Chapter 5

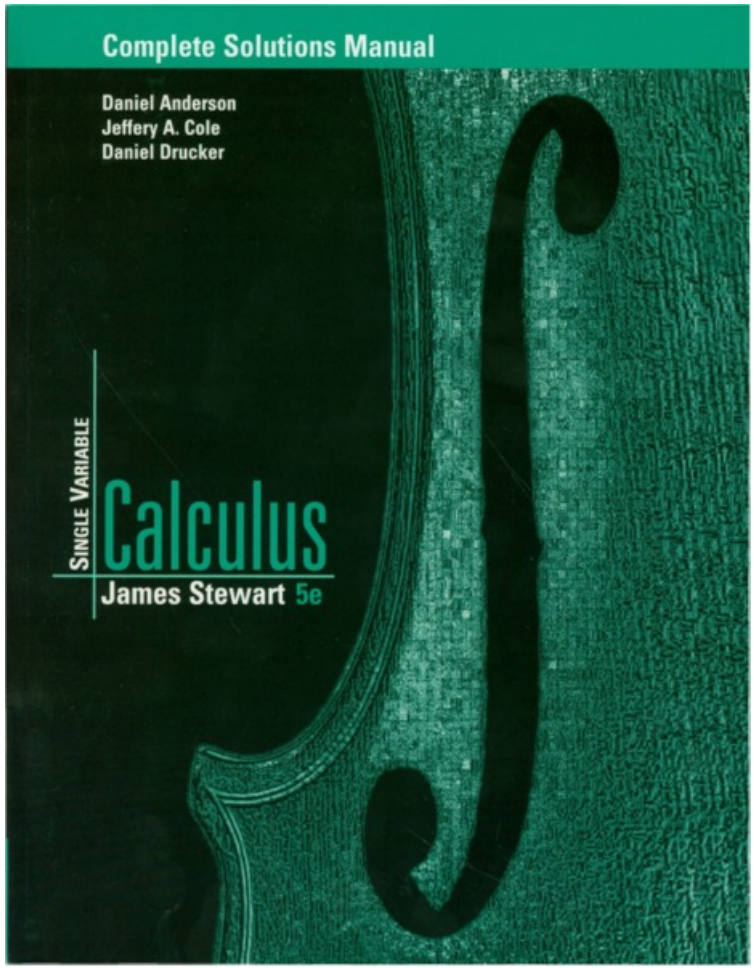
Adapted from the

Complete Solutions Manual

for

James Stewart's

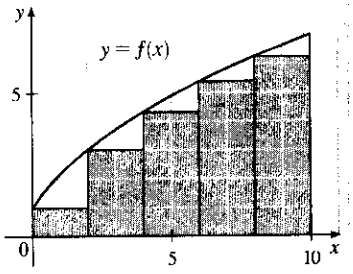
Calculus – 5th Edition



5 □ INTEGRALS

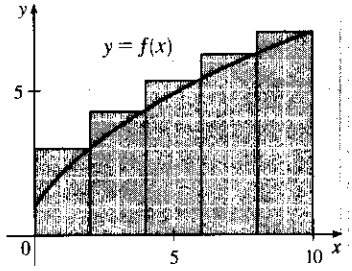
5.1 Areas and Distances

1. (a) Since f is *increasing*, we can obtain a *lower* estimate by using *left* endpoints. We are instructed to use five rectangles, so $n = 5$.



$$L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{b-a}{n} = \tfrac{10-0}{5} = 2]$$
$$= f(x_0)\cdot 2 + f(x_1)\cdot 2 + f(x_2)\cdot 2 + f(x_3)\cdot 2 + f(x_4)\cdot 2$$
$$= 2\,[f(0) + f(2) + f(4) + f(6) + f(8)]$$
$$\approx 2(1 + 3 + 4.3 + 5.4 + 6.3) = 2(20) = 40$$

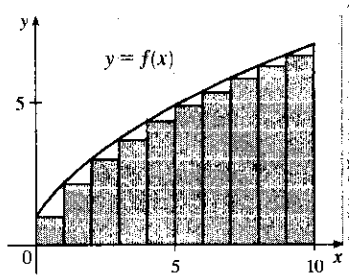
Since f is *increasing*, we can obtain an *upper* estimate by using *right* endpoints.



$$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x$$
$$= 2\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$$
$$= 2\,[f(2) + f(4) + f(6) + f(8) + f(10)]$$
$$\approx 2(3 + 4.3 + 5.4 + 6.3 + 7) = 2(26) = 52$$

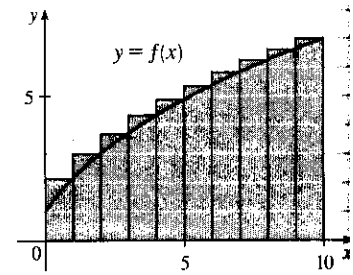
Comparing R_5 to L_5, we see that we have added the area of the rightmost upper rectangle, $f(10)\cdot 2$, to the sum and subtracted the area of the leftmost lower rectangle, $f(0)\cdot 2$, from the sum.

(b) $L_{10} = \sum_{i=1}^{10} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{10-0}{10} = 1]$



$$= 1\,[f(x_0) + f(x_1) + \cdots + f(x_9)]$$
$$= f(0) + f(1) + \cdots + f(9)$$
$$\approx 1 + 2.1 + 3 + 3.7 + 4.3 + 4.9 + 5.4 + 5.8 + 6.3 + 6.7$$
$$= 43.2$$

$$R_{10} = \sum_{i=1}^{10} f(x_i)\,\Delta x = f(1) + f(2) + \cdots + f(10)$$
$$= L_{10} + 1\cdot f(10) - 1\cdot f(0) \quad \left[\begin{array}{l}\text{add rightmost upper rectangle,}\\ \text{subtract leftmost lower rectangle}\end{array}\right]$$
$$= 43.2 + 7 - 1 = 49.2$$



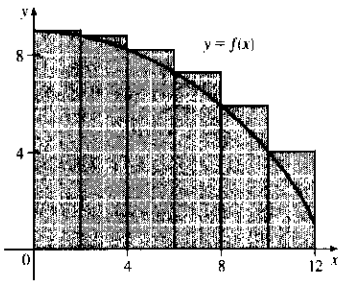
2. (a) (i) $L_6 = \sum_{i=1}^{6} f(x_{i-1})\Delta x \quad [\Delta x = \frac{12-0}{6} = 2]$

$= 2[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$

$= 2[f(0) + f(2) + f(4) + f(6) + f(8) + f(10)]$

$\approx 2(9 + 8.8 + 8.2 + 7.3 + 5.9 + 4.1)$

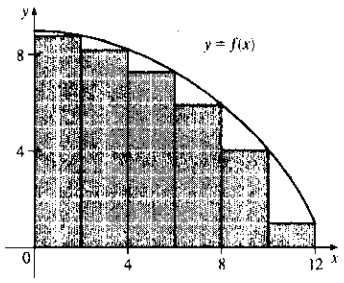
$= 2(43.3) = 86.6$



(ii) $R_6 = L_6 + 2 \cdot f(12) - 2 \cdot f(0)$

$\approx 86.6 + 2(1) - 2(9) = 70.6$

[Add area of rightmost lower rectangle and subtract area of leftmost upper rectangle.]

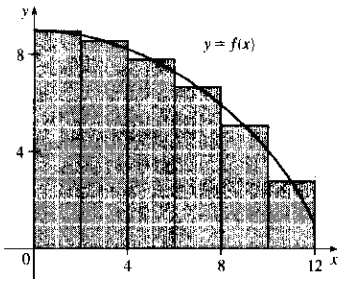


(iii) $M_6 = \sum_{i=1}^{6} f(x_i^*)\,\Delta x$

$= 2[f(1) + f(3) + f(5) + f(7) + f(9) + f(11)]$

$\approx 2(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.8)$

$= 2(39.7) = 79.4$



(b) Since f is *decreasing*, we obtain an *overestimate* by using *left* endpoints; that is, L_6.

(c) Since f is *decreasing*, we obtain an *underestimate* by using *right* endpoints; that is, R_6.

(d) M_6 gives the best estimate, since the area of each rectangle appears to be closer to the true area than the overestimates and underestimates in L_6 and R_6.

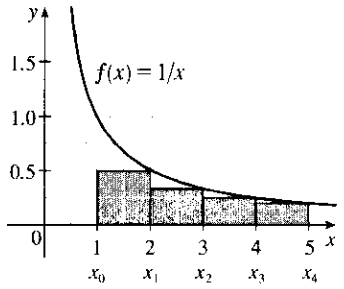
3. (a) $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x \quad [\Delta x = \frac{5-1}{4} = 1]$

$= f(x_1) \cdot 1 + f(x_2) \cdot 1 + f(x_3) \cdot 1 + f(x_4) \cdot 1$

$= f(2) + f(3) + f(4) + f(5)$

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60} = 1.28\overline{3}$

Since f is *decreasing* on $[1, 5]$, an *underestimate* is obtained by using the *right* endpoint approximation, R_4.

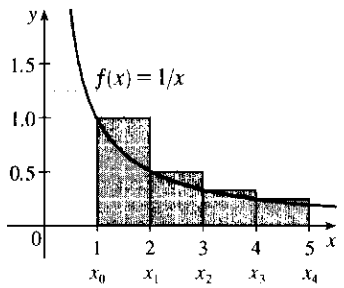


(b) $L_4 = \sum_{i=1}^{4} f(x_{i-1})\,\Delta x$

$= f(1) + f(2) + f(3) + f(4)$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} = 2.08\overline{3}$

L_4 is an overestimate. Alternatively, we could just add the area of the leftmost upper rectangle and subtract the area of the rightmost lower rectangle; that is, $L_4 = R_4 + f(1) \cdot 1 - f(5) \cdot 1$.



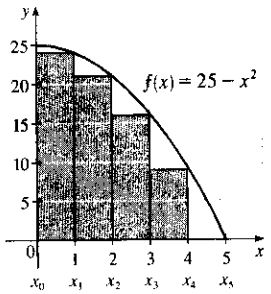
4. (a) $R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x \quad [\Delta x = \frac{5-0}{5} = 1]$

$= f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1 + f(x_5)\cdot 1$

$= f(1) + f(2) + f(3) + f(4) + f(5)$

$= 24 + 21 + 16 + 9 + 0 = 70$

Since f is decreasing on $[0, 5]$, R_5 is an underestimate.

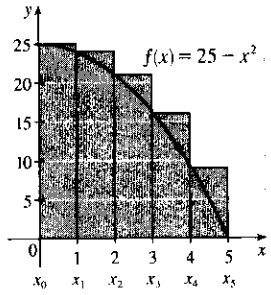


(b) $L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x$

$= f(0) + f(1) + f(2) + f(3) + f(4)$

$= 25 + 24 + 21 + 16 + 9 = 95$

L_5 is an overestimate.



5. (a) $f(x) = 1 + x^2$ and $\Delta x = \dfrac{2-(-1)}{3} = 1 \quad \Rightarrow$

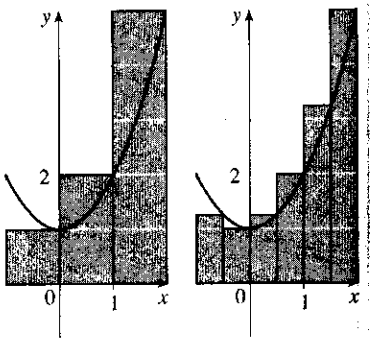
$R_3 = 1\cdot f(0) + 1\cdot f(1) + 1\cdot f(2) = 1\cdot 1 + 1\cdot 2 + 1\cdot 5 = 8.$

$\Delta x = \dfrac{2-(-1)}{6} = 0.5 \quad \Rightarrow$

$R_6 = 0.5[f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$

$= 0.5(1.25 + 1 + 1.25 + 2 + 3.25 + 5)$

$= 0.5(13.75) = 6.875$

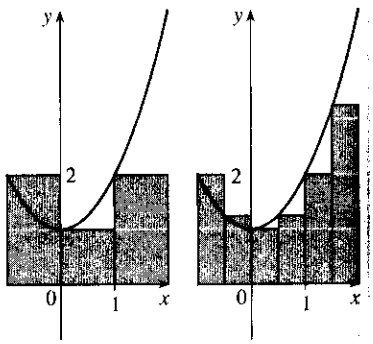


(b) $L_3 = 1\cdot f(-1) + 1\cdot f(0) + 1\cdot f(1) = 1\cdot 2 + 1\cdot 1 + 1\cdot 2 = 5$

$L_6 = 0.5[f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)]$

$= 0.5(2 + 1.25 + 1 + 1.25 + 2 + 3.25)$

$= 0.5(10.75) = 5.375$



(c) $M_3 = 1\cdot f(-0.5) + 1\cdot f(0.5) + 1\cdot f(1.5)$

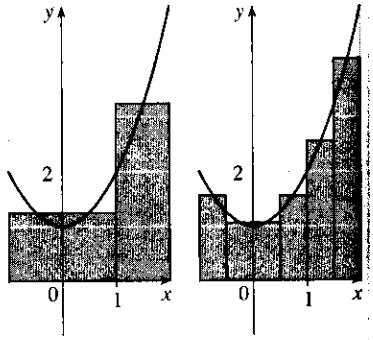
$= 1\cdot 1.25 + 1\cdot 1.25 + 1\cdot 3.25 = 5.75$

$M_6 = 0.5[f(-0.75) + f(-0.25) + f(0.25)$

$\quad + f(0.75) + f(1.25) + f(1.75)]$

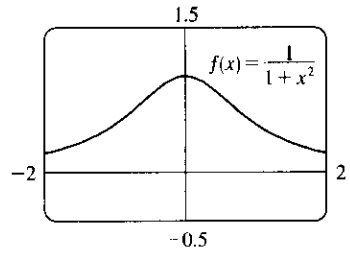
$= 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625)$

$= 0.5(11.875) = 5.9375$



(d) M_6 appears to be the best estimate.

6. (a)



(b) $f(x) = 1/(1+x^2)$ and $\Delta x = \frac{2-(-2)}{4} = 1 \Rightarrow$

(i) $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x$

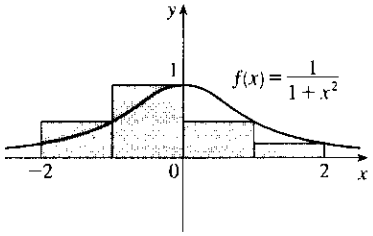
$= f(-1)\cdot 1 + f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1$

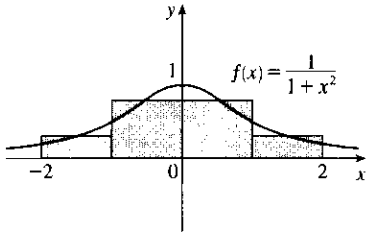
$= \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{5} = \frac{11}{5} = 2.2$

(ii) $M_4 = \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x \quad [\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)]$

$= f(-1.5)\cdot 1 + f(-0.5)\cdot 1 + f(0.5)\cdot 1 + f(1.5)\cdot 1$

$= \frac{4}{13} + \frac{4}{5} + \frac{4}{5} + \frac{4}{13} = \frac{144}{65} \approx 2.2154$





(c) $n = 8$, so $\Delta x = \frac{2-(-2)}{8} = \frac{1}{2}$.

$R_8 = \frac{1}{2}[f(-1.5) + f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$

$= \frac{1}{2}\left[\frac{4}{13} + \frac{1}{2} + \frac{4}{5} + 1 + \frac{4}{5} + \frac{1}{2} + \frac{4}{13} + \frac{1}{5}\right] = \frac{287}{130} \approx 2.2077$

$M_8 = \frac{1}{2}[f(-1.75) + f(-1.25) + f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)]$

$= \frac{1}{2}\left[2\left(\frac{16}{65} + \frac{16}{41} + \frac{16}{25} + \frac{16}{17}\right)\right] \approx 2.2176$

7. Here is one possible algorithm (ordered sequence of operations) for calculating the sums:

1 Let SUM = 0, X_MIN = 0, X_MAX = π, N = 10 (or 30 or 50, depending on which sum we are calculating), DELTA_X = (X_MAX − X_MIN)/N, and RIGHT_ENDPOINT = X_MIN + DELTA_X.

2 Repeat steps 2a, 2b in sequence until RIGHT_ENDPOINT > X_MAX.

2a Add sin (RIGHT_ENDPOINT) to SUM.

2b Add DELTA_X to RIGHT_ENDPOINT.

At the end of this procedure, (DELTA_X) · (SUM) is equal to the answer we are looking for. We find that

$R_{10} = \frac{\pi}{10}\sum_{i=1}^{10} \sin\left(\frac{i\pi}{10}\right) \approx 1.9835$, $R_{30} = \frac{\pi}{30}\sum_{i=1}^{30} \sin\left(\frac{i\pi}{30}\right) \approx 1.9982$, and $R_{50} = \frac{\pi}{50}\sum_{i=1}^{50} \sin\left(\frac{i\pi}{50}\right) \approx 1.9993$.

It appears that the exact area is 2.

Shown below is program SUMRIGHT and its output from a TI-83 Plus calculator. To generalize the program, we have input (rather than assigned) values for Xmin, Xmax, and N. Also, the function, $\sin x$, is assigned to Y_1, enabling us to evaluate any right sum merely by changing Y_1 and running the program.

```
PROGRAM:SUMRIGHT
:0→S
:Prompt Xmin
:Prompt Xmax
:Prompt N
:(Xmax-Xmin)/N→D
:Xmin+D→R
:For(I,1,N)
:S+Y1(R)→S
:R+D→R
:End
:D*S→Z
:Disp Z
```

```
prgmSUMRIGHT
Xmin=?0
Xmax=?π
N=?10
      1.983523537
             Done
```

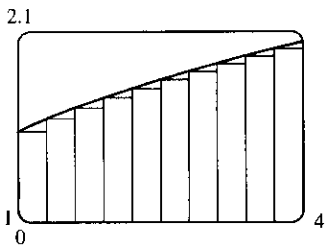
8. We can use the algorithm from Exercise 7 with X_MIN $= 1$, X_MAX $= 2$, and $1/(\text{RIGHT\_ENDPOINT})^2$ instead of sin (RIGHT_ENDPOINT) in step 2a. We find that $R_{10} = \dfrac{1}{10}\displaystyle\sum_{i=1}^{10} \frac{1}{(1+i/10)^2} \approx 0.4640$, $R_{30} = \dfrac{1}{30}\displaystyle\sum_{i=1}^{30} \frac{1}{(1+i/30)^2} \approx 0.4877$, and $R_{50} = \dfrac{1}{50}\displaystyle\sum_{i=1}^{50} \frac{1}{(1+i/50)^2} \approx 0.4926$. It appears that the exact area is $\frac{1}{2}$.

9. In Maple, we have to perform a number of steps before getting a numerical answer. After loading the student package [command: `with(student);`] we use the command `left_sum:=leftsum(x^(1/2),x=1..4,10` [or `30`, or `50`]`);` which gives us the expression in summation notation. To get a numerical approximation to the sum, we use `evalf(left_sum);`. Mathematica does not have a special command for these sums, so we must type them in manually. For example, the first left sum is given by `(3/10)*Sum[Sqrt[1+3(i-1)/10],{i,1,10}]`, and we use the `N` command on the resulting output to get a numerical approximation.

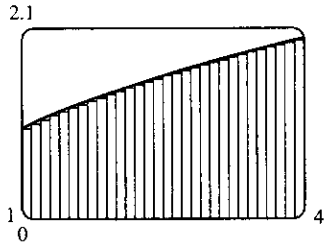
In Derive, we use the `LEFT_RIEMANN` command to get the left sums, but must define the right sums ourselves. (We can define a new function using `LEFT_RIEMANN` with k ranging from 1 to n instead of from 0 to $n-1$.)

(a) With $f(x) = \sqrt{x}$, $1 \le x \le 4$, the left sums are of the form $L_n = \dfrac{3}{n}\displaystyle\sum_{i=1}^{n} \sqrt{1 + \frac{3(i-1)}{n}}$. Specifically, $L_{10} \approx 4.5148$, $L_{30} \approx 4.6165$, and $L_{50} \approx 4.6366$. The right sums are of the form $R_n = \dfrac{3}{n}\displaystyle\sum_{i=1}^{n} \sqrt{1 + \frac{3i}{n}}$. Specifically, $R_{10} \approx 4.8148$, $R_{30} \approx 4.7165$, and $R_{50} \approx 4.6966$.

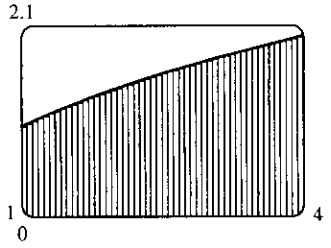
(b) In Maple, we use the `leftbox` and `rightbox` commands (with the same arguments as `leftsum` and `rightsum` above) to generate the graphs.



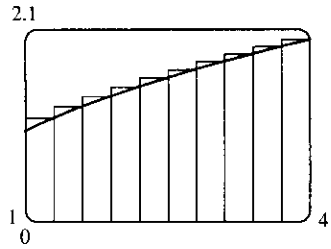
left endpoints, $n = 10$



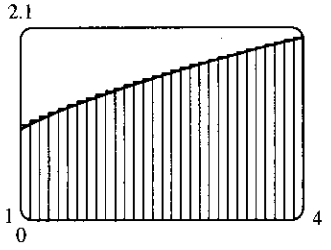
left endpoints, $n = 30$



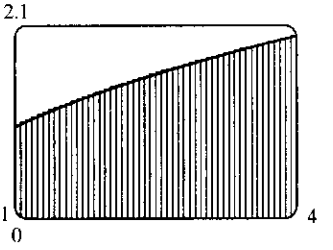
left endpoints, $n = 50$



right endpoints, $n = 10$



right endpoints, $n = 30$



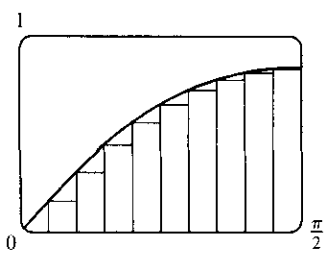
right endpoints, $n = 50$

(c) We know that since $\sqrt{x}$ is an increasing function on $(1, 4)$, all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with $n = 50$ is about $4.637 > 4.6$ and the right sum with $n = 50$ is about $4.697 < 4.7$, we conclude that $4.6 < L_{50} <$ exact area $< R_{50} < 4.7$, so the exact area is between 4.6 and 4.7.

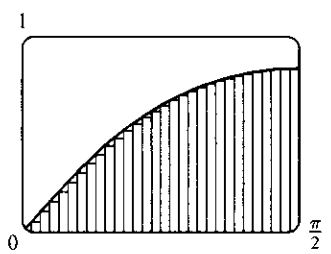
10. See the solution to Exercise 9 for the CAS commands for evaluating the sums.

(a) With $f(x) = \sin(\sin x)$, $0 \le x \le \frac{\pi}{2}$, the left sums are of the form $L_n = \dfrac{\pi}{2n} \displaystyle\sum_{i=1}^{n} \sin\left(\sin \frac{\pi\,(i-1)}{2n}\right)$. In particular, $L_{10} \approx 0.8251$, $L_{30} \approx 0.8710$, and $L_{50} \approx 0.8799$. The right sums are of the form $R_n = \dfrac{\pi}{2n} \displaystyle\sum_{i=1}^{n} \sin\left(\sin \frac{\pi i}{2n}\right)$. In particular, $R_{10} \approx 0.9573$, $R_{30} \approx 0.9150$, and $R_{50} \approx 0.9064$.

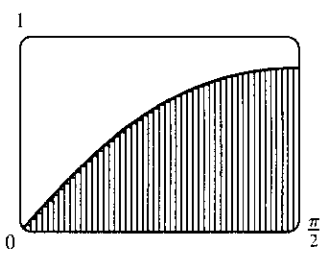
(b) In Maple, we use the `leftbox` and `rightbox` commands (with the same arguments as `leftsum` and `rightsum` above) to generate the graphs.



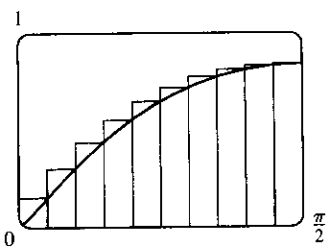
left endpoints, $n = 10$



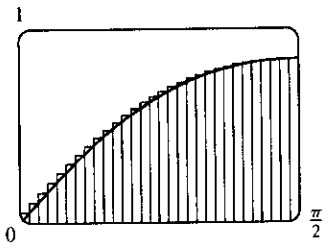
left endpoints, $n = 30$



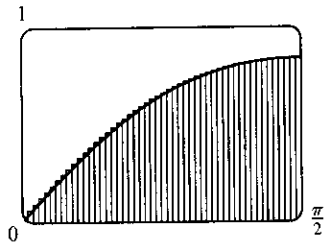
left endpoints, $n = 50$



right endpoints, $n = 10$



right endpoints, $n = 30$



right endpoints, $n = 50$

(c) We know that since $\sin(\sin x)$ is an increasing function on $\left(0, \frac{\pi}{2}\right)$ [this is true because its derivative, $-\cos(\sin x)(-\cos x)$, is positive on that interval], all of the left sums are smaller than the actual area, and all of the right sums are larger than the actual area. Since the left sum with $n = 50$ is about $0.8799 > 0.87$ and the right sum with $n = 50$ is about $0.9064 < 0.91$, we conclude that $0.87 < L_{50} <$ exact area $< R_{50} < 0.91$, so the exact area is between 0.87 and 0.91.

11. Since v is an increasing function, L_6 will give us a lower estimate and R_6 will give us an upper estimate.

$$\begin{aligned} L_6 &= (0 \text{ ft/s})(0.5 \text{ s}) + (6.2)(0.5) + (10.8)(0.5) + (14.9)(0.5) + (18.1)(0.5) + (19.4)(0.5) \\ &= 0.5(69.4) = 34.7 \text{ ft} \end{aligned}$$

$$R_6 = 0.5(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2) = 0.5(89.6) = 44.8 \text{ ft}$$

12. (a) $d \approx L_5 = (30 \text{ ft/s})(12 \text{ s}) + 28 \cdot 12 + 25 \cdot 12 + 22 \cdot 12 + 24 \cdot 12$

$$= (30 + 28 + 25 + 22 + 24) \cdot 12 = 129 \cdot 12 = 1548 \text{ ft}$$

(b) $d \approx R_5 = (28 + 25 + 22 + 24 + 27) \cdot 12 = 126 \cdot 12 = 1512$ ft

(c) The estimates are neither lower nor upper estimates since v is neither an increasing nor a decreasing function of t.

13. Lower estimate for oil leakage: $R_5 = (7.6 + 6.8 + 6.2 + 5.7 + 5.3)(2) = (31.6)(2) = 63.2$ L.

Upper estimate for oil leakage: $L_5 = (8.7 + 7.6 + 6.8 + 6.2 + 5.7)(2) = (35)(2) = 70$ L.

14. We can find an upper estimate by using the final velocity for each time interval. Thus, the distance d traveled after 62 seconds can be approximated by

$$d = \sum_{i=1}^{6} v(t_i)\Delta t_i = (185 \text{ ft/s})(10 \text{ s}) + 319 \cdot 5 + 447 \cdot 5 + 742 \cdot 12 + 1325 \cdot 27 + 1445 \cdot 3 = 54{,}694 \text{ ft}$$

15. For a decreasing function, using left endpoints gives us an overestimate and using right endpoints results in an underestimate. We will use M_6 to get an estimate. $\Delta t = 1$, so

$$\begin{aligned} M_6 &= 1[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5)] \\ &\approx 55 + 40 + 28 + 18 + 10 + 4 = 155 \text{ ft} \end{aligned}$$

For a very rough check on the above calculation, we can draw a line from $(0, 70)$ to $(6, 0)$ and calculate the area of the triangle: $\frac{1}{2}(70)(6) = 210$. This is clearly an overestimate, so our midpoint estimate of 155 is reasonable.

16. For an increasing function, using left endpoints gives us an underestimate and using right endpoints results in an overestimate. We will use M_6 to get an estimate. $\Delta t = \frac{30-0}{6} = 5 \text{ s} = \frac{5}{3600} \text{ h} = \frac{1}{720} \text{ h}$.

$$\begin{aligned} M_6 &= \tfrac{1}{720}[v(2.5) + v(7.5) + v(12.5) + v(17.5) + v(22.5) + v(27.5)] \\ &= \tfrac{1}{720}(31.25 + 66 + 88 + 103.5 + 113.75 + 119.25) = \tfrac{1}{720}(521.75) \approx 0.725 \text{ km} \end{aligned}$$

For a very rough check on the above calculation, we can draw a line from $(0, 0)$ to $(30, 120)$ and calculate the area

of the triangle: $\frac{1}{2}(30)(120) = 1800$. Divide by 3600 to get 0.5, which is clearly an underestimate, making our midpoint estimate of 0.725 seem reasonable. Of course, answers will vary due to different readings of the graph.

17. $f(x) = \sqrt[4]{x},\ 1 \le x \le 16.\quad \Delta x = (16-1)/n = 15/n$ and $x_i = 1 + i\,\Delta x = 1 + 15i/n.$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[4]{1 + \tfrac{15i}{n}} \cdot \tfrac{15}{n}.$$

18. $f(x) = 1 + x^4,\ 2 \le x \le 5.\quad \Delta x = (5-2)/n = 3/n$ and $x_i = 2 + i\,\Delta x = 2 + 3i/n.$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[1 + \left(2 + \frac{3i}{n}\right)^4\right] \cdot \frac{3}{n}.$$

19. $f(x) = x\cos x,\ 0 \le x \le \frac{\pi}{2}.\quad \Delta x = (\frac{\pi}{2} - 0)/n = \frac{\pi}{2}/n$ and $x_i = 0 + i\,\Delta x = \frac{\pi}{2}i/n.$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i\pi}{2n} \cos\left(\frac{i\pi}{2n}\right) \cdot \frac{\pi}{2n}.$$

20. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(5 + \frac{2i}{n}\right)^{10}$ can be interpreted as the area of the region lying under the graph of $y = (5+x)^{10}$ on the interval $[0, 2]$, since for $y = (5+x)^{10}$ on $[0,2]$ with $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$, $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$, and $x_i^* = x_i$, the expression for the area is $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(5 + \dfrac{2i}{n}\right)^{10} \dfrac{2}{n}$. Note that the answer is not unique. We could use $y = x^{10}$ on $[5, 7]$ or, in general, $y = ((5-n) + x)^{10}$ on $[n, n+2]$.

21. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n}$ can be interpreted as the area of the region lying under the graph of $y = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$, since for $y = \tan x$ on $\left[0, \frac{\pi}{4}\right]$ with $\Delta x = \dfrac{\pi/4 - 0}{n} = \dfrac{\pi}{4n}$, $x_i = 0 + i\,\Delta x = \dfrac{i\pi}{4n}$, and $x_i^* = x_i$, the expression for the area is $A = \lim_{n\to\infty} \sum_{i=1}^{n} f\,(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \tan\left(\dfrac{i\pi}{4n}\right) \dfrac{\pi}{4n}$. Note that this answer is not unique, since the expression for the area is the same for the function $y = \tan(x - k\pi)$ on the interval $\left[k\pi, k\pi + \frac{\pi}{4}\right]$, where k is any integer.

22. (a) $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{i}{n}$. $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\dfrac{i}{n}\right)^3 \cdot \dfrac{1}{n}.$

(b) $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^3} \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2 = \lim_{n\to\infty} \frac{(n+1)^2}{4n^2} = \frac{1}{4}\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^2 = \frac{1}{4}$$

23. (a) $y = f(x) = x^5.\ \Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^5 \cdot \frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{32i^5}{n^5} \cdot \frac{2}{n} = \lim_{n\to\infty} \frac{64}{n^6} \sum_{i=1}^{n} i^5.$$

(b) $$\sum_{i=1}^{n} i^5 \overset{\text{CAS}}{=} \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

(c) $$\lim_{n\to\infty} \frac{64}{n^6} \cdot \frac{n^2(n+1)^2(2n^2+2n-1)}{12} = \frac{64}{12}\lim_{n\to\infty} \frac{(n^2+2n+1)(2n^2+2n-1)}{n^2 \cdot n^2}$$
$$= \frac{16}{3}\lim_{n\to\infty}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)\left(2 + \frac{2}{n} - \frac{1}{n^2}\right) = \tfrac{16}{3} \cdot 1 \cdot 2 = \tfrac{32}{3}$$

24. (a) $y = f(x) = x^4 + 5x^2 + x,\ 2 \le x \le 7 \ \Rightarrow\ \Delta x = \dfrac{7-2}{n} = \dfrac{5}{n},\ x_i = 2 + i\,\Delta x = 2 + \dfrac{5i}{n} \ \Rightarrow$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} \left[\left(2 + \frac{5i}{n}\right)^4 + 5\left(2 + \frac{5i}{n}\right)^2 + \left(2 + \frac{5i}{n}\right) \right]$$

(b) $R_n = \dfrac{5}{n} \cdot \dfrac{4723n^4 + 7845n^3 + 3475n^2 - 125}{6n^3}$

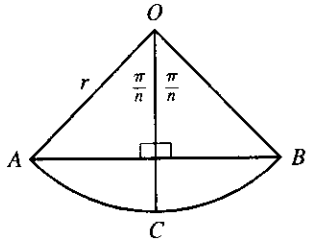
(c) $A = \lim\limits_{n\to\infty} R_n = \dfrac{23{,}615}{6} = 3935.8\overline{3}$

25. $y = f(x) = \cos x.\ \Delta x = \dfrac{b-0}{n} = \dfrac{b}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{bi}{n}$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{bi}{n}\right) \cdot \frac{b}{n} \overset{\text{CAS}}{=} \lim_{n\to\infty} \left[\frac{b \sin\left(b\left(\frac{1}{2n} + 1\right)\right)}{2n \sin\left(\frac{b}{2n}\right)} - \frac{b}{2n} \right] \overset{\text{CAS}}{=} \sin b$$

If $b = \frac{\pi}{2}$, then $A = \sin\frac{\pi}{2} = 1$. The value 1 is the *exact* area, whereas the value obtained in Example 3(b), 1.006, is just an *approximation*.

26. (a)



The diagram shows one of the n congruent triangles, $\triangle AOB$, with central angle $2\pi/n$. O is the center of the circle and AB is one of the sides of the polygon. Radius OC is drawn so as to bisect $\angle AOB$. It follows that OC intersects AB at right angles and bisects AB. Thus, $\triangle AOB$ is divided into two right triangles with legs of length $\frac{1}{2}(AB) = r\sin(\pi/n)$ and $r\cos(\pi/n)$.

$\triangle AOB$ has area $2 \cdot \frac{1}{2}[r\sin(\pi/n)][r\cos(\pi/n)] = r^2\sin(\pi/n)\cos(\pi/n) = \frac{1}{2}r^2\sin(2\pi/n)$, so $A_n = n \cdot \text{area}(\triangle AOB) = \frac{1}{2}nr^2\sin(2\pi/n)$.

(b) To use Equation 3.5.2, $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$, we need to have the same expression in the denominator as we have in the argument of the sine function—in this case, $2\pi/n$.

$\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty} \frac{1}{2}nr^2\sin(2\pi/n) = \lim\limits_{n\to\infty} \frac{1}{2}nr^2 \dfrac{\sin(2\pi/n)}{2\pi/n} \cdot \dfrac{2\pi}{n} = \lim\limits_{n\to\infty} \dfrac{\sin(2\pi/n)}{2\pi/n}\pi r^2$. Let $\theta = \dfrac{2\pi}{n}$.

Then as $n \to \infty$, $\theta \to 0$, so $\lim\limits_{n\to\infty} \dfrac{\sin(2\pi/n)}{2\pi/n}\pi r^2 = \lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}\pi r^2 = (1)\,\pi r^2 = \pi r^2$.

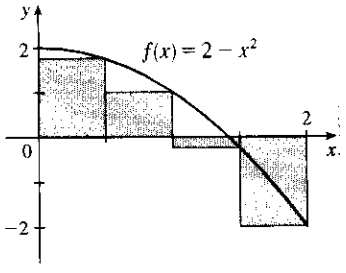
5.2 The Definite Integral

1. $R_4 = \sum\limits_{i=1}^{4} f(x_i)\,\Delta x \quad [x_i^* = x_i \text{ is a right endpoint and } \Delta x = 0.5]$

$= 0.5\,[f(0.5) + f(1) + f(1.5) + f(2)] \quad [f(x) = 2 - x^2]$

$= 0.5\,[1.75 + 1 + (-0.25) + (-2)]$

$= 0.5(0.5) = 0.25$



The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.

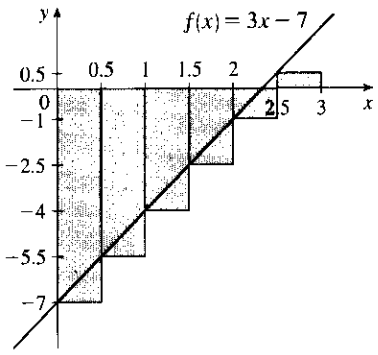
2. $L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \quad [x_i^* = x_{i-1}$ is a left endpoint and $\Delta x = 0.5]$

$= 0.5\,[f(0) + f(0.5) + f(1) + f(1.5) + f(2) + f(2.5)] \quad [f(x) = 3x - 7]$

$= 0.5[-7 + (-5.5) + (-4) + (-2.5) + (-1) + 0.5]$

$= 0.5(-19.5) = -9.75$

The Riemann sum represents the area of the rectangle above the x-axis minus the area of the five rectangles below the x-axis.

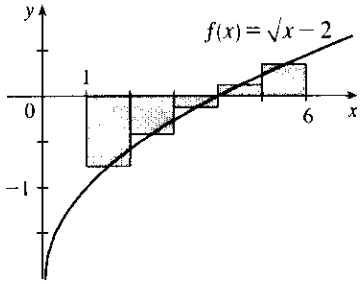


3. $M_5 = \sum_{i=1}^{5} f(\bar{x}_i)\,\Delta x \quad [x_i^* = \bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ is a midpoint and $\Delta x = 1]$

$= 1\,[f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)] \quad [f(x) = \sqrt{x} - 2]$

≈ -0.856759

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis.

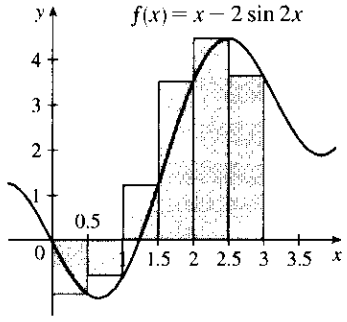


4. (a) $R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x \quad [x_i^* = x_i$ is a right endpoint and $\Delta x = 0.5]$

$= 0.5\,[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3)] \quad [f(x) = x - 2\sin 2x]$

≈ 5.353254

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

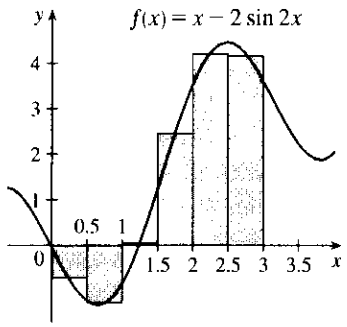


(b) $M_6 = \sum_{i=1}^{6} f(\bar{x}_i)\,\Delta x \quad [x_i^* = \bar{x}_i$ is a midpoint and $\Delta x = 0.5]$

$= 0.5[f(0.25) + f(0.75) + f(1.25) + f(1.75) + f(2.25) + f(2.75)] \quad [f(x) = x - 2\sin 2x]$

≈ 4.458461

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.



5. $\Delta x = (b - a)/n = (8 - 0)/4 = 8/4 = 2.$

(a) Using the right endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(x_i)\,\Delta x = 2[f(2) + f(4) + f(6) + f(8)] \approx 2[1 + 2 + (-2) + 1] = 4.$$

(b) Using the left endpoints to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(x_{i-1})\,\Delta x = 2[f(0) + f(2) + f(4) + f(6)] \approx 2[2 + 1 + 2 + (-2)] = 6.$$

(c) Using the midpoint of each subinterval to approximate $\int_0^8 f(x)\,dx$, we have

$$\sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = 2[f(1)+f(3)+f(5)+f(7)] \approx 2[3+2+1+(-1)] = 10.$$

6. (a) Using the right endpoints to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\begin{aligned}\sum_{i=1}^{6} g(x_i)\,\Delta x &= 1[g(-2)+g(-1)+g(0)+g(1)+g(2)+g(3)]\\ &\approx 1-0.5-1.5-1.5-0.5+2.5=-0.5\end{aligned}$$

(b) Using the left endpoints to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\begin{aligned}\sum_{i=1}^{6} g(x_{i-1})\,\Delta x &= 1[g(-3)+g(-2)+g(-1)+g(0)+g(1)+g(2)]\\ &\approx 2+1-0.5-1.5-1.5-0.5=-1\end{aligned}$$

(c) Using the midpoint of each subinterval to approximate $\int_{-3}^{3} g(x)\,dx$, we have

$$\begin{aligned}\sum_{i=1}^{6} g(\overline{x}_i)\,\Delta x &= 1[g(-2.5)+g(-1.5)+g(-0.5)+g(0.5)+g(1.5)+g(2.5)]\\ &\approx 1.5+0-1-1.75-1+0.5=-1.75\end{aligned}$$

7. Since f is increasing, $L_5 \le \int_0^{25} f(x)\,dx \le R_5$.

$$\begin{aligned}\text{Lower estimate} = L_5 &= \sum_{i=1}^{5} f(x_{i-1})\,\Delta x = 5[f(0)+f(5)+f(10)+f(15)+f(20)]\\ &= 5(-42-37-25-6+15) = 5(-95) = -475\end{aligned}$$

$$\begin{aligned}\text{Upper estimate} = R_5 &= \sum_{i=1}^{5} f(x_i)\,\Delta x = 5[f(5)+f(10)+f(15)+f(20)+f(25)]\\ &= 5(-37-25-6+15+36) = 5(-17) = -85\end{aligned}$$

8. (a) Using the right endpoints to approximate $\int_0^6 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(x_i)\,\Delta x = 2[f(2)+f(4)+f(6)] = 2(8.3+2.3-10.5) = 0.2$$

(b) Using the left endpoints to approximate $\int_0^6 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(x_{i-1})\,\Delta x = 2[f(0)+f(2)+f(4)] = 2(9.3+8.3+2.3) = 39.8$$

(c) Using the midpoint of each interval to approximate $\int_0^6 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(\overline{x}_i)\,\Delta x = 2[f(1)+f(3)+f(5)] = 2(9.0+6.5-7.6) = 15.8.$$

The estimate using the right endpoints must be less than $\int_0^6 f(x)\,dx$, since if we take x_i^* to be the right endpoint x_i of each interval, then $f(x_i) \le f(x)$ for all x on $[x_{i-1}, x_i]$, which implies that $f(x_i)\,\Delta x \le \int_{x_{i-1}}^{x_i} f(x)\,dx$, and so the sum $\sum_{i=1}^{3}[f(x_i)\,\Delta x] \le \sum_{i=1}^{3}\left[\int_{x_{i-1}}^{x_i} f(x)\,dx\right] = \int_0^6 f(x)\,dx$. Similarly, if we take x_i^* to be the left endpoint x_{i-1} of each interval, then $f(x_{i-1}) \ge f(x)$ for all x on $[x_{i-1}, x_i]$, and so $\sum_{i=1}^{3}[f(x_{i-1})\,\Delta x] \ge \int_0^6 f(x)\,dx$. We cannot say anything about the midpoint estimate.

9. $\Delta x = (10-2)/4 = 2$, so the endpoints are 2, 4, 6, 8, and 10, and the midpoints are 3, 5, 7, and 9. The Midpoint Rule gives $\int_2^{10} \sqrt{x^3+1}\,dx \approx \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = 2\left(\sqrt{3^3+1}+\sqrt{5^3+1}+\sqrt{7^3+1}+\sqrt{9^3+1}\right) \approx 124.1644.$

10. $\Delta x = (\pi-0)/6 = \frac{\pi}{6}$, so the endpoints are 0, $\frac{\pi}{6}$, $\frac{2\pi}{6}$, $\frac{3\pi}{6}$, $\frac{4\pi}{6}$, $\frac{5\pi}{6}$, and $\frac{6\pi}{6}$, and the midpoints are $\frac{\pi}{12}$, $\frac{3\pi}{12}$, $\frac{5\pi}{12}$, $\frac{7\pi}{12}$, $\frac{9\pi}{12}$, and $\frac{11\pi}{12}$. The Midpoint Rule gives

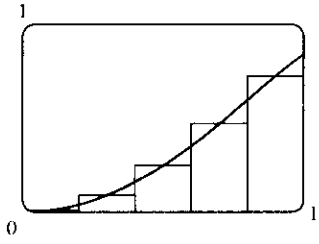
$$\int_0^{\pi} \sec(x/3)\,dx \approx \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x = \tfrac{\pi}{6}\left(\sec\tfrac{\pi}{36}+\sec\tfrac{3\pi}{36}+\sec\tfrac{5\pi}{36}+\sec\tfrac{7\pi}{36}+\sec\tfrac{9\pi}{36}+\sec\tfrac{11\pi}{36}\right) \approx 3.9379.$$

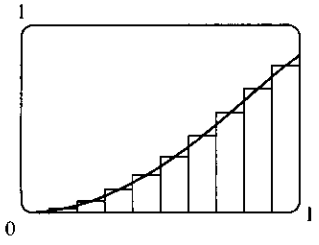
11. $\Delta x = (1-0)/5 = 0.2$, so the endpoints are 0, 0.2, 0.4, 0.6, 0.8, and 1, and the midpoints are 0.1, 0.3, 0.5, 0.7, and 0.9. The Midpoint Rule gives

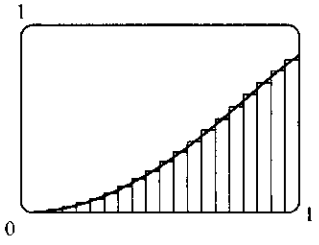
$$\int_0^1 \sin(x^2)\,dx \approx \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x = 0.2\left[\sin(0.1)^2+\sin(0.3)^2+\sin(0.5)^2+\sin(0.7)^2+\sin(0.9)^2\right] \approx 0.3084.$$

12. $\Delta x = \dfrac{5-1}{4} = 1$, so the endpoints are 1, 2, 3, 4, and 5, and the midpoints are 1.5, 2.5, 3.5, and 4.5. The Midpoint Rule gives $\displaystyle\int_1^5 \frac{x-1}{x+1}\,dx \approx \sum_{i=1}^{4} f(\overline{x}_i)\Delta x = 1\left[\frac{1.5-1}{1.5+1}+\frac{2.5-1}{2.5+1}+\frac{3.5-1}{3.5+1}+\frac{4.5-1}{4.5+1}\right] \approx 1.8205.$

13. In Maple, we use the command `with(student);` to load the sum and box commands, then `m:=middlesum(sin(x^2),x=0..1,5);` which gives us the sum in summation notation, then `M:=evalf(m);` which gives $M_5 \approx 0.30843908$, confirming the result of Exercise 11. The command `middlebox(sin(x^2),x=0..1,5)` generates the graph. Repeating for $n = 10$ and $n = 20$ gives $M_{10} \approx 0.30981629$ and $M_{20} \approx 0.31015563$.







14. See the solution to Exercise 5.1.7 for a possible algorithm to calculate the sums. With $\Delta x = (1-0)/100 = 0.01$ and subinterval endpoints 1, 1.01, 1.02, ..., 1.99, 2, we calculate that the left Riemann sum is $L_{100} = \sum_{i=1}^{100} \sin(x_{i-1}^2)\,\Delta x \approx 0.30607$, and the right Riemann sum is $R_{100} = \sum_{i=1}^{100} \sin\left(x_i^2\right)\,\Delta x \approx 0.31448$. Since $f(x) = \sin(x^2)$ is an increasing function, we must have $L_{100} \le \int_0^1 \sin(x^2)\,dx \le R_{100}$, so $0.306 < L_{100} \le \int_0^1 \sin(x^2)\,dx \le R_{100} < 0.315$. Therefore, the approximate value $0.3084 \approx 0.31$ in Exercise 11 must be accurate to two decimal places.

15. We'll create the table of values to approximate $\int_0^\pi \sin x\,dx$ by using the program in the solution to Exercise 5.1.7 with $Y_1 = \sin x$, Xmin $= 0$, Xmax $= \pi$, and $n = 5$, 10, 50, and 100.

The values of R_n appear to be approaching 2.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

16. $\int_0^2 \sqrt{1+x^4}\,dx$ with $n = 5$, 10, 50, and 100.

n	L_n	R_n
5	3.080614	4.329856
10	3.354110	3.978731
50	3.591540	3.716464
100	3.622383	3.684845

The value of the integral lies between 3.622 and 3.685. Note that $f(x) = \sqrt{1+x^4}$ is increasing on $(0, 2)$. We cannot make a similar statement for $\int_{-1}^2 \sqrt{1+x^4}\,dx$ since f is decreasing on $(-1, 0)$.

17. On $[0, \pi]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n x_i \sin x_i\,\Delta x = \int_0^\pi x \sin x\,dx$.

18. On $[1, 5]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n \dfrac{x_i}{1+x_i}\,\Delta x = \displaystyle\int_1^5 \frac{x}{1+x}\,dx$.

19. On $[1, 8]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n \sqrt{2x_i^* + (x_i^*)^2}\,\Delta x = \int_1^8 \sqrt{2x+x^2}\,dx$.

20. On $[0, 2]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n [4 - 3(x_i^*)^2 + 6(x_i^*)^5]\,\Delta x = \int_0^2 (4 - 3x^2 + 6x^5)\,dx$.

21. Note that $\Delta x = \dfrac{5-(-1)}{n} = \dfrac{6}{n}$ and $x_i = -1 + i\,\Delta x = -1 + \dfrac{6i}{n}$.

$$\begin{aligned}
\int_{-1}^5 (1+3x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^n \left[1 + 3\left(-1 + \frac{6i}{n}\right)\right]\frac{6}{n} \\
&= \lim_{n\to\infty} \frac{6}{n} \sum_{i=1}^n \left[-2 + \frac{18i}{n}\right] = \lim_{n\to\infty} \frac{6}{n}\left[\sum_{i=1}^n (-2) + \sum_{i=1}^n \frac{18i}{n}\right] \\
&= \lim_{n\to\infty} \frac{6}{n}\left[-2n + \frac{18}{n}\sum_{i=1}^n i\right] = \lim_{n\to\infty} \frac{6}{n}\left[-2n + \frac{18}{n}\cdot\frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left[-12 + \frac{108}{n^2}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty} \left[-12 + 54\frac{n+1}{n}\right] \\
&= \lim_{n\to\infty} \left[-12 + 54\left(1 + \frac{1}{n}\right)\right] = -12 + 54\cdot 1 = 42
\end{aligned}$$

22. $\displaystyle\int_1^4 (x^2+2x-5)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \qquad [\Delta x = 3/n \text{ and } x_i = 1+3i/n]$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(1+\frac{3i}{n}\right)^2 + 2\left(1+\frac{3i}{n}\right) - 5\right]\left(\frac{3}{n}\right)$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^{n}\left(1+\frac{6i}{n}+\frac{9i^2}{n^2}+2+\frac{6i}{n}-5\right)\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^{n}\left(\frac{9}{n^2}\cdot i^2+\frac{12}{n}\cdot i-2\right)\right]$$

$$= \lim_{n\to\infty}\frac{3}{n}\left[\frac{9}{n^2}\sum_{i=1}^{n} i^2+\frac{12}{n}\sum_{i=1}^{n} i-\sum_{i=1}^{n} 2\right]$$

$$= \lim_{n\to\infty}\left(\frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{36}{n^2}\cdot\frac{n(n+1)}{2}-\frac{6}{n}\cdot n\right)$$

$$= \lim_{n\to\infty}\left(\frac{9}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}+18\cdot\frac{n+1}{n}-6\right)$$

$$= \lim_{n\to\infty}\left[\frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+18\left(1+\frac{1}{n}\right)-6\right] = \tfrac{9}{2}\cdot 1\cdot 2+18\cdot 1-6 = 21$$

23. Note that $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$$\int_0^2 (2-x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(2-\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$= \lim_{n\to\infty}\frac{2}{n}\left[\sum_{i=1}^{n} 2-\frac{4}{n^2}\sum_{i=1}^{n} i^2\right] = \lim_{n\to\infty}\frac{2}{n}\left(2n-\frac{4}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$= \lim_{n\to\infty}\left[4-\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to\infty}\left(4-\frac{4}{3}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty}\left[4-\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 4-\tfrac{4}{3}\cdot 1\cdot 2 = \tfrac{4}{3}$$

24. $\displaystyle\int_0^5 (1+2x^3)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x \qquad [\Delta x = 5/n \text{ and } x_i = 5i/n]$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+2\cdot\frac{125i^3}{n^3}\right)\left(\frac{5}{n}\right) = \lim_{n\to\infty}\frac{5}{n}\left[\sum_{i=1}^{n} 1+\frac{250}{n^3}\sum_{i=1}^{n} i^3\right]$$

$$= \lim_{n\to\infty}\frac{5}{n}\left(1\cdot n+\frac{250}{n^3}\sum_{i=1}^{n} i^3\right) = \lim_{n\to\infty}\left[5+\frac{1250}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty}\left[5+312.5\cdot\frac{(n+1)^2}{n^2}\right] = \lim_{n\to\infty}\left[5+312.5\left(1+\frac{1}{n}\right)^2\right]$$

$$= 5+312.5 = 317.5$$

25. Note that $\Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}$ and $x_i = 1 + i\,\Delta x = 1 + i(1/n) = 1 + i/n$.

$$\int_1^2 x^3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^3\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{n+i}{n}\right)^3$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}(n^3+3n^2i+3ni^2+i^3) = \lim_{n\to\infty}\frac{1}{n^4}\left[\sum_{i=1}^{n}n^3+\sum_{i=1}^{n}3n^2i+\sum_{i=1}^{n}3ni^2+\sum_{i=1}^{n}i^3\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[n\cdot n^3+3n^2\sum_{i=1}^{n}i+3n\sum_{i=1}^{n}i^2+\sum_{i=1}^{n}i^3\right]$$

$$= \lim_{n\to\infty}\left[1+\frac{3}{n^2}\cdot\frac{n(n+1)}{2}+\frac{3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}+\frac{1}{n^4}\cdot\frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty}\left[1+\frac{3}{2}\cdot\frac{n+1}{n}+\frac{1}{2}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n}+\frac{1}{4}\cdot\frac{(n+1)^2}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[1+\frac{3}{2}\left(1+\frac{1}{n}\right)+\frac{1}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\frac{1}{4}\left(1+\frac{1}{n}\right)^2\right] = 1+\frac{3}{2}+\frac{1}{2}\cdot 2+\frac{1}{4} = 3.75$$

26. (a) $\Delta x = (4-0)/8 = 0.5$ and $x_i^* = x_i = 0.5i$.

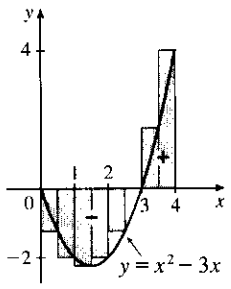
$$\int_0^4 (x^2-3x)\,dx \approx \sum_{i=1}^{8} f(x_i^*)\,\Delta x$$

$$= 0.5\{[0.5^2-3(0.5)]+[1.0^2-3(1.0)]+\cdots$$

$$+[3.5^2-3(3.5)]+[4.0^2-3(4.0)]\}$$

$$= \tfrac{1}{2}\left(-\tfrac{5}{4}-2-\tfrac{9}{4}-2-\tfrac{5}{4}+0+\tfrac{7}{4}+4\right) = -1.5$$

(b)



(c) $$\int_0^4 (x^2-3x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{4i}{n}\right)^2-3\left(\frac{4i}{n}\right)\right]\left(\frac{4}{n}\right)$$

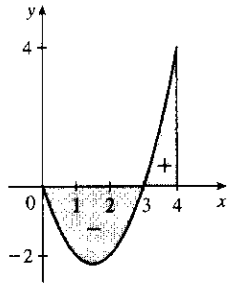
$$= \lim_{n\to\infty}\frac{4}{n}\left[\frac{16}{n^2}\sum_{i=1}^{n}i^2-\frac{12}{n}\sum_{i=1}^{n}i\right]$$

$$= \lim_{n\to\infty}\left[\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}-\frac{48}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-24\left(1+\frac{1}{n}\right)\right]$$

$$= \tfrac{32}{3}\cdot 2-24 = -\tfrac{8}{3}$$

(d) $\int_0^4 (x^2-3x)\,dx = A_1 - A_2$, where A_1 is the area marked + and A_2 is the area marked −.



27. $$\int_a^b x\,dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a+\frac{b-a}{n}i\right] = \lim_{n\to\infty}\left[\frac{a(b-a)}{n}\sum_{i=1}^{n}1+\frac{(b-a)^2}{n^2}\sum_{i=1}^{n}i\right]$$

$$= \lim_{n\to\infty}\left[\frac{a(b-a)}{n}n+\frac{(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2}\right] = a(b-a)+\lim_{n\to\infty}\frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)$$

$$= a(b-a)+\tfrac{1}{2}(b-a)^2 = (b-a)\left(a+\tfrac{1}{2}b-\tfrac{1}{2}a\right) = (b-a)\tfrac{1}{2}(b+a) = \tfrac{1}{2}(b^2-a^2)$$

28. $\displaystyle\int_a^b x^2\,dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a+\frac{b-a}{n}i\right]^2 = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a^2+2a\frac{b-a}{n}i+\frac{(b-a)^2}{n^2}i^2\right]$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\sum_{i=1}^{n}i^2+\frac{2a(b-a)^2}{n^2}\sum_{i=1}^{n}i+\frac{a^2(b-a)}{n}\sum_{i=1}^{n}1\right]$$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{2a(b-a)^2}{n^2}\frac{n(n+1)}{2}+\frac{a^2(b-a)}{n}n\right]$$

$$= \lim_{n\to\infty}\left[\frac{(b-a)^3}{6}\cdot 1\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+a(b-a)^2\cdot 1\cdot\left(1+\frac{1}{n}\right)+a^2(b-a)\right]$$

$$= \frac{(b-a)^3}{3}+a(b-a)^2+a^2(b-a) = \frac{b^3-3ab^2+3a^2b-a^3}{3}+ab^2-2a^2b+a^3+a^2b-a^3$$

$$= \frac{b^3}{3}-\frac{a^3}{3}-ab^2+a^2b+ab^2-a^2b = \frac{b^3-a^3}{3}$$

29. $f(x)=\dfrac{x}{1+x^5}$, $a=2$, $b=6$, and $\Delta x=\dfrac{6-2}{n}=\dfrac{4}{n}$. Using Equation 3, we get $x_i^*=x_i=2+i\,\Delta x=2+\dfrac{4i}{n}$,

so $\displaystyle\int_2^6\frac{x}{1+x^5}\,dx=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5}\cdot\frac{4}{n}$.

30. $f(x)=x^2\sin x$, $[a,b]=[0,2\pi]$, $\Delta x=\dfrac{2\pi-0}{n}=\dfrac{2\pi}{n}$, and $x_i=a+i\,\Delta x=\dfrac{2\pi i}{n}$, so

$$\int_0^{2\pi}x^2\sin x\,dx=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{2\pi i}{n}\right)^2\sin\left(\frac{2\pi i}{n}\right)\right]\cdot\frac{2\pi}{n}.$$

31. $\Delta x=(\pi-0)/n=\pi/n$ and $x_i^*=x_i=\pi i/n$.

$$\int_0^{\pi}\sin 5x\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}(\sin 5x_i)\left(\frac{\pi}{n}\right)=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\sin\frac{5\pi i}{n}\right)\frac{\pi}{n}\overset{\text{CAS}}{=}\pi\lim_{n\to\infty}\frac{1}{n}\cot\left(\frac{5\pi}{2n}\right)\overset{\text{CAS}}{=}\pi\left(\frac{2}{5\pi}\right)=\frac{2}{5}$$

32. $\Delta x=(10-2)/n=8/n$ and $x_i^*=x_i=2+8i/n$.

$$\int_2^{10}x^6\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)^6\left(\frac{8}{n}\right)=8\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)^6$$

$$\overset{\text{CAS}}{=}8\lim_{n\to\infty}\frac{1}{n}\cdot\frac{64(58{,}593n^6+164{,}052n^5+131{,}208n^4-27{,}776n^2+2048)}{21n^5}$$

$$\overset{\text{CAS}}{=}8\left(\tfrac{1{,}249{,}984}{7}\right)=\tfrac{9{,}999{,}872}{7}\approx 1{,}428{,}553.1$$

33. (a) Think of $\int_0^2 f(x)\,dx$ as the area of a trapezoid with bases 1 and 3 and height 2. The area of a trapezoid is $A=\frac{1}{2}(b+B)h$, so $\int_0^2 f(x)\,dx=\frac{1}{2}(1+3)2=4$.

(b) $\int_0^5 f(x)\,dx=\int_0^2 f(x)\,dx+\int_2^3 f(x)\,dx+\int_3^5 f(x)\,dx$

$$\begin{matrix}&\text{trapezoid}&&\text{rectangle}&&\text{triangle}\\ =\ \tfrac{1}{2}(1+3)2&+&3\cdot 1&+&\tfrac{1}{2}\cdot 2\cdot 3&=4+3+3=10\end{matrix}$$

(c) $\int_5^7 f(x)\,dx$ is the negative of the area of the triangle with base 2 and height 3. $\int_5^7 f(x)\,dx=-\frac{1}{2}\cdot 2\cdot 3=-3$.

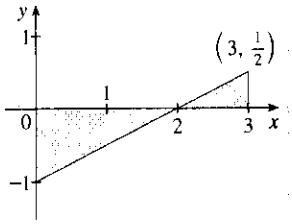
(d) $\int_7^9 f(x)\,dx$ is the negative of the area of a trapezoid with bases 3 and 2 and height 2, so it equals $-\frac{1}{2}(B+b)h = -\frac{1}{2}(3+2)2 = -5$. Thus,
$\int_0^9 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^7 f(x)\,dx + \int_7^9 f(x)\,dx = 10 + (-3) + (-5) = 2.$

34. (a) $\int_0^2 g(x)\,dx = \frac{1}{2}\cdot 4\cdot 2 = 4$ (area of a triangle)

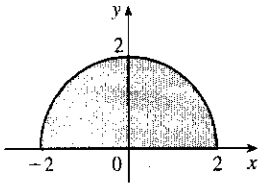
(b) $\int_2^6 g(x)\,dx = -\frac{1}{2}\pi(2)^2 = -2\pi$ (negative of the area of a semicircle)

(c) $\int_6^7 g(x)\,dx = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$ (area of a triangle)
$\int_0^7 g(x)\,dx = \int_0^2 g(x)\,dx + \int_2^6 g(x)\,dx + \int_6^7 g(x)\,dx = 4 - 2\pi + \frac{1}{2} = 4.5 - 2\pi$

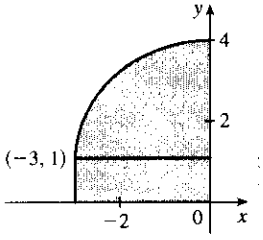
35. $\int_0^3 \left(\frac{1}{2}x - 1\right)dx$ can be interpreted as the area of the triangle above the x-axis minus the area of the triangle below the x-axis; that is,
$\frac{1}{2}(1)\left(\frac{1}{2}\right) - \frac{1}{2}(2)(1) = \frac{1}{4} - 1 = -\frac{3}{4}$.



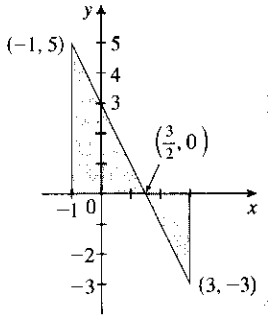
36. $\int_{-2}^2 \sqrt{4-x^2}\,dx$ can be interpreted as the area under the graph of $f(x) = \sqrt{4-x^2}$ between $x = -2$ and $x = 2$. This is equal to half the area of the circle with radius 2, so $\int_{-2}^2 \sqrt{4-x^2}\,dx = \frac{1}{2}\pi\cdot 2^2 = 2\pi$.



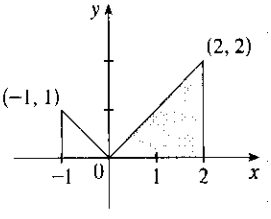
37. $\int_{-3}^0 \left(1 + \sqrt{9-x^2}\right)dx$ can be interpreted as the area under the graph of $f(x) = 1 + \sqrt{9-x^2}$ between $x = -3$ and $x = 0$. This is equal to one-quarter the area of the circle with radius 3, plus the area of the rectangle, so
$\int_{-3}^0 \left(1 + \sqrt{9-x^2}\right)dx = \frac{1}{4}\pi\cdot 3^2 + 1\cdot 3 = 3 + \frac{9}{4}\pi$.



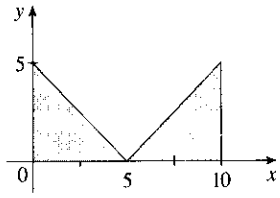
38. $\int_{-1}^3 (3-2x)\,dx$ can be interpreted as the area of the triangle above the x-axis minus the area of the triangle below the x-axis; that is,
$\frac{1}{2}\left(\frac{5}{2}\right)(5) - \frac{1}{2}\left(\frac{3}{2}\right)(3) = \frac{25}{4} - \frac{9}{4} = 4$.



39. $\int_{-1}^2 |x|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$.



40. $\int_0^{10} |x-5|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $2\left(\frac{1}{2}\right)(5)(5) = 25$.



41. $\int_9^4 \sqrt{t}\,dt = -\int_4^9 \sqrt{t}\,dt$ [because we reversed the limits of integration]
$= -\int_4^9 \sqrt{x}\,dx$ [we can use any letter without changing the value of the integral]
$= -\frac{38}{3}$

42. $\int_1^1 x^2 \cos x\,dx = 0$ since the limits of integration are equal.

43. $\int_0^1 (5-6x^2)\,dx = \int_0^1 5\,dx - 6\int_0^1 x^2\,dx = 5(1-0) - 6\left(\frac{1}{3}\right) = 5-2 = 3$

44. $\int_2^5 (1+3x^4)\,dx = \int_2^5 1\,dx + \int_2^5 3x^4\,dx = 1(5-2) + 3\int_2^5 x^4\,dx = 1(3) + 3(618.6) = 1858.8$

45. $\int_1^4 (2x^2-3x+1)\,dx = 2\int_1^4 x^2\,dx - 3\int_1^4 x\,dx + \int_1^4 1\,dx$
$= 2\cdot\frac{1}{3}(4^3-1^3) - 3\cdot\frac{1}{2}(4^2-1^2) + 1(4-1) = \frac{45}{2} = 22.5$

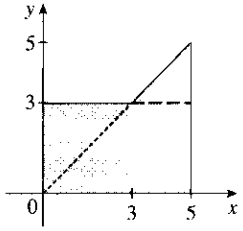
46. $\int_0^{\pi/2}(2\cos x - 5x)\,dx = \int_0^{\pi/2} 2\cos x\,dx - \int_0^{\pi/2} 5x\,dx = 2\int_0^{\pi/2}\cos x\,dx - 5\int_0^{\pi/2} x\,dx$
$= 2(1) - 5\dfrac{(\pi/2)^2 - 0^2}{2} = 2 - \dfrac{5\pi^2}{8}$

47. $\int_{-2}^2 f(x)\,dx + \int_2^5 f(x)\,dx - \int_{-2}^{-1} f(x)\,dx = \int_{-2}^5 f(x)\,dx + \int_{-1}^{-2} f(x)\,dx$ [by Property 5 and reversing limits]
$= \int_{-1}^5 f(x)\,dx$ [Property 5]

48. $\int_1^4 f(x)\,dx = \int_1^5 f(x)\,dx - \int_4^5 f(x)\,dx = 12 - 3.6 = 8.4$

49. $\int_0^9 [2f(x)+3g(x)]\,dx = 2\int_0^9 f(x)\,dx + 3\int_0^9 g(x)\,dx = 2(37) + 3(16) = 122$

50. If $f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3 \end{cases}$, then $\int_0^5 f(x)\,dx$ can be interpreted as the area of the shaded region, which consists of a 5-by-3 rectangle surmounted by an isosceles right triangle whose legs have length 2. Thus, $\int_0^5 f(x)\,dx = 5(3) + \frac{1}{2}(2)(2) = 17$.



51. $0 \leq \sin x < 1$ on $\left[0, \frac{\pi}{4}\right]$, so $\sin^3 x \leq \sin^2 x$ on $\left[0, \frac{\pi}{4}\right]$. Hence, $\int_0^{\pi/4} \sin^3 x\,dx \leq \int_0^{\pi/4} \sin^2 x\,dx$ (Property 7).

52. $5 - x \geq 3 \geq x + 1$ on $[1,2]$, so $\sqrt{5-x} \geq \sqrt{x+1}$ and $\int_1^2 \sqrt{5-x}\,dx \geq \int_1^2 \sqrt{x+1}\,dx$.

53. If $-1 \leq x \leq 1$, then $0 \leq x^2 \leq 1$ and $1 \leq 1+x^2 \leq 2$, so $1 \leq \sqrt{1+x^2} \leq \sqrt{2}$ and $1[1-(-1)] \leq \int_{-1}^1 \sqrt{1+x^2}\,dx \leq \sqrt{2}\,[1-(-1)]$ [Property 8]; that is, $2 \leq \int_{-1}^1 \sqrt{1+x^2}\,dx \leq 2\sqrt{2}$.

54. $\frac{1}{2} \leq \sin x \leq 1$ for $\frac{\pi}{6} \leq x \leq \frac{\pi}{2}$, so $\frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \leq \int_{\pi/6}^{\pi/2} \sin x\,dx \leq 1\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$ [Property 8]; that is, $\frac{\pi}{6} \leq \int_{\pi/6}^{\pi/2} \sin x\,dx \leq \frac{\pi}{3}$.

55. If $1 \leq x \leq 2$, then $\frac{1}{2} \leq \frac{1}{x} \leq 1$, so $\frac{1}{2}(2-1) \leq \int_1^2 \frac{1}{x}\,dx \leq 1(2-1)$ or $\frac{1}{2} \leq \int_1^2 \frac{1}{x}\,dx \leq 1$.

56. If $0 \le x \le 2$, then $0 \le x^3 \le 8$, so $1 \le x^3 + 1 \le 9$ and $1 \le \sqrt{x^3+1} \le 3$. Thus, $1(2-0) \le \int_0^2 \sqrt{x^3+1}\,dx \le 3(2-0)$ that is, $2 \le \int_0^2 \sqrt{x^3+1}\,dx \le 6$.

57. If $\frac{\pi}{4} \le x \le \frac{\pi}{3}$, then $1 \le \tan x \le \sqrt{3}$, so $1\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ or $\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \frac{\pi}{12}\sqrt{3}$.

58. Let $f(x) = x^3 - 3x + 3$ for $0 \le x \le 2$. Then $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$, so f is decreasing on $(0,1)$ and increasing on $(1,2)$. f has the absolute minimum value $f(1) = 1$. Since $f(0) = 3$ and $f(2) = 5$, the absolute maximum value of f is $f(2) = 5$. Thus, $1 \le x^3 - 3x + 3 \le 5$ for x in $[0,2]$. It follows from Property 8 that $1 \cdot (2-0) \le \int_0^2 (x^3 - 3x + 3)\,dx \le 5 \cdot (2-0)$; that is, $2 \le \int_0^2 (x^3 - 3x + 3)\,dx \le 10$.

59. For $-1 \le x \le 1$, $0 \le x^4 \le 1$ and $1 \le \sqrt{1+x^4} \le \sqrt{2}$, so $1[1-(-1)] \le \int_{-1}^{1} \sqrt{1+x^4}\,dx \le \sqrt{2}\,[1-(-1)]$ or $2 \le \int_{-1}^{1} \sqrt{1+x^4}\,dx \le 2\sqrt{2}$.

60. If $\frac{1}{4}\pi \le x \le \frac{3}{4}\pi$, then $\frac{\sqrt{2}}{2} \le \sin x \le 1$ and $\frac{1}{2} \le \sin^2 x \le 1$, so $\frac{1}{2}\left(\frac{3}{4}\pi - \frac{1}{4}\pi\right) \le \int_{\pi/4}^{3\pi/4} \sin^2 x\,dx \le 1\left(\frac{3}{4}\pi - \frac{1}{4}\pi\right)$; that is, $\frac{1}{4}\pi \le \int_{\pi/4}^{3\pi/4} \sin^2 x\,dx \le \frac{1}{2}\pi$.

61. $\sqrt{x^4+1} \ge \sqrt{x^4} = x^2$, so $\int_1^3 \sqrt{x^4+1}\,dx \ge \int_1^3 x^2\,dx = \frac{1}{3}(3^3 - 1^3) = \frac{26}{3}$.

62. $0 \le \sin x \le 1$ for $0 \le x \le \frac{\pi}{2}$, so $x \sin x \le x \quad \Rightarrow \quad \int_0^{\pi/2} x \sin x\,dx \le \int_0^{\pi/2} x\,dx = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right] = \frac{\pi^2}{8}$.

63. Using a regular partition and right endpoints as in the proof of Property 2, we calculate

$$\int_a^b cf(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} cf(x_i)\,\Delta x_i = \lim_{n\to\infty} c \sum_{i=1}^{n} f(x_i)\,\Delta x_i = c \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i = c\int_a^b f(x)\,dx.$$

64. As in the proof of Property 2, we write $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$. Now $f(x_i) \ge 0$ and $\Delta x \ge 0$, so $f(x_i)\,\Delta x \ge 0$ and therefore $\sum_{i=1}^{n} f(x_i)\,\Delta x \ge 0$. But the limit of nonnegative quantities is nonnegative by Theorem 2.3.2, so $\int_a^b f(x)\,dx \ge 0$.

65. Since $-|f(x)| \le f(x) \le |f(x)|$, it follows from Property 7 that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \quad \Rightarrow \quad \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Note that the definite integral is a real number, and so the following property applies: $-a \le b \le a \quad \Rightarrow \quad |b| \le a$ for all real numbers b and nonnegative numbers a.

66. $\left|\int_0^{2\pi} f(x)\sin 2x\,dx\right| \le \int_0^{2\pi} |f(x)\sin 2x|\,dx$ [by Exercise 65] $= \int_0^{2\pi} |f(x)|\,|\sin 2x|\,dx \le \int_0^{2\pi} |f(x)|\,dx$ by Property 7, since $|\sin 2x| \le 1 \quad \Rightarrow \quad |f(x)|\,|\sin 2x| \le |f(x)|$.

67. $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^4}{n^5} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^4 = \int_0^1 x^4\,dx$$

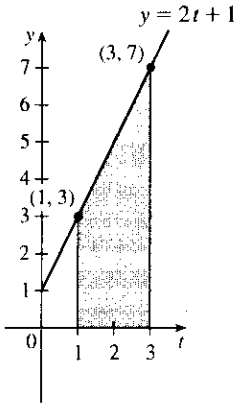
68. $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+(i/n)^2} = \int_0^1 \frac{dx}{1+x^2}$$

69. Choose $x_i = 1 + \dfrac{i}{n}$ and $x_i^* = \sqrt{x_{i-1}x_i} = \sqrt{\left(1 + \dfrac{i-1}{n}\right)\left(1 + \dfrac{i}{n}\right)}$. Then

$$\begin{aligned}
\int_1^2 x^{-2}\,dx &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\left(1 + \frac{i-1}{n}\right)\left(1 + \frac{i}{n}\right)} \\
&= \lim_{n\to\infty} n \sum_{i=1}^{n} \frac{1}{(n+i-1)(n+i)} \\
&= \lim_{n\to\infty} n \sum_{i=1}^{n} \left(\frac{1}{n+i-1} - \frac{1}{n+i}\right) \qquad \text{[by the hint]} \\
&= \lim_{n\to\infty} n \left(\sum_{i=0}^{n-1} \frac{1}{n+i} - \sum_{i=1}^{n} \frac{1}{n+i}\right) \\
&= \lim_{n\to\infty} n \left(\left[\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}\right] - \left[\frac{1}{n+1} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right]\right) \\
&= \lim_{n\to\infty} n \left(\frac{1}{n} - \frac{1}{2n}\right) = \lim_{n\to\infty} \left(1 - \tfrac{1}{2}\right) = \tfrac{1}{2}
\end{aligned}$$

DISCOVERY PROJECT Area Functions

1. (a)



Area of trapezoid $= \frac{1}{2}(b_1 + b_2)h$

$= \frac{1}{2}(3 + 7)2$

$= 10$ square units

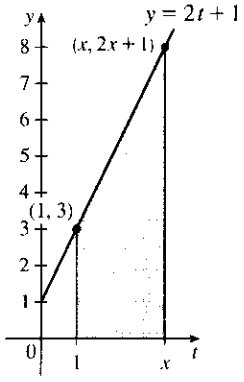
Or:

Area of rectangle + area of triangle

$= b_r h_r + \frac{1}{2} b_t h_t$

$= (2)(3) + \frac{1}{2}(2)(4) = 10$ square units

(b)



As in part (a),

$A(x) = \frac{1}{2}[3 + (2x + 1)](x - 1)$

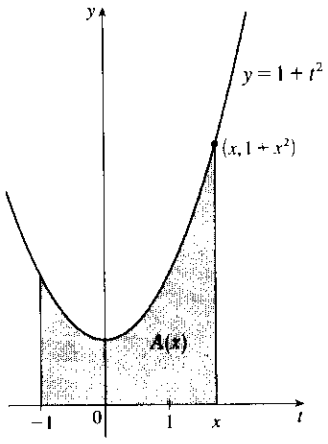
$= \frac{1}{2}(2x + 4)(x - 1)$

$= (x + 2)(x - 1)$

$= x^2 + x - 2$ square units

(c) $A'(x) = 2x + 1$. This is the y-coordinate of the point $(x, 2x + 1)$ on the given line.

2. (a)



(b) $A(x) = \int_{-1}^{x} (1 + t^2)\, dt = \int_{-1}^{x} 1\, dt + \int_{-1}^{x} t^2\, dt$ [Property 2]

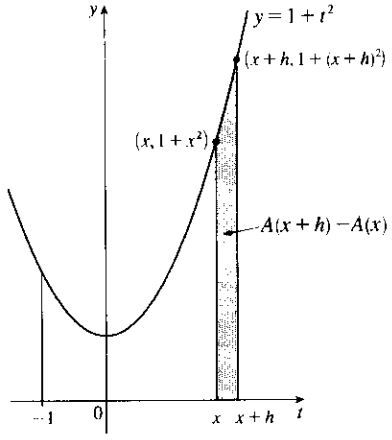
$= 1[x - (-1)] + \dfrac{x^3 - (-1)^3}{3}$ [Property 1 and Exercise 5.2.28]

$= x + 1 + \frac{1}{3}x^3 + \frac{1}{3}$

$= \frac{1}{3}x^3 + x + \frac{4}{3}$

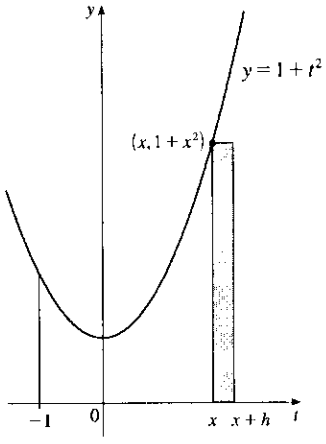
(c) $A'(x) = x^2 + 1$. This is the y-coordinate of the point $(x, 1 + x^2)$ on the given curve.

(d)



$A(x + h) - A(x)$ is the area under the curve $y = 1 + t^2$ from $t = x$ to $t = x + h$.

(e)



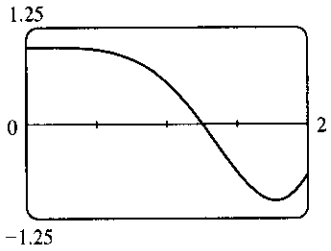
An approximating rectangle is shown in the figure. It has height $1 + x^2$, width h, and area $h(1 + x^2)$, so

$A(x + h) - A(x) \approx h(1 + x^2) \Rightarrow$

$\dfrac{A(x + h) - A(x)}{h} \approx 1 + x^2.$

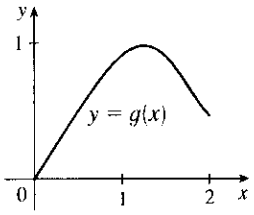
(f) Part (e) says that the average rate of change of A is approximately $1 + x^2$. As h approaches 0, the quotient approaches the instantaneous rate of change—namely, $A'(x)$. So the result of part (c), $A'(x) = x^2 + 1$, is geometrically plausible.

3. (a) $f(x) = \cos(x^2)$

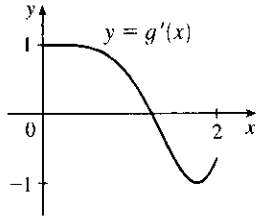


(b) $g(x)$ starts to decrease at that value of x where $\cos(t^2)$ changes from positive to negative; that is, at about $x = 1.25$.

(c) $g(x) = \int_0^x \cos(t^2)\,dt$. Using an integration command, we find that $g(0) = 0$, $g(0.2) \approx 0.200$, $g(0.4) \approx 0.399$, $g(0.6) \approx 0.592$, $g(0.8) \approx 0.768$, $g(1.0) \approx 0.905$, $g(1.2) \approx 0.974$, $g(1.4) \approx 0.950$, $g(1.6) \approx 0.826$, $g(1.8) \approx 0.635$, and $g(2.0) \approx 0.461$.



(d) We sketch the graph of g' using the method of Example 1 in Section 3.2. The graphs of $g'(x)$ and $f(x)$ look alike, so we guess that $g'(x) = f(x)$.



4. In Problems 1 and 2, we showed that if $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$, for the functions $f(t) = 2t + 1$ and $f(t) = 1 + t^2$. In Problem 3 we guessed that the same is true for $f(t) = \cos(t^2)$, based on visual evidence. So we conjecture that $g'(x) = f(x)$ for any continuous function f. This turns out to be true and is proved in Section 5.3 (the Fundamental Theorem of Calculus).

5.3 The Fundamental Theorem of Calculus

1. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it at the end of Section 5.3.

2. (a) $g(x) = \int_0^x f(t)\,dt$, so $g(0) = \int_0^0 f(t)\,dt = 0$.

$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2} \cdot 1 \cdot 1$ [area of triangle] $= \frac{1}{2}$.

$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$ [below the x-axis]
$= \frac{1}{2} - \frac{1}{2} \cdot 1 \cdot 1 = 0$.

$g(3) = g(2) + \int_2^3 f(t)\,dt = 0 - \frac{1}{2} \cdot 1 \cdot 1 = -\frac{1}{2}$.

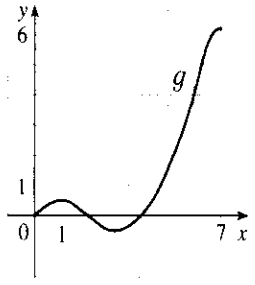
$g(4) = g(3) + \int_3^4 f(t)\,dt = -\frac{1}{2} + \frac{1}{2} \cdot 1 \cdot 1 = 0$.

$g(5) = g(4) + \int_4^5 f(t)\,dt = 0 + 1.5 = 1.5$.

$g(6) = g(5) + \int_5^6 f(t)\,dt = 1.5 + 2.5 = 4$.

(b) $g(7) = g(6) + \int_6^7 f(t)\,dt \approx 4 + 2.2$ [estimate from the graph] $= 6.2$.

(d)



(c) The answers from part (a) and part (b) indicate that g has a minimum at $x = 3$ and a maximum at $x = 7$. This makes sense from the graph of f since we are subtracting area on $1 < x < 3$ and adding area on $3 < x < 7$.

3. (a) $g(x) = \int_0^x f(t)\,dt$.

$g(0) = \int_0^0 f(t)\,dt = 0$

$g(1) = \int_0^1 f(t)\,dt = 1 \cdot 2 = 2$ [rectangle],

$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = g(1) + \int_1^2 f(t)\,dt$

$= 2 + 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 = 5$ [rectangle plus triangle],

$g(3) = \int_0^3 f(t)\,dt = g(2) + \int_2^3 f(t)\,dt = 5 + \frac{1}{2} \cdot 1 \cdot 4 = 7$,

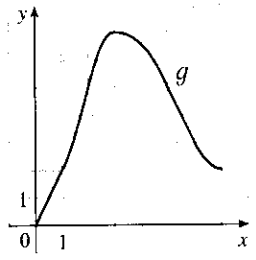
$g(6) = g(3) + \int_3^6 f(t)\,dt$ [the integral is negative since f lies under the x-axis]

$= 7 + \left[-\left(\frac{1}{2} \cdot 2 \cdot 2 + 1 \cdot 2\right)\right] = 7 - 4 = 3$

(b) g is increasing on $(0, 3)$ because as x increases from 0 to 3, we keep adding more area.

(c) g has a maximum value when we start subtracting area; that is, at $x = 3$.

(d)



4. (a) $g(-3) = \int_{-3}^{-3} f(t)\,dt = 0$, $g(3) = \int_{-3}^{3} f(t)\,dt = \int_{-3}^{0} f(t)\,dt + \int_0^3 f(t)\,dt = 0$ by symmetry, since the area above the x-axis is the same as the area below the axis.

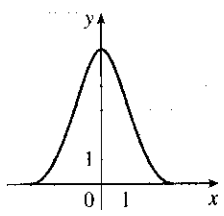
(b) From the graph, it appears that to the nearest $\frac{1}{2}$, $g(-2) = \int_{-3}^{-2} f(t)\,dt \approx 1$, $g(-1) = \int_{-3}^{-1} f(t)\,dt \approx 3\frac{1}{2}$, and $g(0) = \int_{-3}^{0} f(t)\,dt \approx 5\frac{1}{2}$.

(c) g is increasing on $(-3, 0)$ because as x increases from -3 to 0, we keep adding more area.

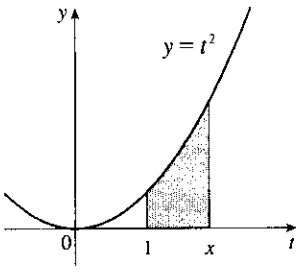
(d) g has a maximum value when we start subtracting area; that is, at $x = 0$.

(f) The graph of $g'(x)$ is the same as that of $f(x)$, as indicated by FTC1.

(e)



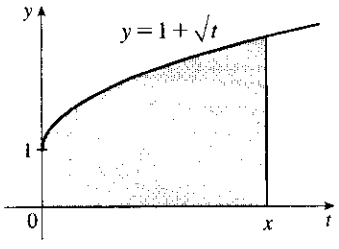
5.



(a) By FTC1 with $f(t) = t^2$ and $a = 1$, $g(x) = \int_1^x t^2\,dt \;\Rightarrow$ $g'(x) = f(x) = x^2$.

(b) Using FTC2, $g(x) = \int_1^x t^2\,dt = \left[\frac{1}{3}t^3\right]_1^x = \frac{1}{3}x^3 - \frac{1}{3} \;\Rightarrow$ $g'(x) = x^2$.

6.



(a) By FTC1 with $f(t) = 1 + \sqrt{t}$ and $a = 0$,

$g(x) = \int_0^x \left(1 + \sqrt{t}\right) dt \;\Rightarrow\; g'(x) = f(x) = 1 + \sqrt{x}$.

(b) Using FTC2, $g(x) = \int_0^x \left(1 + \sqrt{t}\right) dt = \left[t + \frac{2}{3}t^{3/2}\right]_0^x = x + \frac{2}{3}x^{3/2}$

$\Rightarrow\; g'(x) = 1 + x^{1/2} = 1 + \sqrt{x}$.

7. $f(t) = \sqrt{1 + 2t}$ and $g(x) = \int_0^x \sqrt{1 + 2t}\,dt$, so by FTC1, $g'(x) = f(x) = \sqrt{1 + 2x}$.

8. $f(t) = \left(2 + t^4\right)^5$ and $g(x) = \int_1^x \left(2 + t^4\right)^5 dt$, so $g'(x) = f(x) = \left(2 + x^4\right)^5$.

9. $f(t) = t^2 \sin t$ and $g(y) = \int_2^y t^2 \sin t\,dt$, so by FTC1, $g'(y) = f(y) = y^2 \sin y$.

10. $f(x) = \dfrac{1}{x + x^2}$ and $g(u) = \displaystyle\int_3^u \frac{1}{x + x^2}\,dx$, so $g'(u) = f(u) = \dfrac{1}{u + u^2}$.

11. $F(x) = \int_x^2 \cos(t^2)\,dt = -\int_2^x \cos(t^2)\,dt \quad \Rightarrow \quad F'(x) = -\cos(x^2)$

12. $f(\theta) = \tan\theta$ and $F(x) = \int_x^{10} \tan\theta\,d\theta = -\int_{10}^x \tan\theta\,d\theta$, so by FTC1, $F'(x) = -f(x) = -\tan x$.

13. Let $u = \dfrac{1}{x}$. Then $\dfrac{du}{dx} = -\dfrac{1}{x^2}$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_2^{1/x} \sin^4 t\,dt = \frac{d}{du}\int_2^u \sin^4 t\,dt \cdot \frac{du}{dx} = \sin^4 u\,\frac{du}{dx} = \frac{-\sin^4(1/x)}{x^2}.$$

14. Let $u = x^2$. Then $\dfrac{du}{dx} = 2x$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_0^{x^2} \sqrt{1+r^3}\,dr = \frac{d}{du}\int_0^u \sqrt{1+r^3}\,dr \cdot \frac{du}{dx} = \sqrt{1+u^3}(2x) = 2x\sqrt{1+(x^2)^3} = 2x\sqrt{1+x^6}.$$

15. Let $u = \sqrt{x}$. Then $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_3^{\sqrt{x}} \frac{\cos t}{t}\,dt = \frac{d}{du}\int_3^u \frac{\cos t}{t}\,dt \cdot \frac{du}{dx} = \frac{\cos u}{u} \cdot \frac{1}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{\cos\sqrt{x}}{2x}.$$

16. Let $u = \cos x$. Then $\dfrac{du}{dx} = -\sin x$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_1^{\cos x} (t+\sin t)\,dt = \frac{d}{du}\int_1^u (t+\sin t)\,dt \cdot \frac{du}{dx}$$
$$= (u+\sin u)\cdot(-\sin x) = -\sin x\,[\cos x + \sin(\cos x)]$$

17. Let $w = 1-3x$. Then $\dfrac{dw}{dx} = -3$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{dw}\dfrac{dw}{dx}$, so

$$y' = \frac{d}{dx}\int_{1-3x}^1 \frac{u^3}{1+u^2}\,du = \frac{d}{dw}\int_w^1 \frac{u^3}{1+u^2}\,du \cdot \frac{dw}{dx}$$
$$= -\frac{d}{dw}\int_1^w \frac{u^3}{1+u^2}\,du \cdot \frac{dw}{dx} = -\frac{w^3}{1+w^2}(-3) = \frac{3(1-3x)^3}{1+(1-3x)^2}$$

18. Let $u = \dfrac{1}{x^2}$. Then $\dfrac{du}{dx} = -\dfrac{2}{x^3}$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_{1/x^2}^0 \sin^3 t\,dt = \frac{d}{du}\int_u^0 \sin^3 t\,dt \cdot \frac{du}{dx} = -\frac{d}{du}\int_0^u \sin^3 t\,dt \cdot \frac{du}{dx} = -\sin^3 u\left(-\frac{2}{x^3}\right) = \frac{2\sin^3(1/x^2)}{x^3}.$$

19. $\displaystyle\int_{-1}^3 x^5\,dx = \left[\frac{x^6}{6}\right]_{-1}^3 = \frac{3^6}{6} - \frac{(-1)^6}{6} = \frac{729-1}{6} = \frac{364}{3}$

20. $\int_{-2}^5 6\,dx = [6x]_{-2}^5 = 6[5-(-2)] = 6(7) = 42$

21. $\int_2^8 (4x+3)\,dx = \left[\frac{4}{2}x^2 + 3x\right]_2^8 = \left(2\cdot 8^2 + 3\cdot 8\right) - \left(2\cdot 2^2 + 3\cdot 2\right) = 152 - 14 = 138$

22. $\int_0^4 (1+3y-y^2)\,dy = \left[y + \frac{3}{2}y^2 - \frac{1}{3}y^3\right]_0^4 = \left(4 + \frac{3}{2}\cdot 16 - \frac{1}{3}\cdot 64\right) - (0) = \frac{20}{3}$

23. $\int_0^1 x^{4/5}\,dx = \left[\frac{5}{9}x^{9/5}\right]_0^1 = \frac{5}{9} - 0 = \frac{5}{9}$

24. $\int_1^8 \sqrt[3]{x}\,dx = \int_1^8 x^{1/3}\,dx = \left[\frac{3}{4}x^{4/3}\right]_1^8 = \frac{3}{4}(8^{4/3} - 1^{4/3}) = \frac{3}{4}(2^4-1) = \frac{3}{4}(16-1) = \frac{3}{4}(15) = \frac{45}{4}$

25. $\displaystyle\int_1^2 \frac{3}{t^4}\,dt = 3\int_1^2 t^{-4}\,dt = 3\left[\frac{t^{-3}}{-3}\right]_1^2 = \frac{3}{-3}\left[\frac{1}{t^3}\right]_1^2 = -1\left(\frac{1}{8} - 1\right) = \frac{7}{8}$

26. $\int_{-2}^{3} x^{-5}\,dx$ does not exist because the function $f(x) = x^{-5}$ has an infinite discontinuity at $x = 0$; that is, f is discontinuous on the interval $[-2, 3]$.

27. $\displaystyle\int_{-5}^{5} \frac{2}{x^3}\,dx$ does not exist because the function $f(x) = \dfrac{2}{x^3}$ has an infinite discontinuity at $x = 0$; that is, f is discontinuous on the interval $[-5, 5]$.

28. $\int_{\pi}^{2\pi} \cos\theta\,d\theta = \left[\sin\theta\right]_{\pi}^{2\pi} = \sin 2\pi - \sin\pi = 0 - 0 = 0$

29. $\int_0^2 x(2 + x^5)\,dx = \int_0^2 (2x + x^6)\,dx = \left[x^2 + \frac{1}{7}x^7\right]_0^2 = \left(4 + \frac{128}{7}\right) - (0 + 0) = \frac{156}{7}$

30. $\displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx = \int_1^4 x^{-1/2}\,dx = \left[\frac{x^{1/2}}{1/2}\right]_1^4 = \left[2x^{1/2}\right]_1^4 = 2\sqrt{4} - 2\sqrt{1} = 4 - 2 = 2$

31. $\int_0^{\pi/4} \sec^2 t\,dt = \left[\tan t\right]_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1 - 0 = 1$

32. $\int_0^1 (3 + x\sqrt{x})\,dx = \int_0^1 \left(3 + x^{3/2}\right)dx = \left[3x + \frac{2}{5}x^{5/2}\right]_0^1 = \left[\left(3 + \frac{2}{5}\right) - 0\right] = \frac{17}{5}$

33. $\int_{\pi}^{2\pi} \csc^2\theta\,d\theta$ does not exist because the function $f(\theta) = \csc^2\theta$ has infinite discontinuities at $\theta = \pi$ and $\theta = 2\pi$; that is, f is discontinuous on the interval $[\pi, 2\pi]$.

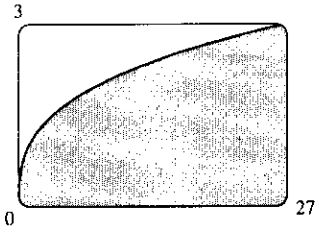
34. $\int_0^{\pi/6} \csc\theta\cot\theta\,d\theta$ does not exist because the function $f(\theta) = \csc\theta\cot\theta$ has an infinite discontinuity at $\theta = 0$; that is, f is discontinuous on the interval $\left[0, \frac{\pi}{6}\right]$.

35. $\int_0^2 f(x)\,dx = \int_0^1 x^4\,dx + \int_1^2 x^5\,dx = \left[\frac{1}{5}x^5\right]_0^1 + \left[\frac{1}{6}x^6\right]_1^2 = \left(\frac{1}{5} - 0\right) + \left(\frac{64}{6} - \frac{1}{6}\right) = 10.7$

36. $\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{0} x\,dx + \int_0^{\pi} \sin x\,dx = \left[\frac{1}{2}x^2\right]_{-\pi}^{0} - \left[\cos x\right]_0^{\pi} = \left(0 - \frac{\pi^2}{2}\right) - (\cos\pi - \cos 0)$

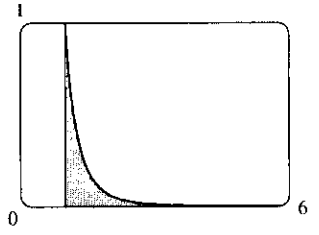
$$= -\frac{\pi^2}{2} - (-1 - 1) = 2 - \frac{\pi^2}{2}$$

37. From the graph, it appears that the area is about 60. The actual area is $\int_0^{27} x^{1/3}dx = \left[\frac{3}{4}x^{4/3}\right]_0^{27} = \frac{3}{4}\cdot 81 - 0 = \frac{243}{4} = 60.75$. This is $\frac{3}{4}$ of the area of the viewing rectangle.

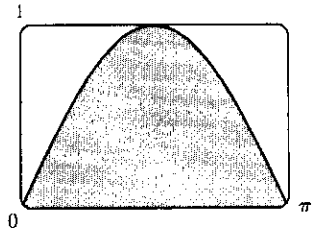


38. From the graph, it appears that the area is about $\frac{1}{3}$. The actual area is

$$\int_1^6 x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_1^6 = \left[\frac{-1}{3x^3}\right]_1^6 = -\frac{1}{3\cdot 216} + \frac{1}{3} = \frac{215}{648} \approx 0.3318.$$

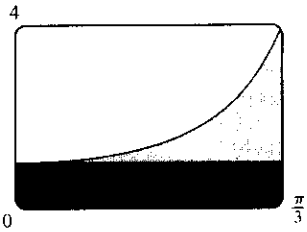


39. It appears that the area under the graph is about $\frac{2}{3}$ of the area of the viewing rectangle, or about $\frac{2}{3}\pi \approx 2.1$. The actual area is $\int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} = (-\cos\pi) - (-\cos 0) = -(-1) + 1 = 2$.

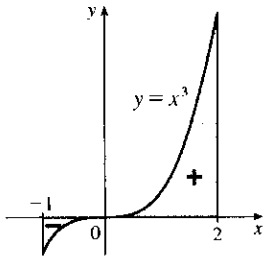


40. Splitting up the region as shown, we estimate that the area under the graph is $\frac{\pi}{3} + \frac{1}{4}\left(3 \cdot \frac{\pi}{3}\right) \approx 1.8$. The actual area is

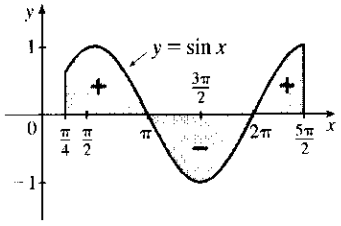
$\int_0^{\pi/3} \sec^2 x\, dx = [\tan x]_0^{\pi/3} = \sqrt{3} - 0 = \sqrt{3} \approx 1.73.$



41. $\int_{-1}^{2} x^3\, dx = \left[\frac{1}{4}x^4\right]_{-1}^{2} = 4 - \frac{1}{4} = \frac{15}{4} = 3.75$



42. $\int_{\pi/4}^{5\pi/2} \sin x\, dx = [-\cos x]_{\pi/4}^{5\pi/2} = 0 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$



43. $g(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\, du = \int_{2x}^{0} \frac{u^2-1}{u^2+1}\, du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\, du = -\int_{0}^{2x} \frac{u^2-1}{u^2+1}\, du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\, du \quad \Rightarrow$

$g'(x) = -\frac{(2x)^2-1}{(2x)^2+1} \cdot \frac{d}{dx}(2x) + \frac{(3x)^2-1}{(3x)^2+1} \cdot \frac{d}{dx}(3x) = -2 \cdot \frac{4x^2-1}{4x^2+1} + 3 \cdot \frac{9x^2-1}{9x^2+1}$

44. $g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\, dt = \int_{\tan x}^{1} \frac{dt}{\sqrt{2+t^4}} + \int_{1}^{x^2} \frac{dt}{\sqrt{2+t^4}} = -\int_{1}^{\tan x} \frac{dt}{\sqrt{2+t^4}} + \int_{1}^{x^2} \frac{dt}{\sqrt{2+t^4}} \quad \Rightarrow$

$g'(x) = \frac{-1}{\sqrt{2+\tan^4 x}} \frac{d}{dx}(\tan x) + \frac{1}{\sqrt{2+x^8}} \frac{d}{dx}(x^2) = -\frac{\sec^2 x}{\sqrt{2+\tan^4 x}} + \frac{2x}{\sqrt{2+x^8}}$

45. $y = \int_{\sqrt{x}}^{x^3} \sqrt{t} \sin t\, dt = \int_{\sqrt{x}}^{1} \sqrt{t} \sin t\, dt + \int_{1}^{x^3} \sqrt{t} \sin t\, dt = -\int_{1}^{\sqrt{x}} \sqrt{t} \sin t\, dt + \int_{1}^{x^3} \sqrt{t} \sin t\, dt \quad \Rightarrow$

$y' = -\sqrt[4]{x}\,(\sin\sqrt{x}\,) \cdot \frac{d}{dx}(\sqrt{x}\,) + x^{3/2} \sin(x^3) \cdot \frac{d}{dx}(x^3) = -\frac{\sqrt[4]{x} \sin\sqrt{x}}{2\sqrt{x}} + x^{3/2} \sin(x^3)\,(3x^2)$

$= 3x^{7/2} \sin(x^3) - \frac{\sin\sqrt{x}}{2\sqrt[4]{x}}$

46. $y = \int_{\cos x}^{5x} \cos(u^2)\, du = \int_{0}^{5x} \cos(u^2)\, du - \int_{0}^{\cos x} \cos(u^2)\, du \quad \Rightarrow$

$y' = \cos(25x^2) \cdot \frac{d}{dx}(5x) - \cos(\cos^2 x) \cdot \frac{d}{dx}(\cos x) = \cos(25x^2) \cdot 5 - \cos(\cos^2 x) \cdot (-\sin x)$

$= 5\cos(25x^2) + \sin x \cos(\cos^2 x)$

47. $F(x) = \int_{1}^{x} f(t)\, dt \quad \Rightarrow \quad F'(x) = f(x) = \int_{1}^{x^2} \frac{\sqrt{1+u^4}}{u}\, du \left[\text{since } f(t) = \int_{1}^{t^2} \frac{\sqrt{1+u^4}}{u}\, du\right] \quad \Rightarrow$

$F''(x) = f'(x) = \frac{\sqrt{1+(x^2)^4}}{x^2} \cdot \frac{d}{dx}(x^2) = \frac{\sqrt{1+x^8}}{x^2} \cdot 2x = \frac{2\sqrt{1+x^8}}{x}$. So $F''(2) = \sqrt{1+2^8} = \sqrt{257}$.

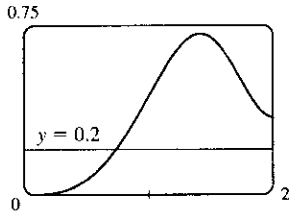
48. For the curve to be concave upward, we must have $y'' > 0$. $y = \int_{0}^{x} \frac{1}{1+t+t^2}\, dt \quad \Rightarrow \quad y' = \frac{1}{1+x+x^2} \quad \Rightarrow$

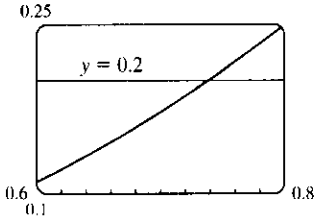
$y'' = \frac{-(1+2x)}{(1+x+x^2)^2}$. For this expression to be positive, we must have $(1+2x) < 0$, since $(1+x+x^2)^2 > 0$ for all x. $(1+2x) < 0 \quad \Leftrightarrow \quad x < -\frac{1}{2}$. Thus, the curve is concave upward on $\left(-\infty, -\frac{1}{2}\right)$.

49. (a) The Fresnel function $S(x) = \int_0^x \sin(\frac{\pi}{2}t^2)\,dt$ has local maximum values where $0 = S'(x) = \sin(\frac{\pi}{2}x^2)$ and S' changes from positive to negative. For $x > 0$, this happens when $\frac{\pi}{2}x^2 = (2n-1)\pi$ [odd multiples of π] $\Leftrightarrow$ $x^2 = 2(2n-1)$ $\Leftrightarrow$ $x = \sqrt{4n-2}$, n any positive integer. For $x < 0$, S' changes from positive to negative where $\frac{\pi}{2}x^2 = 2n\pi$ [even multiples of π] $\Leftrightarrow$ $x^2 = 4n$ $\Leftrightarrow$ $x = -2\sqrt{n}$. S' does not change sign at $x = 0$.

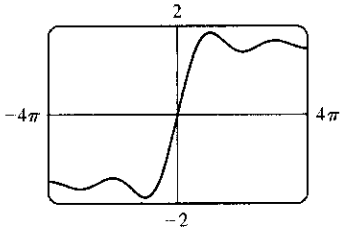
(b) S is concave upward on those intervals where $S''(x) > 0$. Differentiating our expression for $S'(x)$, we get $S''(x) = \cos(\frac{\pi}{2}x^2)(2\frac{\pi}{2}x) = \pi x\cos(\frac{\pi}{2}x^2)$. For $x > 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) > 0$ $\Leftrightarrow$ $0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ or $\left(2n-\frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n+\frac{1}{2}\right)\pi$, n any integer $\Leftrightarrow$ $0 < x < 1$ or $\sqrt{4n-1} < x < \sqrt{4n+1}$, n any positive integer. For $x < 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) < 0$ $\Leftrightarrow$ $\left(2n-\frac{3}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n-\frac{1}{2}\right)\pi$, n any integer $\Leftrightarrow$ $4n-3 < x^2 < 4n-1$ $\Leftrightarrow$ $\sqrt{4n-3} < |x| < \sqrt{4n-1}$ $\Rightarrow$ $\sqrt{4n-3} < -x < \sqrt{4n-1}$ $\Rightarrow$ $-\sqrt{4n-3} > x > -\sqrt{4n-1}$, so the intervals of upward concavity for $x < 0$ are $\left(-\sqrt{4n-1}, -\sqrt{4n-3}\right)$, n any positive integer. To summarize: S is concave upward on the intervals $(0,1)$, $(-\sqrt{3},-1)$, $(\sqrt{3},\sqrt{5})$, $(-\sqrt{7},-\sqrt{5})$, $(\sqrt{7},3)$, ...

(c) In Maple, we use `plot({int(sin(Pi*t^2/2),t=0..x),0.2},x=0..2);`. Note that Maple recognizes the Fresnel function, calling it `FresnelS(x)`. In Mathematica, we use `Plot[{Integrate[Sin[Pi*t^2/2],{t,0,x}],0.2},{x,0,2}]`. In Derive, we load the utility file `FRESNEL` and plot `FRESNEL_SIN(x)`. From the graphs, we see that $\int_0^x \sin(\frac{\pi}{2}t^2)\,dt = 0.2$ at $x \approx 0.74$.





50. (a) In Maple, we should start by setting `si:=int(sin(t)/t,t=0..x);`. In Mathematica, the command is `si=Integrate[Sin[t]/t,{t,0,x}]`. Note that both systems recognize this function; Maple calls it `Si(x)` and Mathematica calls it `SinIntegral[x]`. In Maple, the command to generate the graph is `plot(si,x=-4*Pi..4*Pi);`. In Mathematica, it is `Plot[si,{x,-4*Pi,4*Pi}]`. In Derive, we load the utility file `EXP_INT` and plot `SI(x)`.



(b) $\mathrm{Si}(x)$ has local maximum values where $\mathrm{Si}'(x)$ changes from positive to negative, passing through 0. From the Fundamental Theorem we know that $\mathrm{Si}'(x) = \dfrac{d}{dx}\displaystyle\int_0^x \frac{\sin t}{t}\,dt = \frac{\sin x}{x}$, so we must have $\sin x = 0$ for a maximum, and for $x > 0$ we must have $x = (2n-1)\pi$, n any positive integer, for Si' to be changing from positive to negative at x. For $x < 0$, we must have $x = 2n\pi$, n any positive integer, for a maximum, since the denominator of $\mathrm{Si}'(x)$ is negative for $x < 0$. Thus, the local maxima occur at $x = \pi, -2\pi, 3\pi, -4\pi, 5\pi, -6\pi, \ldots$.

(c) To find the first inflection point, we solve $\mathrm{Si}''(x) = \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2} = 0$. We can see from the graph that the first inflection point lies somewhere between $x = 3$ and $x = 5$. Using a root finder gives the value $x \approx 4.4934$. To find the y-coordinate of the inflection point, we evaluate $\mathrm{Si}(4.4934) \approx 1.6556$. So the coordinates of the first

inflection point to the right of the origin are about $(4.4934, 1.6556)$. Alternatively, we could graph $S''(x)$ and estimate the first positive x-value at which it changes sign.

(d) It seems from the graph that the function has horizontal asymptotes at $y \approx 1.5$, with $\lim_{x\to\pm\infty} \mathrm{Si}(x) \approx \pm 1.5$ respectively. Using the limit command, we get $\lim_{x\to\infty} \mathrm{Si}(x) = \frac{\pi}{2}$. Since $\mathrm{Si}(x)$ is an odd function, $\lim_{x\to-\infty} \mathrm{Si}(x) = -\frac{\pi}{2}$. So $\mathrm{Si}(x)$ has the horizontal asymptotes $y = \pm\frac{\pi}{2}$.

(e) We use the `fsolve` command in Maple (or `FindRoot` in Mathematica) to find that the solution is $x \approx 1.1$. Or, as in Exercise 49(c), we graph $y = \mathrm{Si}(x)$ and $y = 1$ on the same screen to see where they intersect.

51. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 1, 3, 5, 7$, and 9. g has local maxima at $x = 1$ and 5 (since $f = g'$ changes from positive to negative there) and local minima at $x = 3$ and 7. There is no local maximum or minimum at $x = 9$, since f is not defined for $x > 9$.

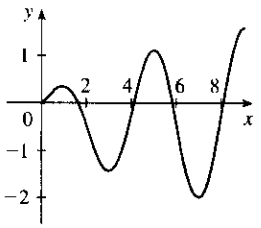
(b) We can see from the graph that $\left|\int_0^1 f\,dt\right| < \left|\int_1^3 f\,dt\right| < \left|\int_3^5 f\,dt\right| < \left|\int_5^7 f\,dt\right| < \left|\int_7^9 f\,dt\right|$.

So $g(1) = \left|\int_0^1 f\,dt\right|$, $g(5) = \int_0^5 f\,dt = g(1) - \left|\int_1^3 f\,dt\right| + \left|\int_3^5 f\,dt\right|$, and

$g(9) = \int_0^9 f\,dt = g(5) - \left|\int_5^7 f\,dt\right| + \left|\int_7^9 f\,dt\right|$. Thus, $g(1) < g(5) < g(9)$, and so the absolute maximum of $g(x)$ occurs at $x = 9$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on (approximately) $(\frac{1}{2}, 2)$, $(4, 6)$ and $(8, 9)$. So g is concave downward on these intervals.

(d)



52. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 2, 4, 6, 8$, and 10. g has local maxima at $x = 2$ and 6 (since $f = g'$ changes from positive to negative there) and local minima at $x = 4$ and 8. There is no local maximum or minimum at $x = 10$, since f is not defined for $x > 10$.

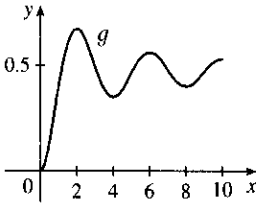
(b) We can see from the graph that $\left|\int_0^2 f\,dt\right| > \left|\int_2^4 f\,dt\right| > \left|\int_4^6 f\,dt\right| > \left|\int_6^8 f\,dt\right| > \left|\int_8^{10} f\,dt\right|$.

So $g(2) = \left|\int_0^2 f\,dt\right|$, $g(6) = \int_0^6 f\,dt = g(2) - \left|\int_2^4 f\,dt\right| + \left|\int_4^6 f\,dt\right|$, and

$g(10) = \int_0^{10} f\,dt = g(6) - \left|\int_6^8 f\,dt\right| + \left|\int_8^{10} f\,dt\right|$. Thus, $g(2) > g(6) > g(10)$, and so the absolute maximum of $g(x)$ occurs at $x = 2$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on $(1, 3)$, $(5, 7)$ and $(9, 10)$. So g is concave downward on these intervals.

(d)



53. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4} = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$

54. $\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n}{n}}\right) = \lim_{n\to\infty} \frac{1-0}{n} \sum_{i=1}^{n} \sqrt{\frac{i}{n}} = \int_0^1 \sqrt{x}\,dx = \left[\frac{2x^{3/2}}{3}\right]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$

55. Suppose $h < 0$. Since f is continuous on $[x+h, x]$, the Extreme Value Theorem says that there are numbers u and v in $[x+h, x]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x+h, x]$. By Property 8 of integrals, $m(-h) \le \int_{x+h}^{x} f(t)\,dt \le M(-h)$; that is, $f(u)(-h) \le -\int_{x}^{x+h} f(t)\,dt \le f(v)(-h)$. Since $-h > 0$, we can divide this inequality by $-h$: $f(u) \le \dfrac{1}{h}\displaystyle\int_{x}^{x+h} f(t)\,dt \le f(v)$. By Equation 2, $\dfrac{g(x+h)-g(x)}{h} = \dfrac{1}{h}\displaystyle\int_{x}^{x+h} f(t)\,dt$ for $h \neq 0$, and hence $f(u) \le \dfrac{g(x+h)-g(x)}{h} \le f(v)$, which is Equation 3 in the case where $h < 0$.

56. $\dfrac{d}{dx}\displaystyle\int_{g(x)}^{h(x)} f(t)\,dt = \frac{d}{dx}\left[\int_{g(x)}^{a} f(t)\,dt + \int_{a}^{h(x)} f(t)\,dt\right]$ (where a is in the domain of f)

$$= \frac{d}{dx}\left[-\int_{a}^{g(x)} f(t)\,dt\right] + \frac{d}{dx}\left[\int_{a}^{h(x)} f(t)\,dt\right] = -f(g(x))\,g'(x) + f(h(x))\,h'(x)$$

$$= f(h(x))\,h'(x) - f(g(x))\,g'(x)$$

57. (a) Let $f(x) = \sqrt{x} \;\Rightarrow\; f'(x) = 1/(2\sqrt{x}) > 0$ for $x > 0 \;\Rightarrow\; f$ is increasing on $(0, \infty)$. If $x \ge 0$, then $x^3 \ge 0$, so $1 + x^3 \ge 1$ and since f is increasing, this means that $f(1+x^3) \ge f(1) \;\Rightarrow\; \sqrt{1+x^3} \ge 1$ for $x \ge 0$. Next let $g(t) = t^2 - t \;\Rightarrow\; g'(t) = 2t - 1 \;\Rightarrow\; g'(t) > 0$ when $t \ge 1$. Thus, g is increasing on $(1, \infty)$. And since $g(1) = 0$, $g(t) \ge 0$ when $t \ge 1$. Now let $t = \sqrt{1+x^3}$, where $x \ge 0$. $\sqrt{1+x^3} \ge 1$ (from above) $\;\Rightarrow\; t \ge 1 \;\Rightarrow\; g(t) \ge 0 \;\Rightarrow\; \left(1+x^3\right) - \sqrt{1+x^3} \ge 0$ for $x \ge 0$. Therefore, $1 \le \sqrt{1+x^3} \le 1 + x^3$ for $x \ge 0$.

(b) From part (a) and Property 7: $\int_0^1 1\,dx \le \int_0^1 \sqrt{1+x^3}\,dx \le \int_0^1 \left(1+x^3\right) dx \;\Leftrightarrow$ $[x]_0^1 \le \int_0^1 \sqrt{1+x^3}\,dx \le \left[x + \frac{1}{4}x^4\right]_0^1 \;\Leftrightarrow\; 1 \le \int_0^1 \sqrt{1+x^3}\,dx \le 1 + \frac{1}{4} = 1.25$.

58. (a) If $x < 0$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0$.

If $0 \le x \le 1$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \left[\frac{1}{2}t^2\right]_0^x = \frac{1}{2}x^2$.

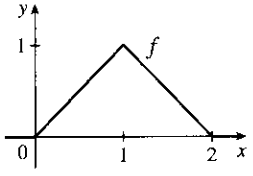
If $1 < x \le 2$, then

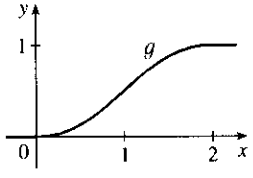
$$g(x) = \int_0^x f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt$$
$$= g(1) + \int_1^x (2-t)\,dt = \tfrac{1}{2}(1)^2 + \left[2t - \tfrac{1}{2}t^2\right]_1^x$$
$$= \tfrac{1}{2} + \left(2x - \tfrac{1}{2}x^2\right) - \left(2 - \tfrac{1}{2}\right) = 2x - \tfrac{1}{2}x^2 - 1.$$

If $x > 2$, then $g(x) = \int_0^x f(t)\,dt = g(2) + \int_2^x 0\,dt = 1 + 0 = 1$. So

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } 0 \le x \le 1 \\ 2x - \frac{1}{2}x^2 - 1 & \text{if } 1 < x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$

(b)





(c) f is not differentiable at its corners at $x = 0$, 1, and 2. f is differentiable on $(-\infty, 0)$, $(0, 1)$, $(1, 2)$ and $(2, \infty)$. g is differentiable on $(-\infty, \infty)$.

59. Using FTC1, we differentiate both sides of $6 + \int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ to get $\frac{f(x)}{x^2} = 2\,\frac{1}{2\sqrt{x}} \quad \Rightarrow \quad f(x) = x^{3/2}$.

To find a, we substitute $x = a$ in the original equation to obtain $6 + \int_a^a \frac{f(t)}{t^2}\,dt = 2\sqrt{a} \quad \Rightarrow \quad 6 + 0 = 2\sqrt{a} \quad \Rightarrow$

$3 = \sqrt{a} \quad \Rightarrow \quad a = 9$.

60. By the Fundamental Theorem of Calculus, $\int_1^2 (h')'(u)\,du = h'(2) - h'(1) = 5 - 2 = 3$. The other information is unnecessary.

61. (a) Let $F(t) = \int_0^t f(s)\,ds$. Then, by FTC1, $F'(t) = f(t) =$ rate of depreciation, so $F(t)$ represents the loss in value over the interval $[0, t]$.

(b) $C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = \frac{A + F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0, t]$, assuming that there has been only one overhaul during that time period. The company wants to minimize average expenditure.

(c) $C(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right]$. Using FTC1, we have $C'(t) = -\frac{1}{t^2}\left[A + \int_0^t f(s)\,ds\right] + \frac{1}{t}f(t)$.

$C'(t) = 0 \quad \Rightarrow \quad t\,f(t) = A + \int_0^t f(s)\,ds \quad \Rightarrow \quad f(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = C(t)$.

62. (a) $C(t) = \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds$. Using FTC1 and the Product Rule, we have

$C'(t) = \frac{1}{t}[f(t) + g(t)] - \frac{1}{t^2}\int_0^t [f(s) + g(s)]\,ds$. Set $C'(t) = 0$:

$\frac{1}{t}[f(t) + g(t)] - \frac{1}{t^2}\int_0^t [f(s) + g(s)]\,ds = 0 \quad \Rightarrow \quad [f(t) + g(t)] - \frac{1}{t}\int_0^t [f(s) + g(s)]\,ds = 0 \quad \Rightarrow$

$[f(t) + g(t)] - C(t) = 0 \quad \Rightarrow \quad C(t) = f(t) + g(t)$.

(b) For $0 \le t \le 30$, we have $D(t) = \int_0^t \left(\frac{V}{15} - \frac{V}{450}s\right)ds = \left[\frac{V}{15}s - \frac{V}{900}s^2\right]_0^t = \frac{V}{15}t - \frac{V}{900}t^2$.

So $D(t) = V \quad \Rightarrow \quad \frac{V}{15}t - \frac{V}{900}t^2 = V \quad \Rightarrow \quad 60t - t^2 = 900 \quad \Rightarrow \quad t^2 - 60t + 900 = 0 \quad \Rightarrow$

$(t - 30)^2 = 0 \quad \Rightarrow \quad t = 30$. So the length of time T is 30 months.

(c) $C(t) = \frac{1}{t}\int_0^t \left(\frac{V}{15} - \frac{V}{450}s + \frac{V}{12{,}900}s^2\right)ds = \frac{1}{t}\left[\frac{V}{15}s - \frac{V}{900}s^2 + \frac{V}{38{,}700}s^3\right]_0^t$

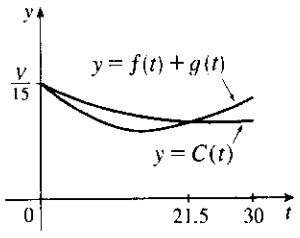
$= \frac{1}{t}\left(\frac{V}{15}t - \frac{V}{900}t^2 + \frac{V}{38{,}700}t^3\right) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2 \quad \Rightarrow$

$C'(t) = -\frac{V}{900} + \frac{V}{19{,}350}t = 0$ when $\frac{1}{19{,}350}t = \frac{1}{900} \quad \Rightarrow \quad t = 21.5$.

$C(21.5) = \frac{V}{15} - \frac{V}{900}(21.5) + \frac{V}{38{,}700}(21.5)^2 \approx 0.05472V$, $C(0) = \frac{V}{15} \approx 0.06667V$, and

$C(30) = \frac{V}{15} - \frac{V}{900}(30) + \frac{V}{38{,}700}(30)^2 \approx 0.05659V$, so the absolute minimum is $C(21.5) \approx 0.05472V$.

(d) As in part (c), we have $C(t) = \frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2$, so $C(t) = f(t) + g(t)$



$$\Leftrightarrow \quad \frac{V}{15} - \frac{V}{900}t + \frac{V}{38{,}700}t^2 = \frac{V}{15} - \frac{V}{450}t + \frac{V}{12{,}900}t^2 \quad \Leftrightarrow$$

$$t^2\left(\frac{1}{12{,}900} - \frac{1}{38{,}700}\right) = t\left(\frac{1}{450} - \frac{1}{900}\right) \quad \Leftrightarrow$$

$t = \dfrac{1/900}{2/38{,}700} = \dfrac{43}{2} = 21.5$. This is the value of t that we obtained as the critical number of C in part (c), so we have verified the result of (a) in this case.

63. $\displaystyle\int_1^9 \frac{1}{2x}\,dx = \frac{1}{2}\int_1^9 \frac{1}{x}\,dx = \tfrac{1}{2}\Big[\ln|x|\Big]_1^9 = \tfrac{1}{2}(\ln 9 - \ln 1) = \tfrac{1}{2}\ln 9 - 0 = \ln 9^{1/2} = \ln 3$

64. $\displaystyle\int_0^1 10^x\,dx = \left[\frac{10^x}{\ln 10}\right]_0^1 = \frac{10}{\ln 10} - \frac{1}{\ln 10} = \frac{9}{\ln 10}$

65. $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}}\,dt = 6\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-t^2}}\,dt = 6\Big[\sin^{-1}t\Big]_{1/2}^{\sqrt{3}/2} = 6\Big[\sin^{-1}\left(\tfrac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\tfrac{1}{2}\right)\Big]$

$= 6\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 6\left(\frac{\pi}{6}\right) = \pi$

66. $\displaystyle\int_0^1 \frac{4}{t^2+1}\,dt = 4\int_0^1 \frac{1}{1+t^2}\,dt = 4\Big[\tan^{-1}t\Big]_0^1 = 4\big(\tan^{-1}1 - \tan^{-1}0\big) = 4\left(\tfrac{\pi}{4} - 0\right) = \pi$

67. $\int_{-1}^1 e^{u+1}\,du = \Big[e^{u+1}\Big]_{-1}^1 = e^2 - e^0 = e^2 - 1$ [or start with $e^{u+1} = e^u e^1$]

68. $\displaystyle\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 \left(4u^{-3} + u^{-1}\right)du = \left[\frac{4}{-2}u^{-2} + \ln|u|\right]_1^2 = \left[\frac{-2}{u^2} + \ln u\right]_1^2$

$= \left(-\frac{1}{2} + \ln 2\right) - (-2 + \ln 1) = \frac{3}{2} + \ln 2$

5.4 Indefinite Integrals and the Net Change Theorem

1. $\dfrac{d}{dx}\left[\sqrt{x^2+1} + C\right] = \dfrac{d}{dx}\left[\left(x^2+1\right)^{1/2} + C\right] = \frac{1}{2}\left(x^2+1\right)^{-1/2}\cdot 2x = \dfrac{x}{\sqrt{x^2+1}}$

2. $\dfrac{d}{dx}\left[x\sin x + \cos x + C\right] = x\cos x + (\sin x)\cdot 1 - \sin x = x\cos x$

3. $\dfrac{d}{dx}\left[\dfrac{x}{a^2\sqrt{a^2-x^2}} + C\right] = \dfrac{1}{a^2}\dfrac{\sqrt{a^2-x^2} - x(-x)/\sqrt{a^2-x^2}}{a^2-x^2} = \dfrac{1}{a^2}\dfrac{\left(a^2-x^2\right)+x^2}{\left(a^2-x^2\right)^{3/2}} = \dfrac{1}{\sqrt{\left(a^2-x^2\right)^3}}$

4. $\dfrac{d}{dx}\left[-\dfrac{\sqrt{x^2+a^2}}{a^2x} + C\right] = -\dfrac{1}{a^2}\dfrac{d}{dx}\left[\dfrac{\sqrt{x^2+a^2}}{x}\right] = -\dfrac{x\left(x/\sqrt{x^2+a^2}\right) - \sqrt{x^2+a^2}\cdot 1}{a^2x^2}$

$= -\dfrac{x^2 - \left(x^2+a^2\right)}{a^2x^2\sqrt{x^2+a^2}} = \dfrac{1}{x^2\sqrt{x^2+a^2}}$

5. $\displaystyle\int x^{-3/4}\,dx = \frac{x^{-3/4+1}}{-3/4+1} + C = \frac{x^{1/4}}{1/4} + C = 4x^{1/4} + C$

6. $\displaystyle\int \sqrt[3]{x}\,dx = \int x^{1/3}\,dx = \frac{x^{4/3}}{4/3} + C = \frac{3}{4}x^{4/3} + C$

7. $\displaystyle\int (x^3 + 6x + 1)\,dx = \frac{x^4}{4} + 6\frac{x^2}{2} + x + C = \tfrac{1}{4}x^4 + 3x^2 + x + C$

8. $\displaystyle\int x(1 + 2x^4)\,dx = \int (x + 2x^5)\,dx = \frac{x^2}{2} + 2\frac{x^6}{6} + C = \tfrac{1}{2}x^2 + \tfrac{1}{3}x^6 + C$

9. $\displaystyle\int (1 - t)(2 + t^2)\,dt = \int (2 - 2t + t^2 - t^3)\,dt = 2t - 2\frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + C = 2t - t^2 + \tfrac{1}{3}t^3 - \tfrac{1}{4}t^4 + C$

10. $\displaystyle\int \left(u^2 + 1 + \frac{1}{u^2}\right) du = \int (u^2 + 1 + u^{-2})\,du = \frac{u^3}{3} + u + \frac{u^{-1}}{-1} + C = \tfrac{1}{3}u^3 + u - \frac{1}{u} + C$

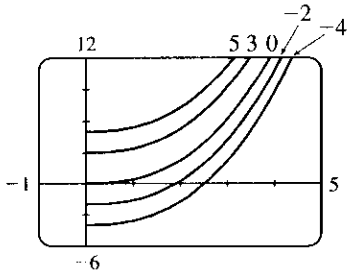
11. $\displaystyle\int (2 - \sqrt{x})^2\,dx = \int (4 - 4\sqrt{x} + x)\,dx = 4x - 4\frac{x^{3/2}}{3/2} + \frac{x^2}{2} + C = 4x - \tfrac{8}{3}x^{3/2} + \frac{1}{2}x^2 + C$

12. $\displaystyle\int (\sin\theta + 3\cos\theta)\,d\theta = -\cos\theta + 3\sin\theta + C$

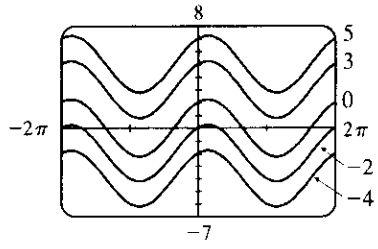
13. $\displaystyle\int \frac{\sin x}{1 - \sin^2 x}\,dx = \int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\,dx = \int \sec x\,\tan x\,dx = \sec x + C$

14. $\displaystyle\int \frac{\sin 2x}{\sin x}\,dx = \int \frac{2\sin x\cos x}{\sin x}\,dx = \int 2\cos x\,dx = 2\sin x + C$

15. $\int x\sqrt{x}\,dx = \int x^{3/2}\,dx = \frac{2}{5}x^{5/2} + C.$
The members of the family in the figure correspond to $C = 5, 3, 0, -2,$ and -4.



16. $\int (\cos x - 2\sin x)\,dx = \sin x + 2\cos x + C.$
The members of the family in the figure correspond to $C = 5, 3, 0, -2,$ and -4.



17. $\displaystyle\int_0^2 (6x^2 - 4x + 5)\,dx = \left[6\cdot\tfrac{1}{3}x^3 - 4\cdot\tfrac{1}{2}x^2 + 5x\right]_0^2 = \left[2x^3 - 2x^2 + 5x\right]_0^2 = (16 - 8 + 10) - 0 = 18$

18. $\displaystyle\int_1^3 (1 + 2x - 4x^3)\,dx = \left[x + 2\cdot\tfrac{1}{2}x^2 - 4\cdot\tfrac{1}{4}x^4\right]_1^3 = \left[x + x^2 - x^4\right]_1^3$
$= (3 + 9 - 81) - (1 + 1 - 1) = -69 - 1 = -70$

19. $\displaystyle\int_{-3}^0 (5y^4 - 6y^2 + 14)\,dy = \left[5(\tfrac{1}{5}y^5) - 6(\tfrac{1}{3}y^3) + 14y\right]_{-3}^0 = \left[y^5 - 2y^3 + 14y\right]_{-3}^0$
$= 0 - (-243 + 54 - 42) = 231$

20. $\displaystyle\int_{-2}^0 (u^5 - u^3 + u^2)\,du = \left[\tfrac{1}{6}u^6 - \tfrac{1}{4}u^4 + \tfrac{1}{3}u^3\right]_{-2}^0 = 0 - \left(\tfrac{32}{3} - 4 - \tfrac{8}{3}\right) = -4$

21. $\displaystyle\int_{-2}^2 (3u + 1)^2\,du = \int_{-2}^2 (9u^2 + 6u + 1)\,du = \left[9\cdot\tfrac{1}{3}u^3 + 6\cdot\tfrac{1}{2}u^2 + u\right]_{-2}^2 = \left[3u^3 + 3u^2 + u\right]_{-2}^2$
$= (24 + 12 + 2) - (-24 + 12 - 2) = 38 - (-14) = 52$

22. $\displaystyle\int_0^4 (2v + 5)(3v - 1)\,dv = \int_0^4 (6v^2 + 13v - 5)\,dv = \left[6\cdot\tfrac{1}{3}v^3 + 13\cdot\tfrac{1}{2}v^2 - 5v\right]_0^4 = \left[2v^3 + \tfrac{13}{2}v^2 - 5v\right]_0^4$
$= (128 + 104 - 20) - 0 = 212$

23. $\displaystyle\int_1^4 \sqrt{t}\,(1 + t)\,dt = \int_1^4 (t^{1/2} + t^{3/2})\,dt = \left[\tfrac{2}{3}t^{3/2} + \tfrac{2}{5}t^{5/2}\right]_1^4 = \left(\tfrac{16}{3} + \tfrac{64}{5}\right) - \left(\tfrac{2}{3} + \tfrac{2}{5}\right) = \tfrac{14}{3} + \tfrac{62}{5} = \tfrac{256}{15}$

24. $\int_0^9 \sqrt{2t}\,dt = \int_0^9 \sqrt{2}\,t^{1/2}\,dt = \left[\sqrt{2}\cdot\frac{2}{3}t^{3/2}\right]_0^9 = \sqrt{2}\cdot\frac{2}{3}\cdot 27 - 0 = 18\sqrt{2}$

25. $\int_{-2}^{-1}\left(4y^3+\frac{2}{y^3}\right)dy = \left[4\cdot\frac{1}{4}y^4 + 2\cdot\frac{1}{-2}y^{-2}\right]_{-2}^{-1} = \left[y^4 - \frac{1}{y^2}\right]_{-2}^{-1} = (1-1) - \left(16-\tfrac{1}{4}\right) = -\tfrac{63}{4}$

26. $\int_1^2 \frac{y+5y^7}{y^3}\,dy = \int_1^2 (y^{-2}+5y^4)\,dy = \left[-y^{-1}+5\cdot\tfrac{1}{5}y^5\right]_1^2 = \left[-\frac{1}{y}+y^5\right]_1^2 = \left(-\tfrac{1}{2}+32\right) - (-1+1) = \tfrac{63}{2}$

27. $\int_0^1 x(\sqrt[3]{x}+\sqrt[4]{x})\,dx = \int_0^1 (x^{4/3}+x^{5/4})\,dx = \left[\tfrac{3}{7}x^{7/3}+\tfrac{4}{9}x^{9/4}\right]_0^1 = \left(\tfrac{3}{7}+\tfrac{4}{9}\right) - 0 = \tfrac{55}{63}$

28. $\int_1^2 \left(x+\frac{1}{x}\right)^2 dx = \int_1^2 \left(x^2+2+x^{-2}\right)dx = \left[\frac{x^3}{3}+2x+\frac{x^{-1}}{-1}\right]_1^2 = \left[\frac{x^3}{3}+2x-\frac{1}{x}\right]_1^2$

$= \left(\tfrac{8}{3}+4-\tfrac{1}{2}\right) - \left(\tfrac{1}{3}+2-1\right) = \tfrac{29}{6}$

29. $\int_1^4 \sqrt{5/x}\,dx = \sqrt{5}\int_1^4 x^{-1/2}\,dx = \sqrt{5}\,[2\sqrt{x}\,]_1^4 = \sqrt{5}\,(2\cdot 2 - 2\cdot 1) = 2\sqrt{5}$

30. $\int_1^9 \frac{3x-2}{\sqrt{x}}\,dx = \int_1^9 (3x^{1/2}-2x^{-1/2})\,dx = \left[3\cdot\tfrac{2}{3}x^{3/2} - 2\cdot 2x^{1/2}\right]_1^9 = \left[2x^{3/2}-4x^{1/2}\right]_1^9$

$= (54-12) - (2-4) = 44$

31. $\int_0^{\pi}(4\sin\theta - 3\cos\theta)\,d\theta = [-4\cos\theta - 3\sin\theta]_0^{\pi} = (4-0) - (-4-0) = 8$

32. $\int_{\pi/4}^{\pi/3} \sec\theta\tan\theta\,d\theta = \left[\sec\theta\right]_{\pi/4}^{\pi/3} = \sec\tfrac{\pi}{3} - \sec\tfrac{\pi}{4} = 2-\sqrt{2}$

33. $\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4}\left(\frac{1}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}\right)d\theta = \int_0^{\pi/4}\left(\sec^2\theta+1\right)d\theta$

$= [\tan\theta+\theta]_0^{\pi/4} = \left(\tan\tfrac{\pi}{4}+\tfrac{\pi}{4}\right) - (0+0) = 1+\tfrac{\pi}{4}$

34. $\int_0^{\pi/3} \frac{\sin\theta+\sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta = \int_0^{\pi/3} \frac{\sin\theta\,(1+\tan^2\theta)}{\sec^2\theta}\,d\theta = \int_0^{\pi/3} \frac{\sin\theta\sec^2\theta}{\sec^2\theta}\,d\theta = \int_0^{\pi/3}\sin\theta\,d\theta$

$= \left[-\cos\theta\right]_0^{\pi/3} = -\tfrac{1}{2} - (-1) = \tfrac{1}{2}$

35. $\int_1^{64} \frac{1+\sqrt[3]{x}}{\sqrt{x}}\,dx = \int_1^{64}\left(\frac{1}{x^{1/2}}+\frac{x^{1/3}}{x^{1/2}}\right)dx = \int_1^{64}(x^{-1/2}+x^{(1/3)-(1/2)})\,dx = \int_1^{64}(x^{-1/2}+x^{-1/6})\,dx$

$= \left[2x^{1/2}+\tfrac{6}{5}x^{5/6}\right]_1^{64} = \left(16+\tfrac{192}{5}\right) - \left(2+\tfrac{6}{5}\right) = 14+\tfrac{186}{5} = \tfrac{256}{5}$

36. $\int_0^1 (1+x^2)^3\,dx = \int_0^1 (1+3x^2+3x^4+x^6)\,dx = \left[x+x^3+\tfrac{3}{5}x^5+\tfrac{1}{7}x^7\right]_0^1 = \left(1+1+\tfrac{3}{5}+\tfrac{1}{7}\right) - 0 = \tfrac{96}{35}$

37. $\int_0^1 \left(\sqrt[4]{x^5}+\sqrt[5]{x^4}\right)dx = \int_0^1 \left(x^{5/4}+x^{4/5}\right)dx = \left[\frac{x^{9/4}}{9/4}+\frac{x^{9/5}}{9/5}\right]_0^1 = \left[\tfrac{4}{9}x^{9/4}+\tfrac{5}{9}x^{9/5}\right]_0^1 = \tfrac{4}{9}+\tfrac{5}{9}-0 = 1$

38. $\int_1^8 \frac{x-1}{\sqrt[3]{x^2}}\,dx = \int_1^8 \left(x^{1/3}-x^{-2/3}\right)dx = \left[\frac{x^{4/3}}{4/3}-\frac{x^{1/3}}{1/3}\right]_1^8 = \left[\frac{3}{4}x^{4/3}-3x^{1/3}\right]_1^8$

$= \left(\tfrac{3}{4}\cdot 16 - 3\cdot 2\right) - \left(\tfrac{3}{4}-3\right) = \tfrac{33}{4}$

39. $\int_{-1}^2 (x-2|x|)\,dx = \int_{-1}^0 [x-2(-x)]\,dx + \int_0^2 [x-2(x)]\,dx = \int_{-1}^0 3x\,dx + \int_0^2 (-x)\,dx = 3\left[\tfrac{1}{2}x^2\right]_{-1}^0 - \left[\tfrac{1}{2}x^2\right]_0^2$

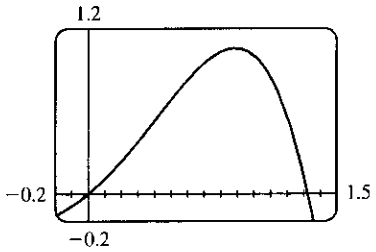
$= 3\left(0-\tfrac{1}{2}\right) - (2-0) = -\tfrac{7}{2} = -3.5$

40. $\int_0^{3\pi/2} |\sin x|\,dx = \int_0^{\pi}\sin x\,dx + \int_{\pi}^{3\pi/2}(-\sin x)\,dx = [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{3\pi/2}$

$= [1-(-1)] + [0-(-1)] = 2+1 = 3$

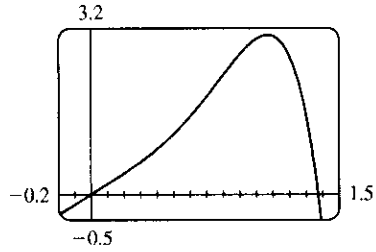
41. The graph shows that $y = x + x^2 - x^4$ has x-intercepts at $x = 0$ and at $x = a \approx 1.32$. So the area of the region that lies under the curve and above the x-axis is

$$\begin{aligned}\int_0^a (x + x^2 - x^4)\,dx &= \left[\tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_0^a \\ &= \left(\tfrac{1}{2}a^2 + \tfrac{1}{3}a^3 - \tfrac{1}{5}a^5\right) - 0 \\ &\approx 0.84\end{aligned}$$



42. The graph shows that $y = 2x + 3x^4 - 2x^6$ has x-intercepts at $x = 0$ and at $x = a \approx 1.37$. So the area of the region that lies under the curve and above the x-axis is

$$\begin{aligned}\int_0^a (2x + 3x^4 - 2x^6)\,dx &= \left[x^2 + \tfrac{3}{5}x^5 - \tfrac{2}{7}x^7\right]_0^a \\ &= \left(a^2 + \tfrac{3}{5}a^5 - \tfrac{2}{7}a^7\right) - 0 \\ &\approx 2.18\end{aligned}$$



43. $A = \int_0^2 (2y - y^2)\,dy = \left[y^2 - \tfrac{1}{3}y^3\right]_0^2 = \left(4 - \tfrac{8}{3}\right) - 0 = \tfrac{4}{3}$

44. $y = \sqrt[4]{x} \;\Rightarrow\; x = y^4$, so $A = \int_0^1 y^4\,dy = \left[\tfrac{1}{5}y^5\right]_0^1 = \tfrac{1}{5}$.

45. If $w'(t)$ is the rate of change of weight in pounds per year, then $w(t)$ represents the weight in pounds of the child at age t. We know from the Net Change Theorem that $\int_5^{10} w'(t)\,dt = w(10) - w(5)$, so the integral represents the increase in the child's weight (in pounds) between the ages of 5 and 10.

46. $\int_a^b I(t)\,dt = \int_a^b Q'(t)\,dt = Q(b) - Q(a)$ by the Net Change Theorem, so it represents the change in the charge Q from time $t = a$ to $t = b$.

47. Since $r(t)$ is the rate at which oil leaks, we can write $r(t) = -V'(t)$, where $V(t)$ is the volume of oil at time t. [Note that the minus sign is needed because V is decreasing, so $V'(t)$ is negative, but $r(t)$ is positive.] Thus, by the Net Change Theorem, $\int_0^{120} r(t)\,dt = -\int_0^{120} V'(t)\,dt = -[V(120) - V(0)] = V(0) - V(120)$, which is the number of gallons of oil that leaked from the tank in the first two hours (120 minutes).

48. By the Net Change Theorem, $\int_0^{15} n'(t)\,dt = n(15) - n(0) = n(15) - 100$ represents the increase in the bee population in 15 weeks. So $100 + \int_0^{15} n'(t)\,dt = n(15)$ represents the total bee population after 15 weeks.

49. By the Net Change Theorem, $\int_{1000}^{5000} R'(x)\,dx = R(5000) - R(1000)$, so it represents the increase in revenue when production is increased from 1000 units to 5000 units.

50. The slope of the trail is the rate of change of the elevation E, so $f(x) = E'(x)$. By the Net Change Theorem, $\int_3^5 f(x)\,dx = \int_3^5 E'(x)\,dx = E(5) - E(3)$ is the change in the elevation E between $x = 3$ miles and $x = 5$ miles from the start of the trail.

51. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for x. Since $f(x)$ is measured in newtons and x is measured in meters, the units for $\int_0^{100} f(x)\,dx$ are newton-meters. (A newton-meter is abbreviated N-m and is called a joule.)

52. The units for $a(x)$ are pounds per foot and the units for x are feet, so the units for da/dx are pounds per foot per foot, denoted (lb/ft)/ft. The unit of measurement for $\int_2^8 a(x)\,dx$ is the product of pounds per foot and feet; that is, pounds.

53. (a) displacement $= \int_0^3 (3t-5)\,dt = \left[\frac{3}{2}t^2 - 5t\right]_0^3 = \frac{27}{2} - 15 = -\frac{3}{2}$ m

(b) distance traveled $= \int_0^3 |3t-5|\,dt = \int_0^{5/3} (5-3t)\,dt + \int_{5/3}^3 (3t-5)\,dt$

$= \left[5t - \frac{3}{2}t^2\right]_0^{5/3} + \left[\frac{3}{2}t^2 - 5t\right]_{5/3}^3 = \frac{25}{3} - \frac{3}{2}\cdot\frac{25}{9} + \frac{27}{2} - 15 - \left(\frac{3}{2}\cdot\frac{25}{9} - \frac{25}{3}\right) = \frac{41}{6}$ m

54. (a) displacement $= \int_1^6 (t^2-2t-8)\,dt = \left[\frac{1}{3}t^3 - t^2 - 8t\right]_1^6 = (72-36-48) - \left(\frac{1}{3} - 1 - 8\right) = -\frac{10}{3}$ m

(b) distance traveled $= \int_1^6 |t^2-2t-8|\,dt = \int_1^6 |(t-4)(t+2)|\,dt$

$= \int_1^4 (-t^2+2t+8)\,dt + \int_4^6 (t^2-2t-8)\,dt = \left[-\frac{1}{3}t^3 + t^2 + 8t\right]_1^4 + \left[\frac{1}{3}t^3 - t^2 - 8t\right]_4^6$

$= \left(-\frac{64}{3} + 16 + 32\right) - \left(-\frac{1}{3} + 1 + 8\right) + (72 - 36 - 48) - \left(\frac{64}{3} - 16 - 32\right) = \frac{98}{3}$ m

55. (a) $v'(t) = a(t) = t+4 \;\Rightarrow\; v(t) = \frac{1}{2}t^2 + 4t + C \;\Rightarrow\; v(0) = C = 5 \;\Rightarrow\; v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s

(b) distance traveled $= \int_0^{10} |v(t)|\,dt = \int_0^{10} \left|\frac{1}{2}t^2 + 4t + 5\right| dt = \int_0^{10} \left(\frac{1}{2}t^2 + 4t + 5\right) dt$

$= \left[\frac{1}{6}t^3 + 2t^2 + 5t\right]_0^{10} = \frac{500}{3} + 200 + 50 = 416\frac{2}{3}$ m

56. (a) $v'(t) = a(t) = 2t+3 \;\Rightarrow\; v(t) = t^2 + 3t + C \;\Rightarrow\; v(0) = C = -4 \;\Rightarrow\; v(t) = t^2 + 3t - 4$

(b) distance traveled $= \int_0^3 |t^2+3t-4|\,dt = \int_0^3 |(t+4)(t-1)|\,dt$

$= \int_0^1 (-t^2-3t+4)\,dt + \int_1^3 (t^2+3t-4)\,dt$

$= \left[-\frac{1}{3}t^3 - \frac{3}{2}t^2 + 4t\right]_0^1 + \left[\frac{1}{3}t^3 + \frac{3}{2}t^2 - 4t\right]_1^3$

$= \left(-\frac{1}{3} - \frac{3}{2} + 4\right) + \left(9 + \frac{27}{2} - 12\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right) = \frac{89}{6}$ m

57. Since $m'(x) = \rho(x)$, $m = \int_0^4 \rho(x)\,dx = \int_0^4 (9 + 2\sqrt{x}\,)\,dx = \left[9x + \frac{4}{3}x^{3/2}\right]_0^4 = 36 + \frac{32}{3} - 0 = \frac{140}{3} = 46\frac{2}{3}$ kg.

58. By the Net Change Theorem, the amount of water that flows from the tank is

$\int_0^{10} r(t)\,dt = \int_0^{10} (200 - 4t)\,dt = \left[200t - 2t^2\right]_0^{10} = (2000 - 200) - 0 = 1800$ liters.

59. Let s be the position of the car. We know from Equation 2 that $s(100) - s(0) = \int_0^{100} v(t)\,dt$. We use the Midpoint Rule for $0 \le t \le 100$ with $n = 5$. Note that the length of each of the five time intervals is 20 seconds $= \frac{20}{3600}$ hour $= \frac{1}{180}$ hour. So the distance traveled is

$$\int_0^{100} v(t)\,dt \approx \tfrac{1}{180}[v(10) + v(30) + v(50) + v(70) + v(90)]$$

$$= \tfrac{1}{180}(38 + 58 + 51 + 53 + 47)$$

$$= \tfrac{247}{180} \approx 1.4 \text{ miles}$$

60. (a) By the Net Change Theorem, the total amount spewed into the atmosphere is $Q(6) - Q(0) = \int_0^6 r(t)\,dt = Q(6)$ since $Q(0) = 0$. The rate $r(t)$ is positive, so Q is an increasing function. Thus, an upper estimate for $Q(6)$ is R_6 and a lower estimate for $Q(6)$ is L_6. $\Delta t = \dfrac{b-a}{n} = \dfrac{6-0}{6} = 1$.

$R_6 = \sum_{i=1}^{6} r(t_i)\,\Delta t = 10 + 24 + 36 + 46 + 54 + 60 = 230$ tonnes.

$L_6 = \sum_{i=1}^{6} r(t_{i-1})\,\Delta t = R_6 + r(0) - r(6) = 230 + 2 - 60 = 172$ tonnes.

(b) $\Delta t = \dfrac{b-a}{n} = \dfrac{6-0}{3} = 2$.

$Q(6) \approx M_3 = 2[r(1) + r(3) + r(5)] = 2(10 + 36 + 54) = 2(100) = 200$ tonnes.

61. From the Net Change Theorem, the increase in cost if the production level is raised from 2000 yards to 4000 yards is $C(4000) - C(2000) = \int_{2000}^{4000} C'(x)\,dx$.

$$\int_{2000}^{4000} C'(x)\,dx = \int_{2000}^{4000} \left(3 - 0.01x + 0.000006x^2\right) dx$$

$$= \left[3x - 0.005x^2 + 0.000002x^3\right]_{2000}^{4000} = 60{,}000 - 2{,}000 = \$58{,}000$$

62. By the Net Change Theorem, the amount of water after four days is

$$\begin{aligned} 25{,}000 + \textstyle\int_0^4 r(t)\,dt &\approx 25{,}000 + M_4 \\ &= 25{,}000 + \tfrac{4-0}{4}\left[r(0.5) + r(1.5) + r(2.5) + r(3.5)\right] \\ &\approx 25{,}000 + [1500 + 1770 + 740 + (-690)] = 28{,}320 \text{ liters} \end{aligned}$$

63. (a) We can find the area between the Lorenz curve and the line $y = x$ by subtracting the area under $y = L(x)$ from the area under $y = x$. Thus,

$$\text{coefficient of inequality} = \frac{\text{area between Lorenz curve and line } y = x}{\text{area under line } y = x} = \frac{\int_0^1 [x - L(x)]\,dx}{\int_0^1 x\,dx}$$

$$= \frac{\int_0^1 [x - L(x)]\,dx}{[x^2/2]_0^1} = \frac{\int_0^1 [x - L(x)]\,dx}{1/2} = 2\int_0^1 [x - L(x)]\,dx$$

(b) $L(x) = \frac{5}{12}x^2 + \frac{7}{12}x \quad \Rightarrow \quad L(50\%) = L\left(\frac{1}{2}\right) = \frac{5}{48} + \frac{7}{24} = \frac{19}{48} = 0.39583\overline{3}$, so the bottom 50% of the households receive at most about 40% of the income. Using the result in part (a),

$$\begin{aligned} \text{coefficient of inequality} &= 2\textstyle\int_0^1 [x - L(x)]\,dx = 2\int_0^1 \left(x - \frac{5}{12}x^2 - \frac{7}{12}x\right) dx \\ &= 2\textstyle\int_0^1 \left(\frac{5}{12}x - \frac{5}{12}x^2\right) dx = 2\int_0^1 \frac{5}{12}\left(x - x^2\right) dx \\ &= \tfrac{5}{6}\left[\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{5}{6}\left(\tfrac{1}{2} - \tfrac{1}{3}\right) = \tfrac{5}{6}\left(\tfrac{1}{6}\right) = \tfrac{5}{36} \end{aligned}$$

64. (a) From Exercise 4.1.66(a), $v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872$.

(b) $h(125) - h(0) = \int_0^{125} v(t)\,dt = \left[0.000365t^4 - 0.03851t^3 + 12.490845t^2 - 21.26872t\right]_0^{125} \approx 206{,}407$ ft

65. $$\int_1^e \frac{x^2 + x + 1}{x}\,dx = \int_1^e \left(x + 1 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + x + \ln|x|\right]_1^e$$

$$= \left(\tfrac{1}{2}e^2 + e + \ln e\right) - \left(\tfrac{1}{2} + 1 + \ln 1\right) = \tfrac{1}{2}e^2 + e - \tfrac{1}{2}$$

66. $$\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + 2x + \ln|x|\right]_4^9$$

$$= \tfrac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4) = \tfrac{85}{2} + \ln\tfrac{9}{4}$$

67. $$\int \left(x^2 + 1 + \frac{1}{x^2 + 1}\right) dx = \frac{x^3}{3} + x + \tan^{-1} x + C$$

68. $B = 3A \quad \Rightarrow \quad \int_0^b e^x\,dx = 3\int_0^a e^x\,dx \quad \Rightarrow \quad [e^x]_0^b = 3\,[e^x]_0^a \quad \Rightarrow \quad e^b - 1 = 3(e^a - 1) \quad \Rightarrow$

$e^b = 3e^a - 2 \quad \Rightarrow \quad b = \ln(3e^a - 2)$

5.5 The Substitution Rule

1. Let $u = 3x$. Then $du = 3\,dx$, so $dx = \frac{1}{3}du$. Thus,

$\int \cos 3x\,dx = \int \cos u\left(\frac{1}{3}\,du\right) = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin 3x + C$. Don't forget that it is often very easy to check an indefinite integration by differentiating your answer. In this case,

$\frac{d}{dx}\left(\frac{1}{3}\sin 3x + C\right) = \frac{1}{3}(\cos 3x)\cdot 3 = \cos 3x$, the desired result.

2. Let $u = 4 + x^2$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$\int x(4+x^2)^{10}\,dx = \int u^{10}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\cdot\frac{1}{11}u^{11} + C = \frac{1}{22}(4+x^2)^{11} + C$.

3. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so

$$\int x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C = \frac{2}{9}(x^3+1)^{3/2} + C.$$

4. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$ and $\dfrac{1}{\sqrt{x}}\,dx = 2\,du$, so

$$\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin u\,(2\,du) = 2(-\cos u) + C = -2\cos\sqrt{x} + C.$$

5. Let $u = 1 + 2x$. Then $du = 2\,dx$ and $dx = \frac{1}{2}\,du$, so

$$\int \frac{4}{(1+2x)^3}\,dx = 4\int u^{-3}\left(\tfrac{1}{2}\,du\right) = 2\frac{u^{-2}}{-2} + C = -\frac{1}{u^2} + C = -\frac{1}{(1+2x)^2} + C.$$

6. Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and $\sin\theta\,d\theta = -du$, so

$\int \cos^4\theta\sin\theta\,d\theta = \int u^4\,(-du) = -\frac{1}{5}u^5 + C = -\frac{1}{5}\cos^5\theta + C$.

7. Let $u = x^2 + 3$. Then $du = 2x\,dx$, so $\int 2x(x^2+3)^4\,dx = \int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}(x^2+3)^5 + C$.

8. Let $u = x^3 + 5$. Then $du = 3x^2dx$ and $x^2dx = \frac{1}{3}\,du$, so

$\int x^2(x^3+5)^9\,dx = \int u^9\left(\frac{1}{3}\,du\right) = \frac{1}{3}\cdot\frac{1}{10}u^{10} + C = \frac{1}{30}(x^3+5)^{10} + C$.

9. Let $u = 3x - 2$. Then $du = 3\,dx$ and $dx = \frac{1}{3}\,du$, so

$\int (3x-2)^{20}\,dx = \int u^{20}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\cdot\frac{1}{21}u^{21} + C = \frac{1}{63}(3x-2)^{21} + C$.

10. Let $u = 2 - x$. Then $du = -dx$ and $dx = -du$, so

$\int (2-x)^6\,dx = \int u^6(-du) = -\frac{1}{7}u^7 + C = -\frac{1}{7}(2-x)^7 + C$.

11. Let $u = 1 + x + 2x^2$. Then $du = (1+4x)\,dx$, so

$$\int \frac{1+4x}{\sqrt{1+x+2x^2}}\,dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+x+2x^2} + C.$$

12. Let $u = x^2 + 1$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int \frac{x}{(x^2+1)^2}\,dx = \int u^{-2}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\cdot\frac{-1}{u} + C = \frac{-1}{2u} + C = \frac{-1}{2(x^2+1)} + C.$$

13. Let $u = 2y + 1$. Then $du = 2\,dy$ and $dy = \frac{1}{2}\,du$, so

$$\int \frac{3}{(2y+1)^5}\,dy = \int 3u^{-5}\left(\tfrac{1}{2}\,du\right) = \frac{3}{2}\cdot\frac{1}{-4}u^{-4} + C = \frac{-3}{8(2y+1)^4} + C.$$

14. Let $u = 5t + 4$. Then $du = 5\,dt$ and $dt = \frac{1}{5}\,du$, so

$$\int \frac{1}{(5t+4)^{2.7}}\,dt = \int u^{-2.7}\left(\tfrac{1}{5}\,du\right) = \frac{1}{5}\cdot\frac{1}{-1.7}u^{-1.7} + C = \frac{-1}{8.5}u^{-1.7} + C = \frac{-2}{17(5t+4)^{1.7}} + C.$$

15. Let $u = 4 - t$. Then $du = -dt$ and $dt = -du$, so

$$\int \sqrt{4-t}\,dt = \int u^{1/2}\,(-du) = -\tfrac{2}{3}u^{3/2} + C = -\tfrac{2}{3}(4-t)^{3/2} + C.$$

16. Let $u = 2y^4 - 1$. Then $du = 8y^3\,dy$ and $y^3\,dy = \frac{1}{8}\,du$, so

$$\int y^3\sqrt{2y^4-1}\,dy = \int u^{1/2}\left(\tfrac{1}{8}\,du\right) = \tfrac{1}{8}\cdot\tfrac{2}{3}u^{3/2} + C = \tfrac{1}{12}(2y^4-1)^{3/2} + C.$$

17. Let $u = \pi t$. Then $du = \pi\,dt$ and $dt = \frac{1}{\pi}\,du$, so

$$\int \sin \pi t\,dt = \int \sin u\left(\tfrac{1}{\pi}\,du\right) = \tfrac{1}{\pi}(-\cos u) + C = -\tfrac{1}{\pi}\cos \pi t + C.$$

18. Let $u = 2\theta$. Then $du = 2\,d\theta$ and $d\theta = \frac{1}{2}\,du$, so

$$\int \sec 2\theta\,\tan 2\theta\,d\theta = \int \sec u \tan u\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\sec u + C = \tfrac{1}{2}\sec 2\theta + C.$$

19. Let $u = \sqrt{t}$. Then $du = \dfrac{dt}{2\sqrt{t}}$ and $\dfrac{1}{\sqrt{t}}\,dt = 2\,du$, so

$$\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt = \int \cos u\,(2\,du) = 2\sin u + C = 2\sin\sqrt{t} + C.$$

20. Let $u = 1 + x^{3/2}$. Then $du = \frac{3}{2}x^{1/2}\,dx$ and $\sqrt{x}\,dx = \frac{2}{3}\,du$, so

$$\int \sqrt{x}\sin(1+x^{3/2})\,dx = \int \sin u\left(\tfrac{2}{3}\,du\right) = \tfrac{2}{3}\cdot(-\cos u) + C = -\tfrac{2}{3}\cos(1+x^{3/2}) + C.$$

21. Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$, so $\int \cos\theta\,\sin^6\theta\,d\theta = \int u^6\,du = \frac{1}{7}u^7 + C = \frac{1}{7}\sin^7\theta + C$.

22. Let $u = 1 + \tan\theta$. Then $du = \sec^2\theta\,d\theta$, so

$$\int (1+\tan\theta)^5\sec^2\theta\,d\theta = \int u^5\,du = \tfrac{1}{6}u^6 + C = \tfrac{1}{6}(1+\tan\theta)^6 + C.$$

23. Let $u = 1 + z^3$. Then $du = 3z^2\,dz$ and $z^2\,dz = \frac{1}{3}\,du$, so

$$\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz = \int u^{-1/3}\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\cdot\tfrac{3}{2}u^{2/3} + C = \tfrac{1}{2}(1+z^3)^{2/3} + C.$$

24. Let $u = ax^2 + 2bx + c$. Then $du = 2(ax+b)\,dx$ and $(ax+b)\,dx = \frac{1}{2}\,du$, so

$$\int \frac{(ax+b)\,dx}{\sqrt{ax^2+2bx+c}} = \int \frac{\frac{1}{2}\,du}{\sqrt{u}} = \tfrac{1}{2}\int u^{-1/2}\,du = u^{1/2} + C = \sqrt{ax^2+2bx+c} + C.$$

25. Let $u = \cot x$. Then $du = -\csc^2 x\,dx$ and $\csc^2 x\,dx = -du$, so

$$\int \sqrt{\cot x}\csc^2 x\,dx = \int \sqrt{u}\,(-du) = -\frac{u^{3/2}}{3/2} + C = -\tfrac{2}{3}(\cot x)^{3/2} + C.$$

26. Let $u = \dfrac{\pi}{x}$. Then $du = -\dfrac{\pi}{x^2}\,dx$ and $\dfrac{1}{x^2}\,dx = -\dfrac{1}{\pi}\,du$, so

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \int \cos u\left(-\frac{1}{\pi}\,du\right) = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin\frac{\pi}{x} + C.$$

27. Let $u = \sec x$. Then $du = \sec x\,\tan x\,dx$, so

$$\int \sec^3 x\,\tan x\,dx = \int \sec^2 x\,(\sec x\,\tan x)\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sec^3 x + C.$$

28. Let $u = x^3 + 1$. Then $x^3 = u - 1$ and $du = 3x^2\,dx$, so

$$\int \sqrt[3]{x^3+1}\,x^5\,dx = \int \sqrt[3]{x^3+1}\cdot x^3\cdot x^2\,dx = \int u^{1/3}(u-1)\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\int (u^{4/3}-u^{1/3})\,du$$
$$= \tfrac{1}{3}\left(\tfrac{3}{7}u^{7/3} - \tfrac{3}{4}u^{4/3}\right) + C = \tfrac{1}{7}(x^3+1)^{7/3} - \tfrac{1}{4}(x^3+1)^{4/3} + C$$

29. Let $u = b + cx^{a+1}$. Then $du = (a+1)cx^a\,dx$, so

$$\int x^a\sqrt{b+cx^{a+1}}\,dx = \int u^{1/2}\frac{1}{(a+1)c}\,du = \frac{1}{(a+1)c}\left(\tfrac{2}{3}u^{3/2}\right) + C = \frac{2}{3c(a+1)}\left(b+cx^{a+1}\right)^{3/2} + C.$$

30. Let $u = \cos t$. Then $du = -\sin t\,dt$ and $\sin t\,dt = -du$, so

$$\int \sin t\,\sec^2(\cos t)\,dt = \int \sec^2 u\cdot(-du) = -\tan u + C = -\tan(\cos t) + C.$$

31. Let $u = x + 2$. Then $du = dx$, so

$$\int \frac{x}{\sqrt[4]{x+2}}\,dx = \int \frac{u-2}{\sqrt[4]{u}}\,du = \int \left(u^{3/4} - 2u^{-1/4}\right)du = \tfrac{4}{7}u^{7/4} - 2\cdot\tfrac{4}{3}u^{3/4} + C$$
$$= \tfrac{4}{7}(x+2)^{7/4} - \tfrac{8}{3}(x+2)^{3/4} + C$$

32. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

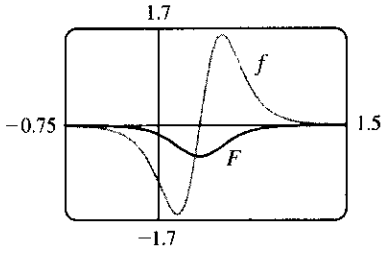
$$\int \frac{x^2}{\sqrt{1-x}}\,dx = \int \frac{(1-u)^2}{\sqrt{u}}\,(-du) = -\int \frac{1-2u+u^2}{\sqrt{u}}\,du = -\int \left(u^{-1/2} - 2u^{1/2} + u^{3/2}\right)du$$
$$= -\left(2u^{1/2} - 2\cdot\tfrac{2}{3}u^{3/2} + \tfrac{2}{5}u^{5/2}\right) + C = -2\sqrt{1-x} + \tfrac{4}{3}(1-x)^{3/2} - \tfrac{2}{5}(1-x)^{5/2} + C$$

In Exercises 33–36, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

33. $f(x) = \dfrac{3x-1}{(3x^2-2x+1)^4}$.

$u = 3x^2 - 2x + 1 \quad\Rightarrow\quad du = (6x-2)\,dx = 2(3x-1)\,dx$, so

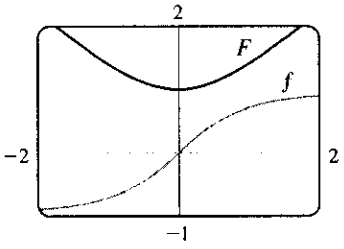
$$\int \frac{3x-1}{(3x^2-2x+1)^4}\,dx = \int \frac{1}{u^4}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-4}\,du$$
$$= -\frac{1}{6}u^{-3} + C = -\frac{1}{6(3x^2-2x+1)^3} + C$$



Notice that at $x = \frac{1}{3}$, f changes from negative to positive, and F has a local minimum.

34. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$. $u = x^2 + 1 \quad\Rightarrow\quad du = 2x\,dx$, so

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{u}}\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int u^{-1/2}\,du$$
$$= u^{1/2} + C = \sqrt{x^2+1} + C.$$

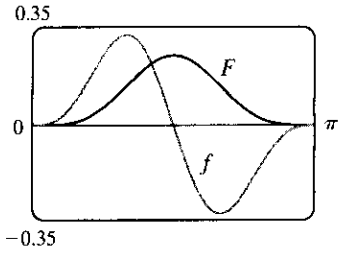


Note that at $x = 0$, f changes from negative to positive and F has a local minimum.

35. $f(x) = \sin^3 x \cos x$. $u = \sin x \quad \Rightarrow \quad du = \cos x\,dx$, so

$$\int \sin^3 x \cos x\,dx = \int u^3\,du = \tfrac{1}{4}u^4 + C = \tfrac{1}{4}\sin^4 x + C$$

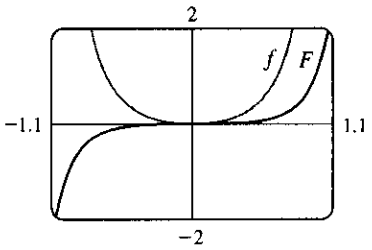
Note that at $x = \frac{\pi}{2}$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period π, so at $x = 0$ and at $x = \pi$, f changes from negative to positive and F has local minima.



36. $f(\theta) = \tan^2\theta \sec^2\theta$. $u = \tan\theta \quad \Rightarrow \quad du = \sec^2\theta\,d\theta$, so

$$\int \tan^2\theta \sec^2\theta\,d\theta = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\tan^3\theta + C$$

Note that f is positive and F is increasing. At $x = 0$, $f = 0$ and F has a horizontal tangent.



37. Let $u = x - 1$, so $du = dx$. When $x = 0$, $u = -1$; when $x = 2$, $u = 1$. Thus, $\int_0^2 (x-1)^{25}\,dx = \int_{-1}^1 u^{25}\,du = 0$ by Theorem 6(b), since $f(u) = u^{25}$ is an odd function.

38. Let $u = 4 + 3x$, so $du = 3\,dx$. When $x = 0$, $u = 4$; when $x = 7$, $u = 25$. Thus,

$$\int_0^7 \sqrt{4+3x}\,dx = \int_4^{25} \sqrt{u}\left(\tfrac{1}{3}\,du\right) = \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_4^{25} = \frac{2}{9}(25^{3/2} - 4^{3/2}) = \frac{2}{9}(125 - 8) = \frac{234}{9} = 26$$

39. Let $u = 1 + 2x^3$, so $du = 6x^2\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 3$. Thus,

$$\int_0^1 x^2\left(1 + 2x^3\right)^5\,dx = \int_1^3 u^5\left(\tfrac{1}{6}\,du\right) = \tfrac{1}{6}\left[\tfrac{1}{6}u^6\right]_1^3 = \tfrac{1}{36}(3^6 - 1^6) = \tfrac{1}{36}(729 - 1) = \tfrac{728}{36} = \tfrac{182}{9}$$

40. Let $u = x^2$, so $du = 2x\,dx$. When $x = 0$, $u = 0$; when $x = \sqrt{\pi}$, $u = \pi$. Thus,

$$\int_0^{\sqrt{\pi}} x\cos\left(x^2\right)dx = \int_0^{\pi} \cos u\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}[\sin u]_0^{\pi} = \tfrac{1}{2}(\sin\pi - \sin 0) = \tfrac{1}{2}(0 - 0) = 0.$$

41. Let $u = t/4$, so $du = \frac{1}{4}\,dt$. When $t = 0$, $u = 0$; when $t = \pi$, $u = \pi/4$. Thus,

$$\int_0^{\pi} \sec^2(t/4)\,dt = \int_0^{\pi/4} \sec^2 u\,(4\,du) = 4\big[\tan u\big]_0^{\pi/4} = 4\big(\tan\tfrac{\pi}{4} - \tan 0\big) = 4(1 - 0) = 4.$$

42. Let $u = \pi t$, so $du = \pi\,dt$. When $t = \frac{1}{6}$, $u = \frac{\pi}{6}$; when $t = \frac{1}{2}$, $u = \frac{\pi}{2}$. Thus,

$$\int_{1/6}^{1/2} \csc\pi t\,\cot\pi t\,dt = \int_{\pi/6}^{\pi/2} \csc u\,\cot u\left(\tfrac{1}{\pi}\,du\right) = \tfrac{1}{\pi}\big[-\csc u\big]_{\pi/6}^{\pi/2} = -\tfrac{1}{\pi}(1 - 2) = \tfrac{1}{\pi}.$$

43. $\int_{-\pi/6}^{\pi/6} \tan^3\theta\,d\theta = 0$ by Theorem 6(b), since $f(\theta) = \tan^3\theta$ is an odd function.

44. $\displaystyle\int_0^2 \frac{dx}{(2x-3)^2}$ does not exist since $\displaystyle f(x) = \frac{1}{(2x-3)^2}$ has an infinite discontinuity at $x = \frac{3}{2}$.

45. Let $u = \cos\theta$, so $du = -\sin\theta\,d\theta$. When $\theta = 0$, $u = 1$; when $\theta = \frac{\pi}{3}$, $u = \frac{1}{2}$. Thus,

$$\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta}\,d\theta = \int_1^{1/2} \frac{-du}{u^2} = \int_{1/2}^1 u^{-2}\,du = \left[-\frac{1}{u}\right]_{1/2}^1 = -1 - (-2) = 1.$$

46. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2\sin x}{1+x^6}\,dx = 0$ by Theorem 6(b), since $\displaystyle f(x) = \frac{x^2\sin x}{1+x^6}$ is an odd function.

47. Let $u = 1 + 2x$, so $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 13$, $u = 27$. Thus,

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \int_1^{27} u^{-2/3}\left(\tfrac{1}{2}\,du\right) = \left[\tfrac{1}{2}\cdot 3u^{1/3}\right]_1^{27} = \tfrac{3}{2}(3 - 1) = 3.$$

48. Let $u = \sin x$, so $du = \cos x\,dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$$\int_0^{\pi/2} \cos x \sin(\sin x)\,dx = \int_0^1 \sin u\,du = \left[-\cos u\right]_0^1 = -(\cos 1 - 1) = 1 - \cos 1.$$

49. Let $u = x - 1$, so $u + 1 = x$ and $du = dx$. When $x = 1$, $u = 0$; when $x = 2$, $u = 1$. Thus,

$$\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 (u^{3/2} + u^{1/2})\,du = \left[\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{2}{5} + \tfrac{2}{3} = \tfrac{16}{15}.$$

50. Let $u = 1 + 2x$, so $x = \frac{1}{2}(u-1)$ and $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 4$, $u = 9$. Thus,

$$\int_0^4 \frac{x\,dx}{\sqrt{1+2x}} = \int_1^9 \frac{\frac{1}{2}(u-1)}{\sqrt{u}}\frac{du}{2} = \tfrac{1}{4}\int_1^9 (u^{1/2} - u^{-1/2})\,du = \tfrac{1}{4}\left[\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9$$

$$= \tfrac{1}{4}\cdot\tfrac{2}{3}\left[u^{3/2} - 3u^{1/2}\right]_1^9 = \tfrac{1}{6}[(27-9) - (1-3)] = \tfrac{20}{6} = \tfrac{10}{3}$$

51. $\displaystyle\int_0^4 \frac{dx}{(x-2)^3}$ does not exist since $f(x) = \dfrac{1}{(x-2)^3}$ has an infinite discontinuity at $x = 2$.

52. Assume $a > 0$. Let $u = a^2 - x^2$, so $du = -2x\,dx$. When $x = 0$, $u = a^2$; when $x = a$, $u = 0$. Thus,

$$\int_0^a x\sqrt{a^2 - x^2}\,dx = \int_{a^2}^0 u^{1/2}\left(-\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^{a^2} u^{1/2}\,du = \tfrac{1}{2}\cdot\left[\tfrac{2}{3}u^{3/2}\right]_0^{a^2} = \tfrac{1}{3}a^3.$$

53. Let $u = x^2 + a^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. When $x = 0$, $u = a^2$; when $x = a$, $u = 2a^2$. Thus,

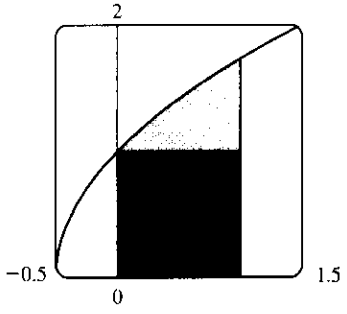
$$\int_0^a x\sqrt{x^2 + a^2}\,dx = \int_{a^2}^{2a^2} u^{1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_{a^2}^{2a^2} = \left[\tfrac{1}{3}u^{3/2}\right]_{a^2}^{2a^2}$$

$$= \tfrac{1}{3}\left[(2a^2)^{3/2} - (a^2)^{3/2}\right] = \tfrac{1}{3}\left(2\sqrt{2} - 1\right)a^3$$

54. $\int_{-a}^a x\sqrt{x^2 + a^2}\,dx = 0$ by Theorem 6(b), since $f(x) = x\sqrt{x^2 + a^2}$ is an odd function.

55. From the graph, it appears that the area under the curve is about $1 + (\text{a little more than } \frac{1}{2}\cdot 1\cdot 0.7)$, or about 1.4. The exact area is given by $A = \int_0^1 \sqrt{2x+1}\,dx$. Let $u = 2x + 1$, so $du = 2\,dx$. The limits change to $2\cdot 0 + 1 = 1$ and $2\cdot 1 + 1 = 3$, and

$$A = \int_1^3 \sqrt{u}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_1^3 = \tfrac{1}{3}\left(3\sqrt{3} - 1\right) = \sqrt{3} - \tfrac{1}{3} \approx 1.399.$$

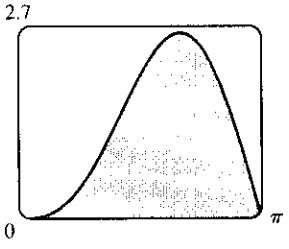


56. From the graph, it appears that the area under the curve is almost $\frac{1}{2}\cdot\pi\cdot 2.6$, or about 4. The exact area is given by

$$A = \int_0^{\pi}(2\sin x - \sin 2x)dx = -2\left[\cos x\right]_0^{\pi} - \int_0^{\pi}\sin 2x\,dx$$

$$= -2(-1-1) - 0 = 4$$

Note: $\int_0^{\pi}\sin 2x\,dx = 0$ since it is clear from the graph of $y = \sin 2x$ that $\int_{\pi/2}^{\pi}\sin 2x\,dx = -\int_0^{\pi/2}\sin 2x\,dx$.



57. First write the integral as a sum of two integrals:

$I = \int_{-2}^{2}(x+3)\sqrt{4-x^2}\,dx = I_1 + I_2 = \int_{-2}^{2} x\sqrt{4-x^2}\,dx + \int_{-2}^{2} 3\sqrt{4-x^2}\,dx$. $I_1 = 0$ by Theorem 6(b), since $f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x = -2$ to $x = 2$. We interpret I_2 as three times the area of a semicircle with radius 2, so $I = 0 + 3 \cdot \frac{1}{2}(\pi \cdot 2^2) = 6\pi$.

58. Let $u = x^2$. Then $du = 2x\,dx$ and the limits are unchanged ($0^2 = 0$ and $1^2 = 1$), so $I = \int_0^1 x\sqrt{1-x^4}\,dx = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du$. But this integral can be interpreted as the area of a quarter-circle with radius 1. So $I = \frac{1}{2} \cdot \frac{1}{4}(\pi \cdot 1^2) = \frac{1}{8}\pi$.

59. The volume of inhaled air in the lungs at time t is

$$V(t) = \int_0^t f(u)\,du = \int_0^t \tfrac{1}{2}\sin\left(\tfrac{2\pi}{5}u\right)du = \int_0^{2\pi t/5} \tfrac{1}{2}\sin v\left(\tfrac{5}{2\pi}\,dv\right) \quad [\text{substitute } v = \tfrac{2\pi}{5}u,\ dv = \tfrac{2\pi}{5}\,du]$$

$$= \tfrac{5}{4\pi}\left[-\cos v\right]_0^{2\pi t/5} = \tfrac{5}{4\pi}\left[-\cos\left(\tfrac{2\pi}{5}t\right) + 1\right] = \tfrac{5}{4\pi}\left[1 - \cos\left(\tfrac{2\pi}{5}t\right)\right] \text{ liters}$$

60. Number of calculators $= x(4) - x(2) = \int_2^4 5000\left[1 - 100(t+10)^{-2}\right]dt$

$$= 5000\left[t + 100(t+10)^{-1}\right]_2^4 = 5000\left[\left(4 + \tfrac{100}{14}\right) - \left(2 + \tfrac{100}{12}\right)\right] \approx 4048$$

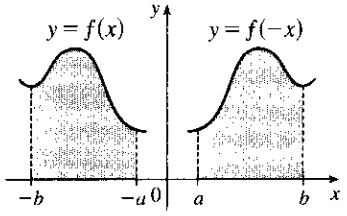
61. Let $u = 2x$. Then $du = 2\,dx$, so $\int_0^2 f(2x)\,dx = \int_0^4 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}(10) = 5$.

62. Let $u = x^2$. Then $du = 2x\,dx$, so $\int_0^3 xf\left(x^2\right)dx = \int_0^9 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}(4) = 2$.

63. (a) Let $u = -x$. Then $du = -dx$, so

$$\int_a^b f(-x)\,dx = \int_{-a}^{-b} f(u)(-du) = \int_{-b}^{-a} f(u)\,du = \int_{-b}^{-a} f(x)\,dx$$

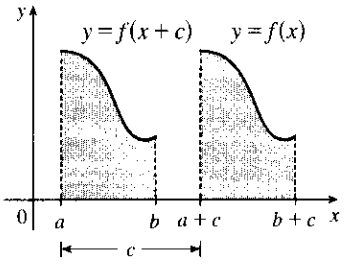
From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f, and the limits of integration, about the y-axis.



(b) Let $u = x + c$. Then $du = dx$, so

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(u)\,du = \int_{a+c}^{b+c} f(x)\,dx$$

From the diagram, we see that the equality follows from the fact that we are translating the graph of f, and the limits of integration, by a distance c.



64. The area under the graph of $y = \sin\sqrt{x}$ from 0 to 4 is $A_1 = \int_0^4 \sin\sqrt{x}\,dx$. The area under the graph of $y = 2x\sin x$ from 0 to 2 is $A_2 = \int_0^2 2x\sin x\,dx$ $[u = x^2,\ du = 2xdx,\ \sqrt{u} = x \text{ for } 0 \le x \le 2]$ $= \int_0^4 \sin\sqrt{u}\,du$. Since the integration variable is immaterial, $A_1 = A_2$.

65. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

$$\int_0^1 x^a(1-x)^b\,dx = \int_1^0 (1-u)^a u^b(-du) = \int_0^1 u^b(1-u)^a\,du = \int_0^1 x^b(1-x)^a\,dx.$$

66. Let $u = \pi - x$. Then $du = -dx$. When $x = \pi$, $u = 0$ and when $x = 0$, $u = \pi$. So

$$\int_0^\pi x f(\sin x)\,dx = -\int_\pi^0 (\pi - u) f(\sin(\pi - u))\,du = \int_0^\pi (\pi - u) f(\sin u)\,du$$
$$= \pi \int_0^\pi f(\sin u)\,du - \int_0^\pi u f(\sin u)\,du = \pi \int_0^\pi f(\sin x)\,dx - \int_0^\pi x f(\sin x)\,dx$$

$$\Rightarrow \quad 2\int_0^\pi x f(\sin x)\,dx = \pi \int_0^\pi f(\sin x)\,dx \quad \Rightarrow \quad \int_0^\pi x f(\sin x)\,dx = \tfrac{\pi}{2} \int_0^\pi f(\sin x)\,dx.$$

67. Let $u = 5 - 3x$. Then $du = -3\,dx$ and $dx = -\frac{1}{3}\,du$, so

$$\int \frac{dx}{5 - 3x} = \int \frac{1}{u}\left(-\tfrac{1}{3}\,du\right) = -\tfrac{1}{3}\ln|u| + C = -\tfrac{1}{3}\ln|5 - 3x| + C.$$

68. Let $u = x^2 + 1$. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int \frac{x}{x^2 + 1}\,dx = \int \frac{\frac{1}{2}\,du}{u} = \tfrac{1}{2}\ln|u| + C = \tfrac{1}{2}\ln\left|x^2 + 1\right| + C = \tfrac{1}{2}\ln\left(x^2 + 1\right) + C \quad [\text{since } x^2 + 1 > 0]$$

or $\ln(x^2 + 1)^{1/2} + C = \ln\sqrt{x^2 + 1} + C$.

69. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\ln x)^3 + C$.

70. Let $u = \tan^{-1} x$. Then $du = \dfrac{dx}{1 + x^2}$, so $\displaystyle\int \frac{\tan^{-1} x}{1 + x^2}\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\tan^{-1} x)^2}{2} + C$.

71. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1 + e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1 + e^x)^{3/2} + C$.

Or: Let $u = \sqrt{1 + e^x}$. Then $u^2 = 1 + e^x$ and $2u\,du = e^x\,dx$, so

$\int e^x\sqrt{1 + e^x}\,dx = \int u \cdot 2u\,du = \frac{2}{3}u^3 + C = \frac{2}{3}(1 + e^x)^{3/2} + C$.

72. Let $u = \cos t$. Then $du = -\sin t\,dt$ and $\sin t\,dt = -du$, so

$\int e^{\cos t}\sin t\,dt = \int e^u\,(-du) = -e^u + C = -e^{\cos t} + C$.

73. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$.

74. Let $u = e^x + 1$. Then $du = e^x\,dx$, so $\displaystyle\int \frac{e^x}{e^x + 1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x + 1) + C$.

75. $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$\int \cot x\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin x| + C$.

76. Let $u = \cos x$. Then $du = -\sin x\,dx$ and $\sin x\,dx = -du$, so

$$\int \frac{\sin x}{1 + \cos^2 x}\,dx = \int \frac{-du}{1 + u^2} = -\tan^{-1} u + C = -\tan^{-1}(\cos x) + C.$$

77. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so

$$\int \frac{1 + x}{1 + x^2}\,dx = \int \frac{1}{1 + x^2}\,dx + \int \frac{x}{1 + x^2}\,dx = \tan^{-1} x + \int \frac{\frac{1}{2}du}{u} = \tan^{-1} x + \tfrac{1}{2}\ln|u| + C$$
$$= \tan^{-1} x + \tfrac{1}{2}\ln\left|1 + x^2\right| + C = \tan^{-1} x + \tfrac{1}{2}\ln\left(1 + x^2\right) + C \quad [\text{since } 1 + x^2 > 0].$$

78. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{1+x^4}\,dx = \int \frac{\frac{1}{2}\,du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}(x^2) + C$.

79. Let $u = 1/x$, so $du = -1/x^2\,dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \frac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^{1/2} e^u\,(-du) = -\left[e^u\right]_1^{1/2} = -(e^{1/2} - e) = e - \sqrt{e}.$$

80. Let $u = -x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} e^u\left(-\tfrac{1}{2}\,du\right) = -\tfrac{1}{2}\left[e^u\right]_0^{-1} = -\tfrac{1}{2}\left(e^{-1} - e^0\right) = \tfrac{1}{2}(1 - 1/e).$$

81. Let $u = \ln x$, so $du = \dfrac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2 - 1) = 2.$$

82. Let $u = \sin^{-1}x$, so $du = \dfrac{dx}{\sqrt{1-x^2}}$. When $x = 0$, $u = 0$; when $x = \frac{1}{2}$, $u = \frac{\pi}{6}$. Thus,

$$\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} u\,du = \left[\frac{u^2}{2}\right]_0^{\pi/6} = \frac{\pi^2}{72}.$$

83. $\dfrac{x\sin x}{1+\cos^2 x} = x\cdot\dfrac{\sin x}{2-\sin^2 x} = xf(\sin x)$, where $f(t) = \dfrac{t}{2-t^2}$. By Exercise 66,

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $u = \cos x$. Then $du = -\sin x\,dx$. When $x = \pi$, $u = -1$ and when $x = 0$, $u = 1$. So

$$\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = -\frac{\pi}{2}\int_1^{-1} \frac{du}{1+u^2} = \frac{\pi}{2}\int_{-1}^{1} \frac{du}{1+u^2} = \frac{\pi}{2}\left[\tan^{-1}u\right]_{-1}^{1}$$

$$= \frac{\pi}{2}\left[\tan^{-1}1 - \tan^{-1}(-1)\right] = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}$$

5 Review

CONCEPT CHECK

1. (a) $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is an expression for a Riemann sum of a function f.
x_i^* is a point in the ith subinterval $[x_{i-1}, x_i]$ and Δx is the length of the subintervals.

(b) See Figure 1 in Section 5.2.

(c) In Section 5.2, see Figure 3 and the paragraph beside it.

2. (a) See Definition 5.2.2.

(b) See Figure 2 in Section 5.2.

(c) In Section 5.2, see Figure 4 and the paragraph above it.

3. See the Fundamental Theorem of Calculus after Example 8 in Section 5.3.

4. (a) See the Net Change Theorem after Example 5 in Section 5.4.

(b) $\int_{t_1}^{t_2} r(t)\,dt$ represents the change in the amount of water in the reservoir between time t_1 and time t_2.

5. (a) $\int_{60}^{120} v(t)\,dt$ represents the change in position of the particle from $t = 60$ to $t = 120$ seconds.

(b) $\int_{60}^{120} |v(t)|\,dt$ represents the total distance traveled by the particle from $t = 60$ to 120 seconds.

(c) $\int_{60}^{120} a(t)\,dt$ represents the change in the velocity of the particle from $t = 60$ to $t = 120$ seconds.

6. (a) $\int f(x)\,dx$ is the family of functions $\{F \mid F' = f\}$. Any two such functions differ by a constant.

(b) The connection is given by the Evaluation Theorem: $\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$ if f is continuous.

7. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it at the end of Section 5.3.

8. See the Substitution Rule (5.5.4). This says that it is permissible to operate with the dx after an integral sign as if it were a differential.

TRUE-FALSE QUIZ

1. True by Property 2 of the Integral in Section 5.2.

2. False. Try $a = 0$, $b = 2$, $f(x) = g(x) = 1$ as a counterexample.

3. True by Property 3 of the Integral in Section 5.2.

4. False. You can't take a variable outside the integral sign. For example, using $f(x) = 1$ on $[0, 1]$,

$$\int_0^1 x\,f(x)\,dx = \int_0^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2} \text{ (a constant) while } x\int_0^1 1\,dx = x\,[x]_0^1 = x \cdot 1 = x \text{ (a variable).}$$

5. False. For example, let $f(x) = x^2$. Then $\int_0^1 \sqrt{x^2}\,dx = \int_0^1 x\,dx = \frac{1}{2}$, but $\sqrt{\int_0^1 x^2\,dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.

6. True by the Net Change Theorem.

7. True by Comparison Property 7 of the Integral in Section 5.2.

8. False. For example, let $a = 0$, $b = 1$, $f(x) = 3$, $g(x) = x$. $f(x) > g(x)$ for each x in $(0, 1)$, but $f'(x) = 0 < 1 = g'(x)$ for $x \in (0, 1)$.

9. True. The integrand is an odd function that is continuous on $[-1, 1]$, so the result follows from Theorem 5.5.6(b).

10. True.
$$\begin{aligned}\int_{-5}^{5}(ax^2 + bx + c)\,dx &= \int_{-5}^{5}(ax^2 + c)\,dx + \int_{-5}^{5} bx\,dx \\ &= 2\int_0^5 (ax^2 + c)\,dx \text{ [by 5.5.6(a)]} \quad + 0 \text{ [by 5.5.6(b)]}\end{aligned}$$

11. False. The function $f(x) = 1/x^4$ is not bounded on the interval $[-2, 1]$. It has an infinite discontinuity at $x = 0$, so it is not integrable on the interval. (If the integral were to exist, a positive value would be expected, by Comparison Property 6 of Integrals.)

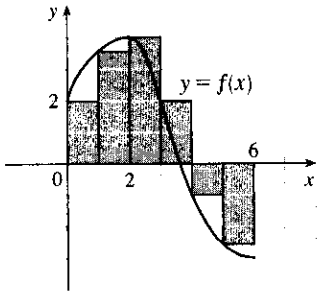
12. False. See the remarks and Figure 4 before Example 1 in Section 5.2, and notice that $y = x - x^3 < 0$ for $1 < x \le 2$.

13. False. For example, the function $y = |x|$ is continuous on $\mathbb{R}$, but has no derivative at $x = 0$.

14. True by FTC1.

EXERCISES

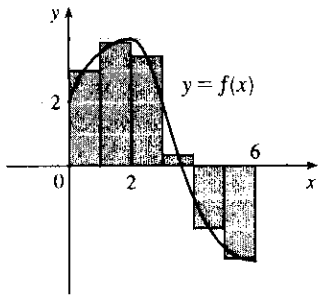
1. (a)



$$L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{6-0}{6} = 1]$$
$$= f(x_0)\cdot 1 + f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1 + f(x_5)\cdot 1$$
$$\approx 2 + 3.5 + 4 + 2 + (-1) + (-2.5) = 8$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

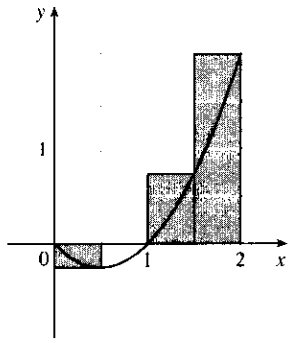
(b)



$$M_6 = \sum_{i=1}^{6} f(\bar{x}_i)\,\Delta x \quad [\Delta x = \tfrac{6-0}{6} = 1]$$
$$= f(\bar{x}_1)\cdot 1 + f(\bar{x}_2)\cdot 1 + f(\bar{x}_3)\cdot 1 + f(\bar{x}_4)\cdot 1 + f(\bar{x}_5)\cdot 1 + f(\bar{x}_6)\cdot 1$$
$$= f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)$$
$$\approx 3 + 3.9 + 3.4 + 0.3 + (-2) + (-2.9) = 5.7$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

2. (a)



$f(x) = x^2 - x$ and $\Delta x = \frac{2-0}{4} = 0.5 \Rightarrow$

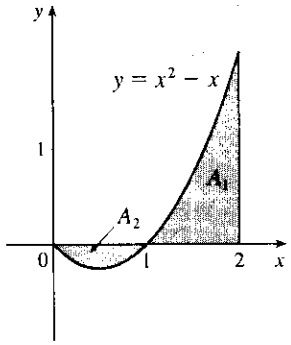
$$R_4 = 0.5f(0.5) + 0.5f(1) + 0.5f(1.5) + 0.5f(2)$$
$$= 0.5(-0.25 + 0 + 0.75 + 2) = 1.25$$

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the area of the rectangle below the x-axis. (The second rectangle vanishes.)

(b) $\int_0^2 (x^2 - x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \quad [\Delta x = 2/n \text{ and } x_i = 2i/n]$

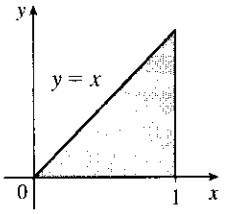
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 - \frac{2}{n}\sum_{i=1}^{n} i\right]$$
$$= \lim_{n\to\infty} \left[\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2}\cdot\frac{n(n+1)}{2}\right]$$
$$= \lim_{n\to\infty} \left[\frac{4}{3}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} - 2\cdot\frac{n+1}{n}\right]$$
$$= \lim_{n\to\infty} \left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 2\left(1+\frac{1}{n}\right)\right] = \tfrac{4}{3}\cdot 1\cdot 2 - 2\cdot 1 = \tfrac{2}{3}$$

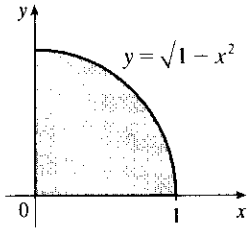
(c) $\int_0^2 (x^2 - x)\,dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_0^2 = \left(\frac{8}{3} - 2\right) = \frac{2}{3}$

(d) 

$\int_0^2 (x^2 - x)\,dx = A_1 - A_2$, where A_1 and A_2 are the areas shown in the diagram.

3. $\int_0^1 \left(x + \sqrt{1-x^2}\,\right)dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1-x^2}\,dx = I_1 + I_2$.





I_1 can be interpreted as the area of the triangle shown in the figure and I_2 can be interpreted as the area of the quarter-circle. Area $= \frac{1}{2}(1)(1) + \frac{1}{4}(\pi)(1)^2 = \frac{1}{2} + \frac{\pi}{4}$.

4. On $[0, \pi]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \sin x_i\,\Delta x = \int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = -(-1) - (-1) = 2$.

5. $\int_0^6 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx \quad \Rightarrow \quad 10 = 7 + \int_4^6 f(x)\,dx \quad \Rightarrow \quad \int_4^6 f(x)\,dx = 10 - 7 = 3$

6. (a) $\int_1^5 \left(x + 2x^5\right)dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x \qquad \left[\Delta x = \dfrac{5-1}{n} = \dfrac{4}{n},\ x_i = 1 + \dfrac{4i}{n}\right]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1 + \frac{4i}{n}\right) + 2\left(1 + \frac{4i}{n}\right)^5\right] \cdot \frac{4}{n}$$

$$= \lim_{n\to\infty} \frac{1305n^4 + 3126n^3 + 2080n^2 - 256}{n^3} \cdot \frac{4}{n} = 5220$$

(b) $\int_1^5 \left(x + 2x^5\right)dx = \left[\frac{1}{2}x^2 + \frac{2}{6}x^6\right]_1^5 = \left(\frac{25}{2} + \frac{15{,}625}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) = 12 + 5208 = 5220$

7. First note that either a or b must be the graph of $\int_0^x f(t)\,dt$, since $\int_0^0 f(t)\,dt = 0$, and $c(0) \neq 0$. Now notice that $b > 0$ when c is increasing, and that $c > 0$ when a is increasing. It follows that c is the graph of $f(x)$, b is the graph of $f'(x)$, and a is the graph of $\int_0^x f(t)\,dt$.

8. (a) By FTC2, we have $\displaystyle\int_0^{\pi/2} \frac{d}{dx}\left(\sin\frac{x}{2}\cos\frac{x}{3}\right)dx = \left[\sin\frac{x}{2}\cos\frac{x}{3}\right]_0^{\pi/2} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - 0 \cdot 1 = \frac{\sqrt{6}}{4}$.

(b) $\dfrac{d}{dx}\displaystyle\int_0^{\pi/2} \sin\frac{x}{2}\cos\frac{x}{3}\,dx = 0$, since the definite integral is a constant.

(c) $\dfrac{d}{dx}\displaystyle\int_x^{\pi/2} \sin\frac{t}{2}\cos\frac{t}{3}\,dt = \frac{d}{dx}\left(-\int_{\pi/2}^{x} \sin\frac{t}{2}\cos\frac{t}{3}\,dt\right) = -\frac{d}{dx}\int_{\pi/2}^{x} \sin\frac{t}{2}\cos\frac{t}{2}\,dt = -\sin\frac{x}{2}\cos\frac{x}{3}$, by FTC1.

9. $\int_1^2 (8x^3 + 3x^2)\,dx = \left[8 \cdot \frac{1}{4}x^4 + 3 \cdot \frac{1}{3}x^3\right]_1^2 = \left[2x^4 + x^3\right]_1^2 = (2 \cdot 2^4 + 2^3) - (2 + 1) = 40 - 3 = 37$

10. $\displaystyle\int_0^T (x^4 - 8x + 7)\,dx = \left[\tfrac{1}{5}x^5 - 4x^2 + 7x\right]_0^T = \left(\tfrac{1}{5}T^5 - 4T^2 + 7T\right) - 0 = \tfrac{1}{5}T^5 - 4T^2 + 7T$

11. $\int_0^1 (1 - x^9)\,dx = \left[x - \frac{1}{10}x^{10}\right]_0^1 = (1 - \frac{1}{10}) - 0 = \frac{9}{10}$

12. Let $u = 1 - x$, so $du = -dx$ and $dx = -du$. When $x = 0$, $u = 1$; when $x = 1$, $u = 0$. Thus,
$\int_0^1 (1 - x)^9\,dx = \int_1^0 u^9(-du) = \int_0^1 u^9\,du = \frac{1}{10}\left[u^{10}\right]_0^1 = \frac{1}{10}(1 - 0) = \frac{1}{10}$.

13. $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du = \int_1^9 (u^{-1/2} - 2u)\,du = \left[2u^{1/2} - u^2\right]_1^9 = (6 - 81) - (2 - 1) = -76$

14. $\int_0^1 (\sqrt[4]{u} + 1)^2\,du = \int_0^1 (u^{1/2} + 2u^{1/4} + 1)\,du = \left[\frac{2}{3}u^{3/2} + \frac{8}{5}u^{5/4} + u\right]_0^1 = (\frac{2}{3} + \frac{8}{5} + 1) - 0 = \frac{49}{15}$

15. Let $u = y^2 + 1$, so $du = 2y\,dy$ and $y\,dy = \frac{1}{2}\,du$. When $y = 0$, $u = 1$; when $y = 1$, $u = 2$. Thus,
$\int_0^1 y(y^2 + 1)^5\,dy = \int_1^2 u^5(\frac{1}{2}\,du) = \frac{1}{2}\left[\frac{1}{6}u^6\right]_1^2 = \frac{1}{12}(64 - 1) = \frac{63}{12} = \frac{21}{4}$.

16. Let $u = 1 + y^3$, so $du = 3y^2\,dy$ and $y^2\,dy = \frac{1}{3}\,du$. When $y = 0$, $u = 1$; when $y = 2$, $u = 9$. Thus,
$\int_0^2 y^2\sqrt{1 + y^3}\,dy = \int_1^9 u^{1/2}(\frac{1}{3}\,du) = \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{2}{9}(27 - 1) = \frac{52}{9}$.

17. $\displaystyle\int_1^5 \frac{dt}{(t-4)^2}$ does not exist because the function $f(t) = \frac{1}{(t-4)^2}$ has an infinite discontinuity at $t = 4$; that is, f is discontinuous on the interval $[1, 5]$.

18. Let $u = 3\pi t$, so $du = 3\pi\,dt$. When $t = 0$, $u = 1$; when $t = 1$, $u = 3\pi$. Thus,
$$\int_0^1 \sin(3\pi t)\,dt = \int_0^{3\pi} \sin u\left(\frac{1}{3\pi}\,du\right) = \frac{1}{3\pi}\left[-\cos u\right]_0^{3\pi} = -\frac{1}{3\pi}(-1 - 1) = \frac{2}{3\pi}.$$

19. Let $u = v^3$, so $du = 3v^2\,dv$. When $v = 0$, $u = 0$; when $v = 1$, $u = 1$. Thus,
$\int_0^1 v^2\cos(v^3)\,dv = \int_0^1 \cos u\,(\frac{1}{3}\,du) = \frac{1}{3}\left[\sin u\right]_0^1 = \frac{1}{3}(\sin 1 - 0) = \frac{1}{3}\sin 1$.

20. $\displaystyle\int_{-1}^1 \frac{\sin x}{1 + x^2}\,dx = 0$ by Theorem 5.5.6(b), since $f(x) = \frac{\sin x}{1 + x^2}$ is an odd function.

21. Let $u = x^2 + 4x$. Then $du = (2x + 4)\,dx = 2(x + 2)\,dx$, so
$$\int \frac{x + 2}{\sqrt{x^2 + 4x}}\,dx = \int u^{-1/2}(\tfrac{1}{2}\,du) = \tfrac{1}{2} \cdot 2u^{1/2} + C = \sqrt{u} + C = \sqrt{x^2 + 4x} + C.$$

22. Let $u = 3t$. Then $du = 3\,dt$, so $\int \csc^2 3t\,dt = \int \csc^2 u\,(\frac{1}{3}\,du) = \frac{1}{3}(-\cot u) + C = -\frac{1}{3}\cot 3t + C$.

23. Let $u = \sin \pi t$. Then $du = \pi\,\cos \pi t\,dt$, so $\int \sin \pi t\,\cos \pi t\,dt = \int u(\frac{1}{\pi}\,du) = \frac{1}{\pi} \cdot \frac{1}{2}u^2 + C = \frac{1}{2\pi}(\sin \pi t)^2 + C$.

24. Let $u = \cos x$. Then $du = -\sin x\,dx$, so
$\int \sin x\,\cos(\cos x)\,dx = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C$.

25. Let $u = 2\theta$. Then $du = 2\,d\theta$, so
$$\int_0^{\pi/8} \sec 2\theta \tan 2\theta\,d\theta = \int_0^{\pi/4} \sec u \tan u\,(\tfrac{1}{2}\,du) = \tfrac{1}{2}\left[\sec u\right]_0^{\pi/4} = \tfrac{1}{2}\left(\sec \tfrac{\pi}{4} - \sec 0\right)$$
$$= \tfrac{1}{2}\left(\sqrt{2} - 1\right) = \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}$$

26. Let $u = 1 + \tan t$, so $du = \sec^2 t\,dt$. When $t = 0$, $u = 1$; when $t = \frac{\pi}{4}$, $u = 2$. Thus,

$$\int_0^{\pi/4}(1+\tan t)^3 \sec^2 t\,dt = \int_1^2 u^3\,du = \left[\tfrac{1}{4}u^4\right]_1^2 = \tfrac{1}{4}\left(2^4 - 1^4\right) = \tfrac{1}{4}(16-1) = \tfrac{15}{4}.$$

27. Since $x^2 - 4 < 0$ for $0 \le x < 2$ and $x^2 - 4 > 0$ for $2 < x \le 3$, we have $\left|x^2-4\right| = -(x^2-4) = 4 - x^2$ for $0 \le x < 2$ and $\left|x^2-4\right| = x^2 - 4$ for $2 < x \le 3$. Thus,

$$\begin{aligned}\int_0^3 \left|x^2-4\right| dx &= \int_0^2 (4-x^2)\,dx + \int_2^3 (x^2-4)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^3 \\ &= \left(8 - \tfrac{8}{3}\right) - 0 + (9-12) - \left(\tfrac{8}{3} - 8\right) = \tfrac{16}{3} - 3 + \tfrac{16}{3} = \tfrac{32}{3} - \tfrac{9}{3} = \tfrac{23}{3}\end{aligned}$$

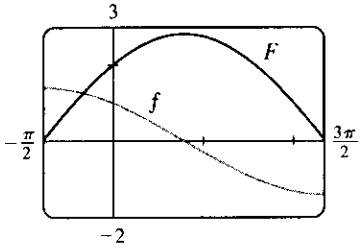
28. Since $\sqrt{x} - 1 < 0$ for $0 \le x < 1$ and $\sqrt{x} - 1 > 0$ for $1 < x \le 4$, we have $\left|\sqrt{x}-1\right| = -(\sqrt{x}-1) = 1 - \sqrt{x}$ for $0 \le x < 1$ and $\left|\sqrt{x}-1\right| = \sqrt{x} - 1$ for $1 < x \le 4$. Thus,

$$\begin{aligned}\int_0^4 \left|\sqrt{x}-1\right| dx &= \int_0^1 \left(1-\sqrt{x}\,\right) dx + \int_1^4 \left(\sqrt{x}-1\right) dx = \left[x - \tfrac{2}{3}x^{3/2}\right]_0^1 + \left[\tfrac{2}{3}x^{3/2} - x\right]_1^4 \\ &= \left(1 - \tfrac{2}{3}\right) - 0 + \left(\tfrac{16}{3} - 4\right) - \left(\tfrac{2}{3} - 1\right) = \tfrac{1}{3} + \tfrac{16}{3} - 4 + \tfrac{1}{3} = 6 - 4 = 2\end{aligned}$$

In Exercises 29 and 30, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

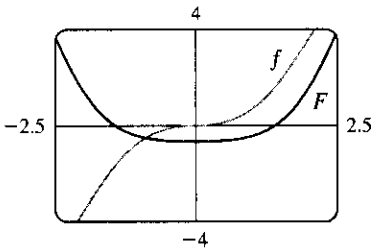
29. Let $u = 1 + \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cos x\,dx}{\sqrt{1+\sin x}} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{1+\sin x} + C.$$



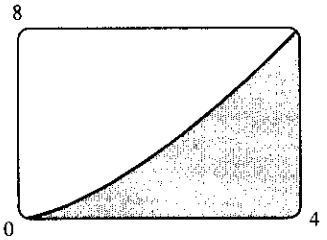
30. Let $u = x^2 + 1$. Then $x^2 = u - 1$ and $x\,dx = \frac{1}{2}\,du$, so

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+1}}\,dx &= \int \frac{(u-1)}{\sqrt{u}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int \left(u^{1/2} - u^{-1/2}\right) du \\ &= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C \\ &= \tfrac{1}{3}\left(x^2+1\right)^{3/2} - \left(x^2+1\right)^{1/2} + C \\ &= \tfrac{1}{3}\left(x^2+1\right)^{1/2}\left[\left(x^2+1\right) - 3\right] + C \\ &= \tfrac{1}{3}\sqrt{x^2+1}\left(x^2-2\right) + C\end{aligned}$$

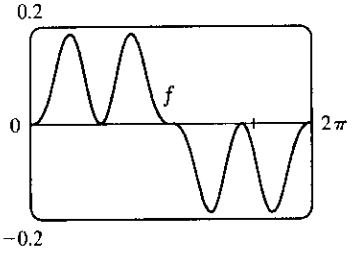


31. From the graph, it appears that the area under the curve $y = x\sqrt{x}$ between $x = 0$ and $x = 4$ is somewhat less than half the area of an 8×4 rectangle, so perhaps about 13 or 14. To find the exact value, we evaluate

$$\int_0^4 x\sqrt{x}\,dx = \int_0^4 x^{3/2}\,dx = \left[\tfrac{2}{5}x^{5/2}\right]_0^4 = \tfrac{2}{5}(4)^{5/2} = \tfrac{64}{5} = 12.8.$$



32. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x \sin^3 x\,dx$ is equal to 0.



To evaluate the integral, we write the integral as

$I = \int_0^{2\pi} \cos^2 x\,(1-\cos^2 x)\sin x\,dx$ and let $u = \cos x \;\Rightarrow$

$du = -\sin x\,dx$. Thus, $I = \int_1^1 u^2(1-u^2)(-du) = 0$.

33. By FTC1, $F(x) = \int_1^x \sqrt{1+t^4}\,dt \;\Rightarrow\; F'(x) = \sqrt{1+x^4}$.

34. $F(x) = \int_\pi^x \tan(s^2)\,ds \;\Rightarrow\; F'(x) = \tan(x^2)$

35. $g(x) = \displaystyle\int_0^{x^3} \frac{t\,dt}{\sqrt{1+t^3}}$. Let $y = g(u)$ and $u = x^3$.

Then $g'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{u}{\sqrt{1+u^3}}3x^2 = \dfrac{x^3}{\sqrt{1+x^9}}3x^2 = \dfrac{3x^5}{\sqrt{1+x^9}}$.

36. Let $u = \cos x$. Then $\dfrac{du}{dx} = -\sin x$. Also, $\dfrac{dg}{dx} = \dfrac{dg}{du}\dfrac{du}{dx}$, so

$$\frac{d}{dx}\int_1^{\cos x} \sqrt[3]{1-t^2}\,dt = \frac{d}{du}\int_1^u \sqrt[3]{1-t^2}\,dt \cdot \frac{du}{dx} = \sqrt[3]{1-u^2}\,(-\sin x)$$

$$= \sqrt[3]{1-\cos^2 x}\,(-\sin x) = -\sin x\,\sqrt[3]{\sin^2 x} = -(\sin x)^{5/3}$$

37. $y = \displaystyle\int_{\sqrt{x}}^{x} \frac{\cos\theta}{\theta}\,d\theta = \int_1^x \frac{\cos\theta}{\theta}\,d\theta + \int_{\sqrt{x}}^1 \frac{\cos\theta}{\theta}\,d\theta = \int_1^x \frac{\cos\theta}{\theta}\,d\theta - \int_1^{\sqrt{x}} \frac{\cos\theta}{\theta}\,d\theta \;\Rightarrow$

$y' = \dfrac{\cos x}{x} - \dfrac{\cos\sqrt{x}}{\sqrt{x}}\dfrac{1}{2\sqrt{x}} = \dfrac{2\cos x - \cos\sqrt{x}}{2x}$

38. $y = \int_{2x}^{3x+1} \sin(t^4)\,dt = \int_{2x}^0 \sin(t^4)\,dt + \int_0^{3x+1} \sin(t^4)\,dt = \int_0^{3x+1} \sin(t^4)\,dt - \int_0^{2x} \sin(t^4)\,dt \;\Rightarrow$

$y' = \sin\left[(3x+1)^4\right]\cdot\dfrac{d}{dx}(3x+1) - \sin\left[(2x)^4\right]\cdot\dfrac{d}{dx}(2x) = 3\sin\left[(3x+1)^4\right] - 2\sin\left[(2x)^4\right]$

39. If $1 \le x \le 3$, then $\sqrt{1^2+3} \le \sqrt{x^2+3} \le \sqrt{3^2+3} \;\Rightarrow\; 2 \le \sqrt{x^2+3} \le 2\sqrt{3}$, so

$2(3-1) \le \int_1^3 \sqrt{x^2+3}\,dx \le 2\sqrt{3}(3-1)$; that is, $4 \le \int_1^3 \sqrt{x^2+3}\,dx \le 4\sqrt{3}$.

40. If $3 \le x \le 5$, then $4 \le x+1 \le 6$ and $\dfrac{1}{6} \le \dfrac{1}{x+1} \le \dfrac{1}{4}$, so $\dfrac{1}{6}(5-3) \le \displaystyle\int_3^5 \frac{1}{x+1}\,dx \le \frac{1}{4}(5-3)$; that is,

$\dfrac{1}{3} \le \displaystyle\int_3^5 \frac{1}{x+1}\,dx \le \frac{1}{2}$.

41. $0 \le x \le 1 \;\Rightarrow\; 0 \le \cos x \le 1 \;\Rightarrow\; x^2\cos x \le x^2 \;\Rightarrow$

$\int_0^1 x^2\cos x\,dx \le \int_0^1 x^2\,dx = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}$ [Property 7].

42. On the interval $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$, x is increasing and $\sin x$ is decreasing, so $\dfrac{\sin x}{x}$ is decreasing. Therefore, the largest value of $\dfrac{\sin x}{x}$ on $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$ is $\dfrac{\sin(\pi/4)}{\pi/4} = \dfrac{\sqrt{2}/2}{\pi/4} = \dfrac{2\sqrt{2}}{\pi}$. By Property 8 with $M = \dfrac{2\sqrt{2}}{\pi}$ we get

$$\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \le \frac{2\sqrt{2}}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

43. Let $f(x) = \sqrt{1+x^3}$ on $[0,1]$. The Midpoint Rule with $n = 5$ gives

$$\begin{aligned}\int_0^1 \sqrt{1+x^3}\,dx &\approx \tfrac{1}{5}[f(0.1)+f(0.3)+f(0.5)+f(0.7)+f(0.9)]\\ &= \tfrac{1}{5}\left[\sqrt{1+(0.1)^3}+\sqrt{1+(0.3)^3}+\cdots+\sqrt{1+(0.9)^3}\right] \approx 1.110\end{aligned}$$

44. (a) displacement $= \int_0^5 (t^2-t)\,dt = \left[\frac{1}{3}t^3-\frac{1}{2}t^2\right]_0^5 = \frac{125}{3}-\frac{25}{2} = \frac{175}{6} = 29.1\overline{6}$ meters

(b) distance traveled $= \int_0^5 |t^2-t|\,dt = \int_0^5 |t(t-1)|\,dt = \int_0^1 (t-t^2)\,dt + \int_1^5 (t^2-t)\,dt$

$$\begin{aligned}&= \left[\tfrac{1}{2}t^2-\tfrac{1}{3}t^3\right]_0^1 + \left[\tfrac{1}{3}t^3-\tfrac{1}{2}t^2\right]_1^5\\ &= \tfrac{1}{2}-\tfrac{1}{3}-0+\left(\tfrac{125}{3}-\tfrac{25}{2}\right)-\left(\tfrac{1}{3}-\tfrac{1}{2}\right) = \tfrac{177}{6} = 29.5 \text{ meters}\end{aligned}$$

45. Note that $r(t) = b'(t)$, where $b(t) =$ the number of barrels of oil consumed up to time t. So, by the Net Change Theorem, $\int_0^3 r(t)\,dt = b(3)-b(0)$ represents the number of barrels of oil consumed from Jan. 1, 2000, through Jan. 1, 2003.

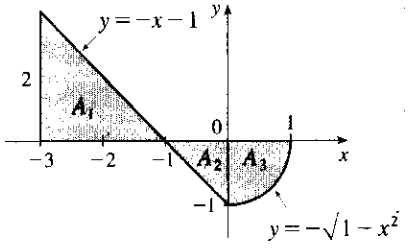
46. Distance covered $= \int_0^{5.0} v(t)\,dt \approx M_5 = \frac{5.0-0}{5}[v(0.5)+v(1.5)+v(2.5)+v(3.5)+v(4.5)]$

$$= 1(4.67+8.86+10.22+10.67+10.81) = 45.23 \text{ m}$$

47. We use the Midpoint Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$. The increase in the bee population was

$$\begin{aligned}\int_0^{24} r(t)\,dt &\approx M_6 = 4[r(2)+r(6)+r(10)+r(14)+r(18)+r(22)]\\ &\approx 4[50+1000+7000+8550+1350+150]\\ &= 4(18{,}100) = 72{,}400\end{aligned}$$

48. $A_1 = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$, $A_2 = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$, and since $y = -\sqrt{1-x^2}$ for $0 \le x \le 1$ represents a quarter-circle with radius 1, $A_3 = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$. So $\int_{-3}^1 f(x)\,dx = A_1 - A_2 - A_3 = 2-\frac{1}{2}-\frac{\pi}{4} = \frac{1}{4}(6-\pi)$.



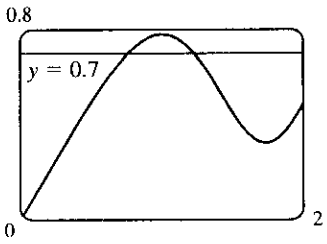
49. By the Fundamental Theorem of Calculus, we know that $F(x) = \int_a^x t^2 \sin(t^2)\,dt$ is an antiderivative of $f(x) = x^2\sin(x^2)$. This integral cannot be expressed in any simpler form. Since $\int_a^a f\,dt = 0$ for any a, we can take $a = 1$, and then $F(1) = 0$, as required. So $F(x) = \int_1^x t^2\sin(t^2)\,dt$ is the desired function.

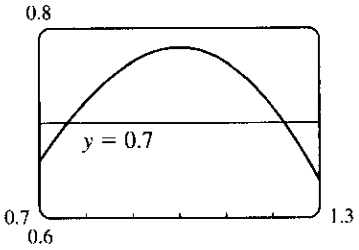
50. (a) C is increasing on those intervals where C' is positive. By the Fundamental Theorem of Calculus, $C'(x) = \dfrac{d}{dx}\left[\int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt\right] = \cos\left(\frac{\pi}{2}x^2\right)$. This is positive when $\frac{\pi}{2}x^2$ is in the interval $\left(\left(2n-\frac{1}{2}\right)\pi, \left(2n+\frac{1}{2}\right)\pi\right)$, n any integer. This implies that $\left(2n-\frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n+\frac{1}{2}\right)\pi \;\Leftrightarrow$ $0 \le |x| \le 1$ or $\sqrt{4n-1} < |x| < \sqrt{4n+1}$, n any positive integer. So C is increasing on the intervals $[-1,1]$, $\left[\sqrt{3},\sqrt{5}\right]$, $\left[-\sqrt{5},-\sqrt{3}\right]$, $\left[\sqrt{7},3\right]$, $\left[-3,-\sqrt{7}\right]$, ...

(b) C is concave upward on those intervals where $C'' > 0$. We differentiate C' to find C'': $C'(x) = \cos\left(\frac{\pi}{2}x^2\right)$ $\Rightarrow$ $C''(x) = -\sin\left(\frac{\pi}{2}x^2\right)\left(\frac{\pi}{2}\cdot 2x\right) = -\pi x\sin\left(\frac{\pi}{2}x^2\right)$. For $x > 0$, this is positive where

$(2n-1)\pi < \frac{\pi}{2}x^2 < 2n\pi$, n any positive integer $\Leftrightarrow$ $\sqrt{2(2n-1)} < x < 2\sqrt{n}$, n any positive integer.
Since there is a factor of $-x$ in C'', the intervals of upward concavity for $x < 0$ are $\left(-\sqrt{2(2n+1)}, -2\sqrt{n}\right)$, n any nonnegative integer. That is, C is concave upward on $(-\sqrt{2}, 0)$, $(\sqrt{2}, 2)$, $(-\sqrt{6}, -2)$, $(\sqrt{6}, 2\sqrt{2})$, ...

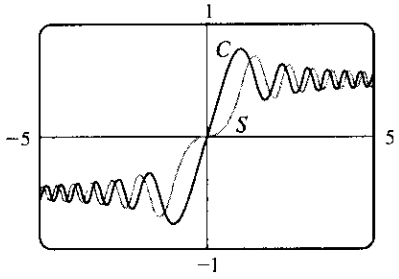
(c)





From the graphs, we can determine that $\int_0^x \cos\left(\frac{\pi}{2}t^2\right) dt = 0.7$ at $x \approx 0.76$ and $x \approx 1.22$.

(d)



The graphs of $S(x)$ and $C(x)$ have similar shapes, except that S's flattens out near the origin, while C's does not. Note that for $x > 0$, C is increasing where S is concave up, and C is decreasing where S is concave down. Similarly, S is increasing where C is concave down, and S is decreasing where C is concave up. For $x < 0$, these relationships are reversed; that is, C is increasing where S is concave down, and S is increasing where C is concave up. See Example 5.3.3 and Exercise 5.3.49 for a discussion of $S(x)$.

51. $\displaystyle\int_0^x f(t)\,dt = x\sin x + \int_0^x \frac{f(t)}{1+t^2}\,dt \;\Rightarrow\; f(x) = x\cos x + \sin x + \frac{f(x)}{1+x^2}$ (by differentiation) $\Rightarrow$

$\displaystyle f(x)\left(1 - \frac{1}{1+x^2}\right) = x\cos x + \sin x \;\Rightarrow\; f(x)\left(\frac{x^2}{1+x^2}\right) = x\cos x + \sin x \;\Rightarrow$

$\displaystyle f(x) = \frac{1+x^2}{x^2}(x\cos x + \sin x)$

52. From the given equation, $\int_a^x f(t)\,dt = \sin x - \frac{1}{2}$. Differentiating both sides using FTC1 gives $f(x) = \cos x$. We put $x = a$ into the first equation to get $0 = \sin a - \frac{1}{2}$, so $a = \frac{\pi}{6}$ satisfies the given equation.

53. Let $u = f(x)$ and $du = f'(x)\,dx$. So $2\int_a^b f(x)f'(x)\,dx = 2\int_{f(a)}^{f(b)} u\,du = \left[u^2\right]_{f(a)}^{f(b)} = [f(b)]^2 - [f(a)]^2$.

54. Let $F(x) = \displaystyle\int_2^x \sqrt{1+t^3}\,dt$. Then $F'(2) = \lim_{h\to 0}\frac{F(2+h)-F(2)}{h} = \lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt$, and

$F'(x) = \sqrt{1+x^3}$, so $\displaystyle\lim_{h\to 0}\frac{1}{h}\int_2^{2+h}\sqrt{1+t^3}\,dt = F'(2) = \sqrt{1+2^3} = \sqrt{9} = 3$.

55. Let $u = 1 - x$. Then $du = -dx$, so $\int_0^1 f(1-x)\,dx = \int_1^0 f(u)(-du) = \int_0^1 f(u)\,du = \int_0^1 f(x)\,dx$.

56. $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right] = \lim_{n\to\infty}\frac{1-0}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^9 = \int_0^1 x^9\,dx = \left[\frac{x^{10}}{10}\right]_0^1 = \frac{1}{10}$

The limit is based on Riemann sums using right endpoints and subintervals of equal length.

PROBLEMS PLUS

1. Differentiating both sides of the equation $x \sin \pi x = \int_0^{x^2} f(t)\,dt$ (using FTC1 and the Chain Rule for the right side) gives $\sin \pi x + \pi x \cos \pi x = 2x f(x^2)$. Letting $x = 2$ so that $f(x^2) = f(4)$, we obtain $\sin 2\pi + 2\pi \cos 2\pi = 4f(4)$, so $f(4) = \frac{1}{4}(0 + 2\pi \cdot 1) = \frac{\pi}{2}$.

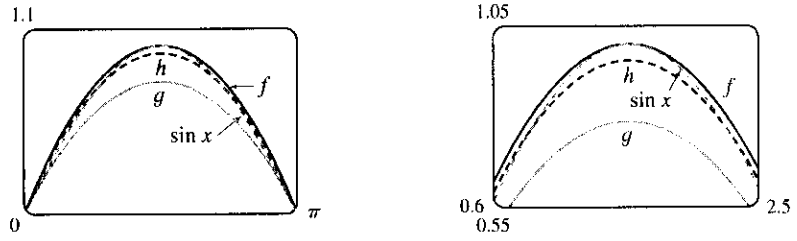
2. (a) Let $f(x) = ax^2 + bx + c$. $f(0) = f(\pi) = 0$, so we know that $f(0) = 0 \;\Rightarrow\; c = 0$, and $f(\pi) = 0 \;\Leftrightarrow\; a\pi^2 + b\pi = 0 \;\Leftrightarrow\; b = -a\pi$. So $f(x) = ax^2 - a\pi x$. Now we want the maximum value of $f(x)$ on $[0, \pi]$ to be the same as that of $\sin x$, that is, 1. So we find the value of x at which f has a maximum by differentiating and setting $f'(x) = 0 \;\Leftrightarrow\; 2ax - a\pi = 0 \;\Leftrightarrow\; x = \frac{\pi}{2}$. Now $f\left(\frac{\pi}{2}\right) = a\left(\frac{\pi}{2}\right)^2 - a\pi\left(\frac{\pi}{2}\right) = -\frac{1}{4}\pi^2 a$. We set this equal to 1, in order to find a: $-\frac{1}{4}\pi^2 a = 1 \;\Leftrightarrow\; a = -\frac{4}{\pi^2} \;\Rightarrow\; b = -a\pi = \frac{4}{\pi}$. Thus, the desired function is $f(x) = -\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x$.

Alternate solution (without calculus): Use $f(x) = ax(x - \pi)$.

(b) Once again, $g(0) = g(\pi) = 0 \;\Rightarrow\; g(x) = ax^2 - a\pi x$. We want $g'(0) = \left[\frac{d}{dx}(\sin x)\right]_0 = \cos 0 = 1$. We calculated $g'(x)$ in part (a), so we set $g'(0) = 1 \;\Leftrightarrow\; 2a(0) - a\pi = 1 \;\Leftrightarrow\; a = -\frac{1}{\pi}$. We also want $g'(\pi) = \left[\frac{d}{dx}(\sin x)\right]_\pi = \cos \pi = -1$, so we check that $g'(\pi) = -1$ with $a = -\frac{1}{\pi}$: $g'(\pi) = 2\left(-\frac{1}{\pi}\right)\pi - \left(-\frac{1}{\pi}\right)\pi = -1$. Thus, the desired function is $g(x) = -\frac{1}{\pi}x^2 + x$.

(c) Again, $h(x) = ax^2 - a\pi x$. Now we want the area under the curves of $h(x)$ and $\sin x$ to be the same; that is, $\int_0^\pi h(x)\,dx = \int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = (1) - (-1) = 2$. We integrate h between 0 and π and set the result equal to 2: $\int_0^\pi \left(ax^2 - a\pi x\right)dx = \left[\frac{1}{3}ax^3 - \frac{1}{2}a\pi x^2\right]_0^\pi = \frac{1}{3}a\pi^3 - \frac{1}{2}a\pi^3 = -\frac{1}{6}a\pi^3 = 2 \;\Leftrightarrow\; a = -12/\pi^3$. Thus, the desired function is $h(x) = -\dfrac{12}{\pi^3}x^2 + \dfrac{12}{\pi^2}x$.

(d)



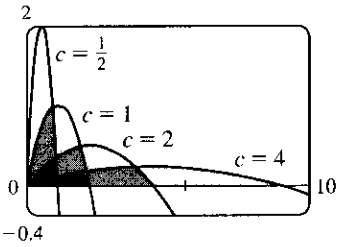
3. For $1 \le x \le 2$, we have $x^4 \le 2^4 = 16$, so $1 + x^4 \le 17$ and $\dfrac{1}{1+x^4} \ge \dfrac{1}{17}$. Thus, $\displaystyle\int_1^2 \frac{1}{1+x^4}\,dx \ge \int_1^2 \frac{1}{17}\,dx = \frac{1}{17}$. Also $1 + x^4 > x^4$ for $1 \le x \le 2$, so $\dfrac{1}{1+x^4} < \dfrac{1}{x^4}$ and $\displaystyle\int_1^2 \frac{1}{1+x^4}\,dx < \int_1^2 x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_1^2 = -\frac{1}{24} + \frac{1}{3} = \frac{7}{24}$. Thus, we have the estimate $\dfrac{1}{17} \le \displaystyle\int_1^2 \frac{1}{1+x^4}\,dx \le \frac{7}{24}$.

4. By FTC2, $\int_0^1 f'(x)\,dx = f(1) - f(0) = 1 - 0 = 1$.

5. Such a function cannot exist. $f'(x) > 3$ for all x means that f is differentiable (and hence continuous) for all x. So by FTC2, $\int_1^4 f'(x)\,dx = f(4) - f(1) = 7 - (-1) = 8$. However, if $f'(x) > 3$ for all x, then $\int_1^4 f'(x)\,dx \geq 3 \cdot (4-1) = 9$ by Comparison Property 8 in Section 5.2.

Another solution: By the Mean Value Theorem, there exists a number $c \in (1, 4)$ such that $f'(c) = \dfrac{f(4) - f(1)}{4-1} = \dfrac{7-(-1)}{3} = \dfrac{8}{3} \quad \Rightarrow \quad 8 = 3f'(c)$. But $f'(x) > 3 \quad \Rightarrow \quad 3f'(c) > 9$, so such a function cannot exist.

6. (a)

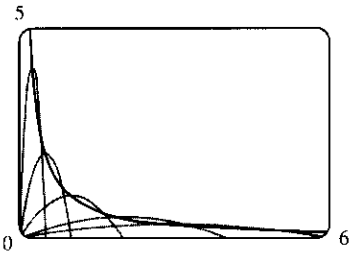


From the graph of $f(x) = \dfrac{2cx - x^2}{c^3}$, it appears that the areas are equal; that is, the area enclosed is independent of c.

(b) We first find the x-intercepts of the curve, to determine the limits of integration: $y = 0 \quad \Leftrightarrow \quad 2cx - x^2 = 0 \quad \Leftrightarrow \quad x = 0$ or $x = 2c$. Now we integrate the function between these limits to find the enclosed area:

$$A = \int_0^{2c} \frac{2cx - x^2}{c^3}\,dx = \frac{1}{c^3}\left[cx^2 - \tfrac{1}{3}x^3\right]_0^{2c} = \frac{1}{c^3}\left[c(2c)^2 - \tfrac{1}{3}(2c)^3\right] = \frac{1}{c^3}\left[4c^3 - \tfrac{8}{3}c^3\right] = \tfrac{4}{3},$$ a constant.

(c)



The vertices of the family of parabolas seem to determine a branch of a hyperbola.

(d) For a particular c, the vertex is the point where the maximum occurs. We have seen that the x-intercepts are 0 and $2c$, so by symmetry, the maximum occurs at $x = c$, and its value is $\dfrac{2c(c) - c^2}{c^3} = \dfrac{1}{c}$. So we are interested in the curve consisting of all points of the form $\left(c, \dfrac{1}{c}\right)$, $c > 0$. This is the part of the hyperbola $y = 1/x$ lying in the first quadrant.

7. $f(x) = \displaystyle\int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}\,dt$, where $g(x) = \int_0^{\cos x} \left[1 + \sin(t^2)\right] dt$. Using FTC1 and the Chain Rule (twice) we have $f'(x) = \dfrac{1}{\sqrt{1+[g(x)]^3}}\,g'(x) = \dfrac{1}{\sqrt{1+[g(x)]^3}}\left[1 + \sin(\cos^2 x)\right](-\sin x)$. Now $g\left(\frac{\pi}{2}\right) = \int_0^0 \left[1 + \sin(t^2)\right] dt = 0$, so $f'\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{1+0}}(1 + \sin 0)(-1) = 1 \cdot 1 \cdot (-1) = -1$.

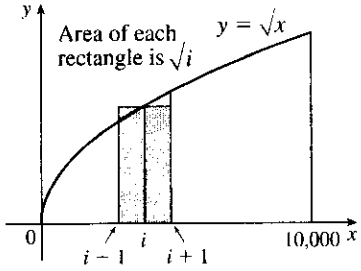
8. If $f(x) = \int_0^x x^2 \sin(t^2)\,dt = x^2 \int_0^x \sin(t^2)\,dt$, then $f'(x) = x^2 \sin(x^2) + 2x \int_0^x \sin(t^2)\,dt$, by the Product Rule and FTC1.

9. $f(x) = 2 + x - x^2 = (-x+2)(x+1) = 0 \quad \Leftrightarrow \quad x = 2$ or $x = -1$. $f(x) \geq 0$ for $x \in [-1, 2]$ and $f(x) < 0$ everywhere else. The integral $\int_a^b (2 + x - x^2)\,dx$ has a maximum on the interval where the integrand is positive, which is $[-1, 2]$. So $a = -1$, $b = 2$. (Any larger interval gives a smaller integral since $f(x) < 0$ outside $[-1, 2]$. Any smaller interval also gives a smaller integral since $f(x) \geq 0$ in $[-1, 2]$.)

10. This sum can be interpreted as a Riemann sum, with the right endpoints of the subintervals as sample points and with $a = 0$, $b = 10{,}000$, and $f(x) = \sqrt{x}$. So we approximate

$$\sum_{i=1}^{10{,}000} \sqrt{i} \approx \lim_{n\to\infty} \frac{10{,}000}{n} \sum_{i=1}^{n} \sqrt{\frac{10{,}000i}{n}} = \int_0^{10{,}000} \sqrt{x}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^{10{,}000} = \tfrac{2}{3}\,(1{,}000{,}000) \approx 666{,}667.$$

Alternate method: We can use graphical methods as follows:



From the figure we see that $\int_{i-1}^{i} \sqrt{x}\,dx < \sqrt{i} < \int_{i}^{i+1} \sqrt{x}\,dx$, so $\int_0^{10{,}000} \sqrt{x}\,dx < \sum_{i=1}^{10{,}000} \sqrt{i} < \int_1^{10{,}001} \sqrt{x}\,dx$. Since $\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$, we get $\int_0^{10{,}000} \sqrt{x}\,dx = 666{,}666.\overline{6}$ and $\int_1^{10{,}001} \sqrt{x}\,dx = \frac{2}{3}\left[(10{,}001)^{3/2} - 1\right] \approx 666{,}766$.

Hence, $666{,}666.\overline{6} < \sum_{i=1}^{10{,}000} \sqrt{i} < 666{,}766$. We can estimate the sum by averaging these bounds:

$\sum_{i=1}^{10{,}000} \approx \frac{666{,}666.\overline{6} + 666{,}766}{2} \approx 666{,}716$. The actual value is about 666,716.46.

11. (a) We can split the integral $\int_0^n [\![x]\!]\,dx$ into the sum $\sum_{i=1}^{n} \left[\int_{i-1}^{i} [\![x]\!]\,dx\right]$. But on each of the intervals $[i-1, i)$ of integration, $[\![x]\!]$ is a constant function, namely $i - 1$. So the ith integral in the sum is equal to $(i-1)[i-(i-1)] = (i-1)$. So the original integral is equal to $\sum_{i=1}^{n} (i-1) = \sum_{i=1}^{n-1} i = \dfrac{(n-1)n}{2}$.

(b) We can write $\int_a^b [\![x]\!]\,dx = \int_0^b [\![x]\!]\,dx - \int_0^a [\![x]\!]\,dx$.

Now $\int_0^b [\![x]\!]\,dx = \int_0^{[\![b]\!]} [\![x]\!]\,dx + \int_{[\![b]\!]}^b [\![x]\!]\,dx$. The first of these integrals is equal to $\frac{1}{2}([\![b]\!] - 1)\,[\![b]\!]$, by part (a), and since $[\![x]\!] = [\![b]\!]$ on $[[\![b]\!], b]$, the second integral is just $[\![b]\!]\,(b - [\![b]\!])$. So

$\int_0^b [\![x]\!]\,dx = \frac{1}{2}([\![b]\!] - 1)\,[\![b]\!] + [\![b]\!]\,(b - [\![b]\!]) = \frac{1}{2}\,[\![b]\!]\,(2b - [\![b]\!] - 1)$ and similarly

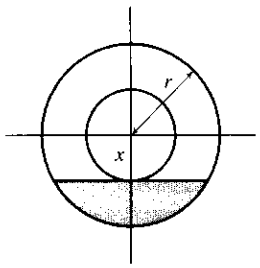
$\int_0^a [\![x]\!]\,dx = \frac{1}{2}\,[\![a]\!]\,(2a - [\![a]\!] - 1)$. Therefore, $\int_a^b [\![x]\!]\,dx = \frac{1}{2}\,[\![b]\!]\,(2b - [\![b]\!] - 1) - \frac{1}{2}\,[\![a]\!]\,(2a - [\![a]\!] - 1)$.

12. By FTC1, $\dfrac{d}{dx} \int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4}\,du\right) dt = \int_1^{\sin x} \sqrt{1+u^4}\,du$. Again using FTC1,

$$\frac{d^2}{dx^2} \int_0^x \left(\int_1^{\sin t} \sqrt{1+u^4}\,du\right) dt = \frac{d}{dx} \int_1^{\sin x} \sqrt{1+u^4}\,du = \sqrt{1+\sin^4 x}\,\cos x.$$

13. Differentiating the equation $\int_0^x f(t)\,dt = [f(x)]^2$ using FTC1 gives $f(x) = 2f(x)f'(x) \;\Rightarrow$ $f(x)[2f'(x) - 1] = 0$, so $f(x) = 0$ or $f'(x) = \frac{1}{2}$. $f'(x) = \frac{1}{2} \;\Rightarrow\; f(x) = \frac{1}{2}x + C$. To find C we substitute into the original equation to get $\int_0^x \left(\frac{1}{2}t + C\right) dt = \left(\frac{1}{2}x + C\right)^2 \;\Leftrightarrow\; \frac{1}{4}x^2 + Cx = \frac{1}{4}x^2 + Cx + C^2$. It follows that $C = 0$, so $f(x) = \frac{1}{2}x$. Therefore, $f(x) = 0$ or $f(x) = \frac{1}{2}x$.

14.



Let x be the distance between the center of the disk and the surface of the liquid. The wetted circular region has area $\pi r^2 - \pi x^2$ while the unexposed wetted region (shaded in the diagram) has area $2\int_x^r \sqrt{r^2 - t^2}\,dt$, so the exposed wetted region has area $A(x) = \pi r^2 - \pi x^2 - 2\int_x^r \sqrt{r^2 - t^2}\,dt$, $0 \le x \le r$. By FTC1, we have $A'(x) = -2\pi x + 2\sqrt{r^2 - x^2}$.

Now $A'(x) > 0 \;\Rightarrow\; -2\pi x + 2\sqrt{r^2 - x^2} > 0 \;\Rightarrow\; \sqrt{r^2 - x^2} > \pi x \;\Rightarrow\; r^2 - x^2 > \pi^2 x^2 \;\Rightarrow$ $r^2 > \pi^2 x^2 + x^2 \;\Rightarrow\; r^2 > x^2(\pi^2 + 1) \;\Rightarrow\; x^2 < \dfrac{r^2}{\pi^2 + 1} \;\Rightarrow\; x < \dfrac{r}{\sqrt{\pi^2 + 1}}$, and we'll call this value x^*.

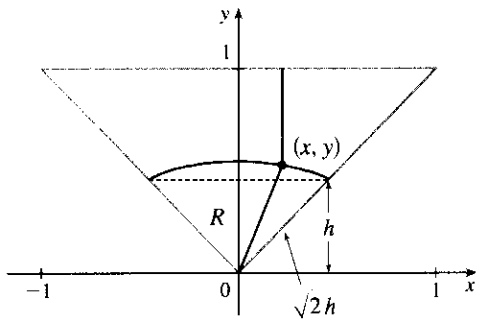
Since $A'(x) > 0$ for $0 < x < x^*$ and $A'(x) < 0$ for $x^* < x < r$, we have an absolute maximum when $x = x^*$.

15. Note that $\dfrac{d}{dx}\left(\int_0^x \left[\int_0^u f(t)\,dt\right] du\right) = \int_0^x f(t)\,dt$ by FTC1, while

$$\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] = \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\left[\int_0^x f(u)u\,du\right]$$
$$= \int_0^x f(u)\,du + xf(x) - f(x)x = \int_0^x f(u)\,du$$

Hence, $\int_0^x f(u)(x-u)\,du = \int_0^x \left[\int_0^u f(t)\,dt\right] du + C$. Setting $x = 0$ gives $C = 0$.

16.



We restrict our attention to the triangle shown. A point in this triangle is closer to the side shown than to any other side, so if we find the area of the region R consisting of all points in the triangle that are closer to the center than to that side, we can multiply this area by 4 to find the total area. We find the equation of the set of points which are equidistant from the center and the side: the distance of the point (x, y) from the side is $1 - y$, and its distance from the center is $\sqrt{x^2 + y^2}$.

So the distances are equal if $\sqrt{x^2 + y^2} = 1 - y \;\Leftrightarrow\; x^2 + y^2 = 1 - 2y + y^2 \;\Leftrightarrow\; y = \frac{1}{2}(1 - x^2)$. Note that the area we are interested in is equal to the area of a triangle plus a crescent-shaped area. To find these areas, we

have to find the y-coordinate h of the horizontal line separating them. From the diagram, $1-h=\sqrt{2}h \quad \Leftrightarrow$ $h=\frac{1}{1+\sqrt{2}}=\sqrt{2}-1$. We calculate the areas in terms of h, and substitute afterward.

The area of the triangle is $\frac{1}{2}(2h)(h)=h^2$, and the area of the crescent-shaped section is $\int_{-h}^{h}\left[\frac{1}{2}\left(1-x^2\right)-h\right]dx=2\int_0^h\left(\frac{1}{2}-h-\frac{1}{2}x^2\right)dx=2\left[\left(\frac{1}{2}-h\right)x-\frac{1}{6}x^3\right]_0^h=h-2h^2-\frac{1}{3}h^3$. So the area of the whole region is

$$4\left[\left(h-2h^2-\tfrac{1}{3}h^3\right)+h^2\right]=4h\left(1-h-\tfrac{1}{3}h^2\right)=4\left(\sqrt{2}-1\right)\left[1-\left(\sqrt{2}-1\right)-\tfrac{1}{3}\left(\sqrt{2}-1\right)^2\right]$$

$$=4\left(\sqrt{2}-1\right)\left(1-\tfrac{1}{3}\sqrt{2}\right)=\tfrac{4}{3}\left(4\sqrt{2}-5\right)$$

17.
$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}}+\frac{1}{\sqrt{n}\sqrt{n+2}}+\cdots+\frac{1}{\sqrt{n}\sqrt{n+n}}\right)\\
&=\lim_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{n}{n+1}}+\sqrt{\frac{n}{n+2}}+\cdots+\sqrt{\frac{n}{n+n}}\right)\\
&=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}}+\frac{1}{\sqrt{1+2/n}}+\cdots+\frac{1}{\sqrt{1+1}}\right)\\
&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f\left(\frac{i}{n}\right)\qquad\left[\text{where } f(x)=\frac{1}{\sqrt{1+x}}\right]\\
&=\int_0^1\frac{1}{\sqrt{1+x}}\,dx=\left[2\sqrt{1+x}\,\right]_0^1=2\left(\sqrt{2}-1\right)
\end{aligned}$$