta y}{\Delta x} = \dfrac{a^2 + a + 3}{a - 2}$.

Therefore, $2a + 1 = \dfrac{a^2 + a + 3}{a - 2}$. Solving this equation for a we get $a^2 + a + 3 = 2a^2 - 3a - 2 \quad \Leftrightarrow$ $a^2 - 4a - 5 = (a-5)(a+1) = 0 \quad \Leftrightarrow \quad a = 5$ or -1. If $a = -1$, the point is $(-1, 0)$ and the slope is -1, so the equation is $y - 0 = (-1)(x+1)$ or $y = -x - 1$. If $a = 5$, the point is $(5, 30)$ and the slope is 11, so the equation is $y - 30 = 11(x - 5)$ or $y = 11x - 25$.

73. (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) $y = \sqrt{x}\,(x^4 + x + 1)(2x - 3)$. Using part (a), we have

$$\begin{aligned} y' &= \frac{1}{2\sqrt{x}}\,(x^4 + x + 1)(2x - 3) + \sqrt{x}\,(4x^3 + 1)(2x - 3) + \sqrt{x}\,(x^4 + x + 1)(2) \\ &= (x^4 + x + 1)\,\frac{2x - 3}{2\sqrt{x}} + \sqrt{x}\left[(4x^3 + 1)(2x - 3) + 2(x^4 + x + 1)\right] \end{aligned}$$

74. (a) Putting $f = g = h$ in part (a), we have

$$\frac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2 f'(x).$$

(b) $y = \left(x^4 + 3x^3 + 17x + 82\right)^3 \;\Rightarrow\; y' = 3\left(x^4 + 3x^3 + 17x + 82\right)^2\left(4x^3 + 9x^2 + 17\right)$

75. $y = f(x) = ax^3 + bx^2 + cx + d \;\Rightarrow\; f'(x) = 3ax^2 + 2bx + c$. The point $(-2, 6)$ is on f, so $f(-2) = 6 \;\Rightarrow$ $-8a + 4b - 2c + d = 6$ **(1)**. The point $(2, 0)$ is on f, so $f(2) = 0 \;\Rightarrow\; 8a + 4b + 2c + d = 0$ **(2)**. Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \;\Rightarrow\; 12a - 4b + c = 0$ **(3)** and $f'(2) = 0 \;\Rightarrow\; 12a + 4b + c = 0$ **(4)**. Subtracting equation **(3)** from **(4)** gives $8b = 0 \;\Rightarrow\; b = 0$. Adding **(1)** and **(2)** gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From **(3)** we have $c = -12a$, so **(2)** becomes $8a + 4(0) + 2(-12a) + 3 = 0 \;\Rightarrow\; 3 = 16a \;\Rightarrow\; a = \frac{3}{16}$. Now $c = -12a = -12\left(\frac{3}{16}\right) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

76. (a) $s(0) = 100{,}000$ subscribers and $n(0) = 1.2$ phone lines per subscriber. $s'(0) = 1000$ subscribers/month and $n'(0) = 0.01$ phone line per subscriber/month.

(b) The total number of lines is given by $L(t) = s(t)n(t)$. To find $L'(0)$, we first find $L'(t)$ using the Product Rule. $L'(t) = s(t)n'(t) + n(t)s'(t) \;\Rightarrow$
$L'(0) = s(0)n'(0) + n(0)s'(0) = 100{,}000(0.01) + 1.2(1000) = 2200$ phone lines/month.

77. If $P(t)$ denotes the population at time t and $A(t)$ the average annual income, then $T(t) = P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \;\Rightarrow$

$$\begin{aligned} T'(1999) &= P(1999)A'(1999) + A(1999)P'(1999) = (961{,}400)(\$1400/\text{yr}) + (\$30{,}593)(9200/\text{yr}) \\ &= \$1{,}345{,}960{,}000/\text{yr} + \$281{,}455{,}600/\text{yr} = \$1{,}627{,}415{,}600/\text{yr} \end{aligned}$$

So the total personal income was rising by about \$1.627 billion per year in 1999.

The term $P(t)A'(t) \approx \$1.346$ billion represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$281$ million represents the portion of the rate of change of total income due to increasing population.

78. (a) $f(20) = 10{,}000$ means that when the price of the fabric is \$20/yard, 10,000 yards will be sold.
$f'(20) = -350$ means that as the price of the fabric increases past \$20/yard, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per yard).

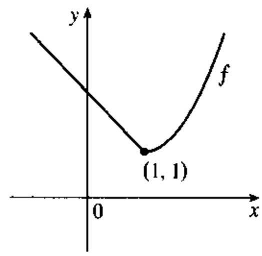
(b) $R(p) = pf(p) \;\Rightarrow\; R'(p) = pf'(p) + f(p) \cdot 1 \;\Rightarrow$
$R'(20) = 20f'(20) + f(20) \cdot 1 = 20(-350) + 10{,}000 = 3000$. This means that as the price of the fabric increases past \$20/yard, the total revenue is increasing at \$3000/(\$/yard). Note that the Product Rule indicates that we will lose \$7000/(\$/yard) due to selling less fabric, but that that loss is more than made up for by the additional revenue due to the increase in price.

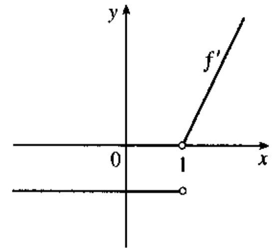
79. $f(x) = 2 - x$ if $x \le 1$ and $f(x) = x^2 - 2x + 2$ if $x > 1$. Now we compute the right- and left-hand derivatives defined in Exercise 3.2.44:

$$f'_-(1) = \lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{2-(1+h)-1}{h} = \lim_{h\to 0^-} \frac{-h}{h} = \lim_{h\to 0^-} -1 = -1 \text{ and}$$

$$f'_+(1) = \lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{(1+h)^2-2(1+h)+2-1}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0.$$

Thus, $f'(1)$ does not exist since $f'_-(1) \ne f'_+(1)$, so f is not differentiable at 1. But $f'(x) = -1$ for $x < 1$ and $f'(x) = 2x - 2$ if $x > 1$.





80. $g(x) = \begin{cases} -1-2x & \text{if } x < -1 \\ x^2 & \text{if } -1 \le x \le 1 \\ x & \text{if } x > 1 \end{cases}$

$$\lim_{h\to 0^-} \frac{g(-1+h)-g(-1)}{h} = \lim_{h\to 0^-} \frac{[-1-2(-1+h)]-1}{h} = \lim_{h\to 0^-} \frac{-2h}{h} = \lim_{h\to 0^-} (-2) = -2 \text{ and}$$

$$\lim_{h\to 0^+} \frac{g(-1+h)-g(-1)}{h} = \lim_{h\to 0^+} \frac{(-1+h)^2-1}{h} = \lim_{h\to 0^+} \frac{-2h+h^2}{h} = \lim_{h\to 0^+} (-2+h) = -2,$$

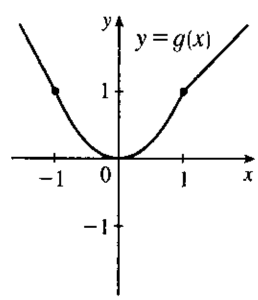
so g is differentiable at -1 and $g'(-1) = -2$.

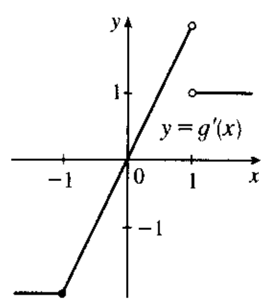
$$\lim_{h\to 0^-} \frac{g(1)-g(1)}{h} = \lim_{h\to 0^-} \frac{(1)^2-1}{h} = \lim_{h\to 0^-} \frac{2h+h^2}{h} = \lim_{h\to 0^-} (2+h) = 2 \text{ and}$$

$$\lim_{h\to 0^+} \frac{g(1)-g(1)}{h} = \lim_{h\to 0^+} \frac{(1)-1}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = 1, \text{ so } g'(1) \text{ does not exist.}$$

Thus, g is differentiable except when $x = 1$, and

$$g'(x) = \begin{cases} -2 & \text{if } x < -1 \\ 2x & \text{if } -1 \le x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$





81. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \iff |x| < 3 \iff -3 < x < 3$. So

$$f(x) = \begin{cases} x^2-9 & \text{if } x \le -3 \\ -x^2+9 & \text{if } -3 < x < 3 \\ x^2-9 & \text{if } x \ge 3 \end{cases} \Rightarrow$$

$$f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

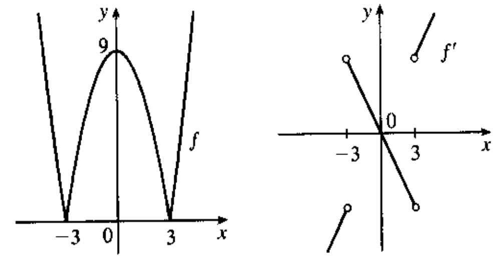
To show that $f'(3)$ does not exist we investigate $\lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 3.2.44.

$$f'_-(3) = \lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-} \frac{[-(3+h)^2+9]-0}{h} = \lim_{h\to 0^-}(-6-h) = -6 \quad \text{and}$$

$$f'_+(3) = \lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+} \frac{[(3+h)^2-9]-0}{h} = \lim_{h\to 0^+} \frac{6h+h^2}{h} = \lim_{h\to 0^+}(6+h) = 6.$$

Since the left and right limits are different, $\lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$ does not exist, that is, $f'(3)$ does not exist. Similarly, $f'(-3)$ does not exist. Therefore, f is not differentiable at 3 or at -3.

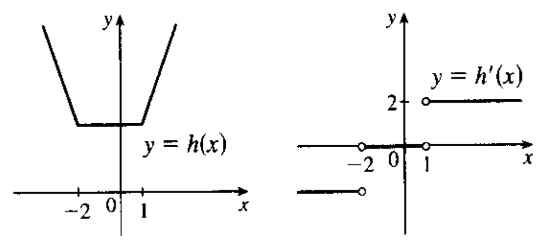
(b)



82. If $x \geq 1$, then $h(x) = |x-1| + |x+2| = x - 1 + x + 2 = 2x + 1$.
If $-2 < x < 1$, then $h(x) = -(x-1) + x + 2 = 3$.
If $x \leq -2$, then $h(x) = -(x-1) - (x+2) = -2x - 1$. Therefore,

$$h(x) = \begin{cases} -2x-1 & \text{if } x \leq -2 \\ 3 & \text{if } -2 < x < 1 \\ 2x+1 & \text{if } x \geq 1 \end{cases} \quad \Rightarrow \quad h'(x) = \begin{cases} -2 & \text{if } x < -2 \\ 0 & \text{if } -2 < x < 1 \\ 2 & \text{if } x > 1 \end{cases}$$

To see that $h'(1) = \lim_{x\to 1} \frac{h(x)-h(1)}{x-1}$ does not exist, observe that $\lim_{x\to 1^-} \frac{h(x)-h(1)}{x-1} = \lim_{x\to 1^-} \frac{3-3}{3-1} = 0$ but $\lim_{x\to 1^+} \frac{h(x)-h(1)}{x-1} = \lim_{x\to 1^+} \frac{2x-2}{x-1} = 2$. Similarly, $h'(-2)$ does not exist.



83. $y = f(x) = ax^2 \quad \Rightarrow \quad f'(x) = 2ax$. So the slope of the tangent to the parabola at $x = 2$ is $m = 2a(2) = 4a$. The slope of the given line, $2x + y = b \quad \Leftrightarrow \quad y = -2x + b$, is seen to be -2, so we must have $4a = -2 \quad \Leftrightarrow$ $a = -\frac{1}{2}$. So when $x = 2$, the point in question has y-coordinate $-\frac{1}{2} \cdot 2^2 = -2$. Now we simply require that the given line, whose equation is $2x + y = b$, pass through the point $(2, -2)$: $2(2) + (-2) = b \quad \Leftrightarrow \quad b = 2$. So we must have $a = -\frac{1}{2}$ and $b = 2$.

84. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so $f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(4x + b) = 8 + b$. Hence, $b = -4$.

85. $F = f/g \quad \Rightarrow \quad f = Fg \quad \Rightarrow \quad f' = F'g + Fg' \quad \Rightarrow \quad F' = \frac{f' - Fg'}{g} = \frac{f' - (f/g)g'}{g} = \frac{f'g - fg'}{g^2}$

86. (a) $xy = c \quad \Rightarrow \quad y = \frac{c}{x}$. Let $P = \left(a, \frac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\frac{c}{a^2}$. Its equation is $y - \frac{c}{a} = -\frac{c}{a^2}(x-a)$ or $y = -\frac{c}{a^2}x + \frac{2c}{a}$, so its y-intercept is $\frac{2c}{a}$. Setting $y = 0$ gives $x = 2a$, so the x-intercept is $2a$. The midpoint of the line segment joining $\left(0, \frac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \frac{c}{a}\right) = P$.

(b) We know the x- and y-intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

87. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of a derivative,

$f'(1) = \lim\limits_{x\to 1} \dfrac{f(x) - f(1)}{x - 1} = \lim\limits_{x\to 1} \dfrac{x^{1000} - 1}{x - 1}$. But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so $f'(1) = 1000(1)^{999} = 1000$. So $\lim\limits_{x\to 1} \dfrac{x^{1000} - 1}{x - 1} = 1000$.

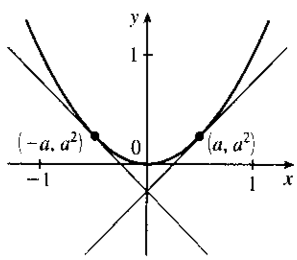
Solution 2: Note that $(x^{1000} - 1) = (x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)$. So

$$\lim_{x\to 1} \frac{x^{1000} - 1}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1)}{x - 1}$$

$$= \lim_{x\to 1} (x^{999} + x^{998} + x^{997} + \cdots + x^2 + x + 1) = \underbrace{1 + 1 + 1 + \cdots + 1 + 1 + 1}_{1000 \text{ ones}}$$

$$= 1000, \text{ as above.}$$

88.



In order for the two tangents to intersect on the y-axis, the points of tangency must be at equal distances from the y-axis, since the parabola $y = x^2$ is symmetric about the y-axis. Say the points of tangency are (a, a^2) and $(-a, a^2)$, for some $a > 0$. Then since the derivative of $y = x^2$ is $dy/dx = 2x$, the left-hand tangent has slope $-2a$ and equation $y - a^2 = -2a(x + a)$, or $y = -2ax - a^2$, and similarly the right-hand tangent line has equation $y - a^2 = 2a(x - a)$, or $y = 2ax - a^2$. So the two lines intersect at $(0, -a^2)$. Now if the lines are perpendicular, then the product of their slopes is -1, so $(-2a)(2a) = -1 \;\Leftrightarrow\; a^2 = \frac{1}{4} \;\Leftrightarrow\; a = \frac{1}{2}$. So the lines intersect at $(0, -\frac{1}{4})$.

3.4 Rates of Change in the Natural and Social Sciences

1. (a) $s = f(t) = t^2 - 10t + 12 \;\Rightarrow\; v(t) = f'(t) = 2t - 10$

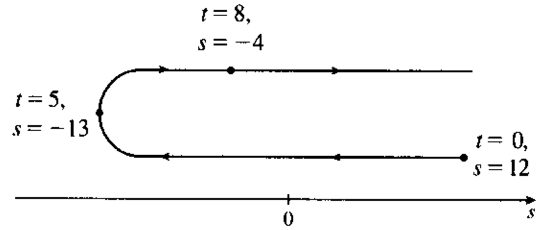
(b) $v(3) = 2(3) - 10 = -4$ ft/s

(c) The particle is at rest when $v(t) = 0 \;\Leftrightarrow\; 2t - 10 = 0 \;\Leftrightarrow\; t = 5$ s.

(d) The particle is moving in the positive direction when $v(t) > 0 \;\Leftrightarrow\; 2t - 10 > 0 \;\Leftrightarrow\; 2t > 10 \;\Leftrightarrow\; t > 5$.

(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0, 5]$ and $[5, 8]$ separately. $|f(5) - f(0)| = |-13 - 12| = 25$ ft and $|f(8) - f(5)| = |-4 - (-13)| = 9$ ft. The total distance traveled during the first 8 s is $25 + 9 = 34$ ft.

(f)



2. (a) $s = f(t) = t^3 - 9t^2 + 15t + 10 \quad \Rightarrow \quad v(t) = f'(t) = 3t^2 - 18t + 15 = 3(t-1)(t-5)$

(b) $v(3) = 3(2)(-2) = -12$ ft/s

(c) $v(t) = 0 \quad \Leftrightarrow \quad t = 1$ s or 5 s

(d) $v(t) > 0 \quad \Leftrightarrow \quad 0 \le t < 1$ or $t > 5$

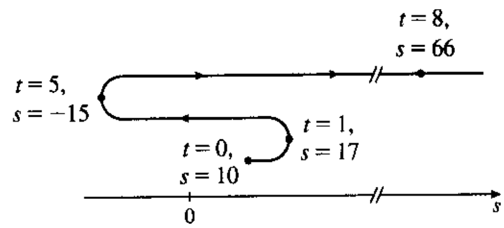
(e) $|f(1) - f(0)| = |17 - 10| = 7$,

$|f(5) - f(1)| = |-15 - 17| = 32$, and

$|f(8) - f(5)| = |66 - (-15)| = 81$.

Total distance $= 7 + 32 + 81 = 120$ ft.

(f)



3. (a) $s = f(t) = t^3 - 12t^2 + 36t \quad \Rightarrow \quad v(t) = f'(t) = 3t^2 - 24t + 36$

(b) $v(3) = 27 - 72 + 36 = -9$ ft/s

(c) The particle is at rest when $v(t) = 0$. $3t^2 - 24t + 36 = 0 \quad \Rightarrow \quad 3(t-2)(t-6) = 0 \quad \Rightarrow \quad t = 2$ s or 6 s.

(d) The particle is moving in the positive direction when $v(t) > 0$. $3(t-2)(t-6) > 0 \quad \Leftrightarrow \quad 0 \le t < 2$ or $t > 6$.

(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0, 2]$, $[2, 6]$, and $[6, 8]$ separately.

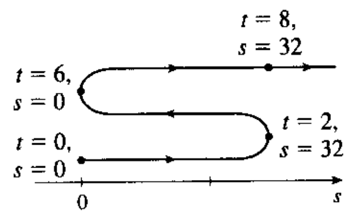
$|f(2) - f(0)| = |32 - 0| = 32$.

$|f(6) - f(2)| = |0 - 32| = 32$.

$|f(8) - f(6)| = |32 - 0| = 32$.

The total distance is $32 + 32 + 32 = 96$ ft.

(f)



4. (a) $s = f(t) = t^4 - 4t + 1 \quad \Rightarrow \quad v(t) = f'(t) = 4t^3 - 4$

(b) $v(3) = 4(3)^3 - 4 = 104$ ft/s

(c) It is at rest when $v(t) = 4(t^3 - 1) = 4(t-1)(t^2 + t + 1) = 0 \quad \Leftrightarrow \quad t = 1$ s.

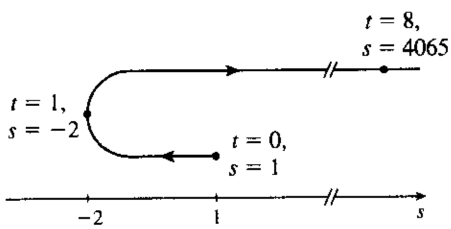
(d) It moves in the positive direction when $4(t^3 - 1) > 0 \quad \Leftrightarrow \quad t > 1$.

(e) Distance in positive direction $= |f(8) - f(1)| = |4065 - (-2)| = 4067$ ft

Distance in negative direction $= |f(1) - f(0)| = |-2 - 1| = 3$ ft

Total distance traveled $= 4067 + 3 = 4070$ ft

(f)



5. (a) $s = \dfrac{t}{t^2 + 1} \quad \Rightarrow \quad v(t) = s'(t) = \dfrac{(t^2+1)(1) - t(2t)}{(t^2+1)^2} = \dfrac{1 - t^2}{(t^2+1)^2}$

(b) $v(3) = \dfrac{1 - (3)^2}{(3^2+1)^2} = \dfrac{1 - 9}{10^2} = \dfrac{-8}{100} = -\dfrac{2}{25}$ ft/s

(c) It is at rest when $v = 0 \quad \Leftrightarrow \quad 1 - t^2 = 0 \quad \Leftrightarrow \quad t = 1$ s $\quad [t \ne -1$ since $t \ge 0]$.

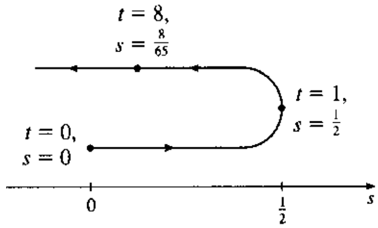
(d) It moves in the positive direction when $v > 0 \quad \Leftrightarrow \quad 1 - t^2 > 0 \quad \Leftrightarrow \quad t^2 < 1 \quad \Leftrightarrow \quad 0 \le t < 1$.

(e) Distance in positive direction $= |s(1) - s(0)| = \left|\frac{1}{2} - 0\right| = \frac{1}{2}$ ft

Distance in negative direction $= |s(8) - s(1)| = \left|\frac{8}{65} - \frac{1}{2}\right| = \frac{49}{130}$ ft

Total distance traveled $= \frac{1}{2} + \frac{49}{130} = \frac{57}{65}$ ft

(f)



6. (a) $s = \sqrt{t}\,(3t^2 - 35t + 90) = 3t^{5/2} - 35t^{3/2} + 90t^{1/2} \Rightarrow$

$v(t) = s'(t) = \frac{15}{2}t^{3/2} - \frac{105}{2}t^{1/2} + 45t^{-1/2} = \frac{15}{2}t^{-1/2}(t^2 - 7t + 6) = \dfrac{15}{2\sqrt{t}}(t-1)(t-6)$

(b) $v(3) = \frac{15}{2\sqrt{3}}(2)(-3) = -15\sqrt{3}$ ft/s

(c) It is at rest when $v = 0 \Leftrightarrow t = 1$ s or 6 s.

(d) It moves in the positive direction when $v > 0 \Leftrightarrow (t-1)(t-6) > 0 \Leftrightarrow 0 \le t < 1$ or $t > 6$.

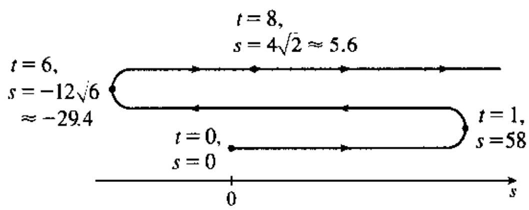
(e) Distance in positive direction $= |s(1) - s(0)| + |s(8) - s(6)| = |58 - 0| + \left|4\sqrt{2} - (-12\sqrt{6})\right|$

$= 58 + 4\sqrt{2} + 12\sqrt{6} \approx 93.05$ ft

Distance in negative direction $= |s(6) - s(1)| = \left|-12\sqrt{6} - 58\right| = 58 + 12\sqrt{6} \approx 87.39$ ft

Total distance traveled $= 58 + 4\sqrt{2} + 12\sqrt{6} + 58 + 12\sqrt{6} = 116 + 4\sqrt{2} + 24\sqrt{6} \approx 180.44$ ft

(f)



7. $s(t) = t^3 - 4.5t^2 - 7t \Rightarrow v(t) = s'(t) = 3t^2 - 9t - 7 = 5 \Leftrightarrow 3t^2 - 9t - 12 = 0 \Leftrightarrow$ $3(t-4)(t+1) = 0 \Leftrightarrow t = 4$ or -1. Since $t \ge 0$, the particle reaches a velocity of 5 m/s at $t = 4$ s.

8. (a) $s = 5t + 3t^2 \Rightarrow v(t) = \dfrac{ds}{dt} = 5 + 6t$, so $v(2) = 5 + 6(2) = 17$ m/s.

(b) $v(t) = 35 \Rightarrow 5 + 6t = 35 \Rightarrow 6t = 30 \Rightarrow t = 5$ s.

9. (a) $h = 10t - 0.83t^2 \Rightarrow v(t) = \dfrac{dh}{dt} = 10 - 1.66t$, so $v(3) = 10 - 1.66(3) = 5.02$ m/s.

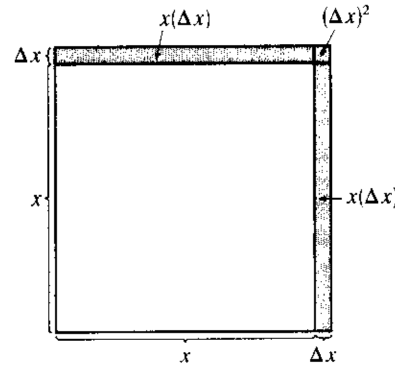
(b) $h = 25 \Rightarrow 10t - 0.83t^2 = 25 \Rightarrow 0.83t^2 - 10t + 25 = 0 \Rightarrow t = \dfrac{10 \pm \sqrt{17}}{1.66} \approx 3.54$ or 8.51.

The value $t_1 = (10 - \sqrt{17})/1.66$ corresponds to the time it takes for the stone to rise 25 m and $t_2 = (10 + \sqrt{17})/1.66$ corresponds to the time when the stone is 25 m high on the way down. Thus, $v(t_1) = 10 - 1.66\left[(10 - \sqrt{17})/1.66\right] = \sqrt{17} \approx 4.12$ m/s.

10. (a) At maximum height the velocity of the ball is 0 ft/s. $v(t) = s'(t) = 80 - 32t = 0 \Leftrightarrow 32t = 80 \Leftrightarrow$ $t = \frac{5}{2}$. So the maximum height is $s\left(\frac{5}{2}\right) = 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 = 200 - 100 = 100$ ft.

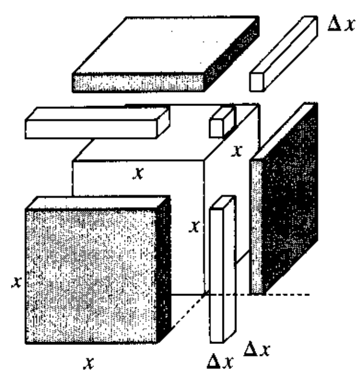
(b) $s(t) = 80t - 16t^2 = 96 \Leftrightarrow 16t^2 - 80t + 96 = 0 \Leftrightarrow 16(t^2 - 5t + 6) = 0 \Leftrightarrow 16(t-3)(t-2) = 0.$ So the ball has a height of 96 ft on the way up at $t = 2$ and on the way down at $t = 3$. At these times the velocities are $v(2) = 80 - 32(2) = 16$ ft/s and $v(3) = 80 - 32(3) = -16$ ft/s, respectively.

11. (a) $A(x) = x^2 \Rightarrow A'(x) = 2x$. $A'(15) = 30\ \text{mm}^2/\text{mm}$ is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.



(b) The perimeter is $P(x) = 4x$, so $A'(x) = 2x = \frac{1}{2}(4x) = \frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx. From the figure, $\Delta A = 2x(\Delta x) + (\Delta x)^2$. If Δx is small, then $\Delta A \approx 2x(\Delta x)$ and so $\Delta A/\Delta x \approx 2x$.

12. (a) $V(x) = x^3 \Rightarrow \dfrac{dV}{dx} = 3x^2$. $\left.\dfrac{dV}{dx}\right|_{x=3} = 3(3)^2 = 27\ \text{mm}^3/\text{mm}$ is the rate at which the volume is increasing as x increases past 3 mm.



(b) The surface area is $S(x) = 6x^2$, so $V'(x) = 3x^2 = \frac{1}{2}(6x^2) = \frac{1}{2}S(x)$. The figure suggests that if Δx is small, then the change in the volume of the cube is approximately half of its surface area (the area of 3 of the 6 faces) times Δx. From the figure, $\Delta V = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$. If Δx is small, then $\Delta V \approx 3x^2(\Delta x)$ and so $\Delta V/\Delta x \approx 3x^2$.

13. (a) Using $A(r) = \pi r^2$, we find that the average rate of change is:

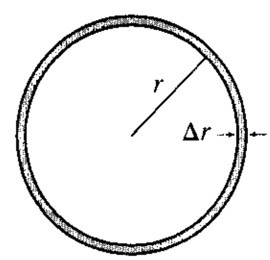
(i) $\dfrac{A(3) - A(2)}{3 - 2} = \dfrac{9\pi - 4\pi}{1} = 5\pi$

(ii) $\dfrac{A(2.5) - A(2)}{2.5 - 2} = \dfrac{6.25\pi - 4\pi}{0.5} = 4.5\pi$

(iii) $\dfrac{A(2.1) - A(2)}{2.1 - 2} = \dfrac{4.41\pi - 4\pi}{0.1} = 4.1\pi$

(b) $A(r) = \pi r^2 \Rightarrow A'(r) = 2\pi r$, so $A'(2) = 4\pi$.

(c) The circumference is $C(r) = 2\pi r = A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr. Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr, so $\Delta A \approx 2\pi r(\Delta r)$. Algebraically,
$\Delta A = A(r + \Delta r) - A(r) = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r(\Delta r) + \pi(\Delta r)^2$.
So we see that if Δr is small, then $\Delta A \approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r \approx 2\pi r$.



14. After t seconds the radius is $r = 60t$, so the area is $A(t) = \pi(60t)^2 = 3600\pi t^2 \Rightarrow A'(t) = 7200\pi t \Rightarrow$

(a) $A'(1) = 7200\pi \text{ cm}^2/\text{s}$ (b) $A'(3) = 21{,}600\pi \text{ cm}^2/\text{s}$ (c) $A'(5) = 36{,}000\pi \text{ cm}^2/\text{s}$

As time goes by, the area grows at an increasing rate. In fact, the rate of change is linear with respect to time.

15. $S(r) = 4\pi r^2 \Rightarrow S'(r) = 8\pi r \Rightarrow$

(a) $S'(1) = 8\pi \text{ ft}^2/\text{ft}$ (b) $S'(2) = 16\pi \text{ ft}^2/\text{ft}$ (c) $S'(3) = 24\pi \text{ ft}^2/\text{ft}$

As the radius increases, the surface area grows at an increasing rate. In fact, the rate of change is linear with respect to the radius.

16. (a) Using $V(r) = \frac{4}{3}\pi r^3$, we find that the average rate of change is:

(i) $\dfrac{V(8) - V(5)}{8 - 5} = \dfrac{\frac{4}{3}\pi(512) - \frac{4}{3}\pi(125)}{3} = 172\pi \ \mu\text{m}^3/\mu\text{m}$

(ii) $\dfrac{V(6) - V(5)}{6 - 5} = \dfrac{\frac{4}{3}\pi(216) - \frac{4}{3}\pi(125)}{1} = 121.\overline{3}\pi \ \mu\text{m}^3/\mu\text{m}$

(iii) $\dfrac{V(5.1) - V(5)}{5.1 - 5} = \dfrac{\frac{4}{3}\pi(5.1)^3 - \frac{4}{3}\pi(5)^3}{0.1} = 102.01\overline{3}\pi \ \mu\text{m}^3/\mu\text{m}$

(b) $V'(r) = 4\pi r^2$, so $V'(5) = 100\pi \ \mu\text{m}^3/\mu\text{m}$.

(c) $V(r) = \frac{4}{3}\pi r^3 \Rightarrow V'(r) = 4\pi r^2 = S(r)$. By analogy with Exercise 13(c), we can say that the change in the volume of the spherical shell, ΔV, is approximately equal to its thickness, Δr, times the surface area of the inner sphere. Thus, $\Delta V \approx 4\pi r^2(\Delta r)$ and so $\Delta V/\Delta r \approx 4\pi r^2$.

17. The mass is $f(x) = 3x^2$, so the linear density at x is $\rho(x) = f'(x) = 6x$.

(a) $\rho(1) = 6 \text{ kg/m}$ (b) $\rho(2) = 12 \text{ kg/m}$ (c) $\rho(3) = 18 \text{ kg/m}$

Since ρ is an increasing function, the density will be the highest at the right end of the rod and lowest at the left end.

18. $V(t) = 5000\left(1 - \frac{1}{40}t\right)^2 = 5000\left(1 - \frac{1}{20}t + \frac{1}{1600}t^2\right) \Rightarrow V'(t) = 5000\left(-\frac{1}{20} + \frac{1}{800}t\right) = -250\left(1 - \frac{1}{40}t\right)$

(a) $V'(5) = -250\left(1 - \frac{5}{40}\right) = -218.75 \text{ gal/min}$ (b) $V'(10) = -250\left(1 - \frac{10}{40}\right) = -187.5 \text{ gal/min}$

(c) $V'(20) = -250\left(1 - \frac{20}{40}\right) = -125 \text{ gal/min}$ (d) $V'(40) = -250\left(1 - \frac{40}{40}\right) = 0 \text{ gal/min}$

The water is flowing out the fastest at the beginning—when $t = 0$, $V'(t) = -250$ gal/min. The water is flowing out the slowest at the end—when $t = 40$, $V'(t) = 0$. As the tank empties, the water flows out more slowly.

19. The quantity of charge is $Q(t) = t^3 - 2t^2 + 6t + 2$, so the current is $Q'(t) = 3t^2 - 4t + 6$.

(a) $Q'(0.5) = 3(0.5)^2 - 4(0.5) + 6 = 4.75 \text{ A}$ (b) $Q'(1) = 3(1)^2 - 4(1) + 6 = 5 \text{ A}$

The current is lowest when Q' has a minimum. $Q''(t) = 6t - 4 < 0$ when $t < \frac{2}{3}$. So the current decreases when $t < \frac{2}{3}$ and increases when $t > \frac{2}{3}$. Thus, the current is lowest at $t = \frac{2}{3}$ s.

20. (a) $F = \dfrac{GmM}{r^2} = (GmM)r^{-2} \Rightarrow \dfrac{dF}{dr} = -2(GmM)r^{-3} = -\dfrac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.

(b) Given $F'(20{,}000) = -2$, find $F'(10{,}000)$. $-2 = -\dfrac{2GmM}{20{,}000^3} \Rightarrow GmM = 20{,}000^3$.

$$F'(10{,}000) = -\frac{2(20{,}000^3)}{10{,}000^3} = -2 \cdot 2^3 = -16 \text{ N/km}$$

21. (a) To find the rate of change of volume with respect to pressure, we first solve for V in terms of P.

$$PV = C \quad\Rightarrow\quad V = \frac{C}{P} \quad\Rightarrow\quad \frac{dV}{dP} = -\frac{C}{P^2}.$$

(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases. Thus, the volume is decreasing more rapidly at the beginning.

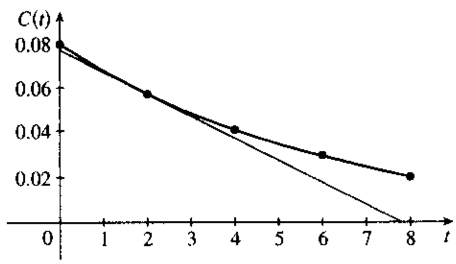
(c) $\beta = -\dfrac{1}{V}\dfrac{dV}{dP} = -\dfrac{1}{V}\left(-\dfrac{C}{P^2}\right) = \dfrac{C}{(PV)P} = \dfrac{C}{CP} = \dfrac{1}{P}$

22. (a) (i) $\dfrac{C(6)-C(2)}{6-2} = \dfrac{0.0295-0.0570}{4} = -0.006875$ (moles/L)/min

(ii) $\dfrac{C(4)-C(2)}{4-2} = \dfrac{0.0408-0.0570}{2} = -0.008$ (moles/L)/min

(iii) $\dfrac{C(2)-C(0)}{2-0} = \dfrac{0.0570-0.0800}{2} = -0.0115$ (moles/L)/min

(b) Slope $= \dfrac{\Delta C}{\Delta t} \approx -\dfrac{0.077}{7.8} \approx -0.01$ (moles/L)/min



23. (a) **1920:** $m_1 = \dfrac{1860-1750}{1920-1910} = \frac{110}{10} = 11$, $m_2 = \dfrac{2070-1860}{1930-1920} = \frac{210}{10} = 21$,

$(m_1+m_2)/2 = (11+21)/2 = 16$ million/year

1980: $m_1 = \dfrac{4450-3710}{1980-1970} = \frac{740}{10} = 74$, $m_2 = \dfrac{5280-4450}{1990-1980} = \frac{830}{10} = 83$,

$(m_1+m_2)/2 = (74+83)/2 = 78.5$ million/year

(b) $P(t) = at^3 + bt^2 + ct + d$ (in millions of people), where $a \approx 0.0012937063$, $b \approx -7.061421911$, $c \approx 12{,}822.97902$, and $d \approx -7{,}743{,}770.396$.

(c) $P(t) = at^3 + bt^2 + ct + d \quad\Rightarrow\quad P'(t) = 3at^2 + 2bt + c$ (in millions of people per year)

(d) $P'(1920) = 3(0.0012937063)(1920)^2 + 2(-7.061421911)(1920) + 12{,}822.97902$

≈ 14.48 million/year [smaller than the answer in part (a), but close to it]

$P'(1980) \approx 75.29$ million/year (smaller, but close)

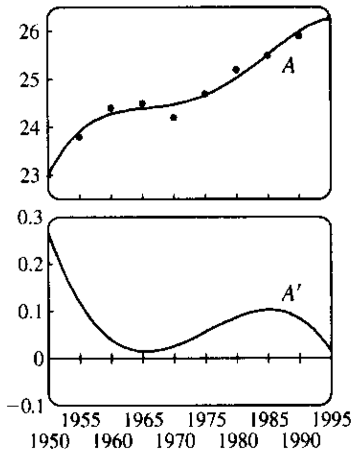
(e) $P'(1985) \approx 81.62$ million/year, so the rate of growth in 1985 was about 81.62 million/year.

24. (a) $A(t) = at^4 + bt^3 + ct^2 + dt + e$, where $a = -5.8275058275396 \times 10^{-6}$, $b = 0.0460458430461$, $c = -136.43277039706$, $d = 179{,}661.02676871$, and $e = -88{,}717{,}597.060767$.

(b) $A(t) = at^4 + bt^3 + ct^2 + dt + e \quad\Rightarrow\quad A'(t) = 4at^3 + 3bt^2 + 2ct + d$

(c) $A'(1990) \approx 0.0833$ years of age per year

(d)



25. (a) $[\mathrm{C}] = \dfrac{a^2kt}{akt+1} \quad\Rightarrow$

rate of reaction $= \dfrac{d\,[\mathrm{C}]}{dt} = \dfrac{(akt+1)(a^2k) - (a^2kt)(ak)}{(akt+1)^2} = \dfrac{a^2k(akt+1-akt)}{(akt+1)^2} = \dfrac{a^2k}{(akt+1)^2}$

(b) If $x = [\mathrm{C}]$, then $a - x = a - \dfrac{a^2kt}{akt+1} = \dfrac{a^2kt + a - a^2kt}{akt+1} = \dfrac{a}{akt+1}$.

So $k(a-x)^2 = k\left(\dfrac{a}{akt+1}\right)^2 = \dfrac{a^2k}{(akt+1)^2} = \dfrac{d\,[\mathrm{C}]}{dt}$ [from part (a)] $= \dfrac{dx}{dt}$.

26. $\dfrac{1}{p} = \dfrac{1}{f} - \dfrac{1}{q} \quad\Leftrightarrow\quad \dfrac{1}{p} = \dfrac{q-f}{fq} \quad\Leftrightarrow\quad p = \dfrac{fq}{q-f}$. So $\dfrac{dp}{dq} = \dfrac{f(q-f) - fq}{(q-f)^2} = -\dfrac{f^2}{(q-f)^2}$.

27. (a) Using $v = \dfrac{P}{4\eta l}\left(R^2 - r^2\right)$ with $R = 0.01$, $l = 3$, $P = 3000$, and $\eta = 0.027$, we have v as a function of r:

$v(r) = \dfrac{3000}{4(0.027)3}\left(0.01^2 - r^2\right)$. $v(0) = 0.\overline{925}$ cm/s, $v(0.005) = 0.69\overline{4}$ cm/s, $v(0.01) = 0$.

(b) $v(r) = \dfrac{P}{4\eta l}\left(R^2 - r^2\right) \quad\Rightarrow\quad v'(r) = \dfrac{P}{4\eta l}(-2r) = -\dfrac{Pr}{2\eta l}$. When $l = 3$, $P = 3000$, and $\eta = 0.027$, we have

$v'(r) = -\dfrac{3000r}{2(0.027)3}$. $v'(0) = 0$, $v'(0.005) = -92.\overline{592}$ (cm/s)/cm, and $v'(0.01) = -185.\overline{185}$ (cm/s)/cm.

(c) The velocity is greatest where $r = 0$ (at the center) and the velocity is changing most where $r = R = 0.01$ cm (at the edge).

28. (a) (i) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-1} \quad\Rightarrow\quad \dfrac{df}{dL} = -\left(\dfrac{1}{2}\sqrt{\dfrac{T}{\rho}}\right)L^{-2} = -\dfrac{1}{2L^2}\sqrt{\dfrac{T}{\rho}}$

(ii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{1/2} \quad\Rightarrow\quad \dfrac{df}{dT} = \dfrac{1}{2}\left(\dfrac{1}{2L\sqrt{\rho}}\right)T^{-1/2} = \dfrac{1}{4L\sqrt{T\rho}}$

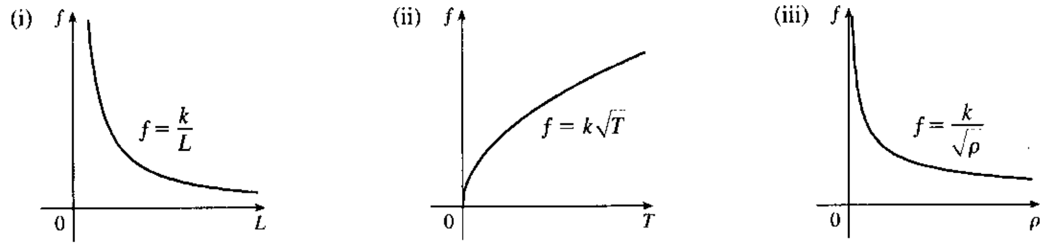
(iii) $f = \dfrac{1}{2L}\sqrt{\dfrac{T}{\rho}} = \left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-1/2} \quad\Rightarrow\quad \dfrac{df}{d\rho} = -\dfrac{1}{2}\left(\dfrac{\sqrt{T}}{2L}\right)\rho^{-3/2} = -\dfrac{\sqrt{T}}{4L\rho^{3/2}}$

(b) *Note:* Illustrating tangent lines on the generic figures may help to explain the results.

(i) $\dfrac{df}{dL} < 0$ and L is decreasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(ii) $\dfrac{df}{dT} > 0$ and T is increasing $\Rightarrow$ f is increasing $\Rightarrow$ higher note

(iii) $\dfrac{df}{d\rho} < 0$ and ρ is increasing $\Rightarrow$ f is decreasing $\Rightarrow$ lower note



29. (a) $C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3 \quad \Rightarrow \quad C'(x) = 3 + 0.02x + 0.0006x^2$

(b) $C'(100) = 3 + 0.02(100) + 0.0006(10{,}000) = 3 + 2 + 6 = \$11/\text{pair}$. $C'(100)$ is the rate at which the cost is increasing as the 100th pair of jeans is produced. It predicts the cost of the 101st pair.

(c) The cost of manufacturing the 101st pair of jeans is

$$\begin{aligned} C(101) - C(100) &= (2000 + 303 + 102.01 + 206.0602) - (2000 + 300 + 100 + 200) \\ &= 11.0702 \approx \$11.07 \end{aligned}$$

30. (a) $C(x) = 84 + 0.16x - 0.0006x^2 + 0.000003x^3 \quad \Rightarrow \quad C'(x) = 0.16 - 0.0012x + 0.000009x^2 \quad \Rightarrow$ $C'(100) = 0.13$. This is the rate at which the cost is increasing as the 100th item is produced.

(b) $C(101) - C(100) = 97.13030299 - 97 \approx \$0.13$.

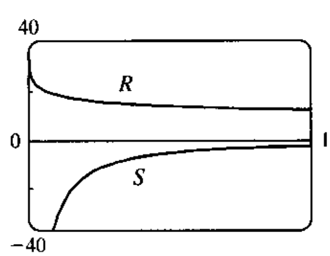
31. (a) $A(x) = \dfrac{p(x)}{x} \quad \Rightarrow \quad A'(x) = \dfrac{xp'(x) - p(x) \cdot 1}{x^2} = \dfrac{xp'(x) - p(x)}{x^2}$. $A'(x) > 0 \quad \Rightarrow \quad A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.

(b) $p'(x)$ is greater than the average productivity $\Rightarrow$ $p'(x) > A(x) \quad \Rightarrow \quad p'(x) > \dfrac{p(x)}{x} \quad \Rightarrow$

$xp'(x) > p(x) \quad \Rightarrow \quad xp'(x) - p(x) > 0 \quad \Rightarrow \quad \dfrac{xp'(x) - p(x)}{x^2} > 0 \quad \Rightarrow \quad A'(x) > 0.$

32. (a) $$\begin{aligned} S = \frac{dR}{dx} &= \frac{(1 + 4x^{0.4})(9.6x^{-0.6}) - (40 + 24x^{0.4})(1.6x^{-0.6})}{(1 + 4x^{0.4})^2} \\ &= \frac{9.6x^{-0.6} + 38.4x^{-0.2} - 64x^{-0.6} - 38.4x^{-0.2}}{(1 + 4x^{0.4})^2} = -\frac{54.4x^{-0.6}}{(1 + 4x^{0.4})^2} \end{aligned}$$

(b)



At low levels of brightness, R is quite large $[R(0) = 40]$ and is quickly decreasing, that is, S is negative with large absolute value. This is to be expected: at low levels of brightness, the eye is more sensitive to slight changes than it is at higher levels of brightness.

33. $PV = nRT \quad \Rightarrow \quad T = \dfrac{PV}{nR} = \dfrac{PV}{(10)(0.0821)} = \dfrac{1}{0.821}(PV)$. Using the Product Rule, we have

$$\frac{dT}{dt} = \frac{1}{0.821}\left[P(t)V'(t) + V(t)P'(t)\right] = \frac{1}{0.821}\left[(8)(-0.15) + (10)(0.10)\right] \approx -0.2436 \text{ K/min.}$$

34. (a) If $dP/dt = 0$, the population is stable (it is constant).

(b) $\dfrac{dP}{dt} = 0 \;\Rightarrow\; \beta P = r_0\left(1 - \dfrac{P}{P_c}\right)P \;\Rightarrow\; \dfrac{\beta}{r_0} = 1 - \dfrac{P}{P_c} \;\Rightarrow\; \dfrac{P}{P_c} = 1 - \dfrac{\beta}{r_0} \;\Rightarrow\; P = P_c\left(1 - \dfrac{\beta}{r_0}\right).$

If $P_c = 10{,}000$, $r_0 = 5\% = 0.05$, and $\beta = 4\% = 0.04$, then $P = 10{,}000(1 - \frac{4}{5}) = 2000$.

(c) If $\beta = 0.05$, then $P = 10{,}000(1 - \frac{5}{5}) = 0$. There is no stable population.

35. (a) If the populations are stable, then the growth rates are neither positive nor negative; that is, $\dfrac{dC}{dt} = 0$ and $\dfrac{dW}{dt} = 0$.

(b) "The caribou go extinct" means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\dfrac{dC}{dt} = aC - bCW$ and $\dfrac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain $0.05C - 0.001CW = 0$ **(1)** and $-0.05W + 0.0001CW = 0$ **(2)**. Adding 10 times **(2)** to **(1)** eliminates the CW-terms and gives us $0.05C - 0.5W = 0 \;\Rightarrow\; C = 10W$. Substituting $C = 10W$ into **(1)** results in $0.05(10W) - 0.001(10W)W = 0 \;\Leftrightarrow\; 0.5W - 0.01W^2 = 0 \;\Leftrightarrow\; 50W - W^2 = 0 \;\Leftrightarrow\; W(50 - W) = 0 \;\Leftrightarrow\; W = 0$ or 50. Since $C = 10W$, $C = 0$ or 500. Thus, the population pairs (C, W) that lead to stable populations are $(0,0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

3.5 Derivatives of Trigonometric Functions

1. $f(x) = x - 3\sin x \;\Rightarrow\; f'(x) = 1 - 3\cos x$

2. $f(x) = x\sin x \;\Rightarrow\; f'(x) = x\cdot\cos x + (\sin x)\cdot 1 = x\cos x + \sin x$

3. $y = \sin x + 10\tan x \;\Rightarrow\; y' = \cos x + 10\sec^2 x$

4. $y = 2\csc x + 5\cos x \;\Rightarrow\; y' = -2\csc x\cot x - 5\sin x$

5. $g(t) = t^3\cos t \;\Rightarrow\; g'(t) = t^3(-\sin t) + (\cos t)\cdot 3t^2 = 3t^2\cos t - t^3\sin t$ or $t^2(3\cos t - t\sin t)$

6. $g(t) = 4\sec t + \tan t \;\Rightarrow\; g'(t) = 4\sec t\tan t + \sec^2 t$

7. $h(\theta) = \theta\csc\theta - \cot\theta \;\Rightarrow\; h'(\theta) = \theta(-\csc\theta\cot\theta) + (\csc\theta)\cdot 1 - (-\csc^2\theta) = \csc\theta - \theta\csc\theta\cot\theta + \csc^2\theta$

8. $y = u(a\cos u + b\cot u) \;\Rightarrow$

$y' = u(-a\sin u - b\csc^2 u) + (a\cos u + b\cot u)\cdot 1 = a\cos u + b\cot u - au\sin u - bu\csc^2 u$

9. $y = \dfrac{x}{\cos x} \;\Rightarrow\; y' = \dfrac{(\cos x)(1) - (x)(-\sin x)}{(\cos x)^2} = \dfrac{\cos x + x\sin x}{\cos^2 x}$

10. $y = \dfrac{1 + \sin x}{x + \cos x} \;\Rightarrow$

$$y' = \frac{(x + \cos x)(\cos x) - (1 + \sin x)(1 - \sin x)}{(x + \cos x)^2} = \frac{x\cos x + \cos^2 x - (1 - \sin^2 x)}{(x + \cos x)^2}$$

$$= \frac{x\cos x + \cos^2 x - (\cos^2 x)}{(x + \cos x)^2} = \frac{x\cos x}{(x + \cos x)^2}$$

11. $f(\theta) = \dfrac{\sec\theta}{1 + \sec\theta} \;\Rightarrow$

$$f'(\theta) = \frac{(1 + \sec\theta)(\sec\theta\tan\theta) - (\sec\theta)(\sec\theta\tan\theta)}{(1 + \sec\theta)^2} = \frac{(\sec\theta\tan\theta)[(1 + \sec\theta) - \sec\theta]}{(1 + \sec\theta)^2} = \frac{\sec\theta\tan\theta}{(1 + \sec\theta)^2}$$

12. $y = \dfrac{\tan x - 1}{\sec x} \quad \Rightarrow$

$$\frac{dy}{dx} = \frac{\sec x \sec^2 x - (\tan x - 1)\sec x \tan x}{\sec^2 x} = \frac{\sec x\left(\sec^2 x - \tan^2 x + \tan x\right)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

Another method: Simplify y first: $y = \sin x - \cos x \quad \Rightarrow \quad y' = \cos x + \sin x$.

13. $y = \dfrac{\sin x}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2 \cos x - (\sin x)(2x)}{(x^2)^2} = \dfrac{x(x\cos x - 2\sin x)}{x^4} = \dfrac{x\cos x - 2\sin x}{x^3}$

14. $y = \csc\theta\,(\theta + \cot\theta) \quad \Rightarrow$

$$\begin{aligned} y' &= \csc\theta\left(1 - \csc^2\theta\right) + (\theta + \cot\theta)(-\csc\theta\cot\theta) = \csc\theta\left(1 - \csc^2\theta - \theta\cot\theta - \cot^2\theta\right) \\ &= \csc\theta\left(-\cot^2\theta - \theta\cot\theta - \cot^2\theta\right) \qquad \left[1 + \cot^2\theta = \csc^2\theta\right] \\ &= \csc\theta\left(-\theta\cot\theta - 2\cot^2\theta\right) = -\csc\theta\cot\theta\,(\theta + 2\cot\theta) \end{aligned}$$

15. $y = \sec\theta\,\tan\theta \quad \Rightarrow \quad y' = \sec\theta\left(\sec^2\theta\right) + \tan\theta\,(\sec\theta\tan\theta) = \sec\theta\left(\sec^2\theta + \tan^2\theta\right)$

Using the identity $1 + \tan^2\theta = \sec^2\theta$, we can write alternative forms of the answer as

$$\sec\theta\left(1 + 2\tan^2\theta\right) \qquad \text{or} \qquad \sec\theta\left(2\sec^2\theta - 1\right)$$

16. Recall that if $y = fgh$, then $y' = f'gh + fg'h + fgh'$. $y = x\sin x\cos x \quad \Rightarrow$

$$\frac{dy}{dx} = \sin x\cos x + x\cos x\cos x + x\sin x\,(-\sin x) = \sin x\cos x + x\cos^2 x - x\sin^2 x$$

17. $\dfrac{d}{dx}(\csc x) = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x} = -\csc x\cot x$

18. $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x} = \sec x\tan x$

19. $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

20. $f(x) = \cos x \quad \Rightarrow$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h} \\ &= \lim_{h\to 0}\left(\cos x\,\frac{\cos h - 1}{h} - \sin x\,\frac{\sin h}{h}\right) = \cos x\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h} \\ &= (\cos x)(0) - (\sin x)(1) = -\sin x \end{aligned}$$

21. $y = \tan x \quad \Rightarrow \quad y' = \sec^2 x \quad \Rightarrow$ the slope of the tangent line at $\left(\frac{\pi}{4}, 1\right)$ is $\sec^2\frac{\pi}{4} = \left(\sqrt{2}\right)^2 = 2$ and an equation of the tangent line is $y - 1 = 2\left(x - \frac{\pi}{4}\right)$ or $y = 2x + 1 - \frac{\pi}{2}$.

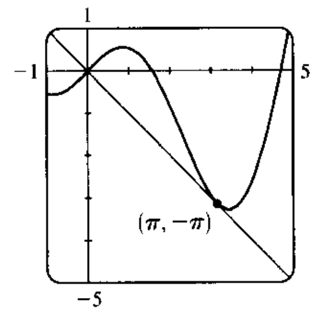
22. $y = (1 + x)\cos x \quad \Rightarrow \quad y' = (1 + x)(-\sin x) + \cos x \cdot 1$. At $(0, 1)$, $y' = 1$, and an equation of the tangent line is $y - 1 = 1(x - 0)$, or $y = x + 1$.

23. $y = x + \cos x \quad \Rightarrow \quad y' = 1 - \sin x$. At $(0, 1)$, $y' = 1$, and an equation of the tangent line is $y - 1 = 1(x - 0)$, or $y = x + 1$.

24. $y = \dfrac{1}{\sin x + \cos x} \quad \Rightarrow \quad y' = -\dfrac{\cos x - \sin x}{(\sin x + \cos x)^2}$ [Reciprocal Rule]. At $(0, 1)$, $y' = -\dfrac{1-0}{(0+1)^2} = -1$, and an equation of the tangent line is $y - 1 = -1(x - 0)$, or $y = -x + 1$.

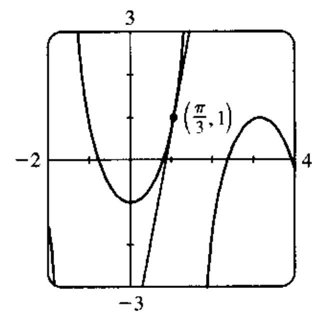
25. (a) $y = x\cos x \quad \Rightarrow \quad y' = x(-\sin x) + \cos x(1) = \cos x - x\sin x$. So the slope of the tangent at the point $(\pi, -\pi)$ is $\cos\pi - \pi\sin\pi = -1 - \pi(0) = -1$, and an equation is $y + \pi = -(x - \pi)$ or $y = -x$.

(b)



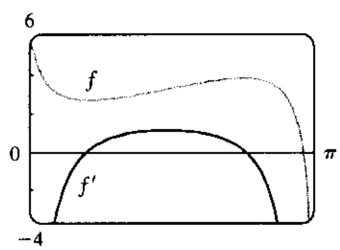
26. (a) $y = \sec x - 2\cos x \quad \Rightarrow \quad y' = \sec x\tan x + 2\sin x \quad \Rightarrow$ the slope of the tangent line at $\left(\frac{\pi}{3}, 1\right)$ is $\sec\frac{\pi}{3}\tan\frac{\pi}{3} + 2\sin\frac{\pi}{3} = 2\cdot\sqrt{3} + 2\cdot\frac{\sqrt{3}}{2} = 3\sqrt{3}$ and an equation is $y - 1 = 3\sqrt{3}\left(x - \frac{\pi}{3}\right)$ or $y = 3\sqrt{3}\,x + 1 - \pi\sqrt{3}$.

(b)



27. (a) $f(x) = 2x + \cot x \quad \Rightarrow \quad f'(x) = 2 - \csc^2 x$

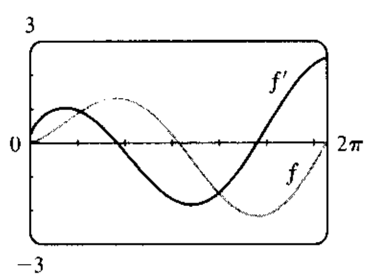
(b)



Notice that $f'(x) = 0$ when f has a horizontal tangent. f' is positive when f is increasing and f' is negative when f is decreasing. Also, $f'(x)$ is large negative when the graph of f is steep.

28. (a) $f(x) = \sqrt{x}\sin x \quad \Rightarrow \quad f'(x) = \sqrt{x}\cos x + (\sin x)\left(\frac{1}{2}x^{-1/2}\right) = \sqrt{x}\cos x + \dfrac{\sin x}{2\sqrt{x}}$

(b)



Notice that $f'(x) = 0$ when f has a horizontal tangent. f' is positive when f is increasing and f' is negative when f is decreasing.

29. $f(x) = x + 2\sin x$ has a horizontal tangent when $f'(x) = 0 \quad \Leftrightarrow \quad 1 + 2\cos x = 0 \quad \Leftrightarrow \quad \cos x = -\frac{1}{2} \quad \Leftrightarrow$ $x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π. This allows us to write the solutions in the more compact equivalent form $(2n + 1)\pi \pm \frac{\pi}{3}$, n an integer.

30. $y = \dfrac{\cos x}{2+\sin x} \Rightarrow$

$y' = \dfrac{(2+\sin x)(-\sin x) - \cos x \cos x}{(2+\sin x)^2} = \dfrac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2} = \dfrac{-2\sin x - 1}{(2+\sin x)^2} = 0$ when

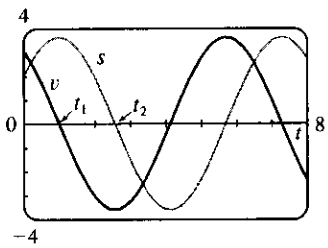
$-2\sin x - 1 = 0 \Leftrightarrow \sin x = -\frac{1}{2} \Leftrightarrow x = \frac{11\pi}{6} + 2\pi n$ or $x = \frac{7\pi}{6} + 2\pi n$, n an integer. So $y = \frac{1}{\sqrt{3}}$ or $y = -\frac{1}{\sqrt{3}}$ and the points on the curve with horizontal tangents are: $\left(\frac{11\pi}{6} + 2\pi n, \frac{1}{\sqrt{3}}\right)$, $\left(\frac{7\pi}{6} + 2\pi n, -\frac{1}{\sqrt{3}}\right)$, n an integer.

31. (a) $x(t) = 8\sin t \Rightarrow v(t) = x'(t) = 8\cos t$

(b) The mass at time $t = \frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right) = 8\sin\frac{2\pi}{3} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$ and velocity $v\left(\frac{2\pi}{3}\right) = 8\cos\frac{2\pi}{3} = 8\left(-\frac{1}{2}\right) = -4$. Since $v\left(\frac{2\pi}{3}\right) < 0$, the particle is moving to the left.

32. (a) $s(t) = 2\cos t + 3\sin t \Rightarrow v(t) = -2\sin t + 3\cos t$

(b)

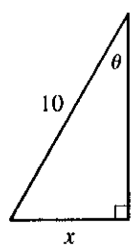


(c) $s = 0 \Rightarrow t_2 \approx 2.55$. So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v = 0 \Rightarrow t_1 \approx 0.98$, $s(t_1) \approx 3.61$ cm. So the mass travels a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s = 0$; that is, when $t = t_2 + n\pi$, n a positive integer.

33.



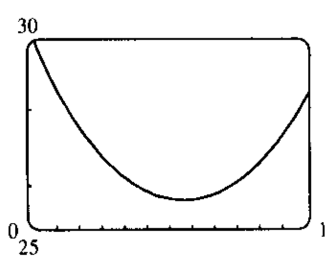
From the diagram we can see that $\sin\theta = x/10 \Leftrightarrow x = 10\sin\theta$. We want to find the rate of change of x with respect to θ; that is, $dx/d\theta$. Taking the derivative of the above expression, $dx/d\theta = 10(\cos\theta)$. So when $\theta = \frac{\pi}{3}$,

$$dx/d\theta = 10\cos\tfrac{\pi}{3} = 10\left(\tfrac{1}{2}\right) = 5 \text{ ft/rad}$$

34. (a) $F = \dfrac{\mu W}{\mu\sin\theta + \cos\theta} \Rightarrow \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{\mu W(\sin\theta - \mu\cos\theta)}{(\mu\sin\theta + \cos\theta)^2}$

(b) $\dfrac{dF}{d\theta} = 0 \Rightarrow \mu W(\sin\theta - \mu\cos\theta) = 0 \Rightarrow \sin\theta = \mu\cos\theta \Rightarrow \tan\theta = \mu \Rightarrow \theta = \tan^{-1}\mu$

(c)



From the graph of $F = \dfrac{0.6(50)}{0.6\sin\theta + \cos\theta}$ for $0 \le \theta \le 1$, we see that $\dfrac{dF}{d\theta} = 0 \Rightarrow \theta \approx 0.54$. Checking this with part (b) and $\mu = 0.6$, we calculate $\theta = \tan^{-1} 0.6 \approx 0.54$. So the value from the graph is consistent with the value in part (b).

35. $\displaystyle \lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{3\sin 3x}{3x}$ [multiply numerator and denominator by 3]

$\displaystyle = 3\lim_{3x\to 0}\frac{\sin 3x}{3x}$ [as $x\to 0,\ 3x\to 0$]

$\displaystyle = 3\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$ [let $\theta = 3x$]

$= 3(1)$ [Equation 2]

$= 3$

36. $$\lim_{x\to 0}\frac{\sin 4x}{\sin 6x} = \lim_{x\to 0}\left(\frac{\sin 4x}{x}\cdot\frac{x}{\sin 6x}\right) = \lim_{x\to 0}\frac{4\sin 4x}{4x}\cdot\lim_{x\to 0}\frac{6x}{6\sin 6x}$$
$$= 4\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{1}{6}\lim_{x\to 0}\frac{6x}{\sin 6x} = 4(1)\cdot\frac{1}{6}(1) = \frac{2}{3}$$

37. $$\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} = \lim_{t\to 0}\left(\frac{\sin 6t}{t}\cdot\frac{1}{\cos 6t}\cdot\frac{t}{\sin 2t}\right) = \lim_{t\to 0}\frac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\lim_{t\to 0}\frac{2t}{2\sin 2t}$$
$$= 6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t} = 6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1) = 3$$

38. $$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0}\frac{\dfrac{\cos\theta-1}{\theta}}{\dfrac{\sin\theta}{\theta}} = \frac{\displaystyle\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}}{\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}} = \frac{0}{1} = 0$$

39. $$\lim_{\theta\to 0}\frac{\sin(\cos\theta)}{\sec\theta} = \frac{\sin\left(\displaystyle\lim_{\theta\to 0}\cos\theta\right)}{\displaystyle\lim_{\theta\to 0}\sec\theta} = \frac{\sin 1}{1} = \sin 1$$

40. $$\lim_{t\to 0}\frac{\sin^2 3t}{t^2} = \lim_{t\to 0}\left(\frac{\sin 3t}{t}\cdot\frac{\sin 3t}{t}\right) = \lim_{t\to 0}\frac{\sin 3t}{t}\cdot\lim_{t\to 0}\frac{\sin 3t}{t}$$
$$= \left(\lim_{t\to 0}\frac{\sin 3t}{t}\right)^2 = \left(3\lim_{t\to 0}\frac{\sin 3t}{3t}\right)^2 = (3\cdot 1)^2 = 9$$

41. $$\lim_{x\to 0}\frac{\cot 2x}{\csc x} = \lim_{x\to 0}\frac{\cos 2x\sin x}{\sin 2x} = \lim_{x\to 0}\cos 2x\left[\frac{(\sin x)/x}{(\sin 2x)/x}\right] = \lim_{x\to 0}\cos 2x\left[\frac{\displaystyle\lim_{x\to 0}[(\sin x)/x]}{2\displaystyle\lim_{x\to 0}[(\sin 2x)/2x]}\right]$$
$$= 1\cdot\frac{1}{2\cdot 1} = \frac{1}{2}$$

42. $$\lim_{x\to\pi/4}\frac{\sin x-\cos x}{\cos 2x} = \lim_{x\to\pi/4}\frac{\sin x-\cos x}{\cos^2 x-\sin^2 x} = \lim_{x\to\pi/4}\frac{\sin x-\cos x}{(\cos x+\sin x)(\cos x-\sin x)}$$
$$= \lim_{x\to\pi/4}\frac{-1}{\cos x+\sin x} = \frac{-1}{\cos\frac{\pi}{4}+\sin\frac{\pi}{4}} = \frac{-1}{\sqrt{2}}$$

43. Divide numerator and denominator by θ. ($\sin\theta$ also works.)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta+\tan\theta} = \lim_{\theta\to 0}\frac{\dfrac{\sin\theta}{\theta}}{1+\dfrac{\sin\theta}{\theta}\cdot\dfrac{1}{\cos\theta}} = \frac{\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}{1+\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\frac{1}{\cos\theta}} = \frac{1}{1+1\cdot 1} = \frac{1}{2}$$

44. $$\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2} = \lim_{x\to 1}\frac{\sin(x-1)}{(x+2)(x-1)} = \lim_{x\to 1}\frac{1}{x+2}\lim_{x\to 1}\frac{\sin(x-1)}{x-1} = \tfrac{1}{3}\cdot 1 = \tfrac{1}{3}$$

45. (a) $\dfrac{d}{dx}\tan x = \dfrac{d}{dx}\dfrac{\sin x}{\cos x} \;\Rightarrow\; \sec^2 x = \dfrac{\cos x\cos x - \sin x\,(-\sin x)}{\cos^2 x} = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$. So $\sec^2 x = \dfrac{1}{\cos^2 x}$.

(b) $\dfrac{d}{dx}\sec x = \dfrac{d}{dx}\dfrac{1}{\cos x} \;\Rightarrow\; \sec x\tan x = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$. So $\sec x\tan x = \dfrac{\sin x}{\cos^2 x}$.

(c) $\dfrac{d}{dx}(\sin x + \cos x) = \dfrac{d}{dx}\dfrac{1+\cot x}{\csc x} \;\Rightarrow$

$$\cos x - \sin x = \frac{\csc x\left(-\csc^2 x\right) - (1+\cot x)(-\csc x\cot x)}{\csc^2 x} = \frac{\csc x\left[-\csc^2 x + (1+\cot x)\cot x\right]}{\csc^2 x}$$

$$= \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x} = \frac{-1+\cot x}{\csc x}$$

So $\cos x - \sin x = \dfrac{\cot x - 1}{\csc x}$.

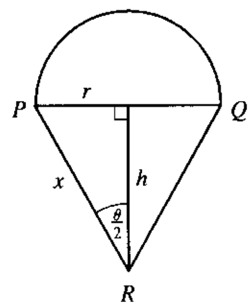
46. Let $|PR| = x$. Then we get the following formulas for r and h in terms of θ and x:

$\sin\dfrac{\theta}{2} = \dfrac{r}{x} \;\Rightarrow\; r = x\sin\dfrac{\theta}{2}$ and $\cos\dfrac{\theta}{2} = \dfrac{h}{x} \;\Rightarrow\; h = x\cos\dfrac{\theta}{2}$.

Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}\pi r^2}{rh} = \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\frac{r}{h} = \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\frac{x\sin(\theta/2)}{x\cos(\theta/2)}$$

$$= \tfrac{1}{2}\pi\lim_{\theta\to 0^+}\tan(\theta/2) = 0.$$



47. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle.

By drawing the bisector of the angle θ, we can see that $\sin\dfrac{\theta}{2} = \dfrac{d/2}{r} \;\Rightarrow\; d = 2r\sin\dfrac{\theta}{2}$.

So $\displaystyle\lim_{\theta\to 0^+}\frac{s}{d} = \lim_{\theta\to 0^+}\frac{r\theta}{2r\sin(\theta/2)} = \lim_{\theta\to 0^+}\frac{2\cdot(\theta/2)}{2\sin(\theta/2)} = \lim_{\theta\to 0}\frac{\theta/2}{\sin(\theta/2)} = 1$. [This is just the reciprocal of the limit $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$ combined with the fact that as $\theta\to 0$, $\frac{\theta}{2}\to 0$ also.]

3.6 The Chain Rule

1. Let $u = g(x) = 4x$ and $y = f(u) = \sin u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\cos u)(4) = 4\cos 4x$.

2. Let $u = g(x) = 4+3x$ and $y = f(u) = \sqrt{u} = u^{1/2}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \frac{1}{2}u^{-1/2}(3) = \dfrac{3}{2\sqrt{u}} = \dfrac{3}{2\sqrt{4+3x}}$.

3. Let $u = g(x) = 1-x^2$ and $y = f(u) = u^{10}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (10u^9)(-2x) = -20x(1-x^2)^9$.

4. Let $u = g(x) = \sin x$ and $y = f(u) = \tan u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\sec^2 u)(\cos x) = \sec^2(\sin x)\cdot\cos x$, or equivalently, $[\sec(\sin x)]^2\cos x$.

5. Let $u = g(x) = \sin x$ and $y = f(u) = \sqrt{u}$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \frac{1}{2}u^{-1/2}\cos x = \dfrac{\cos x}{2\sqrt{u}} = \dfrac{\cos x}{2\sqrt{\sin x}}$.

6. Let $u = g(x) = \sqrt{x}$ and $y = f(u) = \sin u$. Then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = (\cos u)\left(\frac{1}{2}x^{-1/2}\right) = \dfrac{\cos u}{2\sqrt{x}} = \dfrac{\cos\sqrt{x}}{2\sqrt{x}}$.

7. $F(x) = (x^3+4x)^7 \;\Rightarrow\; F'(x) = 7(x^3+4x)^6(3x^2+4)$ [or $7x^6(x^2+4)^6(3x^2+4)$]

8. $F(x) = (x^2-x+1)^3 \;\Rightarrow\; F'(x) = 3(x^2-x+1)^2(2x-1)$

9. $F(x) = \sqrt[4]{1+2x+x^3} = (1+2x+x^3)^{1/4} \quad \Rightarrow$

$$F'(x) = \tfrac{1}{4}(1+2x+x^3)^{-3/4} \cdot \frac{d}{dx}(1+2x+x^3) = \frac{1}{4(1+2x+x^3)^{3/4}} \cdot (2+3x^2)$$

$$= \frac{2+3x^2}{4(1+2x+x^3)^{3/4}} = \frac{2+3x^2}{4\sqrt[4]{(1+2x+x^3)^3}}$$

10. $f(x) = (1+x^4)^{2/3} \quad \Rightarrow \quad f'(x) = \frac{2}{3}(1+x^4)^{-1/3}(4x^3) = \dfrac{8x^3}{3\sqrt[3]{1+x^4}}$

11. $g(t) = \dfrac{1}{(t^4+1)^3} = (t^4+1)^{-3} \quad \Rightarrow \quad g'(t) = -3(t^4+1)^{-4}(4t^3) = -12t^3(t^4+1)^{-4} = \dfrac{-12t^3}{(t^4+1)^4}$

12. $f(t) = \sqrt[3]{1+\tan t} = (1+\tan t)^{1/3} \quad \Rightarrow \quad f'(t) = \frac{1}{3}(1+\tan t)^{-2/3}\sec^2 t = \dfrac{\sec^2 t}{3\sqrt[3]{(1+\tan t)^2}}$

13. $y = \cos(a^3+x^3) \quad \Rightarrow \quad y' = -\sin(a^3+x^3)\cdot 3x^2$ [a^3 is just a constant] $= -3x^2\sin(a^3+x^3)$

14. $y = a^3 + \cos^3 x \quad \Rightarrow \quad y' = 3(\cos x)^2(-\sin x)$ [a^3 is just a constant] $= -3\sin x\cos^2 x$

15. $y = \cot(x/2) \quad \Rightarrow \quad y' = -\csc^2(x/2)\cdot\frac{1}{2} = -\frac{1}{2}\csc^2(x/2)$

16. $y = 4\sec 5x \quad \Rightarrow \quad y' = 4\sec 5x\tan 5x(5) = 20\sec 5x\tan 5x$

17. $g(x) = (1+4x)^5(3+x-x^2)^8 \quad \Rightarrow$

$$\begin{aligned} g'(x) &= (1+4x)^5\cdot 8(3+x-x^2)^7(1-2x) + (3+x-x^2)^8\cdot 5(1+4x)^4\cdot 4 \\ &= 4(1+4x)^4(3+x-x^2)^7\left[2(1+4x)(1-2x)+5(3+x-x^2)\right] \\ &= 4(1+4x)^4(3+x-x^2)^7\left[(2+4x-16x^2)+(15+5x-5x^2)\right] \\ &= 4(1+4x)^4(3+x-x^2)^7(17+9x-21x^2) \end{aligned}$$

18. $h(t) = (t^4-1)^3(t^3+1)^4 \quad \Rightarrow$

$$\begin{aligned} h'(t) &= (t^4-1)^3\cdot 4(t^3+1)^3(3t^2) + (t^3+1)^4\cdot 3(t^4-1)^2(4t^3) \\ &= 12t^2(t^4-1)^2(t^3+1)^3\left[(t^4-1)+t(t^3+1)\right] = 12t^2(t^4-1)^2(t^3+1)^3(2t^4+t-1) \end{aligned}$$

19. $y = (2x-5)^4(8x^2-5)^{-3} \quad \Rightarrow$

$$\begin{aligned} y' &= 4(2x-5)^3(2)(8x^2-5)^{-3} + (2x-5)^4(-3)(8x^2-5)^{-4}(16x) \\ &= 8(2x-5)^3(8x^2-5)^{-3} - 48x(2x-5)^4(8x^2-5)^{-4} \end{aligned}$$

[This simplifies to $8(2x-5)^3(8x^2-5)^{-4}(-4x^2+30x-5)$.]

20. $y = (x^2+1)(x^2+2)^{1/3} \quad \Rightarrow$

$$y' = 2x(x^2+2)^{1/3} + (x^2+1)\left(\tfrac{1}{3}\right)(x^2+2)^{-2/3}(2x) = 2x(x^2+2)^{1/3}\left[1+\frac{x^2+1}{3(x^2+2)}\right]$$

21. $y = x^3\cos nx \quad \Rightarrow \quad y' = x^3(-\sin nx)(n) + \cos nx\,(3x^2) = x^2(3\cos nx - nx\sin nx)$

22. $y = x\sin\sqrt{x} \quad \Rightarrow \quad y' = x\cos\sqrt{x}\cdot\frac{1}{2}x^{-1/2} + \sin\sqrt{x}\cdot 1 = \frac{1}{2}\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}$

23. $y = \sin(x\cos x) \quad \Rightarrow \quad y' = \cos(x\cos x)\cdot[x(-\sin x)+\cos x\cdot 1] = (\cos x - x\sin x)\cos(x\cos x)$

24. $f(x) = \dfrac{x}{\sqrt{7-3x}} \Rightarrow$

$$f'(x) = \frac{\sqrt{7-3x}(1) - x\cdot\frac{1}{2}(7-3x)^{-1/2}\cdot(-3)}{(\sqrt{7-3x})^2} = \frac{\sqrt{7-3x}+\dfrac{3x}{2\sqrt{7-3x}}}{(7-3x)^1}$$

$$= \frac{2(7-3x)+3x}{2(7-3x)^{3/2}} = \frac{14-3x}{2(7-3x)^{3/2}}$$

25. $F(z) = \sqrt{\dfrac{z-1}{z+1}} = \left(\dfrac{z-1}{z+1}\right)^{1/2} \Rightarrow$

$$F'(z) = \frac{1}{2}\left(\frac{z-1}{z+1}\right)^{-1/2}\cdot\frac{d}{dz}\left(\frac{z-1}{z+1}\right) = \frac{1}{2}\left(\frac{z+1}{z-1}\right)^{1/2}\cdot\frac{(z+1)(1)-(z-1)(1)}{(z+1)^2}$$

$$= \frac{1}{2}\frac{(z+1)^{1/2}}{(z-1)^{1/2}}\cdot\frac{z+1-z+1}{(z+1)^2} = \frac{1}{2}\frac{(z+1)^{1/2}}{(z-1)^{1/2}}\cdot\frac{2}{(z+1)^2} = \frac{1}{(z-1)^{1/2}(z+1)^{3/2}}$$

26. $G(y) = \dfrac{(y-1)^4}{(y^2+2y)^5} \Rightarrow$

$$G'(y) = \frac{(y^2+2y)^5\cdot 4(y-1)^3\cdot 1 - (y-1)^4\cdot 5(y^2+2y)^4(2y+2)}{\left[(y^2+2y)^5\right]^2}$$

$$= \frac{2(y^2+2y)^4(y-1)^3[2(y^2+2y)-5(y-1)(y+1)]}{(y^2+2y)^{10}}$$

$$= \frac{2(y-1)^3[(2y^2+4y)+(-5y^2+5)]}{(y^2+2y)^6} = \frac{2(y-1)^3(-3y^2+4y+5)}{(y^2+2y)^6}$$

27. $y = \dfrac{r}{\sqrt{r^2+1}} \Rightarrow$

$$y' = \frac{\sqrt{r^2+1}\,(1) - r\cdot\frac{1}{2}\left(r^2+1\right)^{-1/2}(2r)}{\left(\sqrt{r^2+1}\right)^2} = \frac{\sqrt{r^2+1}-\dfrac{r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2} = \frac{\dfrac{\sqrt{r^2+1}\,\sqrt{r^2+1}-r^2}{\sqrt{r^2+1}}}{\left(\sqrt{r^2+1}\right)^2}$$

$$= \frac{\left(r^2+1\right)-r^2}{\left(\sqrt{r^2+1}\right)^3} = \frac{1}{(r^2+1)^{3/2}} \text{ or } \left(r^2+1\right)^{-3/2}$$

Another solution: Write y as a product and make use of the Product Rule. $y = r\left(r^2+1\right)^{-1/2} \Rightarrow$

$$y' = r\cdot -\tfrac{1}{2}\left(r^2+1\right)^{-3/2}(2r) + \left(r^2+1\right)^{-1/2}\cdot 1$$

$$= \left(r^2+1\right)^{-3/2}\left[-r^2+\left(r^2+1\right)^1\right] = \left(r^2+1\right)^{-3/2}(1) = \left(r^2+1\right)^{-3/2}$$

The step that students usually have trouble with is factoring out $\left(r^2+1\right)^{-3/2}$. But this is no different than factoring out x^2 from x^2+x^5; that is, we are just factoring out a factor with the *smallest* exponent that appears on it. In this case, $-\frac{3}{2}$ is smaller than $-\frac{1}{2}$.

28. $y = \dfrac{\cos\pi x}{\sin\pi x+\cos\pi x} \Rightarrow$

$$y' = \frac{(\sin\pi x+\cos\pi x)(-\pi\sin\pi x)-(\cos\pi x)(\pi\cos\pi x-\pi\sin\pi x)}{(\sin\pi x+\cos\pi x)^2}$$

$$= \frac{-\pi\sin^2\pi x-\pi\sin\pi x\cos\pi x-\pi\cos^2\pi x+\pi\sin\pi x\cos\pi x}{(\sin\pi x+\cos\pi x)^2}$$

$$= \frac{-\pi(\sin^2\pi x+\cos^2\pi x)}{(\sin\pi x+\cos\pi x)^2} = \frac{-\pi}{(\sin\pi x+\cos\pi x)^2} \quad\text{or}\quad \frac{-\pi}{1+2\sin\pi x\cos\pi x}$$

29. $y = \tan(\cos x) \Rightarrow y' = \sec^2(\cos x) \cdot (-\sin x) = -\sin x \sec^2(\cos x)$

30. $y = \dfrac{\sin^2 x}{\cos x} \Rightarrow$

$$y' = \frac{\cos x\,(2\sin x\cos x) - \sin^2 x\,(-\sin x)}{\cos^2 x} = \frac{\sin x\,(2\cos^2 x + \sin^2 x)}{\cos^2 x} = \frac{\sin x\,(1+\cos^2 x)}{\cos^2 x}$$
$$= \sin x\,(1 + \sec^2 x)$$

Another method: $y = \tan x \sin x \Rightarrow y' = \sec^2 x \sin x + \tan x \cos x = \sec^2 x \sin x + \sin x$

31. $y = \sin\sqrt{1+x^2} \Rightarrow y' = \cos\sqrt{1+x^2} \cdot \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x = \left(x \cos\sqrt{1+x^2}\right)/\sqrt{1+x^2}$

32. $y = \tan^2(3\theta) = (\tan 3\theta)^2 \Rightarrow y' = 2(\tan 3\theta) \cdot \dfrac{d}{d\theta}(\tan 3\theta) = 2\tan 3\theta \cdot \sec^2 3\theta \cdot 3 = 6\tan 3\theta \sec^2 3\theta$

33. $y = (1+\cos^2 x)^6 \Rightarrow y' = 6(1+\cos^2 x)^5 \cdot 2\cos x\,(-\sin x) = -12\cos x \sin x\,(1+\cos^2 x)^5$

34. $y = x\sin\dfrac{1}{x} \Rightarrow y' = \sin\dfrac{1}{x} + x\cos\dfrac{1}{x}\left(-\dfrac{1}{x^2}\right) = \sin\dfrac{1}{x} - \dfrac{1}{x}\cos\dfrac{1}{x}$

35. $y = \sec^2 x + \tan^2 x = (\sec x)^2 + (\tan x)^2 \Rightarrow$
$y' = 2(\sec x)(\sec x \tan x) + 2(\tan x)(\sec^2 x) = 2\sec^2 x \tan x + 2\sec^2 x \tan x = 4\sec^2 x \tan x$

36. $y = \cot(x^2) + \cot^2 x = \cot(x^2) + (\cot x)^2 \Rightarrow$
$y' = -\csc^2(x^2) \cdot 2x + 2(\cot x)^1(-\csc^2 x) = -2x\csc^2(x^2) - 2\cot x \csc^2 x$

37. $y = \cot^2(\sin\theta) = [\cot(\sin\theta)]^2 \Rightarrow$
$y' = 2[\cot(\sin\theta)] \cdot \dfrac{d}{d\theta}[\cot(\sin\theta)] = 2\cot(\sin\theta) \cdot [-\csc^2(\sin\theta)\cdot\cos\theta] = -2\cos\theta\,\cot(\sin\theta)\,\csc^2(\sin\theta)$

38. $y = \sin(\sin(\sin x)) \Rightarrow y' = \cos(\sin(\sin x))\,\dfrac{d}{dx}(\sin(\sin x)) = \cos(\sin(\sin x))\,\cos(\sin x)\,\cos x$

39. $y = \sqrt{x+\sqrt{x}} \Rightarrow y' = \frac{1}{2}(x+\sqrt{x})^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right) = \dfrac{1}{2\sqrt{x+\sqrt{x}}}\left(1 + \dfrac{1}{2\sqrt{x}}\right)$

40. $y = \sqrt{x+\sqrt{x+\sqrt{x}}} \Rightarrow y' = \frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2}\left[1 + \frac{1}{2}(x+\sqrt{x})^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right)\right]$

41. $y = \sin\left(\tan\sqrt{\sin x}\right) \Rightarrow$
$$y' = \cos\left(\tan\sqrt{\sin x}\right) \cdot \frac{d}{dx}\left(\tan\sqrt{\sin x}\right) = \cos\left(\tan\sqrt{\sin x}\right)\sec^2\sqrt{\sin x} \cdot \frac{d}{dx}(\sin x)^{1/2}$$
$$= \cos\left(\tan\sqrt{\sin x}\right)\sec^2\sqrt{\sin x} \cdot \tfrac{1}{2}(\sin x)^{-1/2} \cdot \cos x$$
$$= \cos\left(\tan\sqrt{\sin x}\right)\left(\sec^2\sqrt{\sin x}\right)\left(\frac{1}{2\sqrt{\sin x}}\right)(\cos x)$$

42. $y = \sqrt{\cos(\sin^2 x)} \Rightarrow y' = \frac{1}{2}\left(\cos(\sin^2 x)\right)^{-1/2}\left[-\sin(\sin^2 x)\right](2\sin x\cos x) = -\dfrac{\sin(\sin^2 x)\sin x\cos x}{\sqrt{\cos(\sin^2 x)}}$

43. $y = (1+2x)^{10} \Rightarrow y' = 10(1+2x)^9 \cdot 2 = 20(1+2x)^9$. At $(0,1)$, $y' = 20(1+0)^9 = 20$, and an equation of the tangent line is $y - 1 = 20(x-0)$, or $y = 20x + 1$.

44. $y = \sin x + \sin^2 x \Rightarrow y' = \cos x + 2\sin x\cos x$. At $(0,0)$, $y' = 1$, and an equation of the tangent line is $y - 0 = 1(x-0)$, or $y = x$.

45. $y = \sin(\sin x) \Rightarrow y' = \cos(\sin x)\cdot\cos x$. At $(\pi, 0)$, $y' = \cos(\sin\pi)\cdot\cos\pi = \cos(0)\cdot(-1) = 1(-1) = -1$, and an equation of the tangent line is $y - 0 = -1(x-\pi)$, or $y = -x + \pi$.

46. $y = \sqrt{5+x^2} \Rightarrow y' = \frac{1}{2}(5+x^2)^{-1/2}(2x) = x/\sqrt{5+x^2}$. At $(2,3)$, $y' = \frac{2}{3}$, and an equation of the tangent line is $y - 3 = \frac{2}{3}(x-2)$, or $y = \frac{2}{3}x + \frac{5}{3}$.

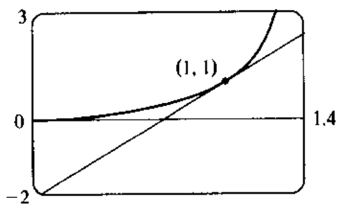
47. (a) $y = f(x) = \tan\left(\frac{\pi}{4}x^2\right) \Rightarrow f'(x) = \sec^2\left(\frac{\pi}{4}x^2\right)\left(2 \cdot \frac{\pi}{4}x\right)$.

The slope of the tangent at $(1,1)$ is thus

$f'(1) = \sec^2 \frac{\pi}{4}\left(\frac{\pi}{2}\right) = 2 \cdot \frac{\pi}{2} = \pi$, and its equation is

$y - 1 = \pi(x-1)$ or $y = \pi x - \pi + 1$.

(b)



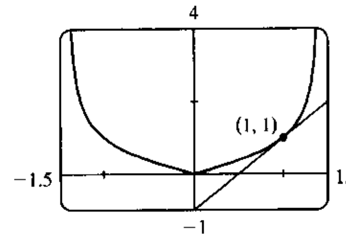
48. (a) For $x > 0$, $|x| = x$, and $y = f(x) = \dfrac{x}{\sqrt{2-x^2}} \Rightarrow$

$$f'(x) = \frac{\sqrt{2-x^2}\,(1) - x\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)}{\left(\sqrt{2-x^2}\right)^2} \cdot \frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}}$$

$$= \frac{(2-x^2)+x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}}$$

So at $(1,1)$, the slope of the tangent line is $f'(1) = 2$ and its equation is $y - 1 = 2(x-1)$ or $y = 2x - 1$.

(b)

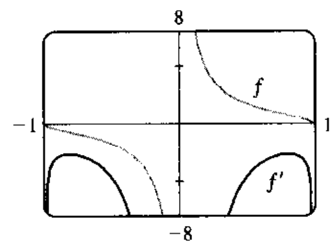


49. (a) $f(x) = \dfrac{\sqrt{1-x^2}}{x} \Rightarrow$

$$f'(x) = \frac{x \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x) - \sqrt{1-x^2}\,(1)}{x^2} \cdot \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}$$

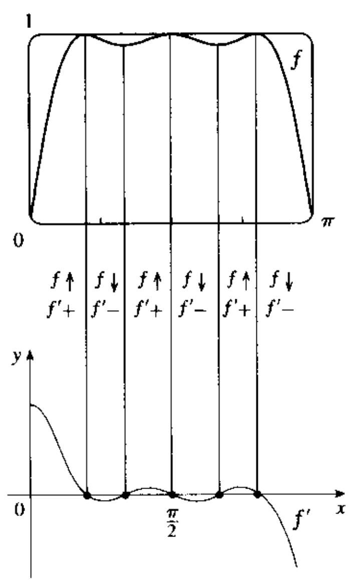
$$= \frac{-x^2 - (1-x^2)}{x^2\sqrt{1-x^2}} = \frac{-1}{x^2\sqrt{1-x^2}}$$

(b)



Notice that all tangents to the graph of f have negative slopes and $f'(x) < 0$ always.

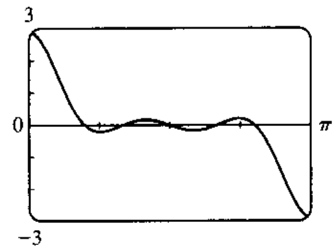
50. (a)



From the graph of f, we see that there are 5 horizontal tangents, so there must be 5 zeros on the graph of f'. From the symmetry of the graph of f, we must have the graph of f' as high at $x = 0$ as it is low at $x = \pi$. The intervals of increase and decrease as well as the signs of f' are indicated in the figure.

(b) $f(x) = \sin(x + \sin 2x) \Rightarrow$

$$f'(x) = \cos(x + \sin 2x) \cdot \frac{d}{dx}(x + \sin 2x)$$
$$= \cos(x + \sin 2x)(1 + 2\cos 2x)$$



51. For the tangent line to be horizontal, $f'(x) = 0$. $f(x) = 2\sin x + \sin^2 x \Rightarrow$ $f'(x) = 2\cos x + 2\sin x \cos x = 0 \Leftrightarrow 2\cos x\,(1 + \sin x) = 0 \Leftrightarrow \cos x = 0$ or $\sin x = -1$, so $x = \frac{\pi}{2} + 2n\pi$ or $\frac{3\pi}{2} + 2n\pi$, where n is any integer. Now $f\left(\frac{\pi}{2}\right) = 3$ and $f\left(\frac{3\pi}{2}\right) = -1$, so the points on the curve with a horizontal tangent are $\left(\frac{\pi}{2} + 2n\pi, 3\right)$ and $\left(\frac{3\pi}{2} + 2n\pi, -1\right)$, where n is any integer.

52. $f(x) = \sin 2x - 2\sin x \Rightarrow f'(x) = 2\cos 2x - 2\cos x = 4\cos^2 x - 2\cos x - 2$, and $4\cos^2 x - 2\cos x - 2 = 0 \Leftrightarrow (\cos x - 1)(4\cos x + 2) = 0 \Leftrightarrow \cos x = 1$ or $\cos x = -\frac{1}{2}$. So $x = 2n\pi$ or $(2n+1)\pi \pm \frac{\pi}{3}$, n any integer.

53. $F(x) = f(g(x)) \Rightarrow F'(x) = f'(g(x)) \cdot g'(x)$, so $F'(3) = f'(g(3)) \cdot g'(3) = f'(6) \cdot g'(3) = 7 \cdot 4 = 28$. Notice that we did not use $f'(3) = 2$.

54. $w = u \circ v \Rightarrow w(x) = u(v(x)) \Rightarrow w'(x) = u'(v(x)) \cdot v'(x)$, so $w'(0) = u'(v(0)) \cdot v'(0) = u'(2) \cdot v'(0) = 4 \cdot 5 = 20$. The other pieces of information, $u(0) = 1$, $u'(0) = 3$, and $v'(2) = 6$, were not needed.

55. (a) $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

(b) $H(x) = g(f(x)) \Rightarrow H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

56. (a) $F(x) = f(f(x)) \Rightarrow F'(x) = f'(f(x)) \cdot f'(x)$, so $F'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.

(b) $G(x) = g(g(x)) \Rightarrow G'(x) = g'(g(x)) \cdot g'(x)$, so $G'(3) = g'(g(3)) \cdot g'(3) = g'(2) \cdot 9 = 7 \cdot 9 = 63$.

57. (a) $u(x) = f(g(x)) \Rightarrow u'(x) = f'(g(x))g'(x)$. So $u'(1) = f'(g(1))g'(1) = f'(3)g'(1)$. To find $f'(3)$, note that f is linear from $(2, 4)$ to $(6, 3)$, so its slope is $\dfrac{3-4}{6-2} = -\dfrac{1}{4}$. To find $g'(1)$, note that g is linear from $(0, 6)$ to $(2, 0)$, so its slope is $\dfrac{0-6}{2-0} = -3$. Thus, $f'(3)g'(1) = \left(-\frac{1}{4}\right)(-3) = \frac{3}{4}$.

(b) $v(x) = g(f(x)) \Rightarrow v'(x) = g'(f(x))f'(x)$. So $v'(1) = g'(f(1))f'(1) = g'(2)f'(1)$, which does not exist since $g'(2)$ does not exist.

(c) $w(x) = g(g(x)) \Rightarrow w'(x) = g'(g(x))g'(x)$. So $w'(1) = g'(g(1))g'(1) = g'(3)g'(1)$. To find $g'(3)$, note that g is linear from $(2, 0)$ to $(5, 2)$, so its slope is $\dfrac{2-0}{5-2} = \dfrac{2}{3}$. Thus, $g'(3)g'(1) = \left(\frac{2}{3}\right)(-3) = -2$.

58. (a) $h(x) = f(f(x)) \Rightarrow h'(x) = f'(f(x))f'(x)$.
So $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1)(-1) = 1$.

(b) $g(x) = f(x^2) \Rightarrow g'(x) = f'(x^2) \cdot \dfrac{d}{dx}(x^2) = f'(x^2)(2x)$.
So $g'(2) = f'(2^2)(2 \cdot 2) = 4f'(4) \approx 4(1.5) = 6$.

59. $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))g'(x)$. So $h'(0.5) = f'(g(0.5))g'(0.5) = f'(0.1)g'(0.5)$. We can estimate the derivatives by taking the average of two secant slopes.

For $f'(0.1)$: $m_1 = \dfrac{14.8 - 12.6}{0.1 - 0} = 22$, $m_2 = \dfrac{18.4 - 14.8}{0.2 - 0.1} = 36$. So $f'(0.1) \approx \dfrac{m_1 + m_2}{2} = \dfrac{22 + 36}{2} = 29$.

For $g'(0.5)$: $m_1 = \dfrac{0.10 - 0.17}{0.5 - 0.4} = -0.7$, $m_2 = \dfrac{0.05 - 0.10}{0.6 - 0.5} = -0.5$. So $g'(0.5) \approx \dfrac{m_1 + m_2}{2} = -0.6$.

Hence, $h'(0.5) = f'(0.1)g'(0.5) \approx (29)(-0.6) = -17.4$.

60. $g(x) = f(f(x)) \Rightarrow g'(x) = f'(f(x))f'(x)$. So $g'(1) = f'(f(1))f'(1) = f'(2)f'(1)$.

For $f'(2)$: $m_1 = \dfrac{3.1 - 2.4}{2.0 - 1.5} = 1.4$, $m_2 = \dfrac{4.4 - 3.1}{2.5 - 2.0} = 2.6$. So $f'(2) \approx \dfrac{m_1 + m_2}{2} = 2$.

For $f'(1)$: $m_1 = \dfrac{2.0 - 1.8}{1.0 - 0.5} = 0.4$, $m_2 = \dfrac{2.4 - 2.0}{1.5 - 1.0} = 0.8$. So $f'(1) \approx \dfrac{m_1 + m_2}{2} = 0.6$.

Hence, $g'(1) = f'(2)f'(1) \approx (2)(0.6) = 1.2$.

61. (a) $F(x) = f(\cos x) \Rightarrow F'(x) = f'(\cos x)\dfrac{d}{dx}(\cos x) = -\sin x f'(\cos x)$

(b) $G(x) = \cos(f(x)) \Rightarrow G'(x) = -\sin(f(x))\, f'(x)$

62. (a) $F(x) = f(x^\alpha) \Rightarrow F'(x) = f'(x^\alpha)\dfrac{d}{dx}(x^\alpha) = f'(x^\alpha)\alpha x^{\alpha-1}$

(b) $G(x) = [f(x)]^\alpha \Rightarrow G'(x) = \alpha\,[f(x)]^{\alpha-1} f'(x)$

63. (a) $f(x) = L(x^4) \Rightarrow f'(x) = L'(x^4) \cdot 4x^3 = (1/x^4) \cdot 4x^3 = 4/x$ for $x > 0$.

(b) $g(x) = L(4x) \Rightarrow g'(x) = L'(4x) \cdot 4 = (1/(4x)) \cdot 4 = 1/x$ for $x > 0$.

(c) $F(x) = [L(x)]^4 \Rightarrow F'(x) = 4[L(x)]^3 \cdot L'(x) = 4[L(x)]^3 \cdot (1/x) = 4[L(x)]^3/x$

(d) $G(x) = L(1/x) \Rightarrow G'(x) = L'(1/x) \cdot (-1/x^2) = (1/(1/x)) \cdot (-1/x^2) = x \cdot (-1/x^2) = -1/x$ for $x > 0$.

64. $r(x) = f(g(h(x))) \Rightarrow r'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$, so
$r'(1) = f'(g(h(1))) \cdot g'(h(1)) \cdot h'(1) = f'(g(2)) \cdot g'(2) \cdot 4 = f'(3) \cdot 5 \cdot 4 = 6 \cdot 5 \cdot 4 = 120$

65. $s(t) = 10 + \frac{1}{4}\sin(10\pi t) \Rightarrow$ the velocity after t seconds is
$v(t) = s'(t) = \frac{1}{4}\cos(10\pi t)(10\pi) = \frac{5\pi}{2}\cos(10\pi t)$ cm/s.

66. (a) $s = A\cos(\omega t + \delta) \Rightarrow$ velocity $= s' = -\omega A\sin(\omega t + \delta)$.

(b) If $A \neq 0$ and $\omega \neq 0$, then $s' = 0 \Leftrightarrow \sin(\omega t + \delta) = 0 \Leftrightarrow \omega t + \delta = n\pi \Leftrightarrow t = \dfrac{n\pi - \delta}{\omega}$, n an integer.

67. (a) $B(t) = 4.0 + 0.35\sin\dfrac{2\pi t}{5.4} \Rightarrow \dfrac{dB}{dt} = \left(0.35\cos\dfrac{2\pi t}{5.4}\right)\left(\dfrac{2\pi}{5.4}\right) = \dfrac{0.7\pi}{5.4}\cos\dfrac{2\pi t}{5.4} = \dfrac{7\pi}{54}\cos\dfrac{2\pi t}{5.4}$

(b) At $t = 1$, $\dfrac{dB}{dt} = \dfrac{7\pi}{54}\cos\dfrac{2\pi}{5.4} \approx 0.16$.

68. $L(t) = 12 + 2.8\sin\left(\frac{2\pi}{365}(t - 80)\right) \Rightarrow L'(t) = 2.8\cos\left(\frac{2\pi}{365}(t - 80)\right)\left(\frac{2\pi}{365}\right)$.
On March 21, $t = 80$, and $L'(80) \approx 0.0482$ hours per day. On May 21, $t = 141$, and $L'(141) \approx 0.02398$, which is approximately one-half of $L'(80)$.

69. (a) Derive gives $g'(t) = \dfrac{45(t-2)^8}{(2t+1)^{10}}$ without simplifying. With either Maple or Mathematica, we first get

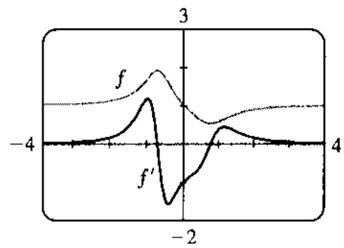
$g'(t) = 9\dfrac{(t-2)^8}{(2t+1)^9} - 18\dfrac{(t-2)^9}{(2t+1)^{10}}$, and the simplification command results in the above expression.

(b) Derive gives $y' = 2(x^3 - x + 1)^3(2x+1)^4(17x^3 + 6x^2 - 9x + 3)$ without simplifying.
With either Maple or Mathematica, we first get
$y' = 10(2x+1)^4(x^3 - x + 1)^4 + 4(2x+1)^5(x^3 - x + 1)^3(3x^2 - 1)$. If we use Mathematica's `Factor` or `Simplify`, or Maple's `factor`, we get the above expression, but Maple's `simplify` gives the polynomial expansion instead. For locating horizontal tangents, the factored form is the most helpful.

70. (a) $f(x) = \left(\dfrac{x^4 - x + 1}{x^4 + x + 1}\right)^{1/2}$. Derive gives $f'(x) = \dfrac{(3x^4 - 1)\sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}}{(x^4 + x + 1)(x^4 - x + 1)}$ whereas either Maple or Mathematica give $f'(x) = \dfrac{3x^4 - 1}{\sqrt{\dfrac{x^4 - x + 1}{x^4 + x + 1}}\,(x^4 + x + 1)^2}$ after simplification.

(b) $f'(x) = 0 \;\Leftrightarrow\; 3x^4 - 1 = 0 \;\Leftrightarrow\; x = \pm\sqrt[4]{\frac{1}{3}} \approx \pm 0.7598$.

(c) $f'(x) = 0$ where f has horizontal tangents. f' has two maxima and one minimum where f has inflection points.



71. (a) If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get
$f'(x) = f'(-x)\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

(b) If f is odd, then $f(x) = -f(-x)$. Differentiating this equation, we get $f'(x) = -f'(-x)(-1) = f'(-x)$, so f' is even.

72.
$$\left[\frac{f(x)}{g(x)}\right]' = \{f(x)\,[g(x)]^{-1}\}' = f'(x)\,[g(x)]^{-1} + (-1)\,[g(x)]^{-2}\,g'(x)f(x)$$
$$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

73. (a)
$$\begin{aligned}\frac{d}{dx}(\sin^n x\cos nx) &= n\sin^{n-1}x\cos x\cos nx + \sin^n x\,(-n\sin nx) && \text{[Product Rule]}\\ &= n\sin^{n-1}x\,(\cos nx\cos x - \sin nx\sin x) && [\text{factor out } n\sin^{n-1}x]\\ &= n\sin^{n-1}x\cos(nx + x) && \text{[Addition Formula for cosine]}\\ &= n\sin^{n-1}x\cos[(n+1)x] && [\text{factor out } x]\end{aligned}$$

(b)
$$\begin{aligned}\frac{d}{dx}(\cos^n x\cos nx) &= n\cos^{n-1}x\,(-\sin x)\cos nx + \cos^n x\,(-n\sin nx) && \text{[Product Rule]}\\ &= -n\cos^{n-1}x\,(\cos nx\sin x + \sin nx\cos x) && [\text{factor out } -n\cos^{n-1}x]\\ &= -n\cos^{n-1}x\sin(nx + x) && \text{[Addition Formula for sine]}\\ &= -n\cos^{n-1}x\sin[(n+1)x] && [\text{factor out } x]\end{aligned}$$

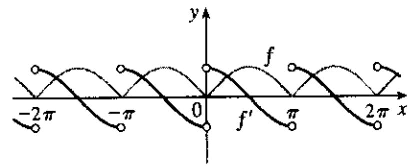
74. "The rate of change of y^5 with respect to x is eighty times the rate of change of y with respect to x" $\Leftrightarrow$
$\dfrac{d}{dx}y^5 = 80\dfrac{dy}{dx} \;\Leftrightarrow\; 5y^4\dfrac{dy}{dx} = 80\dfrac{dy}{dx} \;\Leftrightarrow\; 5y^4 = 80$ (Note that $dy/dx \neq 0$ since the curve never has a horizontal tangent) $\Leftrightarrow\; y^4 = 16 \;\Leftrightarrow\; y = 2$ (since $y > 0$ for all x)

75. Since $\theta^\circ = \left(\frac{\pi}{180}\right)\theta$ rad, we have $\dfrac{d}{d\theta}(\sin\theta^\circ) = \dfrac{d}{d\theta}\left(\sin\frac{\pi}{180}\theta\right) = \frac{\pi}{180}\cos\frac{\pi}{180}\theta = \frac{\pi}{180}\cos\theta^\circ$.

76. (a) $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = x/\sqrt{x^2} = x/|x|$ for $x \neq 0$.
f is not differentiable at $x = 0$.

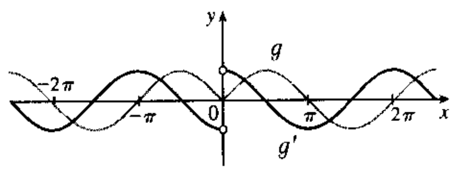
(b) $f(x) = |\sin x| = \sqrt{\sin^2 x} \Rightarrow$

$$f'(x) = \tfrac{1}{2}(\sin^2 x)^{-1/2}\, 2\sin x\cos x = \frac{\sin x}{|\sin x|}\cos x = \begin{cases}\cos x & \text{if } \sin x > 0\\ -\cos x & \text{if } \sin x < 0\end{cases}$$



f is not differentiable when $x = n\pi$, n an integer.

(c) $g(x) = \sin|x| = \sin\sqrt{x^2} \Rightarrow g'(x) = \cos|x| \cdot \dfrac{x}{|x|} = \dfrac{x}{|x|}\cos x = \begin{cases}\cos x & \text{if } x > 0\\ -\cos x & \text{if } x < 0\end{cases}$



g is not differentiable at 0.

77. First note that products and differences of polynomials are polynomials and that the derivative of a polynomial is also a polynomial. When $n = 1$, $f^{(1)}(x) = \left(\dfrac{P(x)}{Q(x)}\right)' = \dfrac{Q(x)P'(x) - P(x)Q'(x)}{[Q(x)]^2} = \dfrac{A_1(x)}{[Q(x)]^{1+1}}$, where $A_1(x) = Q(x)P'(x) - P(x)Q'(x)$. Suppose the result is true for $n = k$, where $k \geq 1$. Then
$f^{(k)}(x) = \dfrac{A_k(x)}{[Q(x)]^{k+1}}$, so

$$\begin{aligned} f^{(k+1)}(x) &= \left(\frac{A_k(x)}{[Q(x)]^{k+1}}\right)' = \frac{[Q(x)]^{k+1}A_k'(x) - A_k(x)\cdot(k+1)[Q(x)]^k\cdot Q'(x)}{\{[Q(x)]^{k+1}\}^2}\\ &= \frac{[Q(x)]^{k+1}A_k'(x) - (k+1)A_k(x)[Q(x)]^kQ'(x)}{[Q(x)]^{2k+2}}\\ &= \frac{[Q(x)]^k\{[Q(x)]^1A_k'(x) - (k+1)A_k(x)Q'(x)\}}{[Q(x)]^k[Q(x)]^{k+2}} = \frac{Q(x)A_k'(x) - (k+1)A_k(x)Q'(x)}{[Q(x)]^{k+2}}\\ &= A_{k+1}(x)/[Q(x)]^{k+2}, \text{ where } A_{k+1}(x) = Q(x)A_k'(x) - (k+1)A_k(x)Q'(x). \end{aligned}$$

We have shown that the formula holds for $n = 1$, and that when it holds for $n = k$ it also holds for $n = k + 1$. Thus, by mathematical induction, the formula holds for all positive integers n.

3.7 Implicit Differentiation

1. (a) $\dfrac{d}{dx}\left(xy+2x+3x^2\right)=\dfrac{d}{dx}(4) \;\Rightarrow\; (x\cdot y'+y\cdot 1)+2+6x=0 \;\Rightarrow\; xy'=-y-2-6x \;\Rightarrow$
$y'=\dfrac{-y-2-6x}{x}$ or $y'=-6-\dfrac{y+2}{x}$.

(b) $xy+2x+3x^2=4 \;\Rightarrow\; xy=4-2x-3x^2 \;\Rightarrow\; y=\dfrac{4-2x-3x^2}{x}=\dfrac{4}{x}-2-3x$, so $y'=-\dfrac{4}{x^2}-3$.

(c) From part (a), $y'=\dfrac{-y-2-6x}{x}=\dfrac{-(4/x-2-3x)-2-6x}{x}=\dfrac{-4/x-3x}{x}=-\dfrac{4}{x^2}-3$.

2. (a) $\dfrac{d}{dx}\left(4x^2+9y^2\right)=\dfrac{d}{dx}(36) \;\Rightarrow\; 8x+18y\cdot y'=0 \;\Rightarrow\; y'=-\dfrac{8x}{18y}=-\dfrac{4x}{9y}$

(b) $4x^2+9y^2=36 \;\Rightarrow\; 9y^2=36-4x^2 \;\Rightarrow\; y^2=\frac{4}{9}\left(9-x^2\right) \;\Rightarrow\; y=\pm\frac{2}{3}\sqrt{9-x^2}$, so
$y'=\pm\frac{2}{3}\cdot\frac{1}{2}\left(9-x^2\right)^{-1/2}(-2x)=\mp\dfrac{2x}{3\sqrt{9-x^2}}$

(c) From part (a), $y'=-\dfrac{4x}{9y}=-\dfrac{4x}{9\left(\pm\frac{2}{3}\sqrt{9-x^2}\right)}=\mp\dfrac{2x}{3\sqrt{9-x^2}}$.

3. (a) $\dfrac{d}{dx}\left(\dfrac{1}{x}+\dfrac{1}{y}\right)=\dfrac{d}{dx}(1) \;\Rightarrow\; -\dfrac{1}{x^2}-\dfrac{1}{y^2}y'=0 \;\Rightarrow\; -\dfrac{1}{y^2}y'=\dfrac{1}{x^2} \;\Rightarrow\; y'=-\dfrac{y^2}{x^2}$

(b) $\dfrac{1}{x}+\dfrac{1}{y}=1 \;\Rightarrow\; \dfrac{1}{y}=1-\dfrac{1}{x}=\dfrac{x-1}{x} \;\Rightarrow\; y=\dfrac{x}{x-1}$, so $y'=\dfrac{(x-1)(1)-(x)(1)}{(x-1)^2}=\dfrac{-1}{(x-1)^2}$.

(c) $y'=-\dfrac{y^2}{x^2}=-\dfrac{[x/(x-1)]^2}{x^2}=-\dfrac{x^2}{x^2(x-1)^2}=-\dfrac{1}{(x-1)^2}$

4. (a) $\dfrac{d}{dx}\left(\sqrt{x}+\sqrt{y}\right)=\dfrac{d}{dx}(4) \;\Rightarrow\; \dfrac{1}{2\sqrt{x}}+\dfrac{1}{2\sqrt{y}}y'=0 \;\Rightarrow\; y'=-\dfrac{\sqrt{y}}{\sqrt{x}}$

(b) $\sqrt{y}=4-\sqrt{x} \;\Rightarrow\; y=\left(4-\sqrt{x}\right)^2=16-8\sqrt{x}+x \;\Rightarrow\; y'=-\dfrac{4}{\sqrt{x}}+1$

(c) $y'=-\dfrac{\sqrt{y}}{\sqrt{x}}=-\dfrac{4-\sqrt{x}}{\sqrt{x}}=-\dfrac{4}{\sqrt{x}}+1$

5. $\dfrac{d}{dx}\left(x^2+y^2\right)=\dfrac{d}{dx}(1) \;\Rightarrow\; 2x+2yy'=0 \;\Rightarrow\; 2yy'=-2x \;\Rightarrow\; y'=-\dfrac{x}{y}$

6. $\dfrac{d}{dx}\left(x^2-y^2\right)=\dfrac{d}{dx}(1) \;\Rightarrow\; 2x-2yy'=0 \;\Rightarrow\; 2x=2yy' \;\Rightarrow\; y'=\dfrac{x}{y}$

7. $\dfrac{d}{dx}\left(x^3+x^2y+4y^2\right)=\dfrac{d}{dx}(6) \;\Rightarrow\; 3x^2+\left(x^2y'+y\cdot 2x\right)+8yy'=0 \;\Rightarrow\; x^2y'+8yy'=-3x^2-2xy$
$\Rightarrow\; \left(x^2+8y\right)y'=-3x^2-2xy \;\Rightarrow\; y'=-\dfrac{3x^2+2xy}{x^2+8y}=-\dfrac{x(3x+2y)}{x^2+8y}$

8. $\dfrac{d}{dx}\left(x^2-2xy+y^3\right)=\dfrac{d}{dx}(c) \;\Rightarrow\; 2x-2(xy'+y\cdot 1)+3y^2y'=0 \;\Rightarrow\; 2x-2y=2xy'-3y^2y' \;\Rightarrow$
$2x-2y=y'\left(2x-3y^2\right) \;\Rightarrow\; y'=\dfrac{2x-2y}{2x-3y^2}$

9. $\frac{d}{dx}\left(x^2y + xy^2\right) = \frac{d}{dx}(3x) \Rightarrow (x^2y' + y\cdot 2x) + (x\cdot 2yy' + y^2\cdot 1) = 3 \Rightarrow$

$x^2y' + 2xyy' = 3 - 2xy - y^2 \Rightarrow y'(x^2 + 2xy) = 3 - 2xy - y^2 \Rightarrow y' = \frac{3 - 2xy - y^2}{x^2 + 2xy}$

10. $\frac{d}{dx}\left(y^5 + x^2y^3\right) = \frac{d}{dx}\left(1 + x^4y\right) \Rightarrow 5y^4y' + x^2\cdot 3y^2y' + y^3\cdot 2x = 0 + x^4y' + y\cdot 4x^3 \Rightarrow$

$y'\left(5y^4 + 3x^2y^2 - x^4\right) = 4x^3y - 2xy^3 \Rightarrow y' = \frac{4x^3y - 2xy^3}{5y^4 + 3x^2y^2 - x^4}$

11. $\frac{d}{dx}\left(x^2y^2 + x\sin y\right) = \frac{d}{dx}(4) \Rightarrow x^2\cdot 2yy' + y^2\cdot 2x + x\cos y\cdot y' + \sin y\cdot 1 = 0 \Rightarrow$

$2x^2yy' + x\cos y\cdot y' = -2xy^2 - \sin y \Rightarrow (2x^2y + x\cos y)y' = -2xy^2 - \sin y \Rightarrow y' = \frac{-2xy^2 - \sin y}{2x^2y + x\cos y}$

12. $\frac{d}{dx}(1 + x) = \frac{d}{dx}\left[\sin(xy^2)\right] \Rightarrow 1 = [\cos(xy^2)](x\cdot 2yy' + y^2\cdot 1) \Rightarrow 1 = 2xy\cos(xy^2)y' + y^2\cos(xy^2)$

$\Rightarrow 1 - y^2\cos(xy^2) = 2xy\cos(xy^2)y' \Rightarrow y' = \frac{1 - y^2\cos(xy^2)}{2xy\cos(xy^2)}$

13. $\frac{d}{dx}(4\cos x\sin y) = \frac{d}{dx}(1) \Rightarrow 4\left[\cos x\cdot\cos y\cdot y' + \sin y\cdot(-\sin x)\right] = 0 \Rightarrow$

$y'(4\cos x\cos y) = 4\sin x\sin y \Rightarrow y' = \frac{4\sin x\sin y}{4\cos x\cos y} = \tan x\tan y$

14. $\frac{d}{dx}\left[y\sin(x^2)\right] = \frac{d}{dx}\left[x\sin(y^2)\right] \Rightarrow y\cos(x^2)\cdot 2x + \sin(x^2)\cdot y' = x\cos(y^2)\cdot 2yy' + \sin(y^2)\cdot 1 \Rightarrow$

$y'\left[\sin(x^2) - 2xy\cos(y^2)\right] = \sin(y^2) - 2xy\cos(x^2) \Rightarrow y' = \frac{\sin(y^2) - 2xy\cos(x^2)}{\sin(x^2) - 2xy\cos(y^2)}$

15. $\frac{d}{dx}\left[\tan(x/y)\right] = \frac{d}{dx}(x + y) \Rightarrow \sec^2(x/y)\cdot\frac{y\cdot 1 - x\cdot y'}{y^2} = 1 + y' \Rightarrow$

$y\sec^2(x/y) - x\sec^2(x/y)\cdot y' = y^2 + y^2y' \Rightarrow y\sec^2(x/y) - y^2 = y^2y' + x\sec^2(x/y) \Rightarrow$

$y\sec^2(x/y) - y^2 = \left[y^2 + x\sec^2(x/y)\right]\cdot y' \Rightarrow y' = \frac{y\sec^2(x/y) - y^2}{y^2 + x\sec^2(x/y)}$

16. $\frac{d}{dx}\left(\sqrt{x + y}\right) = \frac{d}{dx}\left(1 + x^2y^2\right) \Rightarrow \frac{1}{2}(x + y)^{-1/2}(1 + y') = x^2\cdot 2yy' + y^2\cdot 2x \Rightarrow$

$\frac{1}{2\sqrt{x + y}} + \frac{y'}{2\sqrt{x + y}} = 2x^2yy' + 2xy^2 \Rightarrow 1 + y' = 4x^2y\sqrt{x + y}\,y' + 4xy^2\sqrt{x + y} \Rightarrow$

$y' - 4x^2y\sqrt{x + y}\,y' = 4xy^2\sqrt{x + y} - 1 \Rightarrow y'(1 - 4x^2y\sqrt{x + y}) = 4xy^2\sqrt{x + y} - 1 \Rightarrow$

$y' = \frac{4xy^2\sqrt{x + y} - 1}{1 - 4x^2y\sqrt{x + y}}$

17. $\sqrt{xy} = 1 + x^2y \Rightarrow \frac{1}{2}(xy)^{-1/2}(xy' + y\cdot 1) = 0 + x^2y' + y\cdot 2x \Rightarrow \frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} = x^2y' + 2xy$

$\Rightarrow y'\left(\frac{x}{2\sqrt{xy}} - x^2\right) = 2xy - \frac{y}{2\sqrt{xy}} \Rightarrow y'\left(\frac{x - 2x^2\sqrt{xy}}{2\sqrt{xy}}\right) = \frac{4xy\sqrt{xy} - y}{2\sqrt{xy}} \Rightarrow y' = \frac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$

18. $\tan(x-y) = \dfrac{y}{1+x^2} \Rightarrow (1+x^2)\tan(x-y) = y \Rightarrow$

$(1+x^2)\sec^2(x-y)\cdot(1-y') + \tan(x-y)\cdot 2x = y' \Rightarrow$

$(1+x^2)\sec^2(x-y) - (1+x^2)\sec^2(x-y)\cdot y' + 2x\tan(x-y) = y' \Rightarrow$

$(1+x^2)\sec^2(x-y) + 2x\tan(x-y) = \left[1+(1+x^2)\sec^2(x-y)\right]\cdot y' \Rightarrow$

$y' = \dfrac{(1+x^2)\sec^2(x-y) + 2x\tan(x-y)}{1+(1+x^2)\sec^2(x-y)}$

19. $xy = \cot(xy) \Rightarrow y + xy' = -\csc^2(xy)(y+xy') \Rightarrow (y+xy')\left[1+\csc^2(xy)\right] = 0 \Rightarrow$

$y + xy' = 0$ [since $1+\csc^2(xy) > 0$] $\Rightarrow y' = -y/x$

20. $\sin x + \cos y = \sin x\cos y \Rightarrow \cos x - \sin y\cdot y' = \sin x\,(-\sin y\cdot y') + \cos y\cos x \Rightarrow$

$(\sin x\sin y - \sin y)\,y' = \cos x\cos y - \cos x \Rightarrow y' = \dfrac{\cos x\,(\cos y - 1)}{\sin y\,(\sin x - 1)}$

21. $\dfrac{d}{dx}\left\{1 + f(x) + x^2[f(x)]^3\right\} = \dfrac{d}{dx}(0) \Rightarrow f'(x) + x^2\cdot 3[f(x)]^2\cdot f'(x) + [f(x)]^3\cdot 2x = 0$. If $x = 1$, we have

$f'(1) + 1^2\cdot 3[f(1)]^2\cdot f'(1) + [f(1)]^3\cdot 2(1) = 0 \Rightarrow f'(1) + 1\cdot 3\cdot 2^2\cdot f'(1) + 2^3\cdot 2 = 0 \Rightarrow$

$f'(1) + 12f'(1) = -16 \Rightarrow 13f'(1) = -16 \Rightarrow f'(1) = -\frac{16}{13}$.

22. $\dfrac{d}{dx}\left[g(x) + x\sin g(x)\right] = \dfrac{d}{dx}(x^2) \Rightarrow g'(x) + x\cos g(x)\cdot g'(x) + \sin g(x)\cdot 1 = 2x$. If $x = 1$, we have

$g'(1) + 1\cos g(1)\cdot g'(1) + \sin g(1) = 2(1) \Rightarrow g'(1) + \cos 0\cdot g'(1) + \sin 0 = 2 \Rightarrow g'(1) + g'(1) = 2 \Rightarrow$

$2g'(1) = 2 \Rightarrow g'(1) = 1$.

23. $y^4 + x^2y^2 + yx^4 = y + 1 \Rightarrow 4y^3 + \left(x^2\cdot 2y + y^2\cdot 2x\,\dfrac{dx}{dy}\right) + \left(y\cdot 4x^3\,\dfrac{dx}{dy} + x^4\cdot 1\right) = 1 \Rightarrow$

$2xy^2\,\dfrac{dx}{dy} + 4x^3y\,\dfrac{dx}{dy} = 1 - 4y^3 - 2x^2y - x^4 \Rightarrow \dfrac{dx}{dy} = \dfrac{1 - 4y^3 - 2x^2y - x^4}{2xy^2 + 4x^3y}$

24. $(x^2+y^2)^2 = ax^2y \Rightarrow 2(x^2+y^2)\left(2x\,\dfrac{dx}{dy} + 2y\right) = 2ayx\,\dfrac{dx}{dy} + ax^2 \Rightarrow \dfrac{dx}{dy} = \dfrac{ax^2 - 4y(x^2+y^2)}{4x(x^2+y^2) - 2axy}$

25. $x^2 + xy + y^2 = 3 \Rightarrow 2x + xy' + y\cdot 1 + 2yy' = 0 \Rightarrow xy' + 2yy' = -2x - y \Rightarrow$

$y'(x+2y) = -2x - y \Rightarrow y' = \dfrac{-2x-y}{x+2y}$. When $x = 1$ and $y = 1$, we have $y' = \dfrac{-2-1}{1+2\cdot 1} = \dfrac{-3}{3} = -1$, so

an equation of the tangent line is $y - 1 = -1(x-1)$ or $y = -x + 2$.

26. $x^2 + 2xy - y^2 + x = 2 \Rightarrow 2x + 2(xy' + y\cdot 1) - 2yy' + 1 = 0 \Rightarrow 2xy' - 2yy' = -2x - 2y - 1 \Rightarrow$

$y'(2x-2y) = -2x - 2y - 1 \Rightarrow y' = \dfrac{-2x-2y-1}{2x-2y}$. When $x = 1$ and $y = 2$, we have

$y' = \dfrac{-2-4-1}{2-4} = \dfrac{-7}{-2} = \dfrac{7}{2}$, so an equation of the tangent line is $y - 2 = \frac{7}{2}(x-1)$ or $y = \frac{7}{2}x - \frac{3}{2}$.

27. $x^2+y^2=(2x^2+2y^2-x)^2 \Rightarrow 2x+2yy'=2(2x^2+2y^2-x)(4x+4yy'-1)$. When $x=0$ and $y=\frac{1}{2}$, we have $0+y'=2(\frac{1}{2})(2y'-1) \Rightarrow y'=2y'-1 \Rightarrow y'=1$, so an equation of the tangent line is $y-\frac{1}{2}=1(x-0)$ or $y=x+\frac{1}{2}$.

28. $x^{2/3}+y^{2/3}=4 \Rightarrow \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}y'=0 \Rightarrow \dfrac{1}{\sqrt[3]{x}}+\dfrac{y'}{\sqrt[3]{y}}=0 \Rightarrow y'=-\dfrac{\sqrt[3]{y}}{\sqrt[3]{x}}$. When $x=-3\sqrt{3}$ and $y=1$, we have $y'=-\dfrac{1}{(-3\sqrt{3})^{1/3}}=-\dfrac{(-3\sqrt{3})^{2/3}}{-3\sqrt{3}}=\dfrac{3}{3\sqrt{3}}=\dfrac{1}{\sqrt{3}}$, so an equation of the tangent line is $y-1=\frac{1}{\sqrt{3}}(x+3\sqrt{3})$ or $y=\frac{1}{\sqrt{3}}x+4$.

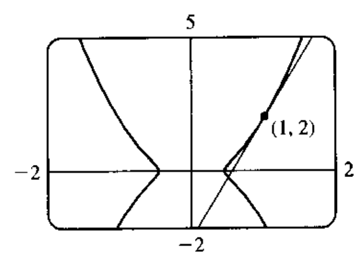
29. $2(x^2+y^2)^2=25(x^2-y^2) \Rightarrow 4(x^2+y^2)(2x+2yy')=25(2x-2yy') \Rightarrow$ $4(x+yy')(x^2+y^2)=25(x-yy') \Rightarrow 4yy'(x^2+y^2)+25yy'=25x-4x(x^2+y^2) \Rightarrow$ $y'=\dfrac{25x-4x(x^2+y^2)}{25y+4y(x^2+y^2)}$. When $x=3$ and $y=1$, we have $y'=\frac{75-120}{25+40}=-\frac{45}{65}=-\frac{9}{13}$, so an equation of the tangent line is $y-1=-\frac{9}{13}(x-3)$ or $y=-\frac{9}{13}x+\frac{40}{13}$.

30. $y^2(y^2-4)=x^2(x^2-5) \Rightarrow y^4-4y^2=x^4-5x^2 \Rightarrow 4y^3y'-8yy'=4x^3-10x$. When $x=0$ and $y=-2$, we have $-32y'+16y'=0 \Rightarrow -16y'=0 \Rightarrow y'=0$, so an equation of the tangent line is $y+2=0(x-0)$ or $y=-2$.

31. (a) $y^2=5x^4-x^2 \Rightarrow 2yy'=5(4x^3)-2x \Rightarrow y'=\dfrac{10x^3-x}{y}$.

So at the point $(1,2)$ we have $y'=\dfrac{10(1)^3-1}{2}=\dfrac{9}{2}$, and an equation of the tangent line is $y-2=\frac{9}{2}(x-1)$ or $y=\frac{9}{2}x-\frac{5}{2}$.

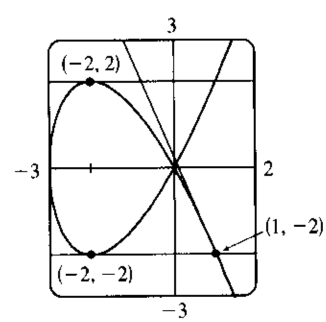
(b)



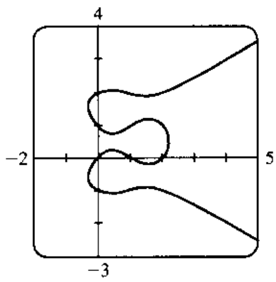
32. (a) $y^2=x^3+3x^2 \Rightarrow 2yy'=3x^2+3(2x) \Rightarrow y'=\dfrac{3x^2+6x}{2y}$. So at the point $(1,-2)$ we have $y'=\dfrac{3(1)^2+6(1)}{2(-2)}=-\dfrac{9}{4}$, and an equation of the tangent line is $y+2=-\frac{9}{4}(x-1)$ or $y=-\frac{9}{4}x+\frac{1}{4}$.

(b) The curve has a horizontal tangent where $y'=0 \Leftrightarrow 3x^2+6x=0 \Leftrightarrow 3x(x+2)=0 \Leftrightarrow x=0$ or $x=-2$. But note that at $x=0$, $y=0$ also, so the derivative does not exist. At $x=-2$, $y^2=(-2)^3+3(-2)^2=-8+12=4$, so $y=\pm 2$. So the two points at which the curve has a horizontal tangent are $(-2,-2)$ and $(-2,2)$.

(c)



33. (a)



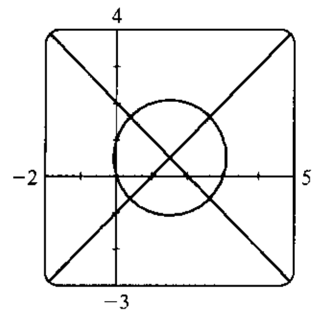
There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

(b) $y' = \dfrac{3x^2 - 6x + 2}{2(2y^3 - 3y^2 - y + 1)} \Rightarrow y' = -1$ at $(0, 1)$ and $y' = \frac{1}{3}$ at $(0, 2)$.

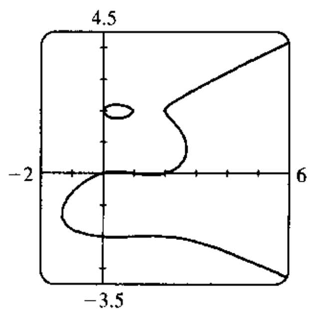
Equations of the tangent lines are $y = -x + 1$ and $y = \frac{1}{3}x + 2$.

(c) $y' = 0 \Rightarrow 3x^2 - 6x + 2 = 0 \Rightarrow x = 1 \pm \frac{1}{3}\sqrt{3}$

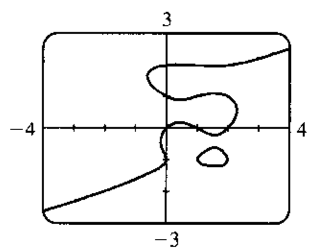
(d) By multiplying the right side of the equation by $x - 3$, we obtain the first graph. By modifying the equation in other ways, we can generate the other graphs.



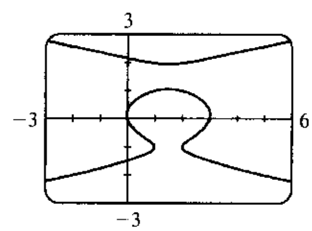
$y(y^2 - 1)(y - 2) = x(x - 1)(x - 2)(x - 3)$



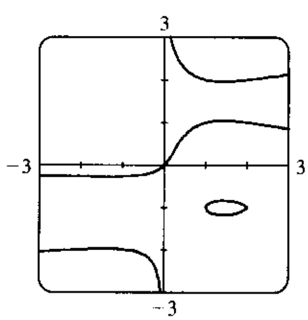
$y(y^2 - 4)(y - 2) = x(x - 1)(x - 2)$



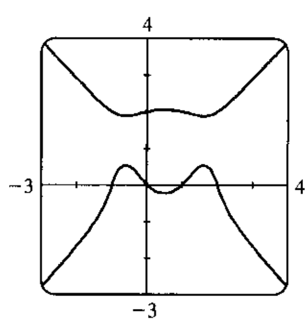
$y(y + 1)(y^2 - 1)(y - 2) = x(x - 1)(x - 2)$



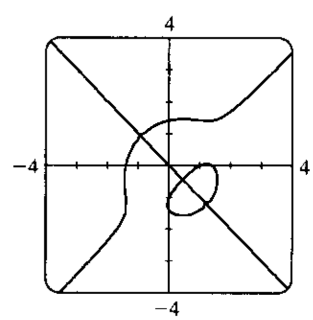
$(y + 1)(y^2 - 1)(y - 2) = (x - 1)(x - 2)$



$x(y + 1)(y^2 - 1)(y - 2) = y(x - 1)(x - 2)$

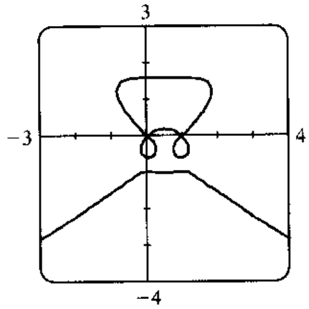


$y(y^2 + 1)(y - 2) = x(x^2 - 1)(x - 2)$



$y(y + 1)(y^2 - 2) = x(x - 1)(x^2 - 2)$

34. (a)



(b) There are 9 points with horizontal tangents: 3 at $x = 0$, 3 at $x = \frac{1}{2}$, and 3 at $x = 1$. The three horizontal tangents along the top of the wagon are hard to find, but by limiting the y-range of the graph (to $[1.6, 1.7]$, for example) they are distinguishable.

35. From Exercise 29, a tangent to the lemniscate will be horizontal if $y' = 0 \;\Rightarrow\; 25x - 4x(x^2 + y^2) = 0 \;\Rightarrow\; x[25 - 4(x^2 + y^2)] = 0 \;\Rightarrow\; x^2 + y^2 = \frac{25}{4}$ **(1)**. (Note that when x is 0, y is also 0, and there is no horizontal tangent at the origin.) Substituting $\frac{25}{4}$ for $x^2 + y^2$ in the equation of the lemniscate, $2(x^2 + y^2)^2 = 25(x^2 - y^2)$, we get $x^2 - y^2 = \frac{25}{8}$ **(2)**. Solving **(1)** and **(2)**, we have $x^2 = \frac{75}{16}$ and $y^2 = \frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4}, \pm\frac{5}{4}\right)$.

36. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \;\Rightarrow\; y' = -\dfrac{b^2 x}{a^2 y} \;\Rightarrow\;$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = \dfrac{-b^2 x_0}{a^2 y_0}(x - x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0 y}{b^2} - \dfrac{y_0^2}{b^2} = -\dfrac{x_0 x}{a^2} + \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the ellipse, we have $\dfrac{x_0 x}{a^2} + \dfrac{y_0 y}{b^2} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1$.

37. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \;\Rightarrow\; y' = \dfrac{b^2 x}{a^2 y} \;\Rightarrow\;$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = \dfrac{b^2 x_0}{a^2 y_0}(x - x_0)$. Multiplying both sides by $\dfrac{y_0}{b^2}$ gives $\dfrac{y_0 y}{b^2} - \dfrac{y_0^2}{b^2} = \dfrac{x_0 x}{a^2} - \dfrac{x_0^2}{a^2}$. Since (x_0, y_0) lies on the hyperbola, we have $\dfrac{x_0 x}{a^2} - \dfrac{y_0 y}{b^2} = \dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$.

38. $\sqrt{x} + \sqrt{y} = \sqrt{c} \;\Rightarrow\; \dfrac{1}{2\sqrt{x}} + \dfrac{y'}{2\sqrt{y}} = 0 \;\Rightarrow\; y' = -\dfrac{\sqrt{y}}{\sqrt{x}} \;\Rightarrow\;$ an equation of the tangent line at (x_0, y_0) is $y - y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0)$. Now $x = 0 \;\Rightarrow\; y = y_0 - \dfrac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0) = y_0 + \sqrt{x_0}\sqrt{y_0}$, so the y-intercept is $y_0 + \sqrt{x_0}\sqrt{y_0}$. And $y = 0 \;\Rightarrow\; -y_0 = -\dfrac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \;\Rightarrow\; x - x_0 = \dfrac{y_0\sqrt{x_0}}{\sqrt{y_0}} \;\Rightarrow\;$ $x = x_0 + \sqrt{x_0}\sqrt{y_0}$, so the x-intercept is $x_0 + \sqrt{x_0}\sqrt{y_0}$. The sum of the intercepts is
$\left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = \left(\sqrt{c}\right)^2 = c$.

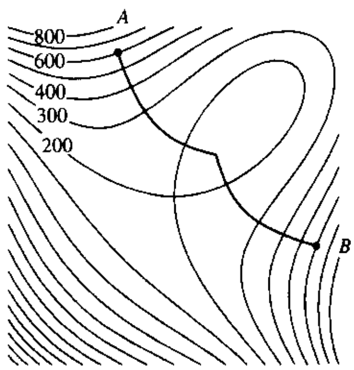
39. If the circle has radius r, its equation is $x^2 + y^2 = r^2 \;\Rightarrow\; 2x + 2yy' = 0 \;\Rightarrow\; y' = -\dfrac{x}{y}$, so the slope of the tangent line at $P(x_0, y_0)$ is $-\dfrac{x_0}{y_0}$. The negative reciprocal of that slope is $\dfrac{-1}{-x_0/y_0} = \dfrac{y_0}{x_0}$, which is the slope of OP, so the tangent line at P is perpendicular to the radius OP.

40. $y^q = x^p \;\Rightarrow\; qy^{q-1}y' = px^{p-1} \;\Rightarrow\; y' = \dfrac{px^{p-1}}{qy^{q-1}} = \dfrac{px^{p-1}y}{qy^q} = \dfrac{px^{p-1}x^{p/q}}{qx^p} = \dfrac{p}{q}x^{(p/q)-1}$

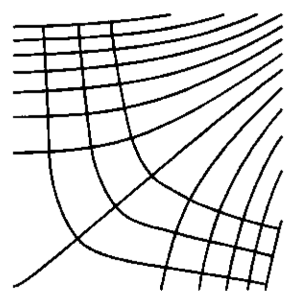
41. $2x^2 + y^2 = 3$ and $x = y^2$ intersect when $2x^2 + x - 3 = 0 \Leftrightarrow (2x+3)(x-1) = 0 \Leftrightarrow x = -\frac{3}{2}$ or 1, but $-\frac{3}{2}$ is extraneous since $x = y^2$ is nonnegative. When $x = 1$, $1 = y^2 \Rightarrow y = \pm 1$, so there are two points of intersection: $(1, \pm 1)$. $2x^2 + y^2 = 3 \Rightarrow 4x + 2yy' = 0 \Rightarrow y' = -2x/y$, and $x = y^2 \Rightarrow 1 = 2yy' \Rightarrow y' = 1/(2y)$. At $(1,1)$, the slopes are $m_1 = -2(1)/1 = -2$ and $m_2 = 1/(2 \cdot 1) = \frac{1}{2}$, so the curves are orthogonal (since m_1 and m_2 are negative reciprocals of each other). By symmetry, the curves are also orthogonal at $(1, -1)$.

42. $x^2 - y^2 = 5$ and $4x^2 + 9y^2 = 72$ intersect when $4x^2 + 9(x^2 - 5) = 72 \Leftrightarrow 13x^2 = 117 \Leftrightarrow x = \pm 3$, so there are four points of intersection: $(\pm 3, \pm 2)$. $x^2 - y^2 = 5 \Rightarrow 2x - 2yy' = 0 \Rightarrow y' = x/y$, and $4x^2 + 9y^2 = 72 \Rightarrow 8x + 18yy' = 0 \Leftrightarrow y' = -4x/9y$. At $(3,2)$, the slopes are $m_1 = \frac{3}{2}$ and $m_2 = -\frac{2}{3}$, so the curves are orthogonal. By symmetry, the curves are also orthogonal at $(3,-2)$, $(-3,2)$ and $(-3,-2)$.

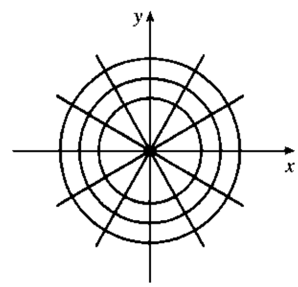
43.



44. The orthogonal family represents the direction of the wind.



45. $x^2 + y^2 = r^2$ is a circle with center O and $ax + by = 0$ is a line through O. $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -x/y$, so the slope of the tangent line at $P_0\,(x_0, y_0)$ is $-x_0/y_0$. The slope of the line OP_0 is y_0/x_0, which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.



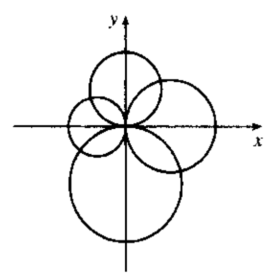
46. The circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ intersect at the origin where the tangents are vertical and horizontal. If (x_0, y_0) is the other point of intersection, then $x_0^2 + y_0^2 = ax_0$ **(1)** and $x_0^2 + y_0^2 = by_0$ **(2)**. Now $x^2 + y^2 = ax \Rightarrow$

$2x + 2yy' = a \Rightarrow y' = \dfrac{a - 2x}{2y}$ and $x^2 + y^2 = by \Rightarrow$

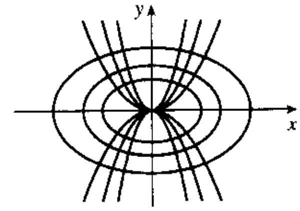
$2x + 2yy' = by' \Rightarrow y' = \dfrac{2x}{b - 2y}$. Thus, the curves are orthogonal at

$(x_0, y_0) \Leftrightarrow \dfrac{a - 2x_0}{2y_0} = -\dfrac{b - 2y_0}{2x_0} \Leftrightarrow 2ax_0 - 4x_0^2 = 4y_0^2 - 2by_0 \Leftrightarrow$

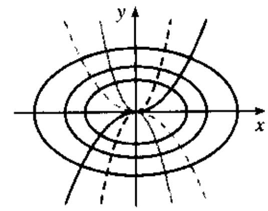
$ax_0 + by_0 = 2(x_0^2 + y_0^2)$, which is true by **(1)** and **(2)**.



47. $y = cx^2 \Rightarrow y' = 2cx$ and $x^2 + 2y^2 = k \Rightarrow 2x + 4yy' = 0 \Rightarrow$ $2yy' = -x \Rightarrow y' = -\dfrac{x}{2(y)} = -\dfrac{x}{2(cx^2)} = -\dfrac{1}{2cx}$, so the curves are orthogonal.



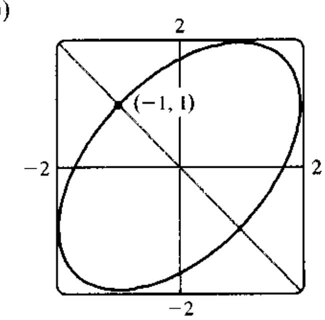
48. $y = ax^3 \Rightarrow y' = 3ax^2$ and $x^2 + 3y^2 = b \Rightarrow 2x + 6yy' = 0 \Rightarrow$ $3yy' = -x \Rightarrow y' = -\dfrac{x}{3(y)} = -\dfrac{x}{3(ax^3)} = -\dfrac{1}{3ax^2}$, so the curves are orthogonal.



49. To find the points at which the ellipse $x^2 - xy + y^2 = 3$ crosses the x-axis, let $y = 0$ and solve for x. $y = 0 \Rightarrow x^2 - x(0) + 0^2 = 3 \Leftrightarrow x = \pm\sqrt{3}$. So the graph of the ellipse crosses the x-axis at the points $(\pm\sqrt{3}, 0)$. Using implicit differentiation to find y', we get $2x - xy' - y + 2yy' = 0 \Rightarrow y'(2y - x) = y - 2x$ $\Leftrightarrow y' = \dfrac{y - 2x}{2y - x}$. So y' at $(\sqrt{3}, 0)$ is $\dfrac{0 - 2\sqrt{3}}{2(0) - \sqrt{3}} = 2$ and y' at $(-\sqrt{3}, 0)$ is $\dfrac{0 + 2\sqrt{3}}{2(0) + \sqrt{3}} = 2$. Thus, the tangent lines at these points are parallel.

50. (a) We use implicit differentiation to find $y' = \dfrac{y - 2x}{2y - x}$ as in Exercise 49. The slope of the tangent line at $(-1, 1)$ is $m = \dfrac{1 - 2(-1)}{2(1) - (-1)} = \dfrac{3}{3} = 1$, so the slope of the normal line is $-\dfrac{1}{m} = -1$, and its equation is $y - 1 = -1(x + 1) \Leftrightarrow y = -x$. Substituting $-x$ for y in the equation of the ellipse, we get $x^2 - x(-x) + (-x)^2 = 3 \Rightarrow$ $3x^2 = 3 \Leftrightarrow x = \pm 1$. So the normal line must intersect the ellipse again at $x = 1$, and since the equation of the line is $y = -x$, the other point of intersection must be $(1, -1)$.

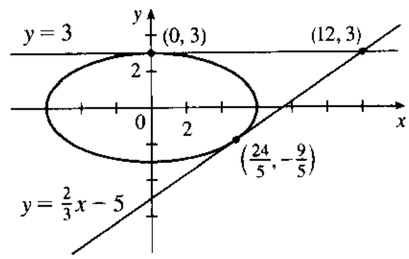
(b)



51. $x^2y^2 + xy = 2 \Rightarrow x^2 \cdot 2yy' + y^2 \cdot 2x + x \cdot y' + y \cdot 1 = 0 \Leftrightarrow y'(2x^2y + x) = -2xy^2 - y \Leftrightarrow$ $y' = -\dfrac{2xy^2 + y}{2x^2y + x}$. So $-\dfrac{2xy^2 + y}{2x^2y + x} = -1 \Leftrightarrow 2xy^2 + y = 2x^2y + x \Leftrightarrow y(2xy + 1) = x(2xy + 1) \Leftrightarrow$ $y(2xy + 1) - x(2xy + 1) = 0 \Leftrightarrow (2xy + 1)(y - x) = 0 \Leftrightarrow xy = -\frac{1}{2}$ or $y = x$. But $xy = -\frac{1}{2} \Rightarrow$ $x^2y^2 + xy = \frac{1}{4} - \frac{1}{2} \neq 2$, so we must have $x = y$. Then $x^2y^2 + xy = 2 \Rightarrow x^4 + x^2 = 2 \Leftrightarrow$ $x^4 + x^2 - 2 = 0 \Leftrightarrow (x^2 + 2)(x^2 - 1) = 0$. So $x^2 = -2$, which is impossible, or $x^2 = 1 \Leftrightarrow x = \pm 1$. Since $x = y$, the points on the curve where the tangent line has a slope of -1 are $(-1, -1)$ and $(1, 1)$.

52. $x^2 + 4y^2 = 36 \Rightarrow 2x + 8yy' = 0 \Rightarrow y' = -\dfrac{x}{4y}$. Let (a, b) be a point on $x^2 + 4y^2 = 36$ whose tangent line passes through $(12, 3)$. The tangent line is then $y - 3 = -\dfrac{a}{4b}(x - 12)$, so $b - 3 = -\dfrac{a}{4b}(a - 12)$. Multiplying both sides by $4b$ gives $4b^2 - 12b = -a^2 + 12a$, so $4b^2 + a^2 = 12(a + b)$. But $4b^2 + a^2 = 36$, so $36 = 12(a + b)$ $\Rightarrow a + b = 3 \Rightarrow b = 3 - a$. Substituting $3 - a$ for b into $a^2 + 4b^2 = 36$ gives $a^2 + 4(3 - a)^2 = 36$ $\Leftrightarrow a^2 + 36 - 24a + 4a^2 = 36 \Leftrightarrow 5a^2 - 24a = 0 \Leftrightarrow a(5a - 24) = 0$, so $a = 0$ or $a = \frac{24}{5}$. If $a = 0$, $b = 3 - 0 = 3$, and if $a = \frac{24}{5}$, $b = 3 - \frac{24}{5} = -\frac{9}{5}$. So the two points on the ellipse are $(0, 3)$ and $\left(\frac{24}{5}, -\frac{9}{5}\right)$.

Using $y - 3 = -\dfrac{a}{4b}(x - 12)$ with $(a, b) = (0, 3)$ gives us the tangent line $y - 3 = 0$ or $y = 3$. With $(a, b) = \left(\frac{24}{5}, -\frac{9}{5}\right)$, we have $y - 3 = -\frac{24/5}{4(-9/5)}(x - 12) \Leftrightarrow y - 3 = \frac{2}{3}(x - 12) \Leftrightarrow$ $y = \frac{2}{3}x - 5$. A graph of the ellipse and the tangent lines confirms our results.



53. $x^2 + 4y^2 = 5 \Rightarrow 2x + 4(2yy') = 0 \Rightarrow y' = -\dfrac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0)/[3 - (-5)] = \frac{1}{8}h$. But the slope of the tangent line through the point (a, b) can be expressed as $y' = -\dfrac{a}{4b}$, or as $\dfrac{b - 0}{a - (-5)} = \dfrac{b}{a + 5}$ [since the line passes through $(-5, 0)$ and (a, b)], so $-\dfrac{a}{4b} = \dfrac{b}{a + 5} \Leftrightarrow$ $4b^2 = -a^2 - 5a \Leftrightarrow a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$ [since (a, b) is on the ellipse], so $5 = -5a \Leftrightarrow$ $a = -1$. Then $4b^2 = -a^2 - 5a = -1 - 5(-1) = 4 \Rightarrow b = 1$, since the point is on the top half of the ellipse. So $\dfrac{h}{8} = \dfrac{b}{a + 5} = \dfrac{1}{-1 + 5} = \dfrac{1}{4} \Rightarrow h = 2$. So the lamp is located 2 units above the x-axis.

3.8 Higher Derivatives

1. $a = f$, $b = f'$, $c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f', since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

2. Where d has horizontal tangents, only c is 0, so $d' = c$. c has negative tangents for $x < 0$ and b is the only graph that is negative for $x < 0$, so $c' = b$. b has positive tangents on $\mathbb{R}$ (except at $x = 0$), and the only graph that is positive on the same domain is a, so $b' = a$. We conclude that $d = f$, $c = f'$, $b = f''$, and $a = f'''$.

3. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.

4. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b' = a$. b is 0 where c has a maximum, so $c' = b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

5. $f(x) = x^5 + 6x^2 - 7x \quad\Rightarrow\quad f'(x) = 5x^4 + 12x - 7 \quad\Rightarrow\quad f''(x) = 20x^3 + 12$

6. $f(t) = t^8 - 7t^6 + 2t^4 \quad\Rightarrow\quad f'(t) = 8t^7 - 42t^5 + 8t^3 \quad\Rightarrow\quad f''(t) = 56t^6 - 210t^4 + 24t^2$

7. $y = \cos 2\theta \quad\Rightarrow\quad y' = -2\sin 2\theta \quad\Rightarrow\quad y'' = -4\cos 2\theta$

8. $y = \theta\sin\theta \quad\Rightarrow\quad y' = \theta\cos\theta + \sin\theta \quad\Rightarrow\quad y'' = \theta(-\sin\theta) + \cos\theta\cdot 1 + \cos\theta = 2\cos\theta - \theta\sin\theta$

9. $F(t) = (1-7t)^6 \quad\Rightarrow\quad F'(t) = 6(1-7t)^5(-7) = -42(1-7t)^5 \quad\Rightarrow$
$F''(t) = -42\cdot 5(1-7t)^4(-7) = 1470(1-7t)^4$

10. $g(x) = \dfrac{2x+1}{x-1} \quad\Rightarrow\quad g'(x) = \dfrac{(x-1)(2) - (2x+1)(1)}{(x-1)^2} = \dfrac{2x-2-2x-1}{(x-1)^2} = \dfrac{-3}{(x-1)^2}$ or $-3(x-1)^{-2}$

$\Rightarrow\quad g''(x) = -3(-2)(x-1)^{-3} = 6(x-1)^{-3}$ or $\dfrac{6}{(x-1)^3}$

11. $h(u) = \dfrac{1-4u}{1+3u} \quad\Rightarrow\quad h'(u) = \dfrac{(1+3u)(-4) - (1-4u)(3)}{(1+3u)^2} = \dfrac{-4-12u-3+12u}{(1+3u)^2} = \dfrac{-7}{(1+3u)^2}$ or

$-7(1+3u)^{-2} \quad\Rightarrow\quad h''(u) = -7(-2)(1+3u)^{-3}(3) = 42(1+3u)^{-3}$ or $\dfrac{42}{(1+3u)^3}$

12. $H(s) = a\sqrt{s} + \dfrac{b}{\sqrt{s}} = as^{1/2} + bs^{-1/2} \quad\Rightarrow$

$H'(s) = a\cdot\frac{1}{2}s^{-1/2} + b\left(-\frac{1}{2}s^{-3/2}\right) = \frac{1}{2}as^{-1/2} - \frac{1}{2}bs^{-3/2} \quad\Rightarrow$

$H''(s) = \frac{1}{2}a\left(-\frac{1}{2}s^{-3/2}\right) - \frac{1}{2}b\left(-\frac{3}{2}s^{-5/2}\right) = -\frac{1}{4}as^{-3/2} + \frac{3}{4}bs^{-5/2}$

13. $h(x) = \sqrt{x^2+1} \quad\Rightarrow\quad h'(x) = \dfrac{1}{2}(x^2+1)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2+1}} \quad\Rightarrow$

$$h''(x) = \frac{\sqrt{x^2+1}\cdot 1 - x\left[\frac{1}{2}(x^2+1)^{-1/2}(2x)\right]}{\left(\sqrt{x^2+1}\right)^2} = \frac{(x^2+1)^{-1/2}\left[(x^2+1) - x^2\right]}{(x^2+1)^1} = \frac{1}{(x^2+1)^{3/2}}$$

14. $y = x^n \quad\Rightarrow\quad y' = nx^{n-1} \quad\Rightarrow\quad y'' = n(n-1)x^{n-2}$

15. $y = (x^3+1)^{2/3} \quad\Rightarrow\quad y' = \frac{2}{3}(x^3+1)^{-1/3}(3x^2) = 2x^2(x^3+1)^{-1/3} \quad\Rightarrow$
$y'' = 2x^2\left(-\frac{1}{3}\right)(x^3+1)^{-4/3}(3x^2) + (x^3+1)^{-1/3}(4x) = 4x(x^3+1)^{-1/3} - 2x^4(x^3+1)^{-4/3}$

16. $y = \dfrac{4x}{\sqrt{x+1}} \quad\Rightarrow$

$$y' = \frac{\sqrt{x+1}\cdot 4 - 4x\cdot\frac{1}{2}(x+1)^{-1/2}}{\left(\sqrt{x+1}\right)^2} = \frac{4\sqrt{x+1} - 2x/\sqrt{x+1}}{x+1} = \frac{4(x+1)-2x}{(x+1)^{3/2}} = \frac{2x+4}{(x+1)^{3/2}} \quad\Rightarrow$$

$$y'' = \frac{(x+1)^{3/2}\cdot 2 - (2x+4)\cdot\frac{3}{2}(x+1)^{1/2}}{\left[(x+1)^{3/2}\right]^2} = \frac{(x+1)^{1/2}[2(x+1) - 3(x+2)]}{(x+1)^3}$$

$$= \frac{2x+2-3x-6}{(x+1)^{5/2}} = \frac{-x-4}{(x+1)^{5/2}}$$

17. $H(t) = \tan 3t \quad\Rightarrow\quad H'(t) = 3\sec^2 3t \quad\Rightarrow$

$H''(t) = 2\cdot 3\sec 3t\,\dfrac{d}{dt}(\sec 3t) = 6\sec 3t\,(3\sec 3t\tan 3t) = 18\sec^2 3t\tan 3t$

18. $g(s) = s^2\cos s \quad\Rightarrow\quad g'(s) = 2s\cos s - s^2\sin s \quad\Rightarrow$
$g''(s) = 2\cos s - 2s\sin s - 2s\sin s - s^2\cos s = \left(2 - s^2\right)\cos s - 4s\sin s$

19. $g(\theta) = \theta\csc\theta \;\Rightarrow\; g'(\theta) = -\theta\csc\theta\cot\theta + \csc\theta \;\Rightarrow$

$$g''(\theta) = (-1)\csc\theta\cot\theta + (-\theta)(-\csc\theta\cot\theta)\cot\theta + (-\theta\csc\theta)(-\csc^2\theta) - \csc\theta\cot\theta$$
$$= -\csc\theta\cot\theta + \theta\csc\theta\cot^2\theta + \theta\csc^3\theta - \csc\theta\cot\theta$$
$$= \csc\theta(\theta\csc^2\theta + \theta\cot^2\theta - 2\cot\theta)$$

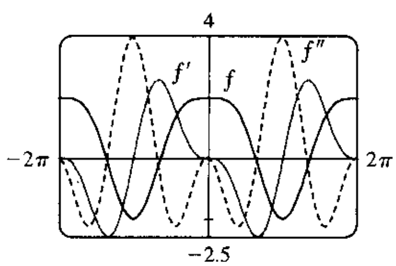
20. $h(x) = \dfrac{x+3}{x^2+2x} \;\Rightarrow$

$$h'(x) = \frac{(x^2+2x)(1) - (x+3)(2x+2)}{(x^2+2x)^2} = \frac{(x^2+2x) - (2x^2+8x+6)}{(x^2+2x)^2} = -\frac{x^2+6x+6}{(x^2+2x)^2} \;\Rightarrow$$

$$h''(x) = -\frac{(x^2+2x)^2(2x+6) - (x^2+6x+6)2(x^2+2x)(2x+2)}{[(x^2+2x)^2]^2}$$
$$= -\frac{2(x^2+2x)\left[(x^2+2x)(x+3) - (x^2+6x+6)(2x+2)\right]}{(x^2+2x)^4}$$
$$= -\frac{2\left[(x^3+5x^2+6x) - (2x^3+14x^2+24x+12)\right]}{(x^2+2x)^3} = \frac{2(x^3+9x^2+18x+12)}{(x^2+2x)^3}$$

21. (a) $f(x) = 2\cos x + \sin^2 x \;\Rightarrow\; f'(x) = 2(-\sin x) + 2\sin x\,(\cos x) = \sin 2x - 2\sin x \;\Rightarrow$
$f''(x) = 2\cos 2x - 2\cos x = 2(\cos 2x - \cos x)$

(b)

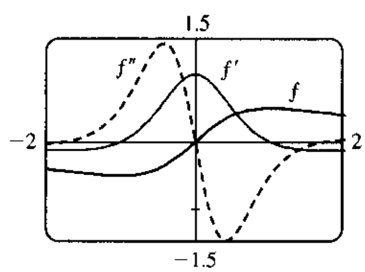


We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

22. (a) $f(x) = \dfrac{x}{x^2+1} \;\Rightarrow\; f'(x) = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \;\Rightarrow$

$$f''(x) = \frac{(x^2+1)^2(-2x) - (1-x^2)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2x(2x^2-2-x^2-1)}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

(b)



We can see that our answers are plausible, since f has horizontal tangents where $f'(x) = 0$, and f' has horizontal tangents where $f''(x) = 0$.

23. $y = \sqrt{2x+3} = (2x+3)^{1/2} \;\Rightarrow\; y' = \frac{1}{2}(2x+3)^{-1/2}\cdot 2 = (2x+3)^{-1/2} \;\Rightarrow$
$y'' = -\frac{1}{2}(2x+3)^{-3/2}\cdot 2 = -(2x+3)^{-3/2} \;\Rightarrow\; y''' = \frac{3}{2}(2x+3)^{-5/2}\cdot 2 = 3(2x+3)^{-5/2}$

24. $y = \dfrac{x}{2x-1} \;\Rightarrow\; y' = \dfrac{(2x-1)(1) - x(2)}{(2x-1)^2} = \dfrac{-1}{(2x-1)^2}$ or $-1(2x-1)^{-2} \;\Rightarrow$
$y'' = -1(-2)(2x-1)^{-3}(2) = 4(2x-1)^{-3} \;\Rightarrow$
$y''' = 4(-3)(2x-1)^{-4}(2) = -24(2x-1)^{-4}$ or $-24/(2x-1)^4$

25. $f(t) = t\cos t \;\Rightarrow\; f'(t) = t(-\sin t) + \cos t \cdot 1 \;\Rightarrow\; f''(t) = t(-\cos t) - \sin t \cdot 1 - \sin t \;\Rightarrow$
$f'''(t) = t\sin t - \cos t \cdot 1 - \cos t - \cos t = t\sin t - 3\cos t$, so $f'''(0) = 0 - 3 = -3$.

26. $g(x) = \sqrt{5 - 2x} \;\Rightarrow\; g'(x) = \frac{1}{2}(5-2x)^{-1/2}(-2) = -(5-2x)^{-1/2} \;\Rightarrow$
$g''(x) = \frac{1}{2}(5-2x)^{-3/2}(-2) = -(5-2x)^{-3/2} \;\Rightarrow\; g'''(x) = \frac{3}{2}(5-2x)^{-5/2}(-2) = -3(5-2x)^{-5/2}$, so
$g'''(2) = -3(1)^{-5/2} = -3$.

27. $f(\theta) = \cot\theta \;\Rightarrow\; f'(\theta) = -\csc^2\theta \;\Rightarrow\; f''(\theta) = -2\csc\theta\,(-\csc\theta\cot\theta) = 2\csc^2\theta\cot\theta \;\Rightarrow$
$f'''(\theta) = 2\left(-2\csc^2\theta\cot\theta\right)\cot\theta + 2\csc^2\theta\left(-\csc^2\theta\right) = -2\csc^2\theta\left(2\cot^2\theta + \csc^2\theta\right) \;\Rightarrow$
$f'''\left(\frac{\pi}{6}\right) = -2(2)^2\left[2\left(\sqrt{3}\right)^2 + (2)^2\right] = -80$

28. $g(x) = \sec x \;\Rightarrow\; g'(x) = \sec x\tan x \;\Rightarrow$
$$g''(x) = \sec x\sec^2 x + \tan x\,(\sec x\tan x) = \sec^3 x + \sec x\tan^2 x = \sec^3 x + \sec x\left(\sec^2 x - 1\right) = 2\sec^3 x - \sec x \;\Rightarrow$$
$g'''(x) = 6\sec^2 x\,(\sec x\tan x) - \sec x\tan x = \sec x\tan x\left(6\sec^2 x - 1\right) \;\Rightarrow$
$g'''\left(\frac{\pi}{4}\right) = \sqrt{2}\,(1)(6\cdot 2 - 1) = 11\sqrt{2}$

29. $9x^2 + y^2 = 9 \;\Rightarrow\; 18x + 2yy' = 0 \;\Rightarrow\; 2yy' = -18x \;\Rightarrow\; y' = -9x/y \;\Rightarrow$
$y'' = -9\left(\dfrac{y\cdot 1 - x\cdot y'}{y^2}\right) = -9\left(\dfrac{y - x(-9x/y)}{y^2}\right) = -9\cdot\dfrac{y^2 + 9x^2}{y^3} = -9\cdot\dfrac{9}{y^3}$ [since x and y must satisfy the original equation, $9x^2 + y^2 = 9$]. Thus, $y'' = -81/y^3$.

30. $\sqrt{x} + \sqrt{y} = 1 \;\Rightarrow\; \dfrac{1}{2\sqrt{x}} + \dfrac{y'}{2\sqrt{y}} = 0 \;\Rightarrow\; y' = -\sqrt{y}/\sqrt{x} \;\Rightarrow$
$$y'' = -\frac{\sqrt{x}\,[1/(2\sqrt{y})]\,y' - \sqrt{y}\,[1/(2\sqrt{x})]}{x} = -\frac{\sqrt{x}\,(1/\sqrt{y})\left(-\sqrt{y}/\sqrt{x}\right) - \sqrt{y}\,(1/\sqrt{x})}{2x} = \frac{1 + \sqrt{y}/\sqrt{x}}{2x}$$
$= \dfrac{\sqrt{x} + \sqrt{y}}{2x\sqrt{x}} = \dfrac{1}{2x\sqrt{x}}$ since x and y must satisfy the original equation, $\sqrt{x} + \sqrt{y} = 1$.

31. $x^3 + y^3 = 1 \;\Rightarrow\; 3x^2 + 3y^2y' = 0 \;\Rightarrow\; y' = -\dfrac{x^2}{y^2} \;\Rightarrow$
$$y'' = -\frac{y^2(2x) - x^2\cdot 2yy'}{(y^2)^2} = -\frac{2xy^2 - 2x^2y(-x^2/y^2)}{y^4} = -\frac{2xy^4 + 2x^4y}{y^6} = -\frac{2xy(y^3 + x^3)}{y^6} = -\frac{2x}{y^5},$$
since x and y must satisfy the original equation, $x^3 + y^3 = 1$.

32. $x^4 + y^4 = a^4 \;\Rightarrow\; 4x^3 + 4y^3y' = 0 \;\Rightarrow\; 4y^3y' = -4x^3 \;\Rightarrow\; y' = -x^3/y^3 \;\Rightarrow$
$$y'' = -\left(\frac{y^3\cdot 3x^2 - x^3\cdot 3y^2y'}{(y^3)^2}\right) = -3x^2y^2\cdot\frac{y - x(-x^3/y^3)}{y^6} = -3x^2\cdot\frac{y^4 + x^4}{y^4y^3} = -3x^2\cdot\frac{a^4}{y^7} = \frac{-3a^4x^2}{y^7}$$

33. $f(x) = x^n \;\Rightarrow\; f'(x) = nx^{n-1} \;\Rightarrow\; f''(x) = n(n-1)x^{n-2} \;\Rightarrow\; \cdots \;\Rightarrow$
$f^{(n)}(x) = n(n-1)(n-2)\cdots 2\cdot 1x^{n-n} = n!$

34. $f(x) = \dfrac{1}{5x - 1} = (5x-1)^{-1} \;\Rightarrow\; f'(x) = -1(5x-1)^{-2}\cdot 5 \;\Rightarrow\; f''(x) = (-1)(-2)(5x-1)^{-3}\cdot 5^2 \;\Rightarrow$
$f'''(x) = (-1)(-2)(-3)(5x-1)^{-4}\cdot 5^3 \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = (-1)^n n!\,5^n(5x-1)^{-(n+1)}$

35. $f(x) = (1+x)^{-1} \;\Rightarrow\; f'(x) = -1(1+x)^{-2}$, $f''(x) = 1\cdot 2(1+x)^{-3}$, $f^{(3)}(x) = -1\cdot 2\cdot 3(1+x)^{-4}$,
$f^{(4)}(x) = 1\cdot 2\cdot 3\cdot 4(1+x)^{-5}, \ldots, f^{(n)}(x) = (-1)^n n!(1+x)^{-(n+1)}$

36. $f(x) = \sqrt{x} = x^{1/2} \Rightarrow f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} \Rightarrow f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-5/2} \Rightarrow$

$f^{(4)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)x^{-7/2} = -\frac{1 \cdot 3 \cdot 5}{2^4}x^{-7/2} \Rightarrow$

$f^{(5)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)x^{-9/2} = \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5}x^{-9/2} \Rightarrow \cdots \Rightarrow$

$f^{(n)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2} - n + 1\right)x^{-(2n-1)/2} = (-1)^{n-1}\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n}x^{-(2n-1)/2}$

37. $f(x) = 1/(3x^3) = \frac{1}{3}x^{-3} \Rightarrow f'(x) = \frac{1}{3}(-3)x^{-4} \Rightarrow f''(x) = \frac{1}{3}(-3)(-4)x^{-5} \Rightarrow$

$f'''(x) = \frac{1}{3}(-3)(-4)(-5)x^{-6} \Rightarrow \cdots \Rightarrow$

$f^{(n)}(x) = \frac{1}{3}(-3)(-4)\cdots[-(n+2)]\,x^{-(n+3)} = \frac{(-1)^n \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot (n+2)}{3x^{n+3}} \cdot \frac{2}{2} = \frac{(-1)^n\,(n+2)!}{6x^{n+3}}$

38. $D \sin x = \cos x \Rightarrow D^2 \sin x = -\sin x \Rightarrow D^3 \sin x = -\cos x \Rightarrow D^4 \sin x = \sin x$. The derivatives of $\sin x$ occur in a cycle of four. Since $74 = 4(18) + 2$, we have $D^{74} \sin x = D^2 \sin x = -\sin x$.

39. Let $f(x) = \cos x$. Then $Df(2x) = 2f'(2x)$, $D^2 f(2x) = 2^2 f''(2x)$, $D^3 f(2x) = 2^3 f'''(2x), \ldots,$ $D^{(n)} f(2x) = 2^n f^{(n)}(2x)$. Since the derivatives of $\cos x$ occur in a cycle of four, and since $103 = 4(25) + 3$, we have $f^{(103)}(x) = f^{(3)}(x) = \sin x$ and $D^{103} \cos 2x = 2^{103} f^{(103)}(2x) = 2^{103} \sin 2x$.

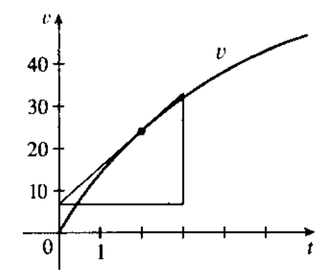
40. Let $f(x) = x \sin x$ and $h(x) = \sin x$, so $f(x) = xh(x)$. Then

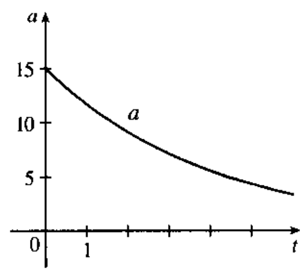
$f'(x) = h(x) + xh'(x)$, $f''(x) = h'(x) + h'(x) + xh''(x) = 2h'(x) + xh''(x)$,

$f'''(x) = 2h''(x) + h''(x) + xh'''(x) = 3h''(x) + xh'''(x), \ldots, f^{(n)}(x) = nh^{(n-1)}(x) + xh^{(n)}(x)$. Since $34 = 4(8) + 2$, we have $h^{(34)}(x) = h^{(2)}(x) = D^2 \sin x = -\sin x$ and $h^{(35)}(x) = -\cos x$. Thus,

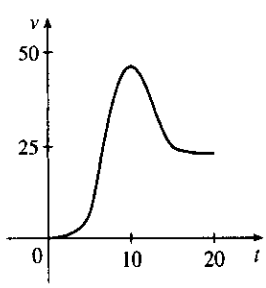
$D^{(35)} x \sin x = 35h^{(34)}(x) + xh^{(35)}(x) = -35 \sin x - x \cos x$.

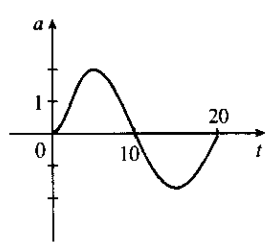
41. By measuring the slope of the graph of $s = f(t)$ at $t = 0, 1, 2, 3, 4$, and 5, and using the method of Example 1 in Section 3.2, we plot the graph of the velocity function $v = f'(t)$ in the first figure. The acceleration when $t = 2$ s is $a = f''(2)$, the slope of the tangent line to the graph of f' when $t = 2$. We estimate the slope of this tangent line to be $a(2) = f''(2) = v'(2) \approx \frac{27}{3} = 9$ ft/s^2. Similar measurements enable us to graph the acceleration function in the second figure.





42. (a) Since we estimate the velocity to be a maximum at $t = 10$, the acceleration is 0 at $t = 10$.





(b) Drawing a tangent line at $t = 10$ on the graph of a, a appears to decrease by 10 ft/s^2 over a period of 20 s. So at $t = 10$ s, the jerk is approximately $-10/20 = -0.5\ \ (\text{ft/s}^2)/\text{s}$ or ft/s^3.

43. (a) $s = 2t^3 - 15t^2 + 36t + 2 \quad\Rightarrow\quad v(t) = s'(t) = 6t^2 - 30t + 36 \quad\Rightarrow\quad a(t) = v'(t) = 12t - 30$

(b) $a(1) = 12 \cdot 1 - 30 = -18\text{ m/s}^2$

(c) $v(t) = 6(t^2 - 5t + 6) = 6(t-2)(t-3) = 0$ when $t = 2$ or 3 and $a(2) = 24 - 30 = -6\text{ m/s}^2$, $a(3) = 36 - 30 = 6\text{ m/s}^2$.

44. (a) $s = 2t^3 - 3t^2 - 12t \quad\Rightarrow\quad v(t) = s'(t) = 6t^2 - 6t - 12 \quad\Rightarrow\quad a(t) = v'(t) = 12t - 6$

(b) $a(1) = 12 \cdot 1 - 6 = 6\text{ m/s}^2$

(c) $v(t) = 6(t^2 - t - 2) = 6(t+1)(t-2) = 0$ when $t = -1$ or 2. Since $t \geq 0$, $t \neq -1$ and $a(2) = 24 - 6 = 18\text{ m/s}^2$.

45. (a) $s = \sin\left(\frac{\pi}{6}t\right) + \cos\left(\frac{\pi}{6}t\right), 0 \leq t \leq 2.$ $v(t) = s'(t) = \cos\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} - \sin\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} = \frac{\pi}{6}\left[\cos\left(\frac{\pi}{6}t\right) - \sin\left(\frac{\pi}{6}t\right)\right]$

$\Rightarrow\quad a(t) = v'(t) = \frac{\pi}{6}\left[-\sin\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6} - \cos\left(\frac{\pi}{6}t\right) \cdot \frac{\pi}{6}\right] = -\dfrac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6}t\right) + \cos\left(\frac{\pi}{6}t\right)\right]$

(b) $a(1) = -\dfrac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6} \cdot 1\right) + \cos\left(\frac{\pi}{6} \cdot 1\right)\right] = -\dfrac{\pi^2}{36}\left[\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\right] = -\dfrac{\pi^2}{72}\left(1 + \sqrt{3}\right) \approx -0.3745\text{ m/s}^2$

(c) $v(t) = 0$ for $0 \leq t \leq 2 \quad\Rightarrow\quad \cos\left(\frac{\pi}{6}t\right) = \sin\left(\frac{\pi}{6}t\right) \quad\Rightarrow\quad 1 = \dfrac{\sin\left(\frac{\pi}{6}t\right)}{\cos\left(\frac{\pi}{6}t\right)} \quad\Rightarrow$

$\tan\left(\frac{\pi}{6}t\right) = 1 \quad\Rightarrow\quad \frac{\pi}{6}t = \tan^{-1}1 \quad\Rightarrow\quad t = \frac{6}{\pi} \cdot \frac{\pi}{4} = \frac{3}{2} = 1.5$ s. Thus,

$a\left(\frac{3}{2}\right) = -\dfrac{\pi^2}{36}\left[\sin\left(\frac{\pi}{6} \cdot \frac{3}{2}\right) + \cos\left(\frac{\pi}{6} \cdot \frac{3}{2}\right)\right] = -\dfrac{\pi^2}{36}\left[\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\right] = -\dfrac{\pi^2}{36}\sqrt{2} \approx -0.3877\text{ m/s}^2.$

46. (a) $s = 2t^3 - 7t^2 + 4t + 1 \quad\Rightarrow\quad v(t) = s'(t) = 6t^2 - 14t + 4 \quad\Rightarrow\quad a(t) = v'(t) = 12t - 14$

(b) $a(1) = 12 - 14 = -2\text{ m/s}^2$

(c) $v(t) = 2(3t^2 - 7t + 2) = 2(3t-1)(t-2) = 0$ when $t = \frac{1}{3}$ or 2 and $a\left(\frac{1}{3}\right) = 12\left(\frac{1}{3}\right) - 14 = -10\text{ m/s}^2$, $a(2) = 12(2) - 14 = 10\text{ m/s}^2$.

47. (a) $s(t) = t^4 - 4t^3 + 2 \quad\Rightarrow\quad v(t) = s'(t) = 4t^3 - 12t^2 \quad\Rightarrow\quad a(t) = v'(t) = 12t^2 - 24t = 12t(t-2) = 0$ when $t = 0$ or 2.

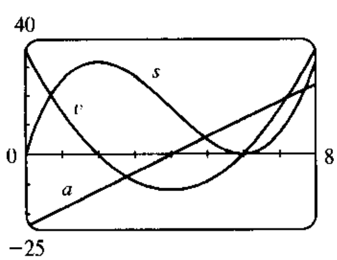
(b) $s(0) = 2$ m, $v(0) = 0$ m/s, $s(2) = -14$ m, $v(2) = -16$ m/s

48. (a) $s(t) = 2t^3 - 9t^2 \quad\Rightarrow\quad v(t) = s'(t) = 6t^2 - 18t \quad\Rightarrow\quad a(t) = v'(t) = 12t - 18 = 0$ when $t = 1.5$.

(b) $s(1.5) = -13.5$ m, $v(1.5) = -13.5$ m/s

49. (a) $s = f(t) = t^3 - 12t^2 + 36t, t \geq 0 \quad\Rightarrow\quad v(t) = f'(t) = 3t^2 - 24t + 36.$ $a(t) = v'(t) = 6t - 24.$ $a(3) = 6(3) - 24 = -6\ (\text{m/s})/\text{s}$ or m/s^2.

(b)

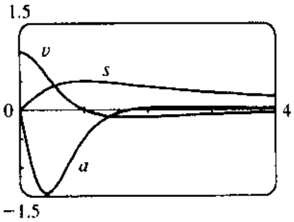


(c) The particle is speeding up when v and a have the same sign. This occurs when $2 < t < 4$ and when $t > 6$. It is slowing down when v and a have opposite signs; that is, when $0 \leq t < 2$ and when $4 < t < 6$.

50. (a) $x(t) = \dfrac{t}{1+t^2} \;\Rightarrow\; v(t) = x'(t) = \dfrac{(1+t^2)(1) - t(2t)}{(1+t^2)^2} = \dfrac{1-t^2}{(1+t^2)^2}$.

$a(t) = v'(t) = \dfrac{2t(t^2-3)}{(1+t^2)^3}$. $a(t) = 0 \;\Rightarrow\; 2t(t^2-3) = 0 \;\Rightarrow\; t = 0$ or $\sqrt{3}$

(b)



(c) v and a have the same sign and the particle is speeding up when $1 < t < \sqrt{3}$. The particle is slowing down and v and a have opposite signs when $0 < t < 1$ and when $t > \sqrt{3}$.

51. (a) $y(t) = A\sin\omega t \;\Rightarrow\; v(t) = y'(t) = A\omega\cos\omega t \;\Rightarrow\; a(t) = v'(t) = -A\omega^2\sin\omega t$

(b) $a(t) = -A\omega^2\sin\omega t = -\omega^2(A\sin\omega t) = -\omega^2 y(t)$, so $a(t)$ is proportional to $y(t)$.

(c) speed $= |v(t)| = A\omega\,|\cos\omega t|$ is a maximum when $\cos\omega t = \pm 1$. But when $\cos\omega t = \pm 1$, we have $\sin\omega t = 0$, and $a(t) = -A\omega^2\sin\omega t = -A\omega^2(0) = 0$.

52. By the Chain Rule, $a(t) = \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = \dfrac{dv}{ds}v(t) = v(t)\dfrac{dv}{ds}$. The derivative dv/dt is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas the derivative dv/ds is the rate of change of the velocity with respect to the displacement.

53. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \;\Rightarrow\; 2a = 2 \;\Rightarrow\; a = 1$.
$P'(2) = 3 \;\Rightarrow\; 2(1)(2) + b = 3 \;\Rightarrow\; 4 + b = 3 \;\Rightarrow\; b = -1$.
$P(2) = 5 \;\Rightarrow\; 1(2)^2 + (-1)(2) + c = 5 \;\Rightarrow\; 2 + c = 5 \;\Rightarrow\; c = 3$. So $P(x) = x^2 - x + 3$.

54. Let $Q(x) = ax^3 + bx^2 + cx + d$. Then $Q'(x) = 3ax^2 + 2bx + c$, $Q''(x) = 6ax + 2b$ and $Q'''(x) = 6a$. Thus, $Q(1) = a + b + c + d = 1$, $Q'(1) = 3a + 2b + c = 3$, $Q''(1) = 6a + 2b = 6$ and $Q'''(1) = 6a = 12$. Solving these four equations in four unknowns a, b, c and d we get $a = 2$, $b = -3$, $c = 3$ and $d = -1$, so $Q(x) = 2x^3 - 3x^2 + 3x - 1$.

55. $y = A\sin x + B\cos x \;\Rightarrow\; y' = A\cos x - B\sin x \;\Rightarrow\; y'' = -A\sin x - B\cos x$. Substituting into $y'' + y' - 2y = \sin x$ gives us $(-3A - B)\sin x + (A - 3B)\cos x = 1\sin x$, so we must have $-3A - B = 1$ and $A - 3B = 0$. Solving for A and B, we add the first equation to three times the second to get $B = -\frac{1}{10}$ and $A = -\frac{3}{10}$.

56. $y = Ax^2 + Bx + C \;\Rightarrow\; y' = 2Ax + B \;\Rightarrow\; y'' = 2A$. We substitute these expressions into the equation $y'' + y' - 2y = x^2$ to get

$$(2A) + (2Ax + B) - 2(Ax^2 + Bx + C) = x^2$$
$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = x^2$$
$$(-2A)x^2 + (2A - 2B)x + (2A + B - 2C) = (1)x^2 + (0)x + (0)$$

The coefficients of x^2 on each side must be equal, so $-2A = 1 \;\Rightarrow\; A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \;\Rightarrow\; A = B = -\frac{1}{2}$ and $2A + B - 2C = 0 \;\Rightarrow\; -1 - \frac{1}{2} - 2C = 0 \;\Rightarrow\; C = -\frac{3}{4}$.

57. $f(x) = xg(x^2) \;\Rightarrow\; f'(x) = x\cdot g'(x^2)\cdot 2x + g(x^2)\cdot 1 = g(x^2) + 2x^2 g'(x^2) \;\Rightarrow$
$f''(x) = g'(x^2)\cdot 2x + 2x^2\cdot g''(x^2)\cdot 2x + g'(x^2)\cdot 4x = 6xg'(x^2) + 4x^3 g''(x^2)$

58. $f(x) = \dfrac{g(x)}{x} \quad \Rightarrow \quad f'(x) = \dfrac{xg'(x) - g(x)}{x^2} \quad \Rightarrow$

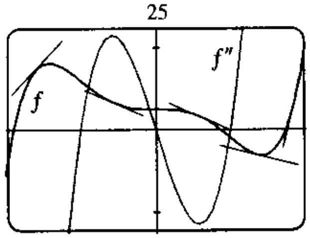
$$f''(x) = \frac{x^2[g'(x) + xg''(x) - g'(x)] - 2x\,[xg'(x) - g(x)]}{x^4} = \frac{x^2 g''(x) - 2xg'(x) + 2g(x)}{x^3}$$

59. $f(x) = g(\sqrt{x}\,) \quad \Rightarrow \quad f'(x) = g'(\sqrt{x}\,) \cdot \frac{1}{2}x^{-1/2} = \dfrac{g'(\sqrt{x}\,)}{2\sqrt{x}} \quad \Rightarrow$

$$f''(x) = \frac{2\sqrt{x} \cdot g''(\sqrt{x}\,) \cdot \frac{1}{2}x^{-1/2} - g'(\sqrt{x}\,) \cdot 2 \cdot \frac{1}{2}x^{-1/2}}{(2\sqrt{x}\,)^2} = \frac{x^{-1/2}\,[\sqrt{x}\,g''(\sqrt{x}\,) - g'(\sqrt{x}\,)]}{4x}$$

$$= \frac{\sqrt{x}\,g''(\sqrt{x}\,) - g'(\sqrt{x}\,)}{4x\sqrt{x}}$$

60.



$f(x) = 3x^5 - 10x^3 + 5 \quad \Rightarrow \quad f'(x) = 15x^4 - 30x^2 \quad \Rightarrow$

$f''(x) = 60x^3 - 60x = 60x(x^2 - 1) = 60x(x + 1)(x - 1)$

So $f''(x) > 0$ when $-1 < x < 0$ or $x > 1$, and on these intervals the graph of f lies above its tangent lines; and $f''(x) < 0$ when $x < -1$ or $0 < x < 1$, and on these intervals the graph of f lies below its tangent lines.

61. (a) $f(x) = \dfrac{1}{x^2 + x} \quad \Rightarrow \quad f'(x) = \dfrac{-(2x+1)}{(x^2+x)^2} \quad \Rightarrow$

$$f''(x) = \frac{(x^2+x)^2(-2) + (2x+1)(2)(x^2+x)(2x+1)}{(x^2+x)^4} = \frac{2(3x^2+3x+1)}{(x^2+x)^3} \quad \Rightarrow$$

$$f'''(x) = \frac{(x^2+x)^3(2)(6x+3) - 2(3x^2+3x+1)(3)(x^2+x)^2(2x+1)}{(x^2+x)^6}$$

$$= \frac{-6(4x^3+6x^2+4x+1)}{(x^2+x)^4} \quad \Rightarrow$$

$$f^{(4)}(x) = \frac{(x^2+x)^4(-6)(12x^2+12x+4) + 6(4x^3+6x^2+4x+1)(4)(x^2+x)^3(2x+1)}{(x^2+x)^8}$$

$$= \frac{24(5x^4+10x^3+10x^2+5x+1)}{(x^2+x)^5}$$

$f^{(5)}(x) = ?$

(b) $f(x) = \dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1} \quad \Rightarrow \quad f'(x) = -x^{-2} + (x+1)^{-2} \quad \Rightarrow \quad f''(x) = 2x^{-3} - 2(x+1)^{-3} \quad \Rightarrow$

$f'''(x) = (-3)(2)x^{-4} + (3)(2)(x+1)^{-4} \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad f^{(n)}(x) = (-1)^n n!\left[x^{-(n+1)} - (x+1)^{-(n+1)}\right]$

62. (a) For $f(x) = \dfrac{7x+17}{2x^2-7x-4}$, a CAS gives us $f'''(x) = \dfrac{-6(56x^4+544x^3-2184x^2+6184x-6139)}{(2x^2-7x-4)^4}$

(b) Using a CAS we get $f(x) = \dfrac{7x+17}{2x^2-7x-4} = \dfrac{-3}{2x+1} + \dfrac{5}{x-4}$. Now we differentiate three times to obtain

$$f'''(x) = \frac{144}{(2x+1)^4} - \frac{30}{(x-4)^4}.$$

63. We will show that for each positive integer n, the nth derivative $f^{(n)}$ exists and equals one of $f, f', f'', f''', \ldots, f^{(p-1)}$. Since $f^{(p)} = f$, the first p derivatives of f are $f', f'', f''', \ldots, f^{(p-1)}$, and f. In particular, our statement is true for $n = 1$. Suppose that k is an integer, $k \geq 1$, for which f is k-times differentiable with $f^{(k)}$ in the set

$S = \{f, f', f'', \ldots, f^{(p-1)}\}$. Since f is p-times differentiable, every member of S [including $f^{(k)}$] is differentiable, so $f^{(k+1)}$ exists and equals the derivative of some member of S. Thus, $f^{(k+1)}$ is in the set $\{f', f'', f''', \ldots, f^{(p)}\}$, which equals S since $f^{(p)} = f$. We have shown that the statement is true for $n = 1$ and that its truth for $n = k$ implies its truth for $n = k + 1$. By mathematical induction, the statement is true for all positive integers n.

64. (a) Use the Product Rule repeatedly: $F = fg \;\Rightarrow\; F' = f'g + fg' \;\Rightarrow$

$$F'' = (f''g + f'g') + (f'g' + fg'') = f''g + 2f'g' + fg''.$$

(b) $F''' = f'''g + f''g' + 2(f''g' + f'g'') + f'g'' + fg''' = f'''g + 3f''g' + 3f'g'' + fg''' \;\Rightarrow$

$$F^{(4)} = f^{(4)}g + f'''g' + 3(f'''g' + f''g'') + 3(f''g'' + f'g''') + f'g''' + fg^{(4)}$$
$$= f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)}$$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g'' + \cdots + \binom{n}{k}f^{(n-k)}g^{(k)} + \cdots + nf'g^{(n-1)} + fg^{(n)}$$

where $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

65. The Chain Rule says that $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{du}\frac{du}{dx}\right) = \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{du}{dx} + \frac{dy}{du}\frac{d}{dx}\left(\frac{du}{dx}\right) \quad \text{[Product Rule]}$$
$$= \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\frac{du}{dx} + \frac{dy}{du}\frac{d^2u}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

66. From Exercise 65, $\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{du^2}\left(\dfrac{du}{dx}\right)^2 + \dfrac{dy}{du}\dfrac{d^2u}{dx^2} \;\Rightarrow$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2\right] + \frac{d}{dx}\left[\frac{dy}{du}\frac{d^2u}{dx^2}\right]$$
$$= \left[\frac{d}{dx}\left(\frac{d^2y}{du^2}\right)\right]\left(\frac{du}{dx}\right)^2 + \left[\frac{d}{dx}\left(\frac{du}{dx}\right)^2\right]\frac{d^2y}{du^2} + \left[\frac{d}{dx}\left(\frac{dy}{du}\right)\right]\frac{d^2u}{dx^2} + \left[\frac{d}{dx}\left(\frac{d^2u}{dx^2}\right)\right]\frac{dy}{du}$$
$$= \left[\frac{d}{du}\left(\frac{d^2y}{du^2}\right)\frac{du}{dx}\right]\left(\frac{du}{dx}\right)^2 + 2\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \left[\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}\right]\left(\frac{d^2u}{dx^2}\right) + \frac{d^3u}{dx^3}\frac{dy}{du}$$
$$= \frac{d^3y}{du^3}\left(\frac{du}{dx}\right)^3 + 3\frac{du}{dx}\frac{d^2u}{dx^2}\frac{d^2y}{du^2} + \frac{dy}{du}\frac{d^3u}{dx^3}$$

APPLIED PROJECT Where Should a Pilot Start Descent?

1. Condition (i) will hold if and only if all of the following four conditions hold:

(α) $P(0) = 0$

(β) $P'(0) = 0$ (for a smooth landing)

(γ) $P'(\ell) = 0$ (since the plane is cruising horizontally when it begins its descent)

(δ) $P(\ell) = h$.

First of all, condition α implies that $P(0) = d = 0$, so $P(x) = ax^3 + bx^2 + cx \Rightarrow P'(x) = 3ax^2 + 2bx + c$. But $P'(0) = c = 0$ by condition β. So $P'(\ell) = 3a\ell^2 + 2b\ell = \ell(3a\ell + 2b)$. Now by condition γ, $3a\ell + 2b = 0 \Rightarrow a = -\frac{2b}{3\ell}$. Therefore, $P(x) = -\frac{2b}{3\ell}x^3 + bx^2$. Setting $P(\ell) = h$ for condition δ, we get $P(\ell) = -\frac{2b}{3\ell}\ell^3 + b\ell^2 = h \Rightarrow -\frac{2}{3}b\ell^2 + b\ell^2 = h \Rightarrow \frac{1}{3}b\ell^2 = h \Rightarrow b = \frac{3h}{\ell^2} \Rightarrow a = -\frac{2h}{\ell^3}$. So $y = P(x) = -\frac{2h}{\ell^3}x^3 + \frac{3h}{\ell^2}x^2$.

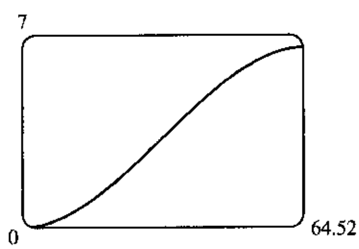
2. By condition (ii), $\frac{dx}{dt} = -v$ for all t, so $x(t) = \ell - vt$. Condition (iii) states that $\left|\frac{d^2y}{dt^2}\right| \leq k$. By the Chain Rule, we have $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = -\frac{2h}{\ell^3}(3x^2)\frac{dx}{dt} + \frac{3h}{\ell^2}(2x)\frac{dx}{dt} = \frac{6hx^2v}{\ell^3} - \frac{6hxv}{\ell^2}$ (for $x \leq \ell$) $\Rightarrow$

$\frac{d^2y}{dt^2} = \frac{6hv}{\ell^3}(2x)\frac{dx}{dt} - \frac{6hv}{\ell^2}\frac{dx}{dt} = -\frac{12hv^2}{\ell^3}x + \frac{6hv^2}{\ell^2}$. In particular, when $t = 0$, $x = \ell$ and so $\left.\frac{d^2y}{dt^2}\right|_{t=0} = -\frac{12hv^2}{\ell^3}\ell + \frac{6hv^2}{\ell^2} = -\frac{6hv^2}{\ell^2}$. Thus, $\left|\frac{d^2y}{dt^2}\right|_{t=0} = \frac{6hv^2}{\ell^2} \leq k$. (This condition also follows from taking $x = 0$.)

3. We substitute $k = 860\text{ mi/h}^2$, $h = 35{,}000\text{ ft} \times \frac{1\text{ mi}}{5280\text{ ft}}$, and $v = 300$ mi/h into the result of part (b):

$$\frac{6\left(35{,}000 \cdot \frac{1}{5280}\right)(300)^2}{\ell^2} \leq 860 \quad\Rightarrow\quad \ell \geq 300\sqrt{6 \cdot \frac{35{,}000}{5280 \cdot 860}} \approx 64.5 \text{ miles.}$$

4. Substituting the values of h and ℓ in Problem 3 into $P(x) = -\frac{2h}{\ell^3}x^3 + \frac{3h}{\ell^2}x^2$ gives us $P(x) = ax^3 + bx^2$, where $a \approx 4.937 \times 10^{-5}$ and $b \approx 4.78 \times 10^{-3}$.



APPLIED PROJECT Building a Better Roller Coaster

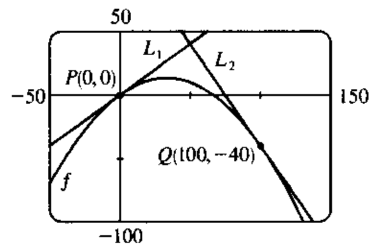
1. (a) $f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b$.

The origin is at P	:	$f(0) = 0$	$\Rightarrow$	$c = 0$
The slope of the ascent is 0.8	:	$f'(0) = 0.8$	$\Rightarrow$	$b = 0.8$
The slope of the drop is -1.6	:	$f'(100) = -1.6$	$\Rightarrow$	$200a + b = -1.6$

(b) $b = 0.8$, so $200a + b = -1.6 \Rightarrow 200a + 0.8 = -1.6 \Rightarrow 200a = -2.4 \Rightarrow a = -\frac{2.4}{200} = -0.012$.

Thus, $f(x) = -0.012x^2 + 0.8x$.

(c) Since L_1 passes through the origin with slope 0.8, it has equation $y = 0.8x$. The horizontal distance between P and Q is 100, so the y-coordinate at Q is $f(100) = -0.012(100)^2 + 0.8(100) = -40$. Since L_2 passes through the point $(100, -40)$ and has slope -1.6, it has equation $y + 40 = -1.6(x - 100)$ or $y = -1.6x + 120$.



(d) The difference in elevation between $P(0, 0)$ and $Q(100, -40)$ is $0 - (-40) = 40$ feet.

2. (a)

Interval	Function	First Derivative	Second Derivative
$(-\infty, 0)$	$L_1(x) = 0.8x$	$L_1'(x) = 0.8$	$L_1''(x) = 0$
$[0, 10)$	$g(x) = kx^3 + lx^2 + mx + n$	$g'(x) = 3kx^2 + 2lx + m$	$g''(x) = 6kx + 2l$
$[10, 90]$	$q(x) = ax^2 + bx + c$	$q'(x) = 2ax + b$	$q''(x) = 2a$
$(90, 100]$	$h(x) = px^3 + qx^2 + rx + s$	$h'(x) = 3px^2 + 2qx + r$	$h''(x) = 6px + 2q$
$(100, \infty)$	$L_2(x) = -1.6x + 120$	$L_2'(x) = -1.6$	$L_2''(x) = 0$

There are 4 values of x (0, 10, 90, and 100) for which we must make sure the function values are equal, the first derivative values are equal, and the second derivative values are equal. The third column in the following table contains the value of each side of the condition — these are found after solving the system in part (b).

At $x =$	Condition	Value	Resulting Equation
0	$g(0) = L_1(0)$	0	$n = 0$
	$g'(0) = L_1'(0)$	$\frac{4}{5}$	$m = 0.8$
	$g''(0) = L_1''(0)$	0	$2l = 0$
10	$g(10) = q(10)$	$\frac{68}{9}$	$1000k + 100l + 10m + n = 100a + 10b + c$
	$g'(10) = q'(10)$	$\frac{2}{3}$	$300k + 20l + m = 20a + b$
	$g''(10) = q''(10)$	$-\frac{2}{75}$	$60k + 2l = 2a$
90	$h(90) = q(90)$	$-\frac{220}{9}$	$729{,}000p + 8100q + 90r + s = 8100a + 90b + c$
	$h'(90) = q'(90)$	$-\frac{22}{15}$	$24{,}300p + 180q + r = 180a + b$
	$h''(90) = q''(90)$	$-\frac{2}{75}$	$540p + 2q = 2a$
100	$h(100) = L_2(100)$	-40	$1{,}000{,}000p + 10{,}000q + 100r + s = -40$
	$h'(100) = L_2'(100)$	$-\frac{8}{5}$	$30{,}000p + 200q + r = -1.6$
	$h''(100) = L_2''(100)$	0	$600p + 2q = 0$

(b) We can arrange our work in a 12×12 matrix as follows.

a	b	c	k	l	m	n	p	q	r	s	constant
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0.8
0	0	0	0	2	0	0	0	0	0	0	0
−100	−10	−1	1000	100	10	1	0	0	0	0	0
−20	−1	0	300	20	1	0	0	0	0	0	0
−2	0	0	60	2	0	0	0	0	0	0	0
−8100	−90	−1	0	0	0	0	729,000	8100	90	1	0
−180	−1	0	0	0	0	0	24,300	180	1	0	0
−2	0	0	0	0	0	0	540	2	0	0	0
0	0	0	0	0	0	0	1,000,000	10,000	100	1	−40
0	0	0	0	0	0	0	30,000	200	1	0	−1.6
0	0	0	0	0	0	0	600	2	0	0	0

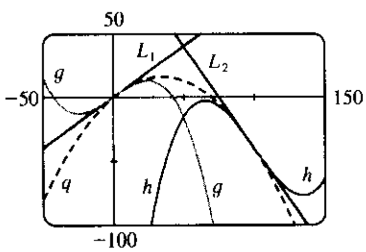
Solving the system gives us the formulas for q, g, and h.

$$\left.\begin{aligned} a &= -0.01\overline{3} = -\tfrac{1}{75} \\ b &= 0.9\overline{3} = \tfrac{14}{15} \\ c &= -0.\overline{4} = -\tfrac{4}{9} \end{aligned}\right\} q(x) = -\tfrac{1}{75}x^2 + \tfrac{14}{15}x - \tfrac{4}{9}$$

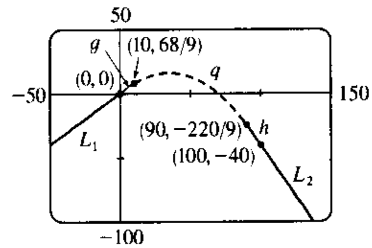
$$\left.\begin{aligned} k &= -0.000\overline{4} = -\tfrac{1}{2250} \\ l &= 0 \\ m &= 0.8 = \tfrac{4}{5} \\ n &= 0 \end{aligned}\right\} g(x) = -\tfrac{1}{2250}x^3 + \tfrac{4}{5}x$$

$$\left.\begin{aligned} p &= 0.000\overline{4} = \tfrac{1}{2250} \\ q &= -0.1\overline{3} = -\tfrac{2}{15} \\ r &= 11.7\overline{3} = \tfrac{176}{15} \\ s &= -324.\overline{4} = -\tfrac{2920}{9} \end{aligned}\right\} h(x) = \tfrac{1}{2250}x^3 - \tfrac{2}{15}x^2 + \tfrac{176}{15}x - \tfrac{2920}{9}$$

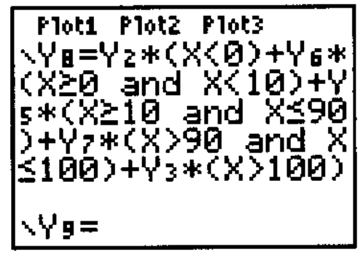
(c) Graph of L_1, q, g, h, and L_2:



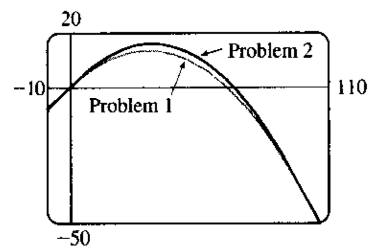
The graph of the five functions as a piecewise-defined function:



This is the piecewise-defined function assignment on a TI-83 Plus calculator, where $Y_2 = L_1$, $Y_6 = g$, $Y_5 = q$, $Y_7 = h$, and $Y_3 = L_2$.



A comparison of the graphs in part 1(c) and part 2(c):



3.9 Related Rates

1. $V = x^3 \quad\Rightarrow\quad \dfrac{dV}{dt} = \dfrac{dV}{dx}\dfrac{dx}{dt} = 3x^2\,\dfrac{dx}{dt}$

2. (a) $A = \pi r^2 \quad\Rightarrow\quad \dfrac{dA}{dt} = \dfrac{dA}{dr}\dfrac{dr}{dt} = 2\pi r\,\dfrac{dr}{dt}$ (b) $\dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt} = 2\pi(30\text{ m})(1\text{ m/s}) = 60\pi\text{ m}^2/\text{s}$

3. $y = x^3 + 2x \quad\Rightarrow\quad \dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = (3x^2+2)(5) = 5(3x^2+2)$. When $x = 2$, $\dfrac{dy}{dt} = 5(14) = 70$.

4. $x^2 + y^2 = 25 \quad\Rightarrow\quad 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0 \quad\Rightarrow\quad x\dfrac{dx}{dt} = -y\dfrac{dy}{dt} \quad\Rightarrow\quad \dfrac{dx}{dt} = -\dfrac{y}{x}\dfrac{dy}{dt}$.
When $y = 4$, $x^2 + 4^2 = 25 \quad\Rightarrow\quad x = \pm 3$. For $\dfrac{dy}{dt} = 6$, $\dfrac{dx}{dt} = -\dfrac{4}{\pm 3}(6) = \mp 8$.

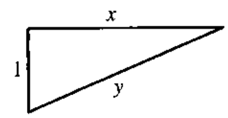
5. $z^2 = x^2 + y^2 \quad\Rightarrow\quad 2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} \quad\Rightarrow\quad \dfrac{dz}{dt} = \dfrac{1}{z}\left(x\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right)$. When $x = 5$ and $y = 12$,
$z^2 = 5^2 + 12^2 \quad\Rightarrow\quad z^2 = 169 \quad\Rightarrow\quad z = \pm 13$. For $\dfrac{dx}{dt} = 2$ and $\dfrac{dy}{dt} = 3$, $\dfrac{dz}{dt} = \dfrac{1}{\pm 13}(5\cdot 2 + 12\cdot 3) = \pm\dfrac{46}{13}$.

6. $y = \sqrt{1+x^3} \quad\Rightarrow\quad \dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \tfrac{1}{2}(1+x^3)^{-1/2}(3x^2)\dfrac{dx}{dt} = \dfrac{3x^2}{2\sqrt{1+x^3}}\dfrac{dx}{dt}$. With $\dfrac{dy}{dt} = 4$ when $x = 2$ and $y = 3$, we have $4 = \dfrac{3(4)}{2(3)}\dfrac{dx}{dt} \quad\Rightarrow\quad \dfrac{dx}{dt} = 2$ cm/s.

7. (a) Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in mi), then we are given that $dx/dt = 500$ mi/h.

(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 2$ mi.

(c)



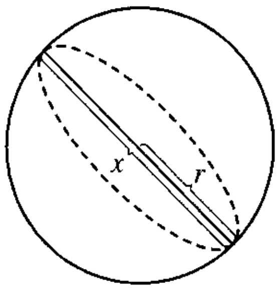
(d) By the Pythagorean Theorem, $y^2 = x^2 + 1 \Rightarrow 2y(dy/dt) = 2x(dx/dt)$.

(e) $\dfrac{dy}{dt} = \dfrac{x}{y}\dfrac{dx}{dt} = \dfrac{x}{y}(500)$. Since $y^2 = x^2 + 1$, when $y = 2$, $x = \sqrt{3}$, so $\dfrac{dy}{dt} = \frac{\sqrt{3}}{2}(500) = 250\sqrt{3} \approx 433$ mi/h.

8. (a) Given: the rate of decrease of the surface area is 1 cm^2/min. If we let t be time (in minutes) and S be the surface area (in cm^2), then we are given that $dS/dt = -1$ cm^2/s.

(b) Unknown: the rate of decrease of the diameter when the diameter is 10 cm. If we let x be the diameter, then we want to find dx/dt when $x = 10$ cm.

(c)



(d) If the radius is r and the diameter $x = 2r$, then $r = \frac{1}{2}x$ and $S = 4\pi r^2 = 4\pi\left(\frac{1}{2}x\right)^2 = \pi x^2 \Rightarrow$

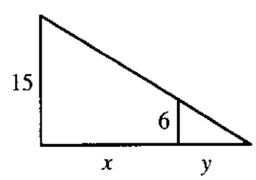
$$\frac{dS}{dt} = \frac{dS}{dx}\frac{dx}{dt} = 2\pi x\frac{dx}{dt}.$$

(e) $-1 = \dfrac{dS}{dt} = 2\pi x\dfrac{dx}{dt} \Rightarrow \dfrac{dx}{dt} = -\dfrac{1}{2\pi x}$. When $x = 10$, $\dfrac{dx}{dt} = -\dfrac{1}{20\pi}$. So the rate of decrease is $\frac{1}{20\pi}$ cm/min.

9. (a) Given: a man 6 ft tall walks away from a street light mounted on a 15-ft-tall pole at a rate of 5 ft/s. If we let t be time (in s) and x be the distance from the pole to the man (in ft), then we are given that $dx/dt = 5$ ft/s.

(b) Unknown: the rate at which the tip of his shadow is moving when he is 40 ft from the pole. If we let y be the distance from the man to the tip of his shadow (in ft), then we want to find $\dfrac{d}{dt}(x+y)$ when $x = 40$ ft.

(c)



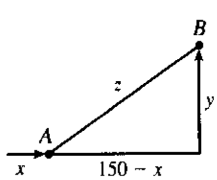
(d) By similar triangles, $\dfrac{15}{6} = \dfrac{x+y}{y} \Rightarrow 15y = 6x + 6y \Rightarrow 9y = 6x \Rightarrow y = \frac{2}{3}x$.

(e) The tip of the shadow moves at a rate of $\dfrac{d}{dt}(x+y) = \dfrac{d}{dt}\left(x + \frac{2}{3}x\right) = \dfrac{5}{3}\dfrac{dx}{dt} = \frac{5}{3}(5) = \frac{25}{3}$ ft/s.

10. (a) Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km/h, and ship B is sailing north at 25 km/h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt = 35$ km/h and $dy/dt = 25$ km/h.

(b) Unknown: the rate at which the distance between the ships is changing at 4:00 P.M. If we let z be the distance between the ships, then we want to find dz/dt when $t = 4$ h.

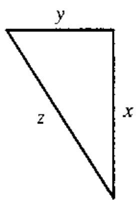
(c)



(d) $z^2 = (150 - x)^2 + y^2 \quad \Rightarrow \quad 2z\,\dfrac{dz}{dt} = 2(150 - x)\left(-\dfrac{dx}{dt}\right) + 2y\,\dfrac{dy}{dt}$

(e) At 4:00 P.M., $x = 4(35) = 140$ and $y = 4(25) = 100 \quad \Rightarrow \quad z = \sqrt{(150 - 140)^2 + 100^2} = \sqrt{10{,}100}$. So

$$\frac{dz}{dt} = \frac{1}{z}\left[(x - 150)\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right] = \frac{-10(35) + 100(25)}{\sqrt{10{,}100}} = \frac{215}{\sqrt{101}} \approx 21.4 \text{ km/h}.$$

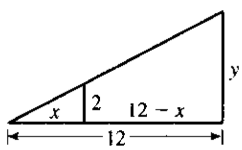
11. 

We are given that $\dfrac{dx}{dt} = 60$ mi/h and $\dfrac{dy}{dt} = 25$ mi/h. $z^2 = x^2 + y^2 \quad \Rightarrow$

$$2z\,\frac{dz}{dt} = 2x\,\frac{dx}{dt} + 2y\,\frac{dy}{dt} \quad \Rightarrow \quad z\,\frac{dz}{dt} = x\,\frac{dx}{dt} + y\,\frac{dy}{dt} \quad \Rightarrow \quad \frac{dz}{dt} = \frac{1}{z}\left(x\,\frac{dx}{dt} + y\,\frac{dy}{dt}\right).$$

After 2 hours, $x = 2\,(60) = 120$ and $y = 2\,(25) = 50 \quad \Rightarrow \quad z = \sqrt{120^2 + 50^2} = 130$,

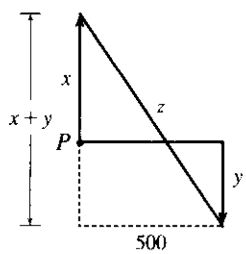
so $\dfrac{dz}{dt} = \dfrac{1}{z}\left(x\,\dfrac{dx}{dt} + y\,\dfrac{dy}{dt}\right) = \dfrac{120(60) + 50(25)}{130} = 65$ mi/h.

12. 

We are given that $\dfrac{dx}{dt} = 1.6$ m/s. By similar triangles, $\dfrac{y}{12} = \dfrac{2}{x} \quad \Rightarrow \quad y = \dfrac{24}{x}$

$\Rightarrow \quad \dfrac{dy}{dt} = -\dfrac{24}{x^2}\dfrac{dx}{dt} = -\dfrac{24}{x^2}(1.6)$. When $x = 8$, $\dfrac{dy}{dt} = -\dfrac{24(1.6)}{64} = -0.6$ m/s,

so the shadow is decreasing at a rate of 0.6 m/s.

13. 

We are given that $\dfrac{dx}{dt} = 4$ ft/s and $\dfrac{dy}{dt} = 5$ ft/s. $z^2 = (x + y)^2 + 500^2 \quad \Rightarrow$

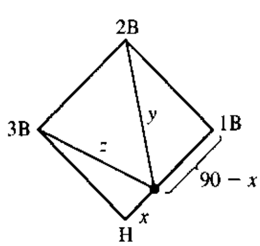
$2z\,\dfrac{dz}{dt} = 2(x + y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. 15 minutes after the woman starts, we have

$x = (4 \text{ ft/s})(20 \text{ min})(60 \text{ s/min}) = 4800$ ft and $y = 5 \cdot 15 \cdot 60 = 4500 \quad \Rightarrow$

$z = \sqrt{(4800 + 4500)^2 + 500^2} = \sqrt{86{,}740{,}000}$, so

$$\frac{dz}{dt} = \frac{x + y}{z}\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = \frac{4800 + 4500}{\sqrt{86{,}740{,}000}}(4 + 5) = \frac{837}{\sqrt{8674}} \approx 8.99 \text{ ft/s}.$$

14. We are given that $\dfrac{dx}{dt} = 24$ ft/s.

(a) 

$y^2 = (90 - x)^2 + 90^2 \quad \Rightarrow \quad 2y\,\dfrac{dy}{dt} = 2(90 - x)\left(-\dfrac{dx}{dt}\right)$.

When $x = 45$, $y = \sqrt{45^2 + 90^2} = 45\sqrt{5}$, so

$$\frac{dy}{dt} = \frac{90 - x}{y}\left(-\frac{dx}{dt}\right) = \frac{45}{45\sqrt{5}}(-24) = -\frac{24}{\sqrt{5}},$$

so the distance from second base is decreasing at a rate of $\frac{24}{\sqrt{5}} \approx 10.7$ ft/s.

(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer — and we do.

$z^2 = x^2 + 90^2 \quad \Rightarrow \quad 2z\,\dfrac{dz}{dt} = 2x\,\dfrac{dx}{dt}$. When $x = 45$, $z = 45\sqrt{5}$, so $\dfrac{dz}{dt} = \dfrac{45}{45\sqrt{5}}(24) = \dfrac{24}{\sqrt{5}} \approx 10.7$ ft/s.

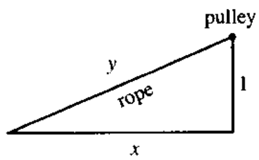
15. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\dfrac{dh}{dt} = 1$ cm/min and $\dfrac{dA}{dt} = 2$ cm^2/min.

Using the Product Rule, we have $\dfrac{dA}{dt} = \dfrac{1}{2}\left(b\,\dfrac{dh}{dt} + h\,\dfrac{db}{dt}\right)$. When $h = 10$ and $A = 100$, we have

$100 = \frac{1}{2}b(10) \;\Rightarrow\; \frac{1}{2}b = 10 \;\Rightarrow\; b = 20$, so $2 = \dfrac{1}{2}\left(20 \cdot 1 + 10\,\dfrac{db}{dt}\right) \;\Rightarrow\; 4 = 20 + 10\,\dfrac{db}{dt} \;\Rightarrow$

$\dfrac{db}{dt} = \dfrac{4 - 20}{10} = -1.6$ cm/min.

16.

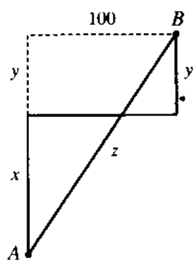


Given $\dfrac{dy}{dt} = -1$ m/s, find $\dfrac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \;\Rightarrow$

$2y\,\dfrac{dy}{dt} = 2x\,\dfrac{dx}{dt} \;\Rightarrow\; \dfrac{dx}{dt} = \dfrac{y}{x}\dfrac{dy}{dt} = -\dfrac{y}{x}$. When $x = 8$, $y = \sqrt{65}$, so

$\dfrac{dx}{dt} = -\dfrac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\dfrac{\sqrt{65}}{8} \approx 1.01$ m/s.

17.



We are given that $\dfrac{dx}{dt} = 35$ km/h and $\dfrac{dy}{dt} = 25$ km/h. $z^2 = (x + y)^2 + 100^2$

$\Rightarrow\; 2z\,\dfrac{dz}{dt} = 2(x + y)\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right)$. At 4:00 P.M., $x = 4(35) = 140$ and

$y = 4(25) = 100 \;\Rightarrow\; z = \sqrt{(140 + 100)^2 + 100^2} = \sqrt{67{,}600} = 260$, so

$\dfrac{dz}{dt} = \dfrac{x + y}{z}\left(\dfrac{dx}{dt} + \dfrac{dy}{dt}\right) = \dfrac{140 + 100}{260}(35 + 25) = \dfrac{720}{13} \approx 55.4$ km/h.

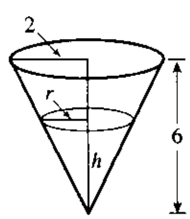
18. Let D denote the distance from the origin $(0, 0)$ to the point on the curve $y = \sqrt{x}$.

$D = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + (\sqrt{x})^2} = \sqrt{x^2 + x} \;\Rightarrow$

$\dfrac{dD}{dt} = \frac{1}{2}\left(x^2 + x\right)^{-1/2}(2x + 1)\,\dfrac{dx}{dt} = \dfrac{2x + 1}{2\sqrt{x^2 + x}}\,\dfrac{dx}{dt}$. With $\dfrac{dx}{dt} = 3$ when $x = 4$,

$\dfrac{dD}{dt} = \dfrac{9}{2\sqrt{20}}(3) = \dfrac{27}{4\sqrt{5}} \approx 3.02$ cm/s.

19.



If $C =$ the rate at which water is pumped in, then $\dfrac{dV}{dt} = C - 10{,}000$, where

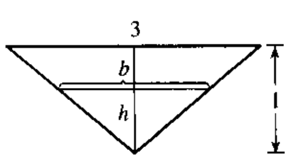
$V = \frac{1}{3}\pi r^2 h$ is the volume at time t. By similar triangles, $\dfrac{r}{2} = \dfrac{h}{6} \;\Rightarrow$

$r = \frac{1}{3}h \;\Rightarrow\; V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3 \;\Rightarrow\; \dfrac{dV}{dt} = \frac{\pi}{9}h^2\,\dfrac{dh}{dt}$.

When $h = 200$ cm, $\dfrac{dh}{dt} = 20$ cm/min, so $C - 10{,}000 = \frac{\pi}{9}(200)^2(20) \;\Rightarrow$

$C = 10{,}000 + \frac{800{,}000}{9}\pi \approx 289{,}253$ cm^3/min.

20.

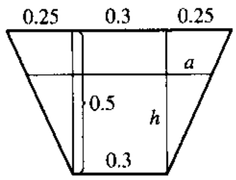


By similar triangles, $\dfrac{3}{1} = \dfrac{b}{h}$, so $b = 3h$. The trough has volume

$V = \frac{1}{2}bh(10) = 5(3h)h = 15h^2 \;\Rightarrow\; 12 = \dfrac{dV}{dt} = 30h\,\dfrac{dh}{dt} \;\Rightarrow$

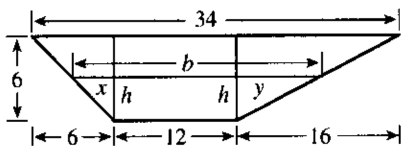
$\dfrac{dh}{dt} = \dfrac{2}{5h}$. When $h = \frac{1}{2}$, $\dfrac{dh}{dt} = \dfrac{2}{5 \cdot \frac{1}{2}} = \dfrac{4}{5}$ ft/min.

21.



The figure is labeled in meters. The area A of a trapezoid is $\frac{1}{2}(\text{base}_1 + \text{base}_2)(\text{height})$, and the volume V of the 10-meter-long trough is $10A$. Thus, the volume of the trapezoid with height h is $V = (10)\frac{1}{2}\left[0.3 + (0.3 + 2a)\right]h$. By similar triangles, $\frac{a}{h} = \frac{0.25}{0.5} = \frac{1}{2}$, so $2a = h \;\Rightarrow\; V = 5(0.6 + h)h = 3h + 5h^2$. Now $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} \;\Rightarrow\; 0.2 = (3 + 10h)\frac{dh}{dt} \;\Rightarrow$ $\frac{dh}{dt} = \frac{0.2}{3 + 10h}$. When $h = 0.3$, $\frac{dh}{dt} = \frac{0.2}{3 + 10(0.3)} = \frac{0.2}{6}$ m/min $= \frac{1}{30}$ m/min or $\frac{10}{3}$ cm/min.

22.



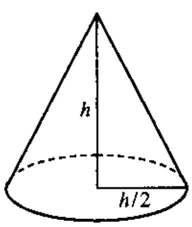
The figure is drawn without the top 3 feet.

$V = \frac{1}{2}(b + 12)h(20) = 10(b + 12)h$ and, from similar triangles, $\frac{x}{h} = \frac{6}{6}$ and $\frac{y}{h} = \frac{16}{6} = \frac{8}{3}$, so $b = x + 12 + y = h + 12 + \frac{8h}{3} = 12 + \frac{11h}{3}$. Thus,

$V = 10\left(24 + \frac{11h}{3}\right)h = 240h + \frac{110h^2}{3}$ and so $0.8 = \frac{dV}{dt} = \left(240 + \frac{220}{3}h\right)\frac{dh}{dt}$. When $h = 5$,

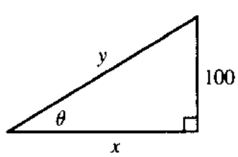
$\frac{dh}{dt} = \frac{0.8}{240 + 5(220/3)} = \frac{3}{2275} \approx 0.00132$ ft/min.

23.



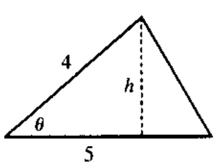
We are given that $\frac{dV}{dt} = 30\ \text{ft}^3/\text{min}$. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$ $\Rightarrow \frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} \;\Rightarrow\; 30 = \frac{\pi h^2}{4}\frac{dh}{dt} \;\Rightarrow\; \frac{dh}{dt} = \frac{120}{\pi h^2}$. When $h = 10$ ft, $\frac{dh}{dt} = \frac{120}{10^2\pi} = \frac{6}{5\pi} \approx 0.38$ ft/min.

24.

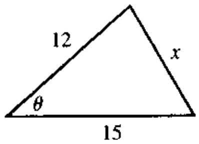


We are given $dx/dt = 8$ ft/s. $\cot\theta = \frac{x}{100} \;\Rightarrow\; x = 100\cot\theta \;\Rightarrow$ $\frac{dx}{dt} = -100\csc^2\theta\,\frac{d\theta}{dt} \;\Rightarrow\; \frac{d\theta}{dt} = -\frac{\sin^2\theta}{100}\cdot 8$. When $y = 200$, $\sin\theta = \frac{100}{200} = \frac{1}{2} \;\Rightarrow\; \frac{d\theta}{dt} = -\frac{(1/2)^2}{100}\cdot 8 = -\frac{1}{50}$ rad/s. The angle is decreasing at a rate of $\frac{1}{50}$ rad/s.

25.



$A = \frac{1}{2}bh$, but $b = 5$ m and $\sin\theta = \frac{h}{4} \;\Rightarrow\; h = 4\sin\theta$, so $A = \frac{1}{2}(5)(4\sin\theta) = 10\sin\theta$. We are given $\frac{d\theta}{dt} = 0.06$ rad/s, so $\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = (10\cos\theta)(0.06) = 0.6\cos\theta$. When $\theta = \frac{\pi}{3}$, $\frac{dA}{dt} = 0.6\left(\cos\frac{\pi}{3}\right) = (0.6)\left(\frac{1}{2}\right) = 0.3\ \text{m}^2/\text{s}$.

26. 

We are given $d\theta/dt = 2°/\text{min} = \frac{\pi}{90}$ rad/min. By the Law of Cosines,

$x^2 = 12^2 + 15^2 - 2(12)(15)\cos\theta = 369 - 360\cos\theta \Rightarrow$

$2x\dfrac{dx}{dt} = 360\sin\theta\dfrac{d\theta}{dt} \Rightarrow \dfrac{dx}{dt} = \dfrac{180\sin\theta}{x}\dfrac{d\theta}{dt}$. When $\theta = 60°$,

$x = \sqrt{369 - 360\cos 60°} = \sqrt{189} = 3\sqrt{21}$, so

$$\frac{dx}{dt} = \frac{180\sin 60°}{3\sqrt{21}}\frac{\pi}{90} = \frac{\pi\sqrt{3}}{3\sqrt{21}} = \frac{\sqrt{7}\,\pi}{21} \approx 0.396 \text{ m/min}.$$

27. Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P\dfrac{dV}{dt} + V\dfrac{dP}{dt} = 0$ $\Rightarrow \dfrac{dV}{dt} = -\dfrac{V}{P}\dfrac{dP}{dt}$. When $V = 600$, $P = 150$ and $\dfrac{dP}{dt} = 20$, so we have $\dfrac{dV}{dt} = -\dfrac{600}{150}(20) = -80$. Thus, the volume is decreasing at a rate of 80 cm^3/min.

28. $PV^{1.4} = C \Rightarrow P\cdot 1.4V^{0.4}\dfrac{dV}{dt} + V^{1.4}\dfrac{dP}{dt} = 0 \Rightarrow \dfrac{dV}{dt} = -\dfrac{V^{1.4}}{P\cdot 1.4V^{0.4}}\dfrac{dP}{dt} = -\dfrac{V}{1.4P}\dfrac{dP}{dt}$. When $V = 400$, $P = 80$ and $\dfrac{dP}{dt} = -10$, so we have $\dfrac{dV}{dt} = -\dfrac{400}{1.4(80)}(-10) = \dfrac{250}{7}$. Thus, the volume is increasing at a rate of $\frac{250}{7} \approx 36 \text{ cm}^3\text{/min}$.

29. With $R_1 = 80$ and $R_2 = 100$, $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2} = \dfrac{1}{80} + \dfrac{1}{100} = \dfrac{180}{8000} = \dfrac{9}{400}$, so $R = \dfrac{400}{9}$. Differentiating $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$ with respect to t, we have $-\dfrac{1}{R^2}\dfrac{dR}{dt} = -\dfrac{1}{R_1^2}\dfrac{dR_1}{dt} - \dfrac{1}{R_2^2}\dfrac{dR_2}{dt} \Rightarrow$

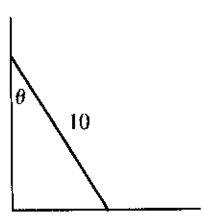
$\dfrac{dR}{dt} = R^2\left(\dfrac{1}{R_1^2}\dfrac{dR_1}{dt} + \dfrac{1}{R_2^2}\dfrac{dR_2}{dt}\right)$. When $R_1 = 80$ and $R_2 = 100$,

$$\frac{dR}{dt} = \frac{400^2}{9^2}\left[\frac{1}{80^2}(0.3) + \frac{1}{100^2}(0.2)\right] = \frac{107}{810} \approx 0.132\ \Omega/\text{s}.$$

30. We want to find $\dfrac{dB}{dt}$ when $L = 18$ using $B = 0.007W^{2/3}$ and $W = 0.12L^{2.53}$.

$$\frac{dB}{dt} = \frac{dB}{dW}\frac{dW}{dL}\frac{dL}{dt} = \left(0.007\cdot\tfrac{2}{3}W^{-1/3}\right)\left(0.12\cdot 2.53\cdot L^{1.53}\right)\left(\frac{20-15}{10{,}000{,}000}\right)$$

$$= \left[0.007\cdot\tfrac{2}{3}\left(0.12\cdot 18^{2.53}\right)^{-1/3}\right]\left(0.12\cdot 2.53\cdot 18^{1.53}\right)\left(\frac{5}{10^7}\right) \approx 1.045\times 10^{-8} \text{ g/yr}$$

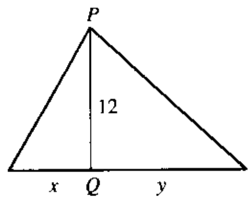
31. 

We are given that $\dfrac{dx}{dt} = 2$ ft/s. $\sin\theta = \dfrac{x}{10} \Rightarrow x = 10\sin\theta \Rightarrow$

$\dfrac{dx}{dt} = 10\cos\theta\dfrac{d\theta}{dt}$. When $\theta = \dfrac{\pi}{4}$, $2 = 10\cos\dfrac{\pi}{4}\dfrac{d\theta}{dt} \Rightarrow$

$$\frac{d\theta}{dt} = \frac{2}{10(1/\sqrt{2})} = \frac{\sqrt{2}}{5} \text{ rad/s}.$$

32.



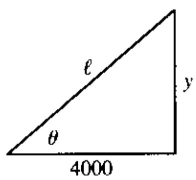
Using Q for the origin, we are given $\frac{dx}{dt} = -2$ ft/s and need to find $\frac{dy}{dt}$ when $x = -5$. Using the Pythagorean Theorem twice, we have $\sqrt{x^2+12^2} + \sqrt{y^2+12^2} = 39$, the total length of the rope. Differentiating with respect to t, we get $\frac{x}{\sqrt{x^2+12^2}}\frac{dx}{dt} + \frac{y}{\sqrt{y^2+12^2}}\frac{dy}{dt} = 0$, so

$\frac{dy}{dt} = -\frac{x\sqrt{y^2+12^2}}{y\sqrt{x^2+12^2}}\frac{dx}{dt}$. Now when $x = -5$, $39 = \sqrt{(-5)^2+12^2} + \sqrt{y^2+12^2} = 13 + \sqrt{y^2+12^2}$ $\Leftrightarrow$

$\sqrt{y^2+12^2} = 26$, and $y = \sqrt{26^2-12^2} = \sqrt{532}$. So when $x = -5$,

$\frac{dy}{dt} = -\frac{(-5)(26)}{\sqrt{532}(13)}(-2) = -\frac{10}{\sqrt{133}} \approx -0.87$ ft/s. So cart B is moving towards Q at about 0.87 ft/s.

33. (a)



By the Pythagorean Theorem, $4000^2 + y^2 = \ell^2$. Differentiating with respect to t, we obtain $2y\frac{dy}{dt} = 2\ell\frac{d\ell}{dt}$. We know that $\frac{dy}{dt} = 600$ ft/s, so when $y = 3000$ ft, $\ell = \sqrt{4000^2+3000^2} = \sqrt{25{,}000{,}000} = 5000$ ft and

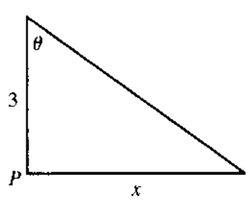
$\frac{d\ell}{dt} = \frac{y}{\ell}\frac{dy}{dt} = \frac{3000}{5000}(600) = \frac{1800}{5} = 360$ ft/s.

(b) Here $\tan\theta = \frac{y}{4000}$ $\Rightarrow$ $\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{y}{4000}\right)$ $\Rightarrow$ $\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{4000}\frac{dy}{dt}$ $\Rightarrow$ $\frac{d\theta}{dt} = \frac{\cos^2\theta}{4000}\frac{dy}{dt}$.

When $y = 3000$ ft, $\frac{dy}{dt} = 600$ ft/s, $\ell = 5000$ and $\cos\theta = \frac{4000}{\ell} = \frac{4000}{5000} = \frac{4}{5}$, so

$\frac{d\theta}{dt} = \frac{(4/5)^2}{4000}(600) = 0.096$ rad/s.

34.

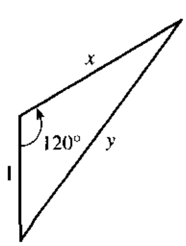


We are given that $\frac{d\theta}{dt} = 4(2\pi) = 8\pi$ rad/min. $x = 3\tan\theta$ $\Rightarrow$

$\frac{dx}{dt} = 3\sec^2\theta\,\frac{d\theta}{dt}$. When $x = 1$, $\tan\theta = \frac{1}{3}$, so $\sec^2\theta = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$ and

$\frac{dx}{dt} = 3\left(\frac{10}{9}\right)(8\pi) = \frac{80\pi}{3} \approx 83.8$ km/min.

35.



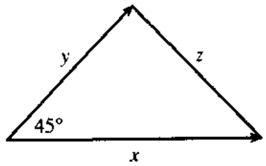
We are given that $\frac{dx}{dt} = 300$ km/h. By the Law of Cosines,

$y^2 = x^2 + 1^2 - 2(1)(x)\cos 120° = x^2 + 1 - 2x\left(-\frac{1}{2}\right) = x^2 + x + 1$, so

$2y\frac{dy}{dt} = 2x\frac{dx}{dt} + \frac{dx}{dt}$ $\Rightarrow$ $\frac{dy}{dt} = \frac{2x+1}{2y}\frac{dx}{dt}$. After 1 minute,

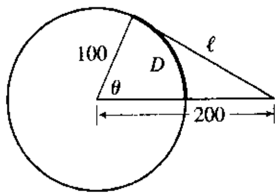
$x = \frac{300}{60} = 5$ km $\Rightarrow$ $y = \sqrt{5^2+5+1} = \sqrt{31}$ km $\Rightarrow$

$\frac{dy}{dt} = \frac{2(5)+1}{2\sqrt{31}}(300) = \frac{1650}{\sqrt{31}} \approx 296$ km/h.

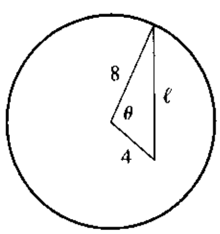
36. 

We are given that $\frac{dx}{dt} = 3$ mi/h and $\frac{dy}{dt} = 2$ mi/h. By the Law of Cosines,
$z^2 = x^2 + y^2 - 2xy\cos 45^\circ = x^2 + y^2 - \sqrt{2}\,xy \;\Rightarrow$
$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - \sqrt{2}\,x\frac{dy}{dt} - \sqrt{2}\,y\frac{dx}{dt}$. After 15 minutes $\left[= \frac{1}{4}\text{ h}\right]$,
we have $x = \frac{3}{4}$ and $y = \frac{2}{4} = \frac{1}{2} \;\Rightarrow\; z^2 = \left(\frac{3}{4}\right)^2 + \left(\frac{2}{4}\right)^2 - \sqrt{2}\left(\frac{3}{4}\right)\left(\frac{2}{4}\right) \;\Rightarrow\; z = \frac{\sqrt{13 - 6\sqrt{2}}}{4}$ and

$\frac{dz}{dt} = \frac{2}{\sqrt{13 - 6\sqrt{2}}}\left[2\left(\frac{3}{4}\right)3 + 2\left(\frac{1}{2}\right)2 - \sqrt{2}\left(\frac{3}{4}\right)2 - \sqrt{2}\left(\frac{1}{2}\right)3\right] = \frac{2}{\sqrt{13 - 6\sqrt{2}}}\,\frac{13 - 6\sqrt{2}}{2} = \sqrt{13 - 6\sqrt{2}}$
≈ 2.125 mi/h.

37. 

Let the distance between the runner and the friend be ℓ. Then by the Law of Cosines,
$\ell^2 = 200^2 + 100^2 - 2 \cdot 200 \cdot 100 \cdot \cos\theta = 50{,}000 - 40{,}000\cos\theta\ (\star)$.
Differentiating implicitly with respect to t, we obtain
$2\ell\frac{d\ell}{dt} = -40{,}000(-\sin\theta)\frac{d\theta}{dt}$. Now if D is the distance run when
the angle is θ radians, then by the formula for the length of an arc on a circle, $s = r\theta$, we have $D = 100\theta$, so
$\theta = \frac{1}{100}D \;\Rightarrow\; \frac{d\theta}{dt} = \frac{1}{100}\frac{dD}{dt} = \frac{7}{100}$. To substitute into the expression for $\frac{d\ell}{dt}$, we must know $\sin\theta$ at the time
when $\ell = 200$, which we find from $(\star)$: $200^2 = 50{,}000 - 40{,}000\cos\theta \;\Leftrightarrow\; \cos\theta = \frac{1}{4} \;\Rightarrow$
$\sin\theta = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$. Substituting, we get $2(200)\frac{d\ell}{dt} = 40{,}000\frac{\sqrt{15}}{4}\left(\frac{7}{100}\right) \;\Rightarrow$
$d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78$ m/s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.

38. 

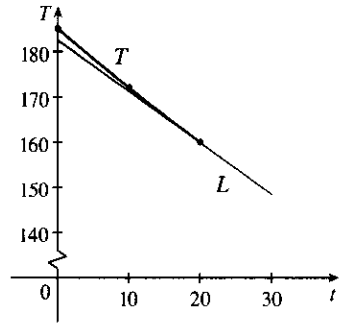
The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12} = \frac{\pi}{6}$ rad/h. The minute hand goes around once an hour, or at the rate of 2π rad/h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of
$d\theta/dt = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ rad/h. Now, to relate θ to ℓ, we use the Law of Cosines: $\ell^2 = 4^2 + 8^2 - 2 \cdot 4 \cdot 8 \cdot \cos\theta = 80 - 64\cos\theta\ (\star)$.
Differentiating implicitly with respect to t, we get $2\ell\frac{d\ell}{dt} = -64(-\sin\theta)\frac{d\theta}{dt}$. At 1:00, the angle between the two hands is one-twelfth of the circle, that is, $\frac{2\pi}{12} = \frac{\pi}{6}$ radians. We use $(\star)$ to find ℓ at 1:00:
$\ell = \sqrt{80 - 64\cos\frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}$. Substituting, we get $2\ell\frac{d\ell}{dt} = 64\sin\frac{\pi}{6}\left(-\frac{11\pi}{6}\right) \;\Rightarrow$
$\frac{d\ell}{dt} = \frac{64\left(\frac{1}{2}\right)\left(-\frac{11\pi}{6}\right)}{2\sqrt{80 - 32\sqrt{3}}} = -\frac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx -18.6$. So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6 mm/h ≈ 0.005 mm/s.

3.10 Linear Approximations and Differentials

1. As in Example 1, $T(0) = 185$, $T(10) = 172$, $T(20) = 160$, and

$$T'(20) \approx \frac{T(10) - T(20)}{10 - 20} = \frac{172 - 160}{-10} = -1.2\ ^\circ\text{F/min}.$$

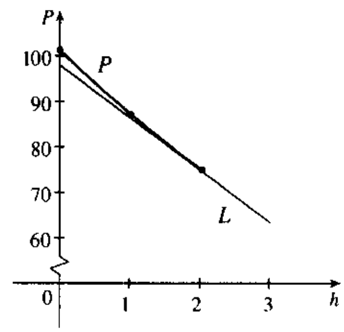
$T(30) \approx T(20) + T'(20)(30 - 20) \approx 160 - 1.2(10) = 148\ ^\circ\text{F}$.



We would expect the temperature of the turkey to get closer to 75 °F as time increases. Since the temperature decreased 13 °F in the first 10 minutes and 12 °F in the second 10 minutes, we can assume that the slopes of the tangent line are increasing through negative values: $-1.3, -1.2, \ldots$. Hence, the tangent lines are under the curve and 148 °F is an underestimate. From the figure, we estimate the slope of the tangent line at $t = 20$ to be $\frac{184 - 147}{0 - 30} = -\frac{37}{30}$. Then the linear approximation becomes $T(30) \approx T(20) + T'(20) \cdot 10 \approx 160 - \frac{37}{30}(10) = 147\frac{2}{3} \approx 147.7$.

2. $P'(2) \approx \dfrac{P(1) - P(2)}{1 - 2} = \dfrac{87.1 - 74.9}{-1} = -12.2$ kilopascals/km.

$P(3) \approx P(2) + P'(2)(3 - 2) \approx 74.9 - 12.2(1) = 62.7$ kPa.

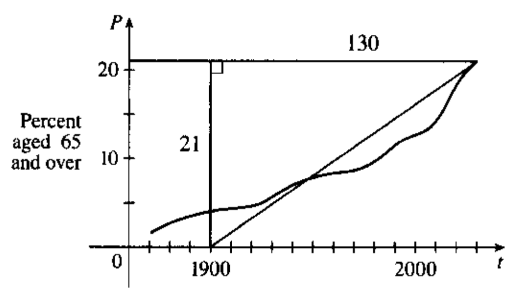


From the figure, we estimate the slope of the tangent line at $h = 2$ to be $\dfrac{98 - 63}{0 - 3} = -\dfrac{35}{3}$. Then the linear approximation becomes

$P(3) \approx P(2) + P'(2) \cdot 1 \approx 74.9 - \frac{35}{3} \approx 63.23$ kPa.

3. Extend the tangent line at the point $(2030, 21)$ to the t-axis. Answers will vary based on this approximation—we'll use $t = 1900$ as our t-intercept. The linearization is then

$$P(t) \approx P(2030) + P'(2030)(t - 2030)$$
$$\approx 21 + \tfrac{21}{130}(t - 2030)$$



$P(2040) = 21 + \frac{21}{130}(2040 - 2030) \approx 22.6\%$

$P(2050) = 21 + \frac{21}{130}(2050 - 2030) \approx 24.2\%$

These predictions are probably too high since the tangent line lies above the graph at $t = 2030$.

4. Let $A = \dfrac{N(1980) - N(1985)}{1980 - 1985} = \dfrac{15.0 - 17.0}{-5} = 0.4$ and $B = \dfrac{N(1990) - N(1985)}{1990 - 1985} = \dfrac{19.3 - 17.0}{5} = 0.46$.

Then $N'(1985) = \lim\limits_{t \to 1985} \dfrac{N(t) - N(1985)}{t - 1985} \approx \dfrac{A + B}{2} = 0.43$ million/year. So

$N(1984) \approx N(1985) + N'(1985)(1984 - 1985) \approx 17.0 + 0.43(-1) = 16.57$ million.

$$N'(2000) \approx \frac{N(1995) - N(2000)}{1995 - 2000} = \frac{22.0 - 24.9}{-5} = 0.58 \text{ million/year}.$$

$N(2006) \approx N(2000) + N'(2000)(2006 - 2000) \approx 24.9 + 0.58(6) = 28.38$ million.

5. $f(x) = x^3 \;\Rightarrow\; f'(x) = 3x^2$, so $f(1) = 1$ and $f'(1) = 3$. With $a = 1$, $L(x) = f(a) + f'(a)(x - a)$ becomes $L(x) = f(1) + f'(1)(x - 1) = 1 + 3(x - 1) = 3x - 2$.

6. $f(x) = 1/\sqrt{2+x} = (2+x)^{-1/2} \;\Rightarrow\; f'(x) = -\frac{1}{2}(2+x)^{-3/2}$ so $f(0) = \frac{1}{\sqrt{2}}$ and $f'(0) = -1/(4\sqrt{2})$. So $L(x) = f(0) + f'(0)(x - 0) = \frac{1}{\sqrt{2}} - \frac{1}{4\sqrt{2}}(x - 0) = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{4}x\right)$.

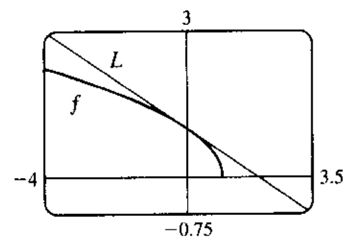
7. $f(x) = \cos x \;\Rightarrow\; f'(x) = -\sin x$, so $f\left(\frac{\pi}{2}\right) = 0$ and $f'\left(\frac{\pi}{2}\right) = -1$. Thus, $L(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) = 0 - 1\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}$.

8. $f(x) = \sqrt[3]{x} = x^{1/3} \;\Rightarrow\; f'(x) = \frac{1}{3}x^{-2/3}$, so $f(-8) = -2$ and $f'(-8) = \frac{1}{12}$. Thus, $L(x) = f(-8) + f'(-8)(x + 8) = -2 + \frac{1}{12}(x + 8) = \frac{1}{12}x - \frac{4}{3}$.

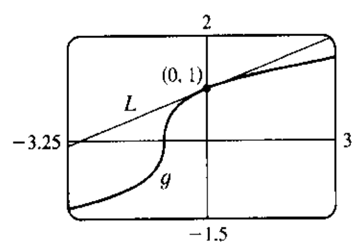
9. $f(x) = \sqrt{1-x} \;\Rightarrow\; f'(x) = \dfrac{-1}{2\sqrt{1-x}}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{2}$. Therefore,

$$\sqrt{1-x} = f(x) \approx f(0) + f'(0)(x - 0) = 1 + \left(-\tfrac{1}{2}\right)(x - 0) = 1 - \tfrac{1}{2}x$$

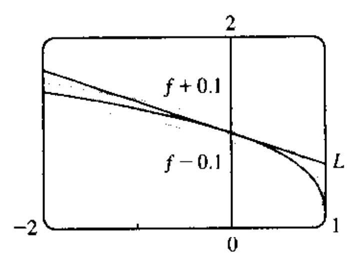
So $\sqrt{0.9} = \sqrt{1 - 0.1} \approx 1 - \frac{1}{2}(0.1) = 0.95$ and $\sqrt{0.99} = \sqrt{1 - 0.01} \approx 1 - \frac{1}{2}(0.01) = 0.995$.



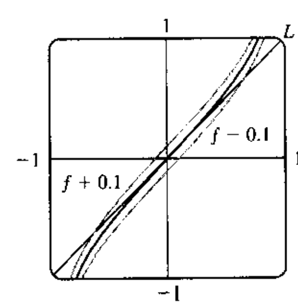
10. $g(x) = \sqrt[3]{1+x} = (1+x)^{1/3} \;\Rightarrow\; g'(x) = \frac{1}{3}(1+x)^{-2/3}$, so $g(0) = 1$ and $g'(0) = \frac{1}{3}$. Therefore, $\sqrt[3]{1+x} = g(x) \approx g(0) + g'(0)(x - 0) = 1 + \frac{1}{3}x$. So $\sqrt[3]{0.95} = \sqrt[3]{1 + (-0.05)} \approx 1 + \frac{1}{3}(-0.05) = 0.98\overline{3}$, and $\sqrt[3]{1.1} = \sqrt[3]{1 + 0.1} \approx 1 + \frac{1}{3}(0.1) = 1.0\overline{3}$.



11. $f(x) = \sqrt[3]{1-x} = (1-x)^{1/3} \;\Rightarrow\; f'(x) = -\frac{1}{3}(1-x)^{-2/3}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{3}$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 - \frac{1}{3}x$. We need $\sqrt[3]{1-x} - 0.1 < 1 - \frac{1}{3}x < \sqrt[3]{1-x} + 0.1$, which is true when $-1.204 < x < 0.706$.



12. $f(x) = \tan x \;\Rightarrow\; f'(x) = \sec^2 x$, so $f(0) = 0$ and $f'(0) = 1$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 0 + 1(x - 0) = x$. We need $\tan x - 0.1 < x < \tan x + 0.1$, which is true when $-0.63 < x < 0.63$.

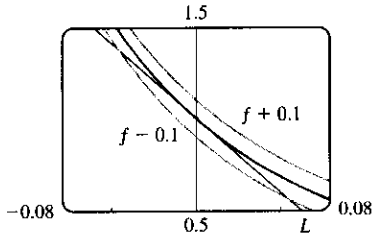


13. $f(x) = \dfrac{1}{(1+2x)^4} = (1+2x)^{-4} \Rightarrow$

$f'(x) = -4(1+2x)^{-5}(2) = \dfrac{-8}{(1+2x)^5}$, so $f(0) = 1$ and $f'(0) = -8$.

Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + (-8)(x-0) = 1 - 8x$.

We need $1/(1+2x)^4 - 0.1 < 1 - 8x < 1/(1+2x)^4 + 0.1$, which is true when $-0.045 < x < 0.055$.

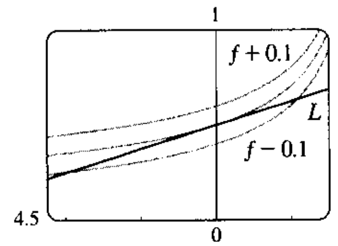


14. $f(x) = \dfrac{1}{\sqrt{4-x}} \Rightarrow f'(x) = \dfrac{1}{2(4-x)^{3/2}}$ so $f(0) = \frac{1}{2}$ and

$f'(0) = \frac{1}{16}$. So $f(x) \approx \frac{1}{2} + \frac{1}{16}(x-0) = \frac{1}{2} + \frac{1}{16}x$. We need

$\dfrac{1}{\sqrt{4-x}} - 0.1 < \frac{1}{2} + \frac{1}{16}x < \dfrac{1}{\sqrt{4-x}} + 0.1$, which is true when

$-3.91 < x < 2.14$.



15. If $y = f(x)$, then the differential dy is equal to $f'(x)\,dx$. $y = x^4 + 5x \Rightarrow dy = (4x^3 + 5)\,dx$.

16. $y = \cos \pi x \Rightarrow dy = -\sin \pi x \cdot \pi\,dx = -\pi \sin \pi x\,dx$

17. $y = x^2 \tan x \Rightarrow dy = (x^2 \sec^2 x + \tan x \cdot 2x)\,dx = (x^2 \sec^2 x + 2x \tan x)\,dx$

18. $y = \sqrt{1+t^2} \Rightarrow dy = \frac{1}{2}(1+t^2)^{-1/2}(2t)\,dt = \dfrac{t}{\sqrt{1+t^2}}\,dt$

19. $y = \dfrac{u+1}{u-1} \Rightarrow dy = \dfrac{(u-1)(1)-(u+1)(1)}{(u-1)^2}\,du = \dfrac{-2}{(u-1)^2}\,du$

20. $y = (1+2r)^{-4} \Rightarrow dy = -4(1+2r)^{-5} \cdot 2\,dr = -8(1+2r)^{-5}\,dr$

21. (a) $y = x^2 + 2x \Rightarrow dy = (2x+2)\,dx$

(b) When $x = 3$ and $dx = \frac{1}{2}$, $dy = [2(3)+2]\left(\frac{1}{2}\right) = 4$.

22. (a) $y = x^3 - 6x^2 + 5x - 7 \Rightarrow dy = (3x^2 - 12x + 5)\,dx$

(b) When $x = -2$ and $dx = 0.1$, $dy = (12 + 24 + 5)(0.1) = 4.1$.

23. (a) $y = \sqrt{4+5x} \Rightarrow dy = \frac{1}{2}(4+5x)^{-1/2} \cdot 5\,dx = \dfrac{5}{2\sqrt{4+5x}}\,dx$

(b) When $x = 0$ and $dx = 0.04$, $dy = \dfrac{5}{2\sqrt{4}}(0.04) = \frac{5}{4} \cdot \frac{1}{25} = \frac{1}{20} = 0.05$.

24. (a) $y = 1/(x+1) \Rightarrow dy = -\dfrac{1}{(x+1)^2}dx$

(b) When $x = 1$ and $dx = -0.01$, $dy = -\dfrac{1}{2^2}(-0.01) = \frac{1}{4} \cdot \frac{1}{100} = \frac{1}{400} = 0.0025$.

25. (a) $y = \tan x \Rightarrow dy = \sec^2 x\,dx$

(b) When $x = \pi/4$ and $dx = -0.1$, $dy = [\sec(\pi/4)]^2(-0.1) = (\sqrt{2})^2(-0.1) = -0.2$.

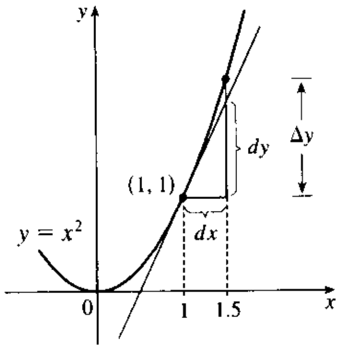
26. (a) $y = \cos x \Rightarrow dy = -\sin x\,dx$

(b) When $x = \pi/3$ and $dx = 0.05$, $dy = -\sin(\pi/3)(0.05) = -0.5\sqrt{3}\,(0.05) = -0.025\sqrt{3} \approx -0.043$.

27. $y = x^2, x = 1, \Delta x = 0.5 \Rightarrow$

$\Delta y = (1.5)^2 - 1^2 = 1.25.$

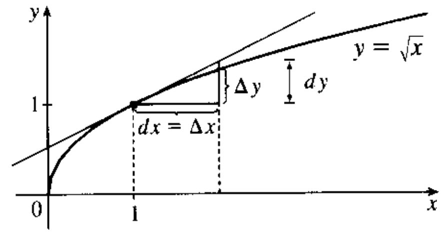
$dy = 2x\,dx = 2(1)(0.5) = 1$



28. $y = \sqrt{x}, x = 1, \Delta x = 1 \Rightarrow$

$\Delta y = \sqrt{2} - \sqrt{1} = \sqrt{2} - 1 \approx 0.414$

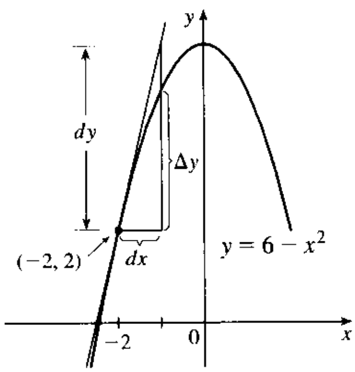
$dy = \dfrac{1}{2\sqrt{x}}dx = \frac{1}{2}(1) = 0.5$



29. $y = 6 - x^2, x = -2, \Delta x = 0.4 \Rightarrow$

$\Delta y = \left(6 - (-1.6)^2\right) - \left(6 - (-2)^2\right) = 1.44$

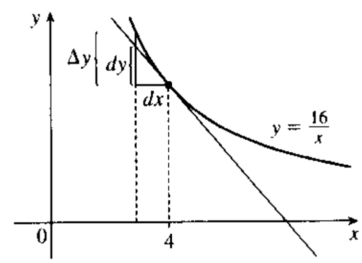
$dy = -2x\,dx = -2(-2)(0.4) = 1.6$



30. $y = \dfrac{16}{x}, x = 4, \Delta x = -1 \Rightarrow$

$\Delta y = \dfrac{16}{3} - \dfrac{16}{4} = \dfrac{4}{3}.$

$dy = -\left(\dfrac{16}{x^2}\right)dx = -\left(\dfrac{16}{4^2}\right)(-1) = 1$



31. $y = f(x) = x^5 \Rightarrow dy = 5x^4\,dx$. When $x = 2$ and $dx = 0.001$, $dy = 5(2)^4(0.001) = 0.08$, so $(2.001)^5 = f(2.001) \approx f(2) + dy = 32 + 0.08 = 32.08$.

32. $y = f(x) = \sqrt{x} \Rightarrow dy = \dfrac{1}{2\sqrt{x}}\,dx$. When $x = 100$ and $dx = -0.2$, $dy = \dfrac{1}{2\sqrt{100}}(-0.2) = -0.01$, so $\sqrt{99.8} = f(99.8) \approx f(100) + dy = 10 - 0.01 = 9.99$.

33. $y = f(x) = x^{2/3} \Rightarrow dy = \dfrac{2}{3\sqrt[3]{x}}\,dx$. When $x = 8$ and $dx = 0.06$, $dy = \dfrac{2}{3\sqrt[3]{8}}(0.06) = 0.02$, so $(8.06)^{2/3} = f(8.06) \approx f(8) + dy = 4 + 0.02 = 4.02$.

34. $y = f(x) = 1/x \Rightarrow dy = (-1/x^2)\,dx$. When $x = 1000$ and $dx = 2$, $dy = [-1/(1000)^2](2) = -0.000\,002$, so $1/1002 = f(1002) \approx f(1000) + dy = 1/1000 - 0.000\,002 = 0.000\,998$

35. $y = f(x) = \tan x \Rightarrow dy = \sec^2 x\,dx$. When $x = 45°$ and $dx = -1°$,

$dy = \sec^2 45°(-\pi/180) = \left(\sqrt{2}\right)^2(-\pi/180) = -\pi/90$, so

$\tan 44° = f(44°) \approx f(45°) + dy = 1 - \pi/90 \approx 0.965.$

36. $y = f(x) = \cos x \Rightarrow dy = -\sin x\,dx$. When $x = \frac{\pi}{6}$ and $dx = \frac{1.5}{180}\pi$,

$dy = -\sin\frac{\pi}{6}\left(\frac{1.5}{180}\pi\right) = -\frac{1}{2}\left(\frac{\pi}{120}\right) = -\frac{\pi}{240}$, so $\cos 31.5° = f\left(\frac{31.5}{180}\pi\right) \approx f\left(\frac{\pi}{6}\right) + dy = \frac{\sqrt{3}}{2} - \frac{\pi}{240} \approx 0.853.$

37. $y = f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$, so $f(0) = 1$ and $f'(0) = 1 \cdot 0 = 0$. The linear approximation of f at 0 is $f(0) + f'(0)(x - 0) = 1 + 0(x) = 1$. Since 0.08 is close to 0, approximating sec 0.08 with 1 is reasonable.

38. If $y = x^6$, $y' = 6x^5$ and the tangent line approximation at $(1, 1)$ has slope 6. If the change in x is 0.01, the change in y on the tangent line is 0.06, and approximating $(1.01)^6$ with 1.06 is reasonable.

39. (a) If x is the edge length, then $V = x^3 \Rightarrow dV = 3x^2\,dx$. When $x = 30$ and $dx = 0.1$, $dV = 3(30)^2(0.1) = 270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3. The relative error is calculated by dividing the change in V, ΔV, by V. We approximate ΔV with dV.

$$\text{Relative error} = \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{3x^2\,dx}{x^3} = 3\frac{dx}{x} = 3\left(\frac{0.1}{30}\right) = 0.01.$$

Percentage error = relative error × 100% = 0.01 × 100% = 1%.

(b) $S = 6x^2 \Rightarrow dS = 12x\,dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2.

$$\text{Relative error} = \frac{\Delta S}{S} \approx \frac{dS}{S} = \frac{12x\,dx}{6x^2} = 2\frac{dx}{x} = 2\left(\frac{0.1}{30}\right) = 0.00\overline{6}.$$

Percentage error = relative error × 100% = $0.00\overline{6}$ × 100% = $0.\overline{6}$%.

40. (a) $A = \pi r^2 \Rightarrow dA = 2\pi r\,dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30\text{ cm}^2$.

(b) $$\text{Relative error} = \frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r\,dr}{\pi r^2} = \frac{2\,dr}{r} = \frac{2(0.2)}{24} = \frac{0.2}{12} = \frac{1}{60} = 0.01\overline{6}.$$

Percentage error = relative error × 100% = $0.01\overline{6}$ × 100% = $1.\overline{6}$%.

41. (a) For a sphere of radius r, the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so $r = C/(2\pi) \Rightarrow S = 4\pi(C/2\pi)^2 = C^2/\pi \Rightarrow dS = (2/\pi)C\,dC$. When $C = 84$ and $dC = 0.5$, $dS = \frac{2}{\pi}(84)(0.5) = \frac{84}{\pi}$, so the maximum error is about $\frac{84}{\pi} \approx 27\text{ cm}^2$. Relative error $\approx \frac{dS}{S} = \frac{84/\pi}{84^2/\pi} = \frac{1}{84} \approx 0.012$

(b) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{C}{2\pi}\right)^3 = \frac{C^3}{6\pi^2} \Rightarrow dV = \frac{1}{2\pi^2}C^2\,dC$. When $C = 84$ and $dC = 0.5$, $dV = \frac{1}{2\pi^2}(84)^2(0.5) = \frac{1764}{\pi^2}$, so the maximum error is about $\frac{1764}{\pi^2} \approx 179\text{ cm}^3$. The relative error is approximately $\frac{dV}{V} = \frac{1764/\pi^2}{(84)^3/(6\pi^2)} = \frac{1}{56} \approx 0.018$.

42. For a hemispherical dome, $V = \frac{2}{3}\pi r^3 \Rightarrow dV = 2\pi r^2\,dr$. When $r = \frac{1}{2}(50) = 25$ m and $dr = 0.05\text{ cm} = 0.0005$ m, $dV = 2\pi(25)^2(0.0005) = \frac{5\pi}{8}$, so the amount of paint needed is about $\frac{5\pi}{8} \approx 2\text{ m}^3$.

43. (a) $V = \pi r^2 h \Rightarrow \Delta V \approx dV = 2\pi rh\,dr = 2\pi rh\,\Delta r$

(b) The error is

$$\begin{aligned}\Delta V - dV &= [\pi(r + \Delta r)^2 h - \pi r^2 h] - 2\pi rh\,\Delta r = \pi r^2 h + 2\pi rh\,\Delta r + \pi(\Delta r)^2 h - \pi r^2 h - 2\pi rh\,\Delta r\\ &= \pi(\Delta r)^2 h\end{aligned}$$

44. $F = kR^4 \Rightarrow dF = 4kR^3\,dR \Rightarrow \frac{dF}{F} = \frac{4kR^3\,dR}{kR^4} = 4\left(\frac{dR}{R}\right)$. Thus, the relative change in F is about 4 times the relative change in R. So a 5% increase in the radius corresponds to a 20% increase in blood flow.

45. (a) $dc = \dfrac{dc}{dx}\,dx = 0\,dx = 0$

(b) $d(cu) = \dfrac{d}{dx}(cu)\,dx = c\,\dfrac{du}{dx}\,dx = c\,du$

(c) $d(u+v) = \dfrac{d}{dx}(u+v)\,dx = \left(\dfrac{du}{dx} + \dfrac{dv}{dx}\right)dx = \dfrac{du}{dx}\,dx + \dfrac{dv}{dx}\,dx = du + dv$

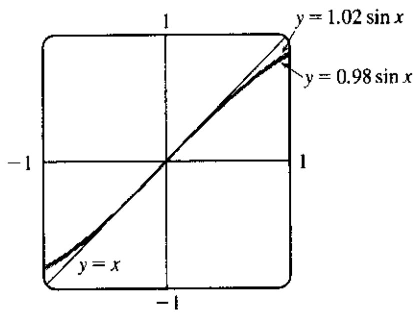
(d) $d(uv) = \dfrac{d}{dx}(uv)\,dx = \left(u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}\right)dx = u\,\dfrac{dv}{dx}\,dx + v\,\dfrac{du}{dx}\,dx = u\,dv + v\,du$

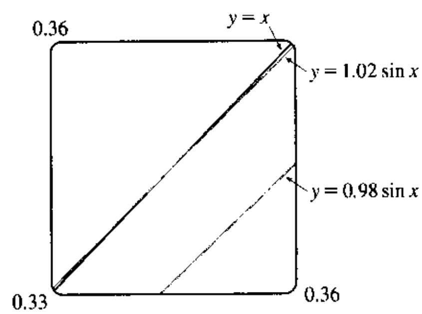
(e) $d\left(\dfrac{u}{v}\right) = \dfrac{d}{dx}\left(\dfrac{u}{v}\right)dx = \dfrac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{v^2}\,dx = \dfrac{v\,\dfrac{du}{dx}\,dx - u\,\dfrac{dv}{dx}\,dx}{v^2} = \dfrac{v\,du - u\,dv}{v^2}$

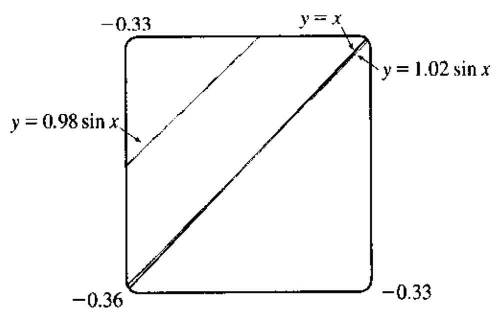
(f) $d\,(x^n) = \dfrac{d}{dx}\,(x^n)\,dx = nx^{n-1}\,dx$

46. (a) $f(x) = \sin x \;\Rightarrow\; f'(x) = \cos x$, so $f(0) = 0$ and $f'(0) = 1$.
Thus, $f(x) \approx f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$.

(b)







We want to know the values of x for which $y = x$ approximates $y = \sin x$ with less than a 2% difference; that is, the values of x for which

$$\left|\frac{x - \sin x}{\sin x}\right| < 0.02 \quad\Leftrightarrow\quad -0.02 < \frac{x - \sin x}{\sin x} < 0.02 \quad\Leftrightarrow$$

$$\begin{cases} -0.02\sin x < x - \sin x < 0.02\sin x & \text{if } \sin x > 0 \\ -0.02\sin x > x - \sin x > 0.02\sin x & \text{if } \sin x < 0 \end{cases} \;\Leftrightarrow\; \begin{cases} 0.98\sin x < x < 1.02\sin x & \text{if } \sin x > 0 \\ 1.02\sin x < x < 0.98\sin x & \text{if } \sin x < 0 \end{cases}$$

In the first figure, we see that the graphs are very close to each other near $x = 0$. Changing the viewing rectangle and using an intersect feature (see the second figure) we find that $y = x$ intersects $y = 1.02\sin x$ at $x \approx 0.344$. By symmetry, they also intersect at $x \approx -0.344$ (see the third figure.). Converting 0.344 radians to degrees, we get $0.344\left(\frac{180^\circ}{\pi}\right) \approx 19.7^\circ \approx 20^\circ$, which verifies the statement.

47. (a) The graph shows that $f'(1) = 2$, so $L(x) = f(1) + f'(1)(x-1) = 5 + 2(x-1) = 2x+3$.
$f(0.9) \approx L(0.9) = 4.8$ and $f(1.1) \approx L(1.1) = 5.2$.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

48. (a) $g'(x) = \sqrt{x^2+5} \;\Rightarrow\; g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95-2) = -4 + 3(-0.05) = -4.15$.
$g(2.05) \approx g(2) + g'(2)(2.05-2) = -4 + 3(0.05) = -3.85$.

(b) The formula $g'(x) = \sqrt{x^2+5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g. Hence, the estimates in part (a) are too small.

LABORATORY PROJECT Taylor Polynomials

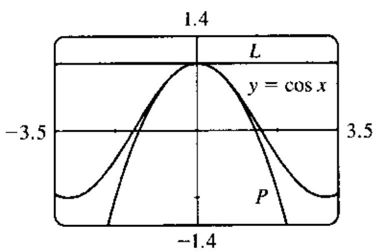
1. We first write the functions described in conditions (i), (ii), and (iii):

$$P(x) = A + Bx + Cx^2 \qquad f(x) = \cos x$$
$$P'(x) = B + 2Cx \qquad f'(x) = -\sin x$$
$$P''(x) = 2C \qquad f''(x) = -\cos x$$

So, taking $a = 0$, our three conditions become

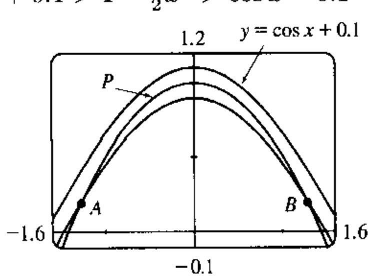
$$P(0) = f(0): \quad A = \cos 0 = 1$$
$$P'(0) = f'(0): \quad B = -\sin 0 = 0$$
$$P''(0) = f''(0): \quad 2C = -\cos 0 = -1 \;\Rightarrow\; C = -\tfrac{1}{2}$$

The desired quadratic function is $P(x) = 1 - \frac{1}{2}x^2$, so the quadratic approximation is $\cos x \approx 1 - \frac{1}{2}x^2$.



The figure shows a graph of the cosine function together with its linear approximation $L(x) = 1$ and quadratic approximation $P(x) = 1 - \frac{1}{2}x^2$ near 0. You can see that the quadratic approximation is much better than the linear one.

2. Accuracy to within 0.1 means that $\left|\cos x - \left(1 - \frac{1}{2}x^2\right)\right| < 0.1 \;\Leftrightarrow$
$-0.1 < \cos x - \left(1 - \frac{1}{2}x^2\right) < 0.1 \;\Leftrightarrow\; 0.1 > \left(1 - \frac{1}{2}x^2\right) - \cos x > -0.1 \;\Leftrightarrow$
$\cos x + 0.1 > 1 - \frac{1}{2}x^2 > \cos x - 0.1 \;\Leftrightarrow\; \cos x - 0.1 < 1 - \frac{1}{2}x^2 < \cos x + 0.1$.



From the figure we see that this is true between A and B. Zooming in or using an intersect feature, we find that the x-coordinates of B and A are about ± 1.26. Thus, the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ is accurate to within 0.1 when $-1.26 < x < 1.26$.

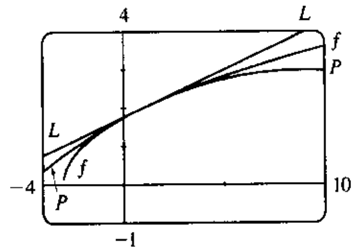
3. If $P(x) = A + B(x-a) + C(x-a)^2$, then $P'(x) = B + 2C(x-a)$ and $P''(x) = 2C$. Applying the conditions (i), (ii), and (iii), we get

$$P(a) = f(a): \quad A = f(a)$$
$$P'(a) = f'(a): \quad B = f'(a)$$
$$P''(a) = f''(a): \quad 2C = f''(a) \quad \Rightarrow \quad C = \tfrac{1}{2}f''(a)$$

Thus, $P(x) = A + B(x-a) + C(x-a)^2$ can be written in the form $P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$.

4. From Example 2 in Section 3.10, we have $f(1) = 2$, $f'(1) = \frac{1}{4}$, and $f'(x) = \frac{1}{2}(x+3)^{-1/2}$. So $f''(x) = -\frac{1}{4}(x+3)^{-3/2} \Rightarrow f''(1) = -\frac{1}{32}$. From Problem 3, the quadratic approximation $P(x)$ is

$$\sqrt{x+3} \approx f(1) + f'(1)(x-1) + \tfrac{1}{2}f''(1)(x-1)^2$$
$$= 2 + \tfrac{1}{4}(x-1) - \tfrac{1}{64}(x-1)^2$$



The figure shows the function $f(x) = \sqrt{x+3}$ together with its linear approximation $L(x) = \frac{1}{4}x + \frac{7}{4}$ and its quadratic approximation $P(x)$. You can see that $P(x)$ is a better approximation than $L(x)$ and this is borne out by the numerical values in the following chart.

	from $L(x)$	actual value	from $P(x)$
$\sqrt{3.98}$	1.9950	1.99499373...	1.99499375
$\sqrt{4.05}$	2.0125	2.01246118...	2.01246094
$\sqrt{4.2}$	2.0500	2.04939015...	2.04937500

5. $T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$. If we put $x = a$ in this equation, then all terms after the first are 0 and we get $T_n(a) = c_0$. Now we differentiate $T_n(x)$ and obtain $T_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1}$. Substituting $x = a$ gives $T_n'(a) = c_1$. Differentiating again, we have $T_n''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a^2) + \cdots + (n-1)nc_n(x-a)^{n-2}$ and so $T_n''(a) = 2c_2$. Continuing in this manner, we get $T_n'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + \cdots + (n-2)(n-1)nc_n(x-a)^{n-3}$ and $T_n'''(a) = 2\cdot 3c_3$. By now we see the pattern. If we continue to differentiate and substitute $x = a$, we obtain $T_n^{(4)}(a) = 2\cdot 3\cdot 4c_4$ and in general, for any integer k between 1 and n,

$$T_n^{(k)}(a) = 2\cdot 3\cdot 4\cdot 5\cdot \cdots \cdot kc_k = k!\,c_k \quad \Rightarrow \quad c_k = \frac{T_n^{(k)}(a)}{k!}$$

Because we want T_n and f to have the same derivatives at a, we require that $c_k = \dfrac{f^{(k)}(a)}{k!}$ for $k = 1, 2, \ldots, n$.

6. $T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$. To compute the coefficients in this equation we need to calculate the derivatives of f at 0:

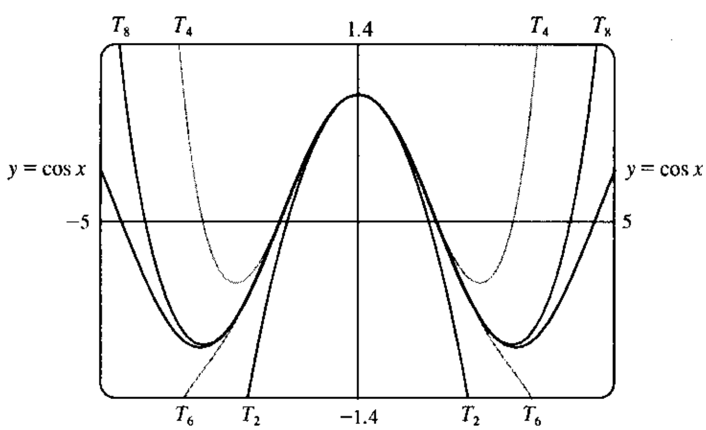
$$\begin{aligned} f(x) &= \cos x & f(0) &= \cos 0 = 1 \\ f'(x) &= -\sin x & f'(0) &= -\sin 0 = 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \end{aligned}$$

We see that the derivatives repeat in a cycle of length 4, so $f^{(5)}(0) = 0$, $f^{(6)}(0) = -1$, $f^{(7)}(0) = 0$, and $f^{(8)}(0) = 1$. From the original expression for $T_n(x)$, with $n = 8$ and $a = 0$, we have

$$\begin{aligned} T_8(x) &= f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(8)}(0)}{8!}(x-0)^8 \\ &= 1 + 0 \cdot x + \frac{-1}{2!}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4 + 0 \cdot x^5 + \frac{-1}{6!}x^6 + 0 \cdot x^7 + \frac{1}{8!}x^8 \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \end{aligned}$$

and the desired approximation is $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$. The Taylor polynomials T_2, T_4, and T_6 consist of the initial terms of T_8 up through degree 2, 4, and 6, respectively. Therefore, $T_2(x) = 1 - \frac{x^2}{2!}$, $T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, and $T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$.

We graph T_2, T_4, T_6, T_8, and f:



Notice that $T_2(x)$ is a good approximation to $\cos x$ near 0, $T_4(x)$ is a good approximation on a larger interval, $T_6(x)$ is a better approximation, and $T_8(x)$ is better still. Each successive Taylor polynomial is a good approximation on a larger interval than the previous one.

3 Review

CONCEPT CHECK

1. See Definition 3.1.2 and the subsequent discussions on the interpretations of the derivative as the slope of a tangent and as a rate of change.

2. (a) A function f is differentiable at a number a if its derivative f' exists at $x = a$; that is, if $f'(a)$ exists.

   (b) See Theorem 3.2.4. This theorem also tells us that if f is *not* continuous at a, then f is *not* differentiable at a.

3. (a) The Power Rule: If n is any real number, then $\frac{d}{dx}(x^n) = nx^{n-1}$. The derivative of a variable base raised to a constant power is the power times the base raised to the power minus one.

   (b) The Constant Multiple Rule: If c is a constant and f is a differentiable function, then $\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$. The derivative of a constant times a function is the constant times the derivative of the function.

   (c) The Sum Rule: If f and g are both differentiable, then $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$. The derivative of a sum of functions is the sum of the derivatives.

   (d) The Difference Rule: If f and g are both differentiable, then $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$. The derivative of a difference of functions is the difference of the derivatives.

   (e) The Product Rule: If f and g are both differentiable, then $\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$. The derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.

   (f) The Quotient Rule: If f and g are both differentiable, then $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$. The derivative of a quotient of functions is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

   (g) The Chain Rule: If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product $F'(x) = f'(g(x))g'(x)$. The derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

4. (a) $y = x^n \Rightarrow y' = nx^{n-1}$
   (b) $y = \sin x \Rightarrow y' = \cos x$
   (c) $y = \cos x \Rightarrow y' = -\sin x$
   (d) $y = \tan x \Rightarrow y' = \sec^2 x$
   (e) $y = \csc x \Rightarrow y' = -\csc x \cot x$
   (f) $y = \sec x \Rightarrow y' = \sec x \tan x$
   (g) $y = \cot x \Rightarrow y' = -\csc^2 x$

5. Implicit differentiation consists of differentiating both sides of an equation involving x and y with respect to x, and then solving the resulting equation for y'.

6. The second derivative of a function f is the rate of change of the first derivative f'. The third derivative is the derivative (rate of change) of the second derivative. If f is the position function of an object, f' is its velocity function, f'' is its acceleration function, and f''' is its jerk function.

7. (a) The linearization L of f at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$.

   (b) If $y = f(x)$, then the differential dy is given by $dy = f'(x)\,dx$.

   (c) See Figure 6 in Section 3.10.

TRUE-FALSE QUIZ

1. False. See the warning after Theorem 3.2.4.

2. True. This is the Sum Rule.

3. False. See the warning before the Product Rule.

4. True. This is the Chain Rule.

5. True by the Chain Rule.

6. False. $\frac{d}{dx} f(\sqrt{x}) = \frac{f'(\sqrt{x})}{2\sqrt{x}}$ by the Chain Rule.

7. False. $f(x) = |x^2 + x| = x^2 + x$ for $x \geq 0$ or $x \leq -1$ and $|x^2 + x| = -(x^2 + x)$ for $-1 < x < 0$. So $f'(x) = 2x + 1$ for $x > 0$ or $x < -1$ and $f'(x) = -(2x + 1)$ for $-1 < x < 0$. But $|2x + 1| = 2x + 1$ for $x \geq -\frac{1}{2}$ and $|2x + 1| = -2x - 1$ for $x < -\frac{1}{2}$.

8. True. $f'(r)$ exists $\Rightarrow$ f is differentiable at r $\Rightarrow$ f is continuous at r $\Rightarrow$ $\lim_{x \to r} f(x) = f(r)$.

9. True. $g(x) = x^5 \Rightarrow g'(x) = 5x^4 \Rightarrow g'(2) = 5(2)^4 = 80$, and by the definition of the derivative, $\lim_{x \to 2} \frac{g(x) - g(2)}{x - 2} = g'(2) = 80$.

10. False. $\frac{d^2y}{dx^2}$ is the second derivative while $\left(\frac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then $\frac{d^2y}{dx^2} = 0$, but $\left(\frac{dy}{dx}\right)^2 = 1$.

11. False. A tangent line to the parabola $y = x^2$ has slope $dy/dx = 2x$, so at $(-2, 4)$ the slope of the tangent is $2(-2) = -4$ and an equation of the tangent line is $y - 4 = -4(x + 2)$. [The given equation, $y - 4 = 2x(x + 2)$, is not even linear!]

12. True. $\frac{d}{dx}(\tan^2 x) = 2 \tan x \sec^2 x$, and $\frac{d}{dx}(\sec^2 x) = 2 \sec x (\sec x \tan x) = 2 \tan x \sec^2 x$.

EXERCISES

1. Estimating the slopes of the tangent lines at $x = 2$, 3, and 5, we obtain approximate values 0.4, 2, and 0.1. The slope of the tangent line at $x = 7$ is negative, so $f'(7) < 0$. Arranging the numbers in increasing order, we have: $f'(7) < 0 < f'(5) < f'(2) < 1 < f'(3)$.

2. $2^6 = 64$, so $f(x) = x^6$ and $a = 2$.

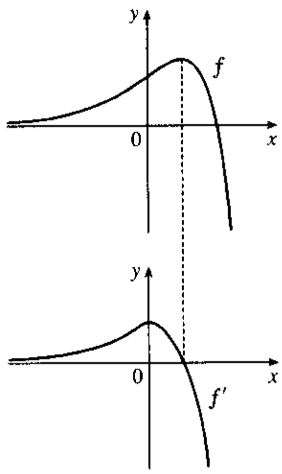
3. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).

   (b) The total cost of paying off the loan is increasing by \$1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

   (c) As r increases, C increases. So $f'(r)$ will always be positive.

For Exercises 4–6, see the hints before Exercise 5 in Section 3.2.

4.



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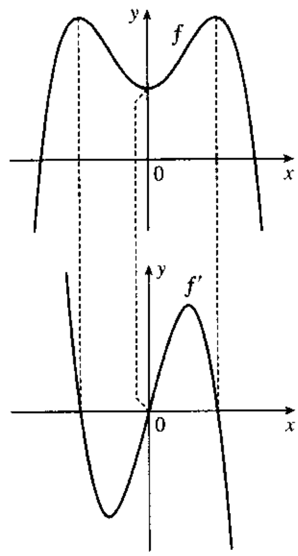

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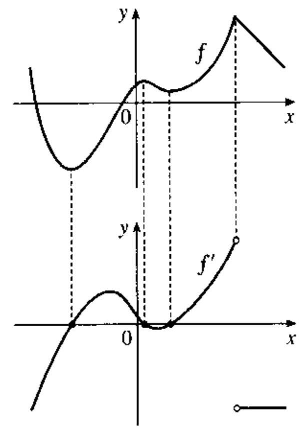


7. The graph of a has tangent lines with positive slope for $x < 0$ and negative slope for $x > 0$, and the values of c fit this pattern, so c must be the graph of the derivative of the function for a. The graph of c has horizontal tangent lines to the left and right of the x-axis and b has zeros at these points. Hence, b is the graph of the derivative of the function for c. Therefore, a is the graph of f, c is the graph of f', and b is the graph of f''.

8. (a) Drawing slope triangles, we obtain the following estimates: $F'(1950) \approx \frac{1.1}{10} = 0.11$,

$F'(1965) \approx \frac{-1.6}{10} = -0.16$, and $F'(1987) \approx \frac{0.2}{10} = 0.02$.

(b) The rate of change of the average number of children born to each woman was increasing by 0.11 in 1950, decreasing by 0.16 in 1965, and increasing by 0.02 in 1987.

(c) There are many possible reasons:

- In the baby-boom era (post-WWII), there was optimism about the economy and family size was rising.
- In the baby-bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.
- In the baby-boomlet era, there was increased economic optimism and a return to more conservative attitudes.

9. $B'(1990)$ is the rate at which the total value of U.S. banknotes in circulation is changing in billions of dollars per year. To estimate the value of $B'(1990)$, we will average the difference quotients obtained using the times $t = 1985$ and $t = 1995$. Let $A = \dfrac{B(1985) - B(1990)}{1985 - 1990} = \dfrac{182.0 - 268.2}{-5} = 17.24$ and

$C = \dfrac{B(1995) - B(1990)}{1995 - 1990} = \dfrac{401.5 - 268.2}{5} = 26.66$. Then

$$B'(1990) = \lim_{t \to 1990} \frac{B(t) - B(1990)}{t - 1990} \approx \frac{A + C}{2} = \frac{17.24 + 26.66}{2} = 21.95 \text{ billions of dollars/year.}$$

10. $f(x) = \dfrac{4-x}{3+x} \quad\Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{4-(x+h)}{3+(x+h)} - \dfrac{4-x}{3+x}}{h}$$
$$= \lim_{h\to 0} \frac{(4-x-h)(3+x)-(4-x)(3+x+h)}{h(3+x+h)(3+x)} = \lim_{h\to 0} \frac{-7h}{h(3+x+h)(3+x)}$$
$$= \lim_{h\to 0} \frac{-7}{(3+x+h)(3+x)} = -\frac{7}{(3+x)^2}$$

11. $f(x) = x^3 + 5x + 4 \quad\Rightarrow$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 + 5(x+h) + 4 - (x^3+5x+4)}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + 5h}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2 + 5) = 3x^2 + 5$$

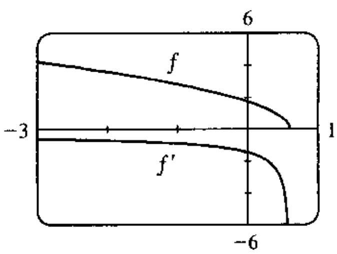
12. (a) $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{3-5(x+h)} - \sqrt{3-5x}}{h} \dfrac{\sqrt{3-5(x+h)} + \sqrt{3-5x}}{\sqrt{3-5(x+h)} + \sqrt{3-5x}}$

$$= \lim_{h\to 0} \frac{[3-5(x+h)]-(3-5x)}{h\left(\sqrt{3-5(x+h)} + \sqrt{3-5x}\right)} = \lim_{h\to 0} \frac{-5}{\sqrt{3-5(x+h)} + \sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}$$

(b) Domain of f: (the radicand must be nonnegative) $3-5x \geq 0 \quad\Rightarrow\quad 5x \leq 3 \quad\Rightarrow\quad x \in \left(-\infty, \frac{3}{5}\right]$

Domain of f': exclude $\frac{3}{5}$ because it makes the denominator zero; $x \in \left(-\infty, \frac{3}{5}\right)$

(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.



13. $y = \left(x^4 - 3x^2 + 5\right)^3 \quad\Rightarrow$

$$y' = 3\left(x^4 - 3x^2 + 5\right)^2 \frac{d}{dx}\left(x^4 - 3x^2 + 5\right) = 3\left(x^4 - 3x^2 + 5\right)^2\left(4x^3 - 6x\right) = 6x\left(x^4 - 3x^2 + 5\right)^2\left(2x^2 - 3\right)$$

14. $y = \cos(\tan x) \quad\Rightarrow\quad y' = -\sin(\tan x)\dfrac{d}{dx}(\tan x) = -\sin(\tan x)(\sec^2 x)$

15. $y = \sqrt{x} + \dfrac{1}{\sqrt[3]{x^4}} = x^{1/2} + x^{-4/3} \quad\Rightarrow\quad y' = \frac{1}{2}x^{-1/2} - \frac{4}{3}x^{-7/3} = \dfrac{1}{2\sqrt{x}} - \dfrac{4}{3\sqrt[3]{x^7}}$

16. $y = \dfrac{3x-2}{\sqrt{2x+1}} \quad\Rightarrow$

$$y' = \frac{\sqrt{2x+1}\,(3) - (3x-2)\frac{1}{2}(2x+1)^{-1/2}(2)}{\left(\sqrt{2x+1}\right)^2} \cdot \frac{(2x+1)^{1/2}}{(2x+1)^{1/2}} = \frac{3(2x+1)-(3x-2)}{(2x+1)^{3/2}} = \frac{3x+5}{(2x+1)^{3/2}}$$

17. $y = 2x\sqrt{x^2+1} \quad\Rightarrow$

$$y' = 2x \cdot \tfrac{1}{2}\left(x^2+1\right)^{-1/2}(2x) + \sqrt{x^2+1}\,(2) = \frac{2x^2}{\sqrt{x^2+1}} + 2\sqrt{x^2+1} = \frac{2x^2 + 2\left(x^2+1\right)}{\sqrt{x^2+1}} = \frac{2\left(2x^2+1\right)}{\sqrt{x^2+1}}$$

18. $y = (x + 1/x^2)^{\sqrt{7}} \quad \Rightarrow \quad y' = \sqrt{7}\,(x + 1/x^2)^{\sqrt{7}-1}(1 - 2/x^3)$

19. $y = \dfrac{t}{1-t^2} \quad \Rightarrow \quad y' = \dfrac{(1-t^2)(1) - t(-2t)}{(1-t^2)^2} = \dfrac{1 - t^2 + 2t^2}{(1-t^2)^2} = \dfrac{t^2+1}{(1-t^2)^2}$

20. $y = \sin(\cos x) \quad \Rightarrow \quad y' = \cos(\cos x)(-\sin x) = -\sin x \cos(\cos x)$

21. $y = \tan\sqrt{1-x} \quad \Rightarrow \quad y' = \left(\sec^2\sqrt{1-x}\right)\left(\dfrac{1}{2\sqrt{1-x}}\right)(-1) = -\dfrac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}$

22. Using the Reciprocal Rule, $g(x) = \dfrac{1}{f(x)} \quad \Rightarrow \quad g'(x) = -\dfrac{f'(x)}{[f(x)]^2}$, we have $y = \dfrac{1}{\sin(x - \sin x)} \quad \Rightarrow$

$y' = -\dfrac{\cos(x - \sin x)(1 - \cos x)}{\sin^2(x - \sin x)}$.

23. $\dfrac{d}{dx}(xy^4 + x^2y) = \dfrac{d}{dx}(x + 3y) \quad \Rightarrow \quad x \cdot 4y^3y' + y^4 \cdot 1 + x^2 \cdot y' + y \cdot 2x = 1 + 3y' \quad \Rightarrow$

$y'(4xy^3 + x^2 - 3) = 1 - y^4 - 2xy \quad \Rightarrow \quad y' = \dfrac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$

24. $y = \sec(1 + x^2) \quad \Rightarrow \quad y' = 2x\sec(1 + x^2)\tan(1 + x^2)$

25. $y = \dfrac{\sec 2\theta}{1 + \tan 2\theta} \quad \Rightarrow$

$$y' = \frac{(1 + \tan 2\theta)(\sec 2\theta\,\tan 2\theta \cdot 2) - (\sec 2\theta)(\sec^2 2\theta \cdot 2)}{(1 + \tan 2\theta)^2} = \frac{2\sec 2\theta\left[(1 + \tan 2\theta)\tan 2\theta - \sec^2 2\theta\right]}{(1 + \tan 2\theta)^2}$$

$$= \frac{2\sec 2\theta\,(\tan 2\theta + \tan^2 2\theta - \sec^2 2\theta)}{(1 + \tan 2\theta)^2} = \frac{2\sec 2\theta\,(\tan 2\theta - 1)}{(1 + \tan 2\theta)^2} \qquad [1 + \tan^2 x = \sec^2 x]$$

26. $\dfrac{d}{dx}(x^2\cos y + \sin 2y) = \dfrac{d}{dx}(xy) \quad \Rightarrow \quad x^2(-\sin y \cdot y') + (\cos y)(2x) + \cos 2y \cdot 2y' = x \cdot y' + y \cdot 1 \quad \Rightarrow$

$y'(-x^2\sin y + 2\cos 2y - x) = y - 2x\cos y \quad \Rightarrow \quad y' = \dfrac{y - 2x\cos y}{2\cos 2y - x^2\sin y - x}$

27. $y = (1 - x^{-1})^{-1} \quad \Rightarrow$

$y' = -1(1 - x^{-1})^{-2}[-(-1x^{-2})] = -(1 - 1/x)^{-2}x^{-2} = -((x-1)/x)^{-2}x^{-2} = -(x-1)^{-2}$

28. $y = (x + \sqrt{x}\,)^{-1/3} \quad \Rightarrow \quad y' = -\frac{1}{3}(x + \sqrt{x}\,)^{-4/3}\left(1 + \dfrac{1}{2\sqrt{x}}\right)$

29. $\sin(xy) = x^2 - y \quad \Rightarrow \quad \cos(xy)(xy' + y \cdot 1) = 2x - y' \quad \Rightarrow \quad x\cos(xy)y' + y' = 2x - y\cos(xy) \quad \Rightarrow$

$y'[x\cos(xy) + 1] = 2x - y\cos(xy) \quad \Rightarrow \quad y' = \dfrac{2x - y\cos(xy)}{x\cos(xy) + 1}$

30. $y = \sqrt{\sin\sqrt{x}} \quad \Rightarrow \quad y' = \frac{1}{2}(\sin\sqrt{x}\,)^{-1/2}(\cos\sqrt{x})\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\cos\sqrt{x}}{4\sqrt{x\sin\sqrt{x}}}$

31. $y = \cot(3x^2 + 5) \quad \Rightarrow \quad y' = -\csc^2(3x^2 + 5)(6x) = -6x\csc^2(3x^2 + 5)$

32. $y = \dfrac{(x + \lambda)^4}{x^4 + \lambda^4} \quad \Rightarrow \quad y' = \dfrac{(x^4 + \lambda^4)(4)(x + \lambda)^3 - (x + \lambda)^4(4x^3)}{(x^4 + \lambda^4)^2} = \dfrac{4(x + \lambda)^3(\lambda^4 - \lambda x^3)}{(x^4 + \lambda^4)^2}$

33. $y = \sin(\tan\sqrt{1 + x^3}\,) \quad \Rightarrow \quad y' = \cos(\tan\sqrt{1 + x^3}\,)(\sec^2\sqrt{1 + x^3}\,)\left[3x^2/(2\sqrt{1 + x^3}\,)\right]$

34. $y = (\sin mx)/x \;\Rightarrow\; y' = (mx\cos mx - \sin mx)/x^2$

35. $y = \tan^2(\sin\theta) = [\tan(\sin\theta)]^2 \;\Rightarrow\; y' = 2[\tan(\sin\theta)]\cdot\sec^2(\sin\theta)\cdot\cos\theta$

36. $x\tan y = y - 1 \;\Rightarrow\; \tan y + (x\sec^2 y)\,y' = y' \;\Rightarrow\; y' = \dfrac{\tan y}{1 - x\sec^2 y}$

37. $y = (x\tan x)^{1/5} \;\Rightarrow\; y' = \frac{1}{5}(x\tan x)^{-4/5}(\tan x + x\sec^2 x)$

38. $y = \dfrac{(x-1)(x-4)}{(x-2)(x-3)} = \dfrac{x^2-5x+4}{x^2-5x+6} \;\Rightarrow$

$$y' = \frac{(x^2-5x+6)(2x-5)-(x^2-5x+4)(2x-5)}{(x^2-5x+6)^2} = \frac{2(2x-5)}{(x-2)^2(x-3)^2}$$

39. $f(t) = \sqrt{4t+1} \;\Rightarrow\; f'(t) = \frac{1}{2}(4t+1)^{-1/2}\cdot 4 = 2(4t+1)^{-1/2} \;\Rightarrow$

$f''(t) = 2(-\frac{1}{2})(4t+1)^{-3/2}\cdot 4 = -4/(4t+1)^{3/2}$, so $f''(2) = -4/9^{3/2} = -\frac{4}{27}$.

40. $g(\theta) = \theta\sin\theta \;\Rightarrow\; g'(\theta) = \theta\cos\theta + \sin\theta\cdot 1 \;\Rightarrow\; g''(\theta) = \theta(-\sin\theta) + \cos\theta\cdot 1 + \cos\theta = 2\cos\theta - \theta\sin\theta$,

so $g''(\pi/6) = 2\cos(\pi/6) - (\pi/6)\sin(\pi/6) = 2\left(\sqrt{3}/2\right) - (\pi/6)(1/2) = \sqrt{3} - \pi/12$.

41. $x^6 + y^6 = 1 \;\Rightarrow\; 6x^5 + 6y^5y' = 0 \;\Rightarrow\; y' = -x^5/y^5 \;\Rightarrow$

$$y'' = -\frac{y^5(5x^4) - x^5(5y^4y')}{(y^5)^2} = -\frac{5x^4y^4\left[y - x(-x^5/y^5)\right]}{y^{10}} = -\frac{5x^4\left[(y^6+x^6)/y^5\right]}{y^6} = -\frac{5x^4}{y^{11}}$$

42. $f(x) = (2-x)^{-1} \;\Rightarrow\; f'(x) = (2-x)^{-2} \;\Rightarrow\; f''(x) = 2(2-x)^{-3} \;\Rightarrow\; f'''(x) = 2\cdot 3(2-x)^{-4} \;\Rightarrow$

$f^{(4)}(x) = 2\cdot 3\cdot 4(2-x)^{-5}$. In general, $f^{(n)}(x) = 2\cdot 3\cdot 4\cdot\cdots\cdot n(2-x)^{-(n+1)} = \dfrac{n!}{(2-x)^{(n+1)}}$.

43. $\displaystyle\lim_{x\to 0}\frac{\sec x}{1-\sin x} = \frac{\sec 0}{1-\sin 0} = \frac{1}{1-0} = 1$

44. $\displaystyle\lim_{t\to 0}\frac{t^3}{\tan^3 2t} = \lim_{t\to 0}\frac{t^3\cos^3 2t}{\sin^3 2t} = \lim_{t\to 0}\cos^3 2t\cdot\frac{1}{8\dfrac{\sin^3 2t}{(2t)^3}} = \lim_{t\to 0}\frac{\cos^3 2t}{8\left(\lim\limits_{t\to 0}\dfrac{\sin 2t}{2t}\right)^3} = \frac{1}{8\cdot 1^3} = \frac{1}{8}$

45. $y = 4\sin^2 x \;\Rightarrow\; y' = 4\cdot 2\sin x\cos x$. At $\left(\frac{\pi}{6}, 1\right)$, $y' = 8\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$, so an equation of the tangent line is

$y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{6}\right)$, or $y = 2\sqrt{3}x + 1 - \pi\sqrt{3}/3$.

46. $y = \dfrac{x^2-1}{x^2+1} \;\Rightarrow\; y' = \dfrac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}$. At $(0,-1)$, $y' = 0$, so an equation of the

tangent line is $y + 1 = 0(x - 0)$, or $y = -1$.

47. $y = \sqrt{1+4\sin x} \;\Rightarrow\; y' = \frac{1}{2}(1+4\sin x)^{-1/2}\cdot 4\cos x = \dfrac{2\cos x}{\sqrt{1+4\sin x}}$. At $(0,1)$, $y' = \dfrac{2}{\sqrt{1}} = 2$, so an

equation of the tangent line is $y - 1 = 2(x - 0)$, or $y = 2x + 1$.

48. $x^2 + 4xy + y^2 = 13 \;\Rightarrow\; 2x + 4(xy' + y\cdot 1) + 2yy' = 0 \;\Rightarrow\; x + 2xy' + 2y + yy' = 0 \;\Rightarrow$

$2xy' + yy' = -x - 2y \;\Rightarrow\; y'(2x+y) = -x - 2y \;\Rightarrow\; y' = \dfrac{-x-2y}{2x+y}$. At $(2,1)$, $y' = \dfrac{-2-2}{4+1} = -\dfrac{4}{5}$, so

an equation of the tangent line is $y - 1 = -\frac{4}{5}(x - 2)$, or $y = -\frac{4}{5}x + \frac{13}{5}$.

49. (a) $f(x) = x\sqrt{5-x} \quad \Rightarrow$

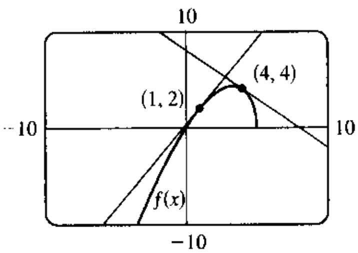
$$f'(x) = x\left[\tfrac{1}{2}(5-x)^{-1/2}(-1)\right] + \sqrt{5-x} = \frac{-x}{2\sqrt{5-x}} + \sqrt{5-x}\cdot\frac{2\sqrt{5-x}}{2\sqrt{5-x}}$$

$$= \frac{-x}{2\sqrt{5-x}} + \frac{2(5-x)}{2\sqrt{5-x}} = \frac{-x+10-2x}{2\sqrt{5-x}} = \frac{10-3x}{2\sqrt{5-x}}$$

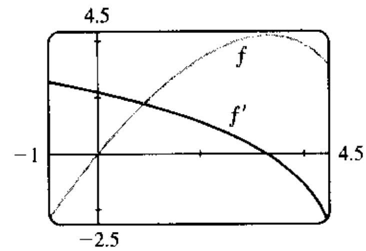
(b) At $(1,2)$: $f'(1) = \frac{7}{4}$. So an equation of the tangent line is $y - 2 = \frac{7}{4}(x-1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.

At $(4,4)$: $f'(4) = -\frac{2}{2} = -1$. So an equation of the tangent line is $y - 4 = -1(x-4)$ or $y = -x + 8$.

(c)



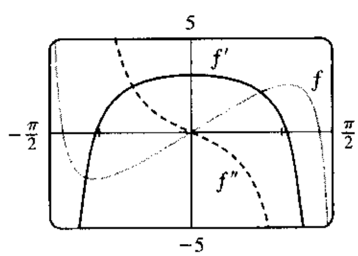
(d)



The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

50. (a) $f(x) = 4x - \tan x \quad \Rightarrow \quad f'(x) = 4 - \sec^2 x \quad \Rightarrow \quad f''(x) = -2\sec x\,(\sec x\,\tan x) = -2\sec^2 x\,\tan x$.

(b)



We can see that our answers are reasonable, since the graph of f' is 0 where f has a horizontal tangent, and the graph of f' is positive where f has tangents with positive slope and negative where f has tangents with negative slope. The same correspondence holds between the graphs of f' and f''.

51. $y = \sin x + \cos x \quad \Rightarrow \quad y' = \cos x - \sin x = 0 \quad \Leftrightarrow \quad \cos x = \sin x$ and $0 \le x \le 2\pi \quad \Leftrightarrow \quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$, so the points are $\left(\frac{\pi}{4}, \sqrt{2}\right)$ and $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

52. $x^2 + 2y^2 = 1 \quad \Rightarrow \quad 2x + 4yy' = 0 \quad \Rightarrow \quad y' = -x/(2y) = 1 \quad \Leftrightarrow \quad x = -2y$. Since the points lie on the ellipse, we have $(-2y)^2 + 2y^2 = 1 \quad \Rightarrow \quad 6y^2 = 1 \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{6}}$. The points are $\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$.

53. $f(x) = (x-a)(x-b)(x-c) \quad \Rightarrow \quad f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$. So

$$\frac{f'(x)}{f(x)} = \frac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}.$$

54. (a) $\cos 2x = \cos^2 x - \sin^2 x \quad \Rightarrow \quad -2\sin 2x = -2\cos x\sin x - 2\sin x\cos x \quad \Leftrightarrow \quad \sin 2x = 2\sin x\cos x$

(b) $\sin(x+a) = \sin x\cos a + \cos x\sin a \quad \Rightarrow \quad \cos(x+a) = \cos x\cos a - \sin x\sin a$.

55. (a) $h(x) = f(x)g(x) \quad \Rightarrow \quad h'(x) = f(x)g'(x) + g(x)f'(x) \quad \Rightarrow$

$h'(2) = f(2)g'(2) + g(2)f'(2) = (3)(4) + (5)(-2) = 12 - 10 = 2$

(b) $F(x) = f(g(x)) \quad \Rightarrow \quad F'(x) = f'(g(x))g'(x) \quad \Rightarrow \quad F'(2) = f'(g(2))g'(2) = f'(5)(4) = 11 \cdot 4 = 44$

56. (a) $P(x) = f(x)g(x) \quad \Rightarrow \quad P'(x) = f(x)g'(x) + g(x)f'(x) \quad \Rightarrow$

$$P'(2) = f(2)g'(2) + g(2)f'(2) = (1)\left(\tfrac{6-0}{3-0}\right) + (4)\left(\tfrac{0-3}{3-0}\right) = (1)(2) + (4)(-1) = 2 - 4 = -2$$

(b) $Q(x) = \dfrac{f(x)}{g(x)} \quad \Rightarrow \quad Q'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \quad \Rightarrow$

$$Q'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(4)(-1) - (1)(2)}{4^2} = \frac{-6}{16} = -\frac{3}{8}$$

(c) $C(x) = f(g(x)) \quad \Rightarrow \quad C'(x) = f'(g(x))g'(x) \quad \Rightarrow$

$$C'(2) = f'(g(2))g'(2) = f'(4)g'(2) = \left(\tfrac{6-0}{5-3}\right)(2) = (3)(2) = 6$$

57. $f(x) = x^2 g(x) \quad \Rightarrow \quad f'(x) = x^2 g'(x) + g(x)(2x) = x\,[xg'(x) + 2g(x)]$

58. $f(x) = g(x^2) \quad \Rightarrow \quad f'(x) = g'(x^2)(2x) = 2xg'(x^2)$

59. $f(x) = [g(x)]^2 \quad \Rightarrow \quad f'(x) = 2[g(x)]^1 \cdot g'(x) = 2g(x)g'(x)$

60. $f(x) = x^a g(x^b) \quad \Rightarrow \quad f'(x) = ax^{a-1}g(x^b) + x^a g'(x^b)(bx^{b-1}) = ax^{a-1}g(x^b) + bx^{a+b-1}g'(x^b)$

61. $f(x) = g(g(x)) \quad \Rightarrow \quad f'(x) = g'(g(x))g'(x)$

62. $f(x) = \sin(g(x)) \quad \Rightarrow \quad f'(x) = \cos(g(x)) \cdot g'(x)$

63. $f(x) = g(\sin x) \quad \Rightarrow \quad f'(x) = g'(\sin x) \cdot \cos x$

64. $f(x) = g(\tan\sqrt{x}) \quad \Rightarrow$

$$f'(x) = g'(\tan\sqrt{x}) \cdot \frac{d}{dx}(\tan\sqrt{x}) = g'(\tan\sqrt{x}) \cdot \sec^2\sqrt{x} \cdot \frac{d}{dx}(\sqrt{x}) = \frac{g'(\tan\sqrt{x})\sec^2\sqrt{x}}{2\sqrt{x}}$$

65. $h(x) = \dfrac{f(x)g(x)}{f(x) + g(x)} \quad \Rightarrow$

$$h'(x) = \frac{[f(x) + g(x)]\,[f(x)g'(x) + g(x)f'(x)] - f(x)g(x)\,[f'(x) + g'(x)]}{[f(x) + g(x)]^2}$$

$$= \frac{[f(x)]^2 g'(x) + f(x)g(x)f'(x) + f(x)g(x)g'(x) + [g(x)]^2 f'(x) - f(x)g(x)f'(x) - f(x)g(x)g'(x)}{[f(x) + g(x)]^2}$$

$$= \frac{f'(x)[g(x)]^2 + g'(x)[f(x)]^2}{[f(x) + g(x)]^2}$$

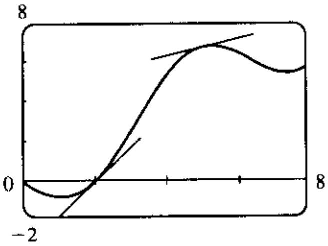
66. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}} \quad \Rightarrow \quad h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{2\sqrt{f(x)/g(x)}\,[g(x)]^2} = \dfrac{f'(x)g(x) - f(x)g'(x)}{2[g(x)]^{3/2}\sqrt{f(x)}}$

67. Using the Chain Rule repeatedly, $h(x) = f(g(\sin 4x)) \quad \Rightarrow$

$$h'(x) = f'(g(\sin 4x)) \cdot \frac{d}{dx}(g(\sin 4x)) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \frac{d}{dx}(\sin 4x)$$

$$= f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4)$$

68. (a)



(b) The average rate of change is larger on $[2, 3]$.

(c) The instantaneous rate of change (the slope of the tangent) is larger at $x = 2$.

(d) $f(x) = x - 2\sin x \quad \Rightarrow \quad f'(x) = 1 - 2\cos x$, so
$f'(2) = 1 - 2\cos 2 \approx 1.8323$ and
$f'(5) = 1 - 2\cos 5 \approx 0.4327$.
So $f'(2) > f'(5)$, as predicted in part (c).

69. f is not differentiable: at $x = -4$ because f is not continuous, at $x = -1$ because f has a corner, at $x = 2$ because f is not continuous, and at $x = 5$ because f has a vertical tangent.

70. (a) $x = \sqrt{b^2 + c^2t^2} \quad \Rightarrow \quad v(t) = x' = \left[1/\left(2\sqrt{b^2 + c^2t^2}\right)\right] 2c^2t = c^2t/\sqrt{b^2 + c^2t^2} \quad \Rightarrow$

$$a(t) = v'(t) = \frac{c^2\sqrt{b^2 + c^2t^2} - c^2t\left(c^2t/\sqrt{b^2 + c^2t^2}\right)}{b^2 + c^2t^2} = \frac{b^2c^2}{(b^2 + c^2t^2)^{3/2}}$$

(b) $v(t) > 0$ for $t > 0$, so the particle always moves in the positive direction.

71. (a) $y = t^3 - 12t + 3 \quad \Rightarrow \quad v(t) = y' = 3t^2 - 12 \quad \Rightarrow \quad a(t) = v'(t) = 6t$

(b) $v(t) = 3(t^2 - 4) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \le t < 2$.

(c) Distance upward $= y(3) - y(2) = -6 - (-13) = 7$,
Distance downward $= y(0) - y(2) = 3 - (-13) = 16$. Total distance $= 7 + 16 = 23$.

72. (a) $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad dV/dh = \frac{1}{3}\pi r^2 \quad [r \text{ constant}]$

(b) $V = \frac{1}{3}\pi r^2 h \quad \Rightarrow \quad dV/dr = \frac{2}{3}\pi r h \quad [h \text{ constant}]$

73. The linear density ρ is the rate of change of mass m with respect to length x. $m = x(1 + \sqrt{x}) = x + x^{3/2} \quad \Rightarrow$ $\rho = dm/dx = 1 + \frac{3}{2}\sqrt{x}$, so the linear density when $x = 4$ is $1 + \frac{3}{2}\sqrt{4} = 4$ kg/m.

74. (a) $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3 \quad \Rightarrow \quad C'(x) = 2 - 0.04x + 0.00021x^2$

(b) $C'(100) = 2 - 4 + 2.1 = \$0.10$/unit. This value represents the rate at which costs are increasing as the hundredth unit is produced, and is the approximate cost of producing the 101st unit.

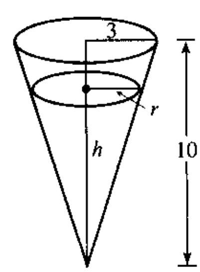
(c) The cost of producing the 101st item is $C(101) - C(100) = 990.10107 - 990 = \$0.10107$, slightly larger than $C'(100)$.

75. If $x =$ edge length, then $V = x^3 \quad \Rightarrow \quad dV/dt = 3x^2\,dx/dt = 10 \quad \Rightarrow \quad dx/dt = 10/(3x^2)$ and $S = 6x^2 \quad \Rightarrow$ $dS/dt = (12x)\,dx/dt = 12x[10/(3x^2)] = 40/x$. When $x = 30$, $dS/dt = \frac{40}{30} = \frac{4}{3}$ cm²/min.

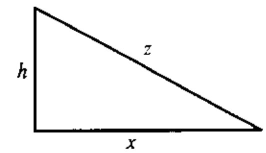
76. Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3}\pi r^2 h$ and, from similar triangles, $\dfrac{r}{h} = \dfrac{3}{10} \quad \Rightarrow \quad V = \dfrac{\pi}{3}\left(\dfrac{3h}{10}\right)^2 h = \dfrac{3\pi}{100}h^3$, so

$$2 = \frac{dV}{dt} = \frac{9\pi}{100}h^2\frac{dh}{dt} \quad \Rightarrow \quad \frac{dh}{dt} = \frac{200}{9\pi h^2} = \frac{200}{9\pi\,(5)^2} = \frac{8}{9\pi} \text{ cm/s}$$

when $h = 5$.



77. Given $dh/dt = 5$ and $dx/dt = 15$, find dz/dt. $z^2 = x^2 + h^2 \;\Rightarrow$

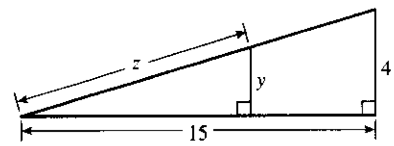


$2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2h\dfrac{dh}{dt} \;\Rightarrow\; \dfrac{dz}{dt} = \dfrac{1}{z}(15x + 5h)$. When $t = 3$,

$h = 45 + 3(5) = 60$ and $x = 15(3) = 45 \;\Rightarrow\; z = \sqrt{45^2 + 60^2} = 75$, so

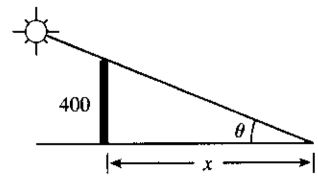
$\dfrac{dz}{dt} = \frac{1}{75}\left[15(45) + 5(60)\right] = 13$ ft/s.

78. We are given $dz/dt = 30$ ft/s. By similar triangles, $\dfrac{y}{z} = \dfrac{4}{\sqrt{241}}$



$\Rightarrow\; y = \dfrac{4}{\sqrt{241}}z$, so $\dfrac{dy}{dt} = \dfrac{4}{\sqrt{241}}\dfrac{dz}{dt} = \dfrac{120}{\sqrt{241}} \approx 7.7$ ft/s.

79. We are given $d\theta/dt = -0.25$ rad/h. $\tan\theta = 400/x \;\Rightarrow$



$x = 400\cot\theta \;\Rightarrow\; \dfrac{dx}{dt} = -400\csc^2\theta\,\dfrac{d\theta}{dt}$. When $\theta = \frac{\pi}{6}$,

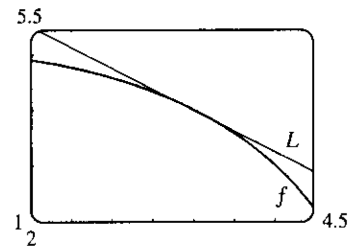
$\dfrac{dx}{dt} = -400(2)^2(-0.25) = 400$ ft/h.

80. (a) $f(x) = \sqrt{25 - x^2} \;\Rightarrow$

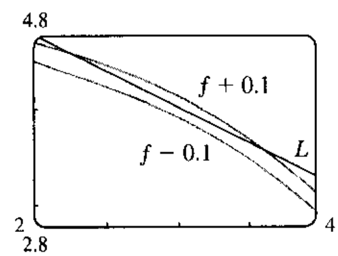
$f'(x) = \dfrac{-2x}{2\sqrt{25 - x^2}} = -x(25 - x^2)^{-1/2}$. So the linear approximation to $f(x)$ near 3 is

$f(x) \approx f(3) + f'(3)(x - 3) = 4 - \frac{3}{4}(x - 3)$.

(b)

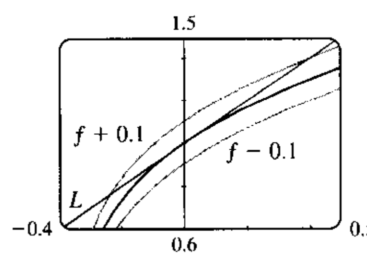


(c) For the required accuracy, we want $\sqrt{25 - x^2} - 0.1 < 4 - \frac{3}{4}(x - 3)$ and $4 - \frac{3}{4}(x - 3) < \sqrt{25 - x^2} + 0.1$. From the graph, it appears that these both hold for $2.24 < x < 3.66$.



81. (a) $f(x) = \sqrt[3]{1 + 3x} = (1 + 3x)^{1/3} \;\Rightarrow\; f'(x) = (1 + 3x)^{-2/3}$, so the linearization of f at $a = 0$ is $L(x) = f(0) + f'(0)(x - 0) = 1^{1/3} + 1^{-2/3}x = 1 + x$. Thus, $\sqrt[3]{1 + 3x} \approx 1 + x \;\Rightarrow$ $\sqrt[3]{1.03} = \sqrt[3]{1 + 3(0.01)} \approx 1 + (0.01) = 1.01$.

(b) The linear approximation is $\sqrt[3]{1 + 3x} \approx 1 + x$, so for the required accuracy we want $\sqrt[3]{1 + 3x} - 0.1 < 1 + x < \sqrt[3]{1 + 3x} + 0.1$. From the graph, it appears that this is true when $-0.23 < x < 0.40$.

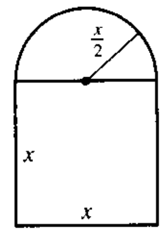


82. $y = x^3 - 2x^2 + 1 \;\Rightarrow\; dy = (3x^2 - 4x)\,dx$. When $x = 2$ and $dx = 0.2$, $dy = \left[3(2)^2 - 4(2)\right](0.2) = 0.8$.

83. $A = x^2 + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \left(1 + \frac{\pi}{8}\right)x^2 \;\Rightarrow\; dA = \left(2 + \frac{\pi}{4}\right)x\,dx$.

When $x = 60$ and $dx = 0.1$, $dA = \left(2 + \frac{\pi}{4}\right)60(0.1) = 12 + \frac{3\pi}{2}$, so

the maximum error is approximately $12 + \frac{3\pi}{2} \approx 16.7\text{ cm}^2$.



84. $\displaystyle \lim_{x\to 1}\frac{x^{17}-1}{x-1} = \left[\frac{d}{dx}x^{17}\right]_{x=1} = 17(1)^{16} = 17$

85. $\displaystyle \lim_{h\to 0}\frac{\sqrt[4]{16+h}-2}{h} = \left[\frac{d}{dx}\sqrt[4]{x}\right]_{x=16} = \tfrac{1}{4}x^{-3/4}\Big|_{x=16} = \frac{1}{4\left(\sqrt[4]{16}\right)^3} = \frac{1}{32}$

86. $\displaystyle \lim_{\theta\to\pi/3}\frac{\cos\theta - 0.5}{\theta - \pi/3} = \left[\frac{d}{d\theta}\cos\theta\right]_{\theta=\pi/3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

87.
$$\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3} = \lim_{x\to 0}\frac{\left(\sqrt{1+\tan x}-\sqrt{1+\sin x}\right)\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}$$
$$= \lim_{x\to 0}\frac{(1+\tan x)-(1+\sin x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)} = \lim_{x\to 0}\frac{\sin x\,(1/\cos x - 1)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)}\cdot\frac{\cos x}{\cos x}$$
$$= \lim_{x\to 0}\frac{\sin x\,(1-\cos x)}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x}\cdot\frac{1+\cos x}{1+\cos x}$$
$$= \lim_{x\to 0}\frac{\sin x\cdot\sin^2 x}{x^3\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$
$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)^3\lim_{x\to 0}\frac{1}{\left(\sqrt{1+\tan x}+\sqrt{1+\sin x}\right)\cos x\,(1+\cos x)}$$
$$= 1^3\cdot\frac{1}{\left(\sqrt{1}+\sqrt{1}\right)\cdot 1\cdot(1+1)} = \frac{1}{4}$$

88. Differentiating the first given equation implicitly with respect to x and using the Chain Rule, we obtain $f(g(x)) = x \;\Rightarrow\; f'(g(x))g'(x) = 1 \;\Rightarrow\; g'(x) = \dfrac{1}{f'(g(x))}$. Using the second given equation to expand the denominator of this expression gives $g'(x) = \dfrac{1}{1+[f(g(x))]^2}$. But the first given equation states that $f(g(x)) = x$, so $g'(x) = \dfrac{1}{1+x^2}$.

89. $\dfrac{d}{dx}[f(2x)] = x^2 \;\Rightarrow\; f'(2x)\cdot 2 = x^2 \;\Rightarrow\; f'(2x) = \frac{1}{2}x^2$. Let $t = 2x$. Then $f'(t) = \frac{1}{2}\left(\frac{1}{2}t\right)^2 = \frac{1}{8}t^2$, so $f'(x) = \frac{1}{8}x^2$.

90. Let (b, c) be on the curve, that is, $b^{2/3} + c^{2/3} = a^{2/3}$. Now $x^{2/3} + y^{2/3} = a^{2/3} \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$,

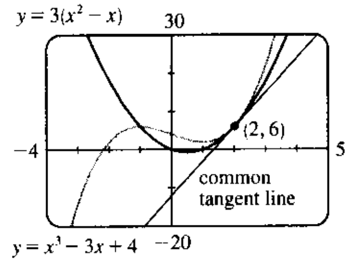
so $\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = -\left(\frac{y}{x}\right)^{1/3}$, so at (b, c) the slope of the tangent line is $-(c/b)^{1/3}$ and an equation of the tangent line is $y - c = -(c/b)^{1/3}(x - b)$ or $y = -(c/b)^{1/3}x + (c + b^{2/3}c^{1/3})$. Setting $y = 0$, we find that the x-intercept is $b^{1/3}c^{2/3} + b = b^{1/3}(c^{2/3} + b^{2/3})$ and setting $x = 0$ we find that the y-intercept is $c + b^{2/3}c^{1/3} = c^{1/3}(c^{2/3} + b^{2/3})$. So the length of the tangent line between these two points is

$$\begin{aligned}\sqrt{[b^{1/3}(c^{2/3}+b^{2/3})]^2 + [c^{1/3}(c^{2/3}+b^{2/3})]^2} &= \sqrt{b^{2/3}(a^{2/3})^2 + c^{2/3}(a^{2/3})^2} \\ &= \sqrt{(b^{2/3}+c^{2/3})a^{4/3}} = \sqrt{a^{2/3}a^{4/3}} \\ &= \sqrt{a^2} = a = \text{constant}\end{aligned}$$

PROBLEMS PLUS

1. Let a be the x-coordinate of Q. Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3}$ $\Rightarrow$ $a = \frac{\sqrt{3}}{2}$. Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

2. $y = x^3 - 3x + 4 \;\Rightarrow\; y' = 3x^2 - 3$, and $y = 3(x^2 - x) \;\Rightarrow\; y' = 6x - 3$. The slopes of the tangents of the two curves are equal when $3x^2 - 3 = 6x - 3$; that is, when $x = 0$ or 2. At $x = 0$, both tangents have slope -3, but the curves do not intersect. At $x = 2$, both tangents have slope 9 and the curves intersect at $(2, 6)$. So there is a common tangent line at $(2, 6)$, $y = 9x - 12$.



3. (a) Put $x = 0$ and $y = 0$ in the equation: $f(0 + 0) = f(0) + f(0) + 0^2 \cdot 0 + 0 \cdot 0^2 \;\Rightarrow\; f(0) = 2f(0)$. Subtracting $f(0)$ from each side of this equation gives $f(0) = 0$.

(b) $$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\left[f(0) + f(h) + 0^2h + 0h^2\right] - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{x\to 0} \frac{f(x)}{x} = 1$$

(c) $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left[f(x) + f(h) + x^2h + xh^2\right] - f(x)}{h} = \lim_{h\to 0} \frac{f(h) + x^2h + xh^2}{h} = \lim_{h\to 0} \left[\frac{f(h)}{h} + x^2 + xh\right] = 1 + x^2$$

4. We find the equation of the parabola by substituting the point $(-100, 100)$, at which the car is situated, into the general equation $y = ax^2$: $100 = a(-100)^2 \;\Rightarrow\; a = \frac{1}{100}$. Now we find the equation of a tangent to the parabola at the point (x_0, y_0). We can show that $y' = a(2x) = \frac{1}{100}(2x) = \frac{1}{50}x$, so an equation of the tangent is $y - y_0 = \frac{1}{50}x_0(x - x_0)$. Since the point (x_0, y_0) is on the parabola, we must have $y_0 = \frac{1}{100}x_0^2$, so our equation of the tangent can be simplified to $y = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(x - x_0)$. We want the statue to be located on the tangent line, so we substitute its coordinates $(100, 50)$ into this equation: $50 = \frac{1}{100}x_0^2 + \frac{1}{50}x_0(100 - x_0) \;\Rightarrow\; x_0^2 - 200x_0 + 5000 = 0 \;\Rightarrow\; x_0 = \frac{1}{2}\left[200 \pm \sqrt{200^2 - 4(5000)}\right] \;\Rightarrow\; x_0 = 100 \pm 50\sqrt{2}$. But $x_0 < 100$, so the car's headlights illuminate the statue when it is located at the point $(100 - 50\sqrt{2}, 150 - 100\sqrt{2}) \approx (29.3, 8.6)$, that is, about 29.3 m east and 8.6 m north of the origin.

5. We use mathematical induction. Let S_n be the statement that $\frac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos(4x + n\pi/2)$.

S_1 is true because

$$\begin{aligned}\frac{d}{dx}(\sin^4 x + \cos^4 x) &= 4\sin^3 x\cos x - 4\cos^3 x\sin x = 4\sin x\cos x\,(\sin^2 x - \cos^2 x)\\ &= -4\sin x\cos x\cos 2x = -2\sin 2x\cos 2x = -\sin 4x = \sin(-4x)\\ &= \cos\left(\tfrac{\pi}{2} - (-4x)\right) = \cos\left(\tfrac{\pi}{2} + 4x\right) = 4^{n-1}\cos\left(4x + n\tfrac{\pi}{2}\right) \text{ when } n = 1\end{aligned}$$

Now assume S_k is true, that is, $\frac{d^k}{dx^k}(\sin^4 x + \cos^4 x) = 4^{k-1}\cos\left(4x + k\frac{\pi}{2}\right)$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}(\sin^4 x + \cos^4 x) &= \frac{d}{dx}\left[\frac{d^k}{dx^k}(\sin^4 x + \cos^4 x)\right] = \frac{d}{dx}\left[4^{k-1}\cos\left(4x + k\tfrac{\pi}{2}\right)\right]\\ &= -4^{k-1}\sin\left(4x + k\tfrac{\pi}{2}\right)\cdot\frac{d}{dx}\left(4x + k\tfrac{\pi}{2}\right) = -4^k\sin\left(4x + k\tfrac{\pi}{2}\right)\\ &= 4^k\sin\left(-4x - k\tfrac{\pi}{2}\right) = 4^k\cos\left(\tfrac{\pi}{2} - \left(-4x - k\tfrac{\pi}{2}\right)\right)\\ &= 4^k\cos\left(4x + (k+1)\tfrac{\pi}{2}\right)\end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\frac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1}\cos\left(4x + n\frac{\pi}{2}\right)$ for every positive integer n, by mathematical induction.

Another proof: First write

$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = 1 - \frac{1}{2}\sin^2 2x = 1 - \frac{1}{4}(1 - \cos 4x) = \frac{3}{4} + \frac{1}{4}\cos 4x$.

Then we have $\frac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = \frac{d^n}{dx^n}\left(\frac{3}{4} + \frac{1}{4}\cos 4x\right) = \frac{1}{4}\cdot 4^n\cos\left(4x + n\frac{\pi}{2}\right) = 4^{n-1}\cos\left(4x + n\frac{\pi}{2}\right)$.

6. If we divide $1 - x$ into x^n by long division, we find that $f(x) = \frac{x^n}{1-x} = -x^{n-1} - x^{n-2} - \cdots - x - 1 + \frac{1}{1-x}$. This can also be seen by multiplying the last expression by $1 - x$ and canceling terms on the right-hand side. So we let $g(x) = 1 + x + x^2 + \cdots + x^{n-1}$, so that $f(x) = \frac{1}{1-x} - g(x) \;\Rightarrow\; f^{(n)}(x) = \left(\frac{1}{1-x}\right)^{(n)} - g^{(n)}(x)$. But g is a polynomial of degree $(n-1)$, so its nth derivative is 0, and therefore $f^{(n)}(x) = \left(\frac{1}{1-x}\right)^{(n)}$. Now

$\frac{d}{dx}(1-x)^{-1} = (-1)(1-x)^{-2}(-1) = (1-x)^{-2}$, $\frac{d^2}{dx^2}(1-x)^{-1} = (-2)(1-x)^{-3}(-1) = 2(1-x)^{-3}$,

$\frac{d^3}{dx^3}(1-x)^{-1} = (-3)\cdot 2(1-x)^{-4}(-1) = 3\cdot 2(1-x)^{-4}$, $\frac{d^4}{dx^4}(1-x)^{-1} = 4\cdot 3\cdot 2(1-x)^{-5}$, and so on. So after n differentiations, we will have $f^{(n)}(x) = \left(\frac{1}{1-x}\right)^{(n)} = \frac{n!}{(1-x)^{n+1}}$.

7. We must find a value x_0 such that the normal lines to the parabola $y = x^2$ at $x = \pm x_0$ intersect at a point one unit from the points $(\pm x_0, x_0^2)$. The normals to $y = x^2$ at $x = \pm x_0$ have slopes $-\frac{1}{\pm 2x_0}$ and pass through $(\pm x_0, x_0^2)$ respectively, so the normals have the equations $y - x_0^2 = -\frac{1}{2x_0}(x - x_0)$ and $y - x_0^2 = \frac{1}{2x_0}(x + x_0)$. The common y-intercept is $x_0^2 + \frac{1}{2}$. We want to find the value of x_0 for which the distance from $\left(0, x_0^2 + \frac{1}{2}\right)$ to $\left(x_0, x_0^2\right)$ equals 1. The square of the distance is $(x_0 - 0)^2 + \left[x_0^2 - \left(x_0^2 + \frac{1}{2}\right)\right]^2 = x_0^2 + \frac{1}{4} = 1 \quad \Leftrightarrow \quad x_0 = \pm\frac{\sqrt{3}}{2}$. For these values of x_0, the y-intercept is $x_0^2 + \frac{1}{2} = \frac{5}{4}$, so the center of the circle is at $\left(0, \frac{5}{4}\right)$.

Another solution: Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$. Solving with the equation of the parabola, $y = x^2$, we get $x^2 + \left(x^2 - a\right)^2 = 1 \quad \Leftrightarrow \quad x^2 + x^4 - 2ax^2 + a^2 = 1$ $\Leftrightarrow \quad x^4 + (1 - 2a)x^2 + a^2 - 1 = 0$. The parabola and the circle will be tangent to each other when this quadratic equation in x^2 has equal roots; that is, when the discriminant is 0. Thus, $(1 - 2a)^2 - 4\left(a^2 - 1\right) = 0 \quad \Leftrightarrow$ $1 - 4a + 4a^2 - 4a^2 + 4 = 0 \quad \Leftrightarrow \quad 4a = 5$, so $a = \frac{5}{4}$. The center of the circle is $\left(0, \frac{5}{4}\right)$.

8. $$\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} = \lim_{x \to a}\left[\frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}\right] = \lim_{x \to a}\left[\frac{f(x) - f(a)}{x - a} \cdot \left(\sqrt{x} + \sqrt{a}\,\right)\right]$$
$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}\left(\sqrt{x} + \sqrt{a}\,\right) = f'(a) \cdot \left(\sqrt{a} + \sqrt{a}\,\right) = 2\sqrt{a}\, f'(a)$$

9. We can assume without loss of generality that $\theta = 0$ at time $t = 0$, so that $\theta = 12\pi t$ rad. [The angular velocity of the wheel is 360 rpm $= 360 \cdot (2\pi \text{ rad})/(60 \text{ s}) = 12\pi$ rad/s.] Then the position of A as a function of time is

$$A = (40\cos\theta, 40\sin\theta) = (40\cos 12\pi t, 40\sin 12\pi t), \text{ so } \sin\alpha = \frac{y}{1.2 \text{ m}} = \frac{40\sin\theta}{120} = \frac{\sin\theta}{3} = \frac{1}{3}\sin 12\pi t.$$

(a) Differentiating the expression for $\sin\alpha$, we get $\cos\alpha \cdot \dfrac{d\alpha}{dt} = \frac{1}{3} \cdot 12\pi \cdot \cos 12\pi t = 4\pi\cos\theta$.

When $\theta = \frac{\pi}{3}$, we have $\sin\alpha = \frac{1}{3}\sin\theta = \frac{\sqrt{3}}{6}$, so $\cos\alpha = \sqrt{1 - \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{11}{12}}$ and

$$\frac{d\alpha}{dt} = \frac{4\pi\cos\frac{\pi}{3}}{\cos\alpha} = \frac{2\pi}{\sqrt{11/12}} = \frac{4\pi\sqrt{3}}{\sqrt{11}} \approx 6.56 \text{ rad/s}.$$

(b) By the Law of Cosines, $|AP|^2 = |OA|^2 + |OP|^2 - 2\,|OA|\,|OP|\cos\theta \quad \Rightarrow$

$120^2 = 40^2 + |OP|^2 - 2 \cdot 40\,|OP|\cos\theta \quad \Rightarrow \quad |OP|^2 - (80\cos\theta)\,|OP| - 12{,}800 = 0 \quad \Rightarrow$

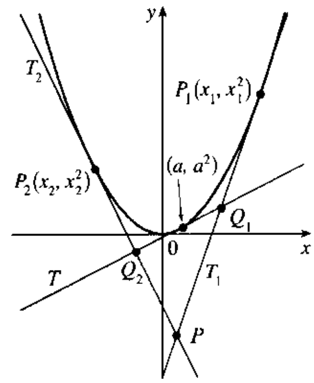
$$|OP| = \tfrac{1}{2}\left(80\cos\theta \pm \sqrt{6400\cos^2\theta + 51{,}200}\right) = 40\cos\theta \pm 40\sqrt{\cos^2\theta + 8}$$
$$= 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\,\right) \text{ cm} \quad (\text{since } |OP| > 0)$$

As a check, note that $|OP| = 160$ cm when $\theta = 0$ and $|OP| = 80\sqrt{2}$ cm when $\theta = \frac{\pi}{2}$.

(c) By part (b), the x-coordinate of P is given by $x = 40\left(\cos\theta + \sqrt{8 + \cos^2\theta}\,\right)$, so

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = 40\left(-\sin\theta - \frac{2\cos\theta\sin\theta}{2\sqrt{8 + \cos^2\theta}}\right) \cdot 12\pi = -480\pi\sin\theta\left(1 + \frac{\cos\theta}{\sqrt{8 + \cos^2\theta}}\right) \text{ cm/s}.$$

In particular, $dx/dt = 0$ cm/s when $\theta = 0$ and $dx/dt = -480\pi$ cm/s when $\theta = \frac{\pi}{2}$.



10. The equation of T_1 is $y - x_1^2 = 2x_1(x - x_1) = 2x_1x - 2x_1^2$ or $y = 2x_1x - x_1^2$. The equation of T_2 is $y = 2x_2x - x_2^2$. Solving for the point of intersection, we get $2x(x_1 - x_2) = x_1^2 - x_2^2 \;\Rightarrow\; x = \frac{1}{2}(x_1 + x_2)$. Therefore, the coordinates of P are $\left(\frac{1}{2}(x_1 + x_2), x_1x_2\right)$. So if the point of contact of T is (a, a^2), then Q_1 is $\left(\frac{1}{2}(a + x_1), ax_1\right)$ and Q_2 is $\left(\frac{1}{2}(a + x_2), ax_2\right)$.

Therefore, $|PQ_1|^2 = \frac{1}{4}(a - x_2)^2 + x_1^2(a - x_2)^2 = (a - x_2)^2\left(\frac{1}{4} + x_1^2\right)$ and $|PP_1|^2 = \frac{1}{4}(x_1 - x_2)^2 + x_1^2(x_1 - x_2)^2 = (x_1 - x_2)^2\left(\frac{1}{4} + x_1^2\right)$. So $\dfrac{|PQ_1|^2}{|PP_1|^2} = \dfrac{(a - x_2)^2}{(x_1 - x_2)^2}$, and similarly $\dfrac{|PQ_2|^2}{|PP_2|^2} = \dfrac{(x_1 - a)^2}{(x_1 - x_2)^2}$. Finally, $\dfrac{|PQ_1|}{|PP_1|} + \dfrac{|PQ_2|}{|PP_2|} = \dfrac{a - x_2}{x_1 - x_2} + \dfrac{x_1 - a}{x_1 - x_2} = 1$.

11. It seems from the figure that as P approaches the point $(0, 2)$ from the right, $x_T \to \infty$ and $y_T \to 2^+$. As P approaches the point $(3, 0)$ from the left, it appears that $x_T \to 3^+$ and $y_T \to \infty$. So we guess that $x_T \in (3, \infty)$ and $y_T \in (2, \infty)$. It is more difficult to estimate the range of values for x_N and y_N. We might perhaps guess that $x_N \in (0, 3)$, and $y_N \in (-\infty, 0)$ or $(-2, 0)$.

In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation of the tangent line: $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1 \;\Rightarrow\; \dfrac{2x}{9} + \dfrac{2y}{4}y' = 0$, so $y' = -\dfrac{4}{9}\dfrac{x}{y}$. So at the point (x_0, y_0) on the ellipse, an equation of the tangent line is $y - y_0 = -\dfrac{4}{9}\dfrac{x_0}{y_0}(x - x_0)$ or $4x_0x + 9y_0y = 4x_0^2 + 9y_0^2$. This can be written as $\dfrac{x_0x}{9} + \dfrac{y_0y}{4} = \dfrac{x_0^2}{9} + \dfrac{y_0^2}{4} = 1$, because (x_0, y_0) lies on the ellipse. So an equation of the tangent line is $\dfrac{x_0x}{9} + \dfrac{y_0y}{4} = 1$.

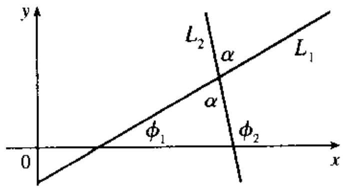
Therefore, the x-intercept x_T for the tangent line is given by $\dfrac{x_0x_T}{9} = 1 \;\Leftrightarrow\; x_T = \dfrac{9}{x_0}$, and the y-intercept y_T is given by $\dfrac{y_0y_T}{4} = 1 \;\Leftrightarrow\; y_T = \dfrac{4}{y_0}$.

So as x_0 takes on all values in $(0, 3)$, x_T takes on all values in $(3, \infty)$, and as y_0 takes on all values in $(0, 2)$, y_T takes on all values in $(2, \infty)$. At the point (x_0, y_0) on the ellipse, the slope of the normal line is $-\dfrac{1}{y'(x_0, y_0)} = \dfrac{9}{4}\dfrac{y_0}{x_0}$, and its equation is $y - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x - x_0)$. So the x-intercept x_N for the normal line is given by $0 - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(x_N - x_0) \;\Rightarrow\; x_N = -\dfrac{4x_0}{9} + x_0 = \dfrac{5x_0}{9}$, and the y-intercept y_N is given by $y_N - y_0 = \dfrac{9}{4}\dfrac{y_0}{x_0}(0 - x_0) \;\Rightarrow\; y_N = -\dfrac{9y_0}{4} + y_0 = -\dfrac{5y_0}{4}$.

So as x_0 takes on all values in $(0, 3)$, x_N takes on all values in $\left(0, \frac{5}{3}\right)$, and as y_0 takes on all values in $(0, 2)$, y_N takes on all values in $\left(-\frac{5}{2}, 0\right)$.

12. $\lim\limits_{x\to 0} \dfrac{\sin(3+x)^2 - \sin 9}{x} = f'(3)$ where $f(x) = \sin x^2$. Now $f'(x) = (\cos x^2)(2x)$, so $f'(3) = 6\cos 9$.

13. (a)



If the two lines L_1 and L_2 have slopes m_1 and m_2 and angles of inclination ϕ_1 and ϕ_2, then $m_1 = \tan\,\phi_1$ and $m_2 = \tan\,\phi_2$. The triangle in the figure shows that $\phi_1 + \alpha + (180^\circ - \phi_2) = 180^\circ$ and so $\alpha = \phi_2 - \phi_1$. Therefore, using the identity for $\tan(x-y)$, we have

$$\tan\alpha = \tan(\phi_2 - \phi_1) = \frac{\tan\phi_2 - \tan\phi_1}{1 + \tan\phi_2\tan\phi_1} \text{ and so}$$

$$\tan\,\alpha = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

(b) (i) The parabolas intersect when $x^2 = (x-2)^2 \;\Rightarrow\; x = 1$. If $y = x^2$, then $y' = 2x$, so the slope of the tangent to $y = x^2$ at $(1,1)$ is $m_1 = 2(1) = 2$. If $y = (x-2)^2$, then $y' = 2(x-2)$, so the slope of the tangent to $y = (x-2)^2$ at $(1,1)$ is $m_2 = 2(1-2) = -2$. Therefore,

$$\tan\alpha = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-2-2}{1 + 2(-2)} = \frac{4}{3} \text{ and so } \alpha = \tan^{-1}\left(\tfrac{4}{3}\right) \approx 53^\circ \text{ (or } 127^\circ).$$

(ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$ intersect when $x^2 - 4x + (x^2 - 3) + 3 = 0 \;\Leftrightarrow\; 2x(x-2) = 0 \;\Rightarrow\; x = 0$ or 2, but 0 is extraneous. If $x = 2$, then $y = \pm 1$. If $x^2 - y^2 = 3$ then $2x - 2yy' = 0 \;\Rightarrow\; y' = x/y$ and $x^2 - 4x + y^2 + 3 = 0 \;\Rightarrow\; 2x - 4 + 2yy' = 0 \;\Rightarrow\; y' = \dfrac{2-x}{y}$. At $(2,1)$ the slopes are $m_1 = 2$ and $m_2 = 0$, so $\tan\alpha = \frac{0-2}{1+2\cdot 0} = -2 \;\Rightarrow\; \alpha \approx 117^\circ$. At $(2,-1)$ the slopes are $m_1 = -2$ and $m_2 = 0$, so $\tan\alpha = \frac{0-(-2)}{1+(-2)(0)} = 2 \;\Rightarrow\; \alpha \approx 63^\circ$ (or 117°).

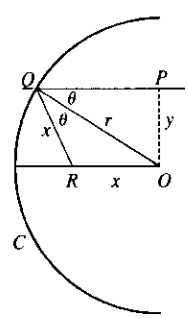
14. $y^2 = 4px \;\Rightarrow\; 2yy' = 4p \;\Rightarrow\; y' = 2p/y \;\Rightarrow$ slope of tangent at $P(x_1, y_1)$ is $m_1 = 2p/y_1$. The slope of FP is $m_2 = \dfrac{y_1}{x_1 - p}$, so by the formula from Problem 13(a),

$$\tan\alpha = \frac{y_1/(x_1-p) - 2p/y_1}{1 + (2p/y_1)[y_1/(x_1-p)]} \cdot \frac{y_1\,(x_1-p)}{y_1\,(x_1-p)} = \frac{y_1^2 - 2p(x_1-p)}{y_1(x_1-p) + 2py_1}$$

$$= \frac{4px_1 - 2px_1 + 2p^2}{x_1y_1 - py_1 + 2py_1} = \frac{2p(p+x_1)}{y_1(p+x_1)} = \frac{2p}{y_1} = \text{slope of tangent at } P = \tan\beta$$

Since $0 \le \alpha, \beta \le \frac{\pi}{2}$, this proves that $\alpha = \beta$.

15. Since $\angle ROQ = \angle OQP = \theta$, the triangle QOR is isosceles, so $|QR| = |RO| = x$. By the Law of Cosines, $x^2 = x^2 + r^2 - 2rx\cos\theta$. Hence, $2rx\cos\theta = r^2$, so $x = \dfrac{r^2}{2r\cos\theta} = \dfrac{r}{2\cos\theta}$. Note that as $y \to 0^+$, $\theta \to 0^+$ (since $\sin\theta = y/r$), and hence $x \to \dfrac{r}{2\cos 0} = \dfrac{r}{2}$. Thus, as P is taken closer and closer to the x-axis, the point R approaches the midpoint of the radius AO.



16. $\displaystyle\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{f(x)-0}{g(x)-0}=\lim_{x\to 0}\frac{f(x)-f(0)}{g(x)-g(0)}=\lim_{x\to 0}\frac{\dfrac{f(x)-f(0)}{x-0}}{\dfrac{g(x)-g(0)}{x-0}}=\frac{\lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0}}{\lim_{x\to 0}\dfrac{g(x)-g(0)}{x-0}}=\frac{f'(0)}{g'(0)}$

17. $\displaystyle\lim_{x\to 0}\frac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}$

$$=\lim_{x\to 0}\frac{\sin a\cos 2x+\cos a\sin 2x-2\sin a\cos x-2\cos a\sin x+\sin a}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin a\,(\cos 2x-2\cos x+1)+\cos a\,(\sin 2x-2\sin x)}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin a\,(2\cos^2 x-1-2\cos x+1)+\cos a\,(2\sin x\cos x-2\sin x)}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin a\,(2\cos x)(\cos x-1)+\cos a\,(2\sin x)(\cos x-1)}{x^2}$$

$$=\lim_{x\to 0}\frac{2(\cos x-1)[\sin a\cos x+\cos a\sin x](\cos x+1)}{x^2(\cos x+1)}$$

$$=\lim_{x\to 0}\frac{-2\sin^2 x\,[\sin(a+x)]}{x^2(\cos x+1)}=-2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2\cdot\frac{\sin(a+x)}{\cos x+1}=-2(1)^2\frac{\sin(a+0)}{\cos 0+1}=-\sin a$$

18. Suppose that $y=mx+c$ is a tangent line to the ellipse. Then it intersects the ellipse at only one point, so the discriminant of the equation $\dfrac{x^2}{a^2}+\dfrac{(mx+c)^2}{b^2}=1 \quad\Leftrightarrow\quad (b^2+a^2m^2)x^2+2mca^2x+a^2c^2-a^2b^2=0$ must be 0; that is,

$$0=(2mca^2)^2-4(b^2+a^2m^2)(a^2c^2-a^2b^2)$$

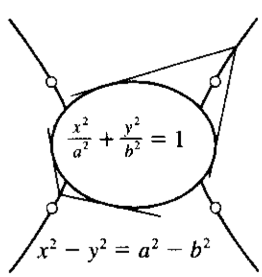
$$=4a^4c^2m^2-4a^2b^2c^2+4a^2b^4-4a^4c^2m^2+4a^4b^2m^2=4a^2b^2(a^2m^2+b^2-c^2)$$

Therefore, $a^2m^2+b^2-c^2=0$.

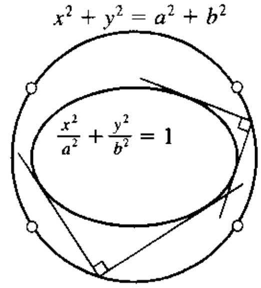
Now if a point (α,β) lies on the line $y=mx+c$, then $c=\beta-m\alpha$, so from above,

$$0=a^2m^2+b^2-(\beta-m\alpha)^2=(a^2-\alpha^2)m^2+2\alpha\beta m+b^2-\beta^2 \quad\Leftrightarrow\quad m^2+\frac{2\alpha\beta}{a^2-\alpha^2}m+\frac{b^2-\beta^2}{a^2-\alpha^2}=0.$$

(a) Suppose that the two tangent lines from the point (α,β) to the ellipse have slopes m and $\dfrac{1}{m}$. Then m and $\dfrac{1}{m}$ are roots of the equation $z^2+\dfrac{2\alpha\beta}{a^2-\alpha^2}z+\dfrac{b^2-\beta^2}{a^2-\alpha^2}=0$. This implies that $(z-m)\left(z-\dfrac{1}{m}\right)=0 \quad\Leftrightarrow$ $z^2-\left(m+\dfrac{1}{m}\right)z+m\left(\dfrac{1}{m}\right)=0$, so equating the constant terms in the two quadratic equations, we get $\dfrac{b^2-\beta^2}{a^2-\alpha^2}=m\left(\dfrac{1}{m}\right)=1$, and hence $b^2-\beta^2=a^2-\alpha^2$. So (α,β) lies on the hyperbola $x^2-y^2=a^2-b^2$.



(b) If the two tangent lines from the point (α, β) to the ellipse have slopes m and $-\frac{1}{m}$, then m and $-\frac{1}{m}$ are roots of the quadratic equation, and so $(z - m)\left(z + \frac{1}{m}\right) = 0$, and equating the constant terms as in part (a), we get $\frac{b^2 - \beta^2}{a^2 - \alpha^2} = -1$, and hence $b^2 - \beta^2 = \alpha^2 - a^2$. So the point (α, β) lies on the circle $x^2 + y^2 = a^2 + b^2$.



19. $y = x^4 - 2x^2 - x \quad \Rightarrow \quad y' = 4x^3 - 4x - 1$. The equation of the tangent line at $x = a$ is $y - (a^4 - 2a^2 - a) = (4a^3 - 4a - 1)(x - a)$ or $y = (4a^3 - 4a - 1)x + (-3a^4 + 2a^2)$ and similarly for $x = b$. So if at $x = a$ and $x = b$ we have the same tangent line, then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-3a^4 + 2a^2 = -3b^4 + 2b^2$. The first equation gives $a^3 - b^3 = a - b \quad \Rightarrow \quad (a - b)(a^2 + ab + b^2) = (a - b)$. Assuming $a \neq b$, we have $1 = a^2 + ab + b^2$. The second equation gives $3(a^4 - b^4) = 2(a^2 - b^2) \quad \Rightarrow$ $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ which is true if $a = -b$. Substituting into $1 = a^2 + ab + b^2$ gives $1 = a^2 - a^2 + a^2 \quad \Rightarrow \quad a = \pm 1$ so that $a = 1$ and $b = -1$ or vice versa. Thus, the points $(1, -2)$ and $(-1, 0)$ have a common tangent line.

As long as there are only two such points, we are done. So we show that these are in fact the only two such points. Suppose that $a^2 - b^2 \neq 0$. Then $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ gives $3(a^2 + b^2) = 2$ or $a^2 + b^2 = \frac{2}{3}$. Thus, $ab = (a^2 + ab + b^2) - (a^2 + b^2) = 1 - \frac{2}{3} = \frac{1}{3}$, so $b = \frac{1}{3a}$. Hence, $a^2 + \frac{1}{9a^2} = \frac{2}{3}$, so $9a^4 + 1 = 6a^2 \quad \Rightarrow$ $0 = 9a^4 - 6a^2 + 1 = (3a^2 - 1)^2$. So $3a^2 - 1 = 0 \quad \Rightarrow \quad a^2 = \frac{1}{3} \quad \Rightarrow \quad b^2 = \frac{1}{9a^2} = \frac{1}{3} = a^2$, contradicting our assumption that $a^2 \neq b^2$.

20. Suppose that the normal lines at the three points (a_1, a_1^2), (a_2, a_2^2), and (a_3, a_3^2) intersect at a common point. Now if one of the a_i is 0 (suppose $a_1 = 0$) then by symmetry $a_2 = -a_3$, so $a_1 + a_2 + a_3 = 0$. So we can assume that none of the a_i is 0.

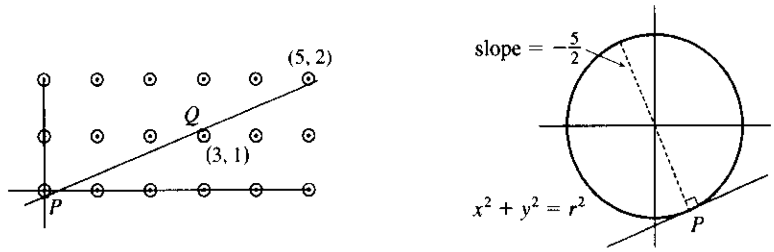
The slope of the tangent line at (a_i, a_i^2) is $2a_i$, so the slope of the normal line is $-\frac{1}{2a_i}$ and its equation is $y - a_i^2 = -\frac{1}{2a_i}(x - a_i)$. We solve for the x-coordinate of the intersection of the normal lines from (a_1, a_1^2) and (a_2, a_2^2): $\quad y = a_1^2 - \frac{1}{2a_1}(x - a_1) = a_2^2 - \frac{1}{2a_2}(x - a_2) \quad \Rightarrow$

$$x\left(\frac{1}{2a_2}-\frac{1}{2a_1}\right)=a_2^2-a_1^2 \quad\Rightarrow\quad x\left(\frac{a_1-a_2}{2a_1a_2}\right)=(-a_1-a_2)(a_1+a_2) \quad\Leftrightarrow\quad x=-2a_1a_2(a_1+a_2) \quad (\star).$$

Similarly, solving for the x-coordinate of the intersections of the normal lines from (a_1, a_1^2) and (a_3, a_3^2) gives $x=-2a_1a_3(a_1+a_3)$ $(\dagger)$.

Equating $(\star)$ and $(\dagger)$ gives $a_2(a_1+a_2)=a_3(a_1+a_3) \quad\Leftrightarrow\quad a_1(a_2-a_3)=a_3^2-a_2^2=-(a_2+a_3)(a_2-a_3)$ $\Leftrightarrow\quad a_1=-(a_2+a_3) \quad\Leftrightarrow\quad a_1+a_2+a_3=0.$

21.

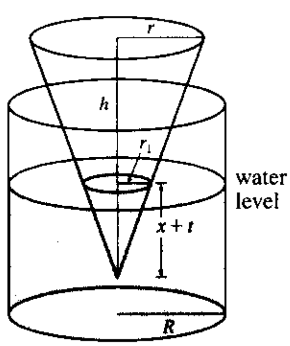


Because of the periodic nature of the lattice points, it suffices to consider the points in the 5×2 grid shown. We can see that the minimum value of r occurs when there is a line with slope $\frac{2}{5}$ which touches the circle centered at $(3,1)$ and the circles centered at $(0,0)$ and $(5,2)$. To find P, the point at which the line is tangent to the circle at $(0,0)$, we simultaneously solve $x^2+y^2=r^2$ and $y=-\frac{5}{2}x \quad\Rightarrow\quad x^2+\frac{25}{4}x^2=r^2 \quad\Rightarrow\quad x^2=\frac{4}{29}r^2 \quad\Rightarrow$ $x=\frac{2}{\sqrt{29}}r$, $y=-\frac{5}{\sqrt{29}}r$. To find Q, we either use symmetry or solve $(x-3)^2+(y-1)^2=r^2$ and $y-1=-\frac{5}{2}(x-3)$. As above, we get $x=3-\frac{2}{\sqrt{29}}r$, $y=1+\frac{5}{\sqrt{29}}r$. Now the slope of the line PQ is $\frac{2}{5}$, so

$$m_{PQ}=\frac{1+\frac{5}{\sqrt{29}}r-\left(-\frac{5}{\sqrt{29}}r\right)}{3-\frac{2}{\sqrt{29}}r-\frac{2}{\sqrt{29}}r}=\frac{1+\frac{10}{\sqrt{29}}r}{3-\frac{4}{\sqrt{29}}r}=\frac{\sqrt{29}+10r}{3\sqrt{29}-4r}=\frac{2}{5} \quad\Rightarrow\quad 5\sqrt{29}+50r=6\sqrt{29}-8r \quad\Leftrightarrow$$

$58r=\sqrt{29} \quad\Leftrightarrow\quad r=\frac{\sqrt{29}}{58}$. So the minimum value of r for which any line with slope $\frac{2}{5}$ intersects circles with radius r centered at the lattice points on the plane is $r=\frac{\sqrt{29}}{58}\approx 0.093$.

22.



Assume the axes of the cone and the cylinder are parallel. Let H denote the initial height of the water. When the cone has been dropping for t seconds, the water level has risen x meters, so the tip of the cone is $x+1t$ meters below the water line. We want to find dx/dt when $x+t=h$ (when the cone is completely submerged). Using similar triangles, $\frac{r_1}{x+t}=\frac{r}{h} \quad\Rightarrow\quad r_1=\frac{r}{h}(x+t)$.

volume of water and cone at time t	=	original volume of water	+	volume of submerged part of cone
$\pi R^2(H+x)$	=	$\pi R^2 H$	+	$\frac{1}{3}\pi r_1^2(x+t)$
$\pi R^2 H+\pi R^2 x$	=	$\pi R^2 H$	+	$\frac{1}{3}\pi\frac{r^2}{h^2}(x+t)^3$
$3h^2R^2x$	=	$r^2(x+t)^3$		

Differentiating implicitly with respect to t gives us

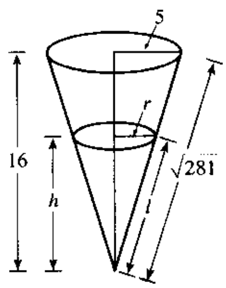
$$3h^2R^2\,\frac{dx}{dt} = r^2\left[3(x+t)^2\,\frac{dx}{dt} + 3(x+t)^2\,\frac{dt}{dt}\right] \quad\Rightarrow$$

$$\frac{dx}{dt} = \frac{r^2(x+t)^2}{h^2R^2 - r^2(x+t)^2} \quad\Rightarrow$$

$$\left.\frac{dx}{dt}\right|_{x+t=h} = \frac{r^2h^2}{h^2R^2 - r^2h^2} = \frac{r^2}{R^2 - r^2}$$

Thus, the water level is rising at a rate of $\dfrac{r^2}{R^2 - r^2}$ cm/s at the instant the cone is completely submerged.

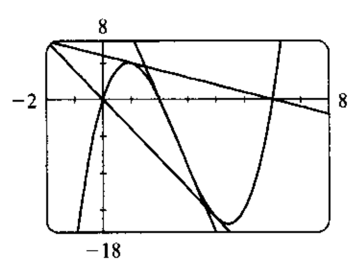
23.



By similar triangles, $\dfrac{r}{5} = \dfrac{h}{16} \;\Rightarrow\; r = \dfrac{5h}{16}$. The volume of the cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\dfrac{5h}{16}\right)^2 h = \dfrac{25\pi}{768}h^3$, so $\dfrac{dV}{dt} = \dfrac{25\pi}{256}h^2\,\dfrac{dh}{dt}$. Now the rate of change of the volume is also equal to the difference of what is being added ($2\text{ cm}^3/\text{min}$) and what is oozing out ($k\pi rl$, where πrl is the area of the cone and k is a proportionality constant). Thus, $\dfrac{dV}{dt} = 2 - k\pi rl$.

Equating the two expressions for $\dfrac{dV}{dt}$ and substituting $h = 10$, $\dfrac{dh}{dt} = -0.3$, $r = \dfrac{5(10)}{16} = \dfrac{25}{8}$, and $\dfrac{l}{\sqrt{281}} = \dfrac{10}{16}$ $\Leftrightarrow$ $l = \frac{5}{8}\sqrt{281}$, we get $\frac{25\pi}{256}(10)^2(-0.3) = 2 - k\pi\frac{25}{8}\cdot\frac{5}{8}\sqrt{281}$ $\Leftrightarrow$ $\dfrac{125k\pi\sqrt{281}}{64} = 2 + \dfrac{750\pi}{256}$. Solving for k gives us $k = \dfrac{256 + 375\pi}{250\pi\sqrt{281}}$. To maintain a certain height, the rate of oozing, $k\pi rl$, must equal the rate of the liquid being poured in; that is, $\dfrac{dV}{dt} = 0$. $k\pi rl = \dfrac{256 + 375\pi}{250\pi\sqrt{281}}\cdot\pi\cdot\dfrac{25}{8}\cdot\dfrac{5\sqrt{281}}{8} = \dfrac{256 + 375\pi}{128} \approx 11.204\text{ cm}^3/\text{min}$.

24. (a) $f(x) = x(x-2)(x-6) = x^3 - 8x^2 + 12x \;\Rightarrow\; f'(x) = 3x^2 - 16x + 12$. The average of the first pair of zeros is $(0+2)/2 = 1$. At $x = 1$, the slope of the tangent line is $f'(1) = -1$, so an equation of the tangent line has the form $y = -1x + b$. Since $f(1) = 5$, we have $5 = -1 + b \;\Rightarrow\; b = 6$ and the tangent has equation $y = -x + 6$. Similarly, at $x = \dfrac{0+6}{2} = 3$, $y = -9x + 18$; at $x = \dfrac{2+6}{2} = 4$, $y = -4x$. From the graph, we see that each tangent line drawn at the average of two zeros intersects the graph of f at the third zero.



(b) A CAS gives $f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$ or $f'(x) = 3x^2 - 2(a+b+c)x + ab + ac + bc$. Using the `Simplify` command, we get

$f'\left(\frac{a+b}{2}\right) = -\frac{(a-b)^2}{4}$ and $f\left(\frac{a+b}{2}\right) = -\frac{(a-b)^2}{8}(a+b-2c)$, so an equation of the tangent line at $x = \frac{a+b}{2}$ is

$$y = -\frac{(a-b)^2}{4}\left(x - \frac{a+b}{2}\right) - \frac{(a-b)^2}{8}(a+b-2c)$$

To find the x-intercept, let $y = 0$ and use the `Solve` command. The result is $x = c$.

Using Derive, we can begin by authoring the expression $(x-a)(x-b)(x-c)$. Now load the utility file Dif_apps. Next we author tangent $(\#1, x, (a+b)/2)$—this is the command to find an equation of the tangent line of the function in #1 whose independent variable is x at the x-value $(a+b)/2$. We then simplify that expression and obtain the equation $y = \#3$. The form in expression #3 makes it easy to see that the x-intercept is the third zero, namely c. In a similar fashion we see that b is the x-intercept for the tangent line at $(a+c)/2$ and a is the x-intercept for the tangent line at $(b+c)/2$.

```
#1:  (x - a)·(x - b)·(x - c)                              Author the function y=

                                          a + b
#2:  TANGENT((x - a)·(x - b)·(x - c), x, -----)           Tangent(#1, x,(a+b)/2)
                                            2

          2            2
       (a  - 2·a·b + b )·(c - x)
#3:  ---------------------------                          0.0s Simp(#2)
                  4
```